Quasi-Local Conserved Quantities in General Relativity

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Zusammenfassung

Aufgrund des Äquivalenzprinzips existiert in der Allgemeinen Relativitätstheorie keine lokale Definition eines Energiebegriffs. Wenngleich Aussicht auf eine mögliche quasi-lokale Definition der Energie bestand, existiert jedoch kein allgemeines Rahmenkonzept innerhalb dessen eine solche Definition einer quasi-lokalen Energie ausreichend verstanden ist. In dieser Arbeit wird versucht ein solches Rahmenkonzept zu schaffen.

Im ersten Teil dieser Arbeit schlagen wir ein allgemeines Rezept zur Definition quasi-lokaler Erhaltungsgröen in der Allgemeinen Relativitätstheorie vor. Unser Startpunkt ist die Konstruktion von Erhaltungsgrößen auf einer Hyperfläche unendlicher lichtartiger Entfernung ("null infinity") durch Wald und Zoupas. Wir zeigen warum ihre Konstruktion im Inneren der Raumzeit nicht einsetzbar ist und deshalb nicht zur Definition quasi-lokaler Erhaltungsgren verwendet werden kann. Dann führen wir eine Modifikation ihrer Konstruktion ein, sodass die Erhaltungsgröen allgemeiner und insbesondere im Inneren der Raumzeit definiert sind. Wir fahren fort unsere Konstruktion auf BMS-Symmetrien anzuwenden. Dies sind asymptotische Symmetrien asymptotisch flacher Raumzeiten, welche bei "null infinity" BMS-Ladungen definieren, welche wiederum die Bondi-Masse beinhalten. Wir diskutieren wie der Begriff einer BMS-Symmetrie in das Innere der Raumzeit fortgesetzt werden kann um dort quasi-lokale erhaltene BMS-Ladungen zu definieren. Wir argumentieren weiterhin, dass die Nullmode dieser Ladung eine vielversprechende Definition quasi-lokaler Energie ist.

Im zweiten Teil dieser Arbeit untersuchen wir den gravitativen Gedächtniseffekt ("gravitational memory"), welcher eine Aussage über die permanente Verschiebung zwischen Geodäten nach dem Passieren eines Strahlungspulses ist. Gegenwrtig ist dieser Effekt am besten auf einem flachen Raumzeithintergrund verstanden, auf welchem man

ausreichende Kontrolle über das Hilfskonstrukt eines "statischen Beobachters" hat, welcher benötigt wird um die Verschiebung zu messen. In einem gekrümmten Hintergrund wie dem eines schwarzen Loches ist es jedoch meist nicht klar wie "gravitational memory" quantifiziert werden kann. Wir schlagen eine neue Methode vor um den "gravitational memory" Effekt zu detektieren. Wir zeigen, dass die Methode im Hintergrund eines schwarzen Loches anwendbar ist. Weiterhin zeigen wir eine Verbindung zwischen unserer Formulierung des "memory" Effekts und BMS-Symmetrien auf. Dies setzt eine zuvor entdeckte Verbindung bei "null infinity" in das Innere der Raumzeit fort.

Abstract

In General Relativity, because of the equivalence principle, a local definition of energy does not exist. The hope has been that it will be possible to define energy quasi-locally. However, there exists no general framework in which a definition of quasi-local energy is sufficiently understood. In this thesis, an attempt is made to provide such a framework.

In the first part of this thesis, we propose a general prescription for defining quasi-local conserved quantities in General Relativity. Our starting point is the construction of conserved quantities by Wald and Zoupas at null infinity. We point out why their construction is not applicable in the bulk of a spacetime and therefore cannot be used to define quasi-local conserved quantities. Then we propose a modification of their prescription so that the conserved quantities are defined more generally, and in particular in the bulk of a spacetime. We proceed by applying our construction to BMS symmetries. These are asymptotic symmetries of asymptotically flat spacetimes, which at null infinity are understood to define BMS charges including the Bondi mass. We discuss how to extend the notion of BMS symmetry into the bulk of the spacetime so as to define quasi-local conserved BMS charges there. We then argue that the zero mode of this charge is a promising definition of quasi-local energy.

In the second part of the thesis, we study the gravitational memory effect, which is a statement about a permanent displacement between geodesics after the passing of a burst of radiation. At present, it is best understood on a flat background, where one has sufficient control over a notion of "static observer", which is used to measure the displacement. However, on a curved background, such as a black hole, it is usually not clear how to quantify the gravitational memory effect. We shall propose a new method to detect gravitational memory. We show that the method is applicable on a black

hole background. Furthermore, we make a connection between our formulation of the memory effect and BMS symmetries. This extends a previously discovered connection at null infinity to the bulk of the spacetime.

Thesis publications

This thesis is based on some of the author's work that was carried out during the period of October 2015 - September 2019 as a Ph.D. student at the Max-Planck-Institut für Physik (Werner-Heisenberg-Institut) München. The corresponding pre-prints are:

[1] **Quasi-local conserved charges in General Relativity**, arXiv:1908.07505.
Henk Bart.

[2] **Gravitational memory in the bulk**, arXiv:1908.07504.
Henk Bart

To a large extent, chapter 2 and chapter 3 are an ad verbatim reproduction of these publications.

Contents

Chapter 1

Introduction

In this chapter, we provide a broad perspective on the topics discussed in chapter 2 and chapter 3. We shall comment on how our work in the bulk of this thesis fits into the existing literature and we explain how the various topics of this thesis are interrelated. This introduction needs not be followed chronologically (or at all). We refer the reader to section 1.6 for an outline of the bulk of the thesis.

1.1 Quasi-local energy

In General Relativity (GR), as a consequence of the equivalence principle, a local energy momentum tensor of the gravitational field does not exist. However, the concept of energy plays an important role, not in the last place because curvature of spacetime is understood as a result of the presence of sources of energy.

Energy in General Relativity certainly exists, but not in a local sense. For example, one may define the total energy of an asymptotically flat spacetime as the ADM energy. Since it was proved by Witten in 1981 [3] that the ADM energy [4] is positive, it has become a useful tool for characterizing the gravitational field. The success of the ADM energy inspired the ambition to define a quasi-local energy. Here the word quasi-local refers to any extended domain of spacetime, which is however finite. The goal being to provide an even better characterization of the gravitational field.

Finding a useful definition of quasi-local energy is relevant for various reasons. For example, it is expected that the laws of black hole mechanics can be formulated using quasi-local quantities. Furthermore, a notion of quasi-local energy is required in formulating and proving certain conjectures in General Relativity, such as the *hoop conjecture*[1]. And lastly, it is important from the point of view of numerical computations. Namely, errors are controlled by conserved quantities, which – on a computer – are always quasi-local.

However, defining a notion of quasi-local energy has proven to be surprisingly difficult. Many quantities have been proposed. But each of them comes with problems. Examples of such problems are that they are only defined on a narrow class of solutions. Or that they are physically ill-behaved. It is even the case that the community does not – at the time of writing – agree on a set of pragmatic criteria that a notion of quasi-local energy would have to satisfy. The present status of the field is nicely summarised by the observation that the Brown-York quasi local energy – the best understood – does not in general vanish on the Minkowski spacetime.

In this section, we shall discuss some aspects of defining quasi-local energy in General Relativity, which will be relevant for the material presented in chapter 2. We refer to [5] for a comprehensive review on the various proposals that have been made and the problems that they encounter.

1.1.1 The factor two

For a person who is being introduced to the study conserved quantities in General Relativity, perhaps the most confusing aspect is a factor of two, which appears in the Komar formula for defining energy, but not in the Komar formula for defining angular momentum. In order to understand the material in chapter 2, it is important to be

[1] The hoop conjecture is a criterion for when a black hole forms under gravitational collapse.

aware of the reason why this discrepancy exists. Let us therefore explain this.

When a spacetime has an isometry k, one may define the Komar integral [6, 7] by

$$Q_K[k] := -\frac{1}{16\pi}\int_S \boldsymbol{\epsilon}_{abcd}\nabla^a k^b, \tag{1.1}$$

where S is any spacelike two-sphere. When $k = \phi$ is a rotation symmetry, $Q_K[\phi]$ is referred to as the angular momentum. When, however, $k = t$ is a time-translation isometry, the quantity $Q_K[t]$ is not referred to as the energy. Namely, in stationary black hole spacetimes, when $Q_K[t]$ is evaluated at spacelike infinity, it is equal to precisely half of the ADM mass, i.e., $m_{\text{ADM}} = 2Q_K[t]$.

What then is the meaning of the quantity (1.1)? It is the Noether charge of the theory given by the action

$$S = \frac{1}{16\pi}\int_M \boldsymbol{R}. \tag{1.2}$$

The Noether charge happens to be a surface integral, because of Noether's second theorem for (internal) gauge symmetries. (Recall that the same is true for the electric charge in electrodynamics.) Now, a mistake that we made, is that we are not actually considering the theory (1.2). Namely, the variational principle for (1.2) does not yield the Einstein equations of motion. In order to derive the Einstein equations from the variational principle, one has to supplement (1.2) with e.g. the Gibbons-Hawking-York boundary term

$$S = \frac{1}{16\pi}\int_M \boldsymbol{R} + \frac{1}{8\pi}\int_{\partial M} \boldsymbol{K}. \tag{1.3}$$

Here $\boldsymbol{K}$ is the trace of the extrinsic curvature of the (here timelike) boundary ∂M. For variations that leave the boundary metric fixed, (1.3) yields the Einstein equations.

The presence of the boundary term in the action (1.3) has profound consequences for defining energy. Namely, as we shall see in chapter 2, on a suitably defined phase space, the Hamiltonian conjugate to the vector field k is given by

$$H[k] = \int_C \boldsymbol{Q}[k] + \frac{1}{8\pi} k \cdot \boldsymbol{K}, \tag{1.4}$$

where C is now a closed spacelike two-surface in ∂M. For unit time translations $k = t$ at spacelike infinity, the boundary term containing $\boldsymbol{K}$ cancels the contribution from the Noether charge to the energy, and it also contains a contribution which yields the ADM mass. For rotations vectors, however, only the Noether charge contributes to the Hamiltonian. This is because the boundary term integrates to zero for vector fields that are tangent to S.

The conclusion of these observations is the following. It is intuitively *wrong* to think that the factor two that appears in (1.1) for defining energy is fundamental. The Komar integral is just the wrong quantity for the energy, and it is a coincidence that this quantity is equal to half of the actual energy. The Hamiltonian (1.4) explains the apparent discrepancy in the treatment of rotation symmetries and translation symmetries. The discrepancy is resolved by recognizing that the boundary term in the action (1.3) contributes to the Hamiltonian. Boundary terms of this type shall play a central role in chapter 2.

Since we shall make use of Hamiltonian functions extensively, we shall now say more about how Hamiltonian functions are treated in a convenient way in General Relativity.

1.2 Covariant phase space

In order to define a Hamiltonian in General Relativity, a $3+1$ split of the spacetime needs to be provided. However, this is from a conceptual point of view undesired, because in relativity theory, one likes to think of space and time as being on equal footing. However, it is in general also possible to talk about Hamiltonian functions in a covariant manner, i.e., Hamiltonians on the covariant phase space. Here, as a preparation for chapter 2, we shall explain why this is possible, and how they are defined. We do this using the free point particle as an example.

Consider the Lagrangian of a free point particle

$$L = \frac{1}{2} m \dot{x}^2. \tag{1.5}$$

Variation of the Lagrangian yields

$$\delta L = -m\ddot{x}\delta x + \frac{\mathrm{d}}{\mathrm{d}t}\big(m\dot{x}\delta x\big). \tag{1.6}$$

The action principle now states that for variations δx which vanish at the beginning and end point of time, denoted by t_i and t_f respectively, $\delta S = 0$ yields the equations of motion. For the example above, we obtain

$$m\ddot{x} = 0. \tag{1.7}$$

This equation describes the dynamics of the point particle. It says that, because there is no force acting on it, its acceleration vanishes.

Here, however, we are not interested in the dynamics of the point particle. Instead, we consider the kinematics of the point particle, contained in the boundary term in (1.6). This boundary term forms a bridge between the Hamiltonian and Lagrangian formulations of the point particle. In order to understand what we mean by this, let us take a closer look at the boundary term in (1.6).

Recall that the problem (1.5) may be formulated with the Hamiltonian

$$H = \frac{1}{2m}p^2. \tag{1.8}$$

The time evolution is described by Hamilton's equations

$$\dot{p} = -\frac{\partial H}{\partial x} \quad \text{and} \quad \dot{x} = \frac{\partial H}{\partial p}. \tag{1.9}$$

Notice now that the second equation yields

$$p = m\dot{x}, \tag{1.10}$$

so that the boundary term $\theta := m\dot{x}\delta x$ in (1.6) is nothing but

$$\theta = p\delta x \tag{1.11}$$

This object is referred to as the (pre-)symplectic potential of the theory. Namely, if the exterior variational derivative is taken another time, one ends up with the symplectic two form

$$\omega := \widetilde{\delta} p \wedge \delta x. \tag{1.12}$$

From here, we may define a vector field X_H by the condition that

$$\omega(X_H, \cdot) = -\delta H. \tag{1.13}$$

In the above example, this is just

$$X_H = \frac{p}{m}\frac{\delta}{\delta x}. \tag{1.14}$$

And using (1.10) this may be rewritten as follows,

$$X_H = \dot{x}\frac{\delta}{\delta x} = \frac{\mathrm{d}}{\mathrm{d}t}. \tag{1.15}$$

The Hamiltonian is conserved along trajectories generated by X_H. Namely,

$$\delta H(X_H) = 0, \tag{1.16}$$

by the antisymmetry of ω. In other words: the Hamiltonian is time-independent.

In the above derivation, we observed that the symplectic potential (an object from the Hamiltonian formulation) can be obtained by variation of the Lagrangian. This is a generic property of Lagrangian theories that we are considering. Notice therefore, that Hamiltonian methods for constructing conserved quantities, can be translated into the Lagrangian formalism. Starting with a Lagrangian, we can construct the symplectic two form (1.12). Then, given some vector field X, we may ask then if there exists a quantity that is conserved along the evolution of X. This question is answered by showing that a solution to

$$\omega(X, \cdot) = -\delta Q, \tag{1.17}$$

does or does not exist. This strategy for defining Hamiltonian functions will be followed in chapter 2.

Equivalent descriptions

The Lagrangian description of the point particle is identical to the Hamiltonian one, except in that different degrees of freedom are used. The degrees of freedom in the Lagrangian case are the worldlines of the particle. The degrees of freedom in the Hamiltonian case are the position and momentum at a given point in space and time.

The equivalence of both descriptions follows from the fact that, given initial conditions, the evolution of the point particle is uniquely determined by the evolution equations. This means that there is a bijective mapping between the *phase space*: the space of positions and momenta, and the *covariant phase space*: the space of worldlines. This mapping allows us to construct a Hamiltonian on the covariant phase space. The same holds true in the case of General Relativity, simply because of the fact that the initial value problem is well-posed.

The derivation similar to the one above, for diffeomorphism covariant field theories, will be repeated in chapter 2.

"Conserved quantities"

The remainer of this section will be a preparation for the next section, where a notion of energy at null infinity on the covariant phase space of General Relativity is reviewed. This notion of energy is significantly more difficult to construct than the ADM energy at spacelike infinity, because – unlike at spacelike infinity – a Hamiltonian function that generates time translations at null infinity does not exist on the phase space that we shall consider. This construction of "conserved charges" at null infinity – that are not actually conserved – will bridge the gap between defining the ADM mass through a Hamiltonian as above, and the starting point for our construction of quasi-local quantities in chapter 2. However, before we will be able to discuss this, we need to introduce asymptotic symmetries at null infinity, referred to as BMS symmetries. Namely, these are the symmetries conjugate to which a quantity such as energy at null infinity will be defined. We therefore proceed with a discussion of BMS symmetries, before we come back to defining energy in section 1.4.

1.3 BMS symmetries

BMS symmetries are asymptotic symmetries of asymptotically flat spacetimes. They are usually defined as vector fields that preserve a given type of asymptotic fall-off conditions of the fields (the metric in the present case). This approach to defining asymptotic symmetries makes it more an art than a science, because of the ambiguities involved in choosing the fall-off conditions. However, notions of asymptotic symmetries have proved to be extremely fruitful as a source of many physical insights. These include the definition of conserved charges at null infinity and a connection with the gravitational memory effect. Here we review these insights.

1.3.1 Definition

An asymptotic symmetry group is defined as

$$\mathrm{ASG} := \frac{\text{allowed gauge transformations}}{\text{trivial gauge transformations}}. \tag{1.18}$$

Is it usually clear what one means by a trivial gauge transformation; namely the actual redundancies of the theory, which do not act on the physical data of the theory. What one means by an allowed gauge transformation depends on the context. We shall assume that a type of boundary conditions or asymptotic fall-off conditions is given. Then an allowed gauge transformation is a transformation that acts on the whole spacetime, but which preserves the given boundary or asymptotic conditions. Examples of asymptotic symmetries are global $U(1)$ transformations in electromagnetism, or global space and time translations in General Relativity.

In choosing the boundary conditions, one must make sure that they are weak enough so that physically interesting solutions are contained in the solution space. On the other hand, the boundary conditions must be strong enough so that the associated conserved charges with the asymptotic symmetries are finite and well-defined. This fine line between physically interesting scenarios, and what is well-defined, makes the study of asymptotic symmetries non-trivial. Even for asymptotically flat spacetimes in General Relativity, the question of specifying the allowed boundary data is not completely understood. It has even been proposed that the definition (1.18) is inadequate, in that the ASG of an asymptotically flat spacetime should also contain transformations which make the spacetime only locally asymptotically flat. The latter transformations are referred to as (singular) *superrotations* [8], which add (cosmic) strings of conical defects to the spacetime. We shall not discuss this topic further in this thesis.

1.3.2 Asymptotically flat spacetimes

Let us proceed by defining BMS symmetries; asymptotic symmetries of asymptotically flat spacetimes. We shall work in Newman-Unti gauge, instead of the perhaps more familiar Bondi gauge, which was used in the original references [9–11]. We do this, because in the bulk of this thesis, the properties of Newman-Unti gauge shall play an important role.

Newman-Unti gauge

Flat spacetime in advanced coordinates near past null infinity ($\mathcal{I}^-$) is described by the Minkowski metric

$$\mathrm{d}s^2 = -\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\gamma_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B, \tag{1.19}$$

where γ_{AB} denotes the round metric on the two-spheres at constant values of r and v. The idea is now to study a gravitational theory where the solution space consists of metrics that asymptote to the flat metric at large radii. As was discussed in the previous section, what one means by "asymptotes" is a matter of taste. Here we shall be interested in spacetimes that contain radiation – the so-called radiative spacetimes – which describe scattering processes from past null infinity to future null infinity. These are included by the fall-off conditions of Newman-Unti (and Bondi) gauge.

Newman-Unti coordinates [12] are based on a null foliation of the spacetime parametrised by the first coordinate v. The second coordinate r is an affine parameter for the null geodesic generators $n_a := -\partial_a v$ in the hypersurfaces Σ_v of constant v. The remaining angular coordinates x^A are defined such that n^a generates light rays at constant angles. The metric in these coordinates takes the form

$$\mathrm{d}s^2 = W\,\mathrm{d}v^2 + 2\,\mathrm{d}r\,\mathrm{d}v + g_{AB}(\,\mathrm{d}x^A - V^A\,\mathrm{d}v)(\,\mathrm{d}x^B - V^B\,\mathrm{d}v). \tag{1.20}$$

Part of the freedom left in the choice of (v, r, x^A) is then used to impose the following fall-off conditions[2]. Namely,

$$g_{AB} = r^2\gamma_{AB} + rC_{AB} + O(1), \tag{1.21}$$

where γ_{AB} is the round metric, $\partial_v D_{AB} = 0$ and C_{AB} is traceless with respect to γ_{AB},

$$\gamma^{AB}C_{AB} = 0. \tag{1.22}$$

Furthermore, $V^A = O(r^{-2})$. And

$$W = -1 + \frac{2m_B + 4\partial_v\beta_0}{r} + O(r^{-2}), \tag{1.23}$$

[2]One way to obtain these expressions is to consider the fall-off conditions in Bondi gauge [8] and to use the relation between the Bondi and Newman-Unti gauges given by Equation (4.5) of [13].

where m_B denotes the *Bondi mass aspect* and

$$\beta_0 := -\frac{1}{32} C_{AB} C^{AB}. \tag{1.24}$$

The inverse metric is given by

$$g^{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -W & V^A \\ 0 & V^B & g^{AB} \end{pmatrix}. \tag{1.25}$$

Degrees of freedom at $\mathcal{I}$

In order to gain some intuition for the asymptotic metric in Newman-Unti gauge, notice that all but one of the asymptotic values of the metric components are entirely determined by the two-tensor C_{AB}. This tensor is referred to as the *asymptotic shear* of the ingoing null normal $n := -\partial_r$. Since it is traceless with respect to the round metric γ_{AB}, it has two independent components. These comprise the radiative degrees of freedom of the metric.

The Bondi mass aspect

The asymptotic value of the remaining metric component g_{vv} in (1.20) is determined by the *Bondi mass aspect* m_B. This quantity is a measure of the amount of mass that is contained in the spacetime in an achronal slice that ends at a cut of null infinity. This quantity is dependent on the coordinate v when radiation "enters" the spacetime at $\mathcal{I}^-$. This quantity, as we shall see momentarily, determines the "conserved charges" associated with BMS symmetries, which we shall define now.

1.3.3 BMS generators

BMS symmetries (in Newman-Unti gauge) are diffeomorphisms that preserve the asymptotic fall-off conditions stated above. The conditions that define BMS generators can be divided into two types. The conditions of the first type preserve the gauge conditions

$$\mathcal{L}_\xi g^{va} = 0, \tag{1.26}$$

and

$$\partial_r \left[\frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} \xi^a) \right] = O(r^{-2}), \tag{1.27}$$

the latter of which preserves[3] the fact that C_{AB} is traceless with respect to γ_{AB}. The conditions of the second type preserve the asymptotic fall-off conditions

$$\begin{aligned} \mathcal{L}_\xi g^{rA} &= O(r^{-2}), \\ \mathcal{L}_\xi g^{AB} &= O(r^{-3}), \\ \mathcal{L}_\xi g^{rr} &= O(r^{-1}). \end{aligned} \tag{1.29}$$

We shall now determine the form that a vector field ξ must take in order that it preserves the conditions above. (We impose the above conditions in the order indicated above.)

Step 1. $\mathcal{L}_\xi g^{vv} = 0 \Rightarrow \partial_r \xi^v = 0$.

Step 2. Using Step 1, $\mathcal{L}_\xi g^{vA} = 0$ implies that

$$\xi^A = -\partial_B f \int^r g^{AB} \, \mathrm{d}r' + Y^A(v, x^B), \tag{1.30}$$

where Y^A is an arbitrary vector that depends on v and x^B.

Step 3. From $\mathcal{L}_\xi g^{rv} = 0$ it follows that

$$\xi^r = -\int^r V^C \partial_C \xi^v \, \mathrm{d}r' - r \partial_v \xi^v + Z(v, x^A) \tag{1.31}$$

[3] To see this, note that the following identity holds true.

$$\nabla_a \xi^a = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} \xi^a). \tag{1.28}$$

where Z is an arbitrary function of v and x^A.

Step 4. From (1.27) it follows that the function Z in Step 3 must be given by

$$Z = \frac{1}{2}\mathring{\Delta} f, \tag{1.32}$$

where $\mathring{\Delta} := \mathring{D}_A \mathring{D}^A$ denotes the spherical laplacian.

Step 5. It holds true that

$$\mathcal{L}_\xi g^{rA} = -\partial_v Y^A + O(r^{-2}) \tag{1.33}$$

From $\mathcal{L}_\xi g^{rA} = O(r^{-2})$ it then follows that $\partial_v Y^B = 0$.

Step 6. It holds true that

$$\mathcal{L}_\xi g^{AB} = 2g^{AB}\partial_v f + \mathring{D}^A Y^B + \mathring{D}^B Y^A + O(r). \tag{1.34}$$

Here we used that $\partial_v g_{AB} = O(r)$. From $\mathcal{L}_\xi g_{AB} = O(r)$ we deduce that $I^A = O(r)$, which fixes the integration constant in (1.30) such that

$$I^A = -\partial_C f \int_\infty^r g^{AC}\, \mathrm{d}r'. \tag{1.35}$$

Furthermore, Y^A must be a conformal Killing vector such that

$$\partial_v f = \frac{1}{2} D_A Y^A \quad \Rightarrow \quad f = T(x^A) + \frac{1}{2} v D_A Y^A. \tag{1.36}$$

Step 7. It is straightforward to check that (1.29) is satisfied for the conditions on ξ^a stated in the previous steps.

Summarizing the previous steps, it follows that BMS symmetries are generated by[4]

[4] In $3+1$ dimensions.

vector fields ξ of the form

$$\begin{cases} \xi^v = f \\ \xi^r = J - r\partial_v f + \frac{1}{2}\mathring{\Delta} f \\ \xi^A = Y^A + I^A \end{cases} \tag{1.37}$$

where

$$\begin{aligned} f &:= T(x^A) + \frac{1}{2} v \mathring{D}_A Y^A, \\ I^A &:= -\partial_B f \int_\infty^r g^{AB}\, \mathrm{d}r', \\ J &:= -\partial_A f \int_\infty^r V^A\, \mathrm{d}r'. \end{aligned} \tag{1.38}$$

Here $T(x^A)$ is an arbitrary function of the angular coordinates, referred to as a *supertranslation*. The operator $\mathring{D}_A$ is the covariant derivative with respect to the round metric γ_{AB}, and Y^A is a conformal Killing vector of γ_{AB}. And $\mathring{\Delta} := \mathring{D}_A \mathring{D}^A$ denotes the spherical Laplacian.

The BMS group

The isometry group of flat spacetime is the ten-dimensional Poincare group. It comprises the Lorentz group, consisting of rotations and boosts, and the translation group. Given the fact that we study spacetimes that asymptote to the flat spacetime at large distances, the a priori expectation is that the asymptotic symmetry group is equal to the Poincare group. This is, however, not true. For the fall-off conditions that were chosen above, the asymptotic symmetry group is infinite-dimensional. Namely, it is given by a semi-direct product of the *supertranslations* $T(x^A)$ with the superrotations Y^A (the latter are generators of the rotations and boosts),

$$\text{BMS group} = \text{supertranslation group} \ltimes \text{Lorentz group}. \tag{1.39}$$

This group contains infinitely many copies of the Poincare group. For instance, when a supertranslation is decomposed into spherical harmonics,

$$T(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{|m| \leq l} b_{lm} Y_{lm}(\theta, \phi), \tag{1.40}$$

a translation subgroup is given by the $l = 0$ (energy) and $l = 1$ (momentum) modes. Any other four dimensional subgroup is also isomorphic to the translation group. And unlike at spacelike infinity, it turns out that there is no way to prefer one translation subgroup over another. One may think of this as a consequence of the fact that the induced metric at null infinity is non-degenerate.

Physical quantities such as "conserved charges" defined at null infinity should be invariant under BMS symmetries. This imposes strong constraints on the form that observables can take. They are even so strong, that there exists no well-defined notion of *angular momentum* at null infinity. We comment on this further in the next section.

1.3.4 The action of supertranslations on physical data

The free data at null infinity is the asymptotic shear tensor C_{AB}. It describes the radiative degrees of freedom. This makes sense, since because C_{AB} is traceless, it has two independent components, which correspond to the two polarisations of gravitational waves. One may verify that a supertranslation acts on C_{AB} in the following way.

$$\delta_T C_{AB} = T \partial_v C_{AB} - 2 \overset{\circ}{\Delta} T. \tag{1.41}$$

The action of BMS symmetries on C_{AB} will be relevant when we consider the gravitational memory effect in section 1.5.

1.4 Quasi-local charges at null infinity

In the previous section, we defined a class of asymptotically flat spacetimes by specifying asymptotic fall-off conditions. We also identified the asymptotic symmetry group as vector fields that act non-trivially at null infinity, but which preserve the Newman-Unti gauge conditions. With the given definitions at hand, we may seek to define – through the usual Noether procedure – a conserved charge associated with the asymptotic symmetry generators ξ. At null infinity, however, this is a non-trivial task. Here we shall explain why this is so.

In the case of spacelike infinity, one may consider unit time translations, compute the Hamiltonian of an arbitrary spacelike slice, which then gives the associated conserved charge associated with the symmetry. The result is referred to as the ADM mass [4], after Arnowitt, Deser and Metzner, who carried out the analysis. This procedure works at spacelike infinity, because a Hamiltonian associated with the symmetry generator exists on the phase space. At null infinity, however, the situation is different, because there a Hamiltonian associated with asymptotic symmetries does not exist. Physically, this is completely obvious, because the physical system enclosed by a sphere at null infinity is not closed: there may be a flow of in- or outgoing radiation in or out of the system.

In order to see formally why a Hamiltonian at null infinity does not exist, let us consider a radiative spacetime that describes e.g. an ingoing gravitational wave. The gravitational wave carries energy, so before the wave enters the spacetime, one would expect the energy to be less than afterwards. In other words, the advanced time derivative of the energy $\partial_v E$ should be non-vanishing. This change of the energy may be written as the variation of a functional on the phase space as $\delta_{\partial_v} E$. However, this requires that the functional E changes under symmetry transformations by ∂_v, which means that E cannot be a Hamiltonian function on the phase space. Namely, recall from section 1.2 that a Hamiltonian on the phase space associated with a symmetry ξ

satisfies the field theory analog of (1.17),

$$\delta H[\xi] = \Omega(\phi, \delta\phi, \mathcal{L}_\xi \phi), \tag{1.42}$$

for variations δ tangent to the phase space. Here Ω is the symplectic two-form. A property of a well-defined symplectic two form Ω is that it vanishes when both entries are symmetries:

$$\Omega(\phi, \mathcal{L}_\eta \phi, \mathcal{L}_\xi \phi) = 0. \tag{1.43}$$

This means that E can indeed only be a Hamiltonian function on the phase space if it is conserved in time. For any quantity that one would intuitively interpret as the energy at null infinity, this fails to be the case for the class of (radiative) spacetimes that we are considering. Therefore, it follows that a notion of energy at null infinity cannot be defined as a Hamiltonian function on the phase space.

We shall next explain how, albeit the fact that it is not conserved, it is still possible to define a notion of energy – or "conserved quantity" – at null infinity.

1.4.1 "Conserved quantities"

For the cases where a Hamiltonian associated with a symmetry generator does not exist, Wald and Zoupas provided a prescription for defining "conserved quantities". Their idea is basically to modify the defining equation of the Hamiltonian (1.42) in a way that is physically motivated. Wald and Zoupas proposed to define a conserved quantity $\mathcal{H}[\xi]$ as a solution to the equation

$$\delta H[\xi] = \Omega(\phi, \delta\phi, \mathcal{L}_\xi \phi) + \int_C \xi \cdot \mathbf{\Theta}(\phi, \delta\phi), \tag{1.44}$$

where $\mathbf{\Theta}(\phi, \delta\phi)$ is a so-called Wald-Zoupas correction term. We shall review the precise definition of this correction term later in subsection 2.3.1. The point that we want to make now is that a solution to (1.44) exists at null infinity for the phase space above that includes radiative spacetimes.

The resulting solution $\mathcal{H}[\xi]$ at a cut C at null infinity is given by [8]

$$\mathcal{H}[\xi] = \frac{1}{8\pi} \int_C \left[2 f m_B + Y^A (N'_A - 2\partial_A \beta_0) \right] \mathrm{d}\Omega^2, \tag{1.45}$$

where m_B denotes the Bondi mass aspect, and N'_A denotes[5] the angular momentum aspect[6]. Consider now the leading order (as $r \to \infty$) of the constraint equations $G_{vA} = 0$, given by [14]

$$\partial_v m_B = \frac{1}{2} D_A D_B N^{AB} - \frac{1}{2} N_{AB} N^{AB}. \tag{1.46}$$

Here $N_{AB} := \partial_v C_{AB}$ denotes the Bondi-news tensor, which quantifies the flux of radiation. Using (1.46), we may rewrite the charge associated with supertranslations ($Y^A = 0$) in (1.45) as

$$\mathcal{H}[T, Y^A = 0] = \frac{1}{8\pi} \int_{\mathcal{I}} N_{AB}\ \delta_T C^{AB}\, \mathrm{d}\Omega^2, \tag{1.47}$$

where the quantity $\delta_T C^{AB}$ denotes the action of a supertranslation T on the asymptotic shear C_{AB} given in (1.41).

Now, it happens to be the case that

$$\boldsymbol{\Theta}(g, \delta g) = \frac{1}{8\pi} N_{AB} \delta C^{AB}. \tag{1.48}$$

This identity tells us that the Wald-Zoupas correction term, for symmetry variations $\delta = \mathcal{L}_\xi$, is the flux of the charge associated with ξ. For obvious reasons, the quantity $\boldsymbol{\Theta}(\phi, \delta\phi)$ is therefore referred to as the Wald-Zoupas flux. Thus, the correction term of Wald and Zoupas in the defining equation of the Hamiltonian in (1.44) is nothing but a compensation for the amount of charge (energy) that was carried away by the flux of radiation.

[5]The notation N'_A has no meaning other than that we will define the usual N_A differently in chapter 2.

[6]The angular momentum aspect N'_A will be defined in chapter 2

Notice, that a Wald-Zoupas flux term is not needed in order to define charges associated with rotations. This is because rotation vector fields are tangent to C. Thus, a non-vanishing flux of angular momentum just redistributes the angular momentum density on C, while the integrated total remains the same.

The idea of correcting the defining equation of the Hamiltonian with a flux term shall form a basis for the work carried out in chapter 2 of this thesis. Although we shall not be using the Wald-Zoupas flux – which is adapted to the special case of null infinity – the basic intuition behind it remains valid.

1.5 Gravitational memory

Now, we come to the second subject of this thesis: the gravitational memory effect. Consider a ring of (initially static) test particles subject to a gravitational plane wave. While the wave passes, the particles oscillate in the $\times$ or $+$ polarisation directions. At the end of this process, the displacement between the particles is different than before the process. This effect is called the gravitational memory effect. It is a statement about a permanent displacement between test particles due to a burst of radiation.

The gravitational memory effect is usually quantified by considering inertial detectors that are initially at rest. Integration of the geodesic deviation twice gives the difference between the initial and final displacements. There are some instances in which this displacement can be expressed in terms of geometric quantities that have a clear physical interpretation. One such place is null infinity. Let us therefore derive the memory effect there.

Consider at null infinity an observer at constant angles x_0^A and at a large constant radius r_0,

$$x^a_{\mathrm{BMS}} := (v, r_0, x_0^A). \tag{1.49}$$

Following [15], we refer to this observer as a BMS observer. One may check that BMS observers are at leading order geodesics. Namely,

$$x^a_{\text{BMS}} = x^a_{\text{geodesic}} + O(r_0^{-1}), \tag{1.50}$$

where x^a_{geodesic} denotes a worldline that solves the geodesic equation. Over a long period of time, the geodesic observers will start to deviate from the BMS ones. But if we consider advanced time lapses that are parmetrically less than r_0, the approximation by BMS observers will suffice.

Consider next the geodesic deviation equation

$$\frac{\mathrm{d}^2 x^A}{\mathrm{d}v^2} = R^A{}_{vBv} x^B. \tag{1.51}$$

Integration of this equation twice, at subleading order in r yields that

$$\Delta x^A = -\frac{d^B}{2r}\Delta C_{AB}, \tag{1.52}$$

where d^A denotes the initial displacement between the BMS detectors and ΔC_{AB} denotes the difference between the asymptotic shears at two different values of the times advanced time v.

Equation (1.52) is historically important in the study of gravitational memory. Namely, through this equation it was realised that the displacement contains a contribution from non-linear effects. This follows directly from the fact that the constraint equation (1.46) contains a term linear in the Bondi news N_{AB} as well as the non-linear term $N_{AB}N^{AB}$. An important implication of this observation is that backreaction in the linearised theory cannot be ignored.

1.5.1 Limitations

At null infinity the memory effect is completely characterised by the analysis above. However, at subleading orders in r, the above description breaks down. This is because the geodesic deviation equation was only solved at the leading non-trivial order in r. However, even if we solved the geodesic deviation equation at subleading orders in r, it would in general have been hard to make a meaningful statement about gravitational memory. This is because the memory effect is understood in terms of a notion of "inertial observers initially at rest". In the bulk of a curved spacetime, it is in general not trivial to construct geodesic observers which are sufficiently at rest.

1.5.2 Memory in the bulk

In chapter 3, we shall make a proposal for detecting gravitational memory in the bulk of the spacetime. Instead of looking for a good notion of "inertial observer initially at rest", we shall use the existing notion of intertial observers at null infinity. We equip these observers with an apparatus that can shoot photons into the bulk. The observers at null infinity can then ask a third observer to measure the displacement between the light rays in the bulk of the spacetime, at different values of the advanced time v. A measurement of the change of this deviation over time is a method for detecting gravitational memory.

Being able to measure the memory effect in the bulk of a spacetime is relevant for several reasons. For example, as opposed to at null infinity, in the bulk of a spacetime one may be able to make a statement about the displacement effect due to timelike matter. Another example concerns the study of black holes. It has been of interest to understand how a black hole stores information degrees of freedom. With statements about gravitational memory in the bulk of a spacetime, it may be possible to understand better what role the gravitational memory effect has to play in the context of the black hole information paradox. With the detection mechanism that we shall propose, it may become possible to answer such questions.

1.5.3 A connection between memory and supertranslations

There is a connection between the gravitational memory effect and the action of supertranslations on a spacetime. The relation is as follows. Suppose that a spacetime at early advanced times v contains no ingoing radiation. Then the asymptotic shear C_{AB} is constant along v. In the literature, therefore, C_{AB} at a given value of v is referred to as a "vacuum". Now, consider the value of C_{AB} at late advanced times v, while somewhere in between the spacetime was subject to ingoing radiation. Then the final value of C_{AB} is related to the initial one by the action (1.41) of a supertranslation. Since the change of the asymptotic shear ΔC_{AB} determines the displacement between geodesics in the formulation of the memory effect (1.52), this establishes a connection between gravitational memory and supertranslations. It was discovered in [15], and we shall be discussing a similar result in chapter 3 of this thesis.

Remark on soft gravitons

The connection between gravitational memory and supertranslations forms just one rib of a triangle. Namely, there is also a connection to soft graviton amplitudes, which has been subject to a lot of attention lately. We shall not be interested in these connections, but refer the reader for completeness to the original reference [15] and a review [14].

1.6 Outline

In chapter 2 we develop a framework for defining quasi-local conserved quantities in General Relativity. We apply our construction to BMS symmetries, so as to define quasi-local BMS charges. Then we compute the BMS charges in the Reissner-Nördstrom spacetime, to observe that our charge at the horizon is given by the irreducible mass of the black hole. This, together with some other checks such as limiting behaviour at null infinity, and the vanishing of the charge in Minkowski space, leads us to the conclusion that the zero mode BMS charge may be a useful definition of quasi-local energy.

In chapter chapter 3, we propose a new method for detecting gravitational memory. It is observed that this method can be connected to the subject of BMS symmetries. This generalises the observation by Strominger and Zhiboedov – explained in subsection 1.5.3 – that BMS symmetries are connected to the memory effect at null infinity, to the bulk of the spacetime.

We use this observation, that the memory effect in the bulk of the spacetime is connected to BMS symmetries, to motivate a choice of BMS gauge that was swept under the rug in chapter 2. Namely, a problem is that BMS symmetries are not uniquely defined in the bulk of the spacetime. The memory effect provides a way to specify such an extension.

Lastly, we show that our method for detecting gravitational memory is applicable in black hole spacetimes. We do this to argue that our method has potentially interesting applications in situations where the usual methods are difficult to apply.

Chapter 2

Quasi-local conserved charges

A general prescription for constructing quasi-local conserved quantities in General Relativity is proposed. The construction is applied to BMS symmetry generators in Newman-Unti gauge, so as to define quasi-local BMS charges. It is argued that the zero mode of this BMS charge is a promising definition of quasi-local energy.

2.1 Introduction and summary of results

Consider a closed spacelike two-surface B in a four dimensional spacetime M. Then a question in General Relativity is:

What is a sensible notion of energy in the region enclosed by B?

Such a notion of energy will be referred to as a *quasi-local energy*.

The history of defining quasi-local energy started with the observation that a local notion of energy and momentum – a stress energy tensor – does not exist for the gravitational field in General Relativity. This follows directly from the equivalence principle. However, quasi-local notions of conserved quantities are not ruled out by the equivalence principle. They are expected to be useful for various reasons[1]. For example, they could provide a more detailed characterisation of states of the gravitational field

[1] See [5] for an overview.

than the globally defined quantities. Furthermore, they are important from the point of view of applications, such as formulating and proving various conjectures[2] in General Relativity, as well as formulating the laws of black hole thermodynamics [16].

Therefore, the hope has been that it will be possible to construct a quasi-local energy. There is some justification for this hope, because a variety of such quantities have appeared in the literature. Examples are the Komar mass [6], Misner-Sharp energy [17], Hawking energy [18], Bartnik mass [19], Brown-York energy [20] and the Wang-Yau mass [21] among many others. See [5] for an overview.

A problem is that the applicability of the known quasi-local quantities breaks down at one point or another. This happens, for instance, because the quantity is only defined in special cases, or because the quantity is (physically) ill-behaved outside of a class of solutions. To indicate the severity of the problem, let us point out that the most well-known notion of quasi-local energy by Brown and York [20] does not in general vanish in the Minkowski spacetime.

The goal of the present chapter is to provide a general framework for constructing quasi-local conserved quantities, and a notion of quasi-local energy that does not suffer from some of the problems referred to above. Our starting point will be the construction of conserved quantities associated with asymptotic symmetries at null infinity by Wald and Zoupas [22]. Though at null infinity, these charges may be thought of as quasi-local charges if one thinks of a cut at null infinity as a sphere (B in Figure 2.1) in a spacelike slice that is sent outwards with the speed of light. The quasi-local region then contains the *Bondi energy*: the total energy of the spacetime minus the energy of the radiation that was sent out at earlier times.

The construction of Wald and Zoupas provides a notion of "conserved quantity" in situations where a Hamiltonian associated with a symmetry generator does not exist. This is the case at null infinity, because unlike at spacelike infinity, the quantity that would be the Hamiltonian is not conserved due to in- or outgoing radiation. The

[2] One example is the *Hoop conjecture*, which is a criterion for when a black hole forms under gravitational collapse. In order to formulate this conjecture more precisely, a good notion of quasi-local energy is needed.

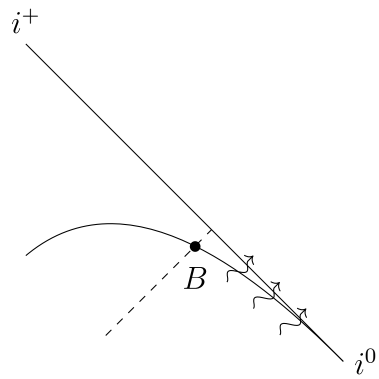

Figure 2.1: The Bondi mass at null infinity may be thought of as a quasi-local charge in the sense that it is given by the total energy of the spacetime minus the energy of the radiation that went out at earlier times. Here we think of the quasi-local region as the region enclosed by a sphere B in a spacelike slice that approaches null infinity.

situation is similar in the bulk of a spacetime; the would-be Hamiltonian is also not conserved due to the presence of radiation or matter. However, there it appears that the construction of Wald and Zoupas cannot be applied. This is because the defining conditions of the Wald-Zoupas charges are tailored to the special case of null infinity.

Nevertheless, given the quasi-local nature of the Wald-Zoupas charges, the hope has been that their construction provides clues about how to define conserved quantities in the bulk of the spacetime. We shall argue that, indeed, a modification of their procedure leads to a well-defined notion of quasi-local conserved charges. The first purpose of the present chapter is thus to modify the construction of Wald and Zoupas so that the "conserved charges" exist more generally, and in particular in the bulk of a spacetime. This then provides a new definition of *quasi-local conserved charges* associated with generators of diffeomorphisms.

Let us outline the technical steps that we will take in terms of the construction of Wald and Zoupas.

> The construction of Wald and Zoupas is essentially a proposal for a correction term $\mathbf{\Theta}(\phi, \delta\phi)$, which is added to the defining equation of the Hamiltonian to guarantee the existence of a solution. Here ϕ denote the fields of the theory and $\delta\phi$ denote variations thereof.

One of the defining conditions of the correction term $\mathbf{\Theta}(\phi, \delta\phi)$ at null infinity is that it vanishes for every *stationary* solution ϕ. This condition makes sense at null infinity, because the quantity $\mathbf{\Theta}(\phi, \mathcal{L}_\xi\phi)$ is equal to the flux of the charge associated with ξ, and at null infinity of stationary spacetimes there is no radiation. In the bulk of a stationary spacetime, however, there may exist other types of matter which in general account for the non-vanishing of the flux. Therefore, the *stationarity condition* is not applicable in the bulk. We propose instead the following defining condition of $\mathbf{\Theta}(\phi, \delta\phi)$.

Consider an auxiliary hypersurface ${}^{(3)}B$. We define $\mathbf{\Theta}(\phi, \delta\phi)$ on ${}^{(3)}B$ by the condition that $\mathbf{\Theta}(\phi, \delta\phi)$, for variations δ that respect a given type of boundary conditions X on ${}^{(3)}B$, integrates to zero on every closed spacelike two-surface B contained in ${}^{(3)}B$. For vector fields tangent to ${}^{(3)}B$, this defines an associated quasi-local conserved charge with respect to boundary conditions X.

This condition is, however, not sufficient, since it defines $\mathbf{\Theta}(\phi, \delta\phi)$ up to a term which is invariant under variations that preserve the boundary conditions X. Moreover, the resulting charge at a closed spacelike two-surface B may be ill-defined, because its definition depends on the choice of auxiliary hypersurface ${}^{(3)}B$. We refer to the freedom in the choice of $\mathbf{\Theta}(\phi, \delta\phi)$ as a choice of *reference term*.

To make our proposal well-defined, we shall impose consistency conditions on the reference term, so that the resulting charge can be interpreted unambigiously as a quasi-local charge at B (independent of the choice of auxiliary ${}^{(3)}B$). We further restrict the freedom in the choice of reference term by introducing *orthogonality* and *zero point conditions*.

A correction term $\mathbf{\Theta}(\phi, \delta\phi)$ that satisfies the conditions mentioned above defines – through the usual procedure [22] – a quasi-local conserved charge.

The second purpose of the present chapter is to apply our construction to BMS symmetries [9–11], so as to define *quasi-local BMS charges*. In the literature, attempts at defining quasi-local BMS charges have been made. See[3] e.g. [28–30,35–39]. However, a drawback of these constructions is that they are not derived from a general framework such as the one developed in this work. In addition, several ambiguities are left untreated, such as the definition of BMS generators in the bulk of a spacetime, which we now comment on.

BMS symmetries are asymptotic symmetries of asymptotically flat spacetimes. They are not a priori defined in the bulk of a spacetime. Therefore, in order to be able to define quasi-local BMS charges, one has to provide a method for extending such symmetries into the bulk. Bulk extensions of BMS generators exist in the literature, such as the extensions in Bondi gauge [9] and Newman-Unti gauge [12]. These are uniquely determined by the requirement that BMS generators in the bulk preserve the given gauge conditions. However, a problem with extensions of this kind is that the gauge choice is essentially arbitrary, and that the generators depend on this gauge. This makes it non-trivial to construct gauge invariant charges.

We do not solve the issue of gauge dependence of the BMS charge. However, we provide in a separate chapter [2] a justification for why the BMS generators in Newman-Unti gauge are physically preferred. Namely, that BMS generators in Newman-Unti gauge are connected to the *gravitational memory effect*.

We show that in Newman-Unti gauge our quasi-local BMS charges have the following properties.

(i) The charges vanish in the Minkowski spacetime.

(ii) The charges coincide asymptotically at null infinity with the BMS charges constructed by Wald and Zoupas.

(iii) At the outer horizon of a Reissner-Nördstrom black hole, the zero mode ($f = 1$) of the gravitational part of the BMS charge is the *irreducible mass* of the black hole.

[3] See also [23–27] for computations of BMS (type) charges in the linearised theory, and [28–34] for discussions about and against their relevance in questions concerning black hole entropy.

Since these are pragmatic criteria that a useful definition of quasi-local energy is expected to satisfy (see e.g. [5] for a list of criteria), we shall put the zero mode BMS charge forward as a new definition of quasi-local energy.

Organisation

In section 2.2 we review the construction of a Hamiltonian on spacetimes with a boundary. This serves as a preparation for section 2.3, where a general prescription for defining quasi-local charges (on a hypersurface ${}^{(3)}B$ which is not necessarily a boundary of the spacetime) in diffeomorphism covariant theories is provided. In section 2.4, we construct quasi-local charges in General Relativity. In section 2.5, we consider BMS generators in Newman-Unti gauge, we evaluate the corresponding charges, and we discuss how the zero mode BMS charge may serve as a definition of quasi-local energy. Possible directions for future work are discussed in section 2.6.

2.2 Quasi-local charges on a boundary

Consider a manifold M with boundary ∂M. Let B be a closed spacelike codimension two-surface in ∂M. Then a quasi-local conserved quantity on B may be defined as a Hamiltonian associated with a vector field that is tangent to ∂M [20, 40]. The construction of a Hamiltonian on the boundary of a spacetime forms a basis for the ideas presented in this chapter. Therefore, we begin with a review of this construction. We follow [40] and we adapt to the notation that boldface symbols are differential forms on the spacetime.

On an n-dimensional manifold M with boundary ∂M, we consider a diffeomorphism covariant theory defined by the action

$$S_X = \int_M \mathbf{L} - \int_{\partial M} \mathbf{B}_X. \tag{2.1}$$

Here $\mathbf{L}$ is a Lagrangian n-form and $\mathbf{B}_X$ is an $(n-1)$-form on ∂M associated with boundary conditions X. For variations δ of the fields ϕ that respect the boundary conditions X, the variation of the action,

$$\delta S_X = \int_M \big[\mathbf{E}(\phi)\delta\phi + \mathrm{d}\boldsymbol{\theta}(\phi,\delta\phi)\big] - \int_{\partial M} \delta \boldsymbol{B}_X, \tag{2.2}$$

yields the equations of motion $\mathbf{E} = 0$ when the boundary term satisfies

$$\delta \boldsymbol{B}_X(\phi) = \overline{\boldsymbol{\theta}}(\phi,\delta\phi)|_{\partial M} - \overline{\mathrm{d}\boldsymbol{\mu}}(\phi,\delta\phi)|_{\partial M}. \tag{2.3}$$

Here $\boldsymbol{\mu}$ is a 2-form and $\overline{\boldsymbol{\theta}}(\phi,\delta\phi)|_{\partial M}$ denotes the pull-back of $\boldsymbol{\theta}(\phi,\delta\phi)$ onto the boundary.

An example of such a theory is General Relativity on a manifold with a timelike boundary ∂M equipped with *canonical boundary conditions* X defined as follows.

Canonical boundary conditions: Boundary conditions X are called *canonical* if the fields whose variation appears in $\boldsymbol{\theta}(\phi,\delta\phi)$ are held fixed on ∂M.

An action for this theory is the Einstein-Hilbert action supplemented with the Gibbons-Hawking-York boundary term [41–43], given by

$$S = \frac{1}{16\pi}\int_M \boldsymbol{R} + \frac{1}{8\pi}\int_{\partial M} \big(\boldsymbol{K} - \boldsymbol{K}_0\big). \tag{2.4}$$

Here $\boldsymbol{R}$ denotes the Ricci scalar and $\boldsymbol{K}$ is the trace of the extrinsic curvature density of the (timelike) boundary ∂M. The boundary three-form $\boldsymbol{K}_0$ is any functional of the boundary metric. It represents an ambiguity of the action for this choice of boundary conditions. The freedom in choosing $\boldsymbol{K}_0$ may be viewed at as a choice of zero point for the Hamiltonian, to which we turn our attention now.

We review the construction of a Hamiltonian on M in the covariant phase space formalism [40, 44]. For a theory of the form (2.2), the symplectic two-form density is given by the variational exterior derivative of the canonical one form $\boldsymbol{\theta}(\phi,\delta\phi)$. Given two independent field variations $\delta_1\phi$ and $\delta_2\phi$, that is,

$$\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi) := \delta_1\boldsymbol{\theta}(\phi, \delta_2\phi) - \delta_2\boldsymbol{\theta}(\phi, \delta_1\phi). \tag{2.5}$$

Consider a foliation of M given by achronal slices Σ_t (labeled by a parameter t), which intersect ∂M orthogonally in compact spacelike $(n-2)$-dimensional surfaces C_t. Then the (pre-)symplectic two form is given by

$$\Omega_{\Sigma_t}(\phi, \delta_1\phi, \delta_2\phi) := \int_{\Sigma_t} \boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi). \tag{2.6}$$

Let ξ be any vector field on M that is tangent to ∂M. Then we say that a real-valued function H_ξ on the covariant phase space is a *Hamiltonian conjugate to* ξ if for all variations of the field that respect the boundary conditions X,

$$\delta H[\xi] = \Omega_{\Sigma_t}(\phi, \delta\phi, \mathcal{L}_\xi\phi). \tag{2.7}$$

Here $\mathcal{L}_\xi$ denotes the Lie-derivative with respect to ξ. As shown in [40], for variations $\delta\phi$ that satisfy the linearized equations of motion, and for on-shell solutions ϕ, it holds true that

$$\Omega_{\Sigma_t}(\phi, \delta\phi, \mathcal{L}_\xi\phi) = \int_{C_t} \delta\mathbf{Q}[\xi] - \xi \cdot \boldsymbol{\theta}(\phi, \delta\phi), \tag{2.8}$$

where $\boldsymbol{Q}[\xi]$ is the *Noether charge two-form.* Then, using (2.3) and the *assumption* that the pull-back $\overline{\boldsymbol{\mu}}|_{C_t}$ of $\boldsymbol{\mu}$ to C_t vanishes, a solution to (2.7) exists and is given by

$$H_X[\xi] = \int_{C_t} \mathbf{Q}[\xi] - \xi \cdot \mathbf{B}_X. \tag{2.9}$$

This is a quasi-local conserved quantity defined on the boundary ∂M.

Notice that $\boldsymbol{B}_X$ is in general determined up to a three-form which depends only on the boundary data X. We shall return to this freedom of choosing a *reference term* ($\boldsymbol{K}_0$ in (2.4)) later.

The following can be said about (2.9). General Relativity satisfies the requirements for the existence of $H_X[\xi]$ for canonical boundary conditions X. When the gravitational reference term $\boldsymbol{K}_0$ in (2.4) is the Hawking-Horowitz-Hunter reference term [45, 46], the Hamiltonian (2.9) conjugate to unit time translations at spacelike infinity is the ADM mass [4] plus possibly additional contributions from long range matter fields [47]. When $\partial\Sigma_t$ is an inner-boundary in the bulk of the spacetime, and ξ is a unit time translation, (2.9) is a generalisation of the Brown-York quasi-local energy [20].

2.3 General definition of quasi-local charges

The function $H_X[\xi]$ constructed in (2.9) is a true Hamiltonian function on the phase space, only if the phase space incorporates the boundary conditions X. There are, however, situations where it is desired to consider a more general class of solutions that violate the boundary conditions X, but where a quantity like $H_X[\xi]$ is still physically meaningful.

One example is null infinity as a boundary of asymptotically flat spacetimes. On the phase space consisting of all asymptotically flat spacetimes, a solution to (2.7) does not exist at null infinity. However, the *Bondi mass* that exists as a Hamiltonian function on the reduced phase space where in- and outgoing radiation is excluded, turns out to be physically relevant on the original phase space too [22]. Only, it is not conserved when radiation enters or leaves through null infinity. This observation indicates that it could be useful have a procedure for constructing "conserved quantities", even though strictly speaking the quantities are not Hamiltonian functions on the phase space.

The goal of this section is to provide a prescription for constructing a "conserved quantity" assocoiated with a vector field ξ on an arbitrary closed spacelike two surface B in M. Our prescription is based on the framework of Wald and Zoupas for constructing "conserved quantities" in diffeomorphism covariant theories in situations where a Hamiltonian does not exist. Their prescription leads to a well-defined notion of conserved charges at null infinity. However, it is not in general applicable in the bulk of a spacetime. We shall provide a modification of their prescription that is applicable in a more general context and in particular in the bulk of a spacetime.

The organisation of this section is as follows. In subsection 2.3.1, we review the construction of Wald and Zoupas and we explain why it is not applicable in the bulk of a spacetime. In subsection 2.3.2, we propose a modification of their construction that is applicable more generally. In subsection 2.3.3 and subsection 2.3.4 we impose consistency conditions. A summary of our proposal is provided in subsection 2.3.5.

2.3.1 The Wald-Zoupas correction term

Consider a hypersurface ${}^{(3)}B$ in M. (Our notation is adapted to the situation where the spacetime dimension is $n = 3+1$.) Let $\boldsymbol{\Theta}$ be a symplectic potential for the pull back $\overline{\boldsymbol{\omega}}$ of $\boldsymbol{\omega}$ onto ${}^{(3)}B$. That is, $\boldsymbol{\Theta}$ satisfies

$$\overline{\boldsymbol{\omega}}(\phi, \delta_1\phi, \delta_2\phi) = \delta_1\boldsymbol{\Theta}(\phi, \delta_2\phi) - \delta_2\boldsymbol{\Theta}(\phi, \delta_1\phi). \tag{2.10}$$

Wald and Zoupas [22] then define a "conserved quantity" $\mathcal{H}[\xi]$ conjugate to a vector field ξ tangent to ${}^{(3)}B$ as a solution to the equation

$$\delta\mathcal{H}[\xi] = \Omega_\Sigma(\phi, \delta\phi, \mathcal{L}_\xi\phi) + \int_B \xi \cdot \boldsymbol{\Theta}. \tag{2.11}$$

Here Ω_Σ is defined by (2.6) in which Σ is an achronal slice with a boundary at $B \subset {}^{(3)}B$. Thus, the idea of Wald and Zoupas is to introduce $\boldsymbol{\Theta}$ as a *correction term* in the defining equation of the Hamiltonian (2.7), such that a solution exists, even in situations where originally it does not.

Note that $\boldsymbol{\Theta}$ must be of the form

$$\boldsymbol{\Theta}(\phi, \delta\phi) = \overline{\boldsymbol{\theta}}(\phi, \delta\phi) - \delta\boldsymbol{W}(\phi), \tag{2.12}$$

where $\overline{\boldsymbol{\theta}}$ is the pull-back of $\boldsymbol{\theta}$ onto ${}^{(3)}B$ and $\boldsymbol{W}$ is an arbitary three-form on ${}^{(3)}B$. It follows that a solution to (2.11) is given by

$$\mathcal{H}[\xi] = \int_B \boldsymbol{Q}[\xi] - \xi \cdot \boldsymbol{W}. \tag{2.13}$$

However, since $\boldsymbol{W}$ is essentially arbitrary, the above prescription is not well-defined. One must impose by hand a sensible condition or procedure to specify $\boldsymbol{W}$.

In order to fix the ambiguity in $\boldsymbol{W}$, Wald and Zoupas [22] imposed the condition that

$$\boldsymbol{\Theta}(\phi, \delta\phi) = 0, \tag{2.14}$$

for every stationary spacetime ϕ and on-shell perturbation $\delta\phi$. They showed that in the limit where ${}^{(3)}B$ approaches null infinity, this condition uniquely defines $\boldsymbol{\Theta}$, and that it gives rise to the Bondi mass as the conserved charge associated with unit time translations.

The motivation to fix $\boldsymbol{\Theta}$ by the requirement that it vanishes on stationary spacetimes is that $\boldsymbol{F}_\xi := \boldsymbol{\Theta}(\phi, \mathcal{L}_\xi \phi)$ is the flux of the charge conjugate to ξ. I.e., for a submanifold $\Delta \subset {}^{(3)}B$,

$$\mathcal{H}[\xi]\big|_{\partial\Delta} = \int_\Delta \boldsymbol{F}_\xi. \tag{2.15}$$

This means that the requirement (2.14) is physically justified at null infinity, because there is no in- or outgoing radiation (Bondi news) in stationary spacetimes.

However, the requirement (2.14) is not physically justified[4] when ${}^{(3)}B$ is a hypersurface in the bulk of a spacetime. Namely, even if ϕ is stationary, when ${}^{(3)}B$ intersects a (stationary) source of matter, one expects that the flux through ${}^{(3)}B$ is non-zero. Such a situation is depicted in Figure 2.2. We shall therefore propose an alternative method to specify $\boldsymbol{\Theta}$, which is also applicable in the bulk of a spacetime.

[4] In [23] the stationarity condition (2.14) is replaced by the requirement that $\Theta(\phi, \delta\phi)$ vanishes on null surfaces with vanishing shear and vanishing expansion.

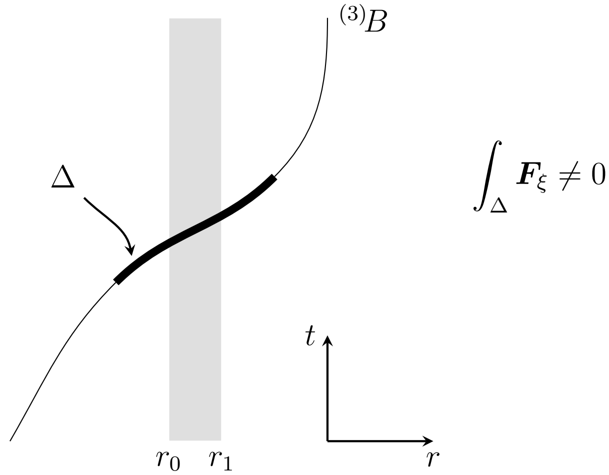

Figure 2.2: A spacetime containing a stationary shell of matter between $r_0 < r < r_1$ (the grey rectangle). Each point in the figure represents a two-sphere at radius r and time t. When the hypersurface $^{(3)}B$ intersects a region containing the matter, one in general expects that the flux $\boldsymbol{F}_\xi$ of the charge associated with a vector ξ tangent to $^{(3)}B$ is non-vanishing.

2.3.2 Correction terms in the bulk

Here we propose a correction term $\boldsymbol{\Theta}(\phi, \delta\phi)$ in the bulk of a spacetime. The idea is the following. Instead of requiring that $\boldsymbol{\Theta}(\phi, \delta\phi)$ vanishes on a given class of spacetimes ϕ, we shall require that it integrates to zero on B for a type of variations $\delta\phi$. How do we define a "type of variation"? We consider boundary conditions X on a $^{(3)}B$ that contains B. A variation of the type X is then defined as a variation of the fields $\delta\phi$ that preserves the boundary conditions X.

The resulting "conserved quantity" is by construction – if it exists – identical to the Hamiltonian (2.9) on the reduced phase space that incorporates the boundary conditions X. The difference with the previous section is that the boundary conditions do not constrain the phase space. They serve only to define the type of variations (processes) for which the flux through $^{(3)}B$ vanishes.

Definition: Let $^{(3)}B$ be a hypersurface in M. Choose $\boldsymbol{\Theta}(\phi, \delta\phi)$ in (2.12) such that there exists an $(n-2)$-form $\boldsymbol{\mu}$, such that for every variation $\delta\phi$ that respects a given choice of boundary conditions X on ${}^{(3)}B$,

$$\mathbf{\Theta}(\phi, \delta\phi) = \overline{\mathrm{d}\boldsymbol{\mu}}(\phi, \delta\phi). \tag{2.16}$$

Here $\overline{\mathrm{d}\boldsymbol{\mu}}$ denotes the pullback of $\mathrm{d}\boldsymbol{\mu}$ to ${}^{(3)}B$. Then at $B \subset {}^{(3)}B$, we shall call a solution $\mathcal{H}_X[\xi]$ to (2.11) a *quasi-local conserved charge with respect to boundary conditions X.*

Different choices of boundary conditions X yield different "conserved charges", each of which has its own physical interpretation. The role of the boundary conditions is to determine what part of the total charge (e.g. energy) is available to an outside observer which respects the boundary conditions X. The situation is similar in statistical thermodynamics, where different ensembles have different free energies. Thus, with the proposed definition, the problem of constructing quasi-local conserved quantities reduces to finding meaningful boundary conditions.

With the proposed choice of $\mathbf{\Theta}(\phi, \delta\phi)$, the "conserved charge" takes the form

$$\mathcal{H}_X[\xi] = \int_B \boldsymbol{Q}[\xi] - \xi \cdot \boldsymbol{B}_X, \tag{2.17}$$

where $\boldsymbol{B}_X$ satisfies

$$\overline{\boldsymbol{\theta}}(\phi, \delta\phi) - \delta\boldsymbol{B}_X(\phi) = \overline{\mathrm{d}\boldsymbol{\mu}}, \tag{2.18}$$

for variations δ that respect the boundary conditions X on ${}^{(3)}B$.

Notice, however, that (2.17) is not yet well-defined. Namely, $\boldsymbol{B}_X$ is defined up to the addition of a three form $\boldsymbol{B}^0_X$ such that

$$\delta\boldsymbol{B}^0_X = 0, \tag{2.19}$$

for variations δ that respect the boundary conditions X. We refer to $\boldsymbol{B}^0_X$ as a *reference term*, which we discuss momentarily.

In the remainder of this chapter, unless stated otherwise, we choose X to be *canonical boundary conditions* as defined in section 2.2. We refer to the corresponding charge as the *canonical quasi-local conserved charge*.

2.3.3 Consistency of the reference term

The correction term $\boldsymbol{\Theta}(\phi, \delta\phi)$ in (2.16) is defined up to a choice of *reference term* $\boldsymbol{B}_X^0$ that satisfies $\delta \boldsymbol{B}_X^0 = 0$ for variations δ that respect the boundary conditions X. One has to remove this freedom by hand.

In the previous section, in e.g. (2.9), there was a similar type of freedom. However, in contrast to the previous section, there are now consistency conditions that restrict the freedom in the choice of reference term $\boldsymbol{B}_X^0$.

Tangent condition

Namely, in contrast to (2.9), the boundary term $\boldsymbol{B}_X$ in (2.17) is defined simultaneously on any auxiliary hypersurface ${}^{(3)}B \supset B$ that is tangent to ξ. Therefore, (2.17) is only well-defined as the charge associated with ξ at B if it is independent of the choice of the *auxiliary background structure* ${}^{(3)}B$. We shall impose this as a consistency condition on the choice of reference term $\boldsymbol{B}_X^0$.

A condition that achieves this is that $\boldsymbol{B}_X$ evaluated at B is identical for all ${}^{(3)}B$ that are tangent to ξ at B. That is, if ${}^{(3)}B$ and ${}^{(3)}B'$ are any two hypersurfaces that are tangent to each other at B, then we require that

$$\boldsymbol{B}_X \overset{B}{=} \boldsymbol{B}'_X \tag{2.20}$$

In this equation, taking the push-forward of $\boldsymbol{B}_X$ into the spacetime is understood. See Figure 2.3 for a situation where this condition should apply.

Linearity condition

In addition, we shall require that the charge (2.17) is linear in ξ. Thus when $\xi = \xi' + \xi''$, we require that

$$\mathcal{H}_X[\xi] = \mathcal{H}_X[\xi'] + \mathcal{H}_X[\xi'']. \tag{2.21}$$

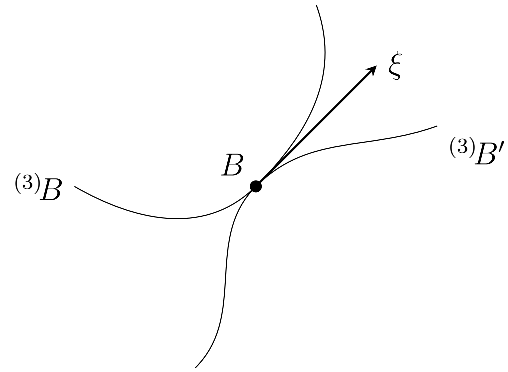

Figure 2.3: An example of two hypersurfaces ${}^{(3)}B$ and ${}^{(3)}B'$ which are tangent at B. Both hypersurfaces define the charge (2.17) associated with ξ at B. The consistency condition (2.20) ensures that the charges are the same for both hypersurfaces.

Since the Noether charge is linear in ξ [48], it will be sufficient to require that

$$\xi \cdot \boldsymbol{B}_X \stackrel{B}{=} \xi' \cdot \boldsymbol{B}'_X + \xi'' \cdot \boldsymbol{B}''_X, \tag{2.22}$$

where $\boldsymbol{B}'_X$ and $\boldsymbol{B}''_X$ denote the boundary terms on hypersurfaces tangent to ξ' and ξ'' respectively. See Figure 2.4 for an example where the condition (2.22) should apply.

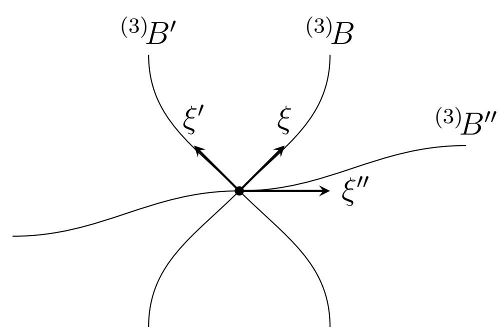

Figure 2.4: The linearity condition is a constraint on the relation between the boundary terms $\boldsymbol{B}_X$, $\boldsymbol{B}'_X$ and $\boldsymbol{B}''_X$ on different hypersurfaces that intersect B.

A sufficient condition so that both (2.20) and (2.22) hold true, is that $\boldsymbol{B}_X$ at B is of the form

$$\boldsymbol{B}_X \stackrel{B}{=} b_X(\phi)\overline{\boldsymbol{V}}, \tag{2.23}$$

where $\overline{\boldsymbol{V}}$ denotes the pull-back of a spacetime three-form $\boldsymbol{V}$ (which may depend on B) onto ${}^{(3)}B$, and b_X is a functional dependent on the boundary data X available on ${}^{(3)}B$, but independent of the choice of ${}^{(3)}B$.

2.3.4 The orthogonality and the zero point conditions

After imposing the condition (2.23), the freedom left in the choice of $\boldsymbol{B}$ is the choice of the three-form $\boldsymbol{V}$ and the functional b_X.

We assume that the kernel of the three-form $\boldsymbol{V}$ is one-dimensional. We may do this if we absorb multiplicative factors into b_X. Then $\boldsymbol{V}$ is determined by a direction $\xi_\perp$ such that

$$\xi_\perp \cdot \boldsymbol{V} = 0. \tag{2.24}$$

There are two natural choices of $\xi_\perp$ at a given closed spacelike two-surface B. Namely, the ingoing and outgoing null directions orthogonal to B denoted by n and l respectively. We shall set

$$\xi_\perp = n, \tag{2.25}$$

and require that (2.24) holds true. We refer to this as the *orthogonality condition*.

To reduce the freedom in the choice of b_X, we shall require that $\boldsymbol{B}_X$ vanishes at B on a reference solution ϕ_0. I.e.,

$$\boldsymbol{B}_X(\phi_0) = 0. \tag{2.26}$$

We refer to this as the *zero point condition*.

2.3.5 Summary of our proposal

Let ξ be a vector field on M. Consider a closed spacelike two-surface B. Pick a hypersurface ${}^{(3)}B$ that contains B and to which ξ is tangent. Denote by $\boldsymbol{\Theta}(\phi, \delta\phi)$ a Wald-Zoupas correction term on ${}^{(3)}B$. That is, a solution to (2.10). The charge $\mathcal{H}_X[\xi]$ will then be defined as a solution to (2.11). Since $\boldsymbol{\Theta}(\phi, \delta\phi)$ is determined up to a total variation $\delta\boldsymbol{W}$ in (2.12), our proposal is a method to specify $\boldsymbol{W}$. At this point, we differ from the original prescription by Wald and Zoupas.

We require to choose $\boldsymbol{W} = \boldsymbol{B}_X$ such that $\boldsymbol{\Theta}(\phi, \delta\phi)$ integrates to zero for variations δ

that respect a given choice of boundary conditions X on ${}^{(3)}B$. This determines $\mathbf{\Theta}(\phi, \delta\phi)$ up to a *reference term* $\boldsymbol{B}_X^0$ which is invariant under variations that respect the boundary conditions X.

We reduce the freedom in the choice of reference term by the consistency condition (2.20). This condition requires that if we had picked a different ${}^{(3)}B$ that contains B and to which ξ is tangent, the quantity $\boldsymbol{B}_X$ at B will be the same. In addition to the consistency condition, we require that the charge is linear in the symmetry generators ξ.

To guarantee consistency and linearity, we impose that the boundary term is of the form $\boldsymbol{B}_X = b_X(\phi)\overline{\boldsymbol{V}}$ (see (2.22)). Here $\boldsymbol{V}$ is a spacetime three-form with a one-dimensional kernel, $\overline{\boldsymbol{V}}$ denotes its pull-back onto ${}^{(3)}B$, and $b_X(\phi)$ is a functional that depends on the boundary data X available on ${}^{(3)}B$, but so that it is independent of the ${}^{(3)}B$ that contain B.

It then remains to limit the freedom in choosing the functional b_X and the three-form $\boldsymbol{V}$. We specify $\boldsymbol{V}$ by requiring that for ingoing lightrays generated by n orthogonal to B, we have $n \cdot \boldsymbol{V} = 0$. The functional b_X is required to be chosen so that the *zero point condition* is satisfied, namely that $\boldsymbol{B}_X$ vanishes at every B on a reference solution ϕ_0.

Existence and uniqueness

In section 2.4 we shall construct a reference term for the Einstein-Hilbert action, which satisfies (2.23), (2.24) and (2.26), and hence yields a well-defined quasi-local conserved charge.

We want to emphasize that we have not in detail investigated uniqueness (and in general existence) of our proposal. Thus we do not guarantee that our proposal is successful outside of the domain studied in the remainder of the present chapter.

Computational remark

In order to evaluate (2.17) for a given symmetry generator ξ on B, it is not required to construct a hypersurface ${}^{(3)}B$ that is tangent to ξ. Namely, property (2.23) guarantees that

$$\xi \cdot \boldsymbol{B}_X \stackrel{B}{=} b_X(\phi)\xi \cdot \boldsymbol{V}, \tag{2.27}$$

so that it is sufficient to determine $\boldsymbol{V}$, and to compute $b_X(\phi)$ on a ${}^{(3)}B \supset B$ that is convenient. For the reference term that we construct for the Einstein-Hilbert action in subsection 2.4.3, it is not even necessary to refer to an auxiliary hypersurface ${}^{(3)}B$.

2.4 Quasi-local charges in General Relativity

Here we construct quasi-local conserved charges for the four-dimensional ($n = 3 + 1$) Einstein-Hilbert action according to the prescription in the previous section. The construction consists of two parts. First, we write down the general form of the charge associated with canonical boundary conditions. Second, we construct a reference term so that the *consistency*, *linearity*, *orthogonality* and *zero point conditions* are satisfied.

2.4.1 The form of the Einstein-Hilbert charges

Consider the Einstein-Hilbert Lagrangian

$$\boldsymbol{L}(g) = \frac{1}{16\pi} R(g)\boldsymbol{\epsilon}(g), \tag{2.28}$$

where $R(g)$ denotes the Ricci scalar and $\boldsymbol{\epsilon}(g)$ is the volume form associated with g. For this theory, a[5] canonical one-form $\boldsymbol{\theta}(g, \delta g)$ obtained through the variation of the Lagrangian in (2.2) is given by [48]

$$\theta_{abc} = \epsilon_{dabc} v^d, \tag{2.29}$$

[5] The canonical one form $\boldsymbol{\theta}$ is defined up to $\boldsymbol{\theta} \mapsto \boldsymbol{\theta} + \mathrm{d}\boldsymbol{Y}$, where $\boldsymbol{Y}(\phi, \delta\phi)$ is a covariant $(n-2)$-form linear in $\delta\phi$.

where

$$v^d = \nabla_a \delta g^{ad} - \nabla^d (g^{ab} \delta g_{ab}). \tag{2.30}$$

The corresponding Noether charge two-form is given by [48]

$$\boldsymbol{Q}[\xi] := -\frac{1}{16\pi} \epsilon_{abcd} \nabla^c \xi^d \, \mathrm{d}x^a \, \mathrm{d}x^b. \tag{2.31}$$

Consider a timelike hypersurface ${}^{(3)}B$. Denote the induced metric on ${}^{(3)}B$ by σ_{ab} and the extrinsic curvature by K_{ab}. Then the pull-back of the canonical one-form (2.29) may be expressed as [40]

$$\overline{\boldsymbol{\theta}}_{abc} = -\frac{1}{16\pi}(K^{de} - \sigma^{de} K) \delta\sigma_{de} \boldsymbol{\epsilon}_{abc} - \delta\left(\frac{1}{8\pi} K \boldsymbol{\epsilon}_{abc}\right) + \frac{1}{16\pi} \mathrm{d}\big(m^c \delta m^d \boldsymbol{\epsilon}_{abcd}\big), \tag{2.32}$$

where $K := \sigma^{ab} K_{ab}$, and m^a is the *outward pointing* unit normal vector to ${}^{(3)}B$, and the induced volume form is $\boldsymbol{\epsilon}_{abc} := m^d \boldsymbol{\epsilon}_{dabc}$.

Now, we write down the form of the correction term defined in subsection 2.3.2 associated with canonical boundary conditions. For the Einstein-Hilbert action, since the variation of the metric appears in $\boldsymbol{\theta}(g, \delta g)$, canonical boundary conditions correspond to fixing the induced metric σ_{ab} at ${}^{(3)}B$. Comparison of (2.32) with (2.18) then immediately tells us that $\boldsymbol{B}_X$ for canonical boundary conditions X is given by

$$\boldsymbol{B} = -\frac{1}{8\pi}(\boldsymbol{K} - \boldsymbol{K}_0), \tag{2.33}$$

where $\boldsymbol{K}_0 = \boldsymbol{K}_0(\sigma)$ is an arbitrary 3-form functional of the induced metric σ_{ab}. Therefore, the canonical charge is given by

$$\mathcal{H}[\xi] = \int_B \boldsymbol{Q}[\xi] + \frac{1}{8\pi} \xi \cdot (\boldsymbol{K} - \boldsymbol{K}_0). \tag{2.34}$$

It remains to construct the *reference term* $\boldsymbol{K}_0$ satisfying the conditions (2.23), (2.24) and (2.26). This is a non-trivial task. To see why, notice that, for instance, the choice

$\boldsymbol{K}_0 = 0$ violates the consistency condition (2.20). The reference term by Brown and York [20] violates the zero point condition[6] (2.26).

The reference term that we shall construct in subsection 2.4.3 is dependent on a specific formulation of the geometry of B and ${}^{(3)}B$, and an expression of $\boldsymbol{K}$ therein. We shall define this formulation now.

2.4.2 The trace of the extrinsic curvature

In this section, we provide an expression of $\boldsymbol{K}$ in terms of the geometry of a foliation of ${}^{(3)}B$ by closed spacelike two surfaces. This expression will be necessary in the next section where we construct a reference term $\boldsymbol{K}_0$ that satisfies the conditions stated in section 2.3. We follow the formalism in [49–52], which we also refer to for technical details.

Evolution vector

We begin by defining an evolution vector of ${}^{(3)}B$.

Let Σ_v be a null foliation of the spacetime, labeled by the parameter v. Denote by B_v the level surfaces of ${}^{(3)}B$ at a constant value of v. Then the *evolution vector* h of ${}^{(3)}B$ is uniquely defined (see [49, 50]) by the conditions that (i) h is tangent to ${}^{(3)}B$, (ii) h is orthogonal to each B_v and (iii) $\mathcal{L}_h v = 1$. We denote half of the norm of h by

$$C := \frac{1}{2} h_a h^a. \tag{2.35}$$

The evolution vector h may be used to define the normalisation of the in- and outgoing null normals orthogonal to B_v denoted by n and l respectively. We normalise them such that $l^a n_a = -1$ and such that

$$h^a = l^a - C n^a. \tag{2.36}$$

[6] This refers to the statement that the Brown-York quasi-local energy does not in general vanish in the Minkowksi spacetime.

It is then natural to define a vector τ normal to ${}^{(3)}B$ by

$$\tau^a := l^a + Cn^a. \tag{2.37}$$

See Figure 2.5 for a pictorial representation of the vectors defined above.

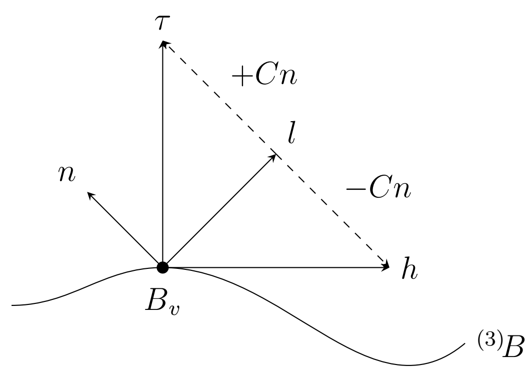

Figure 2.5: A pictorial representation of the vectors h, τ, l and n. The norm of these vectors is determined by the foliation B_v of the hypersurface ${}^{(3)}B$.

Expansion, surface gravity and the twist form

The *expansion* of the induced volume element on B_v along the evolution vector is defined as

$$\theta^{(h)} := \frac{1}{2} q^{cd} q^a_c q^b_d \mathcal{L}_h g_{ab}. \tag{2.38}$$

Here q_{ab} denotes the induced metric on B_v. The expansions $\theta^{(\tau)}$, $\theta^{(l)}$ and $\theta^{(n)}$ are defined similarly. A useful identity is

$$\theta^{(\tau)} = \theta^{(l)} + C\theta^{(n)}. \tag{2.39}$$

Next, we define a connection on the normal bundle of B, referred to as the *twist one-form*, by

$$\omega_a := -n_b \nabla_a l^b. \tag{2.40}$$

The *surface gravity* is defined by

$$\kappa := l^a \omega_a. \tag{2.41}$$

The trace of the extrinsic curvature

We are now in a position to express $\boldsymbol{K}$ in terms of the quantities defined above. Towards this end, we consider the *trace of the extrinsic curvature with respect to* τ, defined by

$$K^{(\tau)} := \sigma^{ab} \sigma_a^c \sigma_b^d \nabla_c \tau_d. \tag{2.42}$$

Here σ_{ab} denotes the induced metric on ${}^{(3)}B$. It is related to the induced metric q_{ab} on B by

$$\sigma^{ab} = q^{ab} + \frac{1}{2C} h^a h^b. \tag{2.43}$$

Using (2.35), (2.36) and (2.41) we may then write

$$K^{(\tau)} = \kappa + \theta^{(\tau)} - \frac{1}{2C} \mathcal{L}_h C. \tag{2.44}$$

Furthermore, we define on ${}^{(3)}B$ the volume form[7]

$$\overline{\boldsymbol{V}} := \mathrm{d}v \wedge \boldsymbol{\epsilon}(q), \tag{2.45}$$

where $\boldsymbol{\epsilon}(q)$ denotes the canonical volume form on B associated with the metric q. Since on a spacelike ${}^{(3)}B$, we have $\boldsymbol{\epsilon}(\sigma) = (2C)^{1/2} \overline{\boldsymbol{V}}$, and (2.42) is related to the usual trace of the extrinsic curvature[8] K by $K^{(\tau)} = (2C)^{1/2} K$, it follows that

$$\boldsymbol{K} = -\left(\kappa + \theta^{(\tau)} - \frac{1}{2C} \mathcal{L}_h C \right) \overline{\boldsymbol{V}}. \tag{2.46}$$

[7]The notation $\overline{\boldsymbol{V}}$ indicates that we shall later view this volume form as the pull-back of a spacetime three-form.

[8]The usual trace of the extrinsic curvature is given by (2.42) where τ is replaced by the outward unit normal vector m.

Here, the minus sign arises because the boundary term in (2.4) has the opposite sign for spacelike hypersurfaces. Notice that κ and $\theta^{(\tau)}$ in (2.46) are defined at surfaces of arbitrary signature. The quantity $C^{-1}\mathcal{L}_h C$ is not defined at points where the boundary becomes null $C = 0$. This issue will be taken care of momentarily.

2.4.3 The reference term

In this section, we construct a reference term $\boldsymbol{K}_0$ in (2.34) so that the consistency, linearity, orthogonality and zero point conditions from section 2.3 are satisfied. For the moment, we assume that ${}^{(3)}B$ is everywhere non-null. At the end, we observe that the resulting $\boldsymbol{B}_X$ is also well-defined at points of ${}^{(3)}B$ which are null.

We shall construct $\boldsymbol{K}_0$ as the trace of the extrinsic curvature density of ${}^{(3)}B$ embedded in a reference spacetime $\widehat{M}$. The embedding is completely determined by the intrinsic geometry of ${}^{(3)}B$, in agreement with our choice of canonical boundary conditions. The reader may find it helpful to consult Figure 2.6 for a pictorial representation of the construction.

Step 1 (Spacetime foliation). The first step is to define a (null) foliation of the spacetime. The purpose of this is that a foliation defines the evolution vector in the previous section, on which our formulation of the geometry is dependent. Since we will need to compare the geometry between hypersurfaces in the original and the reference spacetime, we shall need a sensible way to speak about "the same" foliation in the original as in the reference spacetime. One place where "the same" can be given a meaning is null infinity, where we shall now set up our spacetime foliations.

We introduce in a neighbourhood of past null infinity $\mathcal{I}^-$ a Newman-Unti[9] [12] (or if the reader prefers a Bondi [9]) coordinate system (v, r, x^A). The coordinate v labels a foliation of M by null hypersurfaces Σ_v. The coordinate r parametrises[10] the null

[9] See subsection 2.5.1 for a review of Newman-Unti coordinates.

[10] In Newman-Unti gauge, r is an affine parameter of the geodesics generated by n. In Bondi gauge, r is the *areal* or *luminosity distance*.

geodesic generators of Σ_v. The angular coordinates x^A label the null generators of Σ_v. The asymptotic metric in these coordinates is given by

$$\mathrm{d}s^2 = -\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + (r^2\gamma_{AB} + rC_{AB})\,\mathrm{d}x^A\,\mathrm{d}x^B - 2\overset{\infty}{V}_A\,\mathrm{d}x^A\,\mathrm{d}v + (...)\ , \tag{2.47}$$

where γ_{AB} is the round metric and

$$\overset{\infty}{V}{}^A = \frac{1}{2}D_B C^{AB}, \tag{2.48}$$

and D_A denotes the covariant derivative with respect to γ_{AB}. (See section 2.5 for more precise asymptotic conditions.) There are infinitely many of such Newman-Unti (or Bondi) coordinates (v, r, x^A). We consider the Newman-Unti coordinates such that B is entirely contained in the null hypersurface[11] Σ_0.

We then define $\widehat{\Sigma}_0$ in the reference spacetime $\widehat{M}$ as the level surface of a Newman-Unti coordinate $\widehat{v}$, such that the *angular expansion*, defined by

$$\theta^{(\overset{\infty}{V})} := D_A\overset{\infty}{V}{}^A, \tag{2.49}$$

at $v = 0$ in the original spacetime is identical[12] to the angular expansion in the background spacetime at $\widehat{v} = 0$, i.e.,

$$\widehat{\theta^{(\overset{\infty}{V})}}|_{\widehat{v}=0} = \theta^{(\overset{\infty}{V})}|_{v=0}. \tag{2.51}$$

[11]It is not necessary that the Bondi coordinates cover the surface B. Namely, the null hypersurface Σ_v is defined independent of the coordinate r.

[12]When $\widehat{M}$ is the Minkowski spacetime, a solution to (2.51) exists. *Proof:* In Minkowski space (or more generally a spacetime that admits a canonical Bondi frame [23]),

$$\widehat{C}_{AB} = (D_A D_B - \frac{1}{2}\gamma_{AB}D^2)\widehat{C}, \tag{2.50}$$

where $\widehat{C} = \widehat{C}(x^A)$ can be chosen arbitrarily by a supertranslation. It follows that $\widehat{\theta^{(\overset{\infty}{V})}} = D^2(D^2+2)\widehat{C}$. Thus, if the $\ell = 0$ spherical harmonic coefficient of $\theta^{(V)}$ vanishes, a solution to (2.51) exists. The vanishing of the $\ell = 0$ coefficient follows from the fact that $\theta^{(\overset{\infty}{V})}$ integrates to zero on the (round) two-sphere.

Step 2 (Isometric embedding). We embed B isometrically by a map $i : B \hookrightarrow \widehat{\Sigma}_0$. The image of B is denoted by $\widehat{B} := i[B]$.

Step 3 (Constructing ${}^{(3)}\widehat{B}$). Denote by ${}^{(3)}\widehat{B}$ at this point any hypersurface that contains $\widehat{B}$ and consider its foliation by the null hypersurfaces $\widehat{\Sigma}_{\widehat{v}}$. Let $\widehat{h}$ be the corresponding evolution vector defined in the previous section and denote by $\widehat{n}$ the corresponding ingoing null normal. As defining conditions of ${}^{(3)}\widehat{B}$, we then require that

$$C\theta^{(n)} = i^*\big(\widehat{C}\theta^{(\widehat{n})}\big), \tag{2.52}$$

and

$$C^{-1}\mathcal{L}_h C = i^*\big(\widehat{C}^{-1}\mathcal{L}_{\widehat{h}}\widehat{C}\big). \tag{2.53}$$

Here i^* denotes the pull-back of the map i defined in step 2, and $\widehat{C}$ is defined by (2.35) for the evolution vector $\widehat{h}$. (Notice that $C\theta^{(n)}$ and $C^{-1}\mathcal{L}_h C$ are the quantities in (2.46) which depend on the choice of ${}^{(3)}B$. Therefore, (2.52) and (2.53) will ensure that the resulting reference term satisfies the consistency condition.)

Step 4 (Embedding of ${}^{(3)}B$ into ${}^{(3)}\widehat{B}$). Extend the map $i : {}^{(3)}B \hookrightarrow {}^{(3)}\widehat{B}$ around B such that $i^*(\widehat{v}) = v$. In other words, the extension is defined by identifying the Newman-Unti coordiates v and $\widehat{v}$. This extension is not unique – it may be twisted off B – but since the resulting reference term in (2.55) will not depend on this freedom, we do not fix it.

Step 5 (The reference term). Finally, define at B

$$\boldsymbol{K}_0 :\overset{B}{=} i^*\widehat{\boldsymbol{K}}, \tag{2.54}$$

where $\widehat{\boldsymbol{K}}$ denotes the extrinsic curvature density (2.46) of ${}^{(3)}\widehat{B}$. Comparison of (2.46) and (2.54) with (2.52), (2.53) and (2.39), yields that

$$\boldsymbol{K} - \boldsymbol{K}_0 \overset{B}{=} -\big(\kappa - \widehat{\kappa} + \theta^{(l)} - \widehat{\theta^{(l)}}\big)\overline{\boldsymbol{V}}. \tag{2.55}$$

Here, in our notation we denote by $\widehat{\circ}$ the reference value of the quantity $\circ$.

Notice that, in contrast to (2.46), (2.55) is also well-defined at points where ${}^{(3)}B$ becomes null ($C = 0$). This is because the divergent piece at null surfaces in (2.46) was identified in the reference spacetime by (2.53).

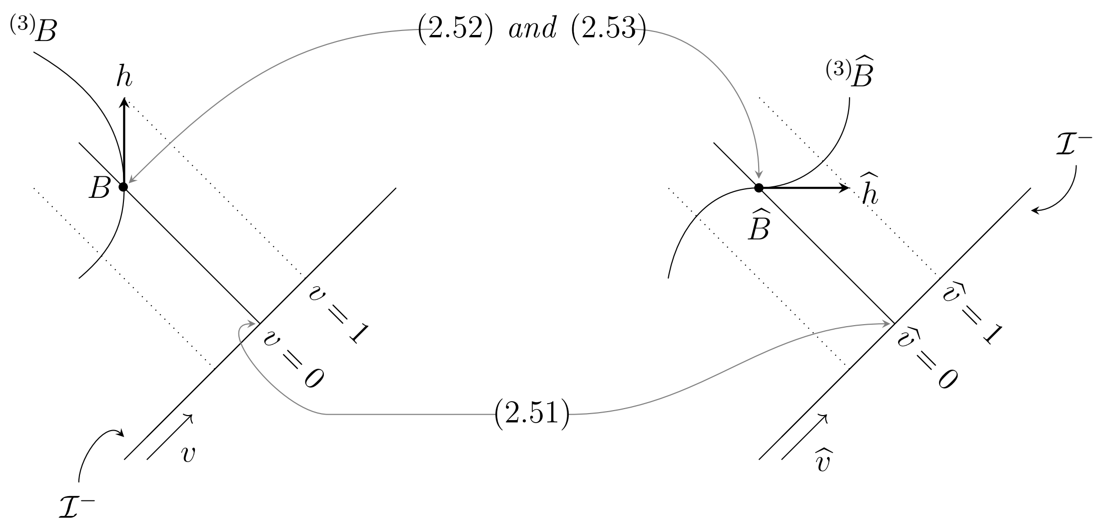

Figure 2.6: Foliations by Newman-Unti (or Bondi) coordinates v and $\widehat{v}$ of the original spacetime M (left) and the reference spacetime $\widehat{M}$ (right). The coordinates v and $\widehat{v}$ are chosen such that the asymptotic shears at the level surfaces of $v = \widehat{v} = 0$ are identical (see (2.51)). The surface B is embedded isometrically into the level surface $\widehat{v} = 0$. Then, a hypersurface ${}^{(3)}\widehat{B}$ is constructed by the conditions ((2.52) and (2.53)). This identification depends on the evolution vectors h and $\widehat{h}$ defined by the foliations of the spacetime.

2.4.4 Consistency, linearity, orthogonality and the zero point

Here we show that (2.55) is of the form (2.23).

First, notice that the volume form (2.45) is at B the pull-back of the spacetime three-form

$$\boldsymbol{V} := -n_a \, \mathrm{d}x^a \wedge \boldsymbol{\epsilon}(q). \tag{2.56}$$

(The values of $\boldsymbol{V}$ outside of B are irrelevant for our purposes.) From this, it follows that

$$n \cdot \boldsymbol{V} = 0, \tag{2.57}$$

so that $\boldsymbol{V}$ satisfies the orthogonality condition (2.24) with respect to the ingoing light direction n. Second, note that $\theta^{(l)}$ and κ depend only on B. This means that the term between brackets in (2.55) is independent of ${}^{(3)}B \supset B$. Therefore, (2.23) is satisfied. This proves consistency and linearity. Lastly, the zero point condition (2.26) is trivially satisfied.

2.4.5 Existence and uniqueness

Here we comment on the existence and uniqueness of the reference term as constructed in (2.55).

As an example, consider Minkowski space as the reference spacetime and suppose that B is contained in a slice Σ_0 for which the asymptotic shear vanishes: $C_{AB}|_{v=0} = 0$. Then the (degenerate) metric on $\widehat{\Sigma}_0$ is given by

$$\mathrm{d}s^2 = 0 \, \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2. \tag{2.58}$$

It follows directly from the *uniformization theorem*[13] that the embedding map i exists and is unique up to isometries of the Minkowski spacetime. It then remains to construct ${}^{(3)}\widehat{B}$. Towards this end, let ${}^{(3)}\widehat{B}$ be located at[14] $\widehat{r} = p(\widehat{v}, \widehat{x}^A)$. Then (2.52) uniquely determines $\partial_{\widehat{v}} p|_B$ and (2.53) uniquely determines $\partial^2_{\widehat{v}} p|_B$. (The function $p|_B$ is determined by the embedding i.) This determines the trace of the extrinsic curvature of the hypersurface ${}^{(3)}\widehat{B}$ at B. Therefore, the reference term exists and is unique.

[13] The uniformization theorem states that every metric on S^2 is conformal to the round metric.

[14] Here $(\widehat{v}, \widehat{r}, \widehat{x}^A)$ denote Newman-Unti coordinates of the Minkowski spacetime with $\widehat{C}_{AB} = 0$.

We leave a more general study of existence and uniqueness for future work.

2.4.6 The canonical Einstein-Hilbert charges

Finally, we evaluate the canonical quasi-local charges for the Einstein-Hilbert Lagrangian.

A given vector field ξ at B may be decomposed as

$$\xi = \alpha l + \beta n + \xi_{||}. \tag{2.59}$$

Here n and l are the in- and outgoing null normals[15] to B, and $\xi_{||}$ is a vector tangent to B. In terms of the decomposition (2.59), the pull-back of the Noether charge two-form (2.31) onto B becomes[16]

$$\overline{\boldsymbol{Q}[\xi]}|_B = \frac{1}{16\pi}\big(\alpha\kappa + \mathcal{L}_l\alpha - \mathcal{L}_n\beta + 2\xi^a_{||}\overline{\omega}_a\big)\boldsymbol{\epsilon}(q), \tag{2.61}$$

where κ is the surface gravity defined in (2.41) and $\overline{\omega}_a$ denotes the pull-back of the twist form (2.40) onto B. Substitution of (2.61) and (2.55) into (2.34) then yields that

$$\mathcal{H}[\xi] = \frac{1}{16\pi}\int_B \Big[\alpha\kappa + \mathcal{L}_l\alpha - \mathcal{L}_n\beta + 2\xi^a_{||}\overline{\omega}_a - 2\alpha\big(\theta^{(l)} - \widehat{\theta^{(l)}} + \kappa - \widehat{\kappa}\big)\Big]\boldsymbol{\epsilon}(q). \tag{2.62}$$

This concludes the construction of canonical quasi-local charges for the Einstein-Hilbert action.

[15]The normalisation of these null normals is defined in subsection 2.4.2 where the parameter v is now a Newman-Unti coordinate.

[16]To derive this expression, we used that

$$\boldsymbol{\epsilon} = \boldsymbol{l} \wedge \boldsymbol{n} \wedge \boldsymbol{\epsilon}(q). \tag{2.60}$$

2.5 Quasi-local BMS charges

In the previous sections, we provided a consistent method to define a conserved charge associated with a symmetry generator ξ at a closed spacelike two-surface B. Our next task is to consider a specific symmetry generator ξ to evaluate the charge. Here our choice[17] will be that ξ is a BMS vector field [9–11].

Usually, BMS symmetries are only considered in the asymptotic region. Namely, they are defined as diffeomorphisms that act non-trivially at null infinity, but which preserve the asymptotically flat boundary conditions. Their action in the bulk is generally considered arbitrary and therefore irrelevant.

However, there do exist ways to extend BMS symmetries into the bulk of a spacetime. For example, when a gauge such as Bondi gauge or Newman-Unti gauge has been fixed, the extension of BMS generators into the bulk is unique by the requirement that they preserve the given gauge conditions.

This does, however, not take away the problem that the gauge fixing method is essentially arbitrary. Different gauge fixing methods lead to different BMS generators. And unfortunately, our charge (2.62) is – in the bulk of the spacetime – not independent[18] of this gauge choice. (This may be verified by comparison of the BMS charge in the Newman-Unti and Bondi gauges.)

In order to define BMS charges, one must make a choice of gauge. Our choice will be Newman-Unti gauge. Our motivation for this choice is that the BMS generators in Newman-Unti gauge are connected to the *gravitational memory effect* in the bulk of a spacetime. This connection is explained in a separate chapter [2], which generalises the observation of Strominger and Zhiboedov [15] that BMS symmetries at null infinity are connected to gravitational memory.

The organisaton of this section is as follows. After a review of BMS symmetries

[17]One other natural choice would be $\xi = l$. However, we do not consider this choice here, because its associated charge does not vanish in the Minkowski spacetime.

[18]This problem also occurs at null infinity. Usually it is assumed that the representatives satisfy the *Geroch-Winicour condition*, which guarantees uniqueness of the charge. See e.g. [22].

in Newman-Unti gauge in subsection 2.5.1, we evaluate the associated *quasi-local BMS charges* in subsection 2.5.2. Then we show in subsection 2.5.3 that the BMS charges vanish in the Minkowski spacetime, and in subsection 2.5.4 that they yield the correct asymptotic behaviour at null infinity. In subsection 2.5.5 we compute the charges in the Vaidya and Reissner-Nördstrom spacetimes, in order to argue in subsection 2.5.6 that the zero mode BMS charge is a promising definition of *quasi-local energy*.

2.5.1 BMS generators in Newman-Unti gauge

Newman-Unti coordinates [12] are based on a null foliation of the spacetime parametrised by the first coordinate v. The second coordinate r is an affine parameter for the null geodesic generators $n_a = -\partial_a v$ in the hypersurfaces Σ_v of constant v. The remaining angular coordinates x^A are defined such that n^a generates light rays at constant angles.

The metric in these coordinates takes the form

$$\mathrm{d}s^2 = W\,\mathrm{d}v^2 + 2\,\mathrm{d}r\,\mathrm{d}v + g_{AB}(\,\mathrm{d}x^A - V^A\,\mathrm{d}v)(\,\mathrm{d}x^B - V^B\,\mathrm{d}v). \tag{2.63}$$

Part of the freedom left in the choice of (v, r, x^A) is then used to impose the following[19] fall-off conditions. Namely,

$$g_{AB} = \big(r^2 - 4\beta_0\big)\gamma_{AB} + rC_{AB} + D_{AB} + O(r^{-1}), \tag{2.64}$$

where γ_{AB} is the round metric, $\partial_v D_{AB} = 0$ and C_{AB} is traceless with respect to γ_{AB},

$$\gamma^{AB}C_{AB} = 0, \tag{2.65}$$

and

$$\beta_0 := -\frac{1}{32}C_{AB}C^{AB}. \tag{2.66}$$

[19] One way to obtain these expressions is to consider the fall-off conditions in Bondi gauge [8] and to use the relation between the Bondi and Newman-Unti gauges given by Equation (4.5) of [13].

(In the previous section C_{AB} was referred to as the asymptotic shear of the geodesic null congruence defined by n^a.) Furthermore,

$$V^A = \overset{\infty}{V}{}^A r^{-2} + O(r^{-3}), \tag{2.67}$$

where

$$\overset{\infty}{V}{}^A := \frac{1}{2} D_B C^{AB}. \tag{2.68}$$

Here D_A is the covariant derivative with respect to the round metric γ_{AB}. And

$$W = -1 + \frac{2m_B + 4\partial_v \beta_0}{r} + O(r^{-2}), \tag{2.69}$$

where m_B denotes the *Bondi mass aspect*. The inverse metric is given by

$$g^{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -W & V^A \\ 0 & V^B & g^{AB} \end{pmatrix}. \tag{2.70}$$

BMS symmetries are diffeomorphisms that preserve the asymptotic fall-off conditions of an asymptotically flat spacetime. In Newman-Unti gauge, they are generated by[20] vector fields ξ of the form [12, 13]

$$\begin{cases} \xi^v = f \\ \xi^r = J - r\partial_v f + \frac{1}{2}\overset{\circ}{\Delta} f \\ \xi^A = Y^A + I^A \end{cases} \tag{2.71}$$

[20] In $3+1$ dimensions.

where

$$
\begin{aligned}
f &:= T(x^A) + \frac{1}{2} v D_A Y^A, \\
I^A &:= -\partial_B f \int_\infty^r g^{AB}\, \mathrm{d}r', \\
J &:= -\partial_A f \int_\infty^r V^A\, \mathrm{d}r'.
\end{aligned}
\tag{2.72}
$$

Here $T(x^A)$ is an arbitrary function of the angular coordinates, referred to as a *supertranslation*, and Y^A is a conformal Killing vector of γ_{AB}. The operator $\mathring{\Delta} := D_A D^A$ denotes the spherical Laplacian.

Domain of applicability

The BMS generators (2.71) are only defined at the points in the spacetime where Newman-Unti coordinates are defined. When the curvature of the spacetime becomes too strong, the lightrays in Σ_v generated by n^a start to intersect[21], at which point the coordinates become ill-defined.

However, Newman-Unti coordinates do cover many interesting situations. We illustrate this with an example. Consider a planet in the vicinity of a black hole. When the energy density of the planet is sufficiently small or the planet is sufficiently close to the black hole, the light rays generated by n^a intersect behind the horizon. This means that the above BMS generators are defined at black hole horizons with sufficiently weakly gravitating matter in the exterior. See Figure 2.7.

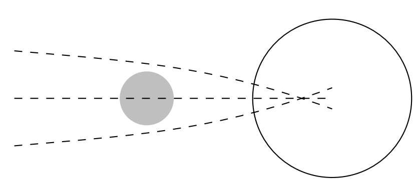

Figure 2.7: A planet in the presence of a black hole. The trajectory of ingoing light rays generated by n^a is deformed by the planet. However, if the energy momentum distribution of the planet is sufficiently weak, the ingoing light rays intersect inside of the trapping region.

[21]The light rays intersect when $\theta^{(n)} = \nabla_a n^a = 0$.

2.5.2 Einstein-Hilbert charges

Here we evaluate (2.62) for the case that ξ is a BMS generator (2.71).

Consider a two-surface B located at *constant* values of v and r in Newman-Unti coordinates. The in- and outgoing null normals n and l respectively are then given by

$$n^a = (0, -1, 0, 0) \quad \text{and} \quad l^a = \left(1, -\frac{W}{2}, V^A\right). \tag{2.73}$$

The BMS vector field (2.71) in the decomposition (2.59) is given by

$$\alpha = f, \tag{2.74}$$

$$\beta = -\xi^r - \frac{\alpha W}{2}, \tag{2.75}$$

$$\xi_{||}^A = \xi^A - \alpha V^A. \tag{2.76}$$

This yields

$$\mathcal{L}_n \beta = -\mathcal{L}_l f - \alpha\kappa, \tag{2.77}$$

where we used that $\kappa = -\frac{1}{2}\partial_r W$. The charge may now be seen to evaluate to

$$\mathcal{H}[\xi] = \frac{1}{8\pi}\int_B \Big[-f\big(\theta^{(l)} - \widehat{\theta^{(l)}} + V^A\omega_A - \widehat{\kappa}\big) + \mathcal{L}_l f + \xi^A\omega_A \Big] \boldsymbol{\epsilon}(q). \tag{2.78}$$

This is the main result of this section.

The quantity $\widehat{\kappa}$ in (2.78) may be interpreted as a reference term for $V^A\omega_A$. To see this, one may verify that in the Minkowski spacetime in Newman-Unti coordinates it holds true that

$$\kappa = V^A\omega_A \quad \text{(in the Minkowski spacetime)}. \tag{2.79}$$

The quantity ξ^A is a geodesic deviation between the light rays generated by n^a and light rays that are BMS deformations thereof. This observation is elaborated in [2].

In the following, we show (i) that the BMS charges (2.78) vanish in the Minkowski spacetime, and (ii) that they coincide asymptotically at null infinity with the BMS charges known in the literature.

2.5.3 Vanishing charges in the Minkowski spacetime

Here we show that the quasi-local BMS charges vanish in the Minkowski spacetime. Since the Minkowski spacetime is also our reference spacetime that satisfies the zero point condition (2.26), it follows from (2.17) that the conserved charge vanishes when the Noether charge vanishes. We now show that the Noether charge associated with BMS generators vanishes in the Minkowski spacetime.

Consider the Minkowksi spacetime given by

$$\mathrm{d}s^2 = -\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\gamma_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B. \tag{2.80}$$

One may verify that on an arbitrary closed two-surface B the pull-back of $\boldsymbol{Q}[\zeta]$ (defined in (2.31)) onto B is a total derivative on B when ζ satisfies

$$\partial_r\zeta^v = 0, \tag{2.81}$$

$$\partial_v\zeta^v - \partial_r\zeta^r = D_A(\,\cdot\,)^A, \tag{2.82}$$

and

$$\zeta^A = \frac{1}{r}\gamma^{AB}\partial_B\lambda, \tag{2.83}$$

where $\partial_v\lambda = \partial_r\lambda = 0$. Furthermore, in Minkowski space, the Noether charge two-form $\boldsymbol{Q}[k]$ also integrates to zero on B when k is an isometry.

Since in Minkowski space in Newman-Unti gauge it holds true that BMS vector fields are a linear combination of the vector fields ζ (supertranslations) and isometries k (rotations and boosts), the corresponding Noether charge vanishes[22].

[22] The same reasoning holds true for BMS generators in Bondi gauge. See [8, 9] for the expression of the BMS generators in Bondi gauge.

2.5.4 Asymptotic behaviour

Here we show that the asymptotic limit of the quasi-local BMS charges (2.78) agrees with the BMS charges constructed by Wald and Zoupas [22] at null infinity. In our notation, we denote by $\mathcal{C}_v$ a cut at null infinity at a constant value of v, i.e., the limit of B as $r \to \infty$.

Notice first that $\omega^A = \frac{1}{2}\partial_r V^A$. Since

$$V^A \omega_A = O(r^{-3}), \tag{2.84}$$

this term, and because of (2.79) also its reference value $\widehat{\kappa}$ in (2.62) do not contribute to the charge as $r \to \infty$. Second, one may verify that the asymptotic value of the term containing $I^A \omega_A + \mathcal{L}_l f$ vanishes.

Next, we compute $\theta^{(l)}$ and its reference value $\widehat{\theta^{(l)}}$. The asymptotic expansion of $\theta^{(l)}$ is given by

$$\theta^{(l)} = \frac{1}{r} + \frac{1}{r^2}\big(D_A \overset{\infty}{V}{}^A - 2m_B\big) + O(r^{-3}), \tag{2.85}$$

The reference value $\widehat{\theta^{(l)}}$ is given by (2.85) where the quantities C_{AB} and m_B are replaced by their reference values $\widehat{C}_{AB}$ and $\widehat{m}_B$. From our construction of the reference term in subsection 2.4.3, it follows that (at the given value of v)

$$\widehat{D_A \overset{\infty}{V}{}^A} = D_A \overset{\infty}{V}{}^A, \tag{2.86}$$

$$\widehat{m}_B = 0, \tag{2.87}$$

so that the asymptotic value of the charge is given by

$$\overset{\infty}{\mathcal{H}}[\xi] = \frac{1}{8\pi}\int_{\mathcal{C}_v} \big(2fm_B + Y^A N_A\big)\boldsymbol{\epsilon}(\gamma). \tag{2.88}$$

Here N_A denotes the *angular momentum aspect* which we define as the subleading coefficient in the asymptotic expansion of ω_A, given by

$$\omega_A = -\frac{1}{2r} D^B C_{AB} + \frac{1}{r^2} N_A + O(r^{-3}). \tag{2.89}$$

(The leading order term of ω_A does not contribute to the charge. This follows from the facts that $D^B C_{AB}$ is a total derivative on $\mathcal{C}_v$, Y^A is a conformal Killing vector and the trace of C_{AB} vanishes.)

The asymptotic charge (2.88) is the desired form. See for comparison[23] e.g. Equation (3.2) of [8].

2.5.5 The charge in spherically symmetric spacetimes

In this section, we compute the BMS charges (2.78) on spherically symmetric spacetimes given by (3.9), where

$$\begin{aligned} W &= W(v,r), \\ V^A &= 0, \\ g_{AB} &= r^2 \gamma_{AB}. \end{aligned} \tag{2.90}$$

Here γ_{AB} is the round metric.

The only contribution to the charge comes from the null expansion $\theta^{(l)}$ and its reference value. For a surface B at constant (v,r), they are given by

$$\theta^{(l)} = -\frac{W}{r}, \tag{2.91}$$

$$\widehat{\theta^{(l)}} = \frac{1}{r}. \tag{2.92}$$

The resulting charges are

$$\mathcal{H}[\xi] = \frac{1}{8\pi} \int_B \frac{f(1+W(v,r))}{r} \boldsymbol{\epsilon}(q). \tag{2.93}$$

[23]The definition of N_A in [8], which we denote by N'_A, is related to ours by $N_A = N'_A - \partial_A \beta_0$, where β_0 was defined in (2.66).

Only the zero mode (time translation) $f = 1$ contributes to the charge, for which it is equal to the *Misner-Sharp* energy [5, 17, 53]

$$E_{\text{MS}}(v, r) := \frac{r}{2}(1 + W(v, r)). \tag{2.94}$$

Vaidya metric

The Vaidya metric is given by the Schwarzschild metric where the mass parameter is made time dependent. That is, (2.90) where

$$W(v, r) = -\left(1 - \frac{2m(v)}{r}\right). \tag{2.95}$$

It describes the formation of a black hole by a spherically symmetric shell of null dust. For B at arbitrary radii, we find that the gravitational part of the canonical charge is given by

$$\mathcal{H}[f = 1] = m(v). \tag{2.96}$$

Reissner-Nördstrom black hole

The Reissner-Nördstrom metric is given by (2.90) where

$$W(v, r) = -\frac{(r - r_-)(r - r_+)}{r^2}, \tag{2.97}$$

where

$$r_\pm := m \pm \sqrt{m^2 - Q^2}. \tag{2.98}$$

We find that

$$\mathcal{H}[f = 1] = \frac{(r_+ + r_-)r - r_+ r_-}{2r}. \tag{2.99}$$

In particular, at $r = r_+$, we have

$$\mathcal{H}[f = 1]|_{r=r_+} = \frac{r_+}{2}. \tag{2.100}$$

This quantity is equal to the *irreducible mass* of the black hole, given by

$$m_{\text{irr}} := \sqrt{\frac{\text{Horizon Area}}{16\pi}}, \quad \text{where} \quad \text{Horizon Area} = 4\pi r_+^2. \tag{2.101}$$

Notice that the Reissner-Nördstrom black hole is a solution to the vacuum Einstein-Maxwell equations, not the Einstein-Hilbert equations. Therefore, the charge also receives a contribution from the gauge field. Although the inclusion of gauge fields in section 2.3 is straight forward, it happens to be the case that the resulting charges are not independent of the electromagnetic gauge. This gauge dependence follows from the fact that the symplectic two form is not gauge invariant [22]. One may have to substantially fix the gauge – or choose appropriate boundary conditions – in order for our procedure to yield gauge invariant quantities. We have left these investigations for the future.

2.5.6 Quasi-local energy

Consider the BMS charge given by (2.78) where ξ is the $Y^A = 0$, $f = 1$ BMS vector field (2.71). Summarizing the observations of the previous section, this charge has the following properties:

1. It vanishes on the Minkowski spacetime,
2. It asymptotes to the Bondi mass at null infinity,
3. On the round spheres in the metric (2.90) it is equal to the Misner-Sharp energy. Specifically, at the outer horizon of a Reissner-Nördstrom black hole, it is the *irreducible mass*.

These properties are contained in a list of pragmatic criteria that a reasonable notion of quasi-local energy is expected to satisfy [5]. We therefore put forward the possibility that the *gravitational part* of the zero mode BMS charge as constructed above may be a useful definition of quasi-local energy.

2.6 Concluding remarks

We provided a general construction of quasi-local "conserved" charges in General Relativity. The construction may be thought of as a modification of the prescription of Wald and Zoupas for defining conserved quantities at null infinity. Our modification is applicable more generally, and in particular in the bulk of a spacetime. We applied our construction to BMS symmetries in the bulk of asymptotically flat spacetimes, so as to define quasi-local BMS charges. We then argued that the zero mode BMS charge is a promising definition of quasi-local energy.

Let us conclude with the following remarks.

(i) Because of computational complexity, we did not consider the Kerr geometry in our examples in section 2.5. However, the expression of the Kerr metrics in Newman-Unti gauge is known [54]. Therefore, our BMS charges are in principle also defined in the Kerr spacetime. It would be useful to check if the zero mode BMS charge at the outer horizon of a Kerr black hole is equal to the irreducible mass.

(ii) In stating that the gravitational part of the zero mode BMS charge at the horizon of a Reissner-Nördstrom black hole is the irreducible mass, we purposefully ignored the contribution from the gauge field to the canonical charge. We did this, because the contribution from the gauge field is dependent on the electromagnetic gauge. In order to construct a gauge invariant quantity, one could consider different boundary conditions, such as fixing the electric charge instead of the gauge field. Another possibility would be to fix the gauge substantially, perhaps similar to the way we fixed Newman-Unti gauge for BMS generators. We have left this for future investigations.

(iii) Our prescription may be used to define quasi-local conserved charges in spacetimes with different asymptotic conditions.

(iv) Our prescription in section 2.3 is formally applicable to diffeomorphism covariant theories in general. However, we have not investigated this in any detail.

Acknowledgements

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I thank an anonymous referee for pointing out that Equation (31) in the first version of [1] is only applicable to spacetimes that admit a canonical Bondi frame.

Chapter 3

Gravitational memory in the bulk

A method for detecting gravitational memory is proposed. It makes use of ingoing null geodesics instead of timelike geodesics in the original formulation by Christodoulou. It is argued that the method is applicable in the bulk of a spacetime. In addition, it is shown that BMS symmetry generators in Newman-Unti gauge have an interpretation in terms of the memory effect. This generalises the connection between BMS supertranslations and gravitational memory, discovered by Strominger and Zhiboedov at null infinity, to the bulk.

3.1 Introduction

In General Relativity, there exists the gravitational memory effect, which was discovered by Zel'dovich and Polnarev [55], then studied by Braginsky and Thorne [56, 57] in the linearised theory, and at null infinity by Christodoulou in the nonlinear theory in [58]. It is a statement about how the relative distance between geodesics permanently changes after the passing of a burst of radiation.

This effect is conceptually nontrivial in the following sense. Usually, one imagines that a ring of test particles subject to a gravitational plane wave oscillates in the $+$ or $\times$ polarisation directions, and then returns to its initial state. The gravitational memory effect states that this is not true; the relative distance between test particles of the ring is permanently changed after the passing of the wave. The effect is often referred to as the Christodoulou memory effect, because Christoudoulou made the observation[1] that gravitational backreaction in the linearised theory cannot be ignored.

Christodoulou formulated the memory effect at null infinity with the help of test particles on timelike geodesics that are initially at rest. This formulation relies on the fact that it is possible at null infinity to define a good notion of "test particles initially at rest". In this chapter, we shall be interested in formulating the memory effect elsewhere[2] in the spacetime. We have in mind a gravitational wave on a black hole background. One could try to set up a ring of timelike test particles around the black hole, wait for the wave to pass by, and measure the relative displacements. However, since timelike geodesics close to a black hole are gravitated inwards, it is difficult to interpret which part of the displacement can be explained in terms of the gravitational memory effect.

Therefore, we shall provide a different formulation of the memory effect. Instead of timelike geodesics, we consider a pair of ingoing null geodesics. We measure the geodesic deviation between the light rays at their affine time[3] (or radius) r. At a later time, we introduce another pair of such light rays. Comparison of the geodesic deviation of the second pair to the first – at the affine time r – is a method for detecting gravitational memory. The setup is sketched in Figure 3.1.

One advantage of this method is clear: an external observer can wait as long as he or she wants to perform the measurement with the second pair of light rays, and still be able to measure the permanent displacement. This is not in general possible in the usual setup with timelike geodesics.

Now, we come to the second point of this chapter. In [15] it was observed that the gravitational memory effect is related to the subject of *BMS symmetries*[4] at null infinity. The relation is that a change of the relative displacement between geodesics due to a

[1] Blanchet and Damour [59] independently obtained the same result.

[2] See [60] for a geometric approach that goes beyond the weak-field analysis. See [29, 34] for statements about the memory effect in black hole spacetimes.

[3] The affine parametrisation of the geodesics has to be chosen in a suitable manner, which will be explained later.

[4] These were defined in [9, 12]. See [8, 14] for a review.

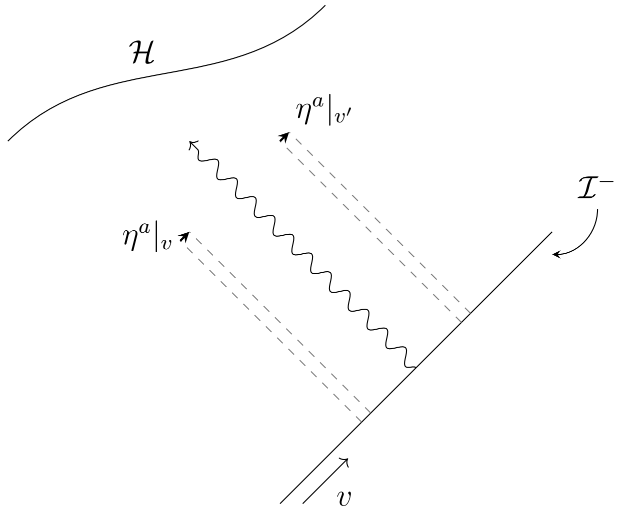

Figure 3.1: A gravitational wave travels towards a (dynamical) black hole horizon $\mathcal{H}$. The passing of the wave changes the geodesic deviation η^a of a (specific) pair of ingoing null geodesics. This geodesic deviation is compared at two different values of v, as a way of detecting gravitational memory in the bulk of the spacetime.

burst of radiation can be understood as the action of a supertranslation. In the present note, we shall observe that our formulation of the gravitational memory effect in the bulk can be understood in terms of the action of BMS supertranslations in Newman-Unti gauge [12]. This generalises the connection [15] between BMS supertranslations and gravitational memory to the bulk of the spacetime.

The organisation of this chapter is as follows. In section 3.2 we review null geodesic generators of null hypersurfaces. In section 3.3 we provide a new formulation of the memory effect in terms of null geodesics. In section 3.4 we discuss the memory effect in relation to BMS symmetries.

3.2 Null geodesics

In this section we show how, starting from a null geodesic generator n^a of a null hypersurface Σ_v, it is possible to construct neighbouring geodesics. In the remainder of this chapter we shall frequently consider the *geodesic deviation* between the original null geodesic n^a and its deformation.

Consider a spacetime M with metric g_{ab}. Denote by $x^a(\tau)$ a path in M parametrised by τ. Then $x^a(\tau)$ is a geodesic when

$$\frac{\partial H}{\partial p_a} = \dot{x}^a \quad \text{and} \quad \frac{\partial H}{\partial x^a} = -\dot{p}_a, \tag{3.1}$$

where

$$H = \frac{1}{2} g^{ab}(x) p_a p_b. \tag{3.2}$$

That is,

$$\dot{x}^a = g^{ab} p_b, \tag{3.3}$$

$$\dot{p}_a = -\frac{1}{2}(\partial_a g^{bc}) p_b p_c. \tag{3.4}$$

Consider now a foliation of the spacetime in terms of null hypersurfaces Σ_v, labelled by the parameter v. Denote by $n_a := -\partial_a v$ the null geodesic generators of Σ_v. Denote by r an affine parameter for the geodesics generated by n^a, and lastly, let x^A be angular coordinates such that the null geodesics are lines at constant angle: $\mathcal{L}_n x^A = 0$. In these coordinates[5] it holds true that

$$g^{rv} = 1 \quad \text{and} \quad g^{vv} = g^{vA} = 0. \tag{3.5}$$

One may then verify that for $\tau = r$ and for a small function $f = f(v, x^A)$, an $O(f^2)$ solution to the equations (3.3) and (3.4) is given by

$$\begin{aligned} x^a &= \int^r g^{ab} p_b + z^a, \\ p_a &= n_a - \partial_a f. \end{aligned} \tag{3.6}$$

Here $z^a = z^a(v, x^A)$.

[5]Newman-Unti coordinates [12] are examples of such coordinates.

Notice that $f = 0$ yields a geodesic generated by n^a. The function z^a determines the ingoing location of the geodesic. The linearised solution (3.6) thus tells us that, starting from a geodesic generated by n^a, we may generate a family of geodesics in its neighbourhood by small deformations f. The geodesics (3.6) are null and affinely parametrised by r.

Geodesic deviation

Consider a geodesic $x^a(r)$ of the type (3.6) and a geodesic $x_0^a(r)$ generated by the vector n^a. The latter is a geodesic of the type (3.6) where $f = 0$. A *deviation vector* between these geodesics is given by

$$\eta^a(r) := x^a(r) - x_0^a(r). \tag{3.7}$$

This quantity (depicted in Figure 3.2) shall play a central role in the remainder of this chapter. Notice, however, that η^a is not yet well-defined. Namely, the right hand side of (3.7) depends on the choice of the affine parameter r. We shall fix this ambiguity in section 3.3.

Notice also that a generator n_f^a of the null geodesic $x^a(r)$ can be constructed from n^a in the following way:

$$n_f^a = n^a + \mathcal{L}_\eta n^a, \tag{3.8}$$

where $\mathcal{L}_\eta$ denotes the Lie derivative with respect to η.

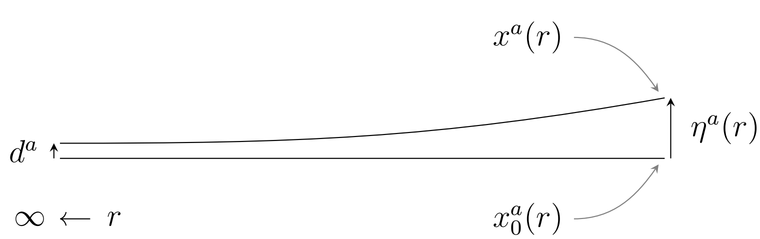

Figure 3.2: A light ray generated by n^a and a deformed light ray generated by n_f^a. The geodesic deviation between the two light rays at affine time r is denoted by $\eta^a(r)$.

3.3 Gravitational memory

The geodesic deviation (3.7) between a light ray generated by n^a and its deformation by the function f can be used to detect gravitational memory at all values of the affine parameter r. The idea is to compare the geodesic deviation $\eta^a(r)$ at different values of v, which we denote by v and v'.

However, for a comparison of $\eta^a(r)$ at different values of v to make sense in the context of quantifying the memory effect, we must impose further restrictions on the choice of the coordinates (v, r, x^A) and the choices of f and z^a. We require the following.

(i) The coordinates (v, r, x^A) are *Newman-Unti coordinates* [12]. Newman-Unti coordinates are of the type (v, r, x^A) above, where in addition the metric satisfies the following conditions.

$$\mathrm{d}s^2 = W\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + g_{AB}(\,\mathrm{d}x^A - V^A\,\mathrm{d}v)(\,\mathrm{d}x^B - V^B\,\mathrm{d}v), \tag{3.9}$$

where

$$W = -1 + O(r^{-1}), \tag{3.10}$$

$$V^A = O(r^{-2}), \tag{3.11}$$

and

$$g_{AB} = r^2\gamma_{AB} + rC_{AB} + O(1). \tag{3.12}$$

Furthermore, C_{AB} is traceless with respect to the round metric γ_{AB},

$$\gamma^{AB}C_{AB} = 0. \tag{3.13}$$

In our notation, capital indices are raised with the metric g_{AB}, except for the indices of γ_{AB} and C_{AB}, which are raised and lowered with the metric γ_{AB}.

(ii) The functions f and ingoing locations z^a are independent of v.

There is a physical motivation for choosing Newman-Unti coordinates. Namely, they have the property that at a large constant radius $r = r_0$, the worldlines (v, r_0, x_0^A) at fixed angles x_0^A are approximately inertial observers. This means that – together with condition (ii) – the setup can be understood as an approximately inertial observer at past null infinity who shoots at two different times "the same" pair of light rays into the spacetime. The quantity η^a can then be compared at the two different values of v by a second observer in the bulk.

Assuming the requirements (i) and (ii), the quantity

$$\Delta\eta^a := \eta^a|_{v'} - \eta^a|_v, \tag{3.14}$$

is now well-defined at every value of r. We argue that this choice of $\Delta\eta^a$ quantifies the memory effect in the bulk of a spacetime.

3.3.1 Memory at null infinity

In order to verify that (3.14) is a formulation of the memory effect, we must show that the known literature about the memory effect at null infinity is correctly reproduced in this formulation.

The memory effect was formulated by Christodoulou in [58] in terms of a permanent relative displacement Δx^A between timelike geodesics that are initially at rest. At null infinity (in $3+1$ dimensions), this displacement may be expressed as[6] [58]

$$\Delta x^A = -\frac{(\delta x_0)^B}{2r}\Delta C^A_B. \tag{3.15}$$

Here $(\delta x_0)^A$ denotes the initial relative separation between the timelike geodesics and ΔC^A_B denotes the difference[7] of the *asymptotic shear* at two different values of v.

[6] Equation (3.15) is obtained by integrating twice the geodesic deviation equation between neighbouring timelike geodesics at subleading order in r.

[7] An ingoing flux of radiation is equivalent to the non-vanishing of the *Bondi news* defined by

Our goal is now to show that our formulation yields the same result. Towards this end, consider the geodesic deviation (3.7) with the same choice of z^A for both geodesics. Then the asymptotic expansion of the angular components of η^a is given by[8]

$$\eta^A = \left(\frac{1}{r}\gamma^{AB} - \frac{1}{2r^2}C^{AB} + O(r^{-3})\right)\partial_B f. \tag{3.17}$$

Next, note that the leading order deviation is

$$d^A := \frac{1}{r}\gamma^{AB}\partial_B f. \tag{3.18}$$

Therefore, we may express the change (3.14) in the geodesic deviation (3.17) as

$$\Delta\eta^A = -\frac{d^B}{2r}\Delta C^A_B + O(r^{-3}). \tag{3.19}$$

Equation (3.19) is the same as (3.15). This shows that our proposal captures the memory effect at null infinity.

3.3.2 Memory in the bulk

Here, we quantify the memory of an accreting black hole. The purpose of this is to illustrate that our method is applicable in the bulk of a spacetime.

We consider a Schwarzschild black hole subject to an ingoing shell of null matter, given by the metric $g = g_0 + h$. Here g_0 denotes the Schwarzschild metric

$$\mathrm{d}s_0^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\gamma_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B, \tag{3.20}$$

$N_{AB} := \partial_v C_{AB}$.

[8]This follows from integrating

$$g^{AB} = \frac{1}{r^2}\gamma^{AB} - \frac{1}{r^3}C^{AB} + O(r^{-4}). \tag{3.16}$$

and h is a perturbation that describes the shell of matter, given by [29]

$$h_{AB} = \theta(v - v_0) r C_{AB}, \tag{3.21}$$

where

$$C_{AB} = -2\Big(\mathring{D}_A \mathring{D}_B C - \frac{1}{2}\gamma_{AB}\mathring{D}^2 C\Big). \tag{3.22}$$

Here $\mathring{D}$ denotes the covariant derivative with respect to γ_{AB} and θ denotes the Heaviside step function. The remaining components[9] of h are given by $h_{vr} = h_{rA} = 0$, (3.23) and (3.24). The linearised solution h is constructed so that, before and after v_0, the metric is diffeomorphic to the Schwarzschild geometry with masses m and $m + \mu$ respectively. The setup is depicted in Figure 3.3.

One may compute that for $v > v_0$,

$$\eta^A = \Big(\frac{1}{r}\gamma^{AB} - \frac{1}{2r^2}C^{AB}\Big)\partial_B f. \tag{3.25}$$

And for $v < v_0$, η^A is given by (3.25) with $C_{AB} = 0$. Consider then the change

$$\Delta\eta^A = -\frac{1}{2r^2}\Delta C^{AB}\partial_B f. \tag{3.26}$$

The deviation $\Delta\eta^A$ quantifies the displacement memory effect in the bulk[10]. Namely, it[11] is defined for all values of r and it vanishes[12] when v and v' are taken both before or both after the incoming shell.

[9]The remaining components of the perturbed metric h_{ab} are given by

$$h_{vv} = \theta(v - v_0)\Big(\frac{2\mu}{r} - \frac{m\mathring{D}^2 C}{r^2}\Big), \tag{3.23}$$

$$h_{vA} = \theta(v - v_0)\partial_A\Big[\Big(1 - \frac{2m}{r} + \frac{1}{2}\mathring{D}^2\Big)C\Big]. \tag{3.24}$$

[10]A similar statement was made in [29], although without the interpretation provided in the present paper.

[11]The apparent discrepancy between the asymptotic behaviour of (3.19) and (3.26) is due to (3.18).

[12]This would not be true without the conditions (i) and (ii) in the beginning of this section.

Note that in general, in addition to $\Delta\eta^A$, one may also consider the deviation $\Delta\eta^r$ in the longitudinal direction.

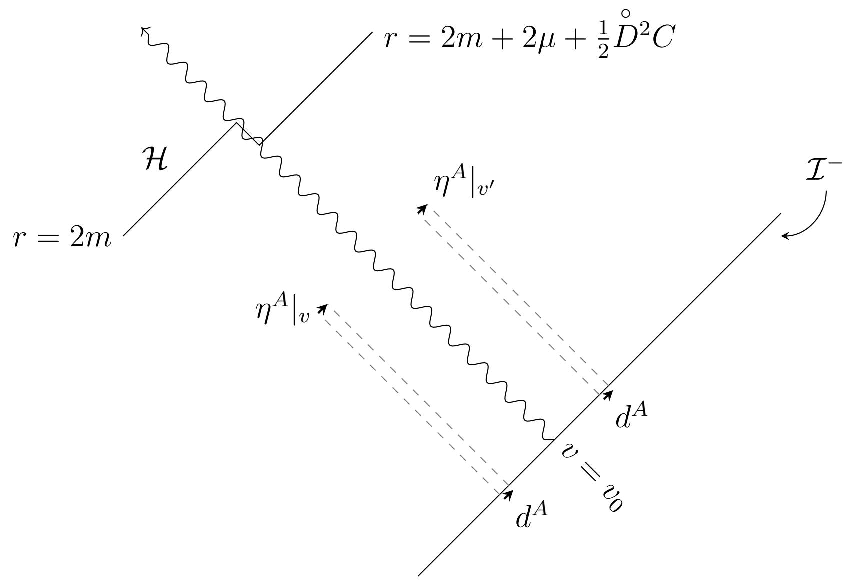

Figure 3.3: A black hole subject to a shell of null matter entering the spacetime at $v = v_0$. The geodesic deviation between a pair of light rays before v_0 and "the same" pair of light rays after v_0 may be compared as a way of detecting gravitational memory in the exterior of the black hole. Here "the same" is given a meaning by the conditions (i) and (ii) in the beginning of this section.

3.4 BMS symmetries

Generators of *BMS symmetries*[13] are vector fields that preserve the asymptotically flat boundary conditions at null infinity. A connection between the gravitational memory effect and BMS generators at null infinity was discovered in [15]. The relation is that a change in the deviation between geodesics may be understood as the action of a BMS supertranslation. Here we show that the formulation of the gravitational memory effect proposed in section 3.3 generalises this connection to the bulk of a spacetime.

Usually, BMS symmetries are only considered in the asymptotic region. Their action in the bulk is considered arbitrary and therefore physically irrelevant. However, when

[13]The original references are [9–11]. See [8, 14, 61] for reviews.

a gauge such as Bondi or Newman-Unti gauge has been fixed, BMS generators have a unique extension into the bulk.

We consider BMS generators in Newman-Unti gauge [12, 13] and observe that they are given by the geodesic deviation vector (3.7), where $x_0^a(r)$ is a geodesic of the type (3.6) where $f = 0$ and $z^a = z_0^a$, and $x^a(r)$ is a geodesic of the type (3.6) where

$$\begin{aligned} f &= T(x^A) + \frac{1}{2} v \mathring{D}_A Y^A, \\ z^v &= f + z_0^v, \\ z^r &= \frac{1}{2} \mathring{\Delta} f + z_0^r, \\ z^A &= Y^A + z_0^A. \end{aligned} \tag{3.27}$$

Here T is a function depending only on x^A, referred to as a *supertranslation*, and Y^A is a conformal Killing vector of γ_{AB}, referred to as a *superrotation*. The operator $\mathring{\Delta} := \mathring{D}_A \mathring{D}^A$ is the spherical Laplacian. This observation tells[14] us that, in Newman-Unti gauge, BMS vector fields are themselves the geodesic deviation between a light ray generated by n^a and a light ray generated by a BMS deformation of n^a. Upon closer inspection of (3.27), the bulk memory in (3.14) may be understood as the action of a supertranslation. This extends the connection between supertranslations and gravitational memory [15] to the bulk.

BMS generators in Bondi gauge [8], where the affine radial coordinate is replaced by the *luminosity* or *areal distance*, are not related to the gravitational memory effect in this way[15].

[14] It could be interesting to see if this observation has implications for defining BMS symmetries in higher dimensions [62].

[15] Consider the metric (3.9) given by $W = -1$, $g_{AB} = r^2 \gamma_{AB}$ and $V^A = V^A(r)$. Then the Newman-Unti coordinate r is also the Bondi coordinate r. The BMS generators in the Bondi and Newman-Unti gauges are related by $\xi_B^a = \xi_{NU}^a + F n^a$, where F is a function of the coordinates including r. Since geodesic deviation vectors are related by a function F which is independent of the affine parameter r, this shows that BMS generators in Bondi gauge are in general not a geodesic deviation.

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