Quantum Scalar Field Theory in AdS and the AdS/CFT Correspondence

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“It is reminiscent of what distinguishes the good theorists from the bad ones. The good ones always make an even number of sign errors, and the bad ones always make an odd number.”

– Anthony Zee

Erstgutachter: Prof. Dr. Ivo Sachs
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Contents

Zusammenfassung

In dieser Arbeit berechnen wir Quantenkorrekturen zu den Zwei- und Vierpunktsfunktionen bis zur zweiten Ordnung in der Kopplungskonstante für eine konform gekoppelte Skalarfeldtheorie mit quartischer Selbstwechselwirkung in der vierdimensionalen Anti–de Sitter-Raumzeit (AdS). Unsere Berechnungen werden durchgeführt, indem die übliche Feynman-Störungstheorie in flacher Raumzeit auf den Poincaré-Patch des Euklidischen AdS verallgemeinert wird. Insbesondere wenden wir keine Kenntnisse in konformer Feldtheorie (CFT) an. Die erhaltenen Ergebnisse für die Zwei- und Vierpunktsfunktionen sind miteinander konsistent. Darüber hinaus argumentieren wir, dass die kritischen Exponenten von Korrelationsfunktionen nahe des dreidimensionalen konformen Randes von AdS die erforderlichen Daten für die Renormierungsbedingungen liefern und somit die üblichen on-shell Bedingungen ersetzen.

Die holographische Vierpunktsfunktion kann systematisch in den konformen Invarianten entwickelt und mit der konformen Block-Entwicklung auf dem Rand von AdS verglichen werden. Dies wird hier in niedriger Ordnung in den konformen Invarianten durchgeführt, wobei gezeigt wird, dass die entsprechenden Expansionskoeffizienten die Daten der konformen Block-Entwicklung eindeutig festlegen. Trotz Feinheiten bei UV- und (manchmal) IR-Divergenzen tritt kein Widerspruch auf. Wir zeigen ferner, dass die resultierende duale Randtheorie, stark eingeschränkt aufgrund der konformen Symmetrie und daher einer Reihe nichttrivialer Bedingungen unterliegend, tatsächlich eine mathematisch und physikalisch konsistente CFT ist. Unsere Theorie liefert daher eine erste explizite Bestätigung einer Quanten-AdS/CFT-Korrespondenz.

Schließlich wird die Struktur der Operatorproduktentwicklung (OPE) der dualen CFT, einer deformierten verallgemeinerten freien Feldtheorie, zusammen mit den Korrekturen sowohl der OPE-Koeffizienten als auch der konformen Dimensionen der primären Operatoren dargelegt. Insbesondere wird das Fehlen des Energie-Impuls-Tensors und jeglicher erhaltener Ströme deutlich. Analytische Ausdrücke für die anomalen Dimensionen werden bei einer Loop-Ordnung gefunden, sowohl für Neumann- als auch für Dirichlet-Randbedingungen.

Abstract

In this thesis we compute quantum corrections to the two- and four-point correlation functions up to second order in the coupling constant for a conformally coupled scalar field theory with quartic selfinteraction in four-dimensional anti–de Sitter space-time (AdS). Our calculations are performed by generalizing the usual flat space-time Feynman perturbation theory to the Poincaré patch of Euclidean AdS. In particular, we do not exert any conformal field theory (CFT) knowledge. The obtained results for the two- and four-point functions are mutually consistent. In addition, we argue that the critical exponents of correlation functions near the three-dimensional conformal boundary of AdS provide the necessary data for the renormalization conditions, thus replacing the usual on-shell condition.

The holographic four-point function can systematically be expanded in the conformal invariants and compared with the conformal block expansion on the boundary of AdS. This is carried out here at low order in the conformal invariants, where the corresponding expansion coefficients are shown to uniquely fix the data for the conformal block expansion. No contradiction arises despite subtleties with UV and (sometimes) IR divergences. We also show that the disclosed boundary dual, subject to a set of nontrivial conditions dictated by the strong constraint of conformal symmetry, is indeed a mathematically and physically consistent CFT. Hence, our theory provides a first explicit confirmation of a quantum AdS/CFT correspondence.

Finally, the operator product expansion (OPE) structure of the dual CFT, a deformed generalized free field theory, is revealed, along with the corrections to both the OPE coefficients and conformal dimensions of primary operators. In particular, the absence of the stress tensor and of any conserved current becomes explicit. Analytic expressions for the anomalous dimensions are found at one loop, both for Neumann and Dirichlet boundary conditions.

I Introduction

1.1 Motivation

Without any doubts, Quantum Field Theory (QFT) is one of the most successful theoretical frameworks in physics at present. It combines field theory with quantum mechanics in a consistent manner and, in contrast to the conventional quantum formulation à la Heisenberg and Schrödinger, it allows for a sensible (special) relativistic formulation. From its inception almost a century ago it fueled theoretical and experimental physicists with deep insights into particle physics. Just to mention a few, the first developed QFT, the theory of Quantum Electrodynamics describing the electromagnetic interaction, was able to explain the phenomenon of spontaneous emission of photons from atoms and to infer the existence of anti–matter. But perhaps, the most impressive achievement of Quantum Electrodynamics is the extraordinary precision in the prediction of the electron magnetic moment [1]. Another significant accomplishment was the implementation of the weak interactions within the framework of QFT. The emergent theory, Quantum Flavordynamics, is however better understood in terms of the Electroweak Theory, a theory unifying the quantum theories of Electrodynamics and Flavordynamics. This unified theory agrees with experiments showing that, at very high energies, the electromagnetic and weak interaction merge into a single electroweak interaction. Also the theory of strong interactions was realized in terms of a QFT and is known today as Quantum Chromodynamics. Although Quantum Chromodynamics was less prolific than the Electroweak Theory, it predicted the asymptotic freedom and has been the missing piece of the puzzle in the formulation of the Standard Model, a theory describing three of the four fundamental interactions between elementary particles. Many predictions of the Standard Model have been met with remarkable experimental precision [2], conveying the impression that we are close to a complete characterization of particle physics.

Moreover, particle physics is not the only subject QFT has been successfully applied to. For instance, QFT has proven to supply a precious formulation of emergent phenomena at macroscopic scales of many-particle systems. Indeed, it found applications in a plethora of fields describing condensed matter, ranging from the simplest crystal lattices to the hardly manageable theories of superconductivity and the fractional quantum hall effect [3]. Notably, inalienable concepts of particle physics like the Higgs mechanism for spontaneous symmetry breaking and the renormalization procedure, were developed respectively improved by the study of Condensed Matter Physics within the framework of QFT. This certifies QFT as a universal concept, and not as a mere descriptive framework.

On the other hand, a slightly older theory than QFT amazed generations of physicists with its puzzling implications. The theory is Einstein's General Relativity (GR), which describes the fourth fundamental interaction, gravitation, the Standard Model is not capable of. It provided us with a flawless theoretical clarification of gravitational phenomena as well as with data of striking precision [4]. Long lasting inconsistencies like the instantaneous action at a distance of Newtonian gravity and the “anomalous” precession of the perihelion of Mercury were solved. Further, it disclosed a rich multitude of physical phenomena, generating new fields of study, the most important being black holes, cosmology, applications to large-scale structures, and the recently observed gravitational waves.

Although GR and QFT are of fundamentally different origin, they have a few major traits in common. Both are relativistic field theories, and both are also gauge theories. QFT is usually formulated as a Yang–Mills theory where gauging the internal symmetry group gives rise to gauge particles of spin-1. In GR instead, the space-time symmetries are gauged and this leads to a spin-2 gauge particle. Despite the similarities, a game-changing difference is at hand; whereas QFT is a quantum theory of fields, GR embodies a classical field. At first sight this does not appear to be an issue. However, when investigating situations of strong gravitational fields (at severe space-time curvature) the incompleteness of GR becomes apparent. For example, quantum theory does not permit particles to inhabit a space smaller than their wavelengths, which is in contrast to the behavior predicted by GR at the singularity of black holes, or at the origin of the universe.

But then what prevents one from quantizing GR, as it was successfully done when passing from classical field theory to QFT, and thus obtaining a theory of Quantum Gravity? Nothing and everything. Nothing, since one could formally treat both theories on the same footing and apply the procedure of quantization to GR in complete analogy with QFT. Everything, since GR is not a Yang–Mills theory; whereas Yang–Mills theories are proven to be renormalizable [5], a naive power counting suggests GR is not. Hence, it is not known how to deal with divergences arising in the perturbative calculations of quantized GR, and the theory loses all its predictive power. It is conceivable that nontrivial cancellations hidden within perturbation theory tame the appearing divergences. But also a more detailed analysis, even if not conclusive, indicates that, at loop level, GR is definitely plagued by noncurable infinities [6].

There have been various attempts to formulate a consistent theory of Quantum Gravity. Perhaps, the most immediate approach is given by simply enlarging the gauge group, in particular by supersymmetry. In fact, it seems that maximally extended Supergravity could establish renormalizability [7, 8]. This proposal is very interesting, but as yet it is unknown how to break supersymmetry to obtain a sensible theory at low energies. Another attempt is given by the asymptotic safety program [9]. It postulates that there is a nontrivial ultraviolet fixed point in the exact renormalization group, but the latter demands the inclusion of irrelevant terms into the bare action. Needless to say, this makes perturbative calculations at loop order extremly complicated to perform (see, for instance, ref. [10]). In four dimensions, there is no evidence of asymptotic safety at present.

While the above approaches aim to a minimal intervention in GR, there have been other approaches of much more exotic nature. The most promising of all, String Theory, solves the problem of nonrenormalizability at the core; in GR the coupling constant, Newton's constant G_N, is dimensionful, whereas the coupling constants of (unbroken) Yang–Mills theories are all dimensionless. Thus, in comparison with Yang–Mills, at each loop order in perturbation theory the divergences of quantized GR intensify. String theory cures the divergences by substituting particles with strings, and this introduces a new length scale, the string tension. However, the price to pay are unusual features like tachyonic exitations, branes, and higher dimensions. Different compactifications of these higher dimensions lead to different predicitions of String Theory, a problem known as the String Theory landscape [11].

For an overview of the principal attempts in the formulation of a consistent theory of Quantum Gravity, we refer to ref. [12]. The vastness of the taken paths, and related, as yet unsurmountable, issues, indicate that a conclusive theory is far from being found. Moreover, none of the candidate theories has delivered accurate predictions.

At this point, it is perhaps better to take a step back and focus on the fundamental requirements a theory of Quantum Gravity has to fulfill. In particular, a substantial improvement of our present understanding of the semiclassical approximation to Quantum Gravity would surely aid further work towards the correct formulation of the full theory itself. This is not a mere academic exercise; in situations where a quantum field theory reaches energies close to the Planck scale, it is expected that ignoring quantum gravitational effects would lead to wrong conclusions. While present particle accelerators are far from being able to induce these new effects of gravity, there is a chance that we might be able to investigate these on astronomical objects in the near future. For instance, the study of quantum gravitational effects in the cosmic microwave background looks promising [13].

But without going too far, there exists a seemingly modest yet rich approach worth to be studied, corresponding to the leading order of the semi–classical approximation to Quantum Gravity. Much like the development of Quantum Electrodynamics benefited substantially by its earliest formulation where a quantum particle was considered immersed in a classical electromagnetic field, studying quantum field theory in classical curved space-times is of immense value to improve our present understanding of the quantization of GR. The reasons are multifold. It delineates what a correct theory of Quantum Gravity should predict at weak space-time curvatures, and it poses far-reaching conceptual as well as formal problems. Solving these problems might hint to the right path to follow, perhaps with a novel framework for GR better suited for quantization.

Even though a myriad of questions[1] are still open, there is general consensus that QFT in flat space-time is well-understood. Yet, a small deformation of the geometry of the flat space-time poses, in general, serious hurdles already at the implementation of a classical field theory on curved space-times. In order to be able to formulate a consistent Cauchy problem of the differential equation governing the dynamics of the field theory under consideration, one usually takes the space-time to be globally hyperbolic. This

[1] especially what concerns nonperturbative effects, as quantum triviality and solitons

requirement is in fact very restrictive, as there exist plenty of non-globally hyperbolic space-times of physical interest. While some approaches introduce new mathematical structures [14], a restricted class of space-times allows for a less radical approach, see, for instance, ref. [15]. In the subsequent process of quantization of a field in curved space-times, one of the major conceptual difficulties one encounters right away is given by the contrasting role of time in QFT and GR. While in QFT time plays a central role, as it is evident from the canonical momentum and the equal time commutation relations in the canonical quantization procedure, in GR the general covariance principle implies it does not. Hence, unless the space-time is stationary[2], the choice of a preferred time coordinate is not unique and, consequently, neither the vacuum state is. Even worse, different choices give rise to, in general, unitarily inequivalent representations of the commutation relations, and therefore to different predictions. Fortunately, the forced designation of one of these unitarily inequivalent constructions can be overcome by formulating the theory via the algebraic approach[3] [18]. Nowadays, it is understood that nonuniqueness of the vacuum state is a prerogative of all space-times; different observers will generally disagree on the results obtained. This insight lead to spectacular predictions of QFT in general space-times, the Unruh effect and the Hawking radiation [19]. The discovery of the Hawking radiation, in particular, posed a wide range of unsolved issues, most notably the information paradox. Contradictions of this weight have been, historically, of invaluable assistance in the progress of physics.[4]

Having said that, the lack of global hyperbolicity and of a preferred time coordinate can be considered as the only serious obstacles introduced by the deformation of the geometry of space-time in the canonical quantization of a field theory; once these are addressed, the quantization procedure can, in principle, be applied straightforwardly. In order to describe dynamical processes of QFT in curved space-times, one generally proceeds with the calculation of correlation functions. These are generically ill-defined already at tree level, as it becomes apparent in the computation of the vacuum energy. In flat space-time, the arising divergence is simply discarded by the normal ordering procedure, which was of no concern since in nongravitational theories only energy differences are measurable. However, in gravitational theories, energy itself is a source of gravity, and cannot be ignored. This is an active field of study, and we refer to ref. [20]. Nonetheless, in the perturbative approach of QFT in curved space-times, the issue is circumvented by considering a particular class of quantum states of free fields, the Hadamard states.

Essentially, the building blocks in the perturbative computation of correlation functions are the (free) propagators, defined as the two-point functions of Hadamard states, of the various fields. Like in flat space-time, the knowledge of the propagator of a free field theory uniquely determines all correlation functions, and hence, the full quantum theory.

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[2] where the presence of the global time-like Killing vector field induces a direction of time

[3] To date, the algebraic approach also seems to yield the most natural framework for an axiomatic formulation of QFT in curved space-times [16, 17].

[4] We remind the reader of the Michelson–Morley experiment, or the ultraviolet catastrophe of black body radiation.

Instead, if interactions are turned on, one generally exerts the perturbative approach of QFT and computes integral expressions of products of propagators. To some extent, these propagators are known in the Schwinger–DeWitt representation [21]. However, analytical expressions are only given for a handful of space-times and spins. In fact, even finding the simplest scalar propagator is a burden, but the difficulties are alleviated if the space-time possesses some symmetries.

Ideally, the preferred space-times to work with are maximally symmetric Lorentzian manifolds. These are solutions of the vacuum Einstein equations and, depending on the sign of the cosmological constant[5], one differentiates in either de Sitter or the anti–de Sitter space-time. The de Sitter space-time (dS) plays a central role in contemporary cosmology for basically two reasons. First, it is believed that the primordial universe has gone through a phase of exponential expansion, called inflation, which is approximately described by a dS [22]. Second, dS models a universe filled by solely dark energy. As our present universe entered the dark energy dominated era about five billion years ago, it is highly probable that it will approach a dS in the far future [23]. On the other hand, anti de–Sitter space-time (AdS) certainly does not offer a realistic physical model since it presents itself with a number of bizarre properties. While the issue of the existence of closed time-like curves can be solved by adopting the universal covering space of AdS, the "box property" is by no means avoidable: an object thrown in any direction by a floating observer in AdS turns always back to the observer. Due to the presence of a time-like conformal boundary, even a light ray gets reflected at spatial infinity and reaches the observer in a finite amount of time. This conformal boundary also renders the space-time non-globally hyperbolic. However, as it was shown in ref. [15], a careful analysis of the information entering and leaving AdS through the conformal boundary allows for a consistent quantization.

Despite its odd properties, AdS has received much attention in mathematical and high-energy physics due to the existence of consistent theories of interacting higher spins, in contrast to flat space-time. In fact, the first complete set of cubic interactions of higher spins have been obtained in AdS [24, 25, 26], and the only presently known fully nonlinear theory of interacting higher spins, described by Vasiliev's equations [27, 28], is formulated around the AdS background. Moreover, it is conjectured that AdS relates to Conformal Field Theory (CFT) on its conformal boundary, as exemplified by the AdS/CFT correspondence [29, 30, 31]. This correspondence is a fascinating modern development in theoretical physics, formally equating a, perhaps consistent, Quantum Gravity in $d+1$ dimensions to a QFT in flat d-dimensional space-time. Furthermore, the conjecture is believed to solve the aforementioned information paradox [32], and it even found application in Condensed Matter Physics, for instance in the description of high-temperature superconductors, see ref. [33]. However, at present, there is no proof of the correspondence in general terms, and only a few, yet striking, agreements were found.

The principal reason is that progress in QFT on curved space-times has been hampered,

[5]The maximally symmetric space-time with vanishing cosmological constant is the usual flat space-time.

in particular, by the absence of a momentum representation for which the Feynman amplitudes can be represented as elementary integrals. The absence of such a representation is particularly limiting at loop level. Indeed, while the short-distance properties of quantum fields in curved space-times can be analyzed systematically, little is known about the influence of curvature at distances of the order of the curvature scale. Consequently, to date, explicit tests of the AdS/CFT correspondence have, to a large extent, been limited to classical fields on AdS, with a handful of examples and new techniques having just begun to appear to tackle the loop corrections, see, for instance, refs. [34, 35, 36, 37, 38, 39, 40, 41, 42, 43].

Moreover, since the coupling of the CFT stress tensor to the graviton is present almost universally, one is confronted, when considering the bulk theory beyond the classical level, with the quantization of gravity together with its perturbative pathologies in the ultraviolet. Of course, if embedded in string theory, these singularities should be resolved.

Indeed, a class of realizations of the AdS/CFT correspondence are formulated in terms of string theories in AdS. The most established one is given by a string theory on the product space $\mathrm{AdS}_5 \times \mathrm{S}^5$ conjectured to be equivalent to a supersymmetric Yang–Mills theory on a four-dimensional boundary. This is a remarkable duality as it provides a stong/weak coupling duality: whenever Yang–Mills theory is strongly coupled, its dual theory is described by weakly coupled strings. However, for the same reason, it is difficult to test the realization. Moreover, world sheet calculations of string theory in AdS are mostly beyond reach at present [44]. Hence, as yet, most of the tests were confined to the string correlation functions at tree-level; that is, CFTs in the leading order in the $1/N$ expansion, where N is the number of colors of the boundary Yang–Mills theory.

Another class of conjectured dualities, that was supposed to be simpler than the usual string-like dualities, is between CFTs with matter in vector representations and theories with massless higher spin fields in the bulk [45, 46]. The simplest example of such a duality is given by a free $O(N)$-vector model, i.e., a bunch of free scalar fields $\varphi^i(x)$, $i = 1, ..., N$ with the $O(N)$-singlet constraint imposed. This duality involves an infinite number of massless higher spin fields in the bulk that are dual to higher spin conserved tensors $J_s \sim \varphi\partial^s\varphi$ in the free scalar CFT. Other options include the critical vector model, the free fermion CFT, the Gross–Neveu model [47, 48] and, more generally, Chern–Simons matter theories [49]. The problem here is twofold. Firstly, higher spin theories reveal some pathological nonlocalities [50, 51, 52] that prevent them from having a bulk definition that is independent of their CFT duals (but can be defined as anti–holographic duals of the corresponding CFTs). Secondly, for massless higher spin fields, ultraviolet pathologies of gravity are amplified with increasing spin, see ref. [53]. At present, it is unclear how they can be resolved, except for the conformal [54, 55, 56] and chiral [57] higher spin theories where the nonlocalities are absent and quantum corrections can be shown to vanish [58, 59].

One possible way to get around these problems is to use the conformal bootstrap, in particular crossing symmetry, to determine the coefficients in the operator product expansion (OPE) and the anomalous dimensions of the corresponding operators in CFT, to make predictions for loop-corrected boundary-to-boundary correlation functions of the dual bulk theory in AdS [60, 39, 41, 40]. Here, the input is the first-order anomalous dimensions for the "double-trace operators" inferred from the tree-level bulk amplitudes which, using crossing symmetry, lead to an equation for the second-order anomalous dimensions of the latter. This has led to explicit results for a class of dual bulk theories in AdS_3 and AdS_5 [39, 41, 40]. However, since there are no closed expressions for the conformal blocks in three dimensions, this approach does not easily generalize to AdS_4, which is the focus of the present work.

Moreover, the above approach does not lead to a verification of the AdS/CFT correspondence beyond the classical field theory, since an explicit and complete calculation of correlation functions in the bulk up to sufficient order is yet missing. The process of renormalization, necessary for handling quantum effects induced by loop amplitudes, is widely uninvestigated in AdS, and there is no certainty that the boundary theory retains its conformal symmetry. The goal of the present thesis is to provide a first explicit confirmation of such a quantum AdS/CFT correspondence by directly computing loop corrected amplitudes in AdS_4 for the simplest interacting model of a scalar field with quartic selfinteraction. On the boundary of AdS, the final results are subject to a set of nontrivial tests dictated by conformal symmetry.

1.2 Research statement and results

In the present thesis we compute loop-corrected correlation functions on the Poincaré patch of AdS without using any particular properties of a CFT dual on the conformal boundary [61, 62]. More precisely, we consider the bulk theory in a loop expansion in position space. There is a convenient representation for loop diagrams in AdS in terms of Mellin amplitudes [34]. A related approach, followed in ref. [35], is to reduce loop diagrams in global coordinates in AdS to a sum over tree-level diagrams using a discrete Mellin space Källén–Lehmann representation with weight function inferred from the OPE in the dual CFT.[6] Alternatively, one may exploit the fact that the data defining the Mellin representation of the loop diagram is already contained in the tree-level data [39].

Here we will not follow this path. Instead, we simply evaluate the loop diagrams in an adapted representation in terms of Schwinger parameters with refinements originally due to Symanzik [63]. In order to avoid pathologies associated to spin-2 and above, we will consider a simple interacting scalar bulk theory. Concretely, we consider an interacting bulk scalar field with action given schematically by

$$S = \int_{\mathrm{AdS}_4} \sqrt{g} \left(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right)$$

on the Poincaré patch of Euclidean AdS_4. This theory is perturbatively renormalizable and thus we will not have to deal with any of the pathologies mentioned above. In

[6] We are not aware of an analogous construction on the Poincaré patch.

particular, we do not quantize the bulk metric but instead treat it as a background. From the point of view of QFT on curved space-time this is a natural truncation. On the other hand, this may not seem so natural to a reader familiar with the AdS/CFT literature where the graviton appears naturally as the bulk field dual to the stress tensor of the CFT. However, if we do not insist on locality on the CFT side, there are many CFTs that do not possess a local stress tensor. Among them, the critical point of Ising-like models with long distance interactions, see, for example, ref. [64]. Such CFTs should also admit an AdS dual description where gravity is frozen to a classical background.

The idea of truncating the bulk theory to an interacting scalar field is not new even in the context of the AdS/CFT correspondence. In particular, this model was considered in ref. [60] as a bulk dual to a CFT with just one low dimensional single-trace operator. The simplest such CFT is the generalized free field [65], which is characterized by the property that the correlation functions of operators factorize similarly to those of the fundamental fields in the Gaußian model. The corresponding bulk dual is just that of a free scalar field ϕ in AdS [31]. If we denote by $\mathcal{O}_\Delta$ the generalized free field of conformal dimension Δ, then, using the standard AdS/CFT dictionary, its two-point function is given by

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle_{\mathrm{CFT}} = \langle \bar{\phi}(x_1)\bar{\phi}(x_2)\rangle_{\mathrm{AdS}} = \frac{N_\phi}{r_{12}^{2\Delta}}, \tag{1.1}$$

where ϕ is the scalar field dual to $\mathcal{O}_\Delta$, $\bar{\phi}$ is its restriction to the boundary of AdS, $r_{12} \equiv |x_1 - x_2|$ and N_ϕ is a normalization constant. Similarly, the four-point function of the generalized free field will be given by

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_3)\mathcal{O}_\Delta(x_4)\mathcal{O}_\Delta(x_2)\rangle_{\mathrm{CFT}} = \frac{N_\phi}{r_{12}^{2\Delta}}\frac{N_\phi}{r_{34}^{2\Delta}} \quad + \quad \text{permutations.} \tag{1.2}$$

Crossing symmetry of the left hand side will be automatically satisfied and it then has an expansion in conformal blocks of the double-trace operators $\mathcal{O}_\Delta\Box^n\partial^l\mathcal{O}_\Delta$ of the corresponding CFT.

Next, we consider a deformation of the generalized free field that does not preserve the factorization property. The simplest such renormalizable deformation is the ϕ^4 theory on Euclidean AdS_4. This deformation should correspond to an interacting CFT with a scalar operator $\mathcal{O}_\Delta$, $m^2 \propto \Delta(\Delta - d)$, but without a local stress tensor. It is clear that any interaction term in an action for the bulk theory will give a crossing-symmetric contribution to the correlation functions on the CFT side by construction. At present, we will take the conformally coupled scalar field in AdS_4. There are two possible choices of boundary conditions for ϕ: $\Delta = 2$ and $\Delta = 1$, both being within the unitarity window [66]. Due to the extremality of the $\Delta = 1$ case we expect some subtleties at the quantum level. For the same reason we do not include the ϕ^3 interaction. We will discuss this in more detail throughout the text.

The first prediction for the CFT that is computable at tree-level in AdS is the anomalous dimensions and OPE coefficients of double-trace operators appearing in the OPE of $\mathcal{O}_\Delta$

with itself. This can be extracted from the exchanges and quartic contact interactions, see, for instance, refs. [67, 68, 60, 69]. We then compute the first quantum corrections to the two- and four-point functions, which includes one- and two-loop diagrams in AdS_4. At a conceptual level, an important implication of this is that the actual loop calculation in the bulk theory is consistent with the duality. In addition, this allows us to extract further CFT data. In particular, we can extract higher order corrections to the anomalous dimension of double-trace operators, as well as their OPE coefficients at next-to-leading order in both the deformation parameter λ and the dimension of the double-trace operators. One result, already noted in ref. [60], is that, while the conformal block expansion of the four-point function of the generalized free field involves primary double-trace operators[7] of all even spin and even dimensions, only the OPE coefficients and dimensions of such operators with spin 0 are affected by the interaction at tree-level.[8] At loop level, however, their dimensions are corrected (see also ref. [39] for AdS_3 and AdS_5).

One of the main results of this thesis is the anomalous dimensions $\Delta_{0,l}$ of the operators of the leading Regge trajectory, i.e., having the form $\mathcal{O}_\Delta \partial^l \mathcal{O}_\Delta$. These are the lowest-twist double-trace primaries appearing in the OPE of $\mathcal{O}_\Delta$ with itself. For $\Delta = 2$ we find

$$\Delta_{0,l} = 4 + l + \gamma\delta_{l,0} + \gamma^2 \begin{cases} \frac{5}{3} & \text{for } l = 0, \\ -\frac{6}{(l+3)(l+2)(l+1)l} & \text{for } l > 0, \end{cases} \tag{1.3}$$

where $\gamma = -\lambda_R/16\pi^2$ and λ_R is the renormalized coupling. For $\Delta = 1$ we have, in turn,

$$\Delta_{0,l} = 2 + l + 2\gamma\delta_{l,0} + \gamma^2 \frac{-4}{2l+1}\psi^{(1)}(l+1) + \gamma^2 \begin{cases} -4 & \text{for } l = 0, \\ -\frac{2}{l(l+1)} & \text{for } l > 0, \end{cases} \tag{1.4}$$

where $\psi^{(1)}(l+1)$ is the trigamma function. Anomalous dimensions for higher twist operators are also computed but they do not seem to have such a simple l-dependence.

An important question that requires clarification at loop level in AdS concerns the dependence on the renormalization scheme. In this thesis we use an ultraviolet cut-off regularization, which manifestly preserves covariance, followed by a nonminimal subtraction. For $\Delta = 1$, there are additional infrared divergences. A convenient covariant regularization of the latter is provided by continuation in Δ. Another issue related to this is the lack of a simple quantum AdS experiment that determines the renormalization conditions in terms of measurable quantities such as the mass of particles, for instance. In the present context we replace the latter by the dimensions of the operators of the dual CFT which seems to be an appropriate replacement in the context of AdS/CFT.

[7] In the present context, based on the identification obtained in the original AdS/CFT conjecture, we denote by double-trace operators all operators which are not dual to a bulk field.

[8] This may come as a surprise since the higher spin primaries do not correspond to conserved currents as they do not saturate the unitarity bound.

1.3 Content of the thesis

The content of this thesis is organized as follows. In chapter II we discuss the basic requirements and properties of a QFT in flat space-time. The main objective is to provide a strong foundation for the material presented in the forthcoming chapters. In particular, the later transition to curved space-times can be best illustrated with the axiomatic formulation of QFT in flat space-time, which is presented in section 2.1 in terms of the Wightman axioms. Another equivalent characterization in terms of the properties of the Wightman functions is discussed in section 2.2. Section 2.3 deals with the procedure of analytic continuation of correlation functions, and with the properties of the resulting Euclidean QFT. Eventually, in section 2.4 we explicate the canonical quantization procedure on the simplest model of a free neutral scalar field in flat space-time and review the resulting QFT in the context of the Wightman axioms.

Chapter III aims to provide a basic understanding of QFT in curved space-times. In section 3.1 we carry on our discussion on the previous example of a free neutral scalar QFT and outline the main difficulties arising in the formulation of the same theory on generally curved space-times. Section 3.2 contains a short discussion on the generalization of the Wightman axioms to curved space-times. This, in particular, tries to elucidate how the previously specified issues could possibly be solved.

Chapter IV contains an overview of the properties of anti–de Sitter space-time. In section 4.1 we first introduce AdS through the ambient space formalism but leave a more detailed discussion for appendix A. Then, we discuss the geometry and causal structure of AdS, and introduce different coordinate patches. In section 4.2 the symmetries of AdS will be presented, and we will pay particular attention on how these act on Poincaré coordinates. The geometrical properties and symmetries of the conformal boundary of AdS are discussed separately in section 4.3.

The knowledge gained in the previous chapters is used in chapter V to present the formulation of QFT in AdS. Specifically, in section 5.1 we explicitly quantize the model of a free neutral scalar introduced earlier on AdS. As argued in [15], a particular care of the conformal boundary of AdS is necessary. In section 5.2 we introduce the Euclidean AdS and review the construction of the scalar propagator on AdS. For the interested reader, higher spin propagators are constructed in appendix D with the aid of the spinor helicity formalism introduced in appendix B and the resulting identities listed in appendix C. We conclude this chapter with section 5.3 containing a direct investigation of the dual CFT appearing on the conformal boundary of AdS.

Chapter VI contains the novel results this thesis aims to present. After shortly specifying the exact model under consideration, we proceed in section 6.1 by listing all the various bulk correlation functions that will be calculated later on. In section 6.2 we derive the one- and two-loop corrections to the two-point function in position space using a Schwinger parameterization. Here we also specify the ultraviolet regularization employed in this thesis. For $\Delta = 1$, we will encounter in addition infrared divergences whose regularization is also discussed there. Section 6.3 contains the computation of the four-point function. The tree-level contribution is well known (e.g., refs. [68, 60]), so we just reproduce their calculations. The one-loop contribution requires an ingenious use of Schwinger parameters. Eventually, the ultraviolet divergences can be absorbed in the renormalized ϕ^4 coupling as expected. It turns out that the $\Delta = 1$ calculation differs from that for $\Delta = 2$ by an extra contribution, which is computationally tedious but manageable as a short-distance expansion on the boundary. Section 6.4 summarizes the final results obtained in the previous sections. Eventually, in section 6.5 we compare the short-distance expansion of the bulk four-point function with the conformal block expansion in conformal field theory. How the short-distance expansion is performed is explained in detail in appendix E. At zeroth order in the bulk coupling λ it is possible to read off the spectrum of double-trace operators. At order λ one determines the anomalous dimensions which vanish for all but the spin-0 double-trace operators at that order. At order λ^2 things become more interesting. Still, for the leading Regge trajectory, we are able to derive a closed formula for the anomalous dimensions. An extensive list of anomalous dimensions and OPE coefficients for various spins and twists are referred to the appendix F.

Eventually, in chapter VII conclusions are drawn and we give possible applications of the derived results and an outlook for further research.

1.4 List of published papers

Parts of this thesis are reproductions of the content of the author's publications. Some of the results presented here have been published in the following papers:

- I. Bertan and I. Sachs, *Loops in Anti-de Sitter Space*, *Phys. Rev. Lett.* **121** (2018) 101601 [`arXiv:1804.01880`]

- I. Bertan, I. Sachs and E. D. Skvortsov, *Quantum ϕ^4 Theory in AdS_4 and its CFT Dual*, *JHEP* **02** (2019) 099 [`arXiv:1810.00907`]

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II QFT in flat space-time

The path to the development of a full theory of quantum gravity passes first through a deep understanding of quantum field theory in curved space-times, where matter is treated in accordance to the principles of quantum field theory but gravity is not. Instead, gravity is treated classically, that is, it is required to satisfy the rules of general relativity. Despite being an approximate theory, the study of QFT in curved space-times has led to deep insights into the nature of quantum gravity, the most striking being the prediction of Hawking radiation [70].

However, a fully satisfactory description of QFT in curved space-times is far from being formulated. Whereas QFT in flat space-time has been remarkably successful and has been confirmed by experiments to within an extraordinary degree of accuracy, it does not provide a straightforward generalization to curved space-times. The main issue is that the (best-understood) particle description of QFT relies crucially on the symmetry properties of space-time, but curved space-times do not even have to exhibit any symmetry.

Here we will discuss the basic requirements and properties of a QFT in flat space-time. This will be a guidance to our later discussion on how quantum field theory could be generalized to curved space-times.

2.1 Axiomatic quantum field theory

By examining the heuristic formulations of QFT in flat space-time, a few common features of the theory emerge. These key properties are resumed by the *Wightman axioms* [71], a first attempt towards a formal description of quantum field theory[1]:

A relativistic (bosonic) quantum field theory in Minkowski space-time $\mathbb{M}_{d,1}$, with $d > 1$ spatial dimensions, is given by a Hilbert space $\mathscr{H}$ being the module of a unitary representation U of the Poincaré group $\mathcal{P}$, and a set of operator-valued distributions (and their adjoints) $\{\Psi_i(x)\}$ over $\mathbb{M}_{d,1}$. The theory should further satisfy [73]

A1: (*Spectrum condition*) The spectrum p^μ of the (selfadjoint) generators P^μ of the subgroup of space-time translations $\mathcal{T} \subset \mathcal{P}$ is confined to the forward cone

$$C_+ := \{x \in \mathbb{M}_{d,1} : x^{\mu^2} \equiv x \cdot x \equiv x_\mu x^\mu \equiv \eta_{\mu\nu} x^\mu x^\nu = -(x^0)^2 + \vec{x}^2 \leq 0, x^0 \geq 0\}.$$

A2: (*Domain condition*) For each test function f defined on $\mathbb{R}^{d+1}$, the domain of the "smeared" operators

$$\Psi_i(f) = \int \mathrm{d}^{d+1}x \; \Psi_i(x) f(x) \tag{2.1}$$

[1] There exist also attempts of different nature, like the algebraic approach leading to the Haag–Kastler axioms (see ref. [72] for an overview).

is a dense subspace $\mathscr{D} \subset \mathscr{H}$ invariant under the action of both $\mathcal{P}$ and $\Psi_i(f)$.

A3: (*Vacuum condition*) There is a, up to a phase, unique ray $|\Omega\rangle \in \mathscr{D}$ invariant under $\mathcal{T}$. This state is also invariant under the Lorentz subgroup $\mathcal{L}$.

A4: (*Completeness condition*) The set of finite linear combinations of vectors of the form $\Psi_{i_1}(f_1)\cdots\Psi_{i_N}(f_N)|\Omega\rangle$ is dense in $\mathscr{H}$.

A5: (*Covariance condition*) The operators transform covariantly under the action of $\mathrm{g} \in \mathcal{P}$, i.e.,

$$U(\mathrm{g})\Psi_i(x)U^{-1}(\mathrm{g}) = M_i(\mathrm{g}^{-1}) \cdot \Psi_i(\mathrm{g}(x))),$$

where M_i is a real-valued tensor representation of the identity component $\mathcal{L}_0$ of the Lorentz group $\mathcal{L} \subset \mathcal{P}$.

A6: (*Microcausality condition*) Take two test functions f, g. If the support of f is space-like to the support g, that is, for each $x \in \operatorname{supp}(f)$ and $y \in \operatorname{supp}(g)$ it follows $(x-y)^{\mu^2} > 0$, then $[\Psi_i(f), \Psi_j(g)] = 0$.

In order to shed light on the Wightman axioms, let us clarify some points. The following discussion might seem too detailed at this stage. However, when discussing QFT in curved space-times, it will allow us to keep an overview on the defining properties of QFT in flat space-time. This further facilitates the detection of relevant differences.

- The physical states are interpreted as the unit rays in $\mathscr{H}$, or equivalently, vectors in the projective space of $\mathscr{H}$. The smeared operators $\Psi_i(f)$ correspond to the quantum fields, and the generator of time translations P^0 is identified with the (unique, see below) Hamiltonian density H. It will later become clear that the spectral condition ensures that all physical states have nonnegative mass and energy.

- In the covariant formulation of gauge theories, the positive definiteness of the Hilbert space inner product is absent. However, these zero modes should resolve after gauge fixing.

- The $\Psi_i(f)$ are (possibly) unbounded operator, thus one must specify their domain. Although there also exist properly unbounded operators defined on the whole Hilbert space[2], in applications only those defined on a proper subspace of $\mathscr{H}$ appear. Since the denseness of the domain is a necessary and sufficient condition for the existence of the adjoint, it is necessary to extend the domain of the unbounded operators to a dense subspace $\mathscr{D} \subset \mathscr{H}$. This is indeed always possible by simply demanding that the kernel of the operator, having domain $\mathscr{K} \subset \mathscr{H}$, contains the complement $\mathscr{K}^\perp$ of $\mathscr{K}$. The statement then follows from $\mathscr{H} = \overline{\mathscr{K}} \oplus \mathscr{K}^\perp$, where $\overline{\mathscr{K}}$ is the closure of $\mathscr{K}$. The restriction of the domain of a bounded operator to some dense subspace $\mathscr{D}$ is not constraining, since the operator has a unique extension to $\mathscr{H}$.

[2]as a pathological result of the axiom of choice

- According to the definition of distributions, the integral $\Psi_i(f)$ in eq. (2.1) denotes a continuous[3] linear functional on the vector space of test functions f. In the treatment of QFTs with renormalizable interactions, our experience dictates us that it appears sufficient to take f as an element of the Schwartz space[4] $\mathscr{S}(\mathbb{R}^{d+1})$ [74, 75]. This restriction is particularly advantageous since the Fourier transformation is an automorphism on both $\mathscr{S}(\mathbb{R}^{d+1})$ and its dual space $\mathscr{S}'(\mathbb{R}^{d+1})$ defined by eq. (2.1), with elements of the latter space called tempered distributions. The Fourier transform of a tempered distribution $\tilde{\Psi}_i(p)$ is the tempered distribution defined via $\tilde{\Psi}_i(f) = \Psi_i(\tilde{f})$, where $\tilde{f}$ is the Fourier transform of $f \in \mathscr{S}(\mathbb{R}^{d+1})$.

- By Wigner's theorem (see, for instance, ref. [76]), in order to preserve the probabilistic interpretation of the theory, every symmetry of the theory is necessarily described by some unitary or anti–unitary representation of $\mathcal{P}$. All symmetries represented by anti–unitary operators involve time-reversion. One might only consider the identity component $\mathcal{P}_0$ of $\mathcal{P}$, excluding de facto all discrete space-time symmetries and consequently the need for anti–unitary operators.

- A particular choice of unitary representation U of $\mathcal{P}_0$ settles the particle content of the theory. More precisely, each unitary irreducible representation (UIR) of $\mathcal{P}_0$ corresponds to a physically relevant particle since, as a consequence of Schur's lemma, irreducibility translates to sharp mass and spin eigenvalues of the respective Casimir operators. It is known that all unitary representations of the Poincaré group can be decomposed into irreducible ones [77]. This is quite surprising, as the complete reducibility property is a virtue of finite groups and compact Lie groups, however $\mathcal{P}_0$ is a noncompact Lie group. Nevertheless, a substantial difference with compact Lie groups exists and is the origin for the necessity of field theory in relativistic quantum mechanics; all the nontrivial UIRs of $\mathcal{P}_0$ are infinite-dimensional representations, corresponding precisely to the field representations. In general, one considers only UIRs whose little group is compact[5]. This guarantees finite-component field representations, since all UIRs of compact Lie groups are finite dimensional. Altogether, these representations describe bosonic massive and massless particles, both of finite spin.

- The Hilbert space $\mathscr{H}$ of a free theory, the Fock space, is separable and thus its basis forms a countable set. In other theories a nonseparable Hilbert space might be necessary, but as long as one considers theories with finite particle content, this is not the case [71]. Working on separable Hilbert spaces guarantees that the generator of each continuous symmetry is a unique selfadjoint operator and

[3] or, equivalently, bounded

[4] This is the space of all infinitely differentiable functions that are rapidly (faster than any power) decreasing at infinity, along with all partial derivatives.

[5] The little group of a massless particle is noncompact but its action on the helicity representation corresponds to that of a compact group, see ref. [78].

therefore an observable. The mathematical formulation is the following. Let U_α with $\alpha \in \mathbb{R}$ be a unitary representation of a one-parameter subgroup of $\mathcal{P}$ on a separable Hilbert space $\mathscr{H}$. Then, according to Stone's theorem[6], there exists a unique selfadjoint operator A such that

$$U_\alpha = e^{-\mathrm{i}\alpha A}. \tag{2.2}$$

The operator A is clearly a selfadjoint representation of one of the generators t_a (or linear combination thereof) of the Lie group $\mathcal{P}$, satisfying the Lie algebra

$$[t_a, t_b] = \mathrm{i} f^c{}_{ab} t_c\,. \tag{2.3}$$

The structure constants $f^c{}_{ab}$ completely characterize the algebra and are independent of the representation of t_a. This allows to study the Lie group $\mathcal{P}$ in a sufficiently small neighborhood of the identity.

- Let us see how the subgroup of space-time translations $\mathcal{T} \subset \mathcal{P}_0$ acts on a quantum field $\Psi_i(x)$. The unitary representation of a space-time translation with the vector $\alpha \in \mathbb{M}_{d,1}$ is given by $U_{\alpha^\mu} = e^{-\mathrm{i}\alpha\cdot P}$, where P^μ are the generators of $\mathcal{T}$. Axiom (A5) yields $\Psi_i(x+\alpha) = U_{\alpha^\mu}\Psi_i(x)U_{-\alpha^\mu}$, which infinitesimally corresponds to the generalization of the Heisenberg equation

$$\mathrm{i}\frac{\partial \Psi_i(x)}{\partial x^\mu} = [P_\mu, \Psi_i(x)], \tag{2.4}$$

where we introduced the distributional derivative

$$\int \mathrm{d}^{d+1}x\ \frac{\partial \Psi_i(x)}{\partial x^\mu} f(x) \equiv -\int \mathrm{d}^{d+1}x\ \Psi_i(x)\frac{\partial f(x)}{\partial x^\mu}. \tag{2.5}$$

Acting with U_{α^μ} on the Fourier transformed quantum field

$$\tilde{\Psi}_i(p) = \int \mathrm{d}^{d+1}x\ e^{-\mathrm{i}x\cdot p}\Psi_i(x), \tag{2.6}$$

leads to

$$U_{\alpha^\mu}\tilde{\Psi}_i(p)U_{-\alpha^\mu} = e^{\mathrm{i}\alpha\cdot p}\tilde{\Psi}_i(p)\,, \qquad -p_\mu\tilde{\Psi}_i(p) = [P_\mu, \tilde{\Psi}_i(p)]. \tag{2.7}$$

Thus, any vector of the form $\tilde{\Psi}_{i_1}(p_1)\cdots\tilde{\Psi}_{i_N}(p_N)|\Omega\rangle$ is an eigenvector of P^μ:

$$P^\mu\tilde{\Psi}_{i_1}(-p_1)\cdots\tilde{\Psi}_{i_N}(-p_N)|\Omega\rangle = (p_1^\mu + \cdots + p_N^\mu)\tilde{\Psi}_{i_1}(-p_1)\cdots\tilde{\Psi}_{i_N}(-p_N)|\Omega\rangle, \tag{2.8}$$

[6] On nonseparable Hilbert spaces, Stone's theorem requires strong continuity of U_α instead of solely weak measurability, see ref. [79].

where we made use of the invariance of the vacuum under $\mathcal{T}$, see axiom (A3). Then, the spectrum condition (A1) requires that for each $\tilde{\Psi}_{i_1}(p_1)\cdots\tilde{\Psi}_{i_N}(p_N)|\Omega\rangle$, $p_1+\cdots+p_N \in (-C_+)$. Furthermore, note that

$$\begin{aligned}(2\pi)^{N(d+1)} &\int \mathrm{d}^{d+1}\alpha\; e^{-\mathrm{i}\alpha\cdot p} U_{-\alpha^\mu} \Psi_{i_1}(x_1)\cdots\Psi_{i_N}(x_N)|\Omega\rangle \\ &= \int \left(\Pi_n \mathrm{d}^{d+1} q_n\right) \delta\left(p^\mu + \sum_n p_n^\mu\right) e^{\mathrm{i}\sum_n x_n\cdot p_n} \tilde{\Psi}_{i_1}(p_1)\cdots\tilde{\Psi}_{i_N}(p_N)|\Omega\rangle,\end{aligned} \tag{2.9}$$

with $n \in \{1,\ldots,N\}$. Since $p_1+\cdots+p_N \in (-C_+)$ it follows that $p \in C_+$. A more general statement is that, for each $|\Psi\rangle \in \mathscr{D}$ (see axiom (A4)), one has

$$\int \mathrm{d}^{d+1}\alpha\; e^{-\mathrm{i}\alpha\cdot p} U_{-\alpha^\mu}|\Psi\rangle = 0 \qquad \text{if} \qquad p \notin C_+. \tag{2.10}$$

- Free fields subject to the (equal time) canonical commutation relations (CCR) trivially satisfy the microlocality condition. On the other hand, all known examples of interacting fields satisfying axiom (A6) are derived, as yet, from free fields.
- Although microcausality at very short distances is not a necessary requirement as long as macrocausality is assured, a rigorous formulation of the latter is difficult [80]. The common idea is that the support of $\Psi_i(x)$ should not grow faster than the speed of light.
- The axioms should apply to free or interacting theories alike.

As a final remark, let us briefly discuss the important physical consequences of dealing with a projective space of $\mathscr{H}$. The unitary representations U of $\mathcal{P}_0$ are more appropriately projective representations, that is, they are only defined up to a phase. Let us choose, for each $\mathrm{g} \in \mathcal{P}_0$, a representative $U(\mathrm{g})$. Then these representations satisfy the composition rule

$$U(\mathrm{g}_1)U(\mathrm{g}_2) = e^{\mathrm{i}\theta(\mathrm{g}_1,\mathrm{g}_2)} U(\mathrm{g}_1 \circ \mathrm{g}_2), \tag{2.11}$$

where $\theta(\mathrm{g}_1,\mathrm{g}_2) \in \mathbb{R}$ and $\mathrm{g}_1, \mathrm{g}_2, \mathrm{g}_1\circ\mathrm{g}_2 \in \mathcal{P}_0$. The associated Lie algebra features an additional term on the right hand side of eq. (2.3), a central charge of the form $\mathrm{i}f_{ab}$. In contrast, ordinary representations are group homomorphisms, for which $\theta(\mathrm{g}_1,\mathrm{g}_2)=0$ for all $\mathrm{g}_1,\mathrm{g}_2$ is satisfied and consequently the central charge is zero. Now, the question arises if is it possible to choose different representatives for $U(\mathrm{g})$ such that eq. (2.11) corresponds to a homomorphism. Bargmann's theorem states that this is indeed possible if $\mathcal{P}_0$ is simply-connected and if its generators can be redefined such to eliminate all central charges from its Lie algebra. However, $\mathcal{P}_0$ does not satisfy the former requirement, which makes it necessary to consider its universal covering group[7].

[7]For Lie groups, the universal cover is the unique simply-connected Lie group that has the same Lie algebra of the original group.

The physical consequences are the following. The ordinary UIRs of the universal cover of $\mathcal{P}_0$ include fermionic particles, in addition to the bosonic particles appearing in the ordinary UIRs of $\mathcal{P}_0$. This allows for the transition to a fermionic (or mixed, by considering direct sums of UIRs) QFT, where one has to impose a grading of the fields into even and odd, and the commutator of odd fields in axiom (A6) has to be replaced with the anticommutator.

2.2 The Wightman functions

To keep things simple, and given the focus on real scalar field theories in this thesis, we henceforth assume that our theory contains only one scalar selfadjoint operator-valued distribution $\Psi(x) = \Psi^\dagger(x)$. Nevertheless, a generalization of sections 2.2 and 2.3 to different operator-valued distributions is straightforward.

Being acquainted with the properties a general QFT has to satisfy, as yet, does not enhance our knowledge on the outputs of the theory. For theoretical particle physicists, the relevant outputs are ideally experimentally verifiable predictions, frequently about scattering. The full data of scattering processes is, by means of the LSZ reduction formula [81], encoded in the N-point functions, i.e., the vacuum expectation values of the product[8] of N operator-valued distributions:

$$\langle \Omega | \Psi(x_1) \cdots \Psi(x_N) | \Omega \rangle. \tag{2.12}$$

In order to postpone the complications concerning pointwise products of distributions, we further take the points $\{x_1, \ldots, x_N\}$ to be distinct. In fact, this generally ill-defined product is what causes the necessity for renormalization in heuristic QFT [82]. However, note that the Wightman axioms comprise renormalized fields by definition, and thus eq. (2.12) is finite.

According to eq. (2.1), let us introduce the *Wightman functions*

$$\mathrm{W}_N(f_1, \ldots, f_N) := \langle \Omega | \Psi(f_1) \cdots \Psi(f_N) | \Omega \rangle, \tag{2.13}$$

which are well-defined[9] multilinear functionals from the product space $\mathscr{S}^N \equiv \mathscr{S}(\mathbb{R}^{d+1}) \times \ldots \times \mathscr{S}(\mathbb{R}^{d+1})$ of Schwartz spaces to the complex numbers $\mathbb{C}$. As it was clarified above, the product of distributions

$$\Psi(x_1) \cdots \Psi(x_N) | \Omega \rangle \tag{2.14}$$

defines a functional which is separately continuous in each test function. Thus, by Schwartz's nuclear theorem [83], the functional can be uniquely extended to a linear and continuous functional which maps $f \in \mathscr{S}_N \equiv \mathscr{S}(\mathbb{R}^{d+1} \times \ldots \times \mathbb{R}^{d+1})$ to $\mathbb{C}$, such that one obtains the same results as integrating with eq. (2.14) whenever f factorizes as

$$f(x_1, \ldots, x_N) = (f_1 \otimes \ldots \otimes f_N)(x_1, \ldots, x_N) \equiv f_1(x_1) \cdots f_N(x_N). \tag{2.15}$$

[8] Here we neglect time-ordering.
[9] see axioms (A2) and (A3)

In particular, also the Wightman functions $\mathrm{W}_N(f_1, \ldots, f_N)$ can be uniquely extended to linear and continuous functionals $\mathrm{W}_N(f)$. From now on, we denote the Wightman functions as $\mathrm{W}_N(f)$ and the corresponding (Wightmann) distributions as $\mathrm{W}_N := \mathrm{W}_N(x_1, \ldots, x_N)$.

Taking into account the Wightman axioms, $\mathrm{W}_N(f)$ are linear and continuous functionals satisfying the following properties:

W1: (*Covariance*) Invariance under transformations $\mathrm{g} \in \mathcal{P}$:

$$\mathrm{W}_N(\mathrm{g}(x_1), \ldots, \mathrm{g}(x_N)) = \mathrm{W}_N(x_1, \ldots, x_N). \tag{2.16}$$

W2: (*Locality*) If $(x_j - x_{j+1})^{\mu^2} > 0$ for some $j \in \{1, \ldots, N-1\}$, then

$$\mathrm{W}_N(x_1, \ldots, x_j, x_{j+1}, \ldots, x_N) = \mathrm{W}_N(x_1, \ldots, x_{j+1}, x_j, \ldots, x_N). \tag{2.17}$$

W3: (*Spectrum property*) For each $\mathrm{W}_N \in \mathscr{S}'_N$ there exists a $\tilde{\mathrm{w}}_{N-1} \in \mathscr{S}'_{N-1}$, with support in the product of forward cones $(C_+)^{N-1}$, such that

$$\mathrm{W}_N(x_1, \ldots, x_N) = (2\pi)^{-(N-1)(d+1)} \int \left(\Pi_j \mathrm{d}^{d+1} q_j\right) \; e^{\mathrm{i} \sum_j u_j \cdot q_j} \tilde{\mathrm{w}}_{N-1}(q_1, \ldots, q_{N-1}), \tag{2.18}$$

where $u_j = x_j - x_{j+1}$ and $j \in \{1, \ldots, N-1\}$.

W4: (*Positive definiteness*) For any sequence $\left(f_{(N)}\right)_{N \in \mathbb{N}}$ with $f_{(N)} \in \mathscr{S}_N$:

$$\sum_{M,N=0}^{k} \mathrm{W}_{M+N}(\bar{f}_{(M)} \otimes f_{(N)}) \geq 0, \tag{2.19}$$

where $\bar{f}_{(M)}(x_1, \ldots, x_M) = f^*_{(M)}(x_M, \ldots, x_1)$ and $f^*_{(M)}$ is the complex conjugate of $f_{(M)}$.

W5: (*Cluster property*) For any space-like vector $\alpha \in \mathbb{M}_{d,1}$ the following holds:

$$\lim_{\lambda \to \infty} \mathrm{W}_N(x_1, \ldots, x_n, x_{n+1} + \lambda\alpha, \ldots, x_N + \lambda\alpha) = \mathrm{W}_n(x_1, \ldots, x_n) \mathrm{W}_{N-n}(x_{n+1}, \ldots, x_N). \tag{2.20}$$

The properties (W1) and (W2) are direct consequences of the axioms (A5) and (A6) respectively, whereas property (W4) simply follows from the the fact that

$$\sum_{M=1}^{k} \Psi(x_1) \cdots \Psi(x_M) |\Omega\rangle \in \mathscr{D}, \tag{2.21}$$

and thus its norm is nonnegative. Property (W3) is slightly more elaborate. Firstly, existence of $\tilde{\mathrm{w}}_{N-1}$ follows from translation invariance, which implies that W_N depends only on the $(N-1)$ difference vectors $u_j = x_j - x_{j+1}$. Indeed, let us write the Wightman distribution W_N in terms of its Fourier transform:

$$\mathrm{W}_N(x_1,\dots,x_N) = (2\pi)^{-N(d+1)} \int \left(\Pi_n \mathrm{d}^{d+1}p_n\right) e^{\mathrm{i}\sum_n x_n\cdot p_n} \tilde{W}_N(p_1,\dots,p_N). \tag{2.22}$$

A change of variables yields

$$\mathrm{W}_N(u_1,\dots,u_{N-1},x_N) = (2\pi)^{-N(d+1)} \int \left(\Pi_n \mathrm{d}^{d+1}q_n\right) e^{\mathrm{i}\sum_j u_j\cdot q_j + \mathrm{i}x_N\cdot q_N} \tilde{W}_N(q_1,\dots,q_N), \tag{2.23}$$

where $q_n = p_1 + \dots + p_n$ and $n \in \{1,\dots,N\}, j \in \{1,\dots N-1\}$. Now, the equation

$$-(p_1^\mu + \dots + p_N^\mu)\tilde{W}_N(p_1,\dots,p_N) = \langle\Omega|P^\mu\tilde{\Psi}(p_1)\cdots\tilde{\Psi}(p_N)|\Omega\rangle = 0, \tag{2.24}$$

has the following solution

$$\tilde{W}_N(q_1,\dots,q_N) = (2\pi)^{(d+1)}\delta(q_N)\tilde{\mathrm{w}}_{N-1}(q_1,\dots,q_{N-1}) \tag{2.25}$$

for some distribution $\tilde{\mathrm{w}}_{N-1} \in \mathscr{S}'_{N-1}$. With this, eq. (2.23) becomes exactly eq. (2.18). Secondly, to find the support of $\tilde{\mathrm{w}}_{N-1}$, let us start with its inverse Fourier transform $\mathrm{w}_{N-1} := \mathrm{w}_{N-1}(u_1,\dots,u_{N-1})$ and Fourier transform the latter with respect to only one of the vectors u_j. This is then proportional to

$$\begin{aligned} &\int \mathrm{d}^{d+1}\alpha\; e^{-\mathrm{i}\alpha\cdot q_j}\mathrm{W}_N(u_1,\dots,u_{j-1},u_j+\alpha,u_{j+1},\dots u_{N-1}) \\ &\quad= \int \mathrm{d}^{d+1}\alpha\; \langle\Omega|\Psi(x_1)\cdots\Psi(x_j)U_{-\alpha^\mu}\Psi(x_{j+1})\cdots\Psi(x_N)|\Omega\rangle, \end{aligned} \tag{2.26}$$

where again q_j is the momentum conjugate of u_j. Then, according to eq. (2.10), we have $\tilde{\mathrm{w}}_{N-1}(q_1,\dots,q_{N-1}) = 0$ whenever any $q_j \notin C_+$.

The derivation of property (W5) is rather lenghty and is beyond the scope of this thesis. It exploits the fact that, for large λ, one can reverse the order in eq. (2.20) to $x_{j+1}+\lambda\alpha,\dots,x_N+\lambda\alpha,x_1,\dots,x_j$. This also reverses the sign of the momentum conjugate q_j to the difference variable u_j. The statement follows applying the spectrum condition on q_j and the uniqueness of the vacuum [84].

The above properties of the Wightman functions together reveal a fundamental peculiarity of quantum field theory, that is, the Wightman functions describe by themselves a well-defined QFT and hence entail all the necessary information. More precisely, the Wightman reconstruction theorem states that

Given any sequence $(\mathrm{W}_N)_{N\in\mathbb{N}}$ of tempered distributions obeying the conditions (W1-W5), there exists a QFT satisfying axioms (A1-A6) for which the W_Ns are the Wightman distributions.

A proof of the theorem can be found in ref. [84]. Let us remark that, by excluding the cluster property (W5), the resulting QFT satisfies anyway all Wightman axioms (A1-A6), with the eventual exclusion of the uniqueness of the vacuum. This is interesting since, in fact, we will later argue that the requirement of the uniqueness of the vacuum is, in general, not possible to satisfy in curved space-times.

In addition to the Wightman reconstruction theorem, several imporant formal properties of QFTs, remarkably the CPT and the spin-statistics theorems, have been proven within the Wightman framework [71]. Yet, its predictive power is limited for various reasons when facing real physical phenomena. First of all, it is a difficult task to even give examples of Wightman quantum field theories for the case of free theories, and no Wightman QFT has been constructed so far for any realistic interaction [85]. Even worse, the canonical commutation relations pose the "choice problem": In nonrelativistic quantum mechanics all selfadjoint representations of the Heisenberg algebra $[q_i, p_j] = \mathrm{i}\delta_{i,j}$ are, within certain technical restricitions, unitarily equivalent (Stone–von Neumann theorem). Therefore, all representations, most importantly Schrödinger and Heisenberg representations, give rise to the same expectation values. In QFT instead, the CCR have uncountably many unitarily inequivalent representations [86]. On which basis does one choose among these inequivalent representations? The obvious response would be to consider only one representation (up to unitary equivalence) as physically relevant and dismiss the remaining. However, this response is proven wrong by Haag's theorem [87, 88], which is a direct consequence of the Wightman axioms[10]. In brief, the theorem states the following:

Take two Wightman QFTs, a free theory[11] $(\mathscr{H}_F, \{\Psi_F(x)\})$ and an "interacting" theory $(\mathscr{H}_I, \{\Psi_I(x)\})$, each satisfying the canonical commutation relations. If there exists a unitary mapping V from $\mathscr{H}_F$ to $\mathscr{H}_I$ such that for each $\Psi_F(x)$ one has $\Psi_I(x) = V\Psi_F(x)V^{-1}$, then also $(\mathscr{H}_I, \{\Psi_I(x)\})$ describes the same free theory[12].

In other words, the attempt to derive the Wightman functions for a truly interacting theory, starting from the Wightman functions of the free theory, fails due to the impossibility to represent nontrivial interactions. Undoubtedly, since all infinite-dimensional separable Hilbert spaces are isomorphic, one can always find a unitary map between $\mathscr{H}_F$ and $\mathscr{H}_I$. However, contrary to the assumptions in Haag's theorem, this map does not identify two unitarily equivalent representations of the CCR. Thus, the existence of inequivalent representations of the CCR are not a mere mathematical subtlety but carries an intricated amount of physical implications. A discussion of these implications is beyond the scope of this thesis and we refer to ref. [89] for an in-depth analysis.

Still, let us note that, owing to the general validity of Haag's theorem, it might seem

[10] and also of the Haag–Kastler's axioms

[11] A free theory in the sense that $\Psi_F(x)$ is a solution of the Klein–Gordon equation, see section 2.4.

[12] If also $\Psi_I(x)$ is a solution of the Klein–Gordon equation, the theorem states that the mass has to be the same as for $\Psi_F(x)$.

natural to consider the canonical formalism of QFT, including the interaction picture and perturbation theory, somehow erroneous. The discrepancy between the canonical formalism and some formal description is indeed reflected in the difficulty of formulating physically relevant QFTs within the above axiomatic setting. Nevertheless, this does not prevent us from using the canonical formalism, due to its, in this view admittedly surprising, exceptional precision.

In spite of the mentioned discordance between the axiomatic and the heuristic approach to QFT, the Wightman axioms are, in their limited domain of application, a great playground to discuss general QFTs in flat space-time and, even further, QFTs in curved space-times. This will be done in the next chapter. Afterwards we will proceed with the heuristic approach, considering first a free scalar field theory in anti–de Sitter space-time and subsequently a quartic deformation of the free theory.

2.3 Analytic continuation of correlation functions

The Wick rotation is widely regarded as a standard tool in quantum field theory, rendering the integrals in the calculation of *correlation functions*[13] more easily manageable. It is strongly bound to the notion of analytic continuation of Wightman functions, relating the Wightman functions to the *Schwinger functions*. On general space-times such a continuation is not always attainable. This section explains the procedure of analytic continuation of correlation functions in flat space-time, in order to later on justify its application in anti–de Sitter space-time. We will essentially follow ref. [90], but a more detailed discussion can for instance be found in refs. [73, 91].

Analytic continuation of tempered distributions.

Let us start from a convex cone C, i.e., a subset of $\mathbb{R}^D$ satisfying $\lambda_1 p_1 + \lambda_2 p_2 \in C$ for each $p_1, p_2 \in C$ and $\lambda_1, \lambda_2 \geq 0$. Its dual

$$C' := \{x \in \mathbb{R}^D : \langle x, p\rangle \leq 0 \text{ for all } p \in C\} \tag{2.27}$$

is also a convex cone. Here $\langle x, p\rangle$ represents any symmetric and nondegenerate bilinear form[14] on $\mathbb{R}^D$ satisfying the inequality

$$\langle x, p\rangle \leq -\frac{1}{2} m \|p\|, \tag{2.28}$$

where m is a nonnegative number dependent on $x \in C'$, and $\|p\|$ denotes the Euclidean length of $p \in C$. Assume the subset $C'^0 = \{x \in C' : m > 0\}$ to be an open, nonempty set. Further, assume that $\tilde{g}(p)$ is a polynomially bounded continuous function with support in C. Polynomial boundedness requires $|\tilde{g}(p)| \leq c(1 + \|p\|)^k$ for some $c \geq 0$ and $k \in \mathbb{N}$.

[13] Which are, dependendly on the context, either Wightman or Schwinger functions. Schwinger functions will be introduced in this section.

[14] Note that this ensures that $\mathbb{R}^D$ is isomorphic to its dual with respect to the defined bilinear form.

Then, the function

$$\mathcal{F}_{\tilde{g}}(\zeta) := (2\pi)^{-D} \int \mathrm{d}^D p \; e^{\mathrm{i}\langle \zeta, p\rangle} \tilde{g}(p), \tag{2.29}$$

of the complex vector ζ in the open tube $\mathrm{T} = \mathbb{R}^D + \mathrm{i}(-C'^0) \subset \mathbb{C}^D$, satisfies the following properties:

- Since $\mathrm{i}\langle \zeta, p\rangle$ has a negative real part, it follows that

$$|\mathcal{F}_{\tilde{g}}(\zeta)| \leq \frac{c}{(2\pi)^D} \int \mathrm{d}^D p \; e^{-\frac{1}{2}m\|p\|}(1+\|p\|)^k \leq \frac{c}{(2\pi)^D} \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \; l! \left(\frac{2}{m}\right)^{l+1} e^{\frac{m}{2}}, \tag{2.30}$$

  for $l = D + k - 1$. In the last step we integrated over the whole $\mathbb{R}^D$ with spherical coordinates.

- $\mathcal{F}_{\tilde{g}}(\zeta)$ is differentiable in ζ since, as a consequence of continuity of $\tilde{g}(p)$, one can take derivatives under the integral.

Thus, $\mathcal{F}_{\tilde{g}}(\zeta)$ is a well-defined holomorphic function on the open tube T. Now, any distribution $\tilde{T} \in \mathscr{S}(\mathbb{R}^D)$ with support in a convex cone can be written in the form $\tilde{T} = P(\mathrm{i}\partial)\tilde{g}(p)$ for some polynomial $P(x) \in \mathbb{C}[x_1, \dots, x_D]$ and some $\tilde{g}(p)$ as above [92]. Then, the function

$$\mathcal{F}_{\tilde{T}}(\zeta) := (2\pi)^{-D} \int \mathrm{d}^D p \; e^{\mathrm{i}\langle \zeta, p\rangle} P(\mathrm{i}\partial)\tilde{g}(p) = P(\zeta)\mathcal{F}_{\tilde{g}}(\zeta) \tag{2.31}$$

is again a holomorphic function on T. In plain words, any tempered distribution with support in a convex cone can be analytically continued on the open tube. Eventually, we are interested in the boundary value $\mathcal{F}_{\tilde{T}}(x)$ of $\mathcal{F}_{\tilde{T}}(\zeta) \equiv \mathcal{F}_{\tilde{T}}(x + \mathrm{i}y)$, but it follows from eq. (2.28) that $0 \notin C'^0$. In fact, for $y = 0$, the integrals (2.29,2.31) generally do not converge. This is akin to claim that a polynomial function does not admit a Fourier transformation. Fortunately, one can circumvent this issue by requiring $\mathcal{F}_{\tilde{T}}(x)$ to converge in the distributional sense:

$$\int \mathrm{d}^D x \; \mathcal{F}_{\tilde{T}}(x)\tilde{f}(x) \equiv \lim_{t \searrow 0} \int \mathrm{d}^D x \; \mathcal{F}_{\tilde{T}}(x + \mathrm{i}ty)\tilde{f}(x) := \int \mathrm{d}^D x \; \tilde{T}(x)\mathcal{F}_{\tilde{f}}(x), \tag{2.32}$$

for each $\tilde{f} \in \mathscr{S}(\mathbb{R}^D)$. The last integral in eq. (2.32) is well-defined if

$$\mathcal{F}_{\tilde{f}}(x) = (2\pi)^{-D} \int \mathrm{d}^D p \; e^{\mathrm{i}\langle x, p\rangle} \tilde{f}(p) \tag{2.33}$$

is an element of $\mathscr{S}(\mathbb{R}^D)$, and in that case $\mathcal{F}_{\tilde{T}}(x)$ is also a tempered distribution.

Let us see how this all applies to our case. With $\mathbb{M}_{d,1}$ as the underlying vector space, one can identify the convex cone C with the forward cone $C_+ \subset \mathbb{M}_{d,1}$, and the bilinear form with the standard scalar product $\langle x, p\rangle = x \cdot p,\ p \in C_+, x \in C'_+$. In this setting, C_+ is selfdual, i.e., $C_+ = C'_+$, and the bilinear form satisfies the inequality

$$\langle x, p\rangle \leq -\frac{1}{2}(x^0 - |\vec{x}|)\|p\|. \tag{2.34}$$

Thus, $m = 0$ corresponds to $x^{\mu^2} = 0$, and therefore

$$C'^0_+ = \{x \in \mathbb{M}_{d,1} : x^{\mu^2} < 0, x^0 > 0\} \tag{2.35}$$

is an open nonempty set. The set $\mathrm{T}_- = \mathbb{M}_{d,1} + \mathrm{i}(-C'^0_+)$ is called open backward tube. Note that, with the standard scalar product, $\mathcal{F}_{\tilde{f}}(x) = f(x) \in \mathscr{S}(\mathbb{R}^{d+1})$ since the Fourier transform is an automorphism on $\mathscr{S}(\mathbb{R}^{d+1})$. Now, consider the Wightman distribution $\mathrm{w}_N \in \mathscr{S}'_N$ in terms of its Fourier transform $\tilde{\mathrm{w}}_N$:

$$\mathrm{w}_N(u_1, \ldots, u_N) = (2\pi)^{-N(d+1)} \int \left(\Pi_n \mathrm{d}^{d+1} q_n\right) e^{\mathrm{i}\sum_n u_n \cdot q_n} \tilde{\mathrm{w}}_N(q_1, \ldots, q_N), \tag{2.36}$$

where $n \in \{1, \ldots, N\}$. By the spectrum property (W3), the tempered distribution $\tilde{\mathrm{w}}_N$ has support in the N-fold forward cone C_+^N. After identifying C_+^N with the convex cone C, it is rather straightforward to verify its self-duality with respect to the extended bilinear form

$$\langle (u_1, \ldots, u_N), (q_1, \ldots, q_N)\rangle_N := \sum_{n=1}^{N} u_n \cdot q_n\,, \tag{2.37}$$

Clearly, this bilinear form is again symmetric, nondegenerate, and satisfies inequality (2.28). Therefore, according to the above discussion, the formula

$$\mathrm{w}_N(\Theta) := (2\pi)^{-N(d+1)} \int \mathrm{d}^{d+1} Q\ e^{\mathrm{i}\langle \Theta, Q\rangle_N} \tilde{\mathrm{w}}_N(Q), \tag{2.38}$$

where $Q \in C_+^N$ and $\Theta = U + \mathrm{i}V \in \mathrm{T}_-^N$, provides a holomorphic function in the variable Θ with boundary value $\mathrm{w}_N(U) \equiv \mathrm{w}_N(u_1, \ldots, u_N)$ (cf. eq. (2.32)). Eventually, owing to the relation $\mathrm{w}_N(U) = \mathrm{W}_{N+1}(X)$ with $X = (x_1, \ldots, x_{N+1})$, $\mathrm{W}_{N+1}(Z) = \mathrm{W}_{N+1}(X + \mathrm{i}Y)$ also provides a holomorphic function on T_-^N.

Extension of the domain of analyticity.

As will be clear at the end of this paragraph, in order to justify the Wick rotation, an extension of the domain of analyticity of w_N appears necessary. A first observation in the context of the Wick rotation is that the scalar product defined on $\mathbb{M}_{d,1}$ can be continued to a complex-bilinear form on $\mathbb{C}^{d+1}$:

$$\langle \zeta, \zeta'\rangle = -\zeta^0 \zeta'^0 + \vec{\zeta} \cdot \vec{\zeta}', \tag{2.39}$$

with $\zeta, \zeta' \in \mathbb{C}^{d+1}$. The subset

$$\mathrm{E} := \{(\mathrm{i}x^0, \vec{x}) \in \mathbb{C}^{d+1} : (x^0, \vec{x}) \in \mathbb{M}_{d,1}\} \tag{2.40}$$

is a vector space of Euclidean points, since, for $e, e' \in \mathrm{E}$,

$$\langle e, e' \rangle = x^0 x'^0 + \vec{x} \cdot \vec{x}\,'. \tag{2.41}$$

Let us now see to which extent a Wightman distribution $\mathrm{W}_{N+1} \in \mathscr{S}'_N$ can be further analytically continued to an open connected domain $\mathrm{U}_N \subset (\mathbb{C}^{d+1})^N$ such that U_N contains a large portion of the N-fold Euclidean space

$$\mathscr{E}_N := \{(e_1 - e_2, \ldots, e_N - e_{N+1}) : (e_1, \ldots, e_{N+1}) \in \mathrm{E}^{N+1}\} \cong \mathrm{E}^N. \tag{2.42}$$

This will be conducted in two steps. Eventually, the restriction of all the analytically continued W_{N+1} to $\mathrm{U}_N \cap \mathscr{E}_N$ defines an Euclidean field theory, as we will see in the next paragraph. The first step exploits covariance of the Wightman functions (W1), which implies the identity

$$\mathrm{w}_N(u_1, \ldots, u_N) = \mathrm{w}_N(\mathrm{g}(u_1), \ldots, \mathrm{g}(u_N)), \tag{2.43}$$

for any $\mathrm{g} \in \mathcal{L}_0$. This identity has a unique analytical continuation to transformations in the (proper) complex Lorentz group $\mathcal{L}_0(\mathbb{C})$[15] [93]. Consequently, w_N can be extended to the so-called N-fold extended tube

$$(\mathrm{T}_-^N)^\epsilon := \bigcup \{(L(\theta_1), \ldots, L(\theta_{N-1})) : (\theta_1, \ldots, \theta_N) \in \mathrm{T}_-^N, L \in \mathcal{L}_0(\mathbb{C})\}. \tag{2.44}$$

It is rather simple to see that $(\mathrm{T}_-)^\epsilon \equiv (\mathrm{T}_-^1)^\epsilon$ contains all Euclidean points E with the only exception of the origin. Still, a big portion of the set E^N is not contained in a general N-fold extended tube $(\mathrm{T}_-^N)^\epsilon$. This takes us to the next step, where locality of the Wightman functions (W2) is used. Note that, while T_- does not contain real points, for any real point $u \in \mathbb{M}_{d,1}$ with $u^{\mu^2} > 0$, one can find an $L \in \mathcal{L}_0(\mathbb{C})$ such that $L(u) \in (-C_+^0)$. Therefore, any real point $\theta = u + \mathrm{i}0$ with $\langle \theta, \theta \rangle > 0$ is contained in the extended tube $(\mathrm{T}_-)^\epsilon$. The converse can also be shown, that is, each $\theta \in \mathbb{M}_{d,1} \cap (\mathrm{T}_-)^\epsilon$ satisfies $\langle \theta, \theta \rangle > 0$.

For general N-fold extended tubes, we have the following due to Jost [73]: An N-tuple of real points $\Theta = (u_1, \ldots, u_N) + \mathrm{i}0$ lies in $(\mathrm{T}_-^N)^\epsilon$ if and only if all convex combinations are space-like:

$$\left(\sum_{n=1}^N \lambda_n u_n^\mu \right)^2 > 0, \qquad \text{with} \quad \sum_{n=1}^N \lambda_n = 1, \lambda_n \geq 0. \tag{2.45}$$

In particular, note that also the difference vectors u_n themselves are space-like, and hence the requirements for locality (W2) are fulfilled.

[15]This is the identity component of the group of complex matrices being orthogonal with respect to the complex scalar product (2.39). Note that $\mathcal{L}(\mathbb{C})$, countrary to $\mathcal{L}$, only contains two connected components with $\det(L) = \pm 1, L \in \mathcal{L}(\mathbb{C})$. As a side note, the CPT theorem is based on this peculiarity of the complex Lorentz group.

Let us now denote a permutation of the set $\{1, \ldots, N+1\}$ with $\sigma \in \mathcal{S}_{N+1}$. The Wightman distribution W_{N+1} and the permuted Wightman distribution

$$^{\sigma}\mathrm{W}_{N+1}(x_1, \ldots, x_{N+1}) := \mathrm{W}_{N+1}(x_{\sigma(1)}, \ldots, x_{\sigma(N+1)}) \tag{2.46}$$

can be analytically continued to their respective N-fold extended tubes $(\mathrm{T}_-^N)^\epsilon$ and $(^\sigma\mathrm{T}_-^N)^\epsilon$. Then, according to the previous step, $(\mathrm{T}_-^N)^\epsilon$ and $(^\sigma\mathrm{T}_-^N)^\epsilon$ have a common nonempty open and connected subset of real points u_n with $u_n^{\mu^2} > 0$. In addition, as a consequence of locality, W_{N+1} and $^\sigma\mathrm{W}_{N+1}$ agree on that common subset. Thus, the analytic domain of W_{N+1} can be further continued to the N-fold permuted extended tube

$$(\mathrm{T}_-^N)^{\epsilon,\sigma} := \bigcup\{(^\sigma\mathrm{T}_-^N)^\epsilon : \sigma \in \mathcal{S}_{N+1}\}. \tag{2.47}$$

This tube contains almost the full N-fold Euclidean space $\mathscr{E}_N$. Indeed, take any $(N+1)$-tuple of noncoincident points $(e_1, \ldots, e_n, \ldots, e_{N+1}) \in \mathrm{E}^{N+1}$, where $e_n = (\mathrm{i}x_n^0, \vec{x}_n)$. Applying a suitable permutation $\sigma \in \mathcal{S}_{N+1}$, one can choose an increasing order of the values x_n^0 appearing inside the zero-components of the points e_n. Thus, without restriction of generality, let us write

$$x_1^0 \leq x_2^0 \leq \cdots \leq x_{N+1}^0. \tag{2.48}$$

Moreover, because of the noncoincidence of the points, there exists a Lorentz transformation $L \in \mathcal{L}_0 \subset \mathcal{L}_0(\mathbb{C})$ such that $L(e_n)^0 \neq L(e_m)^0$ for all $n \neq m$. Accordingly, through a permutation σ, followed by a Lorentz transformation L, the above ordering can always be made strict:

$$x_1'^0 < x_2'^0 < \cdots < x_{N+1}'^0, \tag{2.49}$$

where $e_k' = (\mathrm{i}x_n'^0, \vec{x}\,'_k) = L(e_k)$. The difference points $e_n' - e_{n+1}'$ satisfy $x_n'^0 - x_{n+1}'^0 < 0$ and each of them is therefore element of T_-. We conclude that the difference points $(e_1 - e_2, \ldots, e_N - e_{N+1})$ are indeed contained in $(\mathrm{T}_-^N)^{\epsilon,\sigma}$ for all noncoincident points $(e_1, \ldots, e_{N+1}) \in \mathrm{E}^{N+1}$, or, equivalently, that U_N contains the whole space $\mathscr{E}_N \backslash \Delta_N$ with $\Delta_N = \{(h_1, \ldots, h_n, \ldots, h_N) \in \mathscr{E}_N : h_n \neq 0\}$.

Euclidean quantum field theory.

The Wick rotation corresponds to restricting the analytic continuation of the Wightman distributions W_N to the set $\mathscr{E}_{N-1}\backslash\Delta_{N-1}$. This defines the Schwinger distributions

$$\mathrm{S}_N(x_1, \ldots, x_N) := \mathrm{W}_N((\mathrm{i}x_1^0, \vec{x}_1), \ldots, (\mathrm{i}x_N^0, \vec{x}_N)), \tag{2.50}$$

where the scalar product of the points $x_n \in \mathbb{R}^{d+1}$ complies to the Euclidean scalar product $x_n^{\mu^2} = (x_n^0)^2 + \vec{x}_n^{\,2}$. These analytic distributions satisfy

S1: (*Euclidean covariance*) Invariance under transformations g of the isometry group of the Euclidean space $\mathbb{R}^{d+1}$:

$$\mathrm{S}_N(\mathrm{g}(x_1), \ldots, \mathrm{g}(x_N)) = \mathrm{S}_N(x_1, \ldots, x_N). \tag{2.51}$$

S2: (*Permutation property*) For any $\sigma \in \mathcal{S}_N$ it follows

$$\mathrm{S}_N(x_1, \ldots, x_N) = \mathrm{S}_N(x_{\sigma(1)}, \ldots, x_{\sigma(N)}). \tag{2.52}$$

S3: (*Laplace transformation*) At chronologically ordered points, i.e., for

$$(x_1, \ldots, x_N) \in \mathrm{R}_+^N := \{((x_1^0, \vec{x}_1), \ldots, (x_N^0, \vec{x}_N)) \in \left(\mathbb{R}^{(d+1)}\right)^N : 0 < x_1^0 < \cdots < x_N^0\},$$

the Schwinger distributions are related to the Wightman distributions through

$$\mathrm{S}_N(x_1, \ldots, x_N) = (2\pi)^{-(N-1)(d+1)} \int \left(\Pi_j \mathrm{d}^{d+1} q_j\right) \, e^{\sum_j \left(u_j^0 q_j^0 + \mathrm{i} \vec{u}_j \cdot \vec{q}_j\right)} \tilde{\mathrm{w}}_{N-1}(q_1, \ldots, q_{N-1}), \tag{2.53}$$

where $u_j = x_j - x_{j+1}$ and $j \in \{1, \ldots, N-1\}$.

S4: (*Reflection positivity*) For any sequence $\left(f_{(N)}\right)_{N \in \mathbb{N}}$ with $f_{(N)} \in \mathscr{S}(\mathrm{R}_+^N)$:

$$\sum_{M,N=0}^{k} \mathrm{S}_{\hat{M}+N}(\bar{f}_{(M)} \otimes f_{(N)}) \geq 0, \tag{2.54}$$

where $\mathrm{S}_{\hat{M}+N}(x_1, \ldots, x_M, y_1, \ldots, y_N) = \mathrm{S}_{M+N}(\hat{x}_1, \ldots, \hat{x}_M, y_1, \ldots, y_N)$ with $\hat{x}_m = (-x_m^0, \vec{x}_m)$.

S5: (*Cluster property*) For any Euclidean vector $\alpha \in \mathbb{R}_{d+1}$ the following holds:

$$\lim_{\lambda \to \infty} \mathrm{S}_N(\hat{x}_1, \ldots, \hat{x}_n, x_{n+1} + \lambda\alpha, \ldots, x_N + \lambda\alpha) = \mathrm{S}_n(\hat{x}_1, \ldots, \hat{x}_n)\mathrm{S}_{N-n}(x_{n+1}, \ldots, x_N). \tag{2.55}$$

The above properties are called Osterwalder–Schrader axioms and follow rather directly from properties (W1-W5). As for property (W5) regarding the Wightman functions, the validity of the cluster property for the Schwinger functions (S5) is equivalent to the uniqueness of the vacuum. Moreover, one can derive the properties (W1-W4) directly from the properties (W1-W4) and thus also the Schwinger functions allow to fully reconstruct a quantum field theory in the sense of the Wightman axioms (A1-A6).

Owing to the permutation property (S2), the Schwinger distributions can be regarded as correlation functions of commuting, hence classical, variables. This constitutes the basis for the rich relationship between quantum field theory and classical statistical field theory, see, for instance, ref. [94].

2.4 Free scalar quantum field theory

Here we construct a free neutral scalar quantum field theory from the physicist's viewpoint, similarly as was done in ref. [95]. Thus, the focus lies on the plain formulation of such a theory rather than on the discussion of mathematical subtleties[16]. Nevertheless, the pragmatic approach adopted here suffices to verify the fulfillment of the Wightman axioms, as we will see right afterwards.

[16]The latter will be a central issue in chapter III, where the theory is generalized to curved space-times.

Classical scalar field theory.

The starting point is a real scalar field $\phi(x)$ on $\mathbb{M}_{d,1}$. Under a general coordinate transformation, i.e., a diffeomorphic transformation $x \to x'$, the transformed scalar field ϕ' evaluated in the new coordinates x' retains its value at each point in space-time as given by the original scalar field ϕ in the old coordinates x, and hence

$$\phi(x) \to \phi'(x') = \phi(x). \tag{2.56}$$

As an example, let us consider the isometries $\mathbb{M}_{d,1}$. These are the transformations which leave the line element $\mathrm{d}s = \sqrt{\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}$ invariant. It comprises space-time translations

$$x^\mu \to x'^\mu = x^\mu + \alpha^\mu, \tag{2.57}$$

with $\alpha \in \mathbb{M}_{d,1}$, and space-time rotations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{2.58}$$

where Λ is a matrix satisfying $\Lambda^{\mu\rho}\Lambda^\nu{}_\rho = \eta^{\mu\nu}$. Note that there are $(d+2)(d+1)/2$ independent transformation parameters. Under these transformations, $\phi(x)$ transforms, respectively, as

$$\phi(x) \to \phi(x' - \alpha) \tag{2.59}$$

and

$$\phi(x) \to \phi(\Lambda^{-1} \cdot x'), \tag{2.60}$$

in agreement with eq. (2.56).

A quadratic classical action for a real scalar field $\phi(x)$, being invariant under isometries of $\mathbb{M}_{d,1}$, is given by

$$S_0 = -\frac{1}{2}\int \mathrm{d}^{d+1}x \left[(\partial\phi)\cdot(\partial\phi) + m^2\phi^2\right], \tag{2.61}$$

where $m > 0$ is the mass parameter. The application of the principle of least action, $\delta S_0 = 0$, leads to the Klein–Gordon equation

$$(-\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi(x) = 0. \tag{2.62}$$

This equation is known to pose a well-defined initial-value problem [96], which is a vital requirement, since it ensures the existence and uniqueness (see section 3.1) of advanced and retarded propagators, Green's functions of eq. (2.62) fulfilling particular conditions. We will introduce these functions later.

Let $\mathrm{V}_\mathbb{C}$ be the vector space of smooth complex scalar fields[17] satisfying eq. (2.62). With the purpose to write down the most general solution to eq. (2.62), one has to find a complete, orthonormal set of modes. This would then allow to express any element

[17] Additionally, we require these scalar fields to fall off sufficiently fast at spatial infinity in order to make the various integrals we introduce finite.

$\varphi \in \mathrm{V}_{\mathbb{C}}$ in terms of these modes. However, the notion of orthonormality necessitates a scalar product on $\mathrm{V}_{\mathbb{C}}$, which will be introduced as follows. Assume $\varphi_1, \varphi_2 \in \mathrm{V}_{\mathbb{C}}$. The conserved current

$$J_\mu(\varphi_1, \varphi_2) = -\mathrm{i}\left[\varphi_1^*(x)\partial_\mu\varphi_2(x) - \varphi_2(x)\partial_\mu\varphi_1^*(x)\right] \tag{2.63}$$

naturally defines an indefinite[18], sesquilinear scalar product on $\mathrm{V}_{\mathbb{C}}$:

$$(\varphi_1, \varphi_2) := \int \mathrm{d}^d x \; J^0(\varphi_1, \varphi_2) = \mathrm{i}\int \mathrm{d}^d x \left[\varphi_1^*(x)\partial_t\varphi_2(x) - \varphi_2(x)\partial_t\varphi_1^*(x)\right], \tag{2.64}$$

where $x^0 = t$. By construction, the scalar product is invariant under isometries of $\mathbb{M}_{d,1}$, and also independent on t. Now consider the set $\{u_{\vec{k}}\}$ of complex solutions to eq. (2.62), where

$$u_{\vec{k}}(x) = \frac{1}{\sqrt{2\omega(2\pi)^d}} e^{\mathrm{i}x\cdot k}, \tag{2.65}$$

with $k = (\omega, \vec{k}) \in \mathbb{M}_{d,1}$ satisfying

$$\omega = \sqrt{\vec{k}^{\,2} + m^2}. \tag{2.66}$$

Clearly, $u_{\vec{k}}$ with negative ω would solve eq. (2.62) as well. However, here we insist on the positivity of ω and instead include the set of complex conjugates $\{u_{\vec{k}}^*\}$. Since

$$\begin{aligned}
(u_{\vec{k}_1}, u_{\vec{k}_2}) &= \delta^{(d)}(\vec{k}_1 - \vec{k}_2), \\
(u_{\vec{k}_1}^*, u_{\vec{k}_2}^*) &= -\delta^{(d)}(\vec{k}_1 - \vec{k}_2), \\
(u_{\vec{k}_1}, u_{\vec{k}_2}^*) &= 0,
\end{aligned} \tag{2.67}$$

the modes $u_{\vec{k}}, u_{\vec{k}}^*$ form an orthonormal basis for $\mathrm{V}_{\mathbb{C}}$. Moreover, according to eq. (2.67), $\mathrm{V}_{\mathbb{C}}$ has a decomposition into a direct sum $\mathrm{V}_+ \oplus \mathrm{V}_-$ with, respectively, basis modes $u_{\vec{k}}$ and $u_{\vec{k}}^*$. On the vector space V_+, the scalar product is positive definite, whereas on V_- it is negative definite. A question that one might ask is if, under a continuous transformation of the isometry group of $\mathbb{M}_{d,1}$, the two vector spaces mix. It turns out that this does not happen, as a consequence of the fact that the modes are eigenstates of the operator $\mathrm{i}\partial_t$ appearing in eq. (2.64):

$$\mathrm{i}\partial_t u_{\vec{k}} = \omega u_{\vec{k}}\,, \qquad \mathrm{i}\partial_t u_{\vec{k}}^* = -\omega u_{\vec{k}}^*\,, \tag{2.68}$$

and no continuous transformation of the isometry group of $\mathbb{M}_{d,1}$ can reverse the sign[19] of t. As we will see later, the splitting of $\mathrm{V}_{\mathbb{C}}$ into *positive frequency solutions*[20], spanned by

[18] In nonrelativistic quantum mechanics, the analogue is positive definite, hence J^0 has the interpretation of a probability density. Here such an interpretation is not possible, and thus eq. (2.62) has to be considered a classical equation.

[19] This is easily checked by noting that the matrix Λ describing a continuous space-time rotation obeys $\det(\Lambda) = 1$ and ${\Lambda^0}_0 \geq 1$.

[20] This denomination has historical reasons, given the similarity of the eigenvalue equation with Schrödinger's equation.

basis modes with positive eigenvalues of $\mathrm{i}\partial_t$, and *negative frequency solutions*, spanned by modes with negative eigenvalues of $\mathrm{i}\partial_t$, is vital for a meaningful particle interpretation of the theory, as is affects the definition of the vacuum state and, hence, the Hilbert space.

Here we are especially interested in the real solutions to eq. (2.62), i.e., in the space $\mathrm{V} \subset \mathrm{V}_{\mathbb{C}}$ of real scalar fields ϕ. The set $\{u_{\vec{k}}, u^*_{\vec{k}}\}$ is still a basis of V, but with the requirement that the possible linear combinations of the basis modes are constrained. Indeed, the most general solution $\phi \in \mathrm{V}$, expressed in terms of the basis modes, reads

$$\phi(x) = \int \mathrm{d}^d k [a_{\vec{k}} u_{\vec{k}}(x) + a^*_{\vec{k}} u^*_{\vec{k}}(x)], \tag{2.69}$$

where the coefficient functions $a_{\vec{k}}, a^*_{\vec{k}}$ are related by complex conjugation in order to render ϕ a real scalar field.

Canonical quantization.

The system will be quantized in the canonical quantization scheme, which is one among several possible quantization schemes. Other noteworthy quantization techniques, especially in the prospect of QFT in curved space-times, are the C^* algebra approach and the path integral approach.[21] The former appears in the algebraic approach to QFT mentioned in section 2.1 and is particularly well suited to the rigorous treatment of the functional analysis ubiquitous in QFT. The latter is favorable in the quantization of interacting fields. However, the canonical quantization scheme bears a plain simplicity, as it resembles the standard quantization approach of nonrelativistic quantum mechanics.

To quantize the theory, we promote the classical scalar field $\phi(x)$ to an operator-valued distribution acting on a Hilbert space, and impose the (equal time) canonical commutation relations:

$$\begin{aligned} [\phi(t,\vec{x}_1), \pi(t,\vec{x}_2)] =& \mathrm{i}\delta^{(d)}(\vec{x}_1 - \vec{x}_2), \\ [\phi(t,\vec{x}_1), \phi(t,\vec{x}_2)] =& [\pi(t,\vec{x}_1), \pi(t,\vec{x}_2)] = 0, \end{aligned} \tag{2.70}$$

where the momentum operator π is the canonically conjugate variable to ϕ, that is,

$$\pi \equiv \frac{\delta S_0}{\delta(\partial_t \phi)} = \partial_t \phi. \tag{2.71}$$

Consequently, also the coefficient functions $a_{\vec{k}}, a^*_{\vec{k}}$ appearing in eq. (2.69) are promoted to operator-valued distributions $a_{\vec{k}}, a^\dagger_{\vec{k}}$, and thus the expansion of the quantized scalar field ϕ now reads[22]

$$\phi(x) = \int \mathrm{d}^d k [a_{\vec{k}} u_{\vec{k}}(x) + a^\dagger_{\vec{k}} u^*_{\vec{k}}(x)]. \tag{2.72}$$

[21] Detailed treatments of C^* algebras can be found in refs. [97, 98], and of path integrals in refs. [95, 74].

[22] The field is presented in the Heisenberg representation, where the operators representing observables evolve in time and the states do not.

Note that ϕ is selfadjoint. Plugging eq. (2.72) into eq. (2.70) yields

$$\begin{aligned} [a_{\vec{k}_1}, a^\dagger_{\vec{k}_2}] =& \delta^{(d)}(\vec{k}_1 - \vec{k}_2), \\ [a_{\vec{k}_1}, a_{\vec{k}_2}] =& [a^\dagger_{\vec{k}_1}, a^\dagger_{\vec{k}_2}] = 0. \end{aligned} \tag{2.73}$$

The Hilbert space.

The Hilbert space of the theory will be constructed explicitly by specifying the set of eigenfunctions of the Hamilton operator. The Hamiltonian can be directly derived from eq. (2.61) and is classically given by

$$H = \int \mathrm{d}^d x \, \frac{1}{2} \left[\pi^2 + (\partial_i \phi)^2 + m^2 \phi^2 \right]. \tag{2.74}$$

After quantization, one can express the Hamilton operator in terms of the operators $a_{\vec{k}}, a^\dagger_{\vec{k}}$:

$$H = \int \mathrm{d}^d k \, \frac{\omega}{2} \left(a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^\dagger_{\vec{k}} \right). \tag{2.75}$$

It is then straightforward to compute the following commutation relations:

$$[H, a_{\vec{k}}] = -\omega a_{\vec{k}} \, , \qquad [H, a^\dagger_{\vec{k}}] = \omega a^\dagger_{\vec{k}} \, . \tag{2.76}$$

These make it evident that, given an eigenstate of the Hamilton operator H with energy E, the operators $a_{\vec{k}}, a^\dagger_{\vec{k}}$ will, respectively, lower and raise the energy E by ω. Since the spectrum of the Hamilton operator is bounded from below by construction (see eq. (2.74)), the existence of a state of minimum energy is guaranteed. This state is called vacuum state $|0\rangle$ and it is, up to a phase, the unique state annihilated by all operators $a_{\vec{k}}$.

The similarity of eq. (2.76) with eq. (2.68) is not accidental. In fact, by virtue of (cf. eq. (2.4))

$$\mathrm{i}\partial_t \phi = [\phi, H], \tag{2.77}$$

the Hamilton operator H is related to the generator of space-time translations in the field representation $\mathrm{i}\partial_t$. Furthermore, from

$$a_{\vec{k}} \, \mathrm{i}\frac{\partial}{\partial t} u_{\vec{k}}(x) = [a_{\vec{k}}, H] u_{\vec{k}}(x), \qquad a^\dagger_{\vec{k}} \, \mathrm{i}\frac{\partial}{\partial t} u^*_{\vec{k}}(x) = [a^\dagger_{\vec{k}}, H] u^*_{\vec{k}}(x), \tag{2.78}$$

it is obvious that positive and negative frequency modes $u_{\vec{k}}$ and $u^*_{\vec{k}}$ are identified, respectively, with the *annihilation operators* $a_{\vec{k}}$ and the *creation operators* $a^\dagger_{\vec{k}}$. More precisely, as long as $u_{\vec{k}}$ is a positive frequency mode, $a_{\vec{k}}$ is an annihilation operator, and, analogously, as long as $u^*_{\vec{k}}$ is a negative frequency mode, $a^\dagger_{\vec{k}}$ is a creation operator. Furthermore, recall that the corresponding subspaces V_+ and V_- are invariant under continuous transformations of the isometry group of $\mathbb{M}_{d,1}$. This ensures that the vacuum

$|0\rangle$, defined as the state annihilated by all annihilation operators, is unaffected from such transformations. This invariance of the vacuum has two important physical implications. First, the vacuum is independent of the (inertial) observer[23]. Second, the vacuum does not evolve in time. A more detailed discussion will be given in chapter III.

Let us proceed by computing the energy of the vacuum state:

$$H|0\rangle = \frac{1}{2}\int \mathrm{d}^d k \; \omega \; \delta^{(0)}(0) \; |0\rangle. \tag{2.79}$$

The above formula reveals the problem that the energy of the vacuum state is infinite. One might be worried about the factor $\delta^{(0)}(0)$. However, performing the identical calculation on a space of finite volume L^d gives

$$\frac{1}{2}\int \mathrm{d}^d k \; \omega \; \delta^{(0)}(0) \to \frac{1}{2}\left(\frac{L}{2\pi}\right)^d \sum_{\vec{k}} \omega \,, \tag{2.80}$$

which is still divergent. Hence, the problem lies in the high frequencies, or, equivalently, in the short distances. Indeed, the divergence appears as a consequence of the fact that, being the Hamilton operator defined as a product of distributions at incident points, its definition in eq. (2.74) is faulty (cf. discussion in section 2.2). Luckily, this problem is easily circumvented by considering $:H:$ instead, where the double-dots denote the normal ordering operation, i.e., the prescription of moving all operators $a^\dagger_{\vec{k}}$ on the left of all operators $a_{\vec{k}}$ for each term enclosed in the double-dots. This leads to

$$H \equiv :H:= \int \mathrm{d}^d k \; \omega \; a^\dagger_{\vec{k}} a_{\vec{k}} \,, \tag{2.81}$$

effectively corresponding to a rescaling of the energy, even by an infinite amount, without affecting observable quantities. Indeed, now the energy of the vacuum state results, as expected, in $H|0\rangle = 0$.

Starting from the vacuum state $|0\rangle$, which has the interpretation of a state containing no particles, one can construct one-particle states via

$$|1_{\vec{k}}\rangle = a^\dagger_{1_{\vec{k}}}|0\rangle. \tag{2.82}$$

The denomination of the latter as one-particle states is not casual. Indeed, we interpret the state $|1_{\vec{k}}\rangle$ as the state describing the presence of one particle with momentum $\vec{k}$. Bearing in mind that the vacuum is, up to a phase, invariant under the isometry group of $\mathbb{M}_{d,1}$, this interpretation is valid at each instant and in every inertial frame. Requiring $\langle 0|0\rangle = 1$, the one-particle states satisfy the normalization property

$$\langle 1_{\vec{k}_1}|1_{\vec{k}_2}\rangle = \delta^{(d)}(\vec{k}_1 - \vec{k}_2). \tag{2.83}$$

[23] A noninertial observer would observe particles where an inertial observer would observe none. For example, the vacuum for a uniformly accelerating observer in empty space is seen as a thermal bath of nonvanishing temperature (Unruh effect), see ref. [19].

Since these form a complete set of eigenstates of the Hamilton operator H, they form a basis of the *one-particle Hilbert space* $\mathscr{H}_1$. Therefore, any state $|f\rangle \in \mathscr{H}_1$ can be expanded as

$$|f\rangle = \int \frac{\mathrm{d}^d k}{\sqrt{2\omega(2\pi)^d}}\, f(\vec{k})|1_{\vec{k}}\rangle. \tag{2.84}$$

This makes the inner product on $\mathscr{H}_1$ explicit:

$$\langle g|f\rangle = \int \frac{\mathrm{d}^d k}{2\omega(2\pi)^d} g^*(\vec{k}) f(\vec{k}), \tag{2.85}$$

where also $|g\rangle \in \mathscr{H}_1$. We deduce that $\mathscr{H}_1 = L^2(\mathbb{R}^d, \mathrm{d}^d k/2\omega(2\pi)^d)$, i.e., $\mathscr{H}_1$ is the (separable) space of square-integrable functions[24] over $\mathbb{R}^d$. In particular, note that the measure in eq. (2.85) is invariant under transformations of the subgroup $\mathcal{L}_0 \in \mathcal{P}$, since

$$\frac{\mathrm{d}^{d+1} k}{(2\pi)^d}\, \delta(k^{\mu^2} + m^2)\theta(\omega) = \frac{\mathrm{d}^d k}{2\omega(2\pi)^d}, \tag{2.86}$$

where θ denotes the Heaviside step function. In addition to the one-particle states, let us introduce many-particles states

$$|1_{\vec{k}_1}, 1_{\vec{k}_2}, \ldots, 1_{\vec{k}_j}\rangle = a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} \cdots a^\dagger_{\vec{k}_j} |0\rangle, \tag{2.87}$$

which, in fact, contain one-particle states along with the vacuum state $|0\rangle$. If any $a^\dagger_{\vec{k}}$ appears multiple times, then we set

$$|(n_1)_{\vec{k}_1}, (n_2)_{\vec{k}_2}, \ldots, (n_j)_{\vec{k}_j}\rangle = \frac{1}{\sqrt{n_1! n_2! \cdots n_j!}} (a^\dagger_{\vec{k}_1})^{n_1} (a^\dagger_{\vec{k}_2})^{n_2} \cdots (a^\dagger_{\vec{k}_j})^{n_j} |0\rangle, \tag{2.88}$$

in agreement to the Bose statistics of identical scalar particles. These multi-particles states also satisfy some normalization property:

$$\begin{aligned} &\langle (n'_1)_{\vec{k}'_1}, (n'_2)_{\vec{k}'_2}, \ldots, (n'_i)_{\vec{k}'_i} | (n_1)_{\vec{k}_1}, (n_2)_{\vec{k}_2}, \ldots, (n_j)_{\vec{k}_j}\rangle \\ &= \delta_{ij} \sum_{\sigma \in \mathcal{S}_j} \delta_{n'_1 n_{\sigma(1)}} \cdots \delta_{n'_j n_{\sigma(j)}}\, \delta^{(d)}(\vec{k}'_1 - \vec{k}_{\sigma(1)}) \cdots \delta^{(d)}(\vec{k}'_j - \vec{k}_{\sigma(j)}). \end{aligned} \tag{2.89}$$

Furthermore, it follows that

$$\begin{aligned} &a^\dagger_{\vec{k}_l} |(n_1)_{\vec{k}_1}, \ldots, (n_{l-1})_{\vec{k}_{l-1}}, (n_l)_{\vec{k}_l}, (n_{l+1})_{\vec{k}_{l+1}}, \ldots, (n_j)_{\vec{k}_j}\rangle \\ &= \sqrt{n_l + 1} |(n_1)_{\vec{k}_1}, \ldots, (n_{l-1})_{\vec{k}_{l-1}}, (n_l + 1)_{\vec{k}_l}, (n_{l+1})_{\vec{k}_{l+1}}, \ldots, (n_j)_{\vec{k}_j}\rangle, \end{aligned} \tag{2.90}$$

[24] More precisely, one needs to define equivalence classes; two square integrable functions are equivalent if they only differ on a set of measure zero.

and

$$\begin{aligned} a_{\vec{k}_l}&|(n_1)_{\vec{k}_1},\ldots,(n_{l-1})_{\vec{k}_{l-1}},(n_l)_{\vec{k}_l},(n_{l+1})_{\vec{k}_{l+1}},\ldots,(n_j)_{\vec{k}_j}\rangle \\ &= \delta^{(d)}(0)\sqrt{n_l}|(n_1)_{\vec{k}_1},\ldots,(n_{l-1})_{\vec{k}_{l-1}},(n_l-1)_{\vec{k}_l},(n_{l+1})_{\vec{k}_{l+1}},\ldots,(n_j)_{\vec{k}_j}\rangle. \end{aligned} \tag{2.91}$$

Note that the case $n_l = 0$ is not excluded in the above formulæ. The operator $N_{\vec{k}} := a^\dagger_{\vec{k}} a_{\vec{k}}$ acts on multiple-particles states in the following way:

$$N_{\vec{k}}|(n_1)_{\vec{k}_1},\ldots,(n_j)_{\vec{k}_j}\rangle = \sum_{i=1}^{j} \delta^{(d)}(\vec{k}-\vec{k}_i) n_i |(n_1)_{\vec{k}_1},\ldots,(n_j)_{\vec{k}_j}\rangle, \tag{2.92}$$

and therefore, the so-called *number operator*

$$N_{tot} = \int \mathrm{d}^d k N_{\vec{k}} \tag{2.93}$$

counts the total number of particles in a state $|(n_1)_{\vec{k}_1},\ldots,(n_j)_{\vec{k}_j}\rangle$. Analogously to the one-particle states being a basis for $\mathscr{H}_1$, the states in eq. (2.88) form a basis of another Hilbert space, the Fock space $\mathscr{F}$, defined as the Hilbert space completion of

$$\bigoplus_{n=0}^{\infty} S\left(\mathscr{H}_1^{\otimes n}\right) \equiv \mathbb{C} \oplus \mathscr{H}_1 \oplus S\left(\mathscr{H}_1 \otimes \mathscr{H}_1\right) \oplus S\left(\mathscr{H}_1 \otimes \mathscr{H}_1 \otimes \mathscr{H}_1\right) \oplus \ldots, \tag{2.94}$$

where the operator S symmetrizes each tensor product $\mathscr{H}_1^{\otimes n}$. This space is again separable, since one considers only states with finite particle content.

Symmetries.

Let us introduce the classical stress-tensor:

$$T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}\eta_{\mu\nu}\eta^{\kappa\rho}(\partial_\kappa\phi)(\partial_\rho\phi) - \frac{1}{2}m^2\eta_{\mu\nu}\phi^2; \tag{2.95}$$

which is the Noether current [81] associated to the invariance of S_0 under space-time translations. It determines the charges

$$P_\mu := -\int \mathrm{d}^d x\ T_{0\mu}, \tag{2.96}$$

with

$$T_{00} = \frac{1}{2}\left[(\partial_t\phi)^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right], \qquad T_{0i} = (\partial_t\phi)(\partial_i\phi), \tag{2.97}$$

where $i = 1,\ldots,d$. All these charges are conserved in the sense that $\partial_t P^\mu = 0$, and the charge P^0 agrees with the Hamiltonian H defined in eq. (2.74). Recall that, when considering the quantized charges, normal ordering is required.

Due to its implications, let us stress that H being a conserved charge confirms what stated earlier, that is, the vacuum does not evolve in time. In fact, since H is a conserved charge, the energy of each eigenstate of H is conserved for all values of t. Hence, being $|0\rangle$, up to a phase, the unique eigenstate of H of lowest energy, it does not change in time.

Expressing the $d+1$ charges in terms of the operators $a_{\vec{k}}, a^\dagger_{\vec{k}}$ gives

$$H = \int \mathrm{d}^d k \; \omega \; a^\dagger_{\vec{k}} a_{\vec{k}} \,, \qquad \vec{P} = \int \mathrm{d}^d k \; \vec{k} \; a^\dagger_{\vec{k}} a_{\vec{k}} \,. \tag{2.98}$$

This makes it clear that $\vec{P}$ corresponds to the momentum operator, as expected. By construction, the operators $a_{\vec{k}}, a^\dagger_{\vec{k}}$ act on the Fock space $\mathscr{F}$, and consequently also the Hamilton- and momentum operators given in eq. (2.98) do. The latter operators are selfadjoint, and therefore define unitary operators $U_t = e^{-\mathrm{i}Ht}$ and $U_{\vec{x}} = e^{\mathrm{i}\vec{P}\cdot\vec{x}}$. As we will see later, the operators H and $\vec{P}$ correspond to the generators of the subgroup $\mathcal{T} \subset \mathcal{P}$. Analogously to P^μ in eq. (2.96), the charges $J^{\mu\nu}$, determined by the Noether current associated to the invariance of S_0 under space-time rotations, can be found:

$$J_{\mu\nu} = -\int \mathrm{d}^d x \, (x_\mu T_{0\nu} - x_\nu T_{0\mu}) \,. \tag{2.99}$$

These $d(d+1)/2$ conserved charges, expressed in terms of $a_{\vec{k}}, a^\dagger_{\vec{k}}$, read

$$\begin{aligned} J^{ij} &= \mathrm{i} \int \mathrm{d}^d k \; a^\dagger_{\vec{k}} \left(k^j \frac{\partial}{\partial k^i} - k^i \frac{\partial}{\partial k^j} \right) a_{\vec{k}} \,, \\ J^{i0} &= \mathrm{i} \int \mathrm{d}^d k \; a^\dagger_{\vec{k}} \left(\omega \frac{\partial}{\partial k^i} + \frac{k^i}{2\omega} \right) a_{\vec{k}} \,. \end{aligned} \tag{2.100}$$

Again, they are selfadjoint, and hence define themselves a unitary operator $U_\Omega = e^{-\frac{\mathrm{i}}{2}\Omega_{\mu\nu}J^{\mu\nu}}$, where Ω is a real antisymmetric matrix in $d+1$ dimensions.

By Stone's theorem, the unitary operators defined via the selfadjoint operators $P^\mu, J^{\mu\nu}$ are unitary representations of some Lie group, which is easily revealed by its Lie algebra. Indeed, the commutation relations

$$\begin{aligned} [P^\mu, P^\nu] &= 0, \\ [J^{\mu\nu}, P^\lambda] &= \mathrm{i} \left(\eta^{\mu\lambda} P^\nu - \eta^{\nu\lambda} P^\mu \right) , \\ [J^{\mu\nu}, J^{\lambda\rho}] &= \mathrm{i} \left(\eta^{\mu\lambda} J^{\nu\rho} - \eta^{\nu\lambda} J^{\mu\rho} - \eta^{\mu\rho} J^{\nu\lambda} + \eta^{\nu\rho} J^{\mu\lambda} \right) , \end{aligned} \tag{2.101}$$

describe the Lie algebra of the Poincaré group. So we conclude that we found all $(d+2)(d+1)/2$ generators of $\mathcal{P}$, and P^μ generate the group $\mathcal{T}$ of space-time translations whereas $J^{\mu\nu}$ generate the Lorentz group $\mathcal{L}$ of space-time rotations.

The propagators.

The inhomogeneous Klein–Gordon equation, that is, eq. (2.62) altered by a smooth source term $j(x)$ inserted on the right hand side:

$$(-\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi(x) = j(x), \tag{2.102}$$

remains a well-posed initial value problem. With the aid of Green's function, defined via

$$\left(-\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} + m^2\right) G(x, y; m) = \delta^{(d+1)}(x - y), \tag{2.103}$$

solutions to eq. (2.102) are generated by

$$\phi(x) = \phi_h(x) + \int \mathrm{d}^{d+1}y \; G(x, y; m)j(y). \tag{2.104}$$

The field $\phi_h(x)$ is required to obey the homogeneous Klein–Gordon equation given by eq. (2.62) and is chosen such that $\phi(x)$ satisfies the initial conditions. Most frequently, under a general coordinate transformation $x \to x', y \to y'$, Green's function $G(x, y; m)$ is assumed to transform as $G(x, y; m) \to G'(x', y'; m) = G(x, y; m)$, in analogy to the transformation property of the scalar field. This assumption makes eq. (2.103) invariant under the isometry group of $\mathbb{M}_{d,1}$ and further constrains the relationship between x and y in the dependence of $G(x, y; m)$. For instance, space-time translation symmetry restricts the dependence to the difference vector $x - y$. The strategy adopted here to solve eq. (2.103) is by Fourier transforming Green's function:

$$G(x - y; m) = \frac{1}{(2\pi)^{d+1}} \int \mathrm{d}^{d+1}p \; \tilde{G}(p; m) e^{\mathrm{i}(x-y)\cdot p}, \tag{2.105}$$

which simplifies eq. (2.103) to

$$\tilde{G}(p; m)(p^{\mu^2} + m^2) = 1. \tag{2.106}$$

This equation can be resolved for $\tilde{G}(p; m)$:

$$\tilde{G}_\pm(p; m) = \frac{-1}{(p^0 \pm \mathrm{i}\delta)^2 - \vec{p}^2 - m^2}, \tag{2.107}$$

where $\delta > 0$ is a small parameter coping with the zeros of the expression $p^{\mu^2} + m^2$. The effect of δ can be seen on

$$G_\pm(x - y; m) = -\frac{1}{(2\pi)^{d+1}} \int \mathrm{d}^{d+1}p \; \frac{e^{\mathrm{i}(x-y)\cdot p}}{(p^0 \pm \mathrm{i}\delta)^2 - \vec{p}^2 - m^2}, \tag{2.108}$$

where the integral over p^0 can be carried out using the contour integration method. Hence, δ is equivalent to a slight deformation of the contour in order to avoid the poles of the integrand. This leads to

$$G_\pm(x-y;m) = \pm\mathrm{i}\theta(\pm(x^0-y^0)) \int \frac{\mathrm{d}^d p}{2\omega_{\vec{p}}(2\pi)^d} \left[e^{\mathrm{i}(x-y)\cdot p} - e^{-\mathrm{i}(x-y)\cdot p}\right], \tag{2.109}$$

where now $p = (\omega_{\vec{p}}, \vec{p})$ with $\omega_{\vec{p}} = (\vec{p}^2 + m^2)^{1/2}$. The Green's functions $G_+(x-y;m)$ and $G_-(x-y;m)$ are called, respectively, *advanced and retarded propagator*, having the property that $G_+(x-y;m)$ vanishes for $x^0 > 0$ and $G_-(x-y;m)$ for $x^0 < 0$. Furthermore, both vanish for space-like distances between x and y.

The choice of the integration contour is clearly not restricted to those two cases. For example, another choice is given by

$$\begin{aligned} G_F(x-y;m) &= -\frac{1}{(2\pi)^{d+1}} \int \mathrm{d}^{d+1}p \, \frac{e^{\mathrm{i}(x-y)\cdot p}}{(p^0)^2 - \vec{p}^2 - m^2 + \mathrm{i}\delta} \\ &= \mathrm{i}\theta(x^0-y^0) \int \frac{\mathrm{d}^d p}{2\omega_{\vec{p}}(2\pi)^d}\, e^{\mathrm{i}(x-y)\cdot p} + \mathrm{i}\theta(-x^0+y^0)) \int \frac{\mathrm{d}^d p}{2\omega_{\vec{p}}(2\pi)^d}\, e^{-\mathrm{i}(x-y)\cdot p}, \end{aligned} \tag{2.110}$$

with the same p as above. This Green's function, called *Feynman propagator*, is exponentially falling off for space-like distances. Nevertheless, no matter how the contour is chosen, all propagators differ at most by a contribution corresponding to a solution to eq. (2.62).

Green's functions naturally appear in quantum field theory. An example is given by the field commutator at different space-time points $x, y \in \mathbb{M}_{d,1}$:

$$[\phi(x), \phi(y)] = \int \frac{\mathrm{d}^d k}{2\omega(2\pi)^d} \left[e^{\mathrm{i}(x-y)\cdot k} - e^{-\mathrm{i}(x-y)\cdot k}\right], \tag{2.111}$$

which can be expressed in terms of $G_\pm(x-y)$:

$$[\phi(x), \phi(y)] = -\mathrm{i}\left[G_+(x-y;m) - G_-(x-y;m)\right]. \tag{2.112}$$

The Wightman axioms.

Now we have all the ingredients necessary to a quantum field theory; a Hilbert space $\mathscr{F}$ being the module of a unitary representation of the Poincaré group $\mathcal{P}$, and a self-adjoint operator-valued distribution $\phi(x)$ over $\mathbb{M}_{d,1}$. The Wightman axioms will be verified one by one:

- Applying P^μ on some basis vector of $\mathscr{F}$ yields

$$P^\mu |(n_1)_{\vec{k}_1}, \ldots, (n_j)_{\vec{k}_j}\rangle = \sum_{i=1}^{j} n_i k_i^\mu |(n_1)_{\vec{k}_1}, \ldots, (n_j)_{\vec{k}_j}\rangle, \tag{2.113}$$

where each k_i^μ satisfies $k_i^{\mu^2} = -m^2 \leq 0, k_i^0 \geq 0$. In accordance with the spectrum condition (A1), the spectrum of P^μ is confined to the forward cone C_+. This follows from $\sum_{i=1}^{j} n_i k_i^0 \geq 0$ and

$$- (\sum_{i=1}^{j} n_i k_i^0)^2 + (\sum_{i=1}^{j} n_i \vec{k}_i)^2 \leq -(\sum_{i=1}^{j-1} n_i k_i^0)^2 + (\sum_{i=1}^{j-1} n_i \vec{k}_i)^2 \leq \cdots \leq 0. \tag{2.114}$$

In eq. (2.114) we repeatedly used the fact that one can always choose a coordinate system such that $\vec{k}_j = 0, k_i^0 = m^2 \geq 0$.

- Being $\phi(x)$ defined via the operators $a_{\vec{k}}, a_{\vec{k}}^\dagger$, its domain is the space spanned by the basis vectors $|(n_1)_{\vec{k}_1}, \ldots, (n_j)_{\vec{k}_j}\rangle$. This space is, by construction, a dense subspace of $\mathscr{F}$ which is clearly invariant under $\mathcal{P}$ and $\phi(x)$ itself. This confirms the domain condition (A2).

- As the vacuum condition (A3) requires, the only state in $\mathscr{F}$ invariant under $\mathcal{T}$ is indeed $|0\rangle$. The vacuum state is also invariant under $\mathcal{L}$.

- The completeness condition (A4) is fulfilled, since the set all $\phi(f_1)\cdots\phi(f_N)|0\rangle$ simply corresponds to a basis of $\mathscr{F}$ in position space.

- For space-time translations, the covariance condition (A5) follows directly from the action of the unitary representations of $\mathcal{P}$ on $\phi(x)$. For the subgroup $\mathcal{T}$ one gets

$$U_t \phi(t, \vec{x}) U_{-t} = \phi(0, \vec{x}), \qquad U_{\vec{x}} \phi(t, \vec{x}) U_{-\vec{x}} = \phi(t, 0), \tag{2.115}$$

and for the subgroup $\mathcal{L}$ the result is

$$U_\Omega \phi(x) U_{-\Omega} = \phi({\Lambda^\mu}_\nu x^\nu), \tag{2.116}$$

where

$${\Lambda^\mu}_\nu = e^{-\frac{\mathrm{i}}{2}\Omega_{\alpha\beta}\mathrm{i}\left(\eta^{\alpha\mu}\delta^\beta_\nu - \delta^\alpha_\nu \eta^{\beta\mu}\right)}. \tag{2.117}$$

Since $\Lambda^{\mu\rho}{\Lambda^\nu}_\rho = \eta^{\mu\nu}$, the matrix ${\Lambda^\mu}_\nu$ is indeed an element of $\mathcal{L}$ in the vector representation.

- If $(x-y)$ is space-like, it is possible to find an element in $\mathcal{L}_0$ that transforms $(x-y)$ to $-(x-y)$. By transforming only the second term in eq. (2.111), the field commutator $[\phi(x), \phi(y)]$ vanishes identically, and hence the microcausality condition (A6) is fulfilled. Note that, for time-like $(x-y)$, that is, for $(x-y)$ satisfying $(x-y)^{\mu^2} < 0$, there is no such continuous Lorentz transformation, and the commutator does not vanish.

We thus infer that the quantum field theory of a free neutral scalar field satisfies all the Wightman axioms. More sophisticated quantum field theories in flat space-time are expected to satisfy the axioms as well, even though the direct verification is commonly a highly arduous task.

Generalized free fields.

We conclude this section by mentioning the existence of a particular class of QFTs whose field commutators yield numerical distributions like in the case of a free field theory (see eq. (2.111)). As distinct to the latter, these fields are not necessarily solutions to a linear equation of motion. First introduced by Greenberg [65], the so-called *generalized free fields* are equivalently characterized by correlation functions all factorizing into two-point functions. Another characterization can be indicated with the aid of the Källén–Lehmann representation. Specifically, let us rewrite the two-point function of any (also interacting) scalar QFT as [81]

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \mathrm{d}M^2\ \rho(M^2)G_F(x-y;M) \qquad \text{for } x^0 > y^0, \tag{2.118}$$

where $G_F(x-y;M)$ is the Feynman propagator of a free scalar field of mass M, see eq. (2.110), and $\rho(M^2)$ is a positive spectral density function. Then, a sufficient condition for a field to be generalized free is that its two-point function has support within a finite interval of masses [99].

Generalized free fields appear in various contexts; in $O(N)$- or $U(N)$-symmetric theories they appear in the large N limit [100]. Also restricting a free scalar field of mass m on a time-like hypersurface, as, for instance, the one defined by $x^d = 0$, leads to a generalized free field. The resulting spectral density is given by $\rho(M^2) = 1/\sqrt{M^2 - m^2}$, supported at $M^2 \geq m^2$. Since this corresponds to a continuous superposition of masses in the Källén–Lehmann representation, the theory does not admit a description in terms of associated particles. Therefore, no associated Lagrangian and no (canonical) stress tensor can be found [101]. Nevertheless, as long as the two-point function of the generalized free field theory satisfies the properties (W1-W5), the Hilbert space and the field is obtained uniquely from the Wightman reconstruction theorem discussed in section 2.2. This exemplifies a Wightman theory where the identification between field and particle breaks down.

Generalized free fields emerge in the context of the AdS/CFT correspondence as well. As we will see in section 5.3, they describe the boundary dual of a free scalar field in anti–de Sitter space-time. An exhaustive discussion of generalized free fields, also in connection to the AdS/CFT correspondence, can be found in ref. [101].

III QFT in curved space-times

As it was explicitly displayed in section 2.4, the formulation of a quantum field theory is based on the following main ingredients:

- An equation of motion of the classical theory, specified for example via the classical action, and its solution space.
- A quantization procedure, such as the canonical procedure or the path integral approach.
- The characterization of the Hilbert space.
- The interpretation of the states and of the observables.

It turns out that, independently of the shape of the manifold the quantum field theory is defined on, these ingredients are essential for the construction and the comprehension of a quantum field theory. With section 2.4 as a guideline, the free neutral scalar quantum field theory will be generalized to curved space-times, paying particular attention to the above ingredients. We will however focus on a particular class of space-times, which include many physically interesting curved space-times as well as $\mathbb{M}_{d,1}$. Eventually, we discuss the generalization of the Wightman axioms on this class of space-times.

3.1 Free scalar quantum field theory

In the framework of General Relativity, space-times are represented by pseudo-Riemannian manifolds. These are differentiable manifolds, each endowed with a nondegenerate, symmetric metric. A differential structure allows to define differentiable functions and differentiable curves on the manifold. In particular, each differentiable curve through a point on the manifold defines the directional derivative $\mathrm{d}f/\mathrm{d}s$ of smooth functions f at that point, where s is the parameter along the curve. This naturally induces a vector bundle whose space of sections are tangent spaces; at each point of the manifold, one can attach a tangent space, defined as the space of directional derivatives of all curves passing through that point. Thus, by the chain rule, a basis of each tangent space is given by the set of partial derivatives with respect to the local coordinates. For a detailed discussion of these concepts, we refer to refs. [102, 103].

In the following, let $\mathrm{M}_{d,1}$ be a pseudo-Riemannian manifold in $d+1$ dimensions, where the associated metric $\underline{g_{\mu\nu}}$ presents the same signature as the metric $\eta_{\mu\nu}$ of $\mathbb{M}_{d,1}$. Such manifolds are called *Lorentzian*. In other words, at each point $\mathrm{p} \in \mathrm{M}_{d,1}$, the tangent space is actually given by $\mathbb{M}_{d,1}$. Clearly, the flat space-time $\mathbb{M}_{d,1}$ itself is a $(d+1)$-dimensional Lorentzian manifold. Furthermore, in order to avoid complications concerning calculus, let us assume that $\mathrm{M}_{d,1}$ has a smooth structure and a smooth metric.

Classical scalar field theory.

It is relatively simple to generalize the classical equation of motion to a (smooth) pseudo-Riemannian manifold $\mathrm{M}_{d,1}$ in a local and covariant way. Driven by the general covariance principle, which in plain terms states that there is no preferred coordinate system, the generalization of eq. (2.61) to $\mathrm{M}_{d,1}$ is required to be invariant under general coordinate transformations. Therefore, for a real scalar field $\phi(x)$ on $\mathrm{M}_{d,1}$ transforming as $\phi(x) \to \phi'(x') = \phi(x)$ under a general coordinate transformation $x \to x'$, the quadratic classical action reads[1]

$$S_0 = -\frac{1}{2}\int \mathrm{d}^{d+1}x\sqrt{-g}\left[g^{\underline{\mu\nu}}(\nabla_{\underline{\mu}}\phi)(\nabla_{\underline{\nu}}\phi) + m^2\phi^2\right], \tag{3.1}$$

where g is the determinant of $g_{\underline{\mu\nu}}$ and $\nabla_{\underline{\mu}}$ is any affine connection on $\mathrm{M}_{d,1}$. For the sake of simplicity, we take $\nabla_{\underline{\mu}}$ to be the Levi-Civita connection, which is the unique connection being both metric compatible[2] and torsion free[3]. In particular, the Levi-Civita connection can be expressed in terms of solely partial derivatives, plus the metric and partial derivatives thereof [102].

It is worth to mention that the above action was specified with respect to some coordinate system, but the choice of coordinates is by no means distinguished. Furthermore, on general manifolds $\mathrm{M}_{d,1}$, several coordinate patches with associated $g_{\underline{\mu\nu}}$ may be required to cover the entire manifold.

Setting the variation of the action (3.1) with respect to ϕ equal to zero yields

$$(-\Box + m^2)\phi(x) = 0, \tag{3.2}$$

where $\Box := g^{\underline{\mu\nu}}\nabla_{\underline{\mu}}\nabla_{\underline{\nu}}$ is given by the known formula

$$\Box = \frac{1}{\sqrt{-g}}\partial_{\underline{\mu}}\sqrt{-g}\; g^{\underline{\mu\nu}}\partial_{\underline{\nu}}. \tag{3.3}$$

Unfortunately, the existence and uniqueness properties of the solution to eq. (3.2) are not unconditionally given on $\mathrm{M}_{d,1}$. In order to have a well-posed initial value formulation, as this is the case in $\mathbb{M}_{d,1}$, we set some conditions[4] on $\mathrm{M}_{d,1}$:

- First we restrict attention to time-orientable manifolds. At each point $\mathrm{p} \in \mathrm{M}_{d,1}$, two time-like vectors $X^{\underline{\mu}}, Y^{\underline{\mu}}$ are said to be equivalent if $g_{\underline{\mu\nu}}X^{\underline{\mu}}Y^{\underline{\nu}} < 0$. This is

[1] Note that, while it is simple to formulate an action invariant under general coordinate transformations, this does not render the theory invariant under diffeomorphisms (as General Relativity is). The former transformations are passive transformations, and the metric is transformed accordingly to keep distances between points equal. The latter are considered active transformations, moving points on the manifold. This consequently alters the distances between these points.

[2] $\nabla_{\underline{\sigma}}\, g_{\underline{\mu\nu}} = 0$

[3] $X^{\underline{\mu}}\nabla_{\underline{\mu}}Y^{\underline{\nu}} - Y^{\underline{\mu}}\nabla_{\underline{\mu}}X^{\underline{\nu}} = X^{\underline{\mu}}\partial_{\underline{\mu}}Y^{\underline{\nu}} - Y^{\underline{\mu}}\partial_{\underline{\mu}}X^{\underline{\nu}}$ for any vector fields $X^{\underline{\mu}}, Y^{\underline{\mu}}$ on $\mathrm{M}_{d,1}$

[4] Nevertheless, there exist generalizations of quantum field theories to space-times not admitting well-posed initial value problems, see, for instance, ref. [104].

indeed an equivalence relation defining two equivalence classes, whose vectors are, respectively, referred to with future- and past-directed. Which of the two classes is identified with future-directed vectors is arbitrary. By definition, a time-orientable manifold is a manifold where a continuous designation of future- and past-directed time-like vectors can be made on the whole space-time.

- Second, we consider manifolds which admit a Cauchy surface. Specifically, take a closed set $\Sigma \subset \mathrm{M}_{d,1}$ which is achronal, i.e., no pair of points in $\mathrm{p}, \mathrm{q} \in \mathrm{M}_{d,1}$ can be joined by a time-like curve. The domain of dependence is defined as follows:

$$D(\Sigma) = \{\mathrm{p} \in \mathrm{M}_{d,1} : \text{ every inextensible causal curve through p intersects } \Sigma\},$$

where inextensible curves do not possess endpoints and causal curves have time- or null-like tangent vectors. If $D(\Sigma) = \mathrm{M}_{d,1}$, then Σ is said to be a Cauchy surface of $\mathrm{M}_{d,1}$.

A manifold $\mathrm{M}_{d,1}$ satisfying both requirements is called *globally hyperbolic*. Now we are able to formulate a well-defined initial-value problem for eq. (2.62) making use of the following theorem [103]:

Given a linear, hyperbolic partial differential equation of n-th order for $\phi(x)$ on a globally hyperbolic manifold $\mathrm{M}_{d,1}$, and a set of arbitrary smooth initial data $\{(n^{\underline{\mu}}\nabla_{\underline{\mu}})^k\phi|_\Sigma : k = 0,\ldots,n-1\}$ defined on a smooth Cauchy surface Σ of $\mathrm{M}_{d,1}$, where $n^{\underline{\mu}}$ denotes the future-directed time-like unit vector normal to Σ. Then, there exists a unique solution $\phi(x)$ throughout $\mathrm{M}_{d,1}$. Moreover, a change of the initial data on any closed set $\Sigma' \subset \Sigma$ only influences the solution $\phi(x)$ on $D(\Sigma')$. In addition, $\phi(x)$ is smooth and depends continuously[5] on the initial data.

As eq. (2.62) is a hyperbolic partial differential equation of second order, it poses a well-defined initial-value problem if supplied with $\phi|_\Sigma$ and $n^{\underline{\mu}}\nabla_{\underline{\mu}}\phi|_\Sigma$. Furthermore, the above theorem continues to hold if eq. (2.62) is altered by a smooth source term $j(x)$ inserted on the right hand side. This ensures the existence and uniqueness of advanced and retarded propagators.

In what follows, let us restrict our analysis to a globally hyperbolic manifold $\mathrm{M}_{d,1}$, and let Σ be a Cauchy surface of $\mathrm{M}_{d,1}$ with the future-directed time-like unit vector normal $n^{\underline{\mu}}$. We want to keep things related as closely as possible to section 2.4. Hence, we would now introduce a complete, orthonormal set of modes allowing us to expand the most general solution to eq. (3.2), and afterwards quantize the theory by interpreting the coefficient functions as operators. Let us see where this procedure could possibly break down.

[5] with a suitable Sobolev space topology [96]

A scalar product on $\mathrm{V}_{\mathbb{C}}$, the vector space of smooth complex scalar fields[6] satisfying eq. (3.2), can again be defined via the conserved current in eq. (2.63), this time expressed in a covariant form:

$$J_{\underline{\mu}}(\varphi_1, \varphi_2) = -\mathrm{i}\left[\varphi_1^*(x)\nabla_{\underline{\mu}}\varphi_2(x) - \varphi_2(x)\nabla_{\underline{\mu}}\varphi_1^*(x)\right], \tag{3.4}$$

with $\varphi_1, \varphi_2 \in \mathrm{V}_{\mathbb{C}}$. This leads to the invariant, indefinite, sesquilinear scalar product

$$(\varphi_1, \varphi_2)_\Sigma = \mathrm{i}\int_\Sigma \mathrm{d}^d x\sqrt{|h|}\; n^{\underline{\mu}}\left[\varphi_1^*(x)\nabla_{\underline{\mu}}\varphi_2(x) - \varphi_2(x)\nabla_{\underline{\mu}}\varphi_1^*(x)\right], \tag{3.5}$$

where h is the determinant of the induced metric on Σ:

$$h_{\underline{\mu\nu}} = g_{\underline{\mu\nu}} + n_{\underline{\mu}}n_{\underline{\nu}}. \tag{3.6}$$

It might appear that eq. (3.5) is dependent on Σ. However, this is not the case as can be seen as follows. Let Σ_1 and Σ_2 be two different, nonintersecting Cauchy surfaces. If V is the $(d+1)$-dimensional hypervolume bounded by Σ_1 and Σ_2, one finds

$$(\varphi_1, \varphi_2)_{\Sigma_2} - (\varphi_1, \varphi_2)_{\Sigma_1} = \mathrm{i}\oint_{\partial V} \mathrm{d}^d x\sqrt{|h|}\; n^{\underline{\mu}}\left[\varphi_1^*(x)\nabla_{\underline{\mu}}\varphi_2(x) - \varphi_2(x)\nabla_{\underline{\mu}}\varphi_1^*(x)\right]. \tag{3.7}$$

Eventually, the application of Gauss's law yields

$$(\varphi_1, \varphi_2)_{\Sigma_2} - (\varphi_1, \varphi_2)_{\Sigma_1} = \mathrm{i}\int_V \mathrm{d}^{d+1} x\sqrt{-g}\;\nabla^{\underline{\mu}}\left[\varphi_1^*(x)\nabla_{\underline{\mu}}\varphi_2(x) - \varphi_2(x)\nabla_{\underline{\mu}}\varphi_1^*(x)\right], \tag{3.8}$$

which, as expected, vanishes due to φ_1 and φ_2 being solutions of the Klein–Gordon equation.

Because of global hyperbolicity of $\mathrm{M}_{d,1}$, it is always possible to find a set $\{u_\lambda, u_\lambda^*\}$ of complex solutions to eq. (3.2) and corresponding complex conjugates, satisfying

$$\begin{aligned}
(u_{\lambda_1}, u_{\lambda_2})_\Sigma &= \delta_{\lambda_1\lambda_2},\\
(u_{\lambda_1}^*, u_{\lambda_2}^*)_\Sigma &= -\delta_{\lambda_1\lambda_2},\\
(u_{\lambda_1}, u_{\lambda_2}^*)_\Sigma &= 0,
\end{aligned} \tag{3.9}$$

where λ might denote a discrete or continuous index, but for the moment we adopt the notation pertinent to the discrete case. If the set $\{u_\lambda, u_\lambda^*\}$ is complete, it forms a basis of $\mathrm{V}_{\mathbb{C}}$, and any real scalar field $\phi \in \mathrm{V} \subset \mathrm{V}_{\mathbb{C}}$ can be expanded as

$$\phi(x) = \sum_\lambda \left[a_\lambda u_\lambda(x) + a_\lambda^* u_\lambda^*(x)\right]. \tag{3.10}$$

Furthermore, $\mathrm{V}_{\mathbb{C}}$ has a decomposition into a direct sum $\mathrm{V}_+ \oplus \mathrm{V}_-$ with, respectively, basis modes u_λ and u_λ^*. So far, so good. Though, a main issue arises here. As done in

[6] If Σ is noncompact, then we require these scalar fields to fall off sufficiently fast at spatial infinity.

flat space-time, we would like to interpret V_+ as the vector space of positive frequency solutions and V_- as the vector space of negative frequency solutions, such that these vector spaces are invariant under general coordinate transformations. However, contrary to flat space-time, on $M_{d,1}$ a generic notion for positive/negative frequency solutions does not even exist. For instance, one might be tempted to search for eigenstates of the operator $\mathrm{i}n^{\underline{\mu}}\nabla_{\underline{\mu}}$ appearing in eq. (3.5), which effectively amounts to declare $n^{\underline{\mu}}$ as the direction of time. This is, in principle, allowed. But, owing to the general covariance principle, in $M_{d,1}$ no preferred time direction exists at all, and a different choice of the direction of time would give rise to a different splitting of $V_{\mathbb{C}}$.

So how could this riddle possibly be solved? Actually, for general space-times, this turns out to be a hard task, see ref. [105]. We might scale down our expectations and ask that, at least, there is a splitting of $V_{\mathbb{C}}$ which is preserved during time evolution. As we have seen in section 2.4, this is tantamount to requiring that the Hamiltonian H is a conserved charge, which was itself a consequence of the action being invariant under time translations as part of the isometry group of $\mathbb{M}_{d,1}$. On general space-times $M_{d,1}$, the action is invariant under general coordinate transformations. These are described by an infinite-dimensional Lie group, but surely one does not expect an infinite number of conserved charges. In fact, even if on $M_{d,1}$ the classical stress-tensor

$$T_{\underline{\mu\nu}} = (\nabla_{\underline{\mu}}\phi)(\nabla_{\underline{\nu}}\phi) - \frac{1}{2}g_{\underline{\mu\nu}}g^{\underline{\sigma\rho}}(\nabla_{\underline{\sigma}}\phi)(\nabla_{\underline{\rho}}\phi) - \frac{1}{2}m^2 g_{\underline{\mu\nu}}\phi^2 \tag{3.11}$$

is a well defined conserved current, i.e., $\nabla_{\underline{\mu}}T^{\underline{\mu\nu}} = 0$, a charge is only conserved if it is independent of the choice of the Cauchy surface Σ it is integrated over. Surely, one could construct currents $K^{\underline{\mu}}T_{\underline{\mu\nu}}$, associated to any vector field $K \equiv K^{\underline{\mu}}\partial_{\underline{\mu}}$, whose charge

$$Q_K := -\int_\Sigma \mathrm{d}^d x\sqrt{|h|}\; K^{\underline{\mu}}n^{\underline{\nu}}T_{\underline{\mu\nu}} \tag{3.12}$$

is manifestly covariant. However, Q_K is not conserved unless $K^{\underline{\mu}}T_{\underline{\mu\nu}}$ is conserved, or equivalently, unless[7]

$$\nabla_{\underline{\mu}}K_{\underline{\nu}} + \nabla_{\underline{\nu}}K_{\underline{\mu}} = 0. \tag{3.13}$$

Vector fields satisfying eq. (3.13) are called *Killing vector fields*. Another equipollent characterization is the following. The Lie derivative $\mathscr{L}$ of the metric $g_{\underline{\mu\nu}}$ along a Killing vector field $K^{\underline{\mu}}$ vanishes [102], that is,

$$\mathscr{L}_K g_{\underline{\mu\nu}} \equiv K^{\underline{\sigma}}\partial_{\underline{\sigma}}g_{\underline{\mu\nu}} + g_{\underline{\sigma\nu}}\partial_{\underline{\mu}}K^{\underline{\sigma}} + g_{\underline{\mu\sigma}}\partial_{\underline{\nu}}K^{\underline{\sigma}} = 0, \tag{3.14}$$

where $K = K^{\underline{\mu}}\partial_{\underline{\mu}}$. The isometry group of $M_{d,1}$ can therefore be found by identifying all its Killing vector fields $K^{\underline{\mu}}$.

[7] This is again shown by subtracting Q_K integrated over Σ_1 with Q_K integrated over Σ_2 and using Gauss's law.

A space-time $M_{d,1}$ having a global, nonvanishing, time-like Killing vector field $K_t^{\underline{\mu}}$ is called *stationary*. Thus, stationary symmetry allows to pick a preferred time direction, given by the time-like Killing vector field. Furthermore, such a $K_t \equiv K_t^{\underline{\mu}} \partial_{\underline{\mu}}$ commutes with the Klein-Gordon operator $-\Box + m^2$, and is anti-hermitian[8]:

$$(\varphi_1, K_t \varphi_2) = (-K_t \varphi_1, \varphi_2). \tag{3.15}$$

From the last formula, it follows that the eigenvalues of $\mathrm{i}K_t$ are real.

Now, take a complete orthonormal set of eigenstates of $\mathrm{i}K_t$ with positive eigenvalues, i.e., $\{u_\lambda\}$ with

$$\mathrm{i}\mathscr{L}_{K_t} u_\lambda \equiv \mathrm{i}K_t u_\lambda = \Omega u_\lambda, \qquad \Omega > 0, \tag{3.16}$$

and $(u_{\lambda_1}, u_{\lambda_2})_\Sigma = \delta_{\lambda_1 \lambda_2}$. Then, the corresponding complex conjugates satisfy $\mathrm{i}K_t u_\lambda^* = -\Omega u_\lambda^*$. Furthemore, elements of the joint set $\{u_\lambda, u_\lambda^*\}$ obey again eq. (3.9). This is exactly what we were searching for; the vector space $V_{\mathbb{C}}$ can be decomposed into a vector space of positive frequency solutions V_+ and a vector space of negative frequency solutions V_-. Moreover, a transformation generated by the time-like Killing vector field $K_t^{\underline{\mu}}$ is an isometry of $M_{d,1}$, and its associated conserved charge Q_{K_t} is the Hamiltonian. Thus, the splitting of $V_{\mathbb{C}}$ is unaffected by transformations generated by K_t, and consequently neither the vacuum and the Hilbert space are. However, note that different observers will in general see different vacua, the physical implications thereof will be discussed below.

Canonical quantization.

In order to quantize the system, a decomposition of $M_{d,1}$ in space-like hypersurfaces is needed. Such a space-time foliation can be carried out by requiring that each space-like hypersurface is a level set of a regular[9] scalar field. By regularity of the scalar field, the hypersurfaces do not intersect. The existence of such a scalar field is ascertained by the following theorem [96]:

Let $M_{d,1}$ be a globally hyperbolic manifold. Then, a global time function, i.e., a differentiable function $T(x)$ on $M_{d,1}$ such that $\nabla_{\underline{\mu}} T(x)$ is a past-directed time-like vector field[10]*, can be chosen such that each surface of constant $T(x)$ is a Cauchy surface. Thus, $M_{d,1}$ can be foliated by Cauchy surfaces and the topology of $M_{d,1}$ is $\Sigma \times \mathbb{R}$, where Σ corresponds to a Cauchy surface.*

Let us pick some foliation and label each Cauchy surface by Σ_τ, where τ is the (constant) value of the global time function $T(x)$ on that Cauchy surface. To each Σ_τ, the corresponding normal unit vectors $n^{\underline{\mu}}|_{\Sigma_\tau}$ are related to $T(x)$ by the formula $n_{\underline{\mu}} = -N \nabla_{\underline{\mu}} T(x)$, with a normalization factor $N > 0$ called lapse function.

[8]These two properties are most easily checked by choosing a coordinate system in which $K_t^{\underline{\mu}} = (1, \vec{0})$.
[9]i.e., with nonvanishing gradient
[10]Therefore, the scalar field $T(x)$ increases towards the future.

The system will be quantized in the canonical quantization scheme with respect to the chosen foliation. But, first of all, we have to specify the direction of time. Even for stationary space-times, where a preferred time direction is available, the time-like Killing vector $K_t^{\underline{\mu}}$ does not necessarily have to be parallel to $n^{\underline{\mu}}$. If it is possible to find a foliation such that this is indeed the case, then the space-time is called *static*. Anyway, let us proceed without further assumptions on the space-time other than global hyperbolicity.

The ADM formalism [106] provides a generic and covariant formulation of the time direction, and is shortly presented here. Let us perform a global coordinate transformation $x^{\underline{\mu}}(t, \vec{y})$ such that:

- $(x_\tau)^{\underline{\mu}} := x^{\underline{\mu}}(\tau, \vec{y})$ describes an embedding of Σ_τ in $\mathrm{M}_{d,1}$.
- $(x_{y_0})^{\underline{\mu}} := x^{\underline{\mu}}(t, \vec{y}_0)$ connects points on different Σ_τ having the same spatial value $\vec{y}_0$.
- $t^{\underline{\mu}} := \partial (x_y)^{\underline{\mu}} / \partial t$ obeys $t^{\underline{\mu}} \nabla_{\underline{\mu}} T(x) = 1$.

We choose the time-evolution vector field $t^{\underline{\mu}}$ as our preferred direction of time. The constraint it is subject to can be explained as follows. Consider a point on the Cauchy surface Σ_τ with coordinates $x^{\underline{\mu}}$. By definition, the global time function satisfies $T(x) = \tau$. Further, consider a displacement of the point by the infinitesimal vector $t^{\underline{\mu}} \mathrm{d}\tau$. Requiring that $x^{\underline{\mu}} + t^{\underline{\mu}} \mathrm{d}\tau$ lies in the neighboring Cauchy surface $\Sigma_{\tau + \mathrm{d}\tau}$, leads to

$$T(x^{\underline{\mu}} + t^{\underline{\mu}} \mathrm{d}\tau) = T(x) + t^{\underline{\mu}} \mathrm{d}\tau \nabla_{\underline{\mu}} T(x) \equiv \tau + \mathrm{d}\tau, \tag{3.17}$$

and therefore to $t^{\underline{\mu}} \nabla_{\underline{\mu}} T(x) = 1$ as appointed above. As a result, the time-evolution vector can be decomposed as

$$t^{\underline{\mu}} = N n^{\underline{\mu}} + N^{\underline{\mu}}, \tag{3.18}$$

where the shift vector field $N^{\underline{\mu}}$ is tangential to Σ_τ and parametrizes the freedom in the choice of $t^{\underline{\mu}}$.

Now that the time direction is introduced, the subsequent step is about the formulation of the canonical commutation relations. To this aim, let us compute the canonical momentum π. By the chain rule, it follows that

$$\mathrm{d}x^{\underline{\mu}} = t^{\underline{\mu}} \mathrm{d}t + e_i^{\underline{\mu}} \mathrm{d}y^i, \tag{3.19}$$

where $e_i^{\underline{\mu}} := \partial x^{\underline{\mu}} / \partial y^i$ with $i \in \{1, \ldots, d\}$ is a vector field tangent to the Cauchy surfaces Σ_τ. Hence, the squared line element can be expressed in terms of the new coordinates $(t, \vec{y})$, resulting in

$$\mathrm{d}s^2 = g_{\underline{\mu\nu}} \mathrm{d}x^{\underline{\mu}} \mathrm{d}x^{\underline{\nu}} = (N_i N^i - N^2) \mathrm{d}t^2 + 2 N_i \mathrm{d}t \mathrm{d}y^i + h_{ij} \mathrm{d}y^i \mathrm{d}y^j. \tag{3.20}$$

The quantities $h_{\underline{\mu\nu}}$ and $N^{\underline{\mu}}$ have simply been transformed in the new coordinates, i.e.,

$$h_{ij} = h_{\underline{\mu\nu}} e_i^{\underline{\mu}} e_j^{\underline{\nu}} \equiv g_{\underline{\mu\nu}} e_i^{\underline{\mu}} e_j^{\underline{\nu}}, \qquad h_{ij} N^j = N_i = N_{\underline{\mu}} e_i^{\underline{\mu}}. \tag{3.21}$$

By virtue of eqs. (3.6,3.18,3.19), the action in eq. (3.1) can then be rewritten as

$$S_0 = -\int \frac{\mathrm{d}t}{2} \int \mathrm{d}^d y \; N\sqrt{|h|} \left[-\frac{1}{N^2} \left(\partial_t \phi - N^i \nabla_i \phi\right)^2 + h^{ij} \left(\nabla_i \phi\right) \left(\nabla_j \phi\right) + m^2 \phi^2 \right], \tag{3.22}$$

where we have used $|g| = N^2 |h|$, with h being the determinant of h_{ij}, and

$$t^{\underline{\mu}} \nabla_{\underline{\mu}} \phi(t, \vec{y}) = \partial_t \phi(t, \vec{y}). \tag{3.23}$$

Eventually, the computation of the canonically conjugate variable to ϕ is possible:

$$\pi \equiv \frac{\delta S_0}{\delta \left(\partial_t \phi\right)} = \sqrt{|h|} \; n^{\underline{\mu}} \nabla_{\underline{\mu}} \phi. \tag{3.24}$$

In the coordinate system with coordinates $(t, \vec{y})$, the (equal time) canonical commutation relations take exactly the same form as in eq. (2.70):

$$\begin{aligned} [\phi(t, \vec{y}_1), \pi(t, \vec{y}_2)] =& \mathrm{i}\delta^{(d)}(\vec{y}_1 - \vec{y}_2), \\ [\phi(t, \vec{y}_1), \phi(t, \vec{y}_2)] =& [\pi(t, \vec{y}_1), \pi(t, \vec{y}_2)] = 0. \end{aligned} \tag{3.25}$$

Note that, for any $\varphi_1, \varphi_2 \in \mathrm{V}_{\mathbb{C}}$,

$$[(\varphi_1, \phi)_{\Sigma_\tau}, (\varphi_2, \phi)_{\Sigma_\tau}] = -(\varphi_1, \varphi_2^*)_{\Sigma_\tau} \, . \tag{3.26}$$

Conversely, it is also true that if ϕ satisfies eq. (3.26) for arbitrary $\varphi_1, \varphi_2 \in \mathrm{V}_{\mathbb{C}}$, then ϕ satisfies the canonical commutation relations given in eq. (3.25). Since the scalar product is independent on the choice of Σ_τ, also eq. (3.26) is independent of Σ_τ. Therefore, it is only necessary to impose the canonical commutation relations on one Cauchy surface, and they are automatically satisfied on every Cauchy surface.

Analogously to what was done before, let us promote a_λ, a_λ^* appearing in eq. (3.10) to operator-valued distributions. Then, these appear in the expansion of the quantized (selfadjoint) scalar field ϕ:

$$\phi(t, \vec{y}) = \sum_\lambda [a_\lambda u_\lambda(t, \vec{y}) + a_\lambda^\dagger u_\lambda^*(t, \vec{y})], \tag{3.27}$$

and satisfy

$$\begin{aligned} [a_{\lambda_1}, a_{\lambda_2}^\dagger] =& \delta_{\lambda_1 \lambda_2}, \\ [a_{\lambda_1}, a_{\lambda_2}] =& [a_{\lambda_1}^\dagger, a_{\lambda_2}^\dagger] = 0. \end{aligned} \tag{3.28}$$

The relations in eq. (3.28) can easily be derived by using eq. (3.26) together with

$$a_\lambda = (u_\lambda, \phi)_\Sigma \, , \qquad a_\lambda^\dagger = -(u_\lambda^*, \phi)_\Sigma \, . \tag{3.29}$$

Exactly as in eq. (2.77), one can find the commutator between ϕ and H to be

$$\mathrm{i}\partial_t\phi(x) = [\phi, H]. \tag{3.30}$$

With the assumption that u_λ and u_λ^* are, respectively, positive and negative frequency modes (having eigenvalues $\pm\Omega$) with respect to the operator $\mathrm{i}\partial_t$, one gets

$$a_\lambda\ \mathrm{i}\partial_t u_\lambda(x) = [a_\lambda, H]u_\lambda(x), \qquad a_\lambda^\dagger\ \mathrm{i}\partial_t u_\lambda^*(x) = [a_\lambda^\dagger, H]u_\lambda^*(x), \tag{3.31}$$

and consequently

$$[H, a_\lambda] = -\Omega a_\lambda, \qquad [H, a_\lambda^\dagger] = \Omega a_\lambda^\dagger. \tag{3.32}$$

Therefore, the positive frequency modes u_λ are related with the annihilation operator a_λ, whereas the negative frequency modes u_λ^* are related with the creation operators $a_\lambda^\dagger$, as in flat space-time. This allows us to introduce the vacuum state.

The Hilbert space.

As was done in section 2.4, the vacuum state $|0\rangle$ is introduced as the state annihilated by all operators a_λ. The "one-particle" Hilbert space $\mathscr{H}_1$ is then constructed as the space spanned by $a_\lambda^\dagger|0\rangle$. Consequently, the Fock space $\mathscr{F}$ is given by the Hilbert space completion of the direct sums of symmetrized tensor products $\mathscr{H}_1^{\otimes n}$.

Consider a second, complete orthonormal set of basis modes $\{\bar{u}_\lambda, \bar{u}_\lambda^*\}$ satisfying

$$\begin{aligned}
(\bar{u}_{\lambda_1}, \bar{u}_{\lambda_2})_\Sigma &= \delta_{\lambda_1\lambda_2},\\
(\bar{u}_{\lambda_1}^*, \bar{u}_{\lambda_2}^*)_\Sigma &= -\delta_{\lambda_1\lambda_2},\\
(\bar{u}_{\lambda_1}, \bar{u}_{\lambda_2}^*)_\Sigma &= 0.
\end{aligned} \tag{3.33}$$

Analogously to eq. (3.27), the scalar field ϕ may be expanded in the new basis:

$$\phi(t,\vec{y}) = \sum_\lambda [\bar{a}_\lambda \bar{u}_\lambda(t,\vec{y}) + \bar{a}_\lambda^\dagger \bar{u}_\lambda^*(t,\vec{y})], \tag{3.34}$$

where

$$\begin{aligned}
[\bar{a}_{\lambda_1}, \bar{a}_{\lambda_2}^\dagger] =& \delta_{\lambda_1\lambda_2},\\
[\bar{a}_{\lambda_1}, \bar{a}_{\lambda_2}] =& [\bar{a}_{\lambda_1}^\dagger, \bar{a}_{\lambda_2}^\dagger] = 0.
\end{aligned} \tag{3.35}$$

This defines a new vacuum state $|\bar{0}\rangle$ annihilated by all operators $\bar{a}_\lambda$, and consequently a new Fock space $\bar{\mathscr{F}}$. On a general $\mathrm{M}_{d,1}$, there is no physically reasonable argument to prefer one expansion over the other, and here we discuss the implications of such a choice.

The first query is if the creation and annihilation operators in eq. (3.27) are related to those in eq. (3.34). Since both sets of basis modes are complete, we can express one in terms of the other as

$$\bar{u}_\lambda = \sum_\kappa \left[\alpha_{\lambda\kappa} u_\kappa + \beta_{\lambda\kappa}^* u_\kappa^*\right]. \tag{3.36}$$

With the aid of the scalar product, it is possible to evaluate the so-called Bogolubov coefficients:

$$\begin{aligned} \alpha_{\lambda\kappa} &= (\bar{u}_\lambda, u_\kappa)_\Sigma, & \beta_{\lambda\kappa} &= (\bar{u}^*_\lambda, u_\kappa)_\Sigma, \\ \alpha^*_{\lambda\kappa} &= -(\bar{u}^*_\lambda, u^*_\kappa)_\Sigma, & \beta^*_{\lambda\kappa} &= -(\bar{u}_\lambda, u^*_\kappa)_\Sigma. \end{aligned} \tag{3.37}$$

Equating eq. (3.27) with eq. (3.34) and making use of the orthonormality relations gives

$$a_\kappa = \sum_\lambda \left[\alpha_{\lambda\kappa}\bar{a}_\lambda + \beta_{\lambda\kappa}\bar{a}^\dagger_\lambda\right], \qquad \bar{a}_\lambda = \sum_\kappa \left[\alpha_{\lambda\kappa}a_\kappa - \beta^*_{\lambda\kappa}a^\dagger_\kappa\right]. \tag{3.38}$$

Eventually, the commutation relations given in eqs. (3.28,3.35) lead to the constraints

$$\sum_\lambda \left[\alpha_{\lambda\kappa_1}\alpha^*_{\lambda\kappa_2} - \beta_{\lambda\kappa_1}\beta^*_{\lambda\kappa_2}\right] = \delta_{\kappa_1\kappa_2}, \qquad \sum_\lambda \left[\alpha_{\lambda\kappa_1}\beta_{\lambda\kappa_2} - \alpha_{\lambda\kappa_2}\beta_{\lambda\kappa_1}\right] = 0. \tag{3.39}$$

Transformations given in eq. (3.38) satisfying the above constraints are called *Bogolubov transformation*. In plain words, these are linear transformation of the creation and annihilation operators that preserve the canonical commutation relations, and hence generalize the class of canonical transformations. Note that such transformations may relate unitarily inequivalent representations of the commutation relations, whose existence was already mentioned in section 2.2.

Let us shortly discuss a restricted class of Bogolubov transformations, the one describing general coordinate transformations and time-evolution. For stationary space-times, one requires u_λ to be the positive frequency modes with respect to the time-like Killing vector field $K_t \equiv K^{\underline{\mu}}_t \partial_{\underline{\mu}}$. By virtue of eq. (3.16), the vector space V_+, spanned by these positive frequency modes, is invariant under time-evolution, since the generator of time-translation is given by $\mathrm{i}\mathscr{L}_{K_t}$ in the field representation. Therefore, by eq. (3.36), we expect $\beta^*_{\lambda\kappa} = 0$. It follows that $\bar{a}_\lambda$ is a linear combination of solely a_κ, and therefore $|0\rangle$ keeps its status as a vacuum state during time-evolution. In particular, the basis modes u_κ and $\bar{u}_\lambda$ share a common vacuum state and $|\bar{0}\rangle$ can be chosen to solely differ from $|0\rangle$ by a phase factor. Consequently, the two Fock spaces $\mathscr{F}$ and $\bar{\mathscr{F}}$ are related by a unitary, hence invertible, transformation, cf. eq. (3.39). So, the particle interpretation of $\mathbb{M}_{d,1}$ can be carried over without any essential adjustment to globally hyperbolic, stationary space-times, as long as one sticks with one observer whose trajectory follows along orbits of the Killing vector K_t.

If a space-time is not stationary, as, for instance, a space-time describing the expansion of the universe, then there is no meaningful definition of positive frequency modes. Hence, $\beta_{\lambda\kappa}$ might be nonvanishing under time-evolution, here defined as a translation in direction of some vector $t^{\underline{\mu}}$. It then follows immediately that, for $\beta_{\lambda\kappa} \neq 0$, $|\bar{0}\rangle$ is not annihilated by all a_κ:

$$a_\kappa|\bar{0}\rangle = \sum_\lambda \beta_{\lambda\kappa}\bar{a}^\dagger_\lambda|\bar{0}\rangle \neq 0. \tag{3.40}$$

Since we are working in the Heisenberg picture, if we choose the state $|\bar{0}\rangle$ to be the vacuum state, then it will remain unaltered in the past/future. Thus, it turns out that the vacuum state $|\bar{0}\rangle$ might appear at a different instant of time to be populated with particles, since

$$\langle\bar{0}|N_{tot}|\bar{0}\rangle = \sum_{\kappa}\langle\bar{0}|a_{\kappa}^{\dagger}a_{\kappa}|\bar{0}\rangle = \sum_{\lambda,\kappa}|\beta_{\lambda\kappa}|^{2}, \tag{3.41}$$

where N_{tot} denotes the number operator in the past/future. Therefore, the measurement an observer is performing depends on the instant his measuring device is actually switched on. Something analogous occurs under general coordinate transformations, where a vacuum state in one coordinate system is not necessarily a vacuum state in another coordinate system. The consequence is that the measurement of an observer will generally also depend on the location and state of motion of his measuring device.

In summary, different observers disagree on the particle content of the theory. Accordingly, it is difficult to provide a meaningful particle interpretation. One observer can surely assert the presence of particles, but without specifying the trajectory of the detector, the claim is not very useful. Note that this is also true in flat space-time, but there exist inertial coordinate systems, and on these preferred coordinate systems all measuring devices agree on the measurements performed.

The notion of particles, being heavily dependent on the choice of basis modes which are defined globally[11] on the space-time, is sensitive to the structure of space-time itself. A reasonable particle concept could be possible if these particles were of much smaller wavelength than the curvature scale, but this is clearly not satisfactory as it implies an incomplete set of modes. One has thus to accept that, in a generic situation, it is unknown how to deal with this issue. It seems unlikely that the particle picture will prove meaningful here.

One might ask if there is a way to provide an objective probe. As detectors are of local nature, it seems worthwhile to investigate locally defined, rather than globally defined, quantities, as for example $\langle f|T^{\underline{\mu\nu}}|f\rangle$, with $|f\rangle \in \mathscr{F}$, is. Indeed, for a fixed state $|f\rangle \in \mathscr{F}$, the results for the expectation value of the stress-tensor of different measuring devices are related by the usual tensor transformation.

This however poses the problem of the divergent vacuum energy. In flat space-time this divergence was simply discarded by the normal ordering procedure. However, in curved space-times there is an ambiguity in the choice of the vacuum state with respect to which one should perform the normal ordering, and different choices can lead to inequivalent descriptions. Therefore, it requires a more sophisticated renormalization technique. Since the appearing divergence was due to modes with short wavelengths, and since curved space-times on very small scales look like flat space-time, it should be possible to match the behavior of fields in curved space-times at short distances to those in flat space-time, and thus to subtract off the divergences. For a free theory, this is indeed possible by considering *Hadamard states*, states whose singularity structure at

[11] or at least on a large patch

the coincidence point of its two-point function is the natural generalization to curved space-times of the singularity structure of the two-point function on flat space-time. It can be shown that Hadamard states exist on every globally hyperbolic space-time and are the only states for which the renormalized stress-tensor is well-defined and nonsingular. Furthermore, if the Hadamard condition is satisfied on one Cauchy surface, it is satisfied in its whole domain of dependence. We will not discuss the details there but instead refer to ref. [20].

In conclusion, let us shortly mention that in certain circumstances, it might still be possible to give a (local) notion of particles on nonstationary space-times. This is the case for space-times which are stationary only in limited regions. If each of these regions possesses a Cauchy surface then a vacuum can be defined on with respect of the local time-like Killing vector field. The quantization procedure is applicable since the regions themselves are globally hyperbolic. An example is given by the region outside the horizon of a static black hole, which is globally hyperbolic even though it is not geodesically complete. Another important case is given by a space-time which approaches stationary space-times in the far past and future. The asymptotic regions can either be flat or some other highly symmetric space-times, in order to allow a meaningful particle interpretation. These regions are then called *in* and *out* regions, in analogy to scattering theory of flat space-time where it is assumed that the field interactions approach zero in the far past/future. A useful object in this setting is the S-matrix, a unitary operator relating any state in the Fock space of the far past to any state in the Fock space in the far future. The elements of the S-matrix can be written in terms of the Bogolubov coefficients, see, for instance, ref. [95].

3.2 Generalized Wightman axioms

Here we will give a short glimpse on how it might be possible to extend the Wightman axioms to curved space-times. This is a relatively novel field of study, and is far from being conclusive. In spite of that, it already provides strong hints on the fundamental properties of quantum field theory. Furthermore, in its most notable formulation [16], it implies a curved space-time version of the CPT and spin-statistics theorems. Anyway, an in-depth discussion goes beyond the scope of this thesis. The interested reader is referred to refs. [16, 17].

- Classically, for physically reasonable fields, the stress-tensor satisfies the dominant energy condition, i.e.,
$$K_t^{\underline{\mu}} n^{\underline{\nu}} T_{\underline{\mu\nu}} \geq 0, \tag{3.42}$$
where K_t is a vector field representing time translations. Thus, the total energy is positive, see eq. (3.12), and if K_t is a Killing vector field, then it is also conserved. In quantum field theory the situation drastically changes. It is known that, locally, energy densities can be arbitrarily negative. Nevertheless, in flat space-time, the total energy is positive for all physically reasonable states. In curved space-times this

is not anymore guaranteed, even for space-times with time-translation symmetry[12]. Hence, the lack of an appropriate notion of total energy makes it meaningless to generalize the spectrum condition (A1) in terms of the positivity of such a quantity. In flat space-time, it can be shown that the spectrum condition is equivalent to the *microlocal spectrum condition*, characterizing the short-distance singularity structure of the correlation functions of the quantum fields. This condition has a natural generalization to curved space-times.

- A seemingly meaningful generalization of invariance/covariance under Poincaré transformations of flat space-time is the invariance/covariance under general coordinate transformations of curved space-times. However, this does not provide additional symmetries, since any Poincaré covariant theory can be rephrased in terms of a generally covariant theory. The true symmetry group of space-time transformations is generated by Killing vectors and, in fact, on curved space-times it steps in for the Poincaré group of flat space-time. On the other hand, this does not imply that the requirement of general covariance is of no physical relevance. Contrarily, it dictates that the only background structure is the manifold[13]. Furthermore, it relates quantum fields in different coordinate systems, generalizing the covariance condition (A5).

- Since the quantum fields are smeared over a local region of space-time, one demands that they are locally constructed from the background structure in the sense that the quantum fields in any neighborhood should be covariantly constructed from the background structure within the neighborhood itself. This ensures that a variation of the metric around some point only influences the field locally.

- In the absence of symmetries of the space-time, there is no preferred choice of the vacuum state. Even worse, different vacua may lead to unitarily inequivalent constructions of the theory. For instance, in a free quantum field theory even the restriction to Hadamard states is highly nonunique, and different choices indeed give rise to, in general, unitarily inequivalent constructions. The forced designation of one of these unitarily inequivalent constructions can be overcome by formulating the theory via the algebraic approach [18]. Nevertheless, as already discussed previously, for curved space-times without symmetries there is a general consensus that it is fruitless to seek a preferred vacuum state as much as it is fruitless to seek a preferred coordinate system. Hence, the notion of "vacuum" and "particles" has, in general, to be abandoned. However, in the presence of a global time-like Killing vector field K_t, there exists a preferred class of coordinate systems. Indeed, one may introduce a coordinate t upon which the metric does not depend and with respect to which K_t takes the form $K_t^\mu = (1, 0, \dots, 0)$. Furthermore, K_t may be

[12]One example is the massless scalar field on a $\mathbb{R} \times \mathrm{S}^1$ universe [95].

[13]including its metric, time and space orientation, and spin structure

scaled so that t gives the proper time of at least one observer moving along the orbit of K_t. Relative to that observer, the mode functions can be separated in positive and negative frequency modes. This allows for the introduction of a vacuum state invariant under $\mathcal{L}_{K_t}$. In the presence of further Killing vector fields, one may demand that the vacuum is also invariant under the transformations generated by these. Altoghether, this is a viable generalization of the vacuum condition (A3) to the class of stationary space-times, even though the requirement of uniqueness of the vacuum state is not implementable on general space-times. Sure, in order to render the vacuum state unique one could impose additional constraints, as, for instance, those given in ref. [18]. But the so-defined vacuum state does not always exist.

Bearing all these concepts in mind, the domain condition (A2), the completeness condition (A4) and the microcausality condition (A6) can easily be reformulated on generally curved space-times.

IV The anti–de Sitter space-time

In general, the symmetries of a physical problem are of huge assistance. One can exploit these symmetries to render the problem more manageable. For instance, in most curved space-times, even finding the simplest scalar propagator is a burden but the difficulties are alleviated if the space-time possesses some symmetries. Ideally, the preferred space-times to work with are *maximally symmetric* Lorentzian manifolds, which are space-times admitting the maximal amount of linearly independent global Killing vector fields. In $d+1$ dimensions, there are at most $(d+2)(d+1)/2$ of them. This can easily be seen as follows. Take the following identity which is a direct consequence of the Killing eq. (3.13):

$$\nabla_{\underline{\mu}}\nabla_{\underline{\nu}}K^{\underline{\rho}} = R^{\underline{\rho}}{}_{\underline{\nu\mu\sigma}}K^{\underline{\sigma}}, \tag{4.1}$$

and is therefore valid for any Killing vector field $K^{\underline{\mu}}$. The quantity $R^{\underline{\rho}}{}_{\underline{\nu\mu\sigma}}$ is the Riemann tensor. It is clear that, at any point $\mathrm{p} \in \mathrm{M}_{d,1}$, $K^{\underline{\mu}}$ is uniquely fixed by the $d+1$ values $K^{\underline{\mu}}|_p$ and $(d+1)^2$ values $\nabla_{\underline{\nu}}K^{\underline{\mu}}|_p$, as one can infer from the Taylor expansion of $K^{\underline{\mu}}$ around p. Moreover, the number $(d+1)^2$ of independent values of the derivative of the Killing vector field is further restricted by the Killing eq. (3.13), since only the $d(d+1)/2$ antisymmetric combinations $\nabla_{\underline{\mu}}K_{\underline{\nu}} - \nabla_{\underline{\nu}}K_{\underline{\mu}}$ are nonvanishing. Hence, one has indeed $(d+2)(d+1)/2$ independent values at most.

So how do maximally symmetric space-times look like? For such space-times, it can be shown that the Riemann tensor assumes the form [102]

$$R_{\underline{\mu\nu\rho\sigma}} = -a^2(g_{\underline{\mu\rho}}g_{\underline{\nu\sigma}} - g_{\underline{\mu\sigma}}g_{\underline{\nu\rho}}), \tag{4.2}$$

where a^2 is some real constant, not necessarily positive. Then, the Ricci curvature tensor and the Ricci scalar are, respectively, given by

$$R_{\underline{\mu\nu}} = -da^2 g_{\underline{\mu\nu}}, \qquad R = -d(d+1)a^2. \tag{4.3}$$

Now it is straightforward to verify that maximally symmetric spaces satisfy the vacuum Einstein equations

$$R_{\underline{\mu\nu}} - \frac{1}{2}g_{\underline{\mu\nu}}R + g_{\underline{\mu\nu}}\Lambda_c = 0, \tag{4.4}$$

with the cosmological constant given by

$$\Lambda_c = -\frac{d(d-1)}{2}a^2. \tag{4.5}$$

Thus, one can classify the maximally symmetric space-times by the sign of the cosmological constant; the space-time with vanishing cosmological constant is identified with the flat space-time $\mathbb{M}_{d,1}$, since the Riemann tensor is zero everywhere. Positive and negative cosmological constants give rise to the so-called *de Sitter* and *anti–de Sitter* space-times. These manifolds, despite being quite similar in the above construction, reveal contrasting properties. While de Sitter is a globally hyperbolic but nonstationary space-time, anti–de Sitter space-time, as we will see, is non-globally hyperbolic but stationary, even static. All these properties of the anti–de Sitter space-time will be discussed in this chapter. Let us note that the absence of a Cauchy surface on anti–de Sitter space-time has major implications on its causal structure, eventually leading to the AdS/CFT correspondence.

Working in anti–de Sitter space-time carries a few benefits. First, it provides a great opportunity to validate our further results. Also, it provides insights to the not yet well-understood correspondence itself. Thus, in what follows, we only deal with the anti–de Sitter space-time, hence we assume $a^2 > 0$.

4.1 Geometry of AdS

For an excellent reference on anti–de Sitter space-time, especially what concerns figures depicting different coordinate patches and the conformal structure of AdS_{d+1}, we recommend ref. [107].

The $(d+1)$-dimensional anti–de Sitter space-time AdS_{d+1} can be embedded into a $(d+2)$-dimensional flat ambient space $\mathbb{M}_{d,2}$. AdS_{d+1} is then described by the quadric

$$(X^A)^2 := \eta_{AB}X^AX^B = \eta_{\mu\nu}X^\mu X^\nu - (X^{d+1})^2 = -\frac{1}{a^2}, \tag{4.6}$$

where $A, B = 0, ..., d+1$ and where η_{AB} and $\eta_{\mu\nu}$ are, respectively, the metrics of $\mathbb{M}_{d,2}$ $\mathbb{M}_{d,1}$. This results in a one-sheeted hyperboloid, topologically equivalent to $\mathbb{R}^d \times \mathrm{S}^1$, see figure 1. It is clear that the symmetry group leaving the quadric invariant is given by $O(d,2)$. We will discuss its implications throughly in section 4.2.

The squared line element, restricted to the above quadric, is given by

$$\mathrm{d}s^2 = (\eta_{AB} + a^2 X_A X_B)\mathrm{d}X^A \mathrm{d}X^B = (\mathrm{d}X^\mu)^2 - \frac{a^2(X_\mu \mathrm{d}X^\mu)^2}{1 + a^2(X^\mu)^2}. \tag{4.7}$$

Here we do not further discuss the *ambient space approach* and refer to appendix A. Let us instead adopt immediately a parametrization of the quadric. It appears natural to introduce hyperspherical coordinates for the Euclidean subspace $\mathbb{R}^d$ spanned by X^A for $A = 1, \dots, d$. We choose the main polar angle $\theta \in [0, \pi]$ in direction of X^d, that is,

$$X^d = r\cos\theta. \tag{4.8}$$

The quantity $r \in [0, \infty)$ is the radial coordinate of the hyperspherical coordinates. Hence, it follows that

$$(X^1)^2 + \cdots + (X^{d-1})^2 = r^2 \sin^2\theta \tag{4.9}$$

and

$$(X^1)^2 + \cdots + (X^{d-1})^2 + (X^d)^2 = r^2. \tag{4.10}$$

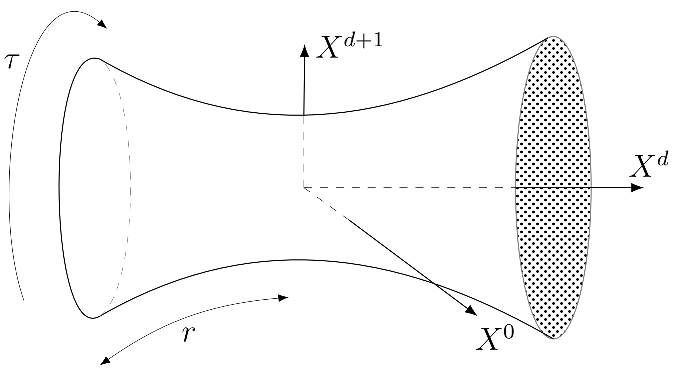


Figure 1: Embedding of the anti de–Sitter space-time.

The full parametrization of the quadric is given by the further identification

$$\begin{aligned} X^0 &= \sqrt{r^2 + \frac{1}{a^2}}\sin(\tau a), \\ X^{d+1} &= \sqrt{r^2 + \frac{1}{a^2}}\cos(\tau a), \end{aligned} \tag{4.11}$$

with $\tau \in [0, 2\pi/a)$. The coordinate patch is a global patch of AdS_{d+1}, covering the whole space-time, as depicted in figure 1. In this coordinate system, the squared infinitesimal line element reads

$$\mathrm{d}s^2 = -(1 + r^2a^2)\mathrm{d}\tau^2 + \frac{1}{1 + r^2a^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{d-1}, \tag{4.12}$$

where $\mathrm{d}\Omega^2_{d-1}$ is the metric on the $(d-1)$-dimensional hypersphere S^{d-1}. Periodicity of the (noncontractible) time coordinate τ has a major drawback, it allows for closed time-like curves. Although it may argued that, by fixing some consistency constraints, one can avoid causality breaking situations and the resulting paradoxes, they are very unpleasant to deal with. Fortunately, there is a simple resolution; we consider the universal covering space[1] of AdS_{d+1}, thus changing the topology of AdS_{d+1} to $\mathbb{R}^{d+1}$. The range of the time coordinate thus is taken to be $(-\infty, \infty)$. In what follows, when we refer to AdS_{d+1}, we intend its universal covering space.

The investigation of the causal structure of AdS_{d+1} is best performed in another coordinate system, related to the previous by

$$\bar{\tau} = \tau a, \qquad \bar{r} = \arctan(ra). \tag{4.13}$$

[1]The universal covering space of S^1 is $\mathbb{R}$.

This leads to the metric

$$\mathrm{d}s^2 = \frac{1}{a^2 \cos^2 \bar{r}} \left(-\mathrm{d}\bar{\tau}^2 + \mathrm{d}\bar{r}^2 + \sin^2 \bar{r}\ \mathrm{d}\Omega^2_{d-1} \right) , \tag{4.14}$$

with $(\bar{\tau}, \bar{r}) \in \mathbb{R} \times [0, \pi/2)$. Thus, in this (also global) coordinate patch, the null geodesics correspond to straight lines. From the corresponding Penrose diagram, see figure 2, it becomes obvious that AdS_{d+1} possesses a time-like *conformal boundary* $\mathcal{J}$ at $\bar{r} = \pi/2$, topologically equivalent to a cylinder $\mathbb{R} \times \mathrm{S}^{d-1}$.

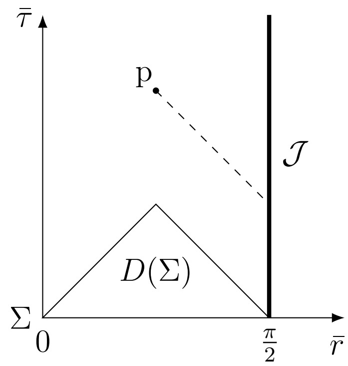


Figure 2: Penrose diagram of anti–de Sitter space-time in the $(\bar{\tau}, \bar{r})$-plane.

Although spatial distance from any point in AdS_{d+1} to the boundary is infinite, null geodesics get there in a finite amount of time and get reflected at a right angle. Moreover, time-like geodesics are sinusoidal, see, for instance, ref. [108]. This so-called *box effect* of anti–de Sitter space-time is a reminiscence of the periodicity of the original time coordinate.

As a consequence, AdS_{d+1} is not globally hyperbolic. Indeed, the achronal hypersurface Σ at, say, $\bar{\tau} = 0$ is not a Cauchy surface, as its domain of dependence $D(\Sigma)$ is not covering the whole space-time. Suppose one is interested in finding a unique solution to a hyperbolic differential equation on AdS_{d+1}. To this end, one may choose initial data on Σ. The field is uniquely determined in $D(\Sigma)$. However, as depicted in figure 2, the field at p is not only influenced by Σ, but also from the boundary $\mathcal{J}$. Hence, in addition to the initial data on Σ, one necessitates also of the data on the boundary itself. This fact lies at the heart of the AdS/CFT correspondence. But whereas AdS_{d+1} is not globally hyperbolic, the union of AdS_{d+1} with its boundary is [109]. This union is called *conformal completion* and will be discussed in section 4.3.

Due to its affinity with the coordinates of Minkowski space-time $\mathbb{M}_{d,1}$, in this thesis we will stick to another patch of AdS_{d+1} which, however, covers only half of the space-time.

It is defined by *Poincaré coordinates* via the parametrization

$$\begin{aligned} X^\alpha =& \frac{x^{\underline{\alpha}}}{a x^{\underline{d}}}, \\ X^d =& \frac{1}{\sqrt{2} a x^{\underline{d}}} \left(1 - \frac{x^{\mu^2}}{2}\right), \\ X^{d+1} =& \frac{1}{\sqrt{2} a x^{\underline{d}}} \left(1 + \frac{x^{\mu^2}}{2}\right), \end{aligned} \tag{4.15}$$

where[2] $\alpha, \underline{\alpha} = 0, \ldots, d-1$ and where

$$x^{\mu^2} = \eta_{\mu\nu} \delta^\mu_{\underline{\mu}} \delta^\nu_{\underline{\nu}} x^{\underline{\mu}} x^{\underline{\nu}} \tag{4.16}$$

as in Minkowski space-time. The coordinate $x^{\underline{0}}$ is identified with the time coordinate t, whereas $x^{\underline{d}} > 0$ is called *radial coordinate*. We will denote the latter by the letter z. The relation between these coordinates and the global coordinates $(\bar{\tau}, \bar{r})$ appearing in eq. (4.14) can be found to be

$$\begin{aligned} t =& \frac{\sqrt{2} \sin \bar{\tau}}{\cos \bar{\tau} + \sin \bar{r} \cos \theta}, \\ z =& \frac{\sqrt{2} \cos \bar{r}}{\cos \bar{\tau} + \sin \bar{r} \cos \theta}. \end{aligned} \tag{4.17}$$

Indeed, it follows from the above relations that the Poincaré patch does not cover the whole anti–de Sitter space-time; in order to have $z > 0$ one has to require the denominator in eq. (4.17) to be positive definite. The other half of AdS_{d+1}, the one with negative definite denominator, is covered by $z < 0$ and is disconnected from $z > 0$ by a singularity at $z = 0$. From now on, let us only consider the coordinate patch defined by $z > 0$. The other Poincaré coordinates are related to the global coordinates via

$$x^{\underline{\alpha}} \propto \frac{\sqrt{2} \sin \bar{r} \sin \theta}{\cos \bar{\tau} + \sin \bar{r} \cos \theta} \qquad \text{for } \underline{\alpha} \neq 0, \tag{4.18}$$

satisfying the constraint

$$\sqrt{(x^{\underline{1}})^2 + \cdots + (x^{\underline{d-1}})^2} = \frac{\sqrt{2} \sin \bar{r} \sin \theta}{\cos \bar{\tau} + \sin \bar{r} \cos \theta}. \tag{4.19}$$

The prerogative of Poincaré coordinates is the simplicity of the metric expressed in these coordinates:

$$\mathrm{d}s^2 = \frac{1}{a^2 z^2} \eta_{\mu\nu} \delta^\mu_{\underline{\mu}} \delta^\nu_{\underline{\nu}} \mathrm{d}x^{\underline{\mu}} \mathrm{d}x^{\underline{\nu}}. \tag{4.20}$$

[2] Henceforth, the indices denoted by the initial letters of the greek alphabet $(\alpha, \beta, \gamma, \ldots)$ run from 0 to $d-1$, as opposed to other greek letters $(\mu, \nu, \rho, \ldots)$ running from 0 to d, the lowercase latin letters $(i, j, k, \ldots)$ running from 1 to d, and the uppercase latin letters $(A, B, C, \ldots)$ running from 0 to $d+1$.

Additionally, it makes the conformal flatness (more will be said in section 4.3) of AdS_{d+1} explicit. Note that, independently of the sign of z, $K_t = \partial_t$ is clearly a (global) time-like Killing vector, hence the anti–de Sitter space-time is stationary. See also section 4.2.

Let us foliate the space-time by choosing the global time function to be $T(x) = t$. The corresponding normal unit vector is given by $n_{\underline{\mu}} = -N\nabla_{\underline{\mu}}T(x)$, with lapse function $N = 1/az$, or explicitly by $n^{\underline{\mu}} = az\,(1,\vec{0})$. Then, since the shift function is identically zero, the anti–de Sitter space-time is also static. Note that this was already manifest in eqs. (4.12,4.14), as we were able to find static global coordinate patches for AdS_{d+1}.

Let us conclude this section by showing that the manifold defined by the quadric (4.6) really corresponds to the anti–de Sitter space-time. With the introduction of the frame field one-form

$$h^{\mu} = h^{\mu}_{\underline{\mu}}\mathrm{d}x^{\underline{\mu}} = \frac{1}{az}\delta^{\mu}_{\underline{\mu}}\mathrm{d}x^{\underline{\mu}}\,, \tag{4.21}$$

which relates the space-time indices to the vector indices of $\mathbb{M}_{d,1}$, the squared line element of AdS_{d+1} in Poincaré coordinates reads

$$\mathrm{d}s^2 = g_{\underline{\mu\nu}}\mathrm{d}x^{\underline{\mu}}\mathrm{d}x^{\underline{\nu}} = \eta_{\mu\nu}h^{\mu}h^{\nu}. \tag{4.22}$$

The spin-connection one-form $\omega^{\mu\nu}$ can be found via the first Cartan structure equation:

$$\mathrm{d}h^{\mu} + \omega^{\mu\nu}\wedge h_{\nu} = 0, \tag{4.23}$$

where $\wedge$ is the exterior product. Note that $\mathrm{d}h^{\mu} = ah^{\mu}\wedge h^{d}$. Therefore, the connection can be written as

$$\omega^{\mu\nu}_{\underline{\mu}} = a\left(h^{\nu}_{\underline{\mu}}\eta^{\mu d} - h^{\mu}_{\underline{\mu}}\eta^{\nu d}\right), \tag{4.24}$$

where the antisymmetry of $\omega^{\mu\nu}$ was exploited. Now that we know the spin-connection, let us compute the second Cartan structure equation:

$$\mathrm{d}\omega^{\mu\nu} + \omega^{\mu\rho}\wedge\omega_{\rho}{}^{\nu} = -a^2h^{\mu}\wedge h^{\nu}. \tag{4.25}$$

Solving for the curvature two-form

$$\frac{1}{2}R_{\mu\nu\rho\sigma}h^{\rho}\wedge h^{\sigma} = -a^2h_{\mu}\wedge h_{\nu} \tag{4.26}$$

leads to

$$R_{\mu\nu\rho\sigma} = -a^2(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}). \tag{4.27}$$

The above result agrees with eq. (4.2) when expressed through space-time indices. Therefore the space-time covered by the Poincaré patch is indeed an open subset of the anti–de Sitter space-time, defined as the maximally symmetric space-time with negative cosmological constant.

4.2 Symmetries

According to eq. (4.6), it immediately follows that the isometry group of anti–de Sitter space-time is given by the Lorentz group of $\mathbb{M}_{d,2}$, and hence is given by $O(d,2)$. This is the group which leaves the quadric invariant. We would like to derive the action of the group $O(d,2)$ on the Poincaré coordinates. To this aim, note that in the embedding space formalism, the $(d+2)(d+1)/2$ generators of the group read $J_{AB} = X_A\partial_B - X_B\partial_A$. Let us define

$$
\begin{aligned}
P_\alpha =& \frac{1}{\sqrt{2}}(J_{(d+1)\alpha} - J_{d\alpha}),\\
L_{\alpha\beta} =& J_{\alpha\beta},\\
K_\alpha =& -\sqrt{2}(J_{(d+1)\alpha} + J_{d\alpha}),\\
D =& J_{(d+1)d}.
\end{aligned}
\tag{4.28}
$$

Then, the above linear combinations of generators act on $x^{\underline{\mu}}$ as follows:

$$
\begin{aligned}
P_\alpha(x^{\underline{\mu}}) &= \delta^{\underline{\mu}}_{\alpha}\,,\\
L_{\alpha\beta}(x^{\underline{\mu}}) &= \eta_{\alpha\sigma}\delta^{\sigma}_{\underline{\sigma}}x^{\underline{\sigma}}\delta^{\underline{\mu}}_{\beta} - \eta_{\beta\sigma}\delta^{\sigma}_{\underline{\sigma}}x^{\underline{\sigma}}\delta^{\underline{\mu}}_{\alpha}\,,\\
K_\alpha(x^{\underline{\mu}}) &= x^{\underline{\mu}^2}\delta^{\underline{\mu}}_{\alpha} - 2\eta_{\alpha\sigma}\delta^{\sigma}_{\underline{\sigma}}x^{\underline{\sigma}}x^{\underline{\mu}}\,,\\
D(x^{\underline{\mu}}) &= x^{\underline{\mu}}\,.
\end{aligned}
\tag{4.29}
$$

In particular, the action of the generators on the coordinate z is

$$
\begin{aligned}
P_\alpha(z) &= 0,\\
L_{\alpha\beta}(z) &= 0,\\
K_\alpha(z) &= -2\eta_{\alpha\sigma}\delta^{\sigma}_{\underline{\sigma}}x^{\underline{\sigma}}z,\\
D(z) &= z.
\end{aligned}
\tag{4.30}
$$

Moreover, it is worth to note that the z-preserving generators, P_α and $L_{\alpha\beta}$, act like a Poincaré group in d dimensions. The derivation of the Killing vector fields of AdS$_{d+1}$ is straightforward:

$$
\begin{aligned}
P_\alpha &= \delta^{\underline{\rho}}_{\alpha}\partial_{\underline{\rho}}\,,\\
L_{\alpha\beta} &= \eta_{\alpha\sigma}\delta^{\sigma}_{\underline{\sigma}}x^{\underline{\sigma}}\delta^{\underline{\rho}}_{\beta}\partial_{\underline{\rho}} - \eta_{\beta\sigma}\delta^{\sigma}_{\underline{\sigma}}x^{\underline{\sigma}}\delta^{\underline{\rho}}_{\alpha}\partial_{\underline{\rho}}\,,\\
K_\alpha &= x^{\underline{\mu}^2}\delta^{\underline{\rho}}_{\alpha}\partial_{\underline{\rho}} - 2\eta_{\alpha\sigma}\delta^{\sigma}_{\underline{\sigma}}x^{\underline{\sigma}}x^{\underline{\rho}}\partial_{\underline{\rho}}\,,\\
D &= x^{\underline{\rho}}\partial_{\underline{\rho}}\,.
\end{aligned}
\tag{4.31}
$$

Indeed, first note that, by acting with these on $x^{\underline{\mu}}$, one immediately recovers eq. (4.29). Furthermore, one can easily verify that, for each value of α, β, these vector fields satisfy eq. (3.14).

A discrete symmetry of the quadric (4.6) is the parity transformation, performed by $X^d \to -X^d$. After a normalization through the operator D, the associated action on the Poincaré coordinates is

$$I(x^{\underline{\mu}}) = \frac{x^{\underline{\mu}}}{x^{\mu 2}}. \tag{4.32}$$

Note that I is an involution, i.e., $I^2 = 1$, and relates P_α and K_α through the identity

$$K_\alpha = I P_\alpha I. \tag{4.33}$$

With the latter symmetry transformation, all the symmetries of the quadric have been found, since all other allowed[3] discrete transformations are equivalent to combinations of the above parity transformation and rotations.

4.3 Conformal boundary

As discussed in section 4.1, the anti–de Sitter space-time possesses a conformal boundary at $\bar{r} = \pi/2$, see also figure 2. Owing to the lack of global hyperbolicity in AdS_{d+1}, we would like to somehow include the boundary into the actual space-time. This is indeed possible by an idea that traces back to Penrose [110] called conformal completion. The general procedure is to attach the boundary points to the manifold, such that the interior of the newly defined manifold is diffeomorphic to the original one. This new manifold will be equipped with a metric which differs from the original one by a *Weyl transformation*:

$$\mathrm{d}s' = \Omega(x)\mathrm{d}s. \tag{4.34}$$

The *conformal factor* $\Omega(x)$ is taken to vanish at the boundary, but beyond that it is an arbitrary function. Hence, the new metric is some representative of the class of conformally equivalent metrics.

Let us illustrate how this works on the Poincaré patch. From eq. (4.17) it is straightforward to see that, for allowed values of $\bar{\tau}, \theta$, setting $\bar{r}$ to $\pi/2$ results in $z = 0$. Hence the topology of AdS_{d+1} at $z = 0$ is given by the Minkowski space-time $\mathbb{M}_{d-1,1}$. However, as yet, this space-time is not equipped with a metric that extends over the space-time with $z = 0$, since the metric (4.20) appears to be singular there. Let us choose the conformal factor as follows:

$$\mathrm{d}s'^2 = a^2 z^2 \mathrm{d}s^2. \tag{4.35}$$

This new metric is regular at the point $z = 0$, turning exactly into the flat metric of $\mathbb{M}_{d-1,1}$. We stress again that one might equally take another representative of the same equivalence class, the class of conformally flat metrics.

So what are the symmetries of the space-time at $z = 0$? Owing to eqs. (4.29,4.30), at $z = 0$, one recovers the full conformal group action. Indeed, at that point the quantities

[3]The transformation $X^{d+1} \to -X^{d+1}$ amounts to change the patch ($z \to -z$) and is therefore not allowed as a symmetry transformation.

$P_\alpha, L_{\alpha\beta}, K_\alpha$ and D correspond exactly to the generators of the conformal group in $\mathbb{M}_{d-1,1}$:

$$\begin{aligned} P_\alpha(x^\delta) &= \delta_\alpha^\delta \,, \\ L_{\alpha\beta}(x^\delta) &= \eta_{\alpha\gamma} x^\gamma \delta_\beta^\delta - \eta_{\beta\gamma} x^\gamma \delta_\alpha^\delta \,, \\ K_\alpha(x^\delta) &= x^{\gamma^2} \delta_\alpha^\delta - 2\eta_{\alpha\gamma} x^\gamma x^\delta \,, \\ D(x^\delta) &= x^\delta \,, \end{aligned} \tag{4.36}$$

as these form a complete set of solutions to the conformal Killing equation in $\mathbb{M}_{d-1,1}$:

$$\partial_\alpha (K_c)_\beta + \partial_\beta (K_c)_\alpha = \frac{2}{d} \eta_{\alpha\beta} \partial_\gamma K_c^\gamma \,. \tag{4.37}$$

Hence, the generators have the interpretation of generating translations (P_α), rotations ($L_{\alpha\beta}$), special conformal transformations (K_α) and dilatations (D). Eventually, I resembles the operation of coordinate inversion.

This, however, poses the problem that the point corresponding to the inversion of the origin of $\mathbb{M}_{d-1,1}$ is not included in the space-time itself. Fortunately, we simply missed a point on the boundary. By fixing $\bar{\tau} = k\pi + \pi/2$ for $k \in \mathbb{N}$ and $\theta = \pi/2$, and afterwards taking the limit $\bar{r} \to \pi/2$, the radial coordinate diverges, i.e., $z \to \infty$. This equips $\mathbb{M}_{d-1,1}$ with an additional and unique point, leading to $\mathbb{M}_{d-1,1} \cup \{\infty\}$. We found the conformal completion of anti–de Sitter space-time. In fact, it is well known that the so-defined *conformal compactification* of Minkowski space-time $\mathbb{M}_{d-1,1}$ has the topology $\mathbb{R} \times \mathrm{S}^{d-1}$ [31], and this cylindric topology coincides exactly with the topology of the boundary of AdS_{d+1}, as discussed in section 4.1.

V Free scalar QFT in (E)AdS

The knowledge gained in the previous chapters III and IV will guide us towards the quantization of a free scalar field on anti–de Sitter space-time. This space-time is static, hence there exists a preferred vacuum state invariant under the respective time-like Killing vector field. However, it is non-globally hyperbolic and therefore the usual quantization procedure is inapplicable. Nevertheless, a consistent quantization scheme can be devised by carefully controlling information entering and leaving the space-time through its boundary [15]. We will not go too much into the details here. Instead, bearing in mind that the conformal completion of anti–de Sitter space-time is globally hyperbolic [109], we will henceforth work with this "improved" space-time. When a distinction between the two space-times has to be made, we will explicitly distinguish between them, and call the interior of AdS_{d+1} the *bulk*.

5.1 Free scalar quantum field theory

Classical scalar field theory.

Let us again start from the action (3.1):

$$S_0 = -\frac{1}{2}\int \mathrm{d}^{d+1}x\sqrt{-g}\,\Big[g^{\underline{\mu\nu}}(\nabla_{\underline{\mu}}\phi)(\nabla_{\underline{\nu}}\phi) + m^2\phi^2\Big], \tag{5.1}$$

where $g = -1/(az)^{2(d+1)}$ in Poincaré coordinates. By means of eq. (3.3), the respective equation of motion can be written as

$$\Big(-a^2z^2\eta^{\mu\nu}\delta^{\underline{\mu}}_{\mu}\delta^{\underline{\nu}}_{\nu}\partial_{\underline{\mu}}\partial_{\underline{\nu}} + (d-1)a^2z\partial_z + m^2\Big)\,\phi = 0. \tag{5.2}$$

With the ansatz

$$\phi = e^{\mathrm{i}k_{\underline{\alpha}}x^{\underline{\alpha}}}J(z), \tag{5.3}$$

the equation of motion (5.2) can further be written as

$$z^2\partial_z^2 J - (d-1)z\partial_z J - \left[z^2k^2 + \frac{m^2}{a^2}\right] J = 0, \tag{5.4}$$

where $k^2 := \eta^{\mu\nu}\delta^{\underline{\alpha}}_{\mu}\delta^{\underline{\beta}}_{\nu}k_{\underline{\alpha}}k_{\underline{\beta}}$. Assuming $k^{\underline{\alpha}}$ to be time-like, i.e., $k^2 < 0$, the above differential equation has a solution in terms of Bessel functions of the first kind. Specifically, one has that

$$J(z) = z^{\frac{d}{2}}J_{\Delta-\frac{d}{2}}(k_z z), \tag{5.5}$$

for

$$\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + \frac{m^2}{a^2}} \tag{5.6}$$

and $k_z \geq 0$, is a solution of eq. (5.4) whenever

$$k_{\underline{0}} = \sqrt{k_{\underline{1}}^2 + \cdots + k_{\underline{d-1}}^2 + k_z^2}. \tag{5.7}$$

As we will see in section 5.3, the quantity Δ corresponds to the *scaling dimension*, simply called (conformal) dimension or weight, of the dual operator living on the boundary of AdS_{d+1}. In order for eq. (5.5) to describe two independent solutions, $\Delta - \frac{d}{2}$ has to be a noninteger. This will not be of great limitation for us, since the explicit model we later work on does agree with this assumption[1]. In case of an integer value for $\Delta - \frac{d}{2}$, one would have to invoke also Bessel functions of the second kind. Clearly, one should also consider the case $k^2 > 0$, for which the solutions can be written in terms of the modified Bessel functions $K_{\Delta-\frac{d}{2}}$ and $I_{\Delta-\frac{d}{2}}$. Note however that it is possible to recover these solutions through the replacement $k_z \to \mathrm{i}k_z$.

We additionally require the mass m to take values within the following range:

$$-\frac{d^2}{4} \leq \frac{m^2}{a^2} < -\frac{d^2}{4} + 1. \tag{5.8}$$

This is based on physical grounds. The lower bound is called Breitenlohner–Freedman bound [111] and is the requisite for having stable solutions[2] in AdS_{d+1}. It implies that Δ is a real number. The upper bound is also necessary as it makes the solutions normalizable. Indeed, the closure equation

$$\int_0^\infty \mathrm{d}z\ z J_{\Delta-\frac{d}{2}}(k_z z) J_{\Delta-\frac{d}{2}}(k'_z z) = \frac{\delta(k_z - k'_z)}{k_z} \tag{5.9}$$

is only valid for[3] $\Delta > (d-2)/2$. Note that, for spatial dimensions $d > 1$ as considered here, this bound requires a negative value of the mass squared, $m^2 < 0$. The minimally coupled scalar, corresponding to the "massless" case $m^2 = 0$, is given by an integer value of $\Delta - \frac{d}{2}$ and will therefore not be covered here. Thus, according to eq. (5.8), the dimension Δ is a positive real number fulfilling

$$\frac{d-2}{2} < \Delta < \frac{d+2}{2}. \tag{5.10}$$

The next step is to find the modes $u_{k_{\underline{\alpha}},k_z}, u^*_{k_{\underline{\alpha}},k_z}$, forming an orthonormal basis for the vector space $\mathrm{V}_{\mathbb{C}}$ of complex solutions to eq. (5.4). With the foliation for AdS_{d+1} introduced in section 4.1, the scalar product (3.5) reads

$$(\varphi_1, \varphi_2) = \mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}x^{\underline{1}} \cdots \mathrm{d}x^{\underline{d-1}} \int_0^\infty \mathrm{d}z\ \frac{1}{(az)^{d-1}} \left[\varphi_1^*(x)\partial_t\varphi_2(x) - \varphi_2(x)\partial_t\varphi_1^*(x)\right], \tag{5.11}$$

[1] Nevertheless, in many cases the scalar modes arising from the Kaluza–Klein reduction of string theory to AdS_{d+1} have integer value $\Delta - \frac{d}{2}$. See, for instance, ref. [31].

[2] An example for an unstable solution is the tachyonic scalar field in flat space-time, corresponding to a particle with negative mass.

[3] This is the unitarity bound for a coupled scalar operator in d-dimensional conformal field theory, see ref. [112].

for any $\varphi_1, \varphi_2 \in \mathrm{V}_{\mathbb{C}}$. By using the closure equation (5.9), it is straightforward to show that the modes

$$u_{k_{\underline{\alpha}},k_z} = \sqrt{\frac{a^{d-1}k_z}{2k_{\underline{0}}(2\pi)^{d-1}}}\, e^{\mathrm{i}k_{\underline{\alpha}}x^{\underline{\alpha}}} z^{\frac{d}{2}} J_{\Delta-\frac{d}{2}}(k_z z) \tag{5.12}$$

indeed satisfy the orthonormality relations given in eq. (3.9). Note that these modes are eigenstates of the operator $\mathrm{i}\mathcal{L}_{K_t} \equiv \partial_t$, hence we have a strict separation of the solution space $\mathrm{V}_{\mathbb{C}}$ in positive and negative frequency modes. Now we can write down the expansion of the scalar field

$$\phi = \int_{-\infty}^{\infty} \mathrm{d}k_{\underline{1}} \cdots \mathrm{d}k_{\underline{d-1}} \int_0^{\infty} \mathrm{d}k_z \left[a_{k_{\underline{\alpha}},k_z} u_{k_{\underline{\alpha}},k_z} + a^*_{k_{\underline{\alpha}},k_z} u^*_{k_{\underline{\alpha}},k_z}\right]. \tag{5.13}$$

Boundary conditions.

As yet we cheated a bit; we omitted to include the boundary of AdS_{d+1} in our considerations. This will be fixed here. First, let us see how ϕ behaves at the boundary. By sending z to 0, eq. (5.12) becomes

$$u_{k_{\underline{\alpha}},k_z} \sim z^{\Delta} \frac{\sqrt{\frac{a^{d-1}}{k_{\underline{0}}(2\pi)^{d-1}}}}{\Gamma(\Delta - \frac{d}{2} + 1)} \left(\frac{k_z}{2}\right)^{\Delta - \frac{d}{2} + \frac{1}{2}} e^{\mathrm{i}k_{\underline{\alpha}}x^{\underline{\alpha}}}. \tag{5.14}$$

For $\Delta \geq 0$, the modes are thus regular on the boundary. For positive values of the dimension Δ, and hence $m \neq 0$, the modes even vanish at the boundary. On the other hand, for $\Delta = 0$ (implying $m = 0$) the modes are finite but nonvanishing at the boundary. The interpretation of this is that particles in AdS_{d+1} have finite probability to reach the conformal boundary. Indeed, as we saw in section 4.1, null geodesics reach the boundary in a finite amount of time, whereas time-like geodesics never reach it.

Moreover, in usual quantum field theory in flat space-time, see section 2.4, a necessary but generally not sufficient requirement for the solutions to be normalizable is to vanish at infinity. The same happens here; the normalizable solutions ϕ, as given in eq. (5.13), all satisfy $\Delta > 0$ and thus vanish at the boundary.

So, in this regard, anti–de Sitter space-time and Minkowski space-time seem to agree. There is however a crucial difference. Whereas sources posed at infinity of $\mathbb{M}_{d,1}$ do not influence the physical processes taking place in the inside, in AdS_{d+1} they do, since a geodesic starting from the boundary can reach the bulk.

Hence, we expect ϕ in AdS_{d+1} to have a universal asymptotic behavior near the boundary. The standard procedure is, as one approaches the boundary, to rescale ϕ by any function of the coordinate z with a pole of order Δ at $z = 0$. This yields a finite, non-vanishing quantity on the boundary. For instance, define the rescaled scalar field of dimension Δ as follows[4]:

$$\phi_0 := z^{-\Delta}\phi. \tag{5.15}$$

[4]Note that such a definition breaks the symmetries of anti de–Sitter space-time. Hence, ϕ_0 will only be employed near the boundary.

Therefore, the boundary value is given by

$$\phi_0 \sim \bar{\phi}_\Delta := \lim_{z\to 0} \phi_0, \tag{5.16}$$

where $\bar{\phi}_\Delta$ is a z-independent function. Until now we considered the small and the large root of eq. (5.6) separately, and in that case, eq. (5.15) just works fine. However, the most general solution of the equation of motion (5.2) is a linear combination of the two fields with different Δ. Without restriction on generality, let us for the moment assume Δ is the small root of eq. (5.6). The solution thus behaves at the boundary as

$$\phi \sim z^\Delta \bar{\phi}_\Delta + z^{d-\Delta}\bar{\phi}_{d-\Delta}, \tag{5.17}$$

where also $\bar{\phi}_{d-\Delta}$ is a z-independent function. Note that ϕ_0 still satisfies $\phi_0 \sim \bar{\phi}_\Delta$ on the boundary and is thus regular. A rescaling of ϕ by $z^{\Delta-d}$ would be singular instead.

Now that it is known how a general ϕ behaves on the boundary, one might ask if there exist constraints $\bar{\phi}_\Delta$ and $\bar{\phi}_{d-\Delta}$ are subject to. Indeed, there are. Reconsider the variation of the action given in eq. (5.1) without omitting the boundary term:

$$\delta S_0 = -\int \mathrm{d}x^{d+1}\sqrt{-g}\delta\phi\left(-\Box + m^2\right)\phi + \frac{1}{2}\int_\partial \frac{\mathrm{d}x^0 \cdots \mathrm{d}x^{d-1}}{(az)^{d-1}}\left[\phi(\partial_z\delta\phi) - (\partial_z\phi)\delta\phi\right], \tag{5.18}$$

where ∂ stands for the boundary of AdS_{d+1}. Thus, contrary to our statement above, the principle of least action requires ϕ not only to obey the equation of motion (5.2), but also to fulfill some boundary conditions such that the second integral in eq. (5.18) vanishes. Such a boundary condition may be given by

$$z\partial_z\phi\big|_\partial = C\phi\big|_\partial, \tag{5.19}$$

with an arbitrary function C, normally taken to be constant. Assuming $C = \Delta$, the above condition can be restated as

$$z^{\Delta-d}\left(z\partial_z - \Delta\right)\phi\big|_\partial = 0. \tag{5.20}$$

The prefactor was included to obtain a finite relation on the boundary. This fixes $\bar{\phi}_{d-\Delta} = 0$, but places no restriction on $\bar{\phi}_\Delta$. Moreover, in terms of ϕ_0, the constraint reads $\partial_z\phi_0|_\partial = 0$ with arbitrary $\phi_0|_\partial$. Thus, eq. (5.20) imposes Neumann boundary conditions on ϕ_0. On the other hand, let us assume $C = d - \Delta$, and therefore

$$z^{-\Delta}\left(z\partial_z - d + \Delta\right)\phi\big|_\partial = 0. \tag{5.21}$$

This now requires $\bar{\phi}_\Delta = 0$ and places no restriction on $\bar{\phi}_{d-\Delta}$. Equivalently, we have that $\phi_0|_\partial = 0$ whereas $\partial_z\phi_0|_\partial$ is arbitrary. These are Dirichlet boundary conditions on ϕ_0. In the particular case where the roots coincide[5], i.e., $\Delta = d/2$, one has the freedom to either impose Neumann or Dirichlet conditions on ϕ_0.

[5]This happens to scalar fields saturating the Breitenlohner–Freedman bound.

Let us make a concluding consideration on boundary conditions. It is clear that eq. (5.19) does not describe the most general boundary condition possible; in fact the Neumann and Dirichlet conditions are the most elementary ones. One could, for instance, choose a nonconstant function C. An even more general boundary condition can be imposed by requiring [113]

$$z\partial_z\phi(x^{\underline{\alpha}}, z)\big|_\partial = \int_\partial \mathrm{d}x'^{\underline{0}} \cdots \mathrm{d}x'^{\underline{d-1}} f(x'^{\underline{\alpha}}, x^{\underline{\alpha}})\phi(x'^{\underline{\alpha}}, z), \tag{5.22}$$

where $f(x'^{\underline{\alpha}}, x^{\underline{\alpha}})$ is symmetric in $x^{\underline{\alpha}}$ and $x'^{\underline{\alpha}}$. Nevertheless, in this thesis we proceed by considering the Neumann and Dirichlet boundary conditions as described above and we simply denote $\bar{\phi}_\Delta$ and $\bar{\phi}_{d-\Delta}$ by $\bar{\phi}$.

Canonical quantization.

By promoting $a_{k_{\underline{\alpha}},k_z}, a^*_{k_{\underline{\alpha}},k_z}$ to operators and imposing the (equal time) canonical commutation relations (3.25), where the canonically conjugate variable (cf. eq. (3.24)) is given by

$$\pi = \frac{\partial_t\phi}{(az)^{d-1}}, \tag{5.23}$$

one gets, as expected,

$$\begin{aligned} [a_{k_{\underline{\alpha}},k_z}, a^\dagger_{k'_{\underline{\alpha}},k'_z}] =& \delta(k_{\underline{1}} - k'_{\underline{1}}) \cdots \delta(k_{\underline{d-1}} - k'_{\underline{d-1}})\delta(k_z - k'_z), \\ [a_{k_{\underline{\alpha}},k_z}, a_{k'_{\underline{\alpha}},k'_z}] =& [a^\dagger_{k_{\underline{\alpha}},k_z}, a^\dagger_{k'_{\underline{\alpha}},k'_z}] = 0. \end{aligned} \tag{5.24}$$

Again, we define the vacuum state as the state annihilated by all operators $a_{k_{\underline{\alpha}},k_z}$ and construct a basis of the Fock space $\mathscr{F}$ by consequent application of $a^\dagger_{k_{\underline{\alpha}},k_z}$ on that state. Although the vacuum may not be unique, this is not of concern to us. What is fundamental is that it is invariant under all Killing vectors (4.31) of AdS_{d+1}.

Symmetries.

The symmetry properties of AdS_{d+1} were already discussed in section 4.2. To each Killing vector in eq. (4.31) there is a conserved charge, defined by eq. (3.12). In particular, also the Hamilton operator, the charge related to the global time-like Killing vector K_t, is a conserved charge.

The propagators.

Computing the field commutator at different space-time points $x_1, x_2 \in \mathrm{AdS}_{d+1}$ gives

$$\begin{aligned} [\phi(x_1), \phi(x_2)] =& -\mathrm{i}a^{d-1} z_1^{\frac{d}{2}} z_2^{\frac{d}{2}} \int_0^\infty \mathrm{d}k_z\ k_z J_{\Delta-\frac{d}{2}}(k_z z_1) J_{\Delta-\frac{d}{2}}(k_z z_2) \\ &\times [G_+(x_1 - x_2; k_z) - G_-(x_1 - x_2; k_z)], \end{aligned} \tag{5.25}$$

where $G_\pm(x_1 - x_2; k_z)$ are the advanced and retarded propagators of flat $\mathbb{M}_{d-1,1}$ space-time, cf. eq. (2.112). In fact, all Green functions in Poincaré coordinates can be derived from the Green functions of Minkowski space-time, and, in particular, also the Feynman propagator of anti–de Sitter space-time [114]. Given that the chosen boundary conditions are either Neumann or Dirichlet, the singularities of these propagators are of the Hadamard form [115, 116].

Let us assume for the moment that $z_1 = z_2$ take a fixed value. One may argue that the field commutator (5.25) satisfies the microcausality condition (A6) within the leaf defined by the fixed radial coordinate. So it appears that the restriction to that leaf yields a description of a particle propagating in $(d-1)$-dimensional Minkowski space-time. Moreover, it seems reasonable to interpret k_z as the mass of the particle since, according to eq. (5.7), also the spectrum condition (A1) is then straightforwardly satisfied. However, this interpretation turns out to be wrong, as k_z takes any nonnegative value and therefore does not yield a discrete particle spectrum. Nevertheless, the idea is not completely invalid, as we will explain now.

Brane restriction.

The Poincaré coordinates of anti–de Sitter space-time make it evident that each value of z defines a time-like hypersurface. One might ask if it is possible to restrict quantum fields of AdS_{d+1} on these branes at fixed z. More precisely, can we construct a one-parameter family of quantum field theories, each living on the corresponding space-time submanifold with the topology $\mathbb{R} \times \mathrm{S}^{d-1}$?

Let us first address this question in flat Minkowski space-time, where it is already a nontrivial statement since quantum fields are distributions which only become operators by smearing them with appropriate test functions. It was shown in ref. [117] that it is only necessary to smear the field operators in the time-like direction alone in order to get meaningful operators. Hence, quantum fields restricted on time-like hypersurfaces exist as distributions in a space-time of one dimension less, whereas on space-like hypersurfaces they do not.

These restricted quantum fields on a time-like hypersurface obviously inherit many properties from the Wightman axioms (A1-A6) satisfied by the original quantum field. The first thing to note is that the Hilbert space for both theories is the same. Thus, the domain condition (A2), the vacuum condition (A3), and the completeness condition (A4) instantly follow. Moreover, also the spectrum condition (A1) is preserved, since the forward cone C_+ is contained in the forward cone of the restricted quantum field. The same applies for the microcausality condition (A6); space-like separations on the hypersurface are also space-like separations on the original space-time, hence the field commutator vanishes. Only the covariance condition (A5) requires a limitation. It is expected that only the hypersurface-preserving subgroup acts on the restricted fields.

It is thus clear that a quantum field on Minkowski space-time, restricted on a time-like hypersurface, describes a full quantum field theory in the Wightman sense. This result has been generalized to the class of warped manifolds, which also includes the anti–de Sitter space-time [118].

Therefore, we expect the brane theory at fixed values of z to satisfy the Wightman axioms of a Poincaré covariant quantum field theory in d-dimensional Minkowski space-time $\mathbb{M}_{d-1,1}$. None of these theories are conformally covariant since the family parameter z sets a scale. But the brane restriction on the limiting brane $z = 0$ is different. For scalar fields, it was shown that the boundary field $\bar{\phi}$ defines also a Wightman quantum field theory in d-dimensional Minkowski space-time $\mathbb{M}_{d-1,1}$ [119]. However, this quantum field theory comes with an enhanced symmetry group, the conformal group, inherited from the isometry group of AdS_{d+1}. As we will see in the next section, the theory of a free massive scalar field in AdS_{d+1} induces a fully consistent conformal field theory on the boundary, yielding a simple example of the AdS/CFT correspondence.

As a concluding remark we recall that the analytic continuation of the Wightman distributions illustrated in section 2.3 was performed using the Wightman axioms, in particular axioms (A1,A5,A6). Since each brane of AdS_{d+1} allows a description of a quantum field theory in terms of the Wightman axioms, one can analytically continue the Wightman distributions at each brane separately. Thus, the Wick rotation is well-defined if performed in the variables $x^{\underline{\alpha}}$ alone [120].

5.2 Correlation functions

From now on until chapter VII, we will deal exclusively with Euclidean anti–de Sitter space (EAdS, also called *hyperbolic space*) $\mathbb{H}_{d+1}$. Specifically, we employ a massive scalar field theory on $\mathbb{H}_{d+1}$ with Euclidean action

$$S_0[\phi] = \frac{1}{2}\int \mathrm{d}^{d+1}x\sqrt{g}\left[g^{\underline{\mu\nu}}(\nabla_{\underline{\mu}}\phi)(\nabla_{\underline{\nu}}\phi) + m^2\phi^2\right]. \tag{5.26}$$

This theory is related by a Wick rotation to the scalar quantum field theory on AdS_{d+1} given by the action (5.1). The reason to use the Euclidean counterpart is multifold. First, in $\mathbb{H}_{d+1}$ causality is not a concern. Second, we will make extensive use of the Schwinger parametrization trick, which requires Euclidean signature of the metric to be rigorous. And third, the *conformal coordinates*, the Euclidean version of the Poincaré coordinates to AdS_{d+1}, cover the whole space $\mathbb{H}_{d+1}$ instead of only half of it.

Geometry of hyperbolic space.

Let us shortly review some properties of Euclidean anti-de Sitter space. Analogously to AdS_{d+1}, the hyperbolic space $\mathbb{H}_{d+1}$, can be embedded into a flat ambient space, this time however in Minkowski space-time $\mathbb{M}_{d+1,1}$. Precisely, $\mathbb{H}_{d+1}$ is one of the sheets of the two-sheeted hyperboloid satisfying the quadric (see figure 3)

$$(X^0)^2 + (X^1)^2 + \cdots + (X^d)^2 - (X^{d+1})^2 = -\frac{1}{a^2}. \tag{5.27}$$

Note that the quadric (4.6) defining AdS_{d+1} is related to the above quadric (5.27) via the substitution $X^0 \to \mathrm{i}X^0$. Thus, in terms of the Poincaré coordinates, this is indeed tantamount to Wick-rotating the time coordinate $t \to \mathrm{i}t$. As asserted in section 5.1, this

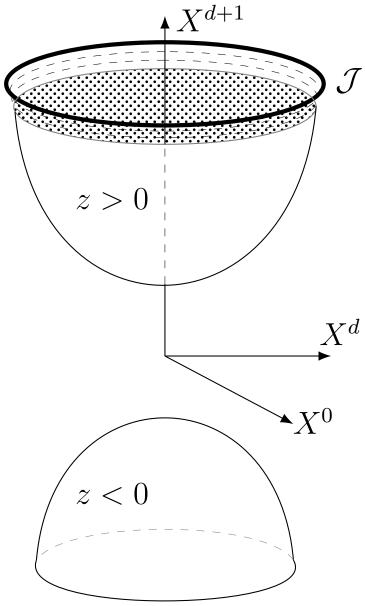


Figure 3: Euclidean anti de–Sitter space and its boundary $\mathcal{J}$ at infinity.

Wick rotation is well-defined since it is applied to a nonradial coordinate.
The isometry group of $\mathbb{H}_{d+1}$ is given by $O(d+1,1)$. Moreover, it is topologically equivalent to a $(d+1)$-dimensional open ball B_{d+1} [31], and thus the conformal completion corresponds to the topological closure of the open ball. Therefore, the conformal boundary, given by the limit points of B_{d+1}, is topologically equivalent to a d-dimensional sphere S^d. In order to relate this to the discussion of section 4.3, recall that, by means of the stereographic projection, the sphere S^d is homeomorphic to $\mathbb{R}^d \cup \{\infty\}$.

We will generally proceed without altering the notation when passing from Lorentzian quantities to their Euclidean counterparts. Thus, from now on any appearing quantity is intended as an Euclidean quantity unless otherwise stated. Owing to the distinguished significance of the coordinate z, let us, in what follows, denote a point $x \in \mathbb{H}_{d+1}$ as $x = (z, x^{\underline{i}})$ or, alternatively, as $x = (z, x^i)$. In other words, let us identify $x^{\underline{0}}$ with z. Furthermore, let us denote

$$x^{i^2} = \delta_{ij}\delta^i_{\underline{i}}\delta^j_{\underline{j}}x^{\underline{i}}x^{\underline{j}}. \tag{5.28}$$

In this notation, the squared line element in conformal coordinates reads

$$\mathrm{d}s^2 = \frac{1}{a^2 z^2}(\mathrm{d}z^2 + \mathrm{d}x^{i^2}), \tag{5.29}$$

and hence $g = 1/(az)^{2(d+1)}$.

We will make extensive use of a dimensionless quantity, K, related to the geodesic distance ρ (see appendix A for its derivation in AdS_{d+1}) as follows

$$K := \frac{1}{\cosh a\rho}. \tag{5.30}$$

Expressed in conformal coordinates, it reads

$$K = \frac{2zw}{(x^i - y^i)^2 + z^2 + w^2}. \tag{5.31}$$

This quantity is invariant under the Lorentz group $O(d+1,1)$. For further reference we note that, at the conformal boundary, we have

$$K^\Delta \sim z^\Delta \bar{K}^\Delta, \tag{5.32}$$

where

$$\bar{K} = \frac{2w}{(x^i - y^i)^2 + w^2} \tag{5.33}$$

is the usual bulk-to-boundary propagator [31]. Clearly, this quantity vanishes at the boundary and needs to be rescaled, as explained in section V. Nevertheless, in what follows, we will leave the procedure implicit.

Later on, it will also be useful to know how K behaves in the *flat space limit*. This limit corresponds to sending the parameter a to 0, as one can easily see from eq. (4.7). By introducing d-dimensional hyperspherical coordinates with radius

$$R := \sqrt{(X^A)^2 + (X^{d+1})^2}, \tag{5.34}$$

we can rewrite the squared line element as

$$\mathrm{d}s^2 = \frac{1}{1 + a^2 R^2}\mathrm{d}R^2 + R^2 \mathrm{d}\Omega_d^2, \tag{5.35}$$

where $\mathrm{d}\Omega_d^2$ is the metric on a d-dimensional hypersphere S^d. To simplify matters, let us settle down to a standard point $Y^{d+1} = \frac{1}{a}$, $Y^0 = \ldots = Y^d = 0$. Then, it is straightforward to show that the relation between the radial coordinate R and K is given by

$$K = \frac{1}{\sqrt{a^2 R^2 + 1}}, \tag{5.36}$$

and hence the flat space limit yields

$$K = 1 - \frac{1}{2}a^2 R^2 + \mathcal{O}(a^4). \tag{5.37}$$

The scalar propagator.

Let us now compute the propagator $\Lambda(x, y; m)$ for the scalar field. By definition, $\Lambda(x, y; m)$ satisfies the equation of motion following from the variation of the action (5.26):

$$\left(-\Box + m^2\right)\Lambda(x, y; m) = \frac{1}{\sqrt{g}}\delta^{(d+1)}(x - y), \tag{5.38}$$

where m is the mass of the scalar field, see eq. (5.26). Making use of the fact that Λ is a function of the geodesic distance (or equivalently K), we find that

$$\Box = a^2 K^2 (1-K^2)\frac{\partial^2}{\partial K^2} - a^2 K\left(d-1+2K^2\right)\frac{\partial}{\partial K}. \tag{5.39}$$

A derivation of the above formula is given in appendices B and C. Right now, it suffices to note that $\Box$ is proportional to $J_{AB}J^{AB}$, and the proportionality factor can be fixed by taking the flat space limit $a \to 0$. Instead of giving right away the general solution of eq. (5.38), let us first study the asymptotic behavior of the above differential equation at the boundary of $\mathbb{H}_{d+1}$. In the limit, $K \to 0$, the homogeneous part of eq. (5.38) becomes an Euler differential equation:

$$\left(-a^2K^2\frac{\partial^2}{\partial K^2} + a^2K(d-1)\frac{\partial}{\partial K} + m^2\right)\Lambda(K;m) = 0. \tag{5.40}$$

One can easily check that $\Lambda(K;m) = K^\Delta$, with Δ given in eq. (5.6), solves the above equation. Note that to each value of the mass m, there usually correspond two different values of Δ. In this regard, bearing in mind the relation between m and Δ, let us henceforth express the propagator by $\Lambda(x,y;\Delta)$ or $\Lambda(K;\Delta)$, i.e., as a function of Δ instead of m. The solutions to eq. (5.38) are well-known (e.g., ref. [121]) and can be expressed in terms of hypergeometric functions ${}_2F_1$:

$$\Lambda(K;\Delta) = C_\Delta K^\Delta\ {}_2F_1\left[\frac{\Delta}{2},\frac{\Delta+1}{2};\Delta+1-\frac{d}{2};K^2\right]. \tag{5.41}$$

As expected, in the limit $K \to 0$, the above solutions are proportional to K^Δ. Moreover, these solutions correspond to the propagators satisfying the Neumann and Dirichlet boundary conditions. The coefficient C_Δ is fixed by eq. (5.38). In fact, using hyperspherical coordinates, see eq. (5.34), and integrating eq. (5.38) over a small ball containing $R=0$, leads to

$$C_\Delta = \frac{a^{d-1}\Gamma(\frac{\Delta}{2})\Gamma(\frac{\Delta+1}{2})}{4\pi^{\frac{d+1}{2}}\Gamma(\Delta+1-\frac{d}{2})}. \tag{5.42}$$

In the flat space limit, $\Lambda(K;\Delta)$ reduces[6] to the Green's function of the Laplacian in $\mathbb{R}^{d+1}$:

$$\Lambda(K;\Delta) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2(d-1)\pi^{\frac{d+1}{2}}R^{d-1}} + \mathcal{O}(a). \tag{5.43}$$

The higher-point functions.

The correlation functions (see section 2.3) of our free scalar field theory can be written entirely in terms of the scalar propagator (5.41). To this aim, let us introduce the generating functional:

$$Z_0[j] := \int \mathcal{D}\phi\ e^{-S_0[\phi]+\int \mathrm{d}^{d+1}x\ j(x)\phi(x)}, \tag{5.44}$$

[6] up to a sign, due to eq. (5.38)

where j is the source function, and the integral runs over all possible field configurations $\phi(x)$. In the case under consideration, $Z_0[j]$ can be found to be [81]

$$Z_0[j] = Z_0[0]e^{\frac{1}{2}\int \mathrm{d}^{d+1}x\ \mathrm{d}^{d+1}y\ j(x)\Lambda(x,y;\Delta)j(y)}. \tag{5.45}$$

The connected correlation functions are generated via the relation

$$\langle\phi(x_1)\cdots\phi(x_N)\rangle_0 = \frac{1}{Z_0[0]}\frac{\delta^N}{\delta j(x_1)\cdot\delta j(x_N)}Z_0[j]\big|_{j=0}, \tag{5.46}$$

and hence, the two- and four-point functions read, respectively,

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\rangle_0 =&\Lambda(x_1,x_2;\Delta),\\ \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_0 =&\Lambda(x_1,x_2;\Delta)\Lambda(x_3,x_4;\Delta)\\ &+\Lambda(x_1,x_3;\Delta)\Lambda(x_2,x_4;\Delta)+\Lambda(x_1,x_4;\Delta)\Lambda(x_2,x_3;\Delta).\end{aligned} \tag{5.47}$$

5.3 The holographic correlators and the CFT dual

In order to find the dual to the theory of a free massive scalar on $\mathbb{H}_{d+1}$, let us first take the correlators in eq. (5.47) and send them to the boundary. The resulting *Witten two- and four-point functions* are then given by

$$\begin{aligned}\langle\bar\phi(x_1)\bar\phi(x_2)\rangle_0 =&\frac{2^\Delta C_\Delta}{r_{12}^{2\Delta}},\\ \langle\bar\phi(x_1)\bar\phi(x_2)\bar\phi(x_3)\bar\phi(x_4)\rangle_0 =&\frac{4^\Delta C_\Delta^2}{(r_{12}r_{34})^{2\Delta}}\left[1+v^\Delta+\frac{v^\Delta}{(1-Y)^\Delta}\right],\end{aligned} \tag{5.48}$$

where $r_{ij}^2 := (x^i-x^j)^2$ and where we introduced the *conformal invariants*

$$v = \frac{r_{12}^2 r_{34}^2}{r_{14}^2 r_{23}^2}, \qquad Y = 1-\frac{r_{13}^2 r_{24}^2}{r_{14}^2 r_{23}^2}. \tag{5.49}$$

The original prescription of the AdS/CFT correspondence is given in the following terms: consider the partition function of the fields ϕ, with fixed boundary values $\bar\phi$, and interpret these boundary values as the source for the operator $\mathcal{O}_\Delta$ of the conformal field theory [31]. Although this proposal has been proven reliable in many cases, we will use a more immediate, yet equivalent, prescription [122]. Specifically, we interpret the boundary limit $\bar\phi$ of the bulk field ϕ as the dual to the conformal field theory operator $\mathcal{O}_\Delta$. In this framework, the boundary limit of the bulk correlation functions, the Witten correlators, correspond directly to the correlation functions of the dual operator $\mathcal{O}_\Delta$. Concerning our case of a free massive scalar field theory, we identify

$$\langle\bar\phi(x_1)\cdots\bar\phi(x_N)\rangle_0 \equiv \langle\mathcal{O}_\Delta(x_1)\cdots\mathcal{O}_\Delta(x_N)\rangle, \tag{5.50}$$

and, therefore, the dual operator $\mathcal{O}_\Delta$ is a real scalar operator satisfying

$$\begin{aligned}\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle =& \frac{2^\Delta C_\Delta}{r_{12}^{2\Delta}},\\ \langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\mathcal{O}_\Delta(x_4)\rangle =& \frac{4^\Delta C_\Delta^2}{(r_{12}r_{34})^{2\Delta}}\left[1+v^\Delta+\frac{v^\Delta}{(1-Y)^\Delta}\right].\end{aligned} \tag{5.51}$$

An immediate consequence is that $\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle$ does not satisfy a linear second order differential equation, unless $\Delta = (d-2)/2$. This can be seen by dimensional analysis. Hence, the CFT operator $\mathcal{O}_\Delta$ is not, in general, a free operator. However, as one can see, for example, from the four-point function in eq. (5.48), it does obey a factorization property:

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\mathcal{O}_\Delta(x_4)\rangle = \langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle\langle \mathcal{O}_\Delta(x_3)\mathcal{O}_\Delta(x_4)\rangle + \text{permutations}.$$

It can easily be checked that, by means of this factorization property, all correlation functions are fully determined by the two-point function. Hence, these operators are generalized free fields, which were already introduced at the end of section 2.4.

It would be of great interest to know which kind of theory of generalized free fields emerges at the boundary. While it is very difficult to indicate the explicit model, a detailed study can reveal much, or even all, of its properties. The most promptly evident one is that we are indeed dealing with a conformal field theory, as the correlation functions for the scalar operator $\mathcal{O}_\Delta$ of dimension Δ are exactly of the form dictated by conformal invariance [123]; the correlation functions (5.51) behave covariantly under the conformal transformations generated by P_i, L_{ij}, K_i and D given in eq. (4.36). Note that the dimension Δ gives the scaling behavior under dilatations of the operator $\mathcal{O}_\Delta$ at the origin:

$$[D,\mathcal{O}_\Delta(0)] = \Delta\mathcal{O}_\Delta(0). \tag{5.52}$$

Additionally, P_i and K_i act, respectively, as raising and lowering operators on the CFT operators at the origin:

$$[D,[P_i,\mathcal{O}_\Delta(0)]] = (\Delta+1)[P_i,\mathcal{O}_\Delta(0)], \qquad [D,[K_i,\mathcal{O}_\Delta(0)]] = (\Delta-1)[K_i,\mathcal{O}_\Delta(0)]. \tag{5.53}$$

So how do we examine the conformal field theory on the boundary? The main target may be to find the full spectrum of the theory, but it suffices to find all the *primaries*, labeled by $\mathcal{O}_I$ with index I, contained in the spectrum of our CFT. Primaries of dimension Δ_I are operators of lowest conformal weight:

$$[D,\mathcal{O}_I(0)] = \Delta_I\mathcal{O}_I(0), \qquad [K_i,\mathcal{O}_I(0)] = 0. \tag{5.54}$$

Note that these conditions are preserved by the action of L_{ij} on $\mathcal{O}_I(0)$ and hence the space of primaries carries representations of the subgroup $SO(d)$, labeled by the spin quantum number l. Examples of primaries are the unit operator and $\mathcal{O}_\Delta$ itself. The remaining operators of the theory, the *descendants*, can be constructed out of the primaries by acting repeatedly with the momentum generator P_i on them.

A full characterization of all primaries can be realized by comparing the correlation functions with the *operator product expansion* (OPE). The OPE between two primaries reads [123]

$$\mathcal{O}_I(x_1)\mathcal{O}_J(x_2) = \sum_K \mathcal{A}^{1/2}_{IJK} \left(\frac{\mathcal{O}_K(x_1)}{r_{12}^{\Delta_I+\Delta_J-\Delta_K}} + \text{descendants} \right), \tag{5.55}$$

where $\mathcal{A}^{1/2}_{IJK}$ are called[7] *OPE coefficients*. Now, applying the OPE on the two- and four point functions given in eq. (5.51), we can infer that the conformal field theory contains a tower of conformal primaries of conformal dimension $2\Delta + 2n + l$ of all even spins l. Schematically, these operators are of the form $:\mathcal{O}_\Delta \Box^n \partial^l \mathcal{O}_\Delta:$. We will discuss this in detail in section 6.5, where an explicit derivation will be given via *conformal blocks* [124] instead of the operator product expansion.

Our boundary theory satisfies *crossing symmetry*, which originates from the associativity of the operator product expansion applied on the correlation functions, and[8] *unitarity*. The latter follows from the requirement of a positive definite inner product and reflects in the following constraints for any primary [123]:

$$\Delta_I \geq \begin{cases} \frac{d-2}{2} & \text{for } l = 0, \\ d+l-2 & \text{for } l > 0, \end{cases} \tag{5.56}$$

with exception of the unit operator, whose dimension is zero. If the primary satisfies the above inequality, then also the related descendants do.

Hence, a free massive scalar field theory in AdS_{d+1} with $\Delta \geq (d-2)/2$ defines a consistent conformal field theory on the boundary. The main scope of this thesis will be to test the validity of the AdS/CFT correspondence on a deformation of the above theory. At tree level, hence by considering solely classical contributions to the interaction, the duality is already successfully verified [125]. It will become clear in the next chapter that the dual theory is not anymore described by generalized free fields, since the factorization property is absent. Another novelty is that the dimensions of the operators receive corrections called *anomalous dimensions*, and we will derive them explicitly.

[7] Interestingly, a data set containing the OPE coefficient and the dimension of each primary uniquely fixes the conformal field theory.

[8] In Euclidean space, unitarity is also called reflection positivity.

VI Interacting scalar QFT in EAdS

In the previous chapter we discussed the AdS/CFT correspondence in the setting of a free scalar field theory in (Euclidean) anti–de Sitter space, which was shown to lead to a consistent CFT dual on its boundary. The verification of the duality was possible since the free theory is analytically solvable. As soon as an interaction is turned on, full analytical solutions may not be known, as it is the case for the model considered here. Hence, a verification of the AdS/CFT correspondence can, at best, only be performed perturbatively. In this chapter we will compute the two- and four-point functions up to second order in the coupling constant. The obtained results will then be compared with a general conformal field theory at the boundary.

Consider a scalar field with a quartic self–interaction propagating on a four-dimensional Euclidean anti–de Sitter $\mathbb{H}_4$ background, described by the action

$$S[\phi] = \int \mathrm{d}^4x\sqrt{g}\left(\frac{1}{2}g^{\underline{\mu\nu}}(\nabla_{\underline{\mu}}\phi)(\nabla_{\underline{\nu}}\phi) + \frac{m^2}{2}\phi^2 - \lambda\frac{\phi^4}{4!}\right), \tag{6.1}$$

where[1] $\lambda > 0$ is a small dimensionless coupling constant. The classical equation of motion is given by

$$(-\Box + m^2)\phi = \lambda\frac{\phi^3}{6}. \tag{6.2}$$

We take the scalar field to be conformally coupled, i.e., we require the action to be invariant under Weyl transformations. Let us write the conformal factor as $\Omega(x) = e^{2\omega(x)}$, with some scalar function $\omega(x)$. Then, the metric changes infinitesimally as

$$\delta g^{\underline{\mu\nu}} = 2\omega(x)g^{\underline{\mu\nu}}. \tag{6.3}$$

Assuming m to be dependent on the metric, the requirement of Weyl invariance can be written as

$$0 = \delta S = \int \mathrm{d}^4x\left[\frac{\sqrt{g}}{2}T_{\underline{\mu\nu}}\,\delta g^{\underline{\mu\nu}} + \frac{\delta S[\phi]}{\delta\phi}\,\delta\phi\right] = \int \mathrm{d}^4x\sqrt{g}\,\omega(x)T^{\underline{\mu}}_{\underline{\mu}}, \tag{6.4}$$

where $\frac{\delta S[\phi]}{\delta\phi} = 0$ by means of eq. (6.2). In other words, the trace of the (Euclidean) stress-tensor

$$T_{\underline{\mu\nu}} = \frac{2}{\sqrt{g}}\frac{\delta S[\phi]}{\delta g^{\underline{\mu\nu}}} \tag{6.5}$$

has to vanish on shell. This leads to the constraint

$$g^{\underline{\mu\nu}}\frac{\delta m^2}{\delta g^{\underline{\mu\nu}}}\phi^2 = \frac{1}{2}\Box\phi^2 + m^2\phi^2. \tag{6.6}$$

[1] Although the sign of the interaction term is inverted in comparison to standard literature, it does not affect any of our conclusions. The usual sign can be restored by systematically substituting $\lambda \to -\lambda$.

By using the following identity for the variation of the Ricci tensor

$$\frac{g^{\underline{\rho\sigma}}\,\delta R_{\underline{\rho\sigma}}}{\delta g^{\underline{\mu\nu}}}\phi^2 = \left(g_{\underline{\mu\nu}}\Box - \nabla_{\underline{\mu}}\nabla_{\underline{\nu}}\right)\phi^2, \tag{6.7}$$

which can be derived by double partial integrating the Palatini identity, it is straightforward to show that eq. (6.6) admits the solution $m^2 = \frac{R}{6}$. In $\mathbb{H}_4$, the Ricci scalar takes the value $R = -12a^2$, see eq. (4.3). Therefore, in what follows, we set $m^2 = -2a^2$, corresponding to the dimensions $\Delta = 1, 2$ as can be seen from eq. (5.6). Note that these values for Δ satisfy inequality (5.10). Moreover, the propagator (5.41) simplifies to

$$\Lambda(K;\Delta) = \frac{a^2 K^{\Delta}}{4\pi^2(1-K^2)} \qquad \text{for } \Delta = 1, 2. \tag{6.8}$$

One may ask why we bother taking a conformally coupled scalar. Although this is certainly not obligatory, there are at least three good reasons for this. First, it preserves the scale-invariance property of the classical[2] ϕ^4 theory of a massless scalar field in Minkowski space-time $\mathbb{M}_{3,1}$, and hence, it can be thought as the most meaningful generalization of masslessness in curved space-times. Second, due to Weyl invariance, it has the nice property that, if ϕ is a classical solution in one space-time, it is also a solution in any space-time differing by a Weyl transformation from the original one. And last but not least, it greatly simplifies the form of the propagator, see eq. (6.8).

6.1 Correlation functions

In analogy to what was done in eq. (5.44), let us introduce the generating functional:

$$Z[j] := \int \mathcal{D}\phi\; e^{-S[\phi] + \int \mathrm{d}^4x\; j(x)\phi(x)}, \tag{6.9}$$

which can be rewritten as

$$Z[j] = e^{\int \mathrm{d}^4x \sqrt{g}\, \frac{\lambda}{4!}\left(\frac{\delta}{\delta j(x)}\right)^4} Z_0[j], \tag{6.10}$$

where $Z_0[j]$ is the generating functional of the four-dimensional free theory, given by imposing $d = 3$ in eq. (5.45). Therefore, it follows directly that

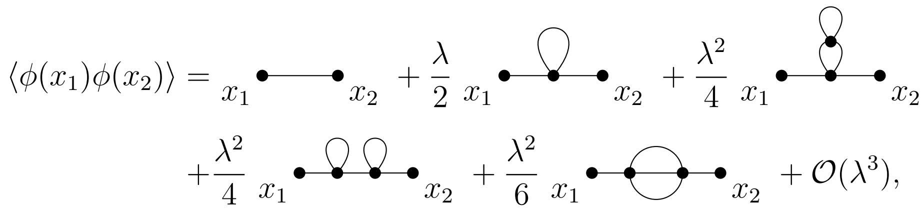
$$Z[0] = e^{\int \mathrm{d}^4x \sqrt{g}\; \frac{3\lambda}{4!}\Lambda(x,x;\Delta)^2}. \tag{6.11}$$

The interacting, connected correlation functions are related to $Z[j]$ via

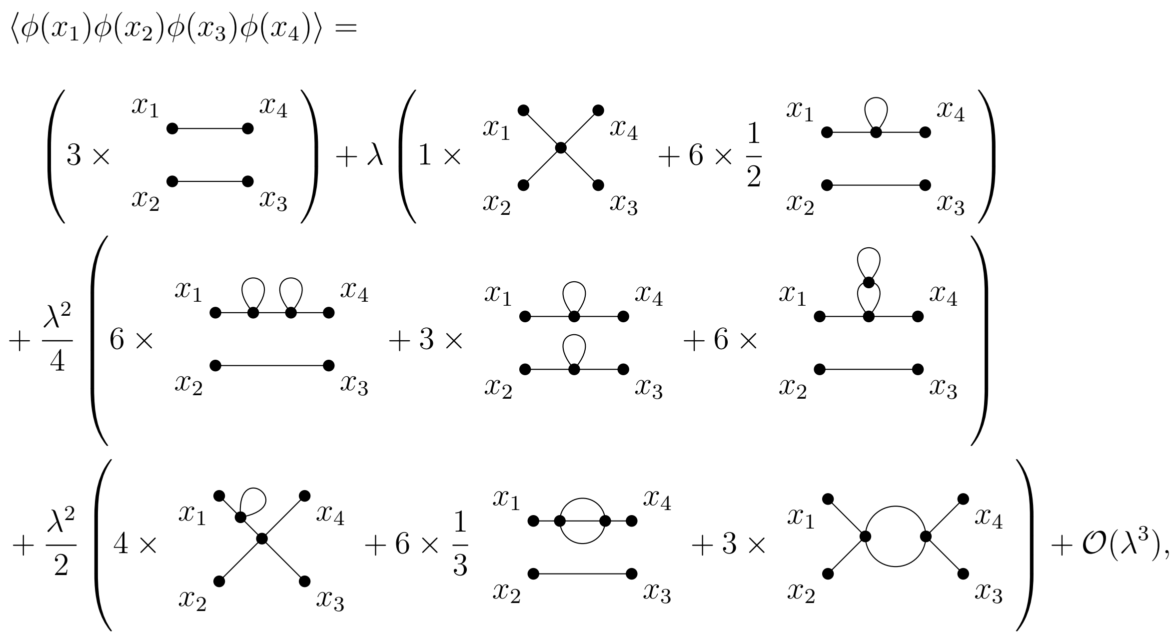
$$\langle \phi(x_1)\cdots\phi(x_N)\rangle = \frac{1}{Z[0]}\frac{\delta^N}{\delta j(x_1)\cdots\delta j(x_N)} Z[j]\big|_{j=0}. \tag{6.12}$$

[2] Quantum effects spoil Weyl invariance, as one can see from the nonzero beta function [81].

This leads to expressions for the two- and four-point functions, given diagrammatically by

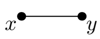


(6.13)



(6.14)

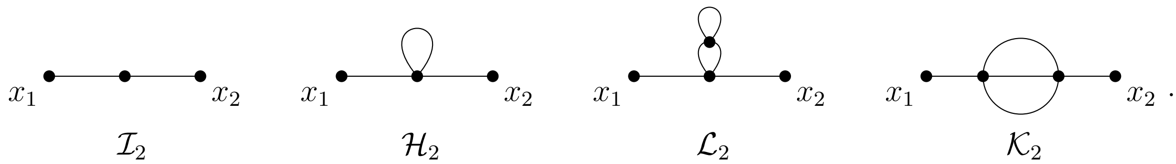
where we adopted the following Feynman rules:

- each line  corresponds to the free two-point function $\Lambda(x, y; \Delta)$,
- each four-vertex  stands for an integral $\int \mathrm{d}^4x\sqrt{g}$ over the vertex point x.

One of the purposes of this work is to calculate all the above diagrams. In the following two sections, 6.2 and 6.3, we will focus on the one-particle irreducible diagrams, since all the other diagrams are simple products and/or permutations of the latter. At the beginning of each of these sections, there will be a figure displaying all the computed diagrams in the given section. The reader not interested in the details of the calculation can skip to section 6.5, where a summary of the obtained results is available.

6.2 Two-point function

In this section, we compute the one-particle irreducible diagrams that contribute to the two-point function (6.13), or explicitly



(6.15)

As a warm-up we compute the mass shift diagram $\mathcal{I}_2$, and proceed afterwards in the calculation of the tadpole diagram $\mathcal{H}_2$ and the double tadpole diagram $\mathcal{L}_2$. Eventually, we discuss the technically more challenging sunset diagram $\mathcal{K}_2$. For $\Delta = 1$ we will encounter infrared divergences which are absent for $\Delta = 2$.

6.2.1 The mass shift diagram

Strictly speaking, this diagram is not needed in our analysis. Nevertheless, it will appear as a counterterm of other diagrams. It is therefore convenient to have its form in order to identify such terms. The mass shift diagram $\mathcal{I}_2$ depicted in figure (6.15) corresponds to the following integral

$$\mathcal{I}_2 = \int \mathrm{d}^4x \sqrt{g(x)}\Lambda(x_1, x; \Delta)\Lambda(x_2, x; \Delta). \tag{6.16}$$

Let us for the moment set the two points x_1, x_2 to $x_1 = (z_1, 0), x_2 = (z_2, 0)$ (the covariant form will be restored later). By further denoting $x^2 \equiv {x^i}^2$, the integral is then

$$\mathcal{I}_2 = \frac{(4z_1z_2)^\Delta}{(4\pi^2)^2} \int \mathrm{d}^4x \frac{z^{2\Delta-4}(x^2+z^2+z_1^2)^{2-\Delta}(x^2+z^2+z_2^2)^{2-\Delta}}{\Pi_{i=1}^2[x^2+(z+z_i)^2][x^2+(z-z_i)^2]}, \tag{6.17}$$

where we have used that $\sqrt{|g(x)|} = \frac{1}{a^4z^4}$. As already mentioned, the above integral features an IR divergence for $\Delta = 1$, but not for $\Delta = 2$. To continue, we introduce a dimensionless regulator $\sigma > 0$ which does not affect the $\Delta = 2$ case:

$$\mathcal{I}_2 = \frac{(4z_1z_2)^\Delta}{(4\pi^2)^2} \int \mathrm{d}^4x \frac{(z^2+\sigma^2 f^2(z_1,z_2))^{\Delta-2}(x^2+z^2+z_1^2)^{2-\Delta}(x^2+z^2+z_2^2)^{2-\Delta}}{\Pi_{i=1}^2[x^2+(z+z_i)^2][x^2+(z-z_i)^2]}, \tag{6.18}$$

where $f(z_1, z_2)$ is a nonnegative function which will be determined later by imposing covariance.

$\Delta = 1$.
The integral (6.18) can be integrated directly. Performing the z-integral and then integrating over x using three-dimensional spherical coordinates yields in the limit of small σ

$$\mathcal{I}_2 = \frac{z_1 z_2 (z_1+z_2)^{-1}}{4\pi\sigma f(z_1,z_2)} + \frac{z_1 z_2}{4\pi^2 (z_1^2 - z_2^2)^2}\left[(z_1^2+z_2^2)\log\frac{16 z_1^2 z_2^2}{(z_1^2-z_2^2)^2} + 2 z_1 z_2 \log\frac{(z_1-z_2)^2}{(z_1+z_2)^2}\right]. \tag{6.19}$$

Then, the covariant form of the IR-finite contribution reads

$$\mathcal{I}_2^{\text{regular}} = \frac{1}{8\pi^2}\left[\frac{K}{1-K^2}\log 4 + \frac{K}{1-K^2}\log\frac{K^2}{1-K^2} + \frac{K^2}{1-K^2}\log\frac{1-K}{1+K}\right]. \tag{6.20}$$

On the other hand, the IR-divergent term is not generally covariant for a generic choice of $f(z_1, z_2)$. To determine this function we note that a covariant regularization can be obtained by continuation in Δ, for instance. However, for noninteger values of Δ the scalar propagator on AdS is complicated. Still we can proceed, using the fact that for any covariant infrared regularization, $\mathcal{I}_2$ has to obey the inhomogenenous differential equation

$$(-\Box_{x_1} - 2a^2)\,\mathcal{I}_2 = \Lambda(x_1, x_2; \Delta). \tag{6.21}$$

For $\Delta = 1$, the most general covariant solution is

$$\mathcal{I}_2 = c_1\frac{K}{1-K^2} + c_2\frac{K^2}{1-K^2} + \frac{1}{8\pi^2}\left[\frac{K}{1-K^2}\log\frac{K^2}{1-K^2} + \frac{K^2}{1-K^2}\log\frac{1-K}{1+K}\right], \tag{6.22}$$

where the homogeneous part with constants c_1, c_2 corresponds to free $\Delta = 1, 2$ propagators. Now consider the boundary limit:

$$\mathcal{I}_2 \sim \left(c_1 + \frac{\log K}{4\pi^2}\right) K + c_2 K^2 + \mathcal{O}(K^3). \tag{6.23}$$

The divergent term, whenever an AdS invariant IR regulator is available, should enter in c_1 or c_2. The term proportional to c_2 produces a fall-off behaviour corresponding to the $\Delta = 2$ boundary condition. Thus, for $\Delta = 1$, we set $c_2 = 0$. The divergent part should therefore be parameterized by c_1. Comparing eq. (6.22) with eq. (6.19) uniquely fixes $f(z_1, z_2)$ up to a scale as

$$f(z_1, z_2) = \pi\frac{(z_1-z_2)^2(z_1+z_2)}{z_1^2+z_2^2}. \tag{6.24}$$

The covariant form of eq. (6.19) then reads

$$\mathcal{I}_2 = \frac{1}{8\pi^2}\left[\frac{K}{1-K^2}\left(\log 4 + \frac{1}{\sigma}\right) + \frac{K}{1-K^2}\log\frac{K^2}{1-K^2} + \frac{K^2}{1-K^2}\log\frac{1-K}{1+K}\right],$$

with boundary limit

$$\mathcal{I}_2 \sim I_2 = \frac{K}{8\pi^2\sigma} + \frac{\log 2K}{4\pi^2}K + \mathcal{O}(K^3). \tag{6.25}$$

$\Delta = 2$.

For $\Delta = 2$, where there are no IR issues, the evaluation of the integral (6.17) can again be carried out straightforwardly:

$$\mathcal{I}_2 = -\frac{2z_1z_2}{8\pi^2(z_1^2 - z_2^2)^2}\left[(z_1^2 + z_2^2)\log\frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} + 2z_1z_2\log\frac{16z_1^2z_2^2}{(z_1^2 - z_2^2)^2}\right], \tag{6.26}$$

which corresponds to the covariant expression

$$\mathcal{I}_2 = -\frac{1}{8\pi^2}\left[\frac{K^2}{1 - K^2}\log 4 + \frac{K}{1 - K^2}\log\frac{1 - K}{1 + K} + \frac{K^2}{1 - K^2}\log\frac{K^2}{1 - K^2}\right]. \tag{6.27}$$

This solution features two properties. Firstly, it solves as well the differential equation (6.21). Secondly, its behavior at the boundary exhibits a clear similarity to the regular part of the same expression for $\Delta = 1$ (cf. eq. (6.25)):

$$\mathcal{I}_2 \sim I_2 = \frac{1 - \log 2K}{4\pi^2}K^2 + \mathcal{O}(K^3). \tag{6.28}$$

As expected, the leading terms of both eqs. (6.25), (6.28) are of order K^Δ.

6.2.2 The tadpole diagram

The tadpole diagram $\mathcal{H}_2$ given in figure (6.15) has the integral expression

$$\mathcal{H}_2 = \int \mathrm{d}^4x\sqrt{g(x)}\Lambda(x, x_1; \Delta)\Lambda(x, x_2; \Delta)\Lambda(x, x; \Delta). \tag{6.29}$$

The expression $\Lambda(x, x; \Delta)$ is clearly ultraviolet divergent, and needs to be regularized. In principle, we can set it to zero by hand, but we would like to introduce the regulator that we systematically use later on. In position space, UV divergences result in the limit of colliding points, where $K \to 1$. The following "rescaling" is AdS-invariant and resolves the short distance singularity of $1/(1 - K)$-like expressions

$$K \to \frac{K}{1 + \epsilon}. \tag{6.30}$$

With the help of eq. (5.37), we find that in the flat space limit the ϵ regularization takes the form

$$1 - \frac{K}{1 + \epsilon} = \frac{\frac{a^2r^2}{2} + \epsilon}{1 + \epsilon} + \mathcal{O}(a^4). \tag{6.31}$$

From this it becomes clear that the above regularization carves out a small ϵ-ball around the point and then rescales it by $1/(1 + \epsilon)$. This regularization procedure will be used systematically on every UV-divergent integral encountered in this work. More precisely, for propagators representing internal lines, we rescale each K as in eq. (6.30), i.e.,

$$\Lambda(x,y;\Delta) \to (1+\epsilon)^{2-\Delta}\frac{a^2 K_{xy}^{\Delta}}{4\pi^2(1+K_{xy}+\epsilon)(1-K_{xy}+\epsilon)}, \tag{6.32}$$

while for propagators representing external legs, only those K's appearing in the numerator are rescaled. Here K_{xy} stands for K as a function of the points x and y.

Returning to the tadpole diagram, the propagator at coincident points is given by

$$\Lambda(x,x;\Delta) = \frac{a^2}{4\pi^2}\frac{(1+\epsilon)^{2-\Delta}}{\epsilon(2+\epsilon)}. \tag{6.33}$$

Therefore, the tadpole diagram reduces to the mass-shift diagram (6.16) times a divergent prefactor

$$\mathcal{H}_2 = \frac{a^2}{8\pi^2}\left(\frac{1}{\epsilon}+\frac{3}{2}-3\Delta\right)\mathcal{I}_2 + \mathcal{O}(\epsilon). \tag{6.34}$$

After sending x_1 and x_2 to the boundary, eq. (6.34) simplifies to

$$\mathcal{H}_2 \sim H_2 = \frac{a^2}{8\pi^2}\left(\frac{1}{\epsilon}+\frac{3}{2}-3\Delta\right)I_2 + \mathcal{O}(\epsilon), \tag{6.35}$$

where I_2 is given in eq. (6.25) and eq. (6.28).

6.2.3 The double tadpole diagram

The double tadpole diagram $\mathcal{L}_2$ in figure (6.15) corresponds to the following integral

$$\mathcal{L}_2 = \int \mathrm{d}^4x\sqrt{g(x)}\int \mathrm{d}^4y\sqrt{g(y)}\Lambda(x_1,x;\Delta)\Lambda(x_2,x;\Delta)\Lambda(x,y;\Delta)^2\Lambda(y,y;\Delta), \tag{6.36}$$

which, containing two loops, requires again a regularization. Adopting the regularization described in section 6.2.2, $\mathcal{L}_2$ takes the form

$$\mathcal{L}_2 = \frac{a^2}{(4\pi^2)^3}\frac{(1+\epsilon)^{6-5\Delta}}{\epsilon(2+\epsilon)}\int \mathrm{d}^4x\sqrt{g(x)}\Lambda(x_1,x;\Delta)\Lambda(x_2,x;\Delta)\int\frac{\mathrm{d}^4y}{w^4}\frac{K_{xy}^{2\Delta}}{[(1+\epsilon)^2-K_{xy}^2]^2}.$$

Let us first consider the integral over y. By the substitution $(w,y^i)\to(w,y^i+x^i)$, it displays manifest independence on the "nonradial" coordinates x^i. Then, a simple rescaling argument can be used to show that the integral also does not depend on the radial coordinate z. Indeed, any rescaling $z\to\theta z$, $\theta>0$ can be undone by a substitution of the form $y^i\to\theta y^i$, $w\to\theta w$. In particular, we can set z to any value[3] $z_0>0$. Thus, the nested integral (6.36) factorizes as

$$\mathcal{L}_2 = \frac{a^2}{(4\pi^2)^3}\frac{(1+\epsilon)^{6-5\Delta}}{\epsilon(2+\epsilon)}\,\mathcal{M}_2\times\mathcal{I}_2\,, \tag{6.37}$$

[3]We might as well set $z_0=1$, but since the integral has to be regularized for $\Delta=1$, we keep z_0 and show that the scaling symmetry survives regularization.

where $\mathcal{I}_2$ is the mass shift computed in section 6.2.1 and where

$$\mathcal{M}_2 = \int \frac{\mathrm{d}^4y}{w^4}\frac{K^{2\Delta}_{x_0y}}{[(1+\epsilon)^2 - K^2_{x_0y}]^2} = (2z_0)^{2\Delta}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4y}{w^{4-2\Delta}}\frac{Q^{4-2\Delta}[y^2+(w-z_0)^2+\epsilon Q]^{-2}}{[y^2+(w+z_0)^2+\epsilon Q]^2}. \tag{6.38}$$

In the above formula we defined $Q = y^2 + w^2 + z_0^2$. For $\Delta = 1$, $\mathcal{M}_2$ exhibits another IR divergence. In order to introduce the same IR-regulator as in section 6.2.1, note that the integral above is essentially the mass shift diagram (6.16) evaluated by setting $x_1 = x_2 = (z_0, 0)$ after rescaling K as in eq. (6.30) to regulate the resulting UV divergence. It is not hard to see that $f(z_1, z_2)$ given in eq. (6.24) generalizes to

$$f(z_1,z_2) = \pi\frac{[(z_1-z_2)^2+\epsilon(z_1^2+z_2^2)][(z_1+z_2)^2+\epsilon(z_1^2+z_2^2)]}{(z_1^2+z_2^2)(z_1+z_2)}, \tag{6.39}$$

which, by colliding the points, reduces to z_0 up to a coefficient. Thus, we regulate the integral as follows:

$$\mathcal{M}_2 = (2z_0)^{2\Delta}\int\frac{\mathrm{d}^4y}{(w^2+\sigma^2z_0^2)^{2-\Delta}}\frac{Q^{4-2\Delta}}{[y^2+(w+z_0)^2+\epsilon Q]^2[y^2+(w-z_0)^2+\epsilon Q]^2},$$

where the coefficient in front of σ is irrelevant due to the scale invariance in z_0 (which survives the regularization) explained above. Owing to the $w \to -w$ symmetry of the integral, let us double the integration domain:

$$\mathcal{M}_2 = \frac{(2z_0)^{2\Delta}}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4y}{(w^2+\sigma^2z_0^2)^{2-\Delta}}\frac{Q^{4-2\Delta}}{[y^2+(w+z_0)^2+\epsilon Q]^2[y^2+(w-z_0)^2+\epsilon Q]^2}.$$

$\boldsymbol{\Delta = 1}$.

For $\Delta = 1$, the integral (6.38) becomes

$$\mathcal{M}_2 = 2z_0^2\int_{-\infty}^{\infty}\frac{\mathrm{d}^4y}{w^2+\sigma^2z_0^2}\frac{Q^2}{[y^2+(w+z_0)^2+\epsilon Q]^2[y^2+(w-z_0)^2+\epsilon Q]^2}, \tag{6.40}$$

or equivalently $\mathcal{M}_2 = \mathcal{M}_2^+ + \mathcal{M}_2^-$, where

$$\mathcal{M}_2^{\pm} = \frac{z_0^2}{(1+\epsilon)^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4y}{w^2+\sigma^2z_0^2}\frac{1}{[y^2+(w+z_0)^2+\epsilon Q][y^2+(w\pm z_0)^2+\epsilon Q]}. \tag{6.41}$$

In the last step we used the $w \to -w$ symmetry and the identity

$$2(1+\epsilon)Q\frac{[y^2+(w-z_0)^2+\epsilon Q]^{-1}}{[y^2+(w+z_0)^2+\epsilon Q]} = \frac{1}{[y^2+(w+z_0)^2+\epsilon Q]} + \frac{1}{[y^2+(w-z_0)^2+\epsilon Q]}.$$

The easiest way to deal with integrals of this form is to implement Schwinger parameters. Introducing one Schwinger parameter for each of the three factors in the denominator, the integral (6.41) reads

$$\mathcal{M}_2^{\pm} = \frac{z_0^2}{(1+\epsilon)^2}\int_{-\infty}^{\infty}\mathrm{d}^4y\int_0^{\infty}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\, e^{-(t_1+t_2)(1+\epsilon)Q-(t_1\pm t_2)2wz_0-t_3(w^2+\sigma^2z_0^2)}. \tag{6.42}$$

Now, the spatial integral is a straightforward Gaussian integral resulting in

$$\mathcal{M}_2^{\pm} = \frac{\pi^2 z_0^2}{(1+\epsilon)^3} \int_0^{\infty} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3 \frac{e^{-z_0^2(1+\epsilon)\frac{(t_1+t_2+t_3)^2 - \frac{(t_1 \pm t_2)^2}{(1+\epsilon)^2} + (\sigma^2-1)t_3(t_1+t_2+t_3)}{t_1+t_2+t_3}}}{(t_1+t_2)^{\frac{3}{2}}\sqrt{t_1+t_2+t_3}}, \tag{6.43}$$

where we substituted $t_3 \to (1+\epsilon)t_3$. Let us introduce the following coordinates: $t_i = s s_i$, which simply correspond to a rescaling of our original coordinates t_i by a factor s. Accordingly, one can rewrite the measure as

$$\Pi_{i=1}^n \mathrm{d}t_i = \Pi_{i=1}^n \mathrm{d}s_i \int_0^{\infty} \mathrm{d}s \; s^{n-1} \; \delta(1 - \sum_{i=1}^n s_i), \qquad \forall n \in \mathbb{N}, \tag{6.44}$$

assuming that the condition $s = \sum_i t_i$ is satisfied. Hence, eq. (6.43) becomes

$$\mathcal{M}_2^{\pm} = \frac{\pi^2 z_0^2}{(1+\epsilon)^3} \int_0^{\infty} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}s \; \delta(1 - \sum_{i=1}^3 s_i) \; \frac{e^{-s z_0^2(1+\epsilon)\left(1 - \frac{(s_1 \pm s_2)^2}{(1+\epsilon)^2} + (\sigma^2-1)s_3\right)}}{(s_1+s_2)^{\frac{3}{2}}}, \tag{6.45}$$

where we already integrated over s_3. Integrating over s yields

$$\mathcal{M}_2^{\pm} = \frac{\pi^2}{(1+\epsilon)^4} \int_0^{\infty} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \frac{\delta(1 - s_1 - s_2 - s_3)}{(s_1+s_2)^{\frac{3}{2}} \left[1 - \frac{(s_1 \pm s_2)^2}{(1+\epsilon)^2} + (\sigma^2-1)s_3\right]}, \tag{6.46}$$

which further leads, in the limit $\sigma \to 0$, to

$$\mathcal{M}_2^{+} = \frac{\pi^2}{(1+\epsilon)^4} \left(\frac{\pi}{\sigma} - 2 + 2\frac{\operatorname{arccoth}(1+\epsilon)}{1+\epsilon} \right) + \mathcal{O}(\sigma) \tag{6.47}$$

and

$$\mathcal{M}_2^{-} = \frac{\pi^2}{(1+\epsilon)^4} \left(\frac{\pi}{\sigma} - 1 + \frac{\epsilon(2+\epsilon)\log\frac{\epsilon}{2+\epsilon}}{2(1+\epsilon)} \right) + \mathcal{O}(\sigma). \tag{6.48}$$

Eventually, the full double tadpole diagram is given by

$$\mathcal{L}_2 = \frac{a^2\pi^2}{2(4\pi^2)^3} \left(\frac{-3 + \frac{2\pi}{\sigma} - \log\frac{\epsilon}{2}}{\epsilon} + 11 - \frac{7\pi}{\sigma} + \frac{11}{2}\log\frac{\epsilon}{2} \right) \mathcal{I}_2 + \mathcal{O}(\epsilon, \sigma). \tag{6.49}$$

$\boldsymbol{\Delta = 2}$.
By setting $\Delta = 2$ and rescaling z_0 to 1, the integral (6.38) becomes

$$\mathcal{M}_2 = 8 \int_{-\infty}^{\infty} \mathrm{d}^4 y \frac{1}{[y^2 + (w+1)^2 + \epsilon Q]^2 [y^2 + (w-1)^2 + \epsilon Q]^2}. \tag{6.50}$$

Again, it is favourable to introduce Schwinger parameters. This results in

$$\mathcal{M}_2 = 8\int_{-\infty}^{\infty} \mathrm{d}^4y \int_0^{\infty} \mathrm{d}t_1\mathrm{d}t_2\ t_1t_2\ e^{-(t_1+t_2)(1+\epsilon)Q-(t_1-t_2)2w}, \tag{6.51}$$

which allows for a simple spatial integration, yielding

$$\mathcal{M}_2 = 8\pi^2\int_0^{\infty} \mathrm{d}t_1\mathrm{d}t_2\ \frac{t_1t_2}{(t_1+t_2)^2}\ e^{\frac{(t_1-t_2)^2}{t_1+t_2}-(t_1+t_2)(1+\epsilon)^2}. \tag{6.52}$$

After the substitution $t_i \to ss_i$, the integral becomes

$$\mathcal{M}_2 = 8\pi^2\int_0^{\infty} \mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s\ \frac{s\ s_1s_2\ \delta(1-s_1-s_2)}{(s_1+s_2)^2}\ e^{-s\frac{(s_1+s_2)^2(1+\epsilon)^2-(s_1-s_2)^2}{s_1+s_2}}. \tag{6.53}$$

Then, by removing the Schwinger parameter s and subsequently integrating over s_2, one gets

$$\mathcal{M}_2 = 8\pi^2\int_0^1 \mathrm{d}s_1\ \ \frac{s_1(1-s_1)}{[(1+\epsilon)^2-(2s_1-1)^2]^2}. \tag{6.54}$$

Eventually, the evaluation of the last integral yields

$$\mathcal{M}_2 = -\pi^2\frac{2(1+\epsilon)+[2+\epsilon(2+\epsilon)]\log\frac{\epsilon}{2+\epsilon}}{2(1+\epsilon)^3}. \tag{6.55}$$

For small ϵ, the full solution for the double tadpole diagram reads

$$\mathcal{L}_2 = \frac{a^2\pi^2}{2(4\pi^2)^3}\left(\frac{14+13\log\frac{\epsilon}{2}}{2} - \frac{1+\log\frac{\epsilon}{2}}{\epsilon}\right)\mathcal{I}_2 + \mathcal{O}(\epsilon). \tag{6.56}$$

We close this section with a comment on the renormalization scale. When analyzing QFT in AdS typically one encounters a separation of scales into a UV scale which is related to the scale of local physics and an IR scale given by the AdS radius. In the present context, however, we will be interested in boundary-to-boundary correlation functions for which the AdS radius is the only relevant scale and the UV scale is absent. This is also implicit in the choice of the dimensionless regulator ϵ in eq. (6.30), which is related to a dimensionful cut-off Λ through $\Lambda = \epsilon/a$.

6.2.4 The sunset diagram

The sunset diagram $\mathcal{K}_2$ in figure (6.15) instructs us to compute

$$\mathcal{K}_2 = \int \mathrm{d}^4y\sqrt{g(y)}\int \mathrm{d}^4x\sqrt{g(x)}\Lambda(x_1,x;\Delta)\Lambda(x,y;\Delta)^3\Lambda(x_2,y;\Delta). \tag{6.57}$$

Let us split it into two parts, the first one being

$$\mathcal{J}_2 = \int \mathrm{d}^4y\sqrt{g(y)}\Lambda(x,y;\Delta)^3\Lambda(x_2,y;\Delta). \tag{6.58}$$

As we are eventually interested in the anomalous dimensions of the operators on the boundary we can already take the boundary limit for x_2. Then, $\mathcal{J}_2$ takes the following form

$$\mathcal{J}_2 \sim J_2 = (1+\epsilon)^{6-5\Delta} \frac{a^8 z_2^\Delta}{(4\pi^2)^4} \int \mathrm{d}^4 y \sqrt{g(y)} \frac{K_{xy}^{3\Delta}}{(1-K_{xy}+\epsilon)^3(1+K_{xy}+\epsilon)^3} \bar{K}_{x_2 y}^\Delta \,. \tag{6.59}$$

In the last step we also introduced the UV regulator, cf. eq. (6.32). Using translation symmetry, we can shift x and x_2 by $(0, -x_2^i)$, which leads to

$$J_2 = (1+\epsilon)^{6-5\Delta} \frac{a^8 z_2^\Delta}{(4\pi^2)^4} \int \mathrm{d}^4 y \sqrt{g(y)} \frac{K_{x'y}^{3\Delta}}{(1-K_{x'y}+\epsilon)^3(1+K_{x'y}+\epsilon)^3} \bar{K}_{x_2' y}^\Delta \,, \tag{6.60}$$

where $x_2' = (0,0)$. Next, as was done in ref. [126], we use inversion symmetry[4] to simplify the above expression. In particular, we invert every point by itself, which therefore results in sending x_2' to infinity. For the propagators, this gives

$$K_{x'y} = K_{x''y''}, \qquad \bar{K}_{x_2' y} = x_2''^2 \bar{K}_{x_2'' y''} = 2w'', \tag{6.61}$$

where we denoted the inverted points by double primes. Note that the measure of the integral does not change under the inversion. A subsequent variable substitution $(w'', y^{i''}) = (w, y^i + x^{i''})$ eventually gives

$$J_2 = (1+\epsilon)^{6-5\Delta} \frac{a^4 (16 z_2)^\Delta z''^{3\Delta}}{2(4\pi^2)^4} \int_{-\infty}^{\infty} \mathrm{d}^3 y \; \mathrm{d}w \; \frac{Q^{6-3\Delta}[y^2+(z''-w)^2+\epsilon Q]^{-3}}{w^{4-4\Delta}[y^2+(z''+w)^2+\epsilon Q]^3}, \tag{6.62}$$

where $Q = y^2 + z''^2 + w^2$ and we used again the fact that the integrand is symmetric under $w \to -w$.

$\Delta = 1$.

For $\Delta = 1$, the integral (6.62) becomes

$$J_2 = (1+\epsilon) \frac{8 a^4 z_2 z''^3}{(4\pi^2)^4} \int_{-\infty}^{\infty} \mathrm{d}^3 y \; \mathrm{d}w \; \frac{Q^3}{[(y^2+(z''-w)^2+\epsilon Q)(y^2+(z''+w)^2+\epsilon Q)]^3}.$$

A decomposition in partial fractions yields

$$J_2 = \frac{a^4 z_2 z''^3}{(4\pi^2)^4 (1+\epsilon)^2} \int_{-\infty}^{\infty} \mathrm{d}^3 y \; \mathrm{d}w \; \left[\frac{1}{y^2+(z''-w)^2+\epsilon Q} + \frac{1}{y^2+(z''+w)^2+\epsilon Q} \right]^3.$$

Using the $w \to -w$ symmetry, one can write $J_2 = J_2^a + J_2^b$, where

$$J_2^a = \frac{2 a^4 z_2 z''^3}{(4\pi^2)^4 (1+\epsilon)^2} \int_{-\infty}^{\infty} \mathrm{d}^3 y \; \mathrm{d}w \; \frac{1}{[(1+\epsilon)Q + 2z'' w]^3} \tag{6.63}$$

[4] The inversion operator I acts on conformal coordinates as $I(x^i) = \frac{x^i}{x^2+z^2}$, $I(z) = \frac{z}{x^2+z^2}$. See also section 4.2.

and

$$J_2^b = \frac{6a^4 z_2 z''^3}{(4\pi^2)^4(1+\epsilon)^2} \int_{-\infty}^{\infty} \mathrm{d}^3y \ \mathrm{d}w \ \frac{1}{[(1+\epsilon)Q + 2z''w]^2[(1+\epsilon)Q - 2z''w]}. \tag{6.64}$$

The computation of J_2^a is rather simple. Indeed, the introduction of one Schwinger parameter gives

$$J_2^a = \frac{a^4 z_2 z''^3}{(4\pi^2)^4(1+\epsilon)^2} \int_0^{\infty} \mathrm{d}t \ t^2 \int_{-\infty}^{\infty} \mathrm{d}^3y \ \mathrm{d}w \ e^{-t(1+\epsilon)Q - 2tz''w}, \tag{6.65}$$

allowing for the integration over space,

$$J_2^a = \frac{\pi^2 a^4 z_2 z''^3}{(4\pi^2)^4(1+\epsilon)^4} \int_0^{\infty} \mathrm{d}t \ e^{-tz''^2(1+\epsilon)\left[1-\frac{1}{(1+\epsilon)^2}\right]} = \frac{a^4 z_2 z''}{(4\pi^2)^4} \frac{\pi^2}{(1+\epsilon)^3\epsilon(2+\epsilon)}. \tag{6.66}$$

For small values of ϵ, this simplifies to

$$J_2^a = \frac{\pi^2 a^4 z_2 z''}{(4\pi^2)^4} \left(\frac{1}{2\epsilon} - \frac{7}{4} \right) + \mathcal{O}(\epsilon). \tag{6.67}$$

For the calculation of J_2^b, the introduction of two Schwinger parameters is more convenient,

$$J_2^b = \frac{6a^4 z_2 z''^3}{(4\pi^2)^4(1+\epsilon)^2} \int_0^{\infty} \mathrm{d}t_1 \mathrm{d}t_2 \ t_1 \int_{-\infty}^{\infty} \mathrm{d}^3y \ \mathrm{d}w \ e^{-(t_1+t_2)(1+\epsilon)Q - (t_1-t_2)2z''w}. \tag{6.68}$$

Integration over space is now viable and results in

$$J_2^b = \frac{6\pi^2 a^4 z_2 z''^3}{(4\pi^2)^4(1+\epsilon)^4} \int_0^{\infty} \mathrm{d}t_1 \mathrm{d}t_2 \ \frac{t_1}{(t_1+t_2)^2} \ e^{-(t_1+t_2)(1+\epsilon)z''^2\left[1-\frac{(t_1-t_2)^2}{(t_1+t_2)^2(1+\epsilon)^2}\right]}. \tag{6.69}$$

Note that the integral does not change if we swap t_1 and t_2. Therefore one can replace the t_1 in front of the exponential with $(t_1+t_2)/2$. After introducing new coordinates $t_i = ss_i$ as done above, the integration over s_2 leads to

$$J_2^b = \frac{3\pi^2 a^4 z_2 z''^3}{(4\pi^2)^4(1+\epsilon)^4} \int_0^{\infty} \mathrm{d}s \int_0^1 \mathrm{d}s_1 \ e^{-s(1+\epsilon)z''^2\left[1-\frac{(1-2s_1)^2}{(1+\epsilon)^2}\right]}. \tag{6.70}$$

Removing the Schwinger parameter s allows one to integrate over s_1. Therefore, the final result for J_2^b is given by

$$J_2^b = \frac{3\pi^2 a^4 z_2 z''}{2(4\pi^2)^4(1+\epsilon)^4} \log\frac{2+\epsilon}{\epsilon} = -\frac{3\pi^2 a^4 z_2 z''}{2(4\pi^2)^4} \log\frac{\epsilon}{2} + \mathcal{O}(\epsilon). \tag{6.71}$$

$\Delta = 2$.

The integral (6.62) is instead given by

$$J_2 = \frac{a^4 (16 z_2)^2 z''^6}{2(4\pi^2)^4 (1+\epsilon)^4} \int_{-\infty}^{\infty} \mathrm{d}^3 y\ \mathrm{d}w\ \frac{w^4}{[y^2 + (z''+w)^2 + \epsilon Q]^3 [y^2 + (z''-w)^2 + \epsilon Q]^3}.$$

As usual, let us introduce the Schwinger parameters

$$\begin{aligned} J_2 =& \frac{a^4 (16 z_2)^2 z''^6}{8(4\pi^2)^4 (1+\epsilon)^4} \int_0^{\infty} \mathrm{d}t_1 \mathrm{d}t_2\ (t_1 t_2)^2 \int_{-\infty}^{\infty} \mathrm{d}^3 y\ \mathrm{d}w\ w^4\ e^{-(t_1+t_2)(1+\epsilon)Q - (t_1 - t_2) 2 z'' w} \\ =& \frac{a^4 (16 z_2)^2 z''^6 \partial_\gamma^2|_{\gamma=1}}{8(4\pi^2)^4 (1+\epsilon)^6} \int_0^{\infty} \frac{\mathrm{d}t_1 \mathrm{d}t_2\ (t_1 t_2)^2}{(t_1+t_2)^2} \int_{-\infty}^{\infty} \mathrm{d}^3 y\ \mathrm{d}w\ e^{-(t_1+t_2)(1+\epsilon)(y^2 + \gamma w^2 + z''^2) - (t_1 - t_2) 2 z'' w}, \end{aligned}$$

where in the last step we introduced an auxiliary parameter $\gamma > 0$ in order to get rid of the factor w^4 in the numerator. The spatial integration yields

$$J_2 = \frac{a^4 \pi^2 (16 z_2)^2 z''^6}{8(4\pi^2)^4 (1+\epsilon)^8} \partial_\gamma^2|_{\gamma=1} \frac{1}{\sqrt{\gamma}} \int_0^{\infty} \mathrm{d}t_1 \mathrm{d}t_2\ \frac{(t_1 t_2)^2}{(t_1+t_2)^4}\ e^{-(t_1+t_2)(1+\epsilon) z''^2 \left[1 - \frac{(t_1 - t_2)^2}{(t_1+t_2)^2 (1+\epsilon)^2 \gamma}\right]}.$$

Let us now apply the substitution $t_i = s s_i$. After integrating over s_2, we get

$$J_2 = \frac{a^4 \pi^2 (16 z_2)^2 z''^6}{8(4\pi^2)^4 (1+\epsilon)^8} \partial_\gamma^2|_{\gamma=1} \frac{1}{\sqrt{\gamma}} \int_0^{\infty} \mathrm{d}s \int_0^1 \mathrm{d}s_1\ s\ s_1^2 (1-s_1)^2 e^{-s(1+\epsilon) z''^2 \left[1 - \frac{(1-2s_1)^2}{(1+\epsilon)^2 \gamma}\right]}.$$

Integrating over the Schwinger parameter s gives

$$J_2 = \frac{a^4 \pi^2 (16 z_2)^2 z''^2}{8(4\pi^2)^4 (1+\epsilon)^6} \partial_\gamma^2|_{\gamma=1}\ \gamma^{3/2} \int_0^1 \mathrm{d}s_1 \left[\frac{s_1 (1-s_1)}{(1+\epsilon)^2 \gamma - (1-2s_1)^2}\right]^2. \tag{6.72}$$

Differentiating twice by γ, setting $\gamma = 1$, and subsequently integrating over s_1, yields

$$J_2 = \frac{a^4 \pi^2 z_2^2 z''^2}{(4\pi^2)^4} \left[\frac{1}{\epsilon} + \frac{1}{2}(-1 + 6 \log \frac{\epsilon}{2})\right] + \mathcal{O}(\epsilon). \tag{6.73}$$

Recovering the full covariance.

In order to get back the explicit covariant form, let us first note that undoing the inversion and restoring the translation invariance instructs us to replace

$$2z'' \to \bar{K}_{xx_2}, \tag{6.74}$$

which is manifestly covariant. Therefore, it follows that one can write J_2 as

$$J_2 = \frac{a^4 \pi^2}{4(4\pi^2)^4} K_{xx_2}^{\Delta} \left(\frac{1}{\epsilon} + 3(-1)^{\Delta} \log \frac{\epsilon}{2} - \frac{13}{2} + 3\Delta\right). \tag{6.75}$$

Furthermore, it is possible to reconstruct the bulk quantity $\mathcal{J}_2$. Owing to the covariant form of $\mathcal{J}_2$ in eq. (6.58), it has to correspond to the above result modulo some function of K_{xx_2}. In order to find this dependence on K_{xx_2}, note that

$$(-\Box_{x_2} - 2a^2)\mathcal{J}_2 = \Lambda(x, x_2; \Delta)^3. \tag{6.76}$$

The solution to the above differential equation is then given by

$$\begin{aligned}\mathcal{J}_2 = \frac{a^2\pi^2}{4(4\pi^2)^3}\Lambda(x, x_2; \Delta)\Bigg[&\frac{1}{\epsilon} + 3(-1)^\Delta \log\frac{\epsilon}{2} - \frac{13}{2} + 3\Delta \\ &+3(-1)^\Delta K_{xx_2}^{3-2\Delta}\log\left(\frac{1+K_{xx_2}}{1-K_{xx_2}}\right) - 4(\Delta-1) - \frac{2K_{xx_2}^{4-2\Delta}}{1-K_{xx_2}^2}\Bigg],\end{aligned} \tag{6.77}$$

where the integration constants were fixed such that $\mathcal{J}_2 \sim J_2$ of eq. (6.75).

Attaching the missing leg.

The full sunset diagram can be obtained by attaching one more leg to $\mathcal{J}_2$ we just extracted the most singular part of, i.e.,

$$\mathcal{K}_2 = \int \mathrm{d}^4x\sqrt{g(x)}\Lambda(x_1, x; \Delta)\mathcal{J}_2. \tag{6.78}$$

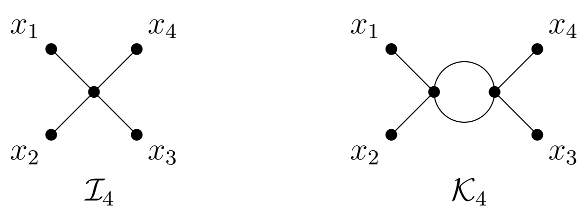
Then the final result can be expressed in terms of the mass shift I_2:

$$\mathcal{K}_2 \sim K_2 = \frac{a^2\pi^2}{4(4\pi^2)^3}\left(\frac{1}{\epsilon} + 3(-1)^\Delta\log\frac{\epsilon}{2} - \frac{13}{2} + 3\Delta\right) I_2. \tag{6.79}$$

It is worth to note that, in contrast to the previously computed diagrams, the sunset diagram is not proportional to $\mathcal{I}_2$. Instead, the proportionality is only given on the boundary.

6.3 Four-point function

In this section we compute the diagrams that contribute to the four-point function. Up to the second order in the coupling constant λ, the one-particle irreducible diagrams are



. (6.80)

The contact cross diagram $\mathcal{I}_4$ is well-known in the literature, see, for instance, ref. [125]. For completeness, we repeat the calculations. Afterwards, we compute the one loop diagram $\mathcal{K}_4$.

6.3.1 The cross diagram

The cross diagram leads to

$$\mathcal{I}_4 = \int \mathrm{d}^4x \sqrt{g(x)} \Lambda(x_1, x; \Delta)\Lambda(x_2, x; \Delta)\Lambda(x_3, x; \Delta)\Lambda(x_4, x; \Delta). \tag{6.81}$$

With all external legs on the boundary the integral reduces to

$$\mathcal{I}_4 \sim \frac{4^{2\Delta} a^4 \Pi_{i=1}^4 z_i^\Delta}{(4\pi^2)^4} I_4, \tag{6.82}$$

with

$$I_4 = \int \mathrm{d}^4x \frac{z^{4\Delta-4}}{\Pi_{i=1}^4 \left[(x-x_i)^2 + z^2\right]^\Delta}. \tag{6.83}$$

As opposed to the rest of this thesis, note the extra factor between $\mathcal{I}_4$ and I_4 which we introduced in order to conform with the literature. The Schwinger parameterization yields

$$I_4 = \frac{\pi^{\frac{3}{2}}\Gamma\left(2\Delta - \frac{3}{2}\right)}{2\Gamma(\Delta)^4} \int_0^\infty \Pi_{i=1}^4 \left(\mathrm{d}t_i \; t_i^{\Delta-1}\right) e^{-\frac{\sum_{i<j} t_i t_j r_{ij}^2}{\sum_i t_i}} \left(\sum_{i=1}^4 t_i\right)^{-2\Delta}. \tag{6.84}$$

Making the redefinition $t_i = s s_i$ and performing the integration over s gives

$$I_4 = \frac{\pi^{\frac{3}{2}}\Gamma\left(2\Delta - \frac{3}{2}\right)\Gamma(2\Delta)}{2\Gamma(\Delta)^4} \int_0^\infty \Pi_{i=1}^4 \left(\mathrm{d}s_i \; s_i^{\Delta-1}\right) \frac{\delta\left(1 - \sum_{i=1}^4 s_i\right)}{\left[\sum_{i<j} s_i s_j r_{ij}^2\right]^{2\Delta}}. \tag{6.85}$$

Now, let us redefine the coordinates as $s_1 = s$ and $s_i \to s s_i$ for $i \geq 2$. The integration over s leads to

$$I_4 = \frac{\pi^{\frac{3}{2}}\Gamma\left(2\Delta - \frac{3}{2}\right)\Gamma(2\Delta)}{2\Gamma(\Delta)^4} \int_0^\infty \frac{\Pi_{i=2}^4 \left(\mathrm{d}s_i \; s_i^{\Delta-1}\right)}{\left[\sum_{i=2}^4 s_i \left(r_{1i}^2 + \sum_{j>i} s_j r_{ij}^2\right)\right]^{2\Delta}}, \tag{6.86}$$

and a subsequent integration over s_2 and s_3 yields

$$I_4 = \frac{\pi^{\frac{3}{2}}\Gamma\left(2\Delta - \frac{3}{2}\right)}{2\Gamma(2\Delta)(r_{23}r_{14})^{2\Delta}} \int_0^\infty \frac{\mathrm{d}s}{s} \, {}_2F_1\left[\Delta, \Delta; 2\Delta; 1 - \left(\frac{\eta+\zeta}{\eta\zeta}\right)^2 - \frac{(\xi s - 1)^2}{\eta\zeta\xi s}\right]. \tag{6.87}$$

Here, we introduced the conformal invariants

$$\xi = \frac{r_{24}r_{34}}{r_{12}r_{13}}, \qquad \eta = \frac{r_{14}r_{23}}{r_{12}r_{34}}, \qquad \zeta = \frac{r_{14}r_{23}}{r_{13}r_{24}}, \tag{6.88}$$

where only two of them are independent and these are related to the previous ones by

$$Y = 1 - \frac{1}{\zeta^2}, \qquad v = \frac{1}{\eta^2}. \tag{6.89}$$

A change of variables $s\xi = e^{2z}$ allows to eliminate the conformal invariant ξ, and eventually we reach the final form:

$$I_4 = \frac{2\pi^{\frac{3}{2}}\Gamma\left(2\Delta - \frac{3}{2}\right)}{\Gamma(2\Delta)(\eta\zeta\ \Pi_{i<j}r_{ij})^{\frac{2}{3}\Delta}} \int_0^\infty \mathrm{d}z\ {}_2F_1\left[\Delta, \Delta; 2\Delta; 1 - \left(\frac{\eta+\zeta}{\eta\zeta}\right)^2 - \frac{4\sinh^2 z}{\eta\zeta}\right]. \tag{6.90}$$

Another way to compute I_4 is the one adopted in ref. [127]. Symanzik was the first to note that eq. (6.84) does not change if one substitutes $\sum_{i=1}^4 t_i$ with $\sum_{i=1}^4 \kappa_i t_i$, where the $\kappa_i \geq 0$ are not all vanishing:

$$I_4 = \frac{\pi^{\frac{3}{2}}\Gamma\left(2\Delta - \frac{3}{2}\right)}{2\Gamma(\Delta)^4} \int_0^\infty \Pi_{i=1}^4\left(\mathrm{d}t_i\ t_i^{\Delta-1}\right) e^{-\frac{\sum_{i<j} t_i t_j r_{ij}^2}{\sum_i \kappa_i t_i}} \left(\sum_{i=1}^4 \kappa_i t_i\right)^{-2\Delta}. \tag{6.91}$$

Let us choose $\kappa_2 = \kappa_3 = \kappa_4 = 0, \kappa_1 = 1$. Then, the Mellin transform of the exponential function:

$$e^{-x} = \frac{1}{2\pi\mathrm{i}} \int_{-\epsilon-\mathrm{i}\infty}^{-\epsilon+\mathrm{i}\infty} \mathrm{d}s\ \Gamma(-s) x^s, \qquad x > 0, \tag{6.92}$$

is used to substitute $e^{-\frac{t_2 t_4 r_{24}^2}{t_1}}$ and $e^{-\frac{t_3 t_4 r_{34}^2}{t_1}}$. With a redefinition $t_i \to 1/t_i$, the integration over t_1 becomes straightforward. Then, a subsequent integration over t_2, t_3 and t_4 yields

$$I_4 = \frac{\pi^{\frac{3}{2}}\Gamma\left(2\Delta - \frac{3}{2}\right)(2\pi\mathrm{i})^{-2}}{2\Gamma(\Delta)^4(\eta\zeta\ \Pi_{i<j}r_{ij})^{\frac{2}{3}\Delta}} \int_{-\epsilon-\mathrm{i}\infty}^{-\epsilon+\mathrm{i}\infty} \mathrm{d}s\mathrm{d}t \frac{1}{\zeta^{2s}\eta^{2t}}\ \Gamma(-s)^2\Gamma(-t)^2\Gamma(s+t+\Delta)^2.$$

Now, using the identity

$$\frac{1}{2\pi\mathrm{i}} \int_{-\epsilon-\mathrm{i}\infty}^{-\epsilon+\mathrm{i}\infty} \mathrm{d}s\ z^s\ \Gamma(-s)^2\Gamma(s+t+\Delta)^2 = \Gamma(t+\Delta)^2 \sum_{m=0}^\infty \frac{\Gamma(t+m+\Delta)^2}{\Gamma(2t+m+2\Delta)m!}(1-z)^m \tag{6.93}$$

the integral becomes

$$I_4 = \frac{\pi^{\frac{3}{2}}\Gamma\left(2\Delta - \frac{3}{2}\right)(2\pi\mathrm{i})^{-1}}{2\Gamma(\Delta)^4(\eta\zeta\ \Pi_{i<j}r_{ij})^{\frac{2}{3}\Delta}} \sum_{m=0}^\infty \frac{(1-\frac{1}{\zeta^2})^m}{m!} \int_{-\epsilon-\mathrm{i}\infty}^{-\epsilon+\mathrm{i}\infty} \mathrm{d}t \frac{1}{\eta^{2t}} \frac{\Gamma(-t)^2\Gamma(t+\Delta)^2\Gamma(t+m+\Delta)^2}{\Gamma(2t+m+2\Delta)}.$$

An integration around the contour with positive real part of t gives

$$\begin{aligned} I_4 =& \frac{\pi^{\frac{3}{2}}\Gamma\left(2\Delta - \frac{3}{2}\right)}{\Gamma(\Delta)^4} \frac{v^\Delta}{(r_{12}r_{34})^{2\Delta}} \sum_{m,n=0}^\infty \frac{Y^m v^n}{m!(n!)^2} \frac{\Gamma(n+\Delta)^2\Gamma(n+m+\Delta)^2}{\Gamma(2n+m+2\Delta)} \\ &\times \left[\psi(n+1) - \frac{1}{2}\log v - \psi(n+\Delta) - \psi(n+\Delta+m) + \psi(2n+2\Delta+m)\right], \end{aligned} \tag{6.94}$$

where for later convenience we used again the conformal invariants v and Y.

6.3.2 The one loop diagram

The one loop diagram $\mathcal{K}_4$ depicted in figure (6.80) is given by the double integral

$$\mathcal{K}_4 = \int \mathrm{d}^4x\sqrt{g(x)}\Lambda(x_1,x;\Delta)\Lambda(x_2,x;\Delta)\mathcal{J}_4, \tag{6.95}$$

where we defined the sub–integral $\mathcal{J}_4$ as

$$\mathcal{J}_4 = \int \mathrm{d}^4y\sqrt{g(y)}\Lambda(x,y;\Delta)^2\Lambda(x_3,y;\Delta)\Lambda(x_4,y;\Delta). \tag{6.96}$$

The latter, by sending z_3 and z_4 to the boundary, takes the form

$$\mathcal{J}_4 \sim J_4 = (1+\epsilon)^{4-6\Delta}\frac{a^8(z_3z_4)^\Delta}{(4\pi^2)^4}\int \mathrm{d}^4y\sqrt{g(y)}\frac{K_{xy}^{2\Delta}}{(1-K_{xy}+\epsilon)^2(1+K_{xy}+\epsilon)^2}\bar{K}_{x_3y}^\Delta\bar{K}_{x_4y}^\Delta.$$

As before, the UV divergences are regularized with the help of ϵ: $K_{xy}\to K_{xy}/(1+\epsilon)$. Furthermore, since the divergence is logarithmic in ϵ, we can safely ignore the prefactor $(1+\epsilon)^{4-6\Delta}$ in what follows. Translating the points x, x_3, and x_4 by $(0,-x_4^i)$ yields

$$J_4 = \frac{a^8(z_3z_4)^\Delta}{(4\pi^2)^4}\int \mathrm{d}^4y\sqrt{g(y)}\frac{K_{x'y}^{2\Delta}}{(1-K_{x'y}+\epsilon)^2(1+K_{x'y}+\epsilon)^2}\bar{K}_{x_3'y}^\Delta\bar{K}_{x_4'y}^\Delta\,, \tag{6.97}$$

where $x_4' = (0,0)$. As detailed in ref. [126], we use the inversion trick to simplify the above expression. For the K's this gives

$$K_{x'y} = K_{x''y''},\qquad \bar{K}_{x_3'y} = \frac{1}{x_3'^2}\bar{K}_{x_3''y''} = \frac{1}{r_{34}^2}\bar{K}_{x_3''y''},\qquad \bar{K}_{x_4'y} = x_4''^2\bar{K}_{x_4''y''} = 2w'', \tag{6.98}$$

where the inverted points are denoted by double primes. One more substitution $(w'',y^{i''}) = (w, y^i + x^{i''})$ eventually yields

$$J_4 = \frac{a^4(16z_3z_4)^\Delta z''^{2\Delta}}{2(4\pi^2)^4r_{34}^{2\Delta}}\int_{-\infty}^{\infty}\mathrm{d}^3y\;\mathrm{d}w\;\frac{Q^{4-2\Delta}w^{4\Delta-4}[y^2+(z''-w)^2+\epsilon Q]^{-2}}{[(x_3'''-y)^2+w^2]^\Delta[y^2+(z''+w)^2+\epsilon Q]^2}\,, \tag{6.99}$$

with $x_3^{i'''} = x_3^{i''} - x^{i''}$, $Q = y^2+z''^2+w^2$. Again, we used the symmetry of the integrand under $w\to -w$.

$\boldsymbol{\Delta = 1}$.

For this value, the integral (6.99) takes the form

$$J_4 = \frac{8a^4z_3z_4z''^2}{(4\pi^2)^4r_{34}^2}\int_{-\infty}^{\infty}\mathrm{d}^3y\;\mathrm{d}w\;\frac{Q^2[(x_3'''-y)^2+w^2]^{-1}}{[(y^2+(z''-w)^2+\epsilon Q)(y^2+(z''+w)^2+\epsilon Q)]^2}. \tag{6.100}$$

The Q in the numerator can be written as

$$\begin{aligned}[2(1+\epsilon)Q]^2 =& [y^2+(z''-w)^2+\epsilon Q]^2 + [y^2+(z''+w)^2+\epsilon Q]^2\\ &+ 2[y^2+(z''+w)^2+\epsilon Q][y^2+(z''-w)^2+\epsilon Q],\end{aligned} \tag{6.101}$$

which allows for a splitting of the integral in a divergent part and a regular part: $J_4 = J_4^d + J_4^r$. Explicitly, one gets

$$J_4^d = \frac{4a^4 z_3 z_4 z''^2}{(4\pi^2)^4 r_{34}^2 (1+\epsilon)^2} \int_{-\infty}^{\infty} \mathrm{d}^3y \; \mathrm{d}w \; \frac{1}{[y^2 + (z''-w)^2 + \epsilon Q]^2 [(x_3''' - y)^2 + w^2]} \tag{6.102}$$

and

$$J_4^r = \frac{4a^4 z_3 z_4 z''^2}{(4\pi^2)^4 r_{34}^2} \int_{-\infty}^{\infty} \mathrm{d}^3y \; \mathrm{d}w \; \frac{1}{[y^2 + (z''-w)^2][y^2 + (z''+w)^2][(x_3''' - y)^2 + w^2]}, \tag{6.103}$$

where for the latter integral, being regular, we already set $\epsilon \to 0$. Additionally, the regular integral can be further simplified to

$$J_4^r = \frac{4a^4 z_3 z_4 z''^2}{(4\pi^2)^4 r_{34}^2} \int_{-\infty}^{\infty} \mathrm{d}^3y \; \mathrm{d}w \; \frac{1}{[y^2 + (z''-w)^2][y^2 + z''^2 + w^2][(x_3''' - y)^2 + w^2]}, \tag{6.104}$$

by using

$$\frac{2Q}{[y^2 + (z''-w)^2][y^2 + (z''+w)^2]} = \frac{1}{y^2 + (z''-w)^2} + \frac{1}{y^2 + (z''+w)^2} \tag{6.105}$$

and the symmetry $w \to -w$ of the integrand of eq. (6.103). The computation of J_4^d is rather simple. Introducing two Schwinger parameters and integrating over space yields

$$J_4^d = \frac{4\pi^2 a^4 z_3 z_4 z''^2}{(4\pi^2)^4 r_{34}^2 (1+\epsilon)^4} \int_0^{\infty} \mathrm{d}t_1 \mathrm{d}t_2 \; t_1 \; e^{-\frac{t_1 t_2 x_3'''^2}{t_1+t_2}} \, e^{-\frac{t_1 (t_2 + \frac{\epsilon}{1+\epsilon} t_1) z''^2}{(1+\epsilon)(t_1+t_2)}} \; e^{-\frac{\epsilon}{1+\epsilon} t_1 z''^2}. \tag{6.106}$$

After the change of variables $t_i = s s_i$, one then finds

$$\begin{aligned} J_4^d &= \frac{4\pi^2 a^4 z_3 z_4 z''^2}{(4\pi^2)^4 r_{34}^2 (1+\epsilon)^4} \int_0^{\infty} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s \; s_1 \delta(1 - s_1 - s_2) \; e^{-s\left[s_1 s_2 x_3'''^2 + \frac{s_1 (s_2 + \frac{\epsilon}{1+\epsilon} s_1) z''^2}{(1+\epsilon)} + \frac{\epsilon}{1+\epsilon} s_1 z''^2\right]} \\ &= -\frac{4\pi^2 a^4 z_3 z_4 z''^2}{(4\pi^2)^4 r_{34}^2 (1+\epsilon)^2} \frac{\log\left(\frac{z''^2 \epsilon (2+\epsilon)}{(1+\epsilon)^2 \left(x_3'''^2 + z''^2\right)}\right)}{x_3'''^2 (\epsilon+1)^2 + z''^2}, \end{aligned}$$

which for small values of ϵ reduces to

$$J_4^d = -\frac{4\pi^2 a^4 z_3 z_4 z''^2}{(4\pi^2)^4 r_{34}^2} \frac{\log\left(\frac{2z''^2}{x_3'''^2 + z''^2}\right) + \log(\epsilon)}{x_3'''^2 + z''^2} + \mathcal{O}(\epsilon). \tag{6.107}$$

The same steps done above, applied on J_4^r, lead to

$$\begin{aligned} J_4^r &= \frac{4\pi^2 a^4 z_3 z_4 z''^2}{(4\pi^2)^4 r_{34}^2} \int_0^{\infty} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \; \frac{\delta(1 - \sum_{i=1}^3 s_i)}{[(s_1+s_2) s_3 x_3'''^2 + (s_2+s_3) s_1 z''^2 + s_2 z''^2]} \\ &= \frac{4\pi^2 a^4 z_3 z_4 z''}{(4\pi^2)^4 r_{34}^2} \int_0^1 \mathrm{d}s_3 \frac{\operatorname{arctanh}\sqrt{\frac{(1-s_3) z''^2}{s_3 x_3'''^2 + z''^2}}}{\sqrt{(1-s_3)(s_3 x_3'''^2 + z''^2)}}. \end{aligned} \tag{6.108}$$

Then, the substitution

$$t = \sqrt{\frac{(1-s_3)z''^2}{s_3 x_3'''^2 + z''^2}} \tag{6.109}$$

yields a simpler form of the integral

$$J_4^r = \frac{8\pi^2 a^4 z_3 z_4}{(4\pi^2)^4 r_{34}^2} \int_0^1 \mathrm{d}t \frac{\operatorname{arctanh} t}{\alpha^2 t^2 + 1}, \tag{6.110}$$

where we defined $\alpha = |x_3'''|/z''$. This integral admits a closed-form solution in terms of the dilogarithm Li_2:

$$J_4^r = \frac{4\pi^2 a^4 z_3 z_4}{(4\pi^2)^4 r_{34}^2} \frac{1}{\alpha} \mathrm{Im}\left\{\mathrm{Li}_2\left(\frac{\mathrm{i}\alpha - 1}{\mathrm{i}\alpha + 1}\right)\right\}. \tag{6.111}$$

However, in order to compute the complete four-point function (6.95), the integral form (6.110) will be better suited.

$\Delta = 2$.

For $\Delta = 2$, the integral (6.99) becomes instead

$$J_4 = \frac{a^4 (16 z_3 z_4)^2 z''^4}{2(4\pi^2)^4 r_{34}^4} \int_{-\infty}^{\infty} \mathrm{d}^3y \; \mathrm{d}w \; \frac{w^4[(x_3''' - y)^2 + w^2]^{-2}}{[(y^2 + (z''-w)^2 + \epsilon Q)(y^2 + (z''+w)^2 + \epsilon Q)]^2}.$$

As usual, let us introduce the Schwinger parameters:

$$\begin{aligned} J_4 = & \frac{a^4 (16 z_3 z_4)^2 z''^4}{2(4\pi^2)^4 r_{34}^4} \int_0^{\infty} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3 \; t_1 t_2 t_3 \int_{-\infty}^{\infty} \mathrm{d}^3y \; \mathrm{d}w \; w^4 \; e^{-(t_1+t_2)\epsilon z''^2} \\ & \times e^{-t_1(z''-w)^2 - t_2(z''+w)^2 - (\epsilon t_1 + \epsilon t_2 + t_3)w^2 - (t_1+t_2)(1+\epsilon)y^2 - t_3(x_3'''-y)^2}. \end{aligned} \tag{6.112}$$

Integrating over the nonradial coordinates, and subsequently substituting $t_1 = s s_1, t_2 = s s_2, t_3 = (1+\epsilon) s s_3$, yields

$$\begin{aligned} J_4 = & \frac{a^4 \pi^{\frac{3}{2}} (16 z_3 z_4)^2 z''^4 (1+\epsilon)^{\frac{3}{2}}}{2(4\pi^2)^4 r_{34}^4} \; \partial_\gamma^2|_{\gamma=0} \int_0^{\infty} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}s \; s^{3/2} \; s_1 s_2 s_3 \int_{-\infty}^{\infty} \mathrm{d}w \; \delta\left(1 - \sum_{i=1}^{3} s_i\right) \\ & \times e^{-s(s_1+s_2)s_3(1+\epsilon)x_3'''^2 - s(s_1(z''-w)^2 + s_2(z''+w)^2 + (\epsilon + s_3 + \gamma)w^2) - s(s_1+s_2)\epsilon z''^2}. \end{aligned}$$

Additionally, above we also introduced an auxiliary parameter $\gamma > 0$ to get rid of the w^4 factor. Then, further evaluation of the integral leads to

$$\begin{aligned} J_4 = & \frac{a^4 \pi^2 (16 z_3 z_4)^2 z''^4 (1+\epsilon)^{\frac{3}{2}}}{2(4\pi^2)^4 r_{34}^4} \; \partial_\gamma^2|_{\gamma=0} \frac{1}{\sqrt{1+\gamma+\epsilon}} \int_0^1 \mathrm{d}s_3 \int_0^{1-s_3} \mathrm{d}s_1 \; s_1 s_3 (1 - s_1 - s_3) \\ & \times \left[(1-s_3)s_3(1+\epsilon)x_3'''^2 + \frac{4 s_1(1-s_1-s_3) + (1-s_3)(\epsilon + s_3 + \gamma)}{1+\gamma+\epsilon} z''^2 + (1-s_3)\epsilon z''^2\right]^{-2}. \end{aligned}$$

Differentiating twice by γ, setting $\gamma = 1$, and integrating over s_1, yields

$$J_4 = \frac{16a^4\pi^2(z_3z_4)^2 z''^4}{(4\pi^2)^4 r_{34}^4} \int_0^1 \mathrm{d}s_3 \, \frac{(1-s_3)s_3(1+\epsilon)}{\left(s_3\left(x_3'''^2(1+\epsilon)^2 + z''^2\right) + z''^2\epsilon(2+\epsilon)\right)^2}. \tag{6.113}$$

Eventually, after integrating over s_3 and taking only the leading orders in ϵ, one gets

$$J_4 = -\frac{16a^4\pi^2(z_3z_4)^2 z''^4}{(4\pi^2)^4 r_{34}^4} \, \frac{2 + \log\left(\frac{2z''^2}{x_3'''^2 + z''^2}\right) + \log\epsilon}{\left(x_3'''^2 + z''^2\right)^2} + \mathcal{O}(\epsilon). \tag{6.114}$$

Recovering the full covariance.

To get back the explicit covariant form, let us first note that

$$\frac{4z''^2}{x_3'''^2 + z''^2} = r_{34}^2 \bar{K}_{xx_3} \bar{K}_{xx_4} \equiv \frac{4}{\alpha^2 + 1}, \tag{6.115}$$

which is manifestly covariant. Therefore, it follows that

$$J_4 = \frac{a^4\pi^2}{(4\pi^2)^4} K_{xx_3}^{\Delta} K_{xx_4}^{\Delta} \left(\log(\alpha^2+1) - \log 2\epsilon + \Delta J_4(\alpha^2)\right), \tag{6.116}$$

where

$$\Delta J_4(\alpha^2) = \begin{cases} 2(\alpha^2+1)\int_0^1 \mathrm{d}t \, \frac{\operatorname{arctanh} t}{\alpha^2 t^2 + 1} & \text{for } \Delta = 1, \\ -2 & \text{for } \Delta = 2. \end{cases} \tag{6.117}$$

Attaching the missing legs.

In order to obtain the complete one loop diagram $\mathcal{K}_4$ given in eq. (6.95), we still need to perform the remaining integral

$$\mathcal{K}_4 = \int \mathrm{d}^4x \sqrt{g(x)} \Lambda(x_1, x, \Delta) \Lambda(x_2, x, \Delta) \mathcal{J}_4(x_3, x_4, x, \Delta). \tag{6.118}$$

Let us send z_1, z_2 to the boundary, reducing $\mathcal{K}_4$ to

$$\mathcal{K}_4 \sim K_4 = \frac{a^4(16z_1z_2z_3z_4)^{\Delta}\pi^2}{(4\pi^2)^6} \int \mathrm{d}^4x \, \frac{z^{4\Delta-4}\left(\log\frac{\alpha^2+1}{2\epsilon} + \Delta J_4(\alpha^2)\right)}{\Pi_{i=1}^4[(x_i - x)^2 + z^2]^{\Delta}}. \tag{6.119}$$

In analogy to what was done above, let us translate $x_k,\ k = 1, \ldots, 4$ by $(0, -x_4^i)$ (denoted by primes), invert all points (denoted by double primes), and then make the substitution $(z'', x^{i''}) = (z, x^i + x_3^{i''})$. Then, the above expression further simplifies to

$$K_4 = \frac{a^4(16\Pi_{i=1}^4 z_i)^{\Delta}\pi^2}{2(4\pi^2)^6(x_1'x_2'x_3')^{2\Delta}} \int_{-\infty}^{\infty} \frac{\mathrm{d}^3x \, \mathrm{d}z}{[x^2+z^2]^{\Delta}} \, \frac{z^{4\Delta-4}\left[\log(\frac{x^2}{z^2}+1) - \log 2\epsilon + \Delta J_4(\frac{x^2}{z^2})\right]}{[(x_1''' - x)^2 + z^2]^{\Delta}[(x_2''' - x)^2 + z^2]^{\Delta}}, \tag{6.120}$$

with $x_1^{i'''} = x_1^{i''} - x_3^{i''}$ and $x_2^{i'''} = x_2^{i''} - x_3^{i''}$. In order to solve K_4, let us first introduce a generating function with parameters $\gamma, t > 0$:

$$\phi_\Delta^{\gamma,t} = \frac{1}{2x_1'^{2\Delta}x_2'^{2\Delta}x_3'^{2\Delta}} \int_{-\infty}^{\infty} \mathrm{d}^3x \ \mathrm{d}z \ \frac{z^{4\Delta-4+2\gamma}}{[(x_1''' - x)^2 + z^2]^\Delta[(x_2''' - x)^2 + z^2]^\Delta[t^2x^2 + z^2]^{\Delta+\gamma}}, \tag{6.121}$$

comprising all different cases in eq. (6.120). Indeed, the integral (6.120) can be written as

$$K_4 = \frac{a^4(16\ \Pi_{i=1}^4 z_i)^\Delta \pi^2}{(4\pi^2)^6} \Big[-\partial_\gamma \phi_\Delta^{\gamma,t=1}|_{\gamma=0} - \log\frac{\epsilon}{2}\ \phi_\Delta^{\gamma=0,t=1} + \Delta K_4 \Big], \tag{6.122}$$

with

$$\Delta K_4 = \begin{cases} 2\int_0^1 \mathrm{d}t\ \phi_{\Delta=1}^{\gamma=0,t} \operatorname{arctanh} t & \text{for } \Delta = 1, \\ -2\phi_{\Delta=2}^{\gamma=0,t=1} & \text{for } \Delta = 2. \end{cases} \tag{6.123}$$

Introducing Schwinger parameters, integrating over spatial coordinates and making the usual substitution $t_i = ss_i$, yields

$$\phi_\Delta^{\gamma,t} = \frac{\pi^{3/2}\Gamma(\Delta)^{-1}\Gamma(2\Delta+\gamma-\frac{3}{2})}{2\Gamma(\Delta+\gamma)(\eta\zeta\ \Pi_{i<j}r_{ij})^{\frac{2}{3}\Delta}} \int_0^\infty \frac{(\Pi_{i=1}^3 \mathrm{d}s_i)}{(s_1s_2s_3)^{1-\Delta}} \frac{s_3^\gamma(\sum_{i=1}^3 s_i)^{\frac{3}{2}-2\Delta-\gamma}\ \delta(1-\sum_{i=1}^3 s_i)}{(1+s_3(t^2-1))^{\frac{3}{2}-\Delta}[\frac{s_1s_2}{\eta^2} + t^2s_3(\frac{s_1}{\zeta^2} + s_2)]^\Delta},$$

where we reintroduced the conformal invariants already defined in eq. (6.88). Making another substitution $s_1 \to ss_1, s_2 \to ss_2, s_3 \to s$ and integrating over s further simplifies the integral to

$$\phi_\Delta^{\gamma,t} = \frac{\pi^{3/2}\Gamma(\Delta)^{-1}\Gamma(2\Delta+\gamma-\frac{3}{2})}{2\Gamma(\Delta+\gamma)(\eta\zeta\ \Pi_{i<j}r_{ij})^{\frac{2}{3}\Delta}} \int_0^\infty \mathrm{d}s_1\mathrm{d}s_2\ \frac{(s_1s_2)^{\Delta-1}(1+s_1+s_2)^{\frac{3}{2}-2\Delta-\gamma}}{(t^2+s_1+s_2)^{\frac{3}{2}-\Delta}[\frac{s_1s_2}{\eta^2} + t^2(\frac{s_1}{\zeta^2} + s_2)]^\Delta}. \tag{6.124}$$

The change of variables $s_1 = t^2sr, s_2 = t^2s(1-r)$ compactifies one integration region, which leads to

$$\phi_\Delta^{\gamma,t} = \frac{\pi^{3/2}\Gamma(\Delta)^{-1}\Gamma(2\Delta+\gamma-\frac{3}{2})}{2\Gamma(\Delta+\gamma)(\eta\zeta\ \Pi_{i<j}r_{ij})^{\frac{2}{3}\Delta}} \int_0^\infty \mathrm{d}s \int_0^1 \mathrm{d}r\ \frac{(sr(1-r))^{\Delta-1}(1+t^2s)^{\frac{3}{2}-2\Delta-\gamma}}{t^{3-2\Delta}(1+s)^{\frac{3}{2}-\Delta}[s\frac{r(1-r)}{\eta^2} + \frac{r}{\zeta^2} + 1 - r]^\Delta}. \tag{6.125}$$

Note that, as one might expect, eq. (6.125) is invariant under the exchanges $x_1 \leftrightarrow x_2$ and $x_3 \leftrightarrow x_4$. For example, $x_1 \leftrightarrow x_2$ yields $\eta \to \frac{\eta}{\zeta}, \zeta \to \frac{1}{\zeta}$, and then invariance of the above formula follows after a change of variables in r.

Term-by-term computation.

Setting $\gamma = 0, t = 1$ in eq. (6.124) should lead to the four-point function at tree level, which was computed earlier. Indeed, integrating over s_1 yields

$$\begin{aligned} \phi_\Delta^{\gamma=0,t=1} =& \frac{\pi^{3/2}\Gamma(2\Delta-\frac{3}{2})\ \eta^{2\Delta}\zeta^{2\Delta}}{2\Gamma(2\Delta)(\eta\zeta\ \Pi_{i<j}r_{ij})^{\frac{2}{3}\Delta}} \int_0^\infty \frac{\mathrm{d}s_2}{s_2^{1-\Delta}} \frac{(1+s_2)^{-\Delta}}{(\eta^2s_2+\zeta^2)^\Delta} \\ &\times\ {}_2F_1\left[\Delta, \Delta; 2\Delta; 1 - \frac{s_2\eta^2\zeta^2}{(1+s_2)\,(\eta^2s_2+\zeta^2)}\right], \end{aligned} \tag{6.126}$$

corresponding to I_4 given in eq. (6.90) up to a Pfaff transformation of the hypergeometric function. This result, together with eq. (6.122), confirms the expectation that the UV divergence can be completely absorbed in the coupling constant λ. The first term in eq. (6.120) is given by

$$-\partial_\gamma \phi_\Delta^{\gamma,t=1}|_{\gamma=0} = I_4 \left(\psi(\Delta) - \psi(2\Delta - \frac{3}{2})\right) + \frac{\pi^{\frac{3}{2}}\Gamma(2\Delta - \frac{3}{2})}{2\Gamma(\Delta)^2} L_4(\eta,\zeta), \tag{6.127}$$

where $\psi(x)$ denotes the digamma function and where

$$L_4(\eta,\zeta) = \frac{1}{(\eta\zeta\ \Pi_{i<j} r_{ij})^{\frac{2}{3}\Delta}} \int_0^\infty \mathrm{d}s \int_0^1 \mathrm{d}r\ \frac{(sr(1-r))^{\Delta-1}\log(1+s)}{(1+s)^\Delta[\frac{sr(1-r)}{\eta^2} + \frac{r}{\zeta^2} + 1 - r]^\Delta}. \tag{6.128}$$

In the $\Delta = 1$ case, eq. (6.120) contains also the term

$$2\int_0^\infty \mathrm{d}t\ \mathrm{arctanh}\ t\ \phi_{\Delta=1}^{\gamma=0,t} = \pi^2 L_4'(\eta,\zeta), \tag{6.129}$$

where

$$L_4'(\eta,\zeta) = \frac{1}{(\eta\zeta\ \Pi_{i<j} r_{ij})^{\frac{2}{3}}} \int_0^1 \mathrm{d}t \int_0^\infty \mathrm{d}s \int_0^1 \mathrm{d}r\ \frac{\mathrm{arctanh}\ t}{t\sqrt{(1+s)(1+t^2 s)}[\frac{sr(1-r)}{\eta^2} + \frac{r}{\zeta^2} + 1 - r]}. \tag{6.130}$$

By means of the relation

$$\frac{\partial}{\partial s}\int_0^1 \mathrm{d}t\ \frac{\mathrm{arctanh}\ t}{t\sqrt{1+t^2 s}} = \frac{\partial}{\partial s}\int_1^\infty \mathrm{d}\lambda\ \frac{\log(1+\lambda s)}{4\lambda\sqrt{1+\lambda s}} = -\frac{1}{4}\frac{\log(1+s)}{s\sqrt{1+s}}, \tag{6.131}$$

which holds for all $s \geq 0$, one can rewrite $L_4'(\eta,\zeta)$ as

$$L_4'(\eta,\zeta) = \frac{1}{(\eta\zeta\ \Pi_{i<j} r_{ij})^{\frac{2}{3}}} \int_1^\infty \mathrm{d}\lambda \int_0^\infty \mathrm{d}s \int_0^1 \mathrm{d}r\ \frac{\log(1+\lambda s)}{4\lambda\sqrt{(1+s)(1+\lambda s)}[\frac{sr(1-r)}{\eta^2} + \frac{r}{\zeta^2} + 1 - r]}. \tag{6.132}$$

Putting everything together, we find that (cf. eq. (6.122))

$$K_4 = \frac{a^4 4^{2\Delta}(\Pi_{i=1}^4 z_i)^\Delta \pi^2}{(4\pi^2)^6}\left[\left(\psi(\Delta) - \psi(2\Delta - \frac{3}{2}) - \log\frac{\epsilon}{2}\right) I_4 + \pi^{\frac{3}{2}}\frac{\Gamma(2\Delta - \frac{3}{2})}{2\Gamma(\Delta)^2} L_4 + \Delta K_4\right] \tag{6.133}$$

with

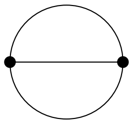
$$\Delta K_4 = \begin{cases} \pi^2 L_4'(\eta,\zeta) & \text{for } \Delta = 1, \\ -2I_4 & \text{for } \Delta = 2. \end{cases} \tag{6.134}$$

The quantities $L_4(\eta,\zeta)$ and $L_4'(\eta,\zeta)$ are given respectively in eq. (6.128) and eqs. (6.130,6.132).

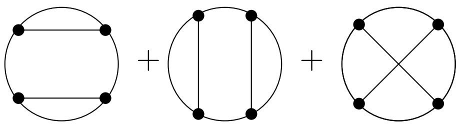
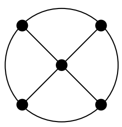
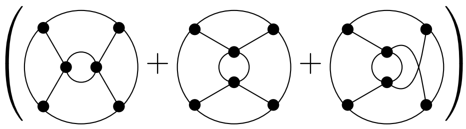
The above result, together with the loop-corrected two-point functions given in section 6.2, are the key new results of this thesis. Eq. (6.133) contains the complete s-channel contribution to the one-loop four-point function in AdS for $\Delta = 1$ as well as $\Delta = 2$. The latter was briefly reported in ref. [61]. The t-channel can be simply recovered by an exchange $x_1 \leftrightarrow x_4$, which turns out to be equivalent to $\eta \leftrightarrow \zeta$. Analogously, the u-channel corresponds to the exchange $x_2 \leftrightarrow x_4$, which, in terms of η and ζ, translates to $\eta \to \frac{1}{\eta}$, $\zeta \to \frac{\zeta}{\eta}$. In the next sections we will relate these results to the conformal block expansion which, in turn, defines the dual conformal field theory. This will be done with the help of a short-distance expansion of eq. (6.133).

6.4 The holographic correlators

Let use summarize the final result of the bulk computation. The Witten two-point function is given by[5]

$$\langle\bar{\phi}(x_1)\bar{\phi}(x_2)\rangle =$$

$$= \frac{N_\phi}{r_{12}^{2\Delta}}, \tag{6.135}$$

where $N_\phi = \frac{2^\Delta a^2}{4\pi^2}$. Then, for the values $\Delta = 1, 2$, the Witten four-point function reads (cf. eq. (6.14))

$$\langle\bar{\phi}(x_1)\bar{\phi}(x_2)\bar{\phi}(x_3)\bar{\phi}(x_4)\rangle =$$

$$+\lambda$$

$$+\frac{\lambda^2}{2}$$

$$+\mathcal{O}\left(\lambda^3\right)$$
$$\begin{aligned} &= \frac{N_\phi^2}{(r_{12}r_{34})^{2\Delta}}\left[1 + v^\Delta + \frac{v^\Delta}{(1-Y)^\Delta} + \lambda\frac{v^\Delta(2\Delta-1)}{4\pi^2}I_0 \right. \\ &\qquad \left. +\lambda^2\frac{3\psi_0}{2}\frac{v^\Delta(2\Delta-1)}{4^3\pi^4}I_0 + \lambda^2\frac{v^\Delta(2\Delta-1)}{4^4\pi^4}K_0\right] + \mathcal{O}\left(\lambda^3\right), \end{aligned} \tag{6.136}$$

[5]The remaining diagrams including tadpoles and sunset diagrams are proportional to the mass shift and thus merely contribute to the relation between renormalized and bare mass of the bulk scalar, as explained in subsection 6.2.4.

where

$$\begin{aligned}\psi_0 &= \psi(\Delta) - \psi(2\Delta - \frac{3}{2}) - \log\frac{\epsilon}{2},\\ I_0 &= \sum_{n,m=0}^{\infty} \frac{Y^m v^n}{m!(n!)^2} \frac{\Gamma(n+\Delta)^2\Gamma(n+m+\Delta)^2}{\Gamma(2n+m+2\Delta)}\psi_{nm},\\ \psi_{nm} &= \psi(n+1) - \frac{1}{2}\log v - \psi(n+\Delta) - \psi(n+\Delta+m) + \psi(2n+2\Delta+m),\end{aligned} \tag{6.137}$$

and

$$K_0 = -12(\Delta-1)I_0 + \sum_{\substack{\pi(x,y,z)\\ x\to v, y\to 1-Y, z\to 1}} \begin{cases} L_0(x,y,z) + 2L_0'(x,y,z) & \text{for } \Delta = 1,\\ L_0(x,y,z) & \text{for } \Delta = 2.\end{cases} \tag{6.138}$$

The above sum runs over the three cyclic permutations π of the variables x, y, z, namely over the s-, u- and t-channel. The appearing quantities are defined as

$$L_0(x,y,z) = \int_0^\infty \mathrm{d}s \int_0^1 \mathrm{d}r \, \frac{(sr(1-r))^{\Delta-1}\log(1+s)}{(1+s)^\Delta[sr(1-r)x + ry + (1-r)z]^\Delta}, \tag{6.139}$$

and

$$L_0'(x,y,z) = \int_0^1 \mathrm{d}t \int_0^\infty \mathrm{d}s \int_0^1 \mathrm{d}r \, \frac{\operatorname{arctanh} t}{t\sqrt{(1+s)(1+t^2 s)}[sr(1-r)x + ry + (1-r)z]}, \tag{6.140}$$

or, equivalently,

$$L_0'(x,y,z) = \int_1^\infty \mathrm{d}\lambda \int_0^\infty \mathrm{d}s \int_0^1 \mathrm{d}r \, \frac{\log(1+\lambda s)}{4\lambda\sqrt{(1+s)(1+\lambda s)}[sr(1-r)x + ry + (1-r)z]}. \tag{6.141}$$

Note that, in this form, the Witten four-point function (6.136) explicitly shows covariance under conformal symmetry.

The logarithmically divergent terms in eq. (6.136) can be absorbed in the coupling constant. Indeed, the renormalized coupling constant obtained through a nonminimal subtraction is given by

$$\lambda = \lambda_R - \frac{3\lambda_R^2}{32\pi^2}\psi_0 + \mathcal{O}(\lambda_R^3). \tag{6.142}$$

Varying the coupling constant with respect to the square root of ϵ leads to the beta function

$$\beta(\lambda) \equiv \sqrt{\epsilon}\frac{\partial\lambda}{\partial\sqrt{\epsilon}} = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3) \tag{6.143}$$

known from standard QFT literature. In what follows, we will use the renormalized coupling constant λ_R. In addition, we will fine-tune the renormalized mass such that the renormalized bulk theory is conformally coupled to geometry. Since the mass has a nonvanishing bulk beta function this is not an bulk RG-invariant statement. One might thus worry that such a theory cannot be dual to a conformal theory on the boundary. This apparent contradiction is, however, resolved by noting that three-dimensional scale transformations on the boundary correspond to an isometry in the four-dimensional bulk.

6.5 The CFT dual

Our final goal is to compare our results on the AdS side, that is, correlations functions evaluated on the boundary, with the double OPE of the full four-point function on the boundary itself. In order to do so, we have to consider the separate limits $v, Y \to 0$ of the whole holographic four-point function (6.136). For the cross diagram I_0, the required expansion is already given to all orders in eq. (6.137). On the other hand, for K_0 given in eq. (6.138), the computation is more elaborate. The adopted procedure is described in detail in appendix E.

We already mentioned in section 5.3 that, in the leading approximation, the bulk scalar is dual to a real scalar operator $\mathcal{O}_\Delta$ of weight Δ. The operator $\mathcal{O}_\Delta$ is a generalized free field and the OPE of $\mathcal{O}_\Delta$ with itself contains double-trace operators $\mathcal{O}_{n,l}$ of all even spins l with conformal dimension $2\Delta + 2n + l$. Schematically, these operators are of the form $\mathcal{O}_{n,l} =: \mathcal{O}_\Delta \Box^n \partial^l \mathcal{O}_\Delta :$, as we will see soon. Let us rewrite the OPE of the operator $\mathcal{O}_\Delta$ with itself as

$$\mathcal{O}_\Delta \mathcal{O}_\Delta = 1 + \sum_{n,l} \mathcal{A}_{n,l}^{1/2}\, \mathcal{O}_{n,l}\,, \tag{6.144}$$

where $\mathcal{A}_{n,l}^{1/2}$ are the OPE coefficients. The cubic vertex is absent in our model, and for this reason $\mathcal{O}_\Delta$ itself does not show up in the OPE. In the leading approximation, the four-point function is given by the disconnected contributions, coming from the product of two-point functions

$$\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle = \frac{1}{(r_{12} r_{34})^{2\Delta}} \left(1 + v^\Delta + \frac{v^\Delta}{(1-Y)^\Delta} \right), \tag{6.145}$$

where we drop N_ϕ^2 from eq. (6.136), being an overall factor. On the other hand, the four-point function has a conformal block expansion[6]

$$\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle = \frac{1}{(r_{12} r_{34})^{2\Delta}} \left(G_0 + \sum_{n,l} \mathcal{A}_{n,l} G_{n,l} \right), \tag{6.146}$$

where G_0 is the contribution of the unit operator and the squares of the OPE coefficients are given in appendix F for any conformal weight ν and any dimension d, see also ref. [36]. Below we consider separately the two cases of interest for us.

[6]We use the recursion relations from ref. [124]. The conformal block $G_{\nu,l}$ of the spin-l operator with weight ν begins with $v^{(\nu-l)/2}(Y^l 2^{-l} + ...)$. See also appendix F.

The connected bulk diagrams result in corrections to the OPE data: the OPE coefficients and anomalous dimensions, which we would like to extract. It is useful to consider the squares of the OPE coefficients $\mathcal{A}_{n,l}$ as functions of the conformal dimension:

$$\mathcal{A}_{\Delta_{n,l}+\gamma^{(1)}_{n,l}+\gamma^{(2)}_{n,l}+\dots} = A_{n,l} + \gamma^{(1)}_{n,l} A^{(1)}_{n,l} + \left(\gamma^{(2)}_{n,l} A^{(1)}_{n,l} + \frac{1}{2}(\gamma^{(1)}_{n,l})^2 A^{(2)}_{n,l}\right) + \dots , \tag{6.147}$$

where it is assumed that $\gamma^{(k)}_{n,l}$ is of order λ_R^k, while $A_{n,l}, A^{(k)}_{n,l}$ are just numbers. Therefore, the conformal block expansion, up to second order in the coupling constant, reads

$$\begin{aligned}\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle &= G_0 + \sum_{n,l} A_{n,l} G_{n,l} + \sum_{n,l} \gamma^{(1)}_{n,l} \left(A^{(1)}_{n,l} G_{n,l} + A_{n,l} G'_{n,l}\right) \\ &+ \sum_{n,l} \left[\gamma^{(2)}_{n,l}(A_{n,l}G'_{n,l} + A^{(1)}_{n,l} G_{n,l}) + \frac{1}{2}(\gamma^{(1)}_{n,l})^2 \left(A^{(2)}_{n,l} G_{n,l} + A_{n,l} G''_{n,l} + 2A^{(1)}_{n,l} G'_{n,l}\right)\right] + \mathcal{O}(\lambda_R^3),\end{aligned}$$

where $G'_{n,l}$ and $G''_{n,l}$ are the derivatives with respect to the conformal dimension evaluated at the free-field value of the double-trace operator's conformal dimension, i.e., at $2\Delta+2n+l$. In what follows we perform the conformal block expansion of the bulk results and extract the OPE data. We first discuss the $\Delta = 2$ case and only afterwards the $\Delta = 1$ case, since the former leads to simpler results.

$\boldsymbol{\Delta = 2}$.

The result of the bulk computation is obtained from eq. (6.136) by setting $\Delta = 2$. Let us rewrite eq. (6.136) as

$$\begin{aligned}&\langle \bar\phi(x_1)\bar\phi(x_2)\bar\phi(x_3)\bar\phi(x_4)\rangle = \\ &= \frac{1}{(r_{12}r_{34})^4}\left[1 + v^2 \sum_{l,m=0}^{\infty} F_{lm}(\log v, \lambda_R)\ Y^m v^l\right],\end{aligned} \tag{6.148}$$

where each F_{lm} can be derived from the results of section 6.4. First, we will focus on the terms with $(l, m) = (0, 0), (1, 0), (0, 1), (0, 2)$ to show how the extraction of OPE data works. Afterwards, we will make more general statements about this data. The conformal blocks $G_{\nu,l}$, where ν is the weight and l the spin of the related operators, which contribute at these orders are

$$\begin{aligned} G_{\nu,0} &= v^{\frac{\nu}{2}}\left[1 + \frac{\nu}{4}Y + \frac{\nu^3(\nu+1)^{-1}}{8(2\nu-1)}v + \frac{\nu(\nu+2)^2}{32(\nu+1)}Y^2\right], \\ G_{\nu,1} &= v^{\frac{\nu-1}{2}}\left[\frac{1}{2}Y + \frac{\nu+1}{8}Y^2\right], \\ G_{\nu,2} &= v^{\frac{\nu-2}{2}}\left[-\frac{1}{3}v + \frac{1}{4}Y^2\right]. \end{aligned} \tag{6.149}$$

Note that the unit operator $\mathbb{1}$ appears in eq. (6.148) in the form of the s-channel disconnected diagram, as its conformal block reads $G_0 = 1$.

For $(l, m) = (0, 0)$, one finds

$$F_{00} = 2 - \frac{\lambda_R}{48\pi^2}(1 + 3\log v) + \frac{\lambda_R^2}{3 \times 2^8 \pi^4}\left(5 + \frac{11}{2}\log v + \frac{3}{4}(\log v)^2\right). \tag{6.150}$$

The only contribution comes from the primary operator $:\mathcal{O}_\Delta^2:$ having conformal dimension $\Delta_{0,0} = 4$. Then, comparing the corresponding conformal block expansion with F_{00} at lowest order in the coupling constant yields $A_{0,0} = 2$. At first order in λ_R, we get

$$\gamma_{0,0}^{(1)} = -\frac{\lambda_R}{16\pi^2}\,, \qquad A_{0,0}^{(1)} = \frac{1}{3}\,, \tag{6.151}$$

whereas at second order in λ_R, we find

$$\gamma_{0,0}^{(1)} = \pm\frac{\lambda_R}{16\pi^2}\,, \qquad \gamma_{0,0}^{(2)} = \frac{5\lambda_R^2}{3 \times 2^8\pi^4}, \qquad A_{0,0}^{(2)} = \frac{20}{9}\,. \tag{6.152}$$

Note that $\gamma_{0,0}^{(1)}$ agrees at different orders in λ_R. This provides an important consitency test for the AdS/CFT duality beyond tree level in the bulk. In previous work, this property was taken as part of the definition of loop diagrams in the bulk (e.g., refs. [39, 35, 128]). It is reassuring to see that it is indeed compatible with an actual bulk calculation. Put differently, this supports the argument that CFT does indeed describe the structure underlying amplitudes of QFT in AdS rather than acting merely as a definition of some bulk theory specified by its correlation functions.

For $(l, m) = (0, 1)$, F_{01} reads

$$F_{01} = 2 - \frac{\lambda_R}{96\pi^2}(5 + 6\log v) + \frac{\lambda_R^2}{12 \times 2^8\pi^4}\left(31 + 25\log v + 3(\log v)^2\right). \tag{6.153}$$

In addition to $:\mathcal{O}_\Delta^2:$, there might be a contribution of a vector operator of dimension 5. However, by comparing F_{01} with the expansions of the conformal blocks, one can infer that the vector operator does not appear in the OPE. This agrees with our expectation based on general CFT arguments.

The term satisfying $(l, m) = (0, 2)$ reads

$$F_{02} = 3 - \frac{9\lambda_R}{160\pi^2}\left(\frac{11}{10} + \log v\right) + \frac{\lambda_R^2}{500 \times 2^9\pi^4}\left(2816 + 1965\log v + 225(\log v)^2\right). \tag{6.154}$$

Bearing in mind that the vector operator of dimension 5 does not appear, the only new operator which contributes is the spin-2 primary of the schematic form $:\mathcal{O}_\Delta\partial^i\partial^j\mathcal{O}_\Delta:$ with $\Delta_{0,2} = 6$. It follows that

$$\begin{aligned} A_{0,2} &= \frac{24}{5}\,, \qquad \gamma_{0,2}^{(1)} = 0, \\ \gamma_{0,2}^{(2)} &= -\frac{\lambda_R^2}{20 \times 2^8\pi^4}\,, \qquad A_{0,2}^{(1)} = \frac{22}{25}\,. \end{aligned} \tag{6.155}$$

Also, here $\gamma^{(1)}_{0,2}$ agrees at both orders in λ_R. An interesting observation is that the spin-2 primary, in spite of not being conserved, does not acquire an anomalous dimension at first order in the coupling λ_R. However, it does get modified at second order. As a consequence to the fact that $\gamma^{(1)}_{0,2} = 0$, we can not provide a value for $A^{(2)}_{0,2}$.

The last term we explicitly consider here corresponds to $(l, m) = (1, 0)$ with

$$F_{10} = \frac{\lambda_R}{120\pi^2}\left(\frac{17}{5} - 6\log v\right) + \frac{\lambda_R^2}{150\times 2^8\pi^4}\left(-\frac{491}{5} + 292\log v + 30(\log v)^2\right), \tag{6.156}$$

which requires a new scalar operator $:\mathcal{O}_\Delta\Box^2\mathcal{O}_\Delta:$ of conformal dimension $\Delta_{1,0} = 6$ and

$$\begin{aligned} A_{1,0} &= \frac{8}{7}\,, \qquad \gamma^{(1)}_{1,0} = -\frac{\lambda_R}{16\pi^2}\,, \qquad A^{(1)}_{1,0} = -\frac{478}{735}\,, \\ \gamma^{(2)}_{1,0} &= \frac{23\lambda_R^2}{15\times 2^7\pi^4}\,, \qquad A^{(2)}_{1,0} = \frac{111392}{77175}\,. \end{aligned} \tag{6.157}$$

Again, there is an agreement of $\gamma^{(1)}_{1,0}$ at different orders in λ_R. Eventually, this gives a complete characterization of all operators of conformal spin $l \leq 2$ entering the OPE.

Now that the adopted strategy is clear, one could proceed in the extraction of the CFT data at higher orders in v and Y. At this point we simply state the results obtained. The zeroth-order OPE coefficients correspond to the disconnected part and follow from the general result discussed above

$$A_{n,l} = \frac{2^{-l-4n}\Gamma\left(l+\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)\Gamma(n+2)\Gamma(l+n+2)\Gamma\left(l+n+\frac{5}{2}\right)\Gamma(l+2n+3)}{\Gamma(l+1)\Gamma(n+1)^2\Gamma\left(l+n+\frac{3}{2}\right)^2\Gamma\left(l+2n+\frac{5}{2}\right)}.$$

The first order anomalous dimensions are easy to extract from I_0 in eq. (6.136):

$$\gamma := \gamma^{(1)}_{n,l=0} = -\frac{\lambda_R}{16\pi^2}, \qquad \gamma^{(1)}_{n,l>0} = 0. \tag{6.158}$$

Only the scalar operators $:\mathcal{O}_\Delta\Box^n\mathcal{O}_\Delta:$ receive anomalous dimensions and, for the simplest quartic interaction as considered here, the anomalous dimension does not depend on n, see also ref. [60]. It is known that such an interaction does not induce anomalous dimensions for the operators with $l > 0$. The OPE coefficients are not so illuminating, but we find them to be in accordance with refs. [60, 36] (note that $\gamma^{(1)}_{n,l}$ does not depend on n and, according to eq. (6.147), is factored out of the OPE coefficients):

$$A^{(1)}_{n,l} = \frac{1}{2}\frac{\partial}{\partial n}A_{n,l}\,, \qquad l = 0\,. \tag{6.159}$$

Additionally, for operators with spin, the OPE coefficients can be determined only at the second order since $\gamma^{(1)}_{n,l>0} = 0$, as is clear from eq. (6.147).

At second order the first few anomalous dimensions read (see appendix F for a more detailed table)

$$
\begin{aligned}
&\gamma^{(2)}_{0,0} = \frac{5}{3}\gamma^2, &&\gamma^{(2)}_{0,2} = -\frac{1}{20}\gamma^2, &&\gamma^{(2)}_{0,4} = -\frac{1}{140}\gamma^2\,, &&\gamma^{(2)}_{0,6} = -\frac{1}{504}\gamma^2\,,\\
&\gamma^{(2)}_{1,0} = \frac{46}{15}\gamma^2\,, &&\gamma^{(2)}_{1,2} = -\frac{107}{1260}\gamma^2\,, &&\gamma^{(2)}_{1,4} = -\frac{19}{1260}\gamma^2\,,\\
&\gamma^{(2)}_{2,0} = \frac{113}{28}\gamma^2\,, &&\gamma^{(2)}_{2,2} = -\frac{269}{2520}\gamma^2\,.
\end{aligned}
$$

Note that the loop correction results in nonvanishing anomalous dimensions for spinning operators as well. Indeed, since there is no operator which saturates the unitarity bound, in our model no local stress tensor and not even a conserved current appears. Therefore, none of the operators is expected to be protected.

While most of the anomalous dimensions are quite complicated, by comparison with the expansion of eq. (6.136) we found a simple formula for the *leading-twist operators*, i.e., for $:\mathcal{O}_\Delta \partial^l \mathcal{O}_\Delta:$ (being in the first Regge trajectory):

$$
\gamma^{(2)}_{0,l} = \gamma^2 \begin{cases} \frac{5}{3} & \text{for } l = 0, \\ -\frac{6}{(l+3)(l+2)(l+1)l} & \text{for } l > 0. \end{cases} \tag{6.160}
$$

The general pattern is that all operators with nonzero spin have negative anomalous dimensions, which should correspond to binding energies in the AdS dual picture. Only the scalar operators have positive anomalous dimensions, which is, however, a second-order effect as compared to $\gamma^{(1)}_{n,l=0}$.

From eq. (6.160) we can easily work out the conformal spin expansion of the anomalous dimensions of the operators belonging to the leading trajectory and notice the latter to be consistent with the general expectations [129, 130, 131]

$$
\gamma^{(2)}_{0,l>0} = \gamma^2 \frac{3}{J^2(1-J^2/2)}, \qquad J^2 = (l+\tau/2)(l+\tau/2-1) = (l+2)(l+1), \tag{6.161}
$$

where the twist τ corresponds to 4.

$\Delta = 1$.

Here we set $\Delta = 1$ in the result of the bulk computation (6.136). The zeroth-order OPE coefficients are then given by[7]

$$
A_{n>0,l} = \frac{4\Gamma\left(l+\frac{3}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\Gamma(l+n+1)\Gamma(l+2n+1)}{2^{l+4n}\Gamma(l+1)\Gamma(n+1)\Gamma\left(l+n+\frac{3}{2}\right)\Gamma\left(l+2n+\frac{1}{2}\right)}\,, \quad A_{n=0,l} = \frac{2\sqrt{\pi}\Gamma(l+1)}{2^l\Gamma\left(l+\frac{1}{2}\right)}\,.
$$

[7] Note that there is small subtlety in taking $\Delta = 1$ in the general formula (F.1), and the $n = 0$ case is special.

The first-order anomalous dimensions are found to be

$$\gamma^{(1)}_{n=0,l=0} = 2\gamma, \qquad \gamma^{(1)}_{n>0,l=0} = \gamma, \qquad \gamma^{(1)}_{n,l>0} = 0. \tag{6.162}$$

We observe a similar pattern as for $\Delta = 2$, except that the anomalous dimension of the very first operator in the OPE, $:\mathcal{O}^2_\Delta:$, jumps from γ to 2γ. The first order OPE coefficients follow eq. (6.159), as expected.

At second order our results are more limited as compared to the $\Delta = 2$ case, the reason being that we did not find an efficient expansion for the integrals $L'_0(x, y, z)$ in eq. (6.140) at high orders in v and Y. Nevertheless, the anomalous dimension of the operators on the first Regge trajectory can be determined to all orders in the spin

$$\gamma^{(2)}_{n=0,l} = \gamma^2 \frac{-4}{2l+1}\psi^{(1)}(l+1) + \gamma^2 \begin{cases} -4 & \text{for } l = 0, \\ -\frac{2}{l(l+1)}, & \text{for } l > 0, \end{cases} \tag{6.163}$$

where $\psi^{(1)}$ is the digamma function, which can be rewritten also as

$$\frac{-4}{2l+1}\psi^{(1)}(l+1) = \frac{1}{2l+1}\left(-\frac{2\pi^2}{3} + 4H^{(2)}_l\right), \tag{6.164}$$

where $H^{(2)}_l = \sum_{k=1}^{l} k^{-2}$ are the generalized harmonic numbers. It is actually in this latter form that the anomalous dimensions emerge from the OPE expansion of the bulk integrals. The last term in eq. (6.163) results from all channels of $L_0(x, y, z)$ and the s-channel of $L'_0(x, y, z)$. In the large spin limit the anomalous dimension behaves as

$$\gamma^{(2)}_{n=0,l}/\gamma^2 = -4/l^2 + 4/l^3 - 10/(3l^4) + \mathcal{O}(1/l^5). \tag{6.165}$$

It can also be seen that the anomalous dimension admits an expansion in terms of the conformal spin. We expect to find a series of the form [129, 130, 131]

$$\gamma^{(2)}_{0,l} = \gamma^2 \sum_{k=1} \frac{Q_k}{J^{2k}}, \qquad J^2 = (l + \tau/2)(l + \tau/2 - 1) = l(l+1), \tag{6.166}$$

where the twist τ is 2 and Q_k are coefficients to be determined. The last term in eq. (6.163) contributes with -2 to Q_1. It is interesting that the first term can also be expanded and the coefficients are related to the Euler-Ramanujan's harmonic number expansion into negative powers of the triangular numbers

$$Q_k = (-)^{k+1} 2^{1-2k} \left(\sum_{j=1}^{k} (-4)^j \binom{k}{j} B_{2j}(\tfrac{1}{2}) + 1 \right), \tag{6.167}$$

where $B_{2j}(x)$ are the Bernoulli polynomials.

The anomalous dimensions of the operators belonging to the subleading Regge trajectories do not have any π^2 (or polygamma) contributions and are listed in appendix F. All of them are negative for $l > 0$ and positive for $l = 0$.

VII Conclusions

In this thesis we gave an analytic derivation of the two-loop correction to bulk/boundary two-point functions for a conformally coupled $\lambda\phi^4$ theory in Euclidean AdS, as well as the one-loop correction for the four-point boundary-to-boundary correlation function, by directly computing the related integrals in position space. The final result can be reduced to a single integral expression which is not given by elementary functions. Nonetheless, it can either be evaluated numerically or, more importantly, be evaluated analytically in a short-distance expansion on the boundary. We have then shown that the theory describes a fully consistent one-parameter family of dual conformal field theories on the boundary of AdS whose OPE coefficients and dimensions are parametrized by the renormalized coupling λ_R. The structure of the dual CFT turns out to be that of a deformed generalized free field of dimension $\Delta = 1$ and $\Delta = 2$. The OPE of the CFT contains an infinite number of further primary double-trace operators which have anomalous dimensions and anomalous OPE coefficients that we are able to compute from our boundary correlation functions. This is the AdS equivalent of determining the masses and branching ratios in flat space-time. In order for the interpretation of our result to work out correctly in terms of a dual CFT, our loop corrected boundary correlation functions have to pass some nontrivial consistency tests. For example, the first order anomalous dimension enters not just at tree-level, but also in the bulk four-point function at one loop multiplying $\log(v)^2$. Similarly, the conformal spin expansion [129, 130, 131] implies a certain asymptotic fall-off behavior of the anomalous dimensions for large spin. All of these conditions are fulfilled by our bulk correlators. In addition, our bulk calculation gives manifestly finite results for all anomalous dimensions in terms of the renormalized bulk coupling, something that is more difficult to achieve in an approach that reconstructs the bulk correlators from the boundary CFT (e.g., refs. [39, 41, 40]).

In summary, the theory considered in this thesis, namely a scalar ϕ^4 theory in AdS_4, leads to a consistent CFT on the conformal boundary, at least up to second order in the coupling constant as was confirmed here. Specifically, the boundary theory is conformally covariant, and satisfies crossing symmetry as well as unitarity. While crossing symmetry follows automatically from the crossing symmetry of the bulk theory, the other two properties are more difficult to achieve. Conformal covariance of the boundary theory can only be obtained by applying a regularization procedure which preserves the AdS symmetry[1] and allows for a well-defined boundary limit. A regulator satisfying these requirements was explicitly constructed here. Unitarity is more demanding. While there is general consensus that a Wightman-alike, and hence finite, QFT in AdS would lead to a unitary CFT on the boundary, this was not a priori ensured for the theory under consideration in this thesis due to appearing IR an UV divergences. At this point, the

[1] at least in the nonradial directions

leading question to raise is if the AdS/CFT correspondence is also satisfied at higher orders. For instance, in case of a truly nonrenormalizable theory in the bulk (e.g., a scalar ϕ^6 theory in AdS_4), it might be necessary to abdicate unitarity since, in order to regularize the theory, one generally introduces a new length scale, the cutoff. This subsequently leads to a family of dual theories parametrized by the cutoff, and the parameter enters in the determination of the conformal weights. It is then not clear if unitarity is satisfied for any value of the cutoff. In the perspective of a nonperturbative analysis of QFTs in AdS, it is conceivable that any effective field theory in the bulk defines an approximate CFT on the boundary[2]. On the other hand, if the AdS/CFT correspondence proves true, a UV complete theory in the bulk should necessarily lead to an exact boundary CFT. Hence, as long as the UV behavior of a scalar ϕ^4 theory in four dimensions remains unknown (we remind the reader of the alleged Landau pole, and of the quantum triviality issue [132]), not even a tentative answer can be given.

Aside from the AdS/CFT correspondence, the scalar field theory with quartic self-interaction in four-dimensional AdS admits a wide spectrum of possible applications. It can be used as a model for a selfinteracting Higgs boson, with the outlook on a full formulation of the Electroweak Theory on AdS. Also, the same selfinteraction appears, among mixed interactions, in the $O(N)$-symmetric vector model. Moreover, for massless spin-1 and spin-$\frac{1}{2}$ particles in AdS, the propagator is again given by eq. (6.8) modulo parallel transport of the polarization vectors. This means that spin-1 and spin-$\frac{1}{2}$ fields lead to similar integrals to those computed here, and hence QED and scalar QED in AdS can be quantized in the same way.

Our results have a direct bearing on higher spin theories in AdS_4 as well. Indeed, all these theories contain a scalar field corresponding to $\Delta = 1, 2$ and need to have a vanishing ϕ^3 bulk coupling[3]. The quartic vertices begin with ϕ^4 and contain infinitely many $(\phi \overleftrightarrow{\nabla}^k \phi)\Box^n(\phi \overleftrightarrow{\nabla}^k \phi)$ vertices [50]. Therefore, our results present a meaningful contribution of the ϕ^4 interaction to the anomalous dimensions and OPE coefficients of the dual vector model [45, 46]. Moreover, this thesis displays a systematic approach which can in principle be applied to the computation of higher spin amplitudes in AdS_4, and offers an example on how to successfully deal with ultraviolet and infrared divergences in the bulk. Implemented with advanced techniques like holographic reconstruction [134, 135], on-shell methods [35, 36], or a combination theoreof [136], it may lead to interesting results.

Another interesting application is in the context of the dS/CFT correspondence [137], where our universe is conjectured to be dual to some three-dimensional Euclidean CFT at early/late times. Even though de Sitter space-time differs from AdS by being nonstationary and by having a space-like boundary, the developed techniques should allow to compute the same amplitudes in dS. The dS/CFT correspondence has appealing

[2] Conformal symmetry could break down when one analyzes operators of large conformal dimension.

[3] In higher spin theories one should also add boundary terms in order to obtain the correct $\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle$ correlator [133].

properties; time is the emergent dimension in the bulk theory, and the duality generates the cosmological constant and hence could explain the dark energy problem. Applied to this framework, our specific model does not possess a dynamical gravity and therefore, besides avoiding UV pathologies, it could lead to an alternative description of the universe where gravity and dark energy emerge directly from the duality. A small perturbation of the dS symmetry could then generate a renormalization group flow of the boundary theory away from the UV fixed point towards the IR fixed point, thus describing the dynamical evolution of the universe.

A concluding remark is given on the interpretation of the boundary-to-boundary correlation functions in the (Euclidean) AdS. In the first place, these fully characterize the bulk theory, much like the scattering amplitudes fully characterize the related QFT in four-dimensional flat space-time. Interestingly, also these scattering amplitudes can be identified with CFT correlators, but on a two-dimensional space-time, the celestial sphere [138]. Additionally, according to the AdS/CFT correspondence, the boundary-to-boundary correlation functions define statistical correlation functions describing second-order phase transitions on the boundary itself. It would be of great interest to detect the explicit theory at the boundary, which in our specific case should correspond to some generalized free field in the zeroth order of the bulk coupling constant λ. One could then analytically or numerically reconstruct the boundary-to-boundary correlation functions of the theory in the bulk, or even provide nonperturbative solutions thereof. Alternatively, an applicable strategy is to extract information about the boundary-to boundary correlation functions by exploiting the known characteristics of the boundary CFT. In flat four-dimensional space-time, a similar approach is given by the S-matrix theory (see, for instance, ref. [76]), where the QFT is bootstrapped by imposing a set of principles to the S-matrix. This is in fact more than an analogy, since the S-matrix theory is related to the AdS/CFT correspondence by the flat space limit [139]. Moreover, the boundary theory of AdS could itself model a physically relevant theory[4], and hence, allow for an experimental determination of the critical exponents of the (universality class of the) statistical theory at the critical point.

[4]For instance, the $O(N)$-model is capable of describing various systems of spins on a lattice [140].

A Ambient space approach

As discussed in section 4.1, the anti–de Sitter space-time AdS_{d+1} can be embedded into a flat space-time $\mathbb{M}_{d,2}$:

$$(X^A)^2 := \eta_{AB} X^A X^B = \eta_{\mu\nu} X^\mu X^\nu - (X^{d+1})^2 = -\frac{1}{a^2}, \tag{A.1}$$

where $A, B = 0, ..., d+1$ and where η_{AB} and $\eta_{\mu\nu}$ are, respectively, the metrics of $\mathbb{M}_{d,2}$ and $\mathbb{M}_{d,1}$. The induced squared line element is given by

$$\mathrm{d}s^2 = (\eta_{AB} + a^2 X_A X_B)\mathrm{d}X^A \mathrm{d}X^B = (\mathrm{d}X^\mu)^2 - \frac{a^2 (X_\mu \mathrm{d}X^\mu)^2}{1 + a^2 (X^\mu)^2}. \tag{A.2}$$

Geodesic distance.

To derive the geodesic distance let us introduce an action on AdS_{d+1} as a measure of the length of a curve $X^A(T)$, constrained to the quadric (4.6):

$$S = \int \mathrm{d}T \left[\frac{1}{2}(\dot{X}^A)^2 + \frac{\kappa^2}{2}\left((X^A)^2 + \frac{1}{a^2}\right)\right], \tag{A.3}$$

where the dot denotes the derivative with respect to the affine parameter T and κ is a Lagrange multiplier. Varying the action gives the equations of motion

$$\ddot{X}_A - \kappa^2 X_A = 0, \qquad (X^A)^2 + \frac{1}{a^2} = 0, \tag{A.4}$$

which have the following general solution:

$$X_A = C_A \exp(\kappa T) + D_A \exp(-\kappa T), \tag{A.5}$$

with the constants C^A and D^A subject to the constraints

$$(C^A)^2 = (D^A)^2 = 0, C_A D^A = -\frac{1}{2a^2}. \tag{A.6}$$

Note that, on shell, $(\dot{X}^A)^2$ satisfies the relation

$$a^2(\dot{X}^A)^2 = \kappa^2. \tag{A.7}$$

Let us take two separate points $X^A \equiv X^A(T_1)$, $Y^A \equiv X^A(T_2)$ on the curve. Then, it follows that

$$X_A Y^A = -\frac{1}{a^2} \frac{e^{a\sqrt{(\dot{X}^A)^2}(T_1 - T_2)} + e^{-a\sqrt{(\dot{X}^A)^2}(T_1 - T_2)}}{2}. \tag{A.8}$$

We can derive the geodesic distance as an integral of the line element:

$$\rho := \int \mathrm{d}s = \int_{T_1}^{T_2} \mathrm{d}T \sqrt{(\dot{X}^A)^2} = \sqrt{(\dot{X}^A)^2}(T_2 - T_1), \tag{A.9}$$

where we integrated over a geodesic curve and used the fact that $(\dot{X}^A)^2$ is a conserved quantity, see eq. (A.7). Eventually, this yields

$$\cosh a\rho = -a^2 X_A Y^A. \tag{A.10}$$

In what follows, we will make use of the quantity

$$u := a^2 X_A Y^A, \tag{A.11}$$

which takes values in $(-\infty, -1]$ and is related to K by

$$K = -\frac{1}{u}. \tag{A.12}$$

Tangent vectors.

Let us for the moment work with $a = 1$. This makes the formulæ simpler and a can be reintroduced by dimensional analysis at any time. One can easily construct a tangent vector V^A to a geodesic curve by taking the gradient of the geodesic distance with respect to X^A, $\partial_A \rho$, and projecting the result onto the tangent space[1] $T_X \mathrm{AdS}_{d+1}$:

$$V^A = -\frac{1}{\sqrt{u^2 - 1}}(Y^A + uX^A). \tag{A.13}$$

Equivalently, one gets another tangent vector by taking the gradient with respect to $Y^{A'}$, where the primed index simply symbolizes that a different point on the quadric is taken:

$$V^{A'} = -\frac{1}{\sqrt{u^2 - 1}}(X^{A'} + uY^{A'}). \tag{A.14}$$

Note that the tangent vectors are automatically normalized $(V^A)^2 = (V^{A'})^2 = 1$.

The vector parallel propagator.

The most general rank-two tensor $G_{AA'}$ that one can construct on AdS_{d+1} with the vectors X_A and $Y_{A'}$ satisfying $(X^A)^2 = (Y^{A'})^2 = -1$ is

$$G_{AA'}(X,Y) = f(u)\eta_{AA'} + g(u)X_A X_{A'} + h(u)X_A Y_{A'} + k(u)Y_A X_A + l(u)Y_A Y_{A'}, \tag{A.15}$$

[1]That is, requiring it to be orthogonal to X^A.

where the coefficients $f(u), g(u), h(u), k(u), l(u)$ are yet to find. These can be uniquely fixed by the following conditions, the defining properties of the *vector parallel propagator*:

$$X^A G_{AA'}(X,Y) = G_{AA'}(X,Y)Y^{A'} = 0, \tag{A.16}$$
$$V^A G_{AA'}(X,Y) = -V_{A'}, \tag{A.17}$$
$$G_{AA'}(X,Y) = G_{A'A}(Y,X), \tag{A.18}$$
$$G_{AA'}(X, Y \to X) = \eta_{AA'} + X_A X_{A'}, \tag{A.19}$$
$$G_{AB'}(X,Y)G^{B'}_{\ C}(Y,X) = G_{AC}. \tag{A.20}$$

Transversality, the constraint in eq. (A.16), restricts our propagator to

$$G_{AA'}(X,Y) = f(u)\left[\eta_{AA'} - \frac{1}{u}Y_A X_{A'}\right] + l(u)\left[X_A X_{A'} + Y_A Y_{A'} + uX_A Y_{A'} + \frac{1}{u}Y_A X_{A'}\right].$$

Furthermore, the *preservation of tangent vectors* (A.17) and the *inverse map condition* (A.20) lead to, respectively,

$$l(u) = \frac{f(u)+u}{1-u^2} \quad \text{and} \quad f(u)^2 = 1. \tag{A.21}$$

The *coincident points limit* (A.19) gives us the boundary condition $f(-1) = 1$ and therefore $f(u) = 1$. Eventually, the parallel propagator becomes

$$G_{AA'} \equiv G_{AA'}(X,Y) = \eta_{AA'} + \frac{1}{1-u}\left[X_A X_{A'} + Y_A Y_{A'} + Y_A X_{A'} + uX_A Y_{A'}\right]. \tag{A.22}$$

Note that the *symmetry condition* (A.18) is manifestly satisfied.

The covariant derivative.

Take any vector T^A lying in the tangent space $T_X\mathrm{AdS}_{d+1}$. As we will see later, the covariant derivative $\nabla_A T_B := G^C_A G^D_B \partial_C T_D$ defines the Levi-Civita connection. First, one can rewrite the above covariant derivative as

$$\nabla_A T_B = \hat{\partial}_A T_B + X_B X^C \hat{\partial}_A T_C = \hat{\partial}_A T_B - X_B\left(\hat{\partial}_A X^C\right) T_C, \tag{A.23}$$

where $\hat{\partial}_A = \left(\delta^B_A + X_A X^B\right)\partial_B$ is the projected ambient space partial derivative. The one-form $e^A := \mathrm{d}X^A$ is the canonical dual basis of $\hat{\partial}_A$, since

$$\mathrm{d}X^A\left(\hat{\partial}_A\right) = \delta^A_B + X^A X_B \tag{A.24}$$

corresponds to the identity element on AdS_{d+1}. This allows for a coordinate-independent reformulation of the covariant derivative:

$$\nabla T_A = \mathrm{d}T_A - X_A e^B T_B. \tag{A.25}$$

The most general covariant derivative which does not affect eq. (A.25) when acting on elements in $T_X\mathrm{AdS}_{d+1}$, but for which both the ambient space metric η_{AB} and the vectors X^A are covariantly constant, is given by

$$\nabla S_A = \mathrm{d}S_A - X_A e^B S_B + e_A X^B S_B \tag{A.26}$$

for any $S_A \in \mathbb{M}_{d,2}$. From

$$\nabla^2 T_A = -e_A e^B T_B \tag{A.27}$$

and $\nabla e^A = 0$ one can see that our connection is metric compatible, torsionless and indeed gives the right curvature form of the AdS_{d+1} space. Therefore, the above definition corresponds to the unique Levi-Civita connection on the anti–de Sitter space-time.

The covariant derivative of the tangent vector in eq. (A.13) is given by

$$\nabla V_A = -\frac{u}{\sqrt{u^2-1}} e^B \left(G_{AB} - V_A V_B\right). \tag{A.28}$$

Note that, as one may expect, it satisfies the geodesic equation $V^B \nabla_B V_A = 0$. One gets a similar expression for the covariant derivative of eq. (A.14):

$$\nabla V_{A'} = -\frac{1}{\sqrt{u^2-1}} e^B \left(G_{BA'} + V_B V_{A'}\right), \tag{A.29}$$

satisfying $V^B \nabla_B V_{A'} = 0$. Another useful relation is the following:

$$\nabla G_{AA'} = -\sqrt{\frac{u+1}{u-1}} e^B \left(G_{BA} V_{A'} + G_{BA'} V_A\right), \tag{A.30}$$

with the property that $V^B \nabla_B G_{AA'} = 0$, where we used the fact that the covariant derivative only acts on the unprimed indices.

The Dirac spinor parallel propagator.

In our conventions, the Clifford algebra on the ambient space $\mathbb{M}_{d,2}$ reads

$$\left\{\gamma^A, \gamma^B\right\} = -2\eta^{AB}. \tag{A.31}$$

It follows that, for X^A with $(X^A)^2 = -1$, the operator $\not{X} := X_A \gamma^A$ squares to one. Additionally, it anticommutes with any $\not{T}$, where $T^A \in T_X\mathrm{AdS}_{d+1}$, and therefore it can be seen as the chirality matrix. Note also that $\left\{\not{X}, \not{Y}\right\} = -2u$. In order to find the *spinor parallel propagator*, we follow the the procedure applied above for the vector parallel propagator. The most general operator dependent on X^A and Y^A one can write down is

$$G(X,Y) = m(u)\mathbf{1} + n(u)\not{X}\not{Y} + o(u)\not{X} + p(u)\not{Y}, \tag{A.32}$$

where $G(X,Y)$ has to satisfy the conditions

$$G(X,Y)\not{Y} = \not{X} G(X,Y), \tag{A.33}$$
$$G(X, Y \to X) = \mathbf{1}, \tag{A.34}$$
$$G(X,Y)G(Y,X) = \mathbf{1}. \tag{A.35}$$

The *preservation of chirality* (A.33) yields

$$G(X,Y) = m(u)\left[\mathbf{1} + \not{X}\not{Y}\right] + o(u)\left[\not{X} + \not{Y}\right], \tag{A.36}$$

whereas the *inverse map condition* (A.35) restricts the coefficients to

$$m(u)^2 = \frac{1}{2(1-u)} \quad \text{and} \quad o(u) = 0. \tag{A.37}$$

Eventually, the *coincident points limit* (A.34) leads to the spinor parallel propagator

$$G(X,Y) = \frac{1}{\sqrt{2(1-u)}}\left[\mathbf{1} + \not{X}\not{Y}\right]. \tag{A.38}$$

To see how the above formula is related to the vector parallel propagator given in eq. (A.22), consider the parallel transport of $\not{T}$ with $T^A \in T_X\mathrm{AdS}_{d+1}$:

$$T^A G_{AA'}\hat{\gamma}^{A'} = T^A G(Y,X)\hat{\gamma}_A G(X,Y), \tag{A.39}$$

where $\hat{\gamma}_A = (\eta_{AB} + X_A X_B)\gamma^B$ are the projected gamma matrices. Inserting eq. (A.38) into the above formula, one can easily reconstruct eq. (A.22). Note that the parallel propagation of normal components X^A vanishes as expected.

The covariant derivative acting on Dirac spinors.

One can make the following ansatz for the covariant derivative of the Dirac spinor:

$$\nabla\psi = \mathrm{d}\psi + A\not{e}\not{X}\psi, \tag{A.40}$$

where A is a complex coefficient which can be found algebraically. The above ansatz is inferred by contracting the connection in eq. (A.26) with the generators of Lorentz transformations in the Dirac representation, $\left[\gamma^A, \gamma^B\right]$. A way to fix the coefficient is to consider the covariant derivative of the gamma matrix. Indeed, the covariant constancy of γ^A, which follows directly from eq. (A.39), yields $A = A^* = -1/2$. Therefore, one concludes that

$$\nabla\psi = \mathrm{d}\psi - \frac{1}{2}\not{e}\not{X}\psi. \tag{A.41}$$

An important property of the above covariant derivative is that it preserves the chirality of the Dirac spinor, since

$$\nabla(\not{X}\psi) = \not{X}\nabla\psi. \tag{A.42}$$

In order to compute the covariant derivative of the spinor parallel propagator given in eq. (A.38), one has to keep in mind that both $\mathbf{1}$ and $\not{X}\not{Y}$ have only one spinor index at the point X^A, and therefore the covariant derivative of these expressions does not vanish. The result is given by

$$\nabla G(X,Y) = -\sqrt{\frac{u+1}{u-1}}\frac{e^A}{2}\left(V_A + \hat{\gamma}_A\not{V}\right)G(X,Y). \tag{A.43}$$

In particular, note that it satisfies the relation $V^A\nabla_A G(X,Y) = 0$.

B Spinor helicity formalism in AdS

In four-dimensional Minkowski space-time $\mathbb{M}_{3,1}$, there exists a powerful tool simplifying the computations of the scattering amplitudes of massless particles with spin, the spinor helicity formalism [141]. In $\mathbb{M}_{3,1}$, one makes use of the fact that its isometry group, the Lorentz group $SO(3,1)$, is, upon complexification, locally isomorphic to $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Thus, all finite-dimensional irreducible representations are labeled by (s_1, s_2), where s_1, s_2 are positive integers or positive half integers. The dimension of the associated representation is $(2s_1+1)(2s_2+1)$. A four-dimensional vector $k^\mu \in \mathbb{M}_{3,1}$, corresponding to the representation $(\frac{1}{2}, \frac{1}{2})$, can be written as having two $SL(2,\mathbb{C})$ indices, $k^{\alpha\dot{\alpha}}$. Each index is related to one of the two inequivalent two-dimensional representations of $SL(2,\mathbb{C})$, the fundamental and the anti–fundamental representation, and hence run over two values. The main advantage of the formalism follows from the possibility to write a four-momentum k^μ, satisfying the mass shell condition ${k^\mu}^2 = 0$, as a product of a spinor with its complex conjugate, i.e., as $k^{\alpha\dot{\alpha}} = \pm\lambda^\alpha\bar{\lambda}^{\dot{\alpha}}$. The sign determines whether k^μ has positive or negative energy.

The same formalism can also be applied on four-dimensional anti–de Sitter space-time AdS_4 [142], even though the situation is slightly different. We do not have full $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ symmetry, but only one $SL(2,\mathbb{C})$ copy corresponding to the $SO(3)$ rotation group in three dimensions. However, this does not prevent us to adopt the same formalism on AdS_4 by using frame fields. Here we see how this works.

The squared line element of AdS_{d+1} is given in Poincaré coordinates by (see section 4.1)

$$\mathrm{d}s^2 = \frac{1}{a^2 z^2}\eta_{\mu\nu}\delta^\mu_{\underline{\mu}}\delta^\nu_{\underline{\nu}}\mathrm{d}x^{\underline{\mu}}\mathrm{d}x^{\underline{\nu}}, \tag{B.1}$$

where $\eta_{\mu\nu}$ is the metric of $\mathbb{M}_{3,1}$. For convenience, let us change notation with respect to the rest of the thesis and write a point $x \in \mathrm{AdS}_4$ as

$$x = (x^0, x^1, z, x^3), \tag{B.2}$$

that is, we position the z-coordinate such that $x^2 = z$. The indices i and $\underline{i}$ are henceforth referred to as running over $0, 1$ and 3. The quantity K then formally reads exactly as in eq. (5.31):

$$K = \frac{2zw}{(x^i - y^i)^2 + z^2 + w^2}, \tag{B.3}$$

where $y = (y^0, y^1, w, y^3)$. Note however that here $(x^i - y^i)^2$ does not evaluate with respect to the Euclidean metric but is related to it by a Wick rotation. Keeping that in mind, since the following main results are functions of K alone, one can recover straightforwardly the expressions for the hyperbolic space $\mathbb{H}_4$ by reinterpreting K as the Euclidean counterpart.

Weyl spinors.

Let us introduce the Pauli matrices:

$$\sigma_\mu^{\alpha\dot\alpha} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \tag{B.4}$$

The epsilon tensor, which will be used to raise and lower the spinor indices, reads

$$\epsilon^{\alpha\beta} = \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{B.5}$$

and an analogous formula applies for dotted indices. We use the same summation convention for both the undotted and the dotted spinor indices, where the contraction applies from the top left to the bottom right. One can easily verify that

$$\sigma_\mu^{\alpha\dot\alpha}\sigma_{\nu\alpha\dot\alpha} = -2\eta_{\mu\nu}, \tag{B.6}$$

which further implies

$$\eta^{\mu\nu}\sigma_\mu^{\alpha\dot\alpha}\sigma_\nu^{\beta\dot\beta} = -2\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}. \tag{B.7}$$

To switch from vector indices to spinor indices, the relations are defined as follows

$$T^{\alpha\dot\alpha} = \sigma_\mu^{\alpha\dot\alpha}T^\mu, \tag{B.8}$$

where $T^\mu = h^\mu_{\underline{\mu}}T^{\underline{\mu}}$ for some vector $T^{\underline{\mu}}$ in AdS_4. The inverse relation is then given by

$$T_\mu = -\frac{1}{2}\sigma_\mu^{\alpha\dot\alpha}T_{\alpha\dot\alpha}. \tag{B.9}$$

Note that

$$\eta_{\mu\nu}T^\mu T^\nu = -\frac{1}{2}T^{\alpha\dot\alpha}T_{\alpha\dot\alpha}. \tag{B.10}$$

Contrary to the definition in eq. (4.21), here it is preferable to introduce a frame field (here vierbein) which directly connects space-time indices to spinor indices:

$$h_{\underline{\mu}}^{\alpha\dot\alpha} = \frac{1}{2z}\sigma_{\underline{\mu}}^{\alpha\dot\alpha}, \tag{B.11}$$

where $\sigma_{\underline{\mu}}^{\alpha\dot\alpha}$ is defined as $\sigma_\mu^{\alpha\dot\alpha}\delta^\mu_{\underline{\mu}}$ and satisfies eqs. (B.6,B.7) for $\eta_{\underline{\mu\nu}} := \eta_{\mu\nu}\delta^\mu_{\underline{\mu}}\delta^\nu_{\underline{\nu}}$. In what follows, we ignore vector indices. The coefficient in eq. (B.11) was chosen such that it normalizes the coefficient in the computation of the curvature of the spin connection, as we will see later. This turns out to be the most natural normalization for spinorial quantities.

With the above vierbein, the metric is given by

$$g_{\underline{\mu\nu}} = h_{\underline{\mu}}^{\alpha\dot\alpha}h_{\underline{\nu}\alpha\dot\alpha} = -\frac{1}{2z^2}\eta_{\underline{\mu\nu}}, \tag{B.12}$$

which can be reinterpreted as $g_{\underline{\mu\nu}}$ having inverse sign and a taking the value $\sqrt{2}$. In any case, a can be reintroduced at any time by dimensional analysis. For the inverse vierbein, we require

$$h^{\underline{\mu}}_{\alpha\dot{\alpha}} h^{\alpha\dot{\alpha}}_{\underline{\nu}} = \delta^{\underline{\mu}}_{\underline{\nu}}\,, \tag{B.13}$$

and this leads to

$$h^{\underline{\mu}}_{\alpha\dot{\alpha}} = -z\sigma_{\underline{\nu}\alpha\dot{\alpha}}\eta^{\underline{\nu\mu}} \equiv g^{\underline{\mu\nu}} h_{\underline{\nu}\alpha\dot{\alpha}}\,. \tag{B.14}$$

Moreover, due to

$$h^{\underline{\mu}\alpha\dot{\alpha}} h^{\beta\dot{\beta}}_{\underline{\mu}} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}, \tag{B.15}$$

we get that

$$T^{\underline{\mu}}T_{\underline{\mu}} \equiv h^{\underline{\mu}\alpha\dot{\alpha}} T_{\alpha\dot{\alpha}} h^{\beta\dot{\beta}}_{\underline{\mu}} T_{\beta\dot{\beta}} = T^{\alpha\dot{\alpha}}T_{\alpha\dot{\alpha}}\,. \tag{B.16}$$

Let us now make a few considerations on the isometry group for fixed radial coordinate z. This corresponds to the Lie group $SO(2,1)$ which is locally isomorphic to the Lie group $SL(2,\mathbb{R})$. For the latter group, the fundamental and the anti–fundamental representations are equivalent. To relate the representations of the two groups, note that a basis of[1] $\mathfrak{sl}(2,\mathbb{R})$ is given by $\{\mathrm{i}\sigma_1, \sigma_2, \mathrm{i}\sigma_3\}$. One can always choose a basis for the fundamental representation of $\mathfrak{sl}(2,\mathbb{C})$ such that, when constrained to the subalgebra $\mathfrak{sl}(2,\mathbb{R})$, the basis corresponds to the one stated above. Indeed, for $\mathfrak{sl}(2,\mathbb{C})$, take the basis $\{\sigma_1, \sigma_2, \sigma_3, \mathrm{i}\sigma_1, \mathrm{i}\sigma_2, \mathrm{i}\sigma_3\}$, where the former three have the interpretation as generators of rotations and the latter three as generators of boosts. Therefore, at any fixed point z, one can directly substitute the undotted indices with $SL(2,\mathbb{R})$ indices:

$$\psi^\alpha := \psi^a \delta_a{}^\alpha, \qquad \psi^a = \psi^\alpha \delta_\alpha{}^a, \tag{B.17}$$

where ψ is a Weyl spinor. A difference arises if one considers the anti–fundamental representation of $\mathfrak{sl}(2,\mathbb{C})$, whose basis is given by $\{\sigma_1, \sigma_2, \sigma_3, -\mathrm{i}\sigma_1, -\mathrm{i}\sigma_2, -\mathrm{i}\sigma_3\}$. Consistency requires that, if N denotes the transformation matrix from dotted to $\mathfrak{sl}(2,\mathbb{R})$ indices, the relations $\{N,\sigma_1\} = \{N,\sigma_3\} = [N,\sigma_2] = 0$ have to be satisfied. Therefore, N is proportional to σ_2 and we write

$$\psi^{\dot{\alpha}} = \mathrm{i}\psi^a \epsilon_a{}^{\dot{\alpha}}, \qquad \psi^a = \mathrm{i}\psi^{\dot{\alpha}}\epsilon_{\dot{\alpha}}{}^a, \tag{B.18}$$

with $\epsilon_1{}^2 = -\epsilon_2{}^1 = 1$. Now that we know how the spinor indices of a nonradial vector $T^{\underline{i}}$ in AdS$_4$ transform under $SO(2,1)$, let us define

$$\tilde{x}^{\alpha\dot{\alpha}} = \sigma^{\alpha\dot{\alpha}}_{\underline{i}} x^{\underline{i}}. \tag{B.19}$$

The quantity K, given explicitly in eq. (B.3), can be used to define a bi-spinor $F^{\alpha\dot{\alpha}}$, which resembles, up to a factor, the tangent vector given in eq. (A.13):

$$F_{\alpha\dot{\alpha}} h^{\alpha\dot{\alpha}} := \mathrm{d}\ln K. \tag{B.20}$$

[1]Recall that the Lie algebra of some Lie group G is denoted by $\mathfrak{g}$.

Note that $\partial_{\alpha\dot\alpha} K = K F_{\alpha\dot\alpha}$. First, one finds

$$F_{\alpha\dot\alpha} h^{\alpha\dot\alpha} = \frac{K}{2z^2 w}\left[\Delta_+ \mathrm{d}z - 2z(x_{\underline{i}} - y_{\underline{i}})\mathrm{d}x^{\underline{i}}\right], \tag{B.21}$$

where $\Delta_\pm = (x^i - y^i)^2 \pm (w^2 - z^2)$. One then can verify that

$$F_{\alpha\dot\alpha} = \frac{K}{2zw}\left[2z(\tilde{x} - \tilde{y})_{\alpha\dot\alpha} - \Delta_+ \mathrm{i}\epsilon_{\alpha\dot\alpha}\right] \tag{B.22}$$

gives exactly eq. (B.21). Because of the symmetry between x and y in the formula for K, the analogue to eq. (A.14) can be read off easily and is given by

$$F_{\alpha'\dot\alpha'} = -\frac{K}{2zw}\left[2w(\tilde{x} - \tilde{y})_{\alpha'\dot\alpha'} + \Delta_- \mathrm{i}\epsilon_{\alpha'\dot\alpha'}\right]. \tag{B.23}$$

Note that $F^{\alpha\dot\alpha} F^{\beta}{}_{\dot\alpha} = (1 - K^2)\epsilon^{\alpha\beta}$. Furthermore, $F^{\alpha\dot\alpha} F_{\alpha\dot\alpha} = 2(K^2 - 1)$.

The Weyl spinor parallel propagator.

To find the parallel propagator $\Pi^{\alpha\alpha'}(x,y)$, one can impose on it the following conditions:

$$\Pi^{\alpha\alpha'}(x,x) = \mathrm{i}\epsilon^{\alpha\alpha'}, \tag{B.24}$$
$$\Pi^{\alpha'\alpha}(y,x) = -\Pi^{\alpha\alpha'}(x,y), \tag{B.25}$$
$$\Pi^{\alpha\beta'}(x,y)\epsilon_{\beta'\gamma'}\Pi^{\gamma'\beta}(y,x) = \epsilon^{\alpha\beta}, \tag{B.26}$$
$$\Pi^{\alpha\alpha'}(x,y)\bar{\Pi}^{\dot\alpha\dot\alpha'}(x,y)F_{\alpha\dot\alpha} = -F^{\alpha'\dot\alpha'}. \tag{B.27}$$

The most general, $\mathfrak{sl}(2,\mathbb{R})$-invariant ansatz has the form

$$\Pi^{\alpha\alpha'}(x,y) = f(x,y)(z+w)\mathrm{i}\epsilon^{\alpha\alpha'} + g(x,y)(\tilde{x} - \tilde{y})^{\alpha\alpha'}, \tag{B.28}$$

where $f(x,y)$ and $g(x,y)$ are real functions which are yet to find. The complex conjugate is given by

$$\bar{\Pi}^{\dot\alpha\dot\alpha'}(x,y) = -f(x,y)(z+w)\mathrm{i}\epsilon^{\dot\alpha\dot\alpha'} + g(x,y)(\tilde{x} - \tilde{y})^{\dot\alpha\dot\alpha'}. \tag{B.29}$$

Using together the *symmetry condition* (B.25) and the *inverse map condition* (B.26), one finds the constraint

$$f(x,y)^2(z+w)^2 + g(x,y)^2(\tilde{x} - \tilde{y})^2 = 1. \tag{B.30}$$

Furthermore, using the *preservation of the tangent vectors* (B.27), the following conditions arise:

$$\begin{aligned} 4z(z+w)(\tilde{x} - \tilde{y})^2 f(x,y)g(x,y) - \Delta_+\left[f(x,y)^2(z+w)^2 - g(x,y)^2(\tilde{x} - \tilde{y})^2\right] &= \Delta_-, \\ (z+w)\Delta_+ f(x,y)g(x,y) + z\left[f(x,y)^2(z+w)^2 - g(x,y)^2(\tilde{x} - \tilde{y})^2\right] &= w. \end{aligned} \tag{B.31}$$

Combining eq. (B.30) with eq. (B.31), one uniquely finds

$$f(x,y)^2 = g(x,y)^2 = f(x,y)g(x,y) = \frac{K}{2zw(K+1)}, \tag{B.32}$$

and thus, the parallel propagator becomes

$$\Pi^{\alpha\alpha'}(x,y) = \sqrt{\frac{K}{2zw(K+1)}}\left[(z+w)\mathrm{i}\epsilon^{\alpha\alpha'} + (\tilde{x}-\tilde{y})^{\alpha\alpha'}\right], \tag{B.33}$$

which automatically satisfies eq. (B.24). At this point is it worth to mention that the parallel propagator for any quantity with multiple spinor indices can be constructed through the application of eq. (B.33) on each of the spinor indices. For example, this allows to find the relation between the Weyl parallel propagator and the vector parallel propagator:

$$\Pi^{\underline{\mu\nu'}}(x,y) = h^{\underline{\mu}}_{\alpha\dot{\alpha}} h^{\underline{\nu'}}_{\alpha'\dot{\alpha}'} \Pi^{\alpha\alpha'}(x,y)\bar{\Pi}^{\dot{\alpha}\dot{\alpha}'}(x,y). \tag{B.34}$$

The covariant derivative acting on Weyl spinors.

Analogously to what done in section 4.1, the spin-connection one-form can be found via the first Cartan structure equation, which reads

$$\mathrm{d}h^{\alpha\dot{\alpha}} + \omega^{\alpha\beta\dot{\alpha}\dot{\beta}} \wedge h_{\beta\dot{\beta}} = 0, \tag{B.35}$$

where

$$\omega^{\alpha\beta\dot{\alpha}\dot{\beta}} = \sigma^{\alpha\dot{\alpha}}_{\mu}\sigma^{\beta\dot{\beta}}_{\nu}\omega^{\mu\nu}. \tag{B.36}$$

Because of its antisymmetry, the spin-connection can be rewritten in the form

$$\omega^{\alpha\beta\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}}\omega^{\alpha\beta} + \epsilon^{\alpha\beta}\omega^{\dot{\alpha}\dot{\beta}}, \tag{B.37}$$

where $\omega^{\alpha\beta}$ and $\omega^{\dot{\alpha}\dot{\beta}}$ are, respectively, the selfdual and the anti-selfdual parts. These quantities are symmetric in the spinor indices. Then, since $\mathrm{d}h^{\alpha\dot{\alpha}} = \frac{1}{z}h^{\alpha\dot{\alpha}} \wedge \mathrm{d}z$, it follows that

$$\omega^{\alpha\beta} = \frac{\mathrm{i}}{2z}\sigma^{\alpha\beta}_{\underline{i}}\mathrm{d}x^{\underline{i}}, \qquad \omega^{\dot{\alpha}\dot{\beta}} = -\frac{\mathrm{i}}{2z}\sigma^{\dot{\alpha}\dot{\beta}}_{\underline{i}}\mathrm{d}x^{\underline{i}}. \tag{B.38}$$

The second Cartan structure equation yields

$$\mathrm{d}\omega^{\alpha\beta} + \omega^{\alpha\gamma} \wedge \omega_{\gamma}{}^{\beta} = h^{\alpha}{}_{\dot{\gamma}} \wedge h^{\beta\dot{\gamma}}, \tag{B.39}$$

corresponding to the curvature two-form of the selfdual part of the spin-connection. As announced earlier, the above quantity is normalized.

Knowing the spin-connection allows us to compute the covariant derivatives of various quantities. For instance, the covariant derivative of $T_{\alpha\dot{\alpha}}$ is given by

$$\nabla T^{\alpha\dot{\alpha}} = \mathrm{d}T^{\alpha\dot{\alpha}} + \omega^{\alpha\beta\dot{\alpha}\dot{\beta}}T_{\beta\dot{\beta}}, \tag{B.40}$$

where $\nabla = \nabla_{\underline{\mu}} \mathrm{d}x^{\underline{\mu}}$. Hence, for Weyl spinors, one gets

$$\nabla\psi^{\alpha} = \mathrm{d}\psi^{\alpha} + \omega^{\alpha\beta}\psi_{\beta}\,, \qquad \nabla\psi^{\dot{\alpha}} = \mathrm{d}\psi^{\dot{\alpha}} + \omega^{\dot{\alpha}\dot{\beta}}\psi_{\dot{\beta}}\,. \tag{B.41}$$

Furthermore, the covariant derivative of $F^{\alpha\dot{\alpha}}$ can be found to be

$$\nabla^{\beta\dot{\beta}}F^{\alpha\dot{\alpha}} := h^{\underline{\mu}\beta\dot{\beta}}\nabla_{\underline{\mu}}F^{\alpha\dot{\alpha}} = 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} + F^{\alpha\dot{\alpha}}F^{\beta\dot{\beta}}. \tag{B.42}$$

Note that $F_{\beta\dot{\beta}}\nabla^{\beta\dot{\beta}}F^{\alpha\dot{\alpha}} = 2K^2F^{\alpha\dot{\alpha}} \neq 0$ since $F^{\alpha\dot{\alpha}}$ is not normalized. The direct computation of the covariant derivative of the Weyl parallel propagator is rather tedious. Let us first determine its general form. The most general ansatz is the following:

$$\nabla^{\beta\dot{\beta}}\Pi^{\alpha\alpha'} = a(x,y)\left(F^{\beta\dot{\beta}}\Pi^{\alpha\alpha'} + b(x,y)F^{\alpha\dot{\beta}}\Pi^{\beta\alpha'}\right), \tag{B.43}$$

where $a(x,y)$ and $b(x,y)$ are some coefficients. Applying the covariant derivative $\nabla^{\delta\dot{\delta}}$ on eq. (B.26) (where eq. (B.25) has also to be considered) and then contracting with $\epsilon_{\alpha\beta}$, one finds the relative coefficient to be $b(x,y) = -2$. This then implies that $F_{\beta\dot{\beta}}\nabla^{\beta\dot{\beta}}\Pi^{\alpha\alpha'} = 0$. The coefficient $a(x,y)$ can be found computationally, and this yields the final form

$$\nabla^{\beta\dot{\beta}}\Pi^{\alpha\alpha'} = \frac{1}{2(K+1)}\left(2F^{\alpha\dot{\beta}}\Pi^{\beta\alpha'} - F^{\beta\dot{\beta}}\Pi^{\alpha\alpha'}\right). \tag{B.44}$$

Using eq. (B.27), one finds the covariant derivative of $F^{\alpha'\dot{\alpha}'}$ to be

$$\nabla^{\beta\dot{\beta}}F^{\alpha'\dot{\alpha}'} = -2K\Pi^{\beta\alpha'}\bar{\Pi}^{\dot{\beta}\dot{\alpha}'} + \frac{K}{K+1}F^{\beta\dot{\beta}}F^{\alpha'\dot{\alpha}'}. \tag{B.45}$$

Note that $F_{\beta\dot{\beta}}\nabla^{\beta\dot{\beta}}F^{\alpha'\dot{\alpha}'} = 2K^2F^{\alpha'\dot{\alpha}'}$.

C Collection of identities

In appendix D, we will compute the bosonic higher spin propagators by using the formalism developed in appendix B. The calculations require the knowledge of a set of identities, which we will give here. Particularly handy is the fact that the structures $K, F^{\alpha\dot{\alpha}}, F^{\alpha'\dot{\alpha}'}, \Pi_{\alpha\alpha'}, \bar{\Pi}^{\dot{\alpha}\alpha'}$ form a closed algebra, and all covariant derivatives do not produce any new object. In what follows, the $\cdot$-product denotes contraction of the (suppressed) spinor indices.

Algebraic Identities.

$$F_{\alpha\dot{\alpha}}F^{\alpha\dot{\alpha}} = 2(K^2-1), \tag{C.1}$$

$$F^{\alpha\dot{\alpha}}F^{\beta}{}_{\dot{\alpha}} = \epsilon^{\alpha\beta}(1-K^2), \tag{C.2}$$

$$F^{\alpha\dot{\alpha}}F_{\alpha}{}^{\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}}(1-K^2), \tag{C.3}$$

$$\Pi^{\nu\alpha'}\Pi_{\nu}{}^{\beta'} = \epsilon^{\alpha'\beta'}, \tag{C.4}$$

$$\Pi_{\alpha\alpha'}\Pi^{\alpha\alpha'} = -2, \tag{C.5}$$

$$\Pi^{\alpha\alpha'} = -\Pi^{\alpha'\alpha}, \tag{C.6}$$

$$\bar{\Pi}^{\dot{\alpha}\dot{\alpha}'} = -\bar{\Pi}^{\dot{\alpha}'\dot{\alpha}}, \tag{C.7}$$

$$F_{\alpha\dot{\alpha}}\Pi^{\alpha\alpha'}\bar{\Pi}^{\dot{\alpha}\dot{\alpha}'} = -F^{\alpha'\dot{\alpha}'}, \tag{C.8}$$

$$F_{\alpha'\dot{\alpha}'}\Pi^{\alpha\alpha'}\bar{\Pi}^{\dot{\alpha}\dot{\alpha}'} = -F^{\alpha\dot{\alpha}}, \tag{C.9}$$

$$\Pi^{\gamma\alpha'}F_{\gamma}{}^{\dot{\alpha}} = -\bar{\Pi}^{\dot{\alpha}\dot{\gamma}'}F^{\alpha'}{}_{\dot{\gamma}'} = \bar{\Pi}^{\dot{\gamma}'\dot{\alpha}}F^{\alpha'}{}_{\dot{\gamma}'}\ , \tag{C.10}$$

$$F^{\alpha}{}_{\dot{\gamma}}\,\bar{\Pi}^{\dot{\gamma}\dot{\gamma}'}F^{\alpha'}{}_{\dot{\gamma}'} = -\Pi^{\alpha\alpha'}(1-K^2), \tag{C.11}$$

$$F_{\gamma}{}^{\dot{\alpha}}\,\Pi^{\gamma\gamma'}F_{\gamma'}{}^{\dot{\alpha}'} = -\bar{\Pi}^{\dot{\alpha}\dot{\alpha}'}(1-K^2), \tag{C.12}$$

$$(F^{\alpha\dot{\gamma}}\bar{\Pi}_{\dot{\gamma}}{}^{\dot{\beta}'})(F^{\gamma\dot{\beta}}\Pi_{\gamma}{}^{\alpha'}) = (1-K^2)\Pi^{\alpha\alpha'}\bar{\Pi}^{\dot{\beta}\dot{\beta}'} - F^{\alpha\dot{\beta}}F^{\alpha'\dot{\beta}'}. \tag{C.13}$$

First derivatives.

$$\nabla_{\alpha\dot{\alpha}}K = KF_{\alpha\dot{\alpha}}\,, \tag{C.14}$$

$$\nabla_{\alpha\dot{\alpha}}F^{\alpha\dot{\alpha}} = 6+2K^2, \tag{C.15}$$

$$\nabla_{\alpha\dot{\alpha}}F_{\beta\dot{\beta}} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + F_{\alpha\dot{\alpha}}F_{\beta\dot{\beta}}\,, \tag{C.16}$$

$$\nabla_{\alpha\dot{\alpha}}F_{\beta'\dot{\beta}'} = \frac{K}{1+K}F_{\alpha\dot{\alpha}}F_{\beta'\dot{\beta}'} - 2K\Pi_{\alpha\beta'}\bar{\Pi}_{\dot{\alpha}\dot{\beta}'}\,, \tag{C.17}$$

$$\nabla_{\alpha\dot{\alpha}}\Pi_{\beta\beta'} = \frac{1}{2(1+K)}\left[2F_{\beta\dot{\alpha}}\Pi_{\alpha\beta'} - F_{\alpha\dot{\alpha}}\Pi_{\beta\beta'}\right], \tag{C.18}$$

$$\nabla_{\alpha\dot{\alpha}}\bar{\Pi}_{\dot{\beta}\dot{\beta}'} = \frac{1}{2(1+K)}\left[2F_{\alpha\dot{\beta}}\bar{\Pi}_{\dot{\alpha}\dot{\beta}'} - F_{\alpha\dot{\alpha}}\bar{\Pi}_{\dot{\beta}\dot{\beta}'}\right], \tag{C.19}$$

$$
\begin{aligned}
(F\cdot\nabla)K &= -2K(1-K^2), && \text{(C.20)}\\
(F\cdot\nabla)F^{\beta\dot\beta} &= 2K^2F^{\beta\dot\beta}\,, && \text{(C.21)}\\
(F\cdot\nabla)F^{\beta'\dot\beta'} &= 2K^2F^{\beta'\dot\beta'}\,, && \text{(C.22)}\\
(F\cdot\nabla)\Pi^{\beta\dot\beta'} &= 0, && \text{(C.23)}\\
\nabla_{\alpha\dot\alpha}F^{\alpha\dot\beta} &= \epsilon_{\dot\alpha}{}^{\dot\beta}\,(3+K^2), && \text{(C.24)}\\
\nabla_{\alpha\dot\alpha}F^{\beta\dot\alpha} &= \epsilon_{\alpha}{}^{\beta}\,(3+K^2). && \text{(C.25)}\\
& && \text{(C.26)}
\end{aligned}
$$

D'Alembertian.

For a scalar function $f(K)$, we have

$$
\nabla^2 f(K) := \nabla_{\alpha\dot\alpha}\nabla^{\alpha\dot\alpha}f(K) = 2K^2(K^2-1)\frac{\partial^2 f(K)}{\partial K^2} + 4K(K^2+1)\frac{\partial f(K)}{\partial K}. \tag{C.27}
$$

Note that the above equation is, up to a sign, in agreement with eq. (5.39) if one chooses $a=\sqrt{2}$ and $d=3$ as we have done. The sign difference comes from the negative overall sign of the metric $g_{\underline{\mu\nu}}$, see eq. (B.12). The fundamental relations involving the D'Alembertian are

$$
\begin{aligned}
\nabla^2 K &= 4K(1+K^2), && \text{(C.28)}\\
\nabla^2 F^{\alpha\dot\alpha} &= F^{\alpha\dot\alpha}2(2K^2+3), && \text{(C.29)}\\
\nabla^2 F^{\alpha'\dot\alpha'} &= 4K^2F^{\alpha'\dot\alpha'}, && \text{(C.30)}\\
\nabla^2\Pi^{\alpha\alpha'} &= \frac{3(1-K)}{2(1+K)}\Pi^{\alpha\alpha'}, && \text{(C.31)}\\
\nabla^2\bar\Pi^{\dot\alpha\dot\alpha'} &= \frac{3(1-K)}{2(1+K)}\bar\Pi^{\dot\alpha\dot\alpha'}. && \text{(C.32)}
\end{aligned}
$$

Mixed-derivatives.

$$
\begin{aligned}
(\nabla K)\cdot(\nabla K) &= -2K^2(1-K^2), && \text{(C.33)}\\
(\nabla K)\cdot(\nabla F_{\beta\dot\beta}) &= 2K^3F_{\beta\dot\beta}\,, && \text{(C.34)}\\
(\nabla K)\cdot(\nabla F_{\beta'\dot\beta'}) &= 2K^3F_{\beta'\dot\beta'}\,, && \text{(C.35)}\\
(\nabla K)\cdot(\nabla\Pi_{\beta\beta'}) &= 0, && \text{(C.36)}\\
(\nabla K)\cdot(\nabla\bar\Pi_{\dot\beta\dot\beta'}) &= 0, && \text{(C.37)}\\
(\nabla F_{\alpha\dot\alpha})\cdot(\nabla F_{\beta\dot\beta}) &= 4\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + 2F_{\alpha\dot\alpha}F_{\beta\dot\beta}(1+K^2), && \text{(C.38)}\\
(\nabla F_{\alpha\dot\alpha})\cdot(\nabla F_{\beta'\dot\beta'}) &= \frac{2K\,(K^2+K+1)}{K+1}F_{\alpha\dot\alpha}F_{\beta'\dot\beta'} - 4K\Pi_{\alpha\beta'}\bar\Pi_{\dot\alpha\dot\beta'}\,, && \text{(C.39)}\\
(\nabla F_{\alpha\dot\alpha})\cdot(\nabla\Pi_{\beta\beta'}) &= \frac{1}{1+K}\left[2F_{\beta\dot\alpha}\Pi_{\alpha\beta'} - F_{\alpha\dot\alpha}\Pi_{\beta\beta'}\right] = 2\nabla_{\alpha\dot\alpha}\Pi_{\beta\beta'}\,, && \text{(C.40)}
\end{aligned}
$$

$$(\nabla\Pi_{\alpha\alpha'})\cdot(\nabla\Pi_{\beta\beta'}) = \frac{(1-K)}{2(1+K)}\left[2\epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'} + \Pi_{\alpha\alpha'}\Pi_{\beta\beta'}\right], \tag{C.41}$$

$$(\nabla\Pi^{\alpha\alpha'})\cdot(\nabla\bar{\Pi}^{\dot{\beta}\dot{\beta}'}) = \frac{1}{2(1+K)^2}\left[2F^{\alpha\dot{\beta}}F^{\alpha'\dot{\beta}'} - (1-K^2)\Pi^{\alpha\alpha'}\bar{\Pi}^{\dot{\beta}\dot{\beta}'}\right]. \tag{C.42}$$

Other random identities.

$$\nabla_{\alpha\dot{\alpha}}(F^{\alpha\dot{\alpha}}F_{\alpha'\dot{\alpha}'}) = (4K^2+6)F_{\alpha'\dot{\alpha}'}\,, \tag{C.43}$$

$$\nabla^{\alpha\dot{\alpha}}(\Pi_{\alpha\alpha'}\bar{\Pi}_{\dot{\alpha}\dot{\alpha}'}) = \frac{3}{K+1}F_{\alpha'\dot{\alpha}'}\,, \tag{C.44}$$

$$\nabla_{\alpha\dot{\alpha}}F_{\beta'\dot{\beta}} = -(1+K)\Pi_{\alpha\beta'}\epsilon_{\dot{\alpha}\dot{\beta}} + \frac{1+2K}{2(1+K)}F_{\alpha\dot{\alpha}}F_{\beta'\dot{\beta}}\,, \tag{C.45}$$

$$(F\cdot\nabla)F_{\beta'\dot{\beta}} = 2K^2F_{\beta'\dot{\beta}}\,, \tag{C.46}$$

$$\nabla_{\alpha\dot{\gamma}}F_{\beta'}{}^{\dot{\gamma}} = -\frac{1}{2}(2K^2+3K+3)\Pi_{\alpha\beta'}\,, \tag{C.47}$$

$$\nabla^2(F_{\alpha\dot{\alpha}}F_{\alpha'\dot{\alpha}'}) = \frac{12K^3+12K^2+10K+6}{K+1}F_{\alpha\dot{\alpha}}F_{\alpha'\dot{\alpha}'} - 8K\Pi_{\alpha\alpha'}\bar{\Pi}_{\dot{\alpha}\dot{\alpha}'}\,, \tag{C.48}$$

$$\nabla^2(\Pi_{\alpha\alpha'}\bar{\Pi}_{\dot{\alpha}\dot{\alpha}'}) = \frac{2}{(K+1)^2}\left[F_{\alpha\dot{\alpha}}F_{\alpha'\dot{\alpha}'} + (1-K^2)\Pi_{\alpha\alpha'}\bar{\Pi}_{\dot{\alpha}\dot{\alpha}'}\right], \tag{C.49}$$

where $F^{\alpha'\dot{\alpha}} = \Pi^{\gamma\alpha'}F_{\gamma}{}^{\dot{\alpha}}$.

D Bosonic higher spin propagators

In section 5.2 we derived the scalar propagator of a massive particle in hyperbolic space $\mathbb{H}_{d+1}$. Here we will firstly repeat the calculation in order to conform with the conventions adopted in appendices B and C. That is, we work in AdS$_4$ space-time with the metric tensor given in eq. (B.12). In addition, we compute the propagators of massless particles of every integer spin.

D.1 Propagation of a scalar particle

In the conventions adopted, the scalar propagator Λ has to fulfill

$$\left(\nabla^2 + m^2\right) \Lambda(x, y; \Delta) = \frac{\mathrm{i}}{\sqrt{-g}} \delta^{(4)}(x - y), \tag{D.1}$$

where $g = -(1/2z)^8$ is the determinant of the AdS$_4$ metric and

$$\Delta = \frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{m^2}{2}}. \tag{D.2}$$

Again, owing to the isotropy of anti–de Sitter space-time, we can take Λ to be a function of only K and Δ. The solutions to the homogeneous part are then given by

$$\Lambda(K; \mu^2) = C_\Delta K^\Delta \, {}_2F_1 \left[\frac{\Delta}{2}, \frac{\Delta + 1}{2}; \frac{2\Delta - 1}{2}; K^2 \right], \tag{D.3}$$

where C_Δ are coefficients, fixed by eq. (D.1). These can be found by noting that the normalization on the right hand side of eq. (D.1) was chosen such that, under a Wick rotation[1] $x^0 \to \mathrm{i}x^0, y^0 \to \mathrm{i}y^0$, one recovers exactly the expression in Euclidean anti–de Sitter space, see eq. (5.38). Hence, by basically repeating the steps of section 5.2, one gets

$$C_\Delta = \frac{\Gamma(\frac{\Delta}{2})\Gamma(\frac{\Delta+1}{2})}{2\pi^2\Gamma(\Delta - \frac{1}{2})}. \tag{D.4}$$

The propagator for the minimally coupled scalar particle is given by

$$\Lambda(K, \Delta = 3) = \frac{K - \mathrm{arctanh}K + K^2 \mathrm{arctanh}K}{2\pi^2 \left(1 - K^2\right)}, \tag{D.5}$$

whereas those for the conformally coupled scalar particle read

$$\Lambda(K, \Delta = 1) = \frac{K}{2\pi^2 \left(1 - K^2\right)}, \qquad \Lambda(K, \Delta = 2) = \frac{K^2}{2\pi^2 \left(1 - K^2\right)}. \tag{D.6}$$

[1] The delta function $\delta(x^0 - y^0)$ changes by a factor $-\mathrm{i}$.

D.2 Propagation of a vector particle

Let us start from the equation of motion of a free massless vector particle $A_{\alpha\dot{\alpha}}$:

$$\nabla^{\beta\dot{\beta}}\nabla_{\beta\dot{\beta}}A_{\alpha\dot{\alpha}} - \nabla^{\beta\dot{\beta}}\nabla_{\alpha\dot{\alpha}}A_{\beta\dot{\beta}} = 0. \tag{D.7}$$

Equivalently, one can write the above equation as

$$\nabla^{\beta\dot{\beta}}\mathcal{F}_{\beta\dot{\beta}\alpha\dot{\alpha}} = 0, \tag{D.8}$$

with

$$\mathcal{F}_{\beta\dot{\beta}\alpha\dot{\alpha}} = \nabla_{[\beta\dot{\beta}}A_{\alpha\dot{\alpha}]} := \nabla_{\beta\dot{\beta}}A_{\alpha\dot{\alpha}} - \nabla_{\alpha\dot{\alpha}}A_{\beta\dot{\beta}} \tag{D.9}$$

being the field strength tensor. Instead of fixing the gauge directly, for instance by the Lorenz gauge condition, let us add an R_ξ-gauge term to the equation of motion:

$$\nabla^{\beta\dot{\beta}}\mathcal{F}_{\beta\dot{\beta}\alpha\dot{\alpha}} + \xi\nabla_{\alpha\dot{\alpha}}\nabla^{\beta\dot{\beta}}A_{\beta\dot{\beta}} = 0. \tag{D.10}$$

Let us now introduce the propagator:

$$\langle A_{\alpha\dot{\alpha}}A_{\alpha'\dot{\alpha}'}\rangle := \Lambda_{\alpha\dot{\alpha}\alpha'\dot{\alpha}'} \tag{D.11}$$

satisfying the equation

$$\nabla^{\beta\dot{\beta}}\langle\mathcal{F}_{\beta\dot{\beta}\alpha\dot{\alpha}}A_{\alpha'\dot{\alpha}'}\rangle + \xi\nabla_{\alpha\dot{\alpha}}\nabla^{\beta\dot{\beta}}\langle A_{\beta\dot{\beta}}A_{\alpha'\dot{\alpha}'}\rangle = \epsilon_{\alpha\alpha'}\epsilon_{\dot{\alpha}\dot{\alpha}'}\frac{\mathrm{i}}{\sqrt{-g}}\delta^{(4)}(x-y), \tag{D.12}$$

where

$$\langle\mathcal{F}_{\beta\dot{\beta}\alpha\dot{\alpha}}A_{\alpha'\dot{\alpha}'}\rangle = \nabla_{[\beta\dot{\beta}}\Lambda_{\alpha\dot{\alpha}]\alpha'\dot{\alpha}'}. \tag{D.13}$$

The most general ansatz is given by

$$\langle A_{\alpha\dot{\alpha}}A_{\alpha'\dot{\alpha}'}\rangle = f(K)\Pi_{\alpha\alpha'}\bar{\Pi}_{\dot{\alpha}\dot{\alpha}'} + \nabla_{\alpha\dot{\alpha}}\nabla_{\alpha'\dot{\alpha}'}g(K), \tag{D.14}$$

where $f(K)$ and $g(K)$ are yet to find. This ansatz has the advantage that the second term is pure gauge [143] and, therefore, in principle it suffices to find $f(K)$. However, here we will follow another route determining $\langle\mathcal{F}_{\beta\dot{\beta}\alpha\dot{\alpha}}A_{\alpha'\dot{\alpha}'}\rangle$ instead. To this aim, let us make the additional ansatzes

$$\begin{aligned}\langle\mathcal{F}_{\beta\dot{\beta}\alpha\dot{\alpha}}A^{\alpha'\dot{\alpha}'}\rangle &= QF_{[\beta\dot{\beta}}\Pi_{\alpha}{}^{\alpha'}\bar{\Pi}_{\dot{\alpha}]}{}^{\dot{\alpha}'},\\ \nabla^{\beta\dot{\beta}}\langle A_{\beta\dot{\beta}}A_{\alpha'\dot{\alpha}'}\rangle &= PF_{\alpha'\dot{\alpha}'}.\end{aligned} \tag{D.15}$$

The relation between f, g and Q, P can be found to be

$$\begin{aligned}Q &= Kf' - \frac{f}{K+1},\\ P &= 4K(3K^2+1)g' + 12K^4g'' + 2K^3g'''(K^2-1) + \frac{3f}{K+1} - Kf',\end{aligned} \tag{D.16}$$

where the prime denotes the derivative with respect to K. Note that Q only depends on f. In terms of Q, P, the homogeneous part of eq. (D.12) is given by

$$\begin{aligned} (K^2-1)KQ' + (K^2+2)Q - \xi K P &= 0, \\ KQ' + \frac{K-2}{K+1}Q + \xi\left(KP' + \frac{K}{K+1}P\right) &= 0, \end{aligned} \tag{D.17}$$

which further leads to a differential equation for solely Q:

$$K^2(K^2-1)Q'' + 2K(2K^2+1)Q' + 2(K^2-1)Q = 0. \tag{D.18}$$

The solution is

$$Q = \frac{CK}{(K+1)^2} + \frac{K^2 D}{(K-1)^2(K+1)^2}, \tag{D.19}$$

where C and D are integration constants. By using eqs. (D.17) together with eq. (D.19), one can find the relation

$$KQ + \xi P = C. \tag{D.20}$$

Note that $\mathcal{F}_{\beta\dot\beta\alpha\dot\alpha}$ is either antisymmetric in the undotted or in the dotted indices. Therefore, it can be split in a selfdual and an anti–selfdual object:

$$\begin{aligned} \mathcal{F}_{\alpha\beta} &:= \epsilon^{\dot\alpha\dot\beta}\mathcal{F}_{\beta\dot\beta\alpha\dot\alpha}, \\ \mathcal{F}_{\alpha\beta} &:= \epsilon^{\dot\alpha\dot\beta}\mathcal{F}_{\beta\dot\beta\alpha\dot\alpha}. \end{aligned} \tag{D.21}$$

These two objects are related to Q as follows

$$\begin{aligned} \langle \mathcal{F}_{\alpha\beta} A_{\alpha'\dot\alpha'} \rangle &= Q F_{\alpha}{}^{\dot\alpha} \Pi_{\alpha\alpha'} \bar\Pi_{\dot\alpha\dot\alpha'}\,, \\ \langle \mathcal{F}_{\dot\alpha\dot\beta} A_{\alpha'\dot\alpha'} \rangle &= Q F^{\alpha}{}_{\dot\alpha} \bar\Pi_{\dot\alpha\dot\alpha'} \Pi_{\alpha\alpha'}\,, \end{aligned} \tag{D.22}$$

where we use the notation that if two indices are identical but uncontracted then one has to symmetrize over them, which is achieved by summing up all necessary permutations without any additional factors. Another quantity one can construct is given by

$$\langle \mathcal{F}_{\beta\dot\beta\alpha\dot\alpha} \mathcal{F}_{\beta'\dot\beta'\alpha'\dot\alpha'} \rangle = \langle \mathcal{F}_{\beta\dot\beta\alpha\dot\alpha} A_{[\alpha'\dot\alpha'} \rangle \overleftarrow{\nabla}_{\beta'\dot\beta']}, \tag{D.23}$$

where the covariant derivative $\overleftarrow{\nabla}_{\beta'\dot\beta'}$ acts on the left. It can be split in four different objects, but due to complex conjugation only the following two are independent:

$$\begin{aligned} \langle \mathcal{F}_{\alpha\beta} \mathcal{F}_{\alpha'\beta'} \rangle &:= \epsilon^{\dot\alpha\dot\beta}\epsilon^{\dot\alpha'\dot\beta'} \langle \mathcal{F}_{\beta\dot\beta\alpha\dot\alpha} \mathcal{F}_{\beta'\dot\beta'\alpha'\dot\alpha'} \rangle, \\ \langle \mathcal{F}_{\alpha\beta} \mathcal{F}_{\dot\alpha'\dot\beta'} \rangle &:= \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha'\beta'} \langle \mathcal{F}_{\beta\dot\beta\alpha\dot\alpha} \mathcal{F}_{\beta'\dot\beta'\alpha'\dot\alpha'} \rangle. \end{aligned} \tag{D.24}$$

In terms of Q, these read

$$\begin{aligned} \langle \mathcal{F}_{\alpha\beta} \mathcal{F}_{\alpha'\beta'} \rangle &= \left[Q'K(K^2-1) + Q(K+1)^2\right] \Pi_{\alpha\alpha'} \Pi_{\alpha\beta'}\,, \\ \langle \mathcal{F}_{\alpha\beta} \mathcal{F}_{\dot\alpha'\dot\beta'} \rangle &= \left(Q'K + Q\frac{K-1}{K+1}\right) \left(F^{\gamma'}{}_{\dot\alpha'} \Pi_{\alpha\gamma'}\right) \left(F_{\alpha}{}^{\dot\gamma} \bar\Pi_{\dot\gamma\dot\beta'}\right). \end{aligned} \tag{D.25}$$

Note that, by using eq. (D.19), the latter expression becomes

$$\langle \mathcal{F}_{\alpha\beta}\mathcal{F}_{\dot{\alpha}'\dot{\beta}'}\rangle = D\left[\frac{K^2}{(1-K)^2(1-K^2)}\right]\left(F^{\gamma'}{}_{\dot{\alpha}'}\Pi_{\alpha\gamma'}\right)\left(F_{\alpha}{}^{\dot{\gamma}}\bar{\Pi}_{\dot{\gamma}\dot{\beta}'}\right), \tag{D.26}$$

which does not depend on how we choose C. However, C affects the behaviour of the first expression in eq. (D.25):

$$\langle \mathcal{F}_{\alpha\beta}\mathcal{F}_{\alpha'\beta'}\rangle = \left[\frac{(4C-D)K^2}{(1+K)^2}\right]\Pi_{\alpha\alpha'}\Pi_{\alpha\beta'}\,. \tag{D.27}$$

Let us take $D = 4C$, such that the latter quantity vanishes. This is physically reasonable, as then the purely (anti)–selfdual correlators are zero[2]. This choice leads to

$$Q = C\frac{K}{(K-1)^2}, \quad P = \frac{C}{\xi}\frac{1-2K}{(K-1)^2}, \tag{D.28}$$

where C is yet to find. To this aim, we perform a Wick rotation and then integrate eq. (D.12) over a covariantly constant test-current $j(x)^{\alpha\dot{\alpha}}$:

$$\int \mathrm{d}^4x\sqrt{g}\; j(x)^{\alpha\dot{\alpha}}\left[\nabla^{\beta\dot{\beta}}QF_{[\beta\dot{\beta}}\Pi_{\alpha\alpha'}\bar{\Pi}_{\dot{\alpha}\dot{\alpha}']} + \xi\nabla_{\alpha\dot{\alpha}}PF_{\alpha'\dot{\alpha}'}\right] = j(y)_{\alpha'\dot{\alpha}'}. \tag{D.29}$$

Partial integration of the left hand side, and subsequent integration of the boundary term over a small ball of radius R (see section 5.2), yields the constraint $C = \frac{1}{4\pi^2}$. Therefore, we conclude that

$$\langle \mathcal{F}_{\beta\dot{\beta}\alpha\dot{\alpha}}A_{\alpha'\dot{\alpha}'}\rangle = \frac{K}{4\pi^2(K-1)^2}F_{[\beta\dot{\beta}}\Pi_{\alpha\alpha'}\bar{\Pi}_{\dot{\alpha}\dot{\alpha}']}. \tag{D.30}$$

If we compare the above expression with eq. (D.19) in the limit $K \to 1$, we can deduce the above results for the alternative choice $C = 0$:

$$\langle \mathcal{F}_{\beta\dot{\beta}\alpha\dot{\alpha}}A_{\alpha'\dot{\alpha}'}\rangle = \frac{K^2}{\pi^2(1-K^2)^2}F_{[\beta\dot{\beta}}\Pi_{\alpha\alpha'}\bar{\Pi}_{\dot{\alpha}\dot{\alpha}']}. \tag{D.31}$$

Therefore

$$\langle \mathcal{F}_{\alpha\beta}\mathcal{F}_{\dot{\alpha}'\dot{\beta}'}\rangle = -\frac{1}{\pi^2}\left(\frac{K^2}{(K-1)^3(K+1)}\right)\left(F^{\gamma'}{}_{\dot{\alpha}'}\Pi_{\alpha\gamma'}\right)\left(F_{\alpha}{}^{\dot{\gamma}}\bar{\Pi}_{\dot{\gamma}\dot{\beta}'}\right), \tag{D.32}$$

and

$$\langle \mathcal{F}_{\alpha\beta}\mathcal{F}_{\alpha'\beta'}\rangle = -\frac{1}{\pi^2}\left(\frac{K}{K+1}\right)^2\Pi_{\alpha\alpha'}\Pi_{\alpha\beta'}. \tag{D.33}$$

Squaring the above quantities yield

$$\begin{aligned}\langle \mathcal{F}_{\alpha\beta}\mathcal{F}_{\dot{\alpha}'\dot{\beta}'}\rangle^2 &= \frac{12K^4}{\pi^4(1-K)^4},\\ \langle \mathcal{F}_{\alpha\beta}\mathcal{F}_{\alpha'\beta'}\rangle^2 &= \frac{12K^4}{\pi^4(1+K)^4}.\end{aligned} \tag{D.34}$$

[2]These correspond to electric-electric respectively magnetic-magnetic correlators, in contrast to eq. (D.26) describing the electric-magentic correlator.

D.3 Propagation of a higher spin particle

We will derive the massless higher spin propagator by making substantial use of the knowledge gained in the vector case. First, let us introduce the spin-s field. Recall that, in four dimensions, it suffices to consider totally symmetric tensors since all other nonvanishing irreducible representations are equivalent to the former, see, for instance, ref. [78]. Being the order of the indices irrelevant, the spin-s field will thus be denoted by $\Phi_{\alpha(s)\dot{\alpha}(s)}$, where $s \in \mathbb{N}$ in the brackets indicates the number of indices of the same type. Again, noncontracted indices denoted by the same letter are understood to be symmetrized. As we take massless fields, one could right away fix the gauge[3] by imposing the *Fierz-Pauli transversality condition* and the *tracelessness condition*, i.e.,

$$\nabla^{\beta\dot{\beta}}\Phi_{\beta\dot{\beta}\alpha(s-1)\dot{\alpha}(s-1)} = 0, \qquad \Phi^{\beta\dot{\beta}}{}_{\beta\dot{\beta}\alpha(s-2)\dot{\alpha}(s-2)} = 0. \tag{D.35}$$

These are necessary conditions on the fields to ensure that these correspond to irreducible representations of the isometry group. However, later we are going to fix again the gauge via R_ξ-gauge, as done in the vector case.

For $m \leq s$, let us introduce the (partial) *Weyl tensor*

$$\mathcal{C}^{(m)}_{\beta(m)\dot{\beta}(m)\alpha(s)\dot{\alpha}(s)} := (\nabla_{[\beta\dot{\beta}})^m \; \Phi_{\alpha(s)\dot{\alpha}(s)]},$$

where the anti–symmetrization is intended to be pairwise in $\beta_i\dot{\beta}_i$ and $\alpha_i\dot{\alpha}_i$ for each $i \in \{1, \dots, m\}$. The full Weyl tensor

$$C_{\beta(s)\dot{\beta}(s)\alpha(s)\dot{\alpha}(s)} := \mathcal{C}^{(s)}_{\beta(s)\dot{\beta}(s)\alpha(s)\dot{\alpha}(s)} \tag{D.36}$$

corresponds to the field strength tensor $\mathcal{F}_{\beta\dot{\beta}\alpha\dot{\alpha}}$ in the $s = 1$ case. Note that, in the spinor helicity formalism, the Weyl tensor is automatically traceless. For a free massless higher spin particle on shell, $\mathcal{C}_{\beta(s)\dot{\beta}(s)\alpha(s)\dot{\alpha}(s)}$ satisfies the well-known equation

$$\nabla^{\beta\dot{\beta}}\mathcal{C}_{\beta(s)\dot{\beta}(s)\alpha(s)\dot{\alpha}(s)} = 0, \tag{D.37}$$

where we did not consider the gauge-fixing term, which will be introduced later. Owing to the anti–symmetry, one can also split the Weyl tensor in its selfdual $\mathcal{C}_{\alpha(2s)}$ and anti–selfdual $\mathcal{C}_{\dot{\alpha}(2s)}$ parts, by contracting the Weyl tensor with s copies of, respectively, $\epsilon_{\dot{\beta}\dot{\alpha}}$ and $\epsilon_{\beta\alpha}$. In terms of these quantities, eq. (D.37) becomes

$$\nabla^{\gamma}{}_{\dot{\gamma}}\mathcal{C}_{\gamma\alpha(2s-1)} = 0. \tag{D.38}$$

The next step is to generalize the notion of the scalar and vector propagators to a tensor propagator $\Lambda_{\alpha(s)\dot{\alpha}(s)\alpha'(s)\dot{\alpha}'(s)}$. Let us define

$$\langle \Phi_{\alpha(s)\dot{\alpha}(s)}\Phi_{\alpha'(s)\dot{\alpha}'(s)} \rangle := \Lambda_{\alpha(s)\dot{\alpha}(s)\alpha'(s)\dot{\alpha}'(s)}. \tag{D.39}$$

[3]Still, there is a residual gauge freedom.

Now it is time to try an ansatz, which will be shown to be consistent by proving that it solves eq. (D.38):

$$\langle \mathcal{C}_{\alpha(2s)} \Phi_{\beta'(s)\dot\beta'(s)} \rangle = Q_s \left(F_\alpha{}^{\dot\delta} \Pi_{\alpha\beta'} \bar\Pi_{\dot\delta\dot\beta'} \right)^s . \tag{D.40}$$

Here the symmetrization has to be implemented over all s factors. Consequently, one gets

$$\langle \mathcal{C}_{\alpha(2s)} \mathcal{C}_{\dot\alpha'(2s)} \rangle = Q_s^{(s)} \left(F_\alpha{}^{\dot\delta} \bar\Pi_{\dot\delta\dot\alpha'} F^{\beta'}{}_{\dot\beta'} \Pi_{\alpha\beta'} \right)^s , \tag{D.41}$$

where $Q_s^{(s)}$ is defined recursively as $Q_s^{(0)} = Q_s$ and

$$Q_s^{(i+1)} = \left(\partial_K Q_s^{(i)}\right) K + Q_s^{(i)} \left(\frac{sK-1}{K+1} + i \right) . \tag{D.42}$$

Analogously, one finds

$$\langle \mathcal{C}_{\alpha(2s)} \mathcal{C}_{\beta'(2s)} \rangle = \tilde Q_s^{(s)} \left(\Pi_{\alpha\alpha'} \Pi_{\alpha\beta'} \right)^s , \tag{D.43}$$

with $\tilde Q_s^{(0)} = Q_s$ and

$$\tilde Q_s^{(i+1)} = \left(\partial_K \tilde Q_s^{(i)}\right) K(K^2-1) + \tilde Q_s^{(i)} (K+1) \left(K(s-i) + \frac{K-1}{K+1} i + 1 \right) . \tag{D.44}$$

By means of eq. (D.38), the above quantities satisfy

$$\nabla^\gamma{}_{\dot\gamma} \langle \mathcal{C}_{\gamma\alpha(2s-1)} \Phi_{\beta'(s)\dot\beta'(s)} \rangle = 0, \tag{D.45}$$

and therefore Q_s fulfills

$$\begin{aligned} &(K^2-1)KQ_s' + (K^2+2)Q_s + (s-1)(K^2+1)Q_s = 0, \\ &KQ_s' + \frac{K-2}{K+1} Q_s + (s-1)\frac{K-1}{K+1} Q_s = 0. \end{aligned} \tag{D.46}$$

Note that, for $s = 1$, the above equations agree with eqs. (D.17) for $\xi = 0$, as one may expect. In order to solve eqs. (D.46), let us introduce the following gauge:

$$\mathcal{R}_{\beta(s-1)\dot\beta(s-1)\alpha(s-1)\dot\alpha(s-1)} \equiv \nabla^{\gamma\dot\gamma} \mathcal{C}^{(s-1)}_{\gamma\dot\gamma\beta(s-1)\dot\beta(s-1)\alpha(s-1)\dot\alpha(s-1)} = 0. \tag{D.47}$$

It is straightforward to see that we can write

$$\langle \mathcal{R}_{\alpha(2s-2)} \Phi_{\beta'(s)\dot\beta'(s)} \rangle = P_s \left(F_\alpha{}^{\dot\delta} \Pi_{\alpha\beta'} \bar\Pi_{\dot\delta\dot\beta'} \right)^{s-1} F_{\beta'\dot\beta'} , \tag{D.48}$$

since it is the only allowed tensor structure one can construct. In analogy to what done in the spin-1 case, the gauge condition can be implemented as a R_ξ-gauge by adding to eq. (D.45) the term

$$\xi \nabla_{\alpha\dot\gamma} \langle \mathcal{R}_{\alpha(2s-2)} \Phi_{\beta'(s)\dot\beta'(s)} \rangle . \tag{D.49}$$

Thus, eqs. (D.46) become

$$\begin{aligned}
&(K^2-1)KQ_s' + (K^2+2)Q_s + (s-1)(K^2+1)Q_s - \xi K P_s = 0, \\
&KQ_s' + \frac{K-2}{K+1}Q_s + (s-1)\frac{K-1}{K+1}Q_s + \xi\left(KP_s' + \frac{K}{K+1}P_s + (s-1)P_s\right) = 0.
\end{aligned} \tag{D.50}$$

Eliminating ξ leads to the differential equation

$$K^2(K^2-1)Q_s'' + 2K(K^2(s+1)+1)Q_s' + (K^2 s + s - 2)(s+1)Q_s = 0, \tag{D.51}$$

whose solution is given by

$$Q_s = \frac{\Phi(s)}{\pi^2}\left(\frac{K}{1-K^2}\right)^{s+1}, \tag{D.52}$$

where $\Phi(s)$, with $\Phi(1)=1$ being a function dependent solely on s. The coefficients were chosen such that $Q_1 \equiv Q$ with $C=0$, cf. eq. (D.31). Indeed, note that P_s is related to Q_s via

$$KQ_s + \xi P_s = 0, \tag{D.53}$$

as expected from eq. (D.20). The spin-s Weyl-field propagator is given by

$$\langle \mathcal{C}_{\alpha(2s)}\Phi_{\beta'(s)\dot\beta'(s)}\rangle = \frac{\Phi(s)}{\pi^2}\left(\frac{K}{1-K^2}\right)^{s+1}\left(F_\alpha{}^{\dot\delta}\Pi_{\alpha\beta'}\bar\Pi_{\dot\delta\dot\beta'}\right)^s, \tag{D.54}$$

exhibiting the right behavior in the flat space limit:

$$\langle \mathcal{C}_{\alpha(2s)}\Phi_{\beta'(s)\dot\beta'(s)}\rangle \sim (-1)^s\frac{\Phi(s)}{4\sqrt{2^s}\pi^2}\frac{1}{R^{s+2}}\left(n_\alpha{}^{\dot\delta}\epsilon_{\alpha\beta'}\epsilon_{\dot\delta\dot\beta'}\right)^s. \tag{D.55}$$

Using eqs. (D.41,D.43), one can derive the Weyl-Weyl propagators

$$\langle \mathcal{C}_{\alpha(2s)}\mathcal{C}_{\dot\alpha'(2s)}\rangle = -\frac{\Phi(s)}{2\pi^2}\frac{(2s)!}{s!}\frac{K^{s+1}}{(K-1)^{2s+1}(K+1)^s}\left(F_\alpha{}^{\dot\delta}\bar\Pi_{\dot\delta\dot\alpha'}F^{\beta'}{}_{\dot\beta'}\Pi_{\alpha\beta'}\right)^s, \tag{D.56}$$

and

$$\langle \mathcal{C}_{\alpha(2s)}\mathcal{C}_{\beta'(2s)}\rangle = -\frac{\Phi(s)}{2\pi^2}\frac{(2s)!}{s!}\frac{K^{s+1}}{(K+1)^{s+1}}\left(\Pi_{\alpha\alpha'}\Pi_{\alpha\beta'}\right)^s. \tag{D.57}$$

Moreover, one can further compute

$$\begin{aligned}
\langle \mathcal{C}_{\alpha(2s)}\mathcal{C}_{\dot\alpha'(2s)}\rangle^2 &= \frac{12^s\tilde\Phi(s)^2}{\pi^4}\left(\frac{K}{1-K}\right)^{2s+2}, \\
\langle \mathcal{C}_{\alpha(2s)}\mathcal{C}_{\beta'(2s)}\rangle^2 &= \frac{12^s\tilde\Phi(s)^2}{\pi^4}\left(\frac{K}{1+K}\right)^{2s+2},
\end{aligned} \tag{D.58}$$

where $\tilde\Phi(s) = \frac{\Phi(s)}{2}\frac{(2s)!}{s!}$.

E Expansions in the conformal invariants

In this appendix we explain how to evaluate the expansion of the integrals L_0 and L_0' given in eqs. (6.139-6.141) in powers of v and Y, that is[1]

$$L_0 = \sum_{m,n=0}^{\infty} L_0^{(n,m)}(\log v)\ v^n Y^m, \tag{E.1}$$

and analogously for L_0'. Here, the coefficient functions $L_0^{(n,m)}(\log v)$ and $L_0'^{(n,m)}(\log v)$ are to be determined. It is possible to obtain these coefficient functions analytically up to reasonably high order with the help of Mathematica. For L_0, the implementation of the code is quite straightforward and works efficiently to high orders. For L_0', however, this turns out to be a much more difficult task. Nevertheless, we able to provide a code which works up to a sufficient order for our purposes. In this setting, the upper bound of computable orders is set by the t- and u-channel of L_0', as will be explained later.

The integral $\mathbf{L_0}$.

We first discuss the simpler integral L_0 given by

$$L_0(x,y,z) = \int_0^\infty \mathrm{d}s \int_0^1 \mathrm{d}r\ \frac{(sr(1-r))^{\Delta-1}\log(1+s)}{(1+s)^\Delta[sr(1-r)x + ry + (1-r)z]^\Delta}. \tag{E.2}$$

The s-channel. The s-channel is given by $L_0(v, 1-Y, 1)$. Let us denote the associated integrand by $l_s(s,r;v,Y)$. Then, for each of the cases $\Delta = 1, 2$, the steps to follow are:

1. Expansion in Y: $\quad l_s(s,r;v,Y) = \sum_{m=0}^{\infty} l_s^{(m)}(s,r;v)\ Y^m$
2. Integration over s: $\quad l_s^{(m)}(r;v) = \int_0^\infty \mathrm{d}s\ l_s^{(m)}(s,r;v)$
3. Expansion in v: $\quad l_s^{(m)}(r;v) = \sum_{n=0}^{\infty} l_s^{(n,m)}(r;\log v)\ v^n$
4. Integration over r: $\quad L_s^{(n,m)}(\log v) = \int_0^1 \mathrm{d}r\ l_s^{(n,m)}(r;\log v)$

The t- and u-channels. For both the t- and u-channels of L_0, although the integrals being different, the employed procedure is the same. The integrals are respectively given by $L_0(1-Y,1,v)$ and $L_0(1,v,1-Y)$. Here, the steps to follow are similar as above, but with the order of integration interchanged:

1. Expansion in Y: $\quad l_{t,u}(s,r;v,Y) = \sum_{m=0}^{\infty} l_{t,u}^{(m)}(s,r;v)\ Y^m$

[1] To do so one has to substitute the variables x, y, z with the conformal invariants $v, 1-Y$ and 1 for each channel in eqs. (6.139-6.141). In order to simplify the notation we will simply take the subscript in L_0 and L_0' as a placeholder for the different channels s, t, u.

2. Integration over r: $l_{t,u}^{(m)}(s;v) = \int_0^1 \mathrm{d}r\; l_{t,u}^{(m)}(s,r;v)$

3. Expansion in v: $l_{t,u}^{(m)}(s;v) = \sum_{n=0}^{\infty} l_{t,u}^{(n,m)}(s;\log v)\; v^n$

4. Integration over s: $L_{t,u}^{(n,m)}(\log v) = \int_0^{\infty} \mathrm{d}s\; l_{t,u}^{(n,m)}(s;\log v)$

The integral $\mathbf{L}_0'$.

The case L_0' brings along various difficulties, most of them associated to the s-channel. The integral to solve is given by

$$L_0'(x,y,z) = \int_0^1 \mathrm{d}t \int_0^{\infty} \mathrm{d}s \int_0^1 \mathrm{d}r\; \frac{\operatorname{arctanh} t}{t\sqrt{(1+s)(1+t^2 s)}[sr(1-r)x - ry + (1-r)z]}, \tag{E.3}$$

or, equivalently, by

$$L_0'(x,y,z) = \int_1^{\infty} \mathrm{d}\lambda \int_0^{\infty} \mathrm{d}s \int_0^1 \mathrm{d}r\; \frac{\log(1+\lambda s)}{4\lambda\sqrt{(1+s)(1+\lambda s)}[sr(1-r)x - ry + (1-r)z]}. \tag{E.4}$$

The s-channel. The code associated to the s-channel integral $L_0'(v, 1-Y, 1)$ is not as simple as the ones above, but leads to a comparable performance when implemented in Mathematica. The basic idea is the following: The s-integral can be split in two parts, an integral from 0 to α and an integral from α to ∞, with $\alpha \gg 1$. Then the integrand to the former integral can be expanded immediately in v and subsequently integrated over, since for $v = 0$ the integral (E.4) only diverges at $s \to \infty$. The integrand of the latter integral is instead expanded for large s (bear in mind that the product $s \times v$ does not have a defined limit), and then integrated over. Eventually, from the sum of the two results α drops out in the limit $\alpha \to \infty$, yielding the final result.

Let us be more precise and redefine the integrand in eq. (E.4) as

$$l_s'(s,r,\lambda;v,Y) = a(s,\lambda)b(s,r;v,Y), \tag{E.5}$$

where

$$a(s,\lambda) = \frac{\log(1+\lambda s)}{4\lambda\sqrt{(1+s)(1+\lambda s)}}, \qquad b(s,r;v,Y) = \frac{1}{sr(1-r)v - rY + 1}. \tag{E.6}$$

The procedure is as follows:

1. Expansion in Y: $b(s,r;v,Y) = \sum_{m=0}^{\infty} b^{(m)}(s,r;v)\; Y^m$

2. Integration region $s \in [\alpha, \infty)$:
   a) Expansion in s: $a(s,\lambda) = \sum_{l=0}^{\infty} a^{(l)}(\log s, \lambda)\; s^{-(l+1)}$
   b) Integration over s: $A^{(m,l)}(\lambda, r, \alpha; v) = \int_{\alpha}^{\infty} \mathrm{d}s\; a^{(l)}(\log s, \lambda) b^{(m)}(s,r;v)\; s^{-(l+1)}$

c) Integration over λ: $A^{(m,l)}(r,\alpha;v) = \int_1^\infty \mathrm{d}\lambda\ A^{(m,l)}(\lambda,r,\alpha;v)$

d) Expansion in v: $A^{(m,l)}(r,\alpha;v) = \sum_{n=0}^\infty A^{(n,m,l)}(r,\alpha;\log v)\ v^n$

e) Integration in r: $A^{(n,m,l)}(\alpha;\log v) = \int_0^1 \mathrm{d}r\ A^{(n,m,l)}(r,\alpha;\log v)$

f) Summation over l: $A^{(n,m)}(\alpha;\log v) = \sum_{l=0}^n A^{(n,m,l)}(\alpha;\log v)$

3. Integration region $s \in [0,\alpha]$ (after the variable substitution $\lambda = t/s$):

   a) Expansion in v: $b^{(m)}(s,r;v) = \sum_{n=0}^\infty b^{(n,m)}(s,r)\ v^n$

   b) Integration in r: $B^{(n,m)}(s,t) = \int_0^1 \mathrm{d}r\ a(s,t) b^{(n,m)}(s,r)$

   c) Integration region $(s,t) \in [0,\alpha] \times [\alpha,\infty)$:

      i. Integration in t: $B_1^{(n,m)}(s,\alpha) = \int_\alpha^\infty \mathrm{d}t\ B^{(n,m)}(s,t)$

      ii. Integration in s: $B_1^{(n,m)}(\alpha) = \int_0^\alpha \mathrm{d}s\ B_1^{(n,m)}(s,\alpha)$

   d) Integration region $(s,t) \in [0,t] \times [0,\alpha]$:

      i. Integration in s: $B_2^{(n,m)}(t,\alpha) = \int_0^t \mathrm{d}s\ B^{(n,m)}(s,t)$

      ii. Integration in t: $B_2^{(n,m)}(\alpha) = \int_0^\alpha \mathrm{d}s\ B_2^{(n,m)}(s,\alpha)$

4. Summation over results: $C^{(n,m)}(\alpha;\log v) = A^{(n,m)}(\alpha;\log v) + B_1^{(n,m)}(\alpha) + B_2^{(n,m)}(\alpha)$

5. Elimination of α: $L_s'^{(n,m)}(\log v) = \lim_{\alpha\to\infty} C^{(n,m)}(\alpha;\log v)$

In step (2f), the sum clearly runs over all natural numbers. However, for $l > n$, the terms $A^{(n,m,l)}(\alpha;\log v)$ vanish in the limit $\alpha \to \infty$ and thus can be dropped. Furthermore, note that after the variable substition $\lambda = t/s$ the integration region is $(s,t) \in [0,\alpha] \times [s,\infty)$, which in turn can be divided into two smaller regions. This was done in step (3c) and step (3d).

The t- and u-channels. The implemented code for the t- and u-channels of L_0' is again straightforward. The integrals are respectively given by $L_0'(1-Y,1,v)$ and $L_0'(1,v,1-Y)$, but in this case we start from eq. (E.3).

1. Expansion in Y: $l_{t,u}'(s,r,t;v,Y) = \sum_{m=0}^\infty l_{t,u}'^{(m)}(s,r,t;v)\ Y^m$

2. Integration over r: $l_{t,u}'^{(m)}(s,t;v) = \int_0^1 \mathrm{d}r\ l_{t,u}'^{(m)}(s,r,t;v)$

3. Expansion in v: $l_{t,u}'^{(m)}(s,t;v) = \sum_{n=0}^\infty l_{t,u}'^{(n,m)}(s,t;\log v)\ v^n$

4. Integration over s: $l_{t,u}'^{(n,m)}(t;\log v) = \int_0^\infty \mathrm{d}s\ l_{t,u}'^{(n,m)}(s,t;\log v)$

5. Integration over t: $L'^{(n,m)}_{t,u}(\log v) = \int_0^1 \mathrm{d}t \; l'^{(n,m)}_{t,u}(t;\log v)$

Unfortunately, the Mathematica code does not run efficiently for these channels. However, by constraining oneself in the computation of solely the coefficients linear in $\log v$ in $L'^{(n,m)}_{t,u}(\log v)$, the code performance improves drastically. Explicitly, after step (3), one proceeds as follows:

4. Expansion in $\log v$: $l'^{(n,m)}_{t,u}(s,t;\log v) = l'^{(n,m,0)}_{t,u}(s,t) + \log v \; l'^{(n,m,1)}_{t,u}(s,t)$
5. Integration over s: $l'^{(n,m,1)}_{t,u}(t) = \int_0^\infty \mathrm{d}s \; l'^{(n,m,1)}_{t,u}(s,t)$
6. Integration over t: $L'^{(n,m,1)}_{t,u} = \int_0^1 \mathrm{d}t \; l'^{(n,m,1)}_{t,u}(t)$

where $L'^{(n,m)}_{t,u}(\log v) = L'^{(n,m,0)}_{t,u} + \log v \; L'^{(n,m,1)}_{t,u}$. The lack of knowledge of $L'^{(n,m,0)}_{t,u}$ is not stringently restrictive for us, since our main interest lies in the anomalous dimensions. Indeed, this only prevents us from deriving the OPE coefficients of higher weight primaries at second order in the coupling constant.

F OPE coefficients

Here we collect some of the OPE coefficients and anomalous dimensions. The (squared) OPE coefficients for the disconnected contribution to the four-point function of a generalized free field of weight ν are[1]

$$A_{n,l}=\frac{\pi\Gamma\left(\frac{d}{2}+l\right)2^{d-4\nu-l-4n+3}\Gamma\left(-\frac{d}{2}+n+\nu+1\right)\Gamma(-d+n+2\nu+1)\Gamma(l+n+\nu)\Gamma(l+2n+2\nu-1)\Gamma\left(-\frac{d}{2}+l+n+2\nu\right)}{\Gamma(\nu)^2\Gamma(l+1)\Gamma(n+1)\Gamma\left(-\frac{d}{2}+\nu+1\right)^2\Gamma\left(\frac{d}{2}+l+n\right)\Gamma\left(-\frac{d}{2}+n+\nu+\frac{1}{2}\right)\Gamma\left(l+n+\nu-\frac{1}{2}\right)\Gamma\left(-\frac{d}{2}+l+2(n+\nu)\right)}, \tag{F.1}$$

where we recall that the conformal blocks are defined via the recursion relations given in ref. [124]. They have the general structure

$$G_{\nu,l}=v^{\frac{\nu-l}{2}}\left(2^{-l}Y^l+Y^{l+1}+...+v(Y^{l-2}+...)+...\right),$$

where only the first coefficient, 2^{-l}, is displayed here, while the others are dropped. This fixes the normalization of the conformal blocks. The recursion begins with the scalar conformal block:

$$G_{\nu,0}=v^{\frac{\nu}{2}}\tilde{G}(\tfrac{\nu}{2},\tfrac{\nu}{2},\nu), \tag{F.2}$$

$$\tilde{G}(b,f,S)=\sum_{n,m}\frac{Y^m v^n}{m!n!}\frac{(b)_{m+n}(S-b)_n(f)_{m+n}(S-f)_n}{\left(S-\frac{3}{2}+1\right)_n(S)_{m+2n}}. \tag{F.3}$$

$\boldsymbol{\Delta}$ = 2.

A few OPE coefficients at order λ_R are given in table 1. Note that, since most of the anomalous dimensions at first order vanish, the OPE coefficients for $l>0$ are determined by matching the OPE expansion at order λ_R^2. Only the first column, $l=0$, comes from the OPE at order λ_R.

At order λ_R^2 the anomalous dimensions of the operators from the first Regge trajectory have a very simple analytic form, see section 6.5. This seems not to be the case for the subleading trajectories and we simply list some of them in table 2. Likewise, few OPE coefficients at order λ_R^2 can be found (note that due to vanishing of the first order anomalous dimensions of the spinning operators most of the second order OPE coefficients will be only fixed at order λ_R^3, and we do not have access to) and these are given in table 3.

$\boldsymbol{\Delta}$ = 1.

Also for $\Delta=1$ the anomalous dimensions of the operators with nonzero spin vanish at order λ_R, and thus only corrections to $A_{n,l=0}$ can be determined at this order. These can be found in table 4. For $l>0$, the lowest order corrections to the OPE coefficients

[1] The coefficients can be found in ref. [36], but we adjusted them to our normalization of conformal blocks.

are determined by matching the OPE expansion at order λ_R^2. However, owing to the complexity of L_4' at order λ_R^2, our results are more limited, see table 5.

Anomalous dimensions of the operators at $n > 0$ trajectories are purely rational (in terms of γ^2) and are given in table 6. Due to $\gamma_{n,l>0}^{(1)} = 0$, again only part of the OPE coefficients can be determined at order λ_R^2, and the first few of them are given in table 7.

$\Delta=2$	$l=0$	$l=2$	$l=4$	$l=6$
$n=0$	$\frac{1}{3}$	$\frac{22}{25}$	$\frac{1066}{1323}$	$\frac{1327184}{2760615}$
$n=1$	$-\frac{478}{735}$	$-\frac{20834676}{22204105}$	$-\frac{4866352}{7966035}$	$-\frac{13064388928}{43517414655}$
$n=2$	$-\frac{11507}{50820}$	$-\frac{14977826}{60144903}$	$-\frac{15701186862675}{107693096012359}$	$-\frac{3223599567312}{47662730167195}$

Table 1: Some of the OPE coefficients $A^{(1)}_{n,l}$

$\Delta=2$	$l=0$	$l=2$	$l=4$	$l=6$	$l=8$	$l=10$
$n=1$	$\frac{46}{15}$	$-\frac{107}{1260}$	$-\frac{19}{1260}$	$-\frac{131}{27720}$	$-\frac{301}{154440}$	$-\frac{19}{20020}$
$n=2$	$\frac{113}{28}$	$-\frac{269}{2520}$	$-\frac{6707}{311850}$	$-\frac{1973}{270270}$	$-\frac{3439}{1081080}$	
$n=3$	$\frac{535}{112}$	$-\frac{6697}{55440}$	$-\frac{19037}{720720}$	$-\frac{143581}{15135120}$		

Table 2: Some anomalous dimensions at order λ_R^2

$\Delta=2$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$l=0$	$\frac{20}{9}$	$\frac{111392}{77175}$	$\frac{27588119}{70436520}$	$\frac{6664117739}{96718146525}$	$\frac{54416659121622349}{5688844669692344160}$

Table 3: Some of the OPE coefficients $A^{(2)}_{n,l}$

$\Delta=1$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$l=0$	-2	$-\frac{1}{3}$	$-\frac{617}{36750}$	$-\frac{20087}{22411620}$	$-\frac{34519}{695674980}$

Table 4: Some of the OPE coefficients $A^{(1)}_{n,0}$

$\Delta=1$	$l=2$	$l=4$
$n=0$	$\frac{144\zeta(3)-31}{18(\pi^2-5)}-\frac{43}{15}$	$\frac{2(604800\zeta(3)-380209)}{3675(120\pi^2-863)}-\frac{10714}{11025}$

Table 5: Some of the OPE coefficients $A^{(1)}_{n,l>0}$

$\Delta=1$	$l=0$	$l=2$	$l=4$	$l=6$	$l=8$
$n=1$	$\frac{5}{2}$	$-\frac{23}{60}$	$-\frac{59}{420}$	$-\frac{37}{504}$	$-\frac{179}{3960}$
$n=2$	$\frac{29}{6}$	$-\frac{373}{1260}$	$-\frac{71}{630}$	$-\frac{1693}{27720}$	
$n=3$	$\frac{367}{60}$	$-\frac{641}{2520}$	$-\frac{15074}{155925}$		

Table 6: Some anomalous dimensions at order λ_R^2

$\Delta=1$	$n=0$	$n=1$	$n=2$
$l=0$	$-4\zeta(3)+\pi^2/2+8$	$\frac{37}{27}$	$\frac{337219}{7717500}$

Table 7: Some of the OPE coefficients $A^{(2)}_{n,l}$

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