Localization

Local Quantum Measurement and Relativity

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Dedicated to Luca

Abstract

This work treats various aspects of the quantum theory of measurement, partially in a relativistic framework. As a basis, measurement (like) processes are defined and carefully analysed within the framework of non relativistic quantum mechanics, without postulating operators as observables. Thereby, the quantum measurement formalism is derived in full generality – including the representation of quantum probabilities by *positive operator valued measures* (POVM's) or the *Kraus representation* of the related state transformations. This approach and the worked out tools will then be used to formulate and analyse certain problems concerning local measurements in relativistic quantum theory.

First, without fixing a particular relativistic quantum theory, the compatibility of the probabilities of quantum measurement formalism for composite, spacelike separated quantum measurements with the (specially) relativistic structure of space-time is examined. In particular, the relations between (i) Lorentz frame independence of outcomes of such measurements (*relativistic consistency*), (ii) the impossibility to send signals faster than light by means of quantum measurement (*no signalling*) and (iii) mathematical commutativity properties of certain operators associated with spacelike separated quantum measurements (*local commutativity*), are analysed and several results revealing these connections are presented.

Next, the so called *localization problem* of relativistic quantum theory is studied. For this purpose, a mathematical framework is developed which entails a variety of strong results relevant for local quantum measurements. Besides encountering some special properties of positive energy wave functions in any relativistic quantum theory (e.g. their strong nonlocal sensitivity with respect to local perturbations) this includes a coherent and comprehensive account of several celebrated theorems such as *Hegerfeldt's theorem*, *Malament's theorem* and generalizations thereof or the *Reeh-Schlieder theorem* and its implications. These assertions are formulated in purely operational terms, e.g. in terms of detector click statistics, to highlight their operational meaning (in contrast to a sometimes alleged ontological meaning). Initially counterintuitive features of these theorems are considered, in particular that under their more or less mild assumptions, detector click probabilities do generically never perfectly vanish (e.g. a positive detector response cannot be excluded even if the initial state was the vacuum, as well as joint detector clicks of remote detectors at spacelike separation even if the initial state belongs to a single particle). A natural understanding is provided by reconsidering the role of the positive energy assumption (*spectrum condition*) with respect to infinite wave function tails, locally caused transformations of wave functions and pair creation phenomena in quantum field theory.

Finally with *Bohmian mechanics* a theory about real particles moving in space is presented and from a dynamical and statistical analysis of Bohmian subsystems, the operational quantum measurement formalism is derived for measurement (like) situations. Relativistic versions of Bohmian mechanics incorporating Lorentz invariance and generalizations to Bohmian quantum field theory are reviewed and the localization problem is discussed from a Bohmian perspective.

Zusammenfassung

Die Arbeit behandelt verschiedene Aspekte der Quantentheorie des Messprozesses, zum Teil in relativistischen Zusammenhängen. Als Grundlage werden Mess(-artige-)prozesse definiert und ausführlich untersucht, wobei darauf verzichtet wird, Operatoren als Observablen zu postulieren. Der Quanten-Messformalismus wird dabei in seiner allgemeinsten Form abgeleitet, inklusive der Kodierung der entsprechenden Wahrscheinlichkeiten durch *positiv-operatorwertige Maße (POVMs)* und der *Kraus Darstellung* der entsprechenden Zustandstransformationen. Dieser Zugang und die erarbeiteten Werkzeuge finden im Folgenden Verwendung, um bestimmte Probleme der Beschreibung lokaler Messungen in der relativistischen Quantenmechanik zu formulieren und zu analysieren.

Ohne eine konkrete relativistische Quantentheorie zu spezifizieren, wird zunächst die Vereinbarkeit des Quanten-Messformalismus zur Beschreibung raumartig getrennter Messungen mit der (speziell) relativistischen Raumzeit-Struktur untersucht. Im Besonderen werden Zusammenhänge zwischen (i) der Unabhängigkeit von Messresultaten vom Lorentz-Bezugssystem (*relativistische Konsistenz*), (ii) der Unmöglichkeit, überlichtschnelle Signale zu senden (*no signalling*), und (iii) mathematischen Kommutativitätseigenschaften bestimmter Operatoren, die raumartig getrennten Messungen zugeordnet sind (*lokale Kommutativität*), analysiert und einige Resultate, die diese Beziehungen aufzeigen, werden dargelegt.

Als nächstes wird das *Lokalisierungsproblem* der relativistischen Quantenmechanik untersucht. Zu diesem Zwecke wird zunächst ein mathematischer Rahmen entwickelt, aus dem sich eine Reihe starker Resultate für die Beschreibung lokaler Messungen ergeben. Neben einigen sehr speziellen Eigenschaften von Wellenfunktionen positiver Energie in jeder relativistischen Quantentheorie (z.B. ihre starke nichtlokale Sensitivität bezüglich lokaler Störungen) beinhaltet dies eine einheitliche und umfassende Darstellung einiger berühmt gewordener Theoreme, wie *Hegerfelds Theorem*, *Malaments Theorem* und seiner Verallgemeinerungen oder dem *Reeh-Schlieder Theorem* und seiner Implikationen. Diese Resultate werden rein operationalistisch formuliert, z.B. bezogen auf die Statistik von Detektor-Klicks, um ihre operationalistische Bedeutung darzulegen (im Gegensatz zu einer manchmal behaupteten ontologischen Bedeutung). Zunächst kontraintuitive Aspekte dieser Theoreme werden genauer betrachtet, im Besonderen die Tatsache, dass unter ihren mehr oder weniger weichen Annahmen Detektor-Klick Wahrscheinlichkeiten grundsätzlich nie perfekt verschwinden können (z.B. ein positiver Ausgang eines Detektorexperiments kann nicht ausgeschlossen werden, selbst wenn der Anfangszustand das Vakuum ist, ebenso wie raumartig getrennte Klicks entfernter Detektoren, selbst wenn der Anfangszustand ein einzelnes Teilchen beschreibt). Diese Vorhersagen lassen sich in natürlicher Weise verstehen, wenn die grundlegende Annahme positiver Energie (*Spektrumsbedingung*) bezüglich ihrer Rolle für unendliche 'Schwänze' (tails) entsprechender Wellenfunktionen, lokal verursachter Transformationen von Wellenfunktionen und Paarerzeugungseffekten in der Quantenfeldtheorie in die Betrachtungen einbezogen wird.

Letztlich wird mit *Bohmscher Mechanik* eine Theorie über reelle Teilchen, die sich im Raum bewegen, dargestellt und aus einer dynamischen und statistischen Analyse Bohmscher Subsysteme der operationalistische Quanten-Messformalismus für Mess(-artige) Situationen abgeleitet. Relativistische Versionen der Bohmschen Mechanik bezüglich Lorentzinvarianz und Verallgemeinerungen zu Bohmscher Quantenfeldtheorie werden besprochen und das Lokalisierungsproblem aus Bohmscher Sicht diskutiert.

Contents

Outlook

This work treats various aspects of the quantum theory of measurement, chapters 2 and 3 in particular in a relativistic framework.

The first chapter shall provide an unconventional but hopefully lucid approach to the general (in the first place non relativistic) quantum measurement formalism[1]. The chapter shall be both, an end in itself giving a special self-contained account of the well established quantum theory of measurement, and a solid and specially tailored foundation for the analysis of the rest of this work. This approach towards the quantum theory of measurement deviates from many presentations of this subject with respect to (one or some of) the following issues:

- The measurement problem is taken seriously and no efforts are made to circumvent it or to hide it in the abstract formalism. It is rather taken as a basis to develop the latter and will be frequently touched and indicated during the analysis.
- Operators are not postulated as observables but the quantum operator formalism is deduced from the other postulates and found to be a practical mathematical toolbox, condensing the statistics and state transitions associated with certain quantum processes which lead in particular to macroscopically well defined outcomes.
- It is not presupposed that all selfadjoint operators acting on a considered Hilbert space correspond to real world measurement (like) processes (and in particular not that operators associated with such processes must form an algebra).

These points will be a golden thread throughout this work. Each of them will find both, justification and helpful applications.

After setting the stage for the present approach and introducing basic notions (like the important notion of *pointer states*) in section 1.1, *measurement (like) processes* will be formally defined in section 1.2. From this definition, basic operators of the quantum theory of measurement are immediately inferred, most importantly linear operators, which transform initial states into non normalized final states and which we shall call *state transformers*. These operators then give rise to *positive operator valued measures (POVMs)*, encoding the statistics of these state transitions arising from the Born rule. In the special case of projective measurements, the associated POVM (which is then a *projection valued measure (PVM)*) can be compactly encoded in a single selfadjoint operator which we call *observable operator*, which corresponds to the usual *observables* of textbooks. The reader who is more interested in later results of this work may read until this section 1.2 to learn the basic notions and formalism used and then directly proceed with the desired sections.

In section 1.3, the conceptual question whether quantum measurement can reveal always preexisting properties of the measured system is examined and answered in the negative by a version of the *Kochen-Specker theorem*, based on a Gedankenexperiment considering a composition of three spin measurements (essentially the GHZ experiment). If the measurements are

[1] This approach is very much influenced by the work of Bell [30], Ludwig [230], Busch et al. [69, 72, 73, 76] and Dürr et al. [126, 127, 129] who worked out many of the basic ideas presented here.

performed at spacelike separation, the very same argument yields a version of the famous *Bell theorem*, stating that the empirically well verified quantum predictions are irreconcilable with any attempt to explain nature by only local direct causes (section 1.3.2). Besides nonlocality of nature, the crucial insight of these arguments is that the wave function dynamics – in particular the reduction of the wave function (collapse) in measurement (like) processes – cannot be interpreted as describing only our limited information (ignorance) or the like, but has to be taken dynamically seriously.

In section 1.4, we shall go back to the general definition of measurement (like) process in section 1.2 and differentiate and technically analyse different kinds of such processes. Important distinctions will be *projective / non-projective* and *reproducible / non-reproducible* (reproducibility refers here to the outcome value upon immediate repetition). The simplest class are projective reproducible measurements, of which *ideal measurements* (the kind of quantum measurements usually presented in textbooks) are a distinguished representative. Generic implementations of non-projective measurements, given by indirect– and approximate measurement schemes, will be developed and discussed.

After discussing more concrete implementations of quantum measurements (the *von Neumann measurement scheme* etc.) in section 1.5, we shall develop the modern operational formulation from the viewpoint of the present approach in section 1.6. This formulation includes *trace preserving and trace reducing completely positive maps (CPMs)* acting on density operators, *quantum channels and instruments*, *purification of mixed states*, *first and second Kraus representations of CPMs*, *Stinespring representation of quantum channels* and *Naimark representation of POVMs*.

Chapter 2 deals with consistency and causality issues in view of quantum nonlocality and relativistic (here Minkowski) structure of space-time. Section 2.1 introduces to the question, what the lack of an absolute time order of spacelike separated events in (special) relativity means for spacelike separated quantum measurements on entangled systems. In section 2.2 these considerations are condensed to a mathematical requirement on the state transformers associated with spacelike separated quantum measurements, whose violation would have inconsistent realities from the viewpoint of different Lorentz frames of reference as a consequence: Displays of measurement results (like 'pointer points onto X') would in general disagree from the viwepoints of different frames. Accordingly, we shall call this requirement *relativistic consistency*. This criterion has not yet been considered as a relativistic requirement as far as I know (in my diploma thesis [24], the notion of relativistic consistency is already proposed in a less worked out version).

A straightforward implication of relativistic consistency conditions are *no signalling* conditions, which are more commonly considered as a relativistic requirement on quantum measurements. The naming 'no signalling' derives from the fact that violation of these conditions would allow for utilizing quantum nonlocality to send signals faster than light by human decisions like performing a given measurement or not. We shall identify relativistic consistency as the more fundamental requirement, since firstly, it has no signalling as a consequence but not vice versa (as will be shown) and secondly, as a fundamental physical requirement it does not rely on anthropocentric concepts like the free will of an experimenter.

The most established mathematical relativistic requirement on operators associated with spacelike separated quantum measurements is *local commutativity* (sometimes also referred to

as *microcausality* or just *causality* or *locality*). If the demand of local commutativity is physically justified in the literature, this is usually done by invoking no signalling arguments. Local commutativity conditions are defined in section 2, which indeed generally imply relativistic consistency and no signalling. In order to require local commutativity as a relativistic physical necessity, at least one of these two physically motivated requirements should in turn imply local commutativity (the implication no signalling $\Rightarrow$ local commutativity would even imply equivalence of all three notions). It will be shown in section 2 that equivalence of relativistic consistency, no signalling and local commutativity is satisfied for wide classes of measurements, like projective measurements (Lüders theorem), but it is not easy to see whether these implications hold or do not hold in general. Partial formal counterexamples can be given (see section 2.3.3 and 2.5), which are though not persuasive as yet.

Chapter 3 treats another basic problem concerning local quantum measurements in a relativistic framework, often referred to as *localization problem*. Section 3.1 introduces to the history of the localization problem, in particular the *Newton-Wigner localization scheme*, *Hegerfeldt's theorem*, *Malament type theorems*[2] and the *Reeh-Schlieder theorem* are qualitatively discussed. As an orientation, the central insights of chapter 3 are also already sketched in this introductory section 3.1.

Section 3.2 provides the mathematical basis for a comprehensive derivation of various results in the following sections. In particular, from the central result of this section (theorem 3.10, respectively its generalizations theorem 3.11 and corollary 3.13) we shall derive Hegerfeldt's theorem, several Malament type theorems and the Reeh-Schlieder theorem (a comparably comprehensive and unified exposition of all these results is not yet known to me). Derivations in section 3.2 are based on mathematical arguments which are close to the usual analyticity arguments as they are very common in the mathematical physics literature of quantum field theory, at the same time this section provides a unique and very elementary approach.

A basic version of Hegerfeld's theorem on the wave function level (without invoking operators) is loosely developed in section 3.3 (the usual operational Hegerfeldt theorem is derived and discussed in appendix B), stating that whenever the Hamiltonian of a (first quantized) quantum theory is bounded from below (in the present considerations the lower bound is usually taken to be zero), causally propagating wave functions cannot be compactly supported in configuration space but must have infinite tails[3]. Moreover, *positive energy localization schemes* are considered in section 3.3, like *Newton-Wigner localization*, *Philips localization* and the *localization scheme of Bracken and Melloy*, showing that positive energy wave functions – although they cannot be perfectly localized (compactly supported) in bounded regions of configurations space – can be nonetheless extremely well localized, such that we can treat them as perfectly localized for all practical purposes. Finally, it is argued in this section that transformations of positive energy wave functions preserving the positive energy property are extremely special, since under the

[2]The theorem which is known as *Malament theorem* (which actually goes back to Schlieder and Jancewicz) covers only the special case of projective measurements. In section 3.4, several generalizations of this theorem are presented, which we collect under the label *Malament type theorems*.

[3]This is a very general and model independent assertion, which holds e.g. for non relativistic Schrödinger theory (where wave functions can be compactly supported but immediately develop infinite tails under the time evolution generated by the positive free Hamiltonian) as well as for positive energy Dirac or Klein-Gordon theory (where wave functions propagate causally but cannot be compactly supported).

positive energy constraint, a local transformation on any small neighborhood already determines the global transformation of the whole wave function (e.g. only local transformations leaving the tails untouched must necessarily lead to contributions of negative energies).

In section 3.4, several Malament type theorems are derived in a somewhat unusual operational setting highlighting their operational meaning. A very general detector formalism is developed first, which associates minimal two element POVMs (click / no click) with bounded spatial regions (the regions covered by the detectors) and general statistical considerations about combined detector experiments are made. The theorems then developed show under a few general assumptions[4] the following: There does not exist a linear set of initial states, for which the joint probability that two detectors are triggered at spacelike separation is precisely zero (or more generally, the joint probability for any given finite number of detectors to click at spacelike separation cannot perfectly vanish). These results apply to first quantized quantum theories as well as to quantum field theories and seem to be counterintuitive at a first glance, if e.g. the considered set of initial states is a (positive energy) one particle Hilbert space or accordingly the one particle sector of Fock space. This initially puzzling feature (which should be associated with negligibly small probabilities in the relevant cases) can be understood by taking infinite tails and the active nature of detectors[5] – inducing particle creation processes[6] with non vanishing probabilities – into account, as will be argued in section 3.4.7.

The 'active nature of detectors' argument receives strong support from a result of *axiomatic/algebraic quantum field theory (AQFT)* which is an immediate consequence of the famous *Reeh-Schlieder theorem* and might be seen as a vacuum version of the Malament type theorems. It asserts that under the assumptions of AQFT, detectors must have non zero click probability, even if the initial state was the vacuum. Thus it is not surprising that a detector has non zero click probability, even if a remote detector was just triggered by a one particle initial state, as the Malament type theorems assert. The Reeh-Schlieder theorem and some of its implications are derived in section 3.5 and for that purpose first a basic introduction into AQFT is given and some considerations are made about its relation to the present approach, in which operators do not play a comparably fundamental role.

Chapter 4 goes beyond operational considerations and provides with *Bohmian mechanics* a physical theory about real particles in motion, whose analysis for measurement (like) situations yields the quantum formalism as a predictive framework predicting their outcomes. The *operational notion of localization* – i.e. a system is *getting localized* by a measuring device – used so far is contrasted with the *ontological notion of localization* of particles which *are* localized by their very definition in section 4.1. Section 4.2 provides the defining equations of motion of Bohmian mechanics. In section 4.3, the dynamics of subsystems is analysed, for which purpose

[4]The assumptions of the Malament type theorems are basically that the click statistics is given by a POVM (i.e. that the quantum measurement formalism applies), space-time translation covariance of the POVM in the Heisenberg picture and positive energy (plus local commutativity for theorem 3.25).

[5]To understand the 'active nature of detectors' argument, it is important to keep in mind that the considered probabilities encoded in POVMs are indivisibly connected with state transformations of the measured system as derived in chapter 1.

[6]In Dirac theory this is very transparent: On the first quantized level, local transformation of a positive energy wave function (as can be expected to happen if a detector is switched on) forces contributions from negative energies. Such processes involve pair creation if they are lifted to Fock space of the second quantized Dirac equation as is well known and can be nicely understood in the Dirac sea picture (see appendix A).

the important notions of *conditional and effective wave functions* are introduced. The conditional wave function has no analogue in standard quantum theory and the effective wave function – if it exists for a given subsystem – corresponds to the collapsed wave function of textbooks quantum theory.

In section 4.4 a statistical analysis based on the Bohmian equations of motion in the spirit of Boltzmann's statistical analysis of classical mechanics is performed. For that purpose, the notion of a *measure of typicality* on the space of microstates (which is configurations space in Bohmian mechanics, in contrast to classical mechanics where the microstates live on phase space) is introduced, telling us which sets of microstates are very large and which are very small. The $|\psi|^2-$measure is dynamically distinguished because it is the only *equivariant* measure on configuration space, meaning roughly speaking that with respect to it large sets stay large under the Bohmian dynamics. By that equivariant measure of typicality – called the *quantum equilibrium measure* – a law of large numbers is proven stating that if a subsystem of the universe has effective wave function φ, its coordinates are typically $|\varphi|^2-$distributed, where typically means for the overwhelming majority of possible configurations of the universe with respect to the quantum equilibrium measure. This statement directly corresponds to Born's rule and it will be shown in section 4.5 that it implies that in measurement (like) situations, Bohmian configurations of measuring devices (pointer positions etc.) typically display the results predicted by the quantum formalism.

In section 4.6, a survey on relativistic Bohmain mechanics is given. It treats separately the fundamental question of Lorentz invariance and possible generalizations to quantum field theories, involving in particular the characteristic feature of particle creation and annihilation. In section 4.7 finally, some brief considerations about the localization problem in Bohmian mechanics are made.

In appendix A the *second quantization of the Dirac equation* is derived from the *Dirac sea model* and pair creation, the lifting of one particle operators to Fock space, the external field problem etc. are discussed. In appendix B, the Hegerfeldt theorem is proven and its meaning with respect to localization is briefly discussed.

Preliminary Remarks

This work presupposes basic knowledge of the reader – on the level of undergraduate courses – in mathematics and physics; in particular acquaintance with the basics of complex and functional analysis, probability theory, special relativity, non relativistic quantum theory, relativistic wave equations, second quantization and quantum field theory is required. Natural units are chosen throughout this work, in particular $\hbar = c = 1$, the signature of the Minkowski metric $g^{\mu\nu}$ is chosen to be $(1, -1, -1, -1)$. The set of bounded linear operators acting on a Hilbert space $\mathcal{H}$ – which is always assumed to be separable in this work – is denoted by $\mathcal{B}(\mathcal{H})$, the space of density operators (positive operators of trace 1) acting on $\mathcal{H}$ is denoted by $\mathcal{S}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. If not stated otherwise, considered pure states $\psi \in \mathcal{H}$ are normalized by $\|\psi\| = 1$. Other notations will be introduced in the text or self evident.

1 Quantum Measurement
The Emergence of POVMs and State Transformers

1.1 Groundwork

To begin with

We start with the postulates of textbooks quantum theory without postulating operators as observables: We assume that a quantum mechanical system is described by a state ψ in some Hilbert space $\mathcal{H}$ and that its unitary time evolution $\psi_t = U_t\psi$ is generated by some Hamiltonian $\mathscr{H}$, e.g. via a Schrödinger type equation[7]

$$i\partial_t\psi = \mathscr{H}\psi \tag{1.1}$$

In addition we assume (of course) that physical processes like measurements have definite outcomes whose probabilities are given by the Borne rule. Based on these ingredients we will show how the operator formalism emerges pretty naturally and in maximal generality from an analysis of a particular class of dynamical processes which we shall call *measurement like*.

Ad Hoc Collapse and the Measurement Problem

The assumption of *definite outcomes* of *measurement like processes* is of course necessary for the theory to be empirially adequate but it is also peculiar since it requires a second dynamical process artificially replacing the unitary evolution: Since apparatuses displaying the outcomes of measurements after interaction with the measured quantum system are also consisting of quantum mechanical constituents (atoms), dynamics of the form (1.1) do in general not produce definite outcomes but rather superpositions of different outcomes like on the right hand side of (1.3) below, where the ϕ_k may be thought of as the states of distinct pointer positions or the like. Thus, in order to end up with a final state of the measuring device corresponding to one definite outcome, we need an additional dynamical principle[8] which picks out the actual pointer state from the sum of several ones (the alternative would be that equation (1.1) is wrong and needs to be modified, in particular the evolution cannot be linear in this case which is proposed in GRW-type collapse theories [148]).

[7] Note that this is not restricted to non relativistic Schrödinger theory, e.g. the wave function ψ might be an element of $L^2(\mathbb{R}^3, d^3x) \otimes \mathbb{C}^4$ and $\mathscr{H} = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m$ the (here free) Dirac Hamiltonian (see appendix A for the meaning of the symbols).

[8] There is a widespread belief that a dynamical mechanism already contained in standard quantum mechanics – namely decoherence – is sufficient to solve this puzzle. I do not doubt that decoherence is essential to explain why quantum interference effects do not show up for macroscopic objects. But this is not the problem here. The point is that – and this is perhaps the right place to use Bohr's and Heisenberg's terminology – decoherence processes do not yield "transitions form the potential to the actual": If there is no fundamental dynamics but the linear Schrödinger evolution, superpositions will evolve to superpositions, no matter if we ignore the (dynamically relevant) environment in the appropriate way (i.e. by tracing it out). If we admit that Schrödinger type dynamics like (1.1) are fundamental, linearity with its grievous consequences is inescapable: We cannot deduce a fundamental nonlinear stochastic dynamics from a fundamental linear deterministic dynamics by simply ignoring a part of the system (in the appropriate way). See also section 1.6.6 below.

In textbooks quantum theory this is done by postulating an additional ad hoc *collapse (or reduction) of the wave function* for measurements together with the Born rule: The joint state of system and apparatus collapses upon measurement onto one summand of the superposition with probability given by the Hilbert space norm of the respective summand squared. It is important to note that this process is random, non-linear[9] and non-unitary, whereas the time evolution given by (1.1) is deterministic, linear and unitary, i.e. something very different!

If now measurements are not distinguished physical processes as compared to other interactions between microscopic and macroscopic systems (whatever these notions precisely mean), this corresponds to introducing a *Heisenberg cut*, arbitrarily inserted somewhere in between what might be ad libitum called *small systems* and *big systems*, *microscopic* and *macroscopic*, at which the laws of physics dramatically change, which is, to say the least, contrived (or straightaway: inconsistent).

But indeed, it is possible to remove the arbitrariness from the reduction postulate, to write down a precise physical theory explaining the (effective) collapse dynamics from a coherent physical dynamics, to give a clear description of what happens in space and time according to physical laws and at the same time to derive the quantum mechanical predictions for the outcomes of measurements from an analysis of the dynamics. A fully worked out example of such a theory is Bohmian mechanics, where this is done by supplementing the dynamics of the wave function ψ with a dynamics of particles governed by ψ (an alternative example would be GRW-type collapse theories equipped with an appropriate ontology, which specifies the predicted events in physical space and time, see e.g. [323, 326, 25, 11]). We will come back to this at the end of this work in chapter 4.

In this chapter and for a large part of this work, we will nevertheless adopt the artificial collapse postulate as a starting point of the analysis, and characterize it by resorting to the vague but workable notion of *'pointer (like) states'*. The latter comprises (at least) all states of macroscopic objects which are subject of our experience. These states are of course not explicitly accessible to us, but the corresponding objects trivially are and do always represent a unique state of affairs (a suitable pointer points onto *a or* onto *b* but never onto *a and b*). The unitary dynamics has to be replaced by the collapse dynamics then whenever a system gets entangled with a collection of pointer (like) states and the associated randomness is described by the Born rule (see below for details). As we shall see, this primitive ad hoc ansatz of a special kind of state transitions (measurement like transitions) alone entails the abstract quantum measurement formalism in its utmost generality as an appropriate and efficient means of its description, including the usual observable operators, positive operator valued measures (POVMs), the Kraus representation of state transformations or the like.

To base this derivation and analysis on ad hoc collapse has the following advantages:

- The present analysis to derive the measurement formalism must be finally valid in any approach towards quantum theory, which regards quantum theory as fundamental in the sense that it holds also for the atoms constituting a measuring device, and in which processes like measurements have definite outcomes[10]. In chapter 4 this will be illustrated with

[9] Collapse is not linear, but linear up to normalization, a fact which will be of some importance later in section 1.6.

[10] Indeed, in this work *many worlds type theories*, where the assumption that measurements have definite outcomes is dropped, are not taken into account. Nonetheless, if it can be managed to justify the Born rule

the example of Bohmian mechanics, where no collapse needs to be postulated but pointers (or the like) consist literally of particles which thus by definition have a definite position and thereby measurements definite outcomes. An operational formulation of quantum theory as derived in the present chapter is that way a straightforward consequence of the Bohmian equations of motion (see chapter 4 for details).

- The derivation of the operator formalism from basic quantum mechanical principles (without postulates about operators) elucidates the point of origin and conceptual status of the textbook *'observable operators'* in a way which can be comprehended even proceeding from textbook quantum theory. This way, a widespread vague uncertainty about the physical meaning of these operators, which underlays many confusions about quantum theory, can be avoided from the outset.

- In particular, the present analysis can offer an instructive physical understanding of several celebrated results in quantum mechanics, which are theorems about operators in the first place. This will be illustrated for the no-go theorems of Kochen & Specker and Bell in section 1.3, the physical meaning of local commutativity in chapter 2 or the Malament type no-go theorems in chapter 3. The present approach provides a plain and demystifying analysis which does not even need to resort to ontology, reconsidering these assertions already from an operational point of view (i.e. talking only about measurement results).

Discrete and Continuous Measurements

In order to keep things as clear and simple as possible, the analysis of measurement like processes shall be developed in chapter 1 preferably in a discrete framework, which corresponds to a countable set of possible outcomes. A generalization to continuous observables is essentially straightforward and can be found in any textbook on quantum theory. On appropriate occasions instructive examples of continuous measurements will be given, too (and the familiarity of the reader with the respective basic mathematical tools is presupposed). It is to be stressed that the state transformations are somewhat delicate for continuous projective measurements, which can be understood to be rather a problem of mathematical idealization (see section 1.5.2 below).

1.1.1 Motivation

In his famous book [334], John von Neumann described a measurement process as a quantum mechanical interaction between two quantum mechanical systems: The measured system and (part of) a measuring device, which is also made out of atoms and thus a quantum mechanical system. To this end, denote the Hilbert space of the measured system by $\mathcal{H}_S$ and the Hilbert space of the (relevant part[11] of the) measuring device by $\mathcal{H}_A$. The apparatus has a ready state

in a many worlds type theory (which is an ambitious task), the present analysis is also valid with respect to a typical branch of the global wave function – e.g. the actual branch associated with my history in my present conciousness. For a nice presentation how many worlds could be formulated in an ontologically transparent way, see [12].

[11] For now, the *relevant part* of the apparatus is everything of it which is involved in the measurement, from the part microscopically interacting with the measured system up to the macroscopic part displaying the measurement

$\phi_0 \in \mathcal{H}_A$ and a couple of final states $\{\phi_k\} \subset \mathcal{H}_A$ (k varies in some given index set) indicating the different possible outcomes of the measurement[12], like positions of a pointer or printouts of numbers etc. We shall assume that the pointer states are (for all practical purposes) mutually orthogonal $\langle \phi_k |\, \phi_l \rangle = \delta_{kl}$ (see the following section for justification of this crucial assumption).

In the first place the measurement is an interaction described by the unitary Schödinger dynamics (1.1) on the full Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A$. We can indicate this interaction by a unitary operator U acting on $\mathcal{H}$ transforming the joint initial state of measured system and apparatus prior to the interaction into the joint final state subsequent to the interaction (if the interaction starts at time $t = t_0$ and ends at time $t = t_1$ and if we denote the interaction Hamiltonian by $\mathscr{H}_{int}$ and assume that the free Hamiltonians of measured system and measuring device can be neglected during interaction, U is generally given by the operator $U = T\left(e^{-i\int_{t_0}^{t_1} \mathscr{H}_{int}(t)\,dt}\right)$, where T is the time ordering operator, see e.g. [299]).

Since a measurement shall tell us something about the measured system, it is standing to reason to assume that the measurement interaction produces a correlation between the state of the measured system and the final state of the apparatus and of course that the measured system and the measuring device were independent systems prior to the measurement. The latter assumption is expressed by setting a product state $\psi\phi_0 \in \mathcal{H}$ as the initial state, where $\psi \in \mathcal{H}_S$ is the initial state of the measured system and $\phi_0 \in \mathcal{H}_A$ is the ready state of the apparatus. The first assumption seems to be implemented in the most direct way if there is a set of states $\{\psi_k\} \subset \mathcal{H}_S$ which are revealed by the pointer states $\{\phi_k\} \subset \mathcal{H}_A$, i.e.

$$\psi_k\phi_0 \xrightarrow{U} \psi_k\phi_k \tag{1.2}$$

for all k (whether zero is amongst the ks depends on the experimental setup: If we descibe e.g. a detection experiment, it is possible that the detector does not click and consequently stays in its ready state). It is easy to construct such interactions explicitly in theory (the Von-Neumann measurement presented in section 1.5.2 gives an example), even more, they build essentially the core of textbook quantum theory and corresponding experiments are also realized in practice with great success.

Nevertheless, the presented scheme is in great disagreement with empirical experience if a state $\psi \in \mathcal{H}_S$ is prepared as the initial state, which is a linear superposition of the states $\{\psi_k\}$. In this case, due to linearity, the unitary evolution gives rise to the transition

$$\psi\phi_0 = \left(\sum_k c_k\psi_k\right)\phi_0 \xrightarrow{U} \sum_k c_k\psi_k\phi_k \tag{1.3}$$

i.e different outcomes displayed by distinct apparatus states are realized at the same time which is clearly not what happens in laboratories! This is the root of the measurement problem. But

result (pointer etc.). In section 1.4.4 we will develop a tool which allows in a sense to look inside the measuring device more closely, in particular to formally separate the microscopic part of the apparatus directly interacting with the measured system from the rest of the device.

[12] Von-Neumann gave a concise argument (see [334] p. 233), that if we account for the fact, that in reality we will not know the state of the apparatus explicitly, all substantial features of the analysis of the measurement process remain the same, i.e. we can, without loss of generality, pretend that the states of the measuring device are exactly known by formally writing down apparatus states ϕ_k.

as mentioned above, we shall for the moment pragmatically postulate as a second dynamical principle[13] an ad hoc non unitary, non linear and stochastic reduction of the state subsequent to the measurement whose statistics is given by the Born rule: The joint state on the right hand side of (1.3) collapses onto the macroscopically unambiguous state $\psi_l\phi_l$ with probability

$$\mathbb{P}^{(\psi\phi_0)}(l) = |\,\langle\psi_l\phi_l|\;U(\psi\phi_0)\rangle\,|^2 = |c_l|^2 \tag{1.4}$$

Note that if we assume that the pointer states are not only orthogonal but disjointly supported in position representation (see the following section), equation (1.4) is simply the usual Born probability to find finally the constituents of the pointer in a configuration in the pointer configuration space which is in the support of ϕ_l (we might loosely speaking characterize such configurations by 'the pointer points onto l'): Say $x \in \mathbb{R}^{3N}$ are the generic coordinates of the measured system (i.e. N particles) and $y \in \mathbb{R}^{3M}$ the generic coordinates of the pointer constituents (i.e. M pointer particles, which will be large in most cases). Then the probability that the 'pointer is found to point onto l' (that its actual configuration $Y \in \mathbb{R}^{3M}$ is found to lie in the support of ϕ_l) subsequent to the measurement is according to Born's rule

$$\begin{aligned}
&\mathbb{P}\left(\textit{pointer points onto } l \mid U(\psi\phi_0)\right) = \mathbb{P}\left(Y \in \operatorname{supp}\phi_l \mid U(\psi\phi_0)\right) = \\
&= \int d^{3N}x \int\limits_{\operatorname{supp}\phi_l} d^{3M}y \left|\sum_k c_k\psi_k(x)\phi_k(y)\right|^2 = \int d^{3N}x \int d^{3M}y\; |c_l\psi_l(x)\phi_l(y)|^2 = \\
&= |c_l|^2 \int |\psi_l(x)|^2\, d^{3N}x \int |\phi_l(y)|^2\, d^{3M}y = |c_l|^2
\end{aligned} \tag{1.5}$$

The final state of the measured system is consequently given by ψ_l and the outcome associated with the final pointer state $\phi_l \in \mathcal{H}_A$ of the apparatus is realized (we tacitly assumed here that the relation between the states of the measured system ψ_l and the pointer states ϕ_l is one to one, an assumption which we will drop later).

Note that this is a surrender with respect to the aspiration to reveal something unambiguous of the measured system prior to the measurement, since the 'measured state' is in general rather produced by the measurement than 'measured'! That this is not only a peculiarity of quantum description but indeed of the actual quantum dynamics is a fact which can be mathematically proven and empirically tested, we will come back to this later in section 1.3.

Observable Operator

It is easy to see (see section 1.4.1 for details) that a dynamics like (1.3) together with the mutual orthogonality of the pointer states ϕ_k and the unitarity of U is only possible if the states ψ_k are also mutually orthogonal (it is sometimes said that only orthogonal states can be perfectly discriminated in experiment). If furthermore the transition (1.3) shall be defined for each initial

[13] Besides the rather vague speculations of the Copenhagen school, there was a great awareness among some of the fathers of quantum theory, that this is a very severe problem: While e.g. von-Neumann (as later Wigner) came to the conclusion, that only conciousness collapses the state (which is, to say the least, ambitious and in many aspects highly problematic, not least for cosmologists), Dirac regarded the measurement problem as a very hard problem whose solution must be postponed to later times. In contrast, Schrödinger and Einstein were cut to the quick by the fact that most physicists went on without struggling for a solution.

$\psi \in \mathcal{H}_S$, the set $\{\psi_k\}$ needs to be a basis of $\mathcal{H}_S$ and with the previous observation it is thus an orthonormal basis of $\mathcal{H}_S$. This also guarantees that the condition that the probabilities (1.4) sum up to unity is equivalent to the condition that the initial state $\psi \in \mathcal{H}_S$ of the measured system in (1.3) has norm one.

In a usual measurement, each outcome indicated by final pointer state ϕ_k is associated with a numerical value $\lambda(k)$ (e.g. in the pointer picture, the pointer orientation picks out some real number on a scale), which is supposed to reflect the value of some physical quantity (in given units), the 'quantity to be measured'[14]. For now we shall assume that the function $\lambda(k) \equiv \lambda_k$ is one to one.

Denote the one dimensional orthogonal projections onto the subspaces spanned by ψ_k by P_k, i.e. P_k acts on all $\varphi \in \mathcal{H}_S$ by $P_k\varphi = \langle\psi_k|\ \varphi\rangle\, \psi_k$ (P_k may be written in the Dirac notation as a dyadic product $P_k = |\psi_k\rangle\, \langle\psi_k|$). We can now calculate the expectation value $\langle\lambda\rangle_\psi$ of the measurement results $\{\lambda_k\}$ as a quadratic form of the initial state ψ of the measured system (recall that orthogonal projections are selfadjoint and satisfy $P^2 = P$):

$$\begin{aligned} \langle\lambda\rangle_\psi &= \sum_k \lambda_k\, \mathbb{P}^\psi(k) = \sum_k \lambda_k\, |c_k|^2 = \sum_k \lambda_k\, \|P_k\psi\|^2 = \\ &= \sum_k \lambda_k\ \langle P_k\psi|\ P_k\psi\rangle = \sum_k \lambda_k\ \langle\psi|\ P_k\psi\rangle = \\ &= \left\langle \psi \middle| \left(\sum_k \lambda_k\, P_k\right) \psi \right\rangle =: \langle\psi|\ \mathcal{A}\psi\rangle \end{aligned} \tag{1.6}$$

where we have defined the *observable operator*

$$\mathcal{A} := \sum_k \lambda_k\, P_k \tag{1.7}$$

giving the expectation of the measurement results as a quadratic form in ψ via

$$\langle\lambda\rangle_\psi = \langle\psi|\ \mathcal{A}\psi\rangle \tag{1.8}$$

We have $\mathcal{A}\psi_k = \lambda_k\psi_k$ for all k, i.e. the eigenvectors of $\mathcal{A}$ are the states ψ_k and the corresponding eigenvalues the numbers λ_k. The operator $\mathcal{A}$ is selfadjoint (given the values λ_k are real numbers) and (1.7) is its spectral representation.

Reconsidering the foregoing lines shows that we got rid of the pointer states, which are not explicitly accessible to us anyway, and are able now to characterize the quantum measurement comprehensively by referring to the measured system alone. The crucial mathematical notion in this regard is however not primarily the observable operator but rather its associated projection valued measure (PVM), which is roughly speaking given by the set of projections $\{P_k\}$, in

[14]But recall that in general actually nothing is being measured at all, in the sense that some pre-existing property was revealed. Rather the outcome is a product of the interaction between the measured system and the measuring device which has a huge impact on the measured system. The possibility that anything substantial is being measured in general quantum measurements, i.e. that pre-existing values beyond the scope of quantum theory are revealed by the measurement, is ruled out by the famous no-go theorems of Kochen-Specker and Bell which will be illustrated in section 1.3.

which the quantum probabilities and state transformations associated with the measurement are straightforwardly encoded.

Probabilities and State Transformation

The composite final states of system and apparatus are the states $\psi_k\phi_k$, which is to say, if the measurement has outcome λ_k,

$$\psi_k = \frac{P_k\psi}{\|P_k\psi\|} \tag{1.9}$$

is the final state of the measured system[15], which happens with probability

$$\mathbb{P}^\psi(\lambda_k) = \|P_k\psi\|^2 \tag{1.10}$$

(note that this transition is of course only possible if $\mathbb{P}^\psi(\lambda_k) \neq 0$, such that 1.10 implies that 1.9 is always well defined).

This already indicates the predictive scope of standard quantum theory of measurements: (i) To assign probabilities to the various outcomes and (ii) to assign post measurement states to the measured system which define the future dynamics, or in other words to describe the measurement as a new state preparation, e.g. in order to make predictions for subsequent measurements. Given an initial state $\psi \in \mathcal{H}_S$ – which can be conceived as the only free variable of a given measurement scheme – according to (1.9) and (1.10), both (i) and (ii) are mathematically represented by the family of projections $\{P_k\}$ which form a PVM[16] acting on $\mathcal{H}_S$ as already mentioned, which thus comprises the probabilistic and dynamical features of the respective measurement scheme.

We see that the central mathematical object associated with the measurement is the associated PVM. The observable operators which are in the center of attention in most textbooks are from this point of view of rather marginal relevance: They are well suited to calculate expectation values (and by a nice mathematical property also the higher moments) and in the light of the spectral theorem, they are a compact expression encoding the PVM which is the actual tool to calculate probabilities and state transformations. We will encounter that for many realistic quantum measurements there is not even something like an associated (reasonable) observable operator, whereas generalizations of a PVM to calculate probabilities and state transformations can be easily found (where, as we shall see, the operators to calculate probabilities need not coincide with the operators implementing the state transformations).

[15]This guarantees *reproducibility* of measurement results: If the measurement had outcome λ_k and is immediately repeated, the initial state of the second measurement is ψ_k leading to apparatus final state ϕ_k, i.e again outcome λ_k with certainty. We will see, that this nice feature is rather special and that for many realistic measurements it does not hold.

[16]If we define $P_{\lambda_k} \equiv P_k$ as an operator valued function of the λ_ks, the PVM is defined on the spectrum $\sigma(\mathcal{A})$ of the associated observable operator $\mathcal{A}$, which is to say it is a mapping from the (measurable) subsets of $\sigma(\mathcal{A})$ to projections acting on $\mathcal{H}_S$. In order to make this precise, some basic properties of PVMs like additivity are needed, which will be made below.

General Measurements

We will see below that dynamics of the form (1.3) defines only a very special case of quantum measurement which will be called *ideal measurement* and analysed in section 1.4.1 more closely. In particular, in many realistic quantum measurement schemes the state of the measured system is transformed in a more intricate way than it is indicated in (1.2) and (1.3) and also the operators encoding the probabilities associated with a quantum measurement need not be projections.

What remains is that every reasonable quantum measurement is an interaction between two (or more) quantum mechanical systems – measured object and apparatus – producing a correlation between them (expressed by entanglement) and resulting in one of a collection of macroscopically accessible and distinguishable states of the apparatus indicating the outcome[17]. The notion of *'macroscopically accessible and distinguishable'* is as mentioned a vague notion though, but we know of course that objects like pointers do exist and share the desired properties. Before starting a more structured analysis, we shall thus finally briefly examine the vague but workable notion of *pointer states* more closely.

1.1.2 Pointer States

What qualifies some physical system to display the result of a measurement? It must in one way or the other give an intersubjective record of facts. May it be that the outcome of some measurement is displayed by the spin degrees of freedom of some (big) spin $\frac{1}{2}$ system, i.e. that the pointer Hilbert space is $\mathcal{H}_A = \mathbb{C}^2$? Certainly not! Ultimately, no one has ever seen a spin. Clearly we can measure spin, e.g. by measuring the position of a spin $\frac{1}{2}$ system after its passage through an inhomogeneous magnetic field. But this does only mean that it may well be (of course) that $\mathcal{H}_S = \mathbb{C}^2$.

Displays are always realized by distributions of matter: Positions of pointers, numbers on computer screens or paper, even the click of a detector is displayed in the displacement of a membrane, the displacement of air molecules transporting the sound, the displacement of the eardrum of the experimenter (and if it is a serious experiment the click statistics should also be recorded somewhere). And according to quantum theory, matter is $|\psi(x)|^2\, d^{3N}x$ distributed, where $\psi \in L^2(\mathbb{R}^{3N}, d^{3N}x)$ is the usual wave function living on configuration space $\mathbb{R}^{3N}$ of N particles. Thus we can always presuppose that $\mathcal{H}_A = L^2(\mathbb{R}^{3N}, d^{3N}x)$ for some N.

Now we come to the notion of *'macroscopically accessible and distinguishable'*. In the sense of the previous paragraph, displays displaying different outcomes differ from one another in being macroscopically distinct distributions of matter and empirical distributions of matter are expected to be in accordance with the quantum theoretical prediction $\varrho(x)\, d^{3N}x = |\psi(x)|^2\, d^{3N}x$. In consequence, states of objects like pointers in different orientations are expected to have (essentially) disjoint support in configuration space, and the separation of the supports is drastically amplified if we do not consider a 'single pointer particle' but a realistic pointer made out of 10^{24} particles or the like, since the separation of distinct configurations of many particle systems grows with the dimension of configuration space, i.e. with the number of particles involved. In

[17] In the formal framework of the quantum theory of measurement one may also define formal measurement processes without interaction and without correlations between the 'measured object' and the measuring device (c.f. the 'no-interaction measurements' in [127]). But we shall not bother with such gimmicks.

this sense one can speak of configurations (or equally supports of states) associated with different pointer positions (or the like) as *macroscopically separated in configuration space*.

So proper pointer states ϕ_k displaying different outcomes of a measurement in particular will be disjointly supported $\mathrm{supp}\{\phi_k\} \cap \mathrm{supp}\{\phi_l\} \approx \emptyset$ for $k \neq l$ to a very high degree of accuracy[18] (i.e. for all practical purposes the $\approx$ can be substituted by $=$, as we will henceforth do). This in turn implies the mutual orthogonality of the pointer states[19]

$$\langle \phi_i |\, \phi_j \rangle = \int \bar{\phi}_i(x)\phi_j(x)\, d^{3N}x = 0 \tag{1.11}$$

The macroscopic separation of the (effective) wave function support in configuration space, which carries over from the pointer states ϕ_k to the states $\psi_k\phi_k$ of measured system plus apparatus, implies that in an interaction like (1.3), the pointer states generate *decoherence* of the measured system, which means roughly speaking that the states ψ_k loose their ability to interfere with one another (the off diagonal elements of the reduced density operator of the measured system vanish). Such decoherence processes are for sure a necessary ingredient to explain the emergence of definite outcomes, which is nonetheless not sufficient since taking alone decoherence into account does not change anything about the final state on the right hand side of (1.3) (we might include the environment of the pointer in the states ϕ_k , or even the rest of the universe...), in particular it does not entail a transition from the superposition to one of its terms associated with an unambiguous final state of the apparatus (see further footnote 8 and section 1.6.6).

Actually, the joint final states of measured system and apparatus need not be of the form $\psi_k\phi_k$ since the measured system might be literally part of the pointer subsequent to the measurement: We may e.g. think of a particle absorbed by a screen yielding a black spot such that it is not clear whether the screen represented by $\mathcal{H}_A$ and the detected particle represented by $\mathcal{H}_S$ form a product state subsequent to the measurement. Thus, to begin with, we just denote the set of final pointer states as $\{\Phi_k\} \subset \mathcal{H}_S \otimes \mathcal{H}_A$ and recover measurement scenarios which lead to a pure final state of the measured system as the special case where the pointer states are of the form $\Phi_k = \psi_k\phi_k$ with $\psi_k \in \mathcal{H}_S$ and $\phi_k \in \mathcal{H}_A$ for all k, in which case we call only the ϕ_ks pointer states and accordingly the states ψ_k the (potential) final states of the measured system[20].

Of course, in the case of a particle hitting a screen we are usually not so much interested in the final state of the measured particle but only in the initial state, the associated Born

[18] We will see later that wave functions in a reasonable setting cannot have compact support (in relativistic quantum theory this is the case if the energy of a wave function is bounded from below, in non relativistic quantum mechanics, an initially compactly supported wave function will instantaneously develop infinite tails), thus there are no wave functions in the universe which do not overlap! But wave functions can be extremely – not to say overwhelmingly – well localized (see section 3.3) and interaction with an environment (decoherence) will do one more thing to keep wave functions of macroscopic objects well localized (see e.g. [192] and references therein), such that it is perfectly justified to treat for all practical purposes wave functions of differently positioned, oriented etc. macroscopic objects as disjointly supported wave functions.

[19] An additional argument is that the pointer states have to be orthogonal if the transition probability shall be zero, that a pointer state labelled with index i 'collapses' onto a distinct pointer state labelled with index $j \neq i$ (e.g. upon a 'measurement of the pointer position') i.e. $\mathbb{P}(\phi_i \to \phi_j) = |\, \langle \phi_i |\, \phi_j \rangle\, |^2 \stackrel{!}{=} 0$.

[20] If the measured system is finally (an entangled) part of the final pointer state Φ_k, one might attribute a final mixed state to the measured system on the density operator level by performing the partial trace $\mathrm{Tr}_{\mathcal{H}_A}[|\Phi_k\rangle\langle\Phi_k|]$. This is an example of a so called *non efficient measurement*, where only mixed final states can be attributed to the measured system even if the initial state was pure (an imprecise resolution of the measurement result which is consistent with several possible final pure states would be another example).

probability distribution in the spatial region of the screen, the empirical appearance of a black spot on the screen and the empirical distribution of black spots for ensembles of identically prepared systems. In other cases, the final state of the measured system is crucial, e.g. if the measurement serves as a preparation for a subsequent measurement. Or the final state of a subsystem of the measured system is crucial, think e.g. of two spin$-\frac{1}{2}$ particles prepared in the singlet state and a subsequent spin measurement on one of them, although the measured particle might be absorbed by a screen behind a Stern-Gerlach device, we can determine from the respective measurement result the final state of the remaining particle and thereby make predictions for future spin measurements on the latter. We may also have in mind more intricate preparation–measurement scenarios e.g. a charged particle in a cloud chamber ionizing an atom – in which case the ion together with a bulk of condensed water molecules form the pointer – where it is crucial that there is a final state of the measured particle which is well localized about the respective region such that it ionizes next an atom nearby and so forth, and produces that way a continuous path.

Measurement (Like) Process and the Emergence of Phenomena

If we take quantum mechanics seriously as a fundamental theory of nature, not just as a computational algorithm making predictions for very artificial situations prepared in laboratories, an analysis of quantum measurement processes is much more relevant and the notion of pointer states is much more general than it might seem at a first glance. After all, when I look at the table in front of me, I measure the position of a bulk of atoms which constitute the table[21]. Experience of the world made out of quantum mechanical constituents is quantum measurement at the end of the day. Hence, the emergence of phenomena in a world guided by quantum theory must be generally of the type as the processes described by the quantum theory of measurement, not just when they happen to emerge in artificial situations in physics laboratories. Physical systems like tables are of course rather feasibly and appropriately described in terms of classical physics. But if quantum mechanics is fundamental, the latter is only (an extremely well) approximation and is to be derived by a classical limit[22] from quantum theory and the collapse dynamics associated with measurement (like) processes should naturally be the dominant dynamics whenever macroscopic objects (whose wave functions are pointer (like) states) are involved[23].

To emphazise this generality, we will repeatedly use the expression *'measurement (like) process'* instead of *measurement* together with *pointer (like) states* which shall have the properties

[21] One might argue that the table is a macroscopic object and thereby rather like a pointer than like a 'measured system'. But we never constrained the state of the measured system to be microscopic or the like, we only considered a quantum description of an interaction producing a correlation between the measured system and a second system (which has some constraints to deserve the name pointer states, as discussed above). And 'me looking at the table' is presumably such a process (which looks presumably rather like (1.2) than (1.3)), where now the pointer states may be chosen to be the photons reflected by the table, or defined by the physical changes which happen on my retina or in my brain when my eye captures the table.

[22] See e.g. chapter VI in [243] for the common treatment of the classical limit of quantum theory or [279] for a comprehensive Bohmian treatment.

[23] This is a bit awkward for unfortunate historical reasons and should be better the other way around: one should first locate a dynamical principle in nature and later strive to understand its role in very special situations like measurements.

established in this section. The pointer (or apparatus) Hilbert space might e.g. be also thought of as the Hilbert space of a macroscopic environment of a considered quantum system.

1.2 Measurement (Like) Process

Next the notion of *measurement (like) processes* will be formally defined and subsequently analysed. As a starting point and if not stated otherwise, this terminology shall actually refer to *discrete measurement (like) processes*.

Definition 1.1 [*Measurement (Like) Process*]

Consider a quantum theory with Hilbert spaces $\mathcal{H}_S, \mathcal{H}_A$ and $\mathcal{H} := \mathcal{H}_S \otimes \mathcal{H}_A$, a ready state $\phi_0 \in \mathcal{H}_A$ and a collection of pointer (like) states $\{\Phi_k\} \subset \mathcal{H}$, where k varies in some discrete index set I which might contain zero or not. If U is a unitary interaction operator acting on $\mathcal{H}$ generated by some interaction Hamiltonian, such that for each initial $\psi \in \mathcal{H}_S$ there is a set of complex numbers $\{c_k\} \subset \mathbb{C}$ such that

$$U(\psi\phi_0) = \sum_{k\in I} c_k \Phi_k \tag{1.12}$$

the sequence

$$\psi\phi_0 \xrightarrow{U} \sum_{k\in I} c_k \Phi_k \xrightarrow{(*)} \Phi_l \tag{1.13}$$

is called a (discrete) measurement (like) process, where the last transition $(*)$ happens for each $l \in I$ with probability

$$\mathbb{P}^{\psi\phi_0}(l) = \left|\langle \Phi_l |\, U(\psi\phi_0)\rangle\right|^2 = |c_l|^2 \tag{1.14}$$

Note that as discussed in the previous section, the pointer (like) states Φ_k are mutually orthogonal, which also ensures that in the process (1.13), one of the outcomes will be realized with certainty provided all states are normalized:

$$\begin{aligned} \sum_{k\in I} \mathbb{P}^{\psi\phi_0}(k) = \sum_{k\in I} |c_k|^2 = \sum_{j,k\in I} \bar{c}_j c_k \delta_{jk} = \sum_{j,k\in I} \bar{c}_j c_k \overbrace{\langle \Phi_j |\, \Phi_k\rangle}^{\delta_{jk}} = \\ = \left\langle \sum_{j\in I} c_j \Phi_j \middle| \sum_{k\in I} c_k \Phi_k \right\rangle = \langle U(\psi\phi_0)|\, U(\psi\phi_0)\rangle = \langle \psi\phi_0|\, \psi\phi_0\rangle = \|\psi\phi_0\|^2 = 1 \end{aligned} \tag{1.15}$$

1.2.1 Efficient Measurements

A very important class of measurement (like) processes is the class of *efficient measurements*, where the measured system can always be associated with a pure final state (given the initial

state was pure). This makes it possible to conceive the measurement again as a preparation procedure and to reduce the whole description to the Hilbert space $\mathcal{H}_S$ of the measured system alone and thus to get rid of the pointer Hilbert space which is not accessible to us anyway.

Definition 1.2 [*Efficient Measurement (Like) Process*]

A measurement (like) process as in definition 1.1 is called efficient if for each $k \in I$ there are states $\psi_k \in \mathcal{H}_S$ and pointer (like) states $\phi_k \in \mathcal{H}_A$ such that

$$\Phi_k = \psi_k \phi_k \tag{1.16}$$

State Transformers

In this case all information about a specific measurement (like) process – the state transformations and the quantum probabilities – can be condensed in a set of operators acting on $\mathcal{H}_S$ which we shall call *state transformers*.

Definition 1.3 [*State Transformers*]

The linear operators $\mathcal{R}_k$ acting on $\mathcal{H}_S$ defined by (1.12), (1.16) *and the relation*

$$\mathcal{R}_k \psi := c_k \psi_k \tag{1.17}$$

are called state transformers. With the state transformers, we can write (1.12) *as*

$$U(\psi\phi_0) = \sum_{k \in I} (\mathcal{R}_k \psi)\phi_k \tag{1.18}$$

and (1.13) *may be written in the following way:*

$$\psi\phi_0 \xrightarrow{U} \sum_{k \in I} (\mathcal{R}_k \psi)\phi_k \xrightarrow{(*)} \frac{(\mathcal{R}_l \psi)\phi_l}{\|\mathcal{R}_l \psi\|} \tag{1.19}$$

Note that the last transition $(*)$ happens with probability $\mathbb{P}^{\psi\phi_0}(l) = |c_l|^2 = \|\mathcal{R}_l\psi\|^2$ such that the right hand side of (1.19) is well defined whenever the related transition has non zero probability. It is easy to check that the operators $\mathcal{R}_k$ are indeed linear, in contrast to the collapse given by the last transition $(*)$ in (1.19) which is obviously non-linear (but linear up to normalization).

POVM

This enables us to write down the probability for 'outcome l' as a quadratic form of the initial wave function $\psi \in \mathcal{H}_S$ in terms of the respective state transformer:

$$\mathbb{P}^{\psi\phi_0}(l) = |c_l|^2 = \left\langle \psi \middle| \mathcal{R}_l^\dagger \mathcal{R}_l \, \psi \right\rangle =: \langle \psi | \, E_l \, \psi \rangle \equiv \mathbb{P}^\psi(l) \tag{1.20}$$

where

$$E_k = \mathcal{R}_k^\dagger \mathcal{R}_k \tag{1.21}$$

are positive operators acting on $\mathcal{H}_S$ since the expressions $\langle \psi | \, E_k \psi \rangle$ are probabilities for all k and for all $\psi \in \mathcal{H}_S$.

Moreover, since $\langle \psi | \, \psi \rangle = 1 = \sum_{k \in I} \mathbb{P}^\psi(k) = \left\langle \psi \middle| \sum_{k \in I} E_k \psi \right\rangle$, these operators sum up to unity

$$\sum_{k \in I} E_k = \mathbb{1}_{\mathcal{H}_S} \tag{1.22}$$

If $J \subseteq I$ is a subset, we can calculate the probability $\mathbb{P}^\psi(k \in J)$ that the outcome is a member of J as

$$\mathbb{P}^\psi(k \in J) = \sum_{k \in J} \mathbb{P}^\psi(k) = \left\langle \psi \middle| \sum_{k \in J} E_k \psi \right\rangle =: \langle \psi | \, E_J \psi \rangle \tag{1.23}$$

where we have defined $E_J := \sum_{k \in J} E_k$. More generally, if $(J_n)_{n \in \mathbb{N}} \subseteq I$ is a family of disjoint subsets $J_n \cap J_m = \emptyset$ for $n \neq m$ we have the property of *additivity*:

$$\sum_n E_{J_n} = E_{\cup_n J_n} \tag{1.24}$$

If the sum in (1.24) extends to infinity, the limit which defines the meaning of the equality sign can be taken in the strong operator topology.

Thus, if we additionally define $E_\emptyset := 0$ as the zero multiplication operator on $\mathcal{H}_S$ (which is to say the probability of 'no outcome' is zero), the family of operators $(E_J)_{J \subseteq I}$ forms a discrete *positive operator valued measure* (POVM) on the set $\Omega = I$. We give the general definition of a POVM which is valid for continuous measurements as well:

Definition 1.4 [*POVM*]

Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{H}$ a (separable) Hilbert space. A positive operator valued measure (POVM) on Ω is a mapping $E : \mathcal{F} \to \mathcal{B}(\mathcal{H})$ from the measurable subsets of Ω to the bounded operators acting on $\mathcal{H}$ with the following properties:

(i) $E_X \geq 0$, i.e. E_X is a (semi-)positive operator for all $X \in \mathcal{F}$

(ii) $E_\emptyset = 0$ and $E_\Omega = \mathbb{1}_{\mathcal{H}}$

(iii) $\sum_k E_{X_k} = E_{\bigcup_k X_k}$ whenever $(X_k)_k$ is a countable family of mutually disjoint (measurable) subsets of Ω, where the limit is to be taken in the strong operator topology if the sum extends to infinity

In particular, for each given $\psi \in \mathcal{H}$ the association $\mathcal{F} \ni X \mapsto \langle\psi|\, E_X\, \psi\rangle$ defines a probability measure on the measure space $(\Omega, \mathcal{F})$.

Properties (i)-(iii) imply $0 \leq E_X \leq \mathbb{1}_{\mathcal{H}}$ (where the latter inequality sign means that $\mathbb{1}_{\mathcal{H}} - E_X$ is a positive operator) for all X, which is easy to see if one comprehends that $\mathbb{1}_{\mathcal{H}} - E_X = E_{X^c}$ with the complement X^c of X in Ω. Such operators were named *effects* by Ludwig.

Definition 1.5 [*Effects*]

A positive operator E acting on some Hilbert space $\mathcal{H}$ with the property that $\mathbb{1}_{\mathcal{H}} - E$ is a positive operator as well (and consequently $\sigma(E) \subseteq [0,1]$), is called an effect. In particular the elements of a POVM are effects.

The Special Case of a PVM

A very important class of measurements are *projective measurements* where the effects constituting the associated POVM are altogether orthogonal projections (actually, in most basic textbooks on quantum theory only projective measurements are discussed). In this case the POVM is a *projection valued measure*:

Definition 1.6 [*PVM*]

A POVM with the additional property

(iv) $(E_X)^2 = E_X \quad$ *for all* $X \in \mathcal{F}$

is called a projection valued measure (PVM). In this case – to indicate the fact that the effects are projections – we write P_X instead of E_X. A measurement (like) process whose associated POVM is a PVM is called projective.

The projection property together with additivity of the effects P_X constituting a PVM has as a crucial technical consequence a very distinguishing property of PVMs in contrast to non projective POVMs, which will repeatedly provide a strong and convenient technical tool when something is to be shown about projective measurements later in this work:

Corollary 1.7

A POVM on some set Ω acting on some Hilbert space $\mathcal{H}$ is a PVM if and only if for all measurable subsets X and X' of Ω the associated effects E_X and $E_{X'}$ obey

$$E_X E_{X'} = E_{X \cap X'} \tag{1.25}$$

or equivalently

$$E_X E_{X'} = 0 \quad \text{whenever} \quad X \cap X' = \emptyset \tag{1.26}$$

Proof: "⇒": Let $\{E_X\} \equiv \{P_X\}$ be a PVM acting on $\mathcal{H}$. We start with proving the latter assertion (1.26), i.e. we prove the mutual orthogonality $P_X P_{X'} = 0$ of projections associated with disjoint subsets $X \cap X' = \emptyset$. To see this, observe first that the disjointness of X and X' entails

$$\begin{gathered} P_X + P_{X'} = P_{X\cup X'} = P^2_{X\cup X'} = (P_X + P_{X'})^2 = \\ = P_X^2 + P_{X'}^2 + P_X P_{X'} + P_{X'} P_X = P_X + P_{X'} + P_X P_{X'} + P_{X'} P_X \end{gathered} \tag{1.27}$$

and comparing the first and the last expression reveals that P_X and $P_{X'}$ anticommute: $P_X P_{X'} + P_{X'} P_X = 0$. But anticommuting projections commute as well and their product is thereby zero, since

$$P_X P_{X'} = P_X P_X P_{X'} = -P_X P_{X'} P_X = P_{X'} P_X P_X = P_{X'} P_X \tag{1.28}$$

i.e.

$$P_X P_{X'} = -P_{X'} P_X = -P_X P_{X'} \tag{1.29}$$

and consequently

$$P_X P_{X'} = 0 \tag{1.30}$$

Indeed, this already implies the seemingly more general condition (1.25) which is not so obvious in the first place (the other way around, i.e. that (1.25) implies (1.26) of course obvious): If X and X' are arbitrary (not necessarily disjoint) measurable subsets of Ω and we decompose X into the disjoint sets $X \cap X'$ and $X \cap X'^c$ (where the superscript c indicates the complement within Ω) and X' into $X \cap X'$ and $X^c \cap X'$, we get the crucial relation

$$\begin{gathered} P_X P_{X'} = \\ = P_{(X\cap X')\cup(X\cap X'^c)} P_{(X\cap X')\cup(X^c\cap X')} = (P_{X\cap X'} + P_{X\cap X'^c})(P_{X\cap X'} + P_{X^c\cap X'}) = \\ = P^2_{X\cap X'} + P_{X\cap X'} P_{X^c\cap X'} + P_{X\cap X'^c} P_{X\cap X'} + P_{X\cap X'^c} P_{X^c\cap X'} = \\ = P_{X\cap X'} \end{gathered} \tag{1.31}$$

where the last three expressions in the third line vanish due to property (1.30) and the facts that $(X \cap X') \cap (X^c \cap X') = \emptyset$ and so on.

"⇐": Now we show that a POVM obeying for all X, X' condition (1.25) (or equivalently (1.26)) is projective, such that (1.25) as well as (1.26) are indeed equivalent to the projection property $E_X^2 = E_X$ for all X. This is easy to see, since

$$E_X + E_{X^c} = E_{X\cup X^c} = E_\Omega = \mathbb{1}_\mathcal{H} \tag{1.32}$$

together with condition (1.25) entails that

$$0 = E_\emptyset = E_{X\cap X^c} = E_X E_{X^c} = E_X(\mathbb{1}_\mathcal{H} - E_X) = E_X - E_X^2 \tag{1.33}$$

such that $E_X^2 = E_X$. ■

Observable Operator

Now we come back to measurement (like) processes and return to the more illustrative discrete framework. As already discussed in section 1.1.1, in a usual measurement, each pointer state ϕ_k is associated with a real number $\lambda(k)$ which can be thought of in the pointer picture

as the number on which the pointer points or more general as the outcome of the measurement, which might be also only a component of a vector valued set of outcomes associated with ϕ_k (we will henceforth loosely refer optionally to both – the elements of the index set $k \in I$ which label the different pointer states and the associated values $\lambda(k)$ – as *'outcomes'*). Then by the identification $E_k \equiv E_{\lambda(k)}$ for all $k \in I$, the associated POVM can be defined on the set $\Omega = \lambda(I)$. Since now each PVM is uniquely associated with a selfadjoint operator by the spectral theorem, we can associate a unique selfadjoint operator acting on the Hilbert space of the measured system with each *projective* measurement, whose spectrum coincides with the set of possible outcomes. We shall call this operator the associated *observable operator*:

Definition 1.8 [*Observable Operator*]

Consider a projective measurement (like) process whose pointer states ϕ_k are associated with values $\lambda(k) \in \mathbb{R}$ (outcomes), respectively, such that the associated PVM $\{P_k\} \equiv \{P_{\lambda(k)}\}$ can be defined on the set $\Omega = \lambda(I)$. This process defines uniquely a selfadjoint operator $\mathcal{A}$ acting on $\mathcal{H}_S$ with spectrum $\sigma(\mathcal{A}) = \lambda(I)$ whose spectral representation is given by

$$\mathcal{A} = \sum_{k \in I} \lambda(k)\, P_{\lambda(k)} \tag{1.34}$$

$\mathcal{A}$ is called the observable operator associated with the measurement (like) process.

If $\psi \in \mathcal{H}_S$ is the initial state of the measured system, $\mathcal{A}$ provides a compact expression for the expectation value $\langle \lambda \rangle_\psi$ of the outcomes of the experiment via

$$\begin{aligned} \langle \lambda \rangle_\psi &= \sum_{k \in I} \lambda(k)\, \mathbb{P}^\psi(k) = \sum_{k \in I} \lambda(k)\, \left\langle \psi \right| P_{\lambda(k)}\, \psi \rangle = \\ &= \left\langle \psi \right| \left(\sum_{k \in I} \lambda(k)\, P_{\lambda(k)} \right) \psi \Big\rangle = \langle \psi |\, \mathcal{A}\, \psi \rangle \end{aligned} \tag{1.35}$$

Furthermore, due to the spectral theorem, $\mathcal{A}$ is itself a compact expression for the underlying PVM.

In general the spectrum of an observable operator can also be continuous or have a continuous and a discrete pure point part (think e.g. of a Hamiltonian with an appropriate potential), such that the general spectral representation can be given by an operator valued Stiltjes integral (see e.g. Reed-Simon [275] for details)

$$\mathcal{A} = \int_{\sigma(\mathcal{A})} \lambda \; dP(\lambda) \tag{1.36}$$

where the operator valued measure $dP(\lambda)$ generates for each $\psi \in \mathcal{H}_S$ the spectral measure $d\mu_\psi = d \langle \psi |\, P(\lambda) \psi \rangle$ on $\sigma(\mathcal{A})$ which can have a pure point and an absolutely continuous part (in pathological situations there might be a singularly continuous part as well).

Relating State Transformers to the POVM

The effects E_k constituting the POVM associated with an efficient measurement like process were defined by the associated state transformers $\mathcal{R}_k$ via the relation

$$E_k = \mathcal{R}_k^\dagger \mathcal{R}_k \tag{1.37}$$

Thus, according to the polar decomposition theorem, we may write (observe that E_k is positive and thereby $\sqrt{E_k}$ is a well defined positive operator)

$$\mathcal{R}_k = U_k \sqrt{E_k} \tag{1.38}$$

where U_k is a partial isometry which might differ for different k. These operators are sometimes called *measurement back-action* in the quantum measurement literature (e.g. [192]) and we shall adopt this terminology in the following, often they are assumed to be unitary or just the identity $U_k = \mathbb{1}_{\mathcal{H}_S}$ for all k, which is actually not true for many realistic measurements. If $U_k = \mathbb{1}_{\mathcal{H}_S}$, i.e. the initial states are transformed by the operators $\mathcal{R}_k = \sqrt{E_k}$, the measurement is called a *Lüders measurement*. We will come back to this in more detail later.

In Terms of Density Operators

The notion of state transformers will be successively generalized during the course of this chapter. As a starting point, note that the state transformers naturally translate to the density operator level: If $\rho = |\psi\rangle\langle\psi| \in \mathcal{S}(\mathcal{H}_S)$ is an initial pure density operator of the measured system, the state transformation associated with outcome l (respectively $\lambda(l)$) can be characterized by a linear mapping of the form

$$\rho \quad \longmapsto \quad \mathcal{W}(\rho \mid l) := \mathcal{R}_l \, \rho \, \mathcal{R}_l^\dagger \tag{1.39}$$

where the right hand side of (1.39) – properly normalized (i.e. divided by its own trace) – is the final density operator of the measured system upon outcome l which happens with probability[24]

$$\begin{aligned} \mathbb{P}^\psi(l) = \langle\psi| \, E_l \, \psi\rangle = \mathrm{Tr}_{\mathcal{H}_S}\left[|\psi\rangle\langle\psi| \, E_l\right] = \\ = \mathrm{Tr}_{\mathcal{H}_S}\left[\rho \, \mathcal{R}_l^\dagger \mathcal{R}_l\right] = \mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{R}_l \, \rho \, \mathcal{R}_l^\dagger\right] = \mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid l)\right] \equiv \mathbb{P}^\rho(l) \end{aligned} \tag{1.41}$$

We will adopt the naming *state transformers* for the mappings $\mathcal{W}(\cdot \mid l)$ acting on density operators, whether state transformers acting on elements of $\mathcal{H}_S$ (wave functions) or on density operators acting on $\mathcal{H}_S$ are meant will be always clear from the context.

Since the mappings $\mathcal{W}(\cdot \mid l)$ and the trace operation are linear and since any mixed state can be written as a (convex) linear combination of pure states, these notions naturally extend from

[24] In calculation (1.41) we exploit the cyclicity of the trace and the fact (which will be repeatedly applied in the following) that for any bounded operator $\mathcal{A}$ acting on some Hilbert space $\mathcal{H}$ we have for any $\psi \in \mathcal{H}$ and associated orthogonal, one dimensional projection $P_{[\psi]} = |\psi\rangle\langle\psi|$

$$\langle\psi| \, \mathcal{A} \, \psi\rangle = \mathrm{Tr}_{\mathcal{H}}\left[P_{[\psi]}\mathcal{A}\right] \tag{1.40}$$

as it is straightforwardly verified.

pure states to all of $\mathcal{S}(\mathcal{H}_S)$, i.e. we can characterize transitions of mixed states to mixed states in the above sense as well.

We will encounter that these notions can be also naturally generalized to measurement (like) processes with state transitions for which the final state of the measured system can only be described as a mixed state even if the initial state was pure, i.e. to non-efficient measurements. Some first basic analysis in this direction will be briefly presented in the following section. Later in section 1.6 we will introduce the notions of *superoperators*, *completely positive maps* and *state transformation valued measures* also known as *instruments* (which generalize the notion of POVMs, only instead of effects based on the much more fundamental state transformers) to account for state transformers from a very general point of view.

1.2.2 Non-Efficient Measurements

Some basic words about non-efficient measurements are in order at this stage, too (we shall repeatedly come back to this issue later). Of course, descriptions of non-efficient measurements force us to analyse the associated state transformations on the density operator level.

Measurements can be non-efficient simply because there is no pure final state of the measured system since the latter is entangled with another system like the apparatus. In this case, the only possibility to obtain a final state of the measured system, if feasible, is to trace out the other system and to associate the reduced density operator of the measured system with the latter, which then must be a mixed state[25]. Instructive examples of such fundamentally non-efficient measurements will be encountered during the analysis of indirect measurements in section 1.4.4.

Coarse Grained Measurements

Another possible root of non-efficient descriptions of measurements is classical noise or ignorance: If we do not know or name the precise outcome, we cannot know or name a final pure state of the measured system but only a mixed state consistent with the information at hand, such that an actual efficient measurement becomes a non-efficient one in its description. Most directly we may describe a coarse grained state transition of initial density operator $\rho \in \mathcal{S}(\mathcal{H}_S)$, such that we attribute a final state to the measured system which is consistent with the constraint that we only know – or take into account – that the outcome k is contained in some set J containing several elements. In view of (1.39) and (1.41) we can in this case weight the possible final states

$$\rho_k = \frac{\mathcal{W}(\rho \mid k)}{\mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid k)\right]} \tag{1.42}$$

consistent with $k \in J$ with their associated quantum probabilities $\mathbb{P}^\rho(k) = \mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid k)\right]$ to

[25] As already explained above, it can be nonetheless possible to associate a pure final state to a subsystem, like in case of one of two entangled spin$-\frac{1}{2}$ particles which is absorbed by a screen after passing the inhomogeneous magnetic field of a Stern-Gerlach device. In this case, it is usually impossible to associate a pure final state (or even any final state) with the two particle system or with the absorbed particle, but a well defined pure state may be attributed to the particle which was not absorbed and whose spin in a given direction was measured by this procedure as well (at least if the measurement was ideal).

obtain the state transformers

$$\begin{aligned}\mathcal{W}(\rho \mid k \in J) = \sum_{k\in J} \mathbb{P}^{\rho}(k)\, \rho_k = \sum_{k\in J} \mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid k)\right]\, \frac{\mathcal{W}(\rho \mid k)}{\mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid k)\right]} = \\ = \sum_{k\in J} \mathcal{W}(\rho \mid k) = \sum_{k\in J} \mathcal{R}_k\, \rho\, \mathcal{R}_k^{\dagger}\end{aligned} \tag{1.43}$$

We can attribute to this transition the probability

$$\begin{aligned}\mathbb{P}^{\rho}(k \in J) = \sum_{k\in J} \mathbb{P}^{\rho}(k) = \mathrm{Tr}_{\mathcal{H}_S}\left[\sum_{k\in J} \mathcal{R}_k\, \rho\, \mathcal{R}_k^{\dagger}\right] = \mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid k \in J)\right] = \\ = \mathrm{Tr}_{\mathcal{H}_S}\left[\sum_{k\in J} \mathcal{R}_k^{\dagger}\mathcal{R}_k\, \rho\right] = \mathrm{Tr}_{\mathcal{H}_S}\left[\sum_{k\in J} E_k\, \rho\right] = \mathrm{Tr}_{\mathcal{H}_S}\left[E_J\, \rho\right]\end{aligned} \tag{1.44}$$

where we have recovered the cumulative effects $E_J = \sum_{k\in J} E_k$ from above as desired, and the final state[26]

$$\rho_{fin} = \frac{\mathcal{W}(\rho \mid k \in J)}{\mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid k \in J)\right]} \tag{1.46}$$

The final density operator (1.46) yields the right quantum probabilities $\mathrm{Tr}_{\mathcal{H}_S}\left[E\, \rho_{fin}\right]$ associated with any effect E for future measurements on the measured system conditional on the fact that the considered measurement was performed with some result $k \in J$.

Non-Selective Measurements

The maximal example in this direction are *non-selective measurements*, which attribute to any given measurement the final state associated with probability 1, i.e. the final state conditional only on the fact that the respective measurement was performed disregarding its actual outcome. Denoting as above the total index set of possible outcomes by I and noting that $\mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid k \in I)\right] = \mathbb{P}^{\rho}(k \in I) = \sum_{k\in I} \mathbb{P}^{\rho}(k) = 1$, we can express the final state of the measured system disregarding the outcome of the measurement by the mixed density operator

$$\rho_{fin} = \frac{\mathcal{W}(\rho \mid k \in I)}{\mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid k \in I)\right]} = \frac{\sum_{k\in I} \mathcal{W}(\rho \mid k)}{1} = \sum_{k\in I} \mathcal{W}(\rho \mid k) = \sum_{k\in I} \mathcal{R}_k\, |\psi\rangle\, \langle\psi|\, \mathcal{R}_k^{\dagger} \tag{1.47}$$

In particular, exclusively in this case the state transformation is linear and the state transformers directly yield the final state of the measured system without further need of normalization (we

[26] Indeed we can interpret (1.46) as a mixed state in the usual way – i.e. as a convex linear combination of the possible final states ρ_k with $k \in J$ weighted with the conditional probabilities $\mathbb{P}^{\rho}(k \mid k \in J)$ (such that they sum up to unity for $k \in J$) – by noting that we can denote the joint probability of $\mathbb{P}^{\rho}(k \wedge k \in J)$ as $\mathbb{P}^{\rho}(k)$ if $k \in J$ and zero otherwise: $\mathbb{P}^{\rho}(k \wedge k \in J) = \mathbb{P}^{\rho}(k)\, \chi_{\{J\}}(k)$ with the indicator function $\chi_{\{J\}}$ of J. Consequently

$$\begin{aligned}\rho_{fin} = \sum_{k\in I} \mathbb{P}^{\rho}(k \mid k \in J)\, \rho_k = \sum_{k\in I} \frac{\mathbb{P}^{\rho}(k \wedge k \in J)}{\mathbb{P}^{\rho}(k \in J)}\, \rho_k = \sum_{k\in I} \frac{\mathbb{P}^{\rho}(k)\, \chi_{\{J\}}(k)}{\mathbb{P}^{\rho}(k \in J)}\, \rho_k = \\ = \frac{\sum_{k\in J} \mathbb{P}^{\rho}(k)\, \rho_k}{\mathbb{P}^{\rho}(k \in J)} = \frac{\mathcal{W}(\rho \mid k \in J)}{\mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid k \in J)\right]}\end{aligned} \tag{1.45}$$

may also include unitary evolution in this scheme, as a processes associated with probability 1 since it is deterministic, such that we identify it with a single state transformer $\mathcal{R} = U$ which is just the associated unitary operator). The density operator (1.47) yields the right quantum probabilities $\mathrm{Tr}_{\mathcal{H}_S}[E\,\rho_{fin}]$ associated with any effect E for future measurements if the outcome of the first measurement is not taken into account but only the fact that it took place. ρ_{fin} can also be associated with the final state ensemble $\{(\psi_k, p_k)\}$ with fractions $p_k = \mathbb{P}^{\psi}(k)$ if the measurement was performed on a large number of systems identically prepared in initial state ψ.

A remarkable fact about non-selective measurements, which however lead to some confusion, is that the density operator (1.47) can be obtained from the unitary part of the measurement (like) process alone by tracing out the pointer states, which tempted some authors to suggest to drop the collapse postulate (and thereby to get rid of the measurement problem) and to explain anything about quantum theory with unitary evolution only. The point is only that actual measurements have actual outcomes and the transition to an actual pointer state cannot be explained this way, also if the environment of the pointer, which can be pictorially perceived as a pointer pointing on the pointer, is taken into account. We will come back to this later in section 1.6.6 in more detail.

It is also to be remarked, that non-selective measurements play a certain role in the description of open quantum systems, where a given system might have uncontrollable measurement like interaction with its environment (this interaction might mix up with unitary interaction as well). Some more details about general aspects of open quantum systems will be given in section 1.6.

State Transformers and POVMs of Non-Efficient Measurements in General

We will prove later in section 1.6, that in a reasonable setting *any* state transformation can be characterized on the density operator level by a linear mapping of the form

$$\rho \quad \longmapsto \quad \sum_k \mathcal{R}_k\,\rho\,\mathcal{R}_k^{\dagger} \tag{1.48}$$

with k running in some discrete set (but now in general different ks need not be associated with different outcomes of a measurement (like) process) and correspondingly a countable set $\{\mathcal{R}_k\}$ of bounded operators acting on the respective Hilbert space, such that the right hand side of (1.48) is the final state of this transformation if properly normalized (this is the Kraus representation of completely positive maps). Moreover, the principles of quantum theory suggest that the probability of such transformations is always given by the trace over the right hand side of (1.48) which is also equal to the normalization to recover the final density operator, as already encountered in the primitive example of coarse grained measurements above.

This entails that the notions of state transformers and POVMs, which we derived for efficient measurements above, can be straightforwardly extended to non-efficient measurements on the density operator level not only in case of coarse grained measurements as presented above, but also when a *single outcome* is associated with state transformations of the measured system which in general transform pure states to mixed states. In particular, according to this observation, say the state transformation associated with outcome α (we shift here to indicate the

outcomes by Greek letters to use Latin letters as dummy indices) of a non-efficient measurement is characterized by a linear mapping of the form $\mathcal{W}(\rho \mid \alpha) = \sum_k \mathcal{R}_{\alpha k}\, \rho\, \mathcal{R}_{\alpha k}^\dagger$ such that

$$\mathcal{S}(\mathcal{H}_S) \ni \rho \quad \longmapsto \quad \rho' = \frac{\mathcal{W}(\rho \mid \alpha)}{\mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid \alpha)\right]} \tag{1.49}$$

which happens with probability

$$\begin{aligned} \mathbb{P}^\rho(\alpha) = \mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid \alpha)\right] = \mathrm{Tr}_{\mathcal{H}_S}\left[\sum_k \mathcal{R}_{\alpha k}\, \rho\, \mathcal{R}_{\alpha k}^\dagger\right] = \\ = \mathrm{Tr}_{\mathcal{H}_S}\left[\sum_k \mathcal{R}_{\alpha k}^\dagger \mathcal{R}_{\alpha k}\, \rho\right] = \mathrm{Tr}_{\mathcal{H}_S}\left[E_\alpha\, \rho\right] \end{aligned} \tag{1.50}$$

where we have used the cyclicity of the trace and defined the effect $E_\alpha := \sum_k \mathcal{R}_{\alpha k}^\dagger \mathcal{R}_{\alpha k}$ associated with outcome α which constitutes together with the analogously defined effects associated with the other outcomes the associated POVM. Note again that a transition of the form (1.49) is in general non-linear and only linear if the associated probability $\mathbb{P}^\rho(\alpha) = \mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid \alpha)\right]$ is unity.

So much for now, we will come back to this in more detail in section 1.6.

1.3 Intermezzo: How (not) to Understand Quantum Theory On the Results of Kochen & Specker and Bell

Before we go on with a structured analysis of different kinds of measurement (like) processes based on the approach developed so far, it makes sense to take one step back in order clarify first some fundamental issues about the physical meaning of this predictive framework. For example, in view of the fact that the quantum formalism does in general not designate any unambiguous pre-measurement properties to the measured system which are reflected by the measurement result, the tempting and seemingly natural idea that quantum measurement reveals preexisting properties pertaining to the measured system which are not represented in the quantum formalism should be scrutinized. This and related issues shall be considered in the following two sections to develop a basic physical understanding of the formalism which provides a conceptual basis for the different problems which shall be tackled in the rest of this work.

The following section essentially presents a modified version of the famous no-go theorems of quantum theory, which are commonly associated with the names of *Kochen & Specker and Bell*. The subsequent section then derives based on this result a version of *Bell's* famous *theorem*, and thereby highlights an empirically well verified intrinsic feature of quantum theory, namely *quantum nonlocality*.

Spin Measurements

The following lines of argument will be illustrated with examples of ideal spin measurements on spin$-\frac{1}{2}$ systems. Ideal measurements and spin measurements will be discussed later in more detail, but the reader is assumed to be familiar with the basic formalism of spin and ideal spin measurements. Some of the notation used is depicted in footnote 28.

The spatial parts of the wave functions are to be thought of as (idealized as) compactly supported wave packets and we thus only explicitly consider the spinor part of the wave function as usual in such examples. *Ideality* of a spin measurement means that it is projective and that the associated PVM also constitutes the associated set of state transformers (later we will see that this can only be an idealization). The measurements are fully characterized by the associated PVMs and accordingly the pointer states are not considered[27].

The described experiments were successfully performed and the quantum predictions verified [252], but with polarized photons and polarization filters, beam splitters and so on instead of fermions and SGMs. The photon experiments are much more feasible and structurally equivalent in theoretical description to the associated spin experiments.

1.3.1 Against Naive Ignorance Interpretations

In this section we shall have a look at a controversial issue about measurement (like) processes, which is related with the much more celebrated measurement problem only indirectly, but which did not create less confusion than the latter.

One merit of the approach towards measurement (like) processes presented in this work is that it is not about operators in the first place at all, but plainly about certain transitions of quantum states (transitions which are in particular associated with definite macroscopic outcomes), and we *derived* that the associated probabilities and mean values, which directly follow from the Born rule, can be compactly expressed in terms of the selfadjoint operators, which are *commonly postulated as 'observables' in the first place*. The present approach does not suggest to take these operators as something else than mathematical tools with which the statistics associated with certain state transitions and associated macroscopic readouts can be practically calculated. The usual approach to postulate selfadjoint operators as observables in contrast, seems to suggest that these operators have a deeper meaning, deeply related with physical properties of the quantum systems to which these operators are attributed; properties which are then revealed by the associated experiments, like it is (more or less) the case for measurements in classical physics, where certain quantities pertaining to a given system can be ascertained by experiment.

Considering a measurement (like) process (for simplicity we consider only efficient measurements)

$$\psi_0\phi_0 \xrightarrow{U} \sum_k c_k\psi_k\phi_k \xrightarrow{(*)} \psi_l\phi_l \tag{1.51}$$

on the other hand, suggests that no property of the measured system is revealed at all by quantum measurement, but rather created (namely being in the final state ψ_l). The measurement result, here represented by the number l, does not represent anything which was pertaining to the system prior to the measurement, but only indicates its final state and the final pointer position of the measuring device.

[27] Actually, the assumption that the state transformers are given by the projections of the associated PVM corresponds to an *ideal which-way detector* behind the respective Stern-Gerlach magnet (SGM) which does not disturb the system more than by collapsing its wave function. More realistically, we would consider screens or absorbing detectors such that the final state of the measured particle cannot be determined. This does not restrict the arguments because we are not interested in the final states of absorbed particles but only in the measurement outcomes and in the state of the remaining system if the actually measured particle was a subsystem, which we can specify no matter if the latter was absorbed by the device or if it passed an ideal which-way detector.

It is now tempting to assume that the quantum description of measurements including the peculiar collapse $(*)$ of the wave function is not to be taken literally as representing a dynamical process of individual measured systems, but rather a means of human description to predict relative frequencies of measurement outcomes for ensembles of identically prepared systems which reveal properties of the individual measured systems whose values might be unknowable in principle prior to the respective measurements. Indeed, one can sometimes hear or read something like *'the wave function and its changes only represent our knowledge about the considered system'*, and even some authors who are well acquainted with the foundations of quantum theory suggestively assert that quantum theory is *only about information* (see e.g. [351]). Some conclude then that the collapse of the wave function does not rise any problems of interpretation, since it does not represent a physical dynamical process of the considered system but it only expresses a sudden change in knowledge of the experimenter.

Now, whatever 'the wave function represents information' might precisely mean, if it refers to *incomplete* information with respect to the values of all observables of individual systems, such a view is untenable, as the following arguments will illustrate. If it refers to *complete* information about the considered system, I see now reason to call it 'information' and not to stick to the notion of 'state' which has much less potential of creating misunderstandings; in particular, in this case the problems associated with wave function collapse are in no way attenuated.

There is actually a third possibility, namely to assume the incompleteness of description of quantum systems by wave functions, but not to naively attribute properties of quantum systems to observable operators and the associated measurements, but to make a reasonable choice for the quantity which completes the physical description, and thereby (among other things) to get rid of the measurement problem. This is the case with Bohmian mechanics (see chapter 4), where the description by a wave function is completed with positions of particles dynamically guided by the wave function in a natural way. The discussion of the measurement process in Bohmian mechanics nicely illustrates how absurd it can be to identify values obtained by quantum measurement with properties of the measured systems in general. This will be briefly illustrated below, an elaborate analysis of the measurement process in Bohmian mechanics will be given in chapter 4.

Illustration

In the following we illustrate the impossibility to identify the outcomes of quantum measurement with preexisting properties of the measured systems by an example of ideal spin measurements[28] on three spin-$\frac{1}{2}$ particles. Consider the three particle initial state

$$\psi = \frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\uparrow\rangle_z + |\downarrow\downarrow\downarrow\rangle_z \right) \tag{1.52}$$

[28] The basic spin formalism is assumend to be known to the reader. The considered Hilbert space of three spin$-\frac{1}{2}$ particles is $\mathcal{H} \cong \mathbb{C}^8$. For $\alpha = x, y, z$ and $i = 1, 2, 3$ the one particle spin operators σ_α^i acting on the three particle Hilbert space we write $\sigma_\alpha^1 = \sigma_\alpha \otimes \mathbb{1}_{\mathbb{C}^2} \otimes \mathbb{1}_{\mathbb{C}^2}$ with associated eigen-projections $P_{\alpha\uparrow}^1 = P_{\alpha\uparrow} \otimes \mathbb{1}_{\mathbb{C}^2} \otimes \mathbb{1}_{\mathbb{C}^2}$ and so on. The thee particle spin states are denoted by $|\uparrow\uparrow\uparrow\rangle_\alpha = |\uparrow\rangle_\alpha \otimes |\uparrow\rangle_\alpha \otimes |\uparrow\rangle_\alpha$ and so on. The outcomes of a spin measurement on particle i (if the reader has worries about numbering identical particles, she might number the SGMs instead) in the $\alpha-$direction are denoted by $S_\alpha^i = +1$ or $\uparrow_\alpha^i$ for 'spin up' and accordingly $S_\alpha^i = -1$ or $\downarrow_\alpha^i$ for 'spin down' (the particle index at the arrows will be omitted if it is clear from the context).

and suppose an ideal spin measurement is performed on each of the particles, one with respect to the $x-$component of the spin and the other two with respect to the $y-$component, respectively. There are obviously three possible sets of SGM orientations which realize this scenario, one for each choice of which of the three SGMs is oriented in the $x-$direction. Later, we will additionally consider a fourth modification of these experiments, where the $x-$component of spin is measured on all of the three particles. We will show that the experimentally well verified [252] predictions of quantum theory for these four experiments are clearly incompatible with the assumption that each particle carries its own spin values (which are only revealed by measurement) prior to the measurements in each individual run[29].

The joint measurement can be characterized by the twelve projections $P^i_{\alpha\uparrow} = |\uparrow\rangle\langle\uparrow|^i_\alpha$ and $P^i_{\alpha\downarrow} = |\downarrow\rangle\langle\downarrow|^i_\alpha$, where $i = 1, 2, 3$ is the particle index, $\alpha = x, y$. These operators are the building blocks of the associated PVMs (and of the state transformers as well). If for example the first SGM measures the $x-$component of spin and the other two the $y-$component, respectively, the probability to obtain 'spin down' in all three measurements – we denote this joint outcome by $(S^1_x = -1) \wedge \left(S^2_y = -1\right) \wedge \left(S^3_y = -1\right)$ or symbolically by $\downarrow_x\downarrow_y\downarrow_y$ – is calculated by

$$\mathbb{P}^\psi\left(\downarrow_x\downarrow_y\downarrow_y\right) = \left\langle\psi\right| P^1_{x\downarrow}P^2_{y\downarrow}P^3_{y\downarrow}\left|\psi\right\rangle \tag{1.53}$$

Since operators associated with measurements on different particles trivially commute, these probabilities do not depend on the time order of the respective measurements[30]. Moreover, as simple calculations show, the marginal probability distribution for the outcomes at one of the tree devices, given the other two measurements were already performed (i.e. the distribution averaged over all possible outcomes of the distant measurements) is precisely the same as the distribution, given the other measurements did not take place before (this sequential independence and its relation with commutativity will be elaborately analysed in chapter 2). But the crucial point for the following arguments is that what is true for the marginal probabilities does in general not hold for the other conditional probabilities, i.e. the distributions associated with a given particle change if measurements are performed on other particles if the outcomes of the latter are not neglected. For example – no matter which of the three devices measures σ_x – the probability to obtain say 'spin up' at one of them is initially $\frac{1}{2}$, but it changes instantaneously to unity if the other two measurements were performed with unequal results. This could be easily comprehended by explicitly calculating the (un-)conditional probabilities (see also calculations (1.58) and (1.60) below), but we shall go a somewhat more indirect way and thereby obtain the promised no-go argument:

To this end, note first that the state (1.52) is a common eigenstate of the three commuting operators

$$\sigma^1_x\sigma^2_y\sigma^3_y \quad \sigma^1_y\sigma^2_x\sigma^3_y \quad \text{and} \quad \sigma^1_y\sigma^2_y\sigma^3_x \tag{1.54}$$

to the eigenvalue -1, respectively. To see that these operators commute, note that for all $i = 1, 2, 3$ the Pauli matrices anticommute $\sigma^i_x\sigma^i_y = -\sigma^i_y\sigma^i_x$ (and the spin operators associated with

[29]Indeed, a strong feature of this argument in contrast to others leading to the same striking conclusion, is that for its experimental verification it would be sufficient in principle to perform each of these four experiments only once, if the measurements could be performed without errors, as the reader will easily comprehend in the following.

[30]To be precise, the form of the state transformers is additionally relevant for this conclusion, see chapter 2 for details about time order independence of measurements.

different particles commute, of course) and reversing the order of the product of two operators in (1.54) involves always two such anticommutations, such that the two minus signs cancel each other out. That (1.52) is indeed a joint eigenstate of these three operators with joint eigenvalue -1 is easily confirmed by noting that $\sigma_x|\uparrow\rangle_z = |\downarrow\rangle_z$, $\sigma_x|\downarrow\rangle_z = |\uparrow\rangle_z$, $i\sigma_y|\uparrow\rangle_z = -|\downarrow\rangle_z$ and $i\sigma_y|\downarrow\rangle_z = |\uparrow\rangle_z$.

Now observe that if a quantum system is in an eigenstate of a given observable operator with respect to a given eigenvalue, this eigenvalue will be the outcome of the associated measurement with certainty and all other probabilities vanish for this initial state (this is easily confirmed by writing the operator in its spectral representation and recalling the relation of the latter with the associated PVM). Moreover, if operators commute they can be jointly diagonalized and thereby, functions of them can be easily translated into functions of the associated eigenvalues, e.g. if we denote the eigenvalues of the spin operators by $s^i_\alpha(\uparrow) = +1$ and $s^i_\alpha(\downarrow) = -1$, the first operator in (1.54) can be written in its spectral representation

$$\sigma^1_x\sigma^2_y\sigma^3_y = \sum_{\lambda_i=\uparrow,\downarrow} s^1_x(\lambda_1)\, s^2_y(\lambda_2)\, s^3_y(\lambda_3)\;\; P^1_{x\lambda_1} P^2_{y\lambda_2} P^3_{y\lambda_3} \tag{1.55}$$

(where obviously each of the two eigenvalues $s^1_x(\lambda_1)\, s^2_y(\lambda_2)\, s^3_y(\lambda_3) = \pm 1$ is fourfold degenerate). In consequence, since ψ is an eigenstate of $\sigma^1_x\sigma^2_y\sigma^3_y$ with eigenvalue -1, the product $S^1_x S^2_y S^3_y$ of the outcomes of the tree spin measurements will be -1 with certainty, which is to say all combinations of the tree outcomes containing exactly two or zero 'spin down's have probability zero (e.g. the joint outcome $\uparrow_x\downarrow_y\downarrow_y$ corresponds to the value $S^1_x S^2_y S^3_y = s^1_y(\uparrow)s^2_y(\downarrow)s^3_y(\downarrow) = +1\cdot(-1)^2 = +1$ and will thus not be realized with certainty, if ψ is the initial state). The same holds for the other two considered possible choices of orientations of the measuring devices with respect to the spin operators $\sigma^1_y\sigma^2_x\sigma^3_y$ and $\sigma^1_y\sigma^2_y\sigma^3_x$.

Now suppose an ensemble of three particle systems in the state ψ, respectively, is prepared and measurements as described above (one time spin$-x$ and two times spin$-y$, respectively) are performed on each of the ensemble members, such that we recover the predictions of quantum theory in the measured relative frequencies. Now we make the natural (but as we will see delicate) assumption that in each individual run of the experiment, each of the particles carries its true spin values already prior to the measurements, which are only revealed by the experiment. We saw that regardless of which of the three devices is chosen to measure σ_x in each single run, the joint outcomes of the three spin measurements will always involve either exactly one or three times 'spin down', i.e. the only realized joint outcomes are of the form $\downarrow\downarrow\downarrow, \downarrow\uparrow\uparrow, \uparrow\downarrow\uparrow$ and $\uparrow\uparrow\downarrow$ where always one of the three arrows represents the outcome of the σ_x-measurement and accordingly the other two the outcomes of the σ_y-measurements.

This does severely restrict the possibilities for the assumed spin values carried by the individual particles in each individual run: There are $2^6 = 64$ possibilities to assign one of the two possible spin values to each of the three particles in $x-$ and $y-$direction, respectively, but only eight of these survive with the prescribed constraint that, whenever a triple of spin values of the three particles is chosen – two times with respect to the $y-$direction and the remaining one with respect to the $x-$direction –, 'spin down' never occurs precisely two times and never occurs not

at all. It is a simple problem to find these eight value sets[31], they are given by

$$\begin{array}{llll} (\downarrow\downarrow\downarrow)_x \wedge (\downarrow\downarrow\downarrow)_y & (\downarrow\uparrow\uparrow)_x \wedge (\downarrow\uparrow\uparrow)_y & (\uparrow\downarrow\uparrow)_x \wedge (\uparrow\downarrow\uparrow)_y & (\uparrow\uparrow\downarrow)_x \wedge (\uparrow\uparrow\downarrow)_y \\ (\downarrow\uparrow\uparrow)_x \wedge (\uparrow\downarrow\downarrow)_y & (\downarrow\downarrow\downarrow)_x \wedge (\uparrow\uparrow\uparrow)_y & (\uparrow\uparrow\downarrow)_x \wedge (\downarrow\downarrow\uparrow)_y & (\uparrow\downarrow\uparrow)_x \wedge (\downarrow\uparrow\downarrow)_y \end{array} \tag{1.57}$$

But actually none of these value sets is consistent with the predictions of quantum theory for a joint spin measurement with initial state ψ in which all three measuring devices are oriented in the $x-$direction. To see this, note first that our initial state ψ given by (1.52) is an eigenstate of the operator $\sigma_x^1\sigma_x^2\sigma_x^3$ as well, but this time with respect to eigenvalue $+1$, which can be easily confirmed in the same way as for the other spin operators above (indeed, with elementary relations of the Pauli matrices it is easy to see, that $\sigma_x^1\sigma_x^2\sigma_x^3$ is minus the product of the three operators (1.54)).

Therefore, with the line of argument from above, we conclude that if the $x-$component of spin is measured on all three particles, 'spin down' will be realized either exactly two times or not at all with certainty and the probabilities of all remaining possibilities will be identically zero. But this is obviously inconsistent with the value sets (1.57), since therein $\downarrow_x$ is always realized either exactly tree times or exactly once. Thus, given quantum theory makes the right predictions for the described experiments – which is experimentally well verified [252] – such value sets cannot exist and the assumption that each particle already carries its own spin values prior to the measurements is not tenable for the considered initial state (note that if these spin measurements could be performed perfectly without errors, only four measurements were required to verify the contradiction of experiment with the assumption of preexisting values!).

Some Background

These arguments can be also given in more generality. In particular, without drawing on a particular initial state and for much more general systems and observable operators (the only restriction is that the dimension of the Hilbert space is greater than two), it can be shown that there do not exist value maps[32] which assign to each observable operator one of its eigenvalues, given these value maps respect primitive functions (like sums or products) of commuting operators (a function of operators is mapped to the function of the respective eigenvalues). The structure of these theorems is always the same: A joint measurement of a given set of commuting

[31] Just start e.g. with the set of SGM-orientations where the first SGM is oriented in $x-$direction and accordingly the other two in the $y-$direction and determine all possibilities to distribute up– and down-arrows in the associated boxes of

$$(\square, -, -)_x \wedge (-, \square, \square)_y \tag{1.56}$$

(where '$-$' indicates the slots irrelevant for this set of orientations), such that down-arrow occurs either exactly one or three times. Then repeat this procedure with the thus emerged four couples of triplets for the other two possible sets of orientations to distribute arrows among the remaining open '$-$' slots with the same constraint.

[32] The concept of value maps is meant here in the following way: A value map assigns to a measurement, represented by some observable operator, its future outcome in an individual run. If the experiment is repeated with the same initial state, the outcome is in general predetermined by another value map leading to a different outcome. Which value map is the actual one of an individual initial system might depend on some internal parameter λ besides the initial state (i.e. λ and ψ can be perceived as indexing the different value maps). The value maps as well as the parameters λ of individual systems might be empirically inaccessible ('hidden') in principle, the theorems discussed nonetheless show that the existence of such value maps can be excluded in principle. See below for some more details about value maps.

operators $A, B, C, \ldots$ is considered and other joint measurements of different sets $A, B', C', \ldots$ containing one operator (A) of the first set but further operators $B', C', \ldots$ which commute with A but not with the operators $B, C, \ldots$ of the first set. Then it is shown that A cannot be consistently assigned to one of its eigenvalues independently of the choice of the set of commuting observables with which it is measured (note that we considered measurements above, where e.g. σ_y^1 was measured either together with σ_x^2 and σ_y^3 or with σ_y^2 and σ_x^3 which all commute with σ_y^1 but not with each other). This is the stuff the famous *no-go theorems* of quantum theory – mostly associated with the names of Kochen & Specker and Bell – are made of.

The first attempt in this direction goes back to John von Neumann, who claimed to prove in his groundbreaking book [334] that the existence of so called *dispersion free states* – that is a completion of the quantum state with possibly empirically inaccessible parameters determining together with the quantum state the outcomes of future measurements – are incompatible with the predictions of quantum theory. Von Neumann's no-go theorem was accepted over three decades as a knock-down argument against so called *hidden variable theories* (although one should have been suspicious at least since 1952, where David Bohm presented his theory [39, 40] and thereby provided a counter example), until John Bell pointed out in [26] that although it is mathematically correct, of course, it is physically irrelevant in a very obvious way: Von Neumann build his theorem on the assumption that the assignment of values to operators provided by dispersion free states respects linearity for non commuting operators, although this assumption is clearly unfounded since roughly speaking a linear combination of two non commuting observable operators corresponds to a completely different experimental setup than it could be given by a combination of the two original measurements, which in particular cannot be performed simultaneously (see [26] for details and an example). After breaking down von Neumann's alleged result, Bell pointed out that a theorem proven by Andrew Gleason in 1957 [151], which was not related with the question of dispersion free states in the first place[33], entailed the mathematical assertion, on which von Neumann grounded his impossibility proof, but without using problematical assumptions about non commuting operators. Based on Gleason's work, Bell and afterwards independently Kochen and Specker proved no-go results [208] of the kind presented here, only for Hilbert spaces with dimension 3 or greater[34], without drawing on a particular initial state and more laborious[35].

Bell's final analysis of these Kochen-Specker-Bell theorems is today common knowledge in the foundations of quantum theory and (essentially) uncontroversially adopted. It in particular refutes the conclusions which previously von Neumann and many others draw from the formally

[33]Roughly speaking, Gleason was able to show that each probability measure which weights the linear subspaces of an at least three dimensional Hilbert space must be at least formally equivalent to the probability measure of the Born rule.

[34]In contrast to higher dimensions, all operators commuting with a given operator acting on a two dimensional Hilbert space commute with each other.

[35] Later Mermin simplified their proofs essentially for Hilbert spaces of dimension 4 or greater [241, 240, 239]. The version given here essentially corresponds to the Greenberger-Horne-Zeilinger (GHZ) version [157] of Bell's second theorem (the famous one) and is thus well suited to prove additionally a more astonishing feature of quantum theory than the non-existence of preexisting values of observables, namely quantum nonlocality. We will come back to this below. The present illustration of the connection between the Kochen-Specker-Bell theorem and Bell's famous nonlocality theorem in the GHZ version is very close to Mermin [241, 240, 239]. He presents these results very nicely and illustrates their connections masterly, but the conclusions he draws eventually are incomprehensible to me.

false result and which later were readopted by Kochen and Specker based on the formally correct result, namely that it rules out the possibility of dispersion free states. Bell illustrated that it is easily possible to supplement quantum theory in such a way that it was possible to predict all measurement outcomes with certainty if all relevant physical data were known (Bell's example of such a theory is Bohmian mechanics). The requirement, on the other hand, which must be met by such a theory according to these theorems in order to be in accordance with the quantum predictions, is that within its predictive framework the value of a measurement outcome cannot, in general, depend on the measured system alone (thus the outcome cannot reflect an intrinsic property of the latter) but all relevant all relevant (sub-)systems – like measuring devices, systems with which the considered system is possibly entangled etc. – must be taken into account. The measuring result cannot generally mirror anything pertaining solely to the system but is the indivisible product of an interaction of all systems involved in the measurement process.

The Kochen-Specker-Bell theorems are usually associated with the notions of *'hidden variables'* and *'contextualtity'*. But since these notions, although in some sense appropriate, seem to have created more confusion than clarity, the present discussion does not draw on them in the first place but rather tries to express as plainly as possible what the theorems say about physics without using picturesque but potentially misleading notions. A few words about these notions are nonetheless in order:

Hidden variables, represented frequently by the letter λ, are hypothetical variables which can be empirically inaccessible in principle, with which the state (wave function) of a considered quantum system has to be supplemented in order to become a dispersion free state, i.e. a state for which the outcomes of future measurements could be predicted with probability 1, if together with ψ also λ and the law by which λ determines the respective outcomes were known. The members of an ensemble of quantum systems prepared in the same initial state ψ are supposed to differ in general with respect to λ, such that the quantum statistics emerges if the predictions of the hidden variable theory are averaged over λ with respect to its empirical distribution.

Hidden variable theories which are ruled out by the Kochen-Specker-Bell theorems are of the following form: There is a law v_λ^ψ (a value map) assigning to each observable operator A (associated with a real world measurement like the spin operators) depending on ψ and λ one of its eigenvalues $v_\lambda^\psi(A) = \alpha \in \sigma(A)$, or equivalently assigning to each orthogonal projection acting on the Hilbert space of the measured system either the probability[36] 1 or 0 (for generalizations of the no-go theorems to POVMs see e.g. [309]). Such theories are called *non-contextual*.

Contextuality, on the other hand, means that the hypothetical values of observables depend on the precise *context* of the measurement. For example, the value of A might depend on the set of commuting observables with which it is simultaneously measured, i.e. the value of A might be different when it is measured together with $B, C, \ldots$ or when it is measured together with $B', C', \ldots$ (where both sets commute with A but not with each other as explained above). In this case, the value maps v_λ^ψ would not only depend on the observable operator A, to which one of its eigenvalues shall be assigned, but on other commuting observables with which together A is possibly measured as well. Hidden variable theories of this kind are called contextual hidden variable theories and the corresponding possibility to attribute properties with contextual values

[36]Note that this is not about determinism in the first place, the primary question is only if observables have values prior to measurement, if these values (or their determining variable λ) are guided by a deterministic law is a different question, in particular, also theories which associate values to observables preexisting only at the immediate instant before measurement which reveals them, are ruled out by the no-go theorems.

to quantum systems is not ruled out by the Kochen-Specker-Bell theorems.

This way of speaking is nonetheless rather misleading, as a closer look at the usual prototype example of a contextual hidden variable theory – namely Bohmian mechanics – shows: On the one hand, Bohmian mechanics (see chapter 4 for details) fits into the picture so far, in particular, the outcomes of measurements are determined by the initial wave function together with the initial configuration $\lambda = X$ of all relevant quantum systems including the apparatus and thereby naturally depend on the context of the measurements and is thus of course in no way ruled out by the no-go theorems, which are an easily understandable result within this theory. But in point of fact both, to call it a hidden variable theory and to call it contextual in the sense that contextual properties were attributed to quantum systems, are inappropriate attributions. Calling X a hidden variable is apparently off the point, since in Bohmian mechanics rather the wave function is hidden in contrast to the particle positions – what we see is not wave functions but objects with positions. Moreover, Bohmian mechanics does not only attribute no non-contextual properties but actually no properties at all to the particles, except having a position and a guiding wave.

As an example, this becomes unambiguously apparent when we have a closer look at spin measurements in a Bohmian world: One and the same spin$-\frac{1}{2}-$particle starting at one and the same initial position with one and the same initial wave function will be measured to have 'spin up' or 'spin down' in a given direction – say the $z-$direction – depending on the orientation of the magnetic field of the measuring device. A very basic analysis of the Bohmian equations of motion[37] shows that, say the particle is prepared in an eigenstate of σ_x and it is measured to have 'spin up' in the $z-$direction, it would have been measured to have 'spin down' in this direction if the polarity of the magnetic field of the SGM would have been inverted, i.e. the particle is deflected in one and the same direction no matter if the inhomogeneous magnetic field of the SGM points in one – or the opposite direction (see [6, 96, 127, 129] for more details).

Thus, spin cannot be regarded as a property of the particle; in Bohmian mechanics spin is nothing else than a degree of freedom in the guiding wave function which has a dynamical effect on the guided particle. This insight does not even need to involve several measurement scenarios with (non-)commuting observables but makes the point clear with a single basic measurement. Analogously, it is easy to see that all other observables except the Bohmian position cannot be reasonably identified with properties of the measured particles. As a consequence, the mystic air about the contextual properties of quantum systems is unromantically destroyed: In Bohmian mechanics, measurement results depend naturally on the experimental setting, since they do not correspond to revealed properties of the measured system (also not to contextual ones) but only to the final states of certain transitions which include all relevant systems, in particular the apparatus settings as well.

[37]This derives simply from the fact that the Bohmian equations of motion are first order PDE's and in consequence due to uniqueness of solutions the Bohmian trajectories cannot intersect in configurations space (which is identical to physical space in case of a single particle). Thus the symmetry line (say $z = 0$) of the experiment constitutes a topological barrier which cannot be crossed by the Bohmian trajectories, which is to say, particles starting at positions with $z > 0$ remain above the symmetry axis and accordingly particles starting at positions with $z < 0$ stay below this line, irrespective of whether the external inhomogeneous magnetic field of the SGM points in the positive or negative $z-$direction (see [6, 96, 127, 129]).

What to Learn from it ?

One way of understanding a fundamental lesson taught by the no-go theorems is in a nutshell:

Take the wave function seriously, not just probabilistically but also dynamically!

Whatever the wave function truly *is*, e.g. the controversial question whether the wave function has ontological or epistemological status, is another question; the only assertion made here is that the wave function and its transformations (in particular the collapse dynamics) *represent* actual transformations of the considered quantum systems leading to empirically accessible facts. This should be clear right from the start if we regard quantum theory not as an effective – but as a fundamental theory of nature as most physicists essentially do, but indeed particularly taking the collapse dynamically seriously is not at all uncontroversial among the physics community.

To see why the Kochen-Specker-Bell theorems strongly suggests to take the wave function dynamically seriously we may consider the correlations between the measurement outcomes of the three measurements of the above example, one measurement of σ_x and two times σ_y. Two outcomes taken together are always perfectly correlated[38] with the third one: As argued above, no matter which of the three devices measures the $x-$component of the spin (such that the remaining two measure the $y-$component), in the state (1.52) the outcome 'spin down' will always occur either precisely one or three times. Consequently, if two of the measurements were performed, we immediately know what the outcome of the remaining measurement will be with certainty: If the outcomes of the first two measurement results agree (two times 'spin up' or two times 'spin down') the third measurement will yield 'spin down', if the first two measurement results disagree the remaining measurement will have outcome 'spin up'. If the considered particle was measured first, quantum theory assigns probability $\frac{1}{2}$ to each of the two possible outcomes (which also equals the marginal probability if the other two spin measurements were performed first, i.e. the probability averaged over the possible outcomes of the latter), if the other two spin measurements were performed first quantum theory assigns the conditional probability (conditional on the outcomes of the first two measurements) 1 to one of the outcomes and 0 to the other one.

Now take your time to think about how these perfect correlations can come about. There are indeed only two possible patterns of explanation: Since the outcome of the third measurement is determined as soon as the first two measurement results are known, it must either have been determined already before the first two measurements took place (e.g. by the preparation of the three particles) or it must be dynamically determined by these two measurements (if anyone has a serious alternative to these two possible explanations, I would be curious to learn about it!).

Now let us have a closer look at the first possible explanation, i.e. suppose the outcome of the third measurement was already determined before the other measurements were performed. Since the choice which of the measurements is performed last was arbitrary, this implies that actually all three measurement outcomes are predetermined. Moreover, since we did not specify which of the three particles is subject to the σ_x-measurement (and thus the remaining two to

[38]The perfect anticorrelations of the singlet state would be of course a more straightforward example, but the no-go argument is in the present case simpler and stronger (the no-go argument with respect to the correlations of the singlet state corresponds to Bell's original version [28] of his famous theorem, the present example to the Greenberger-Horne-Zeilinger version [157] of Bell's theorem).

σ_y-measurements), all six values – the $x-$ and $y-$components of the spin of all three particles – must be predetermined. Finally, in order to comply with the prescribed quantum correlations they must thus be given by one element of the eight value sets of (1.57). But as we have seen above, these value sets are inconsistent with the quantum prediction for σ_x-measurements on all three particles and can therefore not exist! Consequently this pattern of explanation is ruled out[39] and the only remaining explanation is a direct dynamical influence of the first two measurements on the last one, determining the outcome of the latter.

This dynamical influence is directly mirrored in the corresponding collapse dynamics of the wave function: Suppose e.g. the first particle is measured to have 'spin up' in the $x-$direction. The corresponding effect on the remaining particles can be read of from the initial state ψ

[39]Some people argue (in particular in connection with nonlocality, see below) that there is an alternative to this conclusion, namely to insist that logical conclusions which draw on *counterfactuals* (i.e. assertions whose conditional clause is false, like 'If we had measured ... we had obtained...') are not valid [272, 310]. This is inspired by the fact that σ_x^i and σ_y^i do not commute which entails that both associated measurements cannot be performed at the same time. But the present arguments assign to each individual system pre-measurement values to both, σ_x^i and σ_y^i, although always only one of them can be ascertained by experiment at a time. Hence, people argue, we can assign values to each individual system, but always only one of the triplets $(S_x^1, S_y^2, S_y^3), (S_y^1, S_x^2, S_y^3), (S_y^1, S_y^2, S_x^3)$ or (S_x^1, S_x^2, S_x^3), depending on the actual choice of experiment performed in the considered run, but not all four of them at the same time.

This is a valid objection in the first place, but then the question has to be answered, *exactly when* the actual one of the four value triplets is determined. If during the measurements, this is precisely the second explanatory model proposed, namely that the measurements directly influence one another such that e.g. the pre-measurement value ascribed to one wing of the experiment depends on the fact whether the other two measurements already have been performed and if so, on the respective apparatus settings and/or the outcomes (this line of argument in particular entails nonlocality if the measurements are performed at spacelike separation, although one has to cope with the fact then that in a relativistic space-time the phrase 'whether the other two measurements already have been performed' has no frame independent answer if they are performed at spacelike separation, see below).

The alternative would be that the actual one of the four triplets is already determined *before* the measurements took place, e.g. when the particles were prepared. But to argue in this direction means to ascribe to nature a deeply conspiratorial feature, which would make all of experimental physics a farce. This conspiratorial feature namely would be that the initially prepared system already 'knows about' (depends on) the apparatus settings of later measurements and (almost all) physicists agree that this would be inacceptable since the independence of the initial measured system from the measuring device is a basic assumption only giving sense to the venture of inquiring nature by experiment (see e.g. [31]).

So the argument which rules out preexisting values can be formulated without counterfactuals by resorting to the *no-conspiracies requirement*: Suppose an ensemble of many of the three particle systems in the state ψ, respectively, is prepared and the three particles are brought far apart from the preparation device (and possibly from each other), respectively. Then each passes a Stern-Gerlach device, where say a random number generator or an experimenter chooses at the instant before the passing if the SGM is oriented in the $x-$ or the $y-$direction, respectively. Now we consider the outcome triplets $(S_x^1, S_y^2, S_y^3), (S_y^1, S_x^2, S_y^3), (S_y^1, S_y^2, S_x^3)$ or (S_x^1, S_x^2, S_x^3) of the respective subensembles where the devices had the respective orientations and recover the relative frequencies predicted by quantum theory. If the prepared systems were independent of the later choice of the random number generator or the experimenter (no-conspiracies) and if we exclude the possibility of direct dynamical influence among the measurements, all four outcome triplets must obviously have been determined already in advance. But as we have seen, this possibility is inconsistent with the observed quantum statistics, such that if we want to hold on to the no-conspiracies requirement only the 'direct dynamical influence' explanation survives.

properly rewritten:

$$\begin{aligned}\psi = \frac{1}{\sqrt{2}}\Big(|\uparrow\uparrow\uparrow\rangle_z + |\downarrow\downarrow\downarrow\rangle_z\Big) = \frac{1}{2}\Big(\big(|\uparrow\rangle_x + |\downarrow\rangle_x\big)\otimes|\uparrow\uparrow\rangle_z + \big(|\uparrow\rangle_x - |\downarrow\rangle_x\big)\otimes|\downarrow\downarrow\rangle_z\Big) = \\ = \frac{1}{2}\,|\uparrow\rangle_x\otimes\big(|\uparrow\uparrow\rangle_z + |\downarrow\downarrow\rangle_z\big) + \frac{1}{2}\,|\downarrow\rangle_x\otimes\big(|\uparrow\uparrow\rangle_z - |\downarrow\downarrow\rangle_z\big)\end{aligned} \tag{1.58}$$

Thus the outcome 'spin up' in the $x-$direction of a measurement on the first particle (multiply ψ by the state transformer $P^1_{x\uparrow} = |\uparrow\rangle\langle\uparrow|^1_x$ and renormalize) will yield the state of the remaining two particles

$$\psi' = \frac{1}{\sqrt{2}}\Big(|\uparrow\uparrow\rangle_z + |\downarrow\downarrow\rangle_z\Big) \tag{1.59}$$

(which is to be taken in tensor product with $|\uparrow\rangle_x$ of the measured particle if the measurement was really ideal, but (1.59) is equally the resulting state of the remaining two particles in more realistic scenarios, e.g. when the first particle was absorbed by a screen after passing the SGM). The state (1.59) can now again be rewritten as

$$\begin{aligned}\psi' = \frac{1}{\sqrt{2}}\Big(|\uparrow\uparrow\rangle_z + |\downarrow\downarrow\rangle_z\Big) = \\ = \frac{1}{2\sqrt{2}}\Big(\big[|\uparrow\uparrow\rangle_z + |\downarrow\downarrow\rangle_z - i\,|\uparrow\downarrow\rangle + i\,|\downarrow\uparrow\rangle\big] + \big[|\uparrow\uparrow\rangle_z + |\downarrow\downarrow\rangle_z + i\,|\uparrow\downarrow\rangle - i\,|\downarrow\uparrow\rangle\big]\Big) = \\ = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\big(|\uparrow\rangle + i\,|\downarrow\rangle\big)\otimes\frac{1}{\sqrt{2}}\big(|\uparrow\rangle - i\,|\downarrow\rangle\big) + \right. \\ \left. + \frac{1}{\sqrt{2}}\big(|\uparrow\rangle - i\,|\downarrow\rangle\big)\otimes\frac{1}{\sqrt{2}}\big(|\uparrow\rangle + i\,|\downarrow\rangle\big)\right) = \\ = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle_y + |\downarrow\uparrow\rangle_y\Big)\end{aligned} \tag{1.60}$$

Consequently, if the σ_y-measurement of the second particle has outcome 'spin up', the remaining particle is left in the state $|\downarrow\rangle_y$ and accordingly a subsequent measurement on it will find 'spin down' in the $y-$direction with certainty, if the σ_y-measurement of the second particle yields 'spin down', the remaining particle is left in the state $|\uparrow\rangle_y$ and accordingly a subsequent measurement on it will find with certainty 'spin up' in the $y-$direction, as we have already predicted from more general considerations.

Now again, whatever the wave function is regarded to be, the determined outcome of the last measurement reflected in the states $|\uparrow\rangle_y$ or $|\downarrow\rangle_y$, respectively, of the respective particle prior to its measurement, cannot have been determined before the other measurements took place (not even in principle, not even if knowledge about its predetermined value was inaccessible in principle), thus it must have been determined by the latter! In this unpretentious sense the first measurements influence directly the dynamics of the subsystem which is measured last and this dynamical influence finds its formal counterpart in the collapse of the wave function on the level of quantum mechanical description which is thus in this basic sense a fundamental dynamical process.

This becomes even more dramatically apparent (and rises huge challenges to cope with) when the measurements take place in a relativistic space-time at spacelike separation, which we

shall discuss next. But beforehand, one last remark on widespread confusions about terminology: When we say that the outcome of a certain measurement is *determined*, we do not (tacitly) imply that this does correspond to a property of the considered system, but nothing else than that the probability of a certain measurement outcome is unity. If for example a particle is in an eigenstate of say σ_x, the result of a subsequent spin measurement in the $x-$direction is determined, but as the example of spin measurements in Bohmian mechanics above has shown, this need in no way imply that the particle *has a property* 'spin up' or 'spin down'. It simply means that the corresponding spin measurement will yield a certain result with certainty. If this result was not certain before in a fundamental sense (i.e. not only concerning our knowledge), then something must have happened to the considered system, that's all! Equally to call determined future measurement results (when the system is in a respective eigenstate) *'elements of physical reality'* as Einstein, Podolsky and Rosen put it in their famous groundbreaking work, is unrewarding and completely unnecessary for the present analysis and conclusions. This is not about more or less dubious assumptions about 'reality' or 'non-reality' of anything but plainly about upcoming measurements whose results are certain or not[40] in a fundamental way (i.e. not only with respect to our knowledge).

1.3.2 Nonlocality

Now let us start all over again: We consider the experiment(s) described in the previous section, but do not proceed from the prejudice that observables have values prior to their measurements, irrespective of how the respective measurements are performed. What we assume instead is the very different and much more modest requirement inspired by relativity, that direct dynamical influence cannot act faster than light, in particular that causes give rise to their direct effects always locally. This requirement is called the *principle of local causality* or simply *locality* and it is frequently characterized by the phrase *'no action at a distance'*.

Note that wave function collapse is manifestly nonlocal from the outset. Given for example the initial state $\psi = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle)$, the outcome of a spin measurement on each one of the three particles in the $x-$ or the $y-$direction is undetermined (each possible outcome has probability $\frac{1}{2}$). But if two particles are measured to have, say 'spin up' in the $x-$ and the $y-$direction, respectively, the state of the remaining particle collapses instantaneously to $|\downarrow\rangle_y$ (see calculations (1.58) and (1.60) above) and thereby determines the outcome of a subsequent spin measurement in the $y-$direction, no matter how far away the distant particles were measured[41].

[40] At a first glance, one might be suspicious about the notion of 'determined in a fundamental sense' in view of Bohmian mechanics, which is deterministic and thus measurement outcomes are at the end of the day always determined in a fundamental way. This is true of course, but in Bohmian mechanics the outcomes of the considered experiment do not only depend on the initial configuration and initial state of the three measured particles, but also on the experimental setup. In particular, the result of the spin measurement on a given particle (given initial configuration and initial state of the three particles) can e.g. be determined to be 'spin up' in a given direction if the other particles were not measured before and 'spin down' given they already were measured. Consequently, the particular determined result of spin measurement on the considered particle conditional on the spin measurements of the other particles was in general not determined before the latter took place in a fundamental way (i.e. not only concerning our knowledge), but literally determined by them.

[41] Of course, the reduced density operator prior to the distant measurements is identical to the marginal density operator (averaged over all possible outcomes of the distant measurements) subsequent to the distant measurements, which is responsible for the fact that quantum nonlocality cannot be exploited to send superluminal

Note in particular, that the corresponding causal relations can no longer be uniquely decomposed into cause and effect in a relativistic space time in the first place, since spacelike separated events do not have a frame independent absolute time order, such that from the perspective of different frames of reference we would tell different stories about which of the measurements have determined the outcomes of others (for a discussion of how to understand wave function collapse in a relativistic space time see [24]).

But indeed wave function collapse need not entail action at a distance, if it is not to be taken as a truly dynamical process of the considered system(s), but rather as pragmatical means of description. Not least, the particles have previously interacted and were prepared by a common source, so it is not so remarkable in the first place that later measurement outcomes are perfectly correlated, even if the particles are already far away from each other at this instant. A strong correlation between sleeping with one's shoes on and waking up with a headache does not imply that sleeping with ones shoes on causes headache, but might more probably derive from a common cause like going drunk to bed. Conditional on this common cause, the correlations will presumably vanish.

A primitive example which illustrates perfect correlations of spacelike separated events due to common causes in the common past is the *'glove left at home'*: If I reach into my pocket and realize that I have only one glove in my pocket and left the other one at home, the a priori probability assigned to the possibility (A) that the glove left at home is the right hand glove at this time is obviously $\frac{1}{2}$. This probability instantaneously changes to unity as soon as I take the glove in my pocket and realize (B) that it is actually the left hand one. But although $\mathbb{P}(A) = \frac{1}{2} \neq 1 = \mathbb{P}(A \mid B)$ no one would reasonably conclude that something nonlocal is going on here, i.e. that me grabbing into my pocket had any direct influence on the glove left at home. The reason is that there are facts λ (common causes) in the common past of both events – e.g. the right hand glove itself which fell to the ground while I put the left hand glove into my pocket – which already determine A and B if they are taken into account, in particular $\mathbb{P}(A \mid \lambda) = 1 = \mathbb{P}(A \mid B, \lambda)$. Hence the correlations between A and B can be completely accounted for locally by additional data associated with the common past of both events and these correlations disappear as soon as all potentially relevant data are taken into account.

So if we acknowledge the perfect quantum correlations of spacelike separated measurement outcomes and want to dispense with the possibility of direct dynamical influence of measurements at spacelike separation (nonlocality), we can – and indeed must – save the situation by resorting to possible common causes, which might be empirically inaccessible, in the common past (the intersection of the backward light cones) of the respective measurement events.

The argument that the requirement of locality together with the perfect quantum correlations of spacelike separated events makes it necessary to explain the perfect correlations by common causes in the common past of the measurement events goes back to Einstein, Podolsky and Rosen [131] and is accordingly called the *EPR argument*, which also constitutes the first part of Bell's theorem[42]. The second part then is a no-go argument which shows that common causes

signals. But this level of description is by far too coarse-grained to identify and analyse the perfect correlations, which are facts to be analysed though, and is therefore inadequate for the present analysis.

[42]Einstein, Podolsky and Rosen originally formulated their argument in 1935 in terms of position and momentum observables. In 1951 the argument was reformulated and essentially simplified by Bohm [38] in terms of discrete spin variables for the perfect correlations of the spin singlet state. Bohm's version of the EPR argument was also the basis of Bell's work leading to his famous theorem. In the present work the EPR argument is formu-

saving locality with respect to the considered quantum correlations cannot exist, which for the present example we essentially already gave in the forgoing section. We will come back to this below, but first we formulate the EPR argument for the present example.

EPR Argument

Suppose tree spin$-\frac{1}{2}$ particles are prepared in the state (1.52) and firstly that subsequently one of the particles is transferred far away and we measure the spin in the $y-$direction of the remaining two particles. Recognising the two outcomes we can immediately predict with certainty what the outcome of a later (say 'later' with respect to the laboratory frame) spin measurement in the $x-$direction on the distant particle will be, namely 'spin down' if both outcomes agree, respectively 'spin up' if they disagree (see the foregoing section for details). For simplicity, let us therefore condense the two measurements on our wing of the experiment to one single measurement associated with the observable operator $\sigma_y^2\sigma_y^3$ which takes the value $S_y^2 \cdot S_y^3 = +1$ if both measurement results agree and $S_y^2 \cdot S_y^3 = -1$ if they disagree, such that this measurement is in indeed perfectly (anti-)correlated with the distant measurement associated with σ_x^1 taking the values $S_x^1 = \pm 1$.

Now comes the argument: Since distance in space and/or time did not play any role whatsoever, this is equally true if the distant measurement is performed at spacelike separation. So we have two measurements whose outcomes are perfectly (anti-)correlated in each single run of the experiment, also if the measurements are performed at spacelike separation (such experiments have been successfully performed and the quantum predictions have been verified [252]). Now again, take your time to think about this situation: The distant outcome is determined as soon as we recognise the outcome at our wing of the experiment and we want to exclude the possibility that spacelike separated measurements influence one another (locality), how can that be[43]? If the measurement result was not actively determined by measurements at spacelike separation but at the same time it is determined as soon as the outcome of a measurement at spacelike separation is given, the only remaining possibility is that both values were determined already before, namely in the common causal past of the measurements, e.g. in the space-time region where the three particles were prepared by a common source.

The EPR-argument is often neglected or mistaken in discussions of Bell's theorem[44]. This argument does in no way *assume* the existence of predetermined values (hidden variables) of the measurements, but only locality and then deduces that the existence of predetermined values is a necessary condition to maintain locality for the considered experiments: Measurement results should not be actively determined by events at spacelike separation – we know that they are determined by acknowledging an event at spacelike separation – in conclusion they must have been determined already before!

lated essentially for the spin measurement correlations found and analysed by Greenberger, Horn and Zeilinger [157]).

[43] In his very nice paper *'What is the Meaning of the Wave Function?'* [59] Jean Bricmont pointedly comments about this question: *'I believe that if Bells theorem is arguably the most widely misunderstood result in the history of physics, it is precisely because this question is not answered before proceeding further.'*

[44] Bell comments in a footnote of his *'Bertlmann's socks'* article [29] about this widespread misunderstanding, referring to his paper [28] in which he firstly published his famous theorem: *'My own first paper on this subject [...] starts with a summary of the EPR argument from locality to [...] hidden variables. But the commentators have almost universally reported that it begins with [...] hidden variables'.*

Now we can rearrange the experimental setup, such that all three measurements are performed at spacelike separation and refine the previous arguments in order to conclude from locality that all three measurement outcomes must be given by predetermined values (this could be avoided with the usual EPR argument based on the singlet correlations of only two particles, but it is also an easy exercise for the present argument for which the subsequent no-go argument is simpler and stronger).

More formally we can put it like this: In the absence of action at a distance, there must be data λ such that the outcome of each of the three measurements in an individual run of the experiment does not depend on data (apparatus settings and outcomes) of the other measurements performed at spacelike separation if λ is taken into account. In particular, if the experiment is repeated a large number of times – with always ψ as the initial state but λ varying form run to run – the distribution reflecting the empirical relative frequencies of the measurement outcomes on one of the tree wings of the experiment conditional on a given λ must be independent of apparatus settings and outcomes of the spacelike separated measurements at the other wings. There thus exist probability distributions $\mathbb{P}^{\psi}_{\lambda}$ conditional on ψ and given λ, such that the ± 1 valued outcomes S^1_x, S^2_y and S^3_y can be identified with independent random variables with respect to these conditional distributions (i.e. for each given λ we have $\mathbb{P}^{\psi}_{\lambda}(S^1_x = a \mid S^2_y = b, S^3_y = c) = \mathbb{P}^{\psi}_{\lambda}(S^1_x = a)$ and so on).

We might think of λ as whatsoever might there be to provide a local explanation of the predicted and observed quantum correlations, possibly λ is empirically inaccessible in principle. To provide a common cause explanation, λ will somehow represent data associated with the common past of the three measurement events, we might think of it as a hidden variable of a full blown local hidden variables theory, determining together with ψ the outcomes from the outset, or it might just represent the way how the experimenter pushed the button to prepare the particles etc., anything which potentially makes the quantum correlations locally explicable is fine. One would expect that λ varies from one run of the experiment to the other (it thus can be considered as a random variable as well) in such a way, that the relative frequencies of the outcomes (the latter depending on λ in each single run) agree with the predictions of quantum theory when the experiment is repeated a large number of times, i.e. the quantum statistics emerges from the $\mathbb{P}^{\psi}_{\lambda}$ distribution if we average λ out with respect to its empirical distribution.

Denote by $\langle\cdot\rangle^{\psi}_{\lambda}$ the expectation value with respect to the conditional distribution $\mathbb{P}^{\psi}_{\lambda}$. From the previous section we know that $S^1_x \cdot S^2_y \cdot S^3_y = -1$ (which holds in the state ψ for each given λ of course), which together with the statistical independence of the outcome variables S^i_α entails

$$\begin{aligned} -1 = \left\langle S^1_x \cdot S^2_y \cdot S^3_y \right\rangle^{\psi}_{\lambda} = \left\langle S^1_x \right\rangle^{\psi}_{\lambda} \left\langle S^2_y \right\rangle^{\psi}_{\lambda} \left\langle S^3_y \right\rangle^{\psi}_{\lambda} = \\ = \Big(\mathbb{P}^{\psi}_{\lambda}\left(S^1_x = +1\right) - \mathbb{P}^{\psi}_{\lambda}\left(S^1_x = -1\right) \Big) \cdot \\ \cdot \Big(\mathbb{P}^{\psi}_{\lambda}\left(S^2_y = +1\right) - \mathbb{P}^{\psi}_{\lambda}\left(S^2_y = -1\right) \Big) \cdot \\ \cdot \Big(\mathbb{P}^{\psi}_{\lambda}\left(S^3_y = +1\right) - \mathbb{P}^{\psi}_{\lambda}\left(S^3_y = -1\right) \Big) \end{aligned} \tag{1.61}$$

and now it is an easy exercise to conclude that (1.61) can hold if and only if

$$\begin{array}{ll}
\mathbb{P}^{\psi}_{\lambda}\left(S^1_x=-1\right)=1\ \wedge\ \mathbb{P}^{\psi}_{\lambda}\left(S^2_y=-1\right)=1\ \wedge\ \mathbb{P}^{\psi}_{\lambda}\left(S^3_y=-1\right)=1 & \text{or}\\
\mathbb{P}^{\psi}_{\lambda}\left(S^1_x=-1\right)=1\ \wedge\ \mathbb{P}^{\psi}_{\lambda}\left(S^2_y=+1\right)=1\ \wedge\ \mathbb{P}^{\psi}_{\lambda}\left(S^3_y=+1\right)=1 & \text{or}\\
\mathbb{P}^{\psi}_{\lambda}\left(S^1_x=+1\right)=1\ \wedge\ \mathbb{P}^{\psi}_{\lambda}\left(S^2_y=-1\right)=1\ \wedge\ \mathbb{P}^{\psi}_{\lambda}\left(S^3_y=+1\right)=1 & \text{or}\\
\mathbb{P}^{\psi}_{\lambda}\left(S^1_x=+1\right)=1\ \wedge\ \mathbb{P}^{\psi}_{\lambda}\left(S^2_y=+1\right)=1\ \wedge\ \mathbb{P}^{\psi}_{\lambda}\left(S^3_y=-1\right)=1 &
\end{array} \tag{1.62}$$

where the respective converse probabilities are zero of course and for a given λ always one of the four lines in (1.62) must be true. In particular, all individual probabilities in (1.62) are as assumed (for a given λ) always independent of the remaining two measurement outcomes and we see that all outcomes of this experiment are predetermined by λ.

To summarize, if the quantum correlations are locally explicable, there must be parameters λ varying with the individual runs of the considered experiment, such that for each λ one of the four possibilities in (1.62) must hold, which is to say that each λ determines one of the preexisting value sets $(\downarrow_x\downarrow_y\downarrow_y), (\downarrow_x\uparrow_y\uparrow_y), (\uparrow_x\downarrow_y\uparrow_y)$ or $(\uparrow_x\uparrow_y\downarrow_y)$. Alternatively, we can perceive the random variables S^i_α as ±1 valued functions (value maps) of λ whose product is constant $S^1_x(\lambda)S^2_y(\lambda)S^3_y(\lambda)=-1$ for all λ and whose values coincide with the measurement outcomes.

No-Go Argument

Now we can repeat the EPR argument for the apparatus settings represented by the outcome triples (S^1_y, S^2_x, S^3_y) and (S^1_y, S^2_y, S^3_x). Together with the requirement that a freshly prepared system is not affected by (does not depend on) apparatus settings of measurements which will be performed later and far away (the *no-conspiracies* requirement, see footnote 39 and references therein), the requirement of locality then entails that each prepared system can be associated in principle (possibly empirically inaccessibly) with one of the value sets (1.57) which predetermines the outcomes of the spin measurements (if the reader is afraid about counterfactuals, see also footnote 39). Then, as argued above, it is easy to see that each of these value sets is inconsistent with the empirically verified quantum predictions of the experiment associated with the outcome triple (S^1_x, S^2_x, S^3_x), where all three measuring devices are aligned in the same direction. Thus there does not exist any (non-conspiratorial) pattern of explanation presupposing locality which is consistent with experimentally well verified predictions of quantum theory, which is to say: *These experiments are not locally explicable* or straightaway: *Nature is nonlocal*!

1.4 Measurement Schemes

In the following two sections 1.4.1 and 1.4.2 a comprehensive structural analysis of projective measurement (like) processes will be given.

1.4.1 Projective Measurements I
Ideal Measurements

Ideal measurements are at the very basis of the quantum theory of measurement. They correspond to Schrödingers cat gedankenexperiment where the pointer is replaced by a cat. The discrete version of the von-Neumann measurement scheme presented in section 1.5.2 provides

a concrete example. The ideal measurement scheme was already presented in the motivation section 1.1.1, now we will locate and analyse it within the framework of measurement (like) processes developed in section 1.2.

Let $\mathcal{H}_S$ be the Hilbert space of the measured system, $\mathcal{H}_A$ the Hilbert space of the apparatus, $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A$, $\phi_0 \in \mathcal{H}_A$ the pointer ready state and $\{\phi_k\} \subset \mathcal{H}_A$ a set of pointer final states (for the moment we assume that $\phi_j \neq \phi_k$ for $j \neq k$). A unitary interaction U acting on $\mathcal{H}$ correlating each member of a complete set $\{\psi_k\} \subset \mathcal{H}_S$ with an associated pointer state

$$\psi_k\phi_0 \xrightarrow{U} \psi_k\phi_k \tag{1.63}$$

defines an (efficient) measurement (like) process: Let ψ be in $\mathcal{H}_S$. Since the ψ_ks form a complete set there are complex numbers $\{c_k\} \subset \mathbb{C}$ such that $\psi = \sum_k c_k\psi_k$ and consequently

$$\psi\phi_0 = \left(\sum_k c_k\psi_k\right)\phi_0 \xrightarrow{U} \sum_k c_k\psi_k\phi_k \xrightarrow{(*)} \psi_l\phi_l \tag{1.64}$$

where the last transition $(*)$ happens with probability

$$\mathbb{P}^{\psi\phi_0}(l) = |\,\langle\psi_l\phi_l|\,U(\psi\phi_0)\rangle\,|^2 = |c_l|^2 \tag{1.65}$$

The orthogonality of the pointer states together with the unitarity of U entails that the measurement is projective. In order to see this, note that unitarity entails:

$$\begin{aligned}\langle\psi_k\phi_0|\,\psi_l\phi_0\rangle &\overset{\langle\phi_0|\,\phi_0\rangle=1}{=} \langle\psi_k|\,\psi_l\rangle \overset{!}{=} \\ = \langle U(\psi_k\phi_0)|\,U(\psi_l\phi_0)\rangle &= \langle\psi_k\phi_k|\,\psi_l\phi_l\rangle \overset{\langle\phi_k|\,\phi_l\rangle=\delta_{kl}}{=} \delta_{kl}\end{aligned} \tag{1.66}$$

i.e.

$$\langle\psi_k|\,\psi_l\rangle = \delta_{kl} \tag{1.67}$$

Hence we can express the probability that the outcome associated with final pointer state ϕ_l is realized in terms of the one dimensional orthogonal projections P_l (which we may denote by the dyadic product $P_l = |\psi_l\rangle\langle\psi_l|$ in the Dirac notation) by

$$\mathbb{P}^{\psi\phi_0}(l) = |c_l|^2 = \langle\psi|\,P_l\,\psi\rangle \equiv \mathbb{P}^{\psi}(l) \tag{1.68}$$

with the corresponding state transformers

$$\mathcal{R}_k\psi = c_k\psi_k \quad \Longleftrightarrow \quad \mathcal{R}_k = P_k \tag{1.69}$$

Thus the PVM given by the projections $\{P_k\}$ is identical with the set of state transformers. Note that this is consistent with equation (1.21)

$$\mathcal{R}_k^\dagger\mathcal{R}_k = (P_k)^2 = P_k \tag{1.70}$$

and with equation (1.38)

$$P_k = \sqrt{P_k} = \mathbb{1}_{\mathcal{H}_S}\sqrt{\mathcal{R}_k} \tag{1.71}$$

If the pointer states ϕ_k are associated with physical values $\lambda(k)$ (outcomes), we may regard the measurement as a 'measurement of the observable operator'

$$\mathcal{A} := \sum_k \lambda(k)\, P_k \tag{1.72}$$

which gives the right expectation value for the outcomes via $\langle\psi|\, \mathcal{A}\, \psi\rangle$.

Degenerate Ideal Measurement

So far we pretended that distinct states ψ_k of the measured system lead always to distinct pointer states ϕ_k, which is of course not the case in general, where we have to account for degeneration which makes everything a bit more laborious. So let us consider again the previous ideal measurement scheme, with the only modification that now a set of several linearly independent initial states of the measured system can yield one and the same pointer position:

$$\psi_{nk}\phi_0 \xrightarrow{U} \psi_{nk}\phi_n \quad k = 1, ..., d_n \tag{1.73}$$

where d_n is an integer depending on n which might be also infinite. Any linear combination of the states ψ_{nk} for fixed n leads also to the final pointer state ϕ_n with certainty:

$$\left(\sum_{k=1}^{d_n} c_k\psi_{nk}\right)\phi_0 \xrightarrow{U} \sum_{k=1}^{d_n} c_k\psi_{nk}\phi_n = \left(\sum_{k=1}^{d_n} c_k\psi_{nk}\right)\phi_n \tag{1.74}$$

In consequence we can, without loss of generality, assume that the set $\{\psi_{nk}\,|\,k = 1, ..., d_n\}$ is an orthonormal basis[45] of $\mathcal{H}_n = \mathrm{span}(\{\psi_{n1}, ..., \psi_{nd_n}\})$, i.e. $\langle\psi_{nk}|\,\psi_{nl}\rangle = \delta_{kl}$ for all n. The number $d_n = \dim\mathcal{H}_n$ of mutually orthogonal states leading to pointer state ϕ_n is called the *degeneracy* of n.

Hence, for a general superposition we get the measurement (like) process

$$\psi\phi_0 := \left(\sum_n\sum_{k=1}^{d_n} c_{nk}\psi_{nk}\right)\phi_0 \xrightarrow{U} \sum_n \left(\sum_{k=1}^{d_n} c_{nk}\psi_{nk}\right)\phi_n =: \sum_n \psi^{(n)}_{\{c_{nk}\}}\phi_n \xrightarrow{(*)} \psi_l\phi_l \tag{1.75}$$

Here we have defined for each n the *non normalized* state $\psi^{(n)}_{\{c_{nk}\}} := \sum_{k=1}^{d_n} c_{nk}\,\psi_{nk}$ and the final states of the measured system

$$\psi_l := \frac{\psi^{(l)}_{\{c_{lk}\}}}{\|\psi^{(l)}_{\{c_{lk}\}}\|} \tag{1.76}$$

where the last transition $(*)$ in (1.75) happens with probability

$$\begin{aligned}\mathbb{P}^{\psi\phi_0}(l) = |\,\langle\psi_l\phi_l|\; U(\psi\phi_0\rangle)|^2 = \left|\sum_n \frac{\left\langle\psi^{(l)}_{\{c_{lk}\}}\middle|\,\psi^{(n)}_{\{c_{nk}\}}\right\rangle}{\|\psi^{(l)}_{\{c_{lk}\}}\|}\,\delta_{nl}\right|^2 = \\ = \left|\frac{\left\langle\psi^{(l)}_{\{c_{lk}\}}\middle|\,\psi^{(l)}_{\{c_{lk}\}}\right\rangle}{\|\psi^{(l)}_{\{c_{lk}\}}\|}\right|^2 = \left|\left\langle\psi^{(l)}_{\{c_{lk}\}}\middle|\,\psi^{(l)}_{\{c_{lk}\}}\right\rangle\right| = \|\psi^{(l)}_{\{c_{lk}\}}\|^2 = \sum_{k=1}^{d_l}|c_{lk}|^2\end{aligned} \tag{1.77}$$

[45] If $d_n = \infty$ we might worry whether $\mathcal{H}_n$ is a *closed* subspace. But we will see that $\mathcal{H}_n$ is an orthogonal complement (namely $\mathcal{H}_n = (\cup_{m\neq n}\mathcal{H}_m)^\perp$) and thereby closed.

With an analogous line of argument as for the non degenerate ideal measurement above, we can argue that the subspaces $\mathcal{H}_n = \text{span}(\{\psi_{n1}, ..., \psi_{nd_n}\})$ are mutually orthogonal:

$$\begin{aligned} \langle \psi_{nk}\phi_0 |\, \psi_{ml}\phi_0 \rangle &\overset{\langle \phi_0 | \phi_0 \rangle = 1}{=} \langle \psi_{nk} |\, \psi_{ml} \rangle \overset{!}{=} \\ = \langle U(\psi_{nk}\phi_0) |\, U(\psi_{ml}\phi_0) \rangle = \langle \psi_{nk}\phi_n |\, \psi_{ml}\phi_m \rangle &\overset{\langle \phi_n | \phi_m \rangle = \delta_{nm}}{=} \delta_{nm} \langle \psi_{nk} |\, \psi_{ml} \rangle = \delta_{kl}\, \delta_{nm} \end{aligned} \tag{1.78}$$

i.e. in essence

$$\langle \psi_{nk} |\, \psi_{ml} \rangle = \delta_{nm}\, \delta_{kl} \tag{1.79}$$

Consequently, the POVM related to the measurement is again a PVM: If we denote now the orthogonal projection onto $\mathcal{H}_n$ by P_n – i.e. we may write $P_n = \sum_{k=1}^{d_n} |\psi_{nk}\rangle \langle \psi_{nk}|$ – the PVM related to the measurement as well as the set of the respective state transformers is given by the set $\{P_n\}$:

$$\mathbb{P}^{\psi\phi_0}(l) = \sum_{k=1}^{d_k} |c_{lk}|^2 = \langle \psi |\, P_n\, \psi \rangle \equiv \mathbb{P}^{\psi}(l) \tag{1.80}$$

and

$$\mathcal{R}_l \psi = \psi^{(l)}_{\{c_{lk}\}} = P_l\, \psi \tag{1.81}$$

Again, if the pointer states ϕ_n are associated with physical values $\lambda(n)$ (outcomes), we may call the measurement a 'measurement of the observable operator'

$$\mathcal{A} := \sum_n \lambda(n)\, P_n \tag{1.82}$$

whose eigenvalue $\lambda(l)$ is d_l-fold degenerate, given the function $\lambda(n)$ is one to one.

Remarks

Value Function Induced Degeneration: In the previous discussion, degeneration was traced back to the fact that there can be a more than one dimensional subspace $\mathcal{H}_n \subset \mathcal{H}_S$ of the measured system associated with one and the same pointer state $\phi_n \in \mathcal{H}_A$. A *coarse grained readout* of the 'pointer orientation' is another is possible source of degeneration: If distinct pointer states $\{\phi_{n_k}\}$, $k = 1, ..., d$ are associated with one and the same value $\lambda(n_k) = \lambda_0$ for all $k = 1, ..., d$, i.e. if the function $\lambda(n)$ is not injective, we also obtain a degenerate spectrum of the observable operator $\mathcal{A} = \sum_n \lambda(n)\, P_n$.

In this case, the projection $P_{\lambda_0} := \sum_{k=1}^{d} P_{n_k}$ yields the right probability for outcome λ_0 via $\mathbb{P}^{\psi}(\lambda_0) = \langle \psi |\, P_{\lambda_0}\, \psi \rangle$. But we can in general not associate a state transformer acting on ψ with outcome λ_0, though we can define state transformers acting on density operators transforming pure states to mixed states, i.e. we are dealing with a non-efficient measurement then which is not fundamentally non-efficient but due to a coarse choice of the outcome values or due to ignorance.

Alternative Definitions: We have defined ideal measurements by the relations (1.64), respectively (1.75). Alternatively, we might define a measurement of some observable operator $\mathcal{A}$ to be *ideal* iff each eigenstate of $\mathcal{A}$ is left invariant by its associated state transformer:

$$\mathcal{A}\, \psi_{nk} = \lambda(n)\, \psi_{nk} \quad \Longrightarrow \quad \mathcal{R}_n\, \psi_{nk} = \psi_{nk} \tag{1.83}$$

In the quantum measurement literature one can find definitions of *ideal measurement* which differ from the present one at a first glance. On closer inspection it turns out that they are indeed equivalent.

It is commonly stated that ideal measurements are the type of measurements which disturb the state of the measured system as little as possible consistent with the rules of quantum theory. Accordingly, an ideal measurement of an observable operator $\mathcal{A}$ may be also defined in the following way [75]: If ψ is an eigenstate of $\mathcal{A}$ with density operator $\rho_\psi = |\psi\rangle\langle\psi|$, a non selective measurement of $\mathcal{A}$ leaves the density operator ρ_ψ invariant[46]. It is easy to see that this is true for ideal measurements as we have defied them:

$$\sum_n \mathcal{R}_n \rho_\psi \mathcal{R}_n^\dagger = \sum_n P_n |\psi\rangle\langle\psi| P_n = |\psi\rangle\langle\psi| = \rho_\psi \tag{1.84}$$

To understand this, note that ψ is an eigenstate of $\mathcal{A}$ and consequently it lives in one of the subspaces $\mathcal{H}_n$ which we denote, say by $\mathcal{H}_{n'}$. It follows from the orthogonality of these subspaces that $\mathcal{R}_n\psi = P_n\psi = 0$ except in case $n = n'$ where we have $\mathcal{R}_{n'}\psi = P_{n'}\psi = \psi$ which implies (1.84). The other way around, i.e. to see that the property that a density operator corresponding to an eigenstate of $\mathcal{A}$ is left invariant under a non selective measurement of $\mathcal{A}$ implies that the measurement process is of the form (1.64), respectively (1.75), takes a few more steps (see e.g. [73]) but is also straightforward.

Finally, it is interesting to note that continuous measurements can never be ideal measurements! Although a continuous generalization of the first transition $\stackrel{U}{\to}$ in (1.64), respectively (1.75), is easily found and even constructed via an explicit interaction (see the von-Neumann measurement scheme in section 1.5.2), the second transition $\stackrel{(*)}{\to}$ is strictly speaking not possible in this case. Note that, if we define as above a measurement of some observable operator $\mathcal{A}$ to be *ideal* iff all eigenstates of $\mathcal{A}$ are left invariant by the respective state transformers, it is – in view of the fact that continuous observable operators do not have eigenstates – immediately clear that ideal measurements cannot be continuous. We will come back to this later on.

1.4.2 Projective Measurements II Reproducibility and Pauli's Measurements of the First and Second Kind

In an ideal measurement, the set of projections constituting the PVM is identical to the set of state transformers. This has as a consequence that the outcome determines the outcome of a subsequent measurement of the same type in a very obvious way: If the measurement is immediately repeated (where the initial state of the measuring device has of course to be the pointer ready state again), such that we can neglect the free time evolution of the measured

[46] We may view as well the projection $P_{[\psi]} = |\psi\rangle\langle\psi|$ not only as a density operator but as an element of a PVM of an observable operator commuting (compatible) with $\mathcal{A}$ (note that commuting operators can be jointly diagonalized). Then the associated probability $\mathrm{Tr}_{\mathcal{H}_S}\left[P_{[\psi]}\,\rho\right]$ is unity if $\rho = \rho_\psi$ is the density operator of the measured system and (1.84) means then that it will remain unity subsequent to an $\mathcal{A}-$measurement. Following this line of thought, Busch et al. say that in an ideal measurement *'any property which pertains to the system before the measurement and which is compatible with the measured observable should also pertain to the system after the measurement'* [73]. This may be seen as a nice heuristic picture of ideal measurements, but it is important here to be aware that according to the Kochen-Specker-Bell theorems (see section 1.3), we come into serious trouble if we associate projections with properties of the measured system.

system in between the two measurements, the outcome of the second measurement will reproduce the outcome of the former one. This is because the first measurement leaves the state of the measured system in the subspace of states which lead to the pointer state associated with the outcome of the first measurement. Such measurements are called *reproducible* (which actually means that the measurement outcome is reproducible).

Definition 1.9 [*Reproducibility*]

An efficient measurement (like) process is called reproducible if for each k the final states of the measured system associated with final pointer state ϕ_k will lead to ϕ_k again with certainty if the measurement is immediately repeated.

Often realistic quantum measurements are not reproducible. A simple extreme example is the detection of a photon which is literally annihilated by the detector. Irrespective of further specification of the measured quantity (e.g. the photons energy, its polarization or its position...), every possible outcome leaves the measured system in the vacuum state in this case.

How can we understand this schematically in the light of the scheme developed in section 1.2? So suppose again that the considered measurement is discrete. The probabilities of different outcomes $\lambda(k)$ are given by a POVM $\{E_k\}$ which is related with the state transformers $\mathcal{R}_k$ via $\mathcal{R}_k^\dagger \mathcal{R}_k = E_k$ and the state transformers are thus given by $\mathcal{R}_k = U_k \sqrt{E_k}$, where U_k is a unitary operator or at least a partial isometry. Consequently we may – as a simple model – describe e.g. a discrete projective measurement associated with a PVM $\{P_{[\varphi_k]}\}$ which finally annihilates the measured photon by a set of state transformers of the form

$$\mathcal{R}_k = |\Omega\rangle \langle\varphi_k| = |\Omega\rangle \langle\varphi_k| \cdot |\varphi_k\rangle \langle\varphi_k| = U_k P_{[\varphi_k]} \tag{1.85}$$

with the vacuum state Ω and the partial isometries $U_k = |\Omega\rangle \langle\varphi_k|$ with one dimensional initial subspace $\operatorname{span}\{\varphi_k\}$ and final subspace $\operatorname{span}\{\Omega\}$, respectively (observe that $\mathcal{R}_k^\dagger \mathcal{R}_k = P_{[\varphi_k]}$ as desired). Of course, such a measurement is not reproducible, a second measurement on the measured system is not even possible in this case.

We shall work out now that on the one hand, non reproducible measurements need not be that dramatic and on the other hand, that reproducible measurements do not have to be ideal measurements.

Non Ideal Measurements of the First Kind

Inspired by a brief discussion of the quantum measurement process ([213] p. 67 ff.) in a famous paper by Landau and Peirls, in which they went beyond the standard ideal measurement scheme, Pauli ([255] p. 64 ff.) performed a more detailed analysis of different kinds of measurement processes based on an analysis of two quite realistic measurement procedures (the Stern Gerlach experiment and the indirect measurement of the energy of some atomic system by the measurement of the energy of some particle which was scattered off the atomic system).

Pauli ([255] p. 72 f.) defines a *measurement of the fist kind* as a reproducible measurement (Pauli also identifies 'first kindness' with a second a bit more abstract property, which was later

shown to deviate in general from the requirement of reproducibility, but which is taken as the defining property of a first kind measurement by some authors[47]). As argued above, it is clear that ideal measurements are of the first kind. But in fact, any measurement with associated state transformers $\mathcal{R}_n$ which leave the respective subspaces $\mathcal{H}_n$ invariant, is of the first kind. To understand this consider e.g. a projective measurement with state transformers $\{\mathcal{R}_n\}$ of the form $\mathcal{R}_n = U_n P_n$, where P_n is the orthogonal projection onto $\mathcal{H}_n$ and $U_n : \mathcal{H}_n \to \mathcal{H}_n$ is some unitary operator transforming states in $\mathcal{H}_n$ into states in $\mathcal{H}_n$. If $U_n \neq \mathbb{1}_{\mathcal{H}_S}$ the measurement is no longer ideal but still reproducible.

We will consider such projective non ideal measurements of the first kind in more detail next. It should be remarked in advance that in contrast to ideal measurements, measurements of the first and second kind in general need not be projective. Consequently, the following projective measurement (like) processes are not to be taken as definitions but rather as instructive illustrations of measurements of the first and second kind.

To implement a non ideal measurement of the first kind, we generalise the degenerate ideal measurement scheme by properly replacing equation (1.73), i.e. suppose pointer state ϕ_n is triggered under U by d_n linearly independent states $\psi_{n1}, \ldots, \psi_{nd_n}$ of the measured system, which are only now in general not left invariant by the interaction with the apparatus:

$$\psi_{nk}\phi_0 \xrightarrow{U} \varphi_{nk}\phi_n \quad k = 1, ..., d_n \tag{1.86}$$

But although now in general $\varphi_{nk} \neq \psi_{nk}$, we suppose that still $\varphi_{nk} \in \mathcal{H}_n = \text{span}\{\psi_{n1}, ..., \psi_{nd_n}\}$ for all k. As above, linear superpositions of the ψ_{nk} for fixed n lead to the same pointer final state:

$$\left(\sum_{k=1}^{d_n} c_k \psi_{nk}\right) \phi_0 \xrightarrow{U} \left(\sum_{k=1}^{d_n} c_k \varphi_{nk}\right) \phi_n \tag{1.87}$$

such that we can choose the ψ_{nk} to be mutually orthogonal for fixed n (an ONB of $\mathcal{H}_n$) without loss of generality.

For a general initial superposition we thus obtain the measurement (like) process

$$\psi\phi_0 := \left(\sum_n \sum_{k=1}^{d_n} c_{nk}\psi_{nk}\right) \phi_0 \xrightarrow{U} \sum_n \left(\sum_{k=1}^{d_n} c_{nk}\varphi_{nk}\right) \phi_n =: \sum_n \varphi^{(n)}_{\{c_{nk}\}}\phi_n \xrightarrow{(*)} \varphi_l\phi_l \tag{1.88}$$

Here we have introduced for each n the *non normalized* state $\varphi^{(n)}_{\{c_{nk}\}} := \sum_{k=1}^{d_n} c_{nk}\varphi_{nk}$ and the final states of the measured system

$$\varphi_l := \frac{\varphi^{(l)}_{\{c_{lk}\}}}{\|\varphi^{(l)}_{\{c_{lk}\}}\|} \tag{1.89}$$

[47]Pauli claims that reproducibility is equivalent to the property, that the probability of each outcome in a second measurement of the same type is not altered by the first one, if the first measurement is described as a *non selective measurement*. Busch et al. [73] show that this equivalence does only hold in the case of projective measurements, whereas in the general case of measurements associated with arbitrary POVMs it can be only shown that reproducibility implies that a non selective measurement does not change the distribution of outcomes of a second measurement of the same type, but that the latter property does in general not imply reproducibility. Busch et al. take this more abstract property as definition of 'first kindness' of a measurement, which thus in general deviates from the simple definition of a first kind measurement as a reproducible one in the present work in case of non projective measurements.

corresponding to the final pointer states ϕ_l which are realized in the transition $(*)$ in (1.88) with probability

$$\mathbb{P}^{\psi\phi_0}(l) = |\langle\varphi_l\phi_l|\, U(\psi\phi_0)\rangle|^2 = \|\varphi^{(l)}_{\{c_{lk}\}}\|^2 \tag{1.90}$$

But if we set $\psi^{(n)}_{\{c_{nk}\}} := \sum_{k=1}^{d_n} c_{nk}\psi_{nk}$ now – in contrast to ideal measurements – in general we have

$$\varphi^{(n)}_{\{c_{nk}\}} \neq \psi^{(n)}_{\{c_{nk}\}} \equiv P_n\psi \tag{1.91}$$

Now we can analyse the implications of the unitarity of U together with the mutual orthogonality of the pointer states ϕ_n for the states ψ_{nk} and φ_{nk} of the measured system in non ideal projective measurements of the first kind:

$$\begin{aligned} \langle\psi_{nk}\phi_0|\, \psi_{ml}\phi_0\rangle &\overset{\langle\phi_0|\,\phi_0\rangle=1}{=} \langle\psi_{nk}|\, \psi_{ml}\rangle \overset{!}{=} \\ = \langle U(\psi_{nk}\phi_0)|\, U(\psi_{ml}\phi_0)\rangle &= \langle\varphi_{nk}\phi_n|\, \varphi_{ml}\phi_m\rangle \overset{\langle\phi_n|\,\phi_m\rangle=\delta_{kl}}{=} \delta_{nm}\,\langle\varphi_{nk}|\, \varphi_{ml}\rangle \end{aligned} \tag{1.92}$$

i.e. $\langle\psi_{nk}|\, \psi_{ml}\rangle \sim \delta_{nm}$ and hence the subspaces $\mathcal{H}_n$ are mutually orthogonal:

$$\mathcal{H}_n \subseteq (\mathcal{H}_m)^\perp \text{ for } n \neq m \tag{1.93}$$

Since according to assumption $\varphi_{nk} \in \mathcal{H}_n$, this in turn implies $\langle\varphi_{nk}|\, \varphi_{ml}\rangle \sim \delta_{nm}$. If, on the other hand, we set $n = m$ in (1.92) we get $\langle\varphi_{nk}|\, \varphi_{nl}\rangle = \langle\psi_{nk}|\, \psi_{nl}\rangle = \delta_{kl}$. Collecting everything together, we thus have

$$\langle\psi_{nk}|\, \psi_{ml}\rangle = \langle\varphi_{nk}|\, \varphi_{ml}\rangle = \delta_{nm}\,\delta_{kl} \tag{1.94}$$

In consequence, the measurement is projective again:

$$\begin{aligned} \mathbb{P}^{\psi\phi_0}(l) = |\langle\varphi_l\phi_l|\, U(\psi\phi_0)\rangle|^2 = \|\varphi^{(l)}_{\{c_{lk}\}}\|^2 = \sum_{k=1}^{d_l} |c_{lk}|^2 = \\ = \|\psi^{(l)}_{\{c_{lk}\}}\|^2 = \langle\psi|\, P_l\,\psi\rangle \equiv \mathbb{P}^\psi(l) \end{aligned} \tag{1.95}$$

i.e. the related PVM is given by the projections $P_n = \sum_{k=1}^{d_n} |\psi_{nk}\rangle\langle\psi_{nk}|$. But since in general $\varphi^{(n)}_{\{c_{nk}\}} \neq P_n\psi$ these projections do now in general not coincide with the state transformers. So let us define linear operators $U_n : \mathcal{H}_n \to \mathcal{H}_n$ by the action $U_n\psi_{nk} = \varphi_{nk}$ on the basis elements ψ_{nk} of $\mathcal{H}_n$. In order to see that these operators are unitary on $\mathcal{H}_n$, consider two arbitrary states $\psi, \psi' \in \mathcal{H}_n$, $\psi = \sum_{k=1}^{d_n} c_{nk}\psi_{nk}$ and $\psi' = \sum_{l=1}^{d_n} c'_{nl}\psi_{nl}$. Now we may use the unitarity of U by a simple trick:

$$\begin{aligned} \langle\psi|\, \psi'\rangle = \langle\psi\phi_0|\, \psi'\phi_0\rangle = \langle U(\psi\phi_0)|\, U(\psi'\phi_0)\rangle = \\ = \left\langle U\left(\sum_{k=1}^{d_n} c_{nk}\psi_{nk}\phi_0\right) \middle|\, U\left(\sum_{l=1}^{d_n} c'_{nl}\psi_{nl}\phi_0\right)\right\rangle = \\ = \left\langle \sum_{k=1}^{d_n} c_{nk}\varphi_{nk} \middle|\, \sum_{l=1}^{d_n} c'_{nl}\varphi_{nl}\right\rangle \langle\phi_n|\, \phi_n\rangle = \langle U_n\,\psi|\, U_n\,\psi'\rangle \end{aligned} \tag{1.96}$$

The Operators U_n together with the projections P_n define now the desired state transformers

$$\varphi^{(n)}_{\{c_{nk}\}} = U_n\psi^{(n)}_{\{c_{nk}\}} = U_nP_n\psi = \mathcal{R}_n\psi \tag{1.97}$$

i.e.

$$\mathcal{R}_n = U_n P_n \tag{1.98}$$

Note that according to (1.97) and (1.98) the operators U_n do always act subsequent to P_n which always acts first, thus defining them solely on $\mathcal{H}_n$ is not a loss of generality. If we want to express them as operators acting on all of $\mathcal{H}_S$, we may in addition define $U_n\psi = \mathbb{1}_{\mathcal{H}_S}$ for all $\psi \in (\mathcal{H}_n)^\perp$. Due to the orthogonality of the subspaces $\mathcal{H}_n$, this continuation of U_n to all of $\mathcal{H}_S$ is unitary as well.

Reproducibility

Since the final state of the measured system is given by $\varphi_n = \|\varphi^{(n)}_{\{c_{nk}\}}\|^{-1}\varphi^{(n)}_{\{c_{nk}\}} \in \mathcal{H}_n$ if the outcome associated with final pointer state ϕ_n is realized, a subsequent measurement will yield the same outcome with probability

$$\mathbb{P}^{\varphi_n}(n) = \langle\varphi_n|\ P_n\,\varphi_n\rangle = \frac{\left\langle \varphi^{(n)}_{\{c_{nk}\}} \middle|\ \varphi^{(n)}_{\{c_{nk}\}} \right\rangle}{\|\varphi^{(n)}_{\{c_{nk}\}}\|^2} = 1 \tag{1.99}$$

Consequently, the measurement is reproducible. But in contrast to an ideal measurement, in this case the PVM alone (or equivalently the observable operator) does not contain all information about the measurement process. It yields as usual the probabilities of the outcomes but it does not tell us how the state of the measured system looks like subsequent to the measurement beyond the subspace it lives in. In particular, the projection postulate of textbook quantum theory is no longer valid. In consequence, if we only know the PVM associated with the measurement (which is usually the case) we have – unlike the case of an ideal measurement – only limited predictive power for forthcoming events like measurements.

Here is a simple example to illustrate this: Consider a non ideal projective measurement of the first kind with observable operator $\mathcal{A}$ and a subsequent second projective measurement associated with operator $\mathcal{D}$ which commutes with $\mathcal{A}$. Due to the commutativity of $\mathcal{A}$ and $\mathcal{D}$ we can find a joint basis of eigenstates. Suppose now, the joint eigenstates φ_1 and φ_2 span a two dimensional eigenspace of $\mathcal{A}$ associated with one and the same eigenvalue λ_0 (which is thus two fold degenerate) whereas φ_1 and φ_2 are eigenstates of $\mathcal{D}$ belonging to different eigenvalues, μ_1 and μ_2 respectively. Suppose further, the initial state of the $\mathcal{A}-$measurement is φ_1 and consequently the outcome is λ_0 with certainty and the corresponding state transformation 'rotates' the initial state φ_1 onto φ_2, which is possible in a non ideal projective measurement of the first kind. In consequence, the subsequent $\mathcal{D}-$measurement will have outcome μ_2 with certainty. But the operator $\mathcal{A} = \sum_n \lambda(n)\, P^{\mathcal{A}}_n$ – respectively its associated PVM $\{P^{\mathcal{A}}_n\}$ – does not encode whether the corresponding measurement is ideal or not. If it was ideal, the subsequent $\mathcal{D}-$measurement would have outcome μ_1 instead of μ_2 with certainty. The fact whether the measurement is ideal or not is solely encoded in the state transformers $\{\mathcal{R}_n\} = \{U_n P_n\}$.

To summarize, though a degenerate[48] measurement of the first kind associated with a given PVM does reproduce the outcome upon immediate repetition with certainty, it does not uniquely determine the final state of the measured system, which makes in general a crucial difference

[48] Note that a non degenerate measurement of the first kind is always ideal.

for the distribution of outcomes of subsequent measurements, even if all involved operators do commute (i.e. for so called *compatible* observables).

Finally, it shall be remarked that non ideal measurements of the first kind as developed here are not so artificial as they might appear at a first glance, i.e. it is not so unrealistic that eigenstates of the observable operator are disturbed by the measurement, but not so much that the final state of the measured system does not reproduce the outcome upon immediate repetition of the measurement. We may think of measurement of certain degrees of freedom (e.g. spin) of a quantum system which appears as an ideal measurement on the Hilbert space associated with these degrees of freedom, but which disturbs the state on the Hilbert space associated with other degrees of freedom (e.g. spatial ones) even if the system is in an eigenstate of associated observable operator.

Measurements of the Second Kind

The example of the photon detector absorbing the measured photon dramatically illustrates that quantum measurements need not be reproducible. Indeed, also the previous description of non ideal projective measurements of the first kind suggests that reproducible measurements are actually only a very special case: If the state transformers can modify the state beyond just projecting it onto the eigenspace associated with the respective outcome, why should state transformers then in general respect the respective eigenspaces by leaving them invariant, as it is the case for measurements of the first kind. In other words, given a state of the measured system which triggers a certain pointer state with certainty, there is no reason to exclude the possibility that the (in general massively invasive) interaction with the measuring device transforms the state of the measured system to a state which no longer triggers that pointer state with certainty upon immediate repetition. Pauli's example of a measurement of an atoms energy by measurement of the energy of a scattered particle is a nice example of a such a measurement ([255] p. 73 ff.).

So here comes how projective measurements of the second kind generally look like:

Consider a measurement (like) process with a given set of linearly independent initial states $\{\psi_{nk}\}$ of the measured system, each of which triggers a certain pointer state with certainty:

$$\psi_{nk}\phi_0 \xrightarrow{U} \varphi_{nk}\phi_n \tag{1.100}$$

and consequently

$$\psi\phi_0 := \left(\sum_n \sum_{k=1}^{d_n} c_{nk}\psi_{nk}\right)\phi_0 \xrightarrow{U} \sum_n \left(\sum_{k=1}^{d_n} c_{nk}\varphi_{nk}\right)\phi_n =: \sum_n \varphi^{(n)}_{\{c_{nk}\}}\phi_n \xrightarrow{(*)} \varphi_l\phi_l \tag{1.101}$$

Again we have introduced for each n the *non normalized* state $\varphi^{(n)}_{\{c_{nk}\}} = \sum_{k=1}^{d_n} c_{nk}\varphi_{nk}$ and the final states of the measured system

$$\varphi_l = \frac{\varphi^{(l)}_{\{c_{lk}\}}}{\|\varphi^{(l)}_{\{c_{lk}\}}\|} \tag{1.102}$$

corresponding to the final pointer states ϕ_l which are realized in the transition $(*)$ in (1.101) with probability

$$\mathbb{P}^{\psi\phi_0}(l) = |\langle\varphi_l\phi_l|\, U(\psi\phi_0)\rangle|^2 = \|\varphi^{(l)}_{\{c_{lk}\}}\|^2 \tag{1.103}$$

In an ideal measurement we would have $\psi_{nk} = \varphi_{nk}$, in a general measurement of the first kind at least $\varphi_{nk} \in \mathcal{H}_n = \mathrm{span}(\psi_{n1}, ..., \psi_{nd_n})$ for all k. If the states φ_{nk} do in general not live in $\mathcal{H}_n$ the measurement is *no longer reproducible*. This is trivial, since now the state of the measured system subsequent to the measurement is in general no longer in the set of states which trigger the respective pointer state with certainty and consequently, it will in general not reproduce the outcome upon immediate repetition. The state transformers

$$\mathcal{R}_l\psi = \mathcal{R}_l\left(\sum_n\sum_{k=1}^{d_n} c_{nk}\psi_{nk}\right) = \sum_{k=1}^{d_l} c_{lk}\varphi_{lk} \tag{1.104}$$

are in general no longer maps $\mathcal{H}_l \to \mathcal{H}_l$ but transform states in $\mathcal{H}_l$ into arbitrary states in $\mathcal{H}_S$.

Again we choose without loss of generality for each given n the states ψ_{nk} which span $\mathcal{H}_n$ to be mutually orthogonal

$$\langle\psi_{nk}|\, \psi_{nl}\rangle = \delta_{kl} \tag{1.105}$$

for all n. Now we calculate the scalar product $\langle\psi_{nk}|\, \psi_{ml}\rangle$ for arbitrary n, m by using the unitarity of U in (1.100):

$$\begin{aligned}\langle\psi_{nk}|\, \psi_{ml}\rangle = \langle\psi_{nk}\phi_0|\, \psi_{ml}\phi_0\rangle \overset{!}{=} \langle U(\psi_{nk}\phi_0)|\, U(\psi_{ml}\phi_0)\rangle = \\ = \langle\varphi_{nk}\phi_n|\, \varphi_{ml}\phi_m\rangle \overset{\langle\phi_n|\,\phi_m\rangle=\delta_{nm}}{\sim} \delta_{nm}\end{aligned} \tag{1.106}$$

i.e. the subspaces $\mathcal{H}_n$ are again mutually orthogonal and together with (1.105) we have

$$\langle\psi_{nk}|\, \psi_{ml}\rangle = \delta_{nm}\delta_{kl} \tag{1.107}$$

But since now in general $\varphi_{nk} \notin \mathcal{H}_n$ this does not tell us whether the states φ_{nk} are mutually orthogonal or not. In order to investigate whether we can say anything about the orthogonality of these states, we note with the same trick as above that the operators $U_n : \mathcal{H}_n \to \mathcal{H}_S$ defined by $U_n\psi_{nk} = \varphi_{nk}$ preserve the scalar product: Consider the scalar product of two states $\psi, \psi' \in \mathcal{H}_n$, $\psi = \sum_{k=1}^{d_n} c_{nk}\psi_{nk}$ and $\psi' = \sum_{l=1}^{d_n} c'_{nl}\psi_{nl}$

$$\begin{aligned}\langle\psi|\, \psi'\rangle = \langle\psi\phi_0|\, \psi'\phi_0\rangle = \langle U(\psi\phi_0)|\, U(\psi'\phi_0)\rangle = \\ = \left\langle U\left(\sum_{k=1}^{d_n} c_{nk}\psi_{nk}\phi_0\right)\middle|\, U\left(\sum_{l=1}^{d_n} c'_{nl}\psi_{nl}\phi_0\right)\right\rangle = \\ = \left\langle\left(\sum_{k=1}^{d_n} c_{nk}\varphi_{nk}\right)\phi_n\middle|\, \left(\sum_{l=1}^{d_n} c'_{nl}\varphi_{nl}\right)\phi_n\right\rangle = \\ = \left\langle\sum_{k=1}^{d_n} c_{nk}\varphi_{nk}\middle|\, \sum_{l=1}^{d_n} c'_{nl}\varphi_{nl}\right\rangle\langle\phi_n|\, \phi_n\rangle = \langle U_n\psi|\, U_n\psi'\rangle\end{aligned} \tag{1.108}$$

and therefore

$$\delta_{kl} = \langle\psi_{nk}|\, \psi_{nl}\rangle = \langle U_n\psi_{nk}|\, U_n\psi_{nl}\rangle = \langle\varphi_{nk}|\, \varphi_{nl}\rangle \tag{1.109}$$

This shows that for example in a projective measurement (that the present scheme is indeed projective will we shown in a moment) where all states of the measured system are transformed to one and the same final state – like in the simple annihilating detector model defined by the state transformers (1.85) – there cannot be several linearly independent states leading to the same pointer position. In other words, such a measurement cannot be fundamentally degenerate (of course, we may always create degeneration by hand by identifying distinct pointer states with one and the same value).

But on the other hand, we have no reason to expect that the image subspaces $U_n\mathcal{H}_n$ spanned by all possible final states of the measured system compatible with one and the same final pointer state ϕ_n, respectively, are mutually orthogonal for different n (indeed, as the example of the annihilating detector model illustrates, they need not even be distinct), i.e. in general

$$\langle\varphi_{nk}|\,\varphi_{ml}\rangle \neq \delta_{nm}\delta_{kl} \tag{1.110}$$

If we want to continue the operators U_n acting on $\mathcal{H}_n$ to all of $\mathcal{H}_S$ (which is actually not necessary, since – as we shall see in a moment – P_n always acts first), there is no general way to make them unitary now if the dimension of $\mathcal{H}_S$ is infinite (a nice counterexample can be found in footnote 72 below). But we may define $U_n\psi = 0$ for all $\psi \in (\mathcal{H}_n)^\perp$ such that U_n extended to all of $\mathcal{H}_S$ that way becomes a partial isometry with initial subspace $\mathcal{H}_n$.

Although the final states of the measured system associated with different outcomes need not be mutually orthogonal in this case, the measurement is again projective: First observe that

$$\|\varphi^{(l)}_{\{c_{lk}\}}\|^2 = \sum_{k=1}^{d_l} |c_{lk}|^2 = \|\psi^{(l)}_{\{c_{lk}\}}\|^2 \tag{1.111}$$

If we denote now the orthogonal projections onto $\mathcal{H}_n$ by P_n, the probability that the outcome associated with final pointer state ϕ_l is realized is given by

$$\mathbb{P}^{\psi\phi_0}(l) = |\,\langle\varphi_l\phi_l|\,U(\psi\phi_0)\rangle\,|^2 = \|\varphi^{(l)}_{\{c_{lk}\}}\|^2 = \|\psi^{(l)}_{\{c_{lk}\}}\|^2 = \langle\psi|\,P_l\,\psi\rangle \equiv \mathbb{P}^{\psi}(l) \tag{1.112}$$

The corresponding state transformers are given by

$$\mathcal{R}_l\,\psi = \varphi^{(l)}_{\{c_{lk}\}} = U_l\,\psi^{(l)}_{\{c_{lk}\}} = U_l\,P_l\,\psi \tag{1.113}$$

i.e.

$$\mathcal{R}_n = U_nP_n \tag{1.114}$$

Of course we can associate an observable operator $\mathcal{A} = \sum_n \lambda(n)\,P_n$ (with outcome value $\lambda(n)$ associated with pointer state ϕ_n, respectively) in this case with the measurement as well, as for each projective measurement, which does again not encode the state transformations though.

This concludes the analysis of projective measurement schemes.

Indeed, so far each proposal for a realization of a measurement (like) process turned out to be projective. This rises the question how a measurement (like) process, which gives rise to a non projection valued POVM, looks like. We already checked out all possibilities based on the assumption that there is a complete set of states $\{\psi_{nk}\} \subset \mathcal{H}_S$ leading to mutually orthogonal pointer states under U with certainty, i.e.

$$\psi_{nk}\phi_0 \xrightarrow{U} \varphi_{nk}\phi_n \tag{1.115}$$

If we want to adhere to pointer states which are perfectly distinguishable, what remains to derive a non projective process would be to assume that there is at least one pointer state $\phi_{n'}$ such that there does not exist any state $\psi_{n'k} \in \mathcal{H}_S$ satisfying (1.115) (this is essentially equivalent to the assumption that there is no *complete* set of states $\{\psi_{nk}\} \subset \mathcal{H}_S$ satisfying (1.115)). In other words, for all initial states of the measured system $\stackrel{U}{\rightarrow}$ can only lead to a superposition containing $\phi_{n'}$, but only together with other pointer states. This would lead to an effect $0 \leq E_{n'} < 1$, i.e. an effect which does not have eigenvalue 1 (if it had this eigenvalue, the corresponding eigenstate $\psi_{n'}$ would lead to the corresponding pointer state with certainty: $\mathbb{P}^{\psi_{n'}}(n') = \langle \psi_{n'} |\, E_{n'}\, \psi_{n'} \rangle = 1$, i.e. $\psi_{n'}$ would satisfy (1.115)). Such effects which do not have eigenvalue 1 are called *strongly unsharp effects* (strongly unsharp effects will play a certain role later in chapter 3). Since one strongly unsharp effect is enough to sabotage the orthogonality arguments, the other effects associated with the respective POVM – even if they had eigenvalue 1 – need no longer be projections.

But actually the generic measurement schemes leading to non projection valued POVMs do not arise form a direct measurement (like) process in that way but rather from 'fuzzy pointers' in a certain sense. There are essentially two relevant classes of non projective measurement (like) processes. One is the *approximate measurement scheme* where the pointer does not reflect perfectly accurately the result of an actually projective measurement because of measurement error, limited resolution, classical noise or the like. The second is the *indirect measurements scheme*, where an intermediate microscopic 'pointer system' (ancilla or probe system) unitarily interacts with the measured system in the first place, whose 'pointer states' need not be mutually orthogonal and do not involve collapse dynamics, but which must finally measured in a projective measurement again. As we shall see, this does indeed constitute a non projective measurement scheme if it is conceived as a measurement of the system with which the ancilla system initially interacted (moreover we will prove later the Naimark theorem 1.20 which entails that actually any POVM can be implemented by an indirect measurement scheme, at least formally).

We shall now discuss these two non projective measurement schemes one by one.

1.4.3 Non-Projective Measurements I
The Approximate Measurement Scheme

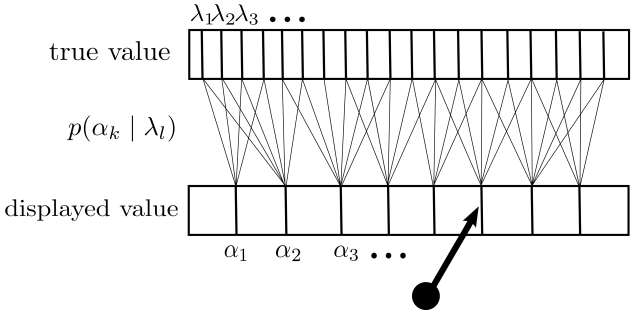

FIGURE 1
APPROXIMATE MEASUREMENT

Approximate measurement POVMs account for the fact that in reality measurements are not perfect. There is always a certain probability that some measurement error occurs, a limited resolution in the measurement result (pixel), classical noise influencing the displayed values, limited capability of the experimenter to read out the precise value of the experiment etc. This can be accounted for by distinguishing the set $\{\lambda_l\}$ of true values somehow realized by the measurement (we will discuss this delicate point in a moment) and a set $\{\alpha_k\}$ of outcomes read off from the display by the experimenter. Furthermore, we assume that the true values λ_l are associated with a complete set $\{P_l\}$ of projections acting on the Hilbert

space $\mathcal{H}$ of the measured system (we drop the index S for now since we do not care about the Hilbert space of the apparatus), i.e. if the measurement was perfect is would be a projective measurement whose PVM is given by the projections P_l. The fact that the measurement is not perfect is reflected in a probability distribution $p(\alpha_k \mid \lambda_l)$ which characterizes the likelihood of read out value α_k, given the true value was λ_l (see Fig. 1). Since we obtain a displayed value in any case, these probabilities sum up to unity, i.e. $\sum_k p(\alpha_k \mid \lambda_l) = 1$, for all l.

We can call this an *approximate measurement* of the observable operator

$$\mathcal{A} = \sum_l \lambda_l \, P_l \tag{1.116}$$

If now $\psi \in \mathcal{H}$ is the initial state of the measured system, the probability to obtain the displayed value $\alpha_{k'}$ is given by

$$\mathbb{P}_p^\psi(\alpha_{k'}) = \sum_l p(\alpha_{k'} \mid \lambda_l)\, \mathbb{P}^\psi(\lambda_l) = \left\langle \psi \middle| \left(\sum_l p(\alpha_{k'} \mid \lambda_l)\, P_l \right) \psi \right\rangle =: \langle \psi |\, E^p_{k'}\, \psi \rangle \tag{1.117}$$

where we have defined the effects

$$E^p_{k'} := \sum_l p(\alpha_{k'} \mid \lambda_l)\, P_l \tag{1.118}$$

It is easy to verify that the effects $E^p_{k'}$ form a POVM and that $(E^p_{k'})^2 \neq E^p_{k'}$ unless there is a one to one correspondence between the elements of the sets $\{\alpha_k\}$ and $\{\lambda_l\}$ for which $p(\alpha_k \mid \lambda_l) = 1$.

If we want to express the final state of the measured system by solely information about the displayed value $\alpha_{k'}$, we are forced to describe the measurement as a non efficient one and thus the final state by a density operator. In particular, if the measurement back action is given by partial isometries U_l, we obtain the mixed final state $\rho_{k'} = \sum_l p(\alpha_{k'} \mid \lambda_l)\, U_l\, P_l\, U_l^\dagger$ (which is accidentally identical to the effect $E^p_{k'}$ in case $U_l = \mathbb{1}_{\mathcal{H}}$).

True Values

Form an orthodox point of view this scheme is not really understandable in the first place. After all, it is commonly taught that it is off-limits to think about true values which are not observed by the observer. In pragmatical operational frameworks, one therefore usually resorts to introducing *classical noise* which allows it to talk about unobserved values since what separates us from access to the latter is something 'non-quantum'; and this changes the rules from radically operationalist to being pragmatic and to give permission to take the existence of an observer independent physical reality into account, at least in the classical regime.

But also in the light of Bohmian mechanics or GRW – where physical reasoning without observers not a taboo but the very foundation – we have to be a bit careful: What does it mean that λ_l is realized while α_k is displayed? The foregoing analysis has shown that according to the quantum formalism, quantum measurements do not, in general, reveal prexisting properties of the measured system, but rather force the measured system to realize a new state which is associated with the outcome. Moreover, this is physically substantiated be the Kochen-Specker-Bell theorems discussed in section 1.3 by showing that such properties do not only have no

formal counterpart in the quantum formalism but that there very existence is in general even inconsistent with the experimentally well verified predictions of quantum theory. Thus, we can only make sense out of the approximate measurement scheme if the measurement is actually perfect with respect to the part of the apparatus which collapses the state – i.e. the final state of the measured system is given by the action of the state transformers $\mathcal{R}_l = U_l P_l$ on the initial state – and only the display ('the last part of the apparatus') somehow picks out the wrong number.

On the other hand, in Bohmian mechanics the position of a particle is indeed a preexisting property which is $|\psi(x)|^2-$distributed and which can be revealed by a position measurement with a certain probability of error. This error might derive from the imperfectness of the measuring device, but we will discuss in section 1.5.2 and later in chapter 3 more fundamental roots of limitations for the perfect exactness of position measurements.

We will illustrate the approximate measurement scheme now by applying it to position measurements, which is also a standard example in the literature (though it has actually a less substantial meaning in an orthodox than in a Bohmian framework, where the 'true position' has a precise and unambiguous meaning). To this end, we switch from discrete to continuous observables for this subsection. The tools we shall develop in the following will prove helpful also for later purposes.

Approximate Position Measurement

We start with the standard position measurement scheme of a single non relativistic particle as it is presented in textbooks: The Hilbert space of the measured system is $\mathcal{H} = L^2(\mathbb{R}^3, d^3x)$. According to the Born rule, the position X of the particle (capital letters shall indicate random variables) is $\rho_X(x)\, d^3x = |\psi(x)|^2\, d^3x$ distributed, i.e. the probability to find a particle with wave function $\psi(x) \in \mathcal{H}$ in some (measurable) spacial region $\Delta \subseteq \mathbb{R}^3$ is given by

$$\mathbb{P}^\psi(X \in \Delta) = \int_\Delta |\psi(x)|^2\, d^3x = \int_{\mathbb{R}^3} \bar{\psi}(x)\, \chi_{\{\Delta\}}(x)\, \psi(x)\, d^3x \equiv \left\langle \psi \right|\, \chi_{\{\Delta\}}(q)\, \left| \psi \right\rangle \tag{1.119}$$

where $\chi_{\{\Delta\}}(x)$ is the indicator function of Δ

$$\chi_{\{\Delta\}}(x) = \begin{cases} 1 & \text{for } x \in \Delta \\ 0 & \text{otherwise} \end{cases} \tag{1.120}$$

and q is the standard position operator defined by $q\,\psi(x) = x\,\psi(x)$. Thus the position PVM on the measurable subsets $\Delta \subseteq \mathbb{R}^3$ of space is given by the indicator functions

$$E_\Delta = \chi_{\{\Delta\}}(q) \tag{1.121}$$

whose argument is the position operator.

The imperfectness of the position measurement shall now be reflected by the deviation from the true $|\psi|^2-$distributed position, represented by a random variable Y which is distributed according to some continuous error distribution ρ_Y about the origin. If ρ_Y is highly peaked about the origin, the measurement is almost accurate. The random variable X' describing the

statistics of the readouts is the correct position X plus the deviation form the correct position Y

$$X' = X + Y \tag{1.122}$$

and if we assume that the random variables X and Y are statistically independent (which is reasonable), we can calculate the distribution $\rho_{X'}$ of X', e.g by the method of characteristic functions[49]: The Fourier transform of the density – the characteristic function of the respective distribution – is given by

$$\widehat{\rho_{X'}}(t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{itx}\, \rho_{X'}(x)\, d^3x = \mathbb{E}(e^{itX'}) \tag{1.123}$$

i.e. $\widehat{\rho_{X'}}$ is the expectation value of the function $e^{itX'}$. Now the statistical independence of the random variables X and Y implies that

$$\widehat{\rho_{X'}}(t) = \mathbb{E}(e^{itX'}) = \mathbb{E}(e^{it(X+Y)}) = \mathbb{E}(e^{itX}\, e^{itY}) = \mathbb{E}(e^{itX})\, \mathbb{E}(e^{itY}) = \widehat{\rho_X}(t)\, \widehat{\rho_Y}(t) \tag{1.124}$$

i.e. the characteristic function with respect to X' is the product of the characteristic functions with respect to X and Y. Using the convolution theorem, we see that the measured position is distributed according to

$$\rho_{X'}(x) = (\rho_X * \rho_Y)(x) = \int |\psi(y)|^2\, \rho_Y(x-y)\, d^3y \tag{1.125}$$

Now we shall find the related POVM: The probability to find the particle by position measurement in a spatial region $\Delta \subseteq \mathbb{R}^3$ is given by[50]

$$\begin{aligned}
\mathbb{P}^\psi(X' \in \Delta) &= \int_\Delta \rho_{X'}(x)\, d^3x = \int_\Delta d^3x \int_{\mathbb{R}^3} d^3y\, \rho_Y(x-y)\, |\psi(y)|^2 = \\
&= \int_{\mathbb{R}^3} \bar\psi(y) \left[\int_{\mathbb{R}^3} \chi_\Delta(x) \rho_Y(x-y)\, d^3x \right] \psi(y)\, d^3y \quad \overset{f(x) := \rho_Y(-x)}{=} \\
&= \int_{\mathbb{R}^3} \bar\psi(y) \left[\int_{\mathbb{R}^3} \chi_\Delta(x) f(y-x)\, d^3x \right] \psi(y)\, d^3y = \\
&= \int_{\mathbb{R}^3} \bar\psi(y) \left[(\chi_\Delta * f)(y) \right] \psi(y)\, d^3y =: \left\langle \psi \middle|\, E^f_\Delta\, \psi \right\rangle
\end{aligned} \tag{1.126}$$

where we have defined the function f by $f(x) = \rho_Y(-x)$[51] and the positive operator

$$E^f_\Delta := (\chi_\Delta * f)(q) \tag{1.127}$$

[49] In this work, the term *'characteristic function'* refers to the Fourier transforms of probability distributions, not to be confused with indicator functions, which sometimes share the same name in the literature.

[50] Note that this is indeed the continuous counterpart of the discrete approximate measurement effects defined in (1.118). In particular, $p(\alpha_k \mid \lambda_l)$ corresponds to the distribution of X' conditional on X which equals the $Y-$distribution, where the argument is written as $Y = X' - X$, i.e. $\rho(X' \mid X) = \rho(X + Y \mid X) \equiv \rho_Y(X' - X)$ and the projections P_l are of course to be identified with the indicator functions $\chi_\Delta(X')$.

[51] If the error distribution ρ_Y is spherically symmetric about the origin (which is reasonable) such that $\rho_Y(x) = \rho_Y(-x)$ for all $x \in \mathbb{R}^3$, we have $f \equiv \rho_Y$.

It is easy to check that the family of operators $\{E^f_\Delta\}$ indexed by the measurable spatial subsets $\Delta \subseteq \mathbb{R}^3$ forms a POVM on $\mathbb{R}^3$. But the effects E^f_Δ are not projections unless $\rho_Y(x) = \delta(x)$, in which case the POVM reduces to the standard position PVM (1.121) given by the indicator functions (to be precise, something like $\rho_Y(x) = \delta(x-a)$ would be also sufficient to obtain a PVM).

The related first moment $\langle X' \rangle_\psi = \mathbb{E}(X')$ is given by

$$\begin{aligned}\langle X' \rangle_\psi &= \int_{\mathbb{R}^3} x \, \left\langle \psi \middle| \, E^f_{d^3x} \, \psi \right\rangle = \int_{\mathbb{R}^3} d^3y \, |\psi(y)|^2 \int_{\mathbb{R}^3} x \, \rho_Y(x-y) \, d^3x \overset{x-y=:\xi}{=} \\ &= \int_{\mathbb{R}^3} y \, |\psi(y)|^2 d^3y \overbrace{\int_{\mathbb{R}^3} \rho_Y(\xi) \, d\xi}^{=1} + \overbrace{\int_{\mathbb{R}^3} |\psi(y)|^2 \, d^3y}^{=1} \int_{\mathbb{R}^3} \xi \, \rho_Y(\xi) \, d\xi = \\ &= \langle X \rangle_\psi + \mathbb{E}(Y)\end{aligned} \tag{1.128}$$

In particular, if $\mathbb{E}(Y) = \int_{\mathbb{R}^3} \xi \, \rho_Y(\xi) \, d\xi = 0$, which is typically the case, the measurement statistics reproduces the right expectation value: $\langle X' \rangle_\psi = \langle X \rangle_\psi$. The second moment is given by

$$\begin{aligned}\left\langle (X')^2 \right\rangle_\psi &= \int_{\mathbb{R}^3} x^2 \, \left\langle \psi \middle| \, E^f_{d^3x} \, \psi \right\rangle = \int_{\mathbb{R}^3} d^3y \, |\psi(y)|^2 \int_{\mathbb{R}^3} x^2 \, \rho_Y(x-y) \, d^3x \overset{x-y=:\xi}{=} \\ &= \int_{\mathbb{R}^3} y^2 \, |\psi(y)|^2 d^3y + \int_{\mathbb{R}^3} \xi^2 \, \rho_Y(\xi) \, d\xi + 2 \left(\int_{\mathbb{R}^3} y \, |\psi(y)|^2 d^3y \right) \left(\int_{\mathbb{R}^3} \xi \, \rho_Y(\xi) \, d\xi \right) \\ &= \left\langle X^2 \right\rangle_\psi + \mathbb{E}(Y^2) + 2 \, \langle X \rangle_\psi \, \mathbb{E}(Y)\end{aligned} \tag{1.129}$$

such that the variance becomes

$$\begin{aligned}&\mathrm{Var}_\psi(X') = \\ &= \left\langle (X')^2 \right\rangle_\psi - (\langle X' \rangle_\psi)^2 = \\ &= \left(\left\langle (X)^2 \right\rangle_\psi - (\left\langle X^2 \right\rangle_\psi)^2 \right) + \left(\mathbb{E}(Y^2) - (\mathbb{E}(Y^2))^2 \right) = \\ &= \mathrm{Var}_\psi(X) + \mathrm{Var}(Y)\end{aligned} \tag{1.130}$$

where we may call $\mathrm{Var}_\psi(X)$ the *quantum fluctuations* and $\mathrm{Var}(Y)$ *classical noise*.

We will encounter in section 1.5.2 that such POVMs need not necessarily arise only from introducing 'classical noise' but can emerge from treating the apparatus as a quantum system as well. Moreover, in a very strict sense finally every (in particular continuous) measurement leads to a POVM of that kind, given the pointer wave function is not a delta function (i.e. in any case).

In chapter 3 we will see that in relativistic quantum theory, without need to choose a particular theory but only given the energy is bounded from below and space-time translation covariance and local commutativity (whose relevance we will investigate in chapter 2) are true, neither a position operator, nor a position PVM and not even a spatial POVM like (1.127) exists on $\mathbb{R}^3$. It should be already clear from the analysis of the operator formalism of quantum theory so far, that this raises the problem of how we can yet account for position measurements (or more

generally account for local measurement like processes at all) – which obviously happen everyday in laboratories – from a quantum theoretical point of view. In other words, these results pose an operational problem, in the first place, instead of an ontological one as claimed by some authors (though, of course, the emergence of measurement results must be explicable proceeding from ontology in a satisfactory physical theory). It is a major aim of this work to give a solution of this problem.

1.4.4 Non-Projective Measurements II
Indirect Measurement: The Ancilla Scheme

On closer inspection of the device system which we represented by an 'apparatus' or a 'pointer' so far (and which usually comprises the environment including the experimenter of the actual device as well), any viable quantum measurement can be finally perceived as an indirect measurement. A typical scenario starts with a microscopic interaction between the measured system and some microscopic part of the device system (think e.g. of a charged particle ionizing an atom in a cloud chamber) followed by an amplification process involving more and more subsystems of the device (e.g. water molecules gathering around the ionized atom), leading to some macroscopic display (e.g. water molecules condensing in a small droplet) which in turn interacts with its environment in the laboratory like with air molecules or photons, the latter interacting with the experimenters eyes or a camera etc (for most actual quantum measurements we should probably insert a computer recording the result in digital form somewhere in this chain). In particular, most of these subsystems which we collectively subsumed as pointers or measuring device, do not directly interact with the measured system but with systems which are correlated by a chain of interactions with a system which originally interacted with the measured system.

A more direct example of an indirect quantum measurement is e.g. a pair of two spin$-\frac{1}{2}$ particles prepared in the singlet state, subsequently separated in space followed by an ideal Stern-Gerlach experiment, say with respect to the $z-$direction, on one of the two particles. From the result we immediately gain information about the remote particle, in particular, we immediately know with certainty the result of a measurement of the $z-$component of its spin, given it is subsequently subjected to such a measurement. And Bell's theorem (see section 1.3) tells us that this potential result of a potential measurement of the remote particle cannot has been fixed before the measurement on the present particle. In other words, the measurement does not yield information about preexisting properties of the remote system, but it is immediately invasive on the remote system, immediately invasive as a direct measurement can be. The same is usually true for what is commonly called an *indirect measurement* in the quantum theory of measurement (actually, the two preceding examples could be also described by the following indirect measurement scheme).

A quantum measurement is called an *indirect measurement* if a third quantum mechanical system is implemented intermediary between the measured system and the macroscopic measuring device. This system interacts unitarily with the measured system and is subsequently projectively measured in an appropriate way. This measurement scheme has become a powerful tool in current experimental quantum physics. One important feature of indirect measurements is that it provides the possibility to measure microscopic systems without affecting them more than absolutely necessary according to quantum theory (i.e. only by collapsing the wave function, think of the indirect spin measurement of the remote singlet particle discussed in the last

paragraph), in particular without destroying them such that the same individual system can be measured again and again.

The theoretical analysis of indirect measurements is very instructive and as we shall see later in section 1.6 also very generic with respect to POVMs, since it can be shown that any POVM can be (at least formally) implemented by an indirect measurement scheme (see the Naimark theorem 1.20).

The Basic Scheme

An indirect quantum measurement consists of two interactions between three systems: First the measured system unitarily interacts with a quantum probe system – sometimes also referred to as *ancilla* or *meter system*, which is to be thought of as another microscopic system – resulting in an entangled state of the measured system and the probe which correlates both systems. This is called the *pre-measurement*. In a second step, the probe system interacts with a macroscopic measuring device producing an outcome (pointer position) and collapses its wave function together, by entanglement, with the wave function of the measured system. We call this step the *readout* and suppose that it is given by a projective measurement. We do not need a microscopic description of the readout in terms of pointer states here, we can simply represent it by the corresponding PVM and the corresponding state transformers.

Pre-Measurement

Denote the Hilbert space of the measured system by $\mathcal{H}_S$, the Hilbert space of the probe by $\mathcal{H}_P$ and $\mathcal{H} := \mathcal{H}_S \otimes \mathcal{H}_P$. The pre-measurement is defined by a unitary operator U representing the interaction between measured system and the probe. The idea is that there is a complete set of states $\{\psi_k\} \subset \mathcal{H}_S$, a ready state of the probe $\phi_0 \in \mathcal{H}_P$ and a set of *marker states* $\{\phi_k\} \subset \mathcal{H}_P$ of the probe which 'mark' the states of the measured system if U acts on $\psi_k\phi_0$, i.e.

$$\psi_k\phi_0 \xrightarrow{U} \psi_k\phi_k \tag{1.131}$$

for all k. As in an ideal measurement interaction, the fact that $\{\psi_k\}$ is a complete set of states in $\mathcal{H}_S$ together with the linearity of U entails now for an arbitrary initial state $\psi \in \mathcal{H}_S$ that there are complex numbers $\{c_k\} \subset \mathbb{C}$ such that

$$\psi\phi_0 = \left(\sum_k c_k\psi_k\right)\phi_0 \xrightarrow{U} \sum_k c_k\psi_k\phi_k \tag{1.132}$$

only now – in contrast to pointer states – the marker states ϕ_k need neither be macroscopically accessible and distinguishable nor mutually orthogonal. This constitutes the pre-measurement.

Readout, Probabilities and State Transformers

The readout is a projective measurement of the probe associated with a collection of orthogonal projections $\{P_\alpha\}$ acting on $\mathcal{H}_P$ – such that their extension $\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha$ onto $\mathcal{H}$ is the PVM corresponding to the readout, i.e. $\mathbb{P}^{(\psi\phi_0)}(\alpha) = \langle U(\psi\phi_0)|\,(\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha)U(\psi\phi_0)\rangle$. Moreover, there is

a set of partial isometries $\{U_\alpha\}$, such that the family of operators $\{(\mathbb{1}_{\mathcal{H}_S} \otimes U_\alpha P_\alpha)\}$ constitutes the set of state transformers on $\mathcal{H}$.

As always, we want to express the probabilities as quadratic forms of the initial state ψ of the measured system alone (the only 'free variable'), i.e. we shall derive operators F_α acting on $\mathcal{H}_S$ such that $\mathbb{P}^{(\psi\phi_0)}(\alpha) = \langle\psi|\, F_\alpha\psi\rangle \equiv \mathbb{P}^\psi(\alpha)$. To this end, we choose an arbitrary ONB $\{\xi_k\}$ of $\mathcal{H}_P$. The probability to obtain α as the outcome of the readout is then given by

$$\begin{aligned}
\mathbb{P}^{(\psi\phi_0)}(\alpha) &= \langle U(\psi\phi_0)|\,(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha)U(\psi\phi_0)\rangle = \left\langle \psi\phi_0\right|\, U^\dagger(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha)\,U\,\psi\phi_0\Big\rangle = \\
&= \sum_k \left\langle \psi\phi_0\right|\, U^\dagger\,(\mathbb{1}\otimes P_\alpha)\,U\,(\mathbb{1}_{\mathcal{H}_S}\otimes|\xi_k\rangle\,\langle\xi_k|)\,\psi\phi_0\Big\rangle = \\
&= \sum_k \langle\xi_k|\,\phi_0\rangle\,\left\langle \psi\phi_0\right|\, U^\dagger\,(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha)\,U\,\psi\,\xi_k\Big\rangle = \\
&= \sum_k \left\langle \psi\,\xi_k\right|\,(\mathbb{1}_{\mathcal{H}_S}\otimes|\phi_0\rangle\,\langle\phi_0|)\,U^\dagger\,(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha)\,U\,\psi\,\xi_k\Big\rangle = \\
&=: \sum_k \left\langle \psi\,\xi_k\right|\,(\mathbb{1}_{\mathcal{H}_S}\otimes\rho_P)\,U^\dagger\,(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha)\,U\,\psi\,\xi_k\Big\rangle = \\
&= \langle\psi|\,\mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S}\otimes\rho_P)\,U^\dagger\,(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha)\,U\right]|\psi\rangle =: \langle\psi|\,F_\alpha\,\psi\rangle \equiv \mathbb{P}^\psi(\alpha)
\end{aligned} \tag{1.133}$$

where we have denoted the initial density operator of the probe by $\rho_p := |\phi_0\rangle\,\langle\phi_0|$ and defined the effects

$$F_\alpha := \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S}\otimes\rho_P)\,U^\dagger\,(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha)\,U\right] \tag{1.134}$$

which form a POVM acting on $\mathcal{H}_S$ as can be straightforwardly verified (observe in particular that $\sum_\alpha F_\alpha = \mathbb{1}_{\mathcal{H}_S}$).

In order to find the associated state transformers $\mathcal{R}_\alpha$ acting on $\mathcal{H}_S$ (recall that the state transformers acting on $\mathcal{H}$ are given by $(\mathbb{1}_{\mathcal{H}_S}\otimes U_\alpha P_\alpha)$), let us suppose for the moment that the P'_αs are altogether one dimensional projections, i.e. that the readout is given by a non degenerate measurement. In this case, for each α there is a state $\varphi_\alpha \in \mathcal{H}_P$ such that we may write P_α as a dyadic product $P_\alpha = |\varphi_\alpha\rangle\,\langle\varphi_\alpha|$ and the readout dis-entangles the state of the measured system and the probe:

$$\begin{aligned}
(\mathbb{1}_{\mathcal{H}_S}\otimes U_\alpha P_\alpha)\,U(\psi\phi_0) &= (\mathbb{1}_{\mathcal{H}_S}\otimes U_\alpha P_\alpha)\sum_k c_k\psi_k\phi_k = \sum_k c_k\,\overbrace{\langle\varphi_\alpha|\,\phi_k\rangle}^{=:b_{\alpha k}}\,\psi_k\,(U_\alpha\varphi_\alpha) = \\
&= \left(\sum_k c_k b_{\alpha k}\psi_k\right)(U_\alpha\varphi_\alpha) = (\mathcal{R}_\alpha\psi)\,(U_\alpha\varphi_\alpha)
\end{aligned} \tag{1.135}$$

So the final state of the measured system is given by $\frac{1}{N}\sum_k c_k b_{\alpha k}\psi_k$ (with $N = \|\sum_k c_k b_{\alpha k}\psi_k\|$ as normalization) if its initial state was $\sum_k c_k\psi_k$ and the readout resulted in outcome α, and the associated state transformers may be written as

$$\mathcal{R}_\alpha\psi = \sum_k c_k b_{\alpha k}\psi_k = \sum_k c_k\,\langle\varphi_\alpha|\,\phi_k\rangle\,\psi_k = \sum_k c_k\,\langle\varphi_\alpha|\,U\phi_0\rangle\,\psi_k = \langle\varphi_\alpha|\,U\phi_0\rangle\sum_k c_k\psi_k \tag{1.136}$$

where $U\phi_0 : \mathcal{H}_S \to \mathcal{H}$ is understood as the operator mapping a state $\psi \in \mathcal{H}_S$ onto the state $U(\psi\phi_0) \in \mathcal{H}$ and $\langle\varphi_\alpha| : \mathcal{H} \to \mathcal{H}_S$ is defied by the 'partial scalar product': $\langle\varphi_\alpha|\,\psi\phi_0\rangle :=$

$\langle\varphi_\alpha|\,\phi_0\rangle\,\psi$. With this notation[52] we can denote the state transformers by

$$\mathcal{R}_\alpha = \langle\varphi_\alpha|\,U\phi_0\rangle \tag{1.138}$$

If the set $\{\psi_k\}$ is an ONB of $\mathcal{H}_S$ (so far we have only assumed that it is a complete set) we may write (1.138) in terms of the more familiar projections $P_{[\psi_k]} = |\psi_k\rangle\,\langle\psi_k|$:

$$\begin{aligned}\mathcal{R}_\alpha = \langle\varphi_\alpha|\,U\phi_0\rangle = \sum_k \langle\varphi_\alpha|\,U\phi_0\rangle\,|\psi_k\rangle\,\langle\psi_k| = \\ = \sum_k \langle\varphi_\alpha|\,\phi_k\rangle\,|\psi_k\rangle\,\langle\psi_k| = \sum_k \langle\varphi_\alpha|\,\phi_k\rangle\,P_{[\psi_k]}\end{aligned} \tag{1.139}$$

As it should be, the state transformers fulfil the relation $\mathcal{R}_\alpha^\dagger\mathcal{R}_\alpha = F_\alpha$:

$$\begin{aligned}\mathcal{R}_\alpha^\dagger\mathcal{R}_\alpha = \langle\phi_0|U^\dagger\varphi_\alpha\rangle\,\langle\varphi_\alpha|\,U\phi_0\rangle = \left\langle\phi_0\right|\,U^\dagger(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha)U\,\phi_0\Big\rangle = \\ = \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S}\otimes\rho_P)U^\dagger(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha)U\right] = F_\alpha\end{aligned} \tag{1.140}$$

What is Ultimately being Measured ?

As always with quantum measurement, in general actually nothing is being measured at all in an indirect measurement – measured in the naive sense that quantities initially pertaining to the measured system can be ascertained by measurement. Interactions produce a correlation between some microscopic system and some macroscopic display which we call measurement result. This process massively changes the state of the measured system in general. Quantum theory tells us the probability of some given result given the initial state of the measured system and it tells us how the final state looks like given a particular result (if no second-kind-like-processes are involved in the pre measurement, which we always assume).

But to get a bit of a feeling what the point of indirect measurement is, we may contemplate for a moment on the special case of an *ideal indirect measurement*: If the set of marker states $\{\phi_k\}$ is identical with the set $\{\varphi_\alpha\}$ defining the projections P_α associated with the readout (this entails in particular that the marker states are mutually orthogonal) the indirect measurement is essentially identical to an ideal measurement of the measured system. In this case, each of the initial states ψ_k leads to a certain outcome of the readout with certainty and the gain of information is exactly the same as in an ideal measurement. The merit is that this is even the case if the readout is not an ideal measurement. The readout can even by destructive to the probe system which can be accounted for by the operators U_α as in the example of the detector absorbing the photon from above. The measured system, on the other hand, survives and is not changed more than it is ultimately necessary according to quantum theory, i.e. it

[52] In the following calculations we will repeatedly make use of this notation. The following fact will be useful: Consider an operator $\mathcal{A}:\mathcal{H}\to\mathcal{H}$, then for some $\phi\in\mathcal{H}_P$ the operator $\langle\phi|\,\mathcal{A}\,\phi\rangle:\mathcal{H}_S\to\mathcal{H}_S$ can be written as (again with some ONB $\{\xi_k\}\subset\mathcal{H}_P$)

$$\langle\phi|\,\mathcal{A}\,\phi\rangle = \sum_k \langle\phi|\,\mathcal{A}\,\xi_k\rangle\,\langle\xi_k|\,\phi\rangle = \sum_k \langle\xi_k|\,\phi\rangle\,\langle\phi|\,\mathcal{A}\,\xi_k\rangle = \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S}\otimes|\phi\rangle\,\langle\phi|)\,\mathcal{A}\right] \tag{1.137}$$

does not suffer from some possibly uncontrollable measurement back action of the measuring device. Note in particular, that the state transformation of the measured system depends solely upon the projections P_α associated with the readout, whereas the measurement back action U_α is entirely 'absorbed' by the probe as it can be nicely read of from the calculation of the total state transformation in (1.135)! In the real world, indirect measurements are thus probably the only practicable way to implement clean ideal measurements in a laboratory.

A second merit is that if the marker states deviate from the φ'_αs and are possibly not even orthogonal, the measurement does not perfectly decohere the measured system which opens the door to explore quantum interference effects beyond the scope of direct quantum measurements.

The scheme is also well suited to describe processes which do not look like indirect quantum measurements at a first glance. Section 1.5.1 will provide an example, where the 'measured system' and the 'probe system' are simply taken to be two different degrees of freedom (spacial and spin degrees of freedom, respectively) of one and the same quantum particle.

Indeed, also direct measurement (like) processes as defined and analysed in section 1.2 can be straightforwardly incorporated into the scheme: Firstly, the pre-measurement interaction need not provide such a 'clean' marking of states as in equations (1.131) and (1.132) (which rather corresponds to an ideal measurement interaction), one way to account for more general pre-measurement interactions will be presented below. Secondly, there is no reason to identify necessarily the probe with another microscopic system as in the standard conception of indirect measurements, we may consider a usual quantum measurement process, identify the probe with the apparatus, the pre-measurement interaction with the usual measurement interaction and the readout with the collapse onto one definite pointer state (if you want, 'upon looking at the pointer'). Then the formalism developed in this section – with mutually orthogonal probe/pointer states, of course – maps one to one to the formalism of measurement (like) processes developed in section 1.2. Therefore one might regard the indirect measurement scheme as the most general scheme of direct and indirect measurement (like) processes without external sources of error (noise) as in the approximate measurement scheme (which nonetheless can be straightforwardly incorporated into the scheme, see e.g. [318]). The other way around, the second representation theorem of Kraus (theorem 1.19) and the Naimark theorem (theorem 1.20) which we shall prove in section 1.6, assert that each state transition in a reasonable framework and each POVM can be implemented by an indirect measurement scheme on a larger Hilbert space (system + probe), which supports the perception of the indirect measurement scheme as a generic scheme for descriptions of measurement (like) processes.

In Terms of the Density Matrix

Indirect quantum measurements are mostly described in terms of density operators instead of wave functions. This has as we shall see indeed some advantages and allows for generalizations. Of course, we can as always express the probabilities of the outcomes of the readout alternatively in terms of the initial density operator:

$$\mathbb{P}^\psi(\alpha) = \langle\psi|\, F_\alpha\, \psi\rangle = \mathrm{Tr}_{\mathcal{H}_S}[F_\alpha \rho_S] = \mathbb{P}^{\rho_S}(\alpha) \tag{1.141}$$

i.e. $\rho_S = |\psi\rangle\langle\psi|$ is the initial density operator of the measured system. Indeed, the right hand side of (1.141) is an expression in terms of density operators not only with respect to the measured system but with respect to the probe as well, since F_α contains the initial density

operator ρ_P of the probe. For the corresponding state transformations of the measured system we have derived above

$$\psi \quad \overset{\alpha}{\longmapsto} \quad \frac{1}{N_\alpha}\mathcal{R}_\alpha\psi \tag{1.142}$$

with the state transformers (1.138) and normalization

$$N_\alpha = \|\mathcal{R}_\alpha\psi\| = \sqrt{\left\langle\psi\right| \mathcal{R}_\alpha^\dagger\mathcal{R}_\alpha\psi\Big\rangle} = \sqrt{\mathrm{Tr}[F_\alpha\rho_S]} \tag{1.143}$$

which implies for the density operator of the measured system the transition

$$\rho_S = |\psi\rangle\langle\psi| \quad \overset{\alpha}{\longmapsto} \quad \frac{\mathcal{R}_\alpha\,\rho_S\,\mathcal{R}_\alpha^\dagger}{\mathrm{Tr}_{\mathcal{H}_S}[F_\alpha\rho_S]} =: \frac{\mathcal{W}(\rho_S \mid \alpha)}{\mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho_S \mid \alpha)\right]} \tag{1.144}$$

where we have introduced the state transformers $\mathcal{W}(\cdot \mid \alpha) = \mathcal{R}_\alpha \cdot \mathcal{R}_\alpha^\dagger$ on the density operator level, which are *linear maps*[53] mapping density operators to density operators modulo normalization (later in section 1.6 we will introduce the notion of superoperators to account for such mappings).

Since $\mathbb{P}^{\rho_S}(\alpha)$ and $\mathcal{W}(\rho_S \mid \alpha)$ are linear in ρ_S and since mixed states can always be written as convex linear combinations of pure states, it is straightforward to conclude that these expressions are valid when the initial state of the measured system ρ_S is not pure, as well.

Now here is an alternative strategy to derive the state transformers (which turns out to be better suited for generalizations): Proceed from the total final density operator

$$\frac{1}{\mathrm{Tr}_{\mathcal{H}_S}[F_\alpha\rho_S]}\,(\mathbb{1}_{\mathcal{H}_S}\otimes U_\alpha P_\alpha)\,U\,(\rho_S\otimes\rho_P)\,U^\dagger\,(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha U_\alpha^\dagger) \tag{1.145}$$

of the measured system and the probe subsequent to the readout with outcome α, discard the normalization (such that we obtain a linear transformation) and now plainly trace out the Hilbert space of the probe[54]

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S}\otimes U_\alpha P_\alpha)\,U\,(\rho_S\otimes\rho_P)\,U^\dagger\,(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha U_\alpha^\dagger)\right] = \\
&= \mathrm{Tr}_{\mathcal{H}_P}[\overbrace{(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha U_\alpha^\dagger)\,(\mathbb{1}_{\mathcal{H}_S}\otimes U_\alpha P_\alpha)}^{=(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha)}\,U\,(\rho_S\otimes\rho_P)\,U^\dagger\,] = \\
&\qquad = \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S}\otimes P_\alpha)\,U\,(\rho_S\otimes\rho_P)\,U^\dagger\right] = \\
&\qquad = \sum_k \langle\xi_k|\,\varphi_\alpha\rangle\,\langle\varphi_\alpha|\,U\,\phi_0\rangle\,\rho_S\,\left\langle\phi_0\right|\,U^\dagger\,\xi_k\Big\rangle = \\
&\qquad = \sum_k \langle\varphi_\alpha|\,U\,\phi_0\rangle\,\rho_S\,\left\langle\phi_0\right|\,U^\dagger\,\xi_k\Big\rangle\,\langle\xi_k|\,\varphi_\alpha\rangle = \\
&= \langle\varphi_\alpha|\,U\,\phi_0\rangle\,\rho_S\,\left\langle\phi_0\right|\,U^\dagger\,\varphi_\alpha\Big\rangle \equiv \mathcal{R}_\alpha\,\rho_S\,\mathcal{R}_\alpha^\dagger = \mathcal{W}(\rho_S \mid \alpha)
\end{aligned} \tag{1.146}$$

[53] But note that the total transformation (1.144) is *non-linear* because of normalization, as always when measurement (like) processes have outcomes which have not probability 1 for all initial states (note that the normalization factor equals the associated probability).

[54] Note that the cyclicity of the trace does in general not hold for partial traces, but indeed it still holds for operators of the form $\mathbb{1}\otimes\mathcal{A}$ if the trace is performed with respect to the Hilbert space on which $\mathcal{A}$ acts, which justifies the equality of the first and second line in calculation (1.146). Moreover, as found in section 1.4.2, U_α if not unitary is at least a partial isometry with initial space $P_\alpha\mathcal{H}_P$, which entails $P_\alpha U_\alpha^\dagger U_\alpha P_\alpha = P_\alpha P_\alpha = P_\alpha$, which yields the expression over the brace in the second line of (1.146).

One of the merits of this strategy is as already indicated above, that it also works if the pre-measurement interaction is not so perfect that it entails such a 'clean' marking of states as in equations (1.131) and (1.132): These details of the pre-measurement were not needed to derive the right state transformers in (1.146). This shows that, given only the central ingredient of calculation (1.146) that the operators ρ_S, ρ_P and P_α are one dimensional projections – i.e. given the initial state of the measured system and the probe ready state are pure and the readout is a non-degenerate measurement of the probe system – the final state of the measured system is always also given by a pure state, irrespective of the details of the pre-measurement: The operators $\mathcal{R}_\alpha$ are linear operators acting on $\mathcal{H}_S$ such that

$$\frac{\mathcal{R}_\alpha \left|\psi\right\rangle \left\langle\psi\right| \mathcal{R}_\alpha^\dagger}{\mathrm{Tr}[\mathcal{R}_\alpha^\dagger \mathcal{R}_\alpha \left|\psi\right\rangle \left\langle\psi\right|]} \tag{1.147}$$

is always a one dimensional projection, i.e. a pure state.

In other words, in the basic scheme of indirect measurements with non-degenerate readout, the measured system and the probe are as pure states always in a product state subsequent to the readout since otherwise a pure final state of the measured system would not exist. And this fact is even independent from the details of the pre-measurement, any unitary $U : \mathcal{H} \to \mathcal{H}$ is principally fine, if it is physically relevant is another question (this is indeed not a big surprise since if $P_\alpha = \left|\varphi_\alpha\right\rangle \left\langle\varphi_\alpha\right|$ with $\varphi_\alpha \in \mathcal{H}_P$ and $\Psi \in \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_P$, there is always a $\psi \in \mathcal{H}_S$ such that $(\mathbb{1}_{\mathcal{H}_S} \otimes U_\alpha P_\alpha)\, \Psi = \psi\, (U_\alpha \varphi_\alpha)$).

Non Efficient Indirect Measurements

This already indicates the preconditions under which indirect measurements are in general no longer efficient: In order to obtain a pure final state of the measured system we assumed that the operators ρ_S, ρ_P and P_α are one dimensional projections. If this does not hold for ρ_S, i.e. if the initial state of the measured system is not pure, it is no surprise when the final state is also mixed, in particular this does not entail a non efficient measurement. On the other hand, the operators ρ_P and P_α pertaining to the actually considered indirect measurement scheme might be more than one-dimensional projections as well. In other words, we may also consider indirect measurement scenarios where the probe is initially in a mixed state – such that ρ_P is a convex linear combination of one dimensional projections – or where the readout is degenerate[55] – such that the P'_αs are in general orthogonal projections onto subspaces of several dimensions.

We can indeed directly carry over the effects $F_\alpha := \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S} \otimes \rho_P)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha)\, U\right]$ to such scenarios, since they are obviously linear in ρ_P and P_α such that we can insert linear combinations of one dimensional projections for the latter to derive the right POVM and thereby probabilities if the probe system is initially not in a pure state or the readout is degenerate. The state transformers, on the other hand, are of course not so nicely form-invariant under these generalizations.

[55] To be precise, the full PVM associated with the readout is given by the operators $\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha$ which are projections of dimension $\dim(\mathcal{H}_S)$ if the P_α are one dimensional projections, i.e. in this sense the readout is always a degenerate measurement. But a (non-)degenerate readout in the present context means that the readout is (non-)degenerate on the level of the probe system, i.e. that the PVM given by the operators P_α acting solely on $\mathcal{H}_P$ represents a (non-)degenerate measurement.

Mixed Probe

This is a rather trivial example of a non-efficient measurement scheme, since it is not a surprise that the final state of an indirect measurement must be in general described by a mixed state even if the initial state was pure, if the probe system is described by a mixed state: Suppose

$$\rho_P = \sum_k p_k \left|\phi_0^{(k)}\right\rangle \left\langle\phi_0^{(k)}\right| \tag{1.148}$$

with $\sum_k p_k = 1$ is an initially mixed density operator of the probe system. A straightforward calculation in perfect analogy to (1.146) yields that the state transformers then look like

$$\mathcal{W}(\rho_S \mid \alpha) = \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S} \otimes U_\alpha P_\alpha)\, U\, (\rho_S \otimes \rho_P)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha U_\alpha^\dagger)\right] = \sum_k \mathcal{R}_{\alpha k}\, \rho_S\, \mathcal{R}_{\alpha k}^\dagger \tag{1.149}$$

where the linear operators $\mathcal{R}_{\alpha k}$ can now be denoted by

$$\mathcal{R}_{\alpha k} = \sqrt{p_k}\left\langle\varphi_\alpha\right| U \left|\phi_0^{(k)}\right\rangle \tag{1.150}$$

It is now easily verified that (1.149) – properly normalized – is in general a mixed state (see also footnote 56 below, where some subtleties in showing that an expression which looks like a mixed state truly is in general a mixed state are discussed).

The fundamental relation between the state transformers $\mathcal{W}(\cdot \mid \alpha)$ and the associated effects F_α is now expressed in terms of the operators $\mathcal{R}_{\alpha k}$ via

$$\begin{aligned}
\sum_k \mathcal{R}_{\alpha k}^\dagger \mathcal{R}_{\alpha k} &= \sum_k p_k \left\langle\phi_0^{(k)}\right| U^\dagger \left|\varphi_\alpha\right\rangle \left\langle\varphi_\alpha\right| U \left|\phi_0^{(k)}\right\rangle = \\
&= \sum_k p_k \left\langle\phi_0^{(k)}\right| U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha)\, U \left|\phi_0^{(k)}\right\rangle = \\
&= \sum_k p_k\, \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S} \otimes \rho_P^{(k)})\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha)\, U\right] = \\
&= \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S} \otimes \rho_P)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha)\, U\right] = F_\alpha
\end{aligned} \tag{1.151}$$

(which obviously reduces to the familiar relation $\mathcal{R}_\alpha^\dagger \mathcal{R}_\alpha = F_\alpha$ in case the measurement is efficient) such that

$$\begin{aligned}
\mathbb{P}^{\rho_S}(\alpha) = \mathrm{Tr}_{\mathcal{H}_\mathcal{S}}\left[F_\alpha \rho_S\right] &= \mathrm{Tr}_{\mathcal{H}_\mathcal{S}}\left[\sum_k \mathcal{R}_{\alpha k}^\dagger \mathcal{R}_{\alpha k}\, \rho_S\right] = \\
&= \mathrm{Tr}_{\mathcal{H}_\mathcal{S}}\left[\sum_k \mathcal{R}_{\alpha k}\, \rho_S\, \mathcal{R}_{\alpha k}^\dagger\right] = \mathrm{Tr}_{\mathcal{H}_\mathcal{S}}\left[\,\mathcal{W}(\rho_S \mid \alpha)\,\right]
\end{aligned} \tag{1.152}$$

Degenerate Readout

Now consider the case in which the readout is a degenerate measurement (and the probe is in an initial pure state again), i.e. the projections P_α are in general no longer one dimensional:

$$P_\alpha = \sum_k \left|\varphi_\alpha^{(k)}\right\rangle \left\langle\varphi_\alpha^{(k)}\right| \quad \text{with} \quad \left\langle\varphi_\alpha^{(k)}\right| \left.\varphi_\alpha^{(k')}\right\rangle = \delta_{kk'} \tag{1.153}$$

where we have chosen the states $\varphi_\alpha^{(l)}$ for fixed α to be an ONB of the subspace $\mathcal{H}_P^{(\alpha)} \subseteq \mathcal{H}_P$ associated with outcome α of the readout.

This entails that the measured system and the probe subsequent to the readout are in general still entangled, which can e.g. be seen by calculating the state transformers (straightforwardly in perfect analogy to (1.146))

$$\mathcal{W}(\rho_S \mid \alpha) = \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S} \otimes U_\alpha P_\alpha)\, U\, (\rho_S \otimes \rho_P)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha U_\alpha^\dagger)\right] = \sum_k \mathcal{R}_{\alpha k}\, \rho_S\, \mathcal{R}_{\alpha k}^\dagger \tag{1.154}$$

where now

$$\mathcal{R}_{\alpha k} = \left\langle \varphi_\alpha^{(k)} \right| U \phi_0 \rangle \tag{1.155}$$

Since the total final state $(\mathbb{1}_{\mathcal{H}_S} \otimes U_\alpha P_\alpha)\, U\, (\rho_S \otimes \rho_P)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha U_\alpha^\dagger)$ is now a pure state (in contrast to the previous case where ρ_P was a mixed state) and since the right hand side of (1.154) – properly normalized – is in general a mixed state[56], the total final state $(\mathbb{1}_{\mathcal{H}_S} \otimes U_\alpha P_\alpha)\, U\, (\rho_S \otimes \rho_P)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha U_\alpha^\dagger)$ can in general no longer be a product state of the measured system and the probe (note that the partial trace leads to a pure state if and only if the total system is in a pure state which is not entangled with respect to the traced out and the remaining system).

As in the previous case, the state transformers $\mathcal{W}(\cdot \mid \alpha)$ relate to the effects F_α via the relation

$$\begin{aligned} \sum_k \mathcal{R}_{\alpha k}^\dagger \mathcal{R}_{\alpha k} &= \sum_k \left\langle \phi_0 \right| U^\dagger \varphi_\alpha^{(k)} \rangle \left\langle \varphi_\alpha^{(k)} \right| U \phi_0 \rangle = \left\langle \phi_0 \right| U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha)\, U\, \phi_0 \rangle = \\ &= \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S} \otimes |\phi_0\rangle\langle\phi_0|)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha)\, U\right] = \\ &= \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S} \otimes \rho_P)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha)\, U\right] = F_\alpha \end{aligned} \tag{1.156}$$

such that

$$\begin{aligned} \mathbb{P}^{\rho_S}(\alpha) &= \mathrm{Tr}_{\mathcal{H}_\mathcal{S}}\left[F_\alpha \rho_S\right] = \mathrm{Tr}_{\mathcal{H}_\mathcal{S}}\left[\sum_k \mathcal{R}_{\alpha k}^\dagger \mathcal{R}_{\alpha k}\, \rho_S\right] = \\ &= \mathrm{Tr}_{\mathcal{H}_\mathcal{S}}\left[\sum_k \mathcal{R}_{\alpha k}\, \rho_S\, \mathcal{R}_{\alpha k}^\dagger\right] = \mathrm{Tr}_{\mathcal{H}_\mathcal{S}}\left[\,\mathcal{W}(\rho_S \mid \alpha)\,\right] \end{aligned} \tag{1.157}$$

[56] The right hand side of (1.154) is in general a mixed state for some fixed α whenever there does not exist a representation of the operators $\mathcal{R}_{\alpha k}$ (i.e. choice of ONB $\{\varphi_\alpha^{(k)}\}$ of $\mathcal{H}_P^{(\alpha)}$) such that $\mathcal{R}_{\alpha k} = 0$ for all but one k and their range is not an identical one dimensional subspace for all k. These two conditions are nicely illustrated in the simplest marker states model from above, i.e. consider an indirect measurement scheme with degenerate readout where the elements of an ONB $\{\psi_l\} \subset \mathcal{H}_S$ are marked in the pre-measurement by some set $\{\phi_l\}$ of 'marker states' of the probe via $\psi_l \phi_0 \stackrel{U}{\to} \psi_l \phi_l$ such that (1.155) becomes $\mathcal{R}_{\alpha k} = \sum_l \left\langle \varphi_\alpha^{(k)} \right| \phi_l \rangle P_{[\psi_l]}$. If now the orthogonal projections $\sum_k \left\langle \varphi_\alpha^{(k)} \right| \phi_l \rangle \varphi_\alpha^{(k)}$ of the marker states onto $\mathcal{H}_P^{(\alpha)}$ are all collinear in a given direction, we may choose the ONB $\{\varphi_\alpha^{(k)}\}$ such that it contains one unit vector $\varphi_\alpha^{(k')}$ in this direction which thus entails $\mathcal{R}_{\alpha k} = 0$ whenever $k \neq k'$. If this is not the case and if $\left\langle \varphi_\alpha^{(k)} \right| \phi_l \rangle \neq \left\langle \varphi_\alpha^{(k')} \right| \phi_l \rangle$ for at least one l and one pair $k \neq k'$, there are pure initial states $\rho_S \in \mathcal{S}(\mathcal{H}_S)$ of the measured system such that (1.154) is a mixed state. This can be made rigorous by resorting to the unitary equivalence of different convex representations of one and the same density operator presented in theorem 1.14 below and by noting that different choices of ONB in $\mathcal{H}_P^{(\alpha)}$ are related by a unitary transformation.

Remarks

- We repeatedly encountered state transformations of density operators characterized by linear maps of the form $\rho_S \mapsto \mathcal{W}(\rho \mid \alpha) = \sum_k \mathcal{R}_{\alpha k}^\dagger \rho_S \mathcal{R}_{\alpha k}$ with linear bounded operators $\{\mathcal{R}_{\alpha k}\}$ acting on $\mathcal{H}_S$, such that the final state is given by $\mathcal{W}(\rho_S \mid \alpha)$ properly normalized, i.e. divided by its own trace. Moreover, these state transformers relate to the effects constituting the associated POVM by $\sum_k \mathcal{R}_{\alpha k}^\dagger \mathcal{R}_{\alpha k} = F_\alpha$ such that the normalization of the final state equals the probability associated with the transformation, i.e. $\mathbb{P}^{\rho_S}(\alpha) = \mathrm{Tr}_{\mathcal{H}_S}[F_\alpha \rho_S] = \mathrm{Tr}_{\mathcal{H}_S}[\mathcal{W}(\rho_S \mid \alpha)]$. We will see in section 1.6 that this structure does not depend on the particular measurement schemes from which it was derived in the present section, but that it is generic and can be deduced from pretty general considerations about reasonable state transformations.

- It is easy to see that in case of non efficient indirect measurements, the representations of the state transformers $\mathcal{W}(\cdot \mid \alpha)$ by sets of operators $\{\mathcal{R}_{\alpha k}\}$ are not unique (in the mixed probe case we can exploit the fact that representations of mixed states as convex linear combinations of pure states are not unique, in the degenerate readout case we may exploit the freedom in choice of an ONB of $\mathcal{H}_P^{(\alpha)}$). In section 1.6 we will see that this non-uniqueness is also generic and that the class of sets $\{\mathcal{R}'_{\alpha l}\}$ of operators representing the same state transformer $\mathcal{W}(\cdot \mid \alpha)$ as a given set $\{\mathcal{R}_{\alpha k}\}$ via $\rho_S \mapsto \mathcal{W}(\rho \mid \alpha) = \sum_k \mathcal{R}_{\alpha k}^\dagger \rho_S \mathcal{R}_{\alpha k} = \sum_l (\mathcal{R}'_{\alpha l})^\dagger \rho_S \mathcal{R}'_{\alpha l}$ for all $\rho_S \in \mathcal{H}_S$ can be uniquely and exhaustively identified.

- As argued above, only the two presented examples lead to non-efficient measurements preceding from the basic indirect measurement scheme, i.e. given the total final state of measured system and the probe subsequent to the readout is given by

$$(\mathbb{1}_{\mathcal{H}_S} \otimes U_\alpha P_\alpha)\, U\, (\rho_S \otimes \rho_P)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes P_\alpha U_\alpha^\dagger) \tag{1.158}$$

  modulo normalization. The indirect measurement scheme can be also modified beyond that, e.g. something like uncontrollable degrees of freedom can be incorporated in the pre-measurement (in this case the total final state looks more complicated than (1.158)), which are to be traced out to get accessible expressions, which leads to a non-efficient indirect measurement scheme as well (see e.g. [318]).

1.5 Modelling Experiments

1.5.1 Stern-Gerlach as Indirect Measurement

Now we are going to apply the abstract formalism of indirect measurement POVMs to a more concrete physical model. We chose a rather unusual but instructive example to demonstrate the applicability of the indirect measurement formalism. Usually, the measured system and the probe are thought of to be two individual physical systems (like particles), separated subsequent to the pre-measurement interaction such that the separate probe system is directly measured in the readout.

Here we use the formal structure of indirect measurements to give an account of the familiar Stern-Gerlach experiment, an account which is a step towards a less idealized description than

the usual text books scheme). We treat the measured system and the probe as two different degrees of freedom of one and the same particle, in particular, we take the measured system to be represented by the spin degrees of freedom of a spin$-\frac{1}{2}-$ particle and the probe to be represented by its spatial degrees of freedom. The Stern-Gerlach experiment is commonly regarded as a measurement of one component of the spin of spin$-\frac{1}{2}-$ particles, but it is finally (as every measurement at the end of the day) a measurement of position (subsequent to the passage of an inhomogeneous magnetic field by the considered particle). From the measured position then conclusions about spin are inferred.

In Fig. 2 we see a postcard which Walther Gerlach sent to Niels Bohr in 1922 to tell him about the experimental discovery of 'space quantisation'. It shows a photography of the result of the original Stern-Gerlach experiment. In [72] Busch et al. nicely explain the procedure of the historic Stern-Gerlach experiment and point out how much this deviates from the common idealized presentations:

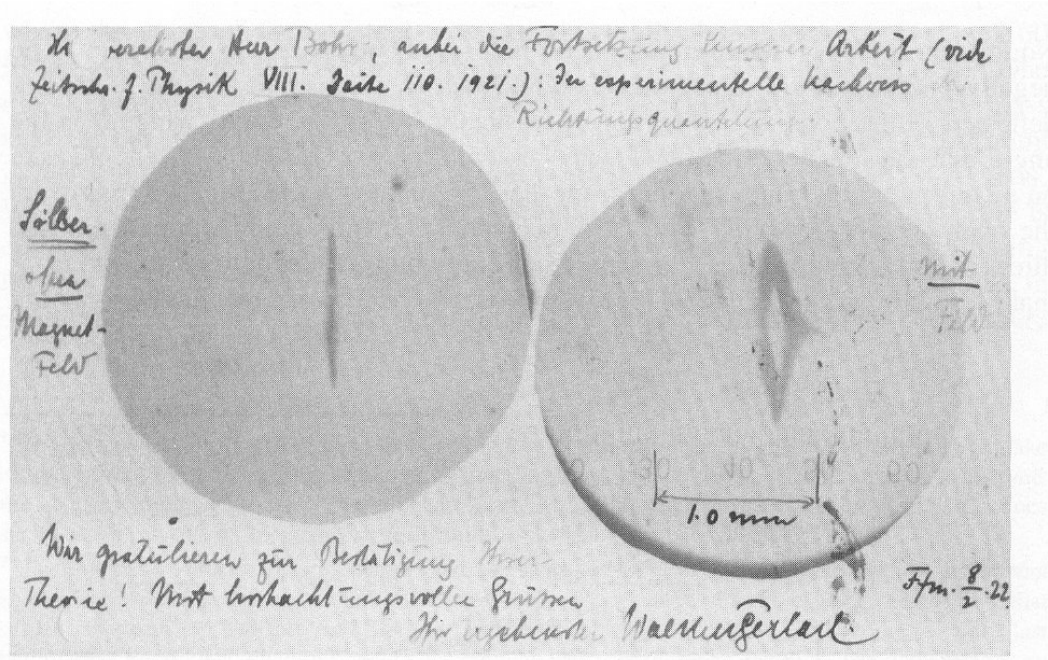

FIGURE 2

Stern-Gerlach: *Postcard Walther Gerlach sent to Niels Bohr on 8 February 1922 with the note: 'Attached the continuation of our work (Zeitschrift für Physik 8 (1921) 110): The experimental proof of directional quantisation. Silver without magnetic field / with magnetic field. We congratulate on the confirmation of your theory.'*

'The following 'laboratory report' of the historic Stern-Gerlach experiment stands quite in contrast to the usual textbook 'caricatures'. A beam of silver atoms, produced in a furnace, is directed through an inhomogeneous magnetic field, eventually impinging on a glass plate. The run time in the original experiment was 8 hours. Comparison was made with a similar experiment with the magnet turned off, run time 4,5 hours. The result of the magnet-off case was a single bar of silver on the glass approximately 1,1 mm long, 0,06-0,1 mm wide. In the magnet-on case, a pair-of-lips

shape appeared on the glass 1,1 mm long, one lip 0,11 mm wide, the other 0,20 mm wide, the maximum gap between the upper and lower lips being approximately of the order of magnitude of the width of the lips. Both lips appeared deflected relative to the position of the bar. Only visual measurements through a microscope were made. No statistics on the distributions were made, nor did one obtain 'two spots' as is stated in some texts. The beam was clearly split into distinguishable but not disjoint beams; yet this was considered to be enough to justify the conclusion that some property had been demonstrated. Gerlach and Stern viewed this property as 'space quantization in a magnetic field'.'

To model (part of) the theoretical description of this experiment by the indirect measurement scheme, we implement the pre-measurement interaction by the passage of the particle through the inhomogeneous magnetic field which couples the spin and the spatial degrees of freedom of the particle, as one can read of from the respective Pauli Hamiltonian. There can be found more and less idealized descriptions of this interaction in the literature, but we shall not bother here with a detailed description of the pre-measurement (for a very thorough analysis the reader is referred to [72]). We simply take it for granted that this interaction splits an incoming wave packet into two wave packets which are deflected upwards respectively downwards. Since the bars respectively lips in Fig 2 are supposed to reflect the respective $|\psi|^2-$distribution (at least if we suppose that the initial wave packets of the different runs of the experiment are more or less identical), we can guess the shape of the support of the wave packets perpendicular to the direction of the velocity of the particles (of course, what we call up and down is left and right on the postcard). The two wave packets are the marker states which mark the respective spin states 'spin up' and 'spin down' of the $z-$component of the spin.

Here we go: We denote $\mathcal{H}_S \cong \mathbb{C}^2$, $\mathcal{H}_P = L^2(\mathbb{R}^3, d^3x)$ and $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_P$. As a basis of $\mathcal{H}_S$ we choose the eigenstates of the operator

$$\sigma_z \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.159}$$

which are in this representation given by $|\uparrow\rangle_z = \binom{1}{0}$ and $|\downarrow\rangle_z = \binom{0}{1}$ to the eigenvalues $+1$and -1, respectively, i.e. for all normalized $\psi \in \mathcal{H}_S$ there are complex numbers c_1 and c_2 with $|c_1|^2 + |c_2|^2 = 1$ such that $\psi = c_1|\uparrow\rangle_z + c_2|\downarrow\rangle_z$. The initial state $\phi_0(x) \in \mathcal{H}_P$ is a wave packet, say moving in $x-$direction[57], and centred about $y = 0$ and $z = 0$. The pre measurement (passage through an inhomogeneous magnetic field) is represented by a unitary operator U which has the effect that

$$|\uparrow\rangle_z \otimes \phi_0(x) \quad \xrightarrow{U} \quad |\uparrow\rangle_z \otimes \phi_1(x) \tag{1.160}$$

and

$$|\downarrow\rangle_z \otimes \phi_0(x) \quad \xrightarrow{U} \quad |\downarrow\rangle_z \otimes \phi_2(x) \tag{1.161}$$

where $\phi_1(x)$ and $\phi_2(x)$ are two wave packets which are deflected in mutually opposite $z-$direction like it is indicated on the right hand side of Fig 2 (where the $z-$ direction goes from the left to the right). Consequently, for arbitrary $\psi = c_1|\uparrow\rangle_z + c_2|\downarrow\rangle_z \in \mathcal{H}_S$ we have

$$\psi\phi_0(x) = \Big(c_1|\uparrow\rangle_z + c_2|\downarrow\rangle_z\Big) \otimes \phi_0(x) \quad \xrightarrow{U} c_1|\uparrow\rangle_z \otimes \phi_1(x) + c_2|\downarrow\rangle_z \otimes \phi_2(x) \tag{1.162}$$

[57] The reader may ignore the abuse of notation here, where x has a double meaning $x = (x, y, z)$ which should not lead to confusion, though.

The read out is a subsequent measurement of position, say in the plane $x = (x_0, y, z)$ (screen) with an associated value function

$$\alpha(z) = \begin{cases} +1 & \text{for } z > 0 \\ -1 & \text{for } z < 0 \end{cases} \tag{1.163}$$

i.e. if the location of the particle is found in the upper, respectively lower half of the screen, we associate with this fact the value $+1$, respectively -1. Let further $\mathcal{P}_\alpha$ with $\alpha \in \{+1, -1\}$ be the projections acting on $\mathcal{H}_S$ corresponding to the location of the resulting dot in the upper, respectively lower half of the screen (i.e. something like indicator functions of the respective regions[58]) such that $\mathbb{1}_{\mathcal{H}_S} \otimes \mathcal{P}_\alpha$ is the PVM associated with the readout. The relevant density operators associated with the probe are denoted by $\rho_p := |\phi_0\rangle\langle\phi_0|$ (the initial state), $\rho_1 := |\phi_1\rangle\langle\phi_1|$ and $\rho_2 := |\phi_2\rangle\langle\phi_2|$. With the measured system we associate the initial density operator $\rho_S := |\psi\rangle\langle\psi|$ and the projections $P_\uparrow := |\uparrow\rangle\langle\uparrow|$ and $P_\downarrow := |\downarrow\rangle\langle\downarrow|$.

Now we calculate the related POVM (note that in a trace containing a trace over $\mathcal{H}_S$, terms proportional to $|\uparrow\rangle\langle\downarrow|_z$ and $|\downarrow\rangle\langle\uparrow|_z$ always vanish, we indicate this fact in the following calculation by the symbol $(*)$):

$$\begin{aligned}
\mathbb{P}^{\psi\phi_0}(\alpha) &= \langle U(\psi\phi_0)|\,(\mathbb{1}_{\mathcal{H}_S}\otimes\mathcal{P}_\alpha)\,U(\psi\phi_0)\rangle = \big\langle \psi\phi_0\big|\,U^\dagger\,(\mathbb{1}_{\mathcal{H}_S}\otimes\mathcal{P}_\alpha)\,U\,\psi\phi_0\big\rangle = \\
&= \mathrm{Tr}_{\mathcal{H}}\left[(\rho_S\otimes\rho_P)\,U^\dagger\,(\mathbb{1}_{\mathcal{H}_S}\otimes\mathcal{P}_\alpha)\,U\right] = \mathrm{Tr}_{\mathcal{H}}\left[U\,(\rho_S\otimes\rho_P)\,U^\dagger\,(\mathbb{1}_{\mathcal{H}_S}\otimes\mathcal{P}_\alpha)\right] \overset{(*)}{=} \\
&= \mathrm{Tr}_{\mathcal{H}}\left[\left(|c_1|^2 P_\uparrow\rho_1 + |c_2|^2 P_\downarrow\rho_2\right)\,(\mathbb{1}_{\mathcal{H}_S}\otimes\mathcal{P}_\alpha)\right] = \\
&= \mathrm{Tr}_{\mathcal{H}_S}\left[|c_1|^2 P_\uparrow\,\mathrm{Tr}_{\mathcal{H}_P}\left[\rho_1\mathcal{P}_\alpha\right] + |c_2|^2 P_\downarrow\,\mathrm{Tr}_{\mathcal{H}_P}\left[\rho_2\mathcal{P}_\alpha\right]\right] = \\
&= \mathrm{Tr}_{\mathcal{H}_S}\left[|c_1|^2\,\langle\phi_1|\,\mathcal{P}_\alpha\,\phi_1\rangle\;P_\uparrow + |c_2|^2\,\langle\phi_2|\,\mathcal{P}_\alpha\,\phi_2\rangle\;P_\downarrow\right] = \\
&= |c_1|^2\,\langle\phi_1|\,\mathcal{P}_\alpha\,\phi_1\rangle + |c_2|^2\,\langle\phi_2|\,\mathcal{P}_\alpha\,\phi_2\rangle = \\
&= \langle\psi|\,[\langle\phi_1|\,\mathcal{P}_\alpha\,\phi_1\rangle\;P_\uparrow + \langle\phi_2|\,\mathcal{P}_\alpha\,\phi_2\rangle\;P_\downarrow]\,\psi\rangle =: \langle\psi|\,F_\alpha\,\psi\rangle \equiv \mathbb{P}^\psi(\alpha)
\end{aligned} \tag{1.164}$$

Thus we have

$$F_\alpha = \langle\phi_1|\,\mathcal{P}_\alpha\,\phi_1\rangle\;P_\uparrow + \langle\phi_2|\,\mathcal{P}_\alpha\,\phi_2\rangle\;P_\downarrow = \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S}\otimes\rho_P)\,U^\dagger\,(\mathbb{1}_{\mathcal{H}_S}\otimes\mathcal{P}_\alpha)\,U\right] \tag{1.165}$$

where we can deduce the last equality in (1.165) from the scheme of section 1.4.4 or directly read it off from the first steps of (1.164) with a trained eye (note that this actually shows in the first place that all diagonal elements $\langle\psi|\,F_\alpha\,\psi\rangle$ of the two expressions for F_α in (1.165) coincide, which is sufficient for the full equality since F_α is selfadjoint and thereby diagonalizable).

If we write $P_{\uparrow\downarrow}$ in (1.165) in the spin$-z$ basis as matrices acting on $\mathbb{C}^2$, the effects F_α read

$$F_\alpha = \begin{pmatrix} \langle\phi_1|\,\mathcal{P}_\alpha\,\phi_1\rangle & 0 \\ 0 & \langle\phi_2|\,\mathcal{P}_\alpha\,\phi_2\rangle \end{pmatrix} \tag{1.166}$$

The idealized textbook presentation of the Stern Gerlach experiment is recovered in the approximation where the wave packet $\phi_1(x)$ is supported entirely in the upper half plane of the screen

[58] This is kept very vaguely since we are actually facing here the problem of arrival time in quantum theory. How the true screen observables look like is controversially debated until today and they are actually not given by projections, for sure. But we will not bother with this important problem which is not the point here (see [72] for a more thorough treatment).

$z > 0$ and $\phi_2(x)$ in the lower half plane $z < 0$ such that $\langle\phi_1|\,\mathcal{P}_{+1}\,\phi_1\rangle = 1$, $\langle\phi_1|\,\mathcal{P}_{-1}\,\phi_1\rangle = 0$, $\langle\phi_2|\,\mathcal{P}_{+1}\,\phi_2\rangle = 0$ and $\langle\phi_2|\,\mathcal{P}_{-1}\,\phi_2\rangle = 1$. In this case the effects F_α become projections

$$F_{+1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |\uparrow\rangle\langle\uparrow|_z \quad \text{and} \quad F_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |\downarrow\rangle\langle\downarrow|_z \tag{1.167}$$

and the measurement projective and might be called a measurement of the observable operator

$$\sum_\alpha \alpha\, F_\alpha = |\uparrow\rangle\langle\uparrow|_z - |\downarrow\rangle\langle\downarrow|_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z \tag{1.168}$$

To apply the indirect measurement scheme to the familiar Stern Gerlach experiment is instructive in order to learn how we can gain information (though – as it is notoriously the case in quantum measurement – not information about preexisting properties) about the measured system by measuring the probe in the readout, even if the marker states are not mutually orthogonal. In the present case, the spin up and spin down wave packets ϕ_1 and ϕ_2 have some small overlap. The non orthogonality of the marker states of course in some sense limits the gain of information. In particular, the experiment does not precisely provide a measurement of the angular momentum in the $z-$direction of the considered particle, whose observable operator is given by $\frac{\hbar}{2}\sigma_z$, but only an approximate measurement of this quantity with a certain probability of error deriving from the overlap of the two final wave packets.

On the other hand, this model is not very well suited to illustrate the state transformations of indirect measurements. If the particle is as usually finally absorbed by a screen, the readout is destructive to both, the probe *and* the measured system – in contrast to the usual intuitive picture of indirect measurements where the measured system is not directly affected by the readout. If we describe the readout as a non-destructive measurement (most easily by assuming $U_\alpha = \mathbb{1}_{\mathcal{H}_P}$, which might be realized by another indirect position measurement by a so called which-way detector), the final spin state of the measured system will have contributions of spin up and spin down states along the $z-$axis – with big weight one state and small weight on the other – depending on the measurement model of the readout either as a superposition or as a statistical mixture[59].

1.5.2 Von Neumann Measurements

In [334] John von Neumann analysed the dynamics resulting from a very simple formal interaction Hamiltonian, which correlates given degrees of freedom of the measured system with the spacial degrees of freedom of another system, where the latter can either be thought of as a macroscopic apparatus (pointer states) or a probe system which is subsequently measured in a readout. In particular, von Neumann ([334] p. 236 f.) considered a formal measurement of the position of the measured system, where roughly speaking a pointer system moves from the origin of the coordinate system to the position where the particle is located.

The von Neumann measurement scheme will be illustrated in the following step by step: First we will present the general scheme in a discrete framework, then we will sketch von Neumann's

[59]Note that the readout is a continuous measurement, such that it is at first problematic to associate a pure final state with the probe. A realistic possibility how this is nonetheless feasible (but at the cost of projectiveness and reproducibility) will be discussed in the following section.

measurement of position (and thereby switch from discrete to a continuous setting) and finally we shall analyse this scheme in the abstract framework developed so far.

General Discrete Framework

To keep things simple, we perform the scheme in only one dimension with one pointer/probe particle varying on a one dimensional scale (generalization to three dimensions is straight forward), i.e. we take the corresponding configuration space to be $\mathbb{R}$. The corresponding coordinate might also be thought of as the center of mass coordinate of some macroscopic object (pointer) varying on a one dimensional scale.

Let $\mathcal{H}_S$ be some Hilbert space, $\mathcal{H}_P = L^2(\mathbb{R}, dy)$ and $\mathcal{A}$ some selfadjoint operator with pure point spectrum acting on $\mathcal{H}_S$ with spectral representation

$$\mathcal{A} = \sum_k \lambda_k P_k \tag{1.169}$$

Now we will model an ideal measurement of the operator $\mathcal{A}$ by interaction with the 'pointer particle'. Let q_P be the standard position operator acting on $\mathcal{H}_P$ and p_P its conjugate momentum operator. Consider now an interaction given by the unitary operator $U = e^{-i\int_0^T \mathscr{H}_{int} dt} = e^{-i\gamma \mathcal{A}\otimes p_P}$ with interaction Hamiltonian $\mathscr{H}_{int}$ where γ is a constant which incorporates the time period T of interaction and the coupling constant of interaction[60].

Let $\psi \in \mathcal{H}_S$ be arbitrary and $\phi_0 \in \mathcal{H}_P$ be the pointer ready state which we assume to be a wave packet which is well localized about the origin $y = 0$, in particular we assume that there is some (small) $\varepsilon > 0$ such that ϕ_0 is supported within the interval $(-\varepsilon, \varepsilon)$. Now we calculate the joint final state $U(\psi\phi_0)$ subsequent to the interaction

$$\begin{aligned} U(\psi\phi_0) = e^{-i\gamma \mathcal{A}\otimes p_P} \left(\sum_k P_k \, \psi \right) \, \phi_0 = \sum_k e^{-i\gamma\lambda_k \, p_P} \, (P_k \, \psi) \, \phi_0 = \\ = \sum_k (P_k \, \psi) \, \left(e^{-i\gamma\lambda_k \, p_P} \, \phi_0 \right) =: \sum_k (P_k \, \psi) \, \phi_k \end{aligned} \tag{1.170}$$

where – since the momentum operator is the infinitesimal generator of spatial translations – ϕ_k in the position representation is given by

$$\phi_k(y) = e^{-i\gamma\lambda_k \, p_P} \, \phi_0(y) = \phi_0(y - \gamma\lambda_k) \tag{1.171}$$

In order to get now a precise measurement, ε has to be small enough (or γ large enough) to uniquely distinguish between the eigenvalues λ_k, i.e. $|\gamma\lambda_k - \gamma\lambda_{k-1}| > 2\varepsilon$ for all k. In this case, the pointer wave functions $\phi_k(y)$ have disjoint support in the disjoint intervals $\Delta_k := (\gamma\lambda_k - \varepsilon, \gamma\lambda_k + \varepsilon)$.

Together with the collapse dynamics ensuring always unambiguous pointer final states, this constitutes an ideal measurement of the observable operator $\mathcal{A}$, no matter if the ϕ's are literal pointer states or the states of a probe particle whose position is subsequently measured in the

[60] Usually it is required that the measurement interaction is *impulsive*, which means roughly speaking that the period T of interaction is small and the coupling constant is large such that the interaction Hamiltonian dominates the free Hamiltonians during the measurement and thereby the latter can be neglected.

readout (in which case it is an ideal indirect measurement): In the first case we have the ideal measurement (like) process:

$$\psi\phi_0 \xrightarrow{U} \sum_k (P_k\,\psi)\;\phi_k \xrightarrow{(*)} \frac{(P_l\,\psi)\,\phi_l}{\|P_l\,\psi\|} \tag{1.172}$$

where the last transition $(*)$ takes place with probability

$$\mathbb{P}^{\psi\phi_0}(l) = |\,\langle (P_l\,\psi)\,\phi_l|\;U(\psi\phi_0\rangle\,|^2 = \langle\psi|\;P_l\,\psi\rangle \equiv \mathbb{P}^{\psi}(l) \tag{1.173}$$

If the final pointer state is ϕ_l (pointer points onto λ_l), the final state of the measured system is given by the eigenstate

$$\frac{(P_l\,\psi)}{\|P_l\,\psi\|} \tag{1.174}$$

of $\mathcal{A}$ with associated eigenvalue λ_l and this outcome will be reproduced with certainty upon immediate repetition.

In the second case the probe particle will be found in the interval $\Delta_l = (\gamma\lambda_k - \varepsilon, \gamma\lambda_k + \varepsilon)$ with probability[61]

$$\begin{aligned}
\mathbb{P}^{\psi}(l) &= \left\langle\psi\right|\,\mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S}\otimes\rho_P)\,U^\dagger(\mathbb{1}_{\mathcal{H}_S}\otimes|\phi_l\rangle\,\langle\phi_l|)\,U\right]\;\left|\psi\right\rangle = \\
&= \mathrm{Tr}_{\mathcal{H}}\left[(\rho_S\otimes\rho_P)\,U^\dagger(\mathbb{1}_{\mathcal{H}_S}\otimes|\phi_l\rangle\,\langle\phi_l|)\,U\right] = \mathrm{Tr}_{\mathcal{H}}\left[U\,(\rho_S\otimes\rho_P)\,U^\dagger(\mathbb{1}_{\mathcal{H}_S}\otimes|\phi_l\rangle\,\langle\phi_l|)\right] = \\
&= \left\langle\phi_l\right|\,\mathrm{Tr}_{\mathcal{H}_S}\left[U\,(\rho_S\otimes\rho_P)\,U^\dagger\right]\;\left|\phi_l\right\rangle = \sum_{j,k}\langle\phi_l|\;\mathrm{Tr}_{\mathcal{H}_S}\left[|(P_j\,\psi)\;\phi_j\rangle\,\langle(P_k\,\psi)\;\phi_k|\right]\;\phi_l\rangle \overset{\langle\phi_k|\,\phi_l\rangle=\delta_{kl}}{=} \\
&= \mathrm{Tr}_{\mathcal{H}_S}\left[|P_l\,\psi\rangle\,\langle P_l\,\psi|\right] = \langle\psi|\;P_l\,\psi\rangle
\end{aligned} \tag{1.175}$$

as above, and the corresponding final state of the measured system is given by

$$\frac{\mathcal{R}_l\psi}{\|\mathcal{R}_l\psi\|} = \frac{\langle\phi_l|\;U\,\phi_0\rangle\,\psi}{\|\,\langle\phi_l|\;U\,\phi_0\rangle\,\psi\|} = \frac{(P_l\,\psi)}{\|P_l\,\psi\|} \tag{1.176}$$

as above.

Position Measurement

Now we come to the measurement of position: Let $\psi \in \mathcal{H}_S = L^2(\mathbb{R}, dx)$ be arbitrary and $\phi_0 \in \mathcal{H}_P = L^2(\mathbb{R}, dy)$ a wave packet localized about the origin as above (the generalization to three dimensions is straight forward). Now we consider the action of the unitary operator $U = e^{-i\gamma q_S\otimes p_P}$, where p_P is as above the momentum operator of the pointer/probe system and q_S

[61] Actually, the PVM associated with the readout is given by the infinite dimensional projections defined by multiplication with the indicator functions $\chi_{\{\Delta_l\}}(y)$ in the position representation. But in order to circumvent the inconveniences related with degenerate readouts we can exchange the projections $\chi_{\{\Delta_l\}}(y)$ with the one dimensional projections $|\phi_l\rangle\,\langle\phi_l|$ by exploiting the fact that in any case $\chi_{\{\Delta_l\}}(y)U(\psi\phi_0) = |\phi_l\rangle\,\langle\phi_l|\,U(\psi\phi_0) = (P_l\psi)\phi_l$.

is the standard position operator of the measured system $q_S\psi(x) = x\psi(x)$. The corresponding interaction has the result

$$\begin{aligned} U(\psi(x)\phi_0(y)) = e^{-i\gamma q_S\otimes p_P}\psi(x)\phi_0(y) = e^{-i\gamma\, x\, p_P}\psi(x)\phi_0(y) = \\ = \psi(x)\, e^{-i\gamma\, x\, p_P}\phi_0(y) = \psi(x)\,\phi_0(y-\gamma x) =: \psi(x)\,\phi_x(y) \end{aligned} \tag{1.177}$$

To keep things simple we set in the following $\gamma = 1$.

Now let us abstract for one moment from the aspiration to design a measurement and consider the interaction (1.177) only as an interaction between two particles, whose position is subsequently simultaneously measured in some reasonable but arbitrary way. We can calculate the probability that this hypothetical joint position measurement of the $x-$ and $y-$system subsequent to the interaction would yield $X \in (x_0-\delta, x_0+\delta) =: \Delta^{\delta}_{x_0}$ and $Y \in (y_0-\delta', y_0+\delta') =: \Delta^{\delta'}_{y_0}$. According to Born's rule this probability is given by

$$\begin{aligned} \mathbb{P}^{\psi\phi_0}(X \in \Delta^{\delta}_{x_0} \wedge Y \in \Delta^{\delta'}_{y_0}) = \int_{\Delta^{\delta}_{x_0}} dx \int_{\Delta^{\delta'}_{y_0}} dy\, |U(\psi(x)\phi_0(y))|^2 = \\ = \int_{x_0-\delta}^{x_0+\delta} dx \int_{y_0-\delta'}^{y_0+\delta'} dy\, |\psi(x)|^2\, |\phi_0(y-x)|^2 \end{aligned} \tag{1.178}$$

Now recall that for a given x the wave function $\phi_x(y) = \phi_0(y-x)$ is narrowly supported about x, i.e. there is some small $\varepsilon > 0$ such that $\operatorname{supp}\phi_x(y) \subseteq (x-\varepsilon, x+\varepsilon)$. This entails that the second integral in (1.178)

$$\int_{y_0-\delta'}^{y_0+\delta'} dy\, |\phi_0(y-x)|^2 \tag{1.179}$$

is non zero if and only if $|y_0 - x| < \delta' + \varepsilon$. Consequently, if a measurement of the position of the $y-$system finds $Y \in \Delta^{\delta'}_{y_0}$, the probability that a hypothetical subsequent measurement of the $x-$system would find X in a distance to y_0 greater than $\delta' + \varepsilon$ is zero. In other words, if a measurement of the position of the $y-$system finds $Y \in \Delta^{\delta'}_{y_0}$, the $x-$system is localized in the interval $\Delta^{\delta'+\varepsilon}_{y_0} = (y_0 - \delta' - \varepsilon\,,\, y_0 + \delta' + \varepsilon)$ with certainty in the operational sense that the probability that a subsequent position measurement finds $X \in \Delta^{\delta}_{x_0}$ is zero unless x_0 is nearby y_0 such that $|x_0 - y_0| < \delta + \delta' + \varepsilon$ (of course, if the coupling factor γ from above is not unity, we have to correspondingly readjust these intervals). This of course entails that upon the result $Y \in \Delta^{\delta'}_{y_0}$, the wave function of the $x-$system vanishes outside of $\Delta^{\delta'+\varepsilon}_{y_0}$ (almost) everywhere (and in a Bohmian framework, that the particle associated with $\mathcal{H}_S$ is truly located within this interval).

To highlight this, we may explicitly write down the strong correlation between X and Y and illustrate the gain of information we obtain about the $x-$coordinate by measuring the $y-$coordinate: We have

$$\begin{aligned} \mathbb{P}^{\psi\phi_0}(Y \in \Delta^{\delta'}_{y_0}) = \int_{-\infty}^{\infty} dx \int_{y_0-\delta'}^{y_0+\delta'} dy\, |\psi(x)|^2\, |\phi_0(y-x)|^2 = \\ = \int_{y_0-\delta'-\varepsilon}^{y_0+\delta'+\varepsilon} dx \int_{y_0-\delta'}^{y_0+\delta'} dy\, |\psi(x)|^2\, |\phi_0(y-x)|^2 = \mathbb{P}^{\psi\phi_0}(X \in \Delta^{\delta'+\varepsilon}_{y_0} \wedge Y \in \Delta^{\delta'}_{y_0}) \end{aligned} \tag{1.180}$$

and consequently

$$\mathbb{P}^{\psi\phi_0}(X \in \Delta_{y_0}^{\delta'+\varepsilon} \mid Y \in \Delta_{y_0}^{\delta'}) = \frac{\mathbb{P}^{\psi\phi_0}(X \in \Delta_{y_0}^{\delta'+\varepsilon} \wedge Y \in \Delta_{y_0}^{\delta'})}{\mathbb{P}^{\psi\phi_0}(Y \in \Delta_{y_0}^{\delta'})} = 1 \tag{1.181}$$

i.e. if we find $Y \in \Delta_{y_0}^{\delta'}$, we can conclude $X \in \Delta_{y_0}^{\delta'+\varepsilon}$ with certainty as already indicated above.

We may choose δ' arbitrarily small, but as long as the pointer/probe wave functions are not supported on a single point, there remains some uncertainty about the $x-$location. This smells like an approximate measurement of the position of the $x-$system, which will be made precise in a moment.

Later we will encounter that the wave functions $\phi_x(y)$ cannot even be *compactly supported* in a reasonable setting, also if the model is properly adjusted such that the $y-$system represents a true macroscopic many particle device like a pointer. In a strict sense, pointer states cannot be expected to be perfectly – but to be extremely well localized in a bounded region. We may thus more appropriately think of the states $\phi_x(y)$ (as pointer states) not to be perfectly localized within $(x-\varepsilon, x+\varepsilon)$ but rather of highly peaked wave packets with width 2ε and infinite tails dropping off very rapidly. In this respect, the analysis so far can in a very strict sense only be an approximation, in particular, the last equality sign in (1.181) is in a strict sense actually always an approximately sign. But it shall be remarked that the infinite tails of macroscopic wave functions will be incredibly strong suppressed in the real world due to decoherence resulting from thermodynamic interaction with an environment (air molecules, photons etc.), such that we should not bother to much with neglecting infinite tails as we reasonably do not bother much with the possibility of violations of the second law of thermodynamics in order to account for phenomena.

Analysis

Now we shall see that this position measurement constitutes indeed an approximate measurement as developed in section 1.4.3. To this end, we acknowledge the findings of the previous analysis, that obtaining a value of the (center-of-mass-)pointer or probe variable Y enables us to draw conclusions about the approximate value of X, and express the probability to find Y in some region Δ as a quadratic form in the wave function ψ of the $x-$system. The probability to find the pointer/probe within Δ is given by

$$\begin{aligned}
\mathbb{P}^{\psi\phi_0}(Y \in \Delta) &= \left\langle U(\psi\phi_0) \right| \left(\mathbb{1}_{\mathcal{H}_S} \otimes \chi_{\{\Delta\}}(q_P) \right) U(\psi\phi_0) \rangle = \\
&= \int_{\mathbb{R}} dx \int_{\Delta} dy \, |\psi(x)|^2 \, |\phi_0(y-x)|^2 = \\
&= \int_{\mathbb{R}} dx \, |\psi(x)|^2 \int_{\mathbb{R}} dy \, \chi_{\{\Delta\}}(y) \, |\phi_0(y-x)|^2 \overset{f(x):=|\phi_0(-x)|^2}{=} \\
&= \int_{\mathbb{R}} dx \, |\psi(x)|^2 \int_{\mathbb{R}} \chi_{\{\Delta\}}(y) \, f(x-y) \, dy = \int_{\mathbb{R}} dx \, |\psi(x)|^2 \left(\chi_{\{\Delta\}} * f \right)(x) = \\
&= \left\langle \psi \right| \left(\chi_{\{\Delta\}} * f \right)(q_S) \, \psi \rangle = \left\langle \psi \right| E_{\Delta}^{f} \, \psi \rangle
\end{aligned} \tag{1.182}$$

i.e. we obtain formally the same POVM as in the case of an approximate measurement, where the measurement error is now distributed according to $\rho_Y = |\phi_0|^2$. This provides us with a

clear interpretation, even if the wave functions $|\phi_x(y)|^2$ are not compactly supported within some interval, but only well localized: The von Neumann scheme with extended pointer states provides an approximate position measurement scheme of the $x-$system, where the distribution of the measurement error is given by the position distribution of the measuring system in its ready state $|\phi_0(y)|^2$. Thus if we find $Y \in \Delta$ we can conclude that approximately $X \in \Delta$ with probability of error (distribution of deviation) given by $|\phi_0(y)|^2$. Note that $(E^f_\Delta)^2 \neq E^f_\Delta$ i.e. the POVM is not projection valued, unless (essentially) $f(x) = \delta(x)$.

If the initial wave function of the y-system has expectation zero $\langle Y \rangle_{\phi_0} = \int_{\mathbb{R}} y\, |\phi_0(y)|^2\, dy = 0$, the measurement statistics reproduces the expectation value of the x-system:

$$\begin{aligned}
\langle Y \rangle_{U(\psi\phi_0)} &= \int_{\mathbb{R}} y\, \mathbb{P}^{\psi\phi_0}(y \in dy) = \int_{\mathbb{R}} dx\, |\psi(x)|^2 \int_{\mathbb{R}} y\, |\phi_0(y-x)|^2\, dy \overset{\xi := y-x}{=} \\
&= \int_{\mathbb{R}} x\, |\psi(x)|^2\, dx \overbrace{\int_{\mathbb{R}} |\phi_0(\xi)|^2\, dy}^{=1} + \overbrace{\int_{\mathbb{R}} |\psi(x)|^2\, dx}^{=1} \overbrace{\int_{\mathbb{R}} \xi\, |\phi_0(\xi)|^2\, dy}^{=0} = \\
&= \int_{\mathbb{R}} x\, |\psi(x)|^2\, dx = \langle X \rangle_\psi
\end{aligned} \tag{1.183}$$

Finally, we shall have a look at the associated state transformations of the $x-$system. If the probe/pointer position is found in some region Δ, we can write the final density operator of the $x-$system as

$$\begin{aligned}
\mathcal{W}(\rho_S|\Delta) &= \mathrm{Tr}_{\mathcal{H}_P}\Big[(\mathbb{1}_{\mathcal{H}_S} \otimes U_\Delta\, \chi_{\{\Delta\}}(q_P))\, U\, (\rho_S \otimes \rho_P)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes \chi_{\{\Delta\}}(q_P)\, U^\dagger_\Delta)\Big] = \\
&= \mathrm{Tr}_{\mathcal{H}_P}\Big[(\mathbb{1}_{\mathcal{H}_S} \otimes U^\dagger_\Delta\, U_\Delta\, \chi_{\{\Delta\}}(q_P))\, U\, (\rho_S \otimes \rho_P)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes \chi_{\{\Delta\}}(q_P))\Big] = \\
&= \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S} \otimes \chi_{\{\Delta\}}(q_P))\, U\, (\rho_S \otimes \rho_P)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes \chi_{\{\Delta\}}(q_P))\right]
\end{aligned} \tag{1.184}$$

with the initial density operators of the $x-$system and the $y-$system $\rho_S = |\psi\rangle\langle\psi|$ and $\rho_P = |\phi_0\rangle\langle\phi_0|$, respectively, the indicator function $\chi_{\{\Delta\}}(q_P)$ of Δ as a function of the position operator q_P acting on $\mathcal{H}_P$ and a possible measurement back action U_Δ acting on $\mathcal{H}_P$ which cancels out in the calculation of $\mathcal{W}(\rho_S|\Delta)$, where the equalities of the three lines in (1.184) are not perfectly trivial but well justified as explained in footnote 54. To get a nice expression for $\mathcal{W}(\rho_S|\Delta)$, we first observe that

$$(\mathbb{1}_{\mathcal{H}_S} \otimes \chi_{\{\Delta\}}(q_P))\, \psi(x)\phi_0(y-x) = \chi_{\{\Delta\}}(y)\, \psi(x)\phi(y-x) \tag{1.185}$$

and consequently

$$\begin{aligned}
\mathcal{W}(\rho_S|\Delta) &= \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}_S} \otimes \chi_{\{\Delta\}}(q_P))\, U\, (\rho_S \otimes \rho_P)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes \chi_{\{\Delta\}}(q_P))\right] = \\
&= \int dy\, \langle y|(\mathbb{1}_{\mathcal{H}_S} \otimes \chi_{\{\Delta\}}(q_P))\, U\, (\rho_S \otimes \rho_P)\, U^\dagger\, (\mathbb{1}_{\mathcal{H}_S} \otimes \chi_{\{\Delta\}}(q_P))|y\rangle = \\
&= \int_\Delta dy\, \phi_0(y-q_S)\, \rho_S\, \bar{\phi}_0(y-q_S)
\end{aligned} \tag{1.186}$$

We may interpret this result as follows: If the measurement of the $y-$coordinate finds $Y = Y_0$,

the wave function $\psi(x)$ of the $x-$system collapses onto the state $\phi_0(Y_0 - x)\psi(x)$[62] (modulo normalization), i.e. we might interpret the corresponding state transformer $\mathcal{R}_{Y_0}$ acting on $\mathcal{H}_S$ as the multiplication by $\phi_0(Y_0 - q_S)$. But in reality, also the $y-$measurement will have a finite resolution. Consequently, if the $y-$measurement finds $Y \in \Delta$, the final state of the $x-$system is in this sense the statistical mixture of all states resulting from 'Y's measured in Δ', which is exactly expressed by the final expression in (1.186) if properly normalized.

Note that these state transformers are additive, i.e. $\mathcal{W}(\rho_S|\Delta \cup \Delta') = \mathcal{W}(\rho_S|\Delta) + \mathcal{W}(\rho_S|\Delta')$ whenever $\Delta \cap \Delta' = \emptyset$, such that we need not bother whether Δ corresponds to some physically determined resolution or if it reflects human ignorance or our arbitrary choice. Note also, that if we interpret the $y-$measurement as the readout of an indirect measurement, it is clear (in the light of the analysis of indirect measurements above) that we have to describe the final state of the $x-$system by a statistical mixture since the readout constitutes a degenerate measurement whenever Δ has finite volume (to be precise, the readout is the yes/no measurement associated with the observable operator $\chi_\Delta(q_P)$, whose eigenvalues 0 (no) and 1 (yes) are both infinitely degenerate for finite Δ).

Continuous Measurements and Extended Pointer States

This model of a position measurement, where a pointer (particle) moves to the position of the measured particle, is of course artificial, but it nonetheless provides some instructive insights. For example, it nicely heuristically illustrates a fact which has been rigorously proven in a serious of publications [98, 251, 227] with considerable technical effort: Continuous measurements cannot be precisely reproducible (but approximate notions of reproducibility have been introduced, which are feasible to implement for continuous measurements and which indeed can be shown to hold for the present example [73]). Note that if a von Neumann measurement of position as developed above finds $Y \in \Delta$ (and we thus conclude approximately $X \in \Delta$), it might well be that upon immediate repetition with a new device we find $Y \notin \Delta$.

And these considerations can be abstracted from the present model: Whenever in a continuous measurement some part of the measuring device picks out the 'measured value' (or a set containing the latter) from a continuous set, the extension of the respective pointer wave function in configuration space (not to be confused with the spatial 'extension of the pointer' in physical space) introduces some uncertainty which is operationally reflected in the general non reproducibility of the exact result upon immediate repetition. This is in particular relevant if this part of the apparatus is itself microscopic such that the corresponding process might be perceived as an indirect measurement[63].

Moreover, this provides a realistic scheme which allows to describe exact state transformations of continuous measurements (e.g. something like multiplication with the 'pointer wave function'

[62] We have to be a bit careful here: The fact that ψ and ϕ are square integrable functions does not guarantee that $\psi\phi$ is in L^2 on non compact domains (we can trace this back to the fact that L^2 is not contained in L^1 on such domains in general). But we presupposed that ϕ_0 is well localized, i.e. drops of very rapidly (e.g. it might live in Schwartz space), such that this fact does indeed not compromise the given interpretation.

[63] One might even guess that in a very strict sense finally all (including discrete) measurements are associated with effects which are similar to E^f (which will be often very close to projections), where f is some measure of uncertainty associated with the measuring device deriving from the (infinite) tails of the associated pointer wave functions and conclude that all measurements are inherently unsharp. But these are rather intellectual games (whenever the 'pointer' is not a microscopic probe), while sources of limited measurement accuracy which do not derive from the quantum nature of the measuring device in the first place are truly relevant to consider instead.

$\mathcal{R}_{Y_0} = \phi(Y_0 - q_S)$ as described above), in contrast to the standard projective measurement scheme. But it is to be remarked that these considerations are presumably not so relevant as – in view of the fact that continuous observables like position (or momentum if you want) are the basic observables in quantum theory – they might appear at a first glance. It can be expected that at the end of the day there are no measurements in the world which have truly a continuous set of real numbers as possible outcomes. For example, in a typical measurement the pointer states finally involve a digital account of the measured value(s), which condemns any such seemingly continuous measurements to have actually only a countable number of possible outcomes, whatever the first part of the measurement process may look like (thus it is not a serious drawback to focus the analysis on discrete measurements), and dots on screens or traces in cloud chamber or the like are of course far from picking out precise values from uncountable sets, anyway.

Outlook

In chapter 3 we will encounter a class of mathematical results which finally entail that in relativistic quantum theory, under very few reasonable assumptions, POVMs on $\mathbb{R}^3$ (say a spacelike submanifold of space-time) describing position measurements do not exist. This is a puzzling result in the first place, since position measurements of course are possible and the statistics of measurement (like) processes is encoded by POVMs in quantum theory as we have derived for efficient measurements above and is (essentially) entailed by the Kraus theorem 1.17 below for the general case. To assume that 'real world POVMs' rather look like $\{E^f_\Delta\}$ does nothing to resolve the puzzle. Actually effects of the form E^f_Δ satisfy all conditions of theorem 3.25, namely they are translation covariant, mutually commutative (note that E^f_Δ are functions of the position operator alone, such that $[E^f_\Delta, E^f_{\Delta'}] = 0$ for all Δ, Δ') and they form an additive POVM on $\mathbb{R}$ (respectively $\mathbb{R}^3$). Theorem 3.25 entails that such effects are not consistent with any covariant quantum theory in which the Hamiltonian is bounded from below, like in the established, empirically strikingly successful QFTs. So we will need additional considerations to see how it is possible to derive probabilities for position measurements in relativistic quantum theory with positive energy, which we shall establish in chapter 3.

1.6 A General Operational Framework

In the previous sections we analysed certain measurement schemes and repeatedly encountered that on the level of density operators an initial state $\rho \in \mathcal{S}(\mathcal{H}_S)$ of the measured system – if we want to describe the corresponding final state as an individual system – undergoes a transformation of the form

$$\mathcal{W}(\rho \mid \alpha) = \sum_m \mathcal{R}_{\alpha m}\, \rho\, \mathcal{R}^\dagger_{\alpha m} \tag{1.187}$$

modulo normalization upon outcome α of the measurement, with a countable set of bounded operators $\{\mathcal{R}_{\alpha m}\}$ which are related with the corresponding effect F_α via

$$F_\alpha = \sum_m \mathcal{R}^\dagger_{\alpha m} \mathcal{R}_{\alpha m} \tag{1.188}$$

such that

$$\sum_{\alpha m} \mathcal{R}_{\alpha m}^{\dagger} \mathcal{R}_{\alpha m} = \mathbb{1}_{\mathcal{H}_S} \tag{1.189}$$

and the respective transition occurs with probability (stemming from the Born rule)

$$\mathbb{P}^{\rho}(\alpha) = \mathrm{Tr}_{\mathcal{H}_S}\left[F_\alpha\, \rho\right] = \mathrm{Tr}_{\mathcal{H}_S}\left[\mathcal{W}(\rho \mid \alpha)\right] \tag{1.190}$$

For efficient measurements this representation was generally derived, where in this case only a single operator $\mathcal{R}_\alpha$ is associated with a single outcome α. For non-efficient measurements, so far we only considered particular examples of measurement schemes which turned out to share this structure. Transitions which are not measurement like, namely unitary ones, can be of course incorporated in this structure, too, where one single unitary operator $\mathcal{R} \equiv U$ is associated with the respective transition which then happens with probability $\mathrm{Tr}_{\mathcal{H}_S}\left[U^{\dagger} U\, \rho\right] = \mathrm{Tr}_{\mathcal{H}_S}\left[\rho\right] = 1$ since unitary evolution is deterministic.

Indeed, physicists like Ludwig, Kraus, Davis and others were able to show that the structure given by the relations (1.187)-(1.190) reflects indeed under a few very general assumptions the generic structure of state transitions independent of measurement schemes or the like as a mathematical fact. Within the framework developed by these authors, fundamental results determining possible representations of state transformers and POVMs can be derived. For example it can be shown that the indirect measurement scheme is generic in so far as any state transformation (consistent with the mentioned mild assumptions) and any POVM can be – at least formally – implemented by an indirect measurement scheme (note that this in particular entails that any non projective measurement can be implemented by a projective measurement scheme on a larger Hilbert space). We shall develop this framework and the basic results therein in the following sections.

1.6.1 Generic State Transitions: Superoperators and Complete Positivity

This section is intended to work out a structural picture, how conceivable physical transitions of quantum states may look like from a rather general perspective, i.e. beyond an analysis of particular schemes. We have already excessively made use of the notion of *state transitions*, which refers to the transformation of an input state at a given starting time $t = 0$, to an output state at a given time later $t = t_1$, without further specifying what happens in between (and without explicitly specifying initial and final time for economic reasons). A related generic notion of time evolution, which is not restricted to such a coarse grained in-out scheme but specifies a state of the system at any single time, is given by master equations (like Lindblad equations in the case of Markovian dynamics), which will be briefly discussed later on, too.

We have encountered so far roughly two kinds of transitions: Unitary ones, due to free time evolution or unitary interaction, and transformations of the kind analysed in detail above, which we called *measurement like*. Proceeding from these transitions as fundamental building blocks we can easily find more examples like transitions of *open quantum systems*, i.e. interaction of a given quantum system with its environment whose details are not explicitly known and the environment is traced out. Some of theses transitions cannot be described as transitions of pure states to pure states like in case of non-efficient measurement (like) processes or any transitions involving a partial trace over an entangled subsystem. As a consequence, in searching for a

general description of state transitions it makes sense to consider these on the level of density operators.

The Positive Trace Class

All state transitions we encountered so far were of the form

$$\rho \mapsto \frac{\mathcal{W}(\rho)}{\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}(\rho)\right]} \tag{1.191}$$

where $\mathcal{W}$ is some linear map acting on density operators and the normalization $\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}(\rho)\right]$ equals the probability $\mathbb{P}^{\rho}$ associated with this transition[64]. In particular, whenever the transition has non zero probability, $\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}(\rho)\right]$ is a positive constant (otherwise we suppose that the transition will not occur, such that we come not into trouble with dividing by zero). The requirement that the final state of the transition, which is given by the right hand side of (1.191), is a density operator again – i.e. that it is a positive operator of trace one – thus translates into the requirement on the linear map $\mathcal{W}$ that it transforms density operators to positive trace class operators, i.e. $\mathcal{W} : \mathcal{S}(\mathcal{H}) \to \mathcal{T}_{+}(\mathcal{H})$ such that $\mathcal{W}(\rho)$ is normalizable to a positive trace one operator by division by a positive constant.

Since each element of the positive trace class $\mathcal{T}_{+}(\mathcal{H})$ becomes a density operator if divided by its own trace, we shall henceforth call the elements of $\mathcal{T}_{+}(\mathcal{H})$ *density operators modulo normalization* and continue the way of speaking to say that a transition of density operators is *characterized* by a linear transformation whenever it is given by a linear mapping up to normalization.

Linearity

One occasionally finds the claim in the literature (e.g. [345] p.17), that *linearity* of state transitions is an inherent feature of quantum theory. This is actually not true for a large class of transitions of standard quantum theory, namely measurements with definite outcomes. Indeed, in view of (1.191) it is only true for transitions which occur with probability 1 for all initial states, like transitions given by unitary evolutions or non selective measurements (later we will introduce the notion of *quantum channels* for such transitions). Transitions associated with non trivial statistical predictions of quantum theory – which actually constitute the great predictive success of this theory – cannot be linear but only linear up to normalization!

So it is rather a fundamental feature of standard quantum theory that all state transitions – including *nonlinear* transformations upon measurement like processes – can be *characterized* by *linear* transformations $\mathcal{W} : \mathcal{S}(\mathcal{H}) \to \mathcal{T}_{+}(\mathcal{H})$ of density operators to positive trace class operators, such that for a given initial $\rho \in \mathcal{S}(\mathcal{H})$ the operator $\mathcal{W}(\rho)$ is the final density operator if properly normalized, i.e. divided by its own trace (this linearity up to normalization is actually not true for evolutions generated by nonlinear Schrödinger equations like they arise in mean field theories!). This is the basic assumption on which the following analysis shall be based.

[64] Or analogously on the wave function level $\psi \mapsto \frac{L(\psi)}{\|L(\psi)\|}$ with a linear map $L : \mathcal{H} \to \mathcal{H}$, where the square of the normalization is equal to the associated probability $\mathbb{P}^{\psi} = \|L(\psi)\|^2$. Note that these quantum probabilities for measurement (like) processes directly stem from the Born rule for the matter distributions of different 'pointer orientations' as derived in the introduction of this chapter.

Superoperators

The mappings $\mathcal{W} : \mathcal{S}(\mathcal{H}) \to \mathcal{T}_+(\mathcal{H})$ characterizing transitions of density operators are linear maps, mapping operators to operators and preserving positivity of operators in their domain $\mathcal{S}(\mathcal{H})$ (such that density operators are never mapped to non positive operators). This motivates the following definitions:

Definition 1.10 [*Superoperators & Positive Superoperators*]

Let $\mathcal{H}$, $\mathcal{H}'$ be Hilbert spaces $\mathcal{X} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{Y} \subseteq \mathcal{B}(\mathcal{H}')$. A linear map $\mathcal{W} : \mathcal{X} \to \mathcal{Y}$ which maps operators to operators is called a superoperator. A superoperator $\mathcal{W} : \mathcal{X} \to \mathcal{Y}$ is called positive, if it maps all positive operators in $\mathcal{X}$ to positive operators in $\mathcal{Y}$, i.e. if $\mathcal{W}(\mathcal{X}_+) \subseteq \mathcal{Y}_+$.

We will be mainly concerned with superoperators $\mathcal{W}$ describing transitions of density operators of a given quantum system, such that mostly $\mathcal{H} = \mathcal{H}'$[65]. As mentioned, the natural physical choice for $\mathcal{X}$ and $\mathcal{Y}$ is $\mathcal{X} = \mathcal{S}(\mathcal{H})$ and $\mathcal{Y} = \mathcal{T}_+(\mathcal{H})$. But for mathematical reasons which will become clear below, it is convenient to define the transition superoperators $\mathcal{W}$ on a larger domain than the density operators, namely on all of the trace class $\mathcal{T}(\mathcal{H}) \supset \mathcal{S}(\mathcal{H})$ (which becomes a Banach space when equipped with the trace norm) which is always possible[66].

So consider a superoperator $\mathcal{W} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ characterizing transitions of density operators which of course implies that $\mathcal{W}$ must be positive: $\mathcal{W}(\mathcal{T}_+(\mathcal{H})) \subseteq \mathcal{T}_+(\mathcal{H})$, i.e. that it maps density operators modulo normalization to density operators modulo normalization.

Complete Positivity

To summarize, a primitive assumption on which the following analysis shall be based is that transitions of quantum states are linear up to normalization which entails that these transitions can be characterized on the density operator level by positive superoperators $\mathcal{W} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$.

Now comes a tiny straightforward sharpening of this assumption which turns out to be mathematically enormously powerful: If $\mathcal{W}$ shall transform density operators to density operators modulo normalization, it is obviously reasonable to require this as well if the considered system is treated as a subsystem of some larger system (and possibly entangled with the complementary system). This translates to the mathematical requirement on $\mathcal{W}$ that it not only preserves positivity, but that it preserves positivity as well if it is extended to some larger Hilbert space in a natural way. This strengthened positivity requirement rigorously defined is called *complete positivity* of a superoperator and provides as mentioned a strong technical tool. Indeed, for the

[65] An example of a superoperator with $\mathcal{H} \neq \mathcal{H}'$ is the partial trace operation $\mathrm{Tr}_{\mathcal{H}_2} : \mathcal{T}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{T}(\mathcal{H}_1)$ with $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{H}' = \mathcal{H}_1$.

[66] An extension to $\mathcal{T}(\mathcal{H})$ is always possible if $\mathcal{W}$ is properly defined on $\mathcal{S}(\mathcal{H})$, since according to assumption $\mathcal{W}$ is linear and, for each $\rho' \in \mathcal{T}_+(\mathcal{H})$ there is a number $a = \mathrm{Tr}_{\mathcal{H}}[\rho']$ such that $\rho' = a\rho$ with $\rho \in \mathcal{S}(\mathcal{H})$ such that we set $\mathcal{W}(\rho') \equiv a\mathcal{W}(\rho)$. Moreover, any $\rho'' \in \mathcal{T}(\mathcal{H})$ can be written as a linear combination of elements of $\mathcal{T}_+(\mathcal{H})$: $\rho'' = A + iB = (A_+ - A_-) + i(B_+ - B_-)$ where A is the symmetric part of ρ'', iB its antisymmetric part, A_+, B_+ the positive parts of A and B, respectively, and A_-, B_- the respective negative parts. It is then easy to see that A_+, A_-, B_+ and B_- are positive trace class operators.

major part of the strong results we shall derive below, positivity alone is not sufficient but only complete positivity!

Before working out a clean definition of complete positivity, its meaning shall be briefly illustrated with the familiar example of ideal spin measurements:

Consider an ideal Stern-Gerlach experiment, first, say on a single spin$-\frac{1}{2}-$particle with the result 'spin up' in a given direction (say the $z-$direction, we drop the indication of direction in this example since we will not consider other orientations of the device) which we associate with the orthonormal basis $\{|\uparrow\rangle; |\downarrow\rangle\} \cong \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ of the Hilbert space $\mathcal{H} \cong \mathbb{C}^2$ (we neglect again the spatial part of the wave function and its extension which was discussed in section 1.5.1). The associated state transition modulo normalization for an ideal spin measurement is given by multiplication of the initial state $\psi \in \mathcal{H}$ with the projection $P_\uparrow = |\uparrow\rangle\langle\uparrow|$ and consequently on the density operator level for an initial density operator $\rho \in \mathcal{S}(\mathcal{H})$ the transition reads

$$\rho \quad \longmapsto \quad \rho' = \frac{\mathcal{W}(\rho)}{\mathcal{N}(\rho)} := \frac{P_\uparrow \rho P_\uparrow}{\mathrm{Tr}_\mathcal{H}\left[P_\uparrow \rho\right]} \tag{1.192}$$

Here, for each $\rho \in \mathcal{S}(\mathcal{H})$ the superoperator $\mathcal{W}$ is defined by $\mathcal{W}(\rho) := P_\uparrow \rho P_\uparrow$ and the normalization $\mathcal{N}$ by $\mathcal{N}(\rho) := \mathrm{Tr}_\mathcal{H}\left[P_\uparrow \rho\right] \equiv \mathrm{Tr}_\mathcal{H}\left[\mathcal{W}(\rho)\right]$ (we see that $\mathcal{W}$ is linear, in contrast to the entire state transformation $\mathcal{N}^{-1}\mathcal{W}$). Of course, $\mathcal{W}$ is a positive superoperator, which can be straightforwardly extended from its domain $\mathcal{S}(\mathcal{H})$ to $\mathcal{T}(\mathcal{H})$.

Now consider how this state transition looks like, if the measured particle is combined and possibly entangled with another spin$-\frac{1}{2}$ particle, like in an EPRB experiment with initial singlet state $\psi = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \in \widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}$ or any other initial $\psi \in \widetilde{\mathcal{H}}$: Suppose a Stern-Gerlach experiment is performed one of the two fermions, say the particle on the left side, associated with the first entry in the kets $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle \ldots$, and the result is: 'spin up'. The transition of any initial density operator $\rho \in \mathcal{S}(\widetilde{\mathcal{H}})$ (like the singlet state) now looks like

$$\rho \quad \longmapsto \quad \rho' = \frac{\mathcal{W} \otimes \mathbb{1}_{\mathcal{S}(\mathcal{H})}(\rho)}{\mathcal{N}(\rho)} = \frac{\left(P_\uparrow \otimes \mathbb{1}_\mathcal{H}\right)\, \rho\, \left(P_\uparrow \otimes \mathbb{1}_\mathcal{H}\right)}{\mathrm{Tr}_\mathcal{H}\left[\left(P_\uparrow \otimes \mathbb{1}_\mathcal{H}\right)\, \rho\right]} \tag{1.193}$$

The superoperator $\mathcal{W} \otimes \mathbb{1}_{\mathcal{S}(\mathcal{H})}$ given by

$$\mathcal{W} \otimes \mathbb{1}_{\mathcal{S}(\mathcal{H})}(\rho) = \left(P_\uparrow \otimes \mathbb{1}_\mathcal{H}\right)\, \rho\, \left(P_\uparrow \otimes \mathbb{1}_\mathcal{H}\right) \quad \rho \in \mathcal{S}(\widetilde{\mathcal{H}}) \tag{1.194}$$

is obviously a positive superoperator again and can be extended from its domain $\mathcal{S}(\widetilde{\mathcal{H}})$ to all of $\mathcal{T}(\widetilde{\mathcal{H}})$.

If $\rho = \rho_1 \otimes \rho_2$ is a product state with $\rho_i \in \mathcal{S}(\mathcal{H})$, we can write the action of $\mathcal{W} \otimes \mathbb{1}_{\mathcal{S}(\mathcal{H})}$ on ρ as $\mathcal{W} \otimes \mathbb{1}_{\mathcal{S}(\mathcal{H})}(\rho_1 \otimes \rho_2) = \mathcal{W}(\rho_1) \otimes \rho_2$, where $\mathcal{W}(\rho_1) = P_\uparrow \rho_1 P_\uparrow$ is the one particle transition (modulo normalization) upon spin-measurement with result 'spin up'. This expresses the requirement, that transformations upon measurement of a subsystem (here associated with ρ_1) should leave the rest of the system (described by ρ_2) undisturbed, given the latter is not entangled and does not interact with the subsystem. But note that in general $\mathcal{W} \otimes \mathbb{1}_{\mathcal{S}(\mathcal{H})}$ does not leave the second system invariant, in particular not in case of entanglement between the two subsystems as for example

its action on the singlet state shows: $\mathcal{W}\otimes\mathbb{1}_{\mathcal{S}(\mathcal{H})}\big(\frac{1}{2}\left(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\right)\left(\langle\uparrow\downarrow|-\langle\downarrow\uparrow|\right)\big)=\frac{1}{2}\,|\uparrow\downarrow\rangle\langle\uparrow\downarrow|$, i.e. the action of $\mathcal{W}\otimes\mathbb{1}_{\mathcal{S}(\mathcal{H})}$ determines the $z-$spin of the second system!

Note that $\mathcal{W}\otimes\mathbb{1}_{\mathcal{S}(\mathcal{H})}(\rho)$ is for each density operator $\rho\in\mathcal{S}(\widetilde{\mathcal{H}})$ again a density operator modulo normalization, i.e. a positive trace class operator, and consequently $\mathcal{W}\otimes\mathbb{1}_{\mathcal{S}(\mathcal{H})}$ is a positive superoperator.

Now let us generalize these last observations to a requirement on superoperators describing admissible physical transitions of density operators acting on some arbitrary (separable) Hilbert space, such that they stay admissible if the respective system is treated as a subsystem of some larger system: Consider a Hilbert space $\mathcal{H}$ and a superoperator $\mathcal{W}:\mathcal{T}(\mathcal{H})\to\mathcal{T}(\mathcal{H})$ encoding a certain kind of physically conceivable transitions of density operators.

Now suppose the system associated with $\mathcal{H}$ is part of a larger system associated with Hilbert space $\widetilde{\mathcal{H}}=\mathcal{H}\otimes\mathcal{H}_N$, where the subscript N of $\mathcal{H}_N$ shall indicate that the remaining system has $N\in\mathbb{N}$ degrees of freedom, i.e. $\dim(\mathcal{H}_N)=N$. If $\{\varphi_k\}\subset\mathcal{H}_N$ is an ONB, we may write a density operator $\rho\in\mathcal{S}(\widetilde{\mathcal{H}})$ as[67]

$$\rho=\sum_{k,l=1}^{N}\rho_{kl}\otimes|\varphi_k\rangle\langle\varphi_l| \tag{1.195}$$

where the matrix elements ρ_{kl} are trace class operators acting in $\mathcal{H}$ (see [212]) and are defined by the partial scalar product (with respect to $\mathcal{H}_N$):

$$\mathcal{T}(\mathcal{H})\ni\rho_{kl}=\langle\varphi_k|\,\rho\,\varphi_l\rangle \tag{1.196}$$

If we take $\mathcal{H}_N$ in the φ_k-representation as $\mathbb{C}^N$, ρ becomes indeed an $N\times N-$matrix with operator valued matrix elements acting in $\mathcal{H}$, i.e.

$$\rho=\begin{pmatrix}\rho_{11}&\dots&\rho_{1N}\\ \vdots&\ddots&\vdots\\ \rho_{N1}&\dots&\rho_{NN}\end{pmatrix}\quad\rho_{kl}\in\mathcal{T}(\mathcal{H}) \tag{1.197}$$

Now we can properly illustrate the action of the superoperator $\mathcal{W}\otimes\mathbb{1}_{\mathcal{T}(\mathcal{H}_N)}$ which acts on the $\mathcal{H}-$system as $\mathcal{W}$ while acting trivially on the $\mathcal{H}_N-$system (but as mentioned this does non mean that it leaves the $\mathcal{H}_N-$system necessarily invariant, as for example the collapse of the singlet state upon spin measurement on one of the two subsystems shows!): $\mathcal{W}\otimes\mathbb{1}_{\mathcal{T}(\mathcal{H}_N)}$ acts as $\mathcal{W}$ on the ρ_{kl} only, i.e.

$$\mathcal{W}\otimes\mathbb{1}_{\mathcal{T}(\mathcal{H}_N)}(\rho)=\sum_{k,l=1}^{N}\mathcal{W}(\rho_{kl})\otimes|\varphi_k\rangle\langle\varphi_l| \tag{1.198}$$

(this is why $\mathcal{W}$ must be defined on $\mathcal{T}(\mathcal{H})$ since in general $\rho_{kl}\notin\mathcal{S}(\mathcal{H})$) or in the $\mathbb{C}^N-$picture

$$\mathcal{W}\otimes\mathbb{1}_{\mathcal{T}(\mathcal{H}_N)}(\rho)=\begin{pmatrix}\mathcal{W}(\rho_{11})&\dots&\mathcal{W}(\rho_{1N})\\ \vdots&\ddots&\vdots\\ \mathcal{W}(\rho_{N1})&\dots&\mathcal{W}(\rho_{NN})\end{pmatrix}\quad\rho_{kl}\in\mathcal{T}(\mathcal{H}) \tag{1.199}$$

[67] This representation of ρ is easily derived, e.g. by multiplying ρ by $\mathbb{1}_{\widetilde{\mathcal{H}}}=\sum_{k=1}^{N}\mathbb{1}_{\mathcal{H}}\otimes|\varphi_k\rangle\langle\varphi_k|$ from the left and from the right.

which in particular implies that the $\mathcal{H}_N-$system is left invariant in case of uncorrelated product states:

$$\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathcal{H}_N)}(\rho_{\mathcal{H}} \otimes \rho_{\mathcal{H}_N}) = \mathcal{W}(\rho_{\mathcal{H}}) \otimes \rho_{\mathcal{H}_N} \tag{1.200}$$

A superoperator $\mathcal{W} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ encoding a transition of density operators takes the form $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathcal{H}_N)} : \mathcal{T}(\mathcal{H} \otimes \mathcal{H}_N) \to \mathcal{T}(\mathcal{H} \otimes \mathcal{H}_N)$ given by (1.198), respectively (1.199), if the $\mathcal{H}-$system is composed with another system with N degrees of freedom and Hilbert space $\mathcal{H}_N$ if both systems do not interact at the respective times and the $\mathcal{H}_N-$system does not directly change in time (i.e. by non-trivial unitary evolution or measurement like processes acting on $\mathcal{H}_N$) while the transformation of the $\mathcal{H}-$system takes place.

A system with $N \in \mathbb{N}$ degrees of freedom might in principle always be added to the $\mathcal{H}-$system. Consequently, the extension $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathcal{H}_N)}$ of $\mathcal{W}$ with respect to each $N-$dimensional Hilbert space $\mathcal{H}_N$ which we can always take as $\mathbb{C}^N$ should map density operators acting on $\mathcal{H} \otimes \mathbb{C}^N$ to density operators modulo normalization acting on $\mathcal{H} \otimes \mathbb{C}^N$, i.e. $\mathcal{W}$ is required to have the property that $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathbb{C}^N)}$ is a positive superoperator for all $N \in \mathbb{N}$. This motivates the following definition:

Definition 1.11 [$\boldsymbol{N}-$*Positivity and CPMs*]

A superoperator $\mathcal{W} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\widetilde{\mathcal{H}})$ is called $N-$positive with respect to some $N \in \mathbb{N}$ if its extension $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathbb{C}^N)} : \mathcal{T}(\mathcal{H} \otimes \mathbb{C}^N) \to \mathcal{T}(\widetilde{\mathcal{H}} \otimes \mathbb{C}^N)$ given by (1.199) is a positive superoperator, i.e. $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathbb{C}^N)}(\rho) \in \mathcal{T}_+(\widetilde{\mathcal{H}} \otimes \mathbb{C}^N)$ for each $\rho \in \mathcal{T}_+(\mathcal{H} \otimes \mathbb{C}^N)$. $\mathcal{W}$ is called a completely positive superoperator or a completely positive mapping (CPM) if it is $N-$positive for all $N \in \mathbb{N}$.

Remarks

∘ Positive but not Completely Positive: It is not immediately obvious that complete positivity is indeed a stronger requirement than positivity. That complete positivity entails positivity is clear since positivity is simply $N-$positivity for $N = 1$ (indeed, one can also show that $N-$positivity entails $n-$positivity for all $n \leq N$, see e.g. [212]).

The other way around, this is not the case. The probably most famous counterexample is *partial transposition*: If for example $\mathcal{H}$ is a Hilbert space of dimension $M \in \mathbb{N}$, operators acting in $\mathcal{H}$ are $M \times M-$matrices in a given representation and we may consider the superoperator given by transposition $\mathcal{W}(\rho') := \rho'^T$ for $\rho' \in \mathcal{T}(\mathcal{H})$. Thus, a given $\rho \in \mathcal{T}(\mathcal{H} \otimes \mathbb{C}^N)$ can be viewed as an $N \times N-$matrix $[\rho_{kl}]$ with $M \times M-$matrix valued matrix elements ρ_{kl}, on which the superoperator $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathbb{C}^N)}$ acts as $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathbb{C}^N)}(\rho) = \mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathbb{C}^N)}([\rho_{kl}]) = [\mathcal{W}(\rho_{kl})] = [\rho_{kl}^T]$. It is easy to see that transposition is a positive superoperator but not completely positive (see e.g. [268]), i.e. there are $\rho \in \mathcal{T}(\mathcal{H} \otimes \mathbb{C}^N)$ such that the operator $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathbb{C}^N)}(\rho)$ is no longer positive. Indeed, although transposition will certainly not characterize a physically realizable transition of density operators, there is some physical relevance from a technical point of view: For density operators $\rho \in \mathcal{S}(\mathcal{H} \otimes \mathbb{C}^N)$ the property *'positive under partial transposition' (PPT)* can be used as an entanglement criterion (sometimes referred to as the Peres-Horodecki criterion), since it is a necessary condition for non-entanglement between the $\mathcal{H}-$system and the $N-$level system

represented by the Hilbert space $\mathbb{C}^N$. In some cases (e.g. in case $N = M = 2$) it is even a sufficient condition for non-entanglement, which is to say that in such cases any entangled density matrix is not PPT (see e.g. [345]).

Another example of a positive but not completely positive superoperator, which is more directly physically interesting, is given by *time reversal*. This operation is given by a transformation of the density operator of the form $\mathcal{S}(\mathcal{H}) \ni \rho \mapsto T\,\rho\,T^{-1}$, where T is an antiunitary operator (essentially complex conjugation). It can be shown in several ways (see [74, 212]) that time reversal is a positive superoperator which is not completely positive, though. Kraus [212] argues that this is unproblematic since (roughly speaking) there are no apparatuses in the world which reverse the direction of time. Busch and Lahti [74] consider this issue not only from such an operational point of view, but take cosmological considerations about the non complete positivity of time reversal into account. Namely, they point out that some authors consider the inversion of the direction of time as a physically reasonable process in connection with a scenario of recollapse of the universe. Busch and Lahti argue that this would not lead to physical inconsistencies related to the non complete positivity of time reversal, because (roughly speaking) the universe cannot be regarded as a subsystem of some larger system.

Also in non-linear quantum mechanics, like Hartree type mean field theories, state evolutions which are given by positive but not completely positive maps occur [10]. These transformations of density operators are not characterized by superoperators as we have defined them though, since they are in general neither linear nor linear up to normalization. The definitions of positivity and complete positivity can be nonetheless straight forwardly extended to non linear maps between sets of operators. Czachor and Kuna [92] argue that this does not lead to physical problems of interpretation (negative probabilities) in the non linear case since the extension $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathbb{C}^N)}$ as given by (1.199) is physically only the appropriate extension of a map acting on density operators in the linear case. Moreover, they provide a more appropriate extension for non linear evolutions which does not lead to non positive density operators and which essentially reduces to $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathbb{C}^N)}$ as defined above in the linear case.

See also the remark on open quantum systems and the initial product assumption subsequent to the proof of the Steinespring theorem 1.18 below, where discussions involving suggestions to take transitions described by non completely positive and even non positive maps (leading to negative probabilities) seriously into account and their refutations are briefly sketched.

∘ The Dual Picture: We know that unitary state evolution can be transferred to unitary transformation of observable operators or effects, i.e. we can change from the Schrödinger picture to the Heisenberg picture by $(U\,\rho\,U^{-1}\ ;\ \mathcal{A}) \mapsto (\rho\ ;\ U^{-1}\mathcal{A}\,U)$, where U is the unitary operator associated with the evolution, $\rho \in \mathcal{S}(\mathcal{H})$ a density operator and $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ an observable operator or an effect. The reason for this freedom in the quantum formalism is of course that this transition leaves the probability functionals $\mathrm{Tr}_{\mathcal{H}}\,[\rho\,\mathcal{A}]$ invariant, i.e. $\mathrm{Tr}_{\mathcal{H}}\,[U\rho U^{-1}\,\mathcal{A}] = \mathrm{Tr}_{\mathcal{H}}\,[\rho\,U^{-1}\mathcal{A}U]$.

This freedom can be generalized to CPMs $\mathcal{W}$ which map density operators to density operators, i.e. which are trace preserving. Physically relevant representatives of trace preserving CPMs (which will be defined as *quantum channels* below) are e.g. unitary evolutions, non selective measurements and open quantum system (traced out environment).

The transition to the generalized Heisenberg picture is then given by the dual[68] superoper-

[68]With respect to each given operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$, we can define a linear functional $\mathrm{Tr}_{\mathcal{A}} : \mathcal{T}(\mathcal{H}) \to \mathbb{C}$ by

tator $\mathcal{W}^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by

$$\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}^*(\mathcal{A})\,\rho\right] \equiv \mathrm{Tr}_{\mathcal{H}}\left[\mathcal{A}\,\mathcal{W}(\rho)\right] \quad \text{for all } \rho \in \mathcal{T}(\mathcal{H}),\ \mathcal{A} \in \mathcal{B}(\mathcal{H}) \tag{1.201}$$

That each trace preserving CPM $\mathcal{W}$ acting on $\mathcal{T}(\mathcal{H})$ can be uniquely associated with a superopertator $\mathcal{W}^*$ acting on $\mathcal{B}(\mathcal{H})$ in this way, can be easily seen e.g. by considering the Kraus representation of $\mathcal{W}$, which will be derived below. Indeed, the definition of complete positivity can be straight forwardly taken over to the dual superoperators $\mathcal{W}^*$ and it turns out that $\mathcal{W}^*$ is completely positive if and only if the associated superoperator $\mathcal{W}$ is completely positive [253, 345].

But observe that this transition to the dual picture does in general make physically no sense if we consider measurements with definite outcomes: In this case $\mathcal{W}(\rho)$ is usually not a density operator anymore (unless the associated probability $\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}(\rho)\right] = 1$) and an expression like $\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{A}\,\mathcal{W}(\rho)\right]$ has no immediate physical interpretation (though we could mathematically define the respective dual map $\mathcal{W}^*$ on $\mathcal{B}(\mathcal{H})$ as well, of course).

∘ Remark on $N = \infty$: One might wonder why complete positivity is only defined through tensor products of the considered Hilbert space $\mathcal{H}$ with finite dimensional Hilbert spaces $\mathcal{H}_N \cong \mathbb{C}^N$ (i.e. that only systems with finite numbers of degrees of freedom are formally added to the considered system), which seems to disregard a combination of the $\mathcal{H}-$system with the most natural candidate of quantum systems, namely those whose Hilbert space is given by some L^2-space of square integrable functions. Indeed, it turns out that complete positivity as defined above already covers this case as well, namely if $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathbb{C}^N)}$ is a positive superoperator on $\mathcal{T}(\mathcal{H} \otimes \mathbb{C}^N)$ for each $N \in \mathbb{N}$ it follows that $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathcal{H}')}$ is a positive superoperator on $\mathcal{T}(\mathcal{H} \otimes \mathcal{H}')$ whenever $\mathcal{H}'$ is a (possibly infinite dimensional) separable Hilbert space. This can be seen by resorting to the Kraus representation of $\mathcal{W}$ which will be derived below: It will be shown that each completely positive $\mathcal{W}$ has an operator sum representation, i.e. there exists a countable set of operators $\{\mathcal{R}_k\} \subset \mathcal{B}(\mathcal{H})$ such that $\mathcal{W}(\rho) = \sum_k \mathcal{R}_k\,\rho\,\mathcal{R}_k^\dagger$ for all $\rho \in \mathcal{T}(\mathcal{H})$. It is easy to see that $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathcal{H}')}$ has thus an operator sum representation [212] given by

$$\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathcal{H}')}(\rho) = \sum_k (\mathcal{R}_k \otimes \mathbb{1}_{\mathcal{H}'})\,\rho\,(\mathcal{R}_k \otimes \mathbb{1}_{\mathcal{H}'})^\dagger, \quad \rho \in \mathcal{T}(\mathcal{H} \otimes \mathcal{H}') \tag{1.202}$$

for each (possibly infinite dimensional) separable Hilbert space $\mathcal{H}'$, which is clearly a positive superoperator as well. Hence, complete positivity of $\mathcal{W}$ as defined above entails positivity of $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathcal{H}')}$ also for infinite dimensional separable $\mathcal{H}'$, such that it is ensured that $\mathcal{W} \otimes \mathbb{1}_{\mathcal{T}(\mathcal{H}')}$ maps density operators to density operators modulo normalization also in case that e.g. the $\mathcal{H}-$system is combined with a system whose Hilbert space is $\mathcal{H}' = L^2$.

∘ CPMs Between Distinct Hilbert Spaces: So far we mainly considered superoperators characterizing transitions of density operators of a given system, in particular the Hilbert space

$\mathcal{T}(\mathcal{H}) \ni \rho \mapsto \mathrm{Tr}_{\mathcal{A}}(\rho) := \mathrm{Tr}_{\mathcal{H}}\left[\mathcal{A}\,\rho\right]$. With respect to these linear functionals, $\mathcal{B}(\mathcal{H})$ can be identified with the dual space of $\mathcal{T}(\mathcal{H})$. The other way around, for each given $\rho \in \mathcal{T}(\mathcal{H})$, we can define the associate linear functional $\mathrm{Tr}_\rho : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ by $\mathcal{B}(\mathcal{H}) \ni \mathcal{A} \mapsto \mathrm{Tr}_\rho(\mathcal{A}) := \mathrm{Tr}_{\mathcal{H}}\left[\mathcal{A}\,\rho\right]$ and with respect to these functionals $\mathcal{T}(\mathcal{H})$ can be identified with the predual of $\mathcal{B}(\mathcal{H})$ [74] (the respective dual space of $\mathcal{B}(\mathcal{H})$ is analogously identified with the set of bounded compact operators).

of the input state was the same as the one of the output state. Nonetheless, CPMs were defined more generally in definition 1.11, allowing also for superoperators $\mathcal{W} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\widetilde{\mathcal{H}})$ with $\mathcal{H} \neq \widetilde{\mathcal{H}}$, which covers additional physically relevant completely positive superoperators. For example the partial trace operation $\mathcal{W} : \mathcal{T}(\mathcal{H} \otimes \mathcal{H}') \to \mathcal{T}(\mathcal{H})$ given by $\rho \mapsto \mathcal{W}(\rho) = \mathrm{Tr}_{\mathcal{H}'}(\rho)$ is a CPM, or if $\rho' \in \mathcal{T}(\mathcal{H}')$, the superoperator $\mathcal{W} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H} \otimes \mathcal{H}')$ given by $\rho \mapsto \mathcal{W}(\rho) = \rho \otimes \rho'$ is completely positive.

∘ Compositions: Compositions and linear combinations with positive coefficients of CPMs are completely positive as well.

1.6.2 Quantum Operations: Channels and Instruments

We call a CPM $\mathcal{W} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ *trace preserving* if $\mathrm{Tr}_{\mathcal{H}}[\mathcal{W}(\rho)] = \mathrm{Tr}_{\mathcal{H}}[\rho]$, *trace decreasing* if $\mathrm{Tr}_{\mathcal{H}}[\mathcal{W}(\rho)] < \mathrm{Tr}_{\mathcal{H}}[\rho]$ for all positive $\rho \in \mathcal{T}_+(\mathcal{H})$ and accordingly *trace non increasing* if it is trace preserving or trace decreasing. CPMs of particular physical interest in quantum theory are trace non increasing, where trace preserving and trace decreasing CPMs constitute two classes of (in general) different physical meaning.

Definition 1.12 [*Quantum Operations & Channels*]

A trace non increasing CPM is called a quantum operation. A trace preserving quantum operation is called a quantum channel.

Quantum operations comprise all transitions of density operators and maps characterizing such transitions, given these transitions are linear (quantum channels) or linear up to normalization[69] (trace decreasing CPMs) and encoded in completely positive superoperators. This relation between linearity and trace properties derives simply from the fact that a trace decreasing transformation of a density operator requires renormalization (i.e. dividing $\mathcal{W}(\rho)$ by its own trace) in order to get back a final density operator, which in turn destroys linearity as already repeatedly stressed.

Quantum Channels

The expression *quantum channels* originates from quantum information theory and refers to the transmission and loss of 'information' by transmission of quantum states, like e.g. by sending a photon through a glass fiber. One can also consider more complex processes involving interaction, entanglement and measurements such that input and output system need not even be one and the same quantum system, like e.g. in experiments of the type which is (unluckily) known under the name *quantum teleportation* [32, 53].

Generic representatives of quantum channels of a single quantum system are unitary transitions, unitary interaction with an environment which is subsequently traced out, non-selective measurement (like) processes or combinations of these transformations.

It is to be remarked that some authors, who believe that decoherence provides a solution of the measurement problem, would regard the third class of quantum channels in the previous list

[69] This condition is e.g. violated in mean field theories.

(non-selective measurements) as a subclass of the second one (traced out entangled environment), since the linearity of non-selective measurements makes it always possible to model the associated transitions by unitary interactions and traced out environments only (see the Stinespring theorem 1.18 below). Here it shall be only remarked that this line of argument breaks down as soon as measurements with definite outcomes are considered, a more detailed discussion of this issue will be given in section 1.6.6 below.

Open Quantum Systems

An analysis of quantum channels deriving from a complex (unitary and/or measurement like) interaction with a traced out environment is the subject of the theory of *open quantum systems*. The associated explicit time evolution can then in general be modelled by master equations (see below). In this case, an initial pure state splits up into a mixture of more and more pure states, which can be described as a state diffusion process in Hilbert space [150]. Following the branches of such splitting up into different possible state histories is the subject of the so called *quantum trajectories* method [93], not to be confused with Bohmian trajectories.

Measurements with Definite Outcomes

If measurement (like) processes with definite outcomes are to be described in terms of superoperators, we are no longer dealing with quantum channels but with CPMs which are in general trace decreasing: Transformations of density operators due to measurement (like) processes are linear up to normalization

$$\mathcal{S}(\mathcal{H}) \ni \rho \quad \longrightarrow \quad \frac{\mathcal{W}(\rho)}{\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}(\rho)\right]} \tag{1.203}$$

and the respective normalization is equal to the associated probability $\mathbb{P}^{\rho} = \mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}(\rho)\right]$. If the outcomes are labelled by the subsets of some real set Ω we thus can introduce in analogy to *positive operator valued measures*, the notion of *state transformation valued measures* (Busch [72]), also known as *instruments* (Davis [98]):

Definition 1.13 [*State Transformation Valued Measure a.k.a. Instrument*]

Let $(\Omega, \mathcal{F})$ be a measure space and denote by $\mathcal{QO}(\mathcal{H})$ the set of all quantum operations associated with Hilbert space $\mathcal{H}$. An instrument or state transformation valued measure or operation valued measure is a mapping $\mathcal{W}(\circ \mid \cdot) : \mathcal{F} \to \mathcal{QO}(\mathcal{H})$ (where the elements of $\mathcal{T}(\mathcal{H})$ are to be plugged into the $\circ-$slot and the elements of $\mathcal{F}$ into the $\cdot-$slot) with

(i) $\mathcal{W}(\rho \mid \emptyset) = 0$ *for all* $\rho \in \mathcal{T}(\mathcal{H})$

(ii) $\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}(\rho \mid \Omega)\right] = \mathrm{Tr}_{\mathcal{H}}\left[\rho\right]$ *for all* $\rho \in \mathcal{T}(\mathcal{H})$

(iii) If (X_k) is a countable family of mutually disjoint subsets of Ω, it follows that

$$\mathcal{W}(\rho \mid \bigcup_k X_k) = \sum_k \mathcal{W}(\rho \mid X_k) \tag{1.204}$$

for all $\rho \in \mathcal{T}(\mathcal{H})$, where the limit is to be taken in the strong operator topology if the sum extends to infinity

In particular, for each given density operator $\rho \in \mathcal{S}(\mathcal{H})$, the association $\mathcal{F} \ni X \mapsto \mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}(\rho \mid X)\right]$ defines a probability measure on the measure space $(\Omega, \mathcal{F})$.

It shall be remarked here that these definitions are sometimes also given with positivity instead of complete positivity, the latter being then an extra constraint, distinguishing certain physically relevant classes of operations, channels and instruments.

Instruments and POVMs

The notion of instruments is indeed strongly related with the notion of POVMs: Each instrument $\mathcal{W} : \mathcal{F} \to \mathcal{QO}(\mathcal{H})$ on $(\Omega, \mathcal{F})$ defines uniquely a POVM $E : \mathcal{F} \to \mathcal{B}(\mathcal{H})$ by the relation

$$\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}(\rho \mid X)\right] = \mathrm{Tr}_{\mathcal{H}}\left[E_X \rho\right] \tag{1.205}$$

for all $\rho \in \mathcal{S}(\mathcal{H})$ and for all $X \in \mathcal{F}$. The other way around, each POVM is related with an infinite number of instruments that way.

This can be understood in the following way: Recall that an operation $\mathcal{W}(\circ \mid X)$ (for a given $X \in \mathcal{F}$) which is associated with a discrete measurement (like) process as defined at the beginning of this chapter, is given by some set of bounded operators $\{\mathcal{R}_{Xk}\} \subset \mathcal{B}(\mathcal{H})$ such that

$$\mathcal{W}(\rho \mid X) = \sum_k \mathcal{R}_{Xk}\, \rho\, \mathcal{R}_{Xk}^\dagger \tag{1.206}$$

for all $\rho \in \mathcal{S}(\mathcal{H})$. The Kraus theorem (which will be proven below, see theorem 1.17) which was already mentioned, tells us that such an operator sum representation (1.206) always exists with a countable set $\{\mathcal{R}_{Xk}\}$ of bounded operators as a mathematical fact, whenever $\mathcal{W}$ is completely positive, in particular if it is a quantum operation. Thus, using the cyclicity of the trace in (1.205), we get uniquely the associated effect by setting

$$E_X \equiv \sum_k \mathcal{R}_{Xk}^\dagger \mathcal{R}_{Xk} \tag{1.207}$$

The other way around, given an effect E_X and $\{\mathcal{R}_{Xk}\} \subset \mathcal{B}(\mathcal{H})$ is a set of state transformers defining a quantum operation $\mathcal{W}(\circ \mid X)$ via (1.206) such that $E_X = \sum_k \mathcal{R}_{Xk}^\dagger \mathcal{R}_{Xk}$, each set $\{\widetilde{\mathcal{R}}_{Xk}\}$ with

$$\widetilde{\mathcal{R}}_{Xk} = U_{Xk} \mathcal{R}_{Xk} \tag{1.208}$$

and the U_{Xk} are unitary operators, defines the same effect via $E_X = \sum_k \widetilde{\mathcal{R}}_{Xk}^\dagger \widetilde{\mathcal{R}}_{Xk}$ (for simplicity, we consider here only unitary U_{Xk} and skip the subtleties concerning the additional freedom of possibly non-unitary partial isometries). This corresponds to the fact that different state transformations can be associated with one and the same POVM. For efficient measurements (i.e. there is only one $\mathcal{R}_{Xk} \equiv \mathcal{R}_X$ associated with a single value X), we have set $\mathcal{R}_X = U_X \sqrt{E_X}$, where the unitary operators U_X were called *measurement back action* and the customary but often unrealistic choice $U_X \equiv \mathbb{1}_{\mathcal{H}}$ defined a so called *Lüders measurement*.

Preparation – Transformation – Measurement

If quantum experiments are considered, each part of the experiment can be associated with a CPM. For example the *preparation* of a pure initial state $\psi \in \mathcal{H}$ from a system previously in some arbitrary state, is given by the CPM associated with the one dimensional projection $P_\psi = |\psi\rangle\langle\psi|$ (i.e. if $\rho \in \mathcal{S}(\mathcal{H})$ represents the system to be prepared, the preparation of the initial state is given by the operation $\rho \mapsto \mathcal{W}(\rho) = P_\psi \, \rho \, P_\psi$, where the right hand side is the initial state if properly normalized). In many experiments, between initial preparation and final measurement, the system undergoes some *transformation*, which is to say a time evolution which might be unitary, (non selective) measurement like, the evolution of an open quantum system or even transmission of certain data to other quantum systems by interaction/entanglement/intermediate measurements (e.g. quantum teleportation type experiments). All of these transformations can be described by quantum operations, usually by quantum channels, except in case of selective measurement like interactions, like selecting only one beam behind a beam splitter by interrupting the other one, in which case the transformation of the quantum states associated with the beam is characterized by a trace decreasing CPM. Finally, the final *measurement* can be described by a state transformation valued measure, both, with respect to the statistical predictions for the outcomes (through the POVM associated with the instrument) as well as with respect to the final states of the measured system subsequent to the experiment.

The division of quantum experiments into the operational building blocks *preparation, transformation* and *measurement* and the association of mathematical objects from the Hilbert space framework with these three categories, respectively, goes back to Ludwig and Kraus. The standard association in the quantum theory of measurement associates with each *preparation* an initial *density operator*, with each *transformation* in between preparation and measurement a trace preserving CPM and with each *measurement* a *POVM*, which is often easier accessible than the true set of state transformations (think e.g. of a particle impinging on a screen).

In operational quantum mechanics, the association of these mathematical objects with the respective categories can be understood as generating equivalence classes of experiments: Distinct experiments associated with the same initial state, the same transformation of the initial state in between preparation and measurement (CPM) or the same POVM are said to belong to the same P-class, T-class or M-class, respectively. Operationalists then sometimes say that two experiments with the same initial state are operationally equivalent with respect to preparation and so on (to decide if there is any valuable gain in considering distinct experiments as operationally equivalent is left to the reader).

This structuring is of course somewhat arbitrary, a preparation might be conceived as a measurement, a measurement as a preparation, a transformation as part of preparation and so on, such that the partitioning of quantum experiments into preparation, transformation and measurement is somewhat artificial or at least admits blurred boundaries. Nonetheless, the mathematical and physical analysis of quantum experiments proceeding from this operational structure, led some authors to develop some helpful technical tools and to establish some nice, not immediately obvious results. These include for example that each mixed density operator can be obtained as a partial trace over some pure state on a larger Hilbert space (purification, which is actually the most trivial of these insights but technically fruitful), that each quantum operation has an operator sum representation (Choi-Kraus theorem), that each quantum channel can be modelled as the partial trace over some unitary evolution on a larger Hilbert space

(Stinespring theorem) or that each POVM and each quantum operation can be implemented by projective measurement schemes on a larger Hilbert space (Naimark theorem and Kraus' second representation theorem). In this connection John Smolin coined the pervasive phrase: *'Going to the Church of the Larger Hilbert Space'*. Whether the larger Hilbert space has direct physical meaning is another question, often it can have, and even if not the associated representations are technically very valuable.

1.6.3 Purification

Before we exploit the technical power of complete positivity to derive strong results applying to possible representations and implementations of transformations of quantum states in the following section, we shall have a closer look on possible representations of the latter in case they are given by mixed states in this section.

Purification means to find a Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$ and a pure state $\Psi \in \widetilde{\mathcal{H}}$ such that a given density operator $\rho \in \mathcal{S}(\mathcal{H})$ emerges as the partial trace of this pure state $\rho = \mathrm{Tr}_{\mathcal{H}'} [|\Psi\rangle \langle\Psi|]$. That purification is always possible is a very trivial fact, as we shall see below, and one might think not even worth to call it a theorem. On the other hand, this trivial fact is technically very powerful and some further results follow impressively easily by 'going to the larger Hilbert space'. Two examples will be given subsequent to the following theorem, the first of which is very fundamental (and will find applications at other places in this work), since it classifies the fundamental relation between all ensembles of states which share a given density operator.

Purification provides also an example of a construction based on a larger Hilbert space, where the latter need not have any direct physical meaning: Of course an ensemble of pure states (e.g. prepare the respective pure states and mix the systems together in the desired fractions) is associated with a mixed density operator which does actually not originate from the partial trace over a pure state on a larger Hilbert space, but such a construction always exists formally which yields the same density operator.

Theorem 1.14 [*Purification of Mixed States and Unitary Equivalence*]

Let

$$\rho = \sum_k p_k |\psi_k\rangle \langle\psi_k| \in \mathcal{S}(\mathcal{H}) \tag{1.209}$$

be a mixed density operator acting on some Hilbert space $\mathcal{H}$ (the ψ_k are normalized but need not be mutually orthogonal and $\sum_k p_k = 1$). Then the following assertions hold:

(i) ***Existence of Purifications:***
There exists some larger Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$, such that ρ emerges as the reduced density operator of a pure state $\Psi \in \widetilde{\mathcal{H}}$:

$$\rho = \mathrm{Tr}_{\mathcal{H}'} [|\Psi\rangle \langle\Psi|] \tag{1.210}$$

Ψ is called a purification of ρ. Actually, any mixed state has infinitely many purifications.

(ii) ***Equivalence of Purifications:***
Given $\Psi \in \widetilde{\mathcal{H}}$ is a purification of ρ. Then $\Phi \in \widetilde{\mathcal{H}}$ is another purification of ρ if and only

if there is some unitary U acting on $\mathcal{H}'$ such that

$$\Phi = (\mathbb{1}_{\mathcal{H}} \otimes U)\ \Psi \tag{1.211}$$

If $\Psi \in \mathcal{H} \otimes \mathcal{H}'$ is a purification and $\Phi \in \mathcal{H} \otimes \mathcal{H}''$, then $U : \mathcal{H}' \to \mathcal{H}''$ needs to be an appropriate (partial) isometry, such that Φ is a purification as well.

Proof:

• (i) As mentioned, the existence of purifications is rather a trivial fact: Choose for $\mathcal{H}'$ any Hilbert space with dimension equal to or greater than the number of non vanishing $p_k > 0$ in (1.209) and an ONB $\{\varphi_k\}$ of $\mathcal{H}'$. Then

$$\Psi := \sum_k \sqrt{p_k}\, \psi_k \varphi_k \tag{1.212}$$

(where superfluous summands are taken as zero if there are more φ_ks than ψ_ks) is obviously a purification of ρ:

$$\mathrm{Tr}_{\mathcal{H}'} \left[|\Psi\rangle \langle\Psi| \right] = \sum_{j,k,l} \sqrt{p_k p_l}\, |\psi_k\rangle \langle\psi_l| \langle\varphi_j|\, \varphi_k\rangle \langle\varphi_l|\, \varphi_j\rangle = \sum_k p_k\, |\psi_k\rangle \langle\psi_k| = \rho \tag{1.213}$$

Since the choice of basis of $\mathcal{H}'$ was arbitrary (and actually the choice of $\mathcal{H}'$ besides its minimal dimension, too), there are infinitely many possibilities to construct in this way distinct purifications of ρ.

• (ii)
"$\Rightarrow$": Let $\Psi, \Phi \in \widetilde{\mathcal{H}}$ be purifications of ρ, i.e.

$$\rho = \mathrm{Tr}_{\mathcal{H}'} \left[|\Psi\rangle \langle\Psi| \right] = \mathrm{Tr}_{\mathcal{H}'} \left[|\Phi\rangle \langle\Phi| \right] \tag{1.214}$$

Now we perform a Schmidt decomposition of Ψ with respect to the tensor product $\mathcal{H} \otimes \mathcal{H}'$:

$$\Psi = \sum_k c_k \eta_k \phi_k \tag{1.215}$$

with non negative coefficients $c_k \geq 0$ and the Schmidt-ONBs $\{\eta_k\} \subset \mathcal{H}$ and $\{\phi_k\} \subset \mathcal{H}'$ associated with Ψ. Now we expand Φ with respect to the η_k ONB of $\mathcal{H}$, i.e. multiply Φ by the identity $\mathbb{1}_{\widetilde{\mathcal{H}}} = \sum_k |\eta_k\rangle \langle\eta_k| \otimes \mathbb{1}_{\mathcal{H}'}$ from the left to obtain

$$\Phi = \sum_k b_k \eta_k \zeta_k \tag{1.216}$$

where we have set for all k for which $\langle\eta_k|\, \Phi\rangle \in \mathcal{H}'$ is non zero

$$b_k \zeta_k := \langle\eta_k|\, \Phi\rangle\,, \quad \zeta_k \in \mathcal{H}', \quad b_k = \| \langle\eta_k|\, \Phi\rangle \|_{\mathcal{H}'} > 0 \tag{1.217}$$

with normalized but not a priori mutually orthogonal states $\zeta_k \in \mathcal{H}'$. Now we evaluate equation (1.214) by performing the partial trace over $\mathcal{H}'$ in the ϕ_k-basis:

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}'}\left[|\Psi\rangle\langle\Psi|\right] &= \sum_{k,l,m} c_k c_l \,|\eta_k\rangle\langle\eta_l|\,\langle\phi_m|\,\phi_k\rangle\langle\phi_l|\,\phi_m\rangle = \sum_{k,l}\delta_{kl}\,c_k c_l\,|\eta_k\rangle\langle\eta_l| \stackrel{!}{=} \\ = \mathrm{Tr}_{\mathcal{H}'}\left[|\Phi\rangle\langle\Phi|\right] &= \sum_{k,l,m} b_k b_l\,|\eta_k\rangle\langle\eta_l|\,\mathrm{Tr}_{\mathcal{H}'}\left[|\zeta_k\rangle\langle\zeta_l|\right] = \sum_{k,l} b_k b_l\,\langle\zeta_l|\,\zeta_k\rangle\,|\eta_k\rangle\langle\eta_l| \end{aligned} \tag{1.218}$$

and comparing the matrix elements in the η_k-basis yields for all k,l with $\langle\eta_k|\,\Phi\rangle \neq 0$ and $\langle\eta_l|\,\Phi\rangle \neq 0$

$$\delta_{kl}\,c_k c_l = b_k b_l\,\langle\zeta_l|\,\zeta_k\rangle \tag{1.219}$$

If we evaluate equation (1.219) for $l = k$ we see that $c_k = b_k$ for all k with $\langle\eta_k|\,\Phi\rangle \neq 0$ (otherwise $c_k = 0$) and thus the ζ_k are mutually orthonormal: $\langle\zeta_l|\,\zeta_k\rangle = \delta_{lk}$. Hence we can complete $\{\zeta_k\} \subset \mathcal{H}'$ to an ONB and define U as the unitary change of basis from $\{\phi_k\}$ to $\{\zeta_k\}$ in $\mathcal{H}'$, i.e.

$$U\phi_k = \zeta_k \quad \text{for all } k \tag{1.220}$$

or more compactly

$$U = \sum_k |\zeta_k\rangle\langle\phi_k| \tag{1.221}$$

This unitary U has the desired property

$$(\mathbb{1}_{\mathcal{H}} \otimes U)\,\Psi = \sum_k c_k\eta_k\,U\phi_k = \sum_k c_k\eta_k\zeta_k = \Phi \tag{1.222}$$

"$\Leftarrow$": Let $\Psi \in \widetilde{\mathcal{H}}$ be a purification of ρ and $\Psi = \sum_k c_k\eta_k\phi_k$ its Schmidt decomposition. Consequently $\rho = \mathrm{Tr}_{\mathcal{H}'}\left[|\Psi\rangle\langle\Psi|\right] = \sum_k c_k^2\,|\eta_k\rangle\langle\eta_k|$ is a possible representation of ρ as a convex linear combination of mutually orthogonal pure states (this is actually simply the spectral representation of the selfadjoint operator ρ). Choose an ONB $\{\phi_k\} \subset \mathcal{H}'$ which contains the ϕ_ks from the Schmidt decomposition above (i.e. complete them to an ONB if necessary).

If now $U \in \mathcal{B}(\mathcal{H}')$ is unitary, it follows that it transforms the ONB $\{\phi_k\}$ into another ONB $\{\zeta_k\}$ of $\mathcal{H}'$ by $U\phi_k = \zeta_k$ for all k. Thus

$$\Phi = (\mathbb{1}_{\mathcal{H}} \otimes U)\;\Psi = \sum_k c_k\eta_k\zeta_k \tag{1.223}$$

is obviously another purification of $\rho = \mathrm{Tr}_{\mathcal{H}'}\left[|\Phi\rangle\langle\Phi|\right] = \sum_k c_k^2\,|\eta_k\rangle\langle\eta_k|$.

The proofs of both implications can be straightforwardly transferred to the case where $U : \mathcal{H}' \to \mathcal{H}''$ is a (partial) isometry if $\mathcal{H}$ and $\mathcal{H}'$ are of different dimension.

■

State Ensembles

Now purification will be applied as a tool to derive further results. The following is very central since it characterizes all ensembles of states which share a given density operator. It was already mentioned several times that resolutions of a given mixed density operator into convex linear combinations of pure states are not unique. This fact will be made more precise in the following corollary:

Corollary 1.15 [*Unitary Mixing of Ensembles*]

Two ensembles $(p_k, \psi_k)_{k=1,...,K}$ and $(q_l, \xi_l)_{l=1,...,L}$ of states in a Hilbert space $\mathcal{H}$ (where K and/or L might be infinite) are represented by the same density operator

$$\rho = \sum_{k=1}^{K} p_k \left|\psi_k\right\rangle \left\langle\psi_k\right| = \sum_{l=1}^{L} q_l \left|\xi_l\right\rangle \left\langle\xi_l\right| \in \mathcal{S}(\mathcal{H}) \tag{1.224}$$

if and only if there is a unitary matrix $[u_{kl}] \in \mathcal{M}_N(\mathbb{C})$ with $N = \sup(K, L)$, such that for all $k = 1, ..., N$

$$\sqrt{p_k}\psi_k = \sum_{l=1}^{N} u_{kl}\sqrt{q_l}\xi_l \tag{1.225}$$

where we set $\sqrt{p_k}\psi_k = 0$ for $k \in \{K+1, ..., N\}$ if $K < L = N$ and $\sqrt{q_l}\xi_l = 0$ for $l \in \{L+1, ..., N\}$ in case $L < K = N$.

Proof: We denote

$$\rho = \sum_{k=1}^{K} p_k \left|\psi_k\right\rangle \left\langle\psi_k\right| \tag{1.226}$$

and

$$\rho' = \sum_{l=1}^{L} q_l \left|\xi_l\right\rangle \left\langle\xi_l\right| \tag{1.227}$$

such that condition (1.224) reads

$$\rho \stackrel{!}{=} \rho' \tag{1.228}$$

"$\Rightarrow$": Choose a Hilbert space $\mathcal{H}'$ with dimension $\dim(\mathcal{H}') = N$ and an ONB $\{\varphi_k\} \subset \mathcal{H}'$. Obviously

$$\Psi = \sum_{k=1}^{K} \sqrt{p_k}\psi_k\varphi_k \in \mathcal{H} \otimes \mathcal{H}' \tag{1.229}$$

is a purification of ρ and

$$\Phi = \sum_{l=1}^{L} \sqrt{q_l}\xi_l\varphi_l \in \mathcal{H} \otimes \mathcal{H}' \tag{1.230}$$

a purification of ρ'. The assumption $\rho = \rho'$ thus entails that Ψ and Φ are purifications of one and the same mixed state. According to theorem 1.14 there thus exists some unitary U acting on $\mathcal{H}'$ such that $\Psi = (\mathbb{1}_{\mathcal{H}} \otimes U)\,\Phi$, or in detail

$$\sum_{m=1}^{K} \sqrt{p_m}\psi_m\varphi_m = \sum_{l=1}^{L} \sqrt{q_l}\xi_l\, U\varphi_l \tag{1.231}$$

Taking the partial scalar product by multiplying equation (1.231) by the $\mathcal{H}'-$dual vectors $\langle\varphi_k|$ from the left yields

$$\sqrt{p_k}\psi_k = \sum_{l=1}^{N} u_{kl}\,\sqrt{p_l}\psi_l \tag{1.232}$$

with the matrix elements

$$u_{kl} = \langle\varphi_k|\,U\,\varphi_l\rangle \tag{1.233}$$

where in the equations (1.232) $\sqrt{p_k}\psi_k = 0$ for $k \in \{K+1, ..., N\}$ if $K < L$ and $\sqrt{q_l}\xi_l = 0$ for $l \in \{L+1, ..., N\}$ in case $L < K$.

"$\Leftarrow$": Sufficiency of 'unitary state-mixing' for obtaining identical density operators is straight forward: Suppose for all $k = 1, \ldots, N$

$$\sqrt{p_k}\psi_k = \sum_{l=1}^{N} u_{kl}\sqrt{q_l}\xi_l \tag{1.234}$$

with unitary $[u_{kl}] \in \mathcal{M}_N(\mathbb{C})$ and as noted above in case $K \neq L$ the vector $(\sqrt{p_1}\psi_1, \ldots, \sqrt{p_K}\psi_K)$, respectively $(\sqrt{q_1}\xi_1, \ldots, \sqrt{q_L}\xi_L)$ with less components is padded with zeros. Consequently

$$\begin{aligned}\sum_{k=1}^{K} p_k\,|\psi_k\rangle\,\langle\psi_k| &= \sum_{k=1}^{N}\sqrt{p_k}\,|\psi_k\rangle\,\langle\psi_k|\,\sqrt{p_k} = \sum_{k,l,m=1}^{N} u_{kl}\sqrt{q_l}\,|\xi_l\rangle\,\langle\xi_m|\,\overline{u}_{km}\sqrt{q_m} = \\ &= \sum_{l,m=1}^{N}\left(\sum_{k=1}^{N} u_{kl}\overline{u}_{km}\right)\sqrt{q_l q_m}\,|\xi_l\rangle\,\langle\xi_m| = \sum_{l=1}^{N} q_l\,|\xi_l\rangle\,\langle\xi_l| = \sum_{l=1}^{L} q_l\,|\xi_l\rangle\,\langle\xi_l|\end{aligned} \tag{1.235}$$

where we have only used that the column vectors of the unitary matrix $[u_{kl}] = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N)$ constitute an ONB of $\mathbb{C}^N$:

$$\langle\boldsymbol{u}_m|\,\boldsymbol{u}_l\rangle_{\mathbb{C}^N} = \sum_{k=1}^{N}\overline{u}_{km}u_{kl} = \delta_{ml} \tag{1.236}$$

Thus $\rho = \rho'$.

■

The minimal number of pure states which can realize a given density operator ρ by linear combination is given by the dimension of its range. In other words, if $\rho = \sum_{k=1}^{N} c_k\,|\eta_k\rangle\,\langle\eta_k|$ is the spectral representation, ρ cannot be written as a linear combination of less than N pure states. The other way around, as corollary 1.15 shows ρ can be written as a linear combination of arbitrarily many pure states, but which cannot be linearly independent (and thereby not mutually orthogonal) if their number exceeds the dimension N of the range of ρ.

Ensemble Preparation at a Distance

Finally, here comes another nice consequence of purification, which is sometimes called the *GHJW-theorem* in the literature, named after Gisin [149] and Hughston, Jozsa and Wootters

[193]. It characterizes in some sense the possibility of nonlocal preparation of states inherent in quantum theory: Think e.g of preparing a spin up state at one wing of an EPRB-experiment by measuring spin down at the other wing, where the initial state was the singlet state $\psi = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. To call this 'preparation' in a strict sense might be admittedly seen questionable, since an experimenter at the device has no influence on the outcome and thus cannot prepare on purpose a spin up state at the other wing, typically half of the times performing this procedure she will unavoidably prepare a spin down state at the other wing as well. In other words, if this 'preparation procedure' is performed on an ensemble of systems in the singlet state, we can associate the 'prepared ensemble' on the distant wing of the experiment with the density operator $\rho = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow| = \frac{1}{2}\mathbb{1}_{\mathcal{H}}$, which describes in this case a real ensemble $\left\{(|\uparrow\rangle; \frac{1}{2}), (|\downarrow\rangle; \frac{1}{2})\right\}$ of pure states, as the perfect anticorrelations of an individual run of the preparation procedure show (see the next paragraph for details).

We did not indicate the orientation of the Stern-Gerlach devices, which is indeed for a given fixed orientation irrelevant, since the singlet state is symmetric with respect to choice of basis, i.e. if we indicate two arbitrary orientations by spatial vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, an easy calculation applying the spin$\frac{1}{2}$ basis transformation laws shows that $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle_{\boldsymbol{a}} - |\downarrow\uparrow\rangle_{\boldsymbol{a}}) = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle_{\boldsymbol{b}} - |\downarrow\uparrow\rangle_{\boldsymbol{b}})$. With respect to the 'preparation procedure' discussed in the previous paragraph, this means that the experimenter can choose to either prepare the ensemble $\left\{(|\uparrow\rangle_{\boldsymbol{a}}; \frac{1}{2}), (|\downarrow\rangle_{\boldsymbol{a}}; \frac{1}{2})\right\}$ or $\left\{(|\uparrow\rangle_{\boldsymbol{b}}; \frac{1}{2}), (|\downarrow\rangle_{\boldsymbol{b}}; \frac{1}{2})\right\}$ from the singlet state on the other wing of the experiment, just by choosing the orientation of the SGM-device. But in consistency with 'no signalling' (see chapter 2), this choice does not alter the statistics of any experiments on the ensemble on the other wing of the experiment, which is expressed by the fact that its density matrix is independent of this choice: $\frac{1}{2}(|\uparrow\rangle\langle\uparrow|_{\boldsymbol{a}} + |\downarrow\rangle\langle\downarrow|_{\boldsymbol{a}}) = \frac{1}{2}(|\uparrow\rangle\langle\uparrow|_{\boldsymbol{b}} + |\downarrow\rangle\langle\downarrow|_{\boldsymbol{b}}) = \frac{1}{2}\mathbb{1}_{\mathcal{H}}$. Nonetheless, the perfect anticorrelations in single runs show, that both ensembles are constituted by distinct pure states: If the experimenter measures 'spin down' with respect to some orientation $\boldsymbol{a}$, she can be sure (within the limits of measurement accuracy), that a spin measurement on the distant particle in the same orientation $\boldsymbol{a}$ will yield 'spin up' with certainty, which was in general not the case if the distant particle was left in one of the states $|\uparrow\rangle_{\boldsymbol{b}}$ or $|\downarrow\rangle_{\boldsymbol{b}}$, but only if it is in the state $|\uparrow\rangle_{\boldsymbol{a}}$. That these perfect correlations in individual runs cannot be used for superluminal signalling can be seen from the fact, that the unconditional density operator of the non selective measurement of an individual system at the distinct wing as well as the one deriving from building the partial trace of the pre-measurement singlet density operator are both given by $\rho = \frac{1}{2}(|\uparrow\rangle\langle\uparrow|_{\boldsymbol{a}} + |\downarrow\rangle\langle\downarrow|_{\boldsymbol{a}})$, such that an experimenter on the distant wing cannot conclude whether the measurement took place or not by any experimental means.

The crucial observation with respect to the following result is this: An experimenter at one wing of the experiment is capable of preparing any ensemble of states at a distance, which is consistent with the density operator $\rho = \frac{1}{2}\mathbb{1}_{\mathcal{H}}$ from the singlet state by choosing the orientation of the measuring device. One might think that this hangs crucially on the distinguished symmetry of the singlet state with respect to basis transformation. But indeed the possibility of nonlocal preparation of any ensemble consistent with a given density operator at a distance, by choosing one of several possible ideal measurements on a given entangled state, is a generic fact, at least on the level of measurement schemes (if these measurements are experimentally realizable is another question).

The possibility to prepare truly *any* ensemble consistent with the density operator $\rho = \frac{1}{2}\mathbb{1}_{\mathcal{H}}$

at a distance from the singlet state could be rigorously proven by making use of the fact that the singlet state is actually a purification of ρ and then using the unitary connections between ensembles sharing the same density operator. That way this fact can be also elegantly proven for the general case:

Corollary 1.16 [*Gisin, Hughston, Jozsa, Wootters*]

Let $\rho \in \mathcal{S}(\mathcal{H})$ be any mixed density operator and $\Psi \in \widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$ a purification of ρ. Then for each ensemble of pure states $\{(\psi_k\,;\,p_k)\}$ which has ρ as its density operator (i.e. $\rho = \sum_k p_k \,|\psi_k\rangle\,\langle\psi_k|$), there exists an ideal measurement scheme associated with $\mathcal{H}'$ which prepares $\{(\psi_k\,;\,p_k)\}$ nonlocally (possibly at spacelike separation) from of an ensemble of systems in the state Ψ.

Proof: Let $\Psi = \sum_n c_n \eta_n \phi_n$ be the Schmidt decomposition of Ψ with mutually orthogonal states $\eta_n \in \mathcal{H}$ and $\phi_n \in \mathcal{H}'$ and positive coefficients $c_n > 0$. Correspondingly $\rho = \sum_n c_n^2 \,|\eta_n\rangle\,\langle\eta_n|$ is the spectral representation of ρ. Since $\rho = \sum_k p_k \,|\psi_k\rangle\,\langle\psi_k| = \sum_n c_n^2 \,|\eta_n\rangle\,\langle\eta_n|$ there is thus according to corollary 1.15 a unitary matrix $[u_{nl}]$ such that $c_n\eta_n = \sum_l u_{nl}\sqrt{p_l}\psi_l$ for all n (where possibly some $c_n\eta_n$ have to be set to zero if there are more $p_l > 0$ than $c_n > 0$, as explained in corollary 1.15). Thus we can rewrite Ψ as

$$\Psi = \sum_n c_n\eta_n\phi_n = \sum_n\sum_l u_{nl}\sqrt{p_l}\psi_l\phi_n = \sum_l \sqrt{p_l}\psi_l \left(\sum_n u_{nl}\phi_n\right) =: \sum_l \sqrt{p_l}\psi_l\xi_l \tag{1.237}$$

where we have defined the states $\xi_l := \sum_n u_{nl}\phi_n \in \mathcal{H}'$ which are mutually orthonormal:

$$\langle\xi_l|\,\xi_m\rangle = \sum_{n,k} \overline{u}_{nl}u_{km}\,\langle\phi_n|\,\phi_k\rangle = \sum_n \overline{u}_{nl}u_{nm} = \delta_{lm} \tag{1.238}$$

where we used again the mutual orthonormality of the column vectors of the unitary matrix $[u_{nl}]$ (see (1.236)). Thus the set $\{\xi_l\} \subset \mathcal{H}'$ either constitutes an ONB of $\mathcal{H}'$ or can be completed to and ONB, such that the set $\{|\xi_l\rangle\,\langle\xi_l|\}$ (possibly completed) of projections forms a PVM on the index set acting on $\mathcal{H}'$, which if associated with an ideal measurement scheme is also identical to the set of associated state transformers.

This measurement scheme is trivially extended to the Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H}\otimes\mathcal{H}'$ by the PVM $\{\mathbb{1}_{\mathcal{H}} \otimes |\xi_l\rangle\,\langle\xi_l|\}$ (which is again identical to the set of state transformers). Correspondingly, the final states of this ideal measurement are $\psi_k\xi_k$ with probability p_k, respectively.

Thus this measurement associated with the $\mathcal{H}'-$part of the total system, if experimentally realizable, prepares the state ψ_k of the $\mathcal{H}-$system with probability p_k, respectively.

■

If we think of the Hilbert space $\mathcal{H}'$ as representing internal degrees of freedom of a given ensemble of systems, we might additionally consider the L^2-space of their spatial degrees of freedom such that the respective wave functions might be supported – and thus the measurements might take place – far away from the location where the state ensemble $\{(\psi_k\,;\,p_k)\}$ is going to be prepared by this procedure, at least in principle (think e.g. of the spin example from above).

1.6.4 Representations of CPMs and POVMs

We shall prove the results in this section only for systems represented by finite dimensional Hilbert spaces, since in this case all proofs can be presented in a very nice and instructive way. The assertions are nonetheless valid with respect to any separable Hilbert space. For the proofs of the general case, which usually resort to mappings between C^*-algebras of operators acting on $\mathcal{H}$, see e.g. [212, 345].

Kraus Representation of CPMs

The notion of completely positive superoperators to characterize transitions of density operators was motivated by familiar examples like unitary evolutions, measurement (like) processes or open quantum systems[70], which are all of the form

$$\mathcal{S}(\mathcal{H}) \ni \rho \quad \longmapsto \quad \mathcal{W}(\rho) = \sum_k \mathcal{R}_k \, \rho \mathcal{R}_k^\dagger \tag{1.239}$$

with bounded operators $\{\mathcal{R}_k\} \subset \mathcal{B}(\mathcal{H})$ obeying

$$\sum_k \mathcal{R}_k^\dagger \mathcal{R}_k \leq \mathbb{1}_\mathcal{H} \tag{1.240}$$

(i.e. $\mathbb{1}_\mathcal{H} - \sum_k \mathcal{R}_k^\dagger \mathcal{R}_k$ is a (semi-)positive operator). The right hand side of (1.239) is called an *operator sum representation* of $\mathcal{W}$. It is easy to check that the trace preserving examples (quantum channels) are exactly those with an operator sum representation (1.239) with $\sum_k \mathcal{R}_k^\dagger \mathcal{R}_k = \mathbb{1}_\mathcal{H}$ whereas the trace decreasing ones obey $\sum_k \mathcal{R}_k^\dagger \mathcal{R}_k < \mathbb{1}_\mathcal{H}$ (in the latter case $\sum_k \mathcal{R}_k^\dagger \mathcal{R}_k$ is usually an effect associated with an outcome of a measurement).

Karl Kraus derived in the 1980's that operator sum representations are generic for quantum operations (a notion which was actually introduced by himself), in particular that CPMs have always an operator sum representation with the mentioned trace properties and that these representations are generically non-unique. Kraus based his proof [212] on the Stinespring theorem [312] (see theorem 1.18 below), but the result was actually already known from a more mathematical framework since the work of Andrzej Jamiolkowski (1971) [196] and Man-Duen Choi (1975) [81]. The following proof is rather oriented on the latter approach and subsequently the Stinespring theorem will be very easily proven by resorting to the operator sum representation of CPMs.

Theorem 1.17 [*Choi-Kraus: Operator-Sum Representation of CPMs*]

Let $\mathcal{W} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ be a superoperator.

(i) If $\mathcal{W}$ is completely positive, it has an operator-sum representation, which is to say there is a countable family of bounded operators $\{\mathcal{R}_k\} \subset \mathcal{B}(\mathcal{H})$ – called Kraus operators – such that

$$\mathcal{W}(\rho) = \sum_k \mathcal{R}_k \, \rho \, \mathcal{R}_k^\dagger \quad \text{for all } \rho \in \mathcal{T}(\mathcal{H}) \tag{1.241}$$

where the limit is to be taken in the trace norm topology if the sum extends to infinity.

[70] To see this for open quantum systems, consider the second part of the proof of theorem 1.18 below.

(ii) If $\mathcal{W}$ has an operator sum representation (1.241)*, it is completely positive.*

(iii) Let $\mathcal{W}$ be completely positive and (1.241) *its operator sum representation. $\mathcal{W}$ is trace preserving if and only if $\sum_k \mathcal{R}_k^\dagger \mathcal{R}_k = \mathbb{1}_\mathcal{H}$ and trace reducing if and only if $\sum_k \mathcal{R}_k^\dagger \mathcal{R}_k < \mathbb{1}_\mathcal{H}$ (i.e. $\mathbb{1}_\mathcal{H} - \sum_k \mathcal{R}_k^\dagger \mathcal{R}_k$ is a positive operator).*

(iv) Operator sum representations (1.241) *of CPMs are not unique: Two sets of Kraus operators $\{\mathcal{R}_k\}$ and $\{\widetilde{\mathcal{R}}_k\}$ represent the same CPM $\mathcal{W}$ (i.e. $\mathcal{W}(\rho) = \sum_l \mathcal{R}_l \rho \mathcal{R}_l^\dagger = \sum_k \widetilde{\mathcal{R}}_k \rho \widetilde{\mathcal{R}}_k^\dagger$ for all $\rho \in \mathcal{T}(\mathcal{H})$) if and only if there is a unitary matrix $[u_{kl}]$ such that*

$$\widetilde{\mathcal{R}}_k := \sum_l u_{kl} \mathcal{R}_l \quad \text{for all } k \tag{1.242}$$

If the sets $\{\widetilde{\mathcal{R}}_k\}$ and $\{\mathcal{R}_k\}$ have a different number of elements, the smaller one is to be padded with zeros in the equations (1.242)*.*

Proof: As mentioned, we prove the theorem for finite dimensional $\mathcal{H}$:

• (i) A possible way to prove this assertion of the Choi-Kraus theorem (which is the actually non trivial part of the theorem) for finite dimensional Hilbert space, resorts to the mathematical trick of characterizing the action of a superoperator $\mathcal{W} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ by a so called *maximally entangled state* $\Phi \in \widetilde{\mathcal{H}}$ on a larger Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$, and an appropriate association of a 'mirror state' $\psi' \in \mathcal{H}'$ with each $\psi \in \mathcal{H}$:

Let $\{\psi_k\}$ be an ONB of $\mathcal{H}$, introduce an arbitrary second finite dimensional Hilbert space $\mathcal{H}'$ with $\dim(\mathcal{H}') \geq \dim(\mathcal{H})$ and ONB $\{\varphi_k\} \subset \mathcal{H}'$ and set $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$. A state $\Phi \in \widetilde{\mathcal{H}}$ with $\mathrm{Tr}_{\mathcal{H}'}\left[|\Phi\rangle\langle\Phi|\right] \sim \mathbb{1}_\mathcal{H}$ is called *maximally entangled*. We make the choice

$$\Phi := \sum_{k=1}^{\dim(\mathcal{H})} \psi_k \varphi_k \tag{1.243}$$

which is not normalized (obviously $\langle\Phi|\,\Phi\rangle = \dim(\mathcal{H})$) and denote the associated dyadic product operator by

$$\Omega := |\Phi\rangle\langle\Phi| \in \mathcal{T}(\widetilde{\mathcal{H}}) \tag{1.244}$$

(consequently $\mathrm{Tr}_{\widetilde{\mathcal{H}}}[\Omega] = \dim(\mathcal{H})$). Finally, with each

$$\psi = \sum_k c_k \psi_k \in \mathcal{H} \tag{1.245}$$

we associate a 'mirror state' $\psi' \in \mathcal{H}'$ by

$$\psi' := \sum_{k=1}^{\dim(\mathcal{H})} \overline{c}_k \varphi_k \in \mathcal{H}' \tag{1.246}$$

Now we come to the actual proof: Since $\mathcal{W}$ is completely positive, $\mathcal{W} \otimes \mathbb{1}_{\mathcal{H}'}(\Omega) \in \mathcal{B}(\widetilde{\mathcal{H}})$ is a positive operator acting on $\widetilde{\mathcal{H}}$, this operator is called the *Choi matrix*. Since it is positive, the Choi matrix has a spectral decomposition

$$\mathcal{W} \otimes \mathbb{1}_{\mathcal{H}'}(\Omega) = \sum_k p_k \left|\phi_k\right\rangle \left\langle\phi_k\right| \tag{1.247}$$

with positive eigenvalues $p_k \geq 0$ (such that $\sqrt{p_k} \geq 0$ exists) and eigenvectors $\phi_k \in \widetilde{\mathcal{H}}$.

For any pure density operator $\rho = \left|\psi\right\rangle \left\langle\psi\right| \in \mathcal{S}(\mathcal{H})$ we can now express $\mathcal{W}(\rho)$ in terms of an operator sum by resorting to its Choi matrix:

$$\begin{aligned}
\mathcal{W}(\rho) &= \sum_{k,j} c_k \overline{c}_j \mathcal{W}(\left|\psi_k\right\rangle \left\langle\psi_j\right|) = \left\langle\psi'\right| \left(\sum_{k,j} \mathcal{W}(\left|\psi_k\right\rangle \left\langle\psi_j\right|) \otimes \left|\varphi_k\right\rangle \left\langle\varphi_j\right| \right) \left|\psi'\right\rangle = \\
&= \left\langle\psi'\right| \left(\sum_{k,j} \Big[\mathcal{W} \otimes \mathbb{1}_{\mathcal{H}'} \Big] (\left|\psi_k\right\rangle \left\langle\psi_j\right| \otimes \left|\varphi_k\right\rangle \left\langle\varphi_j\right|) \right) \left|\psi'\right\rangle = \Big\langle \psi' \Big| \Big[\mathcal{W} \otimes \mathbb{1}_{\mathcal{H}'} \Big] (\Omega)\, \psi' \Big\rangle = \\
&= \sum_k \sqrt{p_k} \left\langle\psi'\right| \left.\phi_k\right\rangle \left\langle\phi_k\right| \left.\psi'\right\rangle \sqrt{p_k} =: \sum_k \mathcal{R}_k \left|\psi\right\rangle \left\langle\psi\right| \mathcal{R}_k^\dagger
\end{aligned} \tag{1.248}$$

where we have defined the linear operators $\mathcal{R}_k$ acting on $\mathcal{H}$ by

$$\mathcal{R}_k \psi := \sqrt{p_k} \left\langle\psi'\right| \left.\phi_k\right\rangle \in \mathcal{H} \quad \text{for all } \psi \in \mathcal{H} \tag{1.249}$$

Thus, for all pure states $\rho \in \mathcal{S}(\mathcal{H})$ we have

$$\mathcal{W}(\rho) = \sum_k \mathcal{R}_k \rho \mathcal{R}_k^\dagger \tag{1.250}$$

and since mixed states can always be written as linear combinations of pure states and calculation (1.248) is linear in ρ, it follows that (1.250) holds for all $\rho \in \mathcal{S}(\mathcal{H})$. Finally, as already explained above, any positive trace class operator $\rho' \in \mathcal{T}_+(\mathcal{H})$ can be written as an element of $\mathcal{S}(\mathcal{H})$ times a positive constant (the trace of ρ) and any trace class operator $\rho'' \in \mathcal{T}(\mathcal{H})$ is a linear combination of four positive trace class operators (the positive and negative parts of the symmetric and antisymmetric parts of ρ'', respectively). Consequently, we have

$$\mathcal{W}(\rho) = \sum_k \mathcal{R}_k \rho \mathcal{R}_k^\dagger \quad \text{for all } \rho \in \mathcal{T}(\mathcal{H}) \tag{1.251}$$

• (ii) This implication is straight forward: Let $\mathcal{W}(\varrho) = \sum_k \mathcal{R}_k \varrho \mathcal{R}_k^\dagger$ for all $\varrho \in \mathcal{T}(\mathcal{H})$. For some $N \in \mathbb{N}$ let $\mathcal{H}'$ be an $N-$dimensional Hilbert space with ONB $\{\varphi_k\}$, $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$ and $\rho \in \mathcal{T}(\widetilde{\mathcal{H}})$. As mentioned above (see (1.195)), we can write $\rho = \sum_{k,l=1}^N \rho_{kl} \otimes \left|\varphi_k\right\rangle \left\langle\varphi_l\right|$, where the

matrix elements $\rho_{kl} = \langle\varphi_k|\,\rho\,\varphi_l\rangle$ are trace class operators acting in $\mathcal{H}$. Consequently,

$$\begin{aligned}\mathcal{W}\otimes\mathbb{1}_{\mathcal{T}(\mathcal{H}_N)}(\rho) &= \sum_{k,l=1}^{N}\mathcal{W}(\rho_{kl})\otimes|\varphi_k\rangle\langle\varphi_l| = \sum_{k,l=1}^{N}\sum_m \mathcal{R}_m\rho_{kl}\mathcal{R}_m^\dagger\otimes|\varphi_k\rangle\langle\varphi_l| = \\ &= \sum_m(\mathcal{R}_m\otimes\mathbb{1}_{\mathcal{H}'})\left(\sum_{k,l=1}^{N}\rho_{kl}\otimes|\varphi_k\rangle\langle\varphi_l|\right)(\mathcal{R}_m\otimes\mathbb{1}_{\mathcal{H}'})^\dagger = \\ &= \sum_m(\mathcal{R}_m\otimes\mathbb{1}_{\mathcal{H}'})\;\rho\;(\mathcal{R}_m\otimes\mathbb{1}_{\mathcal{H}'})^\dagger\end{aligned} \tag{1.252}$$

which is obviously a positive operator whenever ρ is positive, since then for each $\Psi\in\widetilde{\mathcal{H}}$, defining $(\mathcal{R}_k^\dagger\otimes\mathbb{1}_{\mathcal{H}'})\,\Psi =: \Phi_k\in\widetilde{\mathcal{H}}$, we have

$$\langle\Psi|\,\mathcal{W}\otimes\mathbb{1}_{\mathcal{H}'}(\rho)\,\Psi\rangle = \sum_k\left\langle\Psi\right|(\mathcal{R}_k\otimes\mathbb{1}_{\mathcal{H}'})\,\rho\,(\mathcal{R}_k^\dagger\otimes\mathbb{1}_{\mathcal{H}'})\,\Psi\Big\rangle =: \sum_k\langle\Phi_k|\,\rho\,\Phi_k\rangle\geq 0 \tag{1.253}$$

Thus, $\mathcal{W}$ is $N-$positive and since N was arbitrary $\mathcal{W}$ is completely positive.

• (iii) Let $\rho\in\mathcal{T}_+(\mathcal{H})$ with spectral representation $\rho=\sum_k c_k\,|\psi_k\rangle\langle\psi_k|$ (in particular the c_ks are positive) and $\mathcal{W}(\rho)=\sum_k\mathcal{R}_k\rho\mathcal{R}_k^\dagger$. Thus

$$\begin{aligned}&\mathrm{Tr}_{\mathcal{H}}\left[\rho\right]-\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}\left(\rho\right)\right] = \mathrm{Tr}_{\mathcal{H}}\left[\rho\right]-\mathrm{Tr}_{\mathcal{H}}\left[\sum_k\mathcal{R}_k\rho\mathcal{R}_k^\dagger\right] = \\ &\mathrm{Tr}_{\mathcal{H}}\left[\rho\right]-\mathrm{Tr}_{\mathcal{H}}\left[\sum_k\mathcal{R}_k^\dagger\mathcal{R}_k\rho\right] = \mathrm{Tr}_{\mathcal{H}}\left[\left(\mathbb{1}_{\mathcal{H}}-\sum_k\mathcal{R}_k^\dagger\mathcal{R}_k\right)\rho\right] = \\ &= \sum_l c_l\left\langle\psi_l\right|\left(\mathbb{1}_{\mathcal{H}}-\sum_k\mathcal{R}_k^\dagger\mathcal{R}_k\right)\psi_l\Big\rangle\end{aligned} \tag{1.254}$$

and in consequence

→ if $\sum_k\mathcal{R}_k^\dagger\mathcal{R}_k=\mathbb{1}_{\mathcal{H}}$ we have $\mathrm{Tr}_{\mathcal{H}}\left[\rho\right]=\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}\left(\rho\right)\right]$ and – since $\rho\in\mathcal{T}_+(\mathcal{H})$ was arbitrary – $\mathcal{W}$ is trace preserving

→ if $\mathbb{1}_{\mathcal{H}}-\sum_k\mathcal{R}_k^\dagger\mathcal{R}_k$ is a positive operator $\mathrm{Tr}_{\mathcal{H}}\left[\rho\right]>\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}\left(\rho\right)\right]$ for all $\rho\in\mathcal{T}_+(\mathcal{H})$, i.e. $\mathcal{W}$ is trace decreasing

→ if $\mathcal{W}$ is trace preserving $\left\langle\psi\right|\left(\mathbb{1}_{\mathcal{H}}-\sum_k\mathcal{R}_k^\dagger\mathcal{R}_k\right)\psi\Big\rangle=0$ for all $\psi\in\mathcal{H}$ (just set $\rho=|\psi\rangle\langle\psi|$) and thus $\mathbb{1}_{\mathcal{H}}=\sum_k\mathcal{R}_k^\dagger\mathcal{R}_k$

→ if $\mathcal{W}$ is trace decreasing $\left\langle\psi\right|\left(\mathbb{1}_{\mathcal{H}}-\sum_k\mathcal{R}_k^\dagger\mathcal{R}_k\right)\psi\Big\rangle>0$ for all $\psi\in\mathcal{H}$, i.e. $\mathbb{1}_{\mathcal{H}}-\sum_k\mathcal{R}_k^\dagger\mathcal{R}_k$ is a positive operator

• (iv) The choice of ONBs of $\mathcal{H}$ and $\mathcal{H}'$ in the construction of a maximally entangled state, mirror states and the consequential Kraus representation of CPMs in point (i) of this proof was

arbitrary, which reeks of a unitary freedom in the choice Kraus operators representing a CPM. Indeed the freedom of choice of ONBs can be directly translated into a unitary freedom in the operator-sum representation. Even more easily, we can directly apply the unitary equivalence of ensembles yielding the same density operator in theorem 1.15, to see that

$$\widetilde{\mathcal{R}}_k = \sum_l u_{kl}\mathcal{R}_l \quad \text{for all } k \quad \Longleftrightarrow \quad \sum_l \mathcal{R}_l \rho \mathcal{R}_l^\dagger = \sum_k \widetilde{\mathcal{R}}_k \rho \widetilde{\mathcal{R}}_k^\dagger \quad \text{for all } \rho \in \mathcal{S}(\mathcal{H}) \tag{1.255}$$

whenever $[u_{kl}]$ is a unitary matrix (in case of different number of elements of the sets $\{\mathcal{R}_k\}$ and $\{\widetilde{\mathcal{R}}_k\}$, the smaller set is to be padded with zeros). By linearity, the equivalence (1.255) is extended to the whole domain $\mathcal{T}(\mathcal{H})$ of $\mathcal{W}$, as repeatedly explained above. In other words, $\{\mathcal{R}_k\}$ and $\{\widetilde{\mathcal{R}}_k\}$ represent one and the same CPM $\mathcal{W}$ if and only if there is a unitary matrix $[u_{kl}]$ such that $\widetilde{\mathcal{R}}_k = \sum_l u_{kl}\mathcal{R}_l$ for all k.

■

Stinespring Representation of Quantum Channels

The most simple models of open quantum systems start with considering unitary interaction of some system with its environment under the assumption both systems are initially in a product state[71]: Consider a given quantum system in a given initial state $\rho \in \mathcal{S}(\mathcal{H})$ in product $\rho \otimes \rho_0 \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}')$ with its environment (i.e. the latter has initial state $\rho_0 \in \mathcal{S}(\mathcal{H}')$) at a given time, which subsequently unitarily interact – represented by a unitary operator $U \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}')$ – and get thereby possibly entangled. The thus resulting global density operator is $U(\rho \otimes \rho_0)U^{-1}$ and the considered system can be associated with the reduced density operator $\mathrm{Tr}_{\mathcal{H}'}[U(\rho \otimes \rho_0)U^{-1}]$. Denote the associated effective transition by

$$\rho \quad \longmapsto \quad \mathcal{W}(\rho) := \mathrm{Tr}_{\mathcal{H}'}\left[U(\rho \otimes \rho_0)U^{-1}\right] \tag{1.256}$$

Then $\mathcal{W}$ is a completely positive and trace preserving superoperator, i.e. a quantum channel (see the proof of the second implication of Stinespring's theorem below).

Stinespring [312] has shown that such a representation with a given fixed 'environment initial state' is indeed formally generic for quantum channels, i.e. for any given quantum channel $\mathcal{W} : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ we can always find a larger Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$, a state $\rho_0 \in \mathcal{S}(\mathcal{H}')$ and a unitary operator U acting on $\mathcal{H} \otimes \mathcal{H}'$ such that $\mathcal{W}$ can be written as $\mathcal{W}(\rho) := \mathrm{Tr}_{\mathcal{H}'}[U(\rho \otimes \rho_0)U^{-1}]$ for all $\rho \in \mathcal{S}(\mathcal{H})$. In particular we can extend the above model of open quantum systems by including non selective measurement like interactions with the environment (which might be macroscopic and thus containing pointer (like) states) in addition to unitary interaction and nonetheless we can formally always find a representation of the transitions which is of the form (1.256), i.e. containing only unitary interaction (a fact which is also well known from decoherence theory).

As already anticipated above, this led some authors to claim that we can get rid of the collapse postulate and thereby of the measurement problem, since non selective measurements

[71] This assumption is of course questionable in general for open quantum systems, but in certain situations like measurements, where the apparatus might be perceived as an environment of the measured system, appropriate. Complications which arise when the initial product assumption is omitted and ways how to deal with them are briefly presented in the remark subsequent to the proof of the Stinespring theorem below.

are quantum channels and according to Stinespring's theorem we can thus always find a larger Hilbert space (which is usually taken as the Hilbert space of system plus apparatus, or the Hilbert space of system and apparatus plus the environment of the apparatus...), such that the measurement like features only effectively appear for effective descriptions of the measured system (and apparatus, if the environment of the latter is taken into account), whereas the true global dynamics was always unitary. This line of argument is usually substantiated with a decoherence argument (see section 1.6.6 for more details).

But state transitions which are not linear but only linear up to normalization (measurements with outcomes) lead to trace decreasing CPMs which can also according to Stinespring's theorem never be derived from a unitary evolution on a larger Hilbert space! This will be further illustrated by Kraus' second representation theorem 1.19 below, which is the analogue of Stinespring's theorem for trace decreasing CPMs and which shows how to generalize the Stinespring theorem to general quantum operations by incorporating collapse dynamics in an appropriate way.

Theorem 1.18 [*Stinespring: Unitary Extension of trace preserving CPMs*]

A superoperator $\mathcal{W} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ is completely positive and trace preserving (i.e. a quantum channel) if and only if it can be expressed as a unitary transformation U on a larger Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$ with factorized initial condition, reduced to $\mathcal{H}$ by taking the partial trace over $\mathcal{H}'$: For an arbitrary fixed pure state $\rho_0 \in \mathcal{S}(\mathcal{H}')$ there exists a unitary operator U acting on $\widetilde{\mathcal{H}}$ such that for each $\rho \in \mathcal{T}(\mathcal{H})$

$$\mathcal{W}(\rho) = \mathrm{Tr}_{\mathcal{H}'} \left[U \left(\rho \otimes \rho_0 \right) U^\dagger \right] \tag{1.257}$$

and the other way around, each $\mathcal{W} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ of the form (1.257) is a quantum channel.

Proof: As mentioned, we prove the theorem for finite dimensional $\mathcal{H}$:

"$\Rightarrow$" We construct explicitly a possible triple $(\mathcal{H}', U, \rho_0)$ which realizes this implication of theorem 1.18: Let the Kraus representation of $\mathcal{W}$ be given by

$$\mathcal{W}(\rho) = \sum_{k=1}^{N} \mathcal{R}_k \, \rho \, \mathcal{R}_k^\dagger \quad \rho \in \mathcal{T}(\mathcal{H}) \tag{1.258}$$

Choose a Hilbert space $\mathcal{H}'$ with dimension equal to or greater than the number N of Kraus operators in (1.258) (which is actually bounded by $(\dim(\mathcal{H}))^2$ from above, but for the sake of the proof N could be infinite as well), an ONB $\{\varphi_k\} \subset \mathcal{H}'$ and an arbitrary normalized 'initial state' $\varphi_0 \in \mathcal{H}'$. Now consider the Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$, the subspace $W := \{\psi\varphi_0 \mid \psi \in \mathcal{H}\} \subset \widetilde{\mathcal{H}}$ and define a linear operator $\widetilde{U} : W \to \widetilde{\mathcal{H}}$ by

$$\widetilde{U}(\psi\varphi_0) = \sum_k (\mathcal{R}_k\psi)\varphi_k \quad \psi \in \mathcal{H} \tag{1.259}$$

Observe now that $\widetilde{U}$ preserves the scalar product on W, since according to theorem 1.17 the assumption that $\mathcal{W}$ is trace preserving implies $\sum_k \mathcal{R}_k^\dagger \mathcal{R}_k = \mathbb{1}_{\mathcal{H}}$ and consequently for all $\psi, \psi' \in \mathcal{H}$ we have

$$\begin{aligned}\left\langle \widetilde{U}(\psi\varphi_0)\right| \widetilde{U}(\psi'\varphi_0)\Big\rangle = \left\langle \sum_k (\mathcal{R}_k\psi)\varphi_k \right| \sum_l (\mathcal{R}_l\psi')\varphi_l \Big\rangle = \sum_{k,l} \langle \mathcal{R}_k\psi|\ \mathcal{R}_l\psi'\rangle \langle\varphi_k|\ \varphi_l\rangle = \\ = \sum_k \left\langle \psi \right| \mathcal{R}_k^\dagger\mathcal{R}_k\, \psi' \Big\rangle = \langle\psi|\ \psi'\rangle = \langle\psi|\ \psi'\rangle \langle\varphi_0|\ \varphi_0\rangle = \langle\psi\varphi_0|\ \psi'\varphi_0\rangle\end{aligned} \tag{1.260}$$

This in particular implies that $\mathcal{W}$ maps each ONB of the subspace $W \subset \widetilde{\mathcal{H}}$ to an ONB of the subspace $\widetilde{U}(W) \subset \widetilde{\mathcal{H}}$. We can thus extend $\widetilde{U}$ to a unitary operator $U : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ in the following way: Set $U = \widetilde{U}$ on W and now choose an ONB $\{\Phi_l\} \subset W^\perp$ of the orthogonal complement $W^\perp$ of W and an ONB $\{\Phi_l'\} \subset (U(W))^\perp$ and define the action of U on $W^\perp$ by mapping the elements of $\{\Phi_l\}$ bijectively to $\{\Phi_l'\}$, e.g. by the choice $U(\Phi_l) = \Phi_l'$ for all l (it is here where it is crucial to assume a finite dimensional $\mathcal{H}$ for this way of proving the Stinespring theorem, since in case of infinite dimensional $\mathcal{H}$ it is easy to find examples[72] of an operator which preserves the scalar product on a subspace, which can not be unitarily extended to all of $\mathcal{H}$).

Now set $\rho_0 := |\varphi_0\rangle\langle\varphi_0| \in \mathcal{S}(\mathcal{H}')$ and consider some arbitrary pure state $\rho = |\psi\rangle\langle\psi| \in \mathcal{S}(\mathcal{H})$ to see that the triple $(\mathcal{H}', U, \rho_0)$ realizes the desired unitary implementation of $\mathcal{W}$ for pure states:

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}'}\left[U(\rho\otimes\rho_0)U^\dagger\right] = \sum_{k,l} \mathrm{Tr}_{\mathcal{H}'}\left[\mathcal{R}_k |\psi\rangle\langle\psi| \mathcal{R}_l^\dagger \otimes |\varphi_k\rangle\langle\varphi_l|\right] = \\ = \sum_{k,l} \mathcal{R}_k |\psi\rangle\langle\psi| \mathcal{R}_l^\dagger \,\mathrm{Tr}_{\mathcal{H}'}\left[|\varphi_k\rangle\langle\varphi_l|\right] = \sum_{k,l} \mathcal{R}_k |\psi\rangle\langle\psi| \mathcal{R}_l^\dagger \delta_{k,l} = \sum_k \mathcal{R}_k\rho\mathcal{R}_k^\dagger = \mathcal{W}(\rho)\end{aligned} \tag{1.261}$$

By linearity this calculation is easily extended to mixed states, to the positive – and finally to all of the trace class (as explained repeatedly above), such that $\mathrm{Tr}_{\mathcal{H}'}\left[U(\rho\otimes\rho_0)U^\dagger\right] = \mathcal{W}(\rho)$ for all $\rho \in \mathcal{T}(\mathcal{H})$.

"$\Leftarrow$" It remains to show the more trivial implication of the theorem, namely that a superoperator of the form

$$\mathcal{W}(\rho) = \mathrm{Tr}_{\mathcal{H}'}\left[U(\rho\otimes\rho_0)U^\dagger\right] \quad \rho \in \mathcal{T}(\mathcal{H}) \tag{1.262}$$

(with $(\mathcal{H}', U, \rho_0)$ as described in the theorem) is completely positive and trace preserving.

That $\mathcal{W}$ given by (1.262) is trace preserving is straight forward, since

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}(\rho)\right] = \mathrm{Tr}_{\widetilde{\mathcal{H}}}\left[U(\rho\otimes\rho_0)U^\dagger\right] = \mathrm{Tr}_{\widetilde{\mathcal{H}}}\left[U^\dagger U(\rho\otimes\rho_0)\right] = \\ = \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}'}\left[\rho\otimes\rho_0\right] = \mathrm{Tr}_{\mathcal{H}}\left[\rho\right]\mathrm{Tr}_{\mathcal{H}'}\left[\rho_0\right] = \mathrm{Tr}_{\mathcal{H}}\left[\rho\right]\end{aligned} \tag{1.263}$$

for all $\rho \in \mathcal{T}(\mathcal{H})$.

[72] Let $\dim(\mathcal{H}) = \infty$ and $\{\Phi_k\} \subset \mathcal{H}$ an ONB. Now define $\widetilde{U}$ on the subspace W spanned by the Φ_ks with even k by $\widetilde{U}(\Phi_{2k}) = \Phi_k$ for all k. In consequence $\widetilde{U}$ preserves the scalar product on W but $\widetilde{U}(W) = \mathcal{H}$ such that $\widetilde{U}$ cannot be extended to a unitary operator $U : \mathcal{H} \to \mathcal{H}$ since the orthogonal complement $W^\perp$ in $\mathcal{H}$ is infinite dimensional whereas the orthogonal complement $(\widetilde{U}(W))^\perp$ in $\mathcal{H}$ is the empty set. Thus it is impossible to find a one to one correspondence between the elements of ONBs of these two subspaces.

To see that $\mathcal{W}$ is completely positive, it is most easy to show that it has an operator sum representation and is thus completely positive according to the Kraus theorem 1.17: Denote the pure state $\rho_0 \in \mathcal{S}(\mathcal{H}')$ by a dyadic product $\rho_0 = |\varphi_0\rangle\langle\varphi_0|$ with some $\varphi_0 \in \mathcal{H}'$ and let $\{\varphi_k\} \subset \mathcal{H}'$ be an ONB. Hence, for some $\rho \in \mathcal{S}(\mathcal{H})$ we can write $\mathcal{W}(\rho)$ as

$$\begin{aligned}\mathcal{W}(\rho) &= \mathrm{Tr}_{\mathcal{H}'}\left[U(\rho\otimes\rho_0)U^\dagger\right] = \mathrm{Tr}_{\mathcal{H}'}\left[U(\rho\otimes|\varphi_0\rangle\langle\varphi_0|)U^\dagger\right] = \\ &= \sum_k \langle\varphi_k|\, U\,\varphi_0\rangle\ \rho\ \langle\varphi_0|\, U^\dagger\,\varphi_k\rangle =: \sum_k \mathcal{R}_k\,\rho\,\mathcal{R}_k^\dagger\end{aligned} \tag{1.264}$$

where we defined the opreators $\mathcal{R}_k$ as the partial matrix elements $\langle\varphi_k|\, U\,\varphi_0\rangle$ of U with respect to the $\mathcal{H}'-$tensor factor space, i.e.

$$\mathcal{R}_k := \langle\varphi_k|\, U\,\varphi_0\rangle : \mathcal{H}\to\mathcal{H} \tag{1.265}$$

and according to (1.264) these operators permit an operator-sum representation of $\mathcal{W}$. In consequence, $\mathcal{W}$ is completely positive according to theorem 1.17. ■

Remark on Open Quantum Systems and the Initial Product Assumption: There has been a lively debate on the initial product assumption, since on the one hand this assumption is of course in general not appropriate if we consider general (sub-)systems interacting with an environment (experimentally relevant examples arise e.g. in the context of quantum computation, see e.g. [91]). On the other hand it turned out that attempts to drop the initial product assumption are facing unexpected complications. In order to get a state transition, we must establish in some way a map which maps each given initial state to some final state. If the initial product assumption is abandoned, an initial state of the considered system must be given by a reduced density operator. If we now want to model an open quantum system, we must assign to this reduced density operator a global density operator including the environment, unitarily transform this global state and trace out the environment to get back a reduced final state of the considered system. As one might guess, the initial assignment map $\Phi : \mathcal{S}(\mathcal{H}) \mapsto \mathcal{S}(\mathcal{H}\otimes\mathcal{H}')$ assigning to each reduced initial state a global initial state incorporating the environment is notoriously problematic (beyond the fact that it won't be possible to explicitly specify a realistic non trivial assignment map for sufficiently complex environments, of course). Such an assignment is a priori one to many, as for example the non uniqueness of purification illustrates, so it must involve a choice which of several possible global states is assigned to each initial state. Now comes the catastrophe: Given a well defined assignment map Φ which is linear and satisfies the consistency condition $\mathrm{Tr}_{\mathcal{H}'}[\Phi(\rho)] = \rho$ for all $\rho \in \mathcal{S}(\mathcal{H})$, it can be shown [256] that the associated superoperator (and thereby the assignment map itself) $\rho \mapsto \mathrm{Tr}_{\mathcal{H}'}\left[U\Phi(\rho)U^\dagger\right]$ for some unitary U is positive on $\mathcal{S}(\mathcal{H})$ if and only if Φ assigns a product state $\Phi(\rho) = \rho\otimes\rho_0$ for some fixed $\rho_0 \in \mathcal{S}(\mathcal{H}')$ with each $\rho \in \mathcal{S}(\mathcal{H})$. This lead some authors to seriously consider non completely positive and even non positive superoperators (and thereby negative probabilities) as physically relevant. This issue was cleared in a serious of papers, the most prominent of several convincing arguments therein goes back to Jordan, Shaji and Sudarshan [200] and is surprisingly obvious: If we consider some assignment map Φ with the mentioned properties which does not assign product states, it cannot be reasonably defined on all of $\mathcal{S}(\mathcal{H})$. For example pure states in $\mathcal{S}(\mathcal{H})$ in this case must be mapped to non positive operators acting on $\mathcal{H}\otimes\mathcal{H}'$, since $\rho = \mathrm{Tr}_{\mathcal{H}'}[\Phi(\rho)]$

cannot be a pure state if $\Phi(\rho)$ is a density operator (in particular positive) which is not a product state with respect to system and environment. But pure states are of course anyway no admissible initial states if the assignment map defines entangled global initial conditions. That way, one can introduce a physically substantiated compatibility domain in $\mathcal{S}(\mathcal{H})$, on which the considered assignment map is positive as well as completely positive. For more details of these discussions see [91] gives a nice overview.

Second Kraus Representation of CPMs

If we want to include transitions of measurement (like) processes, whose final states are associated with given outcomes, we must resort to trace reducing CPMs and the Stinespring theorem does no longer apply. Instead it tells us that these transitions can actually not be derived from unitary evolution on a larger Hilbert space and the following result completes this insight by telling us the alternative if we want to describe such transitions as a reduced dynamics which originates from a global dynamics on a larger Hilbert space (including e.g. the apparatus, the environment of the apparatus or the rest of the universe), which is to insert the collapse at some stage. It actually states that any state transition can be implemented by an indirect measurement scheme, but note that (as discussed in section 1.4.4) we need not necessarily interpret the probe as another microscopic system as in the standard conception of indirect measurements. Indeed, if we interpret the probe simply as the measuring device and the readout as the collapse onto a definite pointer state (those who want to may assert 'upon looking at the pointer'), we can recover in the indirect measurement scheme the description of direct measurement (like) processes developed in section 1.2, i.e. a unitary interaction of the measured system and the device as in the previous theorem, but followed by projecting out one definite final pointer state, which implements indirectly a collapse of the measured system.

This theorem (with the Stinespring theorem as the limiting case where $P_{\alpha'} = \mathbb{1}_{\mathcal{H}_P}$ and correspondingly $\mathcal{W}$ is trace preserving, see below) is physically the most fundamental of the theorems in this section (it also directly implies the subsequent famous Naimark theorem), but it is actually the less famous one and is often omitted in presentations of these results.

Theorem 1.19 [*Kraus II: Projective Extension of Trace Reducing CPMs*]

A superoperator $\mathcal{W} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ is completely positive and trace reducing if and only if it can be expressed as the state transformer associated with some outcome α' of an indirect measurement scheme, i.e. there is some probe Hilbert space $\mathcal{H}_P$, a pure probe ready state $\rho_0 = |\varphi_0\rangle\langle\varphi_0| \in \mathcal{S}(\mathcal{H}_P)$, a unitary operator U acting on $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}_P$, an orthogonal projection $P_{\alpha'}$ onto a proper subspace of $\mathcal{H}_P$, such that

$$\mathcal{W}(\rho) \equiv \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}} \otimes P_{\alpha'})U(\rho \otimes \rho_0)U^\dagger\right] = \mathcal{W}(\rho \mid \alpha') \tag{1.266}$$

for all $\rho \in \mathcal{T}(\mathcal{H})$. In particular, each instrument can be implemented by a projective measurement scheme, in general on a larger Hilbert space.

REMARK: Note that according to the Stinespring theorem 1.18, $\mathcal{W}$ given by (1.266) is trace preserving if and only if $P_\alpha = \mathbb{1}_{\mathcal{H}_P}$. We might thus merge the two theorems together to a statement about trace non increasing CPMs (quantum operations), with the trace preserving limiting case $P_\alpha = \mathbb{1}_{\mathcal{H}_P}$.

Proof: As mentioned, we prove the theorem for finite dimensional $\mathcal{H}$.

"$\Rightarrow$" Suppose $\mathcal{W}$ is completely positive and trace reducing. According to the Kraus theorem 1.17 there thus exists some set $\{\mathcal{R}_k\}$ of bounded operators with $\sum_k \mathcal{R}_k^\dagger \mathcal{R}_k < \mathbb{1}_{\mathcal{H}}$ such that $\mathcal{W}(\rho) = \sum_k \mathcal{R}_k \rho \mathcal{R}_k^\dagger$ for all $\rho \in \mathcal{T}(\mathcal{H})$.

Now we complete the $\mathcal{R}_k$s in the following way: Consider any decomposition $E = \mathcal{R}^\dagger \mathcal{R}$ of the effect $E := \mathbb{1}_{\mathcal{H}} - \sum_k \mathcal{R}_k^\dagger \mathcal{R}_k$ with respect to some bounded operator $\mathcal{R}$ (e.g. $\mathcal{R} = \sqrt{E}$) such that $\sum_k \mathcal{R}_k^\dagger \mathcal{R}_k + \mathcal{R}^\dagger \mathcal{R} = \mathbb{1}_{\mathcal{H}}$ and consequently according to the Choi-Kraus theorem 1.17 the superoperator $\mathcal{V} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ defined by

$$\mathcal{V}(\rho) = \sum_k \mathcal{R}_k \rho \mathcal{R}_k^\dagger + \mathcal{R} \rho \mathcal{R}^\dagger \tag{1.267}$$

for $\rho \in \mathcal{T}(\mathcal{H})$ is completely positive and trace preserving. Thus, according to the Stinespring theorem 1.18 $\mathcal{V}$ can be implemented by some unitary transformation on a larger Hilbert space, i.e. there is some Hilbert space $\mathcal{H}'$, a pure ready state $\rho_0 = |\varphi_0\rangle \langle\varphi_0| \in \mathcal{S}(\mathcal{H}')$ and a unitary operator U acting on $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$ such that

$$\mathcal{V}(\rho) = \sum_k \mathcal{R}_k \rho \mathcal{R}_k^\dagger + \mathcal{R} \rho \mathcal{R}^\dagger = \mathrm{Tr}_{\mathcal{H}'} \left[U(\rho \otimes \rho_0) U^\dagger \right] \tag{1.268}$$

for all $\rho \in \mathcal{T}(\mathcal{H})$. As in the proof of the Stinespring theorem, we make for U the canonical choice defined by

$$U(\psi\varphi_0) = \sum_k (\mathcal{R}_k \psi)\varphi_k + (\mathcal{R}\psi)\varphi \tag{1.269}$$

where the φ_ks and φ are elements of an ONB of $\mathcal{H}'$. In the foregoing proof it was shown that U as defined with respect to its action on $\psi\varphi_0$ in (1.269) obeys (1.268) with $\rho = |\psi\rangle \langle\psi|$ and, moreover, that it is straightforwardly extended to a unitary operator on all of $\widetilde{\mathcal{H}}$.

Defining now the orthogonal projection

$$P := \sum_k |\varphi_k\rangle \langle\varphi_k| \tag{1.270}$$

we see that for each pure state $\rho = |\psi\rangle \langle\psi| \in \mathcal{S}(\mathcal{H})$

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}'} \left[(\mathbb{1}_{\mathcal{H}} \otimes P) U(\rho \otimes \rho_0) U^\dagger \right] &= \sum_k \langle\varphi_k| \, U(\psi\varphi_0)\rangle \, \langle U(\psi\varphi_0)| \, \varphi_k\rangle \\ \sum_{klm} \langle\varphi_k| \, (\mathcal{R}_l\psi)\varphi_l + (\mathcal{R}\psi)\varphi\rangle \, \langle (\mathcal{R}_m\psi)\varphi_m + (\mathcal{R}\psi)\varphi| \, \varphi_k\rangle &= \\ \sum_k \mathcal{R}_k |\psi\rangle \langle\psi| \mathcal{R}_k^\dagger &= \mathcal{W}(\rho) \end{aligned} \tag{1.271}$$

which is straightforwardly extended by linearity from the pure states $\rho \in \mathcal{S}(\mathcal{H})$ to all of the trace class $\mathcal{T}(\mathcal{H})$ as explained above.

Thus if we construct a projective measurement scheme on $\mathcal{H}'$ associated with a PVM $\{P_\alpha\}$ which associates with some outcome α' the projection $P \equiv P_{\alpha'}$ defined in (1.270), we can identify $\mathcal{H}' \equiv \mathcal{H}_P$ with a probe Hilbert space, implement U as a pre-measurement interaction, such that the triple $(\mathcal{H}_P, U, \varphi_0)$ defines an indirect measurement scheme on $\mathcal{H}$ whose state transformer associated with outcome α' of the readout is for all $\rho \in \mathcal{S}(\mathcal{H})$ given by

$$\mathcal{W}(\rho \mid \alpha') = \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}} \otimes P_{\alpha'})U(\rho \otimes \rho_0)U^\dagger\right] \equiv \mathcal{W}(\rho) \tag{1.272}$$

(1.272) naturally extends by linearity from the density operators acting on $\mathcal{H}$ to all of $\mathcal{T}(\mathcal{H})$ and it thereby yields a representation of the trace reducing CPM $\mathcal{W}$ from which we originally proceeded (note that the readout can be implemented by a measurement scheme which is non degenerate with respect to outcome α' if and only if there is only one single term in the operator sum representation of $\mathcal{W}$ which in turn corresponds to a transition which transforms pure states to pure states, in consistency with the analysis of indirect measurements in section 1.4.4).

"$\Leftarrow$" That $\mathcal{W}$ in (1.266) is completely positive can be for example verified by noting that it admits an operator sum representation: Let $P_\alpha = \sum_k |\varphi_k\rangle\langle\varphi_k|$ with mutually orthogonal $\varphi_k \in \mathcal{H}_P$ be an orthogonal projection onto a proper subspace of $\mathcal{H}_P$ and $\rho_0 = |\varphi_0\rangle\langle\varphi_0| \in \mathcal{S}(\mathcal{H}_P)$ such that

$$\mathcal{W}(\rho) \equiv \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}} \otimes P_{\alpha'})U(\rho \otimes \rho_0)U^\dagger\right] = \sum_k \left\langle \varphi_k \right| U(\rho \otimes \rho_0)U^\dagger \left| \varphi_k \right\rangle = \sum_k \mathcal{R}_k \rho \mathcal{R}_k^\dagger \tag{1.273}$$

where

$$\mathcal{R}_k = \langle \varphi_k | \, U \, \varphi_0 \rangle \tag{1.274}$$

as already shown in section 1.4.4. This entails that $\mathcal{W}$ is completely positive due to the Choi-Kraus theorem 1.17.

To see that $\mathcal{W}$ is trace reducing let $\rho = |\psi\rangle\langle\psi|$ for some (not necessarily normalized) $\psi \in \mathcal{H}$ and consequently

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}(\rho)\right] &= \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}} \otimes P_{\alpha'})U(\rho \otimes \rho_0)U^\dagger\right] = \\
&= \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}_P}\left[U^\dagger(\mathbb{1}_{\mathcal{H}} \otimes P_{\alpha'})U(\rho \otimes \rho_0)\right] = \left\langle \psi\varphi_0 \right| U^\dagger(\mathbb{1}_{\mathcal{H}} \otimes P_{\alpha'})U \left| \psi\varphi_0 \right\rangle = \\
&= \langle U(\psi\varphi_0)| \, (\mathbb{1}_{\mathcal{H}} \otimes P_{\alpha'}) \, U(\psi\varphi_0)\rangle = \langle (\mathbb{1}_{\mathcal{H}} \otimes P_{\alpha'}) \, U(\psi\varphi_0)| \, (\mathbb{1}_{\mathcal{H}} \otimes P_{\alpha'}) \, U(\psi\varphi_0)\rangle = \\
&= \|(\mathbb{1}_{\mathcal{H}} \otimes P_{\alpha'}) \, U(\psi\varphi_0)\|^2 < \| \, U(\psi\varphi_0)\|^2 = \| \, \psi\varphi_0\|^2 = \\
&= \mathrm{Tr}_{\mathcal{H}}\left[|\psi\rangle\langle\psi|\right] \mathrm{Tr}_{\mathcal{H}_P}\left[|\varphi_0\rangle\langle\varphi_0|\right] = \mathrm{Tr}_{\mathcal{H}}\left[\rho\right]
\end{aligned} \tag{1.275}$$

where the inequality sign in the forth line holds whenever P_α is an orthogonal projection onto a proper subspace of $\mathcal{H}_P$, i.e. $P_\alpha \neq \mathbb{1}_{\mathcal{H}_P}$ (otherwise $\mathcal{W}$ is trace preserving). Since we can write any element of $\mathcal{T}_+(\mathcal{H})$ as a linear combination of one dimensional orthogonal projections with positive coefficients, the relation $\mathrm{Tr}_{\mathcal{H}}\left[\mathcal{W}(\rho)\right] < \mathrm{Tr}_{\mathcal{H}}\left[\rho\right]$ found in (1.275) for $\rho = |\psi\rangle\langle\psi|$ is straightforwardly linearly extended to all $\rho \in \mathcal{T}_+(\mathcal{H})$, which concludes the proof.

■

Naimark Representation of POVMs

Proceeding from Kraus' second representation theorem, it is immediately clear that each discrete POVM can be formally implemented by an indirect measurement scheme: Just choose for each effect E_α in the POVM state transformers realizing it – most easily by an efficient measurement via $\mathcal{R}_\alpha^\dagger \mathcal{R}_\alpha = E_\alpha$ – like the associated Lüders state transformers $\mathcal{R}_\alpha = \sqrt{E_\alpha}$, and then implement the associated instrument by an indirect measurement scheme like it was constructed in the proof of the foregoing theorem. Thereby the state transformers take the shape $\mathcal{R}_\alpha = \langle\varphi_\alpha|\, U\, \varphi_0\rangle$ (or on the density operator level the respective sandwich $\rho \mapsto \mathcal{W}(\rho \mid \alpha) = \mathcal{R}_\alpha \rho \mathcal{R}_\alpha^\dagger$), where φ_α defines the one dimensional subspace of the probe Hilbert space associated with outcome α of the non-degenerate readout, U encodes the pre-measurement interaction and φ_0 is the ready state of the probe. Thus we find an indirect measurement scheme representation of the effects constituting the POVM from which we proceeded by

$$E_\alpha = \mathcal{R}_\alpha^\dagger \mathcal{R}_\alpha = \left\langle \varphi_0 \right| U^\dagger \varphi_\alpha \rangle \langle \varphi_\alpha|\, U\, \varphi_0\rangle = \mathrm{Tr}_{\mathcal{H}_P}\left[(\mathbb{1}_{\mathcal{H}} \otimes P_\alpha) U^\dagger (\mathbb{1}_{\mathcal{H}} \otimes \rho_0) U\right] \tag{1.276}$$

where $P_\alpha = |\varphi_\alpha\rangle\langle\varphi_\alpha|$ and $\rho_0 = |\varphi_0\rangle\langle\varphi_0|$.

This is the content of the physically interesting version[73] of the *Naimark theorem*: Given a POVM acting on $\mathcal{H}$ we can always find a larger Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$ such that the POVM emerges from a projective measurement scheme associated with $\widetilde{\mathcal{H}}$ if $\mathcal{H}'$ is traced out and in particular, this is always realizable by an indirect measurement scheme where $\mathcal{H}'$ is identified with the Hilbert space of the probe. This does not necessarily mean that any non projective measurement must be an indirect measurement, as the approximate measurement scheme (see section 1.4.3) with a measurement error, which is not necessarily of particular quantum origin, shows. But indeed, any POVM can be at least formally implemented by a projective measurement scheme on a larger Hilbert space.

In the proof of the Naimark theorem we shall not go the path sketched above and proceed from Kraus' second representation theorem, but go one step back and proceed from the Stinespring theorem in order to directly establish an explicit construction of the indirect measurement scheme (in strong analogy with the foregoing proof) realizing a given POVM. Indeed there can be constructed infinitely many different indirect measurement schemes realizing a given POVM. The construction in the proof below is frequently applied in the literature and called the *canonical Naimark extension*, in many concrete examples it turns out to be more straightforward to develop projective extensions of a given POVM which are not the canonical one (see e.g. the quantum roulette in [253]).

[73] Naimarks original theorem [247, 248] makes a more general but physically less tangible assertion: It states that given a (discrete) POVM $\{F_\alpha\}$ acting on $\mathcal{H}$, there exists always a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace and a PVM $\{P_\alpha\}$ acting on $\mathcal{K}$, such that

$$F_\alpha \psi = \Pi_{\mathcal{H}} P_\alpha \psi \tag{1.277}$$

for all $\psi \in \mathcal{H}$ and for all α, where $\Pi_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. From this the following theorem can then be obtained as a corollary (which was e.g. done by Holevo [191]), but we will prove it more directly. Some authors (e.g. [95]) differentiate theses two cases in terminology: In both cases a POVM is extended to a PVM acting on a larger Hilbert space. If the latter is the direct sum of the original Hilbert space, on which the POVM acts, with another Hilbert space (i.e. contains the original Hilbert space as a subspace) the PVM is called a *Naimark dilation* of the POVM, if the larger Hilbert space is archived by a tensor product with another Hilbert space (the physically interesting case) the PVM acting on the larger Hilbert space is called a *Naimark extension*.

The theorem is usually proven for discrete POVMs only, as we shall do as well (and moreover we will prove it for only finite dimensional $\mathcal{H}$). But from this the analogue statement for POVMs on a continuum can be inferred by partitioning the latter into a countable infinity of disjoint regions and taking a continuum limit in the appropriate way (see [187] and references therein). The theorem could be formulated as an 'if and only if' statement as well, but since it was already shown (and frequently used) above, that any indirect measurement scheme leads to a POVM (see section 1.4.4), we shall only consider the opposite assertion, that any POVM is realizable by an indirect measurement scheme.

Theorem 1.20 [*Naimark, Holevo: Projective Extension of POVMs*]

Let $\{F_\alpha\}$ be a discrete POVM acting on Hilbertspace $\mathcal{H}$. There exists a Hilbert space $\mathcal{H}'$, a pure state $\rho_0 = |\varphi_0\rangle\langle\varphi_0| \in \mathcal{S}(\mathcal{H}')$ and a PVM $\{P_\alpha\}$ acting on $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}'$ such that

$$\mathbb{P}^\rho(\alpha) = \mathrm{Tr}_{\mathcal{H}}\left[F_\alpha\,\rho\right] = \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}'}\left[P_\alpha\,(\rho\otimes\rho_0)\right] = \mathbb{P}^{\rho\otimes\rho_0}(\alpha) \tag{1.278}$$

for all $\rho \in \mathcal{S}(\mathcal{H})$ and for all α.

In particular, there exists an indirect measurement scheme realizing the above claim with $\mathcal{H} \equiv \mathcal{H}_S$ the system Hilbert space, $\mathcal{H}' \equiv \mathcal{H}_P$ the Hilbert space of the probe, the probe ready state $\rho_0 = |\varphi_0\rangle\langle\varphi_0|$, a unitary interaction U describing the premeasurement and a PVM $\{P'_\alpha\}$ acting on $\mathcal{H}'$ associated with the readout, such that $P_\alpha \equiv U^\dagger\,(\mathbb{1}_{\mathcal{H}}\otimes P'_\alpha)\,U$ for all α and

$$F_\alpha = \mathrm{Tr}_{\mathcal{H}'}\left[(\mathbb{1}_{\mathcal{H}}\otimes\rho_0)\,U^\dagger\,(\mathbb{1}_{\mathcal{H}}\otimes P'_\alpha)\,U\right] \tag{1.279}$$

for all α.

Proof: As mentioned, we prove the theorem for finite dimensional $\mathcal{H}$:

We explicitly construct an indirect measurement scheme with the desired properties which is called the *canonical Naimark extension*: First, consider for each effect F_α a decomposition into possible associated (effective measurement) state transformers $F_\alpha = \mathcal{R}_\alpha^\dagger\mathcal{R}_\alpha$ (e.g. the associated Lüders state transformers $\mathcal{R}_\alpha = \sqrt{F_\alpha}$) and the associated non selective measurement transforamtion:

$$\mathcal{S}(\mathcal{H}) \ni \rho \quad \longmapsto \quad \mathcal{W}(\rho) := \sum_\alpha \mathcal{R}_\alpha\,\rho\,\mathcal{R}_\alpha^\dagger \in \mathcal{S}(\mathcal{H}) \tag{1.280}$$

Since this is an operator-sum representation of a superoperator – easily extended to $\mathcal{T}(\mathcal{H})$ by linearity – with $\sum_\alpha \mathcal{R}_\alpha^\dagger\mathcal{R}_\alpha = \mathbb{1}_{\mathcal{H}}$, $\mathcal{W}$ is a trace preserving CPM according to the Kraus theorem 1.17. According to the Stinespring theorem 1.18 there thus exists a Hilbert space $\mathcal{H}'$, a unitary operator U acting on $\mathcal{H}\otimes\mathcal{H}'$ and a pure state $\rho_0 = |\varphi_0\rangle\langle\varphi_0| \in \mathcal{S}(\mathcal{H}')$ such that

$$\mathcal{W}(\rho) = \sum_\alpha \mathcal{R}_\alpha\,\rho\,\mathcal{R}_\alpha^\dagger = \mathrm{Tr}_{\mathcal{H}'}\left[U(\rho\otimes\rho_0)U^\dagger\right] \quad \text{for all } \rho \in \mathcal{T}(\mathcal{H}) \tag{1.281}$$

We choose again the canonical construction of U used in the proof of Stinespring's theorem 1.18,

which acts with respect to a given ONB $\{\varphi_\alpha\} \subset \mathcal{H}'$ for all $\psi \in \mathcal{H}$ on the product state $\psi\varphi_0$ as

$$U(\psi\varphi_0) = \sum_\alpha (\mathcal{R}_\alpha \psi)\, \varphi_\alpha \tag{1.282}$$

As shown in the proof of the Stinespring theorem, U obeys (1.281) and can be extended to a unitary operator on all of $\mathcal{H} \otimes \mathcal{H}'$.

Now we can orthogonally project out the action of a given $\mathcal{R}_\alpha$ of equation (1.282) (understanding again the dual $\langle\varphi_\alpha|$ as a mapping $\langle\varphi_\alpha| : \mathcal{H} \otimes \mathcal{H}' \to \mathcal{H}$):

$$\langle\varphi_{\alpha'}|\, U(\psi\varphi_0)\rangle = \sum_\alpha (\mathcal{R}_\alpha\psi)\, \langle\varphi_{\alpha'}|\, \varphi_\alpha\rangle = \mathcal{R}_{\alpha'}\psi \tag{1.283}$$

Defining now the projections $P'_\alpha := |\varphi_\alpha\rangle\langle\varphi_\alpha|$ we thus get for all $\psi \in \mathcal{H}$ and for all α

$$\begin{aligned}
\langle\psi|\, F_\alpha\, \psi\rangle = \big\langle\psi\big|\, \mathcal{R}_\alpha^\dagger \mathcal{R}_\alpha\, \psi\big\rangle = \langle U(\psi\varphi_0)|\, \varphi_\alpha\rangle\, \langle\varphi_\alpha|\, U(\psi\varphi_0)\rangle = \\
= \langle U(\psi\varphi_0)|\, (\mathbb{1}_\mathcal{H} \otimes P'_\alpha)\, U(\psi\varphi_0)\rangle = \big\langle\psi\varphi_0\big|\, \big(U^\dagger\, (\mathbb{1}_\mathcal{H} \otimes P'_\alpha)\, U\big)\, \psi\varphi_0\big\rangle = \\
= \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}'}\left[(|\psi\rangle\langle\psi| \otimes |\varphi_0\rangle\langle\varphi_0|)\, U^\dagger\, (\mathbb{1}_\mathcal{H} \otimes P'_\alpha)\, U\right] = \\
\mathrm{Tr}_\mathcal{H}\left[|\psi\rangle\langle\psi|\, \mathrm{Tr}_{\mathcal{H}'}\left[(\mathbb{1}_\mathcal{H} \otimes |\varphi_0\rangle\langle\varphi_0|)\, U^\dagger\, (\mathbb{1}_\mathcal{H} \otimes P'_\alpha)\, U\right]\right] = \\
= \big\langle\psi\big|\, \mathrm{Tr}_{\mathcal{H}'}\left[(\mathbb{1}_\mathcal{H} \otimes \rho_0)\, U^\dagger\, (\mathbb{1}_\mathcal{H} \otimes P'_\alpha)\, U\right]\, \psi\big\rangle
\end{aligned} \tag{1.284}$$

which is to say (note that F_α is positive and thus diagonalizable, such that all diagonal matrix elements uniquely determine F_α)

$$F_\alpha = \mathrm{Tr}_{\mathcal{H}'}\left[(\mathbb{1}_\mathcal{H} \otimes \rho_0)\, U^\dagger\, (\mathbb{1}_\mathcal{H} \otimes P'_\alpha)\, U\right] \tag{1.285}$$

and with $P_\alpha := U^\dagger\, (\mathbb{1}_\mathcal{H} \otimes P'_\alpha)\, U$ we get

$$\mathrm{Tr}_\mathcal{H}\left[F_\alpha\, \rho\right] = \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}'}\left[P_\alpha\, (\rho \otimes \rho_0)\right] \tag{1.286}$$

for all $\rho \in \mathcal{S}(\mathcal{H})$.

■

1.6.5 Are all Selfadjoint Operators Observables ?

The standard answer in the literature to the question in the title is 'yes' (apart from operators yielding transitions between different superselection sectors, which are neglected in the present work), which is often justified in one of the following ways:

In quantum theory of textbooks, but also in more advanced approaches like algebraic quantum field theory (AQFT), this is usually taken as a postulate. This is of course not satisfactory from the viewpoint of the present approach, where the operators are rather derived than postulated.

In the quantum theory of measurement, one usually puts a more substantiated argument forth to support an affirmative answer to the question in the title: Firstly, while in other contexts like AQFT one seems to have in mind the type of operators which were named 'observable operators' in the present work, in the quantum theory of measurement one usually addresses the

more general question, whether each POVM (which in particular includes all PVMs and thus observable operators) acting on some Hilbert space associated with a given quantum system corresponds to a real measurement (like) process. The standard argument for an affirmative answer ('all POVMs are observable') is that according to Naimark theorem each POVM can be implemented by an indirect measurement scheme, which in particular only requires a projective measurement on a larger system, and each projective measurement is implementable by the von Neumann scheme (see section 1.5.2), i.e. by coupling the associated observable operator to the conjugate momentum of some pointer observable in the Hamiltonian. Thus for a given quantum system, for any POVM acting on the associated Hilbert space there is an explicit recipe how to construct a measurement scheme involving at most two interactions (pre-measurement and the von Neumann measurement realizing the readout, or only a von Neumann measurement in case of a PVM) whose associated probabilities are encoded in this POVM.

Nonetheless, it might be questioned whether for each quantum system in nature and each arbitrary associated POVM always a probe and a pointer system can be found such that the desired couplings (the pre-measurement interaction and/or the von Neumann coupling) are implementable (at least in principle) by real world interactions[74]. In the present work we shall not presuppose this[75].

The other way around, one and the same observable operator or POVM can be often associated with very different experiments (see e.g. [127]) and very different state transformations can be associated with one and the same observable operator or POVM, as extensively shown in the present chapter. This shows that the question in the title has less physical depth than it suggests at a first glance. An observable operator or more generally a POVM only associates a family of probability distributions (one for each initial state) with the Hilbert space on which it acts in the first place, but it does not encode the set of state transformations associated with these probabilities unless the measurement is ideal and it does in general not represent physical properties of the measured system which are ascertained by an associated experiment as it was demonstrated by the Kochen-Specker-Bell theorems (see section 1.3).

We adopt the phrase *'E is measurable'* as abbreviation for the case that there exists a real world experiment such that the quantum statistics of one of its outcomes is associated with the minimal yes/no POVM $\{E\,;\,\mathbb{1}_{\mathcal{H}} - E\}$ (e.g. on the space $\Omega = \{1, 0\}$), which is always possible if E is an element of any POVM associated with such an experiment. Accordingly we call observable operators measurable if they are associated with a real world experiment (i.e. if all projections of its associated PVM are measurable with respect to one and the same experiment).

For later purposes, it is helpful to note the following: Consider a measurable effect $E \in \mathcal{B}(\mathcal{H})$ as an element of a POVM. As a positive bounded operator, E is selfadjoint such that we may consider its spectral representation and develop a projective measurement scheme which is associated with this effect as its observable operator. This projective measurement – if realizable

[74]For example, the authors in [127] conjecture that it might well be that no real world experiment measuring the observable operator $A = X^2P + PX^2$ exists, where X and P are the standard position and momentum operator, respectively.

[75]Actually, the more interesting fact is not that each POVM is implementable by some measurement scheme, but that according to Kraus second representation theorem 1.19 the same is true for all state transformers, which are much more fundamental objects encoding the associated POVMs (but not the other way around) *and* the transformations of the state of the measured system. That all such implementations correspond to real world processes is equally not presupposed in the present work, of course.

– will have of course in general nothing to do with the measurement from which we proceeded, in particular if E is not a projection[76]. If now E is measurable as an observable operator, we can always associate functions $f(E)$ with one and the same measurement, since the spectral representation $E = \int_{\sigma(E)} f(\lambda) dP_\lambda$ of E turns into $f(E) = \int_{\sigma(f(E))} f(\lambda) dP_\lambda$, i.e. the outcomes λ are just to be substituted by $f(\lambda)$. In later chapters repeatedly the question will arise, whether effect valued functions of a measurable effect must be in any obvious way measurable as well. For example, often the effect E^2 can be associated with the same measurement outcome two times in a row, if the measurement is immediately repeated, but only if the associated state transformers are normal operators (i.e. commute with their adjoints). But we shall see later that this is not the case for a huge class of possible measurements (e.g. for projective measurements of the second kind as can be easily verified, but in this case $E^2 = E$ anyway, such that E^2 is trivially measurable given E is measurable).

Indeed, if the measurement associated with E is not projective and the state transformers are not normal operators, there is no obvious way to infer the measurability of E^2 from the measurability of E as long as it is not presupposed that all effects are measurable in general (i.e. that each theoretical measurement scheme is realizable by some real world experiment). In this case, if E^2 is measurable as well, the respective experimental setup will be presumably very different from the experimental setup to measure E (this is similar to selfadjoint functions of non commuting measurable observable operators like $\sigma_x + \sigma_y$, which might be measurable as well but the experimental setup cannot be composed of the original measurement procedures but requires a totally different experimental device (see section 1.3)).

Measurable effects are supposed to be consistent with physically substantiated requirements on admissible empirical regularities (statistics of outcomes), such that the latter are consistent with physical principles which are supposed to be superior, like e.g. the relativity principle. Thereby we may infer physically substantiated mathematical requirements on measurable effects. For example, in section 2.3.2 we will consider some arbitrary measurable effect E which is thus supposed to be consistent with a given relativistic requirement. In order to prove that this has a certain desired consequence it will turn out that it must be assumed that together with E also E^2 is consistent with this requirement. But of course we have only justification to require E^2 to share this physically motivated property if together with E also E^2 is measurable (in this case locally measurable by the same experimenter), which is as indicated in the previous paragraph in general not obvious as long as it is not presupposed that all effects are measurable in general. We see that in cases like this it makes a considerable difference to obtain certain results whether it is assumed that all effects are measurable or not.

This becomes even more manifest in considering algebraic approaches to quantum theory like axiomatic or algebraic QFT (AQFT), certain results of which will be presented and discussed at the end of chapter 3. In AQFT one proceeds from considering (von Neumann–) algebras of operators acting on some Hilbert space or even from more abstract C^*-algebras from which Hilbert space representations are constructed by the GNS construction (see section 3.5 and references therein). It is then assumed that we can associate each given bounded spatial region – or relativistically more appropriately in the Heisenberg picture each bounded region of space-

[76] Indeed, given an experiment which is associated with a given observable operator, each projection in its spectral representation can be itself interpreted as an observable operator associated with the same measurement procedure, which takes the value 1 if the respective outcome is realized and 0 otherwise.

time – with such an algebra, such that all its selfadjoint elements correspond to measurements which can be performed locally in that region. If it is not presupposed that each selfadjoint operator is necessarily measurable like in the present work, this basic assumption cannot be adopted. But this does not mean that results from the framework of AQFT are then irrelevant, in particular if we assume that operators associated with local measurements are elements of such algebras as QFT suggests and as it is adopted in this work (we shall propose to generate local algebras by the set of state transformers associated with local measurements).

For example, the Reeh-Schlieder theorem which is a fundamental result in AQFT and will be proven in section 3.5 is a somewhat surprising assertion about the existence of at least one element with a certain striking property in each local algebra. Within the present approach this element (or these elements) need not be measurable and accordingly the theorem need not be of physical relevance but might be perceived as a rather technical result. But from the Reeh-Schlieder theorem together with local commutativity (investigated in section 2) a corollary easily follows which is a strong assertion about each element of a local algebra, such that this result is to be taken physically seriously even if it is not presupposed that the totality of the selfadjoint part of each local algebra is observable (measurable).

1.6.6 Remark on Decoherence and the Measurement Problem

Unitary interaction of a quantum system with Hilbert space $\mathcal{H}_S$ with an environment with Hilbert space $\mathcal{H}_E$ of the form

$$\psi\varphi_0 \xrightarrow{U} \sum_k c_k \psi_k \varphi_k \tag{1.287}$$

with $\psi_k \in \mathcal{H}_S$ and mutually orthogonal environment states $\varphi_k \in \mathcal{H}_E$ (like it is the case for pointer like states) yields the system density operator

$$\begin{aligned} \rho_S = \mathrm{Tr}_{\mathcal{H}_E}\left[\left|\sum_k c_k\psi_k\varphi_k\right\rangle\left\langle\sum_l c_l\psi_l\varphi_l\right|\right] = \\ = \sum_{k,l,m} c_k\bar{c}_l|\psi_k\rangle\langle\psi_l|\langle\varphi_m|\,\varphi_k\rangle\langle\varphi_l|\,\varphi_m\rangle = \sum_k |c_k|^2\,|\psi_k\rangle\langle\psi_k| \end{aligned} \tag{1.288}$$

where the environment is traced out (in the second line the partial trace is performed in an ONB containing the φ_ks). Obviously, ρ_S looks like the density operator of the ensemble $\{(\psi_k\,;\,p_k)\}$ weighted with the probabilities $p_k = |c_k|^2$ and is thus identical to the final unconditional density operator $\rho = \sum_k \mathbb{P}^\psi(k)\,|\psi_k\rangle\langle\psi_k|$ of the non selective measurement (like) process

$$\psi\phi_0 \xrightarrow{U} \sum_k c_k\psi_k\phi_k \xrightarrow{(*)} \psi_l\phi_l \tag{1.289}$$

with pointer (like) states ϕ_k, where the last transition $(*)$ happens with probability $\mathbb{P}^\psi(l) = |c_l|^2$.

Thus the final unconditional density operator of a non-selective measurement (like) process based on collapse dynamics and averaging over all possible final states, can be always equally obtained by purely unitary interaction with the apparatus (perceived as the environment of the measured system) without resorting to collapse, by performing the partial trace. Nonetheless,

without collapse the global final state $\sum_k c_k \psi_k \phi_k$ of measured system plus apparatus is trivially still a superposition of different pointer positions, of course, in contradiction to experience.

Decoherence theory now takes the above argument one step further by taking the environment of the pointer into account: In realistic measurements, the pointer – or more generally a macroscopic readout as any macroscopic object – will of course massively interact with its environment, e.g. with air molecules and photons around. The air molecules scattered by the pointer will in turn interact with other air molecules, the walls of the laboratory etc., the photons with the retina of the experimenters eye and so on, such that more and more systems get involved, leading to very different and complex environmental configurations, depending on the position of the pointer (outcome value). This has as a consequence that the states of the environment developing from different pointer positions get effectively mutually orthogonal extremely fast[77] and irreversibly. The irreversibility derives from the enormous number of degrees of freedom involved in the environment, which lead to a tremendous separation of the (effective) supports of the respective wave functions in configuration space, which will be practically impossible to bring to an overlap again forever [192].

So let us supplement the process (1.289) without the collapse dynamics $(*)$ in the last step by the respective environment states ξ_0 and mutually orthogonal $\{\xi_k\}$ in the environmental Hilbert space $\mathcal{H}_E$:

$$\psi \phi_0 \xi_0 \xrightarrow{U} \sum_k c_k \psi_k \phi_k \xi_k \tag{1.290}$$

Of course, realistically the initial environment state ξ_0 will be different for each individual process and in consequence the final ξ_ks as well (this is actually the case for pointer states as well). But the only relevant feature is the mutual (irreversible) orthogonality of the ξ_ks, even if they differ in each individual process (actually, the central argument can also be generalized to the case where environment and pointer do not even form a product state). If we trace out now the unknown environment states, we get the effective density operator of measured system plus apparatus

$$\rho_{S+A} = \mathrm{Tr}_{\mathcal{H}_E} \left[\left| \sum_k c_k \psi_k \phi_k \xi_k \right\rangle \left\langle \sum_l c_l \psi_l \phi_l \xi_l \right| \right] = \sum_k |c_k|^2 \, |\psi_k\rangle \langle \psi_k| \otimes |\phi_k\rangle \langle \phi_k| \tag{1.291}$$

which looks now like the density operator of the ensemble $\{(\psi_k \phi_k \, ; \, p_k)\}$ of the final system states ψ_k in product with the respective pointer states ϕ_k of the measurement like process (1.289), weighted with the respective probabilities $p_k = |c_k|^2$.

Now often people (essentially) say (e.g. [245, 299]) that this means that due to the environment, the superposition of different pointer states has become a classical statistical mixture of these states without need of an extra collapse postulate and consequently, Schrödingers cat – as a very dramatic representative of a pointer – is no longer dead *and* alive like in a superposition, but has become either dead *or* alive only due to interaction with its environment. This claim is,

[77] Calculations within simple models (see e.g. Caldeira and Leggett [78], Joos and Zeh [199, 198], Unruh and Zurek [329, 352]) predict time scales for these processes, which are much smaller than any other time scales which might be relevant for the considered systems: To get an impression, e.g. a dust grain sized pointer of 1μm radius can be expected to cause the environment states of a typical density of air molecules to become orthogonal within 10^{-30} sec if it is displaced by 1mm [245].

to say the least, inconsiderate[78], the true global state $\sum_k c_k \psi_k \phi_k \xi_k$ is trivially still a superposition of different pointer states, of course. Others state a bit more carefully something like the following: For all future observations on measured system and apparatus subsequent to the evolution (1.290), the density operator (1.291) provides the right predictions and these predictions are thus the same as the analogous predictions for observations on members of the ensemble $\{(\psi_k \phi_k \, ; \, p_k)\}$ of definite pointer states, such that both are empirically indistinguishable, i.e. system and apparatus are without collapse not identical to – but operationally indistinguishable (which actually amounts for positivists to the same thing as 'identical') from the collapsed ensemble, such that we can fapp (*for all practical purposes*) pretend that collapse occurred even if the actual global dynamics was linear and unitary.

But crucial the point is that the linear quantum channels do not describe the world we live in properly: Measurement (like) processes create facts like definite outcomes (pointer points onto the value 'X') and the associated transition of the form

$$\sum_k c_k \Psi_k \longrightarrow \Psi_l \tag{1.292}$$

where now the Ψ_ks may represent the measured system or the pointer states or everything which might be relevant (system, apparatus, environment, the rest of the universe...), *cannot be linear and cannot be unitary*. This fact would not change if mixed states were regarded as fundamental (e.g. the universe was fundamentally in a mixed state) as some people do, whenever we describe quantum transitions (possibly only in terms of mixed states) from a state representing several pointer positions to a state representing a definite outcome. This easily follows from the fact that quantum channels are not appropriate to describe measurements with definite outcomes and the necessity to introduce for that purpose on the density operator level the notion *instrument* (or *state transformation valued measure*, see definition 1.13), which amounts to non-linear, non-unitary and stochastic state transformations.

Consequently, if we want to hold on to only unitary dynamics without introducing a second dynamical principle (collapse) or something else besides the wave function (e.g. particles with positions like in Bohmian mechanics), we are inevitably dealing with a many worlds type theory and decoherence contributes nothing to attenuate this fact (but only to destroy interference of the different 'worlds' or branches at macroscopic scales). It is worth noting, that Dieter Zeh, the father of decoherence theory, is very clear about this (see e.g. [350])!

An understanding of decoherence processes is nonetheless of great importance, since they describe the irreversible loss of quantum interference of macroscopic objects (often formalized by the vanishing of the off-diagonal elements (*coherences*) of the effective density operator (1.291) in the pointer basis). Understanding the stable absence of quantum interference in the realm of macroscopic objects is for sure one of the very important building blocks of an understanding of how the classical world emerges when its constituents are guided by quantum theory, but it cannot be the only one.

[78] It is worth noting that according to the unitary state mixing corollary 1.15 a mixed state of the form $\rho = \sum_k p_k \, |\psi_k\rangle \langle\psi_k|$ can – but in no way need to represent an ensemble $\{(\psi_k \, , \, p_k)\}$ which does not contain superpositions of different ψ_ks. For example the density operator $\rho = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)$ can be obtained from the state ensemble $\left\{(|\uparrow\rangle, \frac{1}{2}) \; ; \; (|\downarrow\rangle, \frac{1}{2})\right\}$, but is equally the density operator of the state ensemble $\left\{\left(\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle), \frac{1}{2}\right) \; ; \; \left(\frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle), \frac{1}{2}\right)\right\}$.

2 Local Quantum Measurement I Relativistic Consistency, No-Signalling and Local Commutativity

Now we come to relativistic quantum theory, but we will not specify a particular quantum theory like $N-$particle Dirac theory or QED or another QFT. We just assume that quantum mechanical systems are described in the usual way by states ψ which are vectors in a Hilbert space $\mathcal{H}$ which carries a unitary representation $U(\Lambda, a)$ of the Poincaré group $\{\Lambda, a\}$, where Λ are Lorentz transformations and $a \in \mathbb{R}^4$ are space-time translations. During this chapter we will focus on the subgroup of Lorentz boosts, in which case we simply denote the corresponding unitary operators by $U(\Lambda)$ (where Λ is always only a boost), whereas in the following chapter 3 we will focus on the space-time translation subgroup and denote the associated unitary operators by $U(a)$.

2.1 Quantum Measurements and Relativity

In the following sections, we will investigate step by step, how to deal with the tension between the fact, that spacelike separated events lack a distinguished time order in relativity and the fact, that quantum measurements are invasive on the measured system in a – in general – nonlocal way.

A Single Quantum Measurement and the Relativity Principle

What we call *a single quantum measurement* in the present context might well refer to a plenty (an ensemble) of quantum measurements with identical initial states and experimental setups, whose relative frequencies of outcomes are predicted by quantum theory. The notion of a 'single' measurement shall rather express, loosely speaking, that in each single run the experimental device is macroscopically well localized, in contrast to composite quantum measurements, where two or more (possibly different) measuring devices are triggered (or not) by the measured system in different spatial regions (possibly at spacelike separation) like in EPR type experiments. Such scenarios we shall consider later, but first consider a non-composite (single) quantum measurement in a laboratory (frame):

Let $\psi \in \mathcal{H}$ be the initial state in the laboratory frame Σ. We denote the quantity associated with the measurement (the 'measured quantity') in the laboratory frame by A, which need not be associated with an observable operator. According to quantum measurement formalism developed in the previous chapter, the probability for outcome $A = \alpha$ is given by an effect E^A_α via[79]

$$\mathbb{P}^\psi (A = \alpha) = \left\langle \psi \right| E^A_\alpha \left| \psi \right\rangle \tag{2.1}$$

[79] In the continuous case we may think of α as something like an interval, where we actually mean $A \in \alpha$ when we write $A = \alpha$. With respect to the state transformers this is not so straightforward (see section 1.5.2), thus whenever state transformers are involved we should have discrete measurements in mind. All results in this chapter are nonetheless general enough if we acknowledge that finally any measurement can only have a countable set of possible outcomes (see section 1.5.2) and any state transformation has a Kraus representation with an operator sum over some discrete index set.

The relativity principle demands that the laws of nature are invariant under Lorentz boosts, which implies that a relativistic theory makes identical predictions for outcomes of an experiment, if performed in a boosted frame of reference. Here it is important, of course, that the complete experimental setup – including in particular the preparation device and by that the initial state of the measured system and the measuring device – is boosted. If only a part of the experimental setup is boosted (the initial state or the measuring device), the predictions will differ from the original experiment, of course. We shall assume that not only definite predictions for individual systems but also probabilistic predictions for relative frequencies of outcomes of experiments on ensembles of identically prepared systems must be invariant under Lorentz boosts, which is standing to reason and will be further substantiated below.

For operational predictions of relativistic quantum theory, this invariance is guaranteed by the unitarity of the operators $U(\Lambda)$: Suppose the measurement procedure from above is boosted into another Lorentz frame Σ', moving with respect to the original laboratory frame Σ (we may illustrate this by packing the whole laboratory with all its constituents into a starship, moving with high velocity with respect to the original laboratory space). The preparation device of the initial state is boosted from the point of view of Σ by the Lorentz boost Λ which connects Σ' and Σ by $\Sigma' = \Lambda\Sigma$, such that the initial state of the experiment – which is performed now in Σ' – calculated in Σ is given by

$$\psi' = U(\Lambda)\psi \tag{2.2}$$

Moreover, also the measuring device is boosted with respect to Σ, such that the effect E^A_α associated with outcome $A = \alpha$ must be transformed according to[80]

$$E^{A'}_\alpha = U(\Lambda) E^A_\alpha U^{-1}(\Lambda) \tag{2.5}$$

Consequently, we can calculate the probability to obtain outcome $A = \alpha$ from the viewpoint of frame Σ:

$$\begin{aligned} \mathbb{P}^{\psi'}(A' = \alpha) = \left\langle \psi' \right| E^{A'}_\alpha \left| \psi' \right\rangle = \left\langle \psi \right| U^{-1}(\Lambda)\, U(\Lambda) E^A_\alpha U^{-1}(\Lambda)\, U(\Lambda) \left| \psi \right\rangle = \\ = \left\langle \psi \right| E^A_\alpha \left| \psi \right\rangle = \mathbb{P}^{\psi}(A = \alpha) \end{aligned} \tag{2.6}$$

which is the same as the probability for outcome $A = \alpha$ calculated for the original experiment performed in the original laboratory frame Σ (or calculated for the boosted experiment in Σ'). To summarize, we will make the same predictions for the outcomes if we describe the experiment in a moving frame, as it must be the case if both frames shall agree on each single actual outcome ('pointer points onto α').

We may also view this the other way around: Suppose an experiment A is performed in some laboratory with associated laboratory frame Σ' in which the initial state is given by $\psi \in \mathcal{H}$ and

[80] Equation 2.5 may be motivated by considering the spectral representation

$$E^A_\alpha = \sum_\lambda \lambda \left| \varphi_\lambda \right\rangle \left\langle \varphi_\lambda \right| \tag{2.3}$$

(for simplicity discrete and non degenerate), which is boosted by transforming the eigenstates via

$$\varphi_\lambda \to {\varphi_\lambda}' = U(\Lambda)\, \varphi_\lambda \tag{2.4}$$

the effect associated with outcome $A = \alpha$ is given by E^A_α, i.e. we calculate in Σ' the probability $\mathbb{P}^\psi(A = \alpha) = \langle \psi | E^A_\alpha \psi \rangle$. From the viewpoint of any other Lorentz frame $\Sigma \neq \Sigma'$, the laboratory frame is moving, i.e. Σ and Σ' are connected by some boost Λ via $\Sigma = \Lambda\Sigma'$. Hence, according to equation (2.6), in all frames Σ we will calculate the same probability for outcome α of the experiment performed in Σ', which is the same as if it was performed in Σ, respectively. I.e. we have

$$\mathbb{P}^{\psi'}(A' = \alpha) = \mathbb{P}^\psi(A = \alpha) \tag{2.7}$$

where the left hand side $\mathbb{P}^{\psi'}(A' = \alpha) = \langle \psi' | E^{A'}_\alpha \psi' \rangle$ is the probability for outcome α of the experiment performed in Σ' calculated in (an arbitrary) frame Σ.

What is – as just argued – trivially fulfilled in relativistic quantum theory in case of a single measurement by unitarity of the representatives of boosts on Hilbert space (in the above described sense) is also a trivial necessity under physical considerations; namely it is necessary for the *(relativistic) consistency* of the theory: Displayed facts like 'pointer points onto α in the laboratory' must remain the same displayed facts with respect to a moving frame. Consequently, in the framework of a given (relativistic) theory, the predicted statistics for the outcomes of ensembles of identical experiments in some laboratory must coincide, no matter if evaluated in the laboratory frame or in a moving frame. Otherwise – given these statistics yield the correct predictions for empirical relative frequencies – there would be cases in which 'the pointer points onto α' in the laboratory frame whereas 'the pointer points onto $\alpha' \neq \alpha$ at the same time' from a relatively moving perspective. This is so, trivially because agreement of outcomes in each individual run (from perspectives of different frames) implies identical relative frequencies and consequently, non identical relative frequencies would imply the existence of individual runs with inconsistent outcomes. Hence, a consistent (relativistic) theory must predict identical probabilities for the outcomes of a given experiment performed in a given laboratory (frame) from the perspectives of all frames of reference.

The alert reader may have noticed that this kind of consistency is by no means an exceptionalism of relativistic theories. E.g. in a non Lorentzian ether theory, identical experiments would have in general different outcomes if performed in different frames of reference. But, of course, also such a theory should make the same predictions for a given experiment performed in a given frame form the viewpoint of each other frame, if it is ambitious to be a halfway serious theory. Only in this case, the calculations would involve the relative velocities of the involved frames with respect to a distinguished frame of reference.

Our consistency criterion is simply the demand that we should agree about facts (like 'pointer points onto α') from relatively moving perspectives. All we have ascertained above is that, in the case of a single quantum experiment, this trivial demand is an immediate consequence of the assumption that the Hilbert space $\mathcal{H}$ carries a *unitary* representation of the Poincaré group.

Composite Spacelike Separated Experiments

We will encounter now, that if we consider composite experiments, where two or more measurements are spacelike related, unitary representation of the Poincaré group is no longer sufficient for consistency of measurement outcomes with respect to the different perspectives of relatively moving Lorentz frames (and thereby to maintain the relativity principle). This is due to

- the general dependence of joint distributions of outcomes of composite experiments on the time order of the respective measurements in quantum theory, together with
- the inherent nonlocality of quantum theory

We shall briefly illustrate these two points:

Time Order

Given two successive quantum measurements, we can always find a joint distribution of the outcomes, but this joint distribution in general depends on the time ordering of the measurements. We shall briefly demonstrate this by the example of ideal spin measurements and adopt for this reason the formalism of spin$-\frac{1}{2}-$systems for simplicity from non relativistic quantum theory. The relativistic generalization is straight forward and does not change the conclusions.

Let $\mathcal{H} \cong \mathbb{C}^2$ and σ_i with $i = x, y, z$ be the Pauli matrices with the eigenstates defined by the relations $\sigma_i \,|\, \uparrow_i\rangle = +1|\, \uparrow_i\rangle$ and $\sigma_i \,|\, \downarrow_i\rangle = -1|\, \downarrow_i\rangle$, respectively. The PVM associated with an ideal measurement of the i'th component of spin is given by the two projections $P^i_\uparrow = |\uparrow_i\rangle\langle\uparrow_i|$ and $P^i_\downarrow = |\downarrow_i\rangle\langle\downarrow_i|$, respectively.

One can ask now for joint probabilities, e.g. the probability $\mathbb{P}^\psi(\sigma_x = +1 \wedge \sigma_z = -1)$ that a successive measurement will find *spin up* for the $x-$component of spin and *spin down* for the $z-$component of spin if $\psi \in \mathcal{H}$ is the initial state. We shall demonstrate now the well known fact that expressions like $\mathbb{P}^\psi(\sigma_x = +1 \wedge \sigma_z = -1)$ are indeed in general underdetermined and only well defined if the time order of the successive measurements is given. To this end, we utilize the always well defined conditional probabilities like $\mathbb{P}^\psi\,(\sigma_z = -1 \mid \sigma_x = +1)$: The probability to obtain $\sigma_z = -1$, given a preceding measurement on initial state $\psi \in \mathcal{H}$ yielded $\sigma_x = +1$ (and given the free time evolution between the two measurements can be neglected, which we shall always assume), is obviously given by the probability to obtain $\sigma_z = -1$ if the final state of the first measurement $\psi^x_\uparrow := \frac{P^x_\uparrow \psi}{\|P^x_\uparrow \psi\|}$ is the initial state, i.e.

$$\mathbb{P}^\psi\,(\sigma_z = -1 \mid \sigma_x = +1) = \mathbb{P}^{\psi^x_\uparrow}(\sigma_z = -1) = \frac{\left\langle P^x_\uparrow \psi\right|\, P^z_\downarrow P^x_\uparrow \psi\right\rangle}{\left\langle P^x_\uparrow \psi\right|\, P^x_\uparrow \psi\right\rangle} = \frac{\left\langle \psi\right|\, P^x_\uparrow P^z_\downarrow P^x_\uparrow \psi\right\rangle}{\left\langle \psi\right|\, P^x_\uparrow \psi\right\rangle} \tag{2.8}$$

These conditional probabilities suggest to introduce the notation

$$\mathbb{P}^\psi(\sigma_x = +1 \overrightarrow{\wedge}\, \sigma_z = -1) = \mathbb{P}^\psi\,(\sigma_z = -1 \mid \sigma_x = +1)\ \mathbb{P}^\psi\,(\sigma_x = +1) \tag{2.9}$$

for the joint probability $\mathbb{P}^\psi(\sigma_x = +1 \wedge \sigma_z = -1)$ given the measurement of the $x-$component comes first, and

$$\mathbb{P}^\psi(\sigma_x = +1 \overleftarrow{\wedge}\, \sigma_z = -1) = \mathbb{P}^\psi\,(\sigma_x = +1 \mid \sigma_z = -1)\ \mathbb{P}^\psi\,(\sigma_z = -1) \tag{2.10}$$

given the measurement of the $z-$component precedes the $x-$measurement.

In order to calculate the probabilities (2.9) and (2.10) for some given initial state, say $\psi = |\uparrow_x\rangle$, recall the relations $|\uparrow_x\rangle = \frac{1}{\sqrt{2}}\,(|\uparrow_z\rangle + |\downarrow_z\rangle)$ and $|\downarrow_z\rangle = \frac{1}{\sqrt{2}}\,(|\uparrow_x\rangle - |\downarrow_x\rangle)$. Equation (2.9)

now yields

$$\begin{aligned}\mathbb{P}^{\psi}(\sigma_x = +1 \overrightarrow{\wedge} \sigma_z = -1) &= \mathbb{P}^{\psi}\left(\sigma_z = -1 \mid \sigma_x = +1\right) \mathbb{P}^{\psi}\left(\sigma_x = +1\right) = \\ &= \frac{\left\langle \psi \right| P^x_{\uparrow} P^z_{\downarrow} P^x_{\uparrow} \left| \psi \right\rangle}{\left\langle \psi \right| P^x_{\uparrow} \left| \psi \right\rangle} \left\langle \psi \right| P^x_{\uparrow} \left| \psi \right\rangle = \langle \uparrow_x | P^x_{\uparrow} P^z_{\downarrow} P^x_{\uparrow} | \uparrow_x \rangle = \langle \uparrow_x | P^z_{\downarrow} | \uparrow_x \rangle = \\ &= \frac{1}{2} \left(\langle \uparrow_z | + \langle \downarrow_z | \right) P^z_{\downarrow} \left(| \uparrow_z \rangle + | \downarrow_z \rangle \right) = \frac{1}{2}\end{aligned} \tag{2.11}$$

On the other hand, equation (2.10) yields

$$\begin{aligned}\mathbb{P}^{\psi}(\sigma_x = +1 \overleftarrow{\wedge} \sigma_z = -1) &= \mathbb{P}^{\psi}\left(\sigma_x = +1 \mid \sigma_z = -1\right) \mathbb{P}^{\psi}\left(\sigma_z = -1\right) = \\ &= \frac{\left\langle \psi \right| P^z_{\downarrow} P^x_{\uparrow} P^z_{\downarrow} \left| \psi \right\rangle}{\left\langle \psi \right| P^z_{\downarrow} \left| \psi \right\rangle} \left\langle \psi \right| P^z_{\downarrow} \left| \psi \right\rangle = \langle \uparrow_x | P^z_{\downarrow} P^x_{\uparrow} P^z_{\downarrow} | \uparrow_x \rangle = \\ &= \frac{1}{2} \left(\langle \uparrow_z | + \langle \downarrow_z | \right) P^z_{\downarrow} P^x_{\uparrow} P^z_{\downarrow} \left(| \uparrow_z \rangle + | \downarrow_z \rangle \right) = \frac{1}{2} \langle \downarrow_z | P^x_{\uparrow} | \downarrow_z \rangle = \\ &= \frac{1}{4} \left(\langle \uparrow_x | - \langle \downarrow_x | \right) P^x_{\uparrow} \left(| \uparrow_x \rangle - | \downarrow_x \rangle \right) = \frac{1}{4}\end{aligned} \tag{2.12}$$

such that

$$\mathbb{P}^{\psi}(\sigma_x = +1 \overrightarrow{\wedge} \sigma_z = -1) \neq \mathbb{P}^{\psi}(\sigma_x = +1 \overleftarrow{\wedge} \sigma_z = -1) \tag{2.13}$$

Of course, we know that this is strongly related with the non commutativity of σ_x and σ_z, a relation which we will carefully analyse below.

The fact that joint probability distributions of outcomes of successive quantum measurements depend in general on the time order of the measurements (the common diction is that such observables – in the present case σ_x and σ_z – are not *jointly measurable*), is responsible for the alleged *non classicality* of the quantum probabilities, e.g.

$$\mathbb{P}^{\psi}\left(\sigma_z = -1 \mid \sigma_x = +1\right) \mathbb{P}^{\psi}\left(\sigma_x = +1\right) \neq \mathbb{P}^{\psi}\left(\sigma_x = +1 \mid \sigma_z = -1\right) \mathbb{P}^{\psi}\left(\sigma_z = -1\right) \tag{2.14}$$

In usual probability theory, we would expect that the left and right hand side of (2.14) coincide and are given by $\mathbb{P}^{\psi}(\sigma_x = +1 \wedge \sigma_z = -1) = \mathbb{P}^{\psi}(\sigma_z = -1 \wedge \sigma_x = +1)$ (the absent symmetry (2.14) of joint probabilities would be the crucial ingredient to derive Bayes' law, which is thus in general violated in quantum theory). But note that this 'non classicality' is a pretty trivial fact, if we only acknowledge that measurements change the states of the measured system in a way, such that a change of time order of measurements changes in general the joint probabilities of the outcomes[81].

[81] This observation can be refined by noting that the outcome distribution of some measurement associated with a given observable operator is in general also affected by previous measurements of observables commuting with the latter, given the outcomes of the former measurements are taken into account. This has as a consequence that one cannot in general consistently attribute values to a given observable independently of the set of commuting observables with which it is measured (in particular, if the operators of different sets do not commute with each other but only with the considered observable operator). This is the root of the Kochen-Specker-Bell no-go theorems which were discussed in section 1.3.

Nonlocality

Consider now two quantum measurements performed in spacelike separated regions of space-time, e.g. two spatially separated measurements which are performed almost simultaneously in some laboratory (frame). If the two measured systems are entangled, the quantum probabilities for the outcomes of, say the left hand measurement in general dramatically change, if the right measurement was performed and yielded a given outcome. This is probably most instructively illustrated by the perfect correlations in Bohm's version of the EPR experiment:

Let $\mathcal{H} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ be the Hilbert space of two spin$-\frac{1}{2}-$particles and suppose these particles are prepared in the singlet state and subsequently separated from each other in space. The singlet state is given by $\psi = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, where we make the choice $|\uparrow\downarrow\rangle = |\uparrow_z\rangle \otimes |\downarrow_z\rangle$ and $|\downarrow\uparrow\rangle = |\downarrow_z\rangle \otimes |\uparrow_z\rangle$ (the singlet state is actually form invariant under change of ONB) and the fist ket of the tensor product belongs, say to the left hand particle and the second ket to the particle on the right hand side, respectively. If now an ideal measurement of the $z-$component of the spin of the particle on the right hand side finds 'spin down', the singlet state collapses to the state $|\uparrow\downarrow\rangle$ and a (potential) subsequent measurement of the $z-$component of the spin on the left hand side would find there 'spin up' with certainty. But before the measurement on the right hand side was performed, the quantum probability to find on the left hand side 'spin up' in a (potential) measurement was given by $\mathbb{P}^{\psi}(\uparrow_{\text{left}}) = \|(P^z_{\uparrow} \otimes \mathbb{1})\,\psi\|^2 = \frac{1}{2}$.

So the fact that the outcome of a (potential) measurement on the left hand side is fixed after the measurement on the right had outcome 'spin down' (or 'spin up') is not displayed by the quantum predictions prior to the measurement on the right. Consequently, there are two options on the table: Either the outcome of the (potential) measurement on the left was already fixed before the measurement on the right, and the related quantum probability $\frac{1}{2}$ does only reflect our ignorance about this fact, or the measurement on the right instantaneously fixes the (potential) result on the left. Bells theorem (a version of which for an entangled three particle initial state we have proven in section 1.3) now rules out all possible patterns of explanation of the first type, which rely on the assumption, that the sudden change in the probability (or strong dependence on the remote measurement) is only apparent and may e.g. be traced back to any facts (common causes) which happened in the joint causal past of the two measurement events [29, 27, 24].

Thus we have no reason to expect that the influence a measurement has on the distribution of outcomes of a subsequent measurement vanishes, if the two measurements are performed in regions of space-time which are spacelike separated from each other. Of course, the term 'subsequent measurement' is in this case not clearly defined in relativistic space-time, since the time order of spacelike separated events depends on the Lorentz frame of reference. This has as a consequence, that the (in our example even perfect) correlations between such measurements, which are not – according to Bell's theorem – locally explicable (e.g. by common causes in the common past of the two events), cannot be described as a causal process which can be uniquely decomposed into *cause* and *effect* [24]. This strange feature of such causal processes, which is a fact of nature though, takes admittedly much getting used to, but it is not – a priori – inconsistent. In order to guarantee consistency, we need only to require, that the joint distributions of outcomes of spacelike separated measurements do not depend on the time order of the respective measurements. If this was not the case, observers in relatively moving frames

would in general observe contradictory results of a given measurement performed in a given rest frame. As we shall see below, violation of this consistency requirement would also make it possible for an experimenter to convey superluminal signals by her decision to perform a given measurement or not. We will investigate the consistency requirement and its consequences in the following more deeply.

2.2 Relativistic Consistency and No-Signalling

Preliminary Remarks

For the rest of this work, we will assume that operators associated with measurements can be associated with the respective regions of space-time in which the respective measurements are performed (we switch in the following to the Heisenberg picture, where operators obtain a natural time dependence). In a satisfactory relativistic theory this must be possible, not least since – as we will see – there are important relations between operators associated with different measurements which crucially depend on the respective regions of space-time in which the respective measurements are performed. We shall not bother with the technical details in this chapter, how this association of operators with regions of space-time is precisely established, we just loosely assume that this is done in one way or the other[82]. How this is usually done in relativistic QFT[83], (in standard textbook-, as well as axiomatic/algebraic approaches) will be discussed in section 3.5.

For the first part of this chapter, we only consider efficient measurements transforming pure states to pure states, such that the analysis is feasible on the wave function level only. We will translate the results obtained to state transformations upon non-efficient measurements at the end of this chapter.

Measurement Scheme

Consider an experiment, which consists of two consecutive quantum measurements with associated physical quantities A and B, respectively, in the laboratory frame Σ. For each realizable (i.e. with related non zero probability) pair of outcomes $A = \alpha$ and $B = \beta$, the quantum description of this process involves an initial state ψ in some Hilbert space $\mathcal{H}$, two state transformers $\mathcal{R}^A_\alpha$ and $\mathcal{R}^B_\beta$ and two effects $E^A_\alpha = (\mathcal{R}^A_\alpha)^\dagger \mathcal{R}^A_\alpha$ and $E^B_\beta = (\mathcal{R}^B_\beta)^\dagger \mathcal{R}^B_\beta$ acting on $\mathcal{H}$: For initial state $\psi \in \mathcal{H}$ the $A-$measurement has outcome $A = \alpha$ with probability

$$\mathbb{P}^\psi(A = \alpha) = \left\langle \psi \right| E^A_\alpha \left| \psi \right\rangle \tag{2.15}$$

[82]In addition we will assume that the unitary representation of space-time translations acting on the respective Hilbert space, acts naturally on these 'local' operators (which we will extensively use in the next chapter) in the Heisenberg picture, i.e. if a measurement associated with a given space-time region is performed in a translated space-time region, the operators associated with the transformed measurement are simply given by the action of the related space-time translation on the operators associated with the original experiment.

[83]In non relativistic quantum theory, if association of operators (which are non functions of the position operator) with spatial regions is required, this is usually done by an ad hoc ansatz: For example in an EPRB experiment we have two spatially separated σ_z-measurements on the two particles of the singlet state, respectively, and the associated operators σ^1_z and σ^2_z acting on the spinor part of the wave functions are simply required to act on wave functions whose spatial part has (essentially) support in the respective spatial regions.

in which case the resulting state is given by

$$\psi_\alpha := \frac{\mathcal{R}^A_\alpha \psi}{\|\mathcal{R}^A_\alpha \psi\|} \tag{2.16}$$

which in turn is the initial state of the $B-$measurement (as always, we assume that the free time evolution in between the measurements can be neglected). The $B-$measurement has outcome $B = \beta$ with probability

$$\mathbb{P}^{\psi_\alpha}(B = \beta) = \left\langle \psi_\alpha \right| E^B_\beta \left| \psi_\alpha \right\rangle \tag{2.17}$$

in which case the final state is given by

$$\psi_{\alpha\beta} := \frac{\mathcal{R}^B_\beta \mathcal{R}^A_\alpha \psi}{\|\mathcal{R}^B_\beta \mathcal{R}^A_\alpha \psi\|} \tag{2.18}$$

The associated joint probability $\mathbb{P}^\psi(A = \alpha \overrightarrow{\wedge} B = \beta)$ (again the time order is indicated by the symbol $\overrightarrow{\wedge}$) is given by

$$\begin{aligned} \mathbb{P}^\psi(A = \alpha \overrightarrow{\wedge} B = \beta) &= \mathbb{P}^\psi\left(B = \beta \mid A = \alpha\right) \mathbb{P}^\psi\left(A = \alpha\right) = \mathbb{P}^{\psi_\alpha}(B = \beta)\, \mathbb{P}^\psi\left(A = \alpha\right) = \\ &= \frac{\left\langle \psi \right| (\mathcal{R}^A_\alpha)^\dagger E^B_\beta \mathcal{R}^A_\alpha \left| \psi \right\rangle}{\|\mathcal{R}^A_\alpha \psi\|^2} \left\langle \psi \right| E^A_\alpha \left| \psi \right\rangle = \frac{\left\langle \psi \right| (\mathcal{R}^A_\alpha)^\dagger E^B_\beta \mathcal{R}^A_\alpha \left| \psi \right\rangle}{\left\langle \psi \right| (\mathcal{R}^A_\alpha)^\dagger \mathcal{R}^A_\alpha \left| \psi \right\rangle} \left\langle \psi \right| E^A_\alpha \left| \psi \right\rangle = \\ &= \left\langle \psi \right| (\mathcal{R}^A_\alpha)^\dagger E^B_\beta \mathcal{R}^A_\alpha \left| \psi \right\rangle \end{aligned} \tag{2.19}$$

If now the $A-$measurement and the $B-$measurement are performed in regions of space-time which are spacelike separated from each other, we can describe the history of this experiment from the viewpoint of a Lorentz frame Σ' in which the time order of the two measurements is reversed, i.e. the $B'-$measurement precedes the $A'-$measurement: If Λ is the Lorentz transformation connecting Σ and Σ' by $\Sigma' = \Lambda\Sigma$, from the viewpoint of Σ' the initial state of the experiment is now given by $\psi' = U(\Lambda)\psi$ and conditional on outcome $B' = \beta$, which happens with probability[84]

$$\mathbb{P}^{\psi'}(B' = \beta) = \mathbb{P}^\psi(B = \beta) = \left\langle \psi \right| E^B_\beta \left| \psi \right\rangle \tag{2.20}$$

the new initial state of the subsequent $A'-$measurement is now given by

$$\psi'_\beta := \frac{\mathcal{R}^{B'}_\beta \psi'}{\|\mathcal{R}^{B'}_\beta \psi'\|} = \frac{U(\Lambda)\mathcal{R}^B_\beta U^\dagger(\Lambda) U(\Lambda)\, \psi}{\|U(\Lambda)\mathcal{R}^B_\beta U^\dagger(\Lambda) U(\Lambda)\, \psi\|} = U(\Lambda) \frac{\mathcal{R}^B_\beta\, \psi}{\|\mathcal{R}^B_\beta\, \psi\|} =: U(\Lambda)\psi_\beta \tag{2.21}$$

[84] At this point in the derivation leading to calculation (2.26), there is a subtle point noteworthy: Consistency relation (2.7), which was actually derived for a single measurement from the relativity principle, can be applied here separately to the component parts of the composite measurement (equation (2.20) as well as (2.23) below), because in each frame the predicted statistics of measurements must be independent of any other measurements which are performed afterwards (or not) but only depend on the initial state. So the identity $\mathbb{P}^{\psi'}(B' = \beta) \equiv \mathbb{P}^\psi(B = \beta)$ in equation (2.20) might be seen now as a formal identity of numbers which must still hold although ψ is in this case not the initial state of the $B-$ measurement in frame Σ (which is now ψ_α as discussed above). The analogous argument holds for (2.23).

Here the friction between quantum nonlocality and the lack of an absolute time order of spacelike separated events in relativity becomes visible. In order to ensure that this friction does not mean substantial inconsistency, we require that distributions of matter (which are represented by pointer orientations in an operational predictive framework) are consistent with respect to different Lorentz frames, but not necessarily in the first place compiutational dynamical objects like wave functions (see [24] for more discussions on this point).

where we have set

$$\psi_\beta := \frac{\mathcal{R}^B_\beta\,\psi}{\|\mathcal{R}^B_\beta\,\psi\|} \tag{2.22}$$

Consequently, the subsequent $A'-$measurement has outcome $A'=\alpha$ with probability

$$\mathbb{P}^{\psi'_\beta}(A'=\alpha)=\mathbb{P}^{\psi_\beta}(A=\alpha)=\left\langle\psi_\beta\right|\,E^A_\alpha\,\psi_\beta\rangle \tag{2.23}$$

in which case the final state of the experiment is given by

$$\psi'_{\beta\alpha}:=\frac{\mathcal{R}^{A'}_\alpha\,\psi'_\beta}{\|\mathcal{R}^{A'}_\alpha\,\psi'_\beta\|}=U(\Lambda)\frac{\mathcal{R}^A_\alpha\mathcal{R}^B_\beta\,\psi}{\|\mathcal{R}^A_\alpha\mathcal{R}^B_\beta\,\psi\|}=:U(\Lambda)\,\psi_{\beta\alpha} \tag{2.24}$$

where we have set

$$\psi_{\beta\alpha}:=\frac{\mathcal{R}^A_\alpha\mathcal{R}^B_\beta\,\psi}{\|\mathcal{R}^A_\alpha\mathcal{R}^B_\beta\,\psi\|} \tag{2.25}$$

The associated joint probability $\mathbb{P}^{\psi'}(A'=\alpha\overleftarrow{\wedge}B'=\beta)$ is now given by

$$\begin{aligned}\mathbb{P}^{\psi'}(A'=\alpha\overleftarrow{\wedge}B'=\beta)&=\mathbb{P}^{\psi'}(A'=\alpha\mid B'=\beta)\,\mathbb{P}^{\psi'}(B'=\beta)=\\&=\mathbb{P}^{\psi'_\beta}(A'=\alpha)\,\mathbb{P}^{\psi'}(B'=\beta)=\mathbb{P}^{\psi_\beta}(A=\alpha)\mathbb{P}^{\psi}(B=\beta)=\\&=\frac{\left\langle\psi\right|\,(\mathcal{R}^B_\beta)^\dagger E^A_\alpha\mathcal{R}^B_\beta\,\psi\rangle}{\left\langle\psi\right|\,(\mathcal{R}^B_\beta)^\dagger\mathcal{R}^B_\beta\,\psi\rangle}\left\langle\psi\right|\,E^B_\beta\,\psi\rangle=\\&=\left\langle\psi\right|\,(\mathcal{R}^B_\beta)^\dagger E^A_\alpha\mathcal{R}^B_\beta\,\psi\rangle\equiv\mathbb{P}^{\psi}(A=\alpha\overleftarrow{\wedge}B=\beta)\end{aligned} \tag{2.26}$$

Relativistic Consistency

The demand that ensembles of such an experiment display the same the relative frequencies of each given pair (α,β) of outcomes from the viewpoint of all Lorentz frames implies, that equations (2.19) and (2.26) must coincide:

$$\begin{aligned}\mathbb{P}^{\psi}(A=\alpha\overrightarrow{\wedge}B=\beta)=\left\langle\psi\right|\,(\mathcal{R}^A_\alpha)^\dagger E^B_\beta\,\mathcal{R}^A_\alpha\,\psi\rangle\overset{!}{=}\\=\mathbb{P}^{\psi'}(A'=\alpha\overleftarrow{\wedge}B'=\beta)=\mathbb{P}^{\psi}(A=\alpha\overleftarrow{\wedge}B=\beta)=\left\langle\psi\right|\,(\mathcal{R}^B_\beta)^\dagger E^A_\alpha\mathcal{R}^B_\beta\,\psi\rangle\end{aligned} \tag{2.27}$$

I.e. *the joint probabilities of outcomes of two spacelike separated measurements must be independent of the time order of the respective measurements even in a single frame.* Since furthermore, equation (2.27) must hold for all $\psi\in\mathcal{H}$, we obtain *relativistic consistency* as a mathematical condition[85]

[85] Note that the operators $(\mathcal{R}^A_\alpha)^\dagger E^B_\beta\,\mathcal{R}^A_\alpha$ and $(\mathcal{R}^B_\beta)^\dagger E^A_\alpha\mathcal{R}^B_\beta$ are positive and thereby selfadjoint operators and consequently completely determined by all diagonal matrix elements like $\left\langle\psi\right|\,(\mathcal{R}^A_\alpha)^\dagger E^B_\beta\,\mathcal{R}^A_\alpha\,\psi\rangle$ etc.

Definition 2.1 [*Relativistic Consistency*]

Two (discrete, efficient) quantum measurements performed at spacelike separation, associated with Hilbert space $\mathcal{H}$ and state transformers $\{\mathcal{R}^A_\alpha\}$ and $\{\mathcal{R}^B_\beta\}$ acting on $\mathcal{H}$, are said to be relativistically consistent if

$$(\mathcal{R}^A_\alpha)^\dagger E^B_\beta \mathcal{R}^A_\alpha \stackrel{!}{=} (\mathcal{R}^B_\beta)^\dagger E^A_\alpha \mathcal{R}^B_\beta \tag{2.28}$$

for all α and β, where as always the associated effects are given by $E^A_\alpha = (\mathcal{R}^A_\alpha)^\dagger \mathcal{R}^A_\alpha$ and $E^B_\beta = (\mathcal{R}^B_\beta)^\dagger \mathcal{R}^B_\beta$.

Relativistic consistency is a necessary condition for the requirement that facts like *'pointer points onto X'* do not depend on the Lorentz frame of reference: The predicted relative frequencies of measurement outcomes of the two considered measurements can coincide in all Lorentz frames only if (2.28) holds for all α and β. If this was not the case and the measurements were performed on an ensemble of systems, there would be runs of the experiment where the displayed values disagree from perspectives of different frames, i.e. agents in different frames would perceive not only the respectively Lorentz transformed reality with respect to each other, but a completely different reality[86].

In the following, we will focus on a particular implication of (2.28):

[86]Moreover, we might consider a potential third quantum measurement with associated physical quantity C in the common future (i.e. the intersection of the future light cones) of the two measurements. Also the statistics of outcomes of such future measurements must not depend on the reference frame, i.e. for all initial states $\psi \in \mathcal{H}$ and for all possible triples of outcomes $A = \alpha$, $B = \beta$ and $C = \gamma$ we require

$$\mathbb{P}^{\psi_{\alpha\beta}}(C = \gamma) \stackrel{!}{=} \mathbb{P}^{\psi'_{\beta\alpha}}(C' = \gamma) = \mathbb{P}^{\psi_{\beta\alpha}}(C = \gamma) \tag{2.29}$$

for all measurable quantities C. A sufficient (though not necessary!) condition in order to guarantee that (2.29) holds, is

$$\psi_{\alpha\beta} = \frac{\mathcal{R}^B_\beta \mathcal{R}^A_\alpha \psi}{\sqrt{\left\langle \psi \right| (\mathcal{R}^A_\alpha)^\dagger E^B_\beta \mathcal{R}^A_\alpha \psi \rangle}} \stackrel{!}{=} \psi_{\beta\alpha} = \frac{\mathcal{R}^A_\alpha \mathcal{R}^B_\beta \psi}{\sqrt{\left\langle \psi \right| (\mathcal{R}^B_\beta)^\dagger E^A_\alpha \mathcal{R}^B_\beta \psi \rangle}} \tag{2.30}$$

and with (2.28) we see that the numerators on the left and right hand side of (2.30) coincide, such that

$$\mathcal{R}^B_\beta \mathcal{R}^A_\alpha \psi \stackrel{!}{=} \mathcal{R}^A_\alpha \mathcal{R}^B_\beta \psi \tag{2.31}$$

for all $\psi \in \mathcal{H}$, i.e. the state transformers commute:

$$\mathcal{R}^B_\beta \mathcal{R}^A_\alpha \stackrel{!}{=} \mathcal{R}^A_\alpha \mathcal{R}^B_\beta \tag{2.32}$$

for all α, β. If the state transformers $\mathcal{R}^A_\alpha$ and $\mathcal{R}^B_\beta$ are normal operators we can apply the Fuglede-Putnam theorem (which states that a normal operator commuting with another operator also commutes with its adjoint, see footnote 95) in order to derive from equations (2.32) the relations $\left[E^A_\alpha\,,\,\mathcal{R}^B_\beta\right] = 0$ and $\left[E^B_\beta\,,\,\mathcal{R}^A_\alpha\right] = 0$ for all α, β, which we will identify as the fundamental commutation relations below.

No Signalling

If we sum the relativistic consistency condition (2.28) over all possible outcomes $A = \alpha$, respectively $B = \beta$, we obtain the *no signalling conditions* (to be explained below) of the composite $A-$ and $B-$measurement:

$$\begin{aligned}\sum_{\alpha}(\mathcal{R}_{\alpha}^{A})^{\dagger}E_{\beta}^{B}\,\mathcal{R}_{\alpha}^{A} \stackrel{!}{=} \sum_{\alpha}(\mathcal{R}_{\beta}^{B})^{\dagger}E_{\alpha}^{A}\mathcal{R}_{\beta}^{B} = (\mathcal{R}_{\beta}^{B})^{\dagger}\left(\sum_{\alpha}E_{\alpha}^{A}\right)\mathcal{R}_{\beta}^{B} = \\ = (\mathcal{R}_{\beta}^{B})^{\dagger}\mathbb{1}_{\mathcal{H}}\mathcal{R}_{\beta}^{B} = (\mathcal{R}_{\beta}^{B})^{\dagger}\mathcal{R}_{\beta}^{B} = E_{\beta}^{B}\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}\sum_{\beta}(\mathcal{R}_{\beta}^{B})^{\dagger}E_{\alpha}^{A}\,\mathcal{R}_{\beta}^{B} \stackrel{!}{=} \sum_{\beta}(\mathcal{R}_{\alpha}^{A})^{\dagger}E_{\beta}^{B}\mathcal{R}_{\alpha}^{A} = (\mathcal{R}_{\alpha}^{A})^{\dagger}\left(\sum_{\beta}E_{\beta}^{B}\right)\mathcal{R}_{\alpha}^{A} = \\ = (\mathcal{R}_{\alpha}^{A})^{\dagger}\mathbb{1}_{\mathcal{H}}\mathcal{R}_{\alpha}^{A} = (\mathcal{R}_{\alpha}^{A})^{\dagger}\mathcal{R}_{\alpha}^{A} = E_{\alpha}^{A}\end{aligned} \tag{2.34}$$

Definition 2.2 [*No Signalling*]

Two (discrete, efficient) quantum measurements associated with Hilbert space $\mathcal{H}$ and state transformers $\{\mathcal{R}_{\alpha}^{A}\}$ and $\{\mathcal{R}_{\beta}^{B}\}$ acting on $\mathcal{H}$ satisfy no signalling if

$$E_{\alpha}^{A} \stackrel{!}{=} \sum_{\beta}(\mathcal{R}_{\beta}^{B})^{\dagger}E_{\alpha}^{A}\,\mathcal{R}_{\beta}^{B} \tag{2.35a}$$

$$E_{\beta}^{B} \stackrel{!}{=} \sum_{\alpha}(\mathcal{R}_{\alpha}^{A})^{\dagger}E_{\beta}^{B}\,\mathcal{R}_{\alpha}^{A} \tag{2.35b}$$

for all α and β, where the associated effects are given by $E_{\alpha}^{A} = (\mathcal{R}_{\alpha}^{A})^{\dagger}\mathcal{R}_{\alpha}^{A}$ and $E_{\beta}^{B} = (\mathcal{R}_{\beta}^{B})^{\dagger}\mathcal{R}_{\beta}^{B}$.

So why are equations (2.35) called no signalling conditions? To understand this, note first that since $\mathbb{P}^{\psi}(A = \alpha \overleftarrow{\wedge} B = \beta)$ defines a joint probability distribution (conditional on the time order of the measurements), it also defines marginals, one of which is physically very meaningful, namely the marginal $A-$ distribution given by

$$\mathbb{P}_{marg}^{\psi}(A = \alpha, B) = \sum_{\beta}\mathbb{P}^{\psi}(A = \alpha \overleftarrow{\wedge} B = \beta) = \sum_{\beta}\mathbb{P}^{\psi}\left(A = \alpha \mid B = \beta\right)\mathbb{P}^{\psi}\left(B = \beta\right) \tag{2.36}$$

This is the probability of outcome α of the $A-$measurement averaged over all possible outcomes

β of the preceding $B-$measurement, i.e. it equals the conditional probability

$$\begin{aligned}
&\mathbb{P}^{\psi}\left(A=\alpha \mid B \text{ was measured}\right) \equiv \sum_{\beta} \mathbb{P}^{\psi}\left(A=\alpha \mid B=\beta\right)\ \mathbb{P}^{\psi}\left(B=\beta\right) = \\
&= \sum_{\beta} \mathbb{P}^{\psi_{\beta}}(A=\alpha)\ \mathbb{P}^{\psi}\left(B=\beta\right) = \sum_{\beta} \frac{\left\langle \mathcal{R}^B_{\beta}\,\psi\right|\ E^A_{\alpha}\,\mathcal{R}^B_{\beta}\,\psi\rangle}{\|\mathcal{R}^B_{\beta}\,\psi\|^2}\ \left\langle \psi\right|\ E^B_{\beta}\,\psi\rangle = \\
&= \sum_{\beta} \frac{\left\langle \psi\right|\ (\mathcal{R}^B_{\beta})^{\dagger} E^A_{\alpha}\,\mathcal{R}^B_{\beta}\,\psi\rangle}{\left\langle \psi\right|\ (\mathcal{R}^B_{\beta})^{\dagger}\,\mathcal{R}^B_{\beta}\,\psi\rangle}\ \left\langle \psi\right|\ (\mathcal{R}^B_{\beta})^{\dagger}\,\mathcal{R}^B_{\beta}\,\psi\rangle = \\
&\left\langle \psi\right|\ \sum_{\beta} (\mathcal{R}^B_{\beta})^{\dagger} E^A_{\alpha}\,\mathcal{R}^B_{\beta}\,\psi\ \Big\rangle \overset{(2.35a)}{=} \left\langle \psi\right|\ E^A_{\alpha}\,\psi\rangle = \mathbb{P}^{\psi}\left(A=\alpha\right)
\end{aligned} \tag{2.37}$$

Analogously, from relation (2.35b) we obtain

$$\mathbb{P}^{\psi}\left(B=\beta \mid A \text{ was measured}\right) \equiv \sum_{\alpha} \mathbb{P}^{\psi}\left(B=\beta \mid A=\alpha\right)\ \mathbb{P}^{\psi}\left(A=\alpha\right) \overset{(2.35b)}{=} \mathbb{P}^{\psi}\left(B=\beta\right) \tag{2.38}$$

According to (2.37), no signalling condition (2.35a) entails that the distribution of outcomes α of the $A-$measurement does not depend on whether the $B-$measurement was previously performed or not and analogously, according to (2.38), no signalling condition (2.35b) entails that the distribution of outcomes β of the $B-$measurement does not depend on whether the $A-$measurement was previously performed or not. So the statistics of outcomes of one measurement is not altered by the fact whether the other measurement was performed before or not (but it is important to keep in mind here, that this does not mean that the outcome distribution associated with one measurement is generally not changed by a preceding measurement, conditional on a given outcome of the latter, which is in general false for quantum measurements as the EPR experiment demonstrates, see further footnote 87 below and section 1.3).

Given now a particular quantum measurement and a given initial state $\psi \in \mathcal{H}$, an experimenter has no possibilities whatsoever to control the outcome, the only operational freedom an experimenter has (given measurement and initial state) is to decide to perform the measurement or not. Consequently, if the no signalling conditions are fulfilled for two given measurements, an experimenter controlling one of the experimental devices has no possibilities whatsoever, to utilize the nonlocality of quantum measurements to alter the statistics of outcomes of the other measurement on any given ensemble of initial systems (think e.g. of two entangled beams of particles).

On the other hand, suppose the no signalling conditions were violated for two measurements A and B performed in spacelike separated regions of space-time. In this case it was in general possible for an experimenter controlling, say the device of the $A-$measurement, to alter the outcome statistics of the remote $B-$measurement on some ensemble of entangled initial systems by her decision to perform the $A-$measurement on each member or not. As a consequence, she were able in principle to communicate to a remote experimenter at the $B-$measurement wing her decision to perform the measurement on the members of the considered ensemble or not (if the probabilities for some of the outcomes of the remote $B-$measurement are prior to the $A-$measurement either zero or one and if these probabilities were altered by the decision of the

experimenter to perform the $A-$measurement, it would be even possible to communicate this decision by a single experiment). This would e.g. allow for sending complex binary codes (0 =‘no measurement’, 1 =‘measurement’) faster than light if enough copies of identically prepared initial states are available which can be altogether measured (or not) fast enough.

To summarize, no signalling conditions (2.35a) and (2.35b) for two spacelike separated measurements are necessary and sufficient conditions to prohibit the possibility to exploit quantum nonlocality in order to send superluminal signals by operations which are controllable by human decisions[87]. Moreover, since the time order of spacelike separated events depends on the frame of reference, violation of the no signalling conditions would open the door to send signals into the past (for two-measurement-experiments only from the viewpoint of some frames, but if more measurements of that kind are combined, it would even be possible to send signals into the causal past, i.e. into the backward lightcone (see [24, 236]). No signalling is usually demanded to hold in relativistic quantum theory (in addition to requiring that the elements of the Poincaré group act as unitary operators on Hilbert space) and *local commutativity*, whose relation with no signalling and relativistic consistency will be analysed below, can then be deduced as a mathematical requirement form the no signalling conditions[88].

Expecting no signalling to hold in a relativistic theory is reasonable, nonetheless one may find it unsatisfactory to base on it a fundamental physical principle: As a fundamental postulate, no signalling is usually justified with the possibility of causal paradoxes that would ensue if the principle was violated, allowing for signalling backwards in time. However, these arguments seem to presuppose certain anthropocentric concepts, like the free will of the experimenter that allows her to perform certain operations depending on the signals received ‘from the future’ etc. In fact, it can be argued that backwards in time causation need not, in principle, lead to weird or paradoxical results in a consistent relativistic theory [24, 236, 341].

Therefore, the less anthropocentric requirement of *relativistic consistency* was developed in

[87]To call conditions (2.35) ‘no signalling’ conditions makes also sense if the respective measurements are not spacelike separated, only in this case, they need not be (and often are not) satisfied. In fact, these conditions are also of interest in the theory of quantum measurement without relativistic considerations, where they are sometimes called *non disturbance conditions* [185]. Note however, that this nomenclature is potentially misleading, since the validity of these conditions for two given measurements does in general not imply that the measurements do not disturb one another: Given a particular outcome of the first measurement, the initial state of the second measurement (and by that the prediction for its outcome) in general still dramatically changes. Only if we average over all possible outcomes of the first measurement we see that the predictions for the second measurement are not altered by the fact whether the first measurement was performed or not (consequently the relative frequencies of outcomes of measurements on the members of an ensemble of identically prepared systems are not altered by distant measurements performed previously, but this does not hold if a subensemble belonging to a particular outcome of the first measurement is considered). For example the operators associated with one wing of an EPRB-experiment obey the no signalling conditions with respect to the operators associated with the other wing, nonetheless, as argued above, the two measurements in general are massively disturbing one another in the sense that the predictions for outcomes at one wing conditional on the outcome of the remote measurement are very different from the unconditional predictions (the conditional prediction completely determines one of the two outcomes, whereas the unconditional one attributes to each outcome probability $\frac{1}{2}$), and this sudden change is according to Bell’s theorem not explicable by any lines of argument assuming the absence of direct influence of the remote measurement. See section 1.3 for more details.

[88]Actually, in a huge part of the literature on relativistic quantum theory, local commutativity – sometimes named *mircrocausality* or just *causality* or *locality* – is demanded without deducing it from physical considerations, as if the physical meaning of the mathematical property was primitive or obvious, which is generally not the case, as we shall see below.

the present work to proceed from, and from this requirement no signalling is a straightforward (and reasonable) consequence.

Nonetheless, for most mathematical results in connection with local commutativity, which we shall derive below, no signalling conditions seem to be technically better suited to derive direct implications than relativistic consistency conditions (only theorem 2.10 works directly with relativistic consistency, all other theorems proceed from no signalling). But one should keep in mind that any implication of no signalling is an implication of relativistic consistency as well, since the former is a direct consequence of the latter by taking the respective sums (which can be perceived as coarse graining by averaging one of the two measurements out in the sense described above).

What Comes First ?

So relativistic consistency means that the joint distribution of outcomes of two measurements performed at spacelike separation do not depend on the time order of the measurements and by that they do not depend on the Lorentz frame of reference. No signalling, on the other hand, means that the distribution of outcomes at one wing of the experiment does not depend on the fact whether the measurement at the other wing was previously performed or not (or whether it was previously performed or afterwards). We already derived no signalling mathematically as a consequence of relativistic consistency. It might be tempting now to guess that it is also the other way around and that thus no signalling is equivalent to relativistic consistency, but this is actually false!

To see this, we construct a simple counterexample: Consider an EPRB-like experiment, which consists of two measurements, one of which we call L on the left hand side and another one R on the right hand side. Each measurement has two possible outcomes 'up' and 'down', respectively, i.e. $L, R \in \{\uparrow, \downarrow\}$. In the following, we abbreviate $L = \uparrow$ by $\uparrow_L$ and so forth. Suppose we have a theory, which predicts the statistics of the outcomes, i.e. which provides probabilities for the four possible single measurement outcomes $\uparrow_L, \downarrow_L, \uparrow_R$ and $\downarrow_R$ and if both measurements are performed, eight conditional probabilities for the respective outcomes of one of the measurements given the other one was performed before with a given result. The single measurement probabilities together with the conditional probabilities then yield the joint probabilities, which might depend on the time order of the respective measurements, as it is generally the case for quantum measurements.

Consider now the following distribution:

$$\begin{array}{lll} \mathbb{P}(\uparrow_L) = \frac{1}{2} & \mathbb{P}(\uparrow_L|\uparrow_R) = 0 & \mathbb{P}(\uparrow_R|\uparrow_L) = \frac{1}{2} \\ \mathbb{P}(\downarrow_L) = \frac{1}{2} & \mathbb{P}(\uparrow_L|\downarrow_R) = 1 & \mathbb{P}(\uparrow_R|\downarrow_L) = \frac{1}{2} \\ \mathbb{P}(\uparrow_R) = \frac{1}{2} & \mathbb{P}(\downarrow_L|\uparrow_R) = 1 & \mathbb{P}(\downarrow_R|\uparrow_L) = \frac{1}{2} \\ \mathbb{P}(\downarrow_R) = \frac{1}{2} & \mathbb{P}(\downarrow_L|\downarrow_R) = 0 & \mathbb{P}(\downarrow_R|\downarrow_L) = \frac{1}{2} \end{array} \tag{2.39}$$

The first two columns are exactly the same as in the EPRB-experiment with identical orientation of the two Stern-Gerlach devices and initial singlet state, whereas the third column means that the probabilities of the measurement on the right hand side remain undisturbed if the measurement on the left was previously performed and a given result was obtained. It is easy to check that this distribution is as well defined as any quantum distribution for composite

experiments, in particular all normalization constraints are satisfied but Bayes law is violated. The violation of Bayes law is due to the time order dependence of the joint distribution of outcomes, e.g we have

$$\mathbb{P}^{\psi}(\uparrow_L \overrightarrow{\wedge} \downarrow_R) = \mathbb{P}(\downarrow_R|\uparrow_L)\mathbb{P}(\uparrow_L) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \tag{2.40}$$

and

$$\mathbb{P}^{\psi}(\uparrow_L \overleftarrow{\wedge} \downarrow_R) = \mathbb{P}(\uparrow_L|\downarrow_R)\mathbb{P}(\downarrow_R) = 1\cdot\frac{1}{2} = \frac{1}{2} \tag{2.41}$$

i.e.

$$\mathbb{P}^{\psi}(\uparrow_L \overrightarrow{\wedge} \downarrow_R) \neq \mathbb{P}^{\psi}(\uparrow_L \overleftarrow{\wedge} \downarrow_R) \tag{2.42}$$

and so forth. Consequently, if the two measurements are performed in regions of space-time which are spacelike separated, relativistic consistency is violated! On the other hand, this experiment, if it existed, would not allow for superluminal communication, since all four no signalling conditions hold:

$$\begin{aligned}
\mathbb{P}(\uparrow_L|\uparrow_R)\mathbb{P}(\uparrow_R) + \mathbb{P}(\uparrow_L|\downarrow_R)\mathbb{P}(\downarrow_R) = 0\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = \quad \frac{1}{2} = \mathbb{P}(\uparrow_L) \quad \checkmark \\
\mathbb{P}(\downarrow_L|\uparrow_R)\mathbb{P}(\uparrow_R) + \mathbb{P}(\downarrow_L|\downarrow_R)\mathbb{P}(\downarrow_R) = 1\cdot\frac{1}{2} + 0\cdot\frac{1}{2} = \quad \frac{1}{2} = \mathbb{P}(\downarrow_L) \quad \checkmark \\
\mathbb{P}(\uparrow_R|\uparrow_L)\mathbb{P}(\uparrow_L) + \mathbb{P}(\uparrow_R|\downarrow_L)\mathbb{P}(\downarrow_L) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \quad \frac{1}{2} = \mathbb{P}(\uparrow_R) \quad \checkmark \\
\mathbb{P}(\downarrow_R|\uparrow_L)\mathbb{P}(\uparrow_L) + \mathbb{P}(\downarrow_R|\downarrow_L)\mathbb{P}(\downarrow_L) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \quad \frac{1}{2} = \mathbb{P}(\downarrow_R) \quad \checkmark
\end{aligned} \tag{2.43}$$

Consequently, although the no signalling conditions are satisfied, it should be excluded that these or analogous probabilities arise from any relativistic quantum theory for spacelike separated measurements. If this was the case, the theory would predict inconsistent measurement results (pointer orientations) for the given experiment from the viewpoint of distinct frames of reference.

2.3 The Meaning of Local Commutativity

If operators associated with spacelike separated measurements commute, one usually speaks of *local commutativity* (other common namings are *microcausality*, *causality* or *locality*). In the following we shall carefully scrutinize different conditions of this kind with respect to their physical meaning in connection with relativistic consistency and no signalling. The central results we shall derive in the following are condensed in Fig. 3 (and its associated text) below, which might be also taken as an orientation for the reading.

2.3.1 Sufficiency

If we think of the EPRB example, we know that no signalling is a direct consequence of the fact that operators associated with the left wing of the experiment commute with all operators associated with the right wing of the experiment. These commutativity properties in turn derive from the fact, that the left, respectively right wing of the experiment are associated with different particles (though prepared in the non separable singlet state), such that the joint Hilbert

space is the tensor product $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$ and operators associated with the left wing of the experiment have the form $A \otimes \mathbb{1}_{\mathcal{H}_R}$ and operators associated with the right wing look like $\mathbb{1}_{\mathcal{H}_L} \otimes B$. Thus the mentioned commutativity properties trivially hold: $(A \otimes \mathbb{1}_{\mathcal{H}_R})(\mathbb{1}_{\mathcal{H}_L} \otimes B) = A \otimes B = (\mathbb{1}_{\mathcal{H}_L} \otimes B)(A \otimes \mathbb{1}_{\mathcal{H}_R})$. The fact that the spin operators (and their associated PVMs) associated with one wing of the EPRB-experiment commute with the spin operators (and associated PVMs) associated with the other wing now entails that the inherently nonlocal (according to Bell's theorem) correlations between the respective outcomes cannot be utilized to communicate faster than light. We will see in a moment, that these commutativity properties also imply that the EPRB-probabilities satisfy relativistic consistency.

Consider two effects $E^A_\alpha = (\mathcal{R}^A_\alpha)^\dagger \mathcal{R}^A_\alpha$ and $E^B_\beta = (\mathcal{R}^B_\beta)^\dagger \mathcal{R}^B_\beta$ with associated state transformers[89] $\mathcal{R}^A_\alpha$ and $\mathcal{R}^B_\beta$, respectively, acting on some Hilbert space $\mathcal{H}$, which are associated with particular outcomes $A = \alpha$ and $B = \beta$ of measurements A and B, respectively, as above (in particular, E^A_α and E^B_β are elements of POVMs such that $\sum_\alpha E^A_\alpha = \sum_\beta E^B_\beta = \mathbb{1}_{\mathcal{H}}$). If the measurements are performed at spacelike separation, we expect the relativistic consistency, respectively no signalling conditions to hold. As straightforward compact sufficient conditions (as will be clear in a moment) for these requirements we identify the following *fundamental local commutativity conditions*:

Definition 2.3 [*Local Commutativity*]

Two (discrete, efficient) quantum measurements A and B performed at spacelike separation satisfy local commutativity, if all state transformers associated with the $A-$measurement commute with all effects of the $B-$measurement and vice versa:

$$\left[E^A_\alpha\,,\,\mathcal{R}^B_\beta\right] = 0 \quad \textit{and} \quad \left[E^B_\beta\,,\,\mathcal{R}^A_\alpha\right] = 0 \quad \textit{for all } \alpha, \beta \tag{2.44}$$

It is very easy to see, that the commutativity conditions (2.44) are sufficient to guarantee relativistic consistency and thereby no signalling[90]:

$$\begin{aligned} &(\mathcal{R}^A_\alpha)^\dagger E^B_\beta \mathcal{R}^A_\alpha \overset{(2.44)}{=} (\mathcal{R}^A_\alpha)^\dagger \mathcal{R}^A_\alpha E^B_\beta = E^A_\alpha (\mathcal{R}^B_\beta)^\dagger \mathcal{R}^B_\beta = \\ &= \left(\mathcal{R}^B_\beta E^A_\alpha\right)^\dagger \mathcal{R}^B_\beta \overset{(2.44)}{=} \left(E^A_\alpha \mathcal{R}^B_\beta\right)^\dagger \mathcal{R}^B_\beta = (\mathcal{R}^B_\beta)^\dagger E^A_\alpha \mathcal{R}^B_\beta \end{aligned} \tag{2.46}$$

which is the relativistic consistency condition associated with the pair of outcomes $A = \alpha$ and

[89] It will we helpful below to recall the polar decomposition of the state transformers (see section 1.2), i.e. $\mathcal{R}^A_\alpha = U^A_\alpha \sqrt{E^A_\alpha}$, where U^A_α (sometimes called *measurement back action*) is a partial isometry (possibly unitary or even equal to $\mathbb{1}_{\mathcal{H}}$), and correspondingly with $\mathcal{R}^B_\beta$.

[90] Actually, no signalling is also a trivial direct consequence of conditions (2.44):

$$\sum_\alpha (\mathcal{R}^A_\alpha)^\dagger E^B_\beta \mathcal{R}^A_\alpha \overset{(2.44)}{=} \sum_\alpha (\mathcal{R}^A_\alpha)^\dagger \mathcal{R}^A_\alpha E^B_\beta = \mathbb{1}_{\mathcal{H}} E^B_\beta = E^B_\beta \tag{2.45}$$

and vice versa. But we shall trace the logical path from commutativity to relativistic consistency to no signalling and we will analyse below the question, if and under which circumstances we can continue it again to commutativity, in order to close the chain.

$B = \beta$. Thus, together with (2.33) and (2.34), we have the following implications (see also Fig. 3 below):

Corollary 2.4

Local Commutativity (2.44) ⇒ *Relativistic Consistency* (2.28) ⇒ *No Signalling* (2.35)

If we could close this chain by deriving commutativity again from no signalling, all three conditions were equivalent and commutativity could be demanded as a necessary condition for relativistic consistency and no signalling (note that, although we saw above that relativistic consistency and no signalling as conditions on general joint probability distributions are actually not equivalent, it might well be that the quantum formalism has such an equivalence as a consequence). Whether and under which circumstances this is possible will be the subject of analysis in the following section.

But first a final comment on more commonly required local commutativity conditions which deviate from the present ones.

Corollary 2.5

The fundamental commutativity conditions (2.44) *imply the more familiar commutativity conditions*

$$\left[E^A_\alpha\,,\,E^B_\beta\right] = 0 \quad \textit{for all } \alpha,\beta \tag{2.47}$$

Proof: Note first that conditions (2.44) imply

$$\left[E^A_\alpha\,,\,(\mathcal{R}^B_\beta)^\dagger\right] = 0 \quad \text{and} \quad \left[E^B_\beta\,,\,(\mathcal{R}^A_\alpha)^\dagger\right] = 0 \tag{2.48}$$

by taking (minus) the adjoints. Consequently

$$\begin{aligned} 0 \overset{(2.44)}{=} (\mathcal{R}^B_\beta)^\dagger\,\left[E^A_\alpha\,,\,\mathcal{R}^B_\beta\right] = (\mathcal{R}^B_\beta)^\dagger\,\left(E^A_\alpha\mathcal{R}^B_\beta - \mathcal{R}^B_\beta E^A_\alpha\right) \overset{(2.48)}{=} \\ = E^A_\alpha(\mathcal{R}^B_\beta)^\dagger\mathcal{R}^B_\beta - (\mathcal{R}^B_\beta)^\dagger\mathcal{R}^B_\beta E^A_\alpha = \left[E^A_\alpha\,,\,E^B_\beta\right] \end{aligned} \tag{2.49}$$

■

As a local commutativity condition, mostly the *commutativity of observable operators* associated with spacelike separated measurements is demanded in the literature. Indeed, if the effects $E^A_\alpha \equiv P^A_\alpha$ and $E^B_\beta \equiv P^B_\beta$ are altogether projections, i.e. if the POVMs are PVMs, the secondary commutativity conditions (2.47) are equivalent to the commutativity of the associated observable operators $\hat{A} = \sum_\alpha \lambda_\alpha P^A_\alpha$ and $\hat{B} = \sum_\beta \mu_\beta P^B_\beta$ (that the conditions (2.47) imply $\left[\hat{A}\,,\,\hat{B}\right] = 0$ is trivial, to see the converse implication, recall that commuting selfadjoint operators can be jointly diagonalized, such that it is always possible to find a complete set of mutually orthogonal, one dimensional eigenprojections of $\hat{A}$ and $\hat{B}$).

But one should observe that the conditions $[P^A_\alpha, P^B_\beta] = 0$, though they are implied by the fundamental conditions $[P^A_\alpha, \mathcal{R}^B_\beta] = 0$ and $[P^B_\beta, \mathcal{R}^A_\alpha] = 0$, are in general not equivalent to the latter, unless the measurements are ideal measurements (see section 1.4.1), i.e. unless $\mathcal{R}^A_\alpha = P^A_\alpha$ and $\mathcal{R}^B_\beta = P^B_\beta$ for all α and β. If there is non trivial measurement back action (see footnote 89) represented by partial isometries $U^A_\alpha, U^B_\beta \neq \mathbb{1}_\mathcal{H}$ such that $\mathcal{R}^A_\alpha = U^A_\alpha P^A_\alpha$ and $\mathcal{R}^B_\beta = U^B_\beta P^B_\beta$, we have a priori no reason to expect that $[P^A_\alpha, \mathcal{R}^B_\beta] = 0$ and $[P^B_\beta, \mathcal{R}^A_\alpha] = 0$ given $[P^A_\alpha, P^B_\beta] = 0$. Consequently, the latter commutativity conditions – or equivalently $[\hat{A}, \hat{B}] = 0$ – do in general not imply local commutativity as defined in 2.3 and thus need neither imply relativistic consistency nor no signalling by corollary 2.4.

More generally, in the case of possibly non projective measurements, commutativity of the effects (2.47) implies the fundamental commutativity conditions (2.44) and thereby relativistic consistency and no signalling in general only if[91] $\mathcal{R}^A_\alpha = \sqrt{E^A_\alpha}$ and $\mathcal{R}^B_\beta = \sqrt{E^B_\beta}$. Such measurements are called *Lüders measurements*, see definition 2.8 below (note that ideal measurements are a special case of Lüders measurements since orthogonal projections P satisfy $\sqrt{P} = P$). If there is non trivial measurement back action (see footnote 89), i.e. $\mathcal{R}^A_\alpha = U^A_\alpha \sqrt{E^A_\alpha}$ and $\mathcal{R}^B_\beta = U^B_\beta \sqrt{E^B_\beta}$ with $U^A_\alpha, U^B_\beta \neq \mathbb{1}_\mathcal{H}$, conditions (2.47) need not imply local commutativity as defined in 2.3 and thus are generally not sufficient to guarantee relativistic consistency, respectively no signalling by corollary 2.4.

We summarize the central point of these observations in the following

Corollary 2.6

The commutativity conditions (2.47) *do in general not imply the fundamental commutativity conditions* (2.44) *(only for Lüders measurements like ideal measurements, this implication is generally true).*

2.3.2 Necessity

In this work, relativistic consistency is treated as the fundamental notion of relativistic causality instead of the prevailing notion in the literature, which is no signalling (see e.g. [30, 41, 69, 259, 267]). Nonetheless, in view of the established implications of corollary 2.4 it makes sense to check the necessity of local commutativity with respect to no signalling, since this would entail that local commutativity, relativistic consistency and no signalling were equivalent. For example, this is the case for projective measurements as shall be proven next, which in particular entails that a distribution like (2.39) which respects no signalling but violates relativistic consistency cannot arise from projective measurements. To demand local commutativity for projective measurements is thus perfectly justified from a physical point of view.

[91] That $[E^A_\alpha, E^B_\beta] = 0$ implies $[E^A_\alpha, \sqrt{E^B_\beta}] = 0$ can be seen e.g. by noting that in each ONB, in which E^B_β is diagonal, $\sqrt{E^B_\beta}$ is diagonal as well and that selfadjoint operators are jointly diagonalizable if and only if they commute.

PVMs

For ideal (and later general projective) measurements, it was known for a long time that local commutativity is a necessary condition for no signalling[92]. This result (together with the sufficiency from above) is known as *Lüders theorem* [77, 228, 229]. There are various possible ways to proof this result, some of which do only capture the case of ideal measurements. We will give a proof for general (discrete) projective measurements which will be well suited for generalizations later on. Since the following results are just about relations among (measurement–) operators which hold equally if the associated measurements are not performed at spacelike separation, we can drop the epithet 'local' in the expression 'local commutativity'.

The following result is valid for unbounded observable operators as well (all mathematical arguments are about PVMs and state transformers, which are always bounded operators and the question of possibly unbounded operators does not arise for these arguments), but in order to avoid cumbersome considerations about domains, it shall be formulated here for bounded observable operators.

Theorem 2.7 [*Lüders*]

Consider two discrete projective measurements associated with bounded observable operators $\hat{A} = \sum_\alpha \lambda_\alpha P_\alpha^A$ and $\hat{B} = \sum_\beta \mu_\beta P_\beta^B$ acting on Hilbert space $\mathcal{H}$ (i.e. the families of eigenprojections $\{P_\alpha^A\}$ and $\{P_\beta^B\}$ constitute the associated PVMs) and the associated sets of state transformers $\{\mathcal{R}_\alpha^A\}$ and $\{\mathcal{R}_\beta^B\}$, respectively. Then the following two conditions are equivalent:

(i) Commutativity: $[P_\alpha^A, \mathcal{R}_\beta^B] = 0$ and $[P_\beta^B, \mathcal{R}_\alpha^A] = 0$ for all α, β

(ii) No Signalling: $P_\alpha^A = \sum_\beta (\mathcal{R}_\beta^B)^\dagger P_\alpha^A \mathcal{R}_\beta^B$ for all α and $P_\beta^B = \sum_\alpha (\mathcal{R}_\alpha^A)^\dagger P_\beta^B \mathcal{R}_\alpha^A$ for all β

Remark: It follows from corollary 2.5 that conditions (i) imply $[\hat{A}, \hat{B}] = 0$. Thus theorem 2.7 shows that the no signalling conditions (ii) imply $[\hat{A}, \hat{B}] = 0$ as well. If the measurements are ideal, i.e. if $P_\alpha^A = \mathcal{R}_\alpha^A$ and $P_\beta^B = \mathcal{R}_\beta^B$ for all α and β, the three conditions (i), (ii) and $[\hat{A}, \hat{B}] = 0$ are equivalent (see the remarks subsequent to corollary 2.5 above). For non ideal measurements $[\hat{A}, \hat{B}] = 0$ does in general neither imply (i) nor (ii).

Proof: Implication *(i) ⇒ (ii)* is already given by corollary 2.4 (or more directly by trivial calculation: $\sum_\beta (\mathcal{R}_\beta^B)^\dagger P_\alpha^A \mathcal{R}_\beta^B = \sum_\beta (\mathcal{R}_\beta^B)^\dagger \mathcal{R}_\beta^B P_\alpha^A = P_\alpha^A$).

[92] This result, covering only a special class of measurements, was for a long time the only physical motivation to demand local commutativity (often defined under suggestive names like 'causality' or 'locality') in relativistic quantum theory, which is usually done even without mentioning this result, as if the commutativity of certain operators were an obvious relativistic physical property! Bell [27] gave a further possible physical justification – namely, that local perturbations due to external fields do not propagate faster than light – which is though not necessary to postulate in a relativistic quantum setting based on relativistic (namely hyperbolic) wave equations, such that perturbations of wave functions propagate causally anyway.

In order to show the converse implication $(ii) \Rightarrow (i)$ suppose that for arbitrary β we have

$$P_\beta^B = \sum_\alpha (\mathcal{R}_\alpha^A)^\dagger P_\beta^B \mathcal{R}_\alpha^A \tag{2.50}$$

Now consider the following expression:

$$\begin{aligned}\sum_\alpha \left[P_\beta^B\,,\,\mathcal{R}_\alpha^A\right]^\dagger \left[P_\beta^B\,,\,\mathcal{R}_\alpha^A\right] &= \sum_\alpha \left((\mathcal{R}_\alpha^A)^\dagger P_\beta^B - P_\beta^B (\mathcal{R}_\alpha^A)^\dagger\right)\left(P_\beta^B \mathcal{R}_\alpha^A - \mathcal{R}_\alpha^A P_\beta^B\right) = \\ &= \overbrace{\sum_\alpha (\mathcal{R}_\alpha^A)^\dagger P_\beta^B \mathcal{R}_\alpha^A}^{=P_\beta^B} + P_\beta^B \overbrace{\left(\sum_\alpha P_\alpha^A\right)}^{=\mathbb{1}_\mathcal{H}} P_\beta^B - \\ &- \overbrace{\left(\sum_\alpha (\mathcal{R}_\alpha^A)^\dagger P_\beta^B \mathcal{R}_\alpha^A\right)}^{=P_\beta^B} P_\beta^B - P_\beta^B \overbrace{\left(\sum_\alpha (\mathcal{R}_\alpha^A)^\dagger P_\beta^B \mathcal{R}_\alpha^A\right)}^{=P_\beta^B} = \\ &= 2P_\beta^B - 2P_\beta^B = 0\end{aligned} \tag{2.51}$$

where we have used $\sum_\alpha (\mathcal{R}_\alpha^A)^\dagger \mathcal{R}_\alpha^A = \sum_\alpha P_\alpha^A = \mathbb{1}_\mathcal{H}$ and the projection property $(P_\beta^B)^2 = P_\beta^B$ in addition to (2.50).

Since moreover $\left[P_\beta^B\,,\,\mathcal{R}_\alpha^A\right]^\dagger \left[P_\beta^B\,,\,\mathcal{R}_\alpha^A\right]$ is a positive operator for each α, we see that (2.51) implies that each term in the sum on the left hand side of (2.51) vanishes, i.e.

$$\left[P_\beta^B\,,\,\mathcal{R}_\alpha^A\right]^\dagger \left[P_\beta^B\,,\,\mathcal{R}_\alpha^A\right] = 0 \tag{2.52}$$

for all α, which in turn implies

$$\left[P_\beta^B\,,\,\mathcal{R}_\alpha^A\right] = 0 \tag{2.53}$$

for all α. Consequently, if the no signalling condition (2.50) is satisfied for all β it follows that

$$\left[P_\beta^B\,,\,\mathcal{R}_\alpha^A\right] = 0 \tag{2.54}$$

for all α and β.

Analogously one finds that

$$P_\alpha^A = \sum_\beta (\mathcal{R}_\beta^B)^\dagger P_\alpha^A \mathcal{R}_\beta^B \tag{2.55}$$

for all α implies

$$[P_\alpha^A, \mathcal{R}_\beta^B] = 0 \tag{2.56}$$

for all α and β. ■

The crucial assumption that the measurements are projective measurements can be relaxed in the proof in some ways. Note e.g. that the central argument of the proof, namely that $P_\beta^B = \sum_\alpha (\mathcal{R}_\alpha^A)^\dagger P_\beta^B \mathcal{R}_\alpha^A$ implies $\left[P_\beta^B\,,\,\mathcal{R}_\alpha^A\right] = 0$ goes through, also if we interchange the projection P_β^B with an effect $E_\beta^B \equiv cP_\beta^B$ with $0 \leq c \leq 1$, i.e. with an effect which is proportional to a projection. Further generalizations of the theorem and the proof will be presented below.

The question whether commutativity is a necessary condition in order to guarantee no signalling also in the case of more general (and more realistic) measurements was picked up by Busch et al. in reference [77] and it turned out that it is not easy to answer this question in full generality. Busch at al. were able to give an affirmative answer in two very special cases and motivated by this project, several authors presented extensive investigations in this direction [16, 70, 162, 163, 184, 185, 218, 225, 233, 269, 338, 339]. It turned out, that it is indeed possible to mathematically construct formal sets of state transformers such that given effects obey no signalling with respect these state transformers but at the same time do not obey the related commutativity conditions. Some of these results will be presented and discussed below and we shall investigate the realm in which it is physically justified to demand local commutativity below, but first we shall present one of the two results of Busch et al.

Lüders Measurements

As already mentioned, efficient quantum measurements associated with state transformers which are positive operators were called *(generalized) Lüders measurements* by several authors (see e.g. [73] or the references in the preceiding paragraph).

Definition 2.8 [*Lüders Measurements*]

Efficient quantum measurements associated with only positive state transformers are called Lüders measurements. In this case, the state transformer associated with effect E_α is given by $\mathcal{R}_\alpha = \sqrt{E_\alpha}$ (i.e the measurement back action is trivially given by the identity on $\mathcal{H}$).

A large part of the quantum measurement literature is concerned only with Lüders measurements. The first results covering classes of non projective measurements and showing for them that no signalling entails commutativity considered special cases of Lüders measurements [77]. The more interesting one is the case of a minimal POVM associated with Lüders state transformers. Busch et al. were able to show that each effect which obeys the no signalling condition with respect to such state transformers necessarily commutes with the latter[93]. The proof given here is similar to the proof in [163].

[93] The other case shown by Busch et al. was that an effect which has discrete spectrum of eigenvalues that can be ordered in decreasing order and which obeys the no signalling condition with respect to the state transformers associated with an arbitrary Lüders measurement necessarily commutes with these state transformers and thereby with the associated effects.

Theorem 2.9 [*Busch, Singh*]

Let E be an effect acting on Hilbert space $\mathcal{H}$. Consider a quantum measurement associated with a minimal two element POVM $\{F_1, F_0\}$, where $F_0 = \mathbb{1}_{\mathcal{H}} - F_1$, acting on $\mathcal{H}$ which is associated with generalized Lüders state transformers $\mathcal{R}_1 = \sqrt{F_1}$ and $\mathcal{R}_0 = \sqrt{F_0}$. Then E fulfills the following condition (which is a no signalling condition if E is an element of a POVM) with respect to $\{\mathcal{R}_1, \mathcal{R}_0\}$:

(i) $E = \sqrt{F_1}E\sqrt{F_1} + \sqrt{\mathbb{1}_{\mathcal{H}} - F_1}E\sqrt{\mathbb{1}_{\mathcal{H}} - F_1}$

if and only if

(ii) $[E, \mathcal{R}_1] = [E, \mathcal{R}_0] = [E\,,\, F_1] = [E\,,\, F_0] = 0$

Proof: Implication *(ii)* $\Rightarrow$ *(i)* follows from corollaries 2.4 and 2.5.

In order to get *(i)* $\Rightarrow$ *(ii)*, suppose condition (i) holds and multiply it by $\sqrt{F_1}$ from the left and from the right. We thus get

$$\begin{aligned}
&\sqrt{F_1}E\sqrt{F_1} = F_1EF_1 + \sqrt{\mathbb{1}_{\mathcal{H}} - F_1}\sqrt{F_1}E\sqrt{F_1}\sqrt{\mathbb{1}_{\mathcal{H}} - F_1} = \\
&\overset{(i)}{=} F_1EF_1 + \sqrt{\mathbb{1}_{\mathcal{H}} - F_1}\left(E - \sqrt{\mathbb{1}_{\mathcal{H}} - F_1}E\sqrt{\mathbb{1}_{\mathcal{H}} - F_1}\right)\sqrt{\mathbb{1}_{\mathcal{H}} - F_1} = \\
&= F_1EF_1 - (\mathbb{1}_{\mathcal{H}} - F_1)E(\mathbb{1}_{\mathcal{H}} - F_1) + \sqrt{\mathbb{1}_{\mathcal{H}} - F_1}E\sqrt{\mathbb{1}_{\mathcal{H}} - F_1} = \\
&\overset{(i)}{=} F_1EF_1 - (\mathbb{1}_{\mathcal{H}} - F_1)E(\mathbb{1}_{\mathcal{H}} - F_1) + E - \sqrt{F_1}E\sqrt{F_1} = \\
&= F_1E + EF_1 - \sqrt{F_1}E\sqrt{F_1}
\end{aligned} \tag{2.57}$$

and consequently:

$$0 = F_1E + EF_1 - 2\sqrt{F_1}E\sqrt{F_1} = \left[\sqrt{F_1}, \left[\sqrt{F_1}, E\right]\right] \ . \tag{2.58}$$

For the commutator $[\sqrt{F_1}, E]$ is skew-adjoint, $i[\sqrt{F_1}, E]$ is a self-adjoint operator on $\mathcal{H}$ and $\left[\sqrt{F_1}, i[\sqrt{F_1}, E]\right] = 0$ (which follows immediately from (2.58)) implies for any spectral projection P of $\sqrt{F_1}$ (and by that also of F_1) that $\left[P, i[\sqrt{F_1}, E]\right] = 0$ and thus

$$\left[P, \left[\sqrt{F_1}, E\right]\right] = 0 \ . \tag{2.59}$$

Since now $[P, \sqrt{F_1}] = 0$ holds anyway we may apply the Jacobi-identity in order to permute $\sqrt{F_1}$ into the first slot of the double commutator and then rerun the foregoing argument:

$$\left[\sqrt{F_1}, [E, P]\right] = \left[\sqrt{F_1}, [E, P]\right] + \left[P, \left[\sqrt{F_1}, E\right]\right] + \left[E, \left[P, \sqrt{F_1}\right]\right] = 0 \tag{2.60}$$

and therefore (as above):

$$0 = [P, [E, P]] = 2PEP - EP - PE \ . \tag{2.61}$$

Multiplying (2.61) by P from the right yields

$$EP = PEP = (PEP)^\dagger = (EP)^\dagger = PE \tag{2.62}$$

i.e. $[E, P] = 0$ and consequently

$$0 = [E, F_1] = [E, \sqrt{F_1}] = [E, \mathcal{R}_1] = [E, \sqrt{\mathbb{1}_\mathcal{H} - F_1}] = [E, \mathcal{R}_0] \tag{2.63}$$

■

Let us briefly discuss the physical relevance of this result: Suppose the minimal POVM $\{F_1, F_0\}$ is associated with a local measurement performed in some bounded space-time region $\mathcal{O}$. We might think of a fundamental yes/no measurement[94] like a detector which does either click with probability $\langle\psi|\, F_1\, \psi\rangle$ or not with probability $\langle\psi|\, F_0\, \psi\rangle$. If now the effect E belongs to a POVM associated with a measurement performed in a space-time region $\mathcal{O}'$ which is spacelike separated with respect to $\mathcal{O}$, we should demand that E satisfies the no signalling condition with respect to $\{\mathcal{R}_1, \mathcal{R}_0\}$ (if not, also the related relativistic consistency conditions were violated!). According to theorem (2.9) this entails that E necessarily commutes with $\mathcal{R}_1$ and $\mathcal{R}_0$ and thereby with F_1 and F_0 as well.

We will encounter in chapter 3 that this property, together with very few reasonable additional relativistic requirements (essentially space-time translation covariance and energy bounded from below) entails that it is mathematically not possible to combine two or more such $\{F_1, F_0\}$-detectors to a *here-or-there* measurement, i.e. a measurement for which the probability that more than one detector clicks at the same time is precisely zero. But the latter is exactly what we would expect for a very relevant class of quantum experiments, e.g. for detection experiments with several spacially disjoint particle detectors where the initial state is the state of a single particle or a single bound quantum system like an atom or a molecule or a tennis ball... We shall try to understand and solve this puzzle later.

In view of theorem 2.9, it is indeed possible to derive a more general result for Lüders measurements: We can omit the detour via no signalling and derive directly from the relativistic consistency conditions that local commutativity must hold for two spacelike separated measurements, given the state transformers are selfadjoint operators. This in particular includes arbitrary measurements of the Lüders type which are always associated with positive and thus selfadjoint state transformers. The following result was essentially proven in [163] in a very different context:

[94] One might suggest that this result is in a sense applicable to any Lüders measurement, since any POVM can be decomposed into two element POVMs: Given any effect F_1 from a given POVM, we can always build the minimal yes/no POVM $\{F_1, \mathbb{1}_\mathcal{H} - F_1\}$. But note that if the actual POVM has actually more than two elements $\{F_1, F_2, F_3, \dots\}$ and consequently $\mathbb{1}_\mathcal{H} - F_1 = \sum_{k\geq 2} F_k$, the minimal POVM describes a coarse grained measurement (see section 1.2) and thus $\sqrt{\mathbb{1}_\mathcal{H} - F_1}$ is not the Lüders state transformer associated with the effect $\mathbb{1}_\mathcal{H} - F_1$, but one has rather to describe the associated state transformation on the density operator level, namely the state transformer is then given by $\rho \mapsto \mathcal{W}(\rho) = \sum_{k\geq 2} \sqrt{F_k}\rho\sqrt{F_k}$.

Theorem 2.10 [*Relativistically Consistent Lüders Measurements Locally Commute*]

Let $\mathcal{R}_\alpha^A$ and $\mathcal{R}_\beta^B$ be selfadjoint bounded operators acting on Hilbert space $\mathcal{H}$, $E_\alpha^A := \mathcal{R}_\alpha^A\mathcal{R}_\alpha^A$ and $E_\beta^B := \mathcal{R}_\beta^B\mathcal{R}_\beta^B$, such that

$$\mathcal{R}_\alpha^A E_\beta^B \mathcal{R}_\alpha^A = \mathcal{R}_\beta^B E_\alpha^A \mathcal{R}_\beta^B \tag{2.64}$$

Then

$$\left[E_\alpha^A\,,\,\mathcal{R}_\beta^B\right] = \left[E_\beta^B\,,\,\mathcal{R}_\alpha^A\right] = 0 \tag{2.65}$$

Proof: First we write equation (2.64) as

$$\begin{array}{c} \mathcal{R}_\alpha^A E_\beta^B \mathcal{R}_\alpha^A = \mathcal{R}_\alpha^A\mathcal{R}_\beta^B\mathcal{R}_\beta^B\mathcal{R}_\alpha^A = (\mathcal{R}_\beta^B\mathcal{R}_\alpha^A)^\dagger\mathcal{R}_\beta^B\mathcal{R}_\alpha^A \overset{(2.64)}{=} \\ = \mathcal{R}_\beta^B E_\alpha^A \mathcal{R}_\beta^B = \mathcal{R}_\beta^B\mathcal{R}_\alpha^A\mathcal{R}_\alpha^A\mathcal{R}_\beta^B = \mathcal{R}_\beta^B\mathcal{R}_\alpha^A(\mathcal{R}_\beta^B\mathcal{R}_\alpha^A)^\dagger \end{array} \tag{2.66}$$

Hence, $\mathcal{R}_\beta^B\mathcal{R}_\alpha^A$ and by that its adjoint $\mathcal{R}_\alpha^A\mathcal{R}_\beta^B$ are normal operators (commute with their adjoints).

Due to Putnam's generalization of the Fuglede theorem[95], given two normal operators F, G and a third bounded operator C acting on $\mathcal{H}$, such that $FC = CG$, it follows that $F^\dagger C = CG^\dagger$. Consequently, we set $F := \mathcal{R}_\alpha^A\mathcal{R}_\beta^B$ and $G := \mathcal{R}_\beta^B\mathcal{R}_\alpha^A$ which are as we have seen normal. If now $C := \mathcal{R}_\alpha^A$, obviously we have $FC = CG \equiv \mathcal{R}_\alpha^A\mathcal{R}_\beta^B\mathcal{R}_\alpha^A$ and utilizing the Fuglede-Putnam theorem gives

$$F^\dagger C = \mathcal{R}_\beta^B\mathcal{R}_\alpha^A\mathcal{R}_\alpha^A = CG^\dagger = \mathcal{R}_\alpha^A\mathcal{R}_\alpha^A\mathcal{R}_\beta^B \tag{2.67}$$

i.e.

$$\left[E_\alpha^A\,,\,\mathcal{R}_\beta^B\right] = 0 \tag{2.68}$$

Setting on the other hand $C := \mathcal{R}_\beta^B$ and applying the Fuglede-Putnam theorem to relation $GC = CF \equiv \mathcal{R}_\beta^B\mathcal{R}_\alpha^A\mathcal{R}_\beta^B$, we obtain analogously

$$\left[E_\beta^B\,,\,\mathcal{R}_\alpha^A\right] = 0 \tag{2.69}$$

■

Lüders state transformers are positive operators and thereby selfadjoint. Thus, according to theorem (2.10), two spacelike separated Lüders measurements satisfying relativistic consistency must also satisfy local commutativity. According to corollary 2.4, relativistic consistency has always no signalling as a consequence and thus we obtain with theorem 2.10 that relativistic consistency implies both, (local) commutativity and no signalling, for Lüders measurements. But this does not mean that in the case of Lüders measurements no signalling has (local) commutativity as a logical consequence[96] (but according to theorem 2.9 this is the case for Lüders measurements associated with minimal POVMs).

[95] The theorem of Fuglede [142] gives an affirmative answer to the question posed by von Neumann [333], whether any bounded operator which commutes with some normal operator necessarily commutes with its adjoint as well. Putnam [271] generalized Fuglede's result: $FC = CF \;\Rightarrow\; F^\dagger C = CF^\dagger$ to: $FC = CG \;\Rightarrow\; F^\dagger C = CG^\dagger$ whenever F and G are normal and C is bounded.

[96] Corollary 2.4 together with theorem 2.10 yield for Lüders measurements: *'Local Commutativity ⇔ Relativistic Consistency ⇒ No Signalling'*, but not *'No Signalling ⇒ Local Commutativity'*.

Indeed, formal Lüders measurements associated with more than two formal state transformers $\mathcal{R}_\alpha = \sqrt{E_\alpha}$, $\alpha = 1, 2, 3...$ can be mathematically constructed, such that there are effects F which satisfy the no signalling condition $F = \sum_\alpha \mathcal{R}_\alpha F \mathcal{R}_\alpha$ with respect to the Lüders state transformers but $[F, \mathcal{R}_\alpha] \neq 0$ (see e.g. [338]). But such an effect F cannot be an element of a Lüders POVM which is relativistically consistent with respect the fist measurement, since (the proof of) theorem 2.10 implies $[F, \sqrt{E_\alpha}] \neq 0 \Rightarrow \sqrt{E_\alpha} F \sqrt{E_\alpha} \neq \sqrt{F} E_\alpha \sqrt{F}$.

In the foregoing discussion, we still only considered a very special class of measurements. But realistic measurements like experiments with detectors are presumably often not of the Lüders type, i.e. we can in general not neglect possible measurement back action in realistic detection type scenarios or other local measurements. This will be discussed in more detail below, but beforehand, we shall return to the Lüders theorem 2.7 and generalize its proof from above by relaxing the projection assumption. This yields a result which applies to a class of more general quantum measurements which are not necessarily of the Lüders type. The above proof of theorem 2.7 is well suited for the obvious generalization to exchange the projection property with the more general property that the squared effects also obey the respective no signalling conditions (see below for discussion of the physical relevance of this condition):

Squared Effects No Signalling Conditions

The following theorem is a variation of lemma 3.3 in [16]:

Theorem 2.11 [*Squared Effects No Signalling Conditions*]

Let E be an effect acting on Hilbert space $\mathcal{H}$ associated with the outcome of a measurement and $\{\mathcal{R}_\alpha\}$ a set of state transformers acting on $\mathcal{H}$ with $\sum_\alpha \mathcal{R}_\alpha^\dagger \mathcal{R}_\alpha = \mathbb{1}_\mathcal{H}$. Then the following conditions are equivalent:

(i) Commutativity:

$$[E, \mathcal{R}_\alpha] = [E, \mathcal{R}_\alpha^\dagger] = 0 \text{ for all } \alpha \tag{2.70}$$

(ii) No Signalling Condition for E and E^2 with respect to $\{\mathcal{R}_\alpha\}$:

$$E = \sum_\alpha \mathcal{R}_\alpha^\dagger E \mathcal{R}_\alpha \quad \text{and} \quad E^2 = \sum_\alpha \mathcal{R}_\alpha^\dagger E^2 \mathcal{R}_\alpha \tag{2.71}$$

Proof: To see that (i) implies (ii), note that the commutativity conditions (i) imply also $[E^2, \mathcal{R}_\alpha] = [E^2, \mathcal{R}_\alpha^\dagger] = 0$ for all α such that $\sum_\alpha \mathcal{R}_\alpha^\dagger E^2 \mathcal{R}_\alpha = \sum_\alpha \mathcal{R}_\alpha^\dagger \mathcal{R}_\alpha E^2 = E^2$ (and analogously for E, one might also resort to corollary 2.5).

Now we show that (ii) implies (i): To this end, suppose conditions (ii) hold and consider the

following expression:

$$\begin{aligned}
&\sum_\alpha [E,\mathcal{R}_\alpha]^\dagger [E,\mathcal{R}_\alpha] = \sum_\alpha (\mathcal{R}_\alpha^\dagger E - E\mathcal{R}_\alpha^\dagger)(E\mathcal{R}_\alpha - \mathcal{R}_\alpha E) = \\
&= \sum_\alpha \left(\mathcal{R}_\alpha^\dagger E^2 \mathcal{R}_\alpha + E\mathcal{R}_\alpha^\dagger \mathcal{R}_\alpha E - \mathcal{R}_\alpha^\dagger E \mathcal{R}_\alpha E - E\mathcal{R}_\alpha^\dagger E \mathcal{R}_\alpha\right) = \\
&= \overbrace{\sum_\alpha \mathcal{R}_\alpha^\dagger E^2 \mathcal{R}_\alpha}^{=E^2} + E\overbrace{\left(\sum_\alpha \mathcal{R}_\alpha^\dagger \mathcal{R}_\alpha\right)}^{=\mathbb{1}_{\mathcal{H}}} E - \overbrace{\left(\sum_\alpha \mathcal{R}_\alpha^\dagger E \mathcal{R}_\alpha\right)}^{=E} E - E\overbrace{\left(\sum_\alpha \mathcal{R}_\alpha^\dagger E \mathcal{R}_\alpha\right)}^{=E} = 0
\end{aligned} \tag{2.72}$$

where we have used conditions (ii) and the normalization condition on the operators $\mathcal{R}_\alpha$ in the last step. If we take into account now the fact that $[E,\mathcal{R}_\alpha]^\dagger[E,\mathcal{R}_\alpha]$ is a positive operator for each α, we see that (2.72) can only be true, if each single term in the sum vanishes, i.e.

$$[E,\mathcal{R}_\alpha]^\dagger [E,\mathcal{R}_\alpha] = 0 \tag{2.73}$$

for all α, and consequently

$$[E,\mathcal{R}_\alpha] = 0 \tag{2.74}$$

for all α. Taking (minus) the adjoint of equation (2.74) shows that $[E,\mathcal{R}_\alpha^\dagger] = 0$.

■

Now suppose the state transformers $\{\mathcal{R}_\alpha\}$ are associated with a measurement which is 'performed' in some space-time region $\mathcal{O}$ and E is an element of a POVM associated with a particular outcome of some measurement 'performed' in some spacelike separated space-time region $\mathcal{O}'$. In this case, we should demand that E obeys the no signalling condition with respect to $\{\mathcal{R}_\alpha\}$, i.e. that the first equation in (2.71) is true (recall that violation of no signalling would also imply violation of relativistic consistency according to corollary 2.4). But applying theorem 2.11 in order to justify from this relativistic requirement the demand that the commutativity conditions (2.70) (which are in this case local commutativity conditions) should be true, also the second equation in (2.71) must hold, i.e. also E^2 must obey no signalling with respect to $\{\mathcal{R}_\alpha\}$. How can we understand this?

Now, on the one hand E^2 might fulfil the respective no signalling condition as a mathematical fact, without necessarily physical interpretation. E.g. Prunaru has shown in [269] that if the state transformers $\{\mathcal{R}_\alpha\}$ are normal, mutually commuting operators, i.e. if $[\mathcal{R}_\alpha\,,\,\mathcal{R}_{\alpha'}] = \left[\mathcal{R}_\alpha\,,\,\mathcal{R}_\alpha^\dagger\right] = 0$ for all α,α', the square E^2 of any effect E which obeys the no signalling condition with respect to these state transformers also obeys this no signalling condition. In view of theorem 2.11 this entails that any effect which obeys the no signalling condition with respect to normal, mutually commuting state transformers must obey the respective commutativity conditions (2.70).

But we may also involve physical reasoning: E^2 has a simple physical interpretation, given the associated state transformers are normal operators[97]. In this case, E^2 simply corresponds to

[97]Note that while the square A^2 of an observable operator A has always an immediate physical interpretation (we can simply associate with it the same measurement as associated with A, only the outcomes of that measurement are labelled with the squared values of the $A-$measurement, which can be easily understood by considering the respective spectral representations of A and A^2) this is in general a priori not the case for effects associated with measurements (see also section 1.6.5).

the probability that one and the same outcome (the one associated with E) is realized two times in a row, if the associated measurement is performed and immediately repeated. To see this, think of E as an effect associated with the outcome $B = \beta$ of a measurement B, i.e. with the notation used so far $E \equiv E^B_\beta$ and the associated state transformer is given by $\mathcal{R}^B_\beta$. Consequently, if $\psi \in \mathcal{H}$ is the initial state and given the free time evolution between the two measurements can be neglected, the probability to obtain two times in a row outcome $B = \beta$ is given by

$$\begin{aligned} &\mathbb{P}^\psi(B = \beta \overrightarrow{\wedge} B = \beta) = \mathbb{P}^\psi(B = \beta \mid B = \beta)\, \mathbb{P}^\psi(B = \beta) = \\ &= \frac{\langle \psi |\, (\mathcal{R}^B_\beta)^\dagger E^B_\beta \mathcal{R}^B_\beta\, \psi \rangle}{\langle \psi |\, E^B_\beta\, \psi \rangle}\, \langle \psi |\, E^B_\beta\, \psi \rangle = \langle \psi |\, (\mathcal{R}^B_\beta)^\dagger E^B_\beta \mathcal{R}^B_\beta\, \psi \rangle \end{aligned} \tag{2.75}$$

If now the state transformer $\mathcal{R}^B_\beta$ is a normal operator, we have

$$(\mathcal{R}^B_\beta)^\dagger E^B_\beta \mathcal{R}^B_\beta = (\mathcal{R}^B_\beta)^\dagger (\mathcal{R}^B_\beta)^\dagger \mathcal{R}^B_\beta \mathcal{R}^B_\beta \overset{\mathcal{R}^B_\beta \text{ normal!}}{=} (\mathcal{R}^B_\beta)^\dagger \mathcal{R}^B_\beta\, (\mathcal{R}^B_\beta)^\dagger \mathcal{R}^B_\beta = (E^B_\beta)^2 \tag{2.76}$$

which together with (2.75) entails that

$$\mathbb{P}^\psi(B = \beta \overrightarrow{\wedge} B = \beta) = \langle \psi |\, (E^B_\beta)^2\, \psi \rangle \tag{2.77}$$

i.e. $(E^B_\beta)^2$ is the effect associated with obtaining $B = \beta$ two times in a row.

Thus, in case the state transformers are normal operators, the square of the associated effects have a clear physical interpretation. Moreover, this provides us with a physical justification to demand the no signalling condition for squared effects, given the associated state transformers are normal operators: If two quantum measurements are 'performed' in respectively spacelike separated space-time regions, an effect E^B_β associated with a particular outcome $B = \beta$ of one measurement should satisfy the no signalling condition with respect to the state transformers of the distant measurement (if not, the respective relativistic consistency conditions were also violated). But the $B-$measurement might as well be performed two times in a row, and if the intermediate time interval is short enough, both of these measurements are spacelike separated from the distant measurement. In this case, we can combine the two $B-$measurements formally to a single measurement which is spacelike separated with respect to the distant measurement and the effect associated with outcome 'two times outcome $B = \beta$' is given by $(\mathcal{R}^B_\beta)^\dagger E^B_\beta \mathcal{R}^B_\beta$ which is equal to $(E^B_\beta)^2$ given $\mathcal{R}^B_\beta$ is a normal operator. Thus, also $(E^B_\beta)^2$ must satisfy the no signalling condition with respect to the state transformers of the distant measurement in this case.

To summarize, conditions (ii) of theorem 2.11 (the no signalling conditions for E and E^2 with respect to the state transformers $\{\mathcal{R}_\alpha\}$) must be naturally satisfied, given the respective measurements are spacelike separated and the state transformers associated with E are normal operators. Consequently, according to the theorem, the (local) commutativity conditions (i) must also hold in this case.

This suggests the question, whether it is natural to assume that real measurement state transformers are always given by normal operators. The answer is *no*! E.g. the state transformers of the important class of non reproducible projective measurements (projective second kind measurements) cannot be normal operators. To understand this, recall the reproducibility

property (see chapter 1): A measurement A is said to be reproducible if it reproduces the same outcome upon immediate repetition. We may formalize this as

$$\mathbb{P}^{\psi}\left(A=\alpha \mid A=\alpha\right)=\frac{\left\langle\psi\right|\left(\mathcal{R}_{\alpha}^{A}\right)^{\dagger}E_{\alpha}^{A}\mathcal{R}_{\alpha}^{A}\left|\psi\right\rangle}{\left\langle\psi\right|E_{\alpha}^{A}\left|\psi\right\rangle}\overset{!}{=}1 \tag{2.78}$$

for each initial state $\psi\in\mathcal{H}$ and for all α, i.e.

$$\left\langle\psi\right|\left(\mathcal{R}_{\alpha}^{A}\right)^{\dagger}E_{\alpha}^{A}\mathcal{R}_{\alpha}^{A}\left|\psi\right\rangle\overset{!}{=}\left\langle\psi\right|E_{\alpha}^{A}\left|\psi\right\rangle \tag{2.79}$$

for all $\psi\in\mathcal{H}$ and for all α. Since now $(\mathcal{R}_{\alpha}^{A})^{\dagger}E_{\alpha}^{A}\mathcal{R}_{\alpha}^{A}$ and E_{α}^{A} are positive operators, this implies

$$\left(\mathcal{R}_{\alpha}^{A}\right)^{\dagger}E_{\alpha}^{A}\mathcal{R}_{\alpha}^{A}\overset{!}{=}E_{\alpha}^{A} \tag{2.80}$$

for all α.

On the other hand, we know from equation (2.76) that

$$\left(\mathcal{R}_{\alpha}^{A}\right)^{\dagger}E_{\alpha}^{A}\mathcal{R}_{\alpha}^{A}=\left(E_{\alpha}^{A}\right)^{2} \tag{2.81}$$

given the state transformer $\mathcal{R}_{\alpha}^{A}$ is a normal operator. Hence, if the effect E_{α}^{A} is a projection $E_{\alpha}^{A}\equiv P_{\alpha}^{A}$, this entails

$$\left(\mathcal{R}_{\alpha}^{A}\right)^{\dagger}P_{\alpha}^{A}\mathcal{R}_{\alpha}^{A}=\left(P_{\alpha}^{A}\right)^{2}=P_{\alpha}^{A} \tag{2.82}$$

Comparing with the formalized reproducibility condition (2.80), we see that in case of projective measurements, the assumption of normal state transformers entails the reproducibility of the respective measurement. This in turn has as a consequence, that a projective non repeatable measurement (a projective measurement of the second kind) cannot be associated with state transformers which are altogether normal operators. But in the case of projective measurements, the no signalling conditions trivially imply the 'squared effects no signalling conditions' (which are simply the same conditions) anyway, without need of normality of the associated state transformers. This is why the Lüders theorem 2.7 from above does not need special assumptions on the state transformers in order to derive that for projective measurements no signalling has always (local) commutativity as a consequence, i.e. for all projective measurements local commutativity, no signalling and relativistic consistency are equivalent anyways.

A second relevant class of measurements whose associated state transformers cannot be altogether normal operators, is the class of reproducible non projective measurements: This easily follows from (2.76)

$$\left(\mathcal{R}_{\alpha}^{A}\right)^{\dagger}E_{\alpha}^{A}\mathcal{R}_{\alpha}^{A}=\left(E_{\alpha}^{A}\right)^{2} \tag{2.83}$$

if $\mathcal{R}_{\alpha}^{A}$ is a normal operator, and the reproducibility condition (2.80)

$$\left(\mathcal{R}_{\alpha}^{A}\right)^{\dagger}E_{\alpha}^{A}\mathcal{R}_{\alpha}^{A}\overset{!}{=}E_{\alpha}^{A} \tag{2.84}$$

Combining the latter two equations now entails that reproducible measurements, whose associated state transformers are normal operators, must obey

$$\left(E_{\alpha}^{A}\right)^{2}=E_{\alpha}^{A} \tag{2.85}$$

for all α, i.e. the respective measurements are necessarily projective. Consequently, non projective reproducible measurements cannot be associated with only normal state transformers. Moreover, there is apparently no evidence to guess that the state transformers associated with non repeatable, non projective measurements must normal operators in general.

Therefore, assuming normal state transformers to obtain the 'squared effects no signalling conditions' additional to the usual no signalling conditions by physical argument as done above, and thereby to apply theorem 2.11 in order to have a solid physical justification to demand local commutativity, is not justified for broad classes of potentially relevant quantum measurements.

This is different, of course, if one presupposes that each effect can be measured locally or that all measurements which can be performed locally in a given region are associated with an algebra of operators whose selfadjoint elements are observable operators and effects associated with measurements in the respective regions such that together with each effect E always E^2 is an element of this 'algebra of local observables' as well. But as explained in section 1.6.5, in this work results shall not be derived from such assumptions.

2.3.3 Summary and Sketch of Further Results

The central connections between local commutativity, no signalling and relativistic consistency established so far are sketched in Fig. 3: Lüders theorem 2.7 proves implication (b) for projective measurements, such that with the unconditional implications (indicated by *'always'* in Fig. 3) collected in corollary 2.4 implications (a) and (c) are true as well for projective measurements. Local commutativity as defined in 2.3 (local commutativity I in Fig. 3), no signalling and relativistic consistency are equivalent in this case. This also entails that the prevailing notion of local commutativity – namely the commutativity of observable operators which is included in local commutativity II in Fig. 3 – is necessarily true for projective measurements (the unconditional implication local commutativity I $\Rightarrow$ local commutativity II is given by corollary 2.5). But the latter is sufficient (implication (d)) for local commutativity I and thereby for no signalling and relativistic consistency only in case of Lüders measurements (definition 2.8, corollary 2.6).

Implication (b) – and thereby the equivalence of local commutativity I, no signalling and relativistic consistency – was also shown to hold for special classes of non projective measurements: For efficient measurements associated with minimal two-element POVMs and associated Lüders state transformers (theorem 2.9) or if the no signalling conditions for the effects of one measurement with respect to the state transformers of the other one are satisfied for the respective squared effects as well and vice versa (theorem 2.11), implication (b) must hold. The squared effects property can be physically substantiated by repeated measurement schemes in case the state transformers are normal operators, which is not true for important classes of measurement schemes though (see the discussion subsequent to theorem 2.11).

Implication (c) and thereby the equivalence of local commutativity I with relativistic consistency was proven for Lüders measurements in general by theorem 2.10. Thereby relativistic consistency implies both, local commutativity I and no signalling for Lüders measurements. But this does not entail the equivalence of all three notions, since for Lüders measurements with more than two state transformers no signalling need not have local commutativity I (and thereby relativistic consistency) as a consequence as counterexamples illustrate (see e.g. [185] and below).

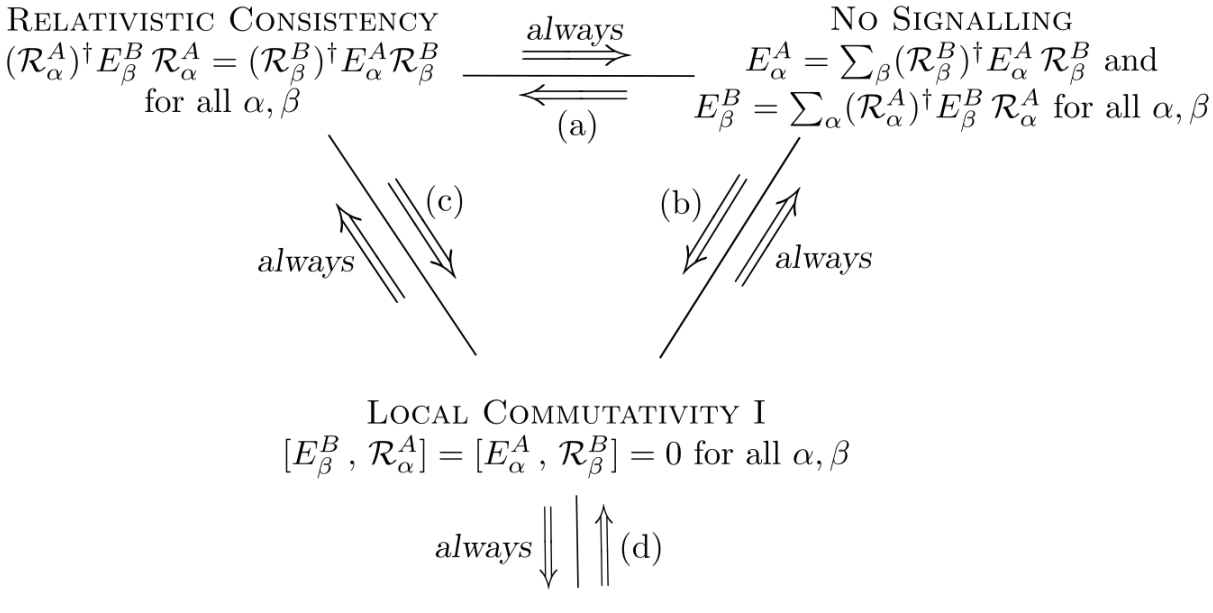

Figure 3

Summery of derived implications: The unconditional implications (indicated by 'always') are given by corollaries 2.4 and 2.5. Implication (b) holds under the assumptions of theorems 2.7 (projective measurements), 2.9 (minimal Lüders POVMs) and 2.11 (squared effects no signalling conditions). Implication (c) holds whenever the state transformers are selfadjoint (theorem 2.10) like in case of Lüders measurements. Implication (d) holds necessarily only for Lüders measurements like ideal measurements (corollary 2.6). See also the first three paragraphs of section 2.3.3.

Is it possible to derive more general results for (in general) non projective measurements whose assumptions are physically motivated without postulates about observables ('E^2 is always locally measurable when E is') and which do not resort to normal state transformers? To me, positive results are only known which treat very special cases. E.g. Heinosaari and Wolf were able to show in [185] that for $\dim(\mathcal{H}) = 2$ (qubit observables) no signalling implies in general local commutativity. But it has been shown, that for $\dim(\mathcal{H}) > 2$ this need not necessarily be the case.

All counterexamples though, which are known to me, do not cover entire constructions of formal measurements obeying the no signalling conditions with respect to each other but violating commutativity. We call two sets of formal state transformers $\{\mathcal{R}^A_\alpha\}$ and $\{\mathcal{R}^B_\beta\}$ (i.e. sets of bounded operators with $\sum_\alpha (\mathcal{R}^A_\alpha)^\dagger \mathcal{R}^A_\alpha = \sum_\beta (\mathcal{R}^B_\beta)^\dagger \mathcal{R}^B_\beta = \mathbb{1}_\mathcal{H}$) *an entire formal counterexample to implication (b)* in Fig. 3 if $E^A_\alpha = \sum_\beta (\mathcal{R}^B_\beta)^\dagger E^A_\alpha \mathcal{R}^B_\beta$ and $E^B_\beta = \sum_\alpha (\mathcal{R}^A_\alpha)^\dagger E^B_\beta \mathcal{R}^A_\alpha$ for all α, β, but there are α, β such that $\left[E^A_\alpha\,,\,\mathcal{R}^B_\beta\right] \neq 0$ or $\left[E^B_\beta\,,\,\mathcal{R}^A_\alpha\right] \neq 0$. This would be a hint that it might be possible in principle that spacelike separated measurements must not necessarily satisfy local commutativity in order to be relativistically adequate (this formal counterexample should be consistent with relativistic consistency as well, of course). Whether the measurements associated with these formal sets of state transformers are finally locally implementable at spacelike separation would then be another question, but at least the relativistic requirement of no signalling would not exclude this possibility.

But as mentioned, all formal counterexamples to implication (b) known to me are not entire: Usually, sets of formal state transformers $\{\mathcal{R}_\alpha\}$ are constructed such that an effect E can be found which satisfies $E = \sum_\alpha (\mathcal{R}_\alpha)^\dagger E \mathcal{R}_\alpha$ (which is one of the no signalling conditions if E is a POVM element associated with a formal measurement) but $[E, \mathcal{R}_\alpha] \neq 0$ for at least one α. We will give a simple explicit counterexample (with $\dim(\mathcal{H}) = 3$) even with a whole POVM instead of a single effect obeying no signalling with respect to a given set of formal state transformers but violating commutativity in the case of non efficient measurements below. But also in this example we will find satisfied no signalling relations for each effect in the POVM with respect to the given formal state transformers but no obvious decomposition of the POVM into another formal set of state transformers such that the converse no signalling relations are satisfied as well (e.g. the associated Lüders state transformers do not work). Without such a decomposition it makes no sense to check relativistic consistency because as long as it violates no signalling (in one direction) it must violate relativistic consistency as well, since the latter implies the former (note also that for each relativistic consistency condition $(\mathcal{R}^A_\alpha)^\dagger E^B_\beta \mathcal{R}^A_\alpha = (\mathcal{R}^B_\beta)^\dagger E^A_\alpha \mathcal{R}^B_\beta$ we need state transformers of both formal measurements in contrast to the single no signalling relations which relate respectively one effect with the state transformers of the other formal measurement).

A very general but abstract result, which specifies the limits of the equivalence between local commutativity and no signalling (in the weak sense sketched in the last paragraph), was presented by Arias, Gheondea and Gudder in [16]. This result is in terms of von Neumann algebras[98]. In order to formulate this result, we define the set of fixed points $\mathcal{Y}_{\{\mathcal{R}_\alpha\}}$ with respect

[98] A von Neumann algebra $\mathcal{A}$ acting on some Hilbert space $\mathcal{H}$ is a *-algebra in the set of bounded operators $\mathcal{B}(\mathcal{H})$ acting on $\mathcal{H}$ containing the identity, which is closed in the weak operator topology. Equivalently, a *-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if the commutant (see below) of the commutant of $\mathcal{A}$ (the so called double commutant of $\mathcal{A}$) is equal to $\mathcal{A}$ (the latter equivalence is the content of von Neumanns double commutant theorem).

to a given formal set of state transformers $\{\mathcal{R}_\alpha\}$ acting on some Hilbert space $\mathcal{H}$ as the subset of the set $\mathcal{B}(\mathcal{H})$ of bounded operators acting on $\mathcal{H}$, which fulfil the no signalling condition with respect to $\{\mathcal{R}_\alpha\}$, i.e.

$$\mathcal{Y}_{\{\mathcal{R}_\alpha\}} := \left\{ E \in \mathcal{B}(\mathcal{H}) \;\middle|\; E = \sum_\alpha (\mathcal{R}_\alpha)^\dagger E \, \mathcal{R}_\alpha \right\} \tag{2.86}$$

The commutant $\mathcal{A}'$ of a subset $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is the set of all operators in $\mathcal{B}(\mathcal{H})$ which commute with all operators in $\mathcal{A}$:

$$\mathcal{A}' := \{B \in \mathcal{B}(\mathcal{H}) \mid [B\,,\,A] = 0 \text{ for all } A \in \mathcal{A}\} \tag{2.87}$$

That now $\{\mathcal{R}_\alpha\}' \subseteq \mathcal{Y}_{\{\mathcal{R}_\alpha\}}$ always holds is trivial and was discussed in section 2.3.1 for effects. A more crucial question is whether or in which cases the converse implication $\mathcal{Y}_{\{\mathcal{R}_\alpha\}} \subseteq \{\mathcal{R}_\alpha\}'$ is true. Arias, Gheondea and Gudder have shown that if $\mathcal{Y}_{\{\mathcal{R}_\alpha\}} \subseteq \{\mathcal{R}_\alpha\}'$ is true for a given formal set of state transformers $\{\mathcal{R}_\alpha\}$ with $\sum_\alpha (\mathcal{R}_\alpha^A)^\dagger \mathcal{R}_\alpha^A = \mathbb{1}_\mathcal{H}$, the commutant $\{\mathcal{R}_\alpha\}'$ of $\{\mathcal{R}_\alpha\}$ must necessarily be an *injective Von-Neumann algebra*.

A von-Neumann algebra $\mathcal{A}$ acting on $\mathcal{H}$ is called *injective* if for each pair $\mathcal{C}_0, \mathcal{C}$ of C*-algebras with $\mathcal{C}_0 \subseteq \mathcal{C}$ and each completely positive map $f : \mathcal{C}_0 \to \mathcal{A}$, there exists a completely positive map $g : \mathcal{C} \to \mathcal{A}$ whose restriction to $\mathcal{C}_0$ is given by f, i.e. $g|_{\mathcal{C}_0} = f$. It can be shown (see e.g. [222]) that an equivalent definition is that a von-Neumann algebra $\mathcal{A}$ acting on some Hilbert space $\mathcal{H}$ is injective if there exists a completely positive map $P : \mathcal{B}(\mathcal{H}) \to \mathcal{A}$ with $P^2 = P$ (i.e. a completely positive projection acting on $\mathcal{B}(\mathcal{H})$, transforming each operator in $\mathcal{B}(\mathcal{H})$ into an element of $\mathcal{A}$).

The result of Arias, Gheondea and Gudder [16] shows that if we find a set of formal state transformers whose commutant is not an injective von-Neumann algebra, there are effects which obey the no signalling condition with respect to these state transformers which do not commute with the latter.

Such sets of formal state transformers have been explicitly constructed [16, 338, 233]. But these constructions seem to be rather abstract and unphysical. Usually, they are based on a standard example of a non injective von-Neumann algebra: The Hilbert space is the space $\mathcal{H} = \ell_2(\mathbb{F}_2)$ of square summable sequences over the free group on two generators $\mathbb{F}_2$. It is possible to find two unitary operators acting on $\mathcal{H}$ (a kind of shift operators) such that the algebra generated by these operators (and thereby its commutant) is not injective [16]. Form these operators it is possible to construct formal state transformers and to find effects acting on $\mathcal{H}$ which obey the no signalling condition with respect to these state transformers but do not commute with the latter. In particular, it has been shown [16] in that way that non of the two restrictions in Busch's theorem 2.9 can be relaxed in order to derive the result: An explicit example of state transformers which are not positive operators (i.e. the Lüders property is dropped) associated with a minimal POVM was given, with respect to which a certain effect obeys the no signalling condition but does not commute with the former and it was shown that there exist general Lüders state transformers (which do not correspond to a minimal POVM) such that there are effects which violate commutativity but obey the respective no signalling condition (but we have shown above, in the context of theorem 2.10, that this cannot be the case, given such an effect belongs to a POVM whose associated state transformers obey the

relativistic consistency conditions with respect to these Lüders state transformers). Counter examples for Lüders measurements with 5− and 3−element POVMs were explicitly constructed in this setting [16, 338].

But the present sketch of the mathematical background without going into details already indicates that these examples are probably rather mathematical games which do not correspond to any real world experiments. We shall present explicitly a less abstract counterexample in the context of non efficient measurements which involves two 2−element POVMs acting on Hilbert space $\mathcal{H} = \mathbb{C}^3$ below.

2.4 Non Efficient Measurements

Finally, we shall generalize the notions of relativistic consistency and no signalling and the central results concerning their connections with (local) commutativity to the case of non efficient measurements. In this case we have to resort to the density operator formalism, since final states of such measurements can in general only be described by mixed states, even if the initial state was pure (see chapter 1).

Consider two non efficient measurements A and B which shall be thought to be 'performed' in spacelike separated space-time regions. The $A-$measurement is associated with a POVM $\{E^A_\alpha\}$ and a set of operators $\{\mathcal{R}^A_{\alpha,k}\}$ acting on some Hilbert space $\mathcal{H}$ with $\sum_{k=1}^{K_\alpha}(\mathcal{R}^A_{\alpha,k})^\dagger\mathcal{R}^A_{\alpha,k} = E^A_\alpha$ (where K_α is for each alpha some integer or infinity, in the latter case the limit is to be taken in the trace norm topology) and equally, the $B-$measurement is associated with a POVM $\{E^B_\beta\}$ and a set of operators $\{\mathcal{R}^B_{\beta,l}\}$ acting on $\mathcal{H}$ with $\sum_{l=1}^{L_\beta}(\mathcal{R}^B_{\beta,l})^\dagger\mathcal{R}^B_{\beta,l} = E^B_\beta$ (L_β some integer or infinity). If a density operator ρ acting on $\mathcal{H}$ is the initial state of the $A-$measurement, outcome α is obtained with probability

$$\mathbb{P}^\rho(A = \alpha) = \mathrm{Tr}_\mathcal{H}\left[E^A_\alpha\,\rho\right] \tag{2.88}$$

in which case the final state is given by

$$\frac{\mathcal{W}_A(\rho|\alpha)}{\mathrm{Tr}_\mathcal{H}\left[\mathcal{W}_A(\rho|\alpha)\right]} = \frac{\sum_{k=1}^{K_\alpha}\mathcal{R}^A_{\alpha,k}\,\rho\,(\mathcal{R}^A_{\alpha,k})^\dagger}{\mathrm{Tr}_\mathcal{H}\left[E^A_\alpha\,\rho\right]} \tag{2.89}$$

which is in general a mixed state also if ρ was pure (given $K_\alpha > 1$).

Analogously we have for outcome $B = \beta$ of the $B-$measurement

$$\mathbb{P}^\rho(B = \beta) = \mathrm{Tr}_\mathcal{H}\left[E^B_\beta\,\rho\right] \tag{2.90}$$

in which case the final state is given by

$$\frac{\mathcal{W}_B(\rho|\beta)}{\mathrm{Tr}_\mathcal{H}\left[\mathcal{W}_B(\rho|\beta)\right]} = \frac{\sum_{l=1}^{L_\beta}\mathcal{R}^B_{\beta,l}\,\rho\,(\mathcal{R}^B_{\beta,l})^\dagger}{\mathrm{Tr}_\mathcal{H}\left[E^B_\beta\,\rho\right]} \tag{2.91}$$

Relativistic Consistency

In order to formulate now the relativistic consistency conditions for this case, we write down the sequential probabilities for outcome $A = \alpha$ given $B = \beta$ was obtained before and vice versa:

$$\begin{aligned}\mathbb{P}^\rho(A = \alpha \overleftarrow{\wedge} B = \beta) &= \mathbb{P}^\rho(A = \alpha | B = \beta)\, \mathbb{P}^\rho(B = \beta) = \\ &= \frac{\mathrm{Tr}_{\mathcal{H}}\left[E^A_\alpha \sum_{l=1}^{L_\beta} \mathcal{R}^B_{\beta,l}\, \rho\, (\mathcal{R}^B_{\beta,l})^\dagger\right]}{\mathrm{Tr}_{\mathcal{H}}\left[E^B_\beta\, \rho\right]}\ \mathrm{Tr}_{\mathcal{H}}\left[E^B_\beta\, \rho\right] = \\ &= \mathrm{Tr}_{\mathcal{H}}\left[\left(\sum_{l=1}^{L_\beta} (\mathcal{R}^B_{\beta,l})^\dagger\, E^A_\alpha\, \mathcal{R}^B_{\beta,l}\right)\ \rho\right]\end{aligned} \tag{2.92}$$

Analogously we find

$$\mathbb{P}^\rho(A = \alpha \overrightarrow{\wedge} B = \beta) = \mathrm{Tr}_{\mathcal{H}}\left[\left(\sum_{k=1}^{K_\alpha} (\mathcal{R}^A_{\alpha,k})^\dagger\, E^B_\beta\, \mathcal{R}^A_{\alpha,k}\right)\ \rho\right] \tag{2.93}$$

As argued above, if the $A-$ and the $B-$measurement are performed at spacelike separation such that there does not exist an absolute (frame independent) time order of the measurements, relativistic consistency requires

$$\mathbb{P}^\rho(A = \alpha \overleftarrow{\wedge} B = \beta) \stackrel{!}{=} \mathbb{P}^\rho(A = \alpha \overrightarrow{\wedge} B = \beta) \tag{2.94}$$

for all initial states ρ, which entails[99]

$$\sum_{l=1}^{L_\beta} (\mathcal{R}^B_{\beta,l})^\dagger\, E^A_\alpha\, \mathcal{R}^B_{\beta,l} \stackrel{!}{=} \sum_{k=1}^{K_\alpha} (\mathcal{R}^A_{\alpha,k})^\dagger\, E^B_\beta\, \mathcal{R}^A_{\alpha,k} \tag{2.96}$$

for all α and β. Equations (2.96) are the generalized relativistic consistency conditions which also apply to non efficient measurements (the relativistic consistency conditions for efficient measurements are just a special case of conditions (2.96) with $K_\alpha = L_\beta = 1$ for all α and β).

[99] The last expressions of equations (2.92) and (2.93) can be read as Hilbert-Schmidt scalar products between (adjoints of) the considered operator sums and density operator $\rho \in \mathcal{S}(\mathcal{H})$. Thus we can choose an ONB $\{\rho_n\} \subset \mathcal{S}(\mathcal{H})$ in the Hilbert space of Hilbert-Schmidt operators acting on $\mathcal{H}$, such that the equality (2.94) for all $\rho \in \mathcal{S}(\mathcal{H})$ entails

$$\left\langle \left(\sum_{l=1}^{L_\beta} (\mathcal{R}^B_{\beta,l})^\dagger\, E^A_\alpha\, \mathcal{R}^B_{\beta,l}\right)^\dagger \middle| \rho_n \right\rangle_{HS} = \left\langle \left(\sum_{k=1}^{K_\alpha} (\mathcal{R}^A_{\alpha,k})^\dagger\, E^B_\beta\, \mathcal{R}^A_{\alpha,k}\right)^\dagger \middle| \rho_n \right\rangle_{HS} \tag{2.95}$$

for all n, which in turn entails (2.96).

No Signalling

The consequent no signalling conditions

$$\sum_{\alpha} \mathbb{P}^{\rho}(A = \alpha \overrightarrow{\wedge} B = \beta) \stackrel{!}{=} \mathbb{P}^{\rho}(B = \beta) \tag{2.97}$$

and

$$\sum_{\beta} \mathbb{P}^{\rho}(A = \alpha \overleftarrow{\wedge} B = \beta) \stackrel{!}{=} \mathbb{P}^{\rho}(A = \alpha) \tag{2.98}$$

for all α, β and ρ follow as operator conditions if we sum equations (2.96) over all α, respectively β and utilize $\sum_{\alpha} E^A_{\alpha} = \sum_{\beta} E^B_{\beta} = \mathbb{1}_{\mathcal{H}}$, as

$$E^A_{\alpha} \stackrel{!}{=} \sum_{\beta} \sum_{l=1}^{L_\beta} (\mathcal{R}^B_{\beta,l})^{\dagger} \, E^A_{\alpha} \, \mathcal{R}^B_{\beta,l} \tag{2.99}$$

for all α and

$$E^B_{\beta} \stackrel{!}{=} \sum_{\alpha} \sum_{k=1}^{K_\alpha} (\mathcal{R}^A_{\alpha,k})^{\dagger} \, E^B_{\beta} \, \mathcal{R}^A_{\alpha,k} \tag{2.100}$$

for all β.

Conditions (2.99) and (2.100) are the generalized no signalling conditions which also apply to non efficient measurements (the no signalling conditions for efficient measurements are again given by the special case $K_\alpha = L_\beta = 1$ for all α and β).

Local Commutativity

The commutativity conditions

$$\left[E^A_{\alpha} \, , \, \mathcal{R}^B_{\beta,l}\right] \quad \text{and} \quad \left[E^B_{\beta} \, , \, \mathcal{R}^A_{\alpha,k}\right] \tag{2.101}$$

for all α, β, k, l are obviously sufficient to guarantee the validity of the no signalling conditions (2.99) and (2.100). That they are also sufficient to guarantee the relativistic consistency conditions (2.96) is also easy to see and in complete analogy to the case of efficient measurements:

$$\begin{aligned} &\sum_{l=1}^{L_\beta} (\mathcal{R}^B_{\beta,l})^{\dagger} \, E^A_{\alpha} \, \mathcal{R}^B_{\beta,l} \stackrel{(2.101)}{=} \sum_{l=1}^{L_\beta} (\mathcal{R}^B_{\beta,l})^{\dagger} \mathcal{R}^B_{\beta,l} \, E^A_{\alpha} = \sum_{k=1}^{K_\alpha} E^B_{\beta} \, (\mathcal{R}^A_{\alpha,k})^{\dagger} \mathcal{R}^A_{\alpha,k} = \\ &= \sum_{k=1}^{K_\alpha} \left(\mathcal{R}^A_{\alpha,k} \, E^B_{\beta}\right)^{\dagger} \mathcal{R}^A_{\alpha,k} \stackrel{(2.101)}{=} \sum_{k=1}^{K_\alpha} \left(E^B_{\beta} \, \mathcal{R}^A_{\alpha,k}\right)^{\dagger} \mathcal{R}^A_{\alpha,k} = \sum_{k=1}^{K_\alpha} (\mathcal{R}^A_{\alpha,k})^{\dagger} \, E^B_{\beta} \, \mathcal{R}^A_{\alpha,k} \end{aligned} \tag{2.102}$$

for all α and β. Note also, that the commutativity conditions (2.101) always entail $\left[E^A_{\alpha} \, , \, E^B_{\beta}\right]$ but in general not vice versa.

The central results concerning necessity of commutativity for no signalling (and by that for relativistic consistency) developed above – the 'squared effects theorem' 2.11 and the Lüders theorem 2.7 which is a special case of the former – also easily generalize to non efficient measurements:

Theorem 2.12

Let E be an effect acting on Hilbert space $\mathcal{H}$ and $\{\mathcal{R}_{\alpha,k}\}$ a set of bounded operators acting on $\mathcal{H}$ such that the effects $E_\alpha = \sum_{k=1}^{K_\alpha} \mathcal{R}_{\alpha,k}^\dagger \mathcal{R}_{\alpha,k}$ obey $\sum_\alpha E_\alpha = \mathbb{1}_\mathcal{H}$. Then the following conditions are equivalent:

(i) Commutativity:

$$[E, \mathcal{R}_{\alpha,k}] = [E, \mathcal{R}_{\alpha,k}^\dagger] = 0 \text{ for all } \alpha \text{ and } k \tag{2.103}$$

(ii) No Signalling Condition for E and E^2 with respect to $\{\mathcal{R}_{\alpha,k}\}$:

$$E = \sum_\alpha \sum_{k=1}^{K_\alpha} \mathcal{R}_{\alpha,k}^\dagger E\, \mathcal{R}_{\alpha,k} \quad \text{and} \quad E^2 = \sum_\alpha \sum_{k=1}^{K_\alpha} \mathcal{R}_{\alpha,k}^\dagger E^2\, \mathcal{R}_{\alpha,k} \tag{2.104}$$

Proof: $(i) \Rightarrow (ii)$ is trivial and was discussed above (note in particular, that $[E, \mathcal{R}_{\alpha,k}] = 0 \Rightarrow [E^2, \mathcal{R}_{\alpha,k}] = 0$ for all $\alpha, k \Rightarrow E^2 = \sum_\alpha \sum_{k=1}^{K_\alpha} \mathcal{R}_{\alpha,k}^\dagger E^2\, \mathcal{R}_{\alpha,k}$).

To prove implication $(ii) \Rightarrow (i)$ is in complete analogy to proof for efficient measurements, theorem 2.11. Hence, we only sketch the central step for the present case: Suppose equations (2.104) hold and consider the following expression

$$\begin{aligned} &\sum_\alpha \sum_{k=1}^{K_\alpha} [E\,,\, \mathcal{R}_{\alpha,k}]^\dagger\, [E\,,\, \mathcal{R}_{\alpha,k}] = \\ &= \sum_\alpha \sum_{k=1}^{K_\alpha} \left(\mathcal{R}_{\alpha,k}^\dagger E^2 \mathcal{R}_{\alpha,k} + E \mathcal{R}_{\alpha,k}^\dagger \mathcal{R}_{\alpha,k} E - \mathcal{R}_{\alpha,k}^\dagger E \mathcal{R}_{\alpha,k} E - E \mathcal{R}_{\alpha,k}^\dagger E \mathcal{R}_{\alpha,k} \right) \overset{(2.104)}{=} 0 \end{aligned} \tag{2.105}$$

where we have used in the last step besides the no signalling conditions (2.104) the normalization property $\sum_\alpha \sum_{k=1}^{K_\alpha} \mathcal{R}_{\alpha,k}^\dagger \mathcal{R}_{\alpha,k} = \sum_\alpha E_\alpha = \mathbb{1}_\mathcal{H}$. As above now, since each term $[E\,,\, \mathcal{R}_{\alpha,k}]^\dagger\, [E\,,\, \mathcal{R}_{\alpha,k}]$ in the sum is a positive operator, equation (2.105) implies that each term in the sum vanishes, i.e.

$$[E\,,\, \mathcal{R}_{\alpha,k}]^\dagger\, [E\,,\, \mathcal{R}_{\alpha,k}] = 0 \tag{2.106}$$

for all α and k, which in turn implies

$$[E\,,\, \mathcal{R}_{\alpha,k}] = 0 \tag{2.107}$$

for all α and k.

■

Observe that if $E \equiv P$ is a projection, $E^2 \equiv P^2 = P \equiv E$ trivially satisfies the no signalling condition with respect to $\{\mathcal{R}_{\alpha,k}\}$ given E satisfies this condition. In consequence, we easily recover from theorem 2.12 the Lüders theorem 2.7 generalized to non efficient measurements, given the two measurements associated with E and $\{\mathcal{R}_{\alpha,k}\}$, respectively, are (non efficient) projective measurements.

Note also, that the discussion subsequent to theorem 2.11 regarding state transformers which are normal operators (or not) can be carried over one to one to the present case of non efficient measurements due to linearity in α of the respective mathematical expressions, by simply supplementing each index α by the index pair α, k in these expressions. It is easy to comprehend, that the mathematical as well as physical arguments stay valid with the obvious generalizations.

Also the above mentioned result of Arias, Gheondea and Gudder (if no signalling implies commutativity, the commutant of the set of state transformers must be an injective von Neumann algebra) is not constrained to efficient measurements. Actually, these authors present their result in terms of unital quantum operations, which are essentially just Kraus representations of completely positive maps, where the unitality property is just the normalization $\sum_\alpha \sum_{k=1}^{K_\alpha} \mathcal{R}_{\alpha,k}^\dagger \mathcal{R}_{\alpha,k} = \sum_\alpha E_\alpha = \mathbb{1}_\mathcal{H}$ of the Kraus operators. Thus the theorem of of Arias, Gheondea and Gudder also applies to the present case of non efficient measurements.

2.5 A Formal Counterexample

Finally, as promised a simple example of explicit formal measurement operators which satisfy no signalling conditions but not so the related commutativity conditions shall be presented (this is based on an example of Heinosaari and Wolf [185]). The example provides two POVMs acting on the Hilbert space $\mathcal{H} = \mathbb{C}^3$ and a decomposition of one of them into operators generating formal state transformers of a non-efficient measurement. With respect to these state transformers, the effects of the other POVM satisfy no-signalling conditions but not (local) commutativity.

Consider the Hilbert space $\mathcal{H} = \mathbb{C}^3$ and the POVMs $A = \{E_+^A, E_-^A\}$ and $B = \{E_+^B, E_-^B\}$ acting on $\mathcal{H}$ given by

$$E_+^A = \frac{1}{4}\begin{pmatrix} 2 & 0 & -\sqrt{2} \\ 0 & 4 & 0 \\ -\sqrt{2} & 0 & 3 \end{pmatrix}, \quad E_-^A = \frac{1}{4}\begin{pmatrix} 2 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 1 \end{pmatrix} \tag{2.108}$$

and

$$E_+^B = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_-^B = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.109}$$

Now consider the following operators:

$$\begin{gathered} \mathcal{R}_{+1}^A = \frac{1}{2}\begin{pmatrix} \sqrt{2} & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{R}_{+2}^A = \frac{1}{10}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sqrt{10} & 2\sqrt{10} \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathcal{R}_{+3}^A = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{R}_{+4}^A = \frac{1}{10}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\sqrt{10} & \sqrt{10} \\ 0 & 0 & 0 \end{pmatrix} \end{gathered} \tag{2.110}$$

and

$$\mathcal{R}_-^A = \frac{1}{2}\begin{pmatrix} \sqrt{2} & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.111}$$

The following facts are straightforwardly calculated:

The operators $\mathcal{R}^A_{+k}$, $\mathcal{R}^A_{+k}$, E^A_+ and E^A_- satisfy the relations $\sum_{k=1}^4 (\mathcal{R}^A_{+k})^\dagger \mathcal{R}^A_{+k} = E^A_+$ and $(\mathcal{R}^A_-)^\dagger \mathcal{R}^A_- = E^A_-$ such that we can associate with them formal state transformers of a formal non efficient measurement associated with the POVM A given by $\mathcal{W}(\rho \mid +) = \sum_{k=1}^4 \mathcal{R}^A_{+k}\, \rho\, (\mathcal{R}^A_{+k})^\dagger$ and $\mathcal{W}(\rho \mid -) = \mathcal{R}^A_-\, \rho\, (\mathcal{R}^A_-)^\dagger$ for all $\rho \in \mathcal{S}(\mathbb{C}^3)$. Moreover, the effects E^B_+ and E^B_- satisfy the no signalling conditions

$$E^B_+ = \sum_{k=1}^4 (\mathcal{R}^A_{+k})^\dagger E^B_+ \mathcal{R}^A_{+k} + (\mathcal{R}^A_-)^\dagger E^B_+ \mathcal{R}^A_- \tag{2.112}$$

and

$$E^B_- = \sum_{k=1}^4 (\mathcal{R}^A_{+k})^\dagger E^B_- \mathcal{R}^A_{+k} + (\mathcal{R}^A_-)^\dagger E^B_- \mathcal{R}^A_- \tag{2.113}$$

with respect to the $A-$measurement.

On the other hand, the related commutativity conditions are violated (note that the effects $E^A_\pm$ do not commute with the effects $E^B_\pm$, such that the latter effects cannot commute with all state transformers associated with A, of course, it can be also easily directly calculated that e.g. $\left[E^B_+\,,\, \mathcal{R}^A_{+1}\right] \neq 0$ etc.). In view of theorem 2.12, this entails that the effects $(E^B_+)^2$ and $(E^B_-)^2$ cannot satisfy the no signalling conditions with respect to the given state transformers of the $A-$measurement, which is easily verified by direct calculations as well, of course.

But note that – as already mentioned – this is not an entirely completed formal counterexample, since if the POVMs were associated with spacelike separated measurements we would demand not only that the $B-$measurement satisfies the no signalling conditions with respect to the $A-$measurement, but also vice versa. And it is not clear whether state transformers associated with the POVM B exist, with respect to which the effects E^A_+ and E^A_- satisfy the no signalling conditions. For example, the Lüders state transformers $\mathcal{R}^B_+ = \sqrt{E^B_+}$ and $\mathcal{R}^B_- = \sqrt{E^B_-}$ associated with the POVM B are easily found[100] and with respect to these, A does not satisfy the no signalling conditions, as the reader may verify by direct calculations. This in particular shows that formal measurement schemes defined by sets of state transformers exist, such that one measurement satisfies no signalling with respect to another one but not vice versa. As long as we lack a formal set of state transformers associated with B with respect to which A satisfies no signalling, it makes no sense to investigate the more fundamental relativistic consistency conditions for this example, since for each single relativistic consistency condition we need state transformers of both measurements and whenever these lead to violations of no signalling (like the Lüders state transformers in footnote 100) relativistic consistency cannot be satisfied since the latter has always no signalling as a consequence.

I did not succeed yet in constructing an entire counterexample with two general formal measurements which satisfy the no signalling conditions or relativistic consistency conditions with respect to each other, but violate (local) commutativity, nor to derive a result which shows the impossibility of such an entire construction. This remains an open task.

[100]

$$\sqrt{E^B_+} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \sqrt{E^B_-} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{2.114}$$

2.6 Résumé

If two quantum measurements are performed at spacelike separation, in relativistic quantum theory the operators associated with these measurements are usually required to mutually commute (this is alternately referred to as *local commutativity* or *microcausality*, or even just as *causality* or *locality*). The present analysis has shown that a general physical justification for these requirements is not as straightforward as suggested in the vast majority of the literature. When physical arguments are given to support these commutativity requirements, these arguments usually are based on the requirement that it should not be possible to send signals faster than light. We supplemented this no signalling requirement (whose justification is usually roughly speaking based on human decisions like the experimenters choice to perform a given measurement or not) with the less anthropocentric requirement, that in any (special) relativistic theory predictions for facts like positions of pointers ('pointer points onto X') should not depend on the Lorentz frame of reference and encountered, that the latter relativistic consistency requirement imposed upon the quantum description of spacelike separated experiments, immediately implies the validity of the no signalling conditions (but in general not vice versa). With respect to the local commutativity requirements, we saw that if they are imposed upon the right operators (namely that the effects associated with one measurement should commute with the state transformers of the other measurement and vice versa) they have relativistic consistency and no signalling as a trivial consequence (commutativity of observable operators or effects associated with spacelike separated measurements, which is frequently demanded in the literature, is only implied by the fundamental commutativity conditions and only in the case of ideal measurements equivalent to the latter). But in order to have a thorough physical justification to demand these commutativity conditions in general, they must not only be sufficient but necessary for the mentioned physical requirements. We have seen that only in special but physically relevant classes of quantum measurements (like projective measurements) such justification can be thoroughly given as yet. Indeed, it is possible to formally develop mathematical constructions of formal state transformers, such that a given effect or even a whole POVM satisfies the no signalling conditions but not the associated local commutativity conditions with respect to the former. An entire formal counterexample with two formal sets of state transformers satisfying no signalling and/or relativistic consistency with respect to each other but not the respective commutativity conditions seems to be outstanding.

In consequence, local commutativity for spacelike separated quantum measurements seems to be a physically well but not entirely motivated demand. There is still work to be done in order to precisely fix the meaning (or scope) of local commutativity as a physically precisely motivated requirement.

3 Local Quantum Measurement II Relativity and Particle Detectors

3.1 Introduction

In this chapter, we shall derive and examine (in a novel way) a class of results in relativistic quantum theory, which have been in the center of lively and controversial debates during the last decades. These results revolve around the (im-)possibility of proper 'localization schemes' in relativistic quantum theory, given it is only allowed for positive kinetic energies, or more generally, given the Hamiltonian is bounded from below.

Newton-Wigner

These discussions started towards the end of the 1940s, most important with the work of Pryce [270] (1948), Newton and Wigner [249] (1949) and Wightman and Schweber [344] (1954), initiated by the fact that the standard position operator does not leave the set of positive energy solutions of relativistic wave equations (like the Klein-Gordon equation or the Dirac equation) invariant: Multiplication of a positive energy wave function $\psi(\boldsymbol{x})$ by $\boldsymbol{x}$ necessarily leads to contributions of infinitely many negative energy eigenstates of the free Hamiltonian.

But states of negative kinetic energy are usually considered as unphysical, since they correspond to an energy spectrum which is unbounded from below and would lead to a catastrophe, as soon as radiation is introduced: Each particle then would continuously minimize its energy by emitting eventually an infinite amount of energy (see e.g. [158] for calculations), which does obviously not happen in the world we live in. This problem was solved for the Dirac equation in the first place heuristically by Dirac's sea picture and in an analogue but more technical language by the method of second quantization, i.e. by the transition from the one particle relativistic wave equation to the associated relativistic quantum field theory (QFT), where only the positive energy wave functions need to be considered[101], but for the price of a variable number of particles and the introduction of associated antiparticles (whose experimentally verified existence was one of the primary impressive predictive successes of relativistic quantum theory). We shall come back to the issue of eliminating the negative energy problem soon, and in greater detail in appendix A.

The work of Price, Newton, Wigner, Wightman and Schweber showed that a few reasonable assumptions which a position operator should have, determine (for each spin, respectively) a unique self adjoint operator in positive energy relativistic quantum theory: The *Newton-Wigner (NW) operator*. Interpreting the NW operator as the (new) observable operator associated with position measurements yields a very small deviation from the distribution associated with the standard position operator (i.e. from Born's rule!), which could not be detected by any experimental means, but which has tremendous and finally untenable consequences from a theoretical point of view. For example, the NW localization scheme does not lead to a conserved probability

[101] Actually, this is only true in a very special sense, since the negative energy wave functions still play an indispensable role in second quantized Dirac theory, they are only made positive energy states by charge conjugation. This corresponds to Dirac's observation that the absence of a particle (hole) in the filled Dirac sea (all negative energy states occupied by particles) appears as a quasi-particle of opposite charge and sign of kinetic energy which is interpreted as an antiparticle. See appendix A.

current and moreover, the new relativistic scheme turns out not to be relativistic: A particle which is localized (in an operational sense, i.e. measured) in a bounded spatial region with probability 1 according to the NW scheme has nonzero probability to be NW localized in any spatial region arbitrarily far apart, an arbitrarily short period of time later, i.e. to move faster than light. Strongly related with this is the fact that a NW localized state in a given Lorentz frame of reference is not NW localized in any other frame, i.e. the scheme is not Lorentz invariant in a very direct sense.

Nonetheless, in view of the fact which we shall encounter, that positive energy wave functions cannot be perfectly localized in bounded regions (i.e. cannot have compact support), the eigenstates of the NW operator – although not form invariant under Lorentz transformations – at least illustrate that positive energy wave functions can be extremely well localized (they are Bessel type functions with width of the order of the respective Compton wavelength, which can be treated as delta functions for all practical purposes). Some more discussion of qualitative aspects of the NW scheme and related schemes will be given in section 3.3.

Hegerfeldt

Later, several authors found out that the mess with the NW operator is generic in relativistic positive energy quantum theory, even if the aspiration to have a self adjoint position operator is abandoned and one is content with a non projective position measurement scheme, i.e. with a POVM on physical space (or more relativistically, on spacelike hyperplanes). One class of theorems in this direction was developed by *Hegerfeldt* [178, 183, 179, 180, 181, 182, 177], who showed that whenever the Hamiltonian is bounded from below, each spatial POVM which associates probability 1 with a given bounded spatial region and a given state, must violate relativistic causality, in the sense that it associates nonzero probability with any (arbitrarily far apart) spatial region immediately afterwards with respect to the time evolution generated by the Hamiltonian (note that the causality violations of the projective NW scheme are a special case of this, since for any projection in a PVM there are states which yield probability 1, namely the eigenstates with eigenvalue 1). A version of Hegerfeld's theorem is stated, briefly discussed and proven on the mathematical basis developed in the following section in appendix B (see also the part on Hegerfeldt's theorem without operators in section 3.3).

One might wonder if this does not contradict the causal propagation speed of waves, which solutions of relativistic wave equations have, due to their hyperbolic character [194]. To see why, let us for simplicity consider the special case of Dirac theory, whose conserved probability density directly corresponds to Born's rule from non relativistic quantum theory[102], only that $|\psi|^2$ now involves a summation over the four spinor components (i.e. it shall be read as abbreviation for $\sum_{k=1}^{4} \left|\psi^{(k)}\right|^2$ where $\psi^{(k)}$ is the k'th spinor component of ψ). So let $\mathcal{H}$ be the Hilbert space of solutions of the free Dirac equation, $\mathcal{H}_+$ its positive energy subspace and P_+ the orthogonal projection onto $\mathcal{H}_+$ in $\mathcal{H}$ (we can write $P_+ = \frac{1}{2}(\mathbb{1}_{\mathcal{H}} + \Pi)$ with the energy sign operator $\Pi =$

[102]The present considerations essentially apply to relativistic wave equations of arbitrary spin, but e.g. for the spin zero case (Klein-Gordon equation) only the positive energy solutions form a Hilbert space but not all solutions [158], such that in this case 'positive energy subspace' is actually an improper notion. Also the analogue of $|\psi|^2$ in Born's rule (the conserved density) is in the positive energy Klein-Gordon case given by the more complex expression $i(\overline{\psi}\partial_t\psi - \psi\partial_t\overline{\psi})$ (from the viewpoint of a particular frame) and strongly related the scalar product looks differently (see e.g. [300]).

$|\mathscr{H}|^{-1}\mathscr{H}$ with the free Dirac Hamiltonian $\mathscr{H} = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m$ and $|\mathscr{H}| = \sqrt{\boldsymbol{p}^2 + m^2}$ in the usual notation). In the standard position measurement scheme it is assumed that a proper position measurement finds a particle in state ψ in some spatial region $\Delta \subset \mathbb{R}^3$ with probability

$$\mathbb{P}^{\psi}(X \in \Delta) = \int_{\Delta} |\psi(\boldsymbol{x})|^2 \, d^3x = \langle \psi | \, P_{\Delta} \, \psi \rangle \tag{3.1}$$

where the action of P_Δ is in position representation simply given by multiplication by the indicator function $\chi_\Delta(\boldsymbol{x})$ of Δ. Now observe that equation (3.1) does not change if we exchange P_Δ on the right hand side by

$$P^+_\Delta := P_+ P_\Delta P_+ \tag{3.2}$$

whenever $\psi \in \mathcal{H}_+$, since $P_+ \mid_{\mathcal{H}_+} = \mathbb{1}_{\mathcal{H}_+}$, where now $P^+_\Delta : \mathcal{H}_+ \to \mathcal{H}_+$. It is now easy to see that P^+_Δ is no longer a projection but that the family of operators $\{P^+_\Delta\}$ with Δ varying in the measurable subsets of $\mathbb{R}^3$ forms a non projective POVM on physical space.

Thus Hegerfeldt's theorem entails that given for some $\psi_0 \in \mathcal{H}_+$ equation (3.1) yields $\mathbb{P}^{\psi_0}(X \in \Delta) = 1$, it follows that $\mathbb{P}^{\psi_t}(X \in \Delta') > 0$ for arbitrarily small $t > 0$ and any spatial region Δ' (of non zero Lebesgue measure) arbitrarily far away from Δ. But this cannot be, since $\int_\Delta |\psi_0(\boldsymbol{x})|^2 \, d^3x = 1 = \int_{\mathbb{R}^3} |\psi_0(\boldsymbol{x})|^2 \, d^3x$ entails that $\psi_0(\boldsymbol{x})$ vanishes almost everywhere outside of Δ and causal propagation of Dirac wave functions [321] thus entails that it vanishes at time t almost everywhere in each region which is separated from Δ by a distance greater than t (or ct if the speed of light c is not set equal to 1), and consequently $\int_{\Delta'} |\psi_t(\boldsymbol{x})|^2 \, d^3x = 0$ for Δ' sufficiently far away from Δ and t sufficiently small.

Since causal propagation and Hegerfeldt's theorem are mathematical facts, it only remains to conclude that the premise that $\mathbb{P}^{\psi}(X \in \Delta) = 1$ for some bounded spatial region Δ and some $\psi \in \mathcal{H}_+$ cannot be satisfied. Indeed, it is straightforward to show in the same way that this holds not only for bounded Δ but for any region whose complement has non zero measure. This amounts to the fact that $\psi \in \mathcal{H}_+ \Longrightarrow \mathrm{supp}(\psi) = \mathbb{R}^3$, in particular positive energy (here Dirac) wave functions cannot have compact support but have always infinite tails. But this is already well known for a long time anyhow, it can be also shown more directly with Paley-Wiener type analyticity arguments [321] which are structurally very close to the proof of Hegerfeldt's theorem and the analyticity arguments we shall derive in the next section. In terms of the quantum measurement formalism, this means that the POVM $\{P^+_\Delta\}$ does not admit perfect localization (i.e. with probability 1) in bounded regions, i.e. 1 is not among the eigenvalues of P^+_Δ whenever Δ is bounded (we shall call such POVMs *strongly unsharp* later).

Malament-Type Theorems

So Hegerfeldt's theorem does not pose a severe problem, it can be read as simply demonstrating the fact that positive energy states have infinite tails in position representation in relativistic quantum theory, which takes a little getting used to but which we have to get used to, anyway (but fapp we can always pretend that there are no tails). But there is another class of theorems, whose physical sense cannot be made transparent that way and which finally suggest, for example, that the standard position PVM of Dirac theory restricted to positive energies as explained above, does not work as a relativistically satisfying POVM for position measurements, either. These theorems will be called *Malament type theorems* in this work and will be derived and

analysed in an unconventional but physically transparent language in section 3.4. The probably most important contributions in this direction were made by Schlieder [292], Jancewicz [197], Malament [234], Busch [77, 69] and Halvorson and Clifton [174] (mathematically, all these works are based on a famous lemma of Borchers [50]).

These results are usually presented as no-go theorems asserting the *impossibility of localization schemes* with certain properties in relativistic positive energy quantum theory. A localization scheme in this sense is defined as the collection of a Hilbert space $\mathcal{H}$ which caries a unitary (strongly continuous) representation of the Poincaré group (or its covering group) whose infinitesimal generator of time translations (the energy operator) has in each Lorentz frame a positive spectrum, and a mapping $\mathcal{O} \mapsto E_{\mathcal{O}}$ of spacelike subsets $\mathcal{O}$ of space-time to effects $E_{\mathcal{O}}$ in the Heisenberg picture acting on $\mathcal{H}$, on which the space-time translation subgroup of the Poincaré group acts naturally, i.e. $E_{\mathcal{O}+x} = U(x)E_{\mathcal{O}}U^{-1}(x)$ for all regions $\mathcal{O}$ and all translations $x \in \mathbb{R}^4$. The latter is referred to as *space-time translation covariance*. A localization scheme usually refers to a POVM on physical space in a given laboratory frame, yielding the probabilities of proper position measurements, i.e. in the laboratory frame the regions $\mathcal{O}$ take the form $\mathcal{O} = (t, \Delta)$, where for each fixed time $t \in \mathbb{R}$ the regions Δ vary in the measurable subsets of $\mathbb{R}^3$ such that $\langle \psi | \, E_{(t,\Delta)} \, \psi \rangle$ is the probability that a proper detector (symbolizing any proper position measurement) covering Δ clicks at time t in the laboratory frame if ψ is the state of the measured system in the Heisenberg picture. The localization effects are then additive for each t, i.e. $E_{(t,\Delta)} + E_{(t,\Delta')} = E_{(t,\Delta\cup\Delta')}$ for $\Delta \cap \Delta' = \emptyset$, and normalized to unity: $E_{(t,\mathbb{R}^3)} = \mathbb{1}_{\mathcal{H}}$ (if all of space was covered with detectors, one of the detectors would click with certainty at time t). Moreover, space-time translation covariance entails that if $x = (s, \boldsymbol{a}) \in \mathbb{R}^4$ is a space-time translation, the probability that a detector covering $\Delta + \boldsymbol{a}$ clicks a time $t + s$ in the laboratory frame is given by $\langle \psi | \, E_{(t,\Delta)+x} \, \psi \rangle$ if ψ is the initial state.

Most Malament type theorems are no-go results about projective localization schemes (PVMs on physical space) and thus do not pose severe problems in view of the discussion of Hegerfeldt's theorem above (there are actually more arguments not to expect realistic position measurements to be exactly projective). But the most general one of these theorems, which was proven by Halvorson and Clifton, is indeed remarkable. It says that space-time translation covariant, positive energy localization schemes as described in the last paragraph are mathematically impossible – not only in the projective case but in general – if local commutativity is assumed to be true (in particular if the effects $E_{\mathcal{O}}$ commute at space-like separation, which would also be a consequence of local commutativity as defined and analysed in the preceding chapter). This for example immediately shows that the spatial positive energy Dirac POVM $\{P^+_\Delta\}$ from above (which can be easily ascertained to be together with the representation of space-time translations of Dirac theory a localization scheme of the considered type) violates local commutativity[103].

Since Malament's *'In Defence of Dogma'* [234], in which he presented a result of Schlieder [292] and Jancewicz [197] with some minor refinements (known as *Malament's theorem*) and equipped with a philosophical interpretation, some authors hold the opinion that this is about no-go theorems which show the impossibility of a particle ontology in relativistic quantum mechanics (Malament and most of his followers consider solely the projective case, but Halvorson and Clifton, who proved the general version, decisively advocate Malaments philosophical stance as well). To identify certain no-go theorems about operators as no-go theorems about ontology,

[103]This can be also shown directly, i.e. in general $P_+P_\Delta P_+P_{\Delta'}P_+ \neq P_+P_{\Delta'}P_+P_\Delta P_+$ for $\Delta \cap \Delta' = \emptyset$.

apparently without need of detailed discussion[104] of this identification, might be traced back to the fact that a physical meaning of operators is usually postulated instead of derived in quantum theory. In this work, we derived the quantum operator formalism as a practical tool to describe the statistics and state transitions of measurement (like) processes, so it should be understandable what these theorems physically assert on the purely operational level[105] (i.e. in terms of measurement outcomes), which shall be done in the present chapter (we shall come to ontology in the subsequent chapter).

Assumptions

So what do we have? We have a class of theorems which somehow seem to be in tension with the possibility of a predictive framework for position measurements in relativistic quantum theory. In view of the fact that there are detectors, bubble chambers or screens and the like in this world, with which the predictions of quantum theory are confirmed by position measurements with great success, we have to clearly state what this 'somehow' precisely means. To this

[104]Malament shortly remarks (see in particular footnote 4 in [234]), that one particular of his assumptions on the considered projections, which he refers to as *localizability*, should hold for a particle ontology but not for a field ontology, such that his no go theorem based on this assumption would rule out a particle ontology and support a field ontology. This assumption is for the considered projections a direct consequence of additivity, which is the assumption which will be questioned and whose physical meaning critically analysed in the present work, too. As Malament remarks correctly (given his projections are associated with an appropriate detector experiment), this assumption excludes initial states which are perfectly incapable of triggering two distant detectors simultaneously. With respect to this assumption, Malament explains: *'In contrast to a particle, a 'field' is spread out throughout all of space and so can, in a sense, be found in two (disjoint) places at one time.'*. Now indeed the well known infinite tails of positive energy wave functions can be argued to be related with violations of Malaments localizability assumption (as we shall do below). Although this is not in contradiction with a particle ontology, Malament might have in mind something like a 'wave function ontology'. But wave functions are not fields on physical space but on configuration space (for Fock space wave functions on the union over all $N \in \mathbb{N}$ of the $N-$particle configuration spaces), such that the connection to physical space is unclear and, moreover, without a separate collapse mechanism like in GRW a wave function ontology is heavily confronted with the measurement problem. It is more likely that Malament thinks of the fields from which QFT obtained its name. But note that these fields are operators and it is easy to make out sets of field operators (or of functions bilinear in the field operators to guarantee local commutativity in case of Fermi fields) which commute pairwise but not crosswise, which is the stuff the Kochen-Specker-Bell theorems (see section 1.3) are made of, which proof an associated 'operator ontology' to be inconsistent. See also the remark on field ontology and references therein at the end of section 4.6.2.

[105]If the probability distributions given by a POVM do not correspond to final states of a measuring device but to 'what there is' without being measured, there is no reason to expect that these operators commute at spacelike separation (note that the physical justification to demand local commutativity – i.e. no signalling, relativistic consistency – as discussed in chapter 2 crucially relies on the assumption that the effects are associated with state transformations upon measurement leading to pointer states of the device reflecting the respective distribution). Moreover, there is even no reason to restrict to a positive energy subspace, if the action of the operators is not associated with state transformations possibly leading to contributions from negative energies. Thus for Dirac theory the standard position PVM (or equivalently in $\mathcal{H}_+$ if you like the associated positive energy POVM) may well describe the statistics of particle positions, it simply reflects a $|\psi|^2-$distribution (so one need not even consider the operators but simply this distribution), without entailing any relativistic oddities or inconsistencies or stability problems. Of course, then one has to understand – if position measurements are supposed to reflect particle positions – in which way the presence of a measuring device changes the situation, which will become apparent already in the present chapter (see also the discussion of local number operators in section 3.5, which is very analogue to the present remarks on the positive energy Dirac POVM). Particle ontology will be subject of the subsequent chapter.

end, we should check the assumptions of the Malament type theorems with respect to position measurements like they are performed with particle detectors.

Very basic assumptions are that the measured system is represented by a state in a (separable) Hilbert space which caries a unitary representation of the Poincaré group[106] (or its covering group), that the statistics of measurements are given by the quantum formalism via POVMs as derived in chapter 1, that the effects (elements) of a POVM in the Heisenberg picture can be associated, in principle, with the space-time region in which the measurement 'takes place' (in the sense that it is performed in the respective spatial region at the respective times in the laboratory frame) and that the unitarily implemented space-time translations act naturally on these effects (space-time translation covariance). One might question one or several of these assumptions, but we shall not do so in this work[107].

Additional assumptions are the positive energy assumption – referred to as the *spectrum condition* – and local commutativity. In chapter 2 we have seen that there is strong evidence to keep local commutativity as a relativistic requirement not only for projective measurements but also in case of general measurements whose statistics are given by non projective POVMs (to guarantee relativistic consistency and no signalling). In the present chapter we shall therefore assume that local commutativity is true. The spectrum condition is satisfied in relativistic QFTs, and these are extremely successful in making very advanced quantum predictions for position measurements (detector responses) in terms of scattering cross sections. We shall therefore also not question that a QFT which satisfies the spectrum condition is in principle capable of making the right predictions for local detector experiments. So we shall assume in the first place that the spectrum condition is valid as well (but we shall argue below that it seems natural to expect violations of the spectrum condition for any local measurement on the first quantized level, which corresponds to particle creation and annihilation with non vanishing probability without violating the spectrum condition if these processes are lifted to the associated QFT).

The only remaining assumption underlying the Malament type theorems is *additivity* $E_{(t,\Delta)} + E_{(t,\Delta')} = E_{(t,\Delta\cup\Delta')}$ for all $\Delta \cap \Delta' = \emptyset$, which can be read as encoding the assumption that the localization scheme corresponds to a POVM on the simultaneity hyperplanes of the laboratory

[106] Actually, the Euclidian group of space-time translations as a subgroup of the Poincaré group is sufficient to derive the results, and since the Euclidian group is a subgroup of the Galilei group as well, these results are actually valid for Galileian space-time and thus for non relativistic quantum theory as well (this is equally true for Hegerfeldt's theorem, which thus illustrates that perfectly localized wave packets immediately develop infinite tails under the Schrödinger time evolution in non relativistic quantum theory, since it is always generated by a Hamiltonian with semibounded spectrum, see appendix B). Only relativistic requirements like *no superluminal wave packet spreading* or *local commutativity* are in non relativistic quantum theory neither required nor true, in general. Halvorson and Clifton show that for the usual derivation of the Malament type theorems (based on Borchers lemma) also not all of the Poincaré group is necessary, but some additional mild relativistic requirement which they refer to as *'no absolute velocity'* and which states that each spacelike translation can be decomposed into the sum of two timelike translations. In the derivation of the Malament type theorems which is developed in this chapter, no such requirement is needed.

[107] One might argue that in a very precise sense (and this precision matters when considering clean mathematical theorems, even if it can be fapp neglected) the apparatus wave functions have infinite tails as well, such that it is not perfectly legitimate to associate a measurement exactly with bounded spatial regions but only approximately. Moreover, the quantum formalism was derived in chapter 1 on basis of the assumption that apparatus wave functions do not overlap, which is not precisely true for the same reason. But both of these points can be accounted for in principle, with approximate measurement POVMs as it was demonstrated with the von Neumann scheme with extended pointer states in section 1.5.2.

frame as explained above (to be precise, the latter assumption would besides additivity entail the normalization $E_{(t,\mathbb{R}^3)} = \mathbb{1}_{\mathcal{H}}$, which plays no role for deriving the Malament type theorems though, and that the localization regions Δ can be arbitrarily small, which is not needed for the way these theorems are proven in this work). But this is actually a very special assumption: In the first place, a position measurement in a given region (represented by a detector covering that region) has two possible outcomes at a time, a positive one (detector clicks / is being triggered) and a negative one (no click). Thus we have a priori a minimal two element POVM $\left\{E_{(t,\Delta)}\ ,\ \mathbb{1}_{\mathcal{H}} - E_{(t,\Delta)}\right\}$ associated with the spatial region Δ and t. That $E_{(t,\Delta)}$ cannot be made straightforwardly into a POVM on the space on which t varies (an arrival time POVM) is well known (see e.g. [335] and references therein), now under the positive energy assumption something similar happens with respect to the space on which Δ varies. But we shall not investigate possible connections between the arrival time problem and the localization problem in relativistic quantum theory (which might be in view of the symmetry between space and time in relativity an interesting project, though) but rather work out and analyse the operational meaning of the latter.

How to Understand?

Indeed, for most possible initial states no one should expect that for any t the detector effects $E_{(t,\Delta)}$ merge to a POVM on $\mathbb{R}^3$, e.g. if these effects act on a Hilbert space which contains a linear manifold of two particle initial states such that it is trivially not excluded that two distant detectors click at the same time (or at spacelike separation) which destroys the additivity property. We shall analyse this property and its physical meaning in the framework of detector experiments in detail in section 3.4 and will show that violation of additivity – which is a consequence of the Malament type theorems under the mentioned assumptions – indeed means that there is no linear manifold of states which is perfectly incapable of triggering two proper detectors at the same time (or more generally at spacelike separation). We shall moreover argue that these results can be generalized to assert that under the assumptions there even cannot be a linear manifold of states which is incapable to trigger more than any given finite number of detectors at the same time (at spacelike separation).

How can we understand this? We shall identify two natural roots of these results: The *infinite tails* of positive energy wave functions and the *active nature detectors*, which is always capable of creating pairs from the vacuum in QFT. Both will be typically associated with extremely small probabilities, but while the active nature of detectors is a physically remarkable issue, the infinite tails might be seen as a rather technical peculiarity in relativistic positive energy quantum theory for which we shall find a couple of arguments why it is irrelevant for all practical purposes.

Infinite Tails

Recall that Hegerfeldt's theorem asserts that any state which is perfectly localized in a bounded spatial region (in the operational sense that a proper position measurement will find it there with probability 1) according to any positive energy localization scheme (space-time translation covariant position POVM), will be localized arbitrarily far away an arbitrarily short time later with (for sure small but) non vanishing probability. We identified the solution of this apparent paradox by noting that according to the standard localization scheme (Born's rule) positive energy states cannot be perfectly localized in a bounded region due to infinite tails,

such that the premise of the disturbing conclusion is never satisfied. But exactly the infinite tails of positive energy wave functions which rescue the situation for Hegerfeldt's theorem pose – if we do not care about perfect localization but only about measurement results – a somewhat milder but similarly strange problem: Suppose the final state of any (non destructive) particle detection experiment with affirmative outcome (click) on a one particle positive energy initial state is still a one particle positive energy state. Of course, we can expect that the wave function in standard position representation is very well localized about the detector region but it must have infinite tails, such that – according to Born's rule – is cannot be perfectly excluded that it triggers a far away detector immediately afterwards[108] and thus, if both events are spacelike separated, simultaneously in some Lorentz frame of reference[109].

[108]This is in prinicple also the case in any Bohmian quantum theory with infinitely extended wave functions (e.g. in positive energy Bohm-Dirac theory [41, 120]), although Bohmian particles will not move faster than light in any relativistic Bohmian model. This will be discussed in chapter 4 and may be for now loosely understood in the following way: The statistical analysis of the Bohmian equations of motion reveals a fundamental limitation (referred to as *absolute uncertainty*) of possible correlations between the configuration of a given (sub-)system and that of its environment and thus a fundamental limitation of possible transfer of 'information' (however 'information' is precisely defined) about the system configuration to any external system like a measuring device or an observer. This limitation is reflected in a $|\psi|^2$ −distribution, where ψ is the mathematically precisely defined so called *conditional wave function* of the subsystem which depends on the configuration of its environment and has in the first place no analogue in standard quantum theory, only in measurement (like) situations it corresponds to the collapsed wave function of the system. Suppose a proper detection experiment has a positive outcome (detector clicks). The conditional wave function of the 'measured particle' will thus be highly peaked about the detector region, but if it has infinite tails, there is a small but non vanishing probability of error, i.e. that the particle is actually somewhere else although the detector was triggered (note that the interaction of the measurement acts primarily on the overlapping wave functions and thereby only indirectly on the particles of measured system and measuring device which are guided by them). Thus if a subsequent further detection experiment finds the particle at its actual position as it is typically the case, there is no superluminal particle velocity involved.

But the probability for such dynamics (the first detection experiment again) will be extremely small, which means loosely speaking that there are only very few initial conditions (initial configurations of the yet untriggered detector and the 'measured particle') leading to such final configurations such that these scenarios will be empirically never realized (note that there are also sets of classical initial conditions of non zero measure on phase space for which the second law of thermodynamics is violated under the Hamiltonian dynamics, see also the remark on Cournot's principle below and section 4.4). In Bohmian mechanics, such scenarios belong to the domain of quantum non-equilibrium (see section 4.4.4), while its empirical success (reproducing the statistics of quantum theory) relies on a statistical analysis which tells us that the world actually is in quantum equilibrium. But Bohmian mechanics (with infinite tails) does not tell us that detectors necessarily can't fail in a fundamental way (but they will much more likely fail for less fundamental reasons) but as will be developed in chapter 4, its virtue lays in providing plain equations of motion which yield an unambiguous description of what happens in physical space and time (and thereby it solves the measurement problem). Only these equations give us a precise mathematical expression for the wave function of a subsystem depending on the configuration of its environment which turns out to have a dynamical and a statistical meaning for the subsystem and conditional on which there can be no further correlations between the subsystem and its environment which fundamentally limits the 'feasible gain of knowledge'. See section 4.7 for more discussion of the infinite tails issue from a Bohmain perspective.

[109]This already indicates, that there must be small deviations from Born's rule for position measurements in relativistic space-time if perfect localization in bounded spatial regions is excluded. More precisely we have (this might be regarded as the simplest no-go theorem about localization): *no perfect localization in bounded regions + relativity* $\Longrightarrow$ *no spatial POVM on* $\mathbb{R}^3$. To see this, suppose a detector was just triggered by a one particle initial state. No perfect localization implies that there is (possibly small but) non vanishing probability that a far away detector is triggered immediately afterwards, i.e. at spacelike separation. Relativity implies thus that there is a Lorentz frame, in which both detectors click simultaneously with non vanishing probability. As will

Typically, the tails of well loclized positive energy wave functions die like $e^{-\frac{|\boldsymbol{x}-\boldsymbol{x}_0|}{\lambda_C}}$ (where $\lambda_C = \frac{1}{m}$ is the respective Compton wavelength) sufficiently far away from the 'center' $\boldsymbol{x}_0$ (see footnote 125 for a rough estimate, how massively localization probabilites can be suppressed for such states already at microscopic distances). Moreover, we shall present a sequence of positive energy wave functions in section 3.3 which are practically already zero outside of the sphere of radius λ_C about their center and by which a delta function can be arbitrarily well approximated. This shows that probabilities for localization in the tails region can be made literally arbitrarily small without violating the spectrum condition. And there is no question that in the relevant situations these probabilities will be extremely small in nature: That final states of local measurements are usually extremely well localized is theoretically well understood by analysis of associated decoherence processes (see e.g. [192]) and of course strongly supported by experience. Consider for example the trajectory of a charged particle in a cloud chamber: We may perceive the joint state of an ionized atom and a condensed bulk of water atoms about it as the pointer state associated with a triggered detector, such that the trace a charged particle leaves in the cloud chamber corresponds to a more or less continuous chain of triggered detectors. That such traces never contain essential discontinuities or sharp 'spikes' (potentially associated with wave function tails) illustrates that the measured states are pretty well localized wave packets (if Born's rule is true). One might reflect similarly about particles in Penning traps or comparable continuous local detections of quantum particles over a period of time like in the fascinating experiments with single electrons for which Dehmelt [105, 106] received his Nobel price (of course, the world of very well localized objects made of atoms surrounding us is a further example standing to reason).

Pair Creation

But besides their great unlikeliness, the operationally peculiar aspect of strange scenarios due to infinite tails (distant detectors triggered by a one particle state) is based on a questionable premise, namely that the final state of local detection of an initial positive energy one particle state is a positive energy one particle state again. This brings us to the active nature of detectors: It is straightforward to argue that one of the two properties – having exclusively positive energy *or* being exactly a one particle state – can be expected to be violated for any final state of a local measurement, depending on whether we look at it from the perspective of a one (or $N-$) particle theory or an associated QFT. To understand this, consider for a moment again Dirac theory, where these things are well understood: If $\psi(\boldsymbol{x})$ is a positive energy Dirac wave function and $\varphi(\boldsymbol{x})$ any other Dirac wave function, it is easy to see that whenever $\varphi(\boldsymbol{x}) = \psi(\boldsymbol{x})$ on any set $\Delta \subset \mathbb{R}^3$ with non vanishing Lebesgue measure but $\varphi \neq \psi$, it follows that φ cannot be of positive energy but must have contributions from the negative energy spectrum (see section 3.3). For example, suppose φ is the result of interaction of $\psi \in \mathcal{H}_+$ with some external potential supported

be shown in section 3.4.2, non vanishing simultaneous click probabilities imply that the probabilities cannot be additive, i.e. cannot belong to a probability measure on $\mathbb{R}^3$ and thus are not given by a spatial POVM on $\mathbb{R}^3$ in this frame (generalization of this argument yields that they cannot be given by a spatial POVM on $\mathbb{R}^3$ in any frame). Thus Born's rule for detector clicks (representing position measurements) cannot precisely hold (of course it must hold for all practical purposes, since it was the starting point to derive the so successful quantum measurement formalism in chapter 1), since it is encoded in a POVM on $\mathbb{R}^3$, namely the PVM of the standard position operator. The same is true for obvious generalizations of Born's rule like an approximate measurement POVM (see sections 1.4.3 and 1.5.2) or any other POVM on $\mathbb{R}^3$. See sections 3.4.7 and 4.7 for discussion of these issues.

in a spatial bounded region which was switched on and off again. As a consequence of the just stated non-invariance of $\mathcal{H}_+$ under local transformations, φ can be no longer an element of $\mathcal{H}_+$ since ψ was disturbed by the local potential only locally and according to causal propagation of Dirac wave functions, local perturbations cannot propagate faster than light. Hence the final wave function $\varphi(\boldsymbol{x})$ equals the initial state $\psi(\boldsymbol{x})$ outside some bounded region, i.e. on a spatial set of infinite Lebesgue measure. The same will be argued to be the case in section 3.3 if φ is not equal to ψ outside some bounded region but is obtained from the latter in a trivial way like by simple suppression, i.e. $\varphi(\boldsymbol{x}) = C\psi(\boldsymbol{x})$ for some small constant C outside some bounded region, like one might expect for the collapse of the wave function due to local measurement respecting the positive energy requirement (and thus infinite tails) for the final state. Thus, in one particle Dirac theory, if local measurements act non trivially on wave functions only locally, final states must have contributions from negative energies[110].

Processes including transitions between negative and positive spectrum of the Hamiltonian in one particle Dirac theory correspond to pair creation and annihilation processes in the associated second quantized theory [159, 264, 290, 321], which can be lucidly understood in the Dirac sea picture[111] (see appendix A on second quantization in the Dirac sea picture). But if the active nature of a detector (which might be in view of chapter 1 represented by the associated state transformers $\mathcal{R}_{(t,\Delta)}$) is always capable of creating particles (for detectors not involving to strong potentials probably with very small probability), it is not at all surprising that the probability that two detectors click at spacelike separation cannot be perfectly zero, even if the initial state was a one particle state in Fock space, since then this action does not leave the one particle sector of Fock space invariant (this seems to be not only a feature of QED but generic in relativistic QFT).

Reeh-Schlieder

But this line of argument would entail that not only two distant detectors have non zero joint click probability at spacelike separation for one particle initial states in QFT, but that equally a single local detector must have non zero click probability in the vacuum as well[112]. And exactly

[110] One might propose to take into account that seemingly local potentials or wave functions of seemingly local measuring devices have actually infinite tails as well, to rescue the situation. In this case, indeed a violation of the positive energy assumption by the considered types of processes ('local' unitary interaction or measurements) follows no longer with necessity but still with great likelihood. The non-invariance of $\mathcal{H}_+$ under all local transformations can be directly obtained by analyticity arguments (we shall go a somewhat simpler but indirect way in section 3.3), since a positive energy wave function can be perceived as the boundary value of an analytic function (see the following section) and analyticity is extremely fragile with respect to 'transformations' since an arbitrarily small piece of an analytic function already determines the whole function on its domain of analyticity by the identity theorem. There thus seems to be little hope that e.g. under collapse like transitions of a positive energy wave function forced by an external measuring device the positive energy property survives, even if these transitions affect each piece of the wave function in a non trivial way (recall for example the 'von Neumann position measurement scheme with extended pointer states' in section 1.5.2, where this transition was morally given by multiplication of the measured state by the pointer wave function).

[111] In case of unitary transformations involving such spectral transitions, there is even a sharp mathematical criterion – the *Shale-Stinespring criterion* (see appendix A) – which determines which class of such processes does not create an infinite amount of pairs in second quantization and thus can be consistently lifted to Fock space at all.

[112] The active nature of detectors is related to a certain absurdity in the notion of 'a detector in the vacuum'. After all, a detector is made of atoms and if it is present, there is not a vacuum. Nonetheless, one might consider

this is a central prediction in the frameworks of axiomatic, respectively algebraic QFT (AQFT), which requires a bit more of assumed structure than the Malament type theorems (in particular that there is a field operator like structure from which the operators associated with local measurements can be constructed). This prediction is a corollary of the Reeh-Schlieder theorem, which we shall prove and analyse in section 3.5 and which is considered as a very important result in AQFT. Its proof relies on exactly the same mathematical structure as the Malament type theorems or the 'no compact support' and the 'no local transformations' properties of positive energy states, as the present derivation of all these results, which is based on a single mathematical theorem developed in the subsequent chapter, will show.

A rigorous analysis of the active nature of detectors with respect to pair creation phenomena and their precise role in the context of the theorems presented in this chapter, as well as rigorous estimates of the magnitudes of associated probabilities, is beyond the scope of this work but might be motivated by it and given somewhere else. The present work instead focusses on a mathematically unified and physically operational presentation of the results and a more or less qualitative discussion of encountered connections and coherencies.

Inquiry into the Improbable and Cournot's Principle

Investigating the all these results, a word of warning is in order, since their analysis constitutes for a large part an inquiry into the improbable and is thereby irrelevant for a physical understanding of how phenomena emerge.

Once Loschmidt [226] argued against Boltzmann and his microscopic derivation of thermodynamics by noting that the time reversal symmetry of the microscopic dynamics reveals an infinity of initial conditions violating the second law of thermodynamics, if the latter is justified microscopically (Loschmidt's famous *Umkehreinwand*): If we take for example any (classical) initial condition on phase space of an ideal gas contained in a gas bottle, which subsequently streams out and is finally distributed homogeneously throughout the room (which typically happens under the Newtonian dynamics), one only needs to reverse all velocities at a given time to obtain an initial condition where the homogeneous gas finally backtracks into the bottle again. Boltzmann [44] replied to this objection that the physical claim in the argument was true of course, but that we need not bother since such initial conditions won't contribute to phenomena since they are (although infinitely many) much to few. Indeed, it is easy to argue that (in Paul and Tatjana Ehrenfest's words [130]) *the overwhelming majority of initial conditions* of a spatially homogeneously distributed ideal gas remain homogeneously distributed under the Newtonian dynamics (apart from minor microscopic fluctuations) such that we need not worry to suddenly suffocate because of macroscopic fluctuations of the air molecules around. This corresponds to the fact that the second law of thermodynamics holds with probability very close to 1 and violations have probability very close to zero. And the golden rule in statistical physics, which has proven to be tremendously successful, is that processes with probability very close to 1 will happen with empirical certainty and processes with probability very close to zero will not

a detector with nothing surrounding it and suppose that this experiment can be well described by the quantum measurement scheme associated with the detector, when the vacuum state Ω is taken as the initial state. But then one has to incorporate that a switched on detector is associated with a state transformation acting on Ω (no matter if it just clicks or not).

be realized by nature (conditional on the thermodynamic non equilibrium universe we obviously live in). This is condensed in the notion of typicality, which refers to everything which need not necessarily happen according to the laws of nature, but which happens with probability very close to 1 (these notions will be made more precise and discussed in more depth in section 4.4).

Despite its vagueness by resorting to unsharp notions like 'very close to 1' or 'very small', the principle of typicality has proven to be strikingly successful in predicting or explaining empirical regularities: In a coin tossing series with many trials, typically about half of the tosses yield 'head' and a long run with exclusively 'heads' typically does not happen, typically ice cubes are melting in water but water at room temperature does not spontaneously lower its energy by thermal radiation to freeze, typically air molecules are distributed more or less homogeneously throughout a room but do not fluctuate altogether into a corner. Indeed, as everyone knows from experience, we can also drop the epithet 'typical' in these assertions, but there is no law of nature in the conventional sense which strictly forbids for example the mentioned atypical behaviour, nonetheless it will not happen.

Indeed, without such a principle it is even impossible to connect mathematical probability theory with the physical world to predict relative frequencies of certain events at all: In order to identify general *'probabilities'* with *'empirical relative frequencies'*, one proves a *law of large numbers* (a concrete example will be presented in section 4.4.4), i.e. one shows that significant deviations of empirical relative frequencies from the associated probabilities do typically not happen, i.e. have very small probability[113]. Thus we only need to require that *events with very small probability do not happen*, or equally that *events with probability very close to* 1 *will happen with empirical certainty*. During the first half of the 19th century this principle was widely known and discussed under the name *Cournot's principle* (named after Antoine-Augustin Cournot [90]), Borel called it *'the only law of chance'* [52] and Lévy referred to it as *'the only connection between probability and the empirical world'* [217], as which it was moreover advocated by Kolmogorov [209], Hadamard [171], Frechét [138] and many others.

Coming back to the issue of this chapter, the results which will be derived and discussed are in a sense structurally reminiscent of Loschmidt's 'result' (i.e. *'under Boltzmann's assumptions, violations of the second law of thermodynamics happen with non vanishing probability'*). We shall show in section 3.19, that (roughly speaking) *under certain assumptions, any quantum initial state can trigger two proper detectors at spacelike separation with non vanishing probability*. To do justice to the theorems, these probabilities can be nonetheless arbitrarily small. For most possible initial states, like states of seversal particles, it is not surprising that these probabilities do not vanish, but when a single electron in a Penning trap is continuously detected by its emitted radiation [105, 106], we need not wonder whether a nearby empty trap might be spontaneously triggered, of course. Probabilities for such scenarios, although being non zero (given certain assumptions are satisfied), will be extremely close to zero.

This is also in accord with the physical explanations discussed above: The infinite tails of positive energy wave functions have been (and will be more explicitly below) argued to be

[113]To sharpen this, one can distinguish between a fundamental *measure of typicality* [129, 154, 216], which does only take values very close to 0 and 1, and an associated probability measure which can be derived from the measure of typicality by the law of large numbers. The measure of typicality must be chosen by physical considerations as an appropriate measure on an appropriate space, like the stationary microcanonical or canonical measure on phase space of initial conditions in classical mechanics, which tells us which sets are huge (occupy almost all of the space) which are very small. See section 4.4

irrelevant for all practical purposes and with respect to the capability of local measuring devices to create particles from the vacuum – although associated probabilities won't be estimated in this work – it is well known that such processes become relevant only at by far higher energies than a usual device can provide [158, 159] (namely at energies of the order of the mass gap $2mc^2$).

3.2 Mathematical Groundwork: Analytic Continuation into the Imaginary Forward Lightcone

The physically interesting results we shall derive below within this chapter, are based on a mathematical fact, namely that functions of the form $f(x) = \langle\varphi|\, U(x)\,\psi\rangle$, where φ, ψ are vectors in some Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot|\,\cdot\rangle$ and $U(x)$ is the unitary representation of space-time translations $x \in \mathbb{R}^4$ in a reasonable relativistic setting (which will be made precise below), have the following mathematical property: If f vanishes in some (arbitrarily small) open region of $\mathbb{R}^4$, it follows that $f(x) = 0$ for all $x \in \mathbb{R}^4$. This result will be derived from the fact that f can be continuously extended to a function $\widetilde{f} : \mathbb{C}^4 \to \mathbb{C}$ which is holomorphic in the domain $\mathcal{D} = \{z = x + iy \in \mathbb{C}^4 \mid x \in \mathbb{R}^4,\ y \in V_+\}$, where $V_+ := \{y \in \mathbb{R}^4 \mid y_0 > 0,\ y_\mu y^\mu > 0\}$ is the open forward light cone (i.e. f is the continuous boundary value of an analytic function with domain of analyticity $\mathcal{D}$).

To show these things, we shall provide a very general relativistic Hilbert space framework below, in which they will prove to be valid. But first, we need to work into some general complex analysis of several complex variables as a basis. The results which will be derived in the following section are well known and have various applications in different fields, in particular in the context of AQFT, nonetheless the way of derivation and its detailed presentation in this work is for the most part unique (at least to my knowledge).

3.2.1 Some Analysis of Several Complex Variables

We start with just stating Bogoliubov's edge-of-the-wedge theorem:

Theorem 3.1 [*Edge-of-the-Wedge*]

Let V be an open convex cone of $\mathbb{R}^n$ with vertex at the origin and $\mathcal{O} \subseteq \mathbb{R}^n$ some open subset of $\mathbb{R}^n$. We define

$$\mathcal{D} := \mathcal{O} + iV = \{z = x + iy \mid x \in \mathcal{O}, y \in V\} \subseteq \mathbb{C}^n \quad \text{and} \quad \overline{\mathcal{D}} = \mathcal{O} - iV \tag{3.3}$$

Then there is a neighbourhood $\mathcal{N} \subseteq \mathbb{C}^n$ of $\mathcal{O}$ such that every continuous function f on $\mathcal{D} \cup \mathcal{O} \cup \overline{\mathcal{D}}$ which is holomorphic on $\mathcal{D} \cup \overline{\mathcal{D}}$ extends to a holomorphic function on $\mathcal{N}$ which coincides with f on $\mathcal{N} \cap (\mathcal{D} \cup \mathcal{O} \cup \overline{\mathcal{D}})$.

The *proof* is rather elaborate but straightforward complex analysis (only the standard techniques have to be generalized to the case of several complex variables) and can be found in the

literature for various variations of the theorem, see e.g. Streater and Wightman [313].

In the present applications we will have $n = 4$ and

$$V = V_+ = \left\{ y \in \mathbb{R}^4 \mid y_\mu y^\mu = y_0^2 - \mathbf{y}^2 > 0 \ , \ y_0 > 0 \right\} \tag{3.4}$$

i.e. V is the open forward light cone with respect to the origin (or $(V_+)^k$ for some $k \in \mathbb{N}$ for later purposes).

Next we shall derive a kind of generalized version of the Schwartz reflection principle for functions of several complex variables from the edge-of-the-wedge theorem, which is extensively used in the context of AQFT and which will be a basic tool also for the present analysis. To this end, we need first to find a suitable version of the *identity theorem for the case of holomorphic functions of several complex variables*: It is well known that, in contrast to the case of only one complex variable, in the case of several complex variables it is no longer sufficient that two holomorphic functions on a joint domain[114] $\mathcal{D} \subseteq \mathbb{C}^n$ coincide on a subset of $\mathcal{D}$ which has an accumulation point in $\mathcal{D}$ in order to conclude that they are one and the same function on all of $\mathcal{D}$. A simple counterexample is given by the two holomorphic functions $f(z, w) = z$ and $g(z, w) = z^2$ on $\mathbb{C}^2$, for which the set on which $f = g$ has obviously an accumulation point at $z = w = 0$ but of course $f \neq g$. The usual generalization of the identity theorem for holomorphic functions on a joint domain $\mathcal{D} \subseteq \mathbb{C}^n$ in the literature requires that $f = g$ on an open subset of $\mathcal{D}$ in order to conclude that $f = g$ on all of $\mathcal{D}$. But this is a too strong constraint for the present purposes and it is indeed possible to weaken this assumption in various ways to get the same conclusion.

We shall need the identity theorem in the following form:

Lemma 3.2

Let f, g be holomorphic functions on a joint domain $\mathcal{D} \subseteq \mathbb{C}^4$. If $\mathcal{D}$ contains a connected real subset $\mathcal{O} \subset \mathcal{D}$ and $\mathcal{O} \subseteq \mathbb{R}^4$ which is open as a subset of $\mathbb{R}^4$ on which $f = g$, it follows that $f = g$ throughout $\mathcal{D}$.

Proof: We can derive this result by applying the usual identity theorem for holomorphic functions of one complex variable successively componentwise:

Without loss of generality we set $g \equiv 0$ (otherwise we take $f - g$ instead of f) and assume $\mathcal{O} = I_0 \times I_1 \times I_2 \times I_3 \subseteq \mathbb{R}^4$ an open box in $\mathbb{R}^4$ and $\mathcal{D} = (I_0 + iI_0') \times (I_1 + iI_1') \times (I_2 + iI_2') \times (I_3 + iI_3') \subseteq \mathbb{C}^4$ an open box in $\mathbb{C}^4$ with the open intervals $I_n = (a_n, b_n)$ for some real numbers $a_n < b_n$ and $I_n' = (a_n', b_n')$ with $a_n' < 0 < b_n'$ (otherwise we can find such boxes within $\mathcal{O}$ and $\mathcal{D}$, respectively). Suppose now $f = 0$ on $\mathcal{O}$ and choose $(x_0, x_1, x_2, x_3) \in \mathcal{O}$. On the axis associated with x_0 the interval I_0 is a neighbourhood of x_0 such that the function $f_0(z) := f(z, x_1, x_2, x_3)$ as a function of $z \in \mathbb{C}$ vanishes for all $z \in I_0$. Furthermore, $f_0(z)$ is holomorphic for all $z \in I_0 + iI_0'$, since in this case $(z, x_1, x_2, x_3) \in \mathcal{D}$ (note that a complex function of several complex variables is complex

[114] When we speak of a 'function on a domain' we always mean 'domain' in a topological sense, i.e. an open and connected set; in contrast, the 'domain of a function' means the set on which the function is defined, which need not be open and connected, of course.

differentiable if and only if it is complex differentiable in each single variable [37, 313]). Thus, x_0 is an accumulation point of the set defined by $f_0 = 0$ and we can apply the identity theorem for holomorphic functions of one complex variable to conclude that $f_0 = 0$ for all $z \in I_0 + iI_0'$.

Now we repeat the argument for the function $f_1(w) := f(z_0, w, x_2, x_3)$ as a function of $w \in \mathbb{C}$, where $z_0 \in I_0 + iI_0'$, $x_2 \in I_2$ and $x_3 \in I_3$ and conclude that $f_1(w) = 0$ for all $w \in I_1 + iI_1'$.

Repeating the argument for the remaining two variables shows that indeed $f \equiv 0$ throughout $\mathcal{D}$.

If $\mathcal{D}$ was actually not a box, but we chose such a box within $\mathcal{D}$, we can now – since $f = 0$ on an open box of $\mathbb{C}^4$ contained in $\mathcal{D}$ – apply the above stated usual form of the identity theorem for functions of several complex variables to conclude that $f \equiv 0$ on all of $\mathcal{D}$.

■

Next we generalize the basic ingredients of the *Schwartz reflection principle* (see e.g. [281] for the usual Schwarz reflection principle) to several complex variables (in the present case four) in a suitable way to combine them afterwards with the edge-of-the-wedge theorem 3.1.

Lemma 3.3

Let $\mathcal{O} \subseteq \mathbb{R}^4$ be some open connected subset of $\mathbb{R}^4$ and $V_+ = \{y \in \mathbb{R}^4 \mid y_\mu y^\mu > 0, y_0 > 0\}$ the open forward light cone. We define

$$\mathcal{D} := \mathcal{O} + iV_+, \quad \text{and} \quad \overline{\mathcal{D}} = \mathcal{O} - iV_+ \tag{3.5}$$

Let f be a continuous function on $\mathcal{D} \cup \mathcal{O}$ which is holomorphic on $\mathcal{D}$ and real valued on $\mathcal{O}$. Then the function F defined by

$$F(z) = \begin{cases} f(z) & z \in \mathcal{D} \cup \mathcal{O} \\ \overline{f(\overline{z})} & z \in \overline{\mathcal{D}} \end{cases} \tag{3.6}$$

is holomorphic on $\mathcal{D} \cup \overline{\mathcal{D}}$ and continuous on $\mathcal{D} \cup \mathcal{O} \cup \overline{\mathcal{D}}$.

Proof: First we show that F is holomorphic on $\mathcal{D} \cup \overline{\mathcal{D}}$, in particular on $\overline{\mathcal{D}}$ since F is holomorphic on $\mathcal{D}$ by assumption anyway: To this end, we pretend for a moment that f is a function of only two complex variables and show that if it is holomorphic on any domain $\mathcal{D} \subset \mathbb{C}^2$, the function g defined by $g(z, w) := \overline{f(\overline{z}, \overline{w})}$ is holomorphic on $\overline{\mathcal{D}}$. For four complex variables, the proof works exactly the same, but is only notationally more cumbersome.

So suppose f is holomorphic on $\mathcal{D} \subset \mathbb{C}^2$ (which means that f is holomorphic in each single variable on $\mathcal{D}$ [37, 313]), i.e. for all fixed w_0 the function $f_{w_0}(z) := f(z, w_0)$ is holomorphic on the domain $\mathcal{D}_{w_0} := \{z \in \mathbb{C} \mid (z, w_0) \in \mathcal{D}\}$ (which might be the empty set depending on the choice of w_0) and the analogous fact holds for the second variable.

Now choose $z_0 \in \mathbb{C}$ such that $(z_0, w_0) \in \overline{\mathcal{D}}$ and $z \in \mathbb{C}$ close enough to z_0 such that $(z, w_0) \in \overline{\mathcal{D}}$ as well (which in particular entails of course $(\overline{z}_0, \overline{w}_0) \in \mathcal{D}$ and $(\overline{z}, \overline{w}_0) \in \mathcal{D}$) and consider the

following difference quotient:

$$\begin{aligned}\frac{g(z,w_0)-g(z_0,w_0)}{z-z_0} &= \frac{\overline{f(\overline{z},\overline{w}_0)}-\overline{f(\overline{z}_0,\overline{w}_0)}}{z-z_0} = \\ &= \overline{\left(\frac{f(\overline{z},\overline{w}_0)-f(\overline{z}_0,\overline{w}_0)}{\overline{z}-\overline{z}_0}\right)} = \overline{\left(\frac{f_{\overline{w}_0}(\overline{z})-f_{\overline{w}_0}(\overline{z}_0)}{\overline{z}-\overline{z}_0}\right)}\end{aligned} \tag{3.7}$$

which exists in the limit $z \to z_0$ since

$$\lim_{z\to z_0} \frac{f_{\overline{w}_0}(\overline{z})-f_{\overline{w}_0}(\overline{z}_0)}{\overline{z}-\overline{z}_0} = f'_{\overline{w}_0}(z)\big|_{z=\overline{z}_0} \tag{3.8}$$

exists according to assumption whenever $(\overline{z}_0,\overline{w}_0) \in \mathcal{D}$. Analogously, we see that the limit

$$\lim_{w\to w_0} \frac{g(z_0,w)-g(z_0,w_0)}{w-w_0} \tag{3.9}$$

exists for all $(z_0,w_0) \in \overline{\mathcal{D}}$, i.e. g is complex differentiable in each single variable on $\overline{\mathcal{D}}$ which is to say g is holomorphic on $\overline{\mathcal{D}}$.

Generalizing this to a function of four complex variables (which is now straight forwardly done) proves that the function F in (3.6) is holomorphic on $\mathcal{D}\cup\overline{\mathcal{D}}$ given f is holomorphic on $\mathcal{D}$.

Now we show that F is continuous on $\mathcal{D}\cup\mathcal{O}\cup\overline{\mathcal{D}}$. We already know that F is continuous on $\mathcal{D}\cup\overline{\mathcal{D}}$ since F is holomorpic there and we know that F is continuous on $\mathcal{D}\cup\mathcal{O}$ by assumption. What remains to show is that F is continuous on $\mathcal{O}$ if $z \in \overline{\mathcal{D}}$ approaches $x \in \mathcal{O}$ (observe that $\mathcal{O}$ is part of the boundary of $\overline{\mathcal{D}}$ since zero is part of the boundary of $-V_+$). So let $(z_n)_{n\in\mathbb{N}} \subset \overline{\mathcal{D}}$ be a sequence of complex four vectors with $\lim_{n\to\infty} z_n = x_0 \in \mathcal{O}$. Then, since $(\overline{z}_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}$ with $\lim_{n\to\infty} \overline{z}_n = \overline{x}_0 = x_0$ and thus by assumption $\lim_{n\to\infty} F(\overline{z}_n) = F(x_0)$ and $F(x_0)$ is real valued by assumption since $x_0 \in \mathcal{O}$,

$$F(x_0) = f(x_0) = f(\overline{x}_0) = \overline{f(\overline{x}_0)} = \overline{\lim_{n\to\infty} f(\overline{z}_n)} = \lim_{n\to\infty} \overline{f(\overline{z}_n)} = \lim_{n\to\infty} F(z_n) \tag{3.10}$$

Thus, F is continuous on $\mathcal{D}\cup\mathcal{O}\cup\overline{\mathcal{D}}$.

■

Now we collect everything together:

Corollary 3.4 [*Edge-of-the-Wedge Reflection Principle*]

Let f be a function on $\mathbb{R}^4$ which has a continuous extension $\widetilde{f}$ to $\mathbb{C}^4$ which is holomorphic on the domain $\mathcal{D} = \mathbb{R}^4 + iV_+$. If there is some open connected subset $\mathcal{O} \subset \mathbb{R}^4$ on which f vanishes, it follows that $f(x) = 0$ for all $x \in \mathbb{R}^4$.

Proof: Let $f(x) = 0$ for all $x \in \mathcal{O}$. By assumption, $\widetilde{f}$ is continuous on $\mathcal{D} \cup \mathbb{R}^4$, holomorphic on $\mathcal{D}$ and real valued (since vanishing) on $\mathcal{O}$. Hence, according to lemma 3.3, the function F defined by

$$F(z) = \begin{cases} \widetilde{f}(z) & z \in \mathcal{D} \cup \mathcal{O} \\ \overline{\widetilde{f}(\overline{z})} & z \in \overline{\mathcal{D}} \end{cases} \tag{3.11}$$

is holomorpic on $\mathcal{D} \cup \overline{\mathcal{D}}$ and continuous on $\mathcal{D} \cup \mathcal{O} \cup \overline{\mathcal{D}}$. According to the edge-of-the-wedge theorem 3.1, there is thus a neighbourhood $\mathcal{N}$ of $\mathcal{O}$ as a subset of $\mathbb{C}^4$, such that F extends to an analytic function $\widetilde{F}$ on $\mathcal{N}$ such that $\widetilde{F} = F$ on $\mathcal{N} \cap (\mathcal{D} \cup \mathcal{O} \cup \overline{\mathcal{D}})$. Since $f = 0$ on $\mathcal{O}$ and consequently $F = 0$ on $\mathcal{O}$ and $\widetilde{F} = 0$ on $\mathcal{O}$, the above version of the identity theorem lemma 3.2 implies now $\widetilde{F} = 0$ on all of $\mathcal{N}$. Since now $0 = \widetilde{F} = F$ on $\mathcal{N} \cap (\mathcal{D} \cup \mathcal{O} \cup \overline{\mathcal{D}})$, it follows again from the identity theorem that $F = 0$ on $\mathcal{D} \cup \mathcal{O} \cup \overline{\mathcal{D}}$ which in particular entails that $\widetilde{f} = 0$ on $\mathcal{D}$. But since $\mathbb{R}^4$ is part of the boundary of $\mathcal{D}$ and by assumption $\widetilde{f}$ is continuous on $\mathcal{D} \cup \mathbb{R}^4$ it follows that $\widetilde{f} = 0$ on $\mathbb{R}^4$ and consequently $f(x) = 0$ for all $x \in \mathbb{R}^4$.

■

Next we specify a general class of functions which satisfy the assumptions of corollary 3.4. To this end, we shall use the concept of integration with respect to a *complex measure*. We shall briefly summarize a few facts about complex measures, which are essentially sufficient to comprehend the following argumentations, for details the reader is referred to the extensive literature on this subject (see e.g. [118, 281]).

A complex measure on a measurable space $(\Omega, \mathcal{A})$ is a countably additive set function $\lambda : \mathcal{A} \to \mathbb{C}$ with $\lambda(\emptyset) = 0$, which is to say for any family $(E_n)_{n \in I} \subset \mathcal{A}$ with countable index set $I \subseteq \mathbb{N}$ of disjoint subsets $E_n \cap E_k = \emptyset$ for $n \neq k$, a complex measure λ is sigma additive

$$\lambda\left(\bigcup_{n \in I} E_n\right) = \sum_{n \in I} \lambda(E_n) \tag{3.12}$$

and the empty set has zero measure. In particular, since a complex measure takes always values in $\mathbb{C}$, complex measures are always finite (for example each probability measure is a complex measure but Lebesgue measure on $\mathbb{R}$ is not).

Important for the present purposes is the fact, that we can always understand a complex measure as a collection of four ordinary finite and positive measures: A complex measure λ can always be decomposed into its real and imaginary parts $\lambda = \lambda_r + i\lambda_i$, each of which is given by a finite signed measure [118, 281]. Thus we can perform a Hahn-Jordan decomposition with respect to λ_r and λ_i to get $\lambda = (\lambda_r^+ - \lambda_r^-) + i(\lambda_i^+ - \lambda_i^-)$, where $\lambda_r^+, \lambda_r^-, \lambda_i^+$ and λ_i^- are finite ordinary positive measures.

Thus we need not establish the theory of integration with respect to a complex measure from scratch, since an integral with respect to a complex measure can always be understood by four ordinary Lebesgue-Stieltjes integrals with respect to ordinary positive finite measures.

This in particular entails, that the dominated convergence theorem is also valid for a complex measure integral. We will need dominated convergence arguments for complex measure integrals in situations of the form: If a sequence of $\lambda-$integrable functions $(f_n)_{n \in \mathbb{N}}$ with $f_n \to f$ $\lambda-$a.e. is uniformly bounded by a constant $|f_n| \leq C$ $\lambda-$ a.e. and for all $n \in \mathbb{N}$, C is an integrable

majorant of $(f_n)_{n\in\mathbb{N}}$ with respect to each measure λ_k^l where $k \in \{r, i\}$ and $l \in \{+, -\}$ separately (recall that the measures λ_k^l are finite):

$$\int C\, d\lambda_k^l = C \int d\lambda_k^l = C\lambda_k^l(\Omega) < \infty \tag{3.13}$$

i.e. for any constant $C \in \mathbb{R}$ we have $C \in L^1(\Omega, d\lambda_k^l)$.

Thus,

$$\begin{aligned} &\lim_{n\to\infty} \int f_n\, d\lambda = \\ &= \lim_{n\to\infty} \left[\left(\int f_n\, d\lambda_r^+ - \int f_n\, d\lambda_r^- \right) + i \left(\int f_n\, d\lambda_i^+ - \int f_n\, d\lambda_i^- \right) \right] = \\ &\left(\int f\, d\lambda_r^+ - \int f\, d\lambda_r^- \right) + i \left(\int f\, d\lambda_i^+ - \int f\, d\lambda_i^- \right) = \int f\, d\lambda \end{aligned} \tag{3.14}$$

which is a finite complex number.

Finally, we need the concept of the *support* $\mathrm{supp}(\lambda)$ *of a (complex) measure*: This is the closure of the complement of the subset of Ω on which λ vanishes. In particular,

$$\int f d\lambda = 0 \tag{3.15}$$

for all $\lambda-$measurable functions f with $\mathrm{supp}(f) \cap \mathrm{supp}(\lambda) = \emptyset$. Consequently, if $\mathrm{supp}(\lambda) =: \mathcal{O}$ and we write an arbitrary $\lambda-$measurable function f as $f = \chi_{\mathcal{O}} f + \chi_{\mathcal{O}^c} f$ with the indicator functions $\chi_{\mathcal{O}}$ of $\mathcal{O}$ and $\chi_{\mathcal{O}^c}$ of the complement of $\mathcal{O}$, we get

$$\int f d\lambda = \int \chi_{\mathcal{O}} f d\lambda + \int \chi_{\mathcal{O}^c} f d\lambda = \int \chi_{\mathcal{O}} f d\lambda = \int_{\mathcal{O}} f d\lambda \tag{3.16}$$

Now we are prepared to specify a general class of functions which satisfy the assumptions of corollary 3.4:

Theorem 3.5 [*Analytic Continuation into the Imaginary Forward Light Cone*]

Let λ be a complex measure on $\mathbb{R}^4$ with support in the closure of the forward light cone $\overline{V}_+$ with respect to the origin. Consider the function $f : \mathbb{R}^4 \to \mathbb{C}$ given by

$$f(x) = \int e^{ipx}\, d^4\lambda(p) \equiv \int_{\overline{V}_+} e^{ipx}\, d^4\lambda(p) \tag{3.17}$$

where $px = p_\mu x^\mu$ is the Minkowski scalar product.

Let $\mathcal{D} := \mathbb{R}^4 + iV_+$. Then the complex extension $\widetilde{f}$ of f given by

$$\widetilde{f}(z) := \int e^{ipz}\, d^4\lambda(p) \equiv \int_{\overline{V}_+} e^{ipz}\, d^4\lambda(p) \tag{3.18}$$

exists as a continuous function on $\mathcal{D} \cup \mathbb{R}^4$ and is holomorphic on $\mathcal{D}$.

REMARK: Of course, this formulation of the theorem is a bit overdetermined: It would be sufficient to assert that a function of the form (3.18) (where λ is a complex measure supported in $\overline{V}_+$) exists as a continuous function on $\mathcal{D}\cup\mathbb{R}^4$ and is holomorphic on $\mathcal{D}$ (which we shall proof in the following). But since the functions we shall be confronted with by physical considerations later are a priori functions on $\mathbb{R}^4$ which can be written in the form (3.17), theorem 3.5 starts with this explicit form, which is though rather didactically motivated than a mathematical necessity.

Proof: First, observe that the integrand[115] on the right hand side of (3.18) is uniformly bounded on $\overline{V}_+$ for each $z\in\mathcal{D}\cup\mathbb{R}^4$: To see this, we denote $z=x+iy$ with $x=(x_0,\boldsymbol{x})^T\in\mathbb{R}^4$ and $y=(y_0,\boldsymbol{y})^T\in V_+\cup\{0\}$, which in particular entails that either $y_0>|\boldsymbol{y}|>0$ or $y_0=|\boldsymbol{y}|=0$. For $p=(p_0,\boldsymbol{p})\in\overline{V}_+$ we have $p_0\geq|\boldsymbol{p}|\geq 0$. If $y=0$ we have $|e^{ipz}|=1$ as a uniform bound. So let $y\in V_+$, i.e. $z\in\mathcal{D}$. Since $\boldsymbol{p}\cdot\boldsymbol{y}\leq|\boldsymbol{p}|\,|\boldsymbol{y}|$, we have for $z\in\mathcal{D}$ and $p\in\overline{V}_+$

$$\left|e^{ipz}\right| = e^{-py} = e^{-p_0y_0+\boldsymbol{p}\cdot\boldsymbol{y}} \leq e^{-p_0y_0+|\boldsymbol{p}||\boldsymbol{y}|} \leq e^{-p_0y_0+p_0|\boldsymbol{y}|} = e^{-(y_0-|\boldsymbol{y}|)p_0} =: e^{-\alpha_y p_0} \tag{3.19}$$

where we have set $\alpha_y := y_0-|\boldsymbol{y}|>0$ and thus we conclude

$$\left|e^{ipz}\right| \leq e^{-\alpha_y p_0} \leq 1 \tag{3.20}$$

which is also true for $y=0$, i.e. $|e^{ipz}|\leq 1$ for all $p\in\overline{V}_+$ and $z\in\mathcal{D}\cup\mathbb{R}^4$.

With these observations at hand, we will now proof the claims about **existence**, **continuity** and **holomorphy** stated in theorem 3.5:

• Equation (3.20) immediately shows the **existence** of $\widetilde{f}$ on $\mathcal{D}\cup\mathbb{R}^4$:

$$\begin{aligned}\left|\widetilde{f}(z)\right| &\leq \sum_{k=r,i}\sum_{l=\pm}\left|\int_{\overline{V}_+} e^{ipz}\; d^4\lambda_k^l(p)\right| \leq \\ &\leq \sum_{k=r,i}\sum_{l=\pm}\int_{\overline{V}_+}\left|e^{ipz}\right| d^4\lambda_k^l(p) \leq \sum_{k=r,i}\sum_{l=\pm}\int d^4\lambda_k^l(p) < \infty\end{aligned} \tag{3.21}$$

whenever $z\in\mathcal{D}\cup\mathbb{R}^4$.

• Similarly easily, the **continuity** of $\widetilde{f}$ on $\mathcal{D}\cup\mathbb{R}^4$ can be checked: Let $(z_n)_{n\in\mathbb{N}}\subset\mathcal{D}\cup\mathbb{R}^4$ be a sequence of complex numbers with $\lim_{n\to\infty}z_n=z_0\in\mathcal{D}\cup\mathbb{R}^4$ and $g_n(p):=e^{ipz_n}$ define a sequence of functions in $L^1(\mathbb{R}^4,d^4\lambda)$. Since $|g_n|\leq 1\in L^1(\mathbb{R}^4,d^4\lambda)$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}g_n(p)=e^{ipz_0}$, we thus have with the dominated convergence theorem:

$$\lim_{n\to\infty}\widetilde{f}(z_n) = \int\lim_{n\to\infty}e^{ipz_n}\;d^4\lambda(p) = \int e^{ipz_0}\;d^4\lambda(p) = \widetilde{f}(z_0) \tag{3.22}$$

and consequently $\widetilde{f}$ is continuous on $\mathcal{D}\cup\mathbb{R}^4$.

• Finally, we show that $\widetilde{f}$ is **holomorphic** on $\mathcal{D}$: To this end, we first proof that $\widetilde{f}(z)\equiv\widetilde{f}(x,y)$ is real differentiable in (each component of) x and y as long as $z=x+iy\in\mathcal{D}$ and will

[115] In the following calculations it is helpful to have in mind, that we may freely identify the integrand e^{ipz} with the function $\chi_{\overline{V}_+}(p)\,e^{ipz}$, where $\chi_{\overline{V}_+}$ is the indicator function of $\overline{V}_+$ with respect to the variable p, in the sense that integrated with respect to a measure which is supported on $\overline{V}_+$ as in the present case, both functions always yield the same value.

then see, that $\widetilde{f}(x,y)$ obeys the Cauchy-Riemann differential equations with respect to (each component of) x and y in this case:

We start with real differentiability of $\widetilde{f}$ with respect to any component of y: Let $y \in V_+$ and choose $\varepsilon > 0$ small enough such that the open ball $\mathcal{B}_\varepsilon(y) \subset V_+$. Consider a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathcal{B}_\varepsilon(y)$ with $\lim_{n\to\infty} y_n = y$, where we assume that only one, say the ν'th component varies with n and the remaining three components are at the constant value they realize in the limit point y. Now consider the following difference quotient for some arbitrary $n \in \mathbb{N}$ and $p \in \overline{V}_+$:

$$\begin{aligned}\left|\frac{e^{ipx-py_n} - e^{ipx-py}}{y_n^\nu - y^\nu}\right| = \left|\frac{e^{-py_n} - e^{-py}}{y_n^\nu - y^\nu}\right| \overset{\exists \xi\in\mathcal{B}_\varepsilon(y)}{=} \left|\partial_{\tilde{y}_\nu} e^{-p\tilde{y}}\right|_{\tilde{y}=\xi} \leq \sup_{\tilde{y}\in\mathcal{B}_\varepsilon(y)} |p^\nu|\, e^{-p\tilde{y}} \leq \\ \leq \sup_{\tilde{y}\in\mathcal{B}_\varepsilon(y)} p_0 e^{-p\tilde{y}} \overset{(3.19)}{\leq} \sup_{\tilde{y}\in\mathcal{B}_\varepsilon(y)} p_0 e^{-\alpha_{\tilde{y}} p_0} \in L^1(\mathbb{R}^4, d^4\lambda)\end{aligned} \tag{3.23}$$

where we have used the mean value theorem (i.e. ξ lays on the line segment connecting y_n and y, which is contained in $\mathcal{B}_\varepsilon(y)$), the assumption that $\mathcal{B}_\varepsilon(y) \subset V_+$ and the fact that $p_0 e^{-\alpha_{\tilde{y}} p_0}$ is bounded for each $\alpha_{\tilde{y}} > 0$ as a function of p on $\overline{V}_+$ (recall that $\alpha_{\tilde{y}} = \tilde{y}_0 - |\tilde{\boldsymbol{y}}| > 0$ for all $\tilde{y} \in V_+$ and that $p_0 \geq 0$ for all $p \in \overline{V}_+$) and thus $\lambda-$ integrable.

Hence we conclude with the dominated convergence theorem that the partial derivative with respect to y^ν exists for all $\nu = 0,1,2,3$ whenever $x + iy \in \mathcal{D}$ and is given by

$$\begin{aligned}\partial_{y_\nu} \widetilde{f}(x,y) = \lim_{n\to\infty} \frac{\widetilde{f}(x,y_n) - \widetilde{f}(x,y)}{y_n^\nu - y^\nu} = \lim_{n\to\infty} \int \frac{e^{ipx-py_n} - e^{ipx-py}}{y_n^\nu - y^\nu} d^4\lambda(p) = \\ = \int \lim_{n\to\infty} \frac{e^{ipx-py_n} - e^{ipx-py}}{y_n^\nu - y^\nu} d^4\lambda(p) = -\int p^\nu\, e^{ipx-py}\, d^4\lambda(p)\end{aligned} \tag{3.24}$$

Similarly, we show now partial differentiability of $\widetilde{f}$ with respect to any component of x. Only in this case, we have to take into account the real and imaginary part of e^{ipx-py}, since now we cannot get rid of the factor e^{ipx} as in the above calculation and the mean value theorem does not apply to complex valued functions but only to its real an imaginary part separately. Now suppose $y \in V_+$, $x \in \mathbb{R}^4$ and $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^4$ is a sequence with $\lim_{n\to\infty} x_n = x$ where as above only the ν'th component of x_n varies with n and the remaining three components are at the constant value they realize in x. Using that $e^{ipx} = \cos(px) + i\sin(px)$ we thus get for some arbitrary $n \in \mathbb{N}$ and $p \in \overline{V}_+$:

$$\begin{aligned}\left|\frac{e^{ipx_n-py} - e^{ipx-py}}{x_n^\nu - x^\nu}\right| \leq e^{-py}\left(\left|\frac{\cos(px_n) - \cos(px)}{x_n^\nu - x^\nu}\right| + \left|\frac{\sin(px_n) - \sin(px)}{x_n^\nu - x^\nu}\right|\right) = \\ = e^{-py}\left(|p^\nu \sin(p\xi)| + |p^\nu \cos(p\xi')|\right) \leq 2\,|p^\nu|\, e^{-yp} \leq 2p_0 e^{-\alpha_y p_0} \in L^1(\mathbb{R}^4, d^4\lambda)\end{aligned} \tag{3.25}$$

where we have used essentially the same arguments as in connection with the calculation (3.23) (i.e. in particular, ξ and ξ' lay on the line segment connecting x_n and x).

Analogously to (3.24), we conclude now with the dominated convergence theorem that the partial derivative with respect to x^ν exists for all $\nu = 0,1,2,3$ whenever $x + iy \in \mathcal{D}$ and is given by

$$\partial_{x_\nu} \widetilde{f}(x,y) = i \int p^\nu\, e^{ipx-py}\, d^4\lambda(p) \tag{3.26}$$

Moreover, comparison of (3.26) with (3.24) yields

$$i\,\partial_{x_\nu}\widetilde{f}(x,y) = \partial_{y_\nu}\widetilde{f}(x,y) \tag{3.27}$$

which are just the Cauchy-Riemann equations with respect to the complex variable $z^\nu = x^\nu + iy^\nu$. Recalling that a continuous function obeying the Cauchy-Riemann equations is complex differentiable in the associated complex variable in the respective region where they are valid and that a function of several complex variables is complex differentiable if and only if it is complex differentiable in each single variable, we conclude that $\widetilde{f}$ is holomorphic on $\mathcal{D}$.

■

Combining theorem 3.5 with corollary 3.4, we get the following:

Corollary 3.6

Let λ be a complex measure on $\mathbb{R}^4$ with support in the closure of the forward light cone $\overline{V}_+$ with respect to the origin. Consider the function $f : \mathbb{R}^4 \to \mathbb{C}$ given by

$$f(x) = \int e^{ipx}\, d^4\lambda(p) \tag{3.28}$$

where $px = p_\mu x^\mu$ is the Minkowski scalar product. If f vanishes on an open connected subset $\mathcal{O} \subset \mathbb{R}^4$ it follows that $f \equiv 0$ on all of $\mathbb{R}^4$.

3.2.2 Hilbert Space, Space-Time Translations and the Spectrum Condition

Next we approach the transition from mathematics to physics by providing a very general Hilbert space framework which underlies most actual relativistic quantum theories. This framework covers concrete models like second quantized Fock space QFTs or their underlying first quantized one particle (or $N-$particle) models, given the latter are constrained to the positive energy subspace of the respective Hilbert space (like positive energy Dirac theory), but also more abstract approaches like axiomatic or algebraic QFTs comprise this general structure.

In the following, we shall switch a bit loosely between rigorous treatment of the mathematical foundations, derivations and results and more or less heuristic physical illustration, which presupposes some knowledge about (relativistic) quantum theory. In particular, some basics of the quantum theory of measurement (chapter 1) as well as the basics of relativistic wave equations, spectral subspaces, Fock space and second quantization are assumed to be known to the reader.

After this framework is established and its physical content loosely illustrated, we will apply corollary 3.6 to it (or to a certain expression in terms of this framework, to be precise), and thereby derive a central result which is the mathematical basis of a collection of strong physical assertions, some of which which will be elaborately derived and analysed afterwards and which form the core of this chapter.

Relativistic Hilbert Space

Definition 3.7 [*Relativistic Hilbert Space*]

We call a Hilbert space $\mathcal{H}$ which carries a unitary representation of the proper, orthochronous Poincaré group (or its covering group in the case of spin) together with the unitary implementation of the latter on $\mathcal{H}$ a relativistic Hilbert space. The unitary operator acting on $\mathcal{H}$ which realizes the Poincaré transformation (Λ, x) is denoted as $U(\Lambda, x)$, where in case of pure space-time translations $(\mathbb{1}, x)$ we write $U(x) := U(\mathbb{1}, x)$.

Space-Time Translations

Let $\mathcal{H}$ be a relativistic Hilbert space. There exists a selfadjoint operator valued four vector (the energy-momentum operator)

$$\widehat{P}^\mu = \int p^\mu \, d^4E(p) \tag{3.29}$$

acting on $\mathcal{H}$ such that each unitary implementation of space-time translation $U(x)$ by some $x \in \mathbb{R}^4$ can be written as

$$U(x) = e^{i\widehat{P}x} = \int e^{ipx} \, d^4E(p) \tag{3.30}$$

where E is a PVM on $\mathbb{R}^4$ acting on $\mathcal{H}$ (for integration with respect to a PVM see Reed, Simon [275]), and $px = p_\mu x^\mu$ and $\widehat{P}x = \widehat{P}_\mu x^\mu$ are Minkowski scalar products. In particular, we have

$$\int d^4E(p) = \mathbb{1}_{\mathcal{H}} \tag{3.31}$$

The right hand side of equation (3.29) is the spectral representation of the energy-momentum operator, which is according to (3.30) the infinitesimal generator of space-time translations. Representation (3.29) in particular entails that all four components of $\widehat{P}^\mu$ can be jointly diagonalized, i.e. are commuting operators.

The fact that $U(x)$ can be written in the form (3.30) is of course well known for concrete models of relativistic quantum theory and is ensured more generally by an immediate generalization of Stone's theorem from unitary strongly continuous representations of one parameter groups to unitary strongly continuous representations of general locally compact abelian groups, which is sometimes called the SNAG-theorem (according to Stone, Naimark, Ambrose and Godement) [232].

$\widehat{P}_0$ is the infinitesimal generator of time translations which we identify with the free Hamiltonian. The present analysis does not need to (and want to) go into the serious problems which come about by trying to introduce interaction in relativistic quantum theories. We are not directly concerned with interactions in the first place, but primary interested in free initial and final states of given experiments. The interactions of the experiments themselves are not considered explicitly, but it is only assumed that they can be in some way or the other associated with state transformers and effects in the sense of the quantum theory of measurement derived

in chapter 1 (this is if you want the implicit dynamical element of the following analysis). Considered space-time translations correspond always to the question what the predictions of the theory are, if a given experiment on a given initial state is performed at a different place and/or time in a given laboratory(-frame) and thus are always associated with the free time evolution.

Note that the present notation (denoting $U(x) = e^{i\,\widehat{P}x}$ and $U^{-1}(x) = U^\dagger(x) = e^{-i\,\widehat{P}x}$) deviates with respect to a sign from the usual notation of time evolution, i.e. if $x_t = (t, 0, 0, 0)^T$ (in a given frame) we have $U(x_t) = e^{i\widehat{P}_0 t} = \left(e^{-i\widehat{P}_0 t}\right)^\dagger = U_t^\dagger$ (with the usual time evolution operator U_t whose infinitesimal generator is the free Hamiltonian $\widehat{P}_0$) and thus $\psi_t = U^\dagger(x_t)\psi_0$ is the (forward) Schrödinger time evolution of some state $\psi_0 \in \mathcal{H}$ and for an operator A acting on $\mathcal{H}$, $A(t) = U(x_t)AU^\dagger(x_t)$ is the related Heisenberg operator.

If we consider, on the other hand, pure spacial translations (in some frame) by $x_{\boldsymbol{a}} = (0, \boldsymbol{a})$ we have $U(x_{\boldsymbol{a}}) = e^{-i\widehat{\boldsymbol{P}}\cdot\boldsymbol{a}}$ and given e.g. $\mathcal{H}$ is a one particle Hilbert space and $\psi_{\boldsymbol{y}_0}(\boldsymbol{y})$ is a one-particle state in $\mathcal{H}$ in position representation, (well) localized about $\boldsymbol{y}_0 \in \mathbb{R}^3$, it follows that $U(x_{\boldsymbol{a}})\psi_{\boldsymbol{y}_0}(\boldsymbol{y}) = \psi_{\boldsymbol{y}_0}(\boldsymbol{y} - \boldsymbol{a}) \equiv \psi_{\boldsymbol{y}_0+\boldsymbol{a}}(\boldsymbol{y})$ is the same state (well) localized about $\boldsymbol{y}_0 + \boldsymbol{a}$ (the corresponding calculations are assumed to be known to the reader). Now consider an experiment performed in some spacial region $\Delta \subset \mathbb{R}^3$ with some associated effect E_Δ acting on $\mathcal{H}$ with the above initial state $\psi_{\boldsymbol{y}_0}$, such that the probability of the related outcome is given by $\langle\psi_{\boldsymbol{y}_0}|\, E_\Delta\, \psi_{\boldsymbol{y}_0}\rangle$. If we consider the same experiment performed in the same region Δ with the same initial state only translated by $-\boldsymbol{a}$, i.e. with initial state $U^\dagger(x_{\boldsymbol{a}})\psi_{\boldsymbol{y}_0} = \psi_{\boldsymbol{y}_0-\boldsymbol{a}}$, we have – according to homogeneity of space – physically the same situation as if the experimental setup was translated by $+\boldsymbol{a}$. This is expressed in the associated probability

$$\langle\psi_{\boldsymbol{y}_0}|\, U(x_{\boldsymbol{a}})\, E_\Delta\, U^\dagger(x_{\boldsymbol{a}})\, \psi_{\boldsymbol{y}_0}\rangle = \langle\psi_{\boldsymbol{y}_0-\boldsymbol{a}}|\, E_\Delta\, \psi_{\boldsymbol{y}_0-\boldsymbol{a}}\rangle \equiv \langle\psi_{\boldsymbol{y}_0}|\, E_{\Delta+\boldsymbol{a}}\, \psi_{\boldsymbol{y}_0}\rangle \tag{3.32}$$

where we can interpret the operator $U(x_{\boldsymbol{a}})\, E_\Delta\, U^\dagger(x_{\boldsymbol{a}}) \equiv E_{\Delta+\boldsymbol{a}}$ as the effect corresponding to E_Δ, given the experiment was performed in the spatial region $\Delta + \boldsymbol{a}$ instead of Δ.

Considering the special case of a one-particle initial wave function was of course only illustration and does not restrict the previous insight to such cases. This may be illustrated by thinking of the action of the translation operator on initial states as the corresponding translation of the preparation device of an experiment and its action on effects as translating the respective measuring device, where the translation of the preparation device in one direction by some amount is physically equivalent (with respect to the statistics of outcomes) to the translation of the measuring device by the same amount in the opposite direction.

All these considerations are tied to a specific frame of reference (in which t is the time variable and Δ is some subset of some simultaneity slice), but we can collect them in the following covariant form: If $E_\mathcal{O}$ is some effect in the Heisenberg picture, which is associated with an experiment which is 'spatio-temporally located'[116] in some region of space-time $\mathcal{O} \subset \mathbb{R}^4$, the effect $U(x)\, E_\mathcal{O}\, U^\dagger(x) \equiv E_{\mathcal{O}+x}$ is the corresponding effect in the Heisenberg picture which is associated with the same experiment, when the measuring device is now 'located' in the space-time region $\mathcal{O} + x$. The analogue statement also holds for state transformers and observable operators, of course.

[116]This notion will be made more precise later, but you may think of an effect associated with a measurement performed in some spatial region Δ at time t_0 in the laboratory frame, such that in the laboratory frame coordinates $(t_0, \Delta) = \{(t, \boldsymbol{x}) \in \mathbb{R}^4 \mid t = t_0,\, \boldsymbol{x} \in \Delta\} \subseteq \mathcal{O}$.

Definition 3.8 [*Space-Time Translation Covariant Family of Operators*]

Let $\mathcal{H}$ be a relativistic Hilbert space. A family of operators $(A_{\mathcal{O}})_{\mathcal{O}\subseteq\mathcal{M}}$ acting on $\mathcal{H}$ and indexed by suitable subsets $\mathcal{O}\subseteq\mathcal{M}$ of space-time is called space-time translation covariant if it obeys $A_{\mathcal{O}+x} = U(x)A_{\mathcal{O}}U^{-1}(x)$ for all index regions $\mathcal{O}$ and $x\in\mathbb{R}^4$, such that $\mathcal{O}+x$ is in the considered set of index regions.

Actually, a family of operators like $(A_{\mathcal{O}})_{\mathcal{O}\subseteq\mathcal{M}}$ (equipped with a natural preorder relation of subsets of $\mathcal{M}$) is a topological net, which is an appropriate generalization of a sequence when the index set in no longer countable, we shall encounter topological nets again towards the end of this chapter).

The Spectrum Condition

Next we introduce an important crucial assumption on the spectrum of the generator of space-time translations, i.e. the energy-momentum operator $\widehat{P}^\mu$:

Definition 3.9 [*Spectrum Condition*]

Let $\mathcal{H}$ be a relativistic Hilbert space. We say that $\mathcal{H}$ obeys the spectrum condition, if the spectrum of the energy-momentum operator is contained in the closure of the forward light cone:

$$\sigma(\widehat{P}) \overset{!}{\subseteq} \overline{V}_+ = \left\{p\in\mathbb{R}^4 \mid p_\mu p^\mu \geq 0,\, p_0\geq 0\right\} \tag{3.33}$$

The spectrum condition in particular entails that the spectral measure E of $\widehat{P}^\mu$ has support in the closure of the forward light cone, i.e. for any $\psi\in\mathcal{H}$ the mapping

$$\mathbb{R}^4 \supseteq \mathcal{O} \mapsto \langle\psi|\, E(\mathcal{O})\,\psi\rangle = \int \chi_{\mathcal{O}}(p)\, d^4\, \langle\psi|\, E(p)\,\psi\rangle \tag{3.34}$$

for any Borel set $\mathcal{O}$ defines a (probability–)measure on $\mathbb{R}^4$ with support in $\overline{V}_+$ (i.e. it vanishes whenever $\mathcal{O}$ and the spectrum of $\widehat{P}^\mu$ are disjoint).

Since in relativistic quantum theory, the energy-momentum operator $\widehat{P}^\mu$ transforms as an (operator valued) four vector under Lorentz transformations, the assumption that $\mathcal{H}$ satisfies the spectrum condition means that the set of eigenvalues of the energy operator $\widehat{P}_0$ is a subset of $\mathbb{R}^+_0$ (positive energy) in each single Lorentz frame of reference. This is usually realized by the assumption that energy and momentum satisfy a *relativistic energy-momentum relation of positive energy* (something like $p_0 = +\sqrt{\boldsymbol{p}^2+m^2}$, see below). The assumption of positive kinetic energy is very crucial as soon as interaction is introduced in the theory. It guarantees that a radiation catastrophe, where charged particles emit an infinite amount of energy – which obviously does not happen in the world we live in – cannot be dynamically possible.

We shall briefly illustrate the spectrum condition with three general examples: First quantized quantum theories of one, respectively N, relativistic positive energy particle(s) of a given rest mass $m>0$ and the primitive Fock space built upon these settings:

The prototype is the mass shell spectrum condition of a single particle: Consider a relativistic wave equation describing a free single particle of mass[117] $m > 0$, whose (possibly spinor valued) positive energy solutions span a Hilbert space $\mathcal{H}_+$, such that the spectrum of $\widehat{P}^\mu$ is (due to the energy momentum relation of a free relativistic particle) given by

$$\sigma_{\mathcal{H}_+}(\widehat{P}^\mu) = \left\{ p^\mu = (p^0, \boldsymbol{p})^T \in \mathbb{R}^4 \mid p^0 = \sqrt{\boldsymbol{p}^2 + m^2} \right\} \tag{3.35}$$

i.e. $p_\mu p^\mu = m^2 > 0$ and $p^0 = \sqrt{\boldsymbol{p}^2 + m^2} > 0$ for all p^μ in the spectrum of $\widehat{P}^\mu$.

Now let $\mathcal{H}_+^{\otimes N}$ be the (possibly symmetrized or antisymmetrized) tensor product of N copies of $\mathcal{H}_+ \equiv \mathcal{H}_+^{\otimes 1}$ (N relativistic positive energy particles of rest mass m). In this case, the spectrum of $\widehat{P}^\mu$ reads

$$\sigma_{\mathcal{H}_+^{\otimes N}}(\widehat{P}^\mu) = \left\{ p^\mu = \sum_{k=1}^{N} p_k^\mu \in \mathbb{R}^4 \,\middle|\, p_k^\mu = (p_k^0, \boldsymbol{p}_k)^T \,,\, p_k^0 = \sqrt{\boldsymbol{p}_k^2 + m^2} \right\} \tag{3.36}$$

i.e. in the standard momentum representation an $N-$particle state $\psi(p_1, ..., p_N)$ obeys

$$\widehat{P}^\mu \, \psi(p_1, ..., p_N) = (p_1^\mu + ... + p_N^\mu) \, \psi(p_1, ..., p_N) \tag{3.37}$$

where each variable $p_k^\mu = (p_k^0, \boldsymbol{p}_k)^T$ satisfies the positive energy mass shell energy momentum relation $p_k^0 = \sqrt{\boldsymbol{p}_k^2 + m^2}$.

Finally, let $\mathcal{F} = \bigoplus_{N \in \mathbb{N}_0} \mathcal{H}_+^{\otimes N}$ be the associated Fock space (but note that Fock spaces emerging from second quantization are in general more intricate[118]), where $\mathcal{H}_+^{\otimes 0} = \mathbb{C}$ represents the vacuum. In this case, we have

$$\sigma_{\mathcal{F}}(\widehat{P}^\mu) = \{0\} \cup \left(\bigcup_{N=1}^{\infty} \sigma_{\mathcal{H}_+^{\otimes N}}(\widehat{P}^\mu) \right) \tag{3.38}$$

where the isolated point $\{0\} \subset \mathbb{R}^4$ on the right hand side of (3.38) corresponds to the eigenvalue of the vacuum vector $\Omega = e^{i\varphi} \oplus 0 \oplus 0... \in \mathcal{F}$ (where $\varphi \in [0, 2\pi]$ is a free phase), which obeys $\widehat{P}^\mu \Omega = 0$. Since the sum of timelike, future oriented four vectors is always a timelike and future oriented four vector, it follows that (3.38) is consistent with the general spectrum condition 3.9, i.e. $\sigma_{\mathcal{F}}(\widehat{P}^\mu) \subseteq \overline{V}_+$. The spectrum of $\widehat{P}^\mu$ for the vacuum sector and the first three particle sectors is sketched in Fig. 4.

Note in particular, that for some $\Psi \in \mathcal{F}$ the probability measure given by

$$\mathbb{R}^4 \supseteq \mathcal{O} \mapsto \langle \Psi | \, E(\mathcal{O}) \, \Psi \rangle = \int \chi_{\mathcal{O}}(p) \, d^4 \, \langle \Psi | \, E(p) \, \Psi \rangle \tag{3.39}$$

on $\mathbb{R}^4$ is in general not absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^4$: A general Fock state Ψ can be written in the form

$$\Psi = \bigoplus_{N \in \mathbb{N}_0} c_N \, \psi^{(N)} \tag{3.40}$$

[117] We avoid discussion of zero mass particles, which require a special treatment.

[118] E.g. the Fock space emerging from second quantization of the Dirac equation is not simply the direct sum of the restricted positive energy $N-$particle subspaces (which would not lead to a stable theory, as soon as radiation is possible); rather, the negative energy subspaces are essential for the construction as well, only made into positive energy subspaces by force via charge conjugation and then represent antiparticles (see appendix A).

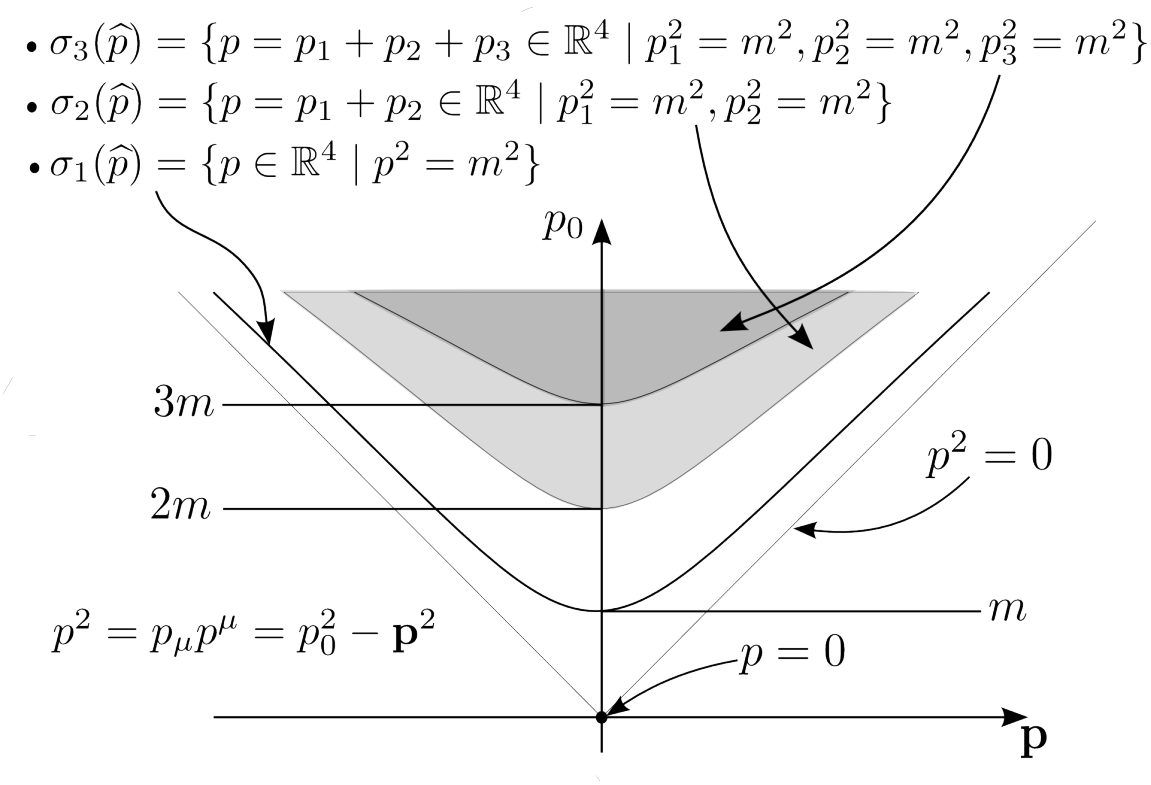

FIGURE 4

where $(c_N)_{N\in\mathbb{N}_0}$ are complex numbers with $\sum_{N=0}^{\infty} |c_N|^2 = 1$, $\psi^{(0)} = e^{i\varphi}$ represents the vacuum (zero particles) and each $\psi^{(N)}$ is a (possibly spinor valued) $N-$particle wave function, respectively. If now $c_0 \neq 0$ and/or $c_1 \neq 0$, the submanifolds $\{0\} \subset \mathbb{R}^4$ (isolated point) and $\sigma_{\mathcal{H}_+^{\otimes 1}}(\widehat{P^\mu}) \subset \mathbb{R}^4$ (three dimensional hyperboloid) which have Lebesgue measure zero as subsets of $\mathbb{R}^4$, will in general have non-zero $\langle\Psi|\, E\, \Psi\rangle$ $-$measure (e.g. we may think of a distributional density proportional to $\Theta(p_0)\delta(p^2 - m^2)$ which puts non zero weight on the hyperboloid $\sigma_{\mathcal{H}_+^{\otimes 1}}(\widehat{P^\mu})$). If on the other hand $c_0 = 0$ and $c_1 = 0$ the measure $\langle\Psi|\, E\, \Psi\rangle$ is absolutely continuous with respect to Lebesgue measure (see Fig. 4).

3.2.3 The Basic Result

Now we go back to the result derived from complex analysis of several complex variables above. If $\mathcal{H}$ is a relativistic Hilbert space and $\psi, \varphi \in \mathcal{H}$, the mapping

$$\mathbb{R}^4 \supseteq \mathcal{O} \mapsto \langle\varphi|\, E(\mathcal{O})\, \psi\rangle = \int \chi_{\mathcal{O}}(p)\, d^4\, \langle\varphi|\, E(p)\, \psi\rangle \tag{3.41}$$

is obviously a countably additive set function (recall that E is a PVM) which maps the measurable subsets $\mathcal{O} \subseteq \mathbb{R}^4$ to complex numbers, i.e. a complex measure which is normalized to

$$\big\langle\varphi\big|\, E(\mathbb{R}^4)\, \psi\big\rangle = \int d^4\, \langle\varphi|\, E(p)\, \psi\rangle = \langle\varphi|\, \psi\rangle \tag{3.42}$$

Moreover, if $\mathcal{H}$ obeys the spectrum condition, this complex measure has support in $\overline{V}_+$. Thus, corollary 3.6 immediately yields the following basic theorem:

Theorem 3.10

Let $\mathcal{H}$ be a relativistic Hilbert space obeying the spectrum condition. Let $\varphi, \psi \in \mathcal{H}$ and consider the function $f_{\varphi\psi}$ on $\mathbb{R}^4$ given by:

$$f_{\varphi\psi}(x) := \langle\varphi|\, U(x)\, \psi\rangle = \int e^{ipx}\, d^4\, \langle\varphi|\, E(p)\, \psi\rangle \tag{3.43}$$

If $f_{\varphi\psi}(x) = 0$ on an open connected subset $\mathcal{O} \subset \mathbb{R}^4$, then $f_{\varphi\psi} \equiv 0$ on all of $\mathbb{R}^4$.

Proof: Since $\langle\varphi|\, E\, \psi\rangle$ is a complex measure, $f_{\varphi\psi}$ is obviously a function which is exactly of the type of functions considered in corollary 3.6. With this corollary, the conclusion of theorem 3.10 immediately follows.

■

Most of the strong results we shall derive in the following directly follow from theorem 3.10. In connection with the Reeh-Schlieder theorem in section 3.5 we will need the following generalization of theorem 3.10:

Theorem 3.11

Let $\mathcal{H}$ be a relativistic Hilbert space obeying the spectrum condition. Let $A_1, \ldots, A_n$ be a set of bounded operators acting on $\mathcal{H}$, $\varphi, \psi \in \mathcal{H}$ and $x_k \in \mathbb{R}^4$ for $k = 1, \ldots, n$. Consider the function $f_{\varphi\psi}$ on $\mathbb{R}^{4n}$ given by:

$$f_{\varphi\psi}(x_1, \ldots, x_n) := \left\langle\varphi\right|\, U(x_1)A_1U^{-1}(x_1)U(x_2)A_2U^{-1}(x_2)\ldots U(x_n)A_n\, \psi\big\rangle \tag{3.44}$$

If $f_{\varphi\psi}(x) = 0$ in a neighborhood $\mathcal{N}(0) \subset \mathbb{R}^{4n}$ of the origin $0 \in \mathbb{R}^{4n}$, it follows that $f_{\varphi\psi} \equiv 0$ on all of $\mathbb{R}^{4n}$.

Sketch of the proof: That U is a representation of the translation group entails the group homomorphism $U(x)U(y) = U(x+y)$ for all $x, y \in \mathbb{R}^4$ and $U^{-1}(x) = U(-x)$. Consequently, we may write (3.44) as

$$\begin{aligned} f_{\varphi\psi}(x_1, \ldots, x_n) &= \langle\varphi|\, U(x_1)A_1U(x_2 - x_1)A_2U(x_3 - x_2)\ldots U(x_n - x_{n-1})A_n\, \psi\rangle \equiv \\ &\equiv \langle\varphi|\, U(\xi_1)A_1U(\xi_2)A_2U(\xi_3)\ldots U(\xi_n)A_n\, \psi\rangle =: g_{\varphi\psi}(\xi_1, \ldots, \xi_n) \end{aligned} \tag{3.45}$$

where we have introduced the n 'relative coordinates' $\xi_1 = x_1$ and $\xi_k = x_k - x_{k-1}$ for $k \geq 2$ to substitute the variables $x_1, \ldots, x_n$ (note that the mapping $(x_1, \ldots, x_n) \mapsto (\xi_1, \ldots, \xi_n)$ is bijective on $\mathbb{R}^{4n}$).

Let now $y_k \in \mathbb{R}^4$ for $k = 1, \ldots, n$ and denote $z_k = x_k + iy_k$ and $\eta_k = \xi_k + i\zeta_k$ with $y_1 = \zeta_1$ and $\zeta_k = y_k - y_{k-1}$ for $k \geq 2$ and consider the complex extension (wherever it exists)

$$\widetilde{f}_{\varphi\psi}(z_1, \ldots, z_n) \equiv \widetilde{g}_{\varphi\psi}(\eta_1, \ldots, \eta_n) \tag{3.46}$$

Observing that the right hand side of

$$\widetilde{g}_{\varphi\psi}(\eta_1, \ldots, \eta_n) = \int \prod_{k=1}^{n} \exp(ip_k\eta_k)\, d^{4n}\, \langle\varphi|\, E(p_1)A_1E(p_2)\ldots E(p_n)A_n\,\psi\rangle \tag{3.47}$$

is a complex measure integral on $\mathbb{R}^{4n}$, it follows straightforwardly in complete analogy to the above arguments (which can be easily generalized from the $4-$dimensional to the $4n-$ dimensional case), that $\widetilde{g}_{\varphi\psi}$ is holomorphic on $\mathcal{D} := \mathbb{R}^{4n} + iV_+^n$ and continuous on $\mathcal{D} \cup \mathbb{R}^{4n}$.

Moreover, it is easy to see that the assumption that there is a neighbourhood $\mathcal{N}(0)$ of $0 \in \mathbb{R}^{4n}$ on which $f_{\varphi\psi}$ vanishes entails that there is a neighbourhood $\widetilde{\mathcal{N}}(0) \subset \mathbb{R}^{4n}$ of the origin, such that $g_{\varphi\psi}(\xi_1, \ldots, \xi_n) = 0$ for all $(\xi_1, \ldots, \xi_n) \in \widetilde{\mathcal{N}}(0)$

If we now apply the obvious generalization from n to $4n$ complex variables of the analyticity arguments (reflection principle etc.) of the preceding section and the Edge-of-the-Wedge theorem 3.1 in $4n$ variables (where the convex cone is now given by $V = V_+^n \subset \mathbb{R}^{4n}$), we conclude that if $f_{\varphi\psi}$ vanishes on $\mathcal{N}(0)$ it follows that $g_{\varphi\psi}(\xi_1, \ldots, \xi_n) = 0$ for all $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^{4n}$ and consequently $f_{\varphi\psi}(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^{4n}$.

■

Remarks

Observe that the product of operators in the sandwich between φ and ψ on the right hand side of (3.44) is not a priori a product of Heisenberg operators of the form $U(x_k)A_kU^{-1}(x_k)$ in which case we would expect the operator $U^{-1}(x_n)$ to appear in between A_n and ψ on the right of (3.44). But this operator would destroy the central line of argument of the (sketch of the) proof: If we proceed in this case in complete analogy to the poof above, we would try to analytically continue the considered function into the region of $\mathbb{C}^{4n}$ where the imaginary parts $y_k \in \mathbb{R}^4$ of the complex variables $z_k = x_k + iy_k \in \mathbb{C}^4$ (where $k = 1, \ldots, n$) are constrained to be in the set

$$(y_1 \in V_+) \cap (y_2 - y_1 \in V_+) \cap (y_3 - y_2 \in V_+) \cap \cdots \cap (y_n - y_{n-1} \in V_+) \cap (y_n \in V_-) \tag{3.48}$$

(note that without the last expression $y_n \in V_-$ on the right, (3.48) exactly means that $\zeta_k \in V_+$ for $k = 1, \ldots, n$ for the 'relative' $\zeta-$coordinates in the proof above). But the set defined by (3.48) is actually the empty set: The last expression $y_n \in V_-$ on the right of (3.48) implies $y_n^0 < 0$, the expression $y_n - y_{n-1} \in V_+$ beforehand that $y_n^0 - y_{n-1}^0 > 0$, i.e. taken together we get $y_{n-1}^0 < 0$ and so on. If we follow this chain back to y_1, we finally arrive at $y_1^0 < 0$ which is in contradiction with $y_1 \in V_+$, though.

In the primary version of the Reeh-Schlieder theorem derived in section 3.5 below, for which theorem 3.11 will be crucial, it is assumed that $\psi = \Omega$ is the vacuum state, which is assumed to be (the uniquely defined state which is) invariant under space-time translations: $U^{-1}(x_n)\Omega = \Omega$. In consequence, (3.44) defines precisely the matrix elements $f_{\varphi\Omega}$ of the product of Heisenberg

operators $U(x_k)A_kU^{-1}(x_k)$ with respect to any state $\varphi \in \mathcal{H}$ and the vacuum Ω, since the troublesome factor $U^{-1}(x_n)$ is absorbed by the vacuum state Ω:

$$\begin{aligned}\big\langle\varphi\big|\, U(x_1)A_1U^{-1}(x_1)U(x_2)A_2U^{-1}(x_2)\dots U(x_n)A_nU^{-1}(x_n)\,\Omega\big\rangle \equiv \\ \equiv \big\langle\varphi\big|\, U(x_1)A_1U^{-1}(x_1)U(x_2)A_2U^{-1}(x_2)\dots U(x_n)A_n\,\Omega\big\rangle\end{aligned} \tag{3.49}$$

Some versions of the Reeh-Schlieder theorem embrace a more general result by observing that applying the above analyticity arguments to the first line of (3.49) does not necessarily require that the state on the right side is the vacuum Ω, although, as argued in the previous paragraphs, it cannot be an arbitrary state. In particular, ψ need not be a state of zero energy as Ω is, it is sufficient that the energy behaves in some sense nicely:

Definition 3.12 [*Analytic for the Energy*]

A state ψ in a relativistic Hilbert space $\mathcal{H}$ is called analytic for the energy, if ψ is in the domain of $(\widehat{P}_0)^n$ for any power $n \in \mathbb{N}$ and the series

$$\sum_n \Big\|(\widehat{P}_0)^n\psi\Big\|_{\mathcal{H}} \frac{z^n}{n!} \tag{3.50}$$

has non zero radius $a > 0$ of convergence.

In particular, any state of bounded energy is analytic for the energy (see e.g. [49, 347] and chapter 3.5). If ψ is analytic for the energy, it is not hard to see, that a function of the form

$$h_{\varphi\psi}(x_1,\dots,x_n) = \big\langle\varphi\big|\, U(x_1)A_1U^{-1}(x_1)U(x_2)A_2U^{-1}(x_2)\dots U(x_n)A_nU^{-1}(x_n)\,\psi\big\rangle \tag{3.51}$$

can be analytically continued into the subset of $\mathbb{C}^{4n}$ in which the imaginary parts of the complex variables are constrained to be in the set defined by (compare with (3.48))

$$(y_1 \in V_+) \cap (y_2 - y_1 \in V_+) \cap (y_3 - y_2 \in V_+) \cap \dots \cap (y_n - y_{n-1} \in V_+) \cap (y_n \in V_- + \widehat{e}_0 a) \tag{3.52}$$

where $\widehat{e}_0$ is the unit vector in time direction an $a > 0$ the radius of convergence of (3.50). This set is indeed non empty (it contains e.g. a neighbourhood of the point defined by $\boldsymbol{y}_k = 0$ for all $k = 0,\dots n$ and $y_1^0 < y_2^0 < \dots < y_n^0 < a$). Moreover, this set has $0 \in \mathbb{R}^{4n}$ in its boundary. This is enough to start the machinery of analyticity arguments to come to the conclusion that theorem 3.11 actually also applies to functions $h_{\varphi\psi}$ of the form (3.51) instead of $f_{\varphi\psi}$ in (3.44), given ψ is analytic for the energy. Thus we obtain the following corollary, which will proof helpful to draw certain parallels between different results of this chapter later in section 3.5:

Corollary 3.13

Let $\mathcal{H}$ be a relativistic Hilbert space obeying the spectrum condition. Let $A_1, \ldots, A_n$ be a set of bounded operators acting on $\mathcal{H}$, $\varphi, \psi \in \mathcal{H}$ with ψ analytic for the energy. Let $x_k \in \mathbb{R}^4$ for $k = 1, \ldots, n$ and consider the function $g_{\varphi\psi}$ on $\mathbb{R}^{4n}$ given by:

$$g_{\varphi\psi}(x_1, \ldots, x_n) := \left\langle \varphi \right| U(x_1)A_1U^{-1}(x_1)U(x_2)A_2U^{-1}(x_2)\ldots U(x_n)A_nU^{-1}(x_n)\left| \psi \right\rangle \qquad (3.53)$$

If $g_{\varphi\psi}(x) = 0$ in a neighbourhood $\mathcal{N}(0) \subset \mathbb{R}^{4n}$ of the origin $0 \in \mathbb{R}^{4n}$, it follows that $g_{\varphi\psi} \equiv 0$ on all of $\mathbb{R}^{4n}$.

3.3 Causal Propagation, Positive Energy, Infinite Tails and Local Transformations

Before analysing implications of theorem 3.10 for the quantum theory of measurement in the following sections, we start with more general physical considerations based on theorem 3.10 and corollary 3.6 in this section and shall thereby introduce important concepts and relations for later discussions. The following arguments are suitable for a relativistic first quantized one-particle quantum theory, the generalization to N particles is trivial by repeating the arguments with respect to the configuration space $\mathbb{R}^{3N}$ instead of $\mathbb{R}^3$ and through second quantization of one particle theories, they are indirectly relevant for the related QFTs as well.

Solutions of relativistic wave equations share two important properties: Causal propagation speed of the waves and the fact that positive energy wave functions are never compactly supported[119] in configuration space but have always infinite tails, although they can drop off extremely rapidly.

The first property derives from the fact that Minkowski space-time metric structure leads to covariant partial differential equations (like the Klein-Gordon or the Dirac equation) which are of the hyperbolic type and by that their solutions propagate in space with finite speed [194]. This mathematical relation is of course deeply related with causality requirements obtained by direct relativistic considerations. For the present purpose we can characterize causal propagation of a wave function ψ_t on $\mathbb{R}^3$ by the following necessary condition: Whenever the support of ψ_0 is contained in a ball of radius r, i.e. $\mathrm{supp}(\psi_0) \subset \mathcal{B}_r(\boldsymbol{a}) \subset \mathbb{R}^3$ for some $\boldsymbol{a} \in \mathbb{R}^3$, the time evolved wave function ψ_t satisfies $\mathrm{supp}(\psi_t) \subset \mathcal{B}_{r+|t|}(\boldsymbol{a})$, i.e. the support stays inside the light cone of each ball (or more general region) in which the wave function is supported at any time.

The second property (no compact support) can be shown directly, e.g. for Dirac theory with Paley-Wiener type analyticity arguments [276, 321] which are mathematically very close to the arguments developed in the previous section. Given causal propagation is true, it is also an immediate consequence of corollary 3.6 and the fact that a positive energy wave function can be perceived as the boundary value of an analytic function: Let ψ_t be a positive energy wave

[119] Actually, what is more, positive energy wave functions ψ can only vanish on sets of zero Lebesgue measure such that $\mathrm{supp}\,\psi = \mathbb{R}^3$ if ψ is a one-particle state. But the stronger notion of 'no compact support' directly corresponds to 'no perfect localization in bounded spatial regions', which is better suited for the present purposes.

function. With $x = (t, \boldsymbol{x}) \in \mathbb{R}^4$ we can write ψ_t covariantly as $\psi_t(\boldsymbol{x}) \equiv \psi(x) = \int e^{ipx} d^4\lambda(p)$ where again $p_\mu x^\mu =: px$ and λ is a complex measure supported in the forward light cone. A positive eneregy mass shell energy momentum relation for example yields a complex measure of the form $d^4\lambda(p) \sim \delta(p_\mu p^\mu - m^2)\Theta(p_0)\hat{\psi}(p)d^4p$, where Θ is the heavyside function and $\hat{\psi}$ a proper wave function in momentum representation. If now ψ_0 is compactly supported, for any $\boldsymbol{a} \in \mathbb{R}^3$ there is some finite $r > 0$ such that $\mathrm{supp}(\psi_0) \subset \mathcal{B}_r(\boldsymbol{a})$ and consequently $\psi_0(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathcal{B}_r^c(\boldsymbol{a})$ (the superscript c denoting the complement). Causal propagation entails then that at any other time ψ_t vanishes in $\mathcal{B}^c_{r+|t|}(\boldsymbol{a})$. If we write again $(t, \boldsymbol{x}) = x \in \mathbb{R}^4$ to denote $\psi_t(\boldsymbol{x}) \equiv \psi(x)$, this means that ψ vanishes in the complement of the forward and backward light cone of $\mathcal{B}_r(\boldsymbol{a})$, which of course contains open connected regions and with corollary 3.6 we conclude $\psi = 0$ on all of $\mathbb{R}^4$. Consequently, positive energy wave functions cannot be compactly supported if they propagate causally. Indeed, with the same arguments it is straightforward to see[120] that the support of such wave functions must be even all of $\mathbb{R}^3$.

Hegerfeldt Theorem without Operators

We can illustrate this connection between causal propagation and 'no compact support' in a Hilbert space framework by theorem 3.10 (which can be seen as a nice preparation for later proofs of this section): Consider a relativistic one-particle quantum theory with Hilbert space $\mathcal{H}$. Suppose $\psi_0 \in \mathcal{H}$ has compact support in position representation and consequently there exists some $r > 0$ such that $\mathrm{supp}(\psi_0) \subset \mathcal{B}_r(\boldsymbol{0})$. Suppose further that ψ_t propagates causally, in particular $\mathrm{supp}(\psi_t) \subset \mathcal{B}_{r+|t|}(\boldsymbol{0})$ for all $t \in \mathbb{R}$.

Now choose $\boldsymbol{a} \in \mathbb{R}^3$ large enough such that there exists some $d > 0$ such that $|\boldsymbol{a}| = 2r + d$ (see Fig. 5(i)). To begin with, this entails that ψ_0 and $\psi_0^{\boldsymbol{a}} := U(0, \boldsymbol{a})\psi_0$ have disjoint spatial support, since $\mathrm{supp}(\psi_0^{\boldsymbol{a}}) \subset \mathcal{B}_r(\boldsymbol{a})$ and $\mathcal{B}_r(\boldsymbol{0}) \cap \mathcal{B}_r(\boldsymbol{a}) = \emptyset$ for $|\boldsymbol{a}| = 2r + d$. Consequently, for a reasonable choice of scalar product in position representation (like the standard L^2 scalar product for possibly spinor valued L^2- functions[121]) we have

$$\langle \psi_0 |\, \psi_0^{\boldsymbol{a}} \rangle = 0 \tag{3.54}$$

Moreover, since $d > 0$ and $\psi_0^{\boldsymbol{a}}$ propagates causally by assumption, we can translate $\psi_0^{\boldsymbol{a}}$ in space and/or time such that its support still does not overlap with the support of ψ_0. In particular for all $(t, \boldsymbol{b}) \in \mathbb{R}^4$ with $|t| + |\boldsymbol{b}| < d$ it is certain that $\mathcal{B}_r(\boldsymbol{0}) \cap \mathcal{B}_{r+|t|}(\boldsymbol{a} + \boldsymbol{b}) = \emptyset$ (see Fig. 5 (ii), where this is depicted for positive t). Thus there is some neighbourhood $\mathcal{N}(0)$ of $0 \in \mathbb{R}^4$ such that

$$\langle \psi_0 |\, U(x)\, \psi_0^{\boldsymbol{a}} \rangle = 0 \quad \text{for all} \quad x \in \mathcal{N}(0) \tag{3.55}$$

If now $\mathcal{H}$ is a relativistic Hilbert space obeying the spectrum condition (e.g. the positive energy subspace of the Dirac equation) (3.55) together with theorem 3.10 entail that

$$\langle \psi_0 |\, U(x)\, \psi_0^{\boldsymbol{a}} \rangle = 0 \quad \text{for all} \quad x \in \mathbb{R}^4 \tag{3.56}$$

[120] Suppose there is some open connected spatial set $\Delta \in \mathbb{R}^3$ on which $\psi_0 = 0$. Thus there exists some $r > 0$ and some $\boldsymbol{a} \in \mathbb{R}^3$ such that the open ball $\mathcal{B}_r(\boldsymbol{a}) \subseteq \Delta$ and consequently $\psi_0(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathcal{B}_r(\boldsymbol{a})$. Causal propagation now entails that there is some $\varepsilon > 0$ such that $\psi_t(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathcal{B}_{\frac{r}{2}}(\boldsymbol{a})$ and for all $|t| < \varepsilon$, i.e. the wave function vanishes on the open connected set $(-t, t) \times \mathcal{B}_{\frac{r}{2}}(\boldsymbol{a})$ of space time, such that the positive energy assumption entails that it vanishes everywhere according to corollary 3.6.

[121] For example the Klein-Gordon scalar product for positive energy wave functions (see e.g. Schweber [300]) is actually not an L^2 scalar product in position representation but also obeys (3.54).

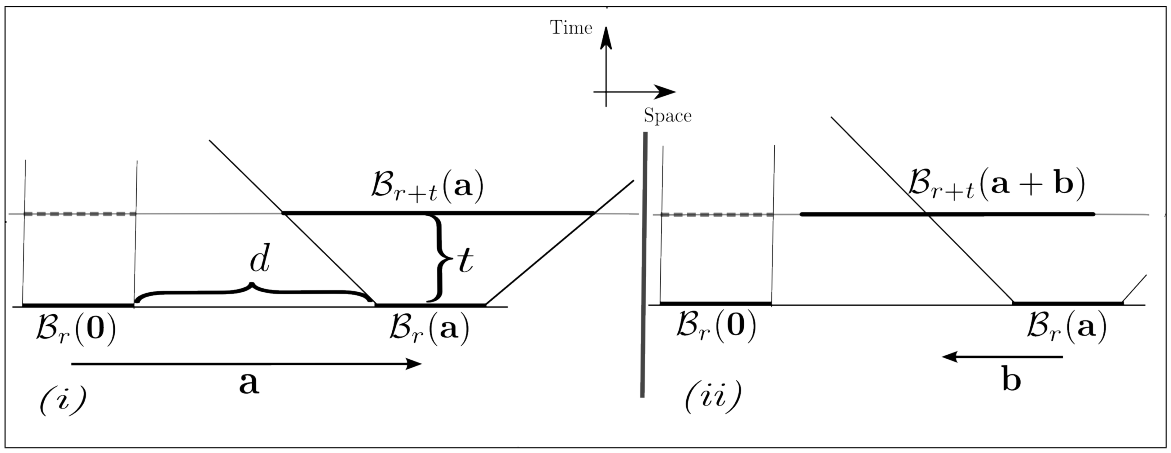

FIGURE 5

(*i*) ***The spatial open balls*** $\mathcal{B}_r(\mathbf{0}), \mathcal{B}_r(\boldsymbol{a}), \mathcal{B}_{r+t}(\boldsymbol{a}) \subset \mathbb{R}^3$ ***as subsets of simultaneity surfaces in a space-time diagram:*** *The wave function* ψ_0 *(see text), which has support contained in* $\mathcal{B}_r(\mathbf{0})$*, is translated by the vector* $\boldsymbol{a} \in \mathbb{R}^3$ *with absolute value* $|\boldsymbol{a}| = 2r + d$ *for some* $d > 0$ *such that* $\psi_0^{\boldsymbol{a}} = U(0, \boldsymbol{a})\psi_0$ *has support in* $\mathcal{B}_r(\boldsymbol{a})$ *which is disjoint from the support of* ψ_0*. Causal propagation implies that the support of* $\psi_0^{\boldsymbol{a}}$ *stays in the (here forward–) light cone of* $\mathcal{B}_r(\boldsymbol{a})$ *under the time evolution, i.e. for positive* t *as depicted we have* $supp(\psi_t^{\boldsymbol{a}}) \subset \mathcal{B}_{r+t}(\boldsymbol{a})$ *which is still disjoint form* $\mathcal{B}_r(\mathbf{0})$ *if* $t > 0$ *is small enough. (*ii*) Small enough spatial translations with respect to* $\psi_t^{\boldsymbol{a}}$ *(with small enough* $t > 0$*) do as well not lead to overlapping support with respect to* ψ_0 *since* $supp(U(t, \boldsymbol{b})\psi_0^{\boldsymbol{a}}) \subset \mathcal{B}_{r+t}(\boldsymbol{a} + \boldsymbol{b})$ *and* $\mathcal{B}_r(\mathbf{0}) \cap \mathcal{B}_{r+t}(\boldsymbol{a} + \boldsymbol{b}) = \emptyset$ *as long as* $|\boldsymbol{b}| + t < d$*.*

In particular, if we set now $x_{-\boldsymbol{a}} = (0, -\boldsymbol{a})$ and observe that $U(x_{-\boldsymbol{a}})\psi_0^{\boldsymbol{a}} = \psi_0$, we obtain

$$0 = \langle \psi_0 |\, U(x_{-\boldsymbol{a}})\, \psi_0^{\boldsymbol{a}} \rangle = \langle \psi_0 |\, \psi_0 \rangle = \|\psi_0\|^2 \tag{3.57}$$

and hence $\psi_0 = 0$. Thus, $\mathcal{H}$ does not contain wave functions with the assumed properties, namely, if causal propagation is true, all wave functions in $\mathcal{H}$ do not have support in a bounded region of space but must have infinite tails.

This is the core of Hegerfeld's theorem, which can be seen as an operational generalization of the argument just given. Hegerfeldt's theorem will be stated and proven in strong analogy to the line of argument from equation (3.54) to (3.57) in appendix B.

Positive Energy Localization Schemes

The following considerations shall be primarily illustrated by the Dirac equation (spin$-\frac{1}{2}$). The central concepts – in particular the Newton-Wigner localization scheme – and implications can be also transferred to relativistic wave equations of other spins like spin-0 (Klein-Gordon equation) or spin-1 (Proca equation) [249].

The assertion that relativistic wave functions must propagate causally and that relativistic positive energy wave functions cannot have compact support is actually not true in a precise sense: In the famous Foldy–Wouthuysen representation [300, 321] of the free Dirac equation for example, in which the first two components of associated spinors correspond to positive energies

whereas the third and forth component correspond to negative energies[122] (which is to say, the free Hamiltonian is given by $\beta\sqrt{\boldsymbol{p}^2+m^2}$, where $\beta = \begin{pmatrix} \mathbb{1}_{\mathbb{C}^2} & 0 \\ 0 & -\mathbb{1}_{\mathbb{C}^2} \end{pmatrix}$ is the $\beta-$Dirac matrix), positive energy wave functions can be perfectly localized in a bounded region of space. Positive energy states which are perfectly localized (compactly supported) in the Foldy–Wouthuysen representation – transformed back to the Dirac representation – are exactly the states which are proposed to be taken as 'localized' according to the Newton-Wigner localization scheme [249] for the Dirac equation. But the previous observations have shown that these states in the Foldy–Wouthuysen representation (where they are compactly supported) cannot propagate causally[123], and indeed, it is well known that they immediately develop infinite tails under the free time evolution [204, 280, 321].

The *Newton-Wigner scheme* was originally intended to provide a position operator in relativistic quantum theory which does not violate the spectrum condition, i.e. which preserves the 'positive energy property' of positive energy states. Recall that the projections constituting the PVM of the standard position operator are indicator functions in the standard position representation (in which Dirac wave functions propagate causally) and thus multiplication of wave functions by them in general results in compact support. Thus, it is an immediate consequence of the present observations that the standard position operator does not leave the positive energy subspace $\mathcal{H}_+$ of Dirac wave functions invariant, i.e. its action is incompatible with the spectrum condition[124].

Wightman has shown in [343] that the NW position operator – which is for a spin$-\frac{1}{2}$ particle given by multiplication by $\boldsymbol{x}$ in the Foldy–Wouthuysen representation and looks more complicated if transformed back to the standard representation – is the unique selfadjoint operator in relativistic quantum theory which does not violate the spectrum condition and which has some very basic properties a reasonable position operator should have (see also [321]). The eigenfunctions of the NW operator are in the standard representation highly peaked Bessel type functions (see below for more details) instead of $\delta-$functions, which can be – for all practical purposes – pretended to be $\delta-$functions though. This entails that e.g. the wave function of a Dirac-particle which is perfectly localized in a bounded region with respect to the NW scheme (i.e. its wave function has support in the Foldy–Wouthuysen representation which is contained in that region) has infinite tails in the standard representation, which amounts to a rejection of the Born rule (the particle could be found in principle outside that region with respect to the Born rule but not if the NW-scheme was true). But the quantitative difference of the two localization schemes – in particular the deviation in localization probabilities – is so small that the disagreement of the two schemes is not empirically accessible.

Nonetheless, conceptually there is a great difference between the two schemes: It is physi-

[122]The recipe for transition to the free Foldy-Wouthuysen representation goes like this: Start with standard position representation, perform a Fourier transformation to momentum space such that the momentum operator $\boldsymbol{p}$ is a real valued vector, diagonalize the 4×4 matrix $\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m$ (the free Dirac Hamiltonian) and go back to position representation by inverse Fourier transformation.

[123]We have just shown that *causal propagation* $\wedge$ *positive energy* $\Longrightarrow$ *no compact support* form which we can deduce *compact support* $\wedge$ *positive energy* $\Longrightarrow$ *no causal propagation*.

[124]But note also that the absence of the standard position operator – when relativistic quantum theory is restricted to only positive energies – does not prevent us from taking $|\psi(\boldsymbol{x})|^2$ as the probability amplitude for particle positions, i.e. to hold onto the Born rule (see also Thaller [321]).

cally inadvisable to regard the states in the Foldy–Wouthuysen representation as a probability amplitude for particle positions, i.e. to assume the NW scheme to be true: After all, as just argued, this would allow particles with non zero positive rest mass to move faster than light in principle, although the related probabilities are extremely small. There are even more arguments not to take the Newton-Wigner localization scheme too seriously, namely the related fact that a NW localized state in a given Lorentz frame is not NW-localized in any other frame [244], that this scheme does not lead to a conserved probability current [23] in contrast to the Born density, or that the minimal coupling of charged particles to an electromagentic field in the Dirac representation – which underlies some impressingly precise new predictions of QED like the Lamb shift – does not work in the Foldy–Wouthuysen representation [89].

But NW localized states within the usual interpretation (i.e. with the Born-rule in the standard position representation) are overwhelmingly well localized Dirac wave functions of positive energy [300], which illustrates that it is fapp possible to pretend that positive energy wave functions can have compact support, but only not from a precise mathematical point of view. The eigenstates of the NW operator are as mentioned Bessel type functions (Hankel functions, to be precise) which are highly peaked about their respective center $\boldsymbol{x}_0$, which have width of the order of the Compton wavelength λ_C (with respect to the considered particle mass) and which drop off like $e^{-\frac{|\boldsymbol{x}-\boldsymbol{x}_0|}{\lambda_C}}$ if the distance from the center $|\boldsymbol{x}-\boldsymbol{x}_0|$ is greater than a few Compton wavelengths[125]. By superposing such states, wave functions can be obtained which can be treated as compactly supported for all practical purposes.

If the aspiration to obtain a new position operator respecting the spectrum condition is abandoned, but only positive energy states which are as close as possible to compactly supported wave functions in relativistic quantum theory are searched for, the NW states are even not the last word but nice refinements are possible:

The *Philips scheme* [263, 143, 202, 144, 203] is the Lorentz invariant modification of the NW scheme, it comes along if in the derivation of the NW scheme the assumption of mutual orthogonality of distantly localized states is dropped if favour of form invariance under Lorentz transformations of localized states (i.e. states localized according to the scheme stay localized according to the scheme under Lorentz transformations). Since orthogonality is dropped, Philips-localized states are not the eigenstates of a selfadjoint operator. These states are Bessel-type functions as the NW-states and are also with respect to width and decrease comparable with the NW-states.

The *localization scheme of Bracken and Melloy* [54, 55, 238] goes even one step further: It shows for the case of Dirac theory that delta functions can be arbitrarily well approximated by positive energy states. In particular, Bracken and Melloy constructed sequences $(\varphi_n)_{n\in\mathbb{N}}$ of positive energy Dirac wave functions which also drop off extremely fast and whose width approaches zero as n goes to infinity[126]. Since this scheme is so surprisingly simple, it shall be

[125]To get an impression, how massively localization probabilities are suppressed for such states already at microscopic distances from the center (e.g. outside the ball $\mathcal{B}_{\lambda_c}(\mathbf{0})$ of radius $r=\lambda_C$ about the center $\boldsymbol{x}_0=\mathbf{0}$), one might calculate

$$\int_{\mathcal{B}_{\lambda_C}(\mathbf{0})} e^{-\frac{|\boldsymbol{x}|}{\lambda_c}}\, d^3x \;\sim\; (\lambda_C)^2 \tag{3.58}$$

which is e.g. for electrons of the order 10^{-24}.

[126]This disproves the wide spread prejudice that positive energy states cannot be squeezed to a width much

briefly sketched here:

We start with a positive energy eigenstate of the free Dirac Hamiltonian in momentum space, a standard choice is for example [158, 300]

$$u^+(\boldsymbol{p}) = \frac{1}{\sqrt{2E(\boldsymbol{p})(E(\boldsymbol{p})+m)}} \begin{pmatrix} E(\boldsymbol{p})+m \\ 0 \\ p_z \\ p_x + ip_y \end{pmatrix} \tag{3.59}$$

where $E(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$. The positive energy eigenequation $\mathscr{H} u^+(\boldsymbol{p}) = E(\boldsymbol{p}) u^+(\boldsymbol{p})$, which u^+ satisfies with respect to the Dirac Hamiltonian $\mathscr{H} = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m$, is for each $\boldsymbol{p}$ just a $4 \times 4-$matrix eigenequation, such that we may multiply u^+ by any (preferably normalizable) function f of $\boldsymbol{p}$ to obtain again a positive energy state in momentum space. In particular, if we choose a sequence of functions f_n whose width increases with increasing n, we can expect that the position representation, i.e. the Fourier transform of $u^+ f_n$ has decreasing width for increasing n. A simple choice is a Gaußian with increasing width, i.e. we may choose

$$f_n(\boldsymbol{p}) = \frac{1}{(\sqrt{\pi} n m)^{\frac{3}{2}}} e^{-\frac{\boldsymbol{p}^2}{2(nm)^2}} = \frac{1}{(\sqrt{\pi} n m)^{\frac{3}{2}}} e^{-\lambda_c^2 \frac{\boldsymbol{p}^2}{2n^2}} \tag{3.60}$$

(where $\lambda_c = \frac{1}{m}$ is the Compton wavelength) such that $\hat{\varphi_n}(\boldsymbol{p}) = u^+(\boldsymbol{p}) f_n(\boldsymbol{p})$ is properly normalized. The corresponding sequence of functions in position representation is given by

$$\varphi_n(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int u^+(\boldsymbol{p}) f_n(\boldsymbol{p}) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} d^3p \tag{3.61}$$

These functions yield a probability distribution

$$\rho_n(\boldsymbol{x}) = \varphi_n^\dagger(\boldsymbol{x}) \varphi_n(\boldsymbol{x}) = \sum_{k=1}^{4} \overline{\varphi_n^{(k)}}(\boldsymbol{x}) \varphi_n^{(k)}(\boldsymbol{x}) \tag{3.62}$$

(where $\varphi_n^{(k)}$ is the k'th spinor component of φ_n) which is practically zero already in the distance of the Compton wave length from the origin and everywhere outside the ball $\mathcal{B}_{\lambda_c}(\boldsymbol{0})$, even for small values of n (see Fig. 6). With n growing ρ_n becomes even more and more localized about the origin. By choosing n appropriately, the effects of infinite tails of positive energy states can be controlled even already on the microscopic level, e.g. we might consider the probabilities according to the Born rule for localization outside of $\mathcal{B}_{\lambda_c}(\boldsymbol{0})$ given by

$$F(n) = \int_{\mathcal{B}_{\lambda_c}^c(\boldsymbol{0})} \rho_n(\boldsymbol{x})\, d^3x \tag{3.63}$$

smaller than the Compton wavelength. Usually it is argued that this leads to an uncertainty in the energy which is large compared with the rest energy m of the considered particle such that particle creation (which corresponds to transitions from the negative to the positive spectrum in the one particle picture) is inevitable. Bracken and Melloy on the other hand show, that the mean energy of states with extremely strongly squeezed width, which is indeed much greater than the associated rest energy if the width is much smaller than λ_C, can be in principle completely realized as kinetic energy, which is to say such states immediately spread with almost the speed of light under the free time evolution [54].

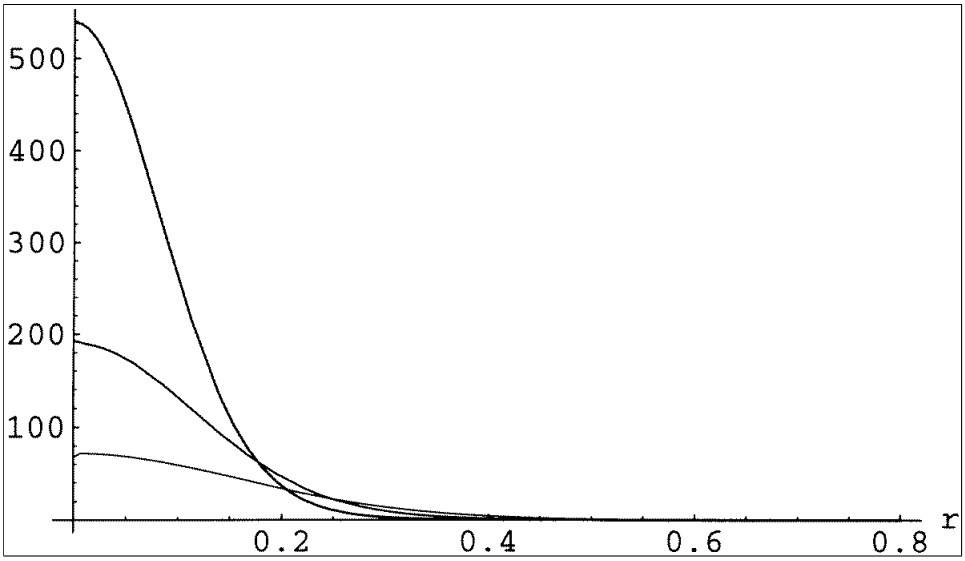

FIGURE 6

Bracken-Melloy: *The spherically symmetric densities ρ_n in* (3.62) *are depicted as functions of $r = \frac{|x|}{\lambda_C}$ for $n = 5, 7$ and* 10. *The plot was numerically determined and is taken from [55].*

where $\mathcal{B}^c_{\lambda_c}(\mathbf{0})$ is the complement of $\mathcal{B}_{\lambda_c}(\mathbf{0})$. And indeed, the results of Bracken and Melloy entail that for each $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that $F(n) < \varepsilon$ for all $n > N$. By superpositions of states like φ_n, one can thus obtain a variety of positive energy wave functions localized in arbitrary spatial regions arbitrarily well, even though never perfectly.

Local Transformations

We can also infer from the the above considerations that local perturbations of positive energy wave functions are not possible[127] in relativistic quantum theory with causal propagation: Consider relativistic wave functions ψ and ψ' and suppose that ψ and ψ' do only differ from one another in a bounded region $\Delta \subset \mathbb{R}^3$; we may think of interaction with a local external potential which is switched on and off again, transforming ψ into ψ', or the like. This entails that the (non normalized) wave function, given by $\varphi(\boldsymbol{x}) := \psi(\boldsymbol{x}) - \psi'(\boldsymbol{x})$ vanishes whenever $\boldsymbol{x} \notin \Delta$, i.e. φ has compact support. According to the above argument (given causal propagation is true), φ can thus not be a positive energy wave function, i.e. given ψ has positive energy, ψ' must have contributions from negative energy eigenstates of the free Hamiltonian.

The very same is true if there is a constant C such that $\psi' = C\psi$ outside of a bounded spatial region Δ: We may think of an arbitrary experiment in some laboratory with initial state ψ. Since in each such experiment, given the experimental device is being triggered by the measured system, the latter is being detected in the laboratory, one might expect that its post measurement wave function ψ' is extremely suppressed in a proper distance from the experimental device, although if ψ' shall be a positive energy state, it cannot have compact support. But the spatial probability distribution of the measured system far away from the device might be expected not to be qualitatively changed beyond being suppressed, i.e. there

[127] I'm very thankful to Roderich Tumulka, who recognized and pointed out this fact to me during discussions!

is some very small constant C such that $\psi' = C\psi$ in some distance from the measuring device (or outside of the laboratory). But the hope to save the positive energy of ψ' must be given up again, since then $\varphi := \psi' - C\psi$ has compact support and thus again (given causal propagation is true) if ψ is assumed to be a positive energy initial state, ψ' must have contributions from negative energy eigenstates.

We can go even one step further and exclude almost all possible global transformations of wave functions – except very special ones – if the spectrum condition shall be rescued: Suppose two positive energy wave functions ψ and ψ' agree on an arbitrarily small open connected spatial region $\Delta \subset \mathbb{R}^3$ such that $\varphi = \psi - \psi'$ vanishes on Δ. Recalling that the causally propagating support of positive energy wave functions must be all of $\mathbb{R}^3$ entails that either φ has contributions from negative energies (and thereby either ψ or ψ' or both) or $\varphi = 0$ such that $\psi = \psi'$. In other words, distinct positive energy states can only coincide on spatial sets of zero Lebesgue measure (provided they propagate causally). This also follows from the identity theorem, if $\psi(t, \boldsymbol{x})$ and $\psi'(t, \boldsymbol{x})$ are perceived as boundary values of analytic functions.

Thus the values a positive energy wave function ψ takes in an arbitrarily small neighbourhood of any point already determines ψ on all of $\mathbb{R}^3$. Thus transformations preserving the spectrum condition are extremely special since the transformation of a positive energy wave function in any region must be precisely in accord with its transformation in any other region. The free unitary time evolution is such a special transformation, but if only in the Hamiltonian there appears interaction with some strictly local, time dependent potential acting locally on the wave function, negative energies will inevitably contribute.

These observations do not leave much hope that local measurements respect the spectrum condition on the one particle level, even if the infinite tails of measuring devices (wave functions, potentials) are taken into account, since transformations upon measurement are very harsh and thorough. One might moreover take into account related decoherence processes, whose strongly localizing impact on wave functions is well examined [192]. But as already remarked in section 3.1, at least in case of Dirac theory we know that physical processes which lead to transitions between positive and negative energy spectrum turn out to be processes which involve pair creation effects if the theory and the respective description of the processes is lifted to Fock space by second quantization (provided they can be lifted at all), see section A in the appendix for details.

3.4 Malament Type Theorems

3.4.1 Prelude on Operators

We start with some rather abstract results and then draw a line to physical analysis of local measurement (like) processes.

Corollary 3.14 [*Products of selfadjoint local operators cannot vanish only locally*]

Let $\mathcal{H}$ be a relativistic Hilbert space which obeys the spectrum condition. Let A, B be bounded selfadjoint linear operators acting on $\mathcal{H}$. If $\mathcal{N} \subseteq \mathbb{R}^4$ is some open subset such that $B \cdot U(x)AU^{-1}(x) = 0$ for all $x \in \mathcal{N}$ it follows $B \cdot U(x)AU^{-1}(x) = 0$ for all $x \in \mathbb{R}^4$.

Proof: For arbitrary $\varphi, \psi \in \mathcal{H}$ the function

$$f(x) := \langle B\,\varphi|\, U(x)\, A\,\psi\rangle = \big\langle \varphi\big|\, \big(B\, U(x)\, A\, U^{-1}(x)\big)\; U(x)\,\psi\big\rangle \tag{3.64}$$

is equal to zero for all $x \in \mathcal{N}$ by assumption. According to theorem 3.10 this implies $f(x) = 0$ for all $x \in \mathbb{R}^4$ and since φ and ψ were arbitrary $(B \cdot U(x)AU^{-1}(x))\; U(x) = 0$ for all $x \in \mathbb{R}^4$ which – multiplied by $U^{-1}(x)$ from the right – yields $B \cdot U(x)AU^{-1}(x) = 0$ for all $x \in \mathbb{R}^4$.

■

Theorem 3.15 [*No PVMs on open subsets of* $\boldsymbol{\mathcal{M}}$]

Let $\mathcal{H}$ be a relativistic Hilbert space which obeys the spectrum condition. Let $\Xi \subset \mathcal{M}$ be some open subset of space-time. There does not exist a space-time translation covariant PVM on Ξ acting on $\mathcal{H}$.

Proof: Suppose there exists a space-time translation covariant PVM on Ξ acting on $\mathcal{H}$ and denote the respective projections by $P_{\mathcal{O}}$. Let $\mathcal{O}' \subset \Xi$ be a non empty measurable subset of Ξ which is without loss of generality (recall that Ξ is open) small enough such that there is an $a \in \mathbb{R}^4$ such that $\mathcal{O}' + a \subset \Xi$ and $\mathcal{O}'$ and $\mathcal{O}' + a$ have mutually disjoint neighbourhoods in Ξ.

The fact that the projections $P_{\mathcal{O}}$ constitute a PVM on Ξ entails according to corollary 1.7 that $P_{\mathcal{O}_1} P_{\mathcal{O}_2} = P_{\mathcal{O}_1 \cap \mathcal{O}_2}$ for all $\mathcal{O}_1, \mathcal{O}_2 \subset \Xi$ and in particular $P_{\mathcal{O}'} P_{\mathcal{O}'+a} = P_\emptyset = 0$. Since now $\mathcal{O}'$ and $\mathcal{O}' + a$ have disjoint neighbourhoods in Ξ, there exists a neighbourhood $\mathcal{N}(0)$ of $0 \in \mathbb{R}^4$ such that $\mathcal{O}' \cap (\mathcal{O}' + a + x) = \emptyset$ and $\mathcal{O}' + a + x \subset \Xi$ for all $x \in \mathcal{N}(0)$. Thus, since the PVM is space-time translation covariant, $P_{\mathcal{O}'} P_{\mathcal{O}'+a+x} = P_{\mathcal{O}'}\, U(x)\, P_{\mathcal{O}'+a}\, U^{-1}(x) = 0$ for all $x \in \mathcal{N}(0)$. The preceding corollary 3.14 then entails that $P_{\mathcal{O}'} P_{\mathcal{O}'+a+x} = 0$ for all $x \in \mathbb{R}^4$. In particular the choice $x = -a$ yields $0 \equiv P_{\mathcal{O}'}^2 = P_{\mathcal{O}'}$.

Now we may cover all of Ξ with translates of $\mathcal{O}'$ and use (sub-)additivity to get $P_\Xi \equiv 0$ which obviously contradicts $P_\Xi \stackrel{!}{=} \mathbb{1}_{\mathcal{H}}$, i.e. a PVM with the assumed properties does not exist[128].

■

Schlieder has shown in [291] in the framework of AQFT that any two projections $P_{\mathcal{O}}, P_{\mathcal{O}'}$ which are elements of local operator algebras $\mathcal{L}_{\mathcal{O}}$ and $\mathcal{L}_{\mathcal{O}'}$ (see section 3.5 below) associated with spacelike separated regions of space-time $\mathcal{O}$ and $\mathcal{O}'$, respectively, can never obey $P_{\mathcal{O}} P_{\mathcal{O}'} = 0$, although they always commute according to a fundamental assumption of AQFT (local commutativity). The central argument in the proof of theorem 3.15 goes in a similar direction as this Schlieder property, it shows that whenever two non zero projections P and P_x are space-time translates of each other for some $x \in \mathbb{R}^4$ in the sense that $P_x = U(x)PU^{-1}(x)$, it is not possible that their product vanishes[129] for variations of x in an arbitrarily small open set.

[128] This is actually a bit sloppy, since it is not taken for granted that a single subset $\mathcal{O}' \subset \Xi$ exists, which is small enough such that we can perform the above argument and which is shaped such that it is possible to exactly cover all of Ξ with unions of translates of $\mathcal{O}'$. But we may choose an arbitrarily small and arbitrarily shaped subset as $\mathcal{O}'$ such that we can approximate Ξ arbitrarily close.

[129] Observe that this entails that mutually anticommuting projections cannot be space-time translates of each other, since as argued in the proof of corollary 1.7, two orthogonal projections can only anticommute if their

Theorem 3.15 is a rather abstract statement though, since it is hard to imagine a meaningful measurement process which corresponds to a PVM on an open subset of space-time. Not least, an open subset of space-time always has a timelike extension, such that the PVM, if it corresponded to a real world measurement, would provide a marginal statistical prediction for a time observable. But we know from the arrival time problem in quantum theory, that such an observable does not exist, at least not as a PVM.

Before formulating and proofing physically more substantial related results, we shall develop a framework of realistic local measurement like processes which can be assumed to be in the predictive scope of relativistic quantum theory:

3.4.2 Detecor Click Statistics

The prototype experiment of the present investigations is that of a general particle detection experiment. In view of the upcoming no-go theorems, we should always have the following in mind: We know that there are particle detectors in the world, that each detector covers a given bounded region of space and that a given detector at a given location in a given experimental setup does either click or not at a given time, where 'click' loosely represents any positive output of the measuring device (like the appearance of a dot on a screen or of a trajectory in a bubble chamber, an electrical signal to a registering computer or just a click etc.). Note however, that at the end of the day presumably every experiment may be described as a detection experiment; if for example the spectrum of a hydrogen atom was measured, the hydrogen atom was detected in the spacial region associated with the laboratory and so on.

Remark: A criticism standing to reason of the basic and natural assumption that detectors cover bounded regions derives from the fact that positive energy wave functions are never perfectly localized in configuration space but have always infinite tails, such that even wave functions of macroscopic devices are spread all over configuration space if the spectrum condition is true, although they will drop of extremely rapidly outside of the region which we identify with their macroscopic spatial position. In particular, the measurement formalism was derived in chapter 1 on basis of the assumption that pointer states have compact and mutually disjoint support in configuration space such that they always represent a unique state of affairs with respect to the display of measurement results, which won't be true in a strict sense (which might be relevant for the results which follow) if the spectrum condition is valid (or more generally that the energy is bounded from below). But it is actually not a big loss of generality to assume that a detector can always be associated with a bounded spatial region, at least approximately, since as demonstrated in section 1.5.2, extended pointer states which are well localized but might be actually extended all over configuration space and thereby identifiable with a bounded spatial region only approximately, can be in principle accounted for by approximate measurement POVMs (see section 1.4.3), and the latter are also covered are by the assertions finally derived in the present analysis (in particular theorem 3.25).

product vanishes: $PP_x = PPP_x = -PP_xP = P_xPP = P_xP$ i.e. anticommuting P and P_x must commute as well and thus $PP_x = -PP_x$ which is only possible if $PP_x = 0$.

Henceforth, we denote the event that a given detector clicks by $\mathscr{D}$ and if the spatial region Δ covered by the detector and the time t at which it clicks in the laboratory frame are specified, we denote the respective click events[130] by $\mathscr{D}_{(t,\Delta)}$.

Assumptions

The results developed below are derived from the following relatively mild assumptions, one or several of which might be questioned or a sensible physical understanding of these results must be given. The way the assumptions are developed and rooted in a physical framework of detector experiments is unusual with respect to other presentations but it is helpful to find a natural possible meaning.

We assume that quantum theory is able to predict the right click statistics of detection type experiments, i.e. if $\mathcal{H}$ is the Hilbert space of possible initial states, there is a positive operator (an effect) D acting on $\mathcal{H}$ such that the click probability is given by

$$\mathbb{P}^{\psi}\left(\mathscr{D}\right) = \langle\psi|\, D\,\psi\rangle \tag{3.65}$$

if $\psi \in \mathcal{H}$ is the initial state. Correspondingly, the probability of 'no click' is given by the operator $D^{\perp} = \mathbb{1}_{\mathcal{H}} - D$ via

$$\mathbb{P}^{\psi}\left(\neg\mathscr{D}\right) = \left\langle\psi\right|\, D^{\perp}\,\psi\rangle \tag{3.66}$$

i.e. the operators $\left\{D,\, D^{\perp}\right\}$ form a minimal POVM associated with the events $\mathscr{D}$: 'click' and $\neg\mathscr{D}$: 'no click'.

Moreover, we assume that $\mathcal{H}$ is a relativistic Hilbert space and that the detector operators D are space-time translation covariant Heisenberg operators: If $D_{(t,\Delta)}$ is the effect in the Heisenberg picture associated with the click probability of a detector covering the spacial region $\Delta \subset \mathbb{R}^3$ at time t in the laboratory frame and $x = (s, \boldsymbol{y}) \in \mathbb{R}^4$, the effect

$$D_{(t,\Delta)+x} := U(x) D_{(t,\Delta)} U^{-1}(x) \tag{3.67}$$

is the effect in the Heisenberg picture associated with the click probability of the same detector if it is displaced such that it covers the spacial region $\Delta + \boldsymbol{y}$ at time $t + s$ in the laboratory frame.

Two or more detectors of the same kind can be combined to an experiment with several detectors (see the following point below). The notion of *'detectors of the same kind'* shall capture all features which we intuitively have about 'detectors of the same kind': In particular, two detectors of the same kind are sensitive to the same kind of quantum systems, i.e. the related effects act on the same Hilbert space, and given both detectors have the same shape and size, they have the same click statistics with respect to the spatial regions they cover and the respective initial states, which is to say both are represented by the same effects $D_{(t,\Delta)}$.

In a strict sense, we neglect in the following that some kinds of detectors – like screens – have a spatial orientation. This fact could be accounted for in the upcoming analysis, by utilizing the

[130] Note, that what is called 'click event' in the present context is actually not an event in the sense of relativistic space-time in a strict sense, since each detector has some spatial extension. But we may think of click events as approximate space-time events, e.g. in the proofs of the theorems below, we may choose arbitrarily large distances between different detectors such that we can neglect their spatial extension.

rotation subgroup additionally to the space time translation subgroup of the Poincaré group, and thereby taking into account the rotations of detectors. But this – besides making things more complicated – would not significantly alter the insights we shall gain.

Definition 3.16 [*Covariant Detector Formalism*]

A covariant quantum formalism $\mathcal{E}^{\mathcal{H}}_{\Delta}$ predicting the statistics of some detection experiment $\mathcal{E}_{\Delta}$ or briefly a covariant detector formalism is defined as follows: Given a yes/no-measurement $\mathcal{E}_{\Delta}$ associated with some bounded spatial region $\Delta \subset \mathbb{R}^3$ (we call Δ the region covered by the detector) at time $t = 0$ in the laboratory frame, the statistics of outcomes can be predicted by a minimal POVM $\left\{D_{(0,\Delta)}, D^{\perp}_{(0,\Delta)}\right\}$ acting on some relativistic Hilbert space $\mathcal{H}$: If $\psi \in \mathcal{H}$ is the initial pure state

$$\mathbb{P}^{\psi}\left(\mathscr{D}_{(0,\Delta)}\right) = \left\langle \psi \right| D_{(0,\Delta)}\, \psi \rangle \tag{3.68}$$

where $\mathscr{D}_{(0,\Delta)}$ denotes the yes-outcome (we say the detector covering Δ clicks at time $t = 0$) and

$$\mathbb{P}^{\psi}\left(\neg\mathscr{D}_{(0,\Delta)}\right) = \left\langle \psi \right| D^{\perp}_{(0,\Delta)}\, \psi \rangle \equiv \left\langle \psi \right| \left(\mathbb{1}_{\mathcal{H}} - D_{(0,\Delta)}\right)\, \psi \rangle \tag{3.69}$$

where $\neg\mathscr{D}_{(0,\Delta)}$ denotes the no-outcome (the detector covering Δ does not click at time $t = 0$).

The POVM is assumed to be space-time translation covariant, i.e. for all $x = (s, \boldsymbol{y}) \in \mathbb{R}^4$, the effect

$$D_{(0,\Delta)+x} := U(x) D_{(0,\Delta)} U^{-1}(x) \tag{3.70}$$

is the effect in the Heisenberg picture, yielding the probability of the yes-outcome of the same experiment at time s if the detector covers $\Delta + \boldsymbol{y}$ in the laboratory frame.

Moreover, we assume that detectors of the same kind can be combined to composite detector experiments and that accordingly joint probability distributions for the joint click statistics of composite detector experiments exist, which are consistent with the no signalling requirement (and thereby with relativistic consistency, see chapter 2), i.e. the click statistics of a given detector is not altered if another detector is simultaneously posed in a spatially separated region (see the following issue on combining detectors for details).

If $\mathcal{H}$ obeys the spectrum condition 3.9 we say that $\mathcal{E}^{\mathcal{H}}_{\Delta}$ obeys the spectrum condition.

Of course, we might straightforwardly generalize this definition to differently shaped detectors of the same kind by letting Δ not only represent a single given spatial region but a collection of differently shaped and sized regions. But for the central lines of argument of the following analysis, a single spatial region together with its translates is essentially sufficient. So whenever in the following we associate two (or more) disjoint spatial detector regions $\Delta, \Delta' \subset \mathbb{R}^3$ with a given detector formalism $\mathcal{E}^{\mathcal{H}}_{\Delta}$, we may always take Δ' as a spatial translate of Δ, i.e. $\Delta' = \Delta + \boldsymbol{a}$ for some vector $\boldsymbol{a} \in \mathbb{R}^3$ which is purely spatial with respect to the laboratory frame.

Combining Detectors

Two or more detectors of the same type can of course be combined to a detection experiment with several detectors. For example, consider the case of two detectors covering disjoint spatial

regions Δ and Δ', respectively, in the laboratory frame. The possible outcomes at a given time t in the laboratory frame are a priori given by the four possible elementary events

$$\Omega_{(t,\Delta)\wedge(t,\Delta')} := \left\{(1_{(t,\Delta)}, 0_{(t,\Delta')})\ ;\ (0_{(t,\Delta)}, 1_{(t,\Delta')})\ ;\ (0_{(t,\Delta)}, 0_{(t,\Delta')})\ ;\ (1_{(t,\Delta)}, 1_{(t,\Delta')})\right\} \tag{3.71}$$

where now $(1_{(t,\Delta)}, 0_{(t,\Delta')})$ denotes the event that the detector covering Δ clicks and the detector covering Δ' does not click at time t in the laboratory frame and so on. With respect to this space of possible outcomes, the event $\mathscr{D}_{(t,\Delta)}\big|_{(t,\Delta')}$ that the detector covering Δ clicks at time t in the laboratory frame – where the subscript $|_{(t,\Delta')}$ indicates the presence of the second detector at Δ' – is no longer an elementary event but given by the disjunction of two possible realizations of the result of the combined detection experiment:

$$\mathscr{D}_{(t,\Delta)}\big|_{(t,\Delta')} = (1_{(t,\Delta)}, 0_{(t,\Delta')}) \vee (1_{(t,\Delta)}, 1_{(t,\Delta')}) \tag{3.72}$$

Analogously, we have

$$\mathscr{D}_{(t,\Delta')}\big|_{(t,\Delta)} = (0_{(t,\Delta)}, 1_{(t,\Delta')}) \vee (1_{(t,\Delta)}, 1_{(t,\Delta')}) \tag{3.73}$$

For each initial state $\psi \in \mathcal{H}$ there should be a joint click statistics which is expressed by some probability measure on $\Omega_{(t,\Delta)\wedge(t,\Delta')}$. In case of a single detector we already assumed the existence of a probability measure $\mathbb{P}^\psi$ on the space of elementary events $\Omega_{(t,\Delta)} = \left\{\mathscr{D}_{(t,\Delta)}, \neg\mathscr{D}_{(t,\Delta)}\right\}$ given by the effects $D_{(t,\Delta)}$ and $\mathbb{1}_{\mathcal{H}} - D_{(t,\Delta)}$ and we sloppily denote the probability measure on $\Omega_{(t,\Delta)\wedge(t,\Delta')}$ by $\mathbb{P}^\psi$ as well, which will cause no confusion if we are aware of the fact that we are actually dealing with two different probability measures on different probability spaces. For the following, we need not specify how the effects which yield the joint probabilities on $\Omega_{(t,\Delta)\wedge(t,\Delta')}$ relate to the single detector formalism[131], but if *no signalling* (see chapter 2) is true – which we always assume – the marginal probability

$$\begin{aligned}\mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)}\big|_{(t,\Delta')}\right) = \mathbb{P}^\psi\left((1_{(t,\Delta)}, 0_{(t,\Delta')}) \vee (1_{(t,\Delta)}, 1_{(t,\Delta')})\right) = \\ = \mathbb{P}^\psi\left((1_{(t,\Delta)}, 0_{(t,\Delta')})\right) + \mathbb{P}^\psi\left((1_{(t,\Delta)}, 1_{(t,\Delta')})\right)\end{aligned} \tag{3.74}$$

equals the probability for a click event when there is only one detector covering Δ present and no detector about Δ' (note that (t,Δ) is spacelike with respect to (t,Δ') for $\Delta \cap \Delta' = \emptyset$), i.e. the statistics at Δ do not depend on the fact whether an apparatus is simultaneously posed at a remote region Δ' or not:

$$\mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)}\big|_{(t,\Delta')}\right) \stackrel{!}{=} \mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)}\right) = \left\langle\psi\right| D_{(t,\Delta)}\left|\psi\right\rangle \tag{3.75}$$

Note that as mentioned above, $\mathbb{P}^\psi$ has two different meanings in (3.75), on the left hand side it is a probability measure on $\Omega_{(t,\Delta)\wedge(t,\Delta')}$ whereas on the right hand side of the first equality sign it is

[131] As comprehensible from the discussions in chapter 2, the effects of the one detector formalism are not sufficient in the first place to specify the two detector effects, but the associated state transformers are needed: For example, if $\mathcal{R}_{(t,\Delta')}$ is the state transformer associated with a detector click in region Δ at time t in the laboratory frame and if local commutativity is true, the effect associated with two triggered detectors in disjoint regions Δ and Δ' at time t is given by $D_{(t,\Delta)\wedge(t,\Delta')} = \mathcal{R}^\dagger_{(t,\Delta')}\, D_{(t,\Delta)}\, \mathcal{R}_{(t,\Delta')} = \mathcal{R}^\dagger_{(t,\Delta)}\, D_{(t,\Delta')}\, \mathcal{R}_{(t,\Delta)}$, which equals using local commutativity once again: $D_{(t,\Delta)} D_{(t,\Delta')}$.

a measure on $\Omega_{(t,\Delta)}$. Motivated by (3.75), we introduce a second somewhat sloppy simplification of notation by setting $\mathscr{D}_{(t,\Delta)}\big|_{(t,\Delta')} \equiv \mathscr{D}_{(t,\Delta)}$, i.e. we denote the event that a detector covering Δ clicks at time t in the laboratory frame always by $\mathscr{D}_{(t,\Delta)}$, no matter if we consider a single detector experiment such that $\mathscr{D}_{(t,\Delta)}$ is an elementary event in the space $\Omega_{(t,\Delta)}$ of possible outcomes, or an experiment with two detectors, such that $\mathscr{D}_{(t,\Delta)}$ is no longer an elementary event of the joint outcome space $\Omega_{(t,\Delta)\wedge(t,\Delta')}$, or if $\mathscr{D}_{(t,\Delta)}$ is associated with the obvious generalization $\Omega_{(t,\Delta_1)\wedge\ldots\wedge(t,\Delta_N)}$ of $\Omega_{(t,\Delta)\wedge(t,\Delta')}$ if more than two detectors are involved. How many detectors are involved will be always clear from the context and according to (3.75) (or its obvious generalization to the case of more detectors), the marginal click probability associated with (t,Δ) in the case of several detectors is always equal to the one detector click probability associated with (t,Δ), therefore no confusions will arise.

Finally, we derive the addition rule for the joint two detector click probabilities: Additivity of $\mathbb{P}^\psi$ understood as a probability measure on $\Omega_{(t,\Delta)\wedge(t,\Delta')}$ yields

$$\begin{aligned}\mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)}\right) &= \mathbb{P}^\psi\left((1_{(t,\Delta)}, 0_{(t,\Delta')}) \vee (1_{(t,\Delta)}, 1_{(t,\Delta')})\right) = \\ &= \mathbb{P}^\psi\left((1_{(t,\Delta)}, 0_{(t,\Delta')})\right) + \mathbb{P}^\psi\left((1_{(t,\Delta)}, 1_{(t,\Delta')})\right)\end{aligned} \tag{3.76}$$

and

$$\mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta')}\right) = \mathbb{P}^\psi\left((0_{(t,\Delta)}, 1_{(t,\Delta')})\right) + \mathbb{P}^\psi\left((1_{(t,\Delta)}, 1_{(t,\Delta')})\right) \tag{3.77}$$

and if we use

$$\mathscr{D}_{(t,\Delta)} \wedge \mathscr{D}_{(t,\Delta')} = (1_{(t,\Delta)}, 1_{(t,\Delta')}) \tag{3.78}$$

and

$$\mathscr{D}_{(t,\Delta)} \vee \mathscr{D}_{(t,\Delta')} = (1_{(t,\Delta)}, 0_{(t,\Delta')}) \vee (0_{(t,\Delta)}, 1_{(t,\Delta')}) \vee (1_{(t,\Delta)}, 1_{(t,\Delta')}) \tag{3.79}$$

such that

$$\mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)} \wedge \mathscr{D}_{(t,\Delta')}\right) = \mathbb{P}^\psi\left((1_{(t,\Delta)}, 1_{(t,\Delta')})\right) \tag{3.80}$$

and

$$\mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)} \vee \mathscr{D}_{(t,\Delta')}\right) = \mathbb{P}^\psi\left((1_{(t,\Delta)}, 0_{(t,\Delta')})\right) + \mathbb{P}^\psi\left((0_{(t,\Delta)}, 1_{(t,\Delta')})\right) + \mathbb{P}^\psi\left((1_{(t,\Delta)}, 1_{(t,\Delta')})\right) \tag{3.81}$$

we arrive at the *general addition rule* for the click probabilities of a two detector experiment:

$$\mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)}\right) + \mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta')}\right) = \mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)} \wedge \mathscr{D}_{(t,\Delta')}\right) + \mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)} \vee \mathscr{D}_{(t,\Delta')}\right) \tag{3.82}$$

Towards Spatial Distributions

If the initial state of the detection experiment describes a 'single quantum system' like an electron, an atom or a bulk of atoms and the detector is an appropriate detector, it seems natural to identify the detection probabilities $\mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)}\right)$ at time t with the respective probabilities of a spacial probability distribution (the probability distribution of the location of the detected system), i.e. to assume that there is some probability distribution $\mathbb{p}$ on the measurable subsets of space which coincides with the detector probabilities at time t if the respective spatial subsets coincide: $\mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)}\right) = \mathbb{p}(\widetilde{\Delta})$ if $\Delta = \widetilde{\Delta}$. (taking quantum theory seriously, we have in mind a spatial distribution close to a $|\psi|^2$ −distribution).

Moreover, if two or more detectors are combined, the joint click statistics should agree with the spatial distribution as well, i.e. $\mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)} \vee \mathscr{D}_{(t,\Delta')}\right) = \mathbb{p}(\Delta \cup \Delta')$. This in particular implies additivity of the click statistics for disjoint detector locations $\Delta \cap \Delta' = \emptyset$:

$$\mathbb{P}^\psi(\mathscr{D}_{(t,\Delta)} \vee \mathscr{D}_{(t,\Delta')}) = \mathbb{p}(\Delta \cup \Delta') = \mathbb{p}(\Delta) + \mathbb{p}(\Delta') = \mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)}\right) + \mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta')}\right) \tag{3.83}$$

Comparison with the addition rule (3.82) then entails that

$$\mathbb{P}^\psi(\mathscr{D}_{(t,\Delta)} \wedge \mathscr{D}_{(t,\Delta')}) \equiv 0 \tag{3.84}$$

i.e. the probability that both detectors click at the same time is zero.

In this case we can thus determine the effect associated with the two detector event $\mathscr{D}_{(t,\Delta)} \vee \mathscr{D}_{(t,\Delta')}$ from the one detector formalism to be $D_{(t,\Delta)} + D_{(t,\Delta')}$, without explicitly specifying the quantum formalism of the joint two detector experiment, which would presumably involve state transformers which are often not so easy getting access to as effects (see also footnote 131). We may alternatively interpret the two detectors covering Δ and Δ', respectively, as one detector covering $\Delta \cup \Delta'$ which has upon condition that (3.83) holds the click probability $\mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta\cup\Delta')}\right) = \mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta)}\right) + \mathbb{P}^\psi\left(\mathscr{D}_{(t,\Delta')}\right)$ and consequently the associated effect $D_{(t,\Delta\cup\Delta')} = D_{(t,\Delta)} + D_{(t,\Delta')}$.

The normalization of the spatial distribution $\mathbb{p}(\mathbb{R}^3) = 1$ then corresponded to the fact, that in case we could cover all of space with such detectors, at each time one of the detectors would click with certainty (in this way we need not bother with the arrival time problem). If then a detector of that type can in principle be of arbitrary size and shape, the family of detector operators $D_{(t,\Delta)}$ for (measurable) subsets $\Delta \subseteq \mathbb{R}^3$ forms a POVM on $\mathbb{R}^3$, or to put it in the language of relativistic space-time, a POVM on the spacelike hyperplane Σ_t associated with time t in the laboratory frame.

The no-go theorems presented below show that such click probabilities (respectively POVMs) do not exist as a prediction of any relativistic, space-time translation covariant quantum theory which obeys the spectrum condition. It is important to note in the following that this non existence has nothing to do with the probability distribution $\mathbb{p}(\Delta)$ beyond its coincidence with the click probabilities for the detector regions and it does not rely on an assumption that detectors of arbitrary size and shape do exist. In particular, we need not bother with the fact that a spatial distribution is defined on arbitrarily small subsets (e.g. beyond the length scale of the Compton wave length) which might be argued not to correspond to a possible resolution of any real world detection experiment even in principle. In fact, the way the no-go theorems are presented in this work will only rely on a single detector covering some arbitrary bounded spatial region Δ and translated copies of it, i.e. all involved detectors may have the same size and shape.

Several Particle Detections

Before we come to non existence, first note that the identification of click probabilities with the respective probabilities of a probability distribution on $\mathbb{R}^3$ is trivially false for most possible initial states and typical detection experiments: If we consider for example some two-particle initial state ψ, the joint probability that two detectors located in disjoint regions click at the same time is of course in general non zero: $\mathbb{P}^\psi(\mathscr{D}_{(t,\Delta)} \wedge \mathscr{D}_{(t,\Delta')}) \overset{i.g.}{>} 0$ for $\Delta \cap \Delta' = \emptyset$. This cannot be consistent with a probability distribution on $\mathbb{R}^3$, of course, since such a distribution always

obeys $\mathbb{p}(\Delta \cap \Delta') = 0$, given $\Delta \cap \Delta' = \emptyset$. As argued above, any joint probability distribution associated with the joint click statistics of two detectors covering Δ and Δ', respectively, obeys the general addition rule

$$\mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta)}) + \mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta')}) = \mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta)} \vee \mathscr{D}_{(t,\Delta')}) + \mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta)} \wedge \mathscr{D}_{(t,\Delta')}) \tag{3.85}$$

If now $\mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta)} \wedge \mathscr{D}_{(t,\Delta')}) \neq 0$ for $\Delta \cap \Delta' = \emptyset$ one might guess that the click probabilities rather coincide with a probability distribution $\mathbb{p}$ on a larger space, e.g. on $\mathbb{R}^3 \times \mathbb{R}^3$ instead of $\mathbb{R}^3$ if ψ is a two particle initial state, or on $\mathbb{R}^{3N}$ in case of an $N-$particle initial state, in which case N disjoint detectors may click at the same time. This is what we already know from ordinary quantum theory, which provides us with a probability distribution on configuration space instead of physical space.

For a two particle initial state, given the detector type is sensitive for both particles, the click probabilities of a single detector would then be given by

$$\mathbb{P}^{\psi}\left(\mathscr{D}_{(t,\Delta)}\right) = \mathbb{p}\left([\Delta \times \mathbb{R}^3] \cup [\mathbb{R}^3 \times \Delta]\right) \tag{3.86}$$

which are do not entail additivity for $\Delta \cap \Delta' = \emptyset$ since

$$\left([\Delta \times \mathbb{R}^3] \cup [\mathbb{R}^3 \times \Delta]\right) \cap \left([\Delta' \times \mathbb{R}^3] \cup [\mathbb{R}^3 \times \Delta']\right) \neq \emptyset \tag{3.87}$$

But the upcoming no-go results can be easily generalized to prove the non existence of click probabilities which are identifiable in the above sense with the respective probabilities of a probability distribution on $\mathbb{R}^{3N}$ for any $N \in \mathbb{N}$ within the framework of any relativistic, space-time translation covariant quantum theory which obeys the spectrum condition.

A 'Single Quantum System'

A basis of the following analysis will be an operational notion of a *'single quantum system'*, which shall capture initial states ψ, for which we suppose that in an appropriate detection experiment at most one of several detectors covering disjoint spacial regions clicks at a given time in the laboratory frame. I.e. if $\Delta \cap \Delta' = \emptyset$ it follows that the joint click statistics is such that at any time t

$$\mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta)} \wedge \mathscr{D}_{(t,\Delta')}) = 0 \tag{3.88}$$

We may think of such initial states as the elements of some one-particle Hilbert space, the one particle sector of Fock space or if you want an irreducible representation of the Poincaré group, for which it seems natural to assume that the click probabilities of appropriate detectors agree with the respective probabilities of a spatial distribution. Not least, experiments on a single quantum particle like an electron are performed in laboratories with great success [105, 106].

We assume henceforth that there is a Hilbert space $\mathcal{H}$ of 'single quantum systems' as possible initial states, which might be only a subspace of the 'true' Hilbert space. With respect to this assumption, $\mathcal{H}$ could even be defined by only a single state, but as mentioned, here we shall rather have in mind something like the one-particle sector of Fock space.

This operational notion of a 'single quantum system' in particular implies additivity of the detection probabilities (probabilities of mutually exclusive events are additive), as argued above, i.e.

$$\mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta)}) + \mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta')}) = \mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta)} \vee \mathscr{D}_{(t,\Delta')}) \tag{3.89}$$

which motivates the following definition:

Definition 3.17 [*Additive Detector Formalism*]

A covariant detector formalism $\mathcal{E}^{\mathcal{H}}_{\Delta}$ is additive, if the detection probabilities associated with two disjoint regions $\Delta \cap \Delta' = \emptyset$ at a given time t in the laboratory frame are always additive:

$$\mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta)}) + \mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta')}) = \mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta)} \vee \mathscr{D}_{(t,\Delta')}) \tag{3.90}$$

for all initial states $\psi \in \mathcal{H}$. If more than two detectors covering disjoint spatial regions $\Delta_1, \Delta_2, ..., \Delta_L$ are involved, additivity generalises to

$$\mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta_1)}) + ... + \mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta_n)}) = \mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta_1)} \vee ... \vee \mathscr{D}_{(t,\Delta_n)}) \tag{3.91}$$

for all $n = 2, ..., L$ and for all initial states $\psi \in \mathcal{H}$.

If $\mathcal{E}^{\mathcal{H}}_{\Delta}$ is additive, we say that $\mathcal{E}^{\mathcal{H}}_{\Delta}$ is incapable of triggering more than one of several detectors instantaneously in the laboratory frame, which expresses the physical meaning of additivity.

Additivity entails that simultaneous detection in disjoint regions in the laboratory (frame) is not possible. We may thus call an additive detection experiment with two detectors a *'here-or-there measurement'* and we will as already mentioned encounter that such measurements are in a strict sense actually not possible in a reasonable relativistic quantum framework. Note that additivity is a necessary condition for the related effects to coincide with the respective effects of a POVM on the spatial $t = const$ hyperplane of the laboratory frame, in which case it translates simply to the additivity property of the respective POVM.

The generalization to $L > 2$ detectors a the end of definition 3.17 will be needed later and in view of the previous considerations it is straightforward to illustrate its physical meaning(fulness): If an initial state is capable of triggering at most one of L detectors at a given time t in the laboratory (frame), only the event $(0_{(t,\Delta_1)}, ..., 0_{(t,\Delta_L)})$ that none of the detectors clicks, and the single click events $(1_{(t,\Delta_1)}, 0_{(t,\Delta_2)}, ..., 0_{(t,\Delta_L)})$ etc. can be elements of the space $\Omega_{(t,\Delta_1)\wedge...\wedge(t,\Delta_L)}$ of elementary events with non zero probability. It is now easy to see, that this is equivalent to the validity of the additivity conditions (3.91) for $L > 2$ in the same way as shown above for the case $L = 2$.

Time and Causal Additivity

Additivity 3.17 as an operational definition of 'a single quantum system' physically means that if a 'single quantum system' is detected in Δ it cannot be detected elsewhere at the same time in the laboratory frame. In a relativistic setting, additivity should thus be generalized in the following way: Given Δ and Δ' have finite distance $d > 0$, a 'single quantum system' which was detected in Δ at time t in the laboratory frame cannot be detected in Δ' at any time t' with $|t - t'| < d$. This guarantees that the system cannot move faster than light, or to put it another way, that the system cannot trigger two separated detectors simultaneously from the viewpoint of any other Lorentz frame. Observing that the time parameter did not play any role in the analysis of combined detector experiments above (in particular, that additivity of probabilities of click events at different times still means that two detectors do not click both at the respective times) thus motivates the following generalization of an additive detector formalism:

Definition 3.18 [*Causal Additivity*]

A covariant detector formalism $\mathcal{E}^{\mathcal{H}}_{\Delta}$ is said to be causally additive if the detection probabilities associated with two spacelike separated regions are always additive:

$$\mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta)}) + \mathbb{P}^{\psi}(\mathscr{D}_{(t',\Delta')}) = \mathbb{P}^{\psi}(\mathscr{D}_{(t,\Delta)} \vee \mathscr{D}_{(t',\Delta')}) \tag{3.92}$$

whenever the space-time region (t, Δ) is spacelike with respect to (t', Δ') and for all initial states $\psi \in \mathcal{H}$. If more than two detectors covering disjoint spatial regions $\Delta_1, \Delta_2, ..., \Delta_L$ are involved, causal additivity generalises to

$$\mathbb{P}^{\psi}(\mathscr{D}_{(t_1,\Delta_1)}) + ... + \mathbb{P}^{\psi}(\mathscr{D}_{(t_n,\Delta_n)}) = \mathbb{P}^{\psi}(\mathscr{D}_{(t_1,\Delta_1)} \vee ... \vee \mathscr{D}_{(t_n,\Delta_n)}) \tag{3.93}$$

for all $n = 2, ..., L$ whenever the space-time regions $(t_1, \Delta_1), ..., (t_N, \Delta_L)$ are mutually spacelike separated and for all initial states $\psi \in \mathcal{H}$.

If $\mathcal{E}^{\mathcal{H}}_{\Delta}$ is causally additive we say that $\mathcal{E}^{\mathcal{H}}_{\Delta}$ is incapable of triggering more than one of several detectors at spacelike separation.

As explained, causal additivity implies that two detectors do never click at the same time in a different Lorentz frame (although one should recognise that regions like (t, Δ) are not purely spatial regions at a given time in a different frame, if Δ is not a point but spatially extended). Technically, we will only need that each pair of detectors which can be arbitrarily far away form each other does not click *'almost instantaneously'* in the laboratory frame, i.e. that there is always some arbitrarily short time interval in which it is certain that at most one of them is triggered by the initial state.

To comprehend the generalization (3.93) to several detectors, which will be needed later to prove theorem 3.25, see the remark subsequent to the definition 3.17.

3.4.3 Projections: Malament & Detectors

Consider a covariant detector formalism $\mathcal{E}^{\mathcal{H}}_{\Delta}$ whose effects yielding the click statistics at laboratory time $t = 0$ are (non trivial) projections $D_{(0,\Delta)} \equiv P_{(0,\Delta)} = P^2_{(0,\Delta)} \neq 0$. Since each (non trivial) projection $P_{(0,\Delta)}$ has eigenvalues 0 and 1, there are states $\psi_{\Delta} \in \mathcal{H}$ (the eigenstates of $P_{(0,\Delta)}$ with eigenvalue 1) such that $\mathbb{P}^{\psi_{\Delta}}(\mathscr{D}_{(0,\Delta)}) = \langle \psi_{\Delta} |\, P_{(0,\Delta)}\, \psi_{\Delta} \rangle = 1$, i.e. the detector clicks with certainty if ψ_{Δ} is the initial state.

Given now $\mathcal{E}^{\mathcal{H}}_{\Delta}$ is additive, for $\Delta \cap \Delta' = \emptyset$ we have

$$\mathbb{P}^{\psi}\left(\mathscr{D}_{(t,\Delta)} \vee \mathscr{D}_{(t,\Delta')}\right) = \mathbb{P}^{\psi}\left(\mathscr{D}_{(0,\Delta)}\right) + \mathbb{P}^{\psi}\left(\mathscr{D}_{(0,\Delta')}\right) \tag{3.94}$$

for all $\psi \in \mathcal{H}$, which in particular entails

$$\mathbb{P}^{\psi_{\Delta}}(\mathscr{D}_{(0,\Delta')}) = \langle \psi_{\Delta} |\, P_{(0,\Delta')}\, \psi_{\Delta} \rangle = 0 \tag{3.95}$$

since $\mathbb{P}^{\psi}\left(\mathscr{D}_{(t,\Delta)} \vee \mathscr{D}_{(t,\Delta')}\right) \leq 1$ for all $\psi \in \mathcal{H}$ and $\mathbb{P}^{\psi_{\Delta}}(\mathscr{D}_{(0,\Delta)}) = 1$. Physically, this can be understood as a trivial consequence of additivity: If the system is perfectly localized within Δ (in the operational sense that a detector covering Δ clicks with certainty) it cannot be 'found' in a disjoint region Δ' at the same time.

Since now

$$0 = \left\langle \psi_\Delta \right| P_{(0,\Delta')}\, \psi_\Delta \left\rangle = \left\langle \psi_\Delta \right| P^2_{(0,\Delta')}\, \psi_\Delta \right\rangle = \left\langle P_{(0,\Delta')}\, \psi_\Delta \right| P_{(0,\Delta')}\, \psi_\Delta \left\rangle = \left\| P_{(0,\Delta')}\, \psi_\Delta \right\|^2 \tag{3.96}$$

it follows that $P_{(0,\Delta')}\,\psi_\Delta = 0$, i.e. the range of $P_{(0,\Delta)}$ (which is spanned by the eigenstates ψ_Δ of $P_{(0,\Delta)}$ with eigenvalue 1, since $P_{(0,\Delta)}$ is a projection) is a subset of the kernel of $P_{(0,\Delta')}$, i.e.

$$P_{(0,\Delta')}P_{(0,\Delta)} = 0 = \left(P_{(0,\Delta')}P_{(0,\Delta)}\right)^\dagger = P_{(0,\Delta)}P_{(0,\Delta')} \tag{3.97}$$

which in particular implies the commutativity of $P_{(0,\Delta)}$ and $P_{(0,\Delta')}$ (which is actually a variant of local commutativity, see chapter 2).

If we additionally assume that $\mathcal{E}^{\mathcal{H}}_\Delta$ is causally additive, the same argument yields

$$P_{(0,\Delta')}P_{(0,\Delta)+x} = P_{(0,\Delta)+x}P_{(0,\Delta')} = 0 \tag{3.98}$$

whenever $x \in \mathbb{R}^4$ is chosen in such a way that $(0,\Delta)+x$ is spacelike with respect to $(0,\Delta')$. If now additionally the spectrum condition is true, this has a very radical consequence:

Theorem 3.19 [*Malament for Detectors*]

Consider a covariant detector formalism $\mathcal{E}^{\mathcal{H}}_\Delta$ which obeys the spectrum condition with associated laboratory frame click effects $D_{(0,\Delta)+x} = U(x)D_{(0,\Delta)}U^\dagger(x)$. If $\mathcal{E}^{\mathcal{H}}_\Delta$ is both

(i) projective, i.e. $D_{(0,\Delta)} \equiv P_{(0,\Delta)} = P^2_{(0,\Delta)}$

(ii) incapable of triggering more than one of several detectors at spacelike separation (see definition 3.18)

it follows that $\mathcal{E}^{\mathcal{H}}_\Delta$ is incapable of triggering any detector:

$$\mathbb{P}^\psi\left(\mathscr{D}_{(0,\Delta)+x}\right) \equiv 0 \quad \textit{for all} \quad x \in \mathbb{R}^4 \quad \textit{and for all} \quad \psi \in \mathcal{H} \tag{3.99}$$

Proof: Let $\boldsymbol{a} \in \mathbb{R}^3$ be large enough such that $\Delta \cap \Delta + \boldsymbol{a} = \emptyset$ and $\mathrm{dist}(\Delta, \Delta+\boldsymbol{a}) =: d > 0$ and suppose $P_{(0,\Delta)} \neq 0$. The assumption that $\mathcal{E}^{\mathcal{H}}_\Delta$ is incapable of triggering more than one of several detectors at spacelike separation means mathematically that $\mathcal{E}^{\mathcal{H}}_\Delta$ is causally additive, which in particular implies that it is additive in the laboratory frame. Thus according to (3.97) we have

$$P_{(0,\Delta)}P_{(0,\Delta+\boldsymbol{a})} = P_{(0,\Delta+\boldsymbol{a})}P_{(0,\Delta)} = 0 \tag{3.100}$$

Moreover, since Δ and $\Delta+\boldsymbol{a}$ are separated by a finite distance $d > 0$, there is a neighbourhood $\mathcal{N}(0)$ of $0 \in \mathbb{R}^4$ such that the space-time regions $(0,\Delta)$ and $(0,\Delta+\boldsymbol{a})+x$ are spacelike separated for all $x \in \mathcal{N}(0)$ (see Fig 7). Causal additivity then implies according to (3.98) that

$$P_{(0,\Delta)}P_{(0,\Delta+\boldsymbol{a})+x} = P_{(0,\Delta+\boldsymbol{a})+x}P_{(0,\Delta)} = 0 \tag{3.101}$$

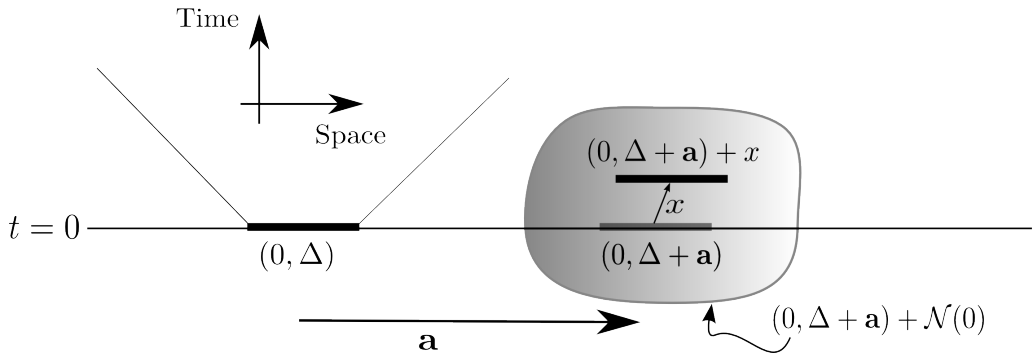

FIGURE 7

Malament Type Theorems Scheme: *Spatial detector region Δ and the spatially translated detector region $\Delta + \boldsymbol{a}$ for large enough $\boldsymbol{a} \in \mathbb{R}^3$ at time $t = 0$ in the laboratory frame, such that there is some neighbourhood $\mathcal{N}(0) \subset \mathbb{R}^4$ of the origin $0 \in \mathbb{R}^4$ such that $(0, \Delta + \boldsymbol{a}) + x$ is spacelike with respect to $(0, \Delta)$ for all $x \in \mathcal{N}(0)$.*

for all $x \in \mathcal{N}(0)$. Since $\mathcal{N}(0)$ is open, corollary 3.14 then implies

$$P_{(0,\Delta)} P_{(0,\Delta+\boldsymbol{a})+x} = P_{(0,\Delta+\boldsymbol{a})+x} P_{(0,\Delta)} = 0 \tag{3.102}$$

for all $x \in \mathbb{R}^4$. If we choose $x = (0, -\boldsymbol{a})$ we thus get

$$0 = P^2_{(0,\Delta)} = P_{(0,\Delta)} \tag{3.103}$$

■

We can also formulate this result as a theorem of non existence of a spatial PVM $\{P_\Delta\}$: A PVM on a given simultaneity plane Σ of space-time entails additivity of an associated detector formalism (i.e. one yielding the same probabilities as the spacial PVM for all detector regions) by definition, the additivity of the detector formalism then is simply a direct consequence of the additivity property of PVMs. If this additivity is not instantaneously lost if one of two projections associated with spatially separated regions is translated a tiny little bit in time (one can drop this assumption if local commutativity is assumed, see the remark on Borchers lemma based approaches below), the above proof of theorem 3.19 immediately shows that all projections associated with bounded spacial regions are zero, such that the family of projections cannot be a subset of elements of a spatial PVM.

Corollary 3.20 [*Malament for a spatial PVM*]

There does not exist a PVM $\{P_\Delta\}$ on a spacelike hyperplane $\Sigma \subset \mathcal{M}$ acting on some relativistic Hilbert space $\mathcal{H}$ which obeys the spectrum condition, if for all $\Delta \cap \Delta' = \emptyset$ with dist$(\Delta, \Delta') > 0$ there is some $\varepsilon > 0$ such that $P_\Delta U(t, \mathbf{0}) P_{\Delta'} U^{-1}(t, \mathbf{0}) = 0$ for all $-\varepsilon \leq t < \varepsilon$.

But note that Malament theorem as formulated in 3.19 requires much less than the whole structure of a spatial PVM. Most importantly, we do not need to define 'localization' in arbitrary measurable subsets of space. To start with a projection associated with a single bounded spatial region is enough to see that it cannot exist, given space-time translation covariance, the spectrum condition and causal additivity are true.

Remarks: The way the Malament theorem is presented and especially proved here deviates from the common approach [234] which is based on a technical result of Borchers [50] – often referred to as *Borchers lemma*[132] – which is a bit elaborate to proof such that usually only the result is stated without elucidating the mathematical structure behind. The present approach, although it has a minor technical disadvantage at a first glance with respect to the common one (see below), is well suited to transparently connect the mathematical structure behind the Malament theorem(s) with the mathematical structure behind other results (e.g. the fact that positive energy wave functions cannot have compact support, the Reeh-Schlieder theorem, Hegerfeld's theorem or the fact that positive energy wave functions cannot transform only locally) and thereby motivates to strive to reveal physical relations between the origin and meaning of these results as well.

The Malament theorem [234] does actually not originate from Malament, but was first loosely formulated by Schlieder [292] and then – essentially in the version presented here – by Jancewicz [197]. The assumption that the product of projections associated with spacelike separated regions vanishes, which was derived here from causal additivity, is in Jancewicz' paper loosely referred to as the assumption that 'localization propagates causally'.

Malament – besides firstly claiming that the theorem has something to say about ontology – added a tiny refinement of the assumptions: He realized that the application of Borchers lemma to projections associated with spatial regions only requires the assumptions that the product of projections associated with disjoint regions on a single hyperplane Σ (i.e. without need of timelike variation) always vanishes plus local commutativity (projections associated with spacelike separated regions commute) to prove the the no-go result, given of course, space-time translation covariance and the spectrum condition are assumed as well. In the language of the present approach this means that we can replace *'causal additivity'* by *'additivity + local commutativity'* which yields – at least mathematically – a stronger theorem, since it has weaker assumptions: As shown above, causal additivity already entails local commutativity in its most trivial form (namely $P_{(t,\Delta)}P_{(t,\Delta')+x} = P_{(t,\Delta')+x}P_{(t,\Delta)} = 0$ whenever (t,Δ) is spacelike with respect to $(t,\Delta')+x$), but not the other way around.

From a relativistic point of view this is not a big advantage though: After all, if the product of projections associated with disjoint spatial regions is assumed to vanish (motivated by a physical argument) at a given time in a given frame, it is consequent to assume that it vanishes in general at spacelike separation as well, which relativistically essentially corresponds to 'simultaneous and spatially disjoint' in another Lorentz frame (at least if we can neglect the spatial extension of the detector regions). Or to put it another way in operational language: If we assume that a given system can only be detected at a single location at a given time in a given frame, relativity

[132] The Borchers lemma states that given a unitary representation of a one parameter group $U(t) = e^{-i\mathscr{H}t}$ acting on some Hilbert space $\mathcal{H}$, whose infinitesimal generator $\mathscr{H}$ is bounded from below, and any two projections P_1, P_2 acting on $\mathcal{H}$ satisfy $P_1P_2 = 0$ and $[U(t)P_1U^\dagger(t)\,,\,P_2] = 0$ for $-\varepsilon < t < \varepsilon$ for some $\varepsilon > 0$, it follows that $U(t)P_1U^\dagger(t)P_2 = 0$ for all $t \in \mathbb{R}$.

suggests that it cannot be detected once again arbitrarily far away an arbitrarily short period of time later (i.e. it cannot move faster than light). Thus, from a physical point of view, the present approach based on causal additivity is essentially as powerful as Malaments approach based on additivity plus local commutativity.

3.4.4 $N-$States

The previous concepts and the Malament theorem can be straightforwardly generalized to a suitable no-go result which is appropriate for more general initial states: We introduced the notion of a 'single quantum system' operationally as an element of a (linearly closed) set of initial states of a suitable detection experiment, which are capable of triggering at most one of several detectors covering disjoint spatial regions at a given time in the laboratory frame, i.e. which are associated with an additive detector formalism. The intuition behind this notion is based on one-particle states or any bound states or the like, for which it seems natural to assume the validity of additivity. We encountered so far that this operational notion of a 'single quantum system' does not work for projective detection measurements in a reasonable relativistic framework.

The analogue intuitive assumption for an $N-$particle initial state or an initial state of N ions or the like, is to assume that it is capable of triggering at most N of $M > N$ spatially disjoint detectors at a given time. We may call such operationally defined states $N-$*states*, which is thus the generalized analogue notion of a 'single quantum system' and it is natural to assume as above that linear combinations of $N-$states are $N-$states as well, in particular that the set of $N-$states forms a Hilbert space $\mathcal{H}$ (which might be a subspace of the 'true' Hilbert space, like the $N-$particle sector of Fock space or the union of the $n-$particle sectors of Fock space for $n \leq N$).

Here we shall only sketch how the generalization of the Malament theorem above (and analogously the Malament type theorems below) can be accomplished: Most easily we combine N detectors in some given spatial detector arrangement to an imaginary $N-$*detector* which is defined to $N-$*click* in case all N constituent detectors click at the same time (to begin with). Then we can copy this combined $N-$detector arrangement and translate the copy by some large spatial vector $\boldsymbol{a} \in \mathbb{R}^3$ far away from the original $N-$detector and make use of the fact that the assumption that upon initial state $\psi \in \mathcal{H}$ at most N of the single detectors can click at the same time in particular entails that never $2N$ single detectors click simultaneously and thus both $N-$detectors will never $N-$click at the same time:

$$\mathbb{P}^{\psi}\left(\left(\mathscr{D}_{(t,\Delta_1)} \wedge ... \wedge \mathscr{D}_{(t,\Delta_N)}\right) \wedge \left(\mathscr{D}_{(t,\Delta_1+\boldsymbol{a})} \wedge ... \wedge \mathscr{D}_{(t,\Delta_N+\boldsymbol{a})}\right)\right) = 0 \tag{3.104}$$

Generalizing now the argument which motivated the definition of an additive detector formalism above, one can straightforwardly derive that (3.104) entails what we might call $N-$*additivity*:

$$\begin{aligned} &\mathbb{P}^{\psi}\left(\mathscr{D}_{(t,\Delta_1)} \wedge ... \wedge \mathscr{D}_{(t,\Delta_N)}\right) + \mathbb{P}^{\psi}\left(\mathscr{D}_{(t,\Delta_1+\boldsymbol{a})} \wedge ... \wedge \mathscr{D}_{(t,\Delta_N+\boldsymbol{a})}\right) = \\ &= \mathbb{P}^{\psi}\left(\left[\mathscr{D}_{(t,\Delta_1)} \wedge ... \wedge \mathscr{D}_{(t,\Delta_N)}\right] \vee \left[\mathscr{D}_{(t,\Delta_1+\boldsymbol{a})} \wedge ... \wedge \mathscr{D}_{(t,\Delta_N+\boldsymbol{a})}\right]\right) \end{aligned} \tag{3.105}$$

Analogously, the additional assumption that the $N-$states in $\mathcal{H}$ are capable of triggering at most

N detectors at spacelike separation (i.e. not only simultaneously) leads to the requirement[133]:

Definition 3.21 [*Causal N-Additivity*]

We say a covariant detector formalism $\mathcal{E}_\Delta^{\mathcal{H}}$ is causally $N-$additive if for each choice of N bounded, mutually disjoint detector regions $\Delta_1, \ldots, \Delta_N \subset \mathbb{R}^3$ and times $t_1, \ldots, t_N$ in the laboratory frame, such that the space-time regions $(t_1, \Delta_1), (t_2, \Delta_2), \ldots, (t_N, \Delta_N)$ are mutually spacelike separated, the additivity condition

$$\begin{aligned} &\mathbb{P}^\psi\left(\mathscr{D}_{(t_1,\Delta_1)} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N)}\right) + \mathbb{P}^\psi\left(\mathscr{D}_{(t_1,\Delta_1+\boldsymbol{a})+x} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N+\boldsymbol{a})+x}\right) = \\ &= \mathbb{P}^\psi\left(\left[\mathscr{D}_{(t_1,\Delta_1)} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N)}\right] \vee \left[\mathscr{D}_{(t_1,\Delta_1+\boldsymbol{a})+x} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N+\boldsymbol{a})+x}\right]\right) \end{aligned} \tag{3.106}$$

holds for all $x \in \mathbb{R}^4$ for which the space-time region $((t_1, \Delta_1 + \boldsymbol{a}) + x) \cup \ldots \cup ((t_N, \Delta_N + \boldsymbol{a}) + x)$ is spacelike with respect to $(t_1, \Delta_1) \cup \ldots \cup (t_N, \Delta_N)$ and for all initial states $\psi \in \mathcal{H}$. A detector formalism $\mathcal{E}_\Delta^{\mathcal{H}}$ must be necessarily causally $N-$additive if $\mathcal{E}_\Delta^{\mathcal{H}}$ is incapable of triggering more than N of several detectors at spacelike separation.

This is all we need to repeat the proof of the Malament theorem, only substituting the single click events $\mathscr{D}_{(0,\Delta)}$ with the generalized multi click events $\mathscr{D}_{(t_1,\Delta_1)} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N)}$ and causal additivity with causal $N-$additivity. The result we may loosely express in the following

Lemma 3.22 [*Malament for N-states*]

Consider a projective covariant detector formalism $\mathcal{E}_\Delta^{\mathcal{H}}$ which obeys the spectrum condition. If $\mathcal{E}_\Delta^{\mathcal{H}}$ is incapable of triggering more than N of several detectors at spacelike separation (see definition 3.21), it is incapable of triggering N detectors at spacelike separation as well:

$$\mathbb{P}^\psi\left(\mathscr{D}_{(t_1,\Delta_1)} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N)}\right) \equiv 0 \quad \textit{for all} \quad \psi \in \mathcal{H} \tag{3.107}$$

whenever the space-time regions $(t_1, \Delta_1), \ldots, (t_N, \Delta_N)$ are mutually spacelike.

This result has the striking consequence that, given a Hilbert space $\mathcal{H}$ represents a class of preparations which are capable of triggering at most N detectors at spacelike separation, each member of $\mathcal{H}$ is necessarily capable of triggering at most $N-1$ detectors at spacelike separation, given the detector statistics can be properly predicted by a covariant detector formalism obeying the spectrum condition whose effects are projections. We conclude that the alleged $N-$states

[133] In contrast to causal additivity, which is equivalent to the physical assumption that two or more detectors never click at spacelike separation, causal $N-$additivity is only a necessary condition for the assumption that never more than N detectors click at spacelike separation, but not sufficient. Indeed, causal $N-$additivity directly implies only the physical requirement that for the related initial states $2N$ or more detectors do never click at spacelike separation (which is obviously a necessary consequence of the assumption that at most N detectors can be triggered).

are actually $(N-1)-$states (which are by definition $N-$states as well, of course) and repeat the argument in order to conclude that they are actually $(N-2)-$states and so on. Finally, we arrive at a Hilbert space of 'single quantum systems' for which we can apply the original Malament theorem 3.19. The implications may be summarized as follows:

$$\begin{aligned}
& \mathbb{P}^{\psi}\left(\text{more than } N \text{ detectors click at spacelike separation}\right) = 0 \quad \forall \psi \in \mathcal{H} \\
\Longrightarrow \quad & \mathbb{P}^{\psi}\left(N \text{ detectors click at spacelike separation}\right) = 0 \quad \forall \psi \in \mathcal{H} \\
\Longrightarrow \quad & \mathbb{P}^{\psi}\left(N-1 \text{ detectors click at spacelike separation}\right) = 0 \quad \forall \psi \in \mathcal{H} \\
& \vdots \\
\Longrightarrow \quad & \mathbb{P}^{\psi}\left(2 \text{ detectors click at spacelike separation}\right) = 0 \quad \forall \psi \in \mathcal{H} \\
\Longrightarrow \quad & \mathbb{P}^{\psi}\left(\text{detector clicks}\right) = 0 \quad \forall \psi \in \mathcal{H}
\end{aligned} \tag{3.108}$$

The other way around, since $N \in \mathbb{N}$ was arbitrary, this shows that – under the assumption of projective detector experiments (which will be weakened and finally completely dropped below) – a covariant detector formalism which obeys the spectrum condition cannot be associated with a Hilbert space $\mathcal{H}$, such that it can be taken for sure that only a given finite number $N \in \mathbb{N}$ of $M > N$ detectors click at spacelike separation for all initial $\psi \in \mathcal{H}$.

Corollary 3.23 [*Malament for N-States*]

Consider a projective covariant detector formalism $\mathcal{E}_{\Delta}^{\mathcal{H}}$ which obeys the spectrum condition. If $\mathcal{E}_{\Delta}^{\mathcal{H}}$ is incapable of triggering more than any given finite number $N \in \mathbb{N}$ of $M > N$ detectors at spacelike separation, it is incapable of triggering any detector:

$$\mathbb{P}^{\psi}\left(\mathscr{D}_{(0,\Delta)+x}\right) \equiv 0 \quad \textit{for all} \quad x \in \mathbb{R}^4 \quad \textit{and for all} \quad \psi \in \mathcal{H} \tag{3.109}$$

3.4.5 Weakly Unsharp Effects: Busch's Result

Busch has shown [69] that the conclusion of the Malament theorem does not only apply to projective detection experiments but does actually extend to any kind of detection experiment for which there exist initial states which trigger a given detector with certainty. Such states may be called *perfectly localized* in an operational sense, and Busch has shown that detection experiments for which such initial states are possible are not reconcilable with a reasonable positive-energy relativistic quantum theory (Busch's result actually assumes local commutativity, while in the present approach commutativity of the relevant effects is a consequence of the assumption of causal additivity).

Reconsidering the proof of the Malament theorem 3.19, we see that one crucial ingredient is that the assumption of (causal) additivity entails that initial states, which trigger a given detector with certainty, are in the kernel of an effect associated with the click probability of a (spacelike) remote detector. This is not only the case for projective measurements, but whenever

initial states which trigger a given detector with certainty exist, i.e. whenever the associated effect has 1 as one of its eigenvalues. But given it is not a projection, the range of such an effect cannot be only spanned by eigenstates with eigenvalue 1, since also eigenstates with eigenvalues $\lambda \in (0,1)$ exist, and consequently, the total range of such an effect is in general no longer a subset of the kernel of an effect which is associated with the click probability of a remote detector, as it was the case for projections which always have eigenvalues $\lambda \in \{0,1\}$. But exactly this property – namely that the range of a given effect associated with the click probability of a given detector is a subset of the kernel of an effect associated with the click probability of a remote detector – yielded the crucial intermediate result that the product of such effects vanishes, which is thus in general no longer true for non projective effects.

It shall be shown next, that with a few additional arguments it is indeed possible to adjust the proof of the Malament theorem to apply for non projective effects which have 1 as an eigenvalue, too.

Effects which do not have 1 as an eigenvalue, such that there do not exist initial states (eigenstates with eigenvalue 1) which yield the associated probability 1, were called *strongly unsharp effects* [69]. Analogously, we call projections (i.e. effects with exclusively eigenvalues 0 and 1) *sharp effects* and non projective effects, which have 1 as an eigenvalue, *weakly unsharp effects*. The following theorem extends the Malament theorem from sharp effects to weakly unsharp effects:

Theorem 3.24 [*Malament for Weakly Unsharp Effects*]

Consider a covariant detector formalism $\mathcal{E}_\Delta^{\mathcal{H}}$ which obeys the spectrum condition with associated laboratory frame click effects $D_{(0,\Delta)+x} = U(x)D_{(0,\Delta)}U^\dagger(x)$. If $\mathcal{E}_\Delta^{\mathcal{H}}$ is incapable of triggering more than one of several detectors at spacelike separation (see definition 3.18) $D_{(0,\Delta)}$ (and by that $D_{(0,\Delta)+x}$ for $x \in \mathbb{R}^4$) cannot have 1 as an eigenvalue, i.e. there do not exist initial states in $\mathcal{H}$ which trigger a detector of that type with certainty:

$$\mathbb{P}^\psi\left(\mathscr{D}_{(0,\Delta)+x}\right) = \left\langle \psi \right| D_{(0,\Delta)+x}\left|\psi\right\rangle < 1 \tag{3.110}$$

for all $x \in \mathbb{R}^4$ and for all $\psi \in \mathcal{H}$.

Remark: In the next section we shall prove essentially the same assertion, only that we can substitute 1 on the right hand side of (3.110) with a positive real number, which can be made inductively arbitrarily small by considering hypothetical composite detector experiments with a large number spatially separated detectors, which amounts to the non existence of *any* (non trivial) causally additive covariant detector formalism satisfying the spectrum condition.

Proof: Suppose there are initial states $\psi_\Delta \in \mathcal{H}$ which are eigenstates of $D_{(0,\Delta)}$ with eigenvalue 1, i.e. which trigger the detector covering Δ at time $t = 0$ (in the laboratory frame) with certainty:

$$\mathbb{P}^{\psi_\Delta}(\mathscr{D}_{(0,\Delta)}) = \left\langle \psi_\Delta \right| D_{(0,\Delta)}\left|\psi_\Delta\right\rangle = 1 \tag{3.111}$$

Let $\boldsymbol{a} \in \mathbb{R}^3$ be large enough such that $\mathrm{dist}(\Delta,\, \Delta + \boldsymbol{a}) = d > 0$. Since $\Delta \cap (\Delta + \boldsymbol{a}) = \emptyset$, additivity

(which is implied by the assumption that the initial states in $\mathcal{H}$ are incapable of triggering more than one of several disjoint detectors simultaneously in the laboratory frame, see definition 3.17) implies

$$1 \geq \mathbb{P}^{\psi_\Delta}(\mathscr{D}_{(0,\Delta)} \vee \mathscr{D}_{(0,\Delta+\boldsymbol{a})}) = \mathbb{P}^{\psi_\Delta}(\mathscr{D}_{(0,\Delta)}) + \mathbb{P}^{\psi_\Delta}(\mathscr{D}_{(0,\Delta+\boldsymbol{a})}) \tag{3.112}$$

which together with (3.111) entails

$$\mathbb{P}^{\psi_\Delta}(\mathscr{D}_{(0,\Delta+\boldsymbol{a})}) = \left\langle \psi_\Delta \right| D_{(0,\Delta+\boldsymbol{a})}\, \psi_\Delta \big\rangle = 0 \tag{3.113}$$

Since $D_{(0,\Delta+\boldsymbol{a})}$ is a positive operator such that its square root $\sqrt{D_{(0,\Delta+\boldsymbol{a})}}$ exists, equation (3.113) may be written as

$$0 = \left\langle \psi_\Delta \right| D_{(0,\Delta+\boldsymbol{a})}\, \psi_\Delta \big\rangle = \big\langle \sqrt{D_{(0,\Delta+\boldsymbol{a})}}\, \psi_\Delta \big| \sqrt{D_{(0,\Delta+\boldsymbol{a})}}\, \psi_\Delta \big\rangle = \big\| \sqrt{D_{(0,\Delta+\boldsymbol{a})}}\, \psi_\Delta \big\|^2 \tag{3.114}$$

and consequently $\sqrt{D_{(0,\Delta+\boldsymbol{a})}}\, \psi_\Delta = 0$ or, multiplied by $\sqrt{D_{(0,\Delta+\boldsymbol{a})}}$ from the left, we get

$$D_{(0,\Delta+\boldsymbol{a})}\, \psi_\Delta = 0 \tag{3.115}$$

Hence, each eigenstate ψ_Δ of $D_{(0,\Delta)}$ with eigenvalue 1 is in the kernel of $D_{(0,\Delta+\boldsymbol{a})}$.

If we denote the orthogonal projection onto the eigenspace of $D_{(0,\Delta)}$ associated with eigenvalue 1 by $P^{(1)}_{(0,\Delta)}$ and the projection onto the kernel of $D_{(0,\Delta+\boldsymbol{a})}$ by $P^{(0)}_{(0,\Delta+\boldsymbol{a})}$, we thus see that

$$P^{(1)}_{(0,\Delta)} \mathcal{H} \subseteq P^{(0)}_{(0,\Delta+\boldsymbol{a})} \mathcal{H} \tag{3.116}$$

The other way around, this implies that each $\psi \in \mathcal{H}$ for which $D_{(0,\Delta+\boldsymbol{a})}\psi \neq 0$ cannot be in the range of $P^{(1)}_{(0,\Delta)}$, i.e. the range of $D_{(0,\Delta+\boldsymbol{a})}$, which is given by all $\psi \in \left(\mathbb{1}_{\mathcal{H}} - P^{(0)}_{(0,\Delta+\boldsymbol{a})}\right) \mathcal{H}$, has no overlap with the range of $P^{(1)}_{(0,\Delta)}$ and thus

$$P^{(1)}_{(0,\Delta)} \left(\mathbb{1}_{\mathcal{H}} - P^{(0)}_{(0,\Delta+\boldsymbol{a})}\right) = 0 \tag{3.117}$$

Now we can repeat this argument with causal additivity ('no multiple clicks at spacelike separation', see definition 3.18) instead of only additivity to obtain

$$P^{(1)}_{(0,\Delta)} \left(\mathbb{1}_{\mathcal{H}} - P^{(0)}_{(0,\Delta+\boldsymbol{a})+x}\right) = 0 \tag{3.118}$$

for all $x \in \mathbb{R}^4$ for which $(0, \Delta + \boldsymbol{a}) + x$ is spacelike with respect to $(0, \Delta)$. Moreover, since Δ and $\Delta + \boldsymbol{a}$ have a finite distance $d > 0$ there is a neighbourhood $\mathcal{N}(0)$ of the origin $0 \in \mathbb{R}^4$, such that $(0, \Delta + \boldsymbol{a}) + x$ is spacelike with respect to $(0, \Delta)$ for all $x \in \mathcal{N}(0)$ (see Fig. 7). Applying corollary 3.14 thus yields

$$P^{(1)}_{(0,\Delta)} \left(\mathbb{1}_{\mathcal{H}} - P^{(0)}_{(0,\Delta+\boldsymbol{a})+x}\right) = 0 \tag{3.119}$$

for all $x \in \mathbb{R}^4$. Choosing $x = (0, -\boldsymbol{a})$ we get

$$P^{(1)}_{(0,\Delta)} \left(\mathbb{1}_{\mathcal{H}} - P^{(0)}_{(0,\Delta)}\right) = 0 \tag{3.120}$$

But since $P^{(0)}_{(0,\Delta)}$ is the projection onto the kernel of $D_{(0,\Delta)}$ and $P^{(1)}_{(0,\Delta)}$ is the projection onto a subsets of its range (the eigenstates with eigenvalue 1) it follows that $P^{(1)}_{(0,\Delta)}\, P^{(0)}_{(0,\Delta)} = 0$ such that (3.120) becomes

$$P^{(1)}_{(0,\Delta)} = 0 \tag{3.121}$$

which contradicts the assumption that there are eigenstates of $D_{(0,\Delta)}$ with eigenvalue 1. Equation (3.121) of course also implies

$$P^{(1)}_{(0,\Delta)+x} = 0 \tag{3.122}$$

for all $x \in \mathbb{R}^4$ such that $\mathbb{P}^{\psi}\left(\mathscr{D}_{(0,\Delta)+x}\right) < 1$ for all $x \in \mathbb{R}^4$ and for all $\psi \in \mathcal{H}$.

■

Taking Born's rule seriously and taking into account the fact, that positive energy states do not have compact support, this is perfectly reasonable: An initial wave function is simply never perfectly localized in a bounded region of space such that there are no initial states which trigger a given detector covering a bounded region of space with certainty. This is the same resolution of the apparent puzzle as found for the seemingly paradoxical conclusion of Hegerfeldt's theorem (see sections 3.1, 3.3 and appendix B): If we drop the assumption that initial states can be perfectly localized in an operational sense (i.e. in the sense that a given detector is triggered by them with absolute certainty) – which is unreasonable in relativistic positive-energy quantum theory anyway – the crucial assumption on which the theorems are based is no longer valid and we need not bother with possible paradoxical conclusions[134]. This corresponds to the observation that the proof of Busch's theorem 3.24 does no longer work – and thus its conclusion is no longer justified – if $D_{(0,\Delta)}$ has no eigenstate with unit eigenvalue but, say, an eigenstate instead with eigenvalue $1-\varepsilon$ for some arbitrarily small $\varepsilon > 0$.

But the problem posed by the Malament type theorem of the following section shows that it is in general not that easy to find a suitable understanding of this issue. In particular, it shows that it is not only impossible to have initial states which trigger precisely one of several detectors with certainty, but that it is indeed impossible in general to have initial states which can trigger at most one detector at a given time (or at most $N \in \mathbb{N}$ detectors in its generalized version to $N-$states), given in addition to the assumption that the click statistics of the considered detectors is predictable by a covariant detector formalism satisfying the spectrum condition, *local commutativity* is true as well.

[134]But recall that this actual non localizability due to infinite tails of positive energy wave functions which technically solves the puzzle, must be seen as a rather academic issue though: As discussed in section 3.3 it is not at odds with the positive energy assumption (spectrum condition) that corresponding wave functions – though they cannot be perfectly localized – are in the domain of physical relevance extremely well localized such that we can treat them as perfectly localized for all practical purposes. So the infinite tails of positive energy wave functions should not encourage us to wonder whether an ion in a Paul trap in a MIT-laboratory might trigger a detector behind the moon as we do not wonder whether the air molecules in a room might suddenly altogether vanish into a bottle, although there are enough initial states (say from the viewpoint of classical mechanics) leading to this scenario that it has actually non-zero (but fapp zero) probability to occur (recall Boltzmann's response to Loschmidt's Umkehreinwand discussed in the concluding remark of section 3.1, see also section 4.4.1 and references therein). But infinite tails, even if they can be neglected for all practical purposes, force us to drop the assumption of perfectly localized states in the sense of the Born rule from a precise mathematical point of view, which is enough to find the conclusions of the Malament (type) theorems 3.19 and 3.24 (even taking into account relativistic causality considerations) to be redundant.

Regarding Busch's theorem 3.24, note finally that it is of course straightforward to derive its $N-$state version as it was sketched to be for the original Malament theorem 3.19 above. This version of Busch's theorem then reads exactly the same as corollary 3.23 only with weakening the assumption of projectiveness of the detector formalism by assuming only that 1 is among the eigenvalues of its effects.

3.4.6 General Effects: Halvorson–Clifton and Detectors

In chapter 2 it was shown that local commutativity as a mathematical condition is sufficient to guarantee the physical requirements of relativistic consistency and no signalling. On the other hand, we encountered that in case of general non projective measurements it is not ultimately clear whether local commutativity is in general a necessary condition for relativistic consistency and no signalling, as well. But we shall not bother with a remaining uncertainty about this issue and assume henceforth that local commutativity holds, as it is commonly done in the physics literature.

The following theorem 3.25 and part of its proof are motivated by an elegant theorem of Halvorson and Clifton [174], which shows that there does not exist a POVM on any given spacelike hyperplane of space-time whose effects are space-time translation covariant operators in the Heisenberg picture in relativistic quantum theory with positive energy. The present theorem 3.25 does not assume the whole structure of a spatial POVM (in order to show its actual non existence) in contrast, but only a single detector POVM (for a single given region covered by a detector of the considered type) which is space-time translation covariant, and the possibility to compose a collection copies of this detector to a spatial detector arrangement, which makes the proof a bit more complicated but is helpful to appreciate the result from a new perspective:

The result of Halvorson and Clifton has been mistaken (in particular by the authors themselves) to be a statement about ontological features of quantum theory. It is not ultimately clear to me, how Halvorson and Clifton (and Malament before) – came to the idea to tacitly assume –as if it were self evident– that the (non–)existence of things like particles with positions is conditional on the (non–)existence of operators acting on some Hilbert space. This illustrates naive realism about operators [96], which one encounters quite often in the quantum mechanics literature. In chapter 1 we saw that the operator formalism of effects and state transformers in quantum theory is appropriate to describe the statistics of outcomes of measurement (like) processes, and consequently statements about such operators are statements about such processes. To highlight this point, the present approach does only utilize operationalist concepts based on a general quantum description of usual experiments with a collection of detectors. In particular, it suggests that the seemingly paradoxical feature of the Halvorson-Clifton theorem is related with a flaw in the intuitive and seemingly natural operationalist concept of a 'single quantum system' developed above, which is supposed to hold e.g. for detection experiments with one single initial particle as they are nowadays frequently performed in many laboratories. Theorem 3.25 then simply shows that – given we want to keep the quantum description of measurement like processes, space-time translation covariance, the spectrum condition and local commutativity – this condition can never hold exactly (though it will certainly hold for all practical purposes if the detector is an appropriate detector). This may be understood by taking into account the fact that a detector is not a passive entity registering what there is, but itself a physical system

interacting with the rest of the world (or with part of it) and thereby changing it (note that each effect is associated with a state transformer which in general corresponds to a rough invasion of the measured system). We shall enlarge upon this towards the end of this chapter.

Secondly, this presentation does not need questionable concepts like detection in arbitrary bounded Borel subsets of space. It is frequently argued for example, that it should be impossible to detect a particle in a region whose extension is much smaller than the Compton wavelength of the measured particle. The following theorem shows that it is enough structure to consider a detector of some arbitrary shape and size and copies thereof which can be freely arranged in space, which seems to be a much more realistic and natural starting point than to assume detection in arbitrary measurable spatial subsets.

Theorem 3.25 [*Malament for General Effects*]

Consider a covariant detector formalism $\mathcal{E}_{\Delta}^{\mathcal{H}}$ which obeys the spectrum condition and local commutativity (see definition 2.3). If $\mathcal{E}_{\Delta}^{\mathcal{H}}$ is incapable of triggering more than one of several detectors at spacelike separation (see definition 3.18) it follows that $\mathcal{E}_{\Delta}^{\mathcal{H}}$ is incapable of triggering any detector:

$$\mathbb{P}^{\psi}\left(\mathscr{D}_{(0,\Delta)+x}\right) \equiv 0 \quad \textit{for all} \quad x \in \mathbb{R}^4 \quad \textit{and for all} \quad \psi \in \mathcal{H} \tag{3.123}$$

Proof: We shall show inductively that considering a proper arrangement of 2^n detectors allows us to conclude that the click probability of each detector of the considered type is uniformly bounded by $(\frac{1}{2})^n$ from above. We proceed step by step:

∘ (i) *First we show that for all initial states $\psi \in \mathcal{H}$ the click probabilities are bounded from above by $\frac{1}{2}$, i.e. we show that $\mathbb{P}^{\psi}(\mathscr{D}_{(0,\Delta)+x}) < \frac{1}{2} + \varepsilon$ for all $\varepsilon > 0$ and for all $x \in \mathbb{R}^4$:*

Choose $\boldsymbol{a} \in \mathbb{R}^3$ large enough such that Δ and $\Delta + \boldsymbol{a}$ are separated by some finite distance $d > 0$. Thus there is some neighbourhood $\mathcal{N}(0)$ of the origin $0 \in \mathbb{R}^4$ such that $(0, \Delta + \boldsymbol{a}) + x$ is spacelike separated with respect to $(0, \Delta)$ for all $x \in \mathcal{N}(0)$ (see Fig. 7). Local commutativity thus entails that the effects associated with the respective detection events commute[135]:

$$D_{(0,\Delta)}\, D_{(0,\Delta+\boldsymbol{a})+x} = D_{(0,\Delta+\boldsymbol{a})+x}\, D_{(0,\Delta)} \quad \text{for all } x \in \mathcal{N}(0) \tag{3.124}$$

But in contrast to projective measurements, we have in the general case a priori no reason to expect that the product on the left and right hand side of (3.124) vanishes. Therefore, we resort to the spectral projections of these effects: Since effects are positive and thereby selfadjoint

[135] See the remark at the end of subsection 2.3.1 and in particular the calculation (2.49), which shows that the commutation relations (2.44) of effects associated with one measurement with state transformers associated with a remote measurement, which we identified as the fundamental local commutativity relations, entail the commutativity of the respective effects (but in general not vice versa).

operators whose spectrum is a subset of $[0,1]$, the respective spectral representations read

$$D_{(0,\Delta)} = \int_{\mathcal{N}([0,1])} \lambda \, dE_{(0,\Delta)}(\lambda) \qquad\qquad D_{(0,\Delta+\boldsymbol{a})+x} = \int_{\mathcal{N}([0,1])} \lambda \, dE_{(0,\Delta+\boldsymbol{a})+x}(\lambda) \tag{3.125}$$

for any neighbourhood $\mathcal{N}([0,1])$ of the spectral interval $[0,1]$. Since as mentioned for $x \in \mathcal{N}(0)$ these operators commute, their spectral projections commute as well (note that commuting operators can be jointly diagonalized). If we denote the spectral projections associated with the spectral interval $[\frac{1}{2}+\varepsilon, 1]$ for some $0 < \varepsilon < \frac{1}{2}$, respectively by

$$F := \int_{\frac{1}{2}+\varepsilon}^{1_+} dE_{(0,\Delta)}(\lambda) \quad \text{and} \quad G(x) := \int_{\frac{1}{2}+\varepsilon}^{1_+} dE_{(0,\Delta+\boldsymbol{a})+x}(\lambda) \tag{3.126}$$

we thus obtain $[F, G(x)] = 0$ for all $x \in \mathcal{N}(0)$, which entails that together with F and $G(x)$, the product $(FG(x))^2 = FG(x)FG(x) = F^2G^2(x) = FG(x)$ is a projection, as well. The notation 1_+ stands for $1_+ := 1 + h$ for any $h > 0$, which is relevant if 1 has nonzero point measure (like e.g. in the case of projections), where we follow the common convention that the Stieltjes integral measure is a measure on half open intervals of the form $[a, b)$ (i.e. the related distribution function is right continuous).

Now we show that $FG(x) = 0$ for all $x \in \mathcal{N}(0)$: To this end, suppose that for some fixed $x \in \mathcal{N}(0)$ we have $FG(x) \neq 0$. Since $FG(x)$ is a projection, this entails that it has at least one eigenstate $0 \neq \psi \in \mathcal{H}$ (as always, the normalization is chosen to be $\|\psi\| = 1$) associated with eigenvalue 1, i.e. $FG(x)\psi = \psi$. Consequently, $F\psi = FFG(x)\psi = FG(x)\psi = \psi$ and analogously, with the commutativity of F and $G(x)$ also $G(x)\psi = \psi$, i.e. ψ is an eigenstate with eigenvalue 1 of F and $G(x)$, as well. This entails that ψ is in the kernel of the respective spectral projections of $D_{(0,\Delta)}$ and $D_{(0,\Delta+\boldsymbol{a})+x}$ associated with the converse spectral interval $[0, \frac{1}{2}+\varepsilon)$, respectively, and consequently

$$\begin{aligned} \left\langle \psi \right| D_{(0,\Delta)} \left| \psi \right\rangle &= \int_{\frac{1}{2}+\varepsilon}^{1_+} \lambda \, d \left\langle \psi \right| E_{(0,\Delta)}(\lambda) \left| \psi \right\rangle \\ \left\langle \psi \right| D_{(0,\Delta+\boldsymbol{a})+x} \left| \psi \right\rangle &= \int_{\frac{1}{2}+\varepsilon}^{1_+} \lambda \, d \left\langle \psi \right| E_{(0,\Delta+\boldsymbol{a})+x}(\lambda) \left| \psi \right\rangle \end{aligned} \tag{3.127}$$

and

$$\int_{\frac{1}{2}+\varepsilon}^{1_+} d \left\langle \psi \right| E_{(0,\Delta)}(\lambda) \left| \psi \right\rangle = \int_{\frac{1}{2}+\varepsilon}^{1_+} d \left\langle \psi \right| E_{(0,\Delta+\boldsymbol{a})+x}(\lambda) \left| \psi \right\rangle = 1 \tag{3.128}$$

But this is in contradiction with the assumption that ψ is incapable of triggering more than one detector at spacelike separation, since then the corresponding causal additivity condition (see definition 3.18) must be satisfied – i.e. $\mathbb{P}^\psi\left(\mathscr{D}_{(0,\Delta)} \vee \mathscr{D}_{(0,\Delta+\boldsymbol{a})+x}\right) = \mathbb{P}^\psi\left(\mathscr{D}_{(0,\Delta)}\right) + \mathbb{P}^\psi\left(\mathscr{D}_{(0,\Delta+\boldsymbol{a})+x}\right)$ – and consequently

$$\begin{aligned} 1 \geq \mathbb{P}^\psi\left(\mathscr{D}_{(0,\Delta)} \vee \mathscr{D}_{(0,\Delta+\boldsymbol{a})+x}\right) = \\ = \mathbb{P}^\psi\left(\mathscr{D}_{(0,\Delta)}\right) + \mathbb{P}^\psi\left(\mathscr{D}_{(0,\Delta+\boldsymbol{a})+x}\right) = \left\langle \psi \right| D_{(0,\Delta)} \left| \psi \right\rangle + \left\langle \psi \right| D_{(0,\Delta+\boldsymbol{a})+x} \left| \psi \right\rangle = \\ = \int_{\frac{1}{2}+\varepsilon}^{1_+} \lambda \, d \left\langle \psi \right| E_{(0,\Delta)}(\lambda) \left| \psi \right\rangle + \int_{\frac{1}{2}+\varepsilon}^{1_+} \lambda \, d \left\langle \psi \right| E_{(0,\Delta+\boldsymbol{a})+x}(\lambda) \left| \psi \right\rangle = \\ = \xi \int_{\frac{1}{2}+\varepsilon}^{1_+} d \left\langle \psi \right| E_{(0,\Delta)}(\lambda) \left| \psi \right\rangle + \xi' \int_{\frac{1}{2}+\varepsilon}^{1_+} d \left\langle \psi \right| E_{(0,\Delta+\boldsymbol{a})+x}(\lambda) \left| \psi \right\rangle \geq \frac{1}{2} + \varepsilon + \frac{1}{2} + \varepsilon \end{aligned} \tag{3.129}$$

which is a contradiction. In the step from the third to the forth line, we applied the mean value theorem, i.e. $\xi, \xi' \in [\frac{1}{2}+\varepsilon, 1]$ and the remaining integrals are equal to 1 according to (3.128).

Hence, if causal additivity holds, it follows that $FG(x) = 0$ for all $x \in \mathcal{N}(0)$. Since

$$G(x) = U(x) \left(\int_{\frac{1}{2}+\varepsilon}^{1_+} dE_{(0,\Delta+\boldsymbol{a})}(\lambda) \right) U^{-1}(x) \tag{3.130}$$

we can now apply corollary 3.14 to conclude that $FG(x) = 0$ for all $x \in \mathbb{R}^4$.

If we set $x_{-\boldsymbol{a}} = (0, -\boldsymbol{a})$ we have $D_{(0,\Delta)} = D_{(0,\Delta+\boldsymbol{a})+x_{-\boldsymbol{a}}}$. Consequently $F = G(x_{-\boldsymbol{a}})$, such that $0 = FG(x_{-\boldsymbol{a}}) = F^2 = F$, which of course also implies $U(x)FU^{-1}(x) = 0$ for all $x \in \mathbb{R}^4$. Thus, the spectrum of $D_{(0,\Delta)+x}$ is a subset of the interval $[0, \frac{1}{2}+\varepsilon)$. This in turn implies (again with the mean value theorem) for all $\varphi \in \mathcal{H}$

$$\begin{aligned} \mathbb{P}^{\varphi}(\mathscr{D}_{(0,\Delta)+x}) = \left\langle \varphi \right| D_{(0,\Delta)+x}\, \varphi \left\rangle \right. = \int_0^{\frac{1}{2}+\varepsilon} \lambda \; d\left\langle \varphi \right| E_{(0,\Delta)+x}(\lambda)\, \varphi \left\rangle \right. = \\ = \xi \int_0^{\frac{1}{2}+\varepsilon} d\left\langle \varphi \right| E_{(t,\Delta)+x}(\lambda)\, \varphi \left\rangle \right. < \tfrac{1}{2}+\varepsilon \end{aligned} \tag{3.131}$$

and consequently, the click probability of a detector of the considered type at any time and at any location in space is smaller than $\frac{1}{2}+\varepsilon$ for all initial states $\varphi \in \mathcal{H}$.

∘ (ii) *Now we show for a choice of* $\boldsymbol{a} \in \mathbb{R}^3$ *and* $\mathcal{N}(0) \subset \mathbb{R}^4$ *as in the first part (i) of the proof (i.e. such that* $(0, \Delta+\boldsymbol{a}) + \mathcal{N}(0)$ *is spacelike with respect to* $(0, \Delta)$*), that* $\mathbb{P}^{\psi}\left(\mathscr{D}_{(0,\Delta)} \vee \mathscr{D}_{(0,\Delta+\boldsymbol{a})+x}\right) < \frac{1}{2}+\varepsilon$ *for all* $x \in \mathcal{N}(0)$*:*

Choose $\boldsymbol{a} \in \mathbb{R}^3$, $\mathcal{N}(0) \subset \mathbb{R}^4$ as above and some $x_0 = (t_0, \boldsymbol{x}_0) \in \mathcal{N}(0)$. Denote the space-time region $\mathcal{X} := (0, \Delta) \cup ((0, \Delta+\boldsymbol{a}) + x_0)$ and define the generalized detection event $\mathscr{D}_{\mathcal{X}} := \mathscr{D}_{(0,\Delta)} \vee \mathscr{D}_{(0,\Delta+\boldsymbol{a})+x_0}$, i.e. the event that the detector covering Δ is triggered at time $t = 0$ or the detector covering $\Delta + \boldsymbol{a} + \boldsymbol{x}_0$ is triggered at time $t = t_0$ in the laboratory frame. Causal additivity thus entails that the effect associated with the generalized detection event $\mathscr{D}_{\mathcal{X}}$ is given by $D_{\mathcal{X}} := D_{(0,\Delta)} + D_{(0,\Delta+\boldsymbol{a})+x_0}$.

Now we perform a translation $\mathcal{X}' := \mathcal{X} + x_{\boldsymbol{b}}$ of $\mathcal{X}$ by a spatial vector $x_{\boldsymbol{b}} = (0, \boldsymbol{b})$, where $\boldsymbol{b} \in \mathbb{R}^3$ is large enough such that there is a neighbourhood $\mathcal{N}'(0)$ of the origin $0 \in \mathbb{R}^4$ such that $\mathcal{X}' + x$ is spacelike with respect to $\mathcal{X}$ for all $x \in \mathcal{N}'(0)$ (see Fig. 8). Thus we can again resort to causal additivity to find that the effect associated with the generalized detection event $\mathscr{D}_{\mathcal{X}'} = \mathscr{D}_{(0,\Delta+\boldsymbol{b})} \vee \mathscr{D}_{(0,\Delta+\boldsymbol{a}+\boldsymbol{b})+x_0}$ is given by $D_{\mathcal{X}'} := D_{(0,\Delta+\boldsymbol{b})} + D_{(0,\Delta+\boldsymbol{a}+\boldsymbol{b})+x_0}$ and the effect associated with the detection event $\mathscr{D}_{\mathcal{X}\cup\mathcal{X}'} := \mathscr{D}_{\mathcal{X}} \vee \mathscr{D}_{\mathcal{X}'}$ (i.e. the 'four detector event' where the detector covering Δ or the detector covering $\Delta + \boldsymbol{b}$ clicks at $t = 0$, respectively, or the detector covering $\Delta + \boldsymbol{a} + \boldsymbol{x}_0$ or the detector covering $\Delta + \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{x}_0$ clicks at $t = t_0$, respectively) is given by $D_{\mathcal{X}\cup\mathcal{X}'} := D_{\mathcal{X}} + D_{\mathcal{X}'}$ (here and henceforth we must resort to the more general causal additivity condition (3.93) for more than two detectors).

If now $\mathcal{X}'$ is exchanged with $\mathcal{X}' + x$ for some $x \in \mathcal{N}'(0)$, all four potential detection events remain mutually spacelike (see Fig. 8), such that causal additivity entails that the effect associated with the generalized detection event $\mathscr{D}_{\mathcal{X}\cup(\mathcal{X}'+x)} := \mathscr{D}_{\mathcal{X}} \vee \mathscr{D}_{\mathcal{X}'+x}$ is given by $D_{\mathcal{X}\cup(\mathcal{X}'+x)} := D_{\mathcal{X}} + D_{\mathcal{X}'+x}$ for all $x \in \mathcal{N}'(0)$.

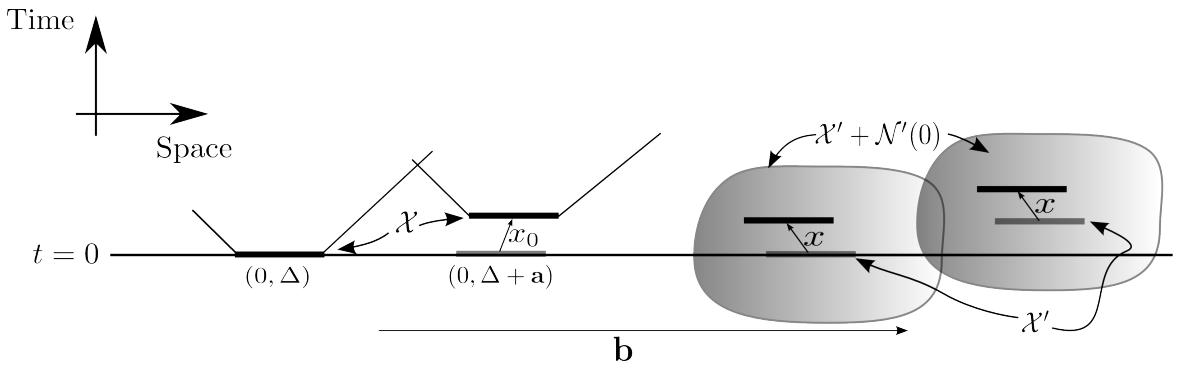

FIGURE 8

Four Detectors: *The space-time region $\mathcal{X} = (0, \Delta) \cup ((0, \Delta + \mathbf{a}) + x_0)$ is defined with $\boldsymbol{a} \in \mathbb{R}^3$ and $x_0 \in \mathcal{N}(0)$ as in part (i) of the proof (see also Fig. 7). $\mathcal{X}' = \mathcal{X} + (0, \mathbf{b})$ with $\boldsymbol{b}$ large enough, such that there is a neighbourhood $\mathcal{N}'(0)$ of the origin $0 \in \mathbb{R}^4$ such that $\mathcal{X}' + x$ is spacelike with respect to $\mathcal{X}$ for all $x \in \mathcal{N}'(0)$. The region $\mathcal{X} \cup \mathcal{X}'$ is associated with four mutually spacelike separated potential detection events of four detectors covering the respective spatial regions at the respective times.*

Now we are prepared to repeat the line of argument of the first part of the proof with the effects $D_{\mathcal{X}}$ and $D_{\mathcal{X}'}$ instead of $D_{(0,\Delta)}$ and $D_{(0,\Delta+\boldsymbol{a})}$: First note that $D_{\mathcal{X}}$ and $D_{\mathcal{X}'+x}$ commute (and thereby all of their spectral projections) for all $x \in \mathcal{N}'(0)$ according to local commutativity. Now redefine F as the spectral projection of the effect $D_{\mathcal{X}}$ associated with the spectral interval $[\frac{1}{2} + \varepsilon, 1]$ and $G(x)$ the spectral projection of the effect $D_{\mathcal{X}'+x}$ associated with the same spectral interval $[\frac{1}{2} + \varepsilon, 1]$ and let $x \in \mathcal{N}'(0)$. The assumption that there exists some (normalized) $\psi \in \mathcal{H}$ which is in the range of $FG(x)$ and thereby, due to the commutativity of F and $G(x)$, in the range of F and of $G(x)$, respectively, as well, leads to the contradiction (see (3.129)):

$$1 \geq \mathbb{P}^{\psi}\left(\mathscr{D}_{\mathcal{X}} \vee \mathscr{D}_{\mathcal{X}'+x}\right) = \langle\psi|\, D_{\mathcal{X}}\, \psi\rangle + \langle\psi|\, D_{\mathcal{X}'+x}\, \psi\rangle \geq \tfrac{1}{2} + \varepsilon + \tfrac{1}{2} + \varepsilon \tag{3.132}$$

It follows that $FG(x) = 0$ for all $x \in \mathcal{N}'(0)$ and consequently, with corollary 3.14 $FG(x) = 0$ for all $x \in \mathbb{R}^4$. Setting $x = x_{-\boldsymbol{b}} = (0, -\boldsymbol{b})$ such that $G(x_{-\boldsymbol{b}}) = F$ then entails $F = 0$, i.e. the spectrum, of $D_{\mathcal{X}} = D_{(0,\Delta)} + D_{(0,\Delta+\boldsymbol{a})+x_0}$ is contained in $[0, \frac{1}{2} + \varepsilon)$ and thus

$$\mathbb{P}^{\varphi}(\mathscr{D}_{\mathcal{X}}) = \mathbb{P}^{\varphi}(\mathscr{D}_{(0,\Delta)} \vee \mathscr{D}_{(0,\Delta+\boldsymbol{a})+x}) < \tfrac{1}{2} + \varepsilon \tag{3.133}$$

for all $x \in \mathcal{N}(0)$ and for all $\varphi \in \mathcal{H}$.

∘ (iii) *Now we show that the click probabilities are bounded from above by $(\frac{1}{2})^2$, i.e. we show that $\mathbb{P}^{\psi}(\mathscr{D}_{(0,\Delta)+x}) < (\frac{1}{2} + \varepsilon)^2$ for all $\varepsilon > 0$, for all $\psi \in \mathcal{H}$ and for all $x \in \mathbb{R}^4$:*

For that purpose, just repeat once again the argument of the first part (i) of the proof, only with redefining F as the spectral projection of $D_{(0,\Delta)}$ associated with the spectral interval $\left[(\frac{1}{2} + \varepsilon)^2, \frac{1}{2} + \varepsilon\right)$ and $G(x)$ as the spectral projection of $D_{(0,\Delta+\boldsymbol{a})+x}$ associated with the same spectral interval $\left[(\frac{1}{2} + \varepsilon)^2, \frac{1}{2} + \varepsilon\right)$. Let $x \in \mathcal{N}(0)$ such that $(0, \Delta)$ and $(0, \Delta + \boldsymbol{a}) + x$ are spacelike separated and thus local commutativity entails that F and $G(x)$ commute and causal additivity that $\mathbb{P}^{\psi}\left(\mathscr{D}_{(0,\Delta)} \vee \mathscr{D}_{(0,\Delta+\boldsymbol{a})+x}\right) = \mathbb{P}^{\psi}\left(\mathscr{D}_{(0,\Delta)}\right) + \mathbb{P}^{\psi}\left(\mathscr{D}_{(0,\Delta+\boldsymbol{a})+x}\right)$ which is according to (3.133)

bounded by $\frac{1}{2}+\varepsilon$ form above. Assuming that there is some (normalized) $\psi \in \mathcal{H}$ which is in the range of $FG(x)$ thus leads to the contradiction (see also (3.129)):

$$\begin{aligned} \tfrac{1}{2}+\varepsilon > \mathbb{P}^{\psi}\left(\mathscr{D}_{(0,\Delta)} \vee \mathscr{D}_{(0,\Delta+\boldsymbol{a})+x}\right) = \\ = \left\langle \psi \right| D_{(0,\Delta)}\left|\psi\right\rangle + \left\langle \psi \right| D_{(0,\Delta+\boldsymbol{a})+x}\left|\psi\right\rangle \geq \left(\tfrac{1}{2}+\varepsilon\right)^2 + \left(\tfrac{1}{2}+\varepsilon\right)^2 \end{aligned} \tag{3.134}$$

Hence $FG(x)=0$ for all $x \in \mathcal{N}(0)$ and thus by corollary 3.14 $FG(x)=0$ for all $x \in \mathbb{R}^4$. Since with $x_{-\boldsymbol{a}}=(0,-\boldsymbol{a})$ we have $G(x_{-\boldsymbol{a}})=F$ it follows that $F=0$ and consequently $U(x)FU^{-1}(x)=0$ for all $x \in \mathbb{R}^4$. Thus the spectrum of $D_{(0,\Delta)+x}$ is a subset of $\left[0,(\frac{1}{2}+\varepsilon)^2\right)$ and thereby

$$\mathbb{P}^{\varphi}(\mathscr{D}_{(0,\Delta)+x}) < \left(\tfrac{1}{2}+\varepsilon\right)^2 \tag{3.135}$$

for all $x \in \mathbb{R}^4$ and for all $\varphi \in \mathcal{H}$.

∘ (iv) *Finally, we argue that the click probabilities are bounded from above by* $(\frac{1}{2})^n$, *i.e.* $\mathbb{P}^{\psi}(\mathscr{D}_{(0,\Delta)+x}) < (\frac{1}{2}+\varepsilon)^n$ *for all* $\varepsilon > 0$, *for all* $n \in \mathbb{N}$, *for all* $\psi \in \mathcal{H}$ *and for all* $x \in \mathbb{R}^4$*:*

Choose $\mathcal{X}$ and $\mathcal{X}'$ as in part (ii) of the proof and consider the generalized detection event (see also Fig. 8)

$$\mathscr{D}_{\mathcal{X}\cup\mathcal{X}'} = \mathscr{D}_{\mathcal{X}} \vee \mathscr{D}_{\mathcal{X}'} = \mathscr{D}_{(0,\Delta)} \vee \mathscr{D}_{(0,\Delta+\boldsymbol{a})+x_0} \vee \mathscr{D}_{(0,\Delta+\boldsymbol{b})} \vee \mathscr{D}_{(0,\Delta+\boldsymbol{a}+\boldsymbol{b})+x_0} \tag{3.136}$$

Now we can translate $\mathcal{X}\cup\mathcal{X}'$ by a spatial vector $x_{\boldsymbol{c}}=(0,\boldsymbol{c})$ where $\boldsymbol{c} \in \mathbb{R}^3$ is large enough such that there is a neighbourhood $\mathcal{N}''(0)$ of the origin $0 \in \mathbb{R}^4$ such that with $\mathcal{X}'' := \mathcal{X}\cup\mathcal{X}'+x_{\boldsymbol{c}}$ the region $\mathcal{X}''+x$ is spacelike with respect to $\mathcal{X}\cup\mathcal{X}'$ for all $x \in \mathcal{N}''(0)$. Considering the generalized eight detector event $\mathscr{D}_{\mathcal{X}\cup\mathcal{X}'\cup\mathcal{X}''} = \mathscr{D}_{\mathcal{X}\cup\mathcal{X}'} \vee D_{\mathcal{X}''}$ and repeating the argument of part (ii) of the proof with $\mathcal{X}\cup\mathcal{X}'$ and $\mathcal{X}''$ (instead of $\mathcal{X}$ and $\mathcal{X}'$) yields $\mathbb{P}^{\psi}\left(\mathscr{D}_{\mathcal{X}\cup\mathcal{X}'}\right) < \frac{1}{2}+\varepsilon$ for all $\varepsilon > 0$ and for all $\psi \in \mathcal{H}$. Repeating the argument of part (iii) of the proof with $\mathcal{X}$ and $\mathcal{X}'$ (instead of $(0,\Delta)$ and $(0,\Delta+\boldsymbol{a})$) then yields $\mathbb{P}^{\psi}\left(\mathscr{D}_{\mathcal{X}}\right) < \left(\frac{1}{2}+\varepsilon\right)^2$ and consequently the assumption that the range of the product of the spectral projections of $D_{(0,\Delta)}$ and $D_{(0,\Delta+\boldsymbol{a})+x}$ (with $x \in \mathcal{N}(0)$) associated with the spectral interval $\left[(\frac{1}{2}+\varepsilon)^3,(\frac{1}{2}+\varepsilon)^2\right)$, respectively, is non-empty leads to a contradiction. Invoking corollary 3.14 then leads to the conclusion that $\mathbb{P}^{\psi}\left(\mathscr{D}_{(0,\Delta)+x}\right) < (\frac{1}{2}+\varepsilon)^3$ for all $\varepsilon > 0$, for all $x \in \mathbb{R}^4$ and for all initial $\psi \in \mathcal{H}$.

Repeating this procedure once more with eight additional detectors yields $\mathbb{P}^{\psi}(\mathscr{D}_{(0,\Delta)+x}) < (\frac{1}{2}+\varepsilon)^4$ for all $\varepsilon > 0$, for all $\psi \in \mathcal{H}$ and for all $x \in \mathbb{R}^4$ and so on. So for any $n \in \mathbb{N}$ we can inductively repeat this recipe to finally arrive at considering a proper arrangement of 2^n spatially separated detectors to find that the click probability of a detector of that type is smaller than $(\frac{1}{2}+\varepsilon)^n$ for all $\varepsilon > 0$, at any location in space and for all initial states which live in a Hilbert (sub-)space in which causal additivity is satisfied.

Since there is no upper bound – in principle – for the number of potential detectors of the considered type which may be arranged in space, we can conclude that the click probabilities of a detector formalism under the assumptions of this theorem must be zero.

■

Consequently, detectors of the considered type will never detect anything and thus do not deserve the name 'detector' in any serious manner.

$N-$States Again

Finally, it shall be sketched how the $N-$state version of the strongest Malament type theorem 3.25 can be derived. Consider a covariant detector formalism $\mathcal{E}_\Delta^{\mathcal{H}}$ and an arbitrary but fixed arrangement of N detectors covering disjoint regions $\Delta_1, \ldots, \Delta_N$ and the associated $N-$click event $\mathscr{D}_{(t_1,\Delta_1)} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N)}$ (as always, the regions (t_k, Δ_k) are denoted in laboratory frame coordinates) such that the regions (t_k, Δ_k) are mutually spacelike separated, and suppose the probability of this $N-$click event is given by the effect $D_{(t_1,\Delta_1)\wedge\ldots\wedge(t_N,\Delta_N)}$ acting on $\mathcal{H}$. In section 3.4.4 it was argued that a detector formalism $\mathcal{E}_\Delta^{\mathcal{H}}$ which is incapable of triggering more than N detectors at spacelike separation must obey *causal $N-$additivity* as defined in definition 3.21. In order to prove the $N-$state version of theorem 3.25 the causal $N-$additivity condition (3.106) needs to be generalized to more translated copies of the $N-$click event $\mathscr{D}_{(t_1,\Delta_1)} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N)}$: If we abbreviate $\mathscr{D} := \mathscr{D}_{(t_1,\Delta_1)} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N)}$ and $\mathscr{D}(x) := \mathscr{D}_{(t_1,\Delta_1+\boldsymbol{a})+x} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N+\boldsymbol{a})+x}$, it is straightforward to derive that given $\mathcal{E}_\Delta^{\mathcal{H}}$ is incapable of triggering more than N detectors at spacelike separation, it must obey

$$\mathbb{P}^\psi\left(\mathscr{D}\right) + \mathbb{P}^\psi\left(\mathscr{D}(x_1)\right) + \cdots + \mathbb{P}^\psi\left(\mathscr{D}(x_L)\right) = \mathbb{P}^\psi\left(\mathscr{D} \vee \mathscr{D}(x_1) \vee \cdots \vee \mathscr{D}(x_L)\right) \tag{3.137}$$

for each corresponding arrangement of $(L+1)N$ detectors whenever all involved click events are mutually spacelike separated (this is the analogue condition to (3.93) in the definition of causal additivity).

Now we can repeat the steps in the proof of theorem 3.25, only substituting the effect $D_{(0,\Delta)}$ by $D_{(t_1,\Delta_1)\wedge\ldots\wedge(t_N,\Delta_N)}$, translating copies of the corresponding $N-$click event $\mathscr{D} = \mathscr{D}_{(t_1,\Delta_1)} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N)}$ in space and time as done with the one-click event $\mathscr{D}_{(0,\Delta)}$ in the above proof and using causal $N-$additivity (in the later steps in its generalized form (3.137)) instead of causal additivity (it is helpful here to reconsider the arguments in section 3.4.4, where it was sketched how to transfer the arguments of the original Malament theorem 3.19 to $N-$states). In the line of argument corresponding to the first step (i) of theorem 3.25 we thus come to the conclusion, that $N-$additivity implies $\mathbb{P}^\psi\left(\mathscr{D}_{(t_1,\Delta_1)+x} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N)+x}\right) < \frac{1}{2} + \varepsilon$ for all $\varepsilon > 0$, for all $x \in \mathbb{R}^4$ and for all $\psi \in \mathcal{H}$. Repeating in complete analogy the rest of the proof, yields then that considering a proper arrangement of $N2^n$ detectors allows to conclude that $\mathbb{P}^\psi\left(\mathscr{D}_{(t_1,\Delta_1)+x} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N)+x}\right) < \left(\frac{1}{2} + \varepsilon\right)^n$ for all $\varepsilon > 0$, for all $x \in \mathbb{R}^4$ and for all $\psi \in \mathcal{H}$ and since there is no upper bound – in principle – of the number of detectors of the considered type which may be arranged in space, we get the following:

Lemma 3.26 [*Malament for N-States and General Effects*]

Consider a covariant detector formalism $\mathcal{E}_\Delta^{\mathcal{H}}$ which obeys the spectrum condition. If $\mathcal{E}_\Delta^{\mathcal{H}}$ is incapable of triggering more than N of several detectors at spacelike separation (see above), it is incapable of triggering N detectors at spacelike separation as well:

$$\mathbb{P}^\psi\left(\mathscr{D}_{(t_1,\Delta_1)} \wedge \ldots \wedge \mathscr{D}_{(t_N,\Delta_N)}\right) \equiv 0 \quad \textit{for all} \quad \psi \in \mathcal{H} \tag{3.138}$$

whenever the space-time regions $(t_1, \Delta_1), \ldots, (t_N, \Delta_N)$ are mutually spacelike.

Thus we find that considered $N-$states are actually $(N-1)-$states, $(N-2)-$states ... are actually $1-$states, to which theorem 3.25 applies (compare with the chain of implications (3.108)). Since $N \in \mathbb{N}$ was arbitrary, we thus finally conclude:

Corollary 3.27 [*Malament for N-States and General Effects*]

Consider a covariant detector formalism $\mathcal{E}_\Delta^{\mathcal{H}}$ which obeys the spectrum condition. If $\mathcal{E}_\Delta^{\mathcal{H}}$ is incapable of triggering more than any given finite number of several detectors at spacelike separation, it is incapable of triggering any detector:

$$\mathbb{P}^\psi\left(\mathscr{D}_{(0,\Delta)+x}\right) \equiv 0 \quad \textit{for all} \quad x \in \mathbb{R}^4 \quad \textit{and for all} \quad \psi \in \mathcal{H} \tag{3.139}$$

3.4.7 How to Understand?

A main purpose of section 3.4 was to develop an operational formulation of the Malament type theorems in order to reveal their operational meaning. A comparably elaborate analysis to develop an understanding of possible solutions is beyond the scope of this work. Nonetheless, as already argued in the introduction to this chapter, it is now easy to find indications why these results are not so unreasonable as they might appear at a first glance. These ideas may be taken as motivation for another, more rigorous analysis in the future.

One possibility would be to question one or several of the primary assumptions of the Malament type theorems which are collected in the basic assumption that the statistics of detector type experiments is predictable in relativistic quantum theory by some covariant detector formalism which satisfies the spectrum condition, plus local commutativity for theorem 3.25 and its $N-$state version 3.27. If these assumptions are taken for granted, theorem 3.25 entails that *causal additivity must be violated*! As argued, this implies that there is no linear manifold of initial states (represented by $\mathcal{H}$) for which the click probabilities of joint detector clicks at spacelike separation is precisely zero, i.e. for all spacelike separated click regions (t_1, Δ_1) and (t_2, Δ_2) we have necessarily

$$\mathbb{P}^\psi\left(\mathscr{D}_{(t_1,\Delta_1)} \wedge \mathscr{D}_{(t_2,\Delta_2)}\right) \neq 0 \quad \text{for all } \psi \in \mathcal{H} \tag{3.140}$$

and an obvious question is how this can match with the experience that it is possible to detect single quantum systems like quantum particles, atoms or tennis balls for which double detections at spacelike separation do not occur.

In this respect it is important to note first that (3.140) says nothing about magnitudes, i.e. the probabilities $\mathbb{P}^\psi\left(\mathscr{D}_{(t_1,\Delta_1)} \wedge \mathscr{D}_{(t_2,\Delta_2)}\right)$ might be bounded from above by some very small number $\varepsilon > 0$, which would be enough to regard their finiteness as irrelevant for all practical purposes (like violations of the second law of thermodynamics, which can have non zero probability in a strict sense as well) as discussed at the beginning of this chapter. Nonetheless, this does not help to coherently physically understand why these probabilities do never perfectly vanish, although this seems still counterintuitive for many systems, even if the related probabilities are extremely small.

In section 3.1, two natural reasons were identified, related to the *infinite tails* of positive energy wave functions and the *active nature of detectors* giving rise to *pair creation* with certain probabilities, which shall be discussed now again in the light of the theorems just developed.

Infinite Tails

To begin with, choose $\mathcal{H}$ to be a positive energy Hilbert space associated with a one particle relativistic wave equation like in positive energy Dirac theory. For Dirac theory it is well understood (see appendix A) that as long as all state transitions respect the spectrum condition, this is equivalent to considering the one particle sector of the associated second quantized QFT, since in this case particle creation processes do not occur.

Now consider the notoriously non zero joint quantum probabilities $\mathbb{P}^\psi\left(\mathscr{D}_{(t_1,\Delta_1)} \wedge \mathscr{D}_{(t_2,\Delta_2)}\right)$ associated with spacelike separated detection regions (t_1,Δ_1) and (t_2,Δ_2). If we pretend that the measurements happen instantaneously (i.e. neglect the duration of interaction and state transformation), denote the state transformers associated with the two click events by $\mathcal{R}_{(t_i,\Delta_i)}$, $i=1,2$, suppose without restricting the generality that $t_1 < t_2$ in the laboratory frame, neglect the free time evolution in between t_1 and t_2 and denote $\psi_{\Delta_1} := \|\mathcal{R}_{(t_1,\Delta_1)}\psi\|^{-1}\mathcal{R}_{(t_1,\Delta_1)}\psi$ (given $\mathbb{P}^\psi\left(\mathscr{D}_{(t_1,\Delta_1)}\right) = \|\mathcal{R}_{(t_1,\Delta_1)}\psi\|^2 > 0$), the considered probability can be written as

$$\begin{aligned}
&\mathbb{P}^\psi\left(\mathscr{D}_{(t_1,\Delta_1)} \wedge \mathscr{D}_{(t_2,\Delta_2)}\right) = \mathbb{P}^\psi\left(\mathscr{D}_{(t_2,\Delta_2)} \mid \mathscr{D}_{(t_1,\Delta_1)}\right)\mathbb{P}^\psi\left(\mathscr{D}_{(t_1,\Delta_1)}\right) = \\
&= \mathbb{P}^{\psi_{\Delta_1}}\left(\mathscr{D}_{(t_2,\Delta_2)}\right)\ \mathbb{P}^\psi\left(\mathscr{D}_{(t_1,\Delta_1)}\right) = \frac{\left\langle \psi \right|\ \mathcal{R}^\dagger_{(t_1,\Delta_1)} D_{(t_2,\Delta_2)} \mathcal{R}_{(t_1,\Delta_1)}\,\psi\right\rangle}{\|\mathcal{R}_{(t_1,\Delta_1)}\psi\|^2}\|\mathcal{R}_{(t_1,\Delta_1)}\psi\|^2 \\
&= \left\langle \psi \right|\ \mathcal{R}^\dagger_{(t_1,\Delta_1)} D_{(t_2,\Delta_2)} \mathcal{R}_{(t_1,\Delta_1)}\,\psi\right\rangle = \left\langle \psi \right|\ \mathcal{R}^\dagger_{(t_1,\Delta_1)} \mathcal{R}^\dagger_{(t_2,\Delta_2)} \mathcal{R}_{(t_2,\Delta_2)} \mathcal{R}_{(t_1,\Delta_1)}\,\psi\right\rangle \\
&= \left\langle \mathcal{R}_{(t_2,\Delta_2)}\mathcal{R}_{(t_1,\Delta_1)}\,\psi \right|\ \mathcal{R}_{(t_2,\Delta_2)}\mathcal{R}_{(t_1,\Delta_1)}\,\psi\right\rangle = \|\mathcal{R}_{(t_2,\Delta_2)}\mathcal{R}_{(t_1,\Delta_1)}\,\psi\|^2
\end{aligned} \tag{3.141}$$

Thus theorem 3.25 tells us that under its primary assumptions (covariant detector formalism, spectrum condition, local commutativity) it follows for all state transformers associated with two spacelike separated click events:

$$\mathcal{R}_{(t_2,\Delta_2)}\mathcal{R}_{(t_1,\Delta_1)}\,\psi \neq 0 \quad \text{for all non zero } \psi \in \mathcal{H} \tag{3.142}$$

This is not very surprising if we acknowledge that all wave functions consistent with the spectrum condition have infinite tails: An intuition behind the expectation (called now into question) that the left hand side of (3.142) should be precisely zero is that $\mathcal{R}_{(t_1,\Delta_1)}$ and $\mathcal{R}_{(t_2,\Delta_2)}$ localize the respective initial states perfectly in the respective regions. For example, for an ideal measurement of the standard position operator q the transformation $\psi \to \mathcal{R}_{(t_1,\Delta_1)}\psi$ would be given by multiplication of ψ by the indicator function $\chi_{\{\Delta_1\}}(x)$ of Δ_1 (at time t_1) in position representation such that $\Delta_1 \cap \Delta_2 = \emptyset$ entails (neglecting the free time evolution[136] in between t_1 and t_2) $\mathcal{R}_{(t_2,\Delta_2)}\mathcal{R}_{(t_1,\Delta_1)}\,\psi = \chi_{\{\Delta_1\}}(q)\chi_{\{\Delta_2\}}(q)\,\psi = 0$. But the action of state transformers of that kind cuts of the tails of ψ and thus violates the spectrum condition in relativistic quantum theory.

[136]Note that for $t_1 \neq t_2$, it can be only justified to neglect the free time evolution here in a relativistic quantum theory, where wave functions propagate causally in position representation. In non relativistic Schrödinger theory, wave functions supported on bounded spatial regions immediately develop infinite tails under the free time evolution.

If we think, on the other hand, of the state transformers as localizing the state very well and consistently with the spectrum condition, we might rather think of multiplication by functions which approximate indicator functions very well (e.g. obtained by convolution of indicator functions with smoothing functions like in case of the approximate position measurement POVMs of section 1.4.3) such that we can treat them as such for all practical purposes, but which have actually infinite tails. In this case, in general, $\mathcal{R}_{(t_2,\Delta_2)}\mathcal{R}_{(t_1,\Delta_1)}\,\psi \neq 0$. Taking into account fundamental finite resolution of all measuring devices, like at least given by infinite tails of the states of measuring devices or 'localizing potentials' involved in the measuring process, this seems to make sense also from the viewpoint of physical considerations. But a closer look reveals that taking into account the infinite tails of measuring devices will not suffice to save the spectrum condition in the one particle picture[137].

Impact of Local Measuring Devices and Pair Creation

In the arguments subsequent to calculation (3.141), it was presupposed that the transformation of a state upon measurement respects the spectrum condition on the one particle level. But as already indicated in the last paragraphs, this premise is not very realistic: It cannot hold if the transformations occur only locally in the region associated with the spatial location of the measuring device, since as encountered in section 3.3 local transformations cannot preserve the positive energy property of wave functions. The same was found to be true in general for global transformations except they are extremely special, since a positive energy wave function is already completely determined by its values in any arbitrarily small neighbourhood. So even if we assume that transformations of wave functions forced by external measuring devices are always global in a non trivial way by taking into account the actual infinite tails of the measuring device wave functions or potentials, there is little hope that the spectrum condition is rescued that way: For that purpose the precise transformation of the wave function of the measured system upon influence of the external measuring device in any spatial region would need to be determined by its actual transformation in any other region. We might moreover take into account decoherence, which is known to massively localize the wave function of a measured system extremely fast [192], to substantiate the conjecture that realistic measurements violate the spectrum condition on the level of first quantization, since their impact on the measured system will be supposedly not of the very sensitive nonlocal nature of transformations preserving the positivity of the energy.

[137] E.g. in section 1.5.2 it was demonstrated that one can derive approximate measurement POVMs (which were considered for non relativistic quantum theory but might e.g. be generalized to Dirac theory by acting on each spinor component separately) on physical space by taking into account the tails of the pointer states. But theorem 3.25 translated to a no-go theorem about POVMs on physical space (i.e. the analogue of corollary 3.20 for general POVMs) shows that such a POVM cannot be given by a space-time translation covariant, causally additive and locally commuting family of operators acting on a relativistic Hilbert space which obeys the spectrum condition. Space time translation covariance and local commutativity hold trivially for approximate measurement POVMs (e.g. local commutativity is trivial, since related effects act simply as multiplication by smoothed out indicator functions in position representation). As derived in section 3.4.2, violation of causal additivity would entail that multiple positive outcomes (say detector clicks) at spacelike separation have non zero probability such that if the formalism is truly relativistic, there are frames in which remote detectors are triggered simultaneously with non vanishing probability. But this is in contradiction with the additivity of the POVM (see section 3.4.2). So either the approximate measurement formalism cannot be made relativistic or it must violate the spectrum condition.

But violations of the spectrum condition in the one (or $N-$) particle picture need not entail violations of the spectrum condition in the associated QFT: For Dirac particles it is well known that transitions between the negative and positive energy spectrum in first quantization correspond to particle creation and annihilation processes in the corresponding QFT, which can be nicely understood and analysed in the Dirac sea picture as shown in appendix A. And if interaction with a detector creates particles with a certain probability (which will be presumably extremely small for usual detectors), the one particle sector of Fock space is not invariant under the state transformations associated with local detection experiments, which makes it appear pretty natural that multiple detections at spacelike separation have non zero probability, even if the initial state was a one particle state.

We shall illustrate this in a general Fock space setting (for simplicity, say for a single particle species[138]): Let $\psi \in \mathcal{F}$ be a one particle state in Fock space $\mathcal{F}$ and $\mathcal{R}_{(t,\Delta)}$ state transformers associated with detector clicks at the space-time locations (t, Δ) (in the laboratory frame) acting on $\mathcal{F}$[139]. Now suppose the non normalized states

$$\mathcal{R}_{(t,\Delta)}\psi \equiv \bigoplus_{N\in\mathbb{N}_0} \varphi^{(N)}_{(t,\Delta),\psi} \tag{3.143}$$

do no longer represent single particles – if $\varphi^{(N)}_{(t,\Delta),\psi} \neq 0$ for more than one $N \in \mathbb{N}_0$ not even any definite number of particles. Here $\varphi^{(N)}_{(t,\Delta),\psi} = [\mathcal{R}_{(t,\Delta)}\psi]_N$ is for each $N > 0$ a non normalized $N-$particle wave function[140] or zero and the vacuum component $\varphi^{(0)}_{(t,\Delta),\psi} = [\mathcal{R}_{(t,\Delta)}\psi]_0$ a

[138]Pair creation requires to have at least particles and antiparticles, but the central aspects of the present considerations about only particles of variable number are easily transferred to the more realistic scenario of particle-antiparticle pairs of variable number.

[139]In which case and how state transformers of the first quantized theory can be lifted to Fock space is as far as I know not yet understood (in contrast to the lift of unitary evolution and observable operators, see appendix A). Here we just assume that state transformers associated with detectors *exist* on $\mathcal{F}$, however they may be constructed or explicitly look like.

[140]It is instructive to construct a toy model, in which the particle creating and the localizing aspects of $\mathcal{R}_{(t,\Delta)}$ are separated by a polar decomposition: For ease of notation we set $t = 0$ and $\mathcal{R}_{(0,\Delta)} \equiv \mathcal{R}_\Delta$. Let $\mathcal{R}_\Delta = \sqrt{K_\Delta}\, U_\Delta$ with $K_\Delta = \mathcal{R}_\Delta \mathcal{R}_\Delta^\dagger$ be the right polar decomposition of $\mathcal{R}_\Delta$ (existence of the right polar decomposition of a bounded operator A follows e.g. from the usual left polar decomposition of its adjoint $A^\dagger = U\sqrt{AA^\dagger}$ such that $A = \sqrt{AA^\dagger}\, U^\dagger$ and the fact that the adjoint $U^\dagger$ of a partial isometry U is again a partial isometry, with initial and final subspace interchanged). Suppose U_Δ creates particles of any number with non vanishing probability, i.e. its action looks like $U_\Delta\, \psi = \bigoplus_{N\in\mathbb{N}_0} \phi^{(N)}_{\Delta,\psi}$, where in general $0 \neq \phi^{(N)}_{\Delta,\psi} = [U_\Delta\, \psi]_N \in \mathcal{H}^{\otimes N}$, even if $\psi \in \mathcal{F}$ was a one particle state (or the vacuum). Suppose further that $\sqrt{K_\Delta}$ incorporates Born's rule in each sector. In particular, it shall act as a direct sum operator $\sqrt{K_\Delta} = \bigoplus_{N\in\mathbb{N}_0} Q^{(N)}_\Delta$ where each $N-$particle operator $Q^{(N)}_\Delta$ localizes an $N-$particle state appropriately: Since it shall be associated with a detector click, it shall be close to the eigenprojections $P^{(N)}_\Delta$ of the standard position operator (indicator function in position representation) corresponding to the region $\bigcup_{k=1}^N \mathbb{R}^3 \times \cdots \times \Delta \times \cdots \times \mathbb{R}^3$ (with Δ at the k'th place, respectively) of $N-$particle configuration space in which at least one of N particles is in Δ, but in contrast to $P^{(N)}_\Delta$ respect the spectrum condition. Being 'close' means that its action shall be indistinguishable from the action of $P^{(N)}_\Delta$ for all practical purposes, e.g. in the sense that $\|(P^{(N)}_\Delta - Q^{(N)}_\Delta)\,\varphi\|_{\mathcal{H}^{\otimes N}} < \varepsilon$ for some small $\varepsilon > 0$ and for all $\varphi \in \mathcal{H}^{\otimes N}$. Such $Q^{(N)}_\Delta$ exist as a consequence of the positive energy localization schemes discussed in section 3.3: One might e.g. build them from the eigenprojections of the Newton-Wigner operator (without interpreting them as describing perfect localization, see section 3.3) which are known to be indistinguishable from the eigenprojections of the standard

$\mathbb{C}-$number ($[\Psi]_N$ denoting the $N-$particle component of the Fock state $\Psi \in \mathcal{F}$). The corresponding $N-$particle Hilbert space (possibly (anti-)symmetrized) shall be denoted by $\mathcal{H}^{\otimes N}$. Then, if we set as always $D_{(t,\Delta)} = \mathcal{R}^{\dagger}_{(t,\Delta)}\mathcal{R}_{(t,\Delta)}$, the probabilities

$$\mathbb{P}^{\psi}\left(\mathscr{D}_{(t,\Delta)}\right) = \left\langle \psi \right| D_{(t,\Delta)}\left|\psi\right\rangle = \left\|\mathcal{R}_{(t,\Delta)}\,\psi\right\|^2_{\mathcal{F}} = \sum_{N=0}^{\infty}\left\|\varphi^{(N)}_{(t,\Delta),\psi}\right\|^2_{\mathcal{H}^{\otimes N}} \tag{3.144}$$

do obviously not display statistics associated with a single particle (e.g. in the sense of the detector formalism developed in section 3.4.2).

Crucial for the present purpose are the joint probabilities $\mathbb{P}^{\psi}\left(\mathscr{D}_{(t_1,\Delta_1)} \wedge \mathscr{D}_{(t_2,\Delta_2)}\right)$. So let $t_1 < t_2$ in the laboratory frame, such that[141] (see calculation (3.141))

$$\mathbb{P}^{\psi}\left(\mathscr{D}_{(t_1,\Delta_1)} \wedge \mathscr{D}_{(t_2,\Delta_2)}\right) = \left\langle \psi \right| \mathcal{R}_{(t_1,\Delta_1)}D_{(t_2,\Delta_2)}\mathcal{R}_{(t_1,\Delta_1)}\,\left|\psi\right\rangle \tag{3.145}$$

Even if the final state of a positive detection at (t_1,Δ_1) given by $\psi_{(t_1,\Delta_1)} := \frac{\mathcal{R}_{(t_1,\Delta_1)}\,\psi}{\|\mathcal{R}_{(t_1,\Delta_1)}\,\psi\|_{\mathcal{F}}}$ is a state of several particles (in contrast to the one particle initial state ψ) as assumed, one could expect, roughly speaking, that it 'looks locally like the vacuum state Ω'[142] in the region (t_2,Δ_2), given it is excluded that the measurement at (t_1,Δ_1) has created particles at spacelike separation[143]. 'Looks like' is an operational notion (in particular it includes 'looks like for an experimenter'), so 'looks locally like the vacuum' should imply that it is indistinguishable from

position operator for all practical purposes [321], or in Dirac theory, one might proceed from approximating the perfectly localized states which arise form the action of the eigenprojections of the standard position operator by (continuous) superpositions of Bracken-Melloy localized positive energy states (which approximate $\delta-$functions arbitrarily well, see section 3.3) and accordingly construct the action of $Q^{(N)}_{\Delta}$ on $\mathcal{H}^{\otimes N}$.

Such state transformers guarantee the validity of Born's rule for position measurements (which is at the end of the day the foundation of all of the so successful predictions of quantum theory, see chapter 1) for all practical purposes and yet the associated POVM should be in accordance with the Malament type theorems. It is in accordance with the relativistic assumptions of these theorems (in particular the spectrum condition) but it will violate (causal) ($N-$)additivity, since non vanishing click probabilities are associated with all particles sectors, even if the initial state described a single particle or the vacuum (see below for discussion of multiple detector clicks).

To make the model half way realistic, U_{Δ} should be chosen in a way, such that for all N with $[\psi]_N = 0$, the sector probabilities $\|[\mathcal{R}_{\Delta}\,\psi]_N\|^2_{\mathcal{H}^{\otimes N}}$ are be negligibly small, of course. But the polar decomposition of a real detector state transformer will nonetheless presumably not provide such a nice separation of particle creation and localization. On the one hand, if the partially isometric (possibly unitary) part U_{Δ} of $\mathcal{R}_{\Delta}$ creates particles, the corresponding states can be expected to be already localized about the detector region (given the detector does not create particles at a distance), and on the other hand, if $\sqrt{K_{\Delta}}$ localizes the states on which it acts, it will presumably create particles with certain probabilities (as already discussed several times in this chapter) and thus not act as a direct sum operator (one could also consider an analogue model, where U_{Δ} is a separate state transformer corresponding to the action of the switched on detector prior to the click event).

[141] Note that symmetry with respect to a frame in which the time order is reversed requires relativistic consistency (see chapter 2) i.e. $\mathcal{R}_{(t_1,\Delta_1)}D_{(t_2,\Delta_2)}\mathcal{R}_{(t_1,\Delta_1)} = \mathcal{R}_{(t_2,\Delta_2)}D_{(t_1,\Delta_1)}\mathcal{R}_{(t_2,\Delta_2)}$, which is e.g. a consequence of local commutativity.

[142] For a possible way how one might mathematically specify the notion that a (sub-)system looks locally like the vacuum from an operational point of view, see [65].

[143] In connection with the Reeh-Schlieder theorem, the possibility is sometimes discussed that, figuratively speaking, 'a measurement on earth creates instantaneously a particle behind the moon', see footnote 153.

the vacuum by any means of measurement in the respective region. Statistical argument[144] thus yields $\mathbb{P}^{\psi_{(t_1,\Delta_1)}}(\mathscr{D}_{(t_2,\Delta_2)}) \stackrel{!}{=} \mathbb{P}^{\Omega}(\mathscr{D}_{(t_2,\Delta_2)})$, i.e.

$$\left\langle \psi_{(t_1,\Delta_1)} \right| D_{(t_2,\Delta_2)}\, \psi_{(t_1,\Delta_1)} \rangle \stackrel{!}{=} \left\langle \Omega \right| D_{(t_2,\Delta_2)}\, \Omega \rangle \tag{3.146}$$

or in terms of state transformers $\left\| \mathcal{R}_{(t_2,\Delta_2)} \psi_{(t_1,\Delta_1)} \right\|^2_{\mathcal{F}} \stackrel{!}{=} \left\| \mathcal{R}_{(t_2,\Delta_2)} \Omega \right\|^2_{\mathcal{F}}$. Thus from (3.146) together with (3.145) and (3.140) we obtain

$$\left\langle \Omega \right| D_{(t_2,\Delta_2)}\, \Omega \rangle \stackrel{!}{>} 0 \tag{3.147}$$

i.e. the detector clicks in the vacuum with non vanishing probability.

Although the above argument involving the locally-looks-like notion has several shortcomings[145], its conclusion (3.147) is exactly what we should expect: If detectors do not leave the one particle sector of Fock space invariant, the very same should be true for the vacuum sector such that detectors have non zero click probability, even if the initial state is the vacuum vector Ω. This amounts to a state transformation 'upon click in the vacuum' of the form

$$\mathcal{R}_{(t,\Delta)}\, \Omega = \bigoplus_{N \in \mathbb{N}_0} \varphi^{(N)}_{(t,\Delta),\Omega} \tag{3.148}$$

(and analogously the state transformation associated with a switched on detector and the complementary event 'no click') where the assumption that the zero particle sector does not contribute to trigger detectors would only entail $\varphi^{(0)}_{(t,\Delta),\Omega} \equiv 0$ (i.e. the vacuum sector does not contribute to the click probabilities $\mathbb{P}^{\Omega}(\mathscr{D}_{(t,\Delta)}) = \left\| \mathcal{R}_{(t,\Delta)}\, \Omega \right\|^2_{\mathcal{F}} = \sum_{N=1}^{\infty} \left\| \varphi^{(N)}_{(t,\Delta),\Omega} \right\|^2_{\mathcal{H}^{\otimes N}}$).

These ideas receive strong support from axiomatic/algebraic QFT, which provides a result which can be seen as a vacuum version of the Malament type theorems and which shall be presented in the following section. This result is a direct consequence of the famous Reeh-Schlieder theorem and asserts, that even if the initial state is taken to be the vacuum of a QFT consistent with some very general assumptions (like the Wightman axioms or the postulates of the Haag-Kastler approach), the probability that a local measuring device is triggered by it cannot be perfectly zero but is always finite.

If detectors have non zero click probability in the vacuum, it is no wonder that detectors have non zero click probability even if a remote detector was just triggered by a one particle initial state. One can interpret this as a consequence of the fact that a 'detector in the vacuum' is a rather unsuitable notion if taken too literally, since a detector is not a device registering

[144] This argument involves ensembles of identically prepared systems which are usually ensembles in space and/or time, while in the present considerations t_1, t_2, Δ_1 and Δ_2 are arbitrary but fixed. The arbitrariness corresponds to translation invariance, which should be exploited to make the statistical argument physically rigorous.

[145] There are several cases which are ignored in the argument: The initial state ψ might be effectively supported in Δ_2 while the detector click at (t_1, Δ_1) was caused by a particle which was created by this detector. In this case, the final state of this detection can be expected to look locally rather like ψ than Ω in Δ_2 (and rather ψ looks like the vacuum in Δ_1). Also infinite tails are ignored, i.e. states which are effectively supported in a bounded region look like the vacuum outside that region for all practical purposes but not in a precise mathematical sense because of their tails. Taking these into account, the equality sign in (3.146) must be replaced by an approximately sign, which is enough to make the conclusion (3.147) no longer justified.

passively what there is, so to speak from outside the world, but an interacting physical part of the world which might be even triggered – in principle – by particle-antiparticle pairs which itself created from the vacuum around (this is a reading of the well known vacuum fluctuations in QFT). This makes it very comprehensible that initial states which are capable of triggering only a given finite number of remote proper detectors do not exist in a strict sense in relativistic quantum theory consistent with the spectrum condition.

3.5 Local Measurements Destroy the Vacuum: Reeh-Schlieder

The theorem of Reeh and Schlieder [277] is a central result in the context of axiomatic, respectively algebraic quantum field theory. We abbreviate such approaches collectively by AQFT[146].

3.5.1 AQFTs and the Present Framework

AQFT strives to work out and analyse a common mathematical structure which is supposed to underlie all QFTs independently of the concrete model. Detailed rigorous analysis of this structure gave rise to well worked out general frameworks and yielded a couple of strong and interesting results within these frameworks ('different frameworks' shall refer to different choices of axioms, whose choice is actually very flexible in AQFT). The probably most important motivation for approaches of this kind to QFT was to put the latter (in contrast to conventional QFT) on a mathematically well defined and transparent basis, in particular to and get rid of the infinity problems which come into play with interactions in conventional QFT. While AQFT has archived as mentioned interesting structural insights, provides a formal framework to derive very general QFT results like the spin-statistics theorem or the PCT-theorem and was very successful in rigorously deriving the (LSZ-) scattering formalism of QFT by the Haag-Ruelle scattering theory [166, 167, 282], it did not succeed in developing or finding a well defined interacting QFT[147]. To get a feeling for the status of AQFT in the realm of relativistic quantum theory, we may refer to Rudolf Haag, one of the founders of AQFT and its probably most prominent agent during the last decades, who states in his standard work *'Local Quantum Physics'* referring to the algebraic approach: *'It has given a frame and a language not a theory'* [167] p. 323.

In the center of attention of AQFT is the consideration of *local operator *algebras* $\mathscr{L}_{\mathcal{O}}$ acting on a Hilbert space $\mathcal{H}$, which are indexed by suitable[148] bounded regions of space-time

[146] AQFT often refers to only algebraic QFT in the literature, but not to axiomatic approaches of the Wightman type (see below), which we shall subsume under the same label here. Algebraic QFT was also called *local quantum physics* by Haag [167].

[147] This is not precisely true, an interacting QFT toy model in two dimensional space-time with an interaction term proportional to Φ^4 (where Φ is the field operator of a scalar field) which satisfies the axioms of AQFT has been constructed already in 1968 by Glimm and Jaffe [152] (and comparable results which have been obtained for models in three dimensional space-time later [153]). But the construction relies on a property – which interaction terms of a scalar field proportional to Φ^4 in two dimensions have – which is called *super-renormalizability* and it can we shown that in four space-time dimensions no super-renormalizable interaction terms do exist [336]. No interacting QFT model in four space-time dimensions which complies with the central AQFT axioms is known as yet.

[148] What the term *suitable* means in this context depends on the precise setting. In some settings, one considers only so called *diamonds* or *double cones*, which are the intersection of the future light cone of some point $x \in \mathcal{M}$ with the past light cone of some point $y \in \mathcal{M}$ laying in the future of x (these are obviously very Lorentz symmetric regions of space-time). In other settings relatively compact regions of space-time are considered. For the present

$\mathcal{O} \subset \mathcal{M}$ (to be precise, in case of algebraic approaches, consideration of operator algebras acting on some Hilbert space refers to considering only a special representation of a more abstract structure, see below). The elements of $\mathscr{L}_{\mathcal{O}}$ (or a subset of these elements) are to be thought of as Heisenberg operators associated with measurements or 'operations' which can be carried out in the respective regions $\mathcal{O}$, i.e. observable operators, effects and/or state transformers from the present point of view. AQFT – in particular the algebraic approach – tries to develop its structural picture of relativistic quantum theory and its predictive scope almost solely proceeding from an analysis of the algebraic properties of operators associated with local measurements, or even on more abstract mathematical objects beyond a Hilbert space framework, objects of which such operators are only a possible choice of representatives. How this might be reconciled with the present approach in the spirit of chapter 1, which decisively rejects to put the operators of the measurement formalism at the basis of physical inquiry, shall be discussed below (for another point of criticism about AQFT, namely its aspiration to be defined on arbitrarily small length scales, see [336] and the discussion of the weak additivity assumption below).

Axiomatic and Algebraic QFTs

To begin with, we take a brief survey through (part of) the basic structure and assumptions of axiomatic QFT of the Wightman type and algebraic QFT. Details which are not immediately important for the present purpose (like irreducible representations of the Poincaré group, superselection sectors, Haag's theorem etc.) are left out of this survey.

Wightman Type QFT: In *axiomatic QFT of the Wightman type* – representing the prototype axiomatic QFT – a rigorous mathematical analysis is based on an assumed Hilbert space setting (in the present language essentially a relativistic Hilbert $\mathcal{H}$ space which satisfies the spectrum condition and which additionally contains a distinguished space-time translation invariant vacuum state, see below) and an assumed associated mathematical structure which is given by field operators defined in a rigorous way: The field operators $\Phi(x)$ can be defined[149] as operator valued tempered distributions, i.e. as mappings from the Schwartz space $\mathscr{S}(\mathbb{R}^4)$ of rapidly decreasing smooth functions on space-time to (in general unbounded) operators acting on a dense domain $\mathcal{D} \subset \mathcal{H}$, the so called *smeared out field operators*. With the usual symbolism of distribution theory, field operators can be denoted by[150]

$$\Phi(f) = \int f(x)\Phi(x)d^4x \tag{3.149}$$

where f varies in the test function space $\mathscr{S}(\mathbb{R}^4)$. The foundational axioms of Wightman type QFT then are besides the assumptions on the Hilbert space setting and the existence of field

purpose arbitrary *bounded* and *open* regions $\mathcal{O} \subset \mathcal{M}$ are fine.

[149] A possibility to rigorously define the field operators $\Phi(x)$ in (3.149) at space-time points x is as sesquilinear forms $\langle\varphi|\ \Phi(x)\ \psi\rangle$ on a dense domain $\mathcal{D} \ni \varphi, \psi$ of $\mathcal{H}$ and accordingly their adjoints by $\langle\varphi|\ \Phi^\dagger(x)\ \psi\rangle = \overline{\langle\psi|\ \Phi(x)\ \varphi\rangle}$ [167].

[150] Equation (3.149) should be read as only abbreviated notation if there are more types of field operators with possibly spinor or tensor components such that f represents actually a collection of test functions (one for each pair of field and spinor/tensor indices) and the integrands in (3.149) look accordingly more complicated and might e.g. contain sums over the different field indices.

operators in the above sense, more or less varying assumptions about properties of the field operators.

A *polynomial in the (smeared) field operators* is then defined to be an operator of the form

$$A = c + \int f_1(x)\Phi(x)d^4x \; + \; \int f_2(x_1,x_2)\Phi(x_1)\Phi(x_2)d^4x_1d^4x_2 \; + \; \ldots \\ \ldots \; + \int f_2(x_1,\ldots,x_n)\Phi(x_1)\ldots\Phi(x_n)d^4x_1\ldots d^4x_n \tag{3.150}$$

with $f_k \in \mathscr{S}(\mathbb{R}^{4k})$ and $c \in \mathbb{C}$. For each open bounded region $\mathcal{O}$ of space-time one considers the set $\mathscr{L}_\mathcal{O}$ of all operators of the form (3.150) and their adjoints with test functions f_k which vanish whenever one of their arguments x_l is outside $\mathcal{O}$ (i.e. which are supported in $\mathcal{O}^k$, respectively). If $A_\mathcal{O}, B_\mathcal{O} \in \mathscr{L}_\mathcal{O}$ and $\alpha \in \mathbb{C}$ it follows obviously that $A_\mathcal{O} + B_\mathcal{O}$, $A_\mathcal{O}B_\mathcal{O}$, $\alpha A_\mathcal{O}$ and $A_\mathcal{O}^\dagger$ are contained in $\mathscr{L}_\mathcal{O}$ as well, i.e. $\mathscr{L}_\mathcal{O}$ is the operator *algebra obtained from the polynomials in the field operators and their adjoints and these algebras are accordingly called the *polynomial algebras*. It is usually suggested to think of a polynomial algebra associated with space time region $\mathcal{O}$ as a collection of operators containing physically relevant operators associated with processes 'taking place' in $\mathcal{O}$ (i.e. in the respective spatial regions at the respective times in each frame). Some common requirements (axioms) on the field operators in Wightman type QFTs translated to the (polynomial) algebras, respectively properties of the latter, will be presented and discussed below.

Algebraic Approaches: In the more abstract algebraic frameworks – the prototype *algebraic QFT* is given by the Haag-Kastler approach [169] – one starts with an abstract family (a topological net) of C^*-algebras $\mathcal{L}_\mathcal{O}$ indexed by suitable bounded subsets of space-time which is assumed to obey some primitive properties (some essential properties will be presented below) and then one can construct a Hilbert space representation with respect to some distinguished state – usually the translation invariant vacuum state – in which the C^*-algebras are represented by families $\mathscr{L}_\mathcal{O}$ of bounded operators and an associated QFT [167]. The construction of a concrete Hilbert space (vacuum) representation proceeding from only abstract C^*-algebras was developed by Gelfand, Naimark and Segal [146, 303] and is accordingly known as the GNS construction. The local algebras $\mathscr{L}_\mathcal{O}$ (representing the abstract algebras $\mathcal{L}_\mathcal{O}$ on $\mathcal{H}$) of operators acting on the Hilbert space of a given representation are $^*-$algebras of bounded operators which are closed in the weak operator topology and which contain by convention always the identity $\mathbb{1}_\mathcal{H}$ (for mathematical convenience), i.e. von Neumann algebras.

A theory of the Wightman type is in some sense more tangible. Not least, proceeding from conventional QFT we are used to express the physically relevant operators as integrals over functions of field operators and suggesting such functions to be operator valued spatial densities (like the Hamilton or charge density) which may be associated with possibly 'locally measurable quantities' if integrated locally is also standard in textbook QFT. Algebraic approaches, on the other hand, are more general[151], which is not so important for the present purpose, though. But it shall be mentioned that algebraic QFTs avoid a technical difficulty right from the start, since

[151] One can show e.g. that different, not unitarily equivalent field operators can be associated with the same abstract net of local algebras (such fields are collected in the same *Borchers class*) and that they thus give rise to the same predictions [167].

the considered algebras on Hilbert spaces are families of only *bounded* operators, the operators generated by polynomials of smeared field operators in contrast are in general unbounded, which requires a lot more technical effort. To obtain bounded operators from unbounded field operators is not a trivial task [47, 117], but possible[152], and we shall not bother with these details here. The relevant operators in the present context are effects and state transformers which are always bounded by unity. So one may have in mind in the following the intuitive picture of obtaining 'local' operators by integrating over field operators but we shall actually consider only families of bounded operators and no domains and the like which is necessary to handle unbounded operators. The technical details how these things can be made rigorous are not needed to work out the arguments of this section (for details see footnote 152 and references [47, 117]).

Apart from the question of unbounded operators, Wightman type QFTs and algebraic QFTs in a vacuum representation share the same basic structure: A relativistic Hilbert space $\mathcal{H}$ obeying the spectrum condition, a unique (up to a phase) *vacuum state* $\Omega \in \mathcal{H}$ which is invariant under space-time translations (i.e. $U(x)\,\Omega = \Omega$ for all $x \in \mathbb{R}^4$) and an association $\mathcal{M} \supset \mathcal{O} \mapsto \mathscr{L}_\mathcal{O}$ between bounded open regions $\mathcal{O}$ of space-time and *algebras $\mathscr{L}_\mathcal{O}$ of operators acting on $\mathcal{H}$ which are supposed to share some basic properties. The algebras $\mathscr{L}_\mathcal{O}$ are usually called *local algebras* and as mentioned we consider here only algebras of bounded operators which are closed in the weak operator topology and contain the identity, i.e. von Neumann algebras. The family $(\mathscr{L}_\mathcal{O})_{\mathcal{O}\subset\mathcal{M}}$ of local algebras is actually a *topological net*, which is the generalization of the notion of a sequence, where the natural numbers as the index set are generalized to arbitrary directed sets (where the directedness is defined here by the inclusions $\mathcal{O}' \subseteq \mathcal{O} \Leftrightarrow \mathcal{O}' \lhd \mathcal{O}$). The basic requirements (axioms) on the net of local algebras $(\mathscr{L}_\mathcal{O})_{\mathcal{O}\subset\mathcal{M}}$ vary among different approaches. Common and central requirements like space-time translation covariance and more specific others, which underlie the theorem of Reeh and Schlieder, are collected in definition 3.28 below and subsequently discussed.

The Reeh-Schlieder theorem then states that in such a framework one can approximate any state in $\mathcal{H}$ by hitting the vacuum state with elements of $\mathscr{L}_\mathcal{O}$ for *any* of the bounded regions $\mathcal{O}$. In other words, $\mathscr{L}_\mathcal{O}\Omega$ is dense in $\mathcal{H}$ or, to put it once more another way, the vacuum is a cyclic vector for any local algebra[153].

[152] For essentially self adjoint unbounded operators in the polynomial algebra, one can consider their spectral projections or bounded functions of the latter, analogously with real and imaginary part of general unbounded operators if these parts are essentially self adjoint. More generally one can make a polar decomposition of unbounded operators, to obtain an isometric operator and a positive self adjoint one which can be spectrally decomposed and thus reduced to (bounded) projections. By taking the weak closure of algebras generated by bounded operators derived in one of these ways, one can obtain (roughly speaking) from the unbounded smeared field operators bounded local von Neumann algebras.

[153] The Reeh-Schlieder theorem is sometimes pictured as implying that a local operation on earth is capable of creating particles behind the moon. From the perspective on the meaning of the quantum operator formalism developed in chapter 1, this is an unjustified conclusion since it appears overconfident to assume that all operators in a von Neumann algebra correspond to 'real world operations' (this will be further discussed in what follows).

Nonetheless, such long distance vacuum correlations with respect to the local operator algebras which are revealed by the Reeh-Schlieder theorem offer a nice playground for a special analysis of (formal) quantum non-locality with respect to the vacuum state, which can be perceived as a maximally entangled state (which by the way matches with the view on the vacuum in the Dirac sea picture) in this respect:

It was shown that such nonlocal vacuum correlations are at least formally very analogous to the correlations considered in connection with Bell's theorem, where now the singlet state is replaced by the vacuum and the spin operators by suitable elements of local algebras (see Redhead [274] for a very nice presentation). It was shown

In the Present Context

The present approach does not, in the first place, consider abstract operator algebras, but state transformers and associated effects in the Heisenberg picture which are associated with realistic measurement (like) processes in the sense of chapter 1, and we assume that in relativistic quantum theory these measurement (like) processes can be (at least approximately) associated with bounded regions of space-time, the regions in which they 'take place' (in the sense that they take place in the respective spatial regions at the respective times in each Lorentz frame). And as discussed in section 1.6.5, in this work it is not presupposed that e.g. all effects are associated with real world measurements (although one can implement each POVM by a formal measurement resorting to the Naimark construction and the von Neumann measurement scheme as explained in chapter 1, if interactions realizing these constructions exist is another question). Analogously we do not presuppose here that all formal state transformers (or more generally CPMs, see chapter 1) and formal observable (i.e. self adjoint) operators need to be of physical relevance.

So if we want to stick in this sense to only operators with direct physical interpretation, we might – as a first guess – consider the von Neumann algebra generated by the state transformers $\mathcal{R}_{\mathcal{O}}$ and their adjoints $\mathcal{R}_{\mathcal{O}}^{\dagger}$ associated with real world measurement (like) processes 'taking place' in a given open bounded space-time region $\mathcal{O}$ (let us for simplicity neglect in the first place non efficient and continuous measurements and unbounded observable operators). This algebra will contain all relevant (bounded) operators associated with these processes, namely the respective effects $E_{\mathcal{O}} = \mathcal{R}_{\mathcal{O}}^{\dagger}\mathcal{R}_{\mathcal{O}}$ and bounded observable operators, which are linear combinations of special effects – namely projections – with real valued coefficients. But the family (net) of von Neumann algebras obtained by repeating this with each open bounded space-time region (where obviously each algebra associated with a given region $\mathcal{O}$ is contained in each algebra associated with a larger region of which $\mathcal{O}$ is a subset) is probably too large and too small at the same time, to be an appropriate candidate to reconcile the aspiration to consider only operators with direct physical interpretation with the local algebra approach of AQFT (from which we shall derive the Reeh-Schlieder theorem and its consequences below):

It is to small, if we do not presuppose that measurement (like) processes can be associated with arbitrarily small regions of space-time, which is not clear in the first place and one might argue that it is not admissible to perform such an association with respect to a region whose spatial extension is e.g. much smaller than the Compton wave length of the measured system.

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that any pair of local algebras associated with arbitrarily far spacelike separated regions $\mathcal{O}$ and $\mathcal{O}'$, respectively, contains effects (projections) which yield maximally correlated probabilities in the following sense: For all $\varepsilon > 0$ and for any projection $P_{A_{\mathcal{O}}} \in \mathscr{L}_{\mathcal{O}}$ with $\mathbb{P}^{\Omega}(A_{\mathcal{O}}) = \langle \Omega |\, P_{A_{\mathcal{O}}}\, \Omega \rangle$ (in particular also in case $\mathbb{P}^{\Omega}(A_{\mathcal{O}}) \ll 1$) there is a projection $P_{B_{\mathcal{O}'}} \in \mathscr{L}_{\mathcal{O}'}$ with $\mathbb{P}^{\Omega}(B_{\mathcal{O}'}) = \big\langle \Omega \big|\, P_{B_{\mathcal{O}'}}\, \Omega \big\rangle \neq 0$ such that $\mathbb{P}^{\Omega}(A_{\mathcal{O}} \mid B_{\mathcal{O}'}) = \big\langle \Omega \big|\, P_{B_{\mathcal{O}'}} P_{A_{\mathcal{O}}} P_{B_{\mathcal{O}'}}\, \Omega \big\rangle > 1 - \varepsilon$ (the probability cannot be perfectly 1, since this would entail that the complementary event had probability zero which is not possible as we shall see below).

Even a CHSH-inequality was derived [317] which is always maximally violated in AQFT by probabilities generated by suitable elements of local algebras associated with arbitrary spacelike separated regions in the vacuum state.

On the other hand, due to so called cluster theorems [139], these correlations (though they are maximal) are extremely suppressed, which may be loosely illustrated by noting that for any pair $A_{\mathcal{O}}$ and $B_{\mathcal{O}'}$ as above, such that $\mathbb{P}^{\Omega}(A_{\mathcal{O}}) \ll 1$ but $\mathbb{P}^{\Omega}(A_{\mathcal{O}} \mid B_{\mathcal{O}'}) > 1 - \varepsilon$ (for some small ε), the probability $\mathbb{P}^{\Omega}(B_{\mathcal{O}'})$ (although it is actually non zero) falls of exponentially with the spacelike Lorentz distance between $\mathcal{O}$ and $\mathcal{O}'$ such rapidly, that it is effectively zero if this distance is much greater than the Compton wavelength of the considered particle type.

But the weak additivity assumption which we shall need to prove the Reeh-Schlieder theorem suggests that an association of non trivial von Neumann algebras with arbitrarily small regions is necessary. Even if such an association can be circumvented (e.g. by introducing some cutoff at small length scales below which no assumptions about associated algebras are made), we still need to assume that a given operator associated with a given process in a given region $\mathcal{O}$ can be obtained by (the weak limits of) linear combinations of products of operators which are members of the local algebras $\mathscr{L}_{\mathcal{O}_k}$ associated with any (possibly not necessarily too fine) partition $\bigcup_k \mathcal{O}_k = \mathcal{O}$ of the original region $\mathcal{O}$, which has apparently no obvious justification if we work with algebras solely generated by real world state transformers.

On the other hand, the algebras generated by state transformers are so large that they will without postulates about the physical meaning of operators always contain elements without a priori physical interpretation. For example, if A and B are non commuting observable operators in such an algebra, AB is a non self adjoint element of the latter, so it cannot be an observable operator nor an effect and if the spectrum of $|AB|$ is not bounded by 1 it cannot be a state transformer either. Thus, given A and B are associated with local measurements, AB has no such direct physical interpretation. One may consider $|AB|$ or $AB + BA$ as candidates for observable operators, but associated measurements won't be derivable from the A and B measurements in any obvious way. Similarly it was argued in chapter 2 that a measurement (one of which outcomes is) associated with an effect E^2 can in general not be derived from a given measurement procedure associated with the effect E in any obvious way, if the state transformers of the $E-$measurement are not normal operators, which is true for important classes of realistic measurements (if the state transformers are normal operators, E^2 yields the sequential probability for obtaining two times in a row the outcome associated with E if the measurement is immediately repeated). Thus, if we do not presuppose that any operator is associated with real world measurement (like) processes (maybe by taking its self adjoint part etc.), an algebra generated by real world state transformers will always contain elements without immediate physical interpretation.

Thus, if we want to investigate the physical meaning of results obtained from the algebraic structure of given sets of operators, we cannot do otherwise than to include operators which are in the first place abstract mathematical objects, not necessarily associated with physical processes (a point which is often not considered in AQFT). This is important here since it entails that any assertion about the existence of an element with a certain property of an algebra which we can prove, does without postulates about the physical meaning of all operators in the algebra not have any immediate physical interpretation, but is an abstract mathematical result in the first place. But the other way around, any assertion we can prove to hold for all elements of such an algebra (or e.g. all effects in such an algebra) is physically relevant whenever the algebra only contains elements (e.g. effects) with direct physical interpretation. The Reeh-Schlieder theorem is an assertion of the first kind (in the algebra there exists at least one element...) while an interesting corollary from it is an assertion of the second kind (all effects of an algebra must ...), such that this corollary retains physical relevance if we expect only some but not all elements of the considered algebras to have a physical meaning. The status of the primary Reeh Schlieder theorem itself is thus not clear in the context of the present approach; it might be regarded as a mere formal result without immediate physical interpretation.

Accordingly, also without a priori postulates about operators it can be useful to consider

abstract operator algebras, as long as we have reason to assume that operators with physical meaning are amongst the members of such algebras. And the algebraic structure provided by the field operators of QFT is indeed suitable, since it generates in a certain sense all operators on the Hilbert space. *Completeness (irreducibility)* of the field operators means roughly speaking that we can approximate every operator acting on the considered Hilbert space arbitrarily close by (integrals over functions of) field operators. It is a property which is satisfied for QFTs obtained by second quantization, it is in varying formulations amongst the axioms of Wightman type QFTs, and the analogue property of the net of local algebras in algebraic approaches is automatically satisfied in a vacuum representation. Often completeness is expressed by the strongly related condition that the vacuum is a cyclic vector for the union of all local algebras, which is an important assumption to derive the Reeh-Schlieder theorem and whose relation with the mentioned completeness assumption will be discussed below.

So in QFT operators are represented by (integrals of functions of) field operators, but what is usually stated and needed for the physical interpretation of the following results goes beyond that, namely that accordingly operators associated with *local* processes are obtained by *local* field operator integrals over the respective regions. One can think e.g. of the potentially measurable energy or charge content in some spatial region, whose associated operators correspond to integrals over the respective region of the Hamilton or charge density operators, which are given in terms of field operators [298, 300]. The spectral projections of such operators would then be the effects and state transformers of associated ideal measurements. More elaborate examples of modellings of effects and state transformers associated with realistic local measurements by field operators would be desirable, but are not known to me[154]. One sometimes reads as motivation to interprete polynomials in the locally smeared field operators as the building blocks of operators associated with 'local operations' in the respective regions, that a field operator $\Phi(x)$ acts at the space-time point x and therefore a smeared field $\Phi(f)$ as in equation (3.149) acts in the region in which f is supported, which is actually not true in general with respect to its action on the elements of the Hilbert space[155] (recall that local transformations of wave functions cannot be

[154]The *Unruh-De Witt detector model* [330, 328, 110] might be seen as the basis for a heuristic attempt in this direction.

[155]We may roughly illustrate this nonlocal action of local field operators by second quantization as follows: First note that in non relativistic second quantization, given an ONB $\{\varphi_k\}$ of the respective one particle Hilbert space $\mathcal{H} = \mathcal{H}^{\otimes 1}$ in position representation, we can represent an (say adjoint) field operator $\Phi^\dagger(\boldsymbol{x})$ acting on the associated Fock space $\mathcal{F} = \left(\bigoplus_{N\in\mathbb{N}_0} \mathcal{H}^{\otimes N}\right)$ (where $\mathcal{H}^{\otimes 0} = \mathbb{C}$) in the Schrödinger picture by $\Phi^\dagger(\boldsymbol{x}) = \sum_k \overline{\varphi_k(\boldsymbol{x})} a^\dagger(\varphi_k)$ (see e.g. [298]) with the creation operators $a^\dagger(\varphi_k)$ of the states φ_k (i.e. $a^\dagger(\varphi_k)$ adds to each non vanishing $N-$particle component of the the Fock space state on which it acts the state φ_k in a proper – possibly symmetrized or antisymmetrized – way to obtain an respective $N+1-$particle component of the Fock state). Consequently, if $f \in \mathcal{H}^{\otimes 1}$ we obtain the smeared field operator

$$\Phi^\dagger(f) = \int f(\boldsymbol{x})\Phi^\dagger(\boldsymbol{x})d^3x = \sum_k a^\dagger(\varphi_k) \int \overline{\varphi_k(\boldsymbol{x})} f(x) d^3x = \sum_k \langle\varphi_k|\, f\rangle\, a^\dagger(\varphi_k) = a^\dagger(f) \tag{3.151}$$

i.e. $\Phi^\dagger(f)$ is the creation operator of the state f (the last equality sign in (3.151) is easily verified). Analogously with $\Phi(\boldsymbol{x})$ and annihilation operators.

If we turn now for example to second quantized Dirac theory, the spectrum condition (whose rescue by second quantization and its meaning for the following is best visualized by the Dirac sea picture, see appendix A) makes the action of the field operators more intricate: If f is an element of the Hilbert space $\mathcal{H} = \mathcal{H}^{\otimes 1} = L^2(\mathbb{R}^3, d^3x) \otimes \mathbb{C}^4$ of the one particle Dirac equation, we can represent the associated smeared field operator by $\Phi(f) = a(P_+ f) + b^\dagger(P_- f)$, where $P_\pm$ are the projections onto the positive, respectively negative energy subspaces

consistent with the spectrum condition and see footnote 155).

But at least, the field operators couple locally to the interaction potential of the apparatus in the interaction Hamiltonian generating the unitary part of the measurement transformation. For example if we describe the interaction potential of the apparatus as an external electromagnetic vector potential A^μ which couples to a fermion system, we get the well known coupling term proportional to $: \overline{\Phi}(\boldsymbol{x})\gamma^\mu\Phi(\boldsymbol{x})A_\mu(\boldsymbol{x}) :$ (the usual symbols and there meaning like the normal ordering are assumed to be known here). Thus it makes sense to consider the field operators in the region where the macroscopic apparatus is located, since its potential will be supported there and consequently a coupling term like above is zero outside that region (possible infinite tails of such potentials or apparatus wave functions will be briefly addressed below, too). Nonetheless, this does not yet show that the associated effects and state transformers, which encode not only the unitary measurement interaction but in addition the collapse associated with particular outcomes, can depend only on the field operators whose arguments lay in the region of the measurement (note however that in the von Neumann scheme of section 1.5.2, which is of course not a QFT scheme, only the effects (the PVM), which are in case of an ideal measurement also the state transformers, appear in the measurement interaction, since the observable operator encoding the PVM couples to the conjugate momentum of the measuring device). In this regard, there seems still much work to be done.

Nonetheless, we shall adopt in the following the doctrine of QFT that operators associated with local measurements can be obtained locally from the field operators, or more generally, that there exists a net of local operator algebras as described, of which the real measurement operators (in particular effects) pertaining to local measurements in the respective regions are elements. The physical interpretation and relevance of the following results are in particular conditional on this preconception.

Assumptions

Before collecting necessary assumptions, a helpful technical remark: The *double commutant* $\mathscr{L}''$ of any set $\mathscr{L} \subseteq \mathcal{B}(\mathcal{H})$ of bounded operators acting on Hilbert space $\mathcal{H}$ is the set of all bounded operators which commute with all operators which commute with all elements of $\mathscr{L}$. Clearly $\mathscr{L} \subseteq \mathscr{L}''$ and moreover, all products and linear combinations of (products of) elements of $\mathscr{L}$ are contained in $\mathscr{L}''$ as well, i.e. the algebra generated by the elements of $\mathscr{L}$ – which is a *algebra if the adjoint of each member of $\mathscr{L}$ is a member of the latter as well – is obviously a subset of $\mathscr{L}''$. Moreover, the identity $\mathbb{1}_\mathcal{H}$ which commutes with all of $\mathcal{B}(\mathcal{H})$ is always contained in $\mathscr{L}''$. Indeed, due to von Neumann's double commutant theorem [167], if $\mathscr{L} \subseteq \mathcal{B}(\mathcal{H})$ is closed under taking adjoints and thus generates a *algebra, $\mathscr{L}''$ is exactly the weak closure of this algebra (plus possibly the identity[156], if the latter was not a member of $\mathscr{L}$) which equals the

$\mathcal{H}_\pm$ of the one particle Dirac equation, $a(g)$ is an annihilation operator of a particle in state $g \in \mathcal{H}_+$ (i.e. it roughly speaking subtracts the wave function g from each $N-$particle wave function of the Fock space state in a proper way to obtain respectively an antisymmetrized $N-1-$particle wave function) and accordingly $b^\dagger(h)$ creates an antiparticle in the state Ch where $h \in \mathcal{H}_-$ and C is the charge conjugation operator (see e.g. [321] and appendix A). Thus, since $P_+ f \in \mathcal{H}_+$ and $P_- f \in \mathcal{H}_-$ must be one particle Dirac wave functions which are spread all over space (in particular also if f had compact support, see section 3.3), the action of $\Phi(f)$ affects the Fock space state in position representation all over space.

[156]The identity operator as a member is not always (but mostly) taken as part of the definition of a von Neumann algebra. To include the identity only makes things technically simpler since it allows to identify the von Neumann algebra generated by a set of operators closed under taking adjoints with its double commutant.

strong closure of this algebra in this case. In particular the weak and strong closure of each *subalgebra of $\mathcal{B}(\mathcal{H})$ coincide. Thus we can always write the von Neumann algebra generated by a given set of bounded operators $\mathscr{L} \subseteq \mathcal{B}(\mathcal{H})$ which is closed under taking adjoints as its double commutant $\mathscr{L}''$.

Next we just state the precise setting of assumptions needed to prove the Reeh-Schlieder theorem and afterwards briefly discuss their physical an mathematical background.

Definition 3.28 [*Notions & Assumptions Underlying the Reeh-Schlieder Theorem*]

Consider a relativistic Hilbert space $\mathcal{H}$ which satisfies the spectrum condition and which contains a unique (up to a phase) state $\Omega \in \mathcal{H}$ – the vacuum state – which is invariant under the Poincaré group (in particular $U(x)\Omega = \Omega$ for all $x \in \mathbb{R}^4$).

We assume that $\mathcal{H}$ carries a mathematical structure $(\mathscr{L}_\mathcal{O})_{\mathcal{O}\subset\mathcal{M}}$ which is given by families $\mathscr{L}_\mathcal{O}$ of bounded operators acting on $\mathcal{H}$, indexed by bounded open regions $\mathcal{O} \subset \mathcal{M}$ of space-time, such that each family $\mathscr{L}_\mathcal{O}$ forms a *algebra containing the identity which is closed in the weak operator topology, i.e. a von Neumann algebra. The structure $(\mathscr{L}_\mathcal{O})_{\mathcal{O}\subset\mathcal{M}}$ is a topological net of von Neumann algebras. We call the algebras $\mathscr{L}_\mathcal{O}$ local algebras and their elements local operators.

We assume moreover that the net $(\mathscr{L}_\mathcal{O})_{\mathcal{O}\subset\mathcal{M}}$ has the following properties:

(i) $(\mathscr{L}_\mathcal{O})_{\mathcal{O}\subset\mathcal{M}}$ is space-time translation covariant, i.e. $\mathscr{L}_{\mathcal{O}+x} = U(x)\mathscr{L}_\mathcal{O}U^{-1}(x)$ for each of the regions $\mathcal{O}$ and for all $x \in \mathbb{R}^4$

(ii) $(\mathscr{L}_\mathcal{O})_{\mathcal{O}\subset\mathcal{M}}$ satisfies isotony, i.e. whenever $\mathcal{O}' \subseteq \mathcal{O}$ it follows that $\mathscr{L}_{\mathcal{O}'} \subseteq \mathscr{L}_\mathcal{O}$

(iii) The vacuum is a cyclic vector for the total algebra $\mathscr{L} := \bigcup_{\mathcal{O}\subset\mathcal{M}} \mathscr{L}_\mathcal{O}$, where the union runs over the bounded open regions $\mathcal{O}$ of space-time, i.e. $\mathscr{L}\,\Omega$ is dense in $\mathcal{H}$

(iv) $(\mathscr{L}_\mathcal{O})_{\mathcal{O}\subset\mathcal{M}}$ satisfies weak additivity: The algebra generated by the union $\bigcup_{x\in\mathbb{R}^4} \mathscr{L}_{\mathcal{O}'+x}$ of all translates of the single local algebra $\mathscr{L}_{\mathcal{O}'}$ is weakly dense in the total algebra $\mathscr{L} = \bigcup_{\mathcal{O}\subset\mathcal{M}} \mathscr{L}_\mathcal{O}$ for each open bounded region $\mathcal{O}' \subset \mathcal{M}$

The basic requirements on the (net of) local algebras vary among different approaches to AQFT. The above collected assumptions reflect the (probably) minimal structure to derive the Reeh-Schlieder theorem and are essentially common to different versions of AQFT, where some are axioms and some primary implications of other axioms, depending on the particular choice of axioms.

Space-time translation covariance and *isotony* are mathematically as well as physically transparent requirements: For the physically relevant members of the local algebras – i.e. state transformers, effects and observable operators – space-time translation covariance was already discussed and extensively used in the previous sections, and it is of course comprehensible to define also the complete abstract structure containing these operators in such a covariant way in a relativistic quantum theory (think e.g. of the transformation properties of field operators under Poincaré transformations in QFT). Isotony encodes physically the simple fact that a measure-

ment spatio-temporally located in a given region is located in a larger region as well, of which the original region is a subset. For the polynomial algebras generated by the (in general unbounded) smeared field operators in Wightman theory, isotony is obviously automatically satisfied since if the support of a test function is contained in a given region it is trivially contained in any larger region as well. The cyclicity assumptions (iii) and (iv) are less intuitive at a first glance:

Cyclicity of the vacuum: Assumption (iii) is a very central property/requirement/axiom in practically every approach to AQFT. It is strongly related with the completeness property already mentioned above: *Completeness*[157] *of the net* roughly means that the elements of the local algebras are appropriate building blocks to express any bounded operator acting on $\mathcal{H}$, possibly as a limit of a sequence. More precisely, the completeness assumption means that we can approximate every bounded operator acting on $\mathcal{H}$ arbitrarily well in the weak operator topology by members of the local algebras, i.e.

$$\left(\bigcup_{\mathcal{O}\subset\mathcal{M}} \mathscr{L}_{\mathcal{O}}\right)'' \stackrel{!}{=} \mathcal{B}(\mathcal{H}) \tag{3.152}$$

where as mentioned the double quotes indicate the double commutant and thereby the weak or equally strong closure of the total algebra $\mathscr{L} := \bigcup_{\mathcal{O}\subset\mathcal{M}} \mathscr{L}_{\mathcal{O}}$ (which is indeed an algebra due to isotony and since for any two bounded open regions $\mathcal{O}, \mathcal{O}' \subset \mathcal{M}$ there is a bounded open region $\widetilde{\mathcal{O}} \subset \mathcal{M}$ which contains both $\mathcal{O}$ and $\mathcal{O}'$). But as conventional, we shall not make use of the completeness assumption in this direct form but rather of the *cyclicity of the vacuum* for the total algebra $\mathscr{L}$, which is an almost trivial consequence of completeness (and indeed for different reasons in different AQFT settings equivalent to completeness, which is not so trivial though, see below):

$$\overline{\bigcup_{\mathcal{O}\subset\mathcal{M}} \mathscr{L}_{\mathcal{O}}\,\Omega} \stackrel{!}{=} \mathcal{H} \tag{3.153}$$

where the bar indicates the usual closure in the norm topology of $\mathcal{H}$. Indeed, it is very easy to see that *any* state in a Hilbert space $\mathcal{H}$ is cyclic for any weakly dense *subalgebra of $\mathcal{B}(\mathcal{H})$:

Lemma 3.29

Any state in a Hilbert space $\mathcal{H}$ is a cyclic vector for any *algebra $\mathscr{L}$ of bounded operators which is weakly dense in $\mathcal{B}(\mathcal{H})$.

Proof: Let $\Omega \in \mathcal{H}$ be (without loss of generality) normalized to 1 (Ω is here arbitrary, not necessarily the vacuum). Of course Ω is cyclic for $\mathcal{B}(\mathcal{H})$, since $\mathcal{B}(\mathcal{H})\Omega$ is not only dense

[157] *Completeness* defined this way is equally one possible definition of *irreducibility*, if the net of operator algebras is perceived as a Hilbert space representation of a more abstract algebraic structure, which makes *Schur's lemma* applicable as a strong technical tool for further analysis of the net. But we shall not enter representation theory of algebras here, which is not necessary do derive the following results. Thus, 'completeness' is a more appropriate notion for the present purpose, since it is immediately understandable and reasonable as an assumption in the Hilbert space framework: All (here bounded) operators acting on $\mathcal{H}$ should be expressible by means of local operators.

in – but actually equal to $\mathcal{H}$: For any $\psi \in \mathcal{H}$ e.g. the one dimensional partial isometrie $A = |\psi\rangle\langle\Omega| \in \mathcal{B}(\mathcal{H})$ obviously satisfies $A\Omega = \psi$. By assumption $\mathscr{L}'' = \mathcal{B}(\mathcal{H})$ and thus there is a sequence of operators $(A_n)_{n\in\mathbb{N}} \subset \mathscr{L}$ which converges weakly to A. By von Neumann's double commutant theorem A is a strong limit of this sequence as well. Thus there is a sequence $(\psi_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ defined by $\psi_n = A_n\Omega$ for each n such that $\|\psi - \psi_n\| = \|(A - A_n)\,\Omega\| \to 0$ and since $\psi \in \mathcal{H}$ was arbitrary it follows that $\overline{\mathscr{L}\,\Omega}{=}\mathcal{H}$.

■

Thus the assumption that the vacuum is cyclic for the total algebra $\mathscr{L}$ – which we will need to prove the Reeh-Schlieder theorem – can be perceived as a necessary condition for the comprehensible requirement that the net of local algebras shall generate all of $\mathcal{B}(\mathcal{H})$ in the above described sense.

The converse statement of lemma 3.29 is in general not true. It is easy to find counterexamples and indeed, as mentioned, the Reeh-Schlieder theorem surprisingly asserts that the vacuum is even cyclic for each of the local algebras $\mathscr{L}_\mathcal{O}$, but as may be conceived, each $\mathscr{L}_\mathcal{O}$ is usually not weakly dense in $\mathcal{B}(\mathcal{H})$. But for the total algebra and the vacuum, the converse statement is indeed true, such that the cylicity of the vacuum is usually stated as a condition which can be regarded as encoding completeness. For the polynomial field algebras of a Wightman type theory, the analogue statement is e.g. shown in [313] p. 141. For a Haag-Kastler type AQFT in a vacuum representation, the cyclicity of the vacuum is a direct consequence of the GNS-construction and the completeness (irreducibility) of the total algebra another strongly related direct consequence (relying additionally on the requirement that the cyclic vacuum – upon which the GNS-construction is build – is a pure state).

It shall be mentioned, that the cyclicity of the vacuum, respectively completeness of the net is often supplemented by an analogue physically motivated, tightened requirement, the *primitive causality* – or *time slice axiom* [167, 170] One should expect that we do not need all open bounded regions of space-time as index regions of the local algebras to approximate all of $\mathcal{B}(\mathcal{H})$, an arbitrarily small neighbourhood $\mathcal{N}(\Sigma)$ of any Cauchy surface ('time slice') Σ should suffice, i.e. $\left(\bigcup_{\mathcal{O}\subset\mathcal{N}(\Sigma)} \mathscr{L}_\mathcal{O}\right)'' \stackrel{!}{=} \mathcal{B}(\mathcal{H})$, where the union runs over all bounded open subsets of $\mathcal{N}(\Sigma)$. If we think e.g. of the smeared field operators of a Wightman theory, one might wonder why a 'smearing in time' to obtain proper operators is required at all, i.e. the operators on Hilbert space should be well defined at a given time in a given frame and accordingly derivable from the field operators at a given time. That at least a small timelike extension of the support of the test functions is necessary is usually motivated by renormalization theory [80] and stated to be a rather technical issue to avoid troubling singularities. One might also argue that each physical process with which a given operator is associated takes some maybe small but always finite period of time.

For a nice analysis focussing on the time slice axiom and its relation to the dynamical aspects of the QFT (in particular the aspect that an underlying relativistic wave equation should determine the fields at any time given a complete set of Cauchy data), see [170].

Weak additivity: Assumption (iv) is the only requirement which is not always among the very basic axioms/properties of the different AQFT settings. For the polynomial algebras of Wightman theory it is automatically fulfilled. In algebraic QFTs in vacuum representation it is

sometimes directly added as an axiom when needed. In other formulations *additivity of the local algebras* is taken as an additional axiom, from which together with isotony, weak additivity can be straightforwardly obtained as a result [175]. Additivity states that any local algebra $\mathscr{L}_{\mathcal{O}}$ can be generated by the local algebras associated with an arbitrary partitioning $\{\mathcal{O}_k\}$ of $\mathcal{O}$, i.e.

$$\mathcal{O} = \bigcup_k \mathcal{O}_k \quad \overset{!}{\Longrightarrow} \quad \mathscr{L}_{\mathcal{O}} = \left(\bigcup_k \mathscr{L}_{\mathcal{O}_k} \right)'' \tag{3.154}$$

This requirement is motivated by the algebras obtained from field operators and one can argue that, although it appears to be a natural requirement, it (and analogously weak additivity) might be problematic. It means in a sense that the structure contained in a given local algebra is already contained in algebras associated with arbitrarily small regions, e.g. that we can identify a non trivial operator algebra with a region, say of spatial diameter $10^{-80}cm$ (a legitimate length scale in AQFT), and that we can generate an algebra containing the operators associated with some usual experiment by many of such algebras. But the UV-problem of interacting QFTs may be interpreted as suggesting that the aspiration to define the mathematical structure of QFT on arbitrarily small length scales may be inconsistent with a well defined interacting theory (at least in four space-time dimensions). There is a very controversial discussion about this issue going on, displaying in particular fundamental criticisms on AQFT approaches in general (see [336] and references therein).

But for the present purpose, we do not need to associate non trivial algebras with arbitrarily small regions, validity of weak additivity for regions associated with real world measurement (like) processes is sufficient[158]. Given such a region $\mathcal{O}'$, weak additivity (not necessarily assumed to hold for arbitrarily small regions) together with the completeness assumption discussed above, means

$$\left(\bigcup_{x \in \mathbb{R}^4} \mathscr{L}_{\mathcal{O}'+x} \right)'' = \left(\bigcup_{\mathcal{O} \subset \mathcal{M}} \mathscr{L}_{\mathcal{O}} \right)'' = \mathcal{B}(\mathcal{H}) \tag{3.155}$$

(but as mentioned, we shall make use only of the first equality in (3.155) together with the cyclicity of the vacuum for the total algebra, but not directly of the second one). Since this is motivated by the structure provided by field operators which work very well – at least for not too small length scales – it seems legitimate to base physical analysis on this assumption.

3.5.2 The Theorem and Implications

Now we prove the Reeh-Schlieder theorem, which (as explained above) we may perceive as a rather formal result in the first place, if we do not proceed from operator postulates in quantum theory.

Theorem 3.30 [*Reeh-Schlieder*]

Let $\mathcal{H}$, Ω and $(\mathscr{L}_{\mathcal{O}})_{\mathcal{O} \subset \mathcal{M}}$ be as in definition 3.28. Then the vacuum Ω is cyclic for each of the local algebras, i.e. $\mathscr{L}_{\mathcal{O}}\,\Omega$ is dense in $\mathcal{H}$ for each $\mathcal{O}$.

[158] Naively, one might for example think of a cutoff at some small length scale, below which it is not assumed that non trivial algebras can be associated with respective regions which generate the larger algebras etc.

Proof: Consider a given local algebra $\mathscr{L}_{\mathcal{O}}$ for some arbitrary but fixed bounded open region $\mathcal{O} \subset \mathcal{M}$. We shall start with two preliminary observations:

(**1**) Since $\mathcal{O}$ is open we can find a proper open subset $\mathcal{O}' \subset \mathcal{O}$ and a neighbourhood $\mathcal{N}(0)$ of the origin $0 \in \mathbb{R}^4$ such that $\mathcal{O}' + x \subset \mathcal{O}$ for all $x \in \mathcal{N}(0)$. Isotony thus entails $\mathscr{L}_{\mathcal{O}'} \subset \mathscr{L}_{\mathcal{O}}$ which together with translation covariance yields $A_{\mathcal{O}'+x} = U(x)A_{\mathcal{O}'}U^{-1}(x) \in \mathscr{L}_{\mathcal{O}}$ for all $A_{\mathcal{O}'} \in \mathscr{L}_{\mathcal{O}'}$ and $x \in \mathcal{N}(0)$. Moreover, for each $N \in \mathbb{N}$ and $A^{(1)}_{\mathcal{O}'}, \ldots, A^{(N)}_{\mathcal{O}'} \in \mathscr{L}_{\mathcal{O}'}$ (which are thereby also elements of $\mathscr{L}_{\mathcal{O}}$), the product

$$A^{(1)}_{\mathcal{O}'+x_1} \cdots A^{(N)}_{\mathcal{O}'+x_N} = U(x_1)A^{(1)}_{\mathcal{O}'}U^{-1}(x_1) \cdots U(x_N)A^{(N)}_{\mathcal{O}'}U^{-1}(x_N) \in \mathscr{L}_{\mathcal{O}} \tag{3.156}$$

for all $x_1, \ldots, x_N \in \mathcal{N}(0)$ since $\mathscr{L}_{\mathcal{O}}$ is an algebra.

(**2**) Since $\mathscr{L}_{\mathcal{O}}\,\Omega \subseteq \mathcal{H}$ is a subspace (since $\mathscr{L}_{\mathcal{O}}$ is an algebra), its closure can be obtained by 'taking twice' its orthogonal complement[159] $\overline{\mathscr{L}_{\mathcal{O}}\,\Omega} = (\mathscr{L}_{\mathcal{O}}\,\Omega)^{\perp\perp}$ and consequently the assumption that $\mathscr{L}_{\mathcal{O}}\,\Omega$ is dense in $\mathcal{H}$ (which we want to prove) is equivalent to

$$(\mathscr{L}_{\mathcal{O}}\,\Omega)^{\perp\perp} = \mathcal{H} \tag{3.157}$$

Noting that trivially $(\mathscr{L}_{\mathcal{O}}\,\Omega)^{\perp\perp\perp} = (\mathscr{L}_{\mathcal{O}}\,\Omega)^{\perp}$ and $\mathcal{H}^{\perp\perp} = \mathcal{H}$, equation (3.157) in turn is equivalent to $(\mathscr{L}_{\mathcal{O}}\,\Omega)^{\perp} = \mathcal{H}^{\perp}$ and thus together with $\mathcal{H}^{\perp} = \{0\}$ we finally get that $\mathscr{L}_{\mathcal{O}}\,\Omega$ *being dense in* $\mathcal{H}$ *is equivalent to* $(\mathscr{L}_{\mathcal{O}}\,\Omega)^{\perp} = \{0\}$.

Now comes the actual proof: Suppose the vacuum is not cyclic for $\mathscr{L}_{\mathcal{O}}$, i.e. $\mathscr{L}_{\mathcal{O}}\,\Omega$ is not dense in $\mathcal{H}$ such that with the previous observation (**2**) we conclude $(\mathscr{L}_{\mathcal{O}}\,\Omega)^{\perp} \neq \{0\}$ and thus there exists some non zero $\psi \in (\mathscr{L}_{\mathcal{O}}\,\Omega)^{\perp}$, i.e. $\langle\psi|\, A_{\mathcal{O}}\,\Omega\rangle = 0$ for all $A_{\mathcal{O}} \in \mathscr{L}_{\mathcal{O}}$. In consequence, with $\mathcal{O}' \subset \mathcal{O}$ and $A^{(1)}_{\mathcal{O}'}, \ldots, A^{(N)}_{\mathcal{O}'} \in \mathscr{L}_{\mathcal{O}'} \subset \mathscr{L}_{\mathcal{O}}$ as in observation (**1**) above, we obtain

$$\Big\langle \psi \Big|\, A^{(1)}_{\mathcal{O}'+x_1} \cdots A^{(N)}_{\mathcal{O}'+x_N}\,\Omega \Big\rangle = \Big\langle \psi \Big|\, U(x_1)A^{(1)}_{\mathcal{O}'}U^{-1}(x_1) \cdots U(x_N)A^{(N)}_{\mathcal{O}'}\,\Omega \Big\rangle = 0 \tag{3.158}$$

for all $x_1, \ldots, x_N \in \mathcal{N}(0)$, where we have used the translation invariance of the vacuum by setting $U^{-1}(x_N)\,\Omega = \Omega$. The spectrum condition together with theorem 3.11 then entails that

$$\Big\langle \psi \Big|\, A^{(1)}_{\mathcal{O}'+x_1} \cdots A^{(N)}_{\mathcal{O}'+x_N}\,\Omega \Big\rangle = 0 \tag{3.159}$$

for all $x_1, \ldots, x_N \in \mathbb{R}^4$, i.e. ψ is in the orthogonal complement of $A^{(1)}_{\mathcal{O}'+x_1} \cdots A^{(N)}_{\mathcal{O}'+x_N}\,\Omega$ for all $x_1, \ldots, x_N \in \mathbb{R}^4$.

Now observe that the algebra generated by the elements of $\bigcup_{x\in\mathbb{R}^4} \mathscr{L}_{\mathcal{O}'+x}$ consists of linear combinations[160] of operators of the form $A^{(1)}_{\mathcal{O}'+x_1} \cdots A^{(N)}_{\mathcal{O}'+x_N}$ with $x_1, \ldots, x_N \in \mathbb{R}^4$. Since the

[159] Here it assumed to be known (as easily verified or looked up in any functional analysis textbook) that orthogonal complements are always closed and that for any subset $X \subseteq \mathcal{H}$ the closure of its linear span can be obtained taking the orthogonal complement of its orthogonal complement: $\overline{\operatorname{span} X} = X^{\perp\perp}$. This of course implies that the orthogonal complement operation acts as an involution on closed subspaces $X = X^{\perp\perp}$.

[160] Note in particular that $\bigcup_{x\in\mathbb{R}^4} \mathscr{L}_{\mathcal{O}'+x}$ itself is not an algebra, since for example with $x_1, x_2 \in \mathbb{R}^4$ such that $(\mathcal{O}' + x_1) \cap (\mathcal{O}' + x_2) = \emptyset$ and $A^{(1)}_{\mathcal{O}'+x_1} \in \mathscr{L}_{\mathcal{O}'+x_1}$ and $A^{(2)}_{\mathcal{O}'+x_2} \in \mathscr{L}_{\mathcal{O}'+x_2}$ it follows that $A^{(1)}_{\mathcal{O}'+x_1}A^{(2)}_{\mathcal{O}'+x_2} \notin \bigcup_{x\in\mathbb{R}^4} \mathscr{L}_{\mathcal{O}'+x}$. This is why we need to consider not only elements of $\bigcup_{x\in\mathbb{R}^4} \mathscr{L}_{\mathcal{O}'+x}$ but operators of the form $A^{(1)}_{\mathcal{O}'+x_1} \cdots A^{(N)}_{\mathcal{O}'+x_N}$, linear combinations of which constitute the algebra generated by the elements of $\bigcup_{x\in\mathbb{R}^4} \mathscr{L}_{\mathcal{O}'+x}$.

matrix elements of these operators with respect to ψ and Ω are according to (3.159) always zero, each operator which is a weak limit of sequences of such operators has correspondingly a vanishing matrix element with respect to ψ and Ω as well. Thus, all states in $\mathcal{H}$ which come about when an element of the weak closure of the algebra generated by $\bigcup_{x\in\mathbb{R}^4}\mathscr{L}_{\mathcal{O}'+x}$ – which is given by its double commutant $\left(\bigcup_{x\in\mathbb{R}^4}\mathscr{L}_{\mathcal{O}'+x}\right)''$ – acts on Ω are orthogonal to ψ, which we may denote symbolically by

$$\left\langle \psi \middle| \left(\bigcup_{x\in\mathbb{R}^4} \mathscr{L}_{\mathcal{O}'+x} \right)'' \Omega \right\rangle = 0 \tag{3.160}$$

On the other hand, according to the weak additivity assumption (iv), the algebra generated by the elements of $\bigcup_{x\in\mathbb{R}^4}\mathscr{L}_{\mathcal{O}'+x}$ is weakly dense in the total algebra $\bigcup_{\mathcal{O}\subset\mathcal{M}}\mathscr{L}_{\mathcal{O}}$, such that the weak closures of both algebras coincide:

$$\left(\bigcup_{\mathcal{O}\subset\mathcal{M}} \mathscr{L}_{\mathcal{O}} \right)'' \equiv \left(\bigcup_{x\in\mathbb{R}^4} \mathscr{L}_{\mathcal{O}'+x} \right)'' \tag{3.161}$$

But for the total algebra, the vacuum is a cyclic vector by assumption (iii) and since the total algebra is a subset of its weak closure, Ω is cyclic for its weak closure as well, such that

$$\{0\} = \left(\left(\bigcup_{\mathcal{O}\subset\mathcal{M}} \mathscr{L}_{\mathcal{O}} \right)'' \Omega \right)^{\perp} = \left(\left(\bigcup_{x\in\mathbb{R}^4} \mathscr{L}_{\mathcal{O}'+x} \right)'' \Omega \right)^{\perp} \tag{3.162}$$

But this is obviously in contradiction with (3.160) which says that $\psi \neq 0$ is in the orthogonal complement of $\left(\bigcup_{x\in\mathbb{R}^4}\mathscr{L}_{\mathcal{O}'+x}\right)''\Omega$. Since the existence of $\psi \neq 0$ which satisfies (3.160) was a consequence of the assumption that ψ is in the orthogonal complement of $\mathscr{L}_{\mathcal{O}}\,\Omega$, it follows that $(\mathscr{L}_{\mathcal{O}}\,\Omega)^{\perp} = \{0\}$ such that $\overline{(\mathscr{L}_{\mathcal{O}}\,\Omega)} = (\mathscr{L}_{\mathcal{O}}\,\Omega)^{\perp\perp} = \{0\}^{\perp} = \mathcal{H}$, i.e. $\mathscr{L}_{\mathcal{O}}\,\Omega$ is dense in $\mathcal{H}$ which is to say the vacuum is cyclic for $\mathscr{L}_{\mathcal{O}}\,\Omega$.

■

Now we come to the announced physically interesting consequence of the Reeh-Schlieder theorem 3.30, which now easily follows, given the net of local algebras is in accordance with another general assumption of AQFTs:

Local Commutativity: Henceforth, we shall assume additionally to the assumptions collected in definition 3.28 that all elements of each local algebra $\mathscr{L}_{\mathcal{O}}$ commute with all elements of local algebras indexed by regions at spacelike separation. Given we suppose that effects and state transformers associated with local real world measurements are always contained in respective local algebras, this assumption is still stronger than local commutativity as defined and analysed in section 2, where it was only required that state transformers of all (real world) measurement (like) processes commute with all effects of measurements at spacelike separation.

But commutativity of the local algebras at spacelike separation is the way local commutativity is usually required in AQFT and as an abstract requirement on the net $(\mathscr{L}_{\mathcal{O}})_{\mathcal{O}\subset\mathcal{M}}$ it is in strong correspondence with the well known (anti-)commutativity requirements on field operators in QFT (and thereby with some of the main results of QFT like the celebrated *spin-statistics*

theorem): The polynomial algebras generated by smeared Bose fields commute at spacelike separation, each polynomial algebra of anitcommuting Fermi fields contains a subalgebra of even products of smeared field operators and these subalgebras commute at spacelike separation as well. It is usually assumed that these subalgebras are the 'observable' ones (while smeared Fermi fields themselves are regarded as 'unobservable'), and we may assume that the effects and state transformers associated with physical measurement (like) processes of systems described by Fermi fields can be obtained from such polynomial subalgebras and thereby commute at spacelike separation as well.

With this additional requirement we can turn now the Reeh-Schlieder theorem as an assertion about abstract algebras into a strong conclusion about each single local measurement (like) processes performed on the vacuum state, given the related effects are contained in local algebras. The following corollary may be considered as a vacuum version of the Malament type theorems presented above: Not only initial states, which are capable of triggering at most one (or N) of several detectors at spacelike separation do not exist, even in the vacuum each local detector has non vanishing click probability if the associated effects are elements of local algebras which are in accord with the assumptions in definition 3.28 and the commutativity requirement discussed in the foregoing paragraphs[161] .

Corollary 3.31 [*Reeh-Schlieder*]

Consider $\mathcal{H}$ and $(\mathscr{L}_{\mathcal{O}})_{\mathcal{O}\subset\mathcal{M}}$ as in theorem 3.30 and suppose the elements of two local algebras with spacelike separated index regions always mutually commute. If $0 \neq E_\alpha$ is an effect acting on $\mathcal{H}$ associated with some outcome α of a measurement (like) process and $E_\alpha \in \mathscr{L}_{\mathcal{O}}$ for any region $\mathcal{O}$, the associated probability cannot be zero in the vacuum:

$$\mathbb{P}^{\Omega}(\alpha) = \langle\Omega|\, E_\alpha\, \Omega\rangle > 0 \tag{3.163}$$

Thus if all effects associated with local measurements are elements of local algebras, each outcome of a local measurement has non zero probability in the vacuum.

Proof: Choose some open bounded region $\mathcal{O}'$ which is spacelike with respect to $\mathcal{O}$ and consider the associated local algebra $\mathscr{L}_{\mathcal{O}'}$. By assumption, the effect $E_\alpha \in \mathscr{L}_{\mathcal{O}}$ commutes with all elements of $\mathscr{L}_{\mathcal{O}'}$, i.e. E_α is an element of the commutant $(\mathscr{L}_{\mathcal{O}'})'$ of $\mathscr{L}_{\mathcal{O}'}$.

Now suppose $\langle\Omega|\, E_\alpha\, \Omega\rangle = 0$. Since E_α is an effect and thereby a positive operator, its square root $\sqrt{E_\alpha}$ exists and thus

$$0 = \langle\Omega|\, E_\alpha\, \Omega\rangle = \left\langle \sqrt{E_\alpha}\Omega \middle|\, \sqrt{E_\alpha}\,\Omega \right\rangle = \left\| \sqrt{E_\alpha}\Omega \right\|^2 \tag{3.164}$$

i.e. $\sqrt{E_\alpha}\Omega = 0$ and – multiplied by $\sqrt{E_\alpha}$ from the left – we get $E_\alpha\Omega = 0$, which together with

[161] To be precise, for the following result it would be basically sufficient to require that all *effects associated with measurements*, contained in local algebras, commute with *all elements* of local algebras indexed by spacelike separated regions.

the (local) commutativity assumption entails

$$E_\alpha A_{\mathcal{O}'}\,\Omega = A_{\mathcal{O}'} E_\alpha\,\Omega = 0 \tag{3.165}$$

for all $A_{\mathcal{O}'} \in \mathscr{L}_{\mathcal{O}'}$, or symbolically

$$E_\alpha \mathscr{L}_{\mathcal{O}'}\,\Omega = 0 \tag{3.166}$$

But by theorem 3.30 $\mathscr{L}_{\mathcal{O}'}\,\Omega \subseteq \mathcal{H}$ is dense in $\mathcal{H}$, i.e. E_α yields zero by its action on a dense subset of $\mathcal{H}$. But this implies that $E_\alpha = 0$ on all of $\mathcal{H}$: Since $\mathscr{L}_{\mathcal{O}'}\,\Omega$ is dense in $\mathcal{H}$, for each $\varphi \in \mathcal{H} \setminus \mathscr{L}_{\mathcal{O}'}\,\Omega$ there is a sequence $(\psi_n)_{n\in\mathbb{N}} \subset \mathscr{L}_{\mathcal{O}'}\,\Omega$ with $\psi_n \overset{n\to\infty}{\to} \varphi$ in the Hilbert space norm. And since $E_\alpha\,\psi_n = 0$ for all $n \in \mathbb{N}$ we have

$$\|E_\alpha\,\varphi\| = \|E_\alpha\,(\varphi - \psi_n)\| \leq \|E_\alpha\|_{\mathrm{Op}}\,\|\varphi - \psi_n\| \overset{n\to\infty}{\to} 0 \tag{3.167}$$

where the operator norm $\|E_\alpha\|_{\mathrm{Op}}$ of E_α is bounded by 1 from above since E_α is an effect. In consequence, also $E_\alpha\,\varphi = 0$, i.e. together with $E_\alpha \mathscr{L}_{\mathcal{O}'}\,\Omega = 0$ we have $E_\alpha = 0$ on all of $\mathcal{H}$ which obviously contradicts the assumption $E_\alpha \neq 0$. Thus, $\langle\Omega|\,E_\alpha\,\Omega\rangle$ cannot be zero but must be positive such that E_α cannot yield zero probability in the vacuum.

■

Substituting $E_\alpha \in \mathscr{L}_{\mathcal{O}}$ in the proof of corollary 3.31 with an arbitrary element $B_{\mathcal{O}} \in \mathscr{L}_{\mathcal{O}}$ (in particular, dropping the positivity assumption), the central argument in the proof – which starts with equation (3.165) and does no longer make use of the positivity of E_α – shows that indeed $B_{\mathcal{O}}\,\Omega = 0$ entails $B_{\mathcal{O}} = 0$ for any element $B_{\mathcal{O}}$ of some local algebra $\mathscr{L}_{\mathcal{O}}$. This means mathematically that local algebras are *separating* for the vacuum Ω (this property is sometimes phrased as 'local operators cannot annihilate the vacuum'). Thus the Reeh-Schlieder theorem 3.30 together with this essential result in the proof of corollary 3.31 can be compactly expressed as: *The vacuum Ω is cyclic and separating for any local algebra $\mathscr{L}_{\mathcal{O}}$.*

Remarks on Local Number Operators

Another nice variation of these results, which aims in particular at pointing out some difficulties in defining a 'local number operator', was given by Redhead [273]:

Corollary 3.32 [*Redhead*]

Consider $\mathcal{H}$ and $(\mathscr{L}_{\mathcal{O}})_{\mathcal{O}\subset\mathcal{M}}$ as in theorem 3.30 and suppose the elements of two local algebras with spacelike separated index regions always mutually commute. Then any (non trivial) orthogonal projection P onto a state or subspace which is parallel or orthogonal to the vacuum cannot be an element of any local algebra.

Proof: The proof of corollary 3.31 shows that given $P \in \mathscr{L}_{\mathcal{O}}$ for any region $\mathcal{O}$, orthogonality to the vacuum $P\Omega = 0$ entails that $P = 0$. If on the other hand $P \in \mathscr{L}_{\mathcal{O}}$ and $P\Omega = \Omega$ it follows firstly that the complementary projection $\mathbb{1}_{\mathcal{H}} - P \in \mathscr{L}_{\mathcal{O}}$ since $\mathscr{L}_{\mathcal{O}}$ is an algebra containing the identity and secondly $(\mathbb{1}_{\mathcal{H}} - P)\Omega = 0$ which as above entails that $\mathbb{1}_{\mathcal{H}} - P = 0$ and thus $P = \mathbb{1}_{\mathcal{H}}$.

■

In particular, if $\mathcal{H}$ is Fock space and if effects and/or state transformers associated with local measurements are elements of local algebras, the projection onto the $N-$particle sector P_N for any $N \in \mathbb{N}_0$ – which projects onto a subspace orthogonal to the vacuum for any $N \neq 0$ and parallel to it for $N = 0$ – cannot be the effect and/or state transformer associated with any local measurement. Moreover, if an observable operator is an element of a local algebra, it can have neither the vacuum nor any state orthogonal to the vacuum as an eigenstate. Corollary 3.32 thus for example entails, that the (unbounded) *number operator*

$$\widehat{N} := \sum_{N=0}^{\infty} N\, P_N \tag{3.168}$$

is not an observable operator which can be associated with any local measurement, given effects (here in particular projections) associated with local measurements are elements of local algebras. This is not so surprising, since (3.168) is rather the spectral representation of a global number operator and not a local one, i.e. roughly speaking a local measurement does never inform us about the actual sector of Fock space we are in since additional particles may always be at another location.

But indeed, the problem with local number operators goes deeper. For example, the standard global QFT number operators of the form $N = \int \Phi^\dagger(\boldsymbol{x})\Phi(\boldsymbol{x})d^3x$ (in the Schrödinger picture) do not provide satisfactory local number operators $N_\Delta = \int_\Delta \Phi^\dagger(\boldsymbol{x})\Phi(\boldsymbol{x})d^3x$ since these violate local commutativity (see e.g. [273, 189]). Halvorson and Clifton [174] also proved a no-go theorem about local number operators based on theorem 3.25. But as the number operators N_Δ above correspond as (potential) observable operators to PVMs, this theorem covers only the projective case[162].

But there is another interesting point in the discussion of these things by Redhead and Halvorson/Clifton: The local number operator no-go theorem of Halvorson and Clifton is based (besides on local commutativity) on the assumption of global particle number conservation. Halvorson and Clifton argue that global number conservation is satisfied in a free field theory such that satisfactory local number operators cannot even be consistently defined in the absence of interactions. Also Redhead [273] states that arguments which make local fluctuations of the particle number responsible for the problems with local number operators like N_Δ were confused because one considers interaction free QFTs in the first place, which do not include 'virtual particles'.

This is one of the situations, where it is important to clarify the precise physical meaning of the considered operators: If only a free system in absence of any interactions is treated, there cannot be measurements either, with which the considered operators can be associated. The operators associated with measurements derived in chapter 1 in particular encoded a unitary interaction between the measured system and the measuring device (together with the collapse), which is reflected in the transformation of the (free) initial states into the respective (free) final states[163] (modulo normalization) by the state transformers $\mathcal{R}_N(\Delta)$, e.g. corresponding here to

[162] Halforson and Clifton claim that their result also includes unsharp observables, i.e. non projective POVMs, but making the spectrum of the observable operators N_Δ continuous such that it does not only include sharp integer numbers, as they do, has of course nothing to do with a non projective measurement which does not admit any reasonable observable operator but only a POVM.

[163] We disregard here serious problems with consistently connecting the Hilbert space of the (asymptotically)

N particles found in the spatial region Δ. So if we have a POVM $\{E_N(\Delta)\}$ which yields for each state a probability distribution and interactions are not considered, there is no decomposition of the POVM into state transformers $E_N(\Delta) = \mathcal{R}_N^\dagger(\Delta)\mathcal{R}_N(\Delta)$ (or more generally for non efficient measurements $E_N(\Delta) = \sum_k \mathcal{R}_{kN}^\dagger(\Delta)\mathcal{R}_{kN}(\Delta)$) which corresponds to a physical transformation associated with these probabilities. And without such state transformations it makes no sense at all to require local commutativity, whose physical justification (relativistic consistency, no signalling), which was carefully analysed in chapter 2, crucially relies on the requirement that associated state transformations do no lead to relativistically inconsistent phenomena.

So a POVM $\{E_N(\Delta)\}$ violating local commutativity may well yield for example a probability distribution of what there is about the considered system devoid of invasion of external measuring devices without creating relativistic oddities (similar to the positive energy position POVM of the Dirac equation discussed at the beginning of this chapter). The statistics of associated measurement results ('pointer positions') in contrast should and need not be encoded in such a POVM, but in a POVM which encodes probabilities for joint transitions of the measured system and the measuring device initiated by interaction of these two systems[164].

Malament-Like Generalization

Finally, we shall have a closer look at the affinity between the Reeh-Schlieder theorem(s) and the Malament type theorems discussed above. It is clear that Malament type theorems and Reeh-Schlieder type theorems have a central part of mathematical structure in common: Part of their assumptions (albeit the Reeh-Schlieder theorem underlies with the local algebras a considerably larger amount of assumed structure), in particular the crucial spectrum condition, as well as the mathematical tools developed at the beginning of this chapter to derive these results, coincide (note that theorem 3.11, on which the Reeh-Schlieder theorem was based is just a straight forward generalization of theorem 3.10, on which Malament type theorems were

free system with the Hilbert space of the interacting system due to Haag's theorem [167, 313]. But note that QFT is very successful in describing transitions of free initial states into free final states with intermediate interaction, only consistently implementing the full time evolution is problematic (even in case of external fields) and many people are still working on finding a proper understanding and a satisfying solution for this problem (see e.g. [103, 215]).

[164] If we consider for example a *Bell type QFT* (see section 4.6.2), which is a QFT of Bohmian particles which can be created and annihilated, N_Δ as above might well describe the statistics of the actual particle number within Δ (it encodes the $|\psi|^2$ quantum equilibrium distribution, see chapter 4). But as the analysis of the present chapter demonstrates, a local number operator associated with corresponding local measurements will probably not be an observable operator but the measurement statistics will be encoded in a POVM, which should be build from a POVM associated with proper position measurements. The latter POVM will be defined on the configuration space of the Bell type QFT (which is the configuration space on which a Fock space wave function lives, namely in the simplest case of one species of distinguishable particles given by $\bigcup_{N=0}^\infty \mathbb{R}^{3N}$, where the actual configuration is always in one of the $\mathbb{R}^{3N}$s, whereas the wave function can be supported is several particle sectors at a time). Moreover, in order to be in accord with the 'no perfect localization in bounded regions' property encountered in this chapter, it must be strongly unsharp in a precise sense (i.e. the effects associated with bounded regions should not have precisely 1 in their spectrum), but it will certainly look like a sharp PVM for all practical purposes for ordinary position measurements. There is no no-go theorem excluding such a POVM (at least to my knowledge) and there is strong evidence that this is also not possible, at least on the basis on which the no-go theorems considered in this chapter were build. These results altogether only excluded initial states which are perfectly incapable of triggering more than any given finite number of remote detectors. In a Bell type QFT in which a detector may create particles from the vacuum with non vanishing probability, there is no reason to expect that such initial states exist.

based). Also from a physical point of view, there is some obvious connection: It was argued that the Malament type theorems tell us that under the respective assumptions an $N-$particle initial state of some detection experiment is not – in principle – incapable of triggering more than N detectors at spacelike separation. And the Reeh-Schlieder corollary 3.31 tells us that under some (more) assumptions including a QFT-structure, the vacuum state Ω as the initial state of a proper detection experiment is not – in principle – incapable of triggering a local detector. Thus, as already pointed out above, corollary 3.31 might be regarded as a vacuum version of the Malament type theorems.

It is even possible to get rid of the restriction to the vacuum state in the Reeh-Schlieder context, and thereby to derive a conclusion which is in essence very close to the conclusion derived from Malament type theorems. This shall be finally briefly illustrated:

The Reeh-Schlieder theorem was mathematically based on theorem 3.11 for whose application to the present framework it was crucial that in the central expression

$$\left\langle \psi \right| A^{(1)}_{\mathcal{O}'+x_1} \cdots A^{(N)}_{\mathcal{O}'+x_N} \Omega \Big\rangle = \left\langle \psi \right| U(x_1) A^{(1)}_{\mathcal{O}'} U^{-1}(x_1) \cdots U(x_N) A^{(N)}_{\mathcal{O}'} \Omega \Big\rangle \tag{3.169}$$

$U^{-1}(x_N)$ does not appear on the right hand side between $A^{(N)}_{\mathcal{O}'}$ and Ω since it is absorbed by the vacuum (translation invariance of the vacuum). Now recall that in the remarks subsequent to theorem 3.11 it was argued, that the assumptions of the theorem can be relaxed such that in the function

$$g_{\varphi\psi}(x_1, \ldots, x_N) := \left\langle \varphi \right| U(x_1) A^{(1)}_{\mathcal{O}'} U^{-1}(x_1) \cdots U(x_N) A^{(N)}_{\mathcal{O}'} U^{-1}(x_N)\, \psi \Big\rangle \tag{3.170}$$

$U^{-1}(x_N)$ need not vanish in order to obtain the same conclusion (namely that given $g_{\varphi\psi}$ vanishes locally it follows that it vanishes everywhere). It is sufficient that ψ is sufficiently moderate with respect to its energy content. More precisely it is sufficient that ψ is *analytic for the energy* (see definition 3.12). ψ was defined to be analytic for the energy iff ψ is in the domain of $(\widehat{P}_0)^n$ (where $\widehat{P}_0$ is the infinitesimal generator of time translations, i.e. a Hamilton operator) for each power $n \in \mathbb{N}$ and the series

$$\sum_n \left\| (\widehat{P}_0)^n \psi \right\| \frac{z^n}{n!} \tag{3.171}$$

has non zero radius $a > 0$ of convergence. We shall discuss the physical meaning of this notion in a moment, but first note, that together with corollary 3.13 we may exchange the vacuum Ω in the Reeh-Schlieder theorem 3.30 with some arbitrary state $\psi \in \mathcal{H}$ which is analytic for the energy and cyclic for the total algebra $\bigcup_{\mathcal{O}\subset\mathcal{M}} \mathscr{L}_{\mathcal{O}}$ to derive the same result. In particular, the central implication corollary 3.31 of the Reeh-Schlieder theorem is straightforwardly generalized to the following

Lemma 3.33

Consider $\mathcal{H}$ and $(\mathscr{L}_{\mathcal{O}})_{\mathcal{O}\subset\mathcal{M}}$ as in theorem 3.30 and suppose the local algebras $\mathscr{L}_{\mathcal{O}}$ commute at spacelike separation. If $\psi \in \mathcal{H}$ is analytic for the energy and cyclic for the total algebra $\bigcup_{\mathcal{O}\subset\mathcal{M}} \mathscr{L}_{\mathcal{O}}$ and $A_{\mathcal{O}'} \in \mathscr{L}'_{\mathcal{O}}$ for some $\mathcal{O}'$ is an effect with $\langle \psi |\, A_{\mathcal{O}'}\, \psi \rangle = 0$ it follows that $A_{\mathcal{O}'} = 0$.

Note that assuming cyclicity of ψ for the total algebra does actually not restrict generality but is according to lemma 3.29 an almost trivial consequence of the well justified completeness assumption[165], i.e. not just the vacuum but actually any $\psi \in \mathcal{H}$ is cyclic for the total algebra $\bigcup_{\mathcal{O}\subset\mathcal{M}} \mathscr{L}_{\mathcal{O}}$, given the latter is weakly dense in $\mathcal{B}(\mathcal{H})$.

A state $\psi \in \mathcal{H}$ is in particular analytic for the energy if the energy contributions in ψ are bounded from above[166] (they are bounded from below by zero due to the spectrum condition anyway), in the sense that there exists some finite spectral interval $[a, b] \subset \mathbb{R}$ such that the associated spectral projection $P_{[a,b]}$ of the energy operator $\widehat{P}_0$ satisfies $P_{[a,b]}\,\psi = \psi$. It is sometimes argued that states without finite energy content in this sense cannot be prepared by any experimental means and are therefore physically irrelevant. Anyhow, it is clear that states with bonded energy contributions form a dense subset of the Hilbert space (in the simplified picture of a discrete spectrum of $\widehat{P}_0$ with $\widehat{P}_0\varphi_k = \epsilon_k\varphi_k$ we may expand any state in the energy eigenbasis $\mathcal{H} \ni \psi = \sum_{k=0}^{\infty} \alpha_k\varphi_k$ with $|\alpha_k| \overset{k\to\infty}{\to} 0$ to see that the states $\psi_N = \sum_{k=0}^{N} \alpha_k\varphi_k$ – which are as argued in footnote 166 analytic for the energy for all $N \in \mathbb{N}$ – converge to ψ as N goes to infinity). And as we shall see now, a *single state* analytic for the energy, which is incapable of triggering more than any given finite number of detectors at spacelike separation is sufficient to contradict the existence of an associated (non trivial) detector formalism on any Hilbert space containing this state, given the effects are elements of local algebras:

Theorem 3.34 [*Reeh-Schlieder Meets Malament*]

Consider $\mathcal{H}$ and $(\mathscr{L}_{\mathcal{O}})_{\mathcal{O}\subset\mathcal{M}}$ as in theorem 3.30 and suppose the local algebras $\mathscr{L}_{\mathcal{O}}$ commute at spacelike separation. Let $\mathcal{E}^{\mathcal{H}}_{\Delta}$ be a covariant detector formalism associated with $\mathcal{H}$ and suppose the effects associated with (possibly composed) detection events are contained in the local algebras (for example $(t, \Delta) \subset \mathcal{O}$ entails $D_{(t,\Delta)} \in \mathscr{L}_{\mathcal{O}}$ and so on). Let $\mathscr{D}$ be any (possibly composed) detection event (e.g. N detectors of a given arrangement click at spacelike separation). If there is any state $\psi \in \mathcal{H}$ analytic for the energy and cyclic for the total algebra with $\mathbb{P}^{\psi}(\mathscr{D}) = 0$ it follows that $\mathbb{P}^{\varphi}(\mathscr{D}) = 0$ for all $\varphi \in \mathcal{H}$.

[165] As explained above, this completeness assumption (irreducibility) can be in turn derived from the cylicity of the vacuum for the total algebra.

[166] To illustrate this, suppose for a moment for simplicity $\widehat{P}_0$ had pure point spectrum and thereby eigenstates $\widehat{P}_0\varphi_k = \epsilon_k\varphi_k$ with $0 \leq \epsilon_k < \infty$ for all $k \in \mathbb{N}$, such that a state ψ whose energy content is bounded can be written as $\psi = \sum_{k=0}^{N} \alpha_k\varphi_k$ with $\sum_{k=0}^{N} |\alpha_k|^2 = 1$ for some finite $N \in \mathbb{N}$. Consequently $(\widehat{P}_0)^n\psi = \sum_{k=0}^{N} \epsilon_k^n\alpha_k\varphi_k$ and thus using $\langle\varphi_k|\,\varphi_l\rangle = \delta_{kl}$ we get

$$\left\|(\widehat{P}_0)^n\psi\right\|^2 \leq \sum_{k=0}^{N} \epsilon_k^n\,|\alpha_k|^2 \leq \sum_{k=0}^{N} \epsilon_k^n < \left(\sum_{k=0}^{N} \epsilon_k\right)^n := C^n \tag{3.172}$$

with $C := \sum_{k=0}^{N} \epsilon_k < \infty$. Hence ψ is analytic for the energy:

$$e^{\sqrt{C}z} = \sum_n C^{\frac{n}{2}}\frac{z^n}{n!} > \sum_n \left\|(\widehat{P}_0)^n\psi\right\| \frac{z^n}{n!} \tag{3.173}$$

Proof: Let D be the effect associated with $\mathscr{D}$. From ψ being analytic for the energy and cyclic for the total algebra, together with $\mathbb{P}^{\psi}(\mathscr{D}) = \langle\psi|\, D\,\psi\rangle = 0$ and lemma 3.33 it follows that $D = 0$ on all of $\mathcal{H}$ and consequently

$$\mathbb{P}^{\varphi}(\mathscr{D}) = \langle\varphi|\, D\,\varphi\rangle = 0 \tag{3.174}$$

for all $\varphi \in \mathcal{H}$. ■

Consequently, under the assumptions of theorem 3.34, if an $N-$particle state with bounded energy content exists (which is the case without question) and is the initial state of a detection experiment with some proper detector arrangement, it must be capable in principle to trigger more than N detectors at spacelike separation (although the related probabilities might be extremely small). This might be compared with the conclusions drawn from the Malament type theorems above.

Final Remarks

A proposal how to understand why these seemingly counterintuitve results are physically reasonable at a closer look was already sketched in the introduction to this chapter and at the end of section 3.4.7: Local transformations of wave functions cause spectral transitions from the negative energy spectrum in the one particle theory and thereby violate the spectrum condition on this level. At least from Dirac theory we know that these processes in the associated QFT do not violate the spectrum condition but cause pair creation processes with a certain probability (it is helpful here to consider second quantization in the Dirac sea picture). In this respect it is not surprising but rather expectable that local detectors can – in principle – always detect something, even if they are placed in a region which looks locally like the vacuum (although one can expect that a customary detector will much more probably click falsely because of its imperfectness than due to particles created by its own potentials from the vacuum, but only probabilities associated with the latter case survive in the 'clean theory').

Indeed, when describing detectors in AQFT one usually circumvents the bothering fact that local detectors have non vanishing click probability in the vacuum by the proposal to describe detector effects by elements of *almost local algebras*: This notion is most easily established in terms of smeared field operators, where one uses well but not perfectly localized test functions from Schwartz space to describe almost local operators (this notion is also straightforwardly generalized to the more abstract algebraic frameworks [167]). One can show that each local algebra contains effects which can be approximated arbitrarily well in the operator norm topology (which means approximation of all associated probabilities by the Cauchy-Schwartz inequality, see footnote 7 in [174]) by almost local operators which yield zero probability in the vacuum [62].

Observing the fact that apparatus wave functions have infinite tails, it seems indeed reasonable to expect that operators associated with measurement (like) processes are actually elements of almost local algebras instead of precisely local ones. If realistic representatives of such operators have such a special form, that they are associated with state transformations perfectly devoid of any likelihood of particle production – and thereby yield zero probabilities in the vacuum – seems to be rather questionable to me, though. The constructions in the literature [62, 174] of almost local effects approximating local ones and having zero probability in the

vacuum looks indeed rather artificial and it was already argued for the analogue problem posed by Malament type theorems in section 3.4.7 that there seems to be little hope to avoid these peculiarities by taking the tails of measuring devices into account[167].

As the development of the mathematical basis for the herein considered results at the beginning of this chapter has shown, by the spectrum condition, all these things stem mathematically from the fact that holomorphic functions are extremely sensible objects with respect to perturbations: Local perturbations preserving analyticity are impossible due to the identity theorem and global transformations preserving analyticity are extremely special since each arbitrarily small piece of an analytic function uniquely determines the whole function also due to the identity theorem. In section 3.3 it was argued that the same is true for positive energy wave functions (which can indeed be perceived boundary values of analytic functions) such that transformations of positive energy wave functions forced by external influence of measuring devices would need to be nonlocal and extremely special to preserve the spectrum condition on the one particle level and thereby to avoid contributions of particle creation and annihilation phenomena with non vanishing probability in the associated QFT[168]. That these arguments extend to the vacuum state in QFT where roughly speaking 'no wave functions are present', is from this point of view nice and consistent.

[167]Recall for example that infinite tails of apparatus wave functions can in principle be accounted for by approximate measurement POVMs, which nonetheless do not exist as POVMs on physical space under the mild assumptions of theorem 3.25 without violating the spectrum condition.

[168]This way one might understand and analyse the structural distinctness between the local operator algebras in AQFT with their intrinsic nonlocal correlations and the operator algebras of non relativistic quantum theory, which is often expressed by the fact that the former are *type III factor* von Neumann algebras, while the latter are *factors of type I* (see e.g. [172, 346]).

4 An Active Notion of Localization

4.1 Passive and Active Notions of Localization

The Malament type theorems analysed in chapter 3 are sometimes interpreted as a proof that relativistic quantum theory was irreconcilable with a particle ontology [174, 234]. Since this is a bit vague, we may rephrase the presumable meaning of this assertion as the claim, that the predictions of (proper) relativistic quantum theory (positive energy etc.) cannot emerge from a theory which describes the motion of particles moving in space. It will be argued in the present work, that this claim is clearly unfounded. Actually, the theorems are theorems about operators acting on Hilbert space, and before serious assertions about their meaning (or non-meaning) for ontology can be made, it first has to be carefully analysed how precisely these operators relate to – or how the operator formalism emerges from – the ontology of a given theory. It was already frequently emphasized in the preceding part of this work that the operator formalism of quantum theory and thereby the results of chapter 3, have an operational meaning in the first place, i.e. primary concern potential measurement results, and their status with respect to the question of *what there is* is a priori unclear; at least if we do not advocate a naive realism about measurements, i.e. the belief that measurements just passively mirror what there is, but take into account that measurement results always result from an interaction between a measuring device a measured system. These differences can involve different notions of localization: What *can be* localized in an operational sense and what *actually is* localized in an ontological sense? If statements about ontology shall be made, based on the operational quantum formalism, it must be made transparent first, how the latter connects to – or can be derived from – the first principles of a theory about ontological facts. This shall be done now for a quantum theory of particles which are always localized in space by their very definition, namely *Bohmian mechanics*.

Bohmian mechanics is a well worked out theory about particles with unambiguous positions moving in space, which can be exhaustively formulated without even mentioning operators. Measurements play no fundamental role in Bohmian mechanics but from an analysis of its equations of motion for measurement (like) situations the quantum formalism is straightforwardly derived in all its generality. Since Bohmian mechanics describes unique matters of fact in physical space (including positions of pointers and the like) it does not suffer from a measurement problem[169]. While in the non-relativistic regime Bohmian mechanics is a mature theory, its generalization to relativistic QFTs is still work in progress and there are still challenges to cope with. But the results discussed in chapter 3 are certainly not among them but appear to be pretty natural from a relativistic Bohmian perspective, which shall become clear from an understanding of how one can gain empirically verifiable predictions from the defining equations of Bohmian mechanics.

In the following sections, Bohmian mechanics shall be defined and it shall be shown how the formalism of orthodox quantum theory arises from its equations. The first and main part of this analysis will be concerned with non-relativistic Bohmian mechanics, where the central notions and tools (in particular regarding the appropriate description of Bohmian subsystems

[169]Nonetheless, as we shall see below, Bohmian mechanics predicts limitations with respect to 'information transfer' about the actual state of facts of subsystems to their environments (like experimenters), as it is e.g. expressed by the Heisenberg uncertainty principle, which is a straightforward corollary in Bohmian mechanics.

and a statistical analysis in the spirit of Boltzmann) for the analysis of this theory will be developed, which also provide the central basis for an understanding of the predictive framework of relativistic versions of Bohmian mechanics. Such generalizations, like with respect to Lorentz invariance or particle creation and annihilation, will be briefly discussed afterwards. At this stage, the status of the theorems of chapter 3 from a Bohmian perspective should be already transparent, such that we can finally make a discussion of these results short.

4.2 Bohmian Mechanics in a Nutshell

Bohmian mechanics describes point particles moving in space guided by a wave function [39, 40, 128, 129]. The wave function is not a function on physical space (except in the case of a single particle) but on configuration space, where also the dynamics is primarily defined as a dynamics of configurations of particles. For example in the simplest case (on which we shall focus in the following) of non relativistic Bohmian mechanics of N distinguishable and spinless particles, the wave function is an element of $L^2(\mathbb{R}^{3N}, d^{3N}q)$ and the configuration space accordingly $\mathcal{Q}_N = \mathbb{R}^{3N}$. The dynamical law describing the particles' motion is then given by a (in general time dependent) velocity vectorfield $v^{\psi_t}(q,t)$ on the configuration space $\mathcal{Q}_N \ni q$ of the particles[170] such that an actual configuration $Q = (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N) \in \mathcal{Q}_N$ evolves in time according to the law

$$\frac{dQ(t)}{dt} = v^{\psi_t}(Q(t), t) \tag{4.1}$$

The vectorfield v^{ψ_t} in turn is generated by a wave function $\psi_t \in \mathcal{H} = L^2(\mathbb{R}^{3N}, d^{3N}q)$ (we abbreviate in the following $d^{3N}q$ by dq) obeying the Schrödinger equation

$$i\partial_t \psi_t(q) = \mathscr{H}\psi_t(q) \tag{4.2}$$

with Hamiltonian $\mathscr{H}$, at time t by the associated Schrödinger current (for simplicity assume that all particles have the same mass m)

$$j^{\psi_t}(q,t) = \frac{1}{2im}\left(\overline{\psi_t}(q)\boldsymbol{\nabla}\psi_t(q) - \psi_t(q)\boldsymbol{\nabla}\overline{\psi_t}(q)\right) \tag{4.3}$$

via

$$v^{\psi_t}(q,t) = \frac{j^{\psi_t}(q,t)}{\rho^{\psi_t}(q,t)} = \frac{1}{m}\,\mathrm{Im}\left(\frac{\boldsymbol{\nabla}\psi_t(q)}{\psi_t(q)}\right) \tag{4.4}$$

with the Born density (whose significance in Bohmian mechanics will be discussed later)

$$\rho^{\psi_t}(q,t) = |\psi_t(q)|^2 \tag{4.5}$$

This of course entails that the motion of the k'th particle in space is defined by the law

$$\frac{d\boldsymbol{Q}_k(t)}{dt} = \boldsymbol{v}^{\psi_t}_{\boldsymbol{q}_k}(Q(t)) = \frac{1}{m}\,\mathrm{Im}\left(\frac{\boldsymbol{\nabla}_k\psi_t(Q(t))}{\psi_t(Q(t))}\right) \tag{4.6}$$

[170] In the following *generic* configurations $q \in \mathcal{Q}$ (i.e. configuration space valued variables, e.g. as arguments of functions on configuration space) are denoted by lower case letters and *actual* configurations $Q \in \mathcal{Q}$ of particles by capital letters.

Note that equation (4.6) in particular implies that the motion of particle k at a given time in general depends not only on its own position but on the actual positions of the other $N-1$ particles at this time as well, which is to say on the whole actual configuration $Q \in \mathcal{Q}_N$ which enters into the velocity field on the right hand side of (4.6). This is how quantum nonlocality acts in Bohmian mechanics. It is easy to see that only in case of a product wave function

$$\psi_t(q) = \prod_{k=1}^{N} \psi_t^{(k)}(\boldsymbol{q}_k) \tag{4.7}$$

the velocity of each particle k does generally not depend on the positions of the other particles, i.e. in that case

$$\boldsymbol{v}_{\boldsymbol{q}_k}^{\psi_t}(Q(t), t) = \boldsymbol{v}_{\boldsymbol{q}_k}^{\psi_t^{(k)}}(\boldsymbol{Q}_k(t), t) \tag{4.8}$$

which nice property is thus in general destroyed by entanglement of the wave function.

Together with an appropriate statistical analysis in the spirit of Boltzmann, which will be discussed in section 4.4, these equations (possibly generalized to particles with distinct masses m_k in the obvious way) define a theory such that the predictions of standard quantum mechanics are predictions of this theory. In particular, this analysis reveals that configurations of Bohmian particles which constitute macroscopic objects (like pointers) displaying the outcomes of quantum experiments, behave in a way such that after many repetitions (with the same initial wave function, respectively), the empirical distributions of the displayed values agree with the statistical predictions of ordinary quantum theory. To show this, an understanding of some basic dynamical aspects of Bohmian subsystems is necessary first, for which purpose the crucial notions of *conditional* and an *effective wave functions* will developed next.

Before we start considering the Bohmian equations more closely, it shall be remarked that generalizations to indistinguishable particles and to spin are easily possible as well [128, 129] (where it turns out, for example, that in Bohmian mechanics spin is not an immediate property of a particle but rather is to be attributed to its guiding wave).

4.3 Subsystems

In standard quantum theory, empirical distributions of outcomes of potential quantum measurements are predicted by the algorithms of quantum theory. These distributions arise if measurements are performed on ensembles of identically prepared systems, i.e. systems prepared in the same wave function. As mentioned, we shall derive this also as a prediction of Bohmian mechanics by an analysis of its defining equations (4.2) and (4.4). But in the first place, the naive meaning of the notion of *'the wave function of a given system'* is usually not directly evident in Bohmian mechanics, and if so, it is rather artificial.

To understand this, note that there is no collapse postulate in Bohmian mechanics, resorting to which an initial wave function of a measured system could be even prepared, rather ubiquitous entanglement in the first place, such that 'the wave function of the considered system' – which is always a subsystem of a larger system, unless it is the whole universe – has in general no a priori meaning. If the joint wave function of the considered subsystem with generic coordinates $x \in \mathbb{R}^n$ and the rest of the world with generic coordinates $y \in \mathbb{R}^m$ with $q = (x, y)$ has product

structure $\phi_t(q) = \psi_t(x)\varphi_t(y)$ at time t, we may at that time regard ψ_t as 'the wave function of the subsystem', since then, as mentioned above, φ_t is indeed irrelevant for the motion of the $x-$system according to the Bohmian dynamics (4.4), i.e. $v_x^{\phi_t}(q,t) = v_x^{\psi_t}(x,t)$. Moreover, if the Hamiltionian contains no interaction between the $x-$ and the $y-$system, it happens that ψ_t satisfies its own autonomous Schrödinger equation. But each interaction between the $x-$ and the $y-$ system typically produces immediately persistent entanglement (as encountered for the measurement like processes in chapter 1) and thus destroys this product structure enduringly and consequently, since interaction is ubiquitous, it is not advisable to base the analysis on the assumption of such a very special product structure – at least if we reject an ad hoc collapse postulate (we may well build a reduced density operator by tracing out (i.e. averaging out) the environment, but this procedure aims at an effective description and will not yield a fundamental understanding of dynamics of subsystems).

It is important here to keep in mind, that spatial separation between a subsystem and 'the rest of the world' does not help at all (not even to obtain good approximations) to treat the subsystem as an independent autonomous system, since in contrast to classical potentials the influence of entanglement is not watered down by increasing distance, as for example the experimentally well confirmed EPRB correlations illustrate. In Bohmian mechanics this is highlighted by the fact that the velocity of a given particle depends in general on the positions of all other (entangled) particles, no matter how far away they are.

Hence we have a priori only the whole universe, with which a wave function ϕ can be associated (which of course no one knows explicitly), and must now understand how an appropriate dynamical and statistical description of subsystems is accomplished in Bohmian mechanics[171]. So we start with a Bohmian universe of N ($\sim 10^{80}$) particles with (of course unknown) universal 'initial' wave function $\phi(q)$ where $q \in \mathbb{R}^{3N} = \mathcal{Q}_N$ (we do not bother with a description of the big bang here of course, so ϕ might be thought of as the wave function of the universe at any time which we set 0). We have a dynamical law for the universal wave function ϕ given by the Schrödinger equation (4.2) which can be expressed by $\phi_t = U_t\phi$ and a dynamics of the particles given by the velocity vector field $v^{\phi_{t'}}$ given by (4.4) which can be expressed by integrating (4.1) in terms of the flow $\Phi_t(Q) = Q(t)$ if Q is the 'initial' configuration of the universe at time 0.

On this basis, we shall focus in the following two sections on dynamical aspects of the Bohmian description of subsystems: In section 4.3.1 we shall see that since we have not only wave functions but also particle positions, Bohmian mechanics provides straightforwardly always a mathematically precisely defined object which is to be regarded as the wave function of a subsystem (the *conditional wave function*) and which has no analogue in ordinary quantum theory and in section 4.3.2 we shall work out that this wave function obtains a very simple and in particular explicitly accessible form (the *effective wave function*) in measurement like situations, where it corresponds to the the collapsed wave function of orthodox quantum theory. With these notions at hand, we are then prepared to develop a framework of statistical predictions for ensembles of subsystems in section 4.4.

[171] Of course, no one believes that Bohmian mechanics as we know it today is the final physical theory which perfectly accurately describes the universe we live in, and when we consider the actual universe as a Bohmian universe, we do not so as well. But as with any proposal for a fundamental physical theory of nature (like e.g. classical mechanics or electrodynamics), it has the natural aspiration to be applicable to the whole universe and not to be restricted to certain regions or to subsystems up to a certain size. That it is probably the limit of another, more fundamental theory, is another issue.

4.3.1 Conditional Wave Function

Let $x \in \mathbb{R}^n$ be the generic coordinates of the considered subsystem, $y \in \mathbb{R}^m$ the generic coordinates of the rest of the world and accordingly $q = (x, y) \in \mathbb{R}^{n+m}$ the generic coordinates of the universe, $\phi_t(q)$ the universal wave function (whatever it may look like) at time t and $Y(t) \in \mathbb{R}^m$ the actual configuration of the rest of the world (whatever it may look like) at that time. Now consider the *conditional wave function of the $x-$system* given by

$$\psi_t^{Y(t)}(x) = \frac{\phi_t(x, Y(t))}{\sqrt{\int |\phi_t(x, Y(t))|^2 \, dx}} \tag{4.9}$$

It can be directly read of from the Bohmian equations of motion (4.1) and (4.4) that this is the appropriate object to write down the dynamics of the subsystem, i.e. if $X(t)$ is the actual configuration of the subsystem at time t and $Q(t) = (X(t), Y(t))$, the motion of the subsystem is given by

$$\frac{dX(t)}{dt} = v_x^{\phi_t}(Q(t), t) = v^{\psi_t^{Y(t)}}(X(t), t) = \frac{1}{m} \operatorname{Im} \left(\frac{\boldsymbol{\nabla}_x \, \psi_t^{Y(t)}(x)}{\psi_t^{Y(t)}(x)} \right) \Bigg|_{x=X(t)} \tag{4.10}$$

Hence in Bohmian mechanics, from a dynamical perspective (and as we shall see later from a statistical point of view as well) the conditional wave function ψ^Y is the proper object to call *the wave function of the subsystem*. But unfortunately, the conditional wave function is in general inaccessible, of course, since no one knows the wave function of the universe and the actual configuration Y of the rest of the world.

Before we shall work out that there exists nonetheless a physically very relevant class of situations (measurement like situations) in which the conditional wave function is explicitly accessible in the following section, a few general facts about the conditional wave function shall be noted: In (spinless[172]) Bohmian mechanics, a subsystem has obviously always a conditional wave function (in contrast to an effective wave function, which will be defined in the next section), which is in general inaccessible in its explicit form. And even if it were, the dynamics of a conditional wave function is in general likewise inaccessible, in particular it is generally not given by some Schrödinger equation. We shall encounter in a moment that in measurement like situations the transformations of conditional wave functions are comprehensible and indeed not given by unitary Schrödinger evolution but by collapse like transitions.

Another point noteworthy about the conditional wave function is the following: In Bohmian mechanics the wave function of course influences the particles by guiding them whereas the wave function is only determined by the Schrödinger equation and not by the actual positions of the particles. This is obviously different with the conditional wave function $\psi^Y(x)$ of a subsystem, which, although it does also not depend on the actual configuration X of the subsystem, in

[172]If the wave function has not only a configurational part but a spinor part in addition (for example $\mathcal{H} = L^2(\mathbb{R}^{3N}, d^{3N}x) \otimes \mathbb{C}^2$ in case of solutions of the Pauli equation), inserting the environmental configuration does in general not yield a unique conditional wave function of the subsystem, in particular if the spin parts of the system and its environment are entangled. In this case one can express the law of motion of the subsystem by its so called conditional density matrix, which changes the general analysis presented here only superficially but not in its essence, while it can complicate computations of actual solutions considerably.

general crucially depends on the positions of the particles of the environment expressed by their actual configuration Y. Thus, loosely speaking, the field through which particles directly feel each other in Bohmian mechanics is not given by the interaction potential in the first place, as in classical physics – note that the interaction Hamiltonian acts in the first place on the wave functions and thereby only indirectly on the particles – but it is rather given by the respective conditional wave functions!

The other way around, there is usually a variety of different configurations $\mathcal{Y}$ which yield one and the same conditional wave function $\psi^Y = \psi^{Y'} \equiv \psi_{\mathcal{Y}}$ iff $Y, Y' \in \mathcal{Y}$. In an important class of physical processes, a variety of environmental configurations $\mathcal{Y}$ even produces a conditional wave function $\psi_{\mathcal{Y}}$ which finally constitutes an autonomous physical subsystem, which can be treated – at least for a period of time – as independent from the rest of the world. This opens the door for more concrete physical reasoning and analysis, i.e. to analyse the (typical) physical behaviour of actual (for a time) autonomous physical (sub)systems as it is usual practice in physical inquiries. How this can be done comes now:

4.3.2 Effective Collapse

Consider a Bohmian system (universe) with generic coordinates $q = (x, y)$, where the $x-$ coordinates belong to a (small) subsystem whose configuration space has dimension n and the $y-$coordinates to the rest of the system (world) with configuration space of dimension m. Suppose further that the associated wave function $\phi_t(q)$ at some time t has the form

$$\phi_t(x, y) = c_1\psi_t(x)\varphi_t(y) + c_2\psi'_t(x)\varphi'_t(y) \tag{4.11}$$

(where $\psi_t, \varphi_t, \psi'_t, \varphi'_t$ are normalized) and suppose φ_t and φ'_t have disjoint support in $\mathbb{R}^m$, which together with $\|\phi_t\| = 1$ implies by orthogonality that $|c_1|^2 + |c_2|^2 = 1$ (we pretend for now that these wave functions are strictly compactly supported which realistically can be only fapp the case, the meaning of infinite tails for the present analysis will be addressed later, but it might be already comprehended in what follows that almost vanishing tails do not alter the obtained results in any significant way).

Now observe that

$$\phi_t(x, Y(t)) = c_1\psi_t(x)\varphi_t(Y(t)) \quad \text{if} \quad Y(t) \in \text{supp}(\varphi_t) \tag{4.12}$$

and thus for any actual configuration $X(t)$ of the $x-$system we have

$$\frac{\boldsymbol{\nabla}\phi_t(X(t), Y(t))}{\phi_t(X(t), Y(t))} = \frac{\boldsymbol{\nabla}\psi_t(X(t))\varphi_t(Y(t))}{\psi_t(X(t))\varphi_t(Y(t))} = \begin{pmatrix} (\psi_t(X(t)))^{-1}\boldsymbol{\nabla}_x\psi_t(X(t)) \\ (\varphi_t(Y(t)))^{-1}\boldsymbol{\nabla}_y\varphi_t(Y(t)) \end{pmatrix} \tag{4.13}$$

$Y(t) \in \text{supp}(\varphi_t)$ thus implies together with the actual form of the Bohmian velocity vector field (4.4) that the Bohmian motion of an actual configuration $Q(t) = (X(t), Y(t))$ is given by the equation

$$\frac{dQ(t)}{dt} = v^{\phi_t}(Q(t), t) = v^{\psi_t\varphi_t}(Q(t), t) = \begin{pmatrix} v_x^{\psi_t}(X(t), t) \\ v_y^{\varphi_t}(Y(t), t) \end{pmatrix} \tag{4.14}$$

and analogously if $Y(t) \in \text{supp}(\varphi'_t)$ we have

$$\frac{dQ(t)}{dt} = v^{\phi_t}(Q(t), t) = v^{\psi'_t\varphi'_t}(Q(t), t) = \begin{pmatrix} v_x^{\psi'_t}(X(t), t) \\ v_y^{\varphi'_t}(Y(t), t) \end{pmatrix} \tag{4.15}$$

More generally, if

$$\phi_t(x,y) = c_1\psi_t(x)\varphi_t(y) + c_2\phi_t^{\perp}(x,y) \tag{4.16}$$

where φ_t and $\phi_t^{\perp}$ have disjoint $y-$support (i.e. the union over all x of the $y-$supports of $\phi_t^{\perp}(x,y)$ is disjoint from the support of $\varphi_t(y)$) and $Y(t) \in \mathrm{supp}(\varphi_t)$, it follows that the motion of the joint configuration $Q(t)$ is given by equation (4.14). Thus, the $x-$ system is at this time indeed independent of the environmental configuration $Y(t)$ and its conditonal wave function is given by

$$\psi_t^{Y_t}(x) = \frac{\phi_t(x,Y(t))}{\sqrt{\int |\phi_t(x,Y(t))|^2 \, dx}} = \psi_t(x) \tag{4.17}$$

(where the second equality sigh holds in general only modulo a constant phase, to be precise) which also obeys an autonomous Schrödinger equation, given the joint Hamiltonian does not contain an interaction potential between the $x-$ and the $y-$ system at time t. In such a case the conditional wave function ψ_t in (4.17) is called the *effective wave function of the $x-$subsystem*.

Are there relevant physical situations in which the appearance of an effective wave function is somehow stable and plays a more than an accidental role? The answer is clearly yes: In the following one should have in mind the analysis of measurement (like) processes of chapter 1, but forget about the ad hoc collapse postulate, which is not an element of Bohmian mechanics, of course. Consider for a moment a contrived Bohmian universe consisting of some 'microscopic' quantum system and some 'macroscopic' measuring device which interact with each other in a measurement type manner.

In chapter 1 the notion of the *pointer states* of the measuring device was introduced, which are associated with displays of the different possible measurement results, and we vaguely characterized them as *macroscopically accessible and distinguishable states*. It was argued that one necessary condition for a collection of wave functions to deserve the name 'pointer states' is that they have (at least fapp) mutually disjoint support, macroscopically separated in configuration space. *Macroscopic separation* means in this context that all configurations constituting the support of a given pointer state are configurations of a macroscopic object (in particular living in a very high dimensional configuration space) like a pointer or a display, macroscopically distinguished from the configurations in the support of the other pointer states (like with pointers pointing onto different numbers).

Let now $x \in \mathbb{R}^n$ be the generic coordinates of the small system with associated Hilbert space $\mathcal{H}_S = L^2(\mathbb{R}^n, d^n x)$ and $y \in \mathbb{R}^m$ the generic coordinates of the measuring device with associated Hilbert space $\mathcal{H}_A = L^2(\mathbb{R}^m, d^m x)$ and $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A$ (in the following we abbreviate again $d^m x$ by dx and $d^n y$ by dy). For simplicity we start with the assumption of an ideal measurement interaction expressed by some unitary operator U (i.e. U transforms the joint initial state prior to the interaction into the joint final state subsequent to the interaction) with two possible measurement results: If φ_0 is the pointer ready state of the apparatus, there are initial states $\psi_k \in \mathcal{H}_S$ where[173] $k = 1, 2$ and associated pointer final states φ_k such that

$$\psi_k\varphi_0 \xrightarrow{U} \psi_k\varphi_k \tag{4.18}$$

[173]If $\mathcal{H}_S$ is more than two dimensional the two outcome measurement is degenerate and each ψ_k can be thought of as representing any state of a larger subspace.

and due to linearity of the Schrödinger equation, if $\psi = c_1\psi_1 + c_2\psi_2$ we have

$$\psi\varphi_0 \xrightarrow{U} c_1\psi_1\varphi_1 + c_2\psi_2\varphi_2 \tag{4.19}$$

In Bohmian mechanics, the joint actual configuration $Q(t) = (X(t), Y(t)) \in \mathbb{R}^{n+m}$ of system and apparatus is at any time guided by the joint wave function (whatever it may look like during the interaction) and arrives at an actual configuration $Q(T) = (X(T), Y(T))$ after the measurement interaction is over at time T and the joint wave function of system and apparatus is given by the right hand side of (4.19). Depending on the initial configuration $Q = (X, Y)$, the configuration $Y(t)$ of the apparatus will either end up in $\mathrm{supp}(\varphi_1)$ or in $\mathrm{supp}(\varphi_2)$ (see Fig 9) which are disjoint and macroscopically separated in the configuration space of the apparatus.

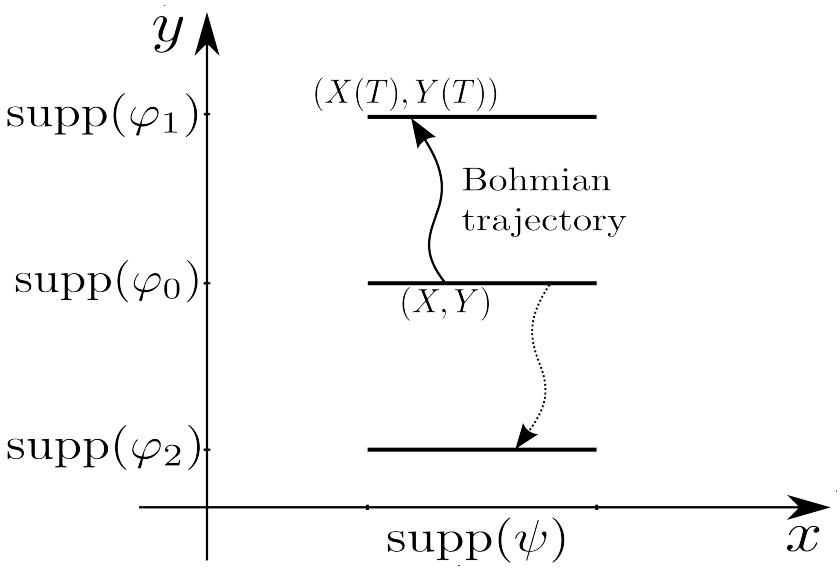

FIGURE 9

Configuration Space Diagram of a Bohmian Measurement (Like) Process: *Depicted is the joint configuration space of the measured system with generic coordinates x – which is thought to be of low dimension – and the measurement apparatus with generic coordinates y – which is macroscopically high dimensional. Prior to the measurement at time $t = 0$ the joint wave function is given by $\psi(x)\varphi_0(y)$ (where φ_0 is the pointer ready state of the apparatus) and the joint actual configuration of measured system and measuring device is (X, Y). During the measurement interaction the joint wave function evolves to the final state $c_1\psi_1\varphi_1 + c_2\psi_2\varphi_2$ at time $t = T$ and guides the particles to the final configuration $(X(T), Y(T))$ with $Y(T) \in \mathrm{supp}(\varphi_1)$ (i.e. 'the pointer points onto 1'). Depending on the initial configuration, the configuration can also be guided into regions of configuration space where the pointer configuration is in $\mathrm{supp}(\varphi_2)$ ('the pointer points onto 2'), which is indicated by the lower dashed trajectory. Obviously, always precisely one of these scenarios is realized (depending on the initial configuration), since there simply is only one configuration! This picture is a reproduction of [129] p. 176.*

The configuration $Y(T)$ of the apparatus subsequent to the measurement constitutes the respective display, we may e.g. think of atoms constituting a pointer pointing onto the number 1 if $Y(T) \in \mathrm{supp}(\varphi_1)$ or onto the number 2 if $Y(T) \in \mathrm{supp}(\varphi_2)$, respectively. Moreover, due to

the disjointness of the supports of φ_1 and φ_2 it follows for all $x \in \mathbb{R}^n$ that

$$U(\psi\varphi_0)(x, Y(T)) = c_1\psi_1(x)\varphi_1(Y(T)) \quad \text{if} \quad Y(T) \in \text{supp}(\varphi_1) \tag{4.20}$$

and

$$U(\psi\varphi_0)(x, Y(T)) = c_2\psi_2(x)\varphi_2(Y(T)) \quad \text{if} \quad Y(T) \in \text{supp}(\varphi_2) \tag{4.21}$$

This entails that the $x-$system has an effective wave function subsequent to the measurement, which is given by ψ_1 if the pointer points onto 1 (i.e. $Y(T) \in \text{supp}(\varphi_1)$) or by ψ_2 given the pointer points onto 2 (i.e. $Y(T) \in \text{supp}(\varphi_2)$). And since, if, say ψ_1 is the effective wave function of the $x-$system, as argued above, ψ_2 as well as the environment have no influence on its dynamics anymore (below we will argue that this holds with respect to ψ_2 fapp forever). Such a transition of the conditional wave function of the measured system is called *effective collapse* in Bohmian mechanics.

Since one can thereby ignore elements of physical description which are no longer dynamically relevant (the empty branches of the wave function), the effective collapse is sometimes misunderstood to be of rather subjective character, i.e. that nothing really collapses but we can pretend that it does by neglecting what is not relevant anymore. But this is manifestly false: Whereas the universal wave function indeed never collapses in Bohmian mechanics, the conditional wave functions of subsystems manifestly do in measurement like situations. For example, in the just described ideal measurement, the conditional wave function was prior to the measurement $c_1\psi_1 + c_2\psi_2$ and if, say the pointer points finally onto 1, it is finally ψ_1, which is of course not unitary but collapse dynamics. And there is nothing subjective about this, the precisely defined mathematical object which primary determines the dynamics of a given subsystem (the conditional wave function) collapses objectively upon measurement (like) interaction with its environment. It collapses, not because a collapse dynamics is among the defining dynamical principles of Bohmian mechanics (which is not the case), but because we have not only wave functions but also configurations of particles, and a closer look at the Bohmian equations straightforwardly reveals that the dynamics of the wave function of a given system conditional on its environmental configuration is in general not given by a Schrödinger equation, but can collapse as well.

We see that a precise and coherent analogue of the vaguely defined collapse upon measurement of ordinary quantum theory arises completely naturally in Bohmian mechanics simply from analysing the equations of motion, since they describe the dynamics of unambiguous physical facts, namely matter moving in space: The wave function defining the dynamics of the 'measured system' subsequent to the measurement is simply ψ_1 if the pointer points onto 1 without any further need of interpretation of an entangled wave function as on the right hand side of (4.19) and without the need of a vaguely defined extra dynamical principle like collapse upon measurement.

Of course, this analysis is in its results perfectly the same if we drop the simplifying assumptions that the measurement has only two outcomes and that it constitutes an ideal measurement. Namely, equations (4.18) (defining an ideal measurement) were only motivation and dispensable for the subsequent analysis, and this analysis is as well easily generalized to a situation where there are more than two possible results of the process, i.e. we may consider instead of (4.19) a general (discrete) measurement (like) process (without ad hoc collapse in the sense of chapter

1) of the form

$$\psi\varphi_0 \xrightarrow{U} \sum_k c_k\psi_k\varphi_k \tag{4.22}$$

to come to the same conclusions with respect to effective collapse: The actual joint configuration of measured system and apparatus is guided by exactly one of the branches $c_k\psi_k\varphi_k$ on the right hand side of (4.22) subsequent to the measurement and if 'the pointer points onto k' – which is to say $Y \in \mathrm{supp}(\varphi_k)$ – ψ_k is the effective final wave function of the measured system[174].

Decoherence

Before we come to statistical reasoning we shall incorporate the rest of the world: We started this analysis of a Bohmian measurement process with considering an artificial Bohmian universe consisting solely of a measured system and a measuring device. Does it any harm to the lines of argument if we start with the more realistic situation of a Bohmian universe of which these two systems are only (small) subsystems? In this case, we must base the analysis on the assumption that the configuration $Y(t)$ is not only the configuration of the apparatus but literally the configuration of the rest of the world (with respect to the measured system) of which the apparatus is only a (small) part. But this makes the separation of environment configurations containing distinct pointer configurations in configuration space (which was a crucial ingredient to establish the notion of effective collapse) only much more effective: Although probably a huge part of the configurations of distant objects will (fapp) not be effected by the measurement process (like the dog peeing at the building wall outside of the laboratory, or the sun or andromeda), for example air molecules and photons will be reflected or scattered by the pointer in different directions depending on the pointer positions, in turn they will interact with more and more systems, scatter with other air molecules, interact with the experimenters eye and thereby brain, the walls of the laboratory etc. Consequently, the wave function of the measured system gets entangled with all these subsystems subsequent to the interaction, i.e. the pointer wave functions $\{\varphi_k\}$ on the right hand side of (4.22) include all these subsystems and the mutual separation of their supports in configuration will dramatically grow in time as more and more systems get involved developing in time very differently depending on 'the pointer position k' (of course, there is a huge variety of configurations $\mathcal{Y}_k$ of the 'rest of the world' compatible with this pointer position and thereby assigning to the measured system effective wave function φ_k). These processes destroy for all practical purposes any possibility of mutual interference (which is essentially to say, to bring their supports to a considerable overlap again) of the branches $c_k\psi_k\varphi_k$ of the post measurement wave function $\sum_k c_k\psi_k\varphi_k$, fapp forever! This is how decoherence acts.

But decoherence alone does in no way suppress any of the branches on the right hand side of (4.22), the superposition stays in all its glory, only it is fapp impossible to bring different

[174] In case of continuous measurements, identifying proper state transformations is problematic (see e.g. the examples of continuous measurements in chapter 1). One can e.g. consider the measurement of a given eigenprojection of an observable operator with continuous spectrum as a discrete yes/no measurement and associate a state transformation with it (to see how this can be generalized to obtain approximations of measurements of the original observable operator, see [127] p. 30). But one can also argue that continuously appearing measurements must be actually discrete on a closer look. For example in [127], the authors remark about realistic quantum measurements: *'Note that to assume there are only finitely, or countably, many outcomes is really no assumption at all, since the outcome should ultimately be converted to digital form, whatever its initial representation may be.'*

branches to interference again. As already discussed in section 1.6.6, in order to single out a particular branch we either need an additional dynamics of the wave function suppressing the other branches – like in GRW – or a dynamical object in addition to the wave function singling out one single actual dynamically relevant branch, like most easily particles with positions as in Bohmian mechanics. The actual configuration $Q = (X(t), Y(t))$ (where $Y(t)$ now may involve all particles potentially relevant to the future of the measurement process) is guided by exactly one branch on the right hand side of (4.22) associated with the actual pointer position, which branch it is depends on the initial configuration. And the fact that decoherence destroys practically any possibility that the actual guiding branch can interfere with one of the other branches at any time subsequent to the measurement dooms the latter to stay dynamically irrelevant.

What happens if Zeilinger's successors can make it one day, to bring the wave function of the alive cat to interference with the wave function of the dead cat? Now (apart from being absurd to think about), from a Bohmian point of view clearly nothing mysterious, only if the actual configuration of the cat is initially guided by the alive cat branch, this experiment would probably kill the cat.

This concludes the Bohmian effective collapse description from a dynamical perspective: Subsequent to a measurement (like) process, the actual configuration of the apparatus will be guided by only one of the wave functions φ_k on the right hand side of (4.22) (which one depends on the initial configuration) and accordingly only ψ_k will be dynamically effective for the future dynamics of the measured system, and decoherence guarantees that we can forget about the non effective branches with clean conscience fapp forever!

Initial Product State

There is one thing one might complain about the description of a Bohmian measurement process above: How is it justified to start with a product wave function of the measured system and the rest of the world in view of the above mentioned fact that product wave functions appear to be very special objects. First note that in order to get an effective wave function there is no need to start with a product wave function, a situation as on the right hand side of (4.22) is quite fine, but it creates an effective product wave function. This is how preparation of initial states works, which are thus initially autonomous physical systems as it is reasonably required for a meaningful experiment, i.e. that we can treat the measurement apparatus and the measured system as independent physical systems prior to the measurement.

4.4 Statistical Analysis

Now we could postulate in addition that the coordinates of Bohmian systems with effective wave function ψ are $|\psi(q)|^2$ −distributed, which is to say that actual configurations of an ensemble of systems with the same effective wave function ψ approach the $|\psi(q)|^2$ −distribution for large enough number of ensemble members. This choice is standing to reason for two reasons: First, $\rho^\psi(q) = |\psi(q)|^2$ is the distinguished density which is associated with the Bohmian current (4.3) via a continuity equation (see below) and in consequence a $|\psi(q)|^2$ −distributed ensemble at time 0 will be $|\psi_t(q)|^2$ −distributed at any other time t if ψ is time evolved by the Schrödinger equation and the configurations by the Bohmian velocity field (see below for details). Secondly, the experimentally well verified predictions of quantum theory straightforwardly follow from

this choice (this will be shown in detail in section 4.5) without running into the conceptual problems of quantum theory, i.e. from a Bohmian perspective, $|\psi|^2$ −distributions of Bohmian configurations are strongly suggested by empirical evidence.

But just postulating statistical laws – in particular for phenomena arising from a microscopically coherent and deterministic theory – is actually unsatisfactory. Indeed, before quantum theory (re)introduced statistical reasoning without a deeper conceptual and mathematical basis, striking developments were made to understand the emergence of statistical empirical regularities as arising from microscopic first principles, probably most impressively represented by Boltzmann's microscopic understanding of the second law of thermodynamics[175]. And since Bohmian mechanics is a microscopically well defined deterministic theory, it makes a lot of sense to perform a statistical analysis of Bohmian mechanics grounded in an understanding of statistical physics in Boltzmann's footsteps[176], in particular by not postulating statistical laws but deriving them from dynamical considerations. This shall be done now.

4.4.1 Measures of Typicality

The central thought is to understand macroscopic phenomena as being constituted by microscopic objects on whose level the fundamental laws of physics are defined, from which the emergence and dynamics of macroscopic phenomena should be explicable. An important notion in this respect is the notion of *microstates* – on which the dynamical notion of *microscopic initial conditions* is based – which describes the objective microscopic fundamental matter of fact whose dynamics is given by fundamental laws of nature. The microstates in Bohmian mechanics are configurations of particles and thus the space of possible microstates is given by configuration space $\mathcal{Q}$. In contrast to Bohmian mechanics, the space of microstates in classical mechanics is given by phase space, i.e. the configurations of particles must be supplemented with their respective momenta. This is due to the fact that the Newtonian equations of motion are partial differential equations of second order in time, while the Bohmian equations of motion are first order, such that an initial configuration is a complete initial condition (set of Cauchy data) to determine the future dynamics. The latter is of course only true if a wave function is given, which determines the motion of the Bohmian particles as a field on configuration space, similar as the classical Hamiltonian determines the motion of Newtonian particles as a field on phase space in the Hamiltonian formulation of classical mechanics.

Another very central notion is the notion of *typicality*[177] which roughly speaking refers to

[175]To estimate how striking the impact of Boltzmann's understanding of statistical physics was, consider the following quote of Schrödinger who stated, referring to Boltzmann's explanation of irreversibility arising from time reversible microscopic laws compared to former ad hoc approaches ('expedients') to account for irreversibility: *'No one who has once understood Boltzmann's theory will ever again have recourse to such expedients. It would be a scientific regression beside which a repudiation of Copernicus in favour of Ptolemy would seem trifling.'* [296].

[176]Besides Boltzmann, who essentially laid the ground (part of the central idea was already loosely formulated by Maxwell and Kelvin before), many great physicists continued working out statistical physics in his way of understanding the emergence of statistical phenomena (Ehrenfest(s), Einstein, Kac, Lebowitz, just to call a few names whose work rely in an essential part on Boltzmann's ideas). A very nice and detailed presentation of Boltzmann's ideas can be found in [129] (where besides classical statistical analysis also the statistical analysis of Bohmian mechanics is worked out), less technical but also highly recomended presentations are [154, 216, 257].

[177] The naming 'typicality' in this context is rather modern, but the concept is indeed very old. It is not only the crucial argument in Boltzmann's physical derivations, but was in a less physically specific version already considered as the crucial concept to connect probability theory with the empirical world before. More

phenomena which can be shown to arise from the overwhelming majority[178] of possible microstates (initial conditions). And in order to get a grip on the vaguely defined notion of the 'overwhelming majority', we need to find an appropriate weight (measure) on the space of microstates which tells us which sets of microstates are large with respect to others and which are small (for simplicity we assume such a measure always to be normalized to 1). Superficially, one might call such a weight properly normalized a *probability measure*, but this is actually an inadequate naming at this stage, since there is no connection to relative frequencies or the like made so far. It is more appropriate to call it a *measure of typicality*, which only needs to tell us which sets of microstates are very large ('probability very close to 1') and which are very small ('probability very close to zero'). Once an appropriate measure of typicality is established, a general probability measure, e.g. to predict empirical relative frequencies for certain events in ensembles of subsystems, can be derived by a law of large numbers. How this is done will be demonstrated later.

Before going more into the subtleties of determining proper measures of typicality, it is important to note that the kind of assertions about certain empirical regularities derived that way obviously differs structurally from the nature of usual physical assertions, since they do not claim necessity but are claims which are shown to hold for the vast majority of possible microstates (initial conditions) within the realm of the given theory, but not for all. There is for example no law of nature (say, from the viewpoint of classical mechanics) which forbids a fluctuation in the thermal velocity distribution of the molecules constituting a stone at the bottom, where all of them collectively move upwards at the same time such that the stone suddenly leaves the ground (and, as it follows from the usual kind of physical assertions, necessarily cools down because of energy conservation). But nonetheless such scenarios do not contribute to phenomena and we can understand why, namely since they are atypical: As can be easily argued, the uncountable infinity of possible empirical velocity profiles[179] which would make the stone spontaneously lift off make a set which is still ridiculously small (practically nothing) within the set of all possible velocity distributions, in particular compared to the set of velocity distributions which leave the stone at its position laying on the ground, to which the overwhelming majority of all microstates belongs. Although notions like the 'overwhelming majority' or 'ridiculously small' are very vague, they allow for a sharp predictive framework of striking empirical success. Similarly as we can assert with empirical certainty that a very long coin tossing series (with a fair coin) does not yield a million times head in a row, the stone will not spontaneously lift off with empirical certainty (which might be roughly compared with a coin tossing series yielding 10^{20} times head (almost) in a row, if we roughly estimate the number of molecules in a stone to be of that order).

To emphasize the trust which this kind of 'empirical certainty' deserves and to get a feeling for the business of properly 'counting microstates' (or more precisely, 'comparing the size

precisely, the observation that only an association of *probabilities very close to zero or one* with empirical events has an immediately relevant meaning, by asserting that events with probability very close to one (zero) will (not) happen with empirical certainty (from this principle an identification of general probabilities with typical empirical relative frequencies can be derived by proving a law of large numbers). This principle was first stated by Bernoulli, who introduced for this purpose the notion of *moral certainty*. Afterwards this principle became known as *Cournot's principle* (see the remark on Cournot's principle in the introduction to chapter 3).

[178] This wording goes back to Paul and Tatiana Ehrenfest [130].

[179] Say, we consider only velocity distributions constrained by a given temperature – i.e. mean kinetic energy per molecule – of the stone.

of very large and very small sets of microstates'), consider as another example the following line of thought (for simplicity from a classical viewpoint again): The maybe about 10^{24} oxygen molecules in the room I am sitting right now are pretty well homogeneously distributed throughout the whole room (with probably minor fluctuations on the microscopic scale) and I need not worry that suddenly all oxygen molecules cluster in the opposite side of the room such that I'm in danger of suffocation. In view of the fact that there are actually uncountably infinitely many configurations of the oxygen molecules, in which all of them are located in the other side of the room and, moreover, uncountably infinitely many initial conditions on phase space such that all oxygen molecules suddenly move into this region under the Newtonian dynamics, one might wonder why I should not be afraid. The answer is again that this uncountable infinity of microstates make a very small set compared with the uncountable infinity of microstates which correspond to a situation in which all molecules are and stay (under the Newtonian dynamics) homogeneously distributed all over the room with fluctuations only on the microscopic level beyond the threshold of my perception (to get an impression how the ratio between the sizes of these sets can be quantitatively estimated and how inconceivably massive it is, see footnote 180 below, see also [129], where these concepts are developed much more carefully than it is possible in the present work).

Since all of these considered sets of microstates have uncountably many elements, we cannot literally count them and thereby compare their 'size' of course[180], but need an appropriate measure. It is natural to start simply with literal volume on phase space given by the Lebesgue measure (in this context often called *Liouville measure*), but apart from the problem that it is not normalizable on phase space it turns out that also with respect to the (Newtonian) dynamics plain Lebesgue measure is not the appropriate choice, since in general, subsets of phase space with equal Lebesgue measure share not the same 'likelihood' that one of their microstates is realized for dynamical reasons. E.g. if energy conservation is taken into account, the singular microcanonical measure which does only give non zero weight to subsets of the respective energy shell in phase space turns out to be the appropriate measure, if a system can exchange energy with its environment the exponential canonical distribution is appropriate and so forth.

All of these measures share an important property with respect to the (here Newtonian) dynamical law: They preserve the weight they give to sets under the Newtonian dynamics, i.e. a given subset of phase space has the same measure as the subset arising from it, if all elements of the original subset are time evolved with the Newtonian dynamics (this is due to the fact that these measures actually are stationary measures, see below). This has as a trivial consequence that huge sets stay huge under the Newtonian dynamics and thus typicality is preserved in

[180] Actually, there is a very coarse grained possibility to count and compare, in order to get an impression about the scales between 'very few' and 'overwhelmingly many': Note that there is only one combinatorial possibility to fill N balls into one actual of two boxes (the remote side of the room), while there are $\binom{N}{N/2}$ possibilities to distribute $N/2$ of the balls in each of the boxes. One may now calculate $\binom{N}{N/2}$ for $N \sim 10^{24}$, i.e. $10^{24} \cdot (10^{24}-1) \cdot (10^{24}-2) \cdots (10^{24}/2+1) \cdot 10^{24}/2$, in order to get am impression what 'overwhelmingly many' means compared to 'very few' in this context.

Similarly we may directly utilize the notion of phase space volume (Lebesgue measure) to compare by noting that the allowed volume of the configurational part of phase space trivially grows by a factor 2 for each molecule, if it is allowed to occupy any position in a two times larger region, such that the total phase space volume of all possible microstates of the gas in the room is $2^{10^{24}}$ times larger than the volume of microstates of a gas occupying only half of the room.

time. Without this property, a reasonable statistical analysis based on the 'size' or 'weight' of sets of microstates is hardly imaginable[181]! Thus – and this is completely independent of the precise form of the microscopic dynamical law (be it Newtonian or e.g. Bohmian) – the dynamical law singles out classes of measures of typicality on the space of microstates which are physically appropriate compared to other measures which cannot provide a coherent physical interpretation. How this basic requirement influences and actually establishes the statistical analysis of Bohmian mechanics (where it turns out that not a whole class but actually only one single measure is singled out by the dynamics in the just described way) comes next:

4.4.2 Equivariance

In Bohmian mechanics the wave function ψ guides particles along trajectories. Moreover, the associated guiding law distinguishes a certain measure valued functional from wave functions to (probability) distributions on configuration space, given by

$$\psi \quad \longmapsto \quad |\psi(q)|^2 \; dq \tag{4.23}$$

This functional is distinguished since it defines a family of *equivariant* measures, which is to say measures that are time invariant under the Bohmian dynamics in the following sense:

Let $\Phi_t : \mathcal{Q} \to \mathcal{Q}$ for $t \in \mathbb{R}$ be the Bohmian flow generated by the velocity field $v^{\psi_{t'}}$ of equation (4.4) obtained by integrating (4.1) with initial wave function ψ, i.e. $\Phi_t(Q) = Q(t)$ is the configuration at time t, if Q is the initial configuration and ψ is the initial wave function at time 0 (note that the flow satisfies the semigroup property $\Phi_t\Phi_s = \Phi_{t+s}$ and $\Phi_{-t} = (\Phi_t)^{-1}$). Let further μ be a (normalized) measure on configuration space $\mathcal{Q}$. The flow Φ_t now naturally defines a time dependence of this measure by

$$\mu_t(\Phi_t(\Delta)) := \mu(\Delta) \quad \text{ or equivalently } \quad \mu_t(\Delta) := \mu(\Phi_{-t}(\Delta)) \tag{4.24}$$

for all (measurable) subsets $\Delta \subseteq \mathcal{Q}$, i.e. the set of the points transported by the flow at time t has the same 'weight' with respect to the time dependent measure as the original set of points. Such a measure dynamically preserves the size of sets, i.e. huge sets of configuration space stay huge under the time evolution and thus, if μ is a measure of typicality, what is typical stays typical. A nice calculation shows (see e.g. [129] p. 21f.) that (4.24) together with $\frac{d\Phi_t}{dt} = v^{\psi_t}$ is equivalent to the continuity equation

$$\partial_t \, \rho_{\mu_t}(q,t) + \boldsymbol{\nabla} \cdot \left(\rho_{\mu_t} v^{\psi_t}\right)(q,t) = 0 \tag{4.25}$$

where $\rho_{\mu_t}(q,t)$ is the density of the measure μ_t, i.e. $\mu_t(\Delta) = \int_\Delta \rho_{\mu_t}(q,t)dq$.

Now in general it is not easy to find a concrete expression to work with for a density obeying (4.25). An important usual way in statistical physics to accomplish this would be to choose

[181] Besides logical reasoning to find that typicality must be independent of time, there are also plain pragmatical reasons to base on this assumption the statistical analysis of a given theory: To give the statistical analysis any practical significance we have to establish a link with empirical regularities which can be potentially observed in the end. This is done by proving a law of large numbers for relative frequencies of particular events in ensembles of subsystems with respect to the considered measure of typicality (which will be done for the statistical analysis of Bohmian mechanics below). With a time dependent notion of typicality (in the sense that a large set of microstates can evolve into a smaller one under the physical dynamics) this would be a hopeless effort, except for one single moment in time (see also [127, 129]).

$\partial_t \rho_{\mu_t}(q,t) = 0$, in which case ρ_{μ_t} would be called the density of a stationary measure, and find out whether the remaining expression $\boldsymbol{\nabla} \cdot \left(\rho_{\mu_t} v^{\psi_t}\right)(q,t) = 0$ can be tackled better (this is e.g. possible in classical mechanics where the Hamiltonian equations of motion are very helpful to find densities – like the trivial Lebesgue-, the microcanonical or the the canonical one – which obey the latter expression with v^{ψ_t} replaced by the Hamiltonian vector field on phase space). The problem in Bohmian mechanics is only that the wave function ψ_t and by that the velocity field v^{ψ_t} are in general explicitely time dependent, such that ρ_{μ_t} cannot be stationary. But fortunately, as is well known, the Schrödinger equation very directly implies a continuity equation with respect to the Bohmian velocity field:

$$\partial_t \rho^{\psi_t}(q,t) + \boldsymbol{\nabla} \cdot j^{\psi_t}(q,t) = \partial_t \rho^{\psi_t}(q,t) + \boldsymbol{\nabla} \cdot \left(\rho^{\psi_t} v^{\psi_t}\right)(q,t) = 0 \tag{4.26}$$

with the probability density of the Born rule

$$\rho^{\psi_t}(q,t) = |\psi_t(q)|^2 \tag{4.27}$$

whose associated probabilities obviously obtain their time dependence from the time dependence of the wave function due to the Schrödinger equation encoded in $\psi_t = U_t \psi$, i.e.

$$\mathbb{P}^{\psi_t}(\Delta) = \int_\Delta \rho^{\psi_t}(q,t)\, dq = \int\limits_\Delta |\psi_t(q)|^2 \; dq \tag{4.28}$$

In consequence, if for all (measurable) $\Delta \subset \mathcal{Q}$, given ψ is the initial wave function at time 0, we identify $\mu_t(\Delta)$ with

$$\mathbb{P}^{\psi}_t(\Delta) = \mathbb{P}^{\psi}(\Phi_{-t}(\Delta)) := \int\limits_{\Phi_{-t}(\Delta)} |\psi(q)|^2 \; dq = \int\limits_\Delta |\psi(\Phi_t(q))|^2 \; dq \tag{4.29}$$

we get

$$\mathbb{P}^{\psi_t} = \mathbb{P}^{\psi}_t \tag{4.30}$$

A probability measure which satisfies (4.30) for all t, where the time dependence on the left hand side is generated by the time dependence of the wave function $\psi_t = U_t \psi$ and the time dependence on the right hand side is generated by the Bohmian flow according to (4.24), is called *equivariant* (actually, equivariance refers to a whole family of probability measures, since it has to hold for all ψ in the Hilbert space). Thus equivariance means that the time dependence of the guiding wave function enters into the measure in such a way that it is constant under the Bohmian flow (this has as a consequence that a $|\psi|^2$ −distributed collection of configurations at some time 0 will be $|\psi_t|^2$ −distributed at any other time t if the wave function is time evolved by the Schrödinger evolution and the configurations by the Bohmian flow, which won't be true for any (reasonable) wave function functional other than $|\psi|^2$ [156]!).

But as mentioned, at this level we do not need the entirety of a probability measure, since there is no relation with empirical relative frequencies yet established, but rather a measure of typicality which tells us which sets of microstates are huge under dynamical considerations, or equivalently which sets of microstates are very small. And the preceding analysis shows that the $|\psi|^2$ −measure $\mathbb{P}^{\psi}$ is a promising candidate for a measure of typicality, since it satisfies

the necessary condition of equivariance, which guarantees that sets of microstates of large/small measure ($\mathbb{P}^\psi$ very close to 1/0) stay large/small in time, such that typicality is a time independent notion for each ψ in the Hilbert space. Moreover, it has been shown in [156] that under few very mild assumptions the $|\psi|^2$−measure is indeed not only one, but *the* only one equivariant measure in Bohmian mechanics! Thus, the equivariant $|\psi|^2$−measure – which is called the *quantum equilibrium measure* in Bohmian mechanics – is singled out by the Bohmian dynamics to be the only physically natural weight on configuration space on which a statistical analysis in the sense of Boltzmann's analysis of statistical mechanics can be based.

4.4.3 Subsystems Again

In section 4.3 we encountered that we must consider the Bohmian equations of motion in the first place for a whole Bohmian universe to derive an appropriate description of the dynamics of subsystems. Indeed, for an appropriate statistical analysis of Bohmian mechanics, we have to go the very same way and start again with a Bohmian universe to derive proceeding from there (statistical) assertions about the behaviour of subsystems.

If one thinks of probabilities in statistical physics as relative frequencies in Gibbs type ensembles, one is lost if one considers the all-embracing system, namely the universe, since it would be unrewarding to consider literally ensembles of universes while we have only the unique one we live in. A measure of typicality in contrast enables us to predict empirical regularities which arise in a typical universe, i.e. which hold for the overwhelming majority of possible microstates (initial conditions) of the universe. And this has proven to be empirically tremendously successful, it provides predictive frameworks and an understanding for a diversity of phenomena within the unique universe we live in. In this sense, it will be shown that in a typical Bohmian universe empirical distributions of final configurations of measuring devices reflecting the outcomes of quantum measurements agree with the quantum predictions. A typical Bohmian universe is said to be in *quantum equilibrium*, which is to say that its microstate belongs to the overwhelming majority of possible microstates with respect to the quantum equilibrium measure (how this overwhelming majority is characterized will be made precise in section 4.4.4).

The $|\psi|^2$−measure of typicality is of course very reminiscent of Borns rule predicting the statistics of position measurements on ensembles of systems with the same wave function. Indeed, we shall derive the analogue Bohmian statement with respect to the empirical distributions of actual positions of Bohmian particles with respect to their effective wave function in section 4.4.4. But in spite of its formal similarity, this is indeed something very different. The fact that the measure of typicality of a Bohmian universe looks formally like the probability measure for empirical distributions in ensembles of subsystems is indeed rather special and should not seduce us to think that these two measures have a comparable meaning. The former makes physically only sense for sets on which it is very close to 1 or 0 (observe that there is e.g. no physical interpretation of a set of possible mircrostates of the universe with 'measure of typicality $\frac{1}{2}$'), the latter will be derived from the former by proving a law of large numbers in section 4.4.4.

Indeed, even if we perceived the measure of typicality as a statistical distribution, one should note that in general also statistical distributions of subsystems differ from the distributions of their encompassing systems: For example, in classical statistical mechanics an appropriate stationary measure for a closed system (and thereby an appropriate measure of typicality for a classical universe, whose energy is conserved) is given by the microcanonical distribution,

while the appropriate statistical description of subsystems is usually given by the canonical or grandcanonical distribution, which look formally quite different from the microcanonical one. Though, if a subsystem is spatially and thereby thermally well isolated from the rest of the world, the microcanonical description can be expected to work very well. In some very special sense, a similar observation can be made for Bohmian subsystems, though it cannot be based on spatial separation because of potential entanglement!

Remark on Thermal Non Equilibrium: In thermodynamics (for simplicity say from a classical point of view), one has to bite the bullet that we apparently do not live in a typical universe, since a typical universe (the overwhelming majority of microstates on phase space e.g. with respect to the stationary microcanonical measure as a measure of typicality) would be in thermodynamic equilibrium, i.e. filled by structureless homogeneously distributed gas or – taking gravitation into account – by structureless clusters of matter[182]. This is very remarkable and Boltzmann and many of his followers were and are very concerned about this fact. Boltzmann first tried to find an explanation in his famous *'fluctuation hypothesis'* [43] (i.e. the currently apparent low entropy state of the part of the universe surrounding us was only a fluctuation from its actual high entropy equilibrium state), but there are very good arguments why the fluctuation hypothesis is untenable and we have to accept that for some reason the initial state of the universe (or at least its microstate in the very far past, say 13 billion years ago) was an extremely special state of low entropy (this was referred to as the *past hypothesis* by Albert [5]).

But within this very small set of microstates belonging to possible universes with such a large amount of structure ($\equiv$ low entropy), the microstate of our actual universe (sloppily treated as a classical universe) does nonetheless belong to a very large subset (almost all), or in other words, conditional on the given atypicality of a universe in thermodynamic non equilibrium, our universe is a thermodynamically typical universe: The second law of thermodynamics is true, an ideal gas subsystem has a Maxwell-Boltzmann velocity distribution (of course to the degree to which the ideal gas and the classical approximation are justified), a long coin tossing series yields approximately half times head and half times tail and Galton boards reveal Gaußian distributions (observe that in a universe in thermal equilibrium for example something like a Galton board would not even exist) etc. This is what Dürr et al. called *typicality within atypicality* [129].

So the thermodynamic arrow of time finds its explanation in typicality, and analogously – as we shall see – find the characteristic statistics of quantum measurements their explanation in typicality in Bohmian mechanics. The universe in thermal non equilibrium is in Bohmian mechanics still in quantum equilibrium, i.e. in equilibrium relative to its wave function, otherwise quantum mechanics would not be so strikingly successful in making very precise predictions. For consideration of thermal non equilibrium from a Bohmian point of view, see e.g. [126].

[182] An appropriate quantity to measure the 'closeness' to thermal equilibrium is the *Boltzmann entropy* – which is best perceived as a function on phase space (which is practically constant on the huge domain of thermal equilibrium) – which agrees with the thermodynamic *Clausius entropy*. In contrast, the *Gibbs entropy* (which is a functional of probability distributions and essentially equals the information theoretic *Shannon entropy* and in quantum theory it is given by the *von Neumann entropy*) which is often used for technical purposes, does only coincide with the Boltzmann-Clausius entropy when the latter is at its maximum, i.e. if the considered system is in thermal equilibrium. The term *entropy* shall refer in the following exclusively to the thermodynamic Boltzmann-Clausius entropy, which is the appropriate quantity to describe transitions of actual microstates towards equilibrium, i.e. to understand the second law of thermodynamics microscopically.

4.4.4 Empirical Distributions

Now we come to the central step to link abstract statistical measures with empirical regularities which can be potentially observed, i.e. to link theory with experience. The path to establish this link is already familiar from probability theory: It is to be shown that the empirical relative frequencies of certain events are typically (i.e. for the overwhelming majority of initial conditions with respect to an appropriate measure) close to their associated probabilities (or equivalently, empirical mean values are close to associated expectation values), i.e. that a law of large numbers holds.

The setting within which we shall prove such a law is the following: Consider a spatial ensemble of M (microscopic) Bohmian subsystems with generic coordinates $q = (x_1, \ldots, x_M)$, where x_i are the generic coordinates of the i'th subsystem with respect to an appropriate coordinate system Σ_i, respectively, each having ψ as an effective wave function with respect to Σ_i (Σ_i is some coordinate frame of $\mathbb{R}^{3N}$ if each subsystem consists of N particles). Let ϕ be the wave function of the universe at that time (whatever it may look like), y the generic coordinates and Y the actual configuration of the rest of the world (whatever it may look like).

To prove a law of large numbers for empirical distributions among the members of such ensembles, we need two technical observations first:

(I) Effective Product Wave Functions: The first important non trivial observation is that (not in general but) in all relevant situations, the effective wave function of the $q-$system is given by the product wave function

$$\Psi(q) = \prod_{i=1}^{M} \psi(x_i) \tag{4.31}$$

This is always the case when Y alone is sufficient to assign to each subsystem of the ensemble the effective wave function ψ, i.e. independently from the x_i-coordinates of the other subsystems, in particular if the number M of subsystems and the number N of components of the coordinates x_i (i.e. the dimension of the frames Σ_i) are not macroscopically large, or if the 'information' that each subsystem has effective wave function ψ is in any way encoded in the environment of the ensemble, e.g. in a preparation device, in an experimenters brain or in the decohering environment. To see this, note that by definition of the effective wave function and the assumption that each x_i-subsystem has effective wave function ψ, we have for all $i = 1, ..., M$ with $q_i := (x_1, ..., x_{i-1}, x_{i+1}, ..., x_M)$ a decomposition of the universal wave function of the form

$$\phi(q, y) = \phi(x_i, q_i, y) = c_1^{(i)} \psi(x_i) \varphi_i(q_i, y) + c_2^{(i)} \phi_i^{\perp}(x_i, q_i, y) \tag{4.32}$$

where φ_i and $\phi_i^{\perp}$ have macroscopically disjoint $(q_i, y)-$support, and the actual configuration (Q_i, Y) of these coordinates is in the support of φ_i.

Now suppose that φ_i and $\phi_i^{\perp}$ have already macroscopically disjoint $y-$support for each i, i.e. the effective wave function of each ensemble member is already determined by the configuration Y of the environment of the ensemble (this is for example the case in one of the above mentioned situations: M and N not macroscopically large or 'information' about the effective

wave functions is encoded in the ensembles environment[183]). Then Y is in the intersection of the $y-$supports of all the φ_i and we have for all i

$$\phi(q, Y) = c_1^{(i)} \psi(x_i) \varphi_i(q_i, Y) \tag{4.33}$$

such that whenever $\phi(x_i, q_i, Y) \neq 0$, the effective wave function $\psi(x_i)$ cannot vanish as well and

$$\frac{\phi(q, Y)}{\psi(x_i)} = f(q_i, Y) \tag{4.34}$$

is a function which no longer depends on x_i. Since this holds for all i, we see (by separation of variables) that $\phi(q, Y)$ is of the form

$$\phi(q, y) = \prod_{i=1}^{M} \psi(x_i) \widetilde{\varphi}(Y) \tag{4.35}$$

such that the effective wave function of our ensemble is indeed given by the product state (4.31).

(II) Conditional Measure: Now we come to the measure with respect to which we want to prove a law of large numbers. It was shown above that we have primarily only one physically meaningful candidate: The equivariant quantum equilibrium measure. But applying the whole quantum equilibrium measure would be too less specific and thereby unrewarding, since the assertion which we want to prove is based on the assumption of an ensemble of systems with effective wave functions as considered in this section, so we should only 'count microstates' which are compatible with the assumption that such an ensemble exists[184]. Thus if $\mathcal{Y}$ is the variety of all possible configurations of the rest of the world such that each of the x_i-systems has effective wave function ψ, the appropriate measure of typicality to count universal microstates compatible with this situation is the conditional measure $\mathbb{P}^{\phi}(\cdot \mid \mathcal{Y}) =: \mathbb{P}^{\phi}_{\mathcal{Y}}$ and fortunately, this conditional measure evaluated on the configurations of our ensemble members is not so hard to find:

To see this, let us for a moment set the number M of ensemble members to 1 such that $x_i \equiv x$ and let $Y \in \mathcal{Y}$ such that $\psi = \psi^Y \equiv \psi_{\mathcal{Y}}$. As well known (see e.g. [254], the conditional density function $\rho(x \mid Y)$ of a density function $\rho(x, y)$ of two continuous random variables, conditional

[183]Here is a simple example of two systems with effective wave functions whose joint effective wave function is not of product form, which is taken from [126] Suppose both systems are macroscopically large (e.g. have configuration space of dimension of the order 10^{20} or greater) and have joint effective wave function $\psi(q) = \psi_1(x_1)\psi_1(x_2) + \psi_2(x_1)\psi_2(x_2)$, where ψ_1 and ψ_2 have macroscopically disjoint support (like pointers pointing to the left or right, respectively). If now both actual configurations X_1 and X_2 are in the support of ψ_1, each of the systems has effective wave function ψ_1. But observe that if the considered systems are subject to interaction with some sufficiently large environment, their joint effective wave function will not involve an 'empty branch' like $\psi_2(x_1)\psi_2(x_2)$.

[184]Actually, we cannot know whether the set of microstates of Bohmian universes in which such an ensemble exists has already small quantum equilibrium measure. Think of thermodynamics, where for example a universe in which a cylinder with a piston exists by which the ideal gas laws can by empirically investigated, has certainly negligibly small microcanonical measure on phase space. But this measure works perfectly well as a measure of typicality, if it is taken conditional on the actual low entropy state of the universe.

on the singular event $y = Y$ is given by [185]

$$\rho(x \mid Y) = \frac{\rho(x, Y)}{\rho(Y)} \tag{4.38}$$

with the marginal density $\rho(y) = \int \rho(x, y)\, dx$ evaluated at $y = Y$ in the denominator[186]. Thus we get

$$\mathbb{P}^{\phi}(X \in \Delta \mid Y) = \frac{\int_{\Delta} |\phi(x, Y) dx|^2}{\int |\phi(x, Y)|^2 \, dx} = \int_{\Delta} \left|\psi^{Y}(x)\right|^2 dx \tag{4.39}$$

which holds of course in general for any conditional wave function ψ^Y of a subsystem. If as considered here the conditional wave function is an effective wave function $\psi^Y = \psi$, we get in particular the familiar expression

$$\mathbb{P}^{\phi}(X \in \Delta \mid Y) = \int_{\Delta} |\psi(x)|^2 \, dx \tag{4.40}$$

Actually, we want to condition not on a single environmental configuration Y (which no one knows), but on the variety $\mathcal{Y}$ of all environmental configurations consistent with ψ being the effective wave function of the considered subsystem. But for this purpose there is indeed not more work to be done, since

$$\mathbb{P}^{\phi}(X \in \Delta \mid Y) \equiv \mathbb{P}^{\phi}(X \in \Delta \mid \mathcal{Y}) \tag{4.41}$$

for any $Y \in \mathcal{Y}$ and for all Δ. To see this, note that the right hand side of (4.40) is for all $Y \in \mathcal{Y}$ independent of Y, such that whenever $Y_1, Y_2 \in \mathcal{Y}$ we have $\mathbb{P}^{\phi}(X \in \Delta \mid Y_1) = \mathbb{P}^{\phi}(X \in \Delta \mid Y_2)$. Pretending now for a moment that $\mathcal{Y}$ is countable such that $\mathcal{Y} = \bigcup_k Y_k$ and defining the marginals

[185] This looks intuitively right, but is indeed a bit tricky since it involves conditioning on sets of zero measure: Y is a single point which has zero weight with respect to a continuous measure and thus if we consider the probabilities $\mathbb{P}$ associated with ρ, both $\mathbb{P}(X \in \Delta \wedge Y)$ and the marginal probability $\mathbb{P}(Y)$ of Y are zero such that the conditional probability

$$\mathbb{P}(X \in \Delta \mid Y) = \frac{\mathbb{P}(X \in \Delta \wedge Y)}{\mathbb{P}(Y)} \tag{4.36}$$

is a priori not well defined. But if we exchange the singular value Y for a sequence of shrinking $\varepsilon-$neighbourhoods $U_\varepsilon(Y)$ of Y (such that everything is well defined) converging to Y, we can give a meaning to (4.36). This is most easily understood by noting that for sufficiently small ε and continuous ρ we have for each x that $\rho(x, y) \approx \rho(x, Y)$ for all $y \in U_\varepsilon(Y)$, such that with the volume $V_\varepsilon(Y) := \int_{U_\varepsilon(Y)} dy$

$$\mathbb{P}(X \in \Delta \mid Y) \approx \frac{\int_{\Delta} dx \int_{U_\varepsilon(Y)} dy\, \rho(x, y)}{\int dx \int_{U_\varepsilon(Y)} dy\, \rho(x, y)} \approx \frac{V_\varepsilon(Y) \int_{\Delta} dx\, \rho(x, Y)}{V_\varepsilon(Y) \int dx\, \rho(x, Y)} = \int_{\Delta} \frac{\rho(x, Y)}{\rho(Y)} dx = \int_{\Delta} \rho(x \mid Y)\, dx \tag{4.37}$$

where the approximations become arbitrarily well for arbitrarily small ε. (4.38) can be be also shown more rigorously and general, e.g. by resorting to weak derivatives and cumulative distribution functions [254].

[186] Note that although conditioning is familiar from probability theory in the first place, it is of course meaningful for any normalized measure: Conditioning simply restricts the measure space to a submanifold and renormalizes it on this manifold to obtain a new measure space with the respective normalization (which works as in the present case even if the submanifold had initially measure zero).

$\mathbb{P}^\phi(Y_k) = \int |\phi(x, Y_k)|^2 \, dx$ and $\mathbb{P}^\phi(\mathcal{Y}) = \sum_k \mathbb{P}^\phi(Y_k)$, it follows that

$$\begin{aligned} \mathbb{P}^\phi(\mathcal{Y})\, \mathbb{P}^\phi(X \in \Delta \mid Y_k) = \sum_i \mathbb{P}^\phi(Y_i)\, \mathbb{P}^\phi(X \in \Delta \mid Y_k) = \\ \sum_i \mathbb{P}^\phi(Y_i)\, \mathbb{P}^\phi(X \in \Delta \mid Y_i) = \sum_i \mathbb{P}^\phi(X \in \Delta \wedge Y_i) = \mathbb{P}^\phi(X \in \Delta \wedge \mathcal{Y}) \end{aligned} \tag{4.42}$$

such that

$$\mathbb{P}^\phi(X \in \Delta \mid Y_k) = \frac{\mathbb{P}^\phi(X \in \Delta \wedge \mathcal{Y})}{\mathbb{P}^\phi(\mathcal{Y})} = \mathbb{P}^\phi(X \in \Delta \mid \mathcal{Y}) \tag{4.43}$$

for all k and of course for all Δ. This calculation is straightforwardly generalized to continuous $\mathcal{Y}$, where it only must be taken care of the fact that each single $Y \in \mathcal{Y}$ has zero measure such that one has to resort to one of the methods explained in footnote 185. Thus the appropriate measure to 'count microstates of the universe' conditional on the existence of a subsystem with effective wave function ψ, evaluated on the configurations of this subsystem, is (denoted now more explicitly to emphasise that it is a measure on the universal configuration space) given by

$$\mathbb{P}^\phi\left((X, Y) : X \in \Delta \mid Y \in \mathcal{Y}\right) =: \mathbb{P}^\phi_{\mathcal{Y}}\left((X, Y) : X \in \Delta\right) = \int_\Delta |\psi(x)|^2 \, dx \tag{4.44}$$

If we return now to our ensemble of M subsystems each having effective wave function ψ with respect to coordinates x_k, respectively, we straightforwardly obtain with the joint effective wave function of the ensemble given by equation (4.31) the analogue result: Let $Q = (X_1, \ldots, X_m)$ and $\mathcal{Y}$ the variety of all configurations of the rest of the universe (with respect to the ensemble) for which all ensemble members have effective wave function ψ, then the quantum equilibrium measure conditional on $\mathcal{Y}$ evaluated on the configurations of the ensemble members is given by

$$\mathbb{P}^\phi_{\mathcal{Y}}\left((Q, Y) : X_1 \in \Delta_1, \ldots, X_M \in \Delta_M\right) = \prod_{k=1}^{M} \int_{\Delta_k} |\psi(x_k)|^2 \, dx \tag{4.45}$$

Empirical Distributions

The central notion now in order to connect this theoretical measure with empirical evidence is the notion of an *empirical distribution*: If $Q = (X_1, \ldots, X_M)$ is the actual configuration of the subsystems, the associated empirical distribution is defined by the (distribution valued) function

$$\rho^Q_{\text{emp}}(x) = \frac{1}{M} \sum_{i=1}^{M} \delta(x - X_i) \tag{4.46}$$

and the associated *relative frequencies* for the events '$X_i \in \Delta$' are given by

$$\mathbb{P}^Q_{\text{emp}}(\Delta) = \int_\Delta \rho^Q_{\text{emp}}(x) \, dx = \int \chi_{\{\Delta\}}(x)\, \rho^Q_{\text{emp}}(x) \, dx = \frac{\#X_i \in \Delta}{M} \tag{4.47}$$

We want to show now that the empirical relative frequencies (4.47) are typically very close to the quantum probabilities

$$\mathbb{P}^\psi(X \in \Delta) = \int_\Delta \rho^\psi(x) dx = \int \chi_{\{\Delta\}}(x)\, |\psi(x)|^2 \, dx \tag{4.48}$$

if M is large. This is indeed now a direct consequence of the results obtained so far: The conditional quantum equilibrium measure (4.45) is a product measure, which can be written in view of (4.48) as

$$\mathbb{P}^{\phi}_{\mathcal{Y}}\left((Q,Y) : X_1 \in \Delta_1, \ldots, X_M \in \Delta_M\right) = \prod_{k=1}^{M} \mathbb{P}^{\psi}(X_k \in \Delta) \tag{4.49}$$

Thus, the X_k form a Bernoulli sequence of random variables, independently, identically distributed with respect to a $|\psi|^2$−distribution, and a weak law of large numbers connects this theoretical distribution with the associated empirical distributions. In particular, within the set of all possible microstates of Bohmian universes in which such an ensemble exists (characterized by the variety of environments $\mathcal{Y}$), the mircrostates for which the empirical distribution of the X_k deviates significantly from the $|\psi|^2$−distribution has quantum equilibrium measure very close to 0 (almost non of the microstates) if M is large, and we can even estimate how small it is (respectively how large M must be, respectively how close to $|\psi|^2$ the empirical distribution typically is): As a direct consequence of the Chebyshev inequality (the usual prove of the weak law of large numbers for independently, identically distributed random variables, see e.g. [119]), we obtain

$$\begin{aligned} &\mathbb{P}^{\phi}_{\mathcal{Y}}\left((Q,Y) : \left|\int \left(\rho^{Q}_{\mathrm{emp}}(x) - \rho^{\psi}(x)\right) f(x)dx\right| > \varepsilon\right) = \\ &= \mathbb{P}^{\phi}_{\mathcal{Y}}\left((Q,Y) : \left|\frac{1}{M}\sum_{i=1}^{M} f(X_i) - \int |\psi(x)|^2 f(x)dx\right| > \varepsilon\right) = \\ &= \eta(\varepsilon, M, f) \end{aligned} \tag{4.50}$$

where for any measurable (usually real valued) function f and for any given (arbitrarily small) $\varepsilon > 0$, the actual value of η dies like M^{-1}, more precisely

$$\eta(\varepsilon, M, f) \leq \frac{\mathrm{Var}(f(X))}{\varepsilon^2 M} \tag{4.51}$$

with the variance $\mathrm{Var}(f(X)) = \int \left(f(x)\right)^2 |\psi(x)|^2 dx - \left(\int f(x)\,|\psi(x)|^2 dx\right)^2$ of the random variable $f(X)$.

If we choose now for f the indicator functions $f(x) = \chi_{\{\Delta\}}(x)$ of (measurable) regions Δ in configuration space, (4.50) yields the desired assertion: If an ensemble of M identically prepared subsystems is given, each having ψ as an effective wave function (relative to some coordinate frame), the conditional quantum equilibrium measure of universal microstates (configurations) for which the empirical relative frequencies (4.47) deviate significantly from the quantum probabilities (4.48) is very small if M is large. More compactly, if M is large the $\mathbb{P}^{\phi}_{\mathcal{Y}}$−measure of the disagreement sets

$$\mathrm{Dis}^{\psi}_{\varepsilon}(\Delta) := \left\{(Q,Y) : \left|\mathbb{P}^{Q}_{\mathrm{emp}}(\Delta) - \mathbb{P}^{\psi}\left(X \in \Delta\right)\right| > \varepsilon\,,\ Y \in \mathcal{Y}\right\} \tag{4.52}$$

is negligibly small.

The actual universal wave function ϕ and the actual configuration Y of the rest of the universe are not specified further beyond their compatibility with ψ being the effective wave function of

the subsystems. Consequently, the smallness of the disagreement sets (4.52) is true irrespective of the fact how the actual universal pair (ϕ, Y) precisely looks like, if only ψ is the effective wave function of each subsystem, respectively, and the number of subsystems is large enough!

This is the strongest possible result we can wish for, since we can take the actual environmental configuration Y – which involves galaxies, solar systems, trees, animals or humans capable of performing quantum experiments – as whatever it is, regardless of how it may look like in detail, as long as it is consistent with the existence of an ensemble of subsystems as considered. In particular, this result holds even independently of the question whether (Q, Y) belongs a set of large quantum equilibrium measure or not. Indeed, we cannot empirically exclude the possibility that the set of all possible Bohmian universes in which an ensemble of M subsystems, each having ψ as effective wave function exists itself has small $|\phi|^2$ −measure, in which case the smallness of the disagreement sets (4.52) with respect to the full $\mathbb{P}^\phi$−measure would be of no explanatory value (think of the analogy to thermodynamics as explained in footnote 184). But what we have shown is that within the set of universes which look like our actual one with respect to the existence of (the possibility to prepare) ensembles of subsystems with effective wave functions ψ, the subsystems coordinates are typically $|\psi|^2$ −distributed. Thereby, Bohmian subsystems behave typically the way we encounter it in quantum experiments, i.e. Bohmian mechanics explains the empirically so well verified quantum phenomena similar to the way Boltzmanian statistical mechanics explains why ice cubes are melting in water. How the measurement formalism arises from quantum equilibrium of subsystems shall be shown in the following section. But beforehand, a few final remarks are in order.

Remarks

Time ensembles: If we think of quantum experiments to empirically recover the quantum statistics, the basic assumption of spatial ensembles at a single time is admittedly very special and rather contrived. After all, quantum experiments are usually repeated many times and the ensembles are thus not (only) spatial ensembles but (also) ensembles in time. But indeed, the analogous analysis to the present one can be handled for time ensembles as well, only for time ensembles it encounters additional challenges which require a special treatment (see [126]).

There is one particular point about the statistical analysis of Bohmian ensembles in time which is worth mentioning: The notion of conditional wave functions is of course more fundamental than the notion of effective wave functions, since a given system has always a precisely defined conditional wave function (which is no longer necessarily so if spin is introduced, see footnote 172), whereas effective wave functions exist only under certain boundary conditions (like they are given in measurement like situations) and in view of infinite tails only approximately to a very high degree of accuracy (i.e. fapp). So the question arises naturally, whether the results derived in this section are valid for systems with general conditional wave functions without assuming effectiveness, as well. Indeed, almost all arguments and derivations in this section, treating a spatial ensemble of Bohmian systems identically prepared in effective wave function ψ, did not make use of the assumption that ψ (as a conditional wave function) is effective. Only the fact that the global ensemble wave function has product structure (4.31), which we derived under very mild assumptions above and which was essential to get independently, identically distributed random variables for which a law of large numbers holds, is usually not true for conditional wave functions which are not effective, as one may easily comprehend. This

blocks the way to extend the present analysis to spatial ensembles with identical conditional wave functions, if it is not assumed that they are effective (but note that this does not restrict the generality with respect to empirically relevant situations like quantum measurements, where prepared initial states as well as final states as well as macroscopic pointer states are always given by effective wave functions). But in the statistical analysis of Bohmian time ensembles a product assumption like (4.31) turns out to play no role, which shows that the coordinates of Bohmian systems with conditional wave function ψ^Y are typically $\left|\psi^Y\right|^2$ −distributed, no matter if ψ^Y happens to be an effective wave function or not.

Absolute uncertainty: The analysis of Bohmian mechanics so far has shown that the dynamical as well as the statistical aspects of the configurations of a Bohmian subsystem are thoroughly determined by its conditional wave function. This entails fundamental limitations of the possible 'transfer of information' from a Bohmian subsystem to its environment – including e.g. an experimenter, whose possible gain of knowledge about a given system is thereby fundamentally restricted. However 'information transfer' or 'gain of knowledge' might be precisely defined in physical terms, it must in some way establish a correlation between the subsystem and its environment, which involves some record of 'the transferred information' somehow encoded in the environmental configuration, like in case of experimental inquiries the position of a pointer, a display on a computer screen, printed information on a sheet of paper or changes upon perception in the experimenters brain etc. But all such records are already adopted into the conditional wave function of the subsystem at that time, so empirical distributions of subsystems selected upon particular such records will not deviate from the distributions of other subsystems with the same conditional wave function (which is actually always an effective wave function when the system is 'observed' by the environment).

To make this more explicit, consider again the ensemble of M subsystems of a Bohmian universe with identical effective wave functions ψ with respect to coordinates x_i, respectively, and the actual configuration Y of the rest of the world as above. Now suppose the members of a given subensemble of this ensemble share a given feature $\mathcal{P}$ of their configurations (like being located in some given proper subset of the support of ψ) which is not encoded in ψ but relative to each system in a given feature of the environmental configuration Y. We may imagine something like a pointer in the neighbourhood of each subsystem which points onto the symbol $\mathcal{P}$, and if the correlation is perfect, the empirical distribution of the subensemble members is loosely denoted constrained by $\mathbb{P}_{emp}(\mathcal{P} \mid \text{ pointer points onto } \mathcal{P}) = 1$ and thereby $\mathcal{P}$ is unambiguously revealed by the environment, but we might equally consider less perfect correlations.

Now observe that any distinguished configurational feature $\mathcal{P}$ of the subensemble members of course distinguishes their empirical distribution from the empirical distribution of the full ensemble. But given the number of the subsensemble members is large enough, this cannot be if quantum equilibrium for subsystems with effective wave functions is empirically adequate, which is obviously the case (given our universe is a Bohmian one): Given any environmental Y which is consistent with ψ being the effective wave function of each subsystem (Y is in the considered case supposed to involve the features encoding properties of the subensemble), the members of the full ensemble as well as the members of the subensemble are independently distributed according to a $|\psi|^2$ −distribution as shown above, i.e. if the ensemble and the subensemble are large enough, both empirical distributions are typically close to the probability distribution

given by $\mathbb{P}^{\phi}(X \in \Delta \mid Y) \equiv \mathbb{P}^{\psi}(X \in \Delta) = \int_{\Delta} |\psi(x)|^2 \, dx$. Thus, whatever the considered features of Y may be, they cannot 'represent information' about the subsystems beyond the information that their coordinates are $|\psi|^2$ −distributed.

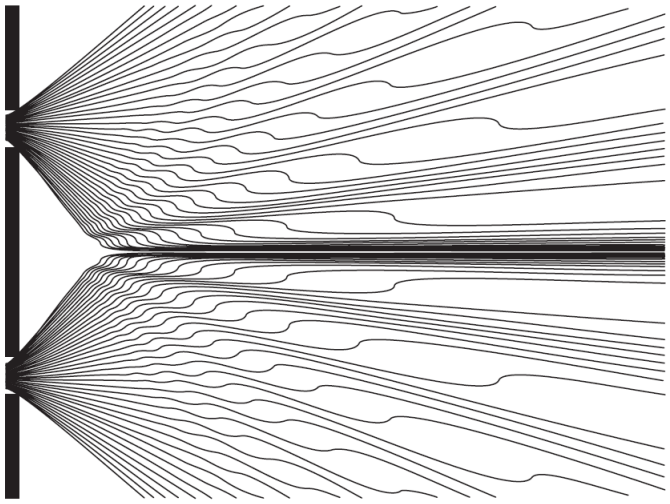

FIGURE 10
BOHMIAN TRAJECTORIES IN A DOUBLE SLIT EXPERIMENT
The vertical density of the trajectories is proportional to $\rho^{\psi}(\boldsymbol{x}) = |\psi(\boldsymbol{x})|^2$, where ψ is the (yet uncollapsed) effective guiding wave function.

This may be visualized by the double slit experiment: If a beam of particles is sent towards a double slit as in the famous quantum double slit experiment, the effective wave function of each particle behind the slits is given by the well known interfering superposition of the two wave packets originating from the slits, and if a given particle hits the screen – which we idealize to represent an at least almost ideal, non destructive position measurement – its effective wave function collapses to a wave packet highly peaked about the spot (in Fig. 10 the Bohmian trajectories inbetween the slits and the screen are depicted, see e.g. [129] for discussion of this experiment from a Bohmian perspective). The pattern of measured as well as unmeasured positions at the screen (or shortly before in case of the yet unmeasured positions) is for large enough particle number given by the well known vertical bars. Now we might consider the subensemble of particles which did not yet hit the screen, whose positions constitute only one particular of the bars. The above line of argument shows now, that in quantum equilibrium this subensemble cannot be selected by any features of the environmental configuration, but if it is, like when the particles have hit the screen, their effective wave functions can no longer be given by the superposition of the two wave packets evolving from the slits.

This important implication of quantum equilibrium, namely the impossibility to gain more information about the configuration of a subsystem than contained in its $|\psi|^2$ −distribution[187], is called *absolute uncertainty* [126].

[187] It follows analogously that Bohmian configurations of subsystems cannot be 'controlled from outside' (e.g. by preparation) more accurately than determined by the $|\psi^Y|^2$ −distribution generated by their actual (prepared) conditional (effective) wave function.

But note that, as it was already implicit in the above discussion of the double slit experiment, absolute uncertainty does not refer to the past: For example, if we perform an appropriate position measurement, we may well *learn* more about the actual configuration of a Bohmian system than encoded in its pre-measurement conditional wave function. If a Bohmian subsystem, e.g. a single particle, with not so well localized conditional wave function is subject to a sensible position measurement, the latter will reveal the (approximate) actual position of the Bohmian particle much more precisely than encoded in the $|\psi|^2$ −distribution generated by its pre-measurement conditional wave function. But the post-measurement conditional (effective) wave function will be highly peaked about the 'measured position' and its width (which might be neglected for all practical purposes) corresponds to a fundamental inaccuracy or uncertainty in the position measurement[188].

That in statistical physics the possible gain of information about the detailed state of a system in equilibrium is very much restricted is of course not a new surprising feature of Bohmian mechanics, but very well known from classical thermodynamics, where it can be understood in perfect analogy to the present considerations from a Boltzmanian point of view[189].

Quantum Non-Equilibrium: In view of the important field of research of non equilibrium thermodynamics one might wonder whether quantum non equilibrium Bohmian mechanics is a field worth considering, and indeed some work on this subject has been published (see e.g. [87] and references therein). But one should note that while we live in a world in thermal non equilibrium, the striking predictive success of quantum theory is an immediate consequence of quantum equilibrium in Bohmian mechanics, such that an analysis of quantum non equilibrium can be done of course, but is equally empirically irrelevant as for example an analysis of violations of the second law of thermodynamics would be. Such things – although in a strict sense not forbidden by any law of nature – obviously do not happen in the world we live in and typicality gives us a very good understanding why they do not happen.

4.5 Measurement Like Process Revisited

It was shown above that within the Bohmian framework we can understand a measurement like process (see chapter 1) like

$$\psi\varphi_0 \xrightarrow{U} \sum_i c_i\psi_i\varphi_i \xrightarrow{(*)} \psi_k\varphi_k \tag{4.53}$$

coherently by analysing the Bohmian equations, where the transition $(*)$ amounts to the effective collapse. What remains to show in order to start the machinery of the quantum theory of measurement developed in chapter 1 without resorting to a vaguely defined ad hoc collapse

[188]Moreover, due to the quantum mechanical dispersion relation well localized wave functions spread very fast in time such that the uncertainty about the actual position grows fast again subsequent to such a measurement resulting in a well localized effective wave function. Together with an understanding of how a measurement must look like whose statistics is given by the momentum operator, this is the root of Heisenberg's uncertainty principle in Bohmian mechanics [129].

[189]If one is afraid whether the impossibility to learn empirically more about a systems configuration than encoded in its conditional (effective) wave function might make the configurations superfluous, one should note first of all that only configurations make it possible to talk about conditional or effective wave functions at all (see also the remarks on infinite tails below).

postulate is to show that the transitions $(*)$ indeed happen with the probabilities predicted by ordinary quantum theory, i.e. the pointer points onto the value k (which is to say $Y(T) \in \mathrm{supp}(\varphi_k)$) with probability $\mathbb{P}^{\psi\varphi_0}(k) = |\langle\psi_k\varphi_k|\, U(\psi\varphi_0)\rangle|^2 = |c_k|^2$ subsequent to the measurement.

This is an easy task by now: We can simply calculate the probability for $Y(T) \in \mathrm{supp}(\varphi_k)$ in quantum equilibrium as derived in the previous section, and exploit the (fapp) mutual disjointness of the supports of the pointer states

$$\mathrm{supp}(\varphi_i) \cap \mathrm{supp}(\varphi_k) = \emptyset \text{ for } i \neq k \tag{4.54}$$

to get

$$\begin{aligned}
\mathbb{P}^{\sum_i c_i\psi_i\varphi_i}(Y \in \mathrm{supp}(\varphi_k)) &= \int d^n x \int\limits_{\mathrm{supp}(\varphi_k)} d^m y \left|\sum_i c_i\psi_i(x)\varphi_i(y)\right|^2 = \\
&= \int d^n x \int\limits_{\mathrm{supp}(\varphi_k)} d^m y \sum_i |c_i\psi_i(x)\varphi_i(y)|^2 + \\
&+ \int d^n x \int\limits_{\mathrm{supp}(\varphi_k)} d^m y \sum_{i\neq j} \overline{c}_i c_j\, \overline{\psi_i}(x)\overline{\varphi_i}(y)\, \psi_j(x)\varphi_j(y) \overset{(4.54)}{=} \\
&= |c_k|^2 \int |\psi_k(x)|^2\, d^n x \int |\varphi_k(y)|\, d^m y = |c_k|^2
\end{aligned} \tag{4.55}$$

This immediately entails that the formalism of the quantum theory of measurement developed in chapter 1 for making predictions for results of measurement (like) processes follows from the Bohmian equations without postulating operators as observables (which was already shown in chapter 1) and without resorting to some vaguely defined ad hoc collapse postulate. In Bohmian mechanics, distributions of matter in space are objective and their dynamics is such that distributions of matter constituting the displays of measurement outcomes behave such that they display the values predicted by orthodox quantum theory.

4.6 Relativistic Bohmian Mechanics

The following two subsections shall give a brief introduction to some of the developments towards relativistically covariant modifications of Bohmian mechanics, respectively Bohmian QFTs. The core of the foregoing analysis of non relativistic Bohmian mechanics – in particular the dynamical notions of conditional and effective wave functions of subsystems and the statistical quantum equilibrium analysis entailing the quantum formalism to predict outcomes of quantum measurements – stay essentially valid. Only the explicit form of the dynamics changes to a relativistically covariant one, quantum nonlocality – which is inherent in the Bohmian law of motion – must be accounted for with respect to the relativistic structure of space-time and if a variable number of particles shall be allowed for – like in a QFT – empirically adequate and preferably natural ways of implementing this must be found[190].

[190] One may also add that for spinor valued wave functions things get a bit more complicated since in this case (in contrast to the spin free case) a subsystem does not need to have always a conditional wave function, as explained in footnote 172. Moreover, one might add an appropriate treatment of identical particles. But these features need not be seen as specifically relativistic (the subtleties of Bohmian description of spin exist already

These sections will not provide a comparably deep analysis of relativistic Bohmian mechanics as the analysis of the non relativistic theory in the previous sections was. It shall be rather illustrated that the relevant building blocks are available and that there are no fundamental obstacles, to formulate covariant Bohmian QFTs whose predictions agree with the relevant (regularized) current QFTs (of course, the problems of these theories, like with consistent and satisfying descriptions of interaction, remain unsolved in the Bohmian versions). In particular, the relativistic considerations and assumptions on which the results of chapters 2 and 3 rely are not fundamentally at odds with Bohmian quantum theory, if properly generalized to relativistic Bohmian QFT. For detailed and rigorous derivations, analysis and discussion of relativistic Bohmian mechanics and Bohmian QFTs the reader is referred to the literature (see e.g. the chapters in part III of [128]).

4.6.1 Lorentz Invariance

Bohmian mechanics can be formulated in relativistically covariant ways as well by starting with a covariant wave equation like the Dirac equation

$$i\partial_t \psi_t(\boldsymbol{q}) = \mathscr{H}_D \psi_t(\boldsymbol{q}) \tag{4.56}$$

with the spinor valued Dirac wave function $\psi_t \in \mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ and Dirac Hamiltonian $\mathscr{H}_D$ (e.g. in the free case $\mathscr{H}_D = -i\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m$) instead of the Schödinger equation. Then the velocity field on configuration space takes a different form as on the right hand side of equation (4.4), but at least in the case of a single Bohmian particle it is always easily found if the covariant wave equation (like the Dirac equation) provides a (covariant) continuity equation

$$\partial_\mu j^\mu = \partial_t \rho^\psi(\boldsymbol{q}, t) + \boldsymbol{\nabla} \cdot \boldsymbol{j}^\psi(\boldsymbol{q}, t) = 0 \tag{4.57}$$

by dividing the 3−current $\boldsymbol{j} \equiv \boldsymbol{j}^\psi$ by its associated density $j_0 \equiv \rho^\psi$:

$$\boldsymbol{v}^\psi = \frac{\boldsymbol{j}^\psi}{\rho^\psi} \tag{4.58}$$

E.g. for the Dirac equation it is well known [158, 300] that the conserved current takes the nice form

$$\boldsymbol{j}^\psi = \psi^\dagger \boldsymbol{\alpha} \psi \tag{4.59}$$

where $\psi^\dagger$ is the adjoint Dirac spinor, and the density

$$\rho^\psi = \psi^\dagger \psi \tag{4.60}$$

such that the Bohmian velocity field is given by

$$\boldsymbol{v}^\psi = \frac{\psi^\dagger \boldsymbol{\alpha} \psi}{\psi^\dagger \psi} \tag{4.61}$$

on the level of the Pauli equation and for a treatment and analysis of identical particles no underlying relativistic wave equation is required) and are not so central for the present purposes (nonetheless spin and identical particles will come into play implicitly below, e.g. by considering the space of antisymmetric solutions of the $N-$particle Dirac equation).

In order to formulate the Bohmian law of motion for N Dirac particles, a well established way is to invoke a distinguished foliation of space-time into spacelike leafs (recall that the Bohmian motion of a given particle in the case of an entangled wave function – which is to say in general – depends on the simultaneous positions of the other particles, but there is of course no absolute simultaneity in relativistic space time). Such a theory is called a *hypersurface Bohm-Dirac model* (HBDM) [120].

A basic element of a HBDM is the Lorentz invariant generalization of the spinor valued one particle Dirac wave function $\psi_t(\boldsymbol{q}) = \psi(t, \boldsymbol{q}) = \psi(x)$ to $N > 1$ particles, which is a *multi time wave function* $\psi(x_1, \ldots, x_N) = \psi((t_1, \boldsymbol{q}_1), \ldots, (t_N, \boldsymbol{q}_N))$ given by a solution of a *multi time Dirac equation* (which is actually a set of N Dirac equations, each of which contains the time derivative with respect to one of the time variables t_k, respectively [221]). The analogue of the wave function at a given time in the above non relativistic analysis is then in a HBDM the multi time wave function restricted to a given leaf of the foliation and the Bohmian positions at a given time in the non-relativistic analysis are replaced by the crossing points of the Bohmian world lines with a given leaf (see [120] for the technical details).

The conceptual status of the foliation with respect to what might be called 'the spirit of relativity' is indeed controversial (see e.g. [121] for discussions of that issue), whereas it is uncontroversial that the theory can be formulated in a perfectly covariant way[191] and that the precise form of the foliation has no phenomenological impact: Although the Bohmian law of motion crucially depends on the foliation and the statistical quantum equilibrium analysis is performed primarily on the leafs of the foliation and it is even a theorem in Bohmian mechanics, that quantum equilibrium cannot hold in all Lorentz frames, it can be shown that quantum equilibrium with respect to the leafs of the foliation guarantees that macroscopic objects like displays displaying the results of quantum experiments reproduce the quantum statistics independently of the Lorentz frame of reference. In particular, there is no possibility whatsoever to detect the 'true foliation' by experiment[192].

For now, this rough introduction shall suffice. To summarize, it is possible to formulate Bohmian mechanics as a relativistically covariant theory of N particles such that the measurement formalism of quantum theory can be derived from an analysis of its equations in the same spirit as in the non relativistic case described above. These predictions for the outcomes of measurement (like) processes are then independent of the Lorentz frame of reference (in the sense discussed at the beginning of chapter 2), in particular, they do not depend on the precise form

[191] In particular, not only the law of motion can be formulated covariantly with respect to a given foliation, but also the foliation itself can be given by a covariant law. E.g. the hyperboloids of constant timelike Lorentz distance from the Big Bang of a Bohmian universe are a Lorentz invariant structure by which the law of motion can be formulated [325], or the frame in which the total mass-energy of the universe is at rest, which is defined by the covariant wave function of the universe and the covariant energy momentum tensor in any relativistic quantum theory [121].

[192] Thus the foliation has nothing to do with – and cannot be used for – synchronization of clocks or the like, which would be apparently at odds with relativity. The physical arena of the theory is still relativistic space-time, which is not touched, and the foliation is rather employed in addition to this structure to account for the nonlocal correlations of entangled particles. The foliation is primarily relevant for subtle physical processes which originate from quantum nonlocality (and it is irrelevant as soon as e.g. decoherence creates effective product wave functions) while the whole phenomenology of special relativity and quantum theory survives, the formal covariance of the theory need not to be touched and the distinguished structure need not to be added as an extra element to the theory as explained in footnote 191 (and more deeply in the references therein).

of the invoked spacelike foliation of space-time which determines the dynamical law and with respect to which the statistical quantum equilibrium analysis is performed [120, 121].

4.6.2 Quantum Field Theory

For Bohmian extensions to QFT, there are two possibilities for the choice of the ontological content of the theory, from which the predictions of QFT shall be derived: Either matter is considered to consist of *point particles* as we have done so far, or of *fields* (which are accordingly guided by wave functionals). For the first possibility there are two further variants possible, namely an ontology of particles which can be created and annihilated in the literal sense (which seems to require a stochastic element in the dynamics which interrupts the deterministic motion) or an ontology of persistent particles from which phenomena involving a variable number of particles can be derived from an effective description as (not actual but) apparent phenomena.

For all these possibilities approaches have been developed. It is remarkable, that in the relativistic case fermionic QFTs seem to fit into a Bohmian framework much more straightforward with an underlying particle ontology, while bosonic QFTs seem to be easier to derive from a Bohmian field ontology (attempts for a field ontology for relativistic fermions and a particle ontology for relativistic bosons have been made, but are not very convincing so far). Since matter (like the matter constituting the final configurations of measuring devices in quantum measurements) seems to be fundamentally constituted exclusively of (spin$-\frac{1}{2}$) fermions (quarks and leptons) and an ontology for matter is at the end of the day all we need to get rid of the ambiguities of quantum theory, we shall focus in the following survey on particle ontology for (relativistic) fermions and conclude only with a short remark on field ontology for QFT.

The term *'relativistic'* refers here to second quantization of relativistic wave equations (or more generally, a Hilbert space which carries a unitary representation of the Poincaré group), while we will no longer consider the question of full Lorentz invariance in the following. As far as predictions for outcomes of quantum measurements are concerned, the predictions of the Bohmian models agree with the predictions of the associated QFTs and are thus trivially as Lorentz invariant as these predictions are (in particular Lorentz invariant for relativistic QFTs). Regarding the explicit dynamics one can take for each $N-$particle aspect of the following models (like associated with the $N-$particle sector of Fock space) an associated HBDM (see the preceding section).

But Lorentz invariance is violated in these models for another reason which lays at the heart of the conceptual problems faced by current QFTs, namely that they need regularization – like an ultraviolet cut-off in the momentum spectrum – to be dynamically well defined, which destroys Lorentz invariance. Finally removing the regularization with renormalization group techniques is as yet only possible for the matrix elements of the $S-$matrix order by order in the associated formal perturbation series, but not for dynamical descriptions of processes which happen in between the asymptotically free in and out states [300]. As soon as actual (non trivial) dynamics is considered, regularization is still inevitable and standard in QFT; and as long as well defined equations of motion for relevant[193] QFTs without need of artificial a priori requirements (e.g. regarding the momentum spectrum) are outstanding, any Bohmian QFT (which goes beyond a black box transforming initial states into final states) requires proper

[193] As 'not relevant' in the present context shall be regarded models like ϕ^4-theory in two space-time dimensions, in the sense that they are not qualified as serious models to describe matter.

regularization for its well definedness as well. Nonetheless, one usually assumes that the effects of proper regularization (e.g. large enough cut-offs) are dynamically negligible for all practical purposes.

The hard questions of consistent and well defined descriptions of interaction are still open (as essentially in standard QFT as well, in particular beyond the regimes of scattering theory), but the following models can be at least formulated for external potentials or simple particle interactions like mediated by Coulomb potentials. For promising ideas how electromagnetic radiation might be implemented without encountering self interaction and ultraviolet divergences see e.g. [220, 327]. The possibilities to include other interactions like weak or strong interaction are roughly addressed in [86] (for discussion of the special problems of gravity and its quantization, see [155]). An encompassing Bohmian standard model is still outstanding, but its building blocks are available in principle.

Bell Type QFT

Bohmian mechanics can be extended to involve an essential feature of relativistic QFTs – namely a description of creation and annihilation of particles, only now in a literal sense. Bell firstly proposed a discrete model for fermions on a spatial lattice, later continuous generalizations where developed by Dürr et al. and subsumed under the label *Bell-type QFTs* [122, 124, 125]. In such theories, the deterministic dynamics of the particle's motion is supplemented by a discrete stochastic jump process changing roughly speaking at random times and positions the number of particles involved in a way, such that the probabilities for particle creation and annihilation of an associated (regularized) QFT are recovered.

The Bell-type QFT scheme associates with a given QFT and its Hamiltonian a $|\psi|^2-$ distributed Markov process on a configuration space of a variable number of particles[194], i.e. the scheme is in the first place independent of a particular model. It can be straightforwardly implemented for non relativistic bosons and fermions and for relativistic fermions, while as yet the relativistic bosonic case faces complications[195]. Bell conjectured that the stochastic element in his model might vanish in the continuum limit, which did not came true for the fermionic Bell-type QFT models considered here, which are supposed to be the continuum limit of Bell's model.

The Hilbert space of the guiding wave functions is (possibly symmetrized or anti-symmetrized) Fock space and for a single particle species, configuration space is given by

$$\mathcal{Q} = \bigcup_{N\in\mathbb{N}_0} \mathcal{Q}_N \tag{4.62}$$

[194]More precisely, the equations of motion of Bell-type QFT define a $|\psi|^2-$distributed Markov process on the configuration space (4.62) (or the Cartesian product of several ones in case of several particle species) which has a continuous deterministic part, corresponding to deterministic Bohmian trajectories, and a stochastic jump part for which one obtains jump rates. For concrete implementations of the Bell-type QFT scheme one can take a closer look at the latter, e.g. for external field QED it turns out that its associated Bell-type QFT involves four kinds of jump processes: Pair creation and pair annihilation of electron positron pairs, and interestingly also spatial jumps of electrons and positrons. Moreover, it is easy to see that the probabilities for creation or annihilation of pairs at a distance significantly larger than the electron Compton wave length are practically zero and the same holds true for spatial jumps across a significantly larger distance.

[195]The different level of difficulty to implement the scheme for relativistic fermions or bosons, respectively, is not grounded in the antisymmetry or symmetry of the wave functions but rather in the Hamiltonians – e.g. Dirac or Klein-Gordon – with which this (anti-)symmetry is associated by the spin statistics theorem.

(for several particle species, one forms the Cartesian product of these $\mathcal{Q}-$spaces associated with the respective particle species). The sectors $\mathcal{Q}_N$ of configuration space $\mathcal{Q}$ are the $N-$particle configuration spaces like $\mathbb{R}^{3N}$ or $\mathbb{R}^{3N}/$permutations (the latter in the case of indistinguishable particles, i.e configurations in which the positions of two or several particles are exchanged, which corresponds physically to one and the same situation, are identified). While the guiding wave functions might well involve superpositions of states of different particle numbers (which probably will be even the generic case, as it is not least suggested by the results obtained in chapter 3), the actual configuration is at any time a configuration of a definite number of particles, i.e. an element of one of the sectors $\mathcal{Q}_N$ of configuration space.

The precise form of the guiding law and the jump law in Bell type QFTs can be found in [122, 125], where rigorous derivations, concrete examples, existence results etc. are given. The predictive empirical content of a given Bell-type QFT agrees with the empirical content of its associated QFT. Concrete models like for external field QED were explicitly constructed [125]. Of course, the development of empirical predictions resort at the end of the day again essentially to the quantum equilibrium analysis presented above (where equivariance must of course be generalized to processes not only involving deterministic time evolution but also the stochastic jumps in the obvious way).

If one should regard the Bell-type QFT pattern in its present form as a serious candidate for a fundamental theory of nature is controversial among physicists working in Bohmian mechanics (examples of possible conceptual objections are discussed in the following paragraph). But at least and in the first place its existence disproves the widespread prejudice that only non relativistic Bohmian mechanics worked very well but that it was irreconcilable in principle with current relativistic QFTs and in particular with their characteristic feature of a variable number of particles. It is a matter of fact that for nowadays regularized fermionic QFTs there exist models of particles moving on trajectories (which do thereby not suffer from a measurement problem) such that the predictions of the QFTs are among the predictions of these models (of course the totality of predictions of the latter is much more comprehensive, since it includes much more than outcomes of quantum measurements). Current Bell-type QFTs might be improved, modified to a more elegant or natural form or identified as effective description of some more fundamental theory (notice that just taking the talk about creation and annihilation of particles, which pervades the language of QFT, seriously, is apparently not unnatural).

Concrete objections concern e.g. the intrinsic stochasticity of Bell-type QFTs and their need for a distinguished initial vacuum reference state. Regarding the first objection, the point is not to desperately cling to determinism, after all, the compelling aspect of Bohmian mechanics is not in the first place its determinism but rather the entire absence of vagueness in its formulation. But Bohmian mechanics provides a Boltzmanian statistical analysis by which the statistical predictions of quantum theory can be derived from an ontology of persistent particles which are guided by a deterministic law, so one might (but not necessarily need to) perceive it as rather ad hoc to superimpose a second dynamical law destroying the persistence of the ontology by a process whose statistical nature is suddenly intrinsic and inexplicable. Regarding the second objection, the actual dynamical content of Bell-type QFT depends crucially on the choice of a distinguished vacuum state which corresponds to the sector of zero particles of configuration space. But one can show [134] that even in rather simple situations like in presence of a slowly varying external potential, the obvious choice of a vacuum (the so called Furry picture, which

corresponds at the one particle level to a division between negative and positive energies according to the actual Hamiltonian at any time) is not independent of the Lorentz frame of reference (i.e. roughly speaking, what appears as vacuum in one frame looks like particles in another frame according to this canonical choice[196]). This notorious non uniqueness of the vacuum state in relativistic QFT is also nicely illustrated by the more famous *Unruh effect*, which suggests that also without external potentials the Minkowski vacuum does in general not 'look like the vacuum' if one switches to an accelerated frame of reference. Also this objection is rather an aesthetic than a logical or physical refutation of the Bell-type QFT scheme, since these 'non uniqueness of the vacuum' arguments do not concern an ontology in the first place but rather are (as all predictions of standard quantum theory) predictions concerning operational notions like detector clicks (e.g. of an Unruh-DeWitt detector in case of the Unruh effect) which can be (and will be) among the predictions of appropriate Bell-type QFT models without contradictions, regardless of the fact that the precise dynamics of these models relies on the choice of a distinguished vacuum state.

Both of these potential deficiencies of the Bell-type QFT scheme (the intrinsic stochaticity and the dependence on a distinguished vacuum state) can be overcome in relativistic fermionic QFT, in principle, if the Dirac sea is taken seriously.

Dirac Sea

The first proposal for a Bohmian Dirac sea model to obtain the predictive framework of relativistic fermionic QFT was formulated loosely by Bohm et al. [41, 42], elaborate investigations in this direction were presented later by Colin [85, 84] and Colin and Struyve [86] and recently by Deckert et al. [104]. The authors of the latter work describe the underlying idea of an ontology of persistent particles, whose effective description involves the phenomenon of pair creation and annihilation as an apparent but not actual phenomenon: *'...there is no particle creation or annihilation. There are only conditions under which particle motion becomes observable or fails to be so. These conditions are not unique; they may even depend on the state of motion of the observer (as in the Unruh effect). We build an ontology of permanent particles on this idea.'*

One starts with the Dirac sea model as explained in appendix A. In order to obtain a physically and mathematically well defined theory, proper regularizations shall be posed in the first place by assuming finite space (making the energy spectrum of the free Dirac equation discrete) and an ultraviolet cutoff (making this discrete spectrum finite), such that $\mathcal{H}_-$ becomes finite dimensional and the Dirac sea $\widetilde{\Omega}$ in (A.2) (see appendix A) a wedge product if finitely many states and thus well defined.

[196] Here one has to be careful what 'looks like' precisely means, because the notion of 'looks like' is defined operationally spoken always relative to a given detector (model) which represents 'the observer who is looking'. In this sense, the Reeh-Schlieder theorem discussed in section 3.5 suggests that the vacuum is actually unobservable (in the sense that a local detector cannot have zero click probability, even if the initial state was the vacuum), such that 'looks like the vacuum' is operationally ill defined. But given a sensible detector model (which for example 'measures the particle content associated with the Furry picture'), it might well be that the click probability associated with a given region of space-time depends on the motion of the detector, i.e. on the laboratory (frame) in which the measurement is performed. According to relativistic consistency (see chapter 2) the fact whether a given detector is triggered or not cannot depend on the frame of reference, but its click probability can without inconsistencies depend on the frame in which the detector is at rest.

It is now a natural option to take this particle picture seriously and perceive the sea as a system of Bohmian particles which is guided by Bohmian equations of motion e.g. of an associated HBDM. The usual arguments why negative (kinetic) energies are physically peculiar and not reasonably interpretable cannot really contest such a Bohmian interpretation, since energy is not a primitive property of a particle but rather a parameter in its equations of motion, and as long these are well defined (which is the case for negative energy electrons) and the model is empirically adequate, there is no reason be afraid. More substantial challenges concern a hard part of mathematical physics which is to be done to put the mean field explanation for the unobservabilty of the sea on a rigorous basis and, related to this, to tackle the apparent arbitrariness to set the cut between observable and unobservable particles depending on the sign of energy, in particular in presence of interactions whose satisfactory comprehensive implementation constitutes another serious challenge (with respect to external electromagnetic potentials, these things are discussed in appendix A). But the immediate fundamental problem with a Bohmian Dirac sea approach is obviously that the model becomes physically absurd and mathematically ill defined when at least one of the regularizations is removed, in which case $\mathcal{H}_-$ becomes infinite dimensional and accordingly the sea consists of an infinite number of negative energy electrons.

The infinity of the sea is strongly related with the infinity problems which plague also standard QED, in particular whenever dynamics beyond the regime of scattering theory is considered. One proposal (see [104]) to tackle this problem, is to develop QED proceeding from *Wheeler-Feynman electrodynamics* [341], which is a reading of Maxwell-Lorentz electrodynamics in which particles are not self-interacting nor emitting radiation into the empty space but rather directly interact along the (future and past) light cones which connect the worldlines of charged distinct particles. In terms of photons, this means that charged particles emit only photons which are in turn absorbed by other charged particles, but never photons which can escape to infinity, i.e. which would irreversibly lower the energy of the considered system of charged particles. Thus a large system of N directly interacting Dirac particles is supposed to stay (typically) stable without running into a radiation catastrophe. This way one might consider a Bohmian Dirac sea model without regularizations but with a sea which consists of a very large but finite number of negative energy particles, whereas deep down in the sea (i.e. in the regime of very low negative energy states) states can be unoccupied without inducing the sea to 'fall down' in a radiation catastrophe. To make this heuristic picture rigorous, there is of course a lot of work to be done.

Concerning the mean field argument of the model, available results, substantiated conjectures and open problems are also set out in [104]. In this work, the naive Dirac sea picture is modified to a more realistic framework. The electrons in the sea interact mutually by Coulomb potentials[197] and possibly with external potentials while radiation is neglected. The vacuum – as primarily a bulk of electrons which is unobservable by its uniform distribution – is characterized by appropriate mean field conditions and it is argued that not only one but rather a whole class of antisymmetrized $N-$particle Dirac wave functions (with large N) is supposed to satisfy these conditions and this class is referred to as the *class of equilibrium states* (this naming stems from an analogy to a classical gas in thermal equilibrium). In absence of external potentials, a natural representative of this class would be the ground state (the state of lowest energy, which obviously exists only in the regularized theory) of the interacting system which – given N is

[197]When coming to the mean field estimates, it is important to note that the electrons do not only repel each other by Coulomb interaction but also by the Fermi pressure arising from the antisymmetry of the wave functions.

taken to be equal to the dimension of the regularized $\mathcal{H}_-$ – can be approximated in the mean field approximation by the Dirac vacuum $\widetilde{\Omega}$ discussed for the naive Dirac sea picture above, i.e. all states in $\mathcal{H}_-$ occupied.

The class of equilibrium states can be argued to be invariant under the time evolution in the absence of external potentials, but if the latter act on an equilibrium state, they cause excitations which are shown in [104] (for the case of excitations of $\widetilde{\Omega}$, i.e. in the realm of the mean field approximation) to behave dynamically like electron/positron pairs as expected. More precisely, the excitations appear in the simplest case (corresponding to the creation of a single electron/positron pair) as an additional charge plus the absence of a charge in the uniformly charged sea, whose dynamics can be described by a wave function $\psi(\boldsymbol{x}, \boldsymbol{y})$ of two positive energy Dirac particles of the same mass – the electron mass – but of opposite charge. The electron (negative charge) tensor component (say, associated with position variable $\boldsymbol{x}$) of this wave function corresponds (in the limits of the mean field approximation) at each time to the state which has to be added to $\widetilde{\Omega}$ and the positron (positive charge) tensor component (accordingly associated with $\boldsymbol{y}$) of the two particle state corresponds to what has to be removed from $\widetilde{\Omega}$ in order to obtain the excited state ('added' and 'removed' is defined by the way creation and annihilation operators act as described in appendix A, in particular, it must be taken care that the resulting states are antisymmetric again). Moreover, if the sea particles are in quantum equilibrium, the apparent electron/positron pairs will be $|\psi(\boldsymbol{x}, \boldsymbol{y})|^2$ −distributed (for more details see [104] and references therein).

Field Ontology

A question standing to reason is whether QFT does not rather suggest a field ontology than an ontology of point particles. Indeed, already in the advent of the Bohmian research program, in the second of his 1952 papers [40], Bohm developed a Bohmian guiding equation for the transversal part $\boldsymbol{a}(\boldsymbol{x}, t)$ of the electromagnetic field, which is accordingly guided[198] by a wave functional $\Psi_t[\boldsymbol{a}]$, which is in turn given by a solution of an appropriate functional Schrödinger equation. Later, further Bohmian quantum theories of fields, in particular for scalar fields, were developed (see [315] and references therein). A statistical quantum equilibrium analysis of such theories is somewhat delicate, since the configuration space of fields is as a function space infinite dimensional and Lebesgue measure is not defined on infinite dimensional spaces. Nonetheless, on a heuristic level the way how to develop a statistical interpretation is rather obvious and can be made rigorous, in principle, with a considerable amount functional analysis and distribution theory [314].

An important point to note is that in a physically serious field ontology (how it is proposed by such Bohmaian approaches) the fields must have a literal meaning of matter fields and are not to

[198]Here it is helpful to note that the velocity field (4.4) can be written as the gradient of the phase of the Schrödinger wave function, i.e. if we write the wave functions in its polar decomposition $\psi_t(\boldsymbol{x}) = R_t(\boldsymbol{x})e^{iS_t(\boldsymbol{x})}$, the velocity field can be written as $v^{\psi_t}(\boldsymbol{x}, t) = m^{-1}\boldsymbol{\nabla} S_t(\boldsymbol{x})$. Analogously, a wave functional Ψ guiding a (for simplicity scalar) field $a(\boldsymbol{x}, t)$ can be written in polar decomposition with (functional valued) phase S, such that an actual field configuration $A(\boldsymbol{x}, t)$ is guided according to

$$\partial_t A(\boldsymbol{x}, t) \sim \left.\frac{\delta S_t[a]}{\delta a}\right|_{a=A(\boldsymbol{x},t)} \tag{4.63}$$

For more details, in particular how functional Schödinger equations for Ψ look like, see e.g. [314, 315].

be confused with the field operators which are usually associated with QFTs. The field operators (which are actually operator valued distributions) are mathematical tools which generate the Fock space from a vacuum vector (see the discussion of field operators in the last point on the Dirac sea) and in particular which generate the algebra of operators acting on this Fock space, since each operator can be obtained or approximated arbitrarily well (in the weak operator topology) by expressions (functions, integrals etc.) of the field operators. This already indicates that any attempts to let the field operators represent ontological entities must fail, for the same reason why naive realism about operators fails generally in quantum theory, which was shown and discussed in section 1.3. The algebraic structure of the field operators makes it impossible to consistently associate with them ontological facts because it is easy to obtain from them operators which commute pairwise but not crosswise (see section 1.3). This is clear since, as mentioned, all operators on Fock space can be obtained from its field operators, but possibly most straightforwardly illustrated directly by the causality requirements on the latter: Since bosonic field operators and 'bilinear' fermionic field operators commute at spacelike separation but in general not at timelike separation, it is easy to find triples of (maybe smeared and/or bilinear) field operators, say $\{A, B, C\}$ such that $[A, B] = [A, C] = 0$ but $[B, C] \neq 0$. But such sets of operators constitute the stuff the Kochen-Specker-Bell theorems of section 1.3 are made of, which prove that a naive realism about these operators (i.e. the assumption that (selfadjoint) operators represent always physically (pre-)existing values) must be inconsistent with the quantum predictions.

4.7 Getting Localized

Bohmian particles are localized in space by the very meaning of *particle*. In appropriate *position measurement (like) processes*, roughly speaking, Bohmian particles reveal their position to their environment to a good degree of accuracy. In particular a configuration of other Bohmian particles guided by a pointer state finally indicates – at least approximately – the Bohmian position(s) of the measured particle(s). As it follows from the absolute uncertainty principle (see section 4.4.4), such processes must localize the effective wave functions of the measured particles sufficiently well, since according to this principle data about the configuration of a system with effective wave function ψ cannot be encoded in the configuration of its environment beyond its $|\psi|^2$ −distribution in quantum equilibrium.

If there is talk of *'localization'* in quantum theory, one usually addresses one (or both) of these two operational aspects, which were illustrated from a Bohmian perspective in the previous paragraph: The values obtained as outcomes of potential position measurements whose probabilities can be obtained from the algorithms of quantum theory, respectively processes of localizing wave functions, e.g. by usual position measurements, but one might also think of ion traps, the localizing effects of decoherence etc.

The prototype is the standard position measurement associated with the standard position operator given by multiplication of the one particle wave functions $\psi(\boldsymbol{x})$ by $\boldsymbol{x}$ in the standard position representation or, equivalently, with its associated PVM given by the indicator functions of the measurable subsets of space. Already at the non relativistic level it is clear that 'a measurement of the position operator' can be only an idealization.

This is immediately clear with respect to the state transformations, since what might be regarded as 'eigenstates' of the standard position operator (delta functions) does not correspond

to admissible final states of a measurement, these 'states' are not even elements of the Hilbert space. Even a coarse grained version of associated state transformations, like given by multiplication of the wave functions by indicator functions (modulo normalization) of small but finite spatial subsets (or by $C_0^\infty-$functions supported on such sets) is not physically admissible, since perfectly cutting off the tails of all wave functions must be associated with an infinite potential well, i.e. an infinite amount of energy must be provided in order to localize an extended wave function perfectly. In the relativistic case, this would moreover violate the spectrum condition as encountered in chapter 3.

On the other hand, the statistical distribution of configurations which is encoded in the standard position operator is exactly the $|\psi|^2-$distribution according to which Bohmian particles are distributed in quantum equilibrium and according to which the outcomes of quantum position measurements must be distributed in general (the Born rule) to derive the quantum formalism (see chapter 1). So at least for all practical purposes, the statistics of proper position measurements should be the statistics encoded in the standard position operator. It is clear that empirical distributions of real world measurements will nonetheless more or less minimally deviate from this prediction because measurements are never perfect but are always subject to certain errors with certain probabilities. Leaving aside usual measurement errors deriving from mechanical imperfectness of measuring devices (false detector clicks etc., which presumably contribute to the vast majority of deviations from the standard scheme in realistic scenarios), there are at least two further aspects fundamentally limiting the accuracy of (position) measurements: Limited resolution of displayed measurement results and the quantum nature of measuring devices[199].

Concerning the first point, an illustrative example is the trajectory of a charged particle revealed by a bubble chamber, where the pointer states correspond to bubbles of water condensed about ionized atoms. The extension of these traces perpendicular to the (effective) particles motion is of course very large on the microscopic level and no one knows the actual trajectory of a Bohmian particle within such a trace (unless one knows the exact initial positions and wave functions of all involved particles and has superhuman capabilities to calculate, but the crucial point is that only the approximate but not the precise trajectory is encoded in the pointer states). Also results of (position) measurements recorded by digital devices are encoded in finite dual numbers and have thereby limited resolution etc. The second point was illustrated by the von Neumann position measurement scheme in section 1.5.2 which, although surely not realistically implementable, is nonetheless instructive since it illustrates the fact that the extension of pointer states (which can be usually neglected for all practical purposes[200]) in the pointer's configuration space fundamentally limits the precision of the displayed results of continuous (position) measurements at microscopic scale. Hence one might expect that empirical distributions of actual outcomes of position measurements (which must agree with Born's rule for all practical purposes) are in a strict sense always given by approximate measurement POVMs (see sections 1.4.3 and 1.5.2). But in relativistic quantum theory, if the spectrum condition is

[199] Picturesquely speaking, one might illustrate these two issues in a simplified way by representing fundamental limitations of (position) measurements by the extension of pointers in physical space and the extension of the pointer states in configuration space, respectively.

[200] One can implement indirect measurements (see section 1.4.4) where the probe states are perceived as generalized (not necessarily mutually orthogonal) pointer states, whose extension in the probe configuration space cannot be neglected. This is in particular the case for weak measurements [318].

supposed to hold, also this is in conflict with the Malament type theorems of section 3.4 (in particularly theorem 3.25).

The Malament type theorems of course are also valid theorems in Bohmian mechanics[201] and its relativistic generalizations. Assuming that POVMs associated with local measurements satisfy space-time translation covariance and for the final Malament type theorem 3.25 that local commutativity holds (which we shall do), it follows that either the spectrum condition or causal additivity must be violated for detector experiments (representing position measurements or more generally local measurements). As derived in section 3.4.2, violation of causal additivity means that two or more remote detectors can be triggered at spacelike separation with non vanishing probability, even if the considered Hilbert space corresponds to a single particle. Relativistic argument straightforwardly yields that in a relativistic theory violation of causal additivity entails a violation of additivity in each frame, since there does always exist a frame in which spacelike separated click events occur simultaneously such that the detector formalism cannot be additive in this frame (generalization of this argument yields that it cannot be additive in any frame). This, however, means that the statistics of detector experiments on a single particle can neither be given by Born's rule nor by obvious generalizations of the latter like an approximate measurement POVM or any other POVM on $\mathbb{R}^3$, which are always additive. But still Born's rule for position measurements was the very starting point to derive the predictively so successful quantum formalism, not only in Bohmian mechanics but also in the operational formulation derived in chapter 1.

If now the infinite tails of positive energy wave functions of the measured system are made responsible for this mess (as discussed in section 3.4.7), one can again argue that associated deviations from Born's rule must be so tiny that they are far from having any relevance for practical purposes (see section 3.3 for arguments how negligibly small the effects of the tails can be expected to be) and that measurement errors deriving from less fundamental sources will certainly play a much bigger role. Moreover, the field in which quantum theory is probably best empirically confirmed is scattering theory, and the derivation of the quantum predictions for scattering theory (transition rates, scattering cross sections etc.) involves a collection of (good) approximations, as one can easily verify by studying any exposition of quantum scattering theory. In this regard – if we call a wave function *effectively compactly supported* if there is a compact region $\Delta \subset \mathbb{R}^3$ and a very small $\varepsilon > 0$ (say $\varepsilon = 10^{-30}$) such that $\int_{\Delta^c} |\psi(\boldsymbol{x})|^2 \, d^3x < \varepsilon$ (where Δ^c is the complement of Δ) – approximating effectively compactly supported wave functions by compactly supported wave functions will certainly not seriously undermine the precession of predictions.

From a Bohmian point of view it is important to note that actual configurations are always in quantum equilibrium, i.e. $|\psi|^2$ −distributed, while potential effects associated with tails (potentially leading to deviations from the $|\psi|^2$ − distribution for position measurements) have so small probability that they belong to the domain of quantum non equilibrium and thus do not contribute to phenomena, like violations of the second law of thermodynamics do not contribute to phenomena because of extremely small associated probabilities (see section 4.4). Investigating strange phenomena associated with quantum non equilibrium is a messy business and unreward-

[201] Recall that actually only the subgroup of space-time translations of the Poincaré group enters into the derivation of the theorems, which is a subgroup of the Galiliei group as well, such that the theorems hold in non relativistic quantum theory / non relativistic Bohmian mechanics as well, where they are easily understood by the instantaneous spreading of wave functions and unboundedness of particle velocities.

ing, not least since the best empirical evidence we have tells us that the world is perfectly in quantum equilibrium. If one is still worried whether potential strange scenarios associated with wave function tails might complicate an explanation of quantum phenomena by an ontology of localized particles[202], one should think about the fact that in Bohmian mechanics the primary object to analyse the dynamics of (measured) subsystems is the conditional wave function which always exists and is always well defined (while effective wave functions can be only good approximations if tails are accounted for) and if we had no particle positions we had no conditional wave functions as well.

Moreover, as discussed in section 3.4.7, strange potential scenarios involving wave function tails become even more irrelevant if the active nature of detectors and the non triviality of the vacuum in QFT is taken into account: The puzzeling fact that it cannot be perfectly excluded that two detectors are triggered at spacelike separation even if the initial state was a one particle state is no longer surprising, if we acknowledge that a detector must have non zero click probability even if the initial state was the vacuum as the Reeh-Schlieder theorem (see section 3.5) asserts. Indeed, resorting to infinite tails arguments to explain such strange potential detector click scenarios usually relies on the assumption that switched on detectors transform one-particle positive energy states into one-particle positive energy states, while it was argued in sections 3.3 and 3.4.7 that locally caused (in a very wide sense) transformations of one-particle wave functions must violate the spectrum condition. In Dirac theory it is well understood that in second quantization such transformations (lifted to Fock space) involve pair creation effects with certain probabilities (see appendix A) such that one must expect that state transformers associated with detectors (or any local measurements) do not leave the one particle sector of Fock space, or the vacuum sector or any subspace with a bounded number of particles (precisely[203]) invariant. Hence, to be precise, one should expect that the statistics of position measurements must be rather given by a spatial POVM on the configuration space associated with Fock space (which is in the simplest case $\bigcup_{N\in\mathbb{N}_0}\mathbb{R}^{3N}$, see e.g. [122, 123, 124, 125] and section 4.6). Such a spatial POVM for position measurements – which is of course also additive and which shall essentially agree with Born's rule in each sector (see e.g. the toy model in footnote 140) – is not excluded by any of the no-go theorems of chapter 3 but can be rather regarded as their natural consequence, from a purely operational as well as from a Bohmian perspective.

Final Remark: One might consider a further line of argument regarding the localization problem which appears to be natural in a relativistic context, namely involving the role of time for position measurements. The symmetry of space and time in relativity suggests that one

[202]Strange scenarios one might have in mind (like two detectors triggered by a one particle initial state at spacelike separation), finally boil down to situations where a detector clicks while the Bohmian initial particle is somewhere else. In other words, there are (very few) initial conditions where the bulk of Bohmian particles constituting a detector evolve to the configuration of a triggered detector under the Bohmian dynamics while the only other Bohmian particle is somewhere else. On the first quantized level one can argue dynamically, that the interaction Hamiltonian of the measurement acts on wave functions (which overlap) and by that only indirectly on the involved particles, consequently Bohmian position measurements with extended wave functions can fail in principle (see also footnote 108). Apart from equilibrium arguments, why we need not bother with such scenarios, they become even less problematic in (Bohmian) QFT, when the detector has non zero probability to evolve to its triggered state even if there was no other initial particle (see below).

[203]Below the energy regime of high energy physics, particle creation and annihilation effects can be neglected of course.

should not consider the issues of position measurements and time measurements as completely detached from each other. In fact, one can argue that a position measurement at a fixed time does presumably not exist, but that the time where e.g. a detector clicks is always random (accordingly – since time is continuous – the probability that a detector clicks at a fixed time should be always zero!). Thus, one might consider the localization problem not at fixed times in given laboratory frames, but rather as a problem of four dimensional (operational) localization on space time which would entangle the spatial localization problem with the famous *arrival time problem*.

The arrival time problem is particularly interesting from a Bohmian perspective, since by the Bohmian trajectories the statistics of arrival times can be directly considered and calculated. It turns out (see e.g. [335]) that the Bohmian arrival time statistics is not given by any POVM since any surface (representing the surface of a detector) can be crossed several times by Bohmian trajectories which destroys additivity of the arrival time probabilities. One can find sets of wave functions which do not involve such a backflow, but it can be shown that such sets are not linear (the superposition of two wave functions belonging to such a set is in general no longer in this set because of interference phenomena) such that a Bohmian arrival time POVM cannot be defined even on a subspace of Hilbert space [335]. This yields an example of quantum measurements which are not covered by the quantum formalism developed in chapter 1, so called *non linear measurements*, which are beyond the scope of this work, though. However, a treatment of the localization problem interconnected with the arrival time problem in the framework of non linear measurements (in particular, considering the possibility that the assumption that spatial detector click statistics is precisely given by POVMs might be dropped) from a Bohmian perspective would be worthwhile. Nonetheless, the previous arguments have shown that the localization problem can be transparently and naturally understood already at the level of the usual quantum formalism of linear measurements by analysing the role of the spectrum condition with respect to infinite tails, locally caused transformations of wave functions and pair creation phenomena.

Appendix

A Second Quantization in the Dirac Sea Picture

Consider the free Dirac equation

$$i\partial_t \psi_t(\boldsymbol{x}) = (\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m)\,\psi_t(\boldsymbol{x}) \tag{A.1}$$

with $\psi_t \in \mathcal{H} = L^2(\mathbb{R}^3, d^3x)\otimes\mathbb{C}^4$, the momentum operator $\boldsymbol{p} = -i\nabla_{\boldsymbol{x}}$, the free Dirac Hamiltonian $\mathscr{H}_0 = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m$ and the 4×4 Dirac matrices $\alpha_k, k = 1,2,3$ and β which satisfy the anticommutation relations $\{\alpha_k, \alpha_l\} = 2\delta_{kl}\mathbb{1}_{\mathbb{C}^4}$, $\{\alpha_k, \beta\} = 0$ and $\beta^2 = \mathbb{1}_{\mathbb{C}^4}$ for all $k,l = 1,2,3$ (see e.g. [321] for details). Denoting by $\psi_{[i]}$ the k'th spinor component of $\psi\in\mathcal{H}$, the scalar product on $\mathcal{H}$ is given by $\langle\psi|\,\varphi\rangle = \sum_{i=1}^4 \langle\psi_{[i]}|\,\varphi_{[i]}\rangle_{L^2}$. The orthogonal projections onto the positive- and negative energy subspacees of $\mathcal{H}$ are given by $P_\pm = \frac{1}{2}(\mathbb{1}_{\mathcal{H}} \pm \Pi)$ with the sign of energy operator $\Pi = \frac{\mathscr{H}_0}{|\mathscr{H}_0|} = \frac{\mathscr{H}_0}{\sqrt{\boldsymbol{p}^2+m^2}}$ (which is a simple multiplication operator in momentum space). The positive and negative energy subspaces of $\mathcal{H}$ are then given by $\mathcal{H}_\pm = P_\pm\mathcal{H}$ and provide a splitting of $\mathcal{H}$ into two orthogonal subspaces: $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

The naive idea of the Dirac sea model goes like this: Let $\{\varphi_k^-\}$ be an ONB of $\mathcal{H}_-$ and $\bigwedge$ denote the antisymmetrized tensor product[204]. Then the Dirac vacuum is formally denoted by

$$\widetilde{\Omega} = \bigwedge_k \varphi_k^- \tag{A.2}$$

Not to get lost into technical problems, we shall pretend for a moment that the one particle Hilbert space $\mathcal{H}$ and thereby $\mathcal{H}_-$ is (maybe high but) finite dimensional such that $\widetilde{\Omega}$ in (A.2) is a wedge product of finitely many states and hence well defined (we will come back to this below, see in particular footnote 205). It is easy to confirm that then (A.2) is due to the antisymmetrized product independent (up to a constant phase) of the actual choice of ONB $\{\varphi_k^-\}\subset\mathcal{H}_-$. The Dirac vacuum $\widetilde{\Omega}$ is the unique state, in which all negative energy states are occupied by Dirac particles, say electrons, and it is assumed that somehow these electrons are homogeneously distributed in a way such that they are unobservable (the sea produces effectively a constant electromagnetic potential with not too bad fluctuations, the Fermi pressure originating from antisymmetrization should play a role here as well), which should be made rigorous with meanfield type arguments (Dirac proposed a Hartree-Fock approximation in terms of density matrices). Thus, the motion of an additional test charge is essentially unaffected by the Dirac vacuum and if this test charge happens to be another electron, its wave function cannot have or obtain contributions from negative energy states due to Pauli's exclusion principle (which is a trivial consequence of the basic property of the wedge product that $\psi_1\wedge\cdots\wedge\psi_N = 0$

[204] For $\psi_1,\cdots,\psi_N\in\mathcal{H}$ and with the permutation group of N elements S_N, the antisymmetrized tensor product (wedge product) is given by $\bigwedge_{k=1}^N\psi_k = \frac{1}{\sqrt{N!}}\sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)\bigotimes_{k=1}^N\psi_{\sigma(k)} \in \mathcal{H}^{\otimes N}$ (in differential geometry the wedge product is usually given without the factor $\frac{1}{\sqrt{N!}}$, which ensures here that $\bigwedge_{k=1}^N\psi_k$ is normalized if the ψ_k are normalized and mutually orthogonal). The antisymmetric subspace of $\mathcal{H}^{\otimes N}$ which is the closure (with respect to the norm induced by the standard scalar product on $\mathcal{H}^{\otimes N}$) of the set of finite linear combinations of the form $\bigwedge_{k=1}^N\psi_k$ is denoted by $\mathcal{H}^{\wedge N}$.

whenever there are indices $k \neq l$ such that $\psi_k = \psi_l$). This prevents a radiation catastrophe, where electrons emit an infinite amount of energy (photons) by occupying lower and lower energy states. If a negative energy electron absorbs enough energy, it can get lifted to the positive energy subspace and become an observable electron. It leaves behind an unoccupied negative energy state (hole) in the sea of negative energy particles, which appears as the absence of a charge in the homogeneously charged sea, which effectively amounts to the presence of a charge of opposite sign. It can be argued that this 'absence of a charge' indeed also dynamically behaves like an electron of positive energy but opposite charge, i.e. a positron, indirectly by the dynamics of sea electrons. Thus this process effectively amounts to the spontaneous appearance of an electron/positron pair from the vacuum, i.e. *pair creation*.

We shall obtain now a convenient effective mathematical net description of the observable particles and holes and thereby of the sea model (without caring about the hard meanfield part of the model), which is known as *'quantization of the Dirac field'* or *'second quantization of the Dirac equation'*. This description is well defined even without the simplifying assumption[205] of finite dimensional $\mathcal{H}_-$. Only the set of one particle operators which can be lifted to Fock space (i.e. which can be *'second quantized'*) will then be restricted by the requirement, roughly speaking, not to create infinitely many pairs (otherwise both, the mathematical description and its sensible physical interpretation, would break down).

Before coming to the net description of the Dirac sea, we shall briefly recall the basics of the formalism allowing for a variable number of electrons (see e.g. [290, 298, 321] for more details): Let $\widetilde{\Phi}^\dagger(\psi)$ be the primary proper creation operator of the state $\psi \in \mathcal{H}$, i.e. $\widetilde{\Phi}^\dagger(\psi)$ adds to each antisymmetrized $N-$particle Dirac wave function the state ψ in a proper way, such that it becomes an antisymmetrized $N+1-$particle Dirac wave function. Observing that the Hilbert space $\mathcal{H}^{\wedge N}$ of antisymmetric $N-$particle Dirac wave functions is the closure of the set of finite linear combinations of states of the form $\psi_1 \wedge \cdots \wedge \psi_N$ with $\psi_1, \ldots, \psi_N \in \mathcal{H}$, the action of $\widetilde{\Phi}^\dagger(\psi) : \mathcal{H}^{\wedge N} \to \mathcal{H}^{\wedge(N+1)}$ (with fixed $\psi \in \mathcal{H}$) is most easily defined by its action on antisymmetric product states via

$$\widetilde{\Phi}^\dagger(\psi)\ \psi_1 \wedge \cdots \wedge \psi_N = \psi \wedge \psi_1 \wedge \cdots \wedge \psi_N \tag{A.3}$$

[205] This assumption is dispensable in principle also for rigorosly defining the state of a Dirac sea (heuristically defined by the wedge product of infinitely many states (A.2)) which can be mathematically represented in a rigorous way by the formalism of infinite wedge spaces [103, 215]. One might also proceed from a finite dimensional $\mathcal{H}_-$ by regularizing the theory by assuming finite space (making the energy spectrum of the free Dirac equation discrete) and an ultraviolet cutoff (making this discrete spectrum finite) and when the effective net description is developed, the regularizations making the sea finite can be dropped again. If the mean field argument of the interacting Dirac sea is treated, these regularizations are also necessary as yet, to avoid infrared and ultraviolet divergences [104].

In [104] it is moreover suggested, that in order to obtain a Dirac sea model which can be taken physically seriously, one might assume a sea of a very high but finite number of Dirac particles without regularizations, such that 'deep down in the sea' states remain unoccupied. It is conjectured that a radiation catastrophe can be omitted, if the electromagnetic interaction is implemented as direct interaction of the Wheeler-Feynman type [341]. In this case radiation into nowhere is impossible (in the language of photons, only photons are emitted by particles which are absorbed by other particles) such that a system of a huge number of directly interacting Dirac particles might be stable, although the Dirac Hamiltonian in unbounded from below. The Dirac sea model with a sea of infinitely many particles could then be regarded, in a sense, as a thermodynamic limit of such a potential theory in the future.

These operators are straightforwardly extended to the Fock space[206] $\widetilde{\mathcal{F}} = \bigoplus_{N\in\mathbb{N}_0} \mathcal{H}^{\wedge N}$, whose states $\Psi \in \widetilde{\mathcal{F}}$ can be represented by infinite tuples of wave functions $\Psi = ([\Psi]_0, [\Psi]_1, [\Psi]_2, \dots)$ with $[\Psi]_N \in \mathcal{H}^{\wedge N}$ where $\mathcal{H}^{\wedge 0} = \mathbb{C}$ and $\mathcal{H}^{\wedge 1} = \mathcal{H}$: By $\widetilde{\Phi}^\dagger(\psi)$ acting on $\Psi \in \widetilde{\mathcal{F}}$ each $N-$particle component $[\Psi]_N \in \mathcal{H}^{\wedge N}$ is shifted to the respective $(N+1)-$particle component $[\Psi]_{N+1} \in \mathcal{H}^{\wedge(N+1)}$ by the prescription (A.3) (with a slight abuse of notation, we denote these operators also by $\widetilde{\Phi}^\dagger(\psi) : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$), where the zero particle component $[\Psi]_0 = c \in \mathbb{C}$ is shifted by the rule $[\widetilde{\Phi}^\dagger(\psi)\,\Psi]_1 = c\,\psi$. The corresponding annihilation operators[207] are (as well known and easily calculated [290, 321]) the adjoints $\widetilde{\Phi}(\psi)$ of the creation operators. While the creation operators $\widetilde{\Phi}^\dagger(\psi)$ are linear in their arguments $\psi \in \mathcal{H}$, the annihilation operators are antilinear.

With the state of zero particles $\Gamma \in \widetilde{F}$ (represented by a tuple $\Gamma = e^{i\alpha}\,(1, 0, 0, \dots)$ with a constant phase $\alpha \in \mathbb{R}$), a general state in $\widetilde{\mathcal{F}}$ can be represented by linear combinations of infinite tuples of the form

$$\widetilde{\Phi}^\dagger(\psi_1) \dots \widetilde{\Phi}^\dagger(\psi_N)\,\Gamma = \left(0, \dots, 0, \bigwedge_{k=1}^{N} \psi_k, 0, \dots\right) \tag{A.4}$$

with $\psi_1, ..., \psi_N \in \mathcal{H}$. The basic vacuum Γ is of course very different from the Dirac vacuum $\widetilde{\Omega}$ in (A.2), to which we shall come back now.

With the linearity of $\widetilde{\Phi}^\dagger$ and $P_+ + P_- = \mathbb{1}_{\mathcal{H}}$, a meaningful decomposition of $\widetilde{\Phi}^\dagger$ is given by

$$\widetilde{\Phi}^\dagger(\psi) = \widetilde{\Phi}^\dagger(P_+\psi) + \widetilde{\Phi}^\dagger(P_-\psi) \tag{A.5}$$

Since in $\widetilde{\Omega}$ in (A.2) all states in $\mathcal{H}_-$ are occupied by electrons, antisymmetry entails that $\widetilde{\Phi}^\dagger(P_-\psi)\ \widetilde{\Omega} = 0$ (Pauli exclusion principle). If the other way around $\xi \in \mathcal{H}_-$ is an unoccupied state (hole) in the sea, this hole vanishes upon the action of $\widetilde{\Phi}^\dagger(\xi)$ which corresponds to the annihilation of a positron in the state $C\xi$, where C is the antiunitary charge conjugation operator (e.g. in the standard representation of the Dirac matrices, the action of C on $\psi \in \mathcal{H}$ can be defined by $C\psi = i\beta\alpha_2\bar{\psi}$, see e.g. [158, 321]). This way we can naturally identify (A.5) with

$$\Phi^\dagger(\psi) = a^\dagger(P_+\psi) + b(P_-\psi) \tag{A.6}$$

where for all $\varphi \in \mathcal{H}_+$, $a^\dagger(\varphi)$ represents the creation of an electron in the state φ and for all $\xi \in \mathcal{H}_-$, $b(\xi)$ represents the annihilation of a positron in the state $C\xi$. Correspondingly, we denote by $a(\varphi)$ the annihilation operator of an electron in the state $\varphi \in \mathcal{H}_+$ and by $b^\dagger(\xi)$ with $\xi \in \mathcal{H}_-$ the creation operator of a positron in the state $C\xi$, which are (as operators acting on the Fock space considered in the following paragraph) indeed the adjoints of the operators $a^\dagger(\varphi)$ and $b(\xi)$, respectively.

[206] $\widetilde{\mathcal{F}}$ is of course a Hilbert space, the scalar product is given by $\langle\Psi|\ \Psi'\rangle_{\widetilde{\mathcal{F}}} = \sum_{N\in\mathbb{N}_0} \langle[\Psi]_N|\ [\Psi']_N\rangle_{\mathcal{H}^{\wedge N}}$.

[207] The action of the annihilation operators can be defined as the inverse action of the creation operators as defined above, i.e. by its action on wedge products, the respective factor wave function is removed. But this is a bit tricky, for the relevant factor state might not be obvious since the decomposition of an antisymmetric product state into its factor wave functions is in general not unique (moreover, if the states in the product are not mutually orthogonal, their parallel parts cancel out by antisymmetrization, which is just a subtel manisfestation of the Pauli exclusion principle). One can equivalently define the general action of the annihilation operators on $\Psi \in \widetilde{\mathcal{F}}$ in position representation by 'integrating the annihilated state out': $[\widetilde{\Phi}(\psi)\,\Psi]_N(x_1, \dots, x_N) := \sqrt{N+1}\sum_{s=1}^{4} \int_{\mathbb{R}^3} \bar{\psi}(x)[\Psi]_{N+1}(x, x_1, \dots, x_N)$, where we have denoted $x = (\boldsymbol{x}, s)$ and accordingly $x_k = (\boldsymbol{x_k}, s_k)$ where $s, s_k = 1, 2, 3, 4$ are the spinor indices of the respective particles.

The appropriate Hilbert space for the effective net description in terms of only positive energy electrons and positrons (holes in the sea) is the Fock space $\mathcal{F} = \mathcal{F}_e \bigotimes \mathcal{F}_p$, where $F_e = \bigoplus_N \mathcal{H}_+^{\wedge N}$ is the electron Fock space and accordingly $F_p = \bigoplus_N (C\mathcal{H}_-)^{\wedge N}$ the Fock space of positrons, where now the state of zero particles $\Omega = e^{i\alpha}(1 \otimes 1, 0 \otimes 0, \dots) \in \mathcal{F}$ is identified with the Dirac vacuum $\widetilde{\Omega}$ in (A.2) of the full description. We can represent $\mathcal{F}$ as the Hilbert space which is spanned by linear combinations of states of the form $a^\dagger(\varphi_1) \dots a^\dagger(\varphi_N)\,\Omega$ and $b^\dagger(\xi_1) \dots a^\dagger(\xi_N)\,\Omega$ (which may be written as tuples in analogy to the right hand side of (A.4)) with $\varphi_1, \dots, \varphi_N \in \mathcal{H}_+$ and $\xi_1, \dots, \xi_N \in \mathcal{H}_-$. The identification of Ω with $\widetilde{\Omega}$ and accordingly the identification of the field operators $\Phi^\dagger$ in (A.6) with $\widetilde{\Phi}^\dagger$ in (A.5) yields together with the antisymmetry of the wave functions on which $\widetilde{\Phi}^\dagger$ acts and a little calculation the well known algebraic structure of the field operators (and thereby of the Fock space) of the net description: With the usual scalar product $\langle \cdot | \cdot \rangle$ on $\mathcal{H}$, we obtain for all $\varphi, \varphi_1, \varphi_2 \in \mathcal{H}_+$ and $\xi, \xi_1, \xi_2 \in \mathcal{H}_-$ the famous relations

$$\begin{gathered} a(\varphi)\Omega = b(\xi)\Omega = 0 \quad \{a(\varphi_1), a^\dagger(\varphi_2)\} = \langle \varphi_1 |\, \varphi_2 \rangle \quad \{b(\xi_1), b^\dagger(\xi_2)\} = \overline{\langle \xi_1 |\, \xi_2 \rangle} \\ \{a(\varphi_1), a(\varphi_2)\} = \{a^\dagger(\varphi_1), a^\dagger(\varphi_2)\} = \{b(\xi_1), b(\xi_2)\} = \{b^\dagger(\xi_1), b^\dagger(\xi_2)\} = 0 \\ \{a(\varphi), b(\xi)\} = \{a^\dagger(\varphi), b^\dagger(\xi)\} = \{a^\dagger(\varphi), b(\xi)\} = \{a(\varphi), b^\dagger(\xi)\} = 0 \end{gathered} \tag{A.7}$$

Note that the crucial relation $b(\xi)\Omega = 0$ expresses the impossibility to add another positive energy electron to the filled sea due to Pauli's exclusion principle, the complex conjugation in the right hand expression of the first line in (A.7) derives from exchanging annihilation operators of negative energy electrons by creation operators of positive energy positrons (in particular, from the antiunitarity of the involved charge conjugation) and the relations in the third line of (A.7) encode the strong relationship between electron and positron creation and annihilation which is explicit in the sea model[208].

Lifting Operators

How to appropriately extend (lift) operators acting on the one particle Hilbert space $\mathcal{H}$ to operators acting on $\mathcal{F}$ depends crucially on their respective physical meaning, e.g. a Hamilton operator acts structurally differently (namely as a sum of the 'particle Hamiltonians') on the $N-$particle component of a Fock vector (with $N > 1$) than a unitary evolution operator (which acts rather as a tensor product), which shall be explained in a moment. In general, it is a simple theorem in second quantized Dirac theory (and an axiom in AQFT) that any operator on $\mathcal{F}$ can be expressed as a function of the field operators (see e.g. [290, 321]) which is the *irreducibility* property of the field operators. For special classes of one particle operators, general simple prescriptions how to obtain their second quantized extensions in terms of field operators can be given:

Extensive Operators: Let us go back for a moment to the simpler Fock space $\widetilde{\mathcal{F}} = \bigoplus_N \mathcal{H}^{\wedge N}$. We call a (bounded or densely defined) one particle observable operator[209] A acting on $\mathcal{H}$ ex-

[208] If electrons and positrons and their creation and annihilation processes were really unrelated, the corresponding operators would mutually commute rather than anticommute.

[209] See sections 1.1.1 and 1.2 and definition 1.8 for the notion of *observable operators* used in this work.

tensive, if it adds up with the involved particles, i.e A is properly extended to $\mathcal{H}^{\wedge N}$ by

$$A_N := \sum_{k=1}^{N} A^{[k]} \quad \text{where} \quad A^{[k]} := \mathbb{1}_{\mathcal{H}} \otimes \cdots \otimes A \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}} \tag{A.8}$$

where at the right hand side of (A.8) A is at the $k-$th of the N places. This is for example the correct extension for the Hamilton-, the charge- or the particle number operator, but e.g. not for the position operator or projections (quantum expectation values of projections are probabilities and these cannot add up with the involved particles, for example they are bounded by 1).

So if we denote a state $\Psi \in \widetilde{\mathcal{F}}$ as a tuple of its $N-$particle components

$$\Psi = \left(c_0 \, , \; c_1\psi^{(1)} \, , \; c_2\psi^{(2)} \, , \; c_3\psi^{(3)} \, , \ldots\right) \tag{A.9}$$

with normalized $N-$particle wave functions $\psi^{(N)}$ and $\sum_{N=0}^{\infty} |c_N|^2 = 1$ (i.e. $c_k\psi^{(k)} = [\Psi]_k$), the correct extension of A to $\widetilde{\mathcal{F}}$ is given by

$$\mathbb{A}\Psi = \left(0 \, , \; c_1 A\psi^{(1)} \, , \; c_2 A_2\psi^{(2)} \, , \; c_3 A_3\psi^{(3)} \, , \ldots\right) \tag{A.10}$$

with A_N given by (A.8). A simple calculation (see e.g. [290, 298]) shows now that we can express $\mathbb{A}$ defined by (A.10) in terms of the field operators by

$$\mathbb{A} = \sum_{k,l} \langle \varphi_k | \, A \, \varphi_l \rangle \, \widetilde{\Phi}^\dagger(\varphi_k)\widetilde{\Phi}(\varphi_l) = \sum_{k} \widetilde{\Phi}^\dagger(A\varphi_k)\widetilde{\Phi}(\varphi_k) \tag{A.11}$$

where $\{\varphi_k\}$ is any ONB of $\mathcal{H}$ (in particular, (A.11) is independent of the choice of ONB).

When we apply now this lifting scheme to the effective net description of the Dirac sea represented by $\mathcal{F}$ and the field operators Φ in (A.6) and their algebraic structure (A.7), it gets interesting: Let $\{\varphi_k^+\}$ be an ONB of $\mathcal{H}_+$ and $\{\varphi_k^-\}$ and ONB of $\mathcal{H}_-$ and consider the analogue of (A.11) in the sea model:

$$\begin{aligned}
\mathbb{A} &= \sum_{k,l} \sum_{m,n=\pm} \langle \varphi_k^m | \, A \, \varphi_l^n \rangle \, \Phi^\dagger(\varphi_k^m)\Phi(\varphi_l^n) = \\
&= \sum_{k,l} \left\langle \varphi_k^+ \right| \, A \, \varphi_l^+ \left\rangle \, a^\dagger(\varphi_k^+)a(\varphi_l^+) + \sum_{k,l} \left\langle \varphi_k^- \right| \, A \, \varphi_l^- \left\rangle \, b(\varphi_k^-)b^\dagger(\varphi_l^-) + \right. \\
&+ \sum_{k,l} \left\langle \varphi_k^+ \right| \, A \, \varphi_l^- \left\rangle \, a^\dagger(\varphi_k^+)b^\dagger(\varphi_l^-) + \sum_{k,l} \left\langle \varphi_k^- \right| \, A \, \varphi_l^+ \left\rangle \, b(\varphi_k^-)a(\varphi_l^+) = \right. \\
&=: \underbrace{\mathbb{A}_{++} + \mathbb{A}_{--}}_{\mathbb{A}_{even}} + \underbrace{\mathbb{A}_{+-} + \mathbb{A}_{-+}}_{\mathbb{A}_{odd}}
\end{aligned} \tag{A.12}$$

where we have defined $\mathbb{A}_{++} := \sum_{k,l} \left\langle \varphi_k^+ \right| \, A \, \varphi_l^+ \left\rangle \, a^\dagger(\varphi_k^+)a(\varphi_l^+)$ and so on. Concerning this decomposition, we make the following observations:

∘ If we set $A_{++} := P_+AP_+$, $A_{--} := P_-AP_-$ and so on, we can associate the decomposition (A.12) of $\mathbb{A}$ with the decomposition $A = A_{++} + A_{--} + A_{+-} + A_{-+}$ of its associated one particle operator, such that $\mathbb{A}_{++} \equiv \sum_{k,l} \left\langle \varphi_k^+ \right| \, A_{++} \, \varphi_l^+ \left\rangle \, a^\dagger(\varphi_k^+)a(\varphi_l^+)$ and so on.

∘ Let us choose now for simplicity an ONB $\{\varphi_k^+\}$ of $\mathcal{H}_+$ in which A_{++} is diagonal[210]. Observing that $a^\dagger(\varphi_k^+)a(\varphi_l^+) =: n(\varphi_k^+)$ is the occupation number operator (see e.g. [298]) 'counting' electrons in the state φ_k^+ (which takes the values 0 or 1 in each particle sector due to antisymmetrization), we see that $\mathbb{A}_{++} := \sum_k \langle \varphi_k^+ |\, A_{++}, \varphi_k^+ \rangle\, n(\varphi_k^+)$ is simply the sum over the quantum expectations $\langle \varphi_k^+ |\, A_{++}, \varphi_k^+ \rangle$ weighted with the respective occupation number operators, which obviously admits a transparent physical interpretation.

∘ The second term $\mathbb{A}_{--}$ of the even part of $\mathbb{A}$ is not quite so straightforward to tackle: Its vacuum expectation value is given by

$$\langle\Omega|\, \mathbb{A}_{--}\, \Omega\rangle = \sum_{k,l} \langle \varphi_k^- |\, A\, \varphi_l^- \rangle \underbrace{\langle b^\dagger(\varphi_k^-)\Omega |\, b^\dagger(\varphi_l^-)\Omega \rangle}_{=\delta_{kl}} = \sum_k \langle \varphi_k^- |\, A\, \varphi_k^- \rangle = \mathrm{Tr}_{\mathcal{H}}\,[A_{--}] \tag{A.13}$$

and it is straightforwardly verified that equation (A.13) equals also $\langle\Omega|\, \mathbb{A}\, \Omega\rangle$, since $\mathbb{A}_{++}, \mathbb{A}_{+-}$ and $\mathbb{A}_{-+}$ do not contribute to the vacuum expectation of $\mathbb{A}$. In particular, if A_{--} is not trace class, not even the vacuum[211] is in the domain of $\mathbb{A}$. Indeed, $\langle\Omega|\, \mathbb{A}_{--}\, \Omega\rangle = \sum_k \langle \varphi_k^- |\, A\, \varphi_k^- \rangle$ is simply the contribution of the infinite Dirac sea to the quantum expectation of A and since it is always a (possibly infinite) constant (namely $\mathrm{Tr}_{\mathcal{H}}\,[A_{--}]$), we may subtract the latter to 'measure' A not relative to the empty sea but relative to the filled sea (i.e. relative to the Dirac vacuum). This is done by the trick of *normal ordering*:

According to the canonical commutation relations (A.7) and the mutual orthogonality of the φ_k^- we have $b(\varphi_k^-)b^\dagger(\varphi_l^-) + b^\dagger(\varphi_l^-)b(\varphi_k^-) = \delta_{kl}$, such that we can write the distracting contribution of the Dirac sea to the expectation of A as (the left hand side of (A.14) should be thought of as multiplied by the identity $\mathbb{1}_{\mathcal{F}}$)

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}}\,[A_{--}] &= \sum_k \langle \varphi_k^- |\, A\, \varphi_k^- \rangle = \sum_{k,l} \langle \varphi_k^- |\, A\, \varphi_l^- \rangle \big(b(\varphi_k^-)b^\dagger(\varphi_l^-) + b^\dagger(\varphi_l^-)b(\varphi_k^-) \big) = \\ &= \sum_{k,l} \langle \varphi_k^- |\, A\, \varphi_l^- \rangle\, b(\varphi_k^-)b^\dagger(\varphi_l^-) + \sum_{k,l} \langle \varphi_k^- |\, A\, \varphi_l^- \rangle\, b^\dagger(\varphi_l^-)b(\varphi_k^-) =: \\ &= \mathbb{A}_{--} - :\mathbb{A}_{--}: = \ \mathbb{A} - :\mathbb{A}: \end{aligned} \tag{A.14}$$

where we have defined the operation of normal ordering $:\cdot:$ which pushes in products of creation- and annihilation operators all creation operators to the left of annihilation operators, where the term is multiplied by a factor -1 for each commutation of a creation- with an annihilation operator. The last equality sign in (A.14) holds because the other parts $\mathbb{A}_{++}, \mathbb{A}_{+-}$ and $\mathbb{A}_{-+}$ are already normally ordered (the latter two trivially, since they do not contain mixed products of creation and annihilation operators). Thus we obtain $:\mathbb{A}: = \mathbb{A} - \mathrm{Tr}_{\mathcal{H}}\,[A_{--}]$ or equally $:\mathbb{A}_{--}: = \mathbb{A}_{--} - \mathrm{Tr}_{\mathcal{H}}\,[A_{--}]$.

Using normal ordering, negative energy states become only relevant with respect to A_{--} (respectively $\mathbb{A}_{--}$) if they are unoccupied, i.e. if they represent positrons. This is now easy to

[210] Here we make simplifying assumption that A_{++} has a purely discrete spectrum which serves only for better illustration, but is not necessary for the central insights.

[211] Not having the vacuum in the domain of an observable operator is not only physically but also technically bad, since Fock space is conveniently described by successive action of creation operators onto the vacuum state. Linear combinations of such vectors form a dense subset of $\mathcal{F}$, which is a dense domain of each operator whose action on the vacuum state is well defined [321].

see, e.g. if we choose a basis $\{\varphi_k^-\}$ in which A_{--} is diagonal (again assuming for a moment that A_{--} has discrete spectrum), the normally ordered operator

$$:\mathbb{A}_{--}: = \sum_k \left\langle \varphi_k^- \right| A\,\varphi_k^- \left\rangle\, b^\dagger(\varphi_k^-)b(\varphi_k^-) = \sum_k \left\langle \varphi_k^- \right| A\,\varphi_k^- \left\rangle\, n(\varphi_k^-) \tag{A.15}$$

is formally completely analogue to the just discussed $\mathbb{A}_{++}$, only now $n(\varphi_k^-)$ is the occupation number operator of positrons in the state $C\varphi_k^-$ (which again takes the values 0 or 1 in each particles sector).

∘ Finally, we come to the odd part $\mathbb{A}_{odd} = \mathbb{A}_{+-} + \mathbb{A}_{-+}$ of $\mathbb{A}$, which derives from the parts $A_{\pm\mp}$ of the first quantized operator, the parts which cause transitions between $\mathcal{H}_-$ and $\mathcal{H}_+$ (i.e. states in $\mathcal{H}_-$ are shifted to $\mathcal{H}_+$ by A_{+-} and vice versa by A_{-+}). Given the operators $\mathbb{A}_{\pm\mp}$ have the Dirac vacuum Ω in their domain, their impact on the latter can be directly read of from their representations in terms of creation and annihilation operators: The operator A_{+-} takes states from $\mathcal{H}_-$ and rotates them into $\mathcal{H}_+$, accordingly $\mathbb{A}_{+-} = \sum_{k,l} \left\langle \varphi_k^+ \right| A_{+-}\,\varphi_l^- \left\rangle\, a^\dagger(\varphi_k^+)b^\dagger(\varphi_l^-)$ creates pairs from the Dirac vacuum: For each pair of indices (k,l) for which $\left\langle \varphi_k^+ \right| A_{+-}\,\varphi_l^- \left\rangle \neq 0$, $\mathbb{A}_{+-}$ contains a term which creates an electron in the state φ_k^+ and a positron in the state $C\varphi_l^-$ if it acts on Ω. This term acting on $a^\dagger(\varphi_k^+)\Omega$ or $b^\dagger(\varphi_l^-)\Omega$ will vanish on the other hand because of the Pauli exclusion principle[212] (in the second case this may be also understood by the fact that a positron in the state $C\varphi_l^-$ in second quantization means that the state φ_l^- is unoccupied in the sea picture and thus there is no particle in that state which might be lifted to $\mathcal{H}_+$).

The action of the operator $\mathbb{A}_{-+} = \sum_{k,l} \left\langle \varphi_k^- \right| A_{-+}\,\varphi_l^+ \left\rangle\, b(\varphi_k^-)a(\varphi_l^+)$ (which is the adjoint of $\mathbb{A}_{+-}$) on the Dirac vacuum Ω is obviously zero, since there are no electrons or positrons in this state which might be annihilated. If there is an electron positron pair present, like in the state $\Psi = b^\dagger(\varphi_n^-)a^\dagger(\varphi_m^+)\Omega \in \mathcal{F}$, using the relations (A.7) we obtain $\mathbb{A}_{-+}\Psi = -\left\langle \varphi_n^- \right| A_{-+}\,\varphi_m^+ \left\rangle\, \Omega$, i.e. $\mathbb{A}_{-+}$ annihilates this pair, given the associated amplitude $\left\langle \varphi_n^- \right| A_{-+}\,\varphi_m^+ \left\rangle$ is nonzero.

But the operators $\mathbb{A}_{\pm\mp}$ and thereby $\mathbb{A}$ need not be well defined on (a dense domain of) Fock space, even if its first quantized counterpart A is bounded. This can be understood as follows: If we denote Fock states of electron-positron pairs by $|\varphi_k^+\varphi_l^-\rangle := a^\dagger(\varphi_k^+)b^\dagger(\varphi_l^-)\Omega$, we can write $\mathbb{A}_{+-}\Omega = \sum_{k,l} \left\langle \varphi_k^+ \right| A_{+-}\,\varphi_l^- \left\rangle\, |\varphi_k^+\varphi_l^-\rangle$, i.e. given there are at all index pairs (n,m) such that $\left\langle \varphi_n^+ \right| A_{+-}\,\varphi_m^- \left\rangle \neq 0$, $\mathbb{A}_{+-}$ creates a superposition of states with exactly only one electron positron pair, respectively (in contrast to physical state transformations, which can create in principle arbitrarily many pairs from the vacuum, see below). But although each pair state is always well defined on $\mathcal{F}$, of course, it is mathematically possible that too many such pairs appear in the superposition with too high amplitude, such that $\mathbb{A}_{+-}\Omega$ would be no longer an element of $\mathcal{F}$:

$$\begin{aligned} \|\mathbb{A}_{+-}\Omega\|_{\mathcal{F}}^2 = \sum_{k,l} \left|\left\langle \varphi_k^+ \right| A_{+-}\,\varphi_l^- \left\rangle\right|^2 = \sum_k \left\langle \varphi_k^+ \right| P_+AP_- \overbrace{\sum_l \left|\varphi_l^-\right\rangle\left\langle\varphi_l^-\right|}^{=P_-} P_-AP_+ \;\; \varphi_k^+ \left\rangle = \\ = \mathrm{Tr}_{\mathcal{H}}\left[P_+AP_-AP_+\right] = \mathrm{Tr}_{\mathcal{H}}\left[A_{+-}A_{+-}^\dagger\right] = \|A_{+-}\|_{HS}^2 \end{aligned} \tag{A.16}$$

[212]Note that the exclusion principle is encoded in the commutation relations (A.7), namely $0 = \{a^\dagger(\varphi_k^+), a^\dagger(\varphi_k^+)\} = 2\left(a^\dagger(\varphi_k^+)\right)^2$ such that φ_k^+ cannot be occupied twice: $\left(a^\dagger(\varphi_k^+)\right)^2 = 0$ (and analogously $\left(b^\dagger(\varphi_l^-)\right)^2 = 0$).

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm (for the equality between the first and the second line, we have used that $\mathrm{Tr}_{\mathcal{H}_+}[X] = \mathrm{Tr}_{\mathcal{H}_+}[P_+X] = \mathrm{Tr}_{\mathcal{H}_+}[P_+X] + \mathrm{Tr}_{\mathcal{H}_-}[P_+X] = \mathrm{Tr}_{\mathcal{H}}[P_+X]$). So if A_{+-} is not a Hilbert-Schmidt operator (i.e. the right hand side of (A.16) infinite), the vacuum is not in the domain of $\mathbb{A}_{+-}$. Since A_{-+} and $\mathbb{A}_{-+}$ are the respective adjoints of A_{+-} and $\mathbb{A}_{+-}$ and thus $\|A_{+-}\|_{HS} = \|A_{-+}\|_{HS}$, A_{-+} must be necessarily Hilbert-Schmidt as well if the vacuum shall be in the domain of $\mathbb{A}_{-+}$.

If on the other hand the odd parts $A_{\pm\mp}$ of A are Hilbert-Schmidt, we can always define its second quantized, normally ordered counterpart $:\mathbb{A}:$ on a dense domain of $\mathcal{F}$: The $N-$electron-$M-$positron sector of $\mathcal{F}$ is spanned by the vectors $\Psi = a^\dagger(\varphi^+_{k_1})\dots a^\dagger(\varphi^+_{k_N})\; b^\dagger(\varphi^-_{l_1})\dots b^\dagger(\varphi^-_{l_M})\Omega$ and one can easily calculate (using the relations (A.7)) how the four parts $\mathbb{A}_{++}, :\mathbb{A}_{--}:, \mathbb{A}_{+-}$ and $\mathbb{A}_{-+}$ of $:\mathbb{A}:$ act on Ψ, in particular, all four resulting states are well defined on $\mathcal{F}$ and $:\mathbb{A}:\Psi$ is their sum. Moreover, the set of finite linear combinations of states of the form Ψ (with N, M running in $\mathbb{N}$) is dense in $\mathcal{F}$ and thereby provides a convenient dense domain of $\mathbb{A}$ [321] (which can be characterized as the domain of the particle number operator).

Saying that $\mathbb{A}_{+-}$ (and thereby $\mathbb{A}$) creates particles is of course not to be taken to literally, since $\mathbb{A}$ does not directly induce a physical state transformation (its associated PVM might do so, but then it has to be treated differently, see below). The role of $\mathbb{A}$ is rather to encode the statistics of an associated measurement, in particular, $\langle\mathbb{A}\rangle_\Psi = \langle\Psi|\,\mathbb{A}\,\Psi\rangle_{\mathcal{F}}$ is the quantum expectation of its outcomes if Ψ is the initial state. Non vanishing odd parts $\mathbb{A}_{\pm\mp}$ of $\mathbb{A}$ are then physically primarily relevant for the value of $\langle\mathbb{A}\rangle_\Psi$, e.g. if Ψ describes a single electron positron pair, i.e. is of the form $\Psi = |\varphi^+_k\varphi^-_l\rangle := a^\dagger(\varphi^+_k)b^\dagger(\varphi^-_l)\Omega$, the part $\mathbb{A}_{+-}$ might also yield contributions from the two pair sector and $\mathbb{A}_{-+}$ from the vacuum sector to the value of $\langle\mathbb{A}\rangle_\Psi$.

Unitary Transformations: A unitary transformation U acting on $\mathcal{H}$ extends (globally[213]) naturally to an $N-$ particle state as a tensor product $U\otimes\cdots\otimes U$, i.e it acts on $\mathcal{H}^{\wedge N}$ as

$$\bigwedge_{k=1}^N \varphi_k \quad\longmapsto\quad \bigwedge_{k=1}^N U\varphi_k \tag{A.17}$$

We may think of U as a Dirac time evolution, possibly with a time dependent external electromagnetic vector potential A_t [103, 215] such that $U \equiv U_t^{A_t}$ (see footnote 218 for details).

Implementing U now on $\mathcal{F}$ is again more intricate if U mixes up the positive and negative energy spectral subspaces: Consider a decomposition of U with respect to the splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, i.e. $U = U_{++} + U_{--} + U_{+-}U_{-+} = U_{even} + U_{odd}$ with $U_{++} = P_+UP_+$, $U_{--} = P_-UP_-$ and so on. Thus U acts on a positive energy state $\varphi\in\mathcal{H}_+$ (representing an electron) by

$$U\varphi = U_{++}\varphi + U_{-+}\varphi =: \varphi^+ + \varphi^- \quad \text{where} \quad \varphi,\varphi^+\in\mathcal{H}_+ \text{ and } \varphi^-\in\mathcal{H}_- \tag{A.18}$$

and on a negative energy state $\xi\in\mathcal{H}_-$ (representing a hole in the sea or equivalently a positron in the state $C\xi$) by

$$U\xi = U_{--}\xi + U_{+-}\xi = \xi^- + \xi^+ \quad \text{where} \quad \xi,\xi^-\in\mathcal{H}_- \text{ and } \xi^+\in\mathcal{H}_+ \tag{A.19}$$

[213] One might also consider local unitary transformations of systems of several particles, e.g. a spatial translation of a subsystem, acting on an $N-$particle state like $\mathbb{1}_{\mathcal{H}}\otimes\cdots\otimes U\otimes\cdots\otimes\mathbb{1}_{\mathcal{H}}$. But such a transformation would destroy antisymmetry of wave functions on which it acts. A proper local transformation of a subsystem preserving antisymmetry of the global wave function would rather look like (A.17), where the generator of U is supported in the region in which also the subsystem wave function is supported, such that U acts as the identity anywhere else.

Representing the states in $\mathcal{F}$ by (limits of) finite linear combinations of states of the form $a^\dagger(\varphi^+_{k_1}) \dots a^\dagger(\varphi^+_{k_N})\, b^\dagger(\varphi^-_{l_1}) \dots b^\dagger(\varphi^-_{l_M})\Omega$, we can implement U in the sense of (A.17) on $\mathcal{F}$ by the instruction

$$\Phi^\dagger(\psi) \quad \mapsto \quad \Phi^\dagger_U := \Phi^\dagger(U\psi) \tag{A.20}$$

Thereby we can compare the particle/antiparticle content[214] of the states in $\mathcal{H}$ before and after U has acted by defining $\Phi^\dagger_U(\psi) := a^\dagger_U(P_+\psi) + b_U(P_-\psi)$ such that (A.20) becomes

$$a^\dagger_U(P_+\psi) + b_U(P_-\psi) = a^\dagger(P_+U\psi) + b(P_-U\psi) \tag{A.21}$$

and choosing once $\varphi \in \mathcal{H}_+$ and once $\xi \in \mathcal{H}_-$ we obtain

$$\begin{aligned} a^\dagger_U(\varphi) &= a^\dagger(U_{++}\varphi) + b(U_{-+}\varphi) \equiv \Phi^\dagger(U\varphi) \quad &&\text{for all } \varphi \in \mathcal{H}_+ \\ b_U(\xi) &= b(U_{--}\xi) + a^\dagger(U_{+-}\xi) \equiv \Phi^\dagger(U\xi) \quad &&\text{for all } \xi \in \mathcal{H}_- \end{aligned} \tag{A.22}$$

In this sense, the implementation of U mixes up creation and annihilation operators of electrons and positrons (this mixing is a Bogoliubov transformation, see below) in a way which is perfectly congruent with the Dirac sea picture, where transitions from $\mathcal{H}_-$ to $\mathcal{H}_+$ correspond to electron creation and from $\mathcal{H}_+$ to $\mathcal{H}_-$ to positron annihilation.

But this is not sufficient to implement U on $\mathcal{F}$, since if we consider for example a one electron state $a^\dagger(\varphi)\,\Omega \in \mathcal{F}$, the transformation to $a^\dagger_U(\varphi)\,\Omega = \big(a^\dagger(U_{++}\varphi) + b(U_{-+}\varphi)\big)\ \Omega = a^\dagger(U_{++}\varphi)\,\Omega$ is in general physically meaningless: To complete the implementation, one has to incorporate the action of U on the pure Dirac vacuum Ω as well, since U acts in general on each single negative energy electron in the sea[215], in particular if the parts U_{--} and U_{+-} of U do not vanish.

The action of U_{--} on the Dirac vacuum is not so interesting, it just stirs, roughly speaking, the unobservable sea (but if there are holes in the sea, U_{--} is indirectly responsible for the positron dynamics, of course). U_{+-} on the other hand is in general capable to lift (a part of) the wave function of each single negative energy electron in the sea to the positive energy subspace, i.e. to create an infinite amount of pairs from the Dirac vacuum. Indeed, it turns out that not each first quantized U is compatible with the net description (second quantization) via $\mathcal{F}$, i.e. U_{+-} acting on each single negative energy electron in the sea $\widetilde{\Omega}$ (A.2) – represented by Ω in the net description – need no longer yield a state which can be represented by an element of $\mathcal{F}$ in the net description, in particular if this action creates an infinite amount of pairs with too high probabilities. The transition probability for a sea electron in the state $\xi \in \mathcal{H}_-$ to be 'found' in $\mathcal{H}_+$ after U has acted (a transition which is strongly related with electron creation processes) is given by $\langle U\,\xi|\ P_+\,U\,\xi\rangle$. One can show [103, 215, 321] that these probabilities must be sufficiently well behaved to obtain a well defined corresponding transformation of the Dirac vacuum Ω on

[214] This comparison corresponds to the fixed Dirac sea defined by the fixed splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, which will be called into question later in the 'presence of U', when U is not a free time evolution but involves for example interaction with an external potential.

[215] This is where the Dirac sea model is very advantageous compared to other, mathematically equivalent formulations like second quantized Dirac theory without an underlying sea picture (like it is usually taught in textbooks) or the so called Feynman-Stückelberg interpretation, in which positrons are interpreted as electrons moving backwards in time. So even if one does not want to regard the Dirac sea model as a physically serious candidate theory of the fermionic part of the world we live in, it is nevertheless a powerful tool to have a clear understanding of the non-triviality of the Dirac vacuum and related processes like pair creation (which in turn might be taken as a hint not to dismiss this model as physically dubious without thought).

$\mathcal{F}$, more precisely they must converge to zero for exceedingly negative energies (deep down in the sea) in such a way, that they sum up to a finite value: Let $\{\varphi_k^-\}$ be an ONB of $\mathcal{H}_-$, then U is implementable on $\mathcal{F}$ only if[216]

$$\sum_k \left\langle U\,\varphi_k^- \middle|\, P_+\,U\,\varphi_k^- \right\rangle = \sum_k \left\langle U_{+-}\,\varphi_k^- \middle|\, U_{+-}\,\varphi_k^- \right\rangle = \mathrm{Tr}_{\mathcal{H}}\left[U_{+-}^\dagger U_{+-}\right] = \|U_{+-}\|_{HS}^2 \overset{!}{<} \infty \tag{A.23}$$

which is to say only if U_{+-} is a Hilbert-Schmidt operator. The analogue statement holds for U_{-+} and the two conditions $\|U_{+-}\|_{HS}^2 < \infty$ and $\|U_{-+}\|_{HS}^2 < \infty$ are not only necessary but together also sufficient for the implementability of U on $\mathcal{F}$.

This again is equivalent to the so called *unitary implementability* of U on $\mathcal{F}$: A unitary operator U acting on $\mathcal{H}$ is called *unitarily implementable* if there exists a unitary operator $\mathbb{U}$ acting on $\mathcal{F}$ such that

$$\Phi_U^\dagger(\psi) = \Phi^\dagger(U\psi) = \mathbb{U}\Phi^\dagger(\psi)\mathbb{U}^{-1} \tag{A.24}$$

for all $\psi \in \mathcal{H}$. This equivalence is the content of the famous *Shale-Stinespring* theorem:

Theorem A.1 [*Shale-Stinespring*]

A unitary operator U acting on $\mathcal{H}$ is unitarily implementable on $\mathcal{F}$ if and only if its odd parts U_{+-}, U_{-+} are Hilbert-Schmidt operators.

For a *proof* of the Shale-Stinespring theorem, see e.g. [103, 215, 321]. If the odd part of U is Hilbert-Schmidt, the action of $\mathbb{U}$ on the Dirac vacuum can be explicitly calculated [321]:

$$\mathbb{U}\,\Omega = C_U \prod_{k=1}^{K} a^\dagger(\phi_k^+) \prod_{l=1}^{L} b^\dagger(\phi_l^-)\; e^{\mathbb{B}}\;\Omega \tag{A.25}$$

where C_U is a constant depending on U (whose constant phase can be chosen freely), $\{\phi_1^+, \ldots, \phi_K^+\}$ and $\{\phi_1^-, \ldots, \phi_L^-\}$ are ONBs of the kernels of $U_{++}^\dagger$ in $\mathcal{H}_+$ and $U_{--}^\dagger$ in $\mathcal{H}_-$, respectively, which can be shown to be always finite dimensional subspaces and $\mathbb{B} = \sum_{k,l} \left\langle \varphi_k^+ \middle|\, B\,\varphi_l^- \right\rangle a^\dagger(\varphi_k^+) b^\dagger(\varphi_l^-)$ with $B := U_{odd} \cdot U_{even}^{-1}$. This demonstrates how lively the action of a unitarily implementable transformation on the Dirac vacuum in general is.

Suppose now U is unitarily implementable and consider the action of its implementation $\mathbb{U}$ on an eigenstate of the number operator:

$$\begin{aligned} &\mathbb{U}\, a^\dagger(\varphi_{k_1}^+)\ \ldots\ a^\dagger(\varphi_{k_N}^+)\, b^\dagger(\varphi_{l_1}^-)\ \ldots\ b^\dagger(\varphi_{l_M}^-)\,\Omega = \\ &= \mathbb{U}\, a^\dagger(\varphi_{k_1}^+)\,\mathbb{U}^{-1}\mathbb{U}\, a^\dagger(\varphi_{k_2}^+)\,\mathbb{U}^{-1}\mathbb{U}\ \ldots\ \mathbb{U}^{-1}\mathbb{U}\, b^\dagger(\varphi_{l_{M-1}}^-)\,\mathbb{U}^{-1}\mathbb{U}\, b^\dagger(\varphi_{l_M}^-)\,\mathbb{U}^{-1}\mathbb{U}\,\Omega = \\ &= a_U^\dagger(\varphi_{k_1}^+)\ \ldots\ a_U^\dagger(\varphi_{k_N}^+)\, b_U^\dagger(\varphi_{l_1}^-)\ \ldots\ b_U^\dagger(\varphi_{l_M}^-)\,\Omega_U \end{aligned} \tag{A.26}$$

where the $a_U^\dagger$ and $b_U^\dagger$ together with their adjoints are the 'mixed up creation and annihilation operators' as defined in (A.22). With a little calculation, one easily finds that these operators

[216] In (A.23) we have used $P_-\xi = \xi$ for all $\xi \in \mathcal{H}_-$ and $P_-\varphi = 0$ for all $\varphi \in \mathcal{H}_+$ and that the latter entails that $\mathrm{Tr}_{\mathcal{H}_-}\left[U_{+-}^\dagger U_{+-}\right] = \mathrm{Tr}_{\mathcal{H}}\left[U_{+-}^\dagger U_{+-}\right]$.

satisfy the canonical anticommutation relations in (A.7) and that Ω_U can be formally treated as a 'new vacuum vector' defined by the action of the 'new annihilation operators', in the sense that

$$a_U(\psi)\,\Omega_U = 0 = b_U(\xi)\,\Omega_U \qquad \Omega_U \sim \mathbb{U}\Omega \tag{A.27}$$

for all $\psi \in \mathcal{H}_+$ and $\xi \in \mathcal{H}_-$. Thus the distinct (but unitarily equivalent) triples (Ω, a, b) and (Ω_U, a_U, b_U) generate identical algebraic structures on the same Fock space $\mathcal{F}$. Such transformations of the field operators, which preserve the canonical anticommutation relations, are called *Bogoliubov transformations*.

If $U \equiv U_t$ is a strongly continuous one parameter group, also a lifting version of Stone's theorem is available[217]. If $U = U_t^{A_t}$ represents time evolution with electromagnetic interactions[218], a whole range of problems appear. Besides the well known problems with the (quantized) radiation field (namely infrared and ultraviolet divergences), which shall not be discussed at this place, there is a very fundamental issue connected with the existence of unitarily inequivalent representations of the canonical anticommutation relations in the interacting theory (well known also in the frameworks of AQFT, e.g. in connection with Haag's theorem [206, 277]) , which comes into play already at the level of interaction of the Dirac vacuum with external electromagnetic potentials.

The crucial (and disastrous) observation was made by Ruijsenaars in 1977 [283, 284]: An electron, minimally coupled to an external electromagnetic field given by a four vector potential $A_t = A_t^\mu = (A_t^0, \boldsymbol{A}_t)$ is as well known described by the Dirac Hamiltonian $\mathscr{H}_{A_t} = \mathscr{H}_0 + \alpha_\mu A_t^\mu$ (where we have set the electron charge $e = -1$, $\mathscr{H}_0 = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m$ for the free Dirac Hamiltonian and $\alpha_0 := \mathbb{1}_{\mathbb{C}^4}$), generating the unitary time evolution $U_t^{A_t}$ (for the present purpose, we always assume an initially yet switched off potential $A_0 = 0$, see footnote 218). Ruijsenaars found that the odd parts $(U_t^{A_t})_{\pm\mp}$ of $U_t^{A_t}$ are Hilbert-Schmidt operators for all t if and only if the magnetic part of A_t^μ vanishes identically, i.e. if and only if $\boldsymbol{A}(t) \equiv \boldsymbol{0}$. This makes it impossible to describe coupling of electrons and positrons to magnetic fields in second quantized Dirac theory (at least by the standard scheme) and moreover, it destroys gauge invariance since a purely electric field

[217] This result was firstly proven by Carey and Ruijsenaars in [79] (see also [321]): Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint with $A_{+-} = (A_{-+})^\dagger$ Hilbert-Schmidt, the generator of the strongly continuous one parameter group $U_t = e^{iAt}$. Then the normally ordered operator $:\widetilde{\mathbb{A}}: = :\sum_{k,l}\,\sum_{m,n=\pm}\langle\varphi_k^m|\,A\,\varphi_l^n\rangle\,\Phi^\dagger(\varphi_k^m)\Phi(\varphi_l^n):$ is essentially selfadjoint on the dense domain of the number operator (the set of finite linear combinations of states of the form $a^\dagger(\varphi_{k_1}^+)\ldots a^\dagger(\varphi_{k_N}^+)\,b^\dagger(\varphi_{l_1}^-)\ldots b^\dagger(\varphi_{l_M}^-)\,\Omega$) and its unique selfadjoint closure $\mathbb{A}$ is the generator of the unitary implementation $\mathbb{U}_t = e^{i\mathbb{A}t}$ of U_t on $\mathcal{F}$.

[218] A unitary time evolution generated by a time dependent Hamiltonian of the form $\mathscr{H}(t) = \mathscr{H}_0 + V(t)$ (for convenience, we shall frequently use the parameter notation $V(t) = V_t$ and $\mathscr{H}(t) = \mathscr{H}_t$), is no longer a one parameter unitary group U_t, but can be instead formalized by a unitary two parameter family $U_I(t,s)$, where instead of the semigroup property $U_{t_2}U_{t_1} = U_{t_1+1_2}$ only the weaker property $U_I(t_2,s)U_I(s,t_1) = U_I(t_2,t_1)$ is satisfied (weaker in the sense, that this property also holds for a one parameter group if we set $U_{t-s} =: U_0(t,s)$, which does not work vice versa). For the present purposes, indexing the time evolution by a single parameter t is fine though, also in the case of time dependent interactions, since in the considered dynamical scenarios we only consider an initial time t_0 which we set to zero and at which we suppose that the interaction is yet switched off ($V_0 = 0$) and a later time $t > 0$ where it is switched on ($V_t \neq 0$) such that we only consider $U_I(t,0) =: U_t^{V_t}$.

In the Schrödinger picture, the time dependently interacting time evolution $U_I(t,s)$ is given by the *Dyson series* $U_I(t,s) = Te^{-i\int_s^t \mathscr{H}(t')dt'}$ with the time ordering operator T (in concrete calculations, it is often more convenient to use the interaction picture, in which the time dependence associated with the free time evolution is carried by operators and the time dependence of states $\psi \in \mathcal{H}$ incorporates the interacting part by $\psi \mapsto U_I^{IP}(t,s)\,\psi = Te^{-i\int_s^t \widetilde{V}(t')dt'}\,\psi$ where $\widetilde{V}(t) = e^{i\mathscr{H}_0 t}V(t)e^{-i\mathscr{H}_0 t}$).

$A_t^\mu = (A_t^0, \mathbf{0})$ can always be brought to the form $A_t^\mu = (A_t^0, \boldsymbol{A}_t)$ with $\boldsymbol{A}_0 \neq \mathbf{0}$ by a proper gauge transformation [103, 215]!

This fundamental problem does usually not appear in QED textbooks, since coupling of the 'Dirac field' to external electromagnetic fields is usually described in the scattering regime, i.e. in terms of asymptotically free in and out states connected by the scattering matrix and the associated transition amplitudes. In contrast to the full time evolution, the asymptotic S-matrix exists for sufficiently well behaved fields in external field QED, which can be roughly understood in the following way: Most of the infinity of sea electrons lifted from $\mathcal{H}_-$ to $\mathcal{H}_+$ by a switched on a magnetic field (which destroy the Hilbert-Schmidt property of $(U_t^{A_t})_{+-}$ and thereby the unitary implementability of $U_t^{A_t}$) evolve back to $\mathcal{H}_-$ when the field is switched off again (this heuristic picture will be substantiated below). So the peculiar infinities appear only when the field is present but not in the asymptotic scattering regime.

If the full time evolution shall be consistently described on Fock space in presence of external magnetic fields one has to break new ground. An option standing to reason is to give up the assumption that the dynamics must be described on a fixed Fock space[219]. The Dirac sea picture developed so far was based on the splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ associated with the positive and negative energy spectral subspaces of the free Dirac Hamiltonian. If now the Dirac Hamiltonian contains an external potential, its spectrum becomes more intricate (bound states appear etc.) and holding on to define the sea with respect to the negative energy spectral subspace of the free Hamiltonian is (to say the least) not mandatory. From a purely technical point of view, one might choose any other splitting $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_b$ of $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ with respect to infinite dimensional subspaces $\mathcal{H}_a$ and $\mathcal{H}_b$ and develop from it an analogue Fock space setting as a net description of a sea model where all states in $\mathcal{H}_b$ are supposed to be initially occupied and invisible, but only holes in $\mathcal{H}_b$ and states in $\mathcal{H}_a$ are described on Fock space. Then $\mathcal{H}_b$ is called a *polarization* [103, 215]. More precisely, a polarization is any closed subspace of $\mathcal{H}$ with infinite dimension and infinite codimension and each polarization $\mathcal{H}'$ corresponds to a Fock space (with vacuum vector Ω') as a particular representation of the algebraic structure of creation and annihilation operators of hypothetical electrons and positrons (the canonical anticommutation relations analogue to (A.7), only that $\mathcal{H}_-$ and $\mathcal{H}_+$ have to be replaced now by the new splitting). A possible way to solve the external field problem of QED is to find a reasonable time dependent polarization defining a time dependent Dirac sea in the presence of time dependent external fields and accordingly implement the time evolution between time dependent Fock spaces.

A trivial toy example of such a time dependent Fock space setting is given by identifying the sea at each time t with the polarization

$$U_t^{A_t}\, \mathcal{H}_- = U_t^{A_t} P_-\, \mathcal{H} = U_t^{A_t} P_- (U_t^{A_t})^\dagger\, \mathcal{H} =: P_-^U(t)\, \mathcal{H} =: \mathcal{H}_-^U(t) \tag{A.28}$$

where we have defined the orthogonal projections $P_-^U(t) := U_t^{A_t} P_- (U_t^{A_t})^\dagger$ onto the new seas (all states in $\mathcal{H}_-^U(t)$ occupied) and used the invariance $\mathcal{H} \equiv (U_t^{A_t})^\dagger \mathcal{H}$. By this polarization, one could then decompose the field operator according to $\Phi(\psi) = c_t^\dagger(P_+^U(t)\,\psi) + d_t(P_-^U(t)\,\psi)$ with $P_+^U(t) := \mathbb{1}_{\mathcal{H}} - P_-^U(t) = U_t^{A_t} P_+ (U_t^{A_t})^\dagger$, interpret $c_t^\dagger(\varphi)$ with $\varphi \in P_+^U(t)\, \mathcal{H} =: \mathcal{H}_+^U(t)$ as the creation operator of an electron in the state φ and $d_t^\dagger(\xi)$ with $\xi \in \mathcal{H}_-^U(t)$ as the annihilation operator of

[219] See [215] for a rigorous treatment of the approach described in the following (implementing the time evolution between time varying Fock spaces) in comparison with an alternative proposal in which the time evolution is being renormalized [214].

a positron in the state $C\xi$ at time t, respectively. These operators correspond to (and act on) Fock spaces $\mathcal{F}_t$ constructed as net descriptions as above, only that now $\mathcal{H}_-$ is substituted by $\mathcal{H}_-^U(t)$ at time t. Correspondingly, the vacuum Ω_t (the state corresponding to the case where all states in $\mathcal{H}_-^U(t)$ are occupied) is at each time t the state in $\mathcal{F}_t$ which satisfies the relations $c_t(\varphi)\Omega_t = d_t(\xi)\Omega_t = 0$ for all $\varphi \in \mathcal{H}_+^U(t)$ and $\xi \in \mathcal{H}_-^U(t)$. Due to construction, these creation and annihilation operators satisfy at each time their analogue of the canonical anticommutation relations (A.7) (i.e. with $\mathcal{H}_+$, $\mathcal{H}_-$ and Ω substituted by the respective new expressions). A feature noteworthy about this construction is that the polarization (and thereby the Fock space) at time t does in general not only depend on the field A_t at that time but also on its whole previous history A_s for all $s < t$ (in contrast to the other polarizations discussed below).

In this model, there would be no external field problem but at the same time it is not a physical solution because it is not empirically adequate. The massive pair creation caused by the magnetic field which bursts the Fock space net description with respect to the original polarization $\mathcal{H}_-$ is replaced by a constant particle number all the time, i.e. no particle creation and annihilation processes would be possible if the time dependent polarization was given by $\mathcal{H}_-^U(t)$: If ψ is in $\mathcal{H}_-$ at time zero at which the external field is yet switched off, $U_t^{A_t}\psi$ is trivially in $\mathcal{H}_-^U(t)$ at each time t and the same holds for states initially in $\mathcal{H}_+$ with respect to $\mathcal{H}_+^U(t)$, of course. Thus there are no transitions between the time dependent sea and its orthogonal complement in $\mathcal{H}$ under the time evolution $U_t^{A_t}$ and consequently its second quantized implementation on the Fock spaces $\mathcal{F}_t$ is devoid of any pair creation and annihilation processes.

A second obvious candidate for a (family of) Fock space(es) in the presence of external fields corresponds to the *Furry picture*, which is indeed frequently applied in strong field QED, given the fields vary sufficiently slowly in time, such that they can be approximated by static fields at each (fixed) time. The polarization on which the Furry picture relies at time t is given by the negative energy subspace of the Hamiltonian $\mathscr{H}_{A_t}$, i.e. we identify the sea with a state in which all eigenstates of $\mathscr{H}_{A_t}$ with negative eigenvalues are occupied and derive the corresponding Fock space as effective net description of unoccupied states (holes≡positrons) in this sea and states build from positive energy eigenstates of $\mathscr{H}_{A_t}$, respectively. The problem with the Furry picture is that it is not Lorentz invariant (see Fierz / Scharf [134]) in the sense that what appears as the vacuum in one Lorentz frame contains electrons and positrons in another one (this is reminiscent of the Unruh effect [330, 328], but with respect to inertial frames instead of accelerated frames).

So there seems to be no canonical, empirically adequate and Lorentz invariant choice of vacuum (or equivalently polarization) in the presence of interactions. A profound solution of this dilemma was developed in [103] (see also [215]). The rough idea is to determine the scope in which the sea must be transformed by the external potential in order that the extent of particles created by the latter relative to the transformed sea stays well defined under the time evolution in its second quantized net description.

A central step in this direction is to collect polarizations $V \subset \mathcal{H}$ into polarization classes $[V]_\sim$ by the equivalence relation $V \sim V'$ if $\|P_V - P_{V'}\|_{HS} < \infty$, where P_V and $P_{V'}$ are the orthogonal projections onto $V$and V', respectively. If U is a unitary operator acting on $\mathcal{H}$, the polarization $U\mathcal{H}_-$ is in the polarization class $[\mathcal{H}_-]_\sim$ of $\mathcal{H}_-$ if and only if U is unitarily implementable on the standard Fock space $\mathcal{F} = \mathcal{F}_{\mathcal{H}_-}$ [103, 215], where we have introduced the notation that a Fock space built on polarization V is denoted by $\mathcal{F}_V$. More generally, polarizations in one and the same polarization class correspond to unitarily equivalent Fock spaces and their respective field

operators and corresponding vacua can be represented on a single Fock space, where they are connected by Bogoliubov transformations as described above. The external field problem, on the other hand, is expressed by the fact that $U_t^{A_t}\mathcal{H}_-$ is no longer in the polarization class $[\mathcal{H}_-]_\sim$ if the magnetic part of A_t does not vanish.

In [103] then a distinguished family of unitary operators $\mathscr{U}_t^{A_t}$ is identified, which preserve the polarization classes of the time evolution in the sense that $[U_t^{A_t}\mathcal{H}_-]_\sim = [\mathscr{U}_t^{A_t}\mathcal{H}_-]_\sim = [\mathscr{U}_t^{A_t}V]_\sim$ for all $V \in [\mathcal{H}_-]_\sim$. These operators are technically distinguished since they depend only on the potential A_t at time t but not on its previous history and the induced polarizations $\mathscr{U}_t^{A_t}V$ with $V \in [\mathcal{H}_-]_\sim$ depend even only on the magnetic part $\boldsymbol{A}_t$ of A_t (in particular $\mathscr{U}_t^{A_t}\mathcal{H}_- = \mathcal{H}_-$ whenever $\boldsymbol{A}_t = 0$). Recalling that in contrast $U_t^{A_t}$ and the polarizations $U_t^{A_t}\mathcal{H}_-$ depend in general on A_s for $0 \geq s \geq t$, this shows that nonetheless the polarization class $[U_t^{A_t}\mathcal{H}_-]_\sim$ is completely determined by $\boldsymbol{A}_t$ and shall be accordingly denoted by $PC[\boldsymbol{A}_t] = [U_t^{A_t}\mathcal{H}_-]_\sim = [\mathscr{U}_t^{A_t}\mathcal{H}_-]_\sim$.

It is now possible to implement the one particle time evolution $U_t^{A_t}$ as unitary operators between time varying Fock spaces (for the explicit constructions see [103, 215]), in particular the time evolution is implementable by unitary operators $\mathbb{U}_t^{A_t}(V_0, V_t) : \mathcal{F}_{V_0} \to \mathcal{F}_{V_t}$ between Fock spaces $\mathcal{F}_{V_0}$ and $\mathcal{F}_{V_t}$ for any $V_0 \in PC[\boldsymbol{A}_0]$ ($= [\mathcal{H}_-]_\sim$ if $A_0 = 0$) and $V_t \in PC[\boldsymbol{A}_t]$. Although the actual orbit of the time dependent polarizations V_t (and thereby of the time dependent Fock spaces $\mathcal{F}_{V_t}$) within $PC[\boldsymbol{A}_t]$ remains undetermined in this framework (the representative $\mathscr{U}_t^{A_t}\mathcal{H}_-$ is technically, but not necessarily physically distinguished), transition probabilities are always well defined between fixed initial and final Fock spaces: For $\Psi_{in} \in \mathcal{F}_{V_0}$ and $\Psi_{out} \in \mathcal{F}_{V_t}$ the expression $\left|\left\langle \Psi_{out}\right|\, \mathbb{U}_t^{A_t}(V_0, V_t)\, \Psi_{in}\right\rangle\right|^2$ does not depend on the orbit $\{V_s \in PC[\boldsymbol{A}_s] \mid 0 < s < t\}$ in between. In particular, if at some time T the external field is switched off again, i.e. $A_T = 0$ the corresponding time evolution can be completely implemented on the standard Fock space $\mathcal{F} = \mathcal{F}_{\mathcal{H}_-}$, i.e. $\mathbb{U}_T^{A_T}(\mathcal{H}_-, \mathcal{H}_-) : \mathcal{F} \to \mathcal{F}$, which corresponds roughly speaking to the implementability of the S-matrix on the standard Fock space.

This does not tell us, of course, how the actual physical vacuum (polarization) looks like when the external field is present, but it shows that Fermions in external fields can be consistently described on Fock space(es) also beyond the regime of asymptotically free states. In [104] the mean field argument argument behind the Dirac sea construction is worked out in some detail and it is argued that one should expect that there is indeed not only one but whole class of *equilibrium sea states* which fulfill proper mean field conditions to be candidates to yield a Dirac vacuum (even in the free case, the sea corresponding to the polarization $\mathcal{H}_-$ is only an obvious candidate). The authors argue in this work that the choice of a proper vacuum depends on how the complex microscopic dynamics of the sea is precisely approximated. Moreover, the operational meaning of this choice crucially depends on the corresponding detector model, i.e. in the words of Deckert et al. [104] *'[...] which excitations of the equilibrium [...] cause clicks'*. See also Lazarovici [215] p.129 ff. for a nice discussion of the ambiguities regarding the choice of the vacuum in presence of external fields.

A comparable analysis *how to lift POVMs and state transformers* from the one particle theory to Fock space is not yet known to me and beyond the scope of this work, but would be very valuable. State transformers should be lifted similarly to unitary transformations (which actually are a special case of state transformers associated with probability 1), but in general more complicated since they can be associated with different outcomes and reduce the norms of

the states on which they act. Once proper lift conditions for state transformers are developed, these could be used to infer lift conditions for the associated POVMs. What is important for the present work is to recognize that state transformers (and accordingly POVMs) associated with local measurements will cause pair creation with certain (albeit certainly usually small) probabilities, which e.g. destroys the possibility of existence of a detector POVM on any configuration space associated with a fixed number of particles, since particle creation destroys additivity of the detector formalism as explained in chapter 3. We also found strong direct indications for this claim in section 3.3, namely that on the one particle level locally caused disturbances of positive energy wave functions must inevitably lead to contributions of negative energies and thus be associated with pair creation as the preceding analysis has shown.

B Hegerfeldt Theorem Faster than Light Spreading Probability Distributions

Theorem B.1 [*Hergerfeldt*]

Let $\mathcal{H}$ be a Hilbert space which carries a unitary representation of space-time translations $U(t,\boldsymbol{a})$ whose infinitesimal generator of time translations is a positive operator (spectrum condition). Let E_Δ be a family of effects acting on $\mathcal{H}$ (i.e. positive operators with spectrum in $[0,1]$) indexed by suitable bounded spatial regions $\Delta \subset \mathbb{R}^3$ such that $\langle\psi|\, E_\Delta\, \psi\rangle = 1$ and $\langle\varphi|\, E_\Delta\, \varphi\rangle = 0$ implies $\langle\psi|\, \varphi\rangle = 0$.

Suppose there exist $\psi \in \mathcal{H}$, $\Delta \subset \mathbb{R}^3$ such that $\langle\psi|\, E_\Delta\, \psi\rangle = 1$. Then for any $\varepsilon > 0$ there does not exist a constant $C_\varepsilon < \infty$ such that $\langle U(t,\boldsymbol{a})\, \psi|\, E_\Delta\, U(t,\boldsymbol{a})\psi\rangle = 0$ for all $|\boldsymbol{a}| > C_\varepsilon$ and $t \in [0,\varepsilon)$.

Before proving the theorem, we briefly discuss its causality implications (see also sections 3.1 and 3.3): The considered effects E_Δ shall describe an operational notion of localization in spatial regions $\Delta \subset \mathbb{R}^3$ in the sense that a proper (position) measurement will 'find' the system (particle, atom etc.) within Δ with probability $\mathbb{P}^\psi(\Delta) = \langle\psi|\, E_\Delta\, \psi\rangle$ if $\psi \in \mathcal{H}$ is the state of the measured system (the obvious example is of course given by the eigenprojections of the standard position operator, further examples will be discussed below). A state $\psi \in \mathcal{H}$ is perfectly localized in Δ in the sense of the operational scheme, if $\mathbb{P}^\psi(\Delta) = \langle\psi|\, E_\Delta\, \psi\rangle = 1$ and accordingly localized outside Δ if $\mathbb{P}^\psi(\Delta) = 0$. Among the assumptions of the theorem is the orthogonality requirement, that states perfectly localized in a given region are orthogonal to states which are localized outside that region. The physical interpretation of the theorem needs moreover translation invariance in the form, that if ψ is perfectly localized in Δ, $U(0,\boldsymbol{a})\,\psi$ is perfectly localized in $\Delta + \boldsymbol{a}$. Requiring that a state perfectly localized in a given bounded region is localized outside any disjoint region (which would be e.g. a consequence of additivity, see section 3.4.2) yields together with translation invariance $\langle\psi|\, E_\Delta\, \psi\rangle = 1 \;\Rightarrow\; \langle U(0,\boldsymbol{a})\, \psi|\, E_\Delta\, U(0,\boldsymbol{a})\, \psi\rangle = 0$ for $|\boldsymbol{a}|$ large enough such that $\Delta + \boldsymbol{a}$ is disjoint from Δ (i.e. the state $U(0,\boldsymbol{a})\,\psi$ is localized outside Δ).

Causal propagation in such an operational framework (however it might be precisely defined) clearly entails that if the probability to find the system in a given region is 1 the probability to find it far away shortly afterwards must be zero. E.g. if $|\boldsymbol{a}|$ is large enough such that Δ and $\Delta + \boldsymbol{a}$ are separated by a finite distance $\mathrm{dist}(\Delta, \Delta+\boldsymbol{a}) =: d > 0$, causal propagation implies that the state $U(t,\boldsymbol{a})\,\psi$ stays localized outside Δ for a finite time, at least as long as $|t| < \frac{d}{c}$ (where c denotes here the speed of light). But this is impossible according to Hegerfeld's theorem under its assumptions.

The primary assumption is the spectrum condition (or more generally, a semibounded Hamiltonian). Different versions of Hegerfeld's theorem (see e.g. [177]) demonstrate that one can go without translation invariance and the orthogonality requirement to derive its causally relevant conclusion: If in a quantum theory the Hamiltonian is bounded from below, causal propagation is incompatible with perfectly localized states, both defined in a very basic operational way as sketched above. In a nutshell, *in any quantum theory, one cannot reconcile positive energy, causal propagation and perfect localization*, but one has to drop at least one of the three.

It is instructive to go through some standard examples for E_Δ to check this assertion in more concrete frameworks: If we choose for E_Δ the eigenprojections P_Δ of the standard position

operator (i.e. multiplication by indicator functions in the standard position representation) in non-relativistc quantum theory (Schrödinger theory), Hegerfeldt's theorem shows that causal propagation is violated since the Schrödinger Hamiltonian is bounded from below and for projections, there do always exist perfectly localized states (their eigenstates with eigenvalue 1). Here, the acausal propagation directly corresponds to the well known instantaneous spreading of initially compactly supported Schrödinger wave functions under the free time evolution. The situation is different for the standard position PVM[220] in Dirac theory, where wave functions always propagate causally (since the Dirac equation is hyperbolic), but compactly supported wave functions do not exist in $\mathcal{H}_+$ (the positive energy spectral subspace) but must have contributions from negative energy eigenstates of the free Hamiltonian, i.e. the spectrum condition is violated in this case. A further standard example is the positive energy position POVM of Dirac theory[221] given by equation (3.2), for which consideration of Hegerfeld's theorem shows that perfect localization in bounded regions is impossible and thereby proves that positive energy Dirac wave functions cannot be compactly supported but have always infinite tails (see section 3.1 or e.g. [321] p. 31). Finally, one can also consider the PVM of the Newton-Wigner position operator (see sections 3.1 and 3.3 and references therein), which respects the spectrum condition, admits perfect localization but violates causal propagation (which corresponds e.g. in Dirac theory to instantaneous spreading of wave functions in the Foldy-Wouthuysen representation).

Proof: We shall prove the theorem by contradiction. Suppose the assumptions of the theorem are satisfied, $\langle\psi|\, E_\Delta\, \psi\rangle = 1$ for some $\psi \in \mathcal{H}$ and $\Delta \subset \mathbb{R}^3$ and $\langle U(t,\boldsymbol{a})\, \psi|\, E_\Delta\, U(t,\boldsymbol{a})\psi\rangle = 0$ for all $t \in [0,\varepsilon)$ and $|\boldsymbol{a}| > C_\varepsilon$. Consequently, by assumption $\langle\psi|\, U(t,\boldsymbol{a})\, \psi\rangle = 0$ for all t and $\boldsymbol{a}$ in the set $\{(t,\boldsymbol{a}) \in \mathbb{R}^4 \mid t \in (0,\varepsilon), |\boldsymbol{a}| > C\}$ which is open and connected. Since the spectrum condition is satisfied by assumption, it follows from theorem 3.10 that $\langle\psi|\, U(t,\boldsymbol{a})\, \psi\rangle = 0$ for all $(t,\boldsymbol{a}) \in \mathbb{R}^4$ (theorem 3.10 requires a relativistic Hilbert space, but its proof does actually not need the whole Poincaré group but only the Euclidean subgroup of space-time translations, which is e.g. a subgroup of the Galilei group as well). In particular, using $U(0,\mathbf{0}) = \mathbb{1}_{\mathcal{H}}$ we obtain $0 = \langle\psi|\, \psi\rangle = \|\psi\|^2$ and thus $\psi = 0$, which is a contradiction to the assumption that $\langle\psi|\, E_\Delta\, \psi\rangle = 1$. ■

[220] One might also consider effects of associated approximate position measurement POVMs (given by convolution of indicator functions with an error distribution, see sections 1.4.3 and 1.5.2) in Schrödinger and Dirac theory, which arise from (possibly fundamental) accuracy limitations of the measuring device. Wheather these admit perfect localization depends on the actual error distribution. In Dirac theory, they also violate the spectrum condition, which follows from theorem 3.25, but multiplication by smoothed out indicator functions violates it possibly more moderate than multiplication by indicator functions (i.e. than the standard position PVM). This can become relvant for the transition to quantum field theory, where violations of the spectrum condition in one particle Dirac theory – if moderate enough – correspond to pair creation processes in second quantization (see appendix A). If violations of the spectrum condition by the action of an operator in the one particle theory are too massive, a corresponding second quantized operator does not exist.

[221] This POVM could be seen as a physically acceptable candidate for describing relaistic position measurements from the viewpoint of Hegerfeld's theorem, but note that it violates local commutativity according tho theorem 3.25.

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, §8, Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation mit dem Titel

Localization - Local Quantum Measurement and Relativity

von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

München, den 13.01.2020

Christian Beck