Superconformal matter coupling in three dimensions and topologically massive gravity

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München 2019

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Dissertation
an der Fakultät für Physik
der Ludwig-Maximilians-Universität München

vorgelegt von Frederik Lauf
aus Bad Sobernheim

München, den 19. August 2019

Erstgutachter: Prof. Dr. Ivo Sachs
Zweitgutachter: Prof. Dr. Peter Mayr
Tag der mündlichen Prüfung: 11. Oktober 2019

Zusammenfassung

Die Kopplung supersymmetrischer skalarer Materie-Multipletts an superkonforme Eichtheorie und Gravitation in drei Raumzeit-Dimensionen wird im Formalismus des $\mathcal{N}$-erweiterten Superraumes beschrieben. Die Formulierungen supersymmetrischer Eichtheorie und konformer Supergravitation in diesem Superraum werden vorgestellt und ein Formalismus für die Analyse von Superfeldkomponenten wird entwickelt. Ein Superfeld-Wirkungsprinzip, welches zu einer allgemeinen Klasse minimaler Multipletts auf der Massenschale führt, wird eingeführt. Darauf folgend wird ein Skalar-Multiplett, welches von einem zwangsbeschränkten Superfeld beschrieben wird und nur aus unter spin($\mathcal{N}$) transformierenden Lorentz-Skalaren und Spinoren besteht, als speziell zwangsbeschränkter Fall des minimalen unbeschränkten Multipletts identifiziert. Seine Superfeldwirkung wird aus dem Wirkungsprinzip für das unbeschränkte skalare Superfeld hergeleitet. Die Analyse wird für eine supersymmetrisch eich- und gravitationskovariante Beschreibung von Superfeldkomponenten verallgemeinert und eine Kopplungsbedingung sowie die Wirkung, welche diese Kopplung beschreibt, werden gefunden. Auf dieser Basis werden alle Eichgruppen für das Skalar-Multiplett, welche von $\mathcal{N} \leq 8$ erweiterter superkonformer Symmetrie erlaubt sind, im flachen Superraum bestimmt und ebenso im gekrümmten Superraum, in welchem das Skalar-Multiplett zusätzlich gravitationell gekoppelt ist. Dies führt zur Konstruktion sämtlicher gravitationell gekoppelter Chern-Simons-Materie-Theorien. Unter der Benutzung des gravitationell gekoppelten Skalar-Multipletts als konformen Kompensator werden die resultierenden kosmologischen topologisch massiven Gravitationen nach den korrespondierenden Parametern $\mu\ell$, also dem Produkt aus der Kopplungskonstante der konformen Gravitation und dem anti-de-Sitter-Radius, klassifiziert. Die Modifikationen von $\mu\ell$ bei der Präsenz erlaubter Eichkopplungen für den skalaren Kompensator werden bestimmt.

Abstract

The coupling of supersymmetric scalar-matter multiplets to superconformal gauge theory and gravity in three-dimensional space-time is described in the formalism of $\mathcal{N}$-extended superspace. The formulations of super-gauge theory and conformal supergravity in this superspace are reviewed and a formalism for analysing superfield components is developed. A superfield action principle giving rise to a general class of minimal on-shell multiplets is introduced. Subsequently, a scalar multiplet described by a constrained superfield and consisting only of Lorentz scalars and spinors transforming under spin($\mathcal{N}$) is recognised as a specially constrained case of the minimal unconstrained multiplet. Its superfield action is deduced from the action principle for the unconstrained scalar superfield. The analysis is generalised to a super-gauge- and supergravity-covariant description of superfield components and a coupling condition for the scalar multiplet as well as the action describing this coupling are obtained. Based on this, all gauge groups for the scalar multiplet consistent with $\mathcal{N} \leq 8$ extended superconformal symmetry are determined in flat superspace, as well as in curved superspace where the scalar multiplet is also coupled to conformal supergravity. This results in the construction of all superconformal Chern-Simons-matter theories coupled to gravity. Using the gravitationally coupled scalar multiplet as a conformal compensator, the resulting cosmological topologically massive supergravities are classified with regard to the corresponding parameters $\mu\ell$, i.e. the product of the conformal-gravity coupling and the anti-de Sitter radius. The modifications of $\mu\ell$ in presence of possible gauge couplings for the scalar compensator are determined.

Contents

1. Introduction

Fields have been recognised as indispensable entities for the formulation of physical theories since the development of electromagnetism by Maxwell. Today, field theories are still the language for the most successful models of fundamental phenomena. Incorporating the principles of special relativity and quantum mechanics, they describe the particles and interactions in the Standard Model of particle physics. A classical field theory for gravity, and thereupon models for cosmology, are implied by the principle of general relativity.

A particular power for the formulation of field theories is provided by the principle of symmetry. A symmetry is given, if the most basic description of a theory is invariant under a specific transformation of its constituting objects.

Above all, the symmetry under space-time transformations is obviously demanded by the principle of relativity. The symmetry transformations associated to space-time can act on two kinds of fields which are called bosonic and fermionic. They are the representations of the space-time symmetry group with integer and half-integer spin, respectively. Physically, they are conceptually distinguished. The latter describe matter fields like leptons and quarks, while the former correspond to force particles like photons and gluons as well as scalar fields like the Higgs boson.

Furthermore, especially in the Standard Model of particle physics, internal symmetry groups acting on the degrees of freedom formed by the fields themselves play an important role. For a manifestly invariant formulation, the degrees of freedom are arranged in multiplets on which a matrix representation of the respective symmetry group can act. These transformations become vital if they are allowed to be space-time-dependent. Such local symmetries, also called gauge symmetries, lead to successful descriptions of force particles interacting with the matter particles and mediating interactions between them.

There is no better outside reason for the existence of such internal symmetries, than that they are, together with space-time symmetry, eligible symmetries of a reasonable scattering matrix, i.e. the matrix describing scattering processes between particles. In this regard, there is only one further allowed kind of symmetry. It is called supersymmetry and concerns the symmetry of space-time itself.

Supersymmetry generalises the symmetry associated to space-time in a way, that the fermionic and bosonic representations transform into each other. Supersymmetric theories therefore always describe systems where both kinds of particles appear together and can be seen as members or aspects, sometimes called superpartners, of one supermultiplet. These supermultiplets can manifestly be described using superfields instead of fields, which are functions of a superspace instead of space-time, resembling the fact that supersymmetry is a generalisation of space-time symmetry. The bosonic and fermionic fields are encapsulated in a superfield as the coefficients of an expansion in even and odd powers of the fermionic superspace coordinates. Together with supersymmetry comes another symmetry, similar to an internal one, naturally acting on the fermionic representations. These degrees of freedom can be regarded as corresponding to a number $\mathcal{N}$ of usual supersymmetries, implying an extension of the structure of possible supermultiplets. These symmetries are called ($\mathcal{N}$-)extended supersymmetries.

Applying the paradigm of supersymmetry leads to various consequences. As for particle physics, it would for example predict corresponding superpartners for each particle of the Standard Model. This has provoked a phenomenological interest in supersymmetry for decades, with no concluding result so far.

Regarding gravity, the principle of general relativity adapted for superspace leads to supersymmetric gravity, with the metric field or graviton and a so-called gravitino as superpartners. An interesting feature of supersymmetry is the automatic implication of supergravity, if the transformations are local, similar to a gauge theory. After all, two (fermionic) local supersymmetry transformations correspond to a (bosonic) local space-time translation, which is the symmetry transformation corresponding to general relativity.

In this thesis, the focus lies on three-dimensional supersymmetric field theories with highly extended supersymmetry. Those can be motivated from the point of view of 11-dimensional space-time: The supermultiplet of 11-dimensional $\mathcal{N} = 1$ supersymmetry contains the graviton as the field with highest spin. Since massless fields with spins higher than two are considered unphysical, this theory qualifies as the highest dimensional and unique supersymmetric field theory and supergravity [1]. The equations of motion of 11-dimensional (11d) supergravity admit a solution describing spatially two-dimensional membranes which are called M2-branes [2]. Their three-dimensional world-volume theory [3], i.e. the theory describing their internal dynamics, naturally involves the coupling to 11d supergravity. The M2-brane world-volume preserves one half of the 32 supercharges needed to parametrise the $\mathcal{N} = 1$ supersymmetry transformations in 11 dimensions, thus gaining the higher amount of $\mathcal{N} = 8$ supersymmetry [4], which is a common effect of this dimensional reduction.

Furthermore, in a suitable scaling limit, the M2-brane solution describes the space $\mathrm{AdS}_4 \times S_7$ [5], i.e. the four-dimensional space of constant negative curvature with 7-spheres at each point. In this case, it additionally displays conformal symmetry [6], which corresponds to invariance under local rescaling of the metric. According to the so-called AdS/CFT correspondence [6], this suggests the existence of a three-dimensional superconformal gauge theory with a number of N internal degrees of freedom, which would be interpreted as the world-volume theory for a stack of N coincident M2-branes.

Another point of view on M2-branes comes from their interplay with superstring theory. The five known formulations of superstring theory are connected by a web of dualities, which means they are equivalent ways for describing the same phenomena, but in different physical regimes. Their low-energy limits are corresponding versions of ten-dimensional supergravity. One of these, the type IIA supergravity, can directly be obtained from the unique 11d supergravity, by compactifying one of the 11 dimensions on a circle whose radius is proportional to the string-coupling constant. Thus, via the duality web, all ten-dimensional supergravities descend from the 11-dimensional one. In turn, 11d supergravity is expected to be the low-energy limit of the so-called M-theory, whose full high-energy formulation is unknown. The superstring theories (as well as 11d supergravity) are thus regarded as certain limits of M-theory and the M2-branes are considered as fundamental for M-theory as strings are for string theory.

The type IIA supergravity also contains membranes which are called D2-branes and describe hypersurfaces for string endpoints. Due to this relation to string dynamics, the low-energy world-volume theory for a stack of N coincident D2-branes is known to be a non-conformal three-dimensional gauge theory for N internal degrees of freedom. The gauge coupling is proportional to the string coupling and thus to the size of the compactified 11th dimension. Therefore, in the limit of infinite coupling strength, the size of the 11th dimension increases again, until the theory corresponds to a stack of M2-branes in 11 dimensions. In this strong-coupling regime, the theory should obtain the conformal symmetry implied by the AdS/CFT correspondence

The three-dimensional field theory meeting the above requirements for the low-energy world-volume theory for a stack of M2-branes is $\mathcal{N} = 8$ superconformal Chern-Simons gauge theory coupled to eight scalar and spinor fields, which are usually called matter fields [7]. Since the Chern-Simons theory is non-dynamical, the scalar fields represent the degrees of freedom of M2-branes in the eight transverse directions. The theory is known as the BLG model [8, 9] and is invariant under the gauge group $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Due to this unique gauge group, it cannot be interpreted as the world-volume theory of a stack of $N \geq 2$ coincident M2-branes [10, 11], as implied by the D2-brane theory and desired by the AdS/CFT correspondence. This is however possible for a theory with only $\mathcal{N} = 6$ supersymmetries and the gauge group $\mathrm{SU}(N) \times \mathrm{SU}(N)$, known as the ABJM model [12]. The reduction of the amount of supersymmetry is achieved by geometric restrictions on the transverse coordinates.

Apart from this motivation for the ABJM model, superconformal Chern-Simons-matter theories with lower amounts $\mathcal{N} \leq 8$ of supersymmetry and different gauge groups are still a matter of interest. Their most crucial feature is that possible gauge groups are restricted by consistency with supersymmetry for $\mathcal{N} \geq 4$. Notable achievements in the classification of Chern-Simons-matter theories with regard to supersymmetry and gauge groups have been made in [13] for $\mathcal{N} = 4$ in the context of four-dimensional supersymmetric gauge theories, in [14, 15] for $\mathcal{N} = 5, 6$ in the context of the geometry of M2-branes, and in demand for formal classifications, in [16] for $\mathcal{N} = 6$ and [17] for all $\mathcal{N}$. A superspace point of view was adopted in [18] for $\mathcal{N} = 4$, [19] for $\mathcal{N} = 5, 6$, [20] for $\mathcal{N} = 6$, [21] for $\mathcal{N} = 6, 8$, and in [22] for $\mathcal{N} = 8$.

As a part of the present thesis, this quest is repeated using the formalism of $\mathcal{N}$-extended superspace. In this approach, the supersymmetric matter is described by a scalar superfield. The advantage of this approach is not only the manifestly supersymmetric formulation, but also the as far as possible unified manner in which the cases of $\mathcal{N}$ are analysed. The scalar superfield is subject to a certain supersymmetric constraint in order to describe a familiar scalar-matter supermultiplet. However, in the presence of a gauge coupling, this constraint is in general inconsistent. Rather, it is valid only under an algebraic condition involving a superfield representing the field-strength multiplet, containing the usual gauge field strength and its superpartners. In general, these field strengths have to be expressed by the scalar-matter current which couples to the gauge fields, according to their Chern-Simons equation of motion. The specific algebraic properties of these matter currents decide on the solvability of the condition for the scalar superfield and thus over the admissibility of the gauge group in question.

The classification of coupled supersymmetric matter gives rise to another application, which is the realisation of certain supergravity theories. Non-supersymmetric three-dimensional gravity, likewise a field theory emerging in curved space, has some distinguished features compared to its analogue in four dimensions. Namely, both its versions of Einstein-Hilbert gravity and conformal gravity each for themselves have no dynamics. The latter is solved by conformally flat space-time and the former completely fixes the geometry of space-time by its equations of motion to be flat or have constant curvature, leaving no room for a locally propagating graviton. Nevertheless, in the presence of a negative cosmological constant, a black hole solution [23] with anti-de Sitter space as the asymptotic limit and in consequence [24] a corresponding two-dimensional conformal field theory in this region with its two propagating modes are supported.

Thanks to the existence of these two models for gravity, a third model can be constructed by adding them together. The result is a dynamical theory known as topologically massive gravity [25], since it gives rise to a new propagating degree of freedom with a mass determined by the coupling of the conformal supergravity. It notoriously requires some finesse regarding the positivity of the occurring energies. As for the massive graviton alone and in absence of the cosmological constant, the sign of the Einstein-Hilbert action must be inverted to ensure a positive energy. This carries with it the downside of negative black-hole masses upon including the negative cosmological constant. A sensible model with no negative energies is given by the so-called chiral gravity [26]. It is characterised by the usual sign of the Einstein-Hilbert action and the value $\mu\ell = 1$, where μ is the conformal-gravity coupling and ℓ is the anti-de Sitter radius related to the negative cosmological constant. Under this specification, the black holes have positive mass while the massive graviton and one of the boundary gravitons disappear, leaving only one mode with positive energy in the boundary conformal field theory [27].

$\mathcal{N}$-extended supergravity in three dimensions can be formulated in $\mathcal{N}$-extended curved superspace. This approach automatically leads to a description of conformal supergravity. In view of the Einstein-Hilbert term of topologically massive gravity, realising non-conformal supergravities requires the coupling to certain fields which are called conformal compensator. These have to display specific properties in order to ensure conformal invariance. In a second step, the conformal symmetry can be broken by fixing an expectation value of the compensator, thus preventing it from preserving conformal invariance by compensating for the transformations of the other fields in the theory.

In order to serve as a conformal compensator, the scalar-matter multiplet coupled to superconformal Chern-Simons theories can further be coupled to conformal supergravity [28, 29, 21]. This carries with it the effect of modifying the spectrum of the admissible gauge groups compared to flat superspace [30, 21]. The above superfield formalism for the coupled scalar-matter multiplet remains applicable, but in addition to the field-strength multiplet, there appears the conformal-supergravity multiplet described by a superfield called super-Cotton tensor. It contains the Cotton tensor, which is an invariant tensor of conformal gravity constructed from the curvature tensor, but instead of curvature, it measures the conformal flatness of a space-time. The impact of curved superspace on the allowed gauge groups is quite particular regarding the number of supersymmetries. In the cases $\mathcal{N} = 4$ and $\mathcal{N} = 5$ it has no effect. For $\mathcal{N} = 6$, it relaxes the restrictions on certain U(1) factors of the gauge groups present in flat superspace [30]. This phenomenon is due to the fact, that the $\mathcal{N} = 6$ super-Cotton tensor can be regarded as being dual to a U(1) field strength. For $\mathcal{N} = 7$ and $\mathcal{N} = 8$ it gives rise to the possibility of matter fields in the fundamental representations of gauge groups which are unrelated to those in flat superspace.

Using the gauge- and supergravity-coupled scalar-matter multiplet as a conformal compensator naturally realises supersymmetric versions of topologically massive gravity. A distinguished feature of the resulting theories is that the value of $\mu\ell$ is always fixed by the superconformal geometry for $\mathcal{N} \geq 4$ [31, 32]. The underlying mechanism is essentially the following. Since the super-Cotton tensor contains the field strength of the gauged SO($\mathcal{N}$) structure group of extended superspace, its value is determined by the Chern-Simons coupling μ via the Chern-Simons equation of motion of conformal supergravity. In the geometry defining anti-de Sitter superspace, the cosmological constant is generated by the value of a torsion superfield transforming inhomogeneously under super-Weyl transformations, which are transformations related to conformal invariance. The presence of a scalar compensator superfield in this background requires a gauge for this superfield relating it to the super-Cotton tensor. Upon giving the scalar compensator its expectation value, this fixes the relation between μ and ℓ.

As is the nature of a conformal compensator, the opposite sign of the Einstein-Hilbert action is generated [29]. It suggests that the nature of negative mass BTZ black holes, being a consequence of this sign, should be addressed by different interpretations. This issue is not a topic of this thesis, but will be addressed again in the conclusion.

The question is then which supergravities may imply the value $\mu\ell = 1$ preferred by the model of chiral gravity. They are the ones with $\mathcal{N} = 4$ [31] and $\mathcal{N} = 6$ [29]. The list of values $\mu\ell$ for all amounts of supersymmetry $4 \leq \mathcal{N} \leq 8$ obtained from the analysis with a compensator not coupled to a gauge group is [32]

$\mathcal{N} =$	4	5	6	7	8
$(\mu\ell)^{-1} =$	1	3/5	1	2	3.

Modifications of these values may occur if the compensator is also coupled to its allowed gauge groups [21, 32]. In this case, a number of its gauge components can be chosen to generate the Einstein-Hilbert coupling constant. The most diverse, yet specific, effects appear for $\mathcal{N} = 6$ with gauge group SU(N) in shape of the formula [32]

$$(\mu\ell)^{-1} = \frac{2}{p} - 1,$$

where p is the number of non-vanishing components, and for $\mathcal{N} = 8$ with the gauge group SO(N) [21] and the formula [33]

$$(\mu\ell)^{-1} = \frac{4}{p} - 1.$$

Both cases additionally allow $\mu\ell = \infty$ corresponding to the solution of Minkowski space. For $\mathcal{N} = 8$ also the value $\mu\ell = 1$ can be generated by choosing two compensator components.

Outline. The present thesis is organised as follows. In Chapter 2, the formalism of three-dimensional extended superspace is introduced to describe supersymmetric scalar matter fields, super-gauge theory and conformal supergravity, which will be coupled together in the subsequent Chapter 3. Regarding supersymmetric scalar matter, the component expansion of scalar superfields is developed and discussed, with the goal of arriving at the case of an on-shell multiplet described by a constrained superfield, which consists of a scalar and a spinor field transforming under the fundamental representation of spin($\mathcal{N}$). An off-shell superfield action principle leading to a general class of minimal on-shell multiplets is proposed, from which the spin($\mathcal{N}$) scalar multiplet and its superfield and component actions follow as a special, constrained case. Subsequently, super-gauge theory and conformal supergravity in the formulations of conventional curved SO($\mathcal{N}$)

superspace and conformal superspace are reviewed. A special focus lies on anti-de Sitter superspace as a supersymmetric background solution of supergravity, which will be relevant for topologically massive supergravity discussed in Chapter 4.

In Chapter 3, the component analysis of scalar superfields is generalised in order to describe gauge-covariant superfield components. This leads to the derivation of a coupling condition for the spin($\mathcal{N}$) scalar multiplet, which is implied by consistency with the covariant constraint on the scalar superfield describing this multiplet or, in other words, by consistency of the supersymmetry transformations of its covariant components. This condition is then fully analysed and solved for $4 \leq \mathcal{N} \leq 8$, resulting in the complete spectrum of allowed gauge groups in flat as well as in curved superspace. To this end, the scalar-matter currents which couple to gauge fields are determined and recast as scalar-superfield currents, corresponding to equations of motion for the superfields describing the gauge and supergravity multiplets.

In Chapter 4, results are combined in order to use the coupled scalar multiplet as a conformal-compensator multiplet for topologically massive supergravities. Requiring consistency with the background of anti-de Sitter superspace leads to a formula for the values of $\mu\ell$ with a single compensator, depending on the case $\mathcal{N}$. Subsequently, the effects on the value of $\mu\ell$ generated by a gauged scalar compensator will be investigated.

Aspects and conventions of the general treatment of symmetry groups used in the main text, as well as some formal expressions belonging to the superfield-component analysis have been relegated to the appendix.

2. Three-dimensional superspace

In this chapter, the relevant formalisms in three-dimensional superspace needed for the later purpose of coupling supersymmetric scalar matter conformally to super-gauge theory and conformal supergravity are reviewed, developed, or elaborated on.

In Section (2.1), the supersymmetry algebra is introduced and properties of the Lorentz group are presented.

In Section (2.2), the representation of supersymmetry on superfields is discussed and a formalism for the analysis of superfield components is developed. An off-shell action principle, giving on shell rise to minimal scalar multiplets, is proposed. Based on this formalism, the constrained scalar superfield transforming under spin($\mathcal{N}$) describing the on-shell multiplet coupled to super-gauge fields and supergravity in the next chapter is analysed and the corresponding superfield action is derived.

In Section (2.3), gauge-covariant derivatives are introduced and the content of the gauge connection is examined. The algebra of covariant derivatives is derived by solving the super-Jacobi identity under the constraint defining the field-strength multiplet.

In Section (2.4), two descriptions of extended conformal supergravity are presented. The first approach is the conventional curved SO($\mathcal{N}$) superspace, which is described by certain Weyl-invariant constraints on the torsions. These will be motivated by investigating the algebra of covariant derivatives in terms of the gauge fields of the local structure group. The super-Jacobi identity will be solved under these constraints in order to derive the field strengths in the supergravity algebra in terms of the super-Cotton tensor and the torsions. Subsequently, anti-de Sitter superspace is introduced as the maximally symmetric background of this geometry. The more briefly presented second approach is conformal superspace, where the whole superconformal group is gauged as the local structure group. It can be translated into conventional superspace and is convenient for obtaining useful relations in a simpler way.

2.1. Supersymmetry algebra

Three-dimensional supersymmetric space-time or superspace is parametrised by the coordinates

$$z_A = (x_a, \theta^I_\alpha), \tag{2.1.1}$$

where θ^I_α are odd supernumbers, i.e.

$$\theta^\alpha_I \theta^\beta_J = -\theta^\beta_J \theta^\alpha_I, \tag{2.1.2}$$

carrying an $\mathrm{SL}(2, \mathbb{R})$ index $\alpha = 1, 2$ and an $\mathrm{SO}(\mathcal{N})$ index $I = 1, ..., \mathcal{N}$. The space-time coordinate x_a carries an $\mathrm{SO}(2, 1)$ index $a = 0, 1, 2$.

The symmetry group of this superspace is the three-dimensional super-Poincaré group, which is generated by the super-Poincaré algebra. The super-Poincaré algebra is obtained from the Poincaré superalgebra

$$[\mathscr{M}^{ab}, \mathscr{M}^{cd}] = -4\eta^{[c[a}\mathscr{M}^{b]d]} \tag{2.1.3a}$$

$$[P_a, P_b] = 0 \tag{2.1.3b}$$

$$\{Q^I_\alpha, Q^J_\beta\} = 2\delta^{IJ} P_{\alpha\beta} \tag{2.1.3c}$$

by requiring anticommuting parameters for the fermionic generators Q^I_α and commuting parameters for the remaining bosonic generators. In other words, an element of the super-Poincaré algebra reads

$$X = \tfrac{1}{2}\omega^{ab}\mathscr{M}_{ab} + \mathrm{i}a^a P_a + \mathrm{i}\varepsilon^\alpha_I Q^I_\alpha, \tag{2.1.4}$$

where

$$\varepsilon^\alpha_I \varepsilon^\beta_J = -\varepsilon^\beta_J \varepsilon^\alpha_I. \tag{2.1.5}$$

The part corresponding to the bosonic part of the Poincaré superalgebra is the symmetry group of non-supersymmetric space-time $\mathrm{SO}(2, 1) \rtimes \mathbb{R}^3$. The part corresponding to the fermionic part is the group of supersymmetry transformations.

Referring to the terminology introduced in Appendix A, the Lorentz group $\mathrm{SO}(2, 1)$ is the pseudo-orthogonal group with

$$\eta^{mn} = \mathrm{diag}(-1, 1, 1)^{mn} \tag{2.1.6}$$

and $\mathrm{SL}(2,\mathbb{R})$ is the symplectic group $\mathrm{Sp}(2)$ with

$$\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\alpha\beta} . \tag{2.1.7}$$

The two-to-one correspondence between these two groups is established by mapping a Lorentz vector to the space of symmetric 2×2-matrices $\mathrm{Sym}(2,\mathbb{R})$ with the basis $S_m = \{S_0, S_1, S_2\}$, where

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad S_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.1.8}$$

This basis is (pseudo-) normalised as

$$\mathrm{tr}(S^m \bar{S}^n) = -2\eta^{mn}, \tag{2.1.9}$$

where $\bar{S}_m = \{S_0, -S_1, -S_2\}$. The components of a Lorentz vector x_m are the expansion coefficients in this basis,

$$X = x_m S^m, \tag{2.1.10}$$

and inversely,

$$-\tfrac{1}{2}\mathrm{tr}(X\bar{S}^m) = -\tfrac{1}{2}\mathrm{tr}(x_n S^n \bar{S}^m) = x^m. \tag{2.1.11}$$

Since the negative determinant of X equals the scalar product of the Lorentz vector, a Lorentz transformation of X must be determinant- and symmetricity-preserving, i.e.

$$X \xrightarrow{\mathrm{LT}} \tilde{X} = AXA^{\mathrm{T}} \tag{2.1.12}$$

with $A \in \mathrm{SL}(2,\mathbb{R})$ being an element of the group of 2×2-matrices with unit determinant. The map between $A \in \mathrm{SL}(2,\mathbb{R})$ and $\Lambda \in \mathrm{SO}(2,1)$ follows from

$$-\tfrac{1}{2}\mathrm{tr}(\tilde{X}\bar{S}^m) = \tilde{x}^m \tag{2.1.13}$$

as

$$-\tfrac{1}{2}\mathrm{tr}(AS_n A^{\mathrm{T}} \bar{S}^m) = \Lambda^m{}_n. \tag{2.1.14}$$

It is a two-to-one map because A and $-A$ are mapped to the same Λ. In other words,

$$\mathrm{SO}(2,1) \cong \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2. \tag{2.1.15}$$

As is apparent, the matrices S_m have two lower indices and the matrices $\bar{S}_m$ have two upper indices,

$$(\bar{S}_m)^{\alpha\beta} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(S_m)_{\gamma\delta}. \tag{2.1.16}$$

A set with one lower and one upper index is defined by

$$\begin{aligned}(\gamma_0)_\alpha{}^\beta &\equiv -\,\varepsilon^{\beta\gamma}(S_0)_{\alpha\gamma}\\ (\gamma_{1,2})_\alpha{}^\beta &\equiv \varepsilon^{\beta\gamma}(S_{1,2})_{\alpha\gamma}.\end{aligned} \tag{2.1.17}$$

These are a basis for traceless matrices and fulfil the Clifford algebra

$$\{\gamma_m, \gamma_n\} = 2\eta_{mn}. \tag{2.1.18}$$

In this context of space-time symmetry, the components of $\mathrm{SL}(2,\mathbb{R})$ vectors (also called spinors) are odd supernumbers with

$$v_\alpha w_\beta = -w_\beta v_\alpha \tag{2.1.19}$$

and the complex conjugation rule

$$(v_\alpha w_\beta)^* = w^*_\beta v^*_\alpha. \tag{2.1.20}$$

An antisymmetric $\mathrm{SO}(2,1)$ tensor of rank two, a vector and a rank-two $\mathrm{SL}(2,\mathbb{R})$ tensor are equivalent to each other by the relations

$$F_{ab} = \varepsilon_{abc}F^c = \varepsilon_{abc}(-\tfrac{1}{2}\varepsilon^{cdf}F_{df}) \tag{2.1.21a}$$

$$F_{\alpha\beta} = (\gamma^a)_{\alpha\beta}F_a = (\gamma^a)_{\alpha\beta}(-\tfrac{1}{2}(\gamma_a)^{\gamma\delta}F_{\gamma\delta}). \tag{2.1.21b}$$

The contraction of two Lorentz vectors can thus be written in the forms

$$F^aG_a = -\tfrac{1}{2}F^{ab}G_{ab} = -\tfrac{1}{2}F^{\alpha\beta}G_{\alpha\beta}. \tag{2.1.22}$$

The action of the Lorentz generators with different label representations is given by

$$\mathscr{M}^{ab}v^{c} = 2\eta^{c[a}v^{b]} \tag{2.1.23a}$$
$$\mathscr{M}^{a}v^{b} = -\varepsilon^{abc}v_{c} \tag{2.1.23b}$$
$$\mathscr{M}^{\alpha\beta}v^{c} = -(\gamma^{cb})^{\alpha\beta}v_{b} \tag{2.1.23c}$$

and

$$\mathscr{M}^{ab}v_{\gamma} = \tfrac{1}{2}(\gamma^{ab})_{\gamma}{}^{\delta}v_{\delta} \tag{2.1.24a}$$
$$\mathscr{M}^{a}v_{\gamma} = \tfrac{1}{2}(\gamma^{a})_{\gamma}{}^{\delta}v_{\delta} \tag{2.1.24b}$$
$$\mathscr{M}^{\alpha\beta}v^{\gamma} = \varepsilon^{\gamma(\alpha}v^{\beta)} \tag{2.1.24c}$$

for Lorentz vectors and spinors, respectively. The commutation relation can be written in the forms

$$[\mathscr{M}^{ab}, \mathscr{M}^{cd}] = -4\eta^{[c[a}\mathscr{M}^{b]d]} \tag{2.1.25a}$$
$$[\mathscr{M}^{a}, \mathscr{M}^{b}] = \varepsilon^{abc}\mathscr{M}_{c} \tag{2.1.25b}$$
$$[\mathscr{M}_{\alpha\beta}, \mathscr{M}_{\gamma\delta}] = -2\varepsilon_{(\gamma(\alpha}\mathscr{M}_{\beta)\delta)}. \tag{2.1.25c}$$

2.2. Superfields

2.2.1. Field representation

Fields $\boldsymbol{A}(x,\theta)$ in superspace are covariant with the coordinates both as finite- and infinite-dimensional representations of the super-Poincaré group. The former correspond to transformations in a finite vector space representing $\mathrm{SL}(2,\mathbb{R})$

$$\boldsymbol{A}'(x',\theta') = \boldsymbol{A}(x,\theta) + \delta\boldsymbol{A}(x,\theta) = \boldsymbol{A}(x,\theta) + \tfrac{1}{2}\omega^{ab}\mathscr{M}^{\mathrm{f}}_{ab}\cdot\boldsymbol{A}(x,\theta) \tag{2.2.1}$$

and the latter to translations in the infinite space of functions

$$\boldsymbol{A}'(x,\theta) = \boldsymbol{A}(x,\theta) + \Delta\boldsymbol{A}(x,\theta). \tag{2.2.2}$$

In this infinitesimal form, the two are related by a Taylor expansion

$$\begin{aligned}\Delta\boldsymbol{A}(x,\theta) = \delta\boldsymbol{A}(x,\theta) &- (a^a + \omega^{ab}x_b)\partial_a\boldsymbol{A}(x,\theta)\\ &- (\varepsilon^\alpha_I + \tfrac{1}{4}\omega^{ab}(\gamma_{ab})^{\alpha\beta}\theta_{\beta I})\partial^I_\alpha\boldsymbol{A}(x,\theta),\end{aligned} \tag{2.2.3}$$

where $\partial_a \equiv \frac{\partial}{\partial x^a}$ and $\partial^I_\alpha \equiv \frac{\partial}{\partial\theta^\alpha_I}$. Comparing the commutator

$$\begin{aligned}[\Delta_1,\Delta_2] =&\omega_1^{ac}\omega_{2c}{}^b(\mathscr{M}^{\mathrm{f}}_{ab} + 2x_{[a}\partial_{b]} - \tfrac{1}{4}(\gamma_{ab})^{\alpha\beta}\theta_{\beta I}\partial^I_\alpha)\\ &+ 2a_{b[1}\omega^{bc}_{2]}\partial_c + 2\varepsilon^{\alpha I}_{[1}(\partial^I_\alpha a^a_{2]})\partial_a\end{aligned} \tag{2.2.4}$$

with the one of two generators $X = \frac{1}{2}\omega^{ab}\mathscr{M}_{ab} + \mathrm{i}a^aP_a + \mathrm{i}\varepsilon^\alpha_I Q^I_\alpha$,

$$[X_1,X_2] =\omega_1^{ac}\omega_{2c}{}^b\mathscr{M}_{ab} + \tfrac{\mathrm{i}}{2}\omega^{ab}_{[1}\varepsilon^{\alpha I}_{2]}(\gamma_{ab})_\alpha{}^\beta Q_{\beta I} - 2\mathrm{i}a^a_{[1}\omega_{2]a}{}^bP_b + 2\varepsilon^{\alpha I}_1\varepsilon^\beta_{2I}P_{\alpha\beta}, \tag{2.2.5}$$

reveals that a_a must have the θ-dependent part, or the “soul”

$$a^{\mathrm{S}}_a = -\mathrm{i}\varepsilon^{\alpha I}\theta^\beta_I(\gamma^a)_{\alpha\beta}, \tag{2.2.6}$$

while

$$\mathscr{M}_{ab} =\mathscr{M}^{\mathrm{f}}_{ab} + 2x_{[a}\partial_{b]} - \tfrac{1}{4}(\gamma_{ab})^{\alpha\beta}\theta_{\beta I}\partial^I_\alpha \tag{2.2.7a}$$

$$P_a =\mathrm{i}\partial_a \tag{2.2.7b}$$

$$Q^I_\alpha =\mathrm{i}\partial^I_\alpha + \theta^{\beta I}(\gamma^a)_{\alpha\beta}\partial_a. \tag{2.2.7c}$$

Furthermore, there is a supercovariant derivative

$$D_\alpha^I = \partial_\alpha^I + \mathrm{i}\theta^{\beta I}(\gamma^a)_{\alpha\beta}\partial_a \tag{2.2.8}$$

commuting with the supersymmetry generator,

$$D_\alpha^I \varepsilon_J^\beta Q_\beta^J \boldsymbol{A} = \varepsilon_J^\beta Q_\beta^J D_\alpha^I \boldsymbol{A}, \tag{2.2.9}$$

and obeying the supersymmetry algebra

$$\{D_\alpha^I, D_\beta^J\} = 2\mathrm{i}\delta^{IJ}\partial_{\alpha\beta}. \tag{2.2.10}$$

It has the useful property [34]

$$(D_\alpha^I \boldsymbol{A})^* = -\, D_\alpha^I \boldsymbol{A}^* \tag{2.2.11a}$$

$$(D_\alpha^I \boldsymbol{A}_\beta)^* = D_\alpha^I \boldsymbol{A}_\beta^* \tag{2.2.11b}$$

and likewise for all $\mathrm{SL}(2,\mathbb{R})$ tensors of even or odd rank.

It can be convenient to combine the supercovariant spinor derivative together with the vector derivative $\partial_a \equiv D_a$ into a supervector

$$D_A = (D_a, D_\alpha^I). \tag{2.2.12}$$

It is subject to the algebra

$$[D_A, D_B\} \equiv D_A D_B - (-1)^{AB} D_B D_A = T_{AB}^C D_C, \tag{2.2.13}$$

where the powers A are 0 if A is a vector index and 1 if it is a spinor index. The torsion T_{AB}^C is constrained by

$$T_{\alpha\beta}^c = 2\mathrm{i}(\gamma^c)_{\alpha\beta}, \tag{2.2.14}$$

while all others are zero.

2.2.2. Component expansion of superfields

A superfield is expanded in powers of θ^I_α as

$$\boldsymbol{A}(x,\theta) = a(x) + \theta^\alpha_I a^I_\alpha(x) + \tfrac{1}{2}\theta^\alpha_I \theta^\beta_J a^{JI}_{\beta\alpha}(x) + \dots . \tag{2.2.15}$$

The component fields are give by the projections

$$(\partial^{I_1}_{\alpha_1}\dots\partial^{I_k}_{\alpha_k}\boldsymbol{A})|_{\theta=0} \equiv \partial^{I_1}_{\alpha_1}\dots\partial^{I_k}_{\alpha_k}\boldsymbol{A}| \equiv a^{I_1\dots I_k}_{\alpha_1\dots\alpha_k}. \tag{2.2.16}$$

This definition requires appropriate normalisation factors in the explicit expansion as indicated above. Since the spinor derivatives anti-commute, the component fields have the symmetry property

$$a^{I_1\dots I_i I_j\dots I_k}_{\alpha_1\dots\alpha_i\alpha_j\dots\alpha_k} = -a^{I_1\dots I_j I_i\dots I_k}_{\alpha_1\dots\alpha_j\alpha_i\dots\alpha_k}. \tag{2.2.17}$$

Supercovariant projections are denoted by

$$D^{I_1}_{\alpha_1}\dots D^{I_k}_{\alpha_k}\boldsymbol{A}| \equiv A^{I_1\dots I_k}_{\alpha_1\dots\alpha_k}. \tag{2.2.18}$$

They are related to the component fields by

$$A^{I_1\dots I_k}_{\alpha_1\dots\alpha_k} = a^{I_1\dots I_k}_{\alpha_1\dots\alpha_k} + \mathfrak{A}^{I_1\dots I_k}_{\alpha_1\dots\alpha_k}. \tag{2.2.19}$$

The fields $\mathfrak{A}^{I_1\dots I_k}_{\alpha_1\dots\alpha_k}$ depend on multiple space-time derivatives of components of correspondingly lower ranks. Explicitly, they are given by

$$\begin{aligned}
\mathfrak{A}^I_\alpha &= 0 && \text{(2.2.20a)}\\
\mathfrak{A}^{IJ}_{\alpha\beta} &= \mathrm{i}\delta^{IJ}\partial_{\alpha\beta}a && \text{(2.2.20b)}\\
\mathfrak{A}^{IJK}_{\alpha\beta\gamma} &= \mathrm{i}\delta^{JK}\partial_{\beta\gamma}a^I_\alpha - \mathrm{i}\delta^{IK}\partial_{\alpha\gamma}a^J_\beta + \mathrm{i}\delta^{IJ}\partial_{\alpha\beta}a^K_\gamma && \text{(2.2.20c)}\\
\mathfrak{A}^{IJKL}_{\alpha\beta\gamma\delta} &= \mathrm{i}\delta^{IJ}\partial_{\alpha\beta}a^{KL}_{\gamma\delta} - \mathrm{i}\delta^{IK}\partial_{\alpha\gamma}a^{JL}_{\beta\delta} + \mathrm{i}\delta^{KL}\partial_{\gamma\delta}a^{IJ}_{\alpha\beta}\\
&\quad + \mathrm{i}\delta^{IL}\partial_{\alpha\delta}a^{JK}_{\beta\gamma} + \mathrm{i}\delta^{JK}\partial_{\beta\gamma}a^{IL}_{\alpha\delta} - \mathrm{i}\delta^{JL}\partial_{\beta\delta}a^{IK}_{\alpha\gamma}\\
&\quad - \delta^{IJ}\delta^{KL}\partial_{\alpha\beta}\partial_{\gamma\delta}a + \delta^{IK}\delta^{JL}\partial_{\alpha\gamma}\partial_{\beta\delta}a - \delta^{IL}\delta^{JK}\partial_{\alpha\delta}\partial_{\beta\gamma}a, && \text{(2.2.20d)}
\end{aligned}$$

and so on. Systematic formulas and further examples are presented in Appendix B.

In the following, this relation will be symbolically denoted by

$$A^{I_1\dots I_k}_{\alpha_1\dots\alpha_k} = a^{I_1\dots I_k}_{\alpha_1\dots\alpha_k} + \mathfrak{A}^{I_1\dots I_k}_{\alpha_1\dots\alpha_k}(\partial a_{(-2)}, \partial\partial a_{(-4)}, \dots), \tag{2.2.21}$$

where $a_{(-l)}$ is of rank $k-l$.

Under infinitesimal supersymmetry transformations, the superfield changes by

$$\delta_\varepsilon \boldsymbol{A} = \mathrm{i}\varepsilon^{\alpha I} Q_{\alpha I} \boldsymbol{A} = \varepsilon^{\alpha I}(-D^I_\alpha + 2\mathrm{i}\theta^{I\beta}\partial_{\alpha\beta})\boldsymbol{A}. \tag{2.2.22}$$

The transformed components are the components of the transformed superfield, i.e.

$$\delta a^{I_1\ldots I_k}_{\alpha_1\ldots\alpha_k} = \partial^{I_1}_{\alpha_1}\ldots\partial^{I_k}_{\alpha_k}\delta\boldsymbol{A}| = D^{I_1}_{\alpha_1}\ldots D^{I_k}_{\alpha_k}\delta\boldsymbol{A}| - \mathfrak{A}^{I_1\ldots I_k}_{\alpha_1\ldots\alpha_k}(\partial\delta a_{(-2)}, \partial\partial\delta a_{(-4)}, \ldots). \tag{2.2.23}$$

Re-expressing the supercovariant projections, it follows

$$\delta a^{I_1\ldots I_k}_{\alpha_1\ldots\alpha_k} = -\varepsilon^{\alpha I} a^{II_1\ldots I_k}_{\alpha\alpha_1\ldots\alpha_k} - \varepsilon^{\alpha I}\mathfrak{A}^{II_1\ldots I_k}_{\alpha\alpha_1\ldots\alpha_k}(\partial a_{(-2)}, \ldots) - \mathfrak{A}^{I_1\ldots I_k}_{\alpha_1\ldots\alpha_k}(\partial\delta a_{(-2)}, \ldots), \tag{2.2.24}$$

iteratively describing the transformations of all superfield components.

It can be shown that these transformations indeed represent the supersymmetry algebra by considering two successive transformations

$$\delta_\eta\delta_\varepsilon a^{I_1..I_k}_{\alpha_1..\alpha_k} = \varepsilon^\alpha_I \eta^\beta_J a^{JII_1..I_k}_{\beta\alpha\alpha_1..\alpha_k} + \varepsilon^\alpha_I\eta^\beta_J \mathfrak{A}^{JII_1..I_k}_{\beta\alpha\alpha_1..\alpha_k}(\partial a_{(-2)}, ..) - \mathfrak{A}^{I_1..I_k}_{\alpha_1..\alpha_k}(\partial\delta_\eta\delta_\varepsilon a_{(-2)}, ..). \tag{2.2.25}$$

Their commutator is

$$[\delta_\eta, \delta_\varepsilon] a^{I_1..I_k}_{\alpha_1..\alpha_k} = (\varepsilon^\alpha_I\eta^\beta_J - \eta^\alpha_I\varepsilon^\beta_J)\mathfrak{A}^{JII_1..I_k}_{\beta\alpha\alpha_1..\alpha_k}(\partial a_{(-2)}, ..) - \mathfrak{A}^{I_1..I_k}_{\alpha_1..\alpha_k}(\partial[\delta_\eta, \delta_\varepsilon] a_{(-2)}, ..). \tag{2.2.26}$$

The combination of the parameters on the right-hand side is either symmetric or anti-symmetric in both types of indices projecting on the corresponding representations in $\mathfrak{A}^{JII_1\ldots I_k}_{\beta\alpha\alpha_1\ldots\alpha_k}$. This is only the symmetric one, given by

$$\mathfrak{A}^{I_1I_2\ldots I_k}_{\alpha_1\alpha_2\ldots\alpha_k} + \mathfrak{A}^{I_2I_1\ldots I_k}_{\alpha_2\alpha_1\ldots\alpha_k} = 2\mathrm{i}\delta^{I_1I_2}\partial_{\alpha_1\alpha_2} a^{I_3\ldots I_k}_{\alpha_3\ldots\alpha_k} + 2\mathrm{i}\delta^{I_1I_2}\partial_{\alpha_1\alpha_2}\mathfrak{A}^{I_3\ldots I_k}_{\alpha_3\ldots\alpha_k}, \tag{2.2.27}$$

as is apparent from its definition. Assuming that the supersymmetry algebra closes on fields of lower ranks inductively leads to its closure on all components,

$$[\delta_\eta, \delta_\varepsilon] a^{I_1\ldots I_k}_{\alpha_1\ldots\alpha_k} = 2\mathrm{i}\varepsilon^\alpha_I\eta^\beta_J\delta^{IJ}\partial_{\alpha\beta} a^{I_1\ldots I_k}_{\alpha_1\ldots\alpha_k}. \tag{2.2.28}$$

It is difficult to read off an explicit solution from the inductive transformation formula.

For example, the transformations of the first five components read

$$\delta a = -\, \varepsilon_I^\alpha a_\alpha^I \tag{2.2.29a}$$

$$\delta a_\alpha^I = -\, \varepsilon_J^\beta a_{\beta\alpha}^{JI} - \mathrm{i}\varepsilon_J^\beta \delta^{JI} \partial_{\beta\alpha} a \tag{2.2.29b}$$

$$\delta a_{\beta\alpha}^{JI} = -\, \varepsilon_K^\gamma a_{\gamma\beta\alpha}^{KJI} - \mathrm{i}\varepsilon_K^\gamma (-\delta^{KI}\partial_{\gamma\alpha} a_\beta^J + \delta^{KJ}\partial_{\gamma\beta} a_\alpha^I) \tag{2.2.29c}$$

$$\delta a_{\gamma\beta\alpha}^{KJI} = -\, \varepsilon_L^\delta a_{\delta\gamma\beta\alpha}^{LKJI} - \mathrm{i}\varepsilon_L^\delta (\delta^{LK}\partial_{\delta\gamma} a_{\beta\alpha}^{JI} - \delta^{LJ}\partial_{\delta\beta} a_{\gamma\alpha}^{KI} + \delta^{LI}\partial_{\delta\alpha} a_{\gamma\beta}^{KJ}) \tag{2.2.29d}$$

$$\begin{aligned}\delta a_{\delta\gamma\beta\alpha}^{LKJI} =& -\, \varepsilon_M^\varepsilon a_{\varepsilon\delta\gamma\beta\alpha}^{MLKJI} \\ & - \mathrm{i}\varepsilon_M^\varepsilon (\delta^{ML}\partial_{\varepsilon\delta} a_{\gamma\beta\alpha}^{KJI} - \delta^{MK}\partial_{\varepsilon\gamma} a_{\delta\beta\alpha}^{LJI} + \delta^{MJ}\partial_{\varepsilon\beta} a_{\delta\gamma\alpha}^{LKI} - \delta^{MI}\partial_{\varepsilon\alpha} a_{\delta\gamma\beta}^{LKJ}).\end{aligned} \tag{2.2.29e}$$

Apparently, each component transforms into the next higher component and first derivatives of the next lower component. Terms with more derivatives drop out and would be inconsistent with the symmetry of the tensor on the left-hand side. Therefore, it can be concluded that

$$\delta a_{\alpha_k \ldots \alpha_1}^{I_k \ldots I_1} = -\, \varepsilon_{I_{k+1}}^{\alpha_{k+1}} \left(a_{\alpha_{k+1}\ldots\alpha_1}^{I_{k+1}\ldots I_1} + \mathrm{i}\delta^{I_{k+1}\{I_k}\partial_{\alpha_{k+1}\{\alpha_k} a_{\alpha_{k-1}\ldots\alpha_1\}}^{I_{k-1}\ldots I_1\}} \right), \tag{2.2.30}$$

where the brackets $\{.\}$ collect the sum of k terms sharing the symmetry of the tensor on the left-hand side, as in the above examples.

The transformation of irreducible $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(\mathcal{N})$ representations contained in the components can easily be derived from the above formula. The particularly common partially reduced field $\overset{(n)}{a}{}_{\alpha_k\ldots\alpha_1}$ defined by

$$\overset{(n)}{a}{}_{\alpha_k\ldots\alpha_1}^{I_k\ldots I_1} = (-\tfrac{1}{2})^n \delta_{J_1 J_2} \varepsilon^{\beta_1\beta_2} \ldots \delta_{J_{2n-1}J_{2n}} \varepsilon^{\beta_{2n-1}\beta_{2n}} a_{\alpha_k\ldots\alpha_1\beta_{2n}\ldots\beta_1}^{I_k\ldots I_1 J_{2n}\ldots J_1} \tag{2.2.31}$$

has the transformation

$$\delta \overset{(n)}{a}{}_{\alpha_k..\alpha_1}^{I_k\ldots I_1} = -\, \varepsilon_{I_{k+1}}^{\alpha_{k+1}} \left(a_{\alpha_{k+1}..\alpha_1}^{I_{k+1}\ldots I_1} + \mathrm{i}\delta^{I_{k+1}\{I_k}\partial_{\alpha_{k+1}\{\alpha_k} a_{\alpha_{k-1}..\alpha_1\}}^{I_{k-1}\ldots I_1} - \mathrm{i}n(-1)^k \partial_{\alpha_{k+1}}{}^{\beta} \overset{(n-1)}{a}{}_{\alpha_k..\alpha_1\beta}^{I_k\ldots I_1 I_{k+1}} \right). \tag{2.2.32}$$

2.2.3. Supercovariant constraints

Since the supercovariant derivatives commute with supersymmetry transformations, they can be used to impose supersymmetrically invariant constraints on superfields. In view of the later purpose of describing scalar multiplets, an important class of scalar constraint equations considered in the following is

$$(D^\alpha_I D^I_\alpha)^n \boldsymbol{A} \equiv (D^2)^n \boldsymbol{A} \equiv \overset{(n)}{\boldsymbol{A}} = 0. \tag{2.2.33}$$

Hiding the SO($\mathcal{N}$) indices, the components of this superfield equation are

$$\overset{(n)}{\boldsymbol{A}} \mid = \overset{(n)}{a} + \overset{(n)}{\mathfrak{A}} = 0 \tag{2.2.34a}$$

$$\partial_{\beta_1} \overset{(n)}{\boldsymbol{A}} \mid = \overset{(n)}{A}_{\beta_1} = \overset{(n)}{a}_{\beta_1} + \overset{(n)}{\mathfrak{A}}_{\beta_1} = 0 \tag{2.2.34b}$$

$$\dots$$

$$\partial_{\beta_{2\mathcal{N}-2n}} .. \partial_{\beta_1} \overset{(n)}{\boldsymbol{A}} \mid = \overset{(n)}{a}_{\beta_{2\mathcal{N}-2n} .. \beta_1} + \overset{(n)}{\mathfrak{A}}_{\beta_{2\mathcal{N}-2n} .. \beta_1} = 0 \tag{2.2.34c}$$

$$\partial_{\beta_{2\mathcal{N}-2n+1}} .. \partial_{\beta_1} \overset{(n)}{\boldsymbol{A}} \mid = \overset{(n)}{\mathfrak{A}}_{\beta_{2\mathcal{N}-2n+1} .. \beta_1} = 0 \tag{2.2.34d}$$

$$\dots$$

$$\partial_{\beta_{2\mathcal{N}}} .. \partial_{\beta_1} \overset{(n)}{\boldsymbol{A}} \mid = \overset{(n)}{\mathfrak{A}}_{\beta_{2\mathcal{N}} .. \beta_1} = 0, \tag{2.2.34e}$$

where components of $\boldsymbol{A}$ being determined by lower components of the equation have to be accordingly substituted in the higher components. This is why the spinor derivatives ∂^I_α on the left-hand-sides can be replaced by supercovariant derivatives.

The equations of the form

$$\overset{(n)}{a}_{\beta_k .. \beta_1} = - \overset{(n)}{\mathfrak{A}}_{\beta_k .. \beta_1} \tag{2.2.35}$$

determine the components $\overset{(n)}{a}_{\beta_k .. \beta_1}$ in terms of derivatives of lower components; however, the equations

$$\overset{(n)}{\mathfrak{A}}_{\beta_{2\mathcal{N}-2n+l} .. \beta_1} = 0 \tag{2.2.36}$$

give rise to higher-order differential relations among the lower components and other representations $\overset{(m<n)}{a}_{\beta_l .. \beta_1}$. This circumstance shows that the constraint sets the superfield partially on shell.

The constraint is invariant under the transformation

$$\boldsymbol{A} \longrightarrow \boldsymbol{A} - \boldsymbol{B}, \tag{2.2.37}$$

if

$$\overset{(m \leq n-1)}{\boldsymbol{B}} = 0. \tag{2.2.38}$$

This gauge freedom can be fixed in the superfield $\hat{\boldsymbol{A}}$ defined by

$$\boldsymbol{A} = \hat{\boldsymbol{A}} + \boldsymbol{B}|_{\boldsymbol{A}}, \tag{2.2.39}$$

where $\boldsymbol{B}|_{\boldsymbol{A}}$ means that the component fields appearing in the constrained superfield $\boldsymbol{B}$ are evaluated at the values of the components of $\boldsymbol{A}$. Concretely, $\hat{\boldsymbol{A}}$ a has those components gauged away which are unconstrained in $\boldsymbol{B}$ and the other components are redefined in terms of the constrained components of $\boldsymbol{B}$ and are not redundant.

In order to illustrate the above in an example, one can consider the case $\mathcal{N} = 2$ and

$$D^2 D^2 \boldsymbol{A} = 0. \tag{2.2.40}$$

This constraint is invariant under

$$\boldsymbol{A} \longrightarrow \boldsymbol{A} - \boldsymbol{B}, \tag{2.2.41}$$

where

$$D^2 \boldsymbol{B} = 0. \tag{2.2.42}$$

The components of $\boldsymbol{B}$ fulfil

$$\tilde{b} = 0 \tag{2.2.43a}$$

$$\tilde{b}^I_\alpha = -\tilde{\mathfrak{B}}^I_\alpha = \mathrm{i}\partial_\alpha{}^\mu b^I_\mu \tag{2.2.43b}$$

$$\tilde{b}^{JI}_{\beta\alpha} = -\tilde{\mathfrak{B}}^{JI}_{\beta\alpha} = \varepsilon_{\beta\alpha}\delta^{JI}\Box b - 2\mathrm{i}\partial_{(\beta}{}^\mu b^{[IJ]}_{\alpha)\mu} \tag{2.2.43c}$$

$$0 = \tilde{\tilde{\mathfrak{B}}}^K_\gamma = 0 \tag{2.2.43d}$$

$$0 = \tilde{\tilde{\mathfrak{B}}}^{LK}_{\delta\gamma} = \partial_\gamma{}^\mu \partial_{(\delta}{}^\nu b^{[KL]}_{\mu)\nu} \tag{2.2.43e}$$

and the gauge shift translates to the components of $\boldsymbol{A}$ as

$$a \longrightarrow a - b \tag{2.2.44a}$$
$$a^I_\alpha \longrightarrow a^I_\alpha - b^I_\alpha \tag{2.2.44b}$$
$$\tilde{a} \longrightarrow \tilde{a} \tag{2.2.44c}$$
$$\tilde{a}^{((JI))} \longrightarrow \tilde{a}^{((JI))} - \tilde{b}^{((JI))} \tag{2.2.44d}$$
$$a^{[JI]}_{(\beta\alpha)} \longrightarrow a^{[JI]}_{(\beta\alpha)} - b^{[JI]}_{(\beta\alpha)}\big|_{0=\partial_\gamma{}^\beta\partial_{(\delta}{}^\alpha b^{[IJ]}_{\beta)\alpha}} \tag{2.2.44e}$$
$$\tilde{a}^I_\alpha \longrightarrow \tilde{a}^I_\alpha - \mathrm{i}\partial_\alpha{}^\mu b^I_\mu \tag{2.2.44f}$$
$$\tilde{a}^{JI}_{\beta\alpha} \longrightarrow \tilde{a}^{JI}_{\beta\alpha} - \varepsilon_{\beta\alpha}\delta^{JI}\Box b - 2\mathrm{i}\partial_{(\beta}{}^\mu b^{[IJ]}_{\alpha)\mu}, \tag{2.2.44g}$$

where $\tilde{a}^{((JI))}$ denotes the traceless part of $-\frac{1}{2}\varepsilon^{\alpha\beta}a^{(JI)}_{\beta\alpha}$. The fields a, a^I_α and $\tilde{a}^{((JI))}$ can be shifted arbitrarily and possibly to zero, while the fields $\tilde{a}$, $\tilde{a}^I_\alpha$ and $\tilde{a}^{JI}_{\beta\alpha}$ are non-redundant. Being interested in a minimal non-redundant multiplet, the field $a^{[JI]}_{(\beta\alpha)}$ can further be required to fulfil the constraint

$$0 = \partial_\gamma{}^\beta\partial_{(\delta}{}^\alpha a^{[IJ]}_{\beta)\alpha}, \tag{2.2.45}$$

in which case it can be shifted to zero as well.

According to (2.2.39), the components of $\boldsymbol{A}$ are then redefined as

$$a = a \tag{2.2.46a}$$
$$a^I_\alpha = a^I_\alpha \tag{2.2.46b}$$
$$\tilde{a} = \hat{\tilde{a}} \tag{2.2.46c}$$
$$\tilde{a}^{((IJ))} = \tilde{a}^{((IJ))} \tag{2.2.46d}$$
$$a^{[JI]}_{(\beta\alpha)} = a^{[JI]}_{(\beta\alpha)} \tag{2.2.46e}$$
$$\tilde{a}^I_\alpha = \hat{\tilde{a}}^I_\alpha + \mathrm{i}\partial_\alpha{}^\mu a^I_\mu \tag{2.2.46f}$$
$$\tilde{a}^{JI}_{\beta\alpha} = \hat{\tilde{a}}^{JI}_{\beta\alpha} + \varepsilon_{\beta\alpha}\delta^{JI}\Box a + 2\mathrm{i}\partial_{(\beta}{}^\mu a^{[IJ]}_{\alpha)\mu}. \tag{2.2.46g}$$

The gauge-fixed superfield $\hat{\boldsymbol{A}}$ is therefore given by

$$\hat{\boldsymbol{A}} = \tfrac{1}{2}\theta^\alpha_I\theta^I_\alpha(\hat{\tilde{a}} + \theta^\gamma_K\hat{\tilde{a}}^K_\gamma + \tfrac{1}{2}\theta^\gamma_K\theta^\delta_L\hat{\tilde{a}}^{KL}_{\delta\gamma}). \tag{2.2.47}$$

Due to (2.2.46g) and the condition (2.2.45), the Lorentz vector $\hat{\tilde{a}}^{[KL]}_{(\delta\gamma)}$ has to fulfil the same relation as the vector $\partial_{(\beta}{}^\mu a^{[IJ]}_{\alpha)\mu}$ in (2.2.45). This can be written as the Maxwell equation and Bianchi identity

$$\partial_{(\alpha}{}^{\gamma}\hat{\hat{a}}^{[KL]}_{\delta)\gamma} = \partial^{\gamma\delta}\hat{\hat{a}}^{[KL]}_{\delta\gamma} = 0. \tag{2.2.48}$$

In summary, the superfields $\boldsymbol{A}$ and $\hat{\boldsymbol{A}}$ fulfil the same constraint equation

$$D^2 D^2 \boldsymbol{A} = D^2 D^2 \hat{\boldsymbol{A}} = 0 \tag{2.2.49}$$

under the assumption of (2.2.48).

2.2.4. Superfield actions and equations of motion

A generalisation of the action known from $\mathcal{N} = 1$ supersymmetry [35] is

$$S = \int \mathrm{d}^3x\ (\mathrm{d}^2\theta)^{\mathcal{N}}\ (D^{\alpha_1}_{I_1}..D^{\alpha_{\mathcal{N}}}_{I_{\mathcal{N}}}\boldsymbol{A})D^{I_1}_{\alpha_1}..D^{I_{\mathcal{N}}}_{\alpha_{\mathcal{N}}}\boldsymbol{A}|. \tag{2.2.50}$$

It is manifestly invariant under supersymmetry, since the integral is over the whole superspace, i.e. the spinorial measure $(\mathrm{d}^2\theta)^{\mathcal{N}} \equiv (\partial^{\alpha}_{I}\partial^{I}_{\alpha})^{\mathcal{N}}$ contains all spinor derivatives and is thus annihilated by every supersymmetry generator. It is worth to note that $(\partial^{\alpha}_{I}\partial^{I}_{\alpha})^{\mathcal{N}}$ may be replaced by $(D^{\alpha}_{I}D^{I}_{\alpha})^{\mathcal{N}}$ up to a total derivative, which is useful for obtaining the component action. While the choice of the measure is unique (assuming that no indices are contracted with those of the integrand), the integrand is highly reducible. A particular choice in view of the present purpose is the completely traced part. For even $\mathcal{N}$ it is

$$S = \int \mathrm{d}^3x\ (\mathrm{d}^2\theta)^{\mathcal{N}}\ [(D^2)^n\boldsymbol{A}](D^2)^n\boldsymbol{A}| \tag{2.2.51}$$

and for odd $\mathcal{N}$

$$S = \int \mathrm{d}^3x\ (\mathrm{d}^2\theta)^{\mathcal{N}}\ [D^{\alpha}_{I}(D^2)^n\boldsymbol{A}]D^{I}_{\alpha}(D^2)^n\boldsymbol{A}|, \tag{2.2.52}$$

where $n = \frac{\mathcal{N}}{2}$ or $n = \frac{\mathcal{N}-1}{2}$ respectively.

The superfield equation of motion can be obtained by partially integrating so that

$$S = \int \mathrm{d}^3x\ (\mathrm{d}^2\theta)^{\mathcal{N}}\ \boldsymbol{A}(D^2)^{\mathcal{N}}\boldsymbol{A}|, \tag{2.2.53}$$

and is given by

$$(D^2)^{\mathcal{N}}\boldsymbol{A} = 0. \tag{2.2.54}$$

It is of the constraint form studied above. It has a redundancy due to the transformation

$$\boldsymbol{A} \longrightarrow \boldsymbol{A} - \boldsymbol{B}, \tag{2.2.55}$$

where

$$(D^2)^{\mathcal{N}-1}\boldsymbol{B} = 0. \tag{2.2.56}$$

Written in components (2.2.34), and omitting the SO($\mathcal{N}$) indices, this constraint on $\boldsymbol{B}$ reads

$$\overset{(\mathcal{N}-1)}{\boldsymbol{B}}| = \overset{(\mathcal{N}-1)}{b} + \overset{(\mathcal{N}-1)}{\mathfrak{B}} = 0 \tag{2.2.57a}$$

$$\partial_{\beta_1}\overset{(\mathcal{N}-1)}{\boldsymbol{B}}| = \overset{(\mathcal{N}-1)}{B}{}_{\beta_1} = \overset{(\mathcal{N}-1)}{b}{}_{\beta_1} + \overset{(\mathcal{N}-1)}{\mathfrak{B}}{}_{\beta_1} = 0 \tag{2.2.57b}$$

$$\partial_{\beta_2}\partial_{\beta_1}\overset{(\mathcal{N}-1)}{\boldsymbol{B}}| = \overset{(\mathcal{N}-1)}{b}{}_{\beta_2\beta_1} + \overset{(\mathcal{N}-1)}{\mathfrak{B}}{}_{\beta_2\beta_1} = 0 \tag{2.2.57c}$$

$$\partial_{\beta_3}..\partial_{\beta_1}\overset{(\mathcal{N}-1)}{\boldsymbol{B}}| = \overset{(\mathcal{N}-1)}{\mathfrak{B}}{}_{\beta_3..\beta_1} = 0 \tag{2.2.57d}$$

$$\dots$$

$$\partial_{\beta_{2\mathcal{N}}}..\partial_{\beta_1}\overset{(\mathcal{N}-1)}{\boldsymbol{B}}| = \overset{(\mathcal{N}-1)}{\mathfrak{B}}{}_{\beta_{2\mathcal{N}}..\beta_1} = 0. \tag{2.2.57e}$$

Accordingly, the components $\overset{(\mathcal{N}-1)}{a}$, $\overset{(\mathcal{N}-1)}{a}{}^{I}_{\alpha}$ and $\overset{(\mathcal{N}-1)}{a}{}^{JI}_{\beta\alpha}$ are not redundant, but can be redefined to invariant fields $\overset{(\hat{\mathcal{N}-1})}{a}$, $\overset{(\hat{\mathcal{N}-1})}{a}{}^{I}_{\alpha}$ and $\overset{(\hat{\mathcal{N}-1})}{a}{}^{JI}_{\beta\alpha}$ as defined by (2.2.39). The other components can be gauged away if they fulfil the superfield equation of motion, because in this case they fulfil the same differential relations as the corresponding components of the gauge parameter field $\boldsymbol{B}$.

In consequence, the superfield equation of motion contains the non-redundant information

$$\overset{(\hat{\mathcal{N}})}{a} = 0 \tag{2.2.58a}$$

$$\mathcal{N}\mathrm{i}\partial_{\alpha}{}^{\mu}\overset{(\hat{\mathcal{N}-1})}{a}{}^{I}_{\mu} = 0 \tag{2.2.58b}$$

$$-\mathrm{i}\mathcal{N}\partial_{(\beta}{}^{\mu}\overset{(\hat{\mathcal{N}-1})}{a}{}^{[IJ]}_{\alpha)\mu} + \delta^{JI}\varepsilon_{\beta\alpha}\Box\overset{(\hat{\mathcal{N}-1})}{a} = 0. \tag{2.2.58c}$$

This system contains the equations of motion of a minimal on-shell multiplet, being defined by the above off-shell action. The Lorentz vector $\overset{(\hat{\mathcal{N}-1})}{a}{}^{JI}_{\beta\alpha}$, even though it appears quadratic in the off-shell action, on shell fulfils the Maxwell equation and in addition the Bianchi identity, which is an effect of the on-shell gauge fixing as demonstrated for the example (2.2.48).

2.2.5. A spin($\mathcal{N}$) on-shell superfield

A superfield $\boldsymbol{Q}_i$ transforming under the fundamental representation of spin($\mathcal{N}$) (see Appendix A) can be subject to the constraint [31, 22, 20, 18, 19]

$$D^I_\alpha \boldsymbol{Q}_i = (\gamma^I)_i{}^j \boldsymbol{Q}_{j\alpha}, \tag{2.2.59}$$

where $\boldsymbol{Q}_{i\alpha}$ is a general superfield carrying an additional Lorentz index and reads

$$\boldsymbol{Q}_{i\alpha} = q_{i\alpha} + \theta^\beta_J q_{i\alpha,\beta}{}^J + \dots . \tag{2.2.60}$$

For chiral representations of spin($\mathcal{N}$), the constraint is

$$D^I_\alpha \boldsymbol{Q} = \Sigma^I \boldsymbol{Q}_\alpha \tag{2.2.61}$$

or

$$D^I_\alpha \boldsymbol{Q} = \bar{\Sigma}^I \boldsymbol{Q}_\alpha, \tag{2.2.62}$$

depending on the chosen handedness. The chiral spin($\mathcal{N}$) indices have been omitted, since they depend on the specific case of $\mathcal{N}$. A discussion of fields transforming under chiral representations will follow in Chapter 3. In the following formal considerations, the form for non-chiral representations will be used.

The first component of the constraint superfield equation is

$$q^I_\alpha = \gamma^I q_\alpha. \tag{2.2.63}$$

Taking a general number of supercovariant projections

$$Q^{J_k..J_1 I}_{\beta_k..\beta_1\alpha} = \gamma^I Q_{\alpha,\beta_k..\beta_1}{}^{J_k..J_1} \tag{2.2.64}$$

and reminding that

$$Q^{I_k..I_1}_{\alpha_k..\alpha_1} = q^{I_k..I_1}_{\alpha_k..\alpha_1} + \mathfrak{Q}^{I_k..I_1}_{\alpha_k..\alpha_1}, \tag{2.2.65}$$

leads to the expression for the components of $\boldsymbol{Q}_i$

$$q^{J_k..J_1 I}_{\beta_k..\beta_1\alpha} = \gamma^I q_{\alpha,\beta_k..\beta_1}{}^{J_k..J_1} + \gamma^I \mathfrak{Q}_{\alpha,\beta_k..\beta_1}{}^{J_k..J_1} - \mathfrak{Q}^{J_k..J_1 I}_{\beta_k..\beta_1\alpha} \tag{2.2.66}$$

in terms of the components of $\boldsymbol{Q}_{i\alpha}$. These are determined by taking the representations of this equation not contained in $q^{J_k..J_1 I}_{\beta_k..\beta_1\alpha}$, i.e. both symmetric or antisymmetric in a particular index pair, leading to

$$\mathfrak{Q}^{J_k..(J_1 I)}_{\beta_k..(\beta_1\alpha)} - \gamma^{(I}\mathfrak{Q}_{(\alpha,|\beta_k..|\beta_1)}{}^{|J_k..|J_1)} = \gamma^{(I} q_{(\alpha,|\beta_k..|\beta_1)}{}^{|J_k..|J_1)} \tag{2.2.67a}$$

$$\mathfrak{Q}^{J_k..[J_1 I]}_{\beta_k..[\beta_1\alpha]} - \gamma^{[I}\mathfrak{Q}_{[\alpha,|\beta_k..|\beta_1]}{}^{|J_k..|J_1]} = \gamma^{[I} q_{[\alpha,|\beta_k..|\beta_1]}{}^{|J_k..|J_1]}. \tag{2.2.67b}$$

In terms of covariant projections, this is solved by

$$Q_{(\alpha,|\beta_k..|\beta_1)}{}^{J_k..J_1} = \mathrm{i}\partial_{\beta_1\alpha}\gamma^{J_1} Q^{J_k..J_2}_{\beta_k..\beta_2} + ... \tag{2.2.68a}$$

$$Q_{[\alpha,|\beta_k..|\beta_1]}{}^{J_k..J_1} = 0, \tag{2.2.68b}$$

where the dots indicate the further permutations implied by the symmetries on the left-hand-side (an example will appear below). Accordingly, the components or super-covariant projections of $\boldsymbol{Q}_i$ are provided by the expressions

$$Q^{J_k..[J_1 I]}_{\beta_k..(\beta_1\alpha)} = \mathrm{i}\partial_{\beta_1\alpha}\gamma^{IJ_1} Q^{J_k..J_2}_{\beta_k..\beta_2} + ... \tag{2.2.69a}$$

$$Q^{J_k..(J_1 I)}_{\beta_k..[\beta_1\alpha]} = 0. \tag{2.2.69b}$$

For the second component this means

$$q^{[JI]}_{(\beta\alpha)} = \mathrm{i}\gamma^{IJ}\partial_{\beta\alpha} q \tag{2.2.70a}$$

$$\tilde{q}^{(JI)} = 0. \tag{2.2.70b}$$

The constraint (2.2.59) therefore defines an on-shell multiplet consisting of q and q_α, subject to the supersymmetry transformations

$$\delta q = -\,\varepsilon^\alpha_I \gamma^I q_\alpha \tag{2.2.71a}$$

$$\delta q_\alpha = -\,\mathrm{i}\varepsilon^\beta_J \gamma^J \partial_{\beta\alpha} q. \tag{2.2.71b}$$

This multiplet corresponds to the special case of a minimal on-shell multiplet (2.2.58) arising from an unconstrained scalar superfield with an attached spin($\mathcal{N}$) index, where

$$\overset{(\mathcal{N})}{a} = 0 \tag{2.2.72a}$$

$$\overset{(\mathcal{N}-1)}{a}{}^{I}_{\alpha} = \gamma^{I} q_{\alpha} \tag{2.2.72b}$$

$$\overset{(\mathcal{N}-1)}{a}{}^{[JI]}_{(\beta\alpha)} = \mathrm{i}\gamma^{IJ}\partial_{\beta\alpha} q. \tag{2.2.72c}$$

In other words, the defining constraint (2.2.59) for $\boldsymbol{Q}_i$ removes the vector $\overset{(\mathcal{N}-1)}{a}{}^{[JI]}_{(\beta\alpha)}$ by identifying it with the derivative of the scalar field, which is consistent with the Maxwell equation and the Bianchi identity fulfilled by this vector.

Since the scalar multiplet is encoded in the lowest components of the constrained superfield $\boldsymbol{Q}_i$, a corresponding superfield action resembles the one for $\mathcal{N} = 1$ [35]. It may be postulated as

$$\begin{aligned} S =& \tfrac{1}{A(\mathcal{N})} \int \mathrm{d}^3x \; \mathrm{d}\theta^{\alpha}_{I} \mathrm{d}\theta^{I}_{\alpha} \; (\overline{D^{\gamma}_{K}\boldsymbol{Q}}) D^{K}_{\gamma}\boldsymbol{Q}| \\ =& \tfrac{1}{A(\mathcal{N})} \int \mathrm{d}^3x \; (\overline{Q^{\gamma}_{K}} Q^{\alpha IK}_{I\alpha\gamma} - \overline{Q^{\alpha I\gamma}_{I\alpha K}} Q^{K}_{\gamma} - 2\overline{Q^{\alpha\gamma}_{IK}} Q^{IK}_{\alpha\gamma}), \end{aligned} \tag{2.2.73}$$

where $A(\mathcal{N})$ is a normalisation factor. Indeed, reminding that

$$Q^{\gamma\beta\alpha}_{KJI} = \mathrm{i}\gamma_I\gamma_J\gamma_K \partial^{\alpha\beta} q^{\gamma} + \mathrm{i}\gamma_J\gamma_K\gamma_I \partial^{\gamma\beta} q^{\alpha} - \mathrm{i}\gamma_I\gamma_K\gamma_J \partial^{\gamma\alpha} q^{\beta} \tag{2.2.74a}$$

$$Q^{\beta\alpha}_{JI} = \mathrm{i}\gamma_I\gamma_J \partial^{\alpha\beta} q, \tag{2.2.74b}$$

the component action follows as

$$S = \tfrac{\mathcal{N}^2}{A(\mathcal{N})} \int \mathrm{d}^3x \; (-2\mathrm{i}\bar{q}^{\gamma}\partial_{\gamma}{}^{\alpha} q_{\alpha} - 2\mathrm{i}q^{\gamma}(\overline{\partial_{\gamma}{}^{\alpha} q_{\alpha}}) - 2(\overline{\partial^{\alpha\beta} q})\partial_{\alpha\beta} q), \tag{2.2.75}$$

where for canonic normalisation it can be chosen $A(\mathcal{N}) = -8\mathcal{N}^2$.

This action is not manifestly supersymmetric, but rather supersymmetric only on shell. It can however be derived from the off-shell actions

$$S = \int \mathrm{d}^3x \; (\mathrm{d}^2\theta)^{\mathcal{N}} \; [(D^2)^n \boldsymbol{A}](D^2)^n \boldsymbol{A}| \tag{2.2.76}$$

and

$$S = \int \mathrm{d}^3x \; (\mathrm{d}^2\theta)^{\mathcal{N}} \; [D_I^\alpha (D^2)^n \boldsymbol{A}] D_\alpha^I (D^2)^n \boldsymbol{A}|. \tag{2.2.77}$$

In the gauge for the minimal multiplet the superfield takes the form

$$\hat{\boldsymbol{A}} \propto (\theta^2)^{\mathcal{N}-1} \overset{\widehat{(\mathcal{N}-1)}}{a} + \dots . \tag{2.2.78}$$

Insertion into the action for even $\mathcal{N}$ (for odd $\mathcal{N}$ the procedure is similar) gives

$$S \propto \int \mathrm{d}^3x \; (\mathrm{d}^2\theta)^2 (\mathrm{d}^2\theta)^{\mathcal{N}-2} \; [(\theta^2)^{\frac{\mathcal{N}}{2}-1} \overset{\widehat{(\mathcal{N}-1)}}{a} + \dots][(\theta^2)^{\frac{\mathcal{N}}{2}-1} \overset{\widehat{(\mathcal{N}-1)}}{a} + \dots]| \tag{2.2.79a}$$

$$\propto \int \mathrm{d}^3x \; (\mathrm{d}^2\theta)^2 \; [\overset{\widehat{(\mathcal{N}-1)}}{a} + \dots][\overset{\widehat{(\mathcal{N}-1)}}{a} + \dots]|. \tag{2.2.79b}$$

The superfield appearing in the square-brackets corresponds to $\boldsymbol{Q}_i$ and can be accordingly replaced so that

$$S \propto \int \mathrm{d}^3x \; (\mathrm{d}^2\theta)^2 \; \overline{\boldsymbol{Q}}\boldsymbol{Q}|, \tag{2.2.80}$$

where the trace over indices of $\boldsymbol{Q}$ is implied. Using the product rule, this form can be brought to the postulated action

$$S \propto \int \mathrm{d}^3x \; \mathrm{d}^2\theta \; (\overline{D_I^\alpha \boldsymbol{Q}}) D_\alpha^I \boldsymbol{Q}|, \tag{2.2.81}$$

where it was used that $D^2\boldsymbol{Q}_i| = 0$, as is implied by (2.2.69b).

Summarising this section, an off-shell action (2.2.51)/(2.2.52) for a general $\mathcal{N}$-extended scalar superfield $\boldsymbol{A}$ was proposed. It contains the three highest component fields of the superfield with canonical kinetic terms, i.e. with not more than two derivatives. The superfield equation of motion (2.2.54) resulting from this action bears redundancies due to its superfield-constraint form involving multiple supercovariant derivatives. In the gauge where the minimal number of fields is kept non-redundant the equations of motion describe a multiplet consisting of a scalar field, to which the canonical dimension one-half can be assigned, a spinor being also an SO($\mathcal{N}$) vector with dimension one, and a field of rank two with dimension three-halves, which contains scalar auxiliary fields vanishing on shell as well as a Lorentz vector displaying the properties of a Maxwell field strength.

This Lorentz vector is considered undesirable for the description of an actual scalar-matter multiplet consisting only of scalars and spinors. Therefore, the scalar superfield $\boldsymbol{Q}_i$ transforming under the fundamental representation of spin($\mathcal{N}$) and being subject to the constraint (2.2.59), which involves the SO($\mathcal{N}$) spin matrices, was introduced. The constraint removes the problematic vector field by identifying it with the derivative of the scalar and decouples the SO($\mathcal{N}$) index from the spinor, leaving an equal number (given by the dimension of the fundamental representation of spin($\mathcal{N}$)) of scalar and spinor fields in the on-shell multiplet.

As compared to the unconstrained superfield $\boldsymbol{A}$, the constrained superfield $\boldsymbol{Q}_i$ (2.2.59) contains the scalar multiplet in its lowest components, rather than in the highest ones. Its superfield action (2.2.73) therefore resembles the one for an unconstrained $\mathcal{N} = 1$ superfield. It can be derived from the off-shell action for the unconstrained superfield $\boldsymbol{A}$ by imposing the on-shell gauge for the minimal multiplet and integrating out the appropriate number of spinor coordinates.

2.3. Local symmetries

2.3.1. Gauge-covariant derivatives

An important generalisation of symmetries is space-time dependence of the transformation parameters, where derivatives of fields representing the symmetry group are not covariant under group transformations, but rather behave as

$$D_A \boldsymbol{A} \longrightarrow D_A \mathrm{e}^{\boldsymbol{X}} \boldsymbol{A} = (D_A \boldsymbol{X}) \mathrm{e}^{\boldsymbol{X}} \boldsymbol{A} + \mathrm{e}^{\boldsymbol{X}} D_A \boldsymbol{A}. \tag{2.3.1}$$

A gauge-covariant derivative is given by

$$\mathscr{D}_A = D_A + \boldsymbol{B}_A, \tag{2.3.2}$$

where B_A is a Lie algebra valued superfield transforming as

$$\boldsymbol{B}_A \longrightarrow \mathrm{e}^{\boldsymbol{X}} \boldsymbol{B}_A \mathrm{e}^{-\boldsymbol{X}} - (D_A \mathrm{e}^{\boldsymbol{X}}) \mathrm{e}^{-\boldsymbol{X}}, \tag{2.3.3}$$

so that in consequence

$$\mathscr{D}_A \boldsymbol{A} \longrightarrow \mathrm{e}^{\boldsymbol{X}} \mathscr{D}_A \boldsymbol{A}. \tag{2.3.4}$$

The supercovariant[1] projections of the spinor gauge field $\boldsymbol{B}^I_\alpha$ transform under infinitesimal gauge transformations as

$$\delta B^{I,J_K..J_1}_{\alpha,\beta_k..\beta_1} = -X^{J_K..J_1 I}_{\beta_k..\beta_1 \alpha} = -x^{J_K..J_1 I}_{\beta_k..\beta_1 \alpha} - \mathfrak{X}^{J_K..J_1 I}_{\beta_k..\beta_1 \alpha}, \tag{2.3.5}$$

which, for the first few projections, means

$$\delta B^I_\alpha = - x^I_\alpha \tag{2.3.6a}$$
$$\delta B^{I,J}_{\alpha,\beta} = - x^{JI}_{\beta\alpha} - \mathrm{i}\delta^{JI} \partial_{\beta\alpha} x \tag{2.3.6b}$$
$$\delta B^{I,KJ}_{\alpha,\gamma\beta} = - x^{KJI}_{\gamma\beta\alpha} - \mathfrak{X}^{KJI}_{\gamma\beta\alpha} \tag{2.3.6c}$$
$$\delta B^{I,LKJ}_{\alpha,\delta\gamma\beta} = - x^{LKJI}_{\delta\gamma\beta\alpha} - \mathfrak{X}^{LKJI}_{\delta\gamma\beta\alpha}. \tag{2.3.6d}$$

[1] Component projections are not of interest, because the spinor gauge field covariantises the supercovariant derivative rather than the spinor derivative in this formalism.

Many of them can be gauged away by these transformations, while

$$\delta B^{(I,J)}_{(\alpha,\beta)} = -\mathrm{i}\delta^{JI}\partial_{\beta\alpha}x \tag{2.3.7a}$$

$$\delta B^{[I,J]}_{[\alpha,\beta]} = 0 \tag{2.3.7b}$$

$$\delta B^{[I,|K|J]}_{[\alpha,|\gamma|\beta]} = 0 \tag{2.3.7c}$$

$$\delta B^{[I,|LK|J]}_{[\alpha,|\delta\gamma|\beta]} = 0. \tag{2.3.7d}$$

This suggests that the trace of $B^{(I,J)}_{(\alpha,\beta)}$ can be identified with the vector gauge field and the other fields correspond to various invariant, i.e. finitely covariant, field strengths. These can be defined in the manifestly covariant formalism of commutators of covariant derivatives, forming the gauge superalgebra described in the following.

2.3.2. Gauge superalgebra

The algebra of covariant derivatives is defined by [35]

$$[\mathscr{D}_A, \mathscr{D}_B\} = \boldsymbol{T}^C_{AB}\mathscr{D}_C + \boldsymbol{F}_{AB}. \tag{2.3.8}$$

The superfields $\boldsymbol{T}^C_{AB}$ and $\boldsymbol{F}_{AB}$ are called torsion and field strength, respectively. The case of two spinor derivatives, written explicitly in terms of the spinor gauge field, reads

$$\{\mathscr{D}^I_\alpha, \mathscr{D}^J_\beta\} = 2\mathrm{i}\delta^{IJ}\partial_{\alpha\beta} + 2D^{(I}_{(\alpha}\boldsymbol{B}^{J)}_{\beta)} + \{\boldsymbol{B}^{(I}_{(\alpha}, \boldsymbol{B}^{J)}_{\beta)}\} + 2D^{[I}_{[\alpha}\boldsymbol{B}^{J]}_{\beta]} + \{\boldsymbol{B}^{[I}_{[\alpha}, \boldsymbol{B}^{J]}_{\beta]}\}. \tag{2.3.9}$$

It implies the identifications

$$2\mathrm{i}\delta^{IJ}\boldsymbol{B}_{\alpha\beta} + \boldsymbol{F}^{(IJ)}_{(\alpha\beta)} = 2D^{(I}_{(\alpha}\boldsymbol{B}^{J)}_{\beta)} + \{\boldsymbol{B}^{(I}_{(\alpha}, \boldsymbol{B}^{J)}_{\beta)}\} \tag{2.3.10a}$$

$$\boldsymbol{F}^{[IJ]}_{[\alpha\beta]} = 2D^{[I}_{[\alpha}\boldsymbol{B}^{J]}_{\beta]} + \{\boldsymbol{B}^{[I}_{[\alpha}, \boldsymbol{B}^{J]}_{\beta]}\}. \tag{2.3.10b}$$

They agree with what is expected from the above analysis of the supercovariant gauge-field projections (2.3.7), by noting that

$$\delta\{\boldsymbol{B}^I_\alpha, \boldsymbol{B}^J_\beta\} = 0. \tag{2.3.11}$$

Conventionally, the trace of $\boldsymbol{F}^{(IJ)}_{(\alpha\beta)}$ can be set to zero, corresponding to a redefinition of $\boldsymbol{B}_{\alpha\beta}$, which leads to [34]

$$\{\mathscr{D}^I_\alpha, \mathscr{D}^J_\beta\} = 2\mathrm{i}\delta^{IJ}\mathscr{D}_{\alpha\beta} + \boldsymbol{F}^{((IJ))}_{(\alpha\beta)} + 2\mathrm{i}\varepsilon_{\alpha\beta}\boldsymbol{F}^{IJ}. \tag{2.3.12}$$

It is common to formulate these identifications equivalently in terms of conventional constraints on the torsion and field strength

$$\boldsymbol{T}^{c}_{\alpha\beta} = 2\mathrm{i}(\gamma^c)_{\alpha\beta} \tag{2.3.13a}$$
$$\boldsymbol{F}^{IJ}_{\alpha\beta} = \boldsymbol{F}^{((IJ))}_{(\alpha\beta)} + 2\mathrm{i}\varepsilon_{\alpha\beta}\boldsymbol{F}^{IJ}, \tag{2.3.13b}$$

with all the other torsion components being zero.

The conventional constraints affect the whole algebra due to the super-Jacobi identity

$$0 = [\mathscr{D}_{[A}, [\mathscr{D}_B, \mathscr{D}_{C)}\}\}, \tag{2.3.14}$$

where $[ABC)$ means antisymmetrisation with the caveat of an additional sign change if two spinor indices are permuted. Unless indices have otherwise been manipulated, it is usually understood that spinor indices are permuted together with their $\mathrm{SO}(\mathcal{N})$ index.

Inserting the commutators in terms of field strengths and torsions, the super-Jacobi identity can be written as

$$\begin{aligned} 0 &= [\mathscr{D}_{[A}, \boldsymbol{T}^D_{BC)}\mathscr{D}_D\} + [\mathscr{D}_{[A}, \boldsymbol{F}_{BC)}\} \\ &= \left(\mathscr{D}_{[A}\boldsymbol{T}^D_{BC)}\right)\mathscr{D}_D - [(-1)^{A(B+C)}]^2\boldsymbol{T}^D_{[BC}[\mathscr{D}_{|D|}, \mathscr{D}_{A)}\} + \mathscr{D}_{[A}\boldsymbol{F}_{BC)} \\ &= \left(\mathscr{D}_{[A}\boldsymbol{T}^D_{BC)} - \boldsymbol{T}^E_{[BC}\boldsymbol{T}^D_{|E|A)}\right)\mathscr{D}_D + \mathscr{D}_{[A}\boldsymbol{F}_{BC)} - \boldsymbol{T}^D_{[BC}\boldsymbol{F}_{|D|A)}, \end{aligned} \tag{2.3.15}$$

where $|A|$ denotes the exclusion of this index from surrounding permutation brackets. It contains four distinct cases of combinations of vector and spinor indices, leading to the conditions on the field strength

$$\mathscr{D}^I_{(\alpha}\boldsymbol{F}^{JK}_{\beta\gamma)} = \boldsymbol{T}_{(\alpha\beta}{}^{IJ\delta L}\boldsymbol{F}^{LK}_{|\delta|\gamma)} + \boldsymbol{T}_{(\alpha\beta}{}^{IJd}\boldsymbol{F}_{|d|\gamma)}{}^{K} \tag{2.3.16a}$$
$$\mathscr{D}_{[a}\boldsymbol{F}_{bc]} = \boldsymbol{T}_{[ab}{}^{\delta L}\boldsymbol{F}^{L}_{|\delta|c]} + \boldsymbol{T}_{[ab}{}^{d}\boldsymbol{F}_{|d|c]} \tag{2.3.16b}$$
$$\mathscr{D}^I_{[\alpha}\boldsymbol{F}_{bc]} = \boldsymbol{T}^{I}_{[\alpha b}{}^{\delta L}\boldsymbol{F}^{L}_{|\delta|c]} + \boldsymbol{T}^{I}_{[\alpha b}{}^{d}\boldsymbol{F}_{|d|c]} \tag{2.3.16c}$$
$$\mathscr{D}^I_{[\alpha}\boldsymbol{F}^{J}_{\beta c)} = \boldsymbol{T}^{J}_{[\beta c}{}^{\delta L}\boldsymbol{F}^{LI}_{|\delta|\alpha)} + \boldsymbol{T}_{[\beta c}{}^{Jd}\boldsymbol{F}_{|d|\alpha)}{}^{I}. \tag{2.3.16d}$$

Due to the vanishing torsions they simplify to

$$\mathscr{D}^I_{(\alpha}\boldsymbol{F}^{JK}_{\beta\gamma)} = 2\mathrm{i}\delta^{IJ}(\gamma^d)_{(\alpha\beta}\boldsymbol{F}_{|d|\gamma)}{}^K \tag{2.3.17a}$$

$$\mathscr{D}_{[a}\boldsymbol{F}_{bc]} = 0 \tag{2.3.17b}$$

$$\mathscr{D}^I_{[\alpha}\boldsymbol{F}_{bc]} = 0 \tag{2.3.17c}$$

$$\mathscr{D}^I_{[\alpha}\boldsymbol{F}^J_{\beta c)} = -2\mathrm{i}\delta^{IJ}(\gamma^d)_{\alpha\beta}\boldsymbol{F}_{dc}. \tag{2.3.17d}$$

The gauge multiplet is defined by the additional constraint $\boldsymbol{F}^{((IJ))}_{(\alpha\beta)} = 0$ [34], in which case

$$\varepsilon_{\alpha\beta}\mathscr{D}^K_\gamma\boldsymbol{F}^{IJ} + \varepsilon_{\beta\gamma}\mathscr{D}^I_\alpha\boldsymbol{F}^{JK} + \varepsilon_{\gamma\alpha}\mathscr{D}^J_\beta\boldsymbol{F}^{KI} = \delta^{IJ}\boldsymbol{F}^K_{\alpha\beta,\gamma} + \delta^{KI}\boldsymbol{F}^J_{\gamma\alpha,\beta} + \delta^{JK}\boldsymbol{F}^I_{\beta\gamma,\alpha} \tag{2.3.18a}$$

$$\mathscr{D}_{[a}\boldsymbol{F}_{bc]} = 0 \tag{2.3.18b}$$

$$\mathscr{D}^I_{[\alpha}\boldsymbol{F}_{bc]} = 0 \tag{2.3.18c}$$

$$\mathscr{D}^I_\alpha\boldsymbol{F}^J_{\beta c} + 2\mathrm{i}\mathscr{D}_c\varepsilon_{\alpha\beta}\boldsymbol{F}^{IJ} - \mathscr{D}^J_\beta\boldsymbol{F}^I_{c\alpha} = -2\mathrm{i}\delta^{IJ}(\gamma^d)_{\alpha\beta}\boldsymbol{F}_{dc}. \tag{2.3.18d}$$

The first line shows that the totally symmetric part $\boldsymbol{F}^k_{(\alpha\beta\gamma)}$ vanishes and implies the relation

$$\mathscr{D}^I_\alpha\boldsymbol{F}^{JK} - \mathscr{D}^{[J}_\alpha\boldsymbol{F}^{K]I} = -\delta^{I[J}(\gamma^d)_\alpha{}^\gamma\boldsymbol{F}_{|d|\gamma}{}^{K]}. \tag{2.3.19}$$

The trace in IJ

$$\mathscr{D}^I_\alpha\boldsymbol{F}^{IK} = -\tfrac{1}{3}(\mathcal{N}-1)(\gamma^d)_\alpha{}^\gamma\boldsymbol{F}_{d\gamma}{}^K \equiv -(\mathcal{N}-1)\boldsymbol{F}^K_\alpha \tag{2.3.20}$$

produces the field strength of dimension three-halves $(\gamma^d)_\alpha{}^\gamma\boldsymbol{F}_{d\gamma}{}^K$. Inserting back, this leads in turn to the consistency relation for the dimension-one field strength

$$\mathscr{D}^I_\alpha\boldsymbol{F}^{JK} = \mathscr{D}^{[I}_\alpha\boldsymbol{F}^{JK]} - \tfrac{2}{\mathcal{N}-1}\delta^{I[J}\mathscr{D}_{\alpha L}\boldsymbol{F}^{K]L}. \tag{2.3.21}$$

The last line of (2.3.18) yields the dimension-two field strength $\boldsymbol{F}_{ab}$ as

$$(\gamma_{[a})^{\alpha\beta}\mathscr{D}^I_\alpha\boldsymbol{F}^I_{\beta b]} = -(\gamma_{ab})^{\alpha\beta}\mathscr{D}^I_\alpha\boldsymbol{F}^I_\beta = 2\mathrm{i}\mathcal{N}\boldsymbol{F}_{ab} \tag{2.3.22}$$

or

$$-(\gamma_{ab})^{\alpha\beta}\mathscr{D}^I_\alpha\mathscr{D}^J_\beta\boldsymbol{F}_{IJ} = 2\mathrm{i}\mathcal{N}(\mathcal{N}-1)\boldsymbol{F}_{ab}. \tag{2.3.23}$$

Summarising, the whole algebra of covariant derivatives can be written as [34]

$$\{\mathscr{D}^I_\alpha, \mathscr{D}^J_\beta\} = 2\mathrm{i}\delta^{IJ}\mathscr{D}_{\alpha\beta} + 2\mathrm{i}\varepsilon_{\alpha\beta}\boldsymbol{F}^{IJ}| \tag{2.3.24a}$$

$$[\mathscr{D}_a, \mathscr{D}^J_\beta] = \tfrac{-1}{\mathcal{N}-1}(\gamma_a)_\beta{}^\gamma\mathscr{D}^K_\gamma\boldsymbol{F}_{KJ}| \tag{2.3.24b}$$

$$[\mathscr{D}_a, \mathscr{D}_b] = \tfrac{\mathrm{i}}{2\mathcal{N}(\mathcal{N}-1)}(\gamma_{ab})^{\alpha\beta}\mathscr{D}^I_\alpha\mathscr{D}^J_\beta\boldsymbol{F}^{IJ}|, \tag{2.3.24c}$$

with the condition

$$\mathscr{D}^I_\alpha\boldsymbol{F}^{JK} = \mathscr{D}^{[I}_\alpha\boldsymbol{F}^{JK]} - \tfrac{2}{\mathcal{N}-1}\delta^{I[J}\mathscr{D}_{\alpha L}\boldsymbol{F}^{K]L}. \tag{2.3.25}$$

The superfield $\boldsymbol{F}^{IJ}$ is called field-strength or gauge multiplet, since it describes all field strengths appearing in the algebra in terms of covariant projections. As seen above, these differ from pure supercovariant projections by those projections of the spinor gauge field $\boldsymbol{B}^I_\alpha$, which can be removed by a gauge transformation (2.3.6).

2.4. Supergravity

2.4.1. Supergravity-covariant derivative

General superspace coordinate transformations

$$z_M \longrightarrow \tilde{z}_M(z) \tag{2.4.1}$$

induce the transformation of a supervector field

$$\boldsymbol{V}^M(z) \longrightarrow \tilde{\boldsymbol{V}}^M(\tilde{z}) = (\partial_N \tilde{z}^M)\boldsymbol{V}^N(z). \tag{2.4.2}$$

At each point, a supervector $\boldsymbol{V}^M$ can be expanded in a standard basis of the tangent space as

$$\boldsymbol{V}^M = \boldsymbol{V}^A \boldsymbol{E}_A{}^M. \tag{2.4.3}$$

The vector $\boldsymbol{V}^A$ transforms under the local structure group of the tangent space, which leaves this expansion invariant and thus relates equivalent bases to each other.

In the conventional curved SO($\mathcal{N}$) superspace, the local structure group is chosen to be $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(\mathcal{N})$ [36]. Consequently, the super-vielbein

$$\boldsymbol{E}_A{}^M = \begin{pmatrix} \boldsymbol{E}_a{}^m & \boldsymbol{E}_{a,J}{}^{\mu} \\ \boldsymbol{E}_{\alpha}^{I,m} & \boldsymbol{E}_{\alpha,J}^{I,\mu} \end{pmatrix} \tag{2.4.4}$$

carries local Lorentz indices a, α and local SO($\mathcal{N}$) indices I, transforming under this local structure group. As a basic principle, the leading component of $\boldsymbol{E}_a{}^m$ is identified with the vielbein of non-supersymmetric space-time,

$$\boldsymbol{E}_a{}^m| = e_a{}^m. \tag{2.4.5}$$

Further, in flat superspace the super-vielbein takes the form [35]

$$\boldsymbol{E}_A{}^B = \begin{pmatrix} \delta_a^b & 0 \\ \mathrm{i}\theta_I^{\gamma}(\gamma^b)_{\gamma\alpha} & \delta_\alpha^\beta \delta_J^I \end{pmatrix} \tag{2.4.6}$$

corresponding to the relation

$$D_A = \boldsymbol{E}_A{}^B \partial_B. \tag{2.4.7}$$

Accordingly, the supergravity covariant derivative is given by [34]

$$\mathscr{D}_A = \boldsymbol{E}_A{}^M \partial_M + \tfrac{1}{2}\boldsymbol{\Omega}_A{}^{mn}\mathscr{M}_{mn} + \tfrac{1}{2}\boldsymbol{\Phi}_A{}^{PQ}\mathscr{N}_{PQ}, \tag{2.4.8}$$

where $\boldsymbol{\Omega}_A$ and $\boldsymbol{\Phi}_A$ are the connection or gauge fields associated with the Lorentz group and SO($\mathcal{N}$), respectively.

2.4.2. Supergravity algebra

The algebra of covariant derivatives is defined by [34]

$$[\mathscr{D}_A, \mathscr{D}_B\} = \boldsymbol{T}_{AB}{}^C\mathscr{D}_C + \tfrac{1}{2}\boldsymbol{R}_{AB}{}^{PQ}\mathscr{N}_{PQ} + \tfrac{1}{2}\boldsymbol{R}_{AB}{}^{mn}\mathscr{M}_{mn}. \tag{2.4.9}$$

The torsion and field strengths are, in terms of the super-vielbein and connections, given by

$$\begin{aligned}\boldsymbol{T}_{AB}{}^C =& \boldsymbol{C}_{AB}{}^C + \tfrac{1}{2}\boldsymbol{\Omega}_A{}^{mn}(\mathscr{M}_{mn})_B{}^C - (-)^{AB}\tfrac{1}{2}\boldsymbol{\Omega}_B{}^{mn}(\mathscr{M}_{mn})_A{}^C \\ &+ \tfrac{1}{2}\boldsymbol{\Phi}_A{}^{PQ}\mathscr{N}_{PQ}\delta_B{}^C - (-)^{AB}\tfrac{1}{2}\boldsymbol{\Phi}_B{}^{PQ}\mathscr{N}_{PQ}\delta_A{}^C \end{aligned} \tag{2.4.10a}$$

$$\begin{aligned}\boldsymbol{R}_{AB} =& \boldsymbol{E}_A\boldsymbol{\Omega}_B - (-)^{AB}\boldsymbol{E}_B\boldsymbol{\Omega}_A - \boldsymbol{C}_{AB}{}^C\boldsymbol{\Omega}_C \\ &+ \boldsymbol{E}_A\boldsymbol{\Phi}_B - (-)^{AB}\boldsymbol{E}_B\boldsymbol{\Phi}_A - \boldsymbol{C}_{AB}{}^C\boldsymbol{\Phi}_C \\ &+ [\boldsymbol{\Omega}_A + \boldsymbol{\Phi}_A, \boldsymbol{\Omega}_B + \boldsymbol{\Phi}_B\}, \end{aligned} \tag{2.4.10b}$$

where the superfields $\boldsymbol{C}_{AB}{}^C$ are defined by

$$[\boldsymbol{E}_A, \boldsymbol{E}_B\} = \left[(\boldsymbol{E}_A\boldsymbol{E}_B{}^M)\boldsymbol{E}_M{}^C - (-1)^{AB}(\boldsymbol{E}_B\boldsymbol{E}_A{}^M)\boldsymbol{E}_M{}^C\right]\boldsymbol{E}_C = \boldsymbol{C}_{AB}{}^C\boldsymbol{E}_C. \tag{2.4.11}$$

Via the Jacobi identity

$$0 = [\boldsymbol{E}_{[A}, [\boldsymbol{E}_B, \boldsymbol{E}_{C)}\}\}, \tag{2.4.12}$$

they are related by

$$\boldsymbol{E}_{[A}\boldsymbol{C}_{BC)}{}^E - \boldsymbol{C}_{[BC}{}^D\boldsymbol{C}_{A)D}{}^E = 0, \tag{2.4.13}$$

where it is as usual understood that SO($\mathcal{N}$) indices are permuted together with their Lorentz spinor index.

The content generated by the fields introduced above can be conventionally reduced by imposing constraints on the torsions [36]. At least in four-dimensional simple supergravity, this procedure (together with chirality-conserving constraints) is known to completely determine the field strengths in terms of these torsions via the super-Jacobi identity. Also in three dimensions, it is sufficient to impose constraints only on the torsions; however, the description of the field strengths needs one additional field emerging in the solution of the super-Jacobi identity, as will be seen below. The torsion constraints are equivalent to choosing connections in terms of vielbeins in shape of the fields ${\boldsymbol{C}_{AB}}^C$ in order to reduce degrees of freedom as much as possible. The specific choices and resulting dependencies are motivated in the following.

Concretely, for the case of two spinor derivatives the torsions are

$$\begin{aligned}\boldsymbol{T}^{IJ\gamma}_{\alpha\beta K}\mathscr{D}^K_\gamma + \boldsymbol{T}^{IJc}_{\alpha\beta}\mathscr{D}_c =& \tfrac{1}{2}[-{\boldsymbol{\Omega}^I_{\alpha,\beta}}^\gamma\delta^J_K - {\boldsymbol{\Omega}^J_{\beta,\alpha}}^\gamma\delta^I_K + \boldsymbol{\Phi}^{I,JK}_\alpha\delta^\gamma_\beta + \boldsymbol{\Phi}^{J,IK}_\beta\delta^\gamma_\alpha]\mathscr{D}^K_\gamma \\ &+ \boldsymbol{C}^{IJ\gamma}_{\alpha\beta K}\mathscr{D}^K_\gamma + \boldsymbol{C}^{IJc}_{\alpha\beta}\mathscr{D}_c. \end{aligned} \tag{2.4.14a}$$

The spinor connections can be chosen to absorb the fields $\boldsymbol{C}^{IJ\gamma}_{\alpha\beta K}$, leaving

$$\boldsymbol{T}^{IJ\gamma}_{\alpha\beta K} = 0 \tag{2.4.15a}$$

$$\boldsymbol{T}^{IJc}_{\alpha\beta} = \boldsymbol{C}^{IJc}_{\alpha\beta}. \tag{2.4.15b}$$

In order to incorporate the case of flat superspace, it is natural to choose

$$\boldsymbol{T}^{IJc}_{\alpha\beta} = \boldsymbol{C}^{IJc}_{\alpha\beta} = 2\mathrm{i}\delta^{IJ}(\gamma^c)_{\alpha\beta}. \tag{2.4.16}$$

In consequence, the vector vielbein is expressed by the spinor vielbeins and spinor connections by virtue of the vielbein algebra

$$\{\boldsymbol{E}^I_\alpha, \boldsymbol{E}^J_\beta\} = 2\mathrm{i}\delta^{IJ}(\gamma^c)_{\alpha\beta}\boldsymbol{E}_c + \boldsymbol{C}^{IJ\gamma}_{\alpha\beta K}\boldsymbol{E}^K_\gamma \tag{2.4.17}$$

with the replacement

$$\boldsymbol{C}^{IJ\gamma}_{\alpha\beta K} = -\tfrac{1}{2}[-{\boldsymbol{\Omega}^I_{\alpha,\beta}}^\gamma\delta^{JK} - {\boldsymbol{\Omega}^J_{\beta,\alpha}}^\gamma\delta^{IK} + \boldsymbol{\Phi}^{I,JK}_\alpha\delta^\gamma_\beta + \boldsymbol{\Phi}^{J,IK}_\beta\delta^\gamma_\alpha]. \tag{2.4.18}$$

The spinor vielbeins, the spinor connections and the vector connections remain as independent objects.

For a vector and a spinor derivative, the torsions are

$$\begin{aligned}\boldsymbol{T}_{a\beta}{}^{c}\mathscr{D}_{c}+\boldsymbol{T}_{a\beta}{}^{\gamma}\mathscr{D}_{\gamma}^{K}=&-\boldsymbol{\Omega}^{J}_{\beta,a}{}^{c}\mathscr{D}_{c}-\tfrac{1}{2}\boldsymbol{\Omega}_{a,\beta}{}^{\gamma}\delta^{JK}\mathscr{D}_{\gamma}^{K}+\boldsymbol{\Phi}_{a}{}^{JK}\delta_{\beta}{}^{\gamma}\mathscr{D}_{\gamma}^{K}\\&+\boldsymbol{C}_{a\beta}{}^{c}\mathscr{D}_{c}+\boldsymbol{C}_{a\beta}{}^{\gamma}\mathscr{D}_{\gamma}^{K}.\end{aligned}\tag{2.4.19a}$$

The constraint

$$\boldsymbol{T}_{a\beta}{}^{c}=0\tag{2.4.20}$$

can be imposed to set

$$\boldsymbol{C}_{a\beta}{}^{c}=\boldsymbol{\Omega}^{J}_{\beta,a}{}^{c}.\tag{2.4.21}$$

This constraint not only relates the spinor Lorentz connections to vielbeins, but also spinor Lorentz and SO($\mathcal{N}$) connections among themselves, due to the Jacobi identity

$$\delta^{JK}(\gamma^{d})_{(\beta\gamma}\boldsymbol{\Omega}^{I}_{\alpha)d}{}^{e}-\boldsymbol{C}^{JK\delta}_{(\beta\gamma I}(\gamma^{e})_{\alpha)\delta}=0.\tag{2.4.22}$$

In order to choose the vector SO($\mathcal{N}$) connections, the constraint

$$\boldsymbol{T}_{a[\beta,\gamma]}^{JK}=0\tag{2.4.23}$$

can be imposed (with $\boldsymbol{T}_{a[\beta,\gamma]}^{[JK]}=0$ or $\boldsymbol{T}_{a\beta,\gamma}^{[JK]}$ being alternative versions), leading to

$$\boldsymbol{C}_{a[\beta,\gamma]}^{J,K}=-\boldsymbol{\Phi}_{a}{}^{JK}\varepsilon_{\beta\gamma}.\tag{2.4.24}$$

The vector Lorentz connection has not been chosen until this point in order to impose the usual constraint on the torsion known from non-supersymmetric gravity. The case of two vector derivatives

$$\boldsymbol{T}_{ab}{}^{c}\mathscr{D}_{c}+\boldsymbol{T}_{ab,K}^{\gamma}\mathscr{D}_{\gamma}^{K}=\boldsymbol{C}_{ab}{}^{c}\mathscr{D}_{c}+\boldsymbol{C}_{ab,K}^{\gamma}\mathscr{D}_{\gamma}^{K}+\boldsymbol{\Omega}_{ab}{}^{c}\mathscr{D}_{c}-\boldsymbol{\Omega}_{ba}{}^{c}\mathscr{D}_{c}\tag{2.4.25a}$$

suggests

$$\boldsymbol{T}_{ab}{}^{c}=0\tag{2.4.26a}$$

$$\boldsymbol{T}_{ab}{}^{\gamma}=\boldsymbol{C}_{ab}{}^{\gamma}\tag{2.4.26b}$$

or equivalently

$$\boldsymbol{C}_{ab}{}^{c}=-2\boldsymbol{\Omega}_{[ab]}{}^{c},\tag{2.4.27}$$

which has a well-known solution for $\boldsymbol{\Omega}_{ab}{}^{c}$ in terms of $\boldsymbol{C}_{ab}{}^{c}$.

Further dependencies are established by the algebra of vielbeins,

$$\{\boldsymbol{E}^I_\alpha, \boldsymbol{E}^J_\beta\} = 2\mathrm{i}\delta^{IJ}(\gamma^c)_{\alpha\beta}\boldsymbol{E}_c + \boldsymbol{C}^{IJ\gamma}_{\alpha\beta K}(\boldsymbol{\Omega}^L_\delta, \boldsymbol{\Phi}^L_\delta)\boldsymbol{E}^K_\gamma, \tag{2.4.28a}$$

$$[\boldsymbol{E}_a, \boldsymbol{E}^J_\beta] = \boldsymbol{\Omega}^J_{\beta,a}{}^c\boldsymbol{E}_c + \boldsymbol{C}_{a\beta}{}^\gamma\boldsymbol{E}^K_\gamma \tag{2.4.28b}$$

$$[\boldsymbol{E}_a, \boldsymbol{E}_b] = -2\boldsymbol{\Omega}_{[ab]}{}^c\boldsymbol{E}_c + \boldsymbol{C}_{ab}{}^\gamma\boldsymbol{E}^K_\gamma \tag{2.4.28c}$$

and by the corresponding Jacobi identity, which contains differential relations between torsions, connections and their superfield components.

2.4.3. Solution to the constrained Jacobi identity

The super-Jacobi identity for the algebra of covariant derivatives

$$0 = [\mathscr{D}_{[A}, [\mathscr{D}_B, \mathscr{D}_{C)}\}\} \tag{2.4.29}$$

can be written as

$$\begin{aligned}0 = &\left(\mathscr{D}_{[A}\boldsymbol{T}_{BC)}{}^D\right)\mathscr{D}_D - \left(\boldsymbol{T}_{[BC}{}^D\boldsymbol{T}_{|D|A)}{}^E\mathscr{D}_E + \tfrac{1}{2}\boldsymbol{T}_{[BC}{}^D\boldsymbol{R}_{|D|A)}{}^{PQ}\mathscr{N}_{PQ} + \tfrac{1}{2}\boldsymbol{T}_{[BC}{}^D\boldsymbol{R}_{|D|A)}{}^{mn}\mathscr{M}_{mn}\right)\\ &+ \tfrac{1}{2}\left(\mathscr{D}_{[A}\boldsymbol{R}_{BC)}{}^{PQ}\right)\mathscr{N}_{PQ} - \tfrac{1}{2}\boldsymbol{R}_{[BC}{}^{PQ}[\mathscr{N}_{PQ}, \mathscr{D}_{A)}]\\ &+ \tfrac{1}{2}\left(\mathscr{D}_{[A}\boldsymbol{R}_{BC)}{}^{mn}\right)\mathscr{M}_{mn} - \tfrac{1}{2}\boldsymbol{R}_{[BC}{}^{mn}[\mathscr{M}_{mn}, \mathscr{D}_{A)}],\end{aligned} \tag{2.4.30}$$

which is equivalent to

$$\begin{aligned}0 = &(\mathscr{D}_{[A}\boldsymbol{T}_{BC)}{}^E)\mathscr{D}_E\\ &- \left(\boldsymbol{T}_{[BC}{}^D\boldsymbol{T}_{|D|A)}{}^E\mathscr{D}_E + \tfrac{1}{2}\boldsymbol{R}_{[BC}{}^{mn}[\mathscr{M}_{mn}, \mathscr{D}_{A)}] + \tfrac{1}{2}\boldsymbol{R}_{[BC}{}^{PQ}[\mathscr{N}_{PQ}, \mathscr{D}_{A)}]\right)\end{aligned} \tag{2.4.31a}$$

$$0 = \mathscr{D}_{[A}\boldsymbol{R}_{BC)}{}^{PQ} - \boldsymbol{T}_{[BC}{}^D\boldsymbol{R}_{|D|A)}{}^{PQ} \tag{2.4.31b}$$

$$0 = \mathscr{D}_{[A}\boldsymbol{R}_{BC)}{}^{mn} - \boldsymbol{T}_{[BC}{}^D\boldsymbol{R}_{|D|A)}{}^{mn}. \tag{2.4.31c}$$

As motivated above, the non-vanishing torsions are subject to the conventional constraints [36]

$$\boldsymbol{T}_{\alpha\beta}^{IJ,c} = 2\mathrm{i}\delta^{IJ}(\gamma^c)_{\alpha\beta} \tag{2.4.32a}$$

$$\boldsymbol{T}_{a\beta,K}^{\;J,\gamma} = (\gamma_a)_\beta{}^\gamma \boldsymbol{K}^{JK} + (\gamma^b)_\beta{}^\gamma \boldsymbol{L}_{ab}{}^{JK} \tag{2.4.32b}$$

$$\boldsymbol{T}_{ab,K}^{\;\;\gamma} \equiv \boldsymbol{\Psi}_{ab,K}^{\;\;\gamma}. \tag{2.4.32c}$$

The special parametrisation of $\boldsymbol{T}_{a\beta,K}^{\;J,\gamma}$ (with $\boldsymbol{T}_{a\beta,K}^{\;J,\beta} = 0$) in terms of the superfields $\boldsymbol{K}^{IJ}$ and $\boldsymbol{L}_{ab}{}^{IJ}$ is conventionally sufficient, but not unique. Since it determines essentially the dimension-one curvatures, the choice of $\boldsymbol{T}_{a\beta,K}^{\;J,\gamma}$ can interact with further conventional redefinitions of these curvatures [37].

In the following, the super-Jacobi identity will be solved under these constraints. The various field strengths are determined and commented.

Dimension-one Lorentz curvature The case $a\beta\gamma$ yields terms proportional to the vector derivative,

$$0 = 2\mathrm{i}(\boldsymbol{T}_{a\beta}^{\delta}(\gamma^d)_{\gamma\delta} + \boldsymbol{T}_{a\gamma}^{\delta}(\gamma^d)_{\beta\delta}) + \boldsymbol{R}_{\beta\gamma}^{JKmd}\eta_{am}, \tag{2.4.33}$$

determining the dimension-one Lorentz curvature [34]

$$\boldsymbol{R}_{\beta\gamma}^{JKad} = -4\mathrm{i}(\gamma^{ad})_{\beta\gamma}\boldsymbol{K}^{JK} - 4\mathrm{i}\varepsilon_{\beta\gamma}\boldsymbol{L}^{adJK}. \tag{2.4.34}$$

Dimension-one $\mathrm{SO}(\mathcal{N})$ curvature The case $\alpha\beta\gamma$ yields only terms proportional to the spinor derivative,

$$0 = \boldsymbol{T}_{(\alpha\beta}^{IJd}\boldsymbol{T}^{\varepsilon}_{|d|\gamma)}\mathscr{D}_\varepsilon + \tfrac{1}{2}\boldsymbol{R}_{(\alpha\beta}^{IJmn}[\mathscr{M}_{mn}, \mathscr{D}^K_{\gamma)}] + \tfrac{1}{2}\boldsymbol{R}_{(\alpha\beta}^{IJ,PQ}[\mathscr{N}_{PQ}, \mathscr{D}^K_{\gamma)}]. \tag{2.4.35}$$

This is equivalent to

$$0 = 2\mathrm{i}\delta^{IJ}(\gamma^d)_{(\alpha\beta}\boldsymbol{T}^{\delta}_{|d|\gamma)} + \tfrac{1}{4}\boldsymbol{R}_{(\alpha\beta}^{IJcd}(\gamma_{cd})_{\gamma)}{}^{\delta}\delta^{KL} + \boldsymbol{R}_{(\alpha\beta}^{IJ,KL}\delta^{\delta}_{\gamma)}. \tag{2.4.36}$$

Inserting the torsion and the dimension-one Lorentz curvature derived above, it follows

$$\begin{aligned}-\boldsymbol{R}_{(\alpha\beta}^{IJ,KL}\delta_{\gamma)}^{\delta} =&2\mathrm{i}(\gamma^d)_{(\alpha\beta}(\gamma_d)_{\gamma)}{}^{\delta}(\delta^{IJ}\boldsymbol{K}^{KL}+\delta^{KL}\boldsymbol{K}^{IJ})\\ &+2\mathrm{i}\delta^{KL}\varepsilon_{(\alpha\beta}(\gamma_d)_{\gamma)}{}^{\delta}\boldsymbol{L}^{dIJ}+2\mathrm{i}\delta^{IJ}(\gamma^e)_{(\alpha\beta}(\gamma^f)_{\gamma)}{}^{\delta}\varepsilon_{efd}\boldsymbol{L}^{dKL},\end{aligned} \tag{2.4.37}$$

which can be identically written as

$$\begin{aligned}-\boldsymbol{R}_{(\alpha\beta}^{IJ,KL}\delta_{\gamma)}^{\delta} =&2\mathrm{i}\delta_{(\gamma}^{\delta}\varepsilon_{\alpha\beta)}(\delta^{KJ}\boldsymbol{K}^{IL}+\delta^{IL}\boldsymbol{K}^{KJ})-2\mathrm{i}\delta_{(\gamma}^{\delta}\varepsilon_{\alpha\beta)}(\delta^{IK}\boldsymbol{K}^{JL}+\delta^{JL}\boldsymbol{K}^{IK})\\ &-4\mathrm{i}\delta_{(\gamma}^{\delta}(\gamma_d)_{\beta\alpha)}\delta^{JL}\boldsymbol{L}^{dIK}\\ &+\mathrm{i}[-2\delta_{(\gamma}^{\delta}(\gamma_d)_{\beta\alpha)}\delta^{IJ}\boldsymbol{L}^{dKL}+4\delta_{(\gamma}^{\delta}(\gamma_d)_{\beta\alpha)}\delta^{KJ}\boldsymbol{L}^{dIL}].\end{aligned} \tag{2.4.38}$$

The dimension-one SO($\mathcal{N}$) curvature is thus [34]

$$\boldsymbol{R}_{\alpha\beta}^{IJ,KL} =2\mathrm{i}\varepsilon_{\alpha\beta}\boldsymbol{W}^{IJKL}+8\mathrm{i}\varepsilon_{\alpha\beta}\delta^{[K[I}\boldsymbol{K}^{J]L]}+8\mathrm{i}(\gamma^d)_{\beta\alpha}\delta^{(I[K}\boldsymbol{L}_d^{L]J)}+2\mathrm{i}(\gamma_d)_{\beta\alpha}\delta^{IJ}\boldsymbol{L}^{dKL}, \tag{2.4.39}$$

where the introduction of the totally antisymmetric tensor field $\boldsymbol{W}^{IJKL}$ is allowed by the Jacobi identity, because

$$\varepsilon_{\alpha\beta}\boldsymbol{W}^{IJKL}\delta_{\gamma}^{\delta}+\varepsilon_{\gamma\alpha}\boldsymbol{W}^{KIJL}\delta_{\beta}^{\delta}+\varepsilon_{\beta\gamma}\boldsymbol{W}^{JKIL}\delta_{\alpha}^{\delta}=\boldsymbol{W}^{IJKL}(2\varepsilon_{\alpha[\beta}\delta_{\gamma]}^{\delta}+\varepsilon_{\beta\gamma}\delta_{\alpha}^{\delta})=0.$$

Dimension-two curvatures The case αbc contains terms proportional to the spinor derivative,

$$0=(\mathscr{D}_{[\alpha}\boldsymbol{T}_{bc]}{}^{\varepsilon})\mathscr{D}_{\varepsilon}-\tfrac{2}{3}\boldsymbol{T}_{\alpha[b}{}^{\delta}\boldsymbol{T}_{|\delta|c]}{}^{\varepsilon}\mathscr{D}_{\varepsilon}-\tfrac{1}{6}\boldsymbol{R}_{bc}{}^{mn}[\mathscr{M}_{mn},\mathscr{D}_{\alpha}]-\tfrac{1}{6}\boldsymbol{R}_{bc}{}^{PQ}[\mathscr{N}_{PQ},\mathscr{D}_{\alpha}], \tag{2.4.40}$$

which are equivalent to

$$0=\mathscr{D}_{[\alpha}\boldsymbol{T}_{bc]}{}^{\varepsilon}-\tfrac{2}{3}\boldsymbol{T}_{\alpha[b}{}^{\delta}\boldsymbol{T}_{|\delta|c]}{}^{\varepsilon}-\tfrac{1}{6}\tfrac{1}{2}\boldsymbol{R}_{bc}{}^{mn}(\gamma_{mn})_{\alpha}{}^{\varepsilon}\delta^{IL}-\tfrac{2}{6}\boldsymbol{R}_{bc}{}^{IL}\delta_{\alpha}^{\varepsilon}. \tag{2.4.41}$$

This equation determines both the Lorentz curvature and the SO($\mathcal{N}$) curvature. The former follows from

$$\begin{aligned}\tfrac{\mathcal{N}}{12}\boldsymbol{R}_{bc}{}^{mn}(\gamma_{mn})_{\alpha}{}^{\varepsilon} =&\mathscr{D}_{[\alpha}\boldsymbol{T}_{bc]}{}^{\varepsilon}-\tfrac{2}{3}\boldsymbol{T}_{\alpha[b}{}^{\delta}\boldsymbol{T}_{|\delta|c]}{}^{\varepsilon}\\ =&\mathscr{D}_{[\alpha}\boldsymbol{T}_{bc]}{}^{\varepsilon}-\tfrac{2}{3}\big((\gamma_{bc})_{\alpha}{}^{\varepsilon}\boldsymbol{K}^{IK}\boldsymbol{K}^{KI}-(\gamma^{mn})_{\alpha}{}^{\varepsilon}\boldsymbol{L}_{m[b}^{IK}\boldsymbol{L}_{c]n}^{KI}\big)\end{aligned} \tag{2.4.42}$$

and reads

$$\boldsymbol{R}_{bc}{}^{mn} = -\tfrac{3}{\mathcal{N}}(\gamma^{mn})_{\varepsilon}{}^{\alpha}\mathscr{D}_{[\alpha}\boldsymbol{T}_{bc]}{}^{\varepsilon} - \tfrac{8}{\mathcal{N}}(\delta^{m}_{[b}\delta^{n}_{c]}\boldsymbol{K}^{IK}\boldsymbol{K}^{KI} - \boldsymbol{L}^{IK}_{m[b}\boldsymbol{L}^{KI}_{c]n}), \tag{2.4.43}$$

while the latter is given by

$$\boldsymbol{R}_{bc}{}^{IL} = \tfrac{3}{2}\mathscr{D}_{[\alpha}\boldsymbol{T}_{bc]}{}^{\alpha} - \boldsymbol{T}_{\alpha[b}{}^{\delta}\boldsymbol{T}_{|\delta|c]}{}^{\alpha} = \tfrac{3}{2}\mathscr{D}_{[\alpha}\boldsymbol{T}_{bc]}{}^{\alpha} + 2\boldsymbol{L}^{K[I}_{d[b}\boldsymbol{L}^{L]K}_{c]d}. \tag{2.4.44}$$

Dimension-three-halves curvatures The curvatures of dimension three-halves are determined by the differential identities

$$0 = \mathscr{D}^{I}_{(\alpha}\boldsymbol{R}^{JK,PQ}_{\beta\gamma)} - 2\mathrm{i}\delta^{JK}(\gamma^{d})_{(\beta\gamma}\boldsymbol{R}_{|d|\alpha)}{}^{I,PQ} \tag{2.4.45a}$$

$$0 = \mathscr{D}^{I}_{(\alpha}\boldsymbol{R}^{JK,mn}_{\beta\gamma)} - 2\mathrm{i}\delta^{JK}(\gamma^{d})_{(\beta\gamma}\boldsymbol{R}_{|d|\alpha)}{}^{I,mn} \tag{2.4.45b}$$

in terms of spinor derivatives of $\boldsymbol{K}^{IJ}$, $\boldsymbol{L}^{IJ}_{a}$ and $\boldsymbol{W}^{IJKL}$. For the second identity, an important consistency relation is obtained by projecting the Jacobi identity on the totally antisymmetric part of the dimension-one curvature, leading to

$$\varepsilon_{\beta\gamma}\mathscr{D}^{I}_{\alpha}\boldsymbol{W}^{JKPQ} - \varepsilon_{\beta\gamma}\mathscr{D}^{[J}_{\alpha}\boldsymbol{W}^{KPQ]I} = 2\delta^{I[J}\boldsymbol{R}_{\alpha[\beta,\gamma]}{}^{K,PQ]}. \tag{2.4.46}$$

The trace in IJ determines the part of the curvature

$$\varepsilon_{\beta\gamma}\mathscr{D}^{I}_{\alpha}\boldsymbol{W}^{IKPQ} = \tfrac{2}{5}(\mathcal{N}-3)\boldsymbol{R}_{\alpha[\beta,\gamma]}{}^{[K,PQ]}, \tag{2.4.47}$$

which in turn yields

$$\mathscr{D}^{I}_{\alpha}\boldsymbol{W}^{JKPQ} = \mathscr{D}^{[I}_{\alpha}\boldsymbol{W}^{JKPQ]} - \tfrac{4}{\mathcal{N}-3}\delta^{I[J}\mathscr{D}_{\alpha L}\boldsymbol{W}^{KPQ]L}. \tag{2.4.48}$$

Remaining relations The remaining differential identities can be regarded as consistency relations with respect to the preceding results [34]. The other remaining identities contain terms proportional to the spinor derivative from the case $a\beta\gamma$, terms proportional to the vector derivative from the case αbc, and the case abc. They relate field strengths of dimension three-halves and two to $\boldsymbol{\Psi}^{\gamma}_{ab}$ and spinor derivatives of $\boldsymbol{K}^{IJ}$ and $\boldsymbol{L}^{IJ}_{a}$. This in turn is a consistency condition with respect to the differential identities which have been used above to define the field strengths of dimension three-halves.

2.4.4. Super-Weyl invariance

The supergravity algebra constructed above describes $\mathcal{N}$-extended conformal supergravity, because it is invariant under super-Weyl transformations [34, 38]. More precisely, the torsion fields $\boldsymbol{K}^{IJ}$ and $\boldsymbol{L}_a^{IJ}$ as well as the superfield $\boldsymbol{W}^{IJKL}$ can be endowed with appropriate transformation properties in order to leave the algebra invariant. They will be derived in the following, by demanding the invariance of the anticommutator of two spinor derivatives.

The infinitesimal super-Weyl transformations of the covariant derivatives can be postulated in the familiar form [34]

$$\delta_{\boldsymbol{\sigma}}\mathscr{D}_\alpha^I = \tfrac{1}{2}\boldsymbol{\sigma}\mathscr{D}_\alpha^I - \check{\boldsymbol{\sigma}}_I^{\beta}\mathscr{M}_{\alpha\beta} - \check{\boldsymbol{\sigma}}_\alpha^J\mathscr{N}^{IJ} \tag{2.4.49a}$$

$$\delta_{\boldsymbol{\sigma}}\mathscr{D}_{\alpha\beta} = \boldsymbol{\sigma}\mathscr{D}_{\alpha\beta} - \mathrm{i}\check{\boldsymbol{\sigma}}^K_{(\alpha}\mathscr{D}^K_{\beta)} - \check{\boldsymbol{\sigma}}_{(\alpha}{}^{\gamma}\mathscr{M}_{\beta)\gamma} - \tfrac{\mathrm{i}}{4}\check{\boldsymbol{\sigma}}^{K,L}_{(\alpha,\beta)}\mathscr{N}_{KL}, \tag{2.4.49b}$$

where $\check{\boldsymbol{\sigma}}_I^\alpha = \mathscr{D}_\alpha^I\boldsymbol{\sigma}$. The anticommutator of covariant spinor derivatives changes by the amount

$$\{\delta_{\boldsymbol{\sigma}}\mathscr{D}_\alpha^I, \mathscr{D}_\beta^J\} + \{\mathscr{D}_\alpha^I, \delta_{\boldsymbol{\sigma}}\mathscr{D}_\beta^J\} = 2\mathrm{i}\delta^{IJ}\delta_{\boldsymbol{\sigma}}\mathscr{D}_{\alpha\beta} + \delta_{\boldsymbol{\sigma}}\boldsymbol{R}^{IJ}_{\alpha\beta}. \tag{2.4.50}$$

Inserting the above expressions leads to

$$\begin{aligned}(\delta_{\boldsymbol{\sigma}} - \boldsymbol{\sigma})\boldsymbol{R}^{IJ}_{\alpha\beta} = &- \check{\boldsymbol{\sigma}}^{I\gamma}_{\alpha J}\mathscr{M}_{\gamma\beta} - \check{\boldsymbol{\sigma}}^{I,K}_{\alpha,\beta}\mathscr{N}^{KJ} - \check{\boldsymbol{\sigma}}^{J\gamma}_{\beta I}\mathscr{M}_{\gamma\alpha} - \check{\boldsymbol{\sigma}}^{J,K}_{\beta,\alpha}\mathscr{N}^{KI} \\ &- 2\mathrm{i}\delta^{IJ}\left(-\check{\boldsymbol{\sigma}}_{(\alpha}{}^{\gamma}\mathscr{M}_{\beta)\gamma} - \tfrac{\mathrm{i}}{4}\check{\boldsymbol{\sigma}}^{K,L}_{(\alpha,\beta)}\mathscr{N}_{KL}\right).\end{aligned} \tag{2.4.51}$$

For the Lorentz field strength this gives

$$\begin{aligned}4\mathrm{i}(\delta_{\boldsymbol{\sigma}} - \boldsymbol{\sigma})(\boldsymbol{K}^{IJ}\mathscr{M}_{\alpha\beta} + \varepsilon_{\alpha\beta}\boldsymbol{L}^{IJ}_{\gamma\delta}\mathscr{M}^{\gamma\delta}) &= - \check{\boldsymbol{\sigma}}^{I\gamma}_{\alpha J}\mathscr{M}_{\gamma\beta} - \check{\boldsymbol{\sigma}}^{J\gamma}_{\beta I}\mathscr{M}_{\gamma\alpha} + 2\mathrm{i}\delta^{IJ}\check{\boldsymbol{\sigma}}_{(\alpha}{}^{\gamma}\mathscr{M}_{\beta)\gamma} \\ &= 2(\mathscr{D}^{\delta(I}\mathscr{D}^{J)}_{\delta}\boldsymbol{\sigma})\mathscr{M}_{\alpha\beta} + \varepsilon_{\alpha\beta}(\mathscr{D}^{\gamma}_{[I}\mathscr{D}^{\delta}_{J]}\boldsymbol{\sigma})\mathscr{M}_{\gamma\delta},\end{aligned} \tag{2.4.52}$$

while the SO($\mathcal{N}$) field strength yields the equality of

$$(\delta_{\boldsymbol{\sigma}} - \boldsymbol{\sigma})\mathrm{i}\left(\varepsilon_{\alpha\beta}\boldsymbol{W}^{IJKL} + 4\varepsilon_{\alpha\beta}\delta^{[K[I}\boldsymbol{K}^{J]L]} + 4\delta^{(I[K}\boldsymbol{L}^{L]J)}_{\alpha\beta} + \delta^{IJ}\boldsymbol{L}^{KL}_{\alpha\beta}\right)\mathscr{N}_{KL} \tag{2.4.53}$$

and

$$\left(2\delta^{K(I}(\mathscr{D}^{J)}_{(\alpha}\mathscr{D}^{L}_{\beta)}\boldsymbol{\sigma}) + \varepsilon_{\alpha\beta}\delta^{K[I}(\mathscr{D}^{J]\gamma}\mathscr{D}^{L}_{\gamma}\boldsymbol{\sigma}) - \tfrac{1}{2}\delta^{IJ}(\mathscr{D}^{K}_{(\alpha}\mathscr{D}^{L}_{\beta)}\boldsymbol{\sigma})\right)\mathscr{N}_{KL}. \tag{2.4.54}$$

The required transformations can then be read off to be [34]

$$\delta_{\boldsymbol{\sigma}}\boldsymbol{W}^{IJKL} = \boldsymbol{\sigma}\boldsymbol{W}^{IJKL} \tag{2.4.55a}$$
$$\delta_{\boldsymbol{\sigma}}\boldsymbol{K}^{IJ} = \boldsymbol{\sigma}\boldsymbol{K}^{IJ} - \tfrac{\mathrm{i}}{4}(\mathscr{D}^{\gamma(I}\mathscr{D}^{J)}_{\gamma}\boldsymbol{\sigma}) \tag{2.4.55b}$$
$$\delta_{\boldsymbol{\sigma}}\boldsymbol{L}^{IJ}_{\alpha\beta} = \boldsymbol{\sigma}\boldsymbol{L}^{IJ}_{\alpha\beta} + \tfrac{\mathrm{i}}{2}(\mathscr{D}^{[I}_{(\alpha}\mathscr{D}^{J]}_{\beta)}\boldsymbol{\sigma}). \tag{2.4.55c}$$

The torsion superfields $\boldsymbol{L}^{IJ}_{\alpha\beta}$ and $\boldsymbol{K}^{IJ}$ transform inhomogeneously. This means that they can be shifted to zero or any other desired value, if a corresponding Weyl gauge is imposed. The superfield $\boldsymbol{W}^{IJKL}$ transforms homogeneously. Due to this property, and because it is supposed to describe conformal supergravity, it is called the super-Cotton tensor. This designation is further justified by its role in conformal superspace [37], which will be reviewed below.

2.4.5. Anti-de Sitter superspace

Anti-de Sitter superspace is defined as a background where the non-Lorentz-scalar part of the torsion $\boldsymbol{L}^{IJ}_{a}$ and covariant derivatives of the Lorentz-scalar part $\boldsymbol{K}^{IJ}$ and of $\boldsymbol{W}^{IJKL}$ vanish. In this case, the algebra of covariant derivatives reads [38]

$$\{\mathscr{D}^{I}_{\alpha}, \mathscr{D}^{J}_{\beta}\} = 2\mathrm{i}\delta^{IJ}\mathscr{D}_{\alpha\beta} + 4\mathrm{i}\boldsymbol{K}^{IJ}\mathscr{M}_{\alpha\beta} + \mathrm{i}\varepsilon_{\alpha\beta}(\boldsymbol{W}^{IJKL} + 4\delta^{K[I}\boldsymbol{K}^{J]L})\mathscr{N}_{KL} \tag{2.4.56a}$$
$$[\mathscr{D}_{a}, \mathscr{D}^{J}_{\beta}] = (\gamma_a)_{\beta}{}^{\gamma}\boldsymbol{K}^{JK}\mathscr{D}^{K}_{\gamma} \tag{2.4.56b}$$
$$[\mathscr{D}_{a}, \mathscr{D}_{b}] = -\tfrac{4}{\mathcal{N}}\boldsymbol{K}^{IJ}\boldsymbol{K}_{IJ}\mathscr{M}_{ab}. \tag{2.4.56c}$$

The fields $\boldsymbol{K}^{IJ}$ and $\boldsymbol{W}^{IJKL}$ are subject to additional algebraic relations. The differential Jacobi identities, which are now algebraic ones, contain the condition

$$\begin{aligned} 0 &= \tfrac{1}{3}\boldsymbol{T}_{\beta\gamma}{}^{d}\boldsymbol{R}_{|d|a}{}^{mn} + \tfrac{2}{3}\boldsymbol{T}_{a(\beta}{}^{\delta}\boldsymbol{R}_{|\delta|\gamma)}{}^{mn} \\ &= \mathrm{i}\tfrac{8}{\mathcal{N}}\delta^{JK}(\gamma^{d})_{\beta\gamma}\delta^{m}_{[d}\delta^{n}_{a]}\boldsymbol{K}^{IL}\boldsymbol{K}_{IL} - 4\mathrm{i}(\gamma_a)_{(\beta}{}^{\delta}(\gamma^{mn})_{\gamma)\delta}\boldsymbol{K}^{JL}\boldsymbol{K}^{LK}, \end{aligned} \tag{2.4.57}$$

which is equivalent to [38]

$$\boldsymbol{K}^{IK}\boldsymbol{K}^{KJ} = \tfrac{1}{\mathcal{N}}\delta^{IJ}\boldsymbol{K}^{KL}\boldsymbol{K}^{LK}. \tag{2.4.58}$$

This means that there is a basis in which $\boldsymbol{K}^{IJ}$ has the form

$$\boldsymbol{K}^{IJ} \equiv \boldsymbol{K}\,\mathrm{diag}(1, ..., 1, -1, ..., -1)^{IJ}, \tag{2.4.59}$$

where a number of p entries is 1 and q entries are -1, in which case the corresponding space is classified as a (p, q) anti-de Sitter superspace.

The covariant constancy further requires the action of the SO($\mathcal{N}$) field strength on $\boldsymbol{K}^{IJ}$ and $\boldsymbol{W}^{IJKL}$ to vanish, since

$$\{\mathscr{D}_\alpha^I, \mathscr{D}_\beta^J\}\boldsymbol{K}^{PQ} = \{\mathscr{D}_\alpha^I, \mathscr{D}_\beta^J\}\boldsymbol{W}^{SPQR} = 0. \tag{2.4.60}$$

In the basis where $\boldsymbol{K}^{IJ}$ is diagonal, this corresponds to the relations

$$0 = -\boldsymbol{W}^{IJL(P}\boldsymbol{K}^{Q)L} \tag{2.4.61a}$$
$$0 = -\boldsymbol{W}^{IJL[S}\boldsymbol{W}^{PQR]L} + 2(\boldsymbol{K}^{L[J}\delta^{I][S} + \delta^{L[J}\boldsymbol{K}^{I][S})\boldsymbol{W}^{PQR]L}. \tag{2.4.61b}$$

The first one implies that for non-vanishing components of $\boldsymbol{W}^{IJKL}$, all indices must be in the same range, either $I = 1, ..., p$ or $I = p + 1, ..., \mathcal{N}$. Then, the second one implies that if there are indices from different ranges, $\boldsymbol{W}^{IJKL}$ must vanish completely. In consequence, the super-Cotton tensor can appear only in an $(\mathcal{N}, 0)$ anti de-Sitter superspace, where $\boldsymbol{K}^{IJ} = \boldsymbol{K}\delta^{IJ}$ [38]. In this case, the above algebra becomes

$$\{\mathscr{D}_\alpha^I, \mathscr{D}_\beta^J\} = 2\mathrm{i}\delta^{IJ}\mathscr{D}_{\alpha\beta} + 4\mathrm{i}\delta^{IJ}\boldsymbol{K}\mathscr{M}_{\alpha\beta} + \mathrm{i}\varepsilon_{\alpha\beta}(\boldsymbol{W}^{IJKL} + 4\delta^{K[I}\delta^{J]L}\boldsymbol{K})\mathscr{N}_{KL} \tag{2.4.62a}$$
$$[\mathscr{D}_a, \mathscr{D}_\beta^J] = (\gamma_a)_\beta{}^\gamma\boldsymbol{K}\mathscr{D}_\gamma^J \tag{2.4.62b}$$
$$[\mathscr{D}_a, \mathscr{D}_b] = -4\boldsymbol{K}^2\mathscr{M}_{ab}, \tag{2.4.62c}$$

where the relation

$$0 = -\boldsymbol{W}^{IJL[S}\boldsymbol{W}^{PQR]L} + 4\boldsymbol{K}\delta^{[I[S}\boldsymbol{W}^{PQR]J]} \tag{2.4.63}$$

holds.

2.4.6. Conformal superspace

In conformal superspace [37], the superconformal group is chosen to be the local structure group. The generators of the superconformal group are the generators of the super-Poincaré group supplemented by the generators for special conformal transformations

K_a, for special superconformal transformations S^I_α and for dilatations D. Accordingly, the superconformal algebra is the Poincaré superalgebra supplemented by the commutation relations

$$\{S^I_\alpha, S^J_\beta\} = 2\delta^{IJ} K_{\alpha\beta} \tag{2.4.64a}$$
$$[K_a, P_b] = 2\eta_{ab} D + 2\mathscr{M}_{ab} \tag{2.4.64b}$$
$$[S^I_\alpha, P_b] = (\gamma_b)_\alpha{}^\beta Q^I_\beta \tag{2.4.64c}$$
$$[K_a, Q^J_\beta] = -(\gamma_a)_\alpha{}^\beta S^J_\beta \tag{2.4.64d}$$
$$[S^I_\alpha, Q^J_\beta] = 2\varepsilon_{\alpha\beta}\delta^{IJ} D - 2\delta^{IJ}\mathscr{M}_{\alpha\beta} - 2\varepsilon_{\alpha\beta}\mathscr{N}^{IJ} \tag{2.4.64e}$$
$$[D, \{P_a, K_a, Q^I_\alpha, S^I_\alpha\}] = \{P_a, -K_a, \tfrac{1}{2}Q^I_\alpha, -\tfrac{1}{2}S^I_\alpha\}. \tag{2.4.64f}$$

The covariant derivatives are denoted by $(\nabla_a, \nabla^I_\alpha)$ and have the same commutation relations as (P_a, Q^I_α) with the rest of the algebra.

The anticommutator of two covariant spinor derivatives has the form

$$\{\nabla^I_\alpha, \nabla^J_\beta\} = 2\mathrm{i}\delta^{IJ}\nabla_{\alpha\beta} + 2\mathrm{i}\varepsilon_{\alpha\beta}\boldsymbol{W}^{IJ}. \tag{2.4.65a}$$

The main idea of conformal superspace is that the algebra of covariant derivatives is given in terms of a super-Cotton tensor $\boldsymbol{W}^{IJKL}$ which coincides with the superfield $\boldsymbol{W}^{IJKL}$ introduced before. It is therefore postulated that

$$\boldsymbol{W}^{IJ} = \tfrac{1}{2}\boldsymbol{W}^{IJKL}\mathscr{N}_{KL} + \tfrac{1}{2(\mathcal{N}-3)}(\nabla^\alpha_K \boldsymbol{W}^{IJKL})S^L_\alpha - \tfrac{1}{4(\mathcal{N}-2)(\mathcal{N}-3)}(\nabla^\alpha_K\nabla^\beta_L \boldsymbol{W}^{IJKL})K_{\alpha\beta}. \tag{2.4.66}$$

This operator proves to be a conformal primary, since

$$
\begin{aligned}
[S^M_\mu, \boldsymbol{W}^{IJ}] =& - \boldsymbol{W}^{IJML} S^L_\mu \\
& + \tfrac{1}{2(\mathcal{N}-3)}[(\{S^M_\mu, \nabla^\alpha_K\}\boldsymbol{W}^{IJKL})S^L_\alpha - (\nabla^\alpha_K \boldsymbol{W}^{IJKL})\{S^M_\mu, S^L_\alpha\}] \\
& - \tfrac{1}{4(\mathcal{N}-2)(\mathcal{N}-3)}[([S^M_\mu, \nabla^K_\alpha \nabla^L_\beta]\boldsymbol{W}^{IJKL})K_c \\
=& - \boldsymbol{W}^{IJML} S^L_\mu \\
& + 2\tfrac{1}{2(\mathcal{N}-3)}[((\delta^{MK} D - \mathcal{N}^{MK})\boldsymbol{W}^{IJKL})S^L_\mu - (\nabla^\alpha_K \boldsymbol{W}^{IJKM})K_{\mu\alpha}] \\
& - 2\tfrac{1}{4(\mathcal{N}-2)(\mathcal{N}-3)}((\varepsilon_{\mu\alpha}\delta^{MK} D - \delta^{MK}\mathscr{M}_{\mu\alpha} - \varepsilon_{\mu\alpha}\mathcal{N}^{MK})\nabla^L_\beta \boldsymbol{W}^{IJKL})K^{\alpha\beta} \\
& + 2\tfrac{1}{4(\mathcal{N}-2)(\mathcal{N}-3)}(\nabla^K_\alpha(\varepsilon_{\mu\beta}\delta^{ML} D - \varepsilon_{\mu\beta}\mathcal{N}^{ML})\boldsymbol{W}^{IJKL})K^{\alpha\beta} \\
=& 0.
\end{aligned} \tag{2.4.67}
$$

The remaining algebra has the form known from super-gauge theory [37]

$$
[\nabla_a, \nabla^I_\alpha] = \tfrac{-1}{\mathcal{N}-1}(\gamma_a)_\alpha{}^\beta [\nabla^J_\beta, \boldsymbol{W}^{JI}] \tag{2.4.68a}
$$

$$
[\nabla_a, \nabla_b] = \tfrac{-\mathrm{i}}{2\mathcal{N}(\mathcal{N}-1)}(\gamma_{ab})^{\alpha\beta}\{\nabla^K_\alpha, [\nabla^L_\beta, \boldsymbol{W}_{KL}]\}. \tag{2.4.68b}
$$

This qualifies $\boldsymbol{W}^{IJKL}$ as a single conformal supergravity multiplet for $\mathcal{N} \geq 4$. The commutator of two vector derivatives is of special interest, since it contains the SO($\mathcal{N}$) field strength [37]

$$
\boldsymbol{F}^{KL}_{ab} = (\gamma_{ab})^{\alpha\beta} \tfrac{-\mathrm{i}}{2(\mathcal{N}-2)(\mathcal{N}-3)} \nabla^I_\alpha \nabla^J_\beta \boldsymbol{W}^{IJKL} \tag{2.4.69}
$$

being expressed as covariant projections of the super-Cotton tensor.

Translating this formalism to the conventional SO($\mathcal{N}$) curved superspace is achieved by de-gauging the special conformal connections via the relation

$$
\mathscr{D}^I_\alpha = \nabla^I_\alpha - \mathfrak{F}_{\alpha I}{}^c K_c - \mathfrak{F}^{I\gamma}_{\alpha K} S^K_\gamma, \tag{2.4.70}
$$

where $\mathfrak{F}_A$ is the special superconformal connection and the dilatation connection has been gauged away as is possible by a superconformal transformation. The algebra can be calculated in these terms, as if acting on a superconformal primary of dimension zero,

$$
\{\mathscr{D}^I_\alpha, \mathscr{D}^J_\beta\} + \mathfrak{F}^{I\gamma}_{\alpha K}\{\nabla^J_\beta, S^K_\gamma\} + \mathfrak{F}^{J\gamma}_{\beta K}\{\nabla^I_\alpha, S^K_\gamma\} = 2\mathrm{i}\delta^{IJ}\mathscr{D}_{\alpha\beta} + \mathrm{i}\varepsilon_{\alpha\beta}\boldsymbol{W}^{IJKL}\mathscr{N}_{KL}, \tag{2.4.71}
$$

which can be written as

$$\begin{aligned}\{\mathscr{D}^I_\alpha, \mathscr{D}^J_\beta\} =& 2\mathrm{i}\delta^{IJ}\mathscr{D}_{\alpha\beta} + \mathrm{i}\varepsilon_{\alpha\beta}\boldsymbol{W}^{IJKL}\mathscr{N}_{KL} \\ &+ 2\mathfrak{F}^{I\gamma}_{\alpha K}(\delta^{KJ}\mathscr{M}_{\gamma\beta} + \varepsilon_{\gamma\beta}\mathscr{N}^{KJ}) + 2\mathfrak{F}^{J\gamma}_{\beta K}(\delta^{KI}\mathscr{M}_{\gamma\alpha} + \varepsilon_{\gamma\alpha}\mathscr{N}^{KI}). \qquad (2.4.72)\end{aligned}$$

Comparison with the geometry of curved SO($\mathcal{N}$) superspace implies the relation

$$\mathfrak{F}^{IK}_{\alpha\gamma} = -\mathrm{i}\varepsilon_{\alpha\gamma}\boldsymbol{K}^{IK} - \mathrm{i}\boldsymbol{L}^{IK}_{\alpha\gamma}. \qquad (2.4.73)$$

This form of $\mathfrak{F}^{IK}_{\alpha\gamma}$ is indeed dictated by the consistent vanishing of the special conformal curvature [37]. The de-gauging procedure leads to the full conventional supergravity algebra and furthermore implies the super-Weyl transformations under which it is invariant [37].

It is of special interest to translate the formula for the SO($\mathcal{N}$) field strength to SO($\mathcal{N}$) curved superspace by evaluating

$$\begin{aligned}\nabla^I_\alpha\nabla^J_\beta\boldsymbol{W}^{IJKL} =& (\mathscr{D}^I_\alpha + \mathfrak{F}_\alpha{}^cK_c + \mathfrak{F}_\alpha{}^\gamma S_\gamma)(\mathscr{D}^J_\beta + \mathfrak{F}_\beta{}^dK_d + \mathfrak{F}_\beta{}^\delta S_\delta)\boldsymbol{W}^{IJKL} \\ =& \left(\mathscr{D}^{[I}_{(\alpha}\mathscr{D}^{J]}_{\beta)} + \mathfrak{F}^{[Ic}_{(\alpha}[K_c, \mathscr{D}^{J]}_{\beta)}] + \mathfrak{F}^{[I\gamma}_{(\alpha M}\{S^M_\gamma, \mathscr{D}^{J]}_{\beta)}\}\right)\boldsymbol{W}^{IJKL} \\ =& \mathscr{D}^{[I}_{(\alpha}\mathscr{D}^{J]}_{\beta)}\boldsymbol{W}^{IJKL} + 2\mathfrak{F}^{IM}_{(\alpha\beta)}(\delta^{MJ}D - \mathscr{N}^{MJ})\boldsymbol{W}^{IJKL}. \qquad (2.4.74)\end{aligned}$$

It differs from the replacement of ∇^I_α by $\mathscr{D}^I_\alpha$ only by a term proportional to $\boldsymbol{L}^{IJ}_a$. Therefore, in the gauge where $\boldsymbol{L}^{IJ}_a$ vanishes, the formula [32]

$$\boldsymbol{F}^{KL}_{ab} = (\gamma_{ab})^{\alpha\beta}\tfrac{-\mathrm{i}}{2(\mathcal{N}-2)(\mathcal{N}-3)}\mathscr{D}^I_\alpha\mathscr{D}^J_\beta\boldsymbol{W}^{IJKL} \qquad (2.4.75)$$

is valid.

To summarise this section, $\mathcal{N}$-extended superconformal gravity was formulated in the language of conventional curved superspace by imposing conventional constraints on the torsions appearing in the algebra of covariant derivatives. These constraints can be motivated by redefinitions in the explicit construction of the torsion fields in terms of gauge connections. The Jacobi identity has been solved under these constraints in order to express the field strengths in terms of the torsion fields and the super-Cotton tensor appearing for $\mathcal{N} \geq 4$. The resulting algebra is invariant under certain super-Weyl transformations, which is the reason why it describes conformal supergravity (a theory has conformal symmetry, if a conformal transformation of the metric can be compensated by local Lorentz and Weyl transformations of the other fields; the algebra must therefore be invariant under the latter). Since the super-Cotton tensor transforms homogeneously under these super-Weyl transformations, i.e. it cannot be gauged to zero, it qualifies as the true conformal-supergravity multiplet.

The formulation in conformal superspace starts from a larger local structure group, namely the superconformal group, and provides an elegant description only in terms of the super-Cotton tensor. The superconformal group has to be eventually de-gauged in order to recover the conventional superspace. Nevertheless, owed to its simplicity, conformal superspace proves to be a powerful tool and was used to derive a relevant formula for the SO($\mathcal{N}$) field strength, which is hard to obtain using conventional superspace alone.

3. Coupled scalar matter

In this chapter, the coupling of scalar-matter multiplets to super-gauge fields and conformal supergravity is discussed and further analysed for the spin($\mathcal{N}$) on-shell scalar multiplet introduced in Chapter 2.

In Section (3.1), the superfield-component formalism from Chapter 2 is adjusted for super-gauge- and supergravity-covariant derivatives in order to describe covariant superfield components.

In Section (3.2), the defining constraint for the spin($\mathcal{N}$) scalar multiplet is altered to a covariant form. Analysing the covariant components, an algebraic relation involving the field-strength multiplet is derived, which is needed for the validity of the covariant defining constraint. Furthermore, the superfield action for the coupled scalar multiplet is presented.

In Section (3.3), the Chern-Simons action for vector gauge fields and the equation of motion for the field strengths in presence of matter currents resulting from the coupled-matter action are given. The equations of motion are recast as dimension-one superfield equations of motion for the gauge multiplet and for the super-Cotton tensor, which contains the SO($\mathcal{N}$) field strength of the local structure group.

In Sections (3.4) and (3.5), these on-shell expressions are used for investigating the coupling condition for the spin($\mathcal{N}$) scalar multiplet derived in Section (3.2). All allowed gauge goups are determined, with regard to the number of supersymmetries both in flat and curved superspace. Thereby, explicit expressions, which are needed for the transformation rules of the scalar multiplet and the component actions, are established.

In Section (3.6), an overview of the allowed gauge groups found in the previous two sections is presented and commented.

3.1. Covariant superfield projections

Generic covariant projections of a superfield are defined by

$$\check{A}_{\alpha_k\ldots\alpha_1} = \mathscr{D}_{\alpha_k}\ldots\mathscr{D}_{\alpha_1}\boldsymbol{A}|. \tag{3.1.1}$$

They are related to superfield components and supercovariant projections by additional terms involving powers of covariant projections of spinor connections $\boldsymbol{B}^I_\alpha$. It is postulated that

$$\check{A}_{\alpha_k\ldots\alpha_1} = \check{a}_{\alpha_k\ldots\alpha_1} + \check{\mathfrak{A}}_{\alpha_k\ldots\alpha_1}, \tag{3.1.2}$$

where $\check{a}^{I_K\ldots I_1}_{\alpha_k\ldots\alpha_1}$ possesses the same symmetry as $a^{I_K\ldots I_1}_{\alpha_k\ldots\alpha_1}$ and is called a gauge-covariant component field.

Explicitly, one can write

$$\begin{aligned}
\boldsymbol{A}| =& a && (3.1.3a)\\
\mathscr{D}^I_\alpha \boldsymbol{A}| =& \check{q}^I_\alpha && (3.1.3b)\\
\mathscr{D}^J_\beta\mathscr{D}^I_\alpha \boldsymbol{A}| =& \check{a}^{JI}_{\beta\alpha} + \mathrm{i}\delta^{JI}\mathscr{D}_{\beta\alpha}a + \tfrac{1}{2}R^{JI}_{\beta\alpha}a && (3.1.3c)\\
\mathscr{D}^K_\gamma\mathscr{D}^J_\beta\mathscr{D}^I_\alpha \boldsymbol{A}| =& \check{a}^{KJI}_{\gamma\beta\alpha} + \check{B}^{I,KJ}_{\alpha,\gamma\beta}a && (3.1.3d)\\
&+ (\mathrm{i}\mathscr{D}^{KJ}_{\gamma\beta} + \tfrac{1}{2}R^{KJ}_{\gamma\beta})\check{a}^I_\alpha - (\mathrm{i}\mathscr{D}^{KI}_{\gamma\alpha} + \tfrac{1}{2}R^{KI}_{\gamma\alpha})\check{a}^J_\beta + (\mathrm{i}\mathscr{D}^{JI}_{\beta\alpha} + \tfrac{1}{2}R^{JI}_{\beta\alpha})\check{a}^K_\gamma\\
&- ((\mathrm{i}\mathscr{D}^{KJ}_{\gamma\beta} + \tfrac{1}{2}R^{KJ}_{\gamma\beta})b^I_\alpha)a + ((\mathrm{i}\mathscr{D}^{KI}_{\gamma\alpha} + \tfrac{1}{2}R^{KI}_{\gamma\alpha})b^J_\beta)a - ((\mathrm{i}\mathscr{D}^{JI}_{\beta\alpha} + \tfrac{1}{2}R^{JI}_{\beta\alpha})b^K_\gamma)a,
\end{aligned}$$

and so on. The field $R^{JI}_{\beta\alpha}$ stands for the field strengths defined by the anticommutator of covariant spinor derivatives, and the covariant components are given by

$$\begin{aligned}
\check{a}^I_\alpha =& a^I_\alpha + b^I_\alpha a && (3.1.4a)\\
\check{a}^{JI}_{\beta\alpha} =& a^{JI}_{\beta\alpha} + b^J_\beta a^I_\alpha - b^I_\alpha a^J_\beta + \check{B}^{[I,J]}_{(\alpha,\beta)}a + \check{B}^{(I,J)}_{[\alpha,\beta]}a && (3.1.4b)\\
\check{a}^{KJI}_{\gamma\beta\alpha} =& a^{KJI}_{\gamma\beta\alpha} + b^K_\gamma a_{\beta\alpha} - b_\beta a_{\gamma\alpha} + b_\alpha a_{\gamma\beta}\\
&+ \check{B}^{[I,J]}_{(\alpha,\beta)}a^K_\gamma - \check{B}^{[I,K]}_{(\alpha,\gamma)}a^J_\beta + \check{B}^{[J,K]}_{(\beta,\gamma)}a^I_\alpha + \check{B}^{(I,J)}_{[\alpha,\beta]}a^K_\gamma - \check{B}^{(I,K)}_{[\alpha,\gamma]}a^J_\beta + \check{B}^{(J,K)}_{[\beta,\gamma]}a^I_\alpha. && (3.1.4c)
\end{aligned}$$

It can be noted that in the gauge corresponding to $b^I_\alpha = 0$ (compare (2.3.6)), components and covariant components coincide. The field $\check{B}^{I,KJ}_{\alpha,\gamma\beta} = \mathscr{D}^K_\gamma\mathscr{D}^J_\beta B^I_\alpha$, appearing in the rank-three projection, contains a gauge-invariant part related to the field strength of dimension three-halves.

3.2. Coupled spin($\mathcal{N}$) superfield

For covariant projections, the constraint (2.2.59) on the spin($\mathcal{N}$) on-shell superfield is modified to [31, 22, 20, 18, 19]

$$\mathscr{D}^I_\alpha \boldsymbol{Q}_i = (\gamma^I)_i{}^j \check{\boldsymbol{Q}}_{j\alpha}. \tag{3.2.1}$$

The leading component of this superfield equation is

$$\check{q}^I_\alpha = \gamma^I \check{q}_\alpha \tag{3.2.2}$$

and the higher covariant projections are of the form

$$\check{q}^{J_k..J_1 I}_{\beta_k..\beta_1\alpha} + \check{\mathfrak{Q}}^{J_k..J_1 I}_{\beta_k..\beta_1\alpha} = \gamma^I \check{q}_{\alpha,\beta_k..\beta_1}{}^{J_k..J_1} + \gamma^I \check{\mathfrak{Q}}_{\alpha,\beta_k..\beta_1}{}^{J_k..J_1}. \tag{3.2.3}$$

The covariant components of $\check{\boldsymbol{Q}}_i$ are provided by this expression, only if the components of $\check{\boldsymbol{Q}}_{\alpha i}$ are determined by taking the representations not contained in $\check{q}^{J_k..J_1 I}_{\beta_k..\beta_1\alpha}$, i.e.

$$\check{\mathfrak{Q}}^{J_k..(J_1 I)}_{\beta_k..(\beta_1\alpha)} - \gamma^{(I}\check{\mathfrak{Q}}_{(\alpha,|\beta_k..|\beta_1)}{}^{|J_k..|J_1)} = \gamma^{(I}\check{q}_{(\alpha,|\beta_k..|\beta_1)}{}^{|J_k..|J_1)} \tag{3.2.4a}$$

$$\check{\mathfrak{Q}}^{J_k..[J_1 I]}_{\beta_k..[\beta_1\alpha]} - \gamma^{[I}\check{\mathfrak{Q}}_{[\alpha,|\beta_k..|\beta_1]}{}^{|J_k..|J_1]} = \gamma^{[I}\check{q}_{[\alpha,|\beta_k..|\beta_1]}{}^{|J_k..|J_1]}. \tag{3.2.4b}$$

A solution of these equations for $\check{q}_{\alpha,\beta_k..\beta_1}{}^{J_k..J_1}$ is therefore a necessary condition for the validity of the covariant defining constraint (3.2.1).

As opposed to the non-coupled case, this solution is not always existent, but requires specific properties of the field-strength multiplets, which will be derived below separately for the gauge- and gravitationally coupled cases. Solutions are given as far as possible; however, the general spectrum of solutions analysed in Sections (3.4) and (3.5) requires the on-shell form of the field-strength and supergravity multiplets derived in Section (3.3).

3.2.1. Gauge-coupled spin($\mathcal{N}$) superfield

Regarding the above expressions for the components of $\check{\boldsymbol{Q}}_{\alpha i}$, the first projection can be recast as the superfield equations

$$\begin{aligned} \mathrm{i}\delta^{JI}\mathscr{D}_{\beta\alpha}\boldsymbol{Q} =& \gamma^{(I}\mathscr{D}^{J)}_{(\beta}\check{\boldsymbol{Q}}_{\alpha)} && (3.2.5a)\\ \tfrac{1}{2}\boldsymbol{F}^{JI}_{\beta\alpha}\boldsymbol{Q} =& \gamma^{[I}\mathscr{D}^{J]}_{[\beta}\check{\boldsymbol{Q}}_{\alpha]}. && (3.2.5b) \end{aligned}$$

The first one is solved by

$$\mathscr{D}^{J}_{(\beta}\check{\boldsymbol{Q}}_{\alpha)} = \mathrm{i}\gamma^{J}\mathscr{D}_{\beta\alpha}\boldsymbol{Q}. \tag{3.2.6}$$

The second one imposes a strong condition on the form of $\boldsymbol{F}^{JI}$. If it does not allow a solution for $\mathscr{D}^{J}_{[\beta}\check{\boldsymbol{Q}}_{\alpha]}$, the covariant components of $\boldsymbol{Q}_i$ are not defined, and the defining constraint (3.2.1) is not valid.

Provided the possible solutions of (3.2.5b), which are obtained in Section (3.4), the existence of all higher covariant components follows by means of covariant projection and proper symmetrisation from

$$\begin{aligned} \check{\boldsymbol{Q}}^{(JI)}_{[\beta\alpha]} =& \gamma^{(I}\check{\boldsymbol{Q}}_{[\alpha,\beta]}^{\;\;J)} && (3.2.7a)\\ \check{\boldsymbol{Q}}^{[JI]}_{(\beta\alpha)} =& \mathrm{i}\gamma^{IJ}\mathscr{D}_{\beta\alpha}\boldsymbol{Q}, && (3.2.7b) \end{aligned}$$

and the transformations of the gauge-covariant components are given by

$$\begin{aligned} \delta q =& -\varepsilon^{\alpha}_{I}\gamma^{I}\check{q}_{\alpha} && (3.2.8a)\\ \delta\check{q}_{\alpha} =& -\varepsilon^{\beta}_{J}(\mathrm{i}\gamma^{J}\mathscr{D}_{\beta\alpha}q + \check{q}_{[\alpha,\beta]}^{\;\;J}), && (3.2.8b) \end{aligned}$$

where $\check{q}_{[\alpha,\beta]}^{\;\;J}$ has to be replaced accordingly.

The superfield action for the gauge-coupled scalar multiplet is obtained from the free action (2.2.73) by replacing the supercovariant derivatives by super-gauge-covariant derivatives [35]

$$\begin{aligned} S_{\mathrm{m}} =& \tfrac{1}{-8\mathcal{N}^2}\int \mathrm{d}^3x\ \mathrm{d}\theta^{\alpha}_{I}\mathrm{d}\theta^{I}_{\alpha}\ (\overline{\mathscr{D}^{\gamma}_{K}\boldsymbol{Q}})\mathscr{D}^{K}_{\gamma}\boldsymbol{Q}| \\ =& \tfrac{1}{-8\mathcal{N}^2}\int \mathrm{d}^3x\ (\overline{\check{Q}^{\gamma}_{K}}\check{Q}^{\alpha IK}_{I\alpha\gamma} - \overline{\check{Q}^{\alpha I\gamma}_{I\alpha K}}\check{Q}^{K}_{\gamma} - 2\overline{\check{Q}^{\alpha\gamma}_{IK}}\check{Q}^{IK}_{\alpha\gamma}), && (3.2.9) \end{aligned}$$

where the integrand is gauge-traced. This coupling is conformal, since the action is scale-invariant [7]. The component action can be obtained by inserting the covariant projections

$$\check{Q}^{\beta\alpha}_{JI} = \mathrm{i}\gamma_I\gamma_J \mathscr{D}^{\beta\alpha} q + \gamma^I \check{\boldsymbol{Q}}_{[\alpha,\beta]}{}^{J} \tag{3.2.10a}$$

$$\begin{aligned}\check{Q}^{\gamma\beta\alpha}_{KJI} =& \mathrm{i}\gamma_I\gamma_J[\mathscr{D}^{\gamma}_K, \mathscr{D}^{\beta\alpha}]\boldsymbol{Q}| + \mathrm{i}\gamma_I\gamma_J\gamma_K\mathscr{D}^{\beta\alpha}\check{q}^{\gamma} + \tfrac{1}{2}\mathscr{D}^{\gamma}_K(\boldsymbol{F}^{\beta\alpha}_{JI}\boldsymbol{Q})| + \mathscr{D}^{\gamma}_K\gamma_{(I}\check{\boldsymbol{Q}}^{[\alpha,\beta]}{}_{J)}| \\ &+ \text{“}\beta\alpha\gamma, JIK\text{”} - \text{“}\gamma\alpha\beta, KIJ\text{”}.\end{aligned} \tag{3.2.10b}$$

These rely on the solutions $\check{\boldsymbol{Q}}_{[\alpha,\beta]}{}^{J}$ of the condition (3.2.5b), which will be found in Section (3.4). Furthermore, it will turn out that the field-strength multiplet $\boldsymbol{F}^{IJ}$ has to be expressed by matter currents which are derived in Section (3.3).

3.2.2. Gravitationally coupled spin($\mathcal{N}$) superfield

As opposed to the gauge-covariant derivatives, the anticommutator of supergravity covariant spinor derivatives contains field-strength terms which are symmetric in Lorentz and SO($\mathcal{N}$) indices. The solvability conditions for the constraint on the spin($\mathcal{N}$) scalar superfield therefore read

$$\mathrm{i}\delta^{JI}\mathscr{D}_{\beta\alpha}\boldsymbol{Q} + \tfrac{1}{2}\boldsymbol{R}^{(JI)}_{(\beta\alpha)}\boldsymbol{Q} = \gamma^{(I}\mathscr{D}^{J)}_{(\beta}\check{\boldsymbol{Q}}_{\alpha)} \tag{3.2.11a}$$

$$\tfrac{1}{2}\boldsymbol{R}^{[JI]}_{[\beta\alpha]}\boldsymbol{Q} = \gamma^{[I}\mathscr{D}^{J]}_{[\beta}\check{\boldsymbol{Q}}_{\alpha]}, \tag{3.2.11b}$$

where

$$\tfrac{1}{2}\boldsymbol{R}^{(JI)}_{(\beta\alpha)} = (2\mathrm{i}\delta^{(J[K}\boldsymbol{L}^{L]I)}_{\beta\alpha} + \tfrac{\mathrm{i}}{2}\delta^{JI}\boldsymbol{L}^{KL}_{\beta\alpha})\mathscr{N}_{KL} \tag{3.2.12a}$$

$$\tfrac{1}{2}\boldsymbol{R}^{[JI]}_{[\beta\alpha]} = \varepsilon_{\beta\alpha}(\tfrac{\mathrm{i}}{2}\boldsymbol{W}^{JIKL} + 2\mathrm{i}\delta^{[K[J}\boldsymbol{K}^{I]L]})\mathscr{N}_{KL}. \tag{3.2.12b}$$

The first one is solved by

$$\mathscr{D}^{J}_{(\beta}\check{\boldsymbol{Q}}_{\alpha)} = \mathrm{i}\gamma^J\mathscr{D}_{\beta\alpha}\boldsymbol{Q} + \tfrac{\mathrm{i}}{4}\gamma^J\boldsymbol{L}^{KL}_{\beta\alpha}\gamma_{KL}\boldsymbol{Q} + \mathrm{i}\boldsymbol{L}^{KJ}_{\beta\alpha}\gamma^K\boldsymbol{Q}. \tag{3.2.13}$$

A solution for $\mathscr{D}^{J}_{[\beta}\check{\boldsymbol{Q}}_{\alpha]}$ can readily be given in terms of the super-Cotton tensor $\boldsymbol{W}^{IJKL}$ and the field $\boldsymbol{K}^{IJ}$ for $\mathcal{N}=4$, $\mathcal{N}=5$ and $\mathcal{N}=6$, as will be shown in the following.

Defining $H^J = \frac{\mathrm{i}}{2}\check{q}_{\beta,}{}^{\beta J}$, one can write

$$\gamma^{[J}H^{I]} = \tfrac{1}{4}(W^{IJKL}\gamma_{KL} + 4K\gamma^{IJ})q. \tag{3.2.14}$$

Generally, a solution can be expected in the form

$$H^I = AW^{IKLM}\gamma_{KLM}q + BW_{KLMN}\gamma^{IKLMN}q - K\gamma^I q \tag{3.2.15}$$

and similarly for chiral representation of spin($\mathcal{N}$) (see Appendix A). In the following, for odd $\mathcal{N}$ the notation $q = \boldsymbol{Q}|$ will be applied, while for even $\mathcal{N}$ the notation $Q = \boldsymbol{Q}|$ will indicate that Q transforms under a chiral representation of spin($\mathcal{N}$).

$\boldsymbol{\mathcal{N} = 4}$ For $\mathcal{N} = 4$, the super-Cotton tensor is dualised as $W^{IJKL} \equiv W\varepsilon^{IJKL}$. Since the left-handed generators are anti-self-dual, it follows that

$$\gamma^{[J}H^{I]} = -\tfrac{1}{2}(W - 2K)\Sigma^{IJ}Q \tag{3.2.16}$$

and

$$H^I = \tfrac{1}{2}(W - 2K)\bar{\Sigma}^I Q. \tag{3.2.17}$$

$\boldsymbol{\mathcal{N} = 5}$ For $\mathcal{N} = 5$, the super-Cotton tensor is $W_{JKLM} \equiv W^I\varepsilon_{IJKLM}$ and

$$\gamma^{[J}H^{I]} = -\tfrac{1}{2}(W_K\gamma^{KIJ} - 2K\gamma^{IJ})q. \tag{3.2.18}$$

The solution is given by

$$H^I = -\tfrac{1}{2}W_K\gamma^{IK}q + \tfrac{1}{2}W^I q - K\gamma^I q. \tag{3.2.19}$$

$\boldsymbol{\mathcal{N} = 6}$ For $\mathcal{N} = 6$, the super-Cotton tensor is $W^{IJKL} = \frac{1}{2}\varepsilon^{IJKLPQ}W_{PQ}$. This theory requires the additional gauging of a U(1) symmetry, which corresponds to the local structure group SO(6) × U(1), rather than SO(6) [21]. The U(1) field-strength superfield is proportional to the super-Cotton tensor by a charge $\tilde{q}$ [21], so that

$$\gamma^{[J}H^{I]} = \tfrac{1}{4}(-\mathrm{i}W^{KL}\gamma_{KLIJ} + 4\tilde{q}W_{IJ} + 4K\gamma_{IJ})q. \tag{3.2.20}$$

Only for $\tilde{q} = -\frac{\mathrm{i}}{2}$ there exists the solution

$$H^I = \tfrac{\mathrm{i}}{4}W_{KL}\bar{\Sigma}^{KLI}Q - \tfrac{\mathrm{i}}{2}W^{IK}\bar{\Sigma}_K Q - K\bar{\Sigma}^I Q. \tag{3.2.21}$$

For $\mathcal{N}=7$ and $\mathcal{N}=8$ (chiral), such solutions exist only in terms of the matter current, which couples to the super-Cotton tensor and is derived in the next section. They will be discussed in Section (3.5).

The superfield action for the gravitationally coupled scalar multiplet is obtained from the free action (2.2.73) by replacing the supercovariant derivatives by supergravity-covariant derivatives and inserting the super-determinant of the super-vielbein $\boldsymbol{E}$ [35], leading to

$$S_{\mathrm{m}} = \tfrac{1}{-8\mathcal{N}^2} \int \mathrm{d}^3x \; \mathrm{d}\theta_I^\alpha \mathrm{d}\theta_\alpha^I \; \boldsymbol{E} \; (\overline{\mathscr{D}_K^\gamma \boldsymbol{Q}}) \mathscr{D}_\gamma^K \boldsymbol{Q}|. \tag{3.2.22}$$

This matter coupling is not conformal, because the action is not locally scale-invariant.

Covariant projections of $\boldsymbol{Q}$ needed to calculate the component action are

$$\begin{aligned}\check{Q}^{JI}_{\beta\alpha} =& \gamma^I (\mathrm{i}\gamma^J \mathscr{D}_{\beta\alpha}\boldsymbol{Q} + \tfrac{\mathrm{i}}{4}\gamma^J \boldsymbol{L}^{KL}_{\beta\alpha}\gamma_{KL}\boldsymbol{Q} + \mathrm{i}\boldsymbol{L}^{KJ}_{\beta\alpha}\gamma^K\boldsymbol{Q})| \\ &+ \tfrac{1}{2}\varepsilon_{\beta\alpha}(\tfrac{\mathrm{i}}{2}\boldsymbol{W}^{JIKL} + 2\mathrm{i}\delta^{[K[J}\boldsymbol{K}^{I]L]})\gamma_{KL}\boldsymbol{Q}| + \gamma^{(I}\mathscr{D}^{J)}_{[\beta}\boldsymbol{Q}_{\alpha]}|\end{aligned} \tag{3.2.23}$$

and

$$\begin{aligned}\check{Q}^{KJI}_{\gamma\beta\alpha} =& \gamma^I \mathscr{D}^K_\gamma (\mathrm{i}\gamma^J \mathscr{D}_{\beta\alpha}\boldsymbol{Q} + \tfrac{\mathrm{i}}{4}\gamma^J \boldsymbol{L}^{KL}_{\beta\alpha}\gamma_{KL}\boldsymbol{Q} + \mathrm{i}\boldsymbol{L}^{KJ}_{\beta\alpha}\gamma^K\boldsymbol{Q})| \\ &+ \tfrac{1}{2}\varepsilon_{\beta\alpha}\mathscr{D}^K_\gamma(\tfrac{\mathrm{i}}{2}\boldsymbol{W}^{JIKL} + 2\mathrm{i}\delta^{[K[J}\boldsymbol{K}^{I]L]})\gamma_{KL}\boldsymbol{Q}| + \mathscr{D}^K_\gamma\gamma^{(I}\mathscr{D}^{J)}_{[\beta}\boldsymbol{Q}_{\alpha]}| \\ &+ \text{“}\beta\alpha\gamma, JIK\text{”} - \text{“}\gamma\alpha\beta, KIJ\text{”}.\end{aligned} \tag{3.2.24}$$

The full component action would also require information about the component structure of $\boldsymbol{E}$.

3.2.3. Gauge- and gravitationally coupled spin($\mathcal{N}$) superfield

In presence of both supergravity and gauge fields, the two respective solutions $\frac{\mathrm{i}}{2}\check{q}_{\beta,}{}^{\beta J} = H^J$ can in many cases be added,

$$H^I = H^I_{\mathrm{SG}} + H^I_{\mathrm{CS}}. \tag{3.2.25}$$

However, as the investigation in Section (3.5) will show, this is not possible for $\mathcal{N}=7$ and $\mathcal{N}=8$, where instead the on-shell gauge sector has to contribute terms to the on-shell supergravity sector in order to solve the coupling condition, causing the gauge couplings to become dependent of the supergravity coupling.

On the other hand, there are more solutions apart from this sum for $\mathcal{N} = 6$, $\mathcal{N} = 7$ and $\mathcal{N} = 8$. They arise in cases where the on-shell supergravity sector can contribute terms to the on-shell gauge sector or vice versa, leading either to a gravity coupling-dependent generalisation of the gauge groups possible in flat superspace, or, as for $\mathcal{N} = 7$ and $\mathcal{N} = 8$, to new gauge groups which are unrelated to those in flat superspace.

Summarising, the coupling of the spin($\mathcal{N}$) on-shell multiplet to gauge theory and conformal supergravity has been described by generalising the analysis of supercovariant projections to gauge covariant projections. Covariance of the defining constraint (3.2.1) requires the strong condition (3.2.5b)/(3.2.11b), which must be fulfilled for the description of a covariant scalar multiplet. Solutions to this condition can be formulated for the coupling to supergravity in terms of the super-Cotton tensor $\boldsymbol{W}^{IJKL}$ and the torsion superfield $\boldsymbol{K}^{IJ}$ in the cases $\mathcal{N} \leq 6$.

The superfield action (3.2.9)/(3.2.22) for the covariantly constrained superfield $\boldsymbol{Q}_i$ can be obtained from the free action (2.2.73) by replacing the supercovariant derivatives by gauge- or supergravity-covariant derivatives and, for the latter, inserting the super-vielbein determinant in the action. These actions are scale-invariant and therefore describe conformal coupling.

3.3. Matter currents

The gauge- and gravitationally coupled spin($\mathcal{N}$) scalar multiplet transforms under the respective gauge group and under the local SO($\mathcal{N}$) structure group of conformal supergravity. In the Lagrangian formalism, it couples to the respective gauge fields as a corresponding current. If the kinetics of the gauge fields are described by Chern-Simons terms, the equations of motion equate these currents to the field strengths of the gauge fields. Since the field strengths are contained in the gauge and conformal-gravity multiplets respectively, these equations of motion can be recast as corresponding superfield equations, in order to obtain on-shell expressions for all fields in these multiplets. They will be constructed in the following.

3.3.1. Chern-Simons-matter current

The Chern-Simons action for a gauge field $B_a = B_a^A T_A = \frac{1}{2} B_a^{ij} T_{ij}$ is given by

$$S_{\mathrm{CS}} = \frac{1}{4\mu} \int \mathrm{d}^3x\ \varepsilon^{mnl} \left(-2B_m^{ij} F_{nl,ij} - \tfrac{4}{3} B_m^{ij} B_n^{jk} B_l^{ki}\right), \tag{3.3.1}$$

where

$$F_{ab}^{ij} = 2\partial_{[a} B_{b]}^{ij} + 2B_{[a}^{ik} B_{b]}^{kj}. \tag{3.3.2}$$

The equation of motion is

$$-\tfrac{1}{\mu} \varepsilon^{abc} F_{ab}^{ij} = \tfrac{2}{\mu} F_{ij}^{c} = 0. \tag{3.3.3}$$

The spin($\mathcal{N}$) scalar multiplet couples to the gauge field via the kinetic terms for q and $\check{q}_{\alpha,i}$ contained in the superfield action (3.2.9)

$$S_{\mathrm{m}} = \tfrac{1}{-8\mathcal{N}^2} \int \mathrm{d}^3x\ \mathrm{d}\theta_I^\alpha \mathrm{d}\theta_\alpha^I\ (\overline{\mathscr{D}_K^\gamma \boldsymbol{Q}}) \mathscr{D}_\gamma^K \boldsymbol{Q}|. \tag{3.3.4}$$

Working out the component form, they are found to be

$$\begin{aligned} S_{\mathrm{m,kin.}} &= -\tfrac{1}{2} \int \mathrm{d}^3x\ \overline{(\mathscr{D}^a q)} \mathscr{D}_a q - \tfrac{\mathrm{i}}{2} \bar{\check{q}}^\alpha \mathscr{D}_\alpha{}^\beta \check{q}_\beta - \tfrac{\mathrm{i}}{2} \check{q}^\alpha \overline{\mathscr{D}_\alpha{}^\beta \check{q}_\beta} \\ &= -\tfrac{1}{2} \int \mathrm{d}^3x\ \overline{(\partial^a q)} \partial_a q - \tfrac{\mathrm{i}}{2} \bar{\check{q}}^\alpha \partial_\alpha{}^\beta \check{q}_\beta - \tfrac{\mathrm{i}}{2} \check{q}^\alpha \overline{\partial_\alpha{}^\beta \check{q}_\beta} + B_{ij}^a j_a^{ij} + \tfrac{1}{4} B_{ij}^a B_a^{kl} \overline{(T^{ij} q)} (T_{kl} q), \end{aligned} \tag{3.3.5}$$

where

$$j_{ij}^a = \tfrac{1}{2} \overline{(\partial^a q)} T_{ij} q + \tfrac{1}{2} \overline{(T_{ij} q)} \partial^a q + \tfrac{\mathrm{i}}{2} (\gamma^a)^{\alpha\beta} \bar{\check{q}}_\alpha T_{ij} \check{q}_\beta. \tag{3.3.6}$$

In presence of this coupling, the equation of motion obtained from the full action $S = S_{\text{CS}} + S_{\text{m}}$ becomes

$$\begin{aligned}\tfrac{2}{\mu}F^a_{ij} =&\tfrac{1}{2}j^a_{ij} + \tfrac{1}{8}B^a_{kl}[\overline{(T^{kl}q)}T_{ij}q + \overline{(T_{ij}q)}T^{kl}q]\\ =&\tfrac{1}{4}\overline{(\mathscr{D}^a q)}T_{ij}q + \tfrac{1}{4}\overline{(T_{ij}q)}\mathscr{D}^a q + \tfrac{\mathrm{i}}{4}(\gamma^a)^{\alpha\beta}\bar{\check{q}}_\alpha T_{ij}\check{q}_\beta,\end{aligned} \tag{3.3.7}$$

which means that the field strength is equal to the covariant matter current.

On the other hand, the dimension-two field strength is contained in the gauge multiplet described by the superfield $\boldsymbol{F}^{IJ}$ as the covariant projection [34]

$$F_a =\tfrac{\mathrm{i}}{2\mathcal{N}(\mathcal{N}-1)}(\gamma_a)^{\alpha\beta}\mathscr{D}^I_\alpha\mathscr{D}^J_\beta\boldsymbol{F}_{IJ}|. \tag{3.3.8}$$

It is desirable to reproduce the equation of motion in presence of a matter current from a superfield equation of motion for $\boldsymbol{F}^{IJ}$. The dimension-one matter current must be proportional to the square of a dimension-one-half scalar superfield [21]. In addition, it has to obey the identity [34]

$$\mathscr{D}^I_\alpha\boldsymbol{F}^{JK} = \mathscr{D}^{[I}_\alpha\boldsymbol{F}^{JK]} - \tfrac{2}{\mathcal{N}-1}\delta^{I[J}\mathscr{D}_{\alpha L}\boldsymbol{F}^{K]L}. \tag{3.3.9}$$

Due to the property of the elements of the SO($\mathcal{N}$) Clifford algebra

$$\gamma^I\gamma^{JK} = \gamma^{IJK} + 2\delta^{I[J}\gamma^{K]} = \gamma^{IJK} - \tfrac{2}{\mathcal{N}-1}\delta^{I[J}\gamma_L\gamma^{K]L}, \tag{3.3.10}$$

it is consistent to write [32, 21]

$$\boldsymbol{F}^{IJ} = a\bar{\boldsymbol{Q}}\gamma^{IJ}\boldsymbol{Q}. \tag{3.3.11}$$

The Lie algebra valued gauge multiplet $\boldsymbol{F}^{IJ} = \frac{1}{2}\boldsymbol{F}^{IJ}_{ij}T^{ij}$ is expanded with the coefficients

$$\boldsymbol{F}^{IJ}_{ij} = a\bar{\boldsymbol{Q}}\gamma^{IJ}T_{ij}\boldsymbol{Q}, \tag{3.3.12}$$

where the trace over gauge indices is implied.

In order to obtain the dimension-two Chern-Simons equation of motion, one has to evaluate

$$\begin{aligned}\tfrac{1}{a}\mathscr{D}^I_{(\alpha}\mathscr{D}^J_{\beta)}\boldsymbol{F}^{ij}_{IJ}| =&(\mathscr{D}^I_{(\alpha}\mathscr{D}^J_{\beta)}\bar{\boldsymbol{Q}})\gamma_{IJ}T^{ij}\boldsymbol{Q}+\bar{\boldsymbol{Q}}\gamma_{IJ}\mathscr{D}^I_{(\alpha}\mathscr{D}^J_{\beta)}T^{ij}\boldsymbol{Q}+2(\mathscr{D}^I_{(\alpha}\bar{\boldsymbol{Q}})\gamma_{IJ}\mathscr{D}^J_{\beta)}T^{ij}\boldsymbol{Q}|\\ =&-\overline{(\mathrm{i}\gamma^{JI}\mathscr{D}_{\alpha\beta}q)}\gamma^{IJ}T^{ij}q+\mathrm{i}\bar{q}\gamma^{IJ}\gamma^{JI}\mathscr{D}_{\alpha\beta}T^{ij}q-2\overline{(\gamma^I\check{q}_{(\alpha)}}\gamma^{IJ}\gamma^JT^{ij}\check{q}_{\beta)}\\ =&-\mathrm{i}\overline{(\mathscr{D}_{\alpha\beta}q)}\gamma^{JI}\gamma^{IJ}T^{ij}q+\mathrm{i}\bar{q}\gamma^{IJ}\gamma^{JI}\mathscr{D}_{\alpha\beta}T^{ij}q-2\bar{\check{q}}_{(\alpha}\gamma^I\gamma^{IJ}\gamma^JT^{ij}\check{q}_{\beta)}\\ =&-\mathrm{i}\mathcal{N}(\mathcal{N}-1)[\overline{(\mathscr{D}_{\alpha\beta}q)}T^{ij}q+\overline{T^{ij}q}\mathscr{D}_{\alpha\beta}q-2\mathrm{i}\bar{\check{q}}_{(\alpha}T^{ij}\check{q}_{\beta)}],\end{aligned}$$

leading to

$$F^{ij}_{ab}=(\gamma_{ab})^{\alpha\beta}a[\overline{(\mathscr{D}_{\alpha\beta}q)}T^{ij}q+\overline{T^{ij}q}\mathscr{D}_{\alpha\beta}q-2\mathrm{i}\bar{\check{q}}_{(\alpha}T^{ij}\check{q}_{\beta)}]. \tag{3.3.13}$$

3.3.2. Supergravity-matter current

The Chern-Simons action for the SO($\mathcal{N}$) gauge field B^{IJ}_a is given by [39]

$$S_{\mathrm{CS}}=\frac{1}{4\mu\kappa^2}\int \mathrm{d}^3x\ e\ \varepsilon^{mnl}\left(-2B^{IJ}_mF_{nl,IJ}-\tfrac{4}{3}B^{IJ}_mB^{JK}_nB^{KI}_l\right), \tag{3.3.14}$$

where

$$F^{IJ}_{ab}=2\partial_{[a}B^{IJ}_{b]}+2B^{IK}_{[a}B^{KJ}_{b]}. \tag{3.3.15}$$

The equation of motion is

$$-\tfrac{1}{\mu\kappa^2}\varepsilon^{abc}F^{IJ}_{ab}=\tfrac{2}{\mu\kappa^2}F^c_{IJ}=0. \tag{3.3.16}$$

The spin($\mathcal{N}$) scalar multiplet couples to the SO($\mathcal{N}$) gauge field via the kinetic terms for q and $\check{q}_{\alpha,i}$ contained in the superfield action (3.2.22)

$$S_{\mathrm{m}}=\tfrac{1}{-8\mathcal{N}^2}\int \mathrm{d}^3x\ \mathrm{d}\theta^\alpha_I\mathrm{d}\theta^I_\alpha\ \boldsymbol{E}\ (\overline{\mathscr{D}^\gamma_K\boldsymbol{Q}})\mathscr{D}^K_\gamma\boldsymbol{Q}|. \tag{3.3.17}$$

They are

$$\begin{aligned}S_{\mathrm{m,kin.}}=&-\tfrac{1}{2}\int \mathrm{d}^3x\ e\ \overline{(\mathscr{D}^aq)}\mathscr{D}_aq-\tfrac{\mathrm{i}}{2}\bar{\check{q}}^\alpha\mathscr{D}_\alpha{}^\beta\check{q}_\beta-\tfrac{\mathrm{i}}{2}\check{q}^\alpha\overline{\mathscr{D}_\alpha{}^\beta\check{q}_\beta} \qquad (3.3.18)\\ =&-\tfrac{1}{2}\int \mathrm{d}^3x\ e\ \overline{(\partial^aq)}\partial_aq-\tfrac{\mathrm{i}}{2}\bar{\check{q}}^\alpha\partial_\alpha{}^\beta\check{q}_\beta-\tfrac{\mathrm{i}}{2}\check{q}^\alpha\overline{\partial_\alpha{}^\beta\check{q}_\beta}+B^a_{IJ}j^{IJ}_a+\tfrac{1}{16}B^a_{IJ}B^{KL}_a\overline{(\gamma^{IJ}q)}(\gamma_{KL}q),\end{aligned}$$

where

$$j^a_{IJ} = \tfrac{1}{4}\overline{(\partial^a q)}\gamma_{IJ}q + \tfrac{1}{4}\overline{(\gamma_{IJ}q)}\partial^a q + \tfrac{\mathrm{i}}{4}(\gamma^a)^{\alpha\beta}\bar{\check{q}}_\alpha\gamma_{IJ}\check{q}_\beta. \tag{3.3.19}$$

For the action $S = S_{\mathrm{CS}} + S_{\mathrm{m}}$ the equation of motion becomes

$$\begin{aligned}\tfrac{2}{\mu}F^a_{IJ} =& \tfrac{1}{2}j^a_{IJ} + \tfrac{1}{32}B^a_{KL}[\overline{(\gamma^{KL}q)}\gamma_{IJ}q + \overline{(\gamma_{IJ}q)}T^{KL}q] \\ =& \tfrac{1}{8}\overline{(\mathscr{D}^a q)}\gamma_{IJ}q + \tfrac{1}{8}\overline{(\gamma_{IJ}q)}\mathscr{D}^a q + \tfrac{\mathrm{i}}{8}(\gamma^a)^{\alpha\beta}\bar{\check{q}}_\alpha\gamma_{IJ}\check{q}_\beta.\end{aligned} \tag{3.3.20}$$

The SO($\mathcal{N}$) field strength F^{IJ}_{ab} is contained in the super-Cotton tensor $\boldsymbol{W}^{IJKL}$ as the covariant projection [32] derived in conformal superspace [37],

$$F^{IJ}_{(\alpha\beta)} = \tfrac{\mathrm{i}}{(\mathcal{N}-2)(\mathcal{N}-3)}\mathscr{D}^K_{(\alpha}\mathscr{D}^L_{\beta)}\boldsymbol{W}^{IJKL}|, \tag{3.3.21}$$

which is valid in the gauge $\boldsymbol{L}^{IJ}_a = 0$. Similarly to the gauge multiplet, the on-shell super-Cotton tensor must fulfil

$$\mathscr{D}^I_\alpha\boldsymbol{W}^{JKPQ} = \mathscr{D}^{[I}_\alpha\boldsymbol{W}^{JKPQ]} - \tfrac{4}{\mathcal{N}-3}\delta^{I[J}\mathscr{D}_{\alpha L}\boldsymbol{W}^{KPQ]L} \tag{3.3.22}$$

and is consistently described by [32, 21]

$$\boldsymbol{W}^{IJKL} = c\bar{\boldsymbol{Q}}\gamma^{IJKL}\boldsymbol{Q}. \tag{3.3.23}$$

The combinatorial number c is found by evaluating

$$\begin{aligned}\mathscr{D}^I_{(\alpha}\mathscr{D}^J_{\beta)}\bar{\boldsymbol{Q}}\gamma^{IJKL}\boldsymbol{Q} =& (\mathscr{D}^I_{(\alpha}\mathscr{D}^J_{\beta)}\bar{\boldsymbol{Q}})\gamma^{IJKL}\boldsymbol{Q} + \bar{\boldsymbol{Q}}\gamma^{IJKL}\mathscr{D}^I_{(\alpha}\mathscr{D}^J_{\beta)}\boldsymbol{Q} + 2(\mathscr{D}^I_{(\alpha}\bar{\boldsymbol{Q}})\gamma^{IJKL}\mathscr{D}^J_{\beta)}\boldsymbol{Q} \\ =& -\overline{(\mathrm{i}\gamma^{JI}\mathscr{D}_{\alpha\beta}q)}\gamma^{IJKL}q + \mathrm{i}\bar{q}\gamma^{IJKL}\gamma^{JI}\mathscr{D}_{\alpha\beta}q - 2\overline{(\gamma^I\check{q}_{(\alpha})}\gamma^{IJKL}\gamma^J\check{q}_{\beta)} \\ =& -\mathrm{i}\overline{(\mathscr{D}_{\alpha\beta}q)}\gamma^{JI}\gamma^{IJKL}q + \mathrm{i}\bar{q}\gamma^{IJKL}\gamma^{JI}\mathscr{D}_{\alpha\beta}q - 2\bar{\check{q}}_{(\alpha}\gamma^I\gamma^{IJKL}\gamma^J\check{q}_{\beta)} \\ =& -\mathrm{i}(\mathcal{N}-3)(\mathcal{N}-2)[\overline{(\mathscr{D}_{\alpha\beta}q)}\gamma^{KL}q + \overline{(\gamma^{KL}q)}\mathscr{D}_{\alpha\beta}q - 2\mathrm{i}\bar{\check{q}}_{(\alpha}\gamma^{KL}\check{q}_{\beta)}],\end{aligned}$$

leading to the conclusion that [31, 32]

$$\boldsymbol{W}^{IJKL} = \tfrac{\mu\kappa^2}{16}\bar{\boldsymbol{Q}}\gamma^{IJKL}\boldsymbol{Q}. \tag{3.3.24}$$

The equations of motion for the gauge and supergravity multiplets in terms of the dimension-one superfields constructed in this section are relevant for two reasons.

Firstly, in order to possibly resolve the coupling condition (3.2.5b), the field-strength multiplet must fulfil the equation of motion and be expressed by matter fields. The corresponding analysis will be performed in the next section. Similarly, the super-Cotton tensor must be expressed by matter fields in order to resolve the coupling condition (3.2.11b) for $\mathcal{N} \geq 6$ as will be shown in Section (3.5).

Secondly, the pre-factor in the determination of the super-Cotton tensor (3.3.24) will furthermore be required for the realisation of topologically massive gravities in Chapter 4.

3.4. All gauge groups for the $\mathrm{spin}(\mathcal{N})$ scalar multiplet

In this section, the coupling condition for the constrained $\mathrm{spin}(\mathcal{N})$ scalar multiplet is analysed in flat superspace for $\mathcal{N} \leq 8$. The results will be adapted to the presence of supergravity in the next section.

For the sake of generality, the superfield $\boldsymbol{Q}_i$ is taken to transform under the bifundamental representation of a flavour group of the form $F \times G$. The gauge algebra acting on $\boldsymbol{Q}_i$ has the form

$$\{\mathscr{D}^I_\alpha, \mathscr{D}^J_\beta\}\boldsymbol{Q}_i = 2\mathrm{i}\delta^{IJ}\mathscr{D}_{\alpha\beta}\boldsymbol{Q}_i + 2\mathrm{i}\varepsilon_{\alpha\beta}\boldsymbol{F}^{IJ}\boldsymbol{Q}_i + 2\mathrm{i}\varepsilon_{\alpha\beta}\boldsymbol{Q}_i\boldsymbol{G}^{IJ}, \tag{3.4.1}$$

where $\boldsymbol{F}^{IJ}$ and $\boldsymbol{G}^{IJ}$ are the right- and left-acting field strengths corresponding to the two group factors [21]. They must allow solutions for H^I of the equation

$$\gamma^{[J}H^{I]} = F^{IJ}q + qG^{IJ}, \tag{3.4.2}$$

where $H^J = \frac{\mathrm{i}}{2}\check{q}_{\beta,}{}^{\beta J}$.

A general ansatz in terms of the off-shell field strengths

$$H^I = AF^{IK}\gamma_K q + BF_{KL}\gamma^{IKL}q + C\gamma_K qG^{IK} + D\gamma^{IKL}qG_{KL} \tag{3.4.3}$$

does not provide such a solution. The field strengths must rather be replaced by their on-shell equations in terms of the scalar fields [21]. Introducing gauge-group indices, the right- and left-acting field-strength terms are expressed by

$$F^{IJ}_A(\tau^A q)_r{}^{\bar r} = a\,\mathrm{tr}(q\gamma^{IJ}\tau_A\bar q)(\tau^A q)_r{}^{\bar r} = a\,q_v{}^{\bar v}\gamma^{IJ}(\tau_A)_w{}^v\bar q_{\bar v}{}^w(\tau^A)_r{}^s q_s{}^{\bar r} \tag{3.4.4a}$$

$$(q\sigma^A)_r{}^{\bar r}G^{IJ}_A = b\,(q\sigma^A)_r{}^{\bar r}\mathrm{tr}(\bar q\bar\gamma^{IJ}\sigma_A q) = b\,q_r{}^{\bar s}(\sigma^A)_{\bar s}{}^{\bar r}\bar q_{\bar v}{}^v\bar\gamma^{IJ}(\sigma_A)_{\bar w}{}^{\bar v}q_v{}^{\bar w}, \tag{3.4.4b}$$

where the order of conjugated fields has been adjusted for convenience. The numbers a, b are the coupling constants and τ_A, σ_A are the generators of the right- and left-acting group factors, respectively. In the case of a fundamental representation the single field-strength term has the form

$$F^{IJ}_A(\tau^A\cdot q)_r = a\,\bar q^v\gamma^{IJ}(\tau_A)_v{}^w q_w(\tau^A)_r{}^s q_s. \tag{3.4.5}$$

It is further useful to define a product of three fields in a bifundamental representation which itself is in the bifundamental representation as

$$\begin{aligned}\{A\bar{B}C\}_{r\bar{r}} \equiv \quad & c_1 A_{r\bar{r}}\bar{B}C + c_2 A_r \bar{B}_{\bar{r}} C + c_3 A_r \bar{B} C_{\ \bar{r}} \\ & + d_1 A_{\ \bar{r}} \bar{B}_{\ r} C + d_2 A \bar{B}_{\bar{r}r} C + d_3 A \bar{B}_{\ r} C_{\ \bar{r}} \\ & + e_1 A_{\ \bar{r}} \bar{B} C_r + e_2 A \bar{B}_{\bar{r}} C_r + e_3 A \bar{B} C_{r\bar{r}} \\ & + C_{rvws}(f_1 A_v \bar{B}_{\ w} C_{s\bar{r}} + f_2 A_v \bar{B}_{\bar{r}w} C_s + f_3 A_{v\bar{r}} \bar{B}_{\ w} C_s) \\ & + C_{\bar{r}\bar{v}\bar{w}\bar{s}}(g_1 A_{\ \bar{v}} \bar{B}_{\bar{w}} C_{r\bar{s}} + g_2 A_{\ \bar{v}} \bar{B}_{\bar{w}r} C_{\ \bar{s}} + g_3 A_{r\bar{v}} \bar{B}_{\bar{w}} C_{\ \bar{s}}),\end{aligned} \tag{3.4.6}$$

where the invisible indices are appropriately contracted. Correspondingly, for a fundamental representation,

$$\{A\bar{B}C\}_\alpha = c_1 A_\beta \bar{B}_\beta C_\alpha + c_2 A_\beta \bar{B}_\alpha C_\beta + c_3 A_\alpha \bar{B}_\beta C_\beta + c_4 C_{\alpha\beta\gamma\delta} A_\beta \bar{B}_\gamma C_\delta. \tag{3.4.7}$$

The summation over Lie-algebra indices A can be evaluated using the known completeness or incompleteness relations for the generators (see Appendix A). The chosen conventions and examples for field-strength terms for a fundamental representation are presented in the table

group factor	$(\tau_A)_{\alpha\beta}(\tau^A)_{\gamma\delta}$	$(\bar{q}\gamma^{IJ}\tau_A q)(\tau^A q)_\alpha$
SO(N)	$2\delta_{\gamma[\alpha}\delta_{\beta]\delta}$	$2(\bar{q}_{[\alpha}\gamma^{IJ}q_{\beta]})q_\beta$
Sp(N)	$2\Omega_{\gamma(\alpha}\Omega_{\beta)\delta}$	$2\Omega^{\beta\gamma}(\bar{q}_{(\alpha}\gamma^{IJ}q_{\gamma)})q_\beta$
U(1)	$-q^2\delta^\beta_\alpha\delta^\delta_\gamma$	$-q^2(\bar{q}^\beta\gamma^{IJ}q_\beta)q_\alpha$
SU(N)	$\frac{1}{N}\delta^\beta_\alpha\delta^\delta_\gamma - \delta^\delta_\alpha\delta^\beta_\gamma$	$\frac{1}{N}(\bar{q}^\beta\gamma^{IJ}q_\beta)q_\alpha - (\bar{q}^\beta\gamma^{IJ}q_\alpha)q_\beta$
U(N)	$-\delta^\delta_\alpha\delta^\beta_\gamma$	$-(\bar{q}^\beta\gamma^{IJ}q_\alpha)q_\beta$
spin(7)	$2\delta_{\alpha[\gamma}\delta_{\delta]\beta} + \frac{3}{2}C_{\alpha\beta\gamma\delta}$	$2(\bar{q}_{[\alpha}\gamma^{IJ}q_{\beta]})q_\beta + \frac{3}{2}C_{\alpha\beta\gamma\delta}(\bar{q}_\beta\gamma^{IJ}q_\gamma)q_\delta$.

The analysis in terms of these tools will be carried out in subsections, each being concerned with scalar fields in fundamental representations of spin($\mathcal{N}$) which have the same dimension. These families are $\mathcal{N}$ = 2(Clifford)/$\mathcal{N}$ = 3/$\mathcal{N}$ = 4(chiral), $\mathcal{N}$ = 4(Clifford)/$\mathcal{N}$ = 5/$\mathcal{N}$ = 6(chiral) and $\mathcal{N}$ = 6(Clifford)/$\mathcal{N}$ = 7/$\mathcal{N}$ = 8(chiral), where the addenda (Clifford) and (chiral) denote the reducible Clifford representation and the irreducible chiral representation for even $\mathcal{N}$.

For better distiction, chiral spinor representations will be denoted by $\boldsymbol{Q}| = Q$, as opposed to $\boldsymbol{Q}| = q$ for the cases of odd $\mathcal{N}$ and reducible Clifford spinors for even $\mathcal{N}$.

3.4.1. $\mathcal{N} = 2$ (Clifford), $\mathcal{N} = 3$ and $\mathcal{N} = 4$ (chiral)

The spin matrices are (compare Appendix A)

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \gamma^* = \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.4.8}$$

and the generators proportional to γ^{12}, γ^{13} and γ^{23} are those of spin(3) = SU(2). The left- and right-handed chiral spin matrices are

$$(\Sigma^I)_{i\bar{i}} = [\mathbb{1}, \mathrm{i}\gamma_{1,2,3}]_{i\bar{i}} \tag{3.4.9a}$$
$$(\bar{\Sigma}^I)^{\bar{i}i} = [\mathbb{1}, -\mathrm{i}\gamma_{1,2,3}]^{\bar{i}i} \tag{3.4.9b}$$

and the corresponding spin group is $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$, where the two factors are associated with the indices i and $\bar{i}$ respectively. Relevant identities are

$$(\gamma^{[J})_{ij}(\gamma^{I]})_{kl} = \varepsilon_{(i(k}(\gamma^{IJ})_{l)j)} \tag{3.4.10}$$

and

$$(\Sigma^{[I})_{i\bar{j}}(\Sigma^{J]})_{k\bar{l}} = -\tfrac{1}{2}\varepsilon_{ik}(\bar{\Sigma}^{IJ})_{\bar{j}\bar{l}} + \tfrac{1}{2}\varepsilon_{\bar{j}\bar{l}}(\Sigma^{IJ})_{ik} \tag{3.4.11a}$$
$$(\Sigma^{[I})_{i\bar{j}}(\bar{\Sigma}^{J]})^{\bar{l}k} = -\tfrac{1}{2}\delta_i^k(\bar{\Sigma}^{IJ})^{\bar{l}}{}_{\bar{j}} + \tfrac{1}{2}\delta_{\bar{j}}^{\bar{l}}(\Sigma^{IJ})_i{}^k. \tag{3.4.11b}$$

The most general expression for H^I in terms of the Clifford spinor is given by

$$\begin{aligned} H_i^I =& (\gamma^I)_i{}^j(A\{q_j\bar{q}^k q_k\} + B\{q^k\bar{q}_j q_k\} + C\{q^k\bar{q}_k q_j\}) \\ &+ 4(\gamma^I)_k{}^l(D\{q^k\bar{q}_l q_i\} + E\{q^k\bar{q}_i q_l\} + F\{q_i\bar{q}^k q_l\}) \end{aligned} \tag{3.4.12}$$

and leads to

$$\begin{aligned}(\gamma^{[J}H^{I]})_i = &- A(\gamma^{IJ}\{q)_i\bar{q}^k q_k\} - B\{q^k(\gamma^{IJ}\bar{q})_i q_k\} - C\{q^k\bar{q}_k(\gamma^{IJ}q)_i\} \\ &+ D\{q_i(\bar{q}\gamma^{IJ}q) + q^k\bar{q}_i(\gamma^{IJ}q)_k - q^k(\gamma^{IJ}\bar{q})_i q_k - (\gamma^{IJ}q)_i\bar{q}^k q_k\} \\ &+ E\{q_i(\bar{q}\gamma^{IJ}q) + (q\gamma^{IJ}\bar{q})q_i - q^k\bar{q}_k(\gamma^{IJ})_i + (\gamma^{IJ}q)_i\bar{q}^k q_k\} \\ &+ F\{q^k\bar{q}_i(\gamma^{IJ}q)_k + (q\gamma^{IJ}\bar{q})q_i + q^k\bar{q}_k(\gamma^{IJ}q)_i + q^k(\gamma^{IJ}\bar{q})_i q_k\}.\end{aligned} \tag{3.4.13}$$

In order to reproduce the on-shell field strengths, the coefficients must be related by $D = -F = -B$, $E - D = A$ and $F - E = C$, so that

$$(\gamma^{[J}H^{I]})_i = (E+F)\{(q\gamma^{IJ}\bar{q})q_i\} + (E-F)\{q_i(\bar{q}\gamma^{IJ}q)\}. \tag{3.4.14}$$

Since the left- and right-acting terms have independent coefficients, any gauge group is possible if one chooses the appropriate coefficients in the expansion (3.4.6) or (3.4.7) of $\{.\}$.

The most general expression for H^I in terms of a left-handed chiral spinor reads

$$H^{I,\bar{m}} = (\bar{\Sigma}^I)^{\bar{m}m}(A\{Q_m\bar{Q}^iQ_i\} + B\{Q^i\bar{Q}_mQ_i\} + C\{Q^i\bar{Q}_iQ_m\}), \tag{3.4.15}$$

leading to

$$(\Sigma^{[J}H^{I]})_k = (\Sigma^{JI})^{im}(A\{Q_m\bar{Q}_kQ_i - Q_m\bar{Q}_iQ_k\} + C\{Q_k\bar{Q}_iQ_m - Q_i\bar{Q}_kQ_m\}), \tag{3.4.16}$$

where B has been set to zero without loss of generality. This case requires the relation $A = C$ in order to cancel the terms where the free spin($\mathcal{N}$) index k is carried by the conjugate field, so that

$$(\Sigma^{[J}H^{I]})_k = -A\{(Q\Sigma^{IJ}\bar{Q})Q_k - Q_k(\bar{Q}\Sigma^{IJ}Q)\}. \tag{3.4.17}$$

The only expansion of $\{.\}$ reproducing group-specific field-strength terms which do not involve an invariant tensor C_{ijkl} is then given by the choice where only $c_3 \neq 0$. This can be written as

$$(\Sigma^{[J}H^{I]})_k = -Ac_3[(Q\Sigma^{IJ}\bar{Q})Q_k - Q_k(\bar{Q}\Sigma^{IJ}Q)], \tag{3.4.18}$$

where the products are understood as matrix products with the bifundamental indices.

The groups which naturally fulfil this requirement are $\mathrm{U}(M) \times \mathrm{U}(N)$ and $\mathrm{SU}(N) \times \mathrm{SU}(N)$ [13] with opposite couplings $a = -b$ or, more generally, $\mathrm{SU}(M) \times \mathrm{SU}(N) \times \mathrm{U}(1)^{\circ}$ where the U(1) charge is constrained by $-q^2 = a(\frac{1}{M} - \frac{1}{N})$ so that it cancels the gauge-traced bilinear terms from the $\mathrm{SU}(N)$ factors. This product can be extended by further pairwise cancelling U(1) factors. The coefficients are $Ac_3 = a$, resulting in the solution

$$H^{I,\bar{m}} = a(\bar{\Sigma}^I)^{\bar{m}m}(Q_m \bar{Q}^i Q_i - Q_i \bar{Q}^i Q_m). \tag{3.4.19}$$

Another combination is $\mathrm{Sp}(M) \times \mathrm{SO}(N)$ [13] with opposite couplings and the reality condition

$$Q_i = \bar{Q}^j \varepsilon_{ji}, \tag{3.4.20}$$

leading to

$$H^{I,\bar{m}} = 2a(\bar{\Sigma}^I)^{\bar{m}m}(Q_m \bar{Q}^i Q_i - Q_i \bar{Q}^i Q_m). \tag{3.4.21}$$

The third possibility is to involve an invariant antisymmetric tensor $C_{[ijkl]}$ in the expansion of $\{.\}$. In this case, the second group factor has to be SU(2) in order to write

$$-C_{rvws} Q_v{}^{\bar{v}} (\bar{Q}_{\bar{v}w} \Sigma^{IJ} Q_s{}^{\bar{r}}) = \tfrac{1}{2} C_{rvws} (Q_v{}^{\bar{v}} \Sigma^{IJ} \bar{Q}_{\bar{v}w}) Q_s{}^{\bar{r}}, \tag{3.4.22}$$

where the reality of Q has been used. Since the left-acting symplectic SU(2) requires the presence of an orthogonal term from the right-acting factor, the generators of the right-acting group must fulfil

$$(\tau^A)_{ij}(\tau_A)_{kl} = 2\delta_{i[k}\delta_{l]j} + \tfrac{3}{2} C_{ijkl}. \tag{3.4.23}$$

This is the case for spin(7) (and its subgroup G_2) [17]. In this case, the solution reads

$$\begin{aligned} H^{I,\bar{m}} =& 2a(\bar{\Sigma}^I)^{\bar{m}m}(Q_m \bar{Q}^i Q_i - Q_i \bar{Q}^i Q_m)_r{}^{\bar{r}} \\ &+ a(\bar{\Sigma}^I)^{\bar{m}m} C_{rvws}(Q_{m,v}{}^{\bar{v}} \bar{Q}_{\bar{v}w} Q_s{}^{\bar{r}} - Q_v{}^{\bar{v}} \bar{Q}_{\bar{v}w} Q_{m,s}{}^{\bar{r}}). \end{aligned} \tag{3.4.24}$$

Finally, the analysis can be repeated for fundamental representations. This leads to

$$(\Sigma^{[J} H^{I]})_{k,\beta} = \mp c_1 \left[(\bar{Q}^{\alpha} \Sigma^{IJ} Q_{\alpha}) Q_{k,\beta} - (\bar{Q}^{\alpha} \Sigma^{IJ} Q_{\beta}) Q_{k,\alpha} \right] - (c_2 \mp c_2)(\bar{Q}_{\beta} \Sigma^{IJ} Q^{\alpha}) Q_{k,\alpha}, \tag{3.4.25}$$

where the lower sign holds for groups with an antisymmetric metric and, without loss of generality, it has been chosen $c_3 = 0$ and $A = 1$. This is consistent with the groups $\mathrm{Sp}(N) \times \mathrm{U}(1)^\circ$ ($q^2 = a$) and $\mathrm{SU}(N) \times \mathrm{U}(1)^\circ$ ($q^2 = \frac{a}{N} - a$),with the U(1) charges being restricted as indicated. For $\mathrm{Sp}(N) \times \mathrm{U}(1)^\circ$ the solution is

$$H^{I,\bar{m}} = \tfrac{3}{2} a (\bar{\Sigma}^I)^{\bar{m}m} (Q_{m,\alpha} \bar{Q}^{i,\alpha} Q_{i,\beta} - Q_{i,\alpha} \bar{Q}^{i,\alpha} Q_{m,\beta}) \tag{3.4.26}$$

and for $\mathrm{SU}(N) \times \mathrm{U}(1)^\circ$ it is

$$H^{I,\bar{m}} = -a (\bar{\Sigma}^I)^{\bar{m}m} (Q_{m,\alpha} \bar{Q}^{i,\alpha} Q_{i,\beta} - Q_{i,\alpha} \bar{Q}^{i,\alpha} Q_{m,\beta}). \tag{3.4.27}$$

These fundamental representations are equivalent to special cases of the above bifundamental ones, namely $\mathrm{Sp}(M) \times \mathrm{SO}(2)$ and $\mathrm{SU}(N) \times \mathrm{SU}(1) \times \mathrm{U}(1)^\circ$, respectively.

3.4.2. $\mathcal{N} = 4$ (Clifford), $\mathcal{N} = 5$ and $\mathcal{N} = 6$ (chiral)

The transition between the $\mathcal{N} = 4$ (chiral) and $\mathcal{N} = 4$ (Clifford) representations is performed by formally defining the Clifford spinor

$$q_a \doteq \begin{pmatrix} Q_i \\ Q^{\bar{i}} \end{pmatrix}, \tag{3.4.28}$$

the Clifford matrices

$$(\gamma^I)_a{}^b \doteq \begin{pmatrix} 0 & (\Sigma^I)_{i\bar{j}} \\ (\bar{\Sigma}^I)^{\bar{i}j} & 0 \end{pmatrix} \quad , \quad (\gamma^*)_a{}^b \doteq \begin{pmatrix} \delta_i^j & 0 \\ 0 & -\delta_{\bar{j}}^{\bar{i}} \end{pmatrix} \tag{3.4.29}$$

and also

$$C^{ab} \doteq \mathrm{diag}(\varepsilon^{ij}, \varepsilon_{\bar{i}\bar{j}}). \tag{3.4.30}$$

These matrices are equal to the SO(5) matrices $(\gamma^I)_i{}^j$, which can be written as

$$\gamma_1 = \sigma_1 \otimes \mathbb{1} \tag{3.4.31a}$$

$$\gamma_{2,3,4} = -\sigma_2 \otimes \hat{\gamma}_{1,2,3} \tag{3.4.31b}$$

$$\gamma^* = \gamma_5 = \sigma_3 \otimes \mathbb{1}. \tag{3.4.31c}$$

The corresponding generators are those of USp(4) with the metric $\varepsilon_{ij} = C_{ab}$, i.e. $\varepsilon_{12} = -\varepsilon_{21} = \varepsilon_{34} = -\varepsilon_{43} = 1$. The $\mathcal{N} = 5$ matrices with lower and upper indices $\tilde{\gamma}^I$ serve as the chiral blocks for $\mathcal{N}=6$, i.e.

$$(\Sigma^I)_{ij} = [\varepsilon, \mathrm{i}\tilde{\gamma}^{1,2,3,4,5}]_{ij} \tag{3.4.32a}$$
$$(\bar{\Sigma}^I)^{ij} = [-\varepsilon, \mathrm{i}\tilde{\gamma}^{1,2,3,4,5}]^{ij}, \tag{3.4.32b}$$

and the corresponding generators are those of SU(4).

Basic identities for these spin matrices are

$$\varepsilon^{ijkl}(\gamma^I)_{kl} = -2(\gamma^I)^{ij} \tag{3.4.33a}$$
$$\varepsilon^{ijkl}(\Sigma^I)_{kl} = -2(\bar{\Sigma}^I)^{ij}, \tag{3.4.33b}$$

$$(\gamma^I)_i{}^j(\gamma_I)_k{}^l = -\delta_i^j\delta_k^l + 2\varepsilon_{ik}\varepsilon^{jl} + 2\delta_i^l\delta_k^j \tag{3.4.34a}$$
$$(\Sigma^I)_{ij}(\bar{\Sigma}_I)^{kl} = -4\delta_{[i}^k\delta_{j]}^l \tag{3.4.34b}$$
$$(\bar{\Sigma}^I)^{ij}(\bar{\Sigma}_I)^{kl} = 2\varepsilon^{ijkl} \tag{3.4.34c}$$

and

$$(\gamma^{[I})_{ij}(\gamma^{J]})^{kl} = 2\delta_{[i}^{[k}(\gamma^{IJ})_{j]}{}^{l]} \tag{3.4.35a}$$
$$(\bar{\Sigma}^{[I})^{ij}(\Sigma^{J]})_{kl} = 2\delta_{[k}^{[i}(\Sigma^{IJ})_{l]}{}^{j]}. \tag{3.4.35b}$$

The most general expression for H_a^I in terms of an $\mathcal{N}=5$ spinor is

$$\begin{aligned} H_k^I =& (\gamma^I)_k{}^l\left[A\{q_l\bar{q}^iq_i\} + B\{q^i\bar{q}_lq_i\} + C\{q^i\bar{q}_iq_l\}\right] \\ &+ 2(\gamma^I)_i{}^j\left[D\{q^i\bar{q}_jq_k\} + E\{q^i\bar{q}_kq_j\} + F\{q_k\bar{q}^iq_j\}\right]. \end{aligned} \tag{3.4.36}$$

Without loss of generality, one can choose $D = F = B = 0$. In order to generate field-strength terms, the relation $A = C = -E$ must hold, so that

$$(\gamma^{[J}H^{I]})_k = E\{(q\gamma^{IJ}\bar{q})q_k - q_k(\bar{q}\gamma^{IJ}q)\}. \tag{3.4.37}$$

The formulation in terms of an $\mathcal{N}=4$ Clifford spinor is equivalent, except for the range of I. For example, it holds that

$$\begin{aligned}(\gamma^{[J}H^{I]})_a \doteq& E\{\begin{pmatrix}(\Sigma^{IJ}Q)_k\bar{Q}_{\bar{j}}Q^{\bar{j}} + Q_k(\bar{Q}\bar{\Sigma}^{IJ}Q) + Q_{\bar{j}}\bar{Q}^{\bar{j}}(\Sigma^{IJ}Q)_k - (Q\bar{\Sigma}^{IJ}\bar{Q})Q_k \\ (Q\Sigma^{IJ}\bar{Q})Q^{\bar{k}} - Q^j\bar{Q}_j(\bar{\Sigma}^{IJ}Q)^{\bar{k}} - Q^{\bar{k}}(\bar{Q}\Sigma^{IJ}Q) - (\bar{\Sigma}^{IJ}Q)^{\bar{k}}\bar{Q}^jQ_j\end{pmatrix}\} + ... \\ =& E\{(q\gamma^{IJ}\bar{q})q_a - q_a(\bar{q}\gamma^{IJ}q) - (\gamma^{IJ}q)_a\bar{q}^cq_c - q^c\bar{q}_c(\gamma^{IJ}q)_a\} + ...\end{aligned}$$

and similarly for other types of terms. In particular, no terms involving γ^* are required.

As is apparent, the allowed gauge groups are the same as in the case of the chiral SO(4) spinor discussed in the previous subsection [15, 17]. The corresponding solutions H^I in terms of the $\mathcal{N}=5$ spinor are listed in the table

gauge group	$H^{I,a}$
U(M) × U(N)	$a(\gamma^I)^{ab}(q_b\bar{q}^cq_c - q_c\bar{q}^cq_b) + 2a(\gamma^I)^{cd}q_c\bar{q}^aq_d$
Sp(M) × SO(N)	$2a(\gamma^I)^{ab}(q_b\bar{q}^cq_c - q_c\bar{q}^cq_b) + 4a(\gamma^I)^{cd}q_c\bar{q}^aq_d$
spin(7) × SU(2)	$-2a(\gamma^I)^{ab}(q_b\bar{q}^cq_c - q_c\bar{q}^cq_b) - 4a(\gamma^I)^{cd}q_c\bar{q}^aq_d$
	$-a(\gamma^I)^{ab}C[(q_b\bar{q}^cq_c - q_c\bar{q}^cq_b)] - 2a(\gamma^I)^{cd}C[q_c\bar{q}^aq_d]$
SU(N) × U(1)	$-a(\gamma^I)^{ab}(q_b^\alpha\bar{q}_\alpha^cq_c^\beta - q_c^\alpha\bar{q}_\alpha^cq_b^\beta) - 2a(\gamma^I)^{cd}q_c^\alpha\bar{q}_\alpha^aq_d^\beta$
Sp(N) × U(1)	$-\frac{3}{2}a(\gamma^I)^{ab}(q_b^\alpha\bar{q}_\alpha^cq_c^\beta - q_c^\alpha\bar{q}_\alpha^cq_b^\beta) - 3a(\gamma^I)^{cd}q_c^\alpha\bar{q}_\alpha^aq_d^\beta$,

where $(C[A\bar{B}C])_{r\bar{r}} \equiv C_{rvws}A_{v\bar{v}}\bar{B}_{\bar{v}w}C_{s\bar{r}}$. They may be written in the compact form

$$H^{I,a} = -\,E(\gamma^I)^{ab}\{(q_b\bar{q}^cq_c - q_c\bar{q}^cq_b) + 2(\gamma^I)^{cd}q_c\bar{q}^aq_d\}, \tag{3.4.38}$$

with the appropriate coefficients specified above.

The most general expression for H^I in terms of a left-handed chiral $\mathcal{N}=6$ spinor is

$$H^{I,k} = 2A(\bar{\Sigma}^I)^{ij}\{Q_i\bar{Q}^kQ_j\} + B(\bar{\Sigma}^I)^{kl}\{Q_l\bar{Q}^iQ_i\} + C(\bar{\Sigma}^I)^{kl}\{Q_i\bar{Q}^iQ_l\}. \tag{3.4.39}$$

The coefficients must be chosen $A=B=-C$, leading to

$$(\Sigma^{[J}H^{I]})_k = A\{(Q\bar{\Sigma}^{IJ}\bar{Q})Q_k + Q_k(\bar{Q}\Sigma^{IJ}Q)\}. \tag{3.4.40}$$

This admits the same gauge groups as for $\mathcal{N}=4$ and $\mathcal{N}=5$, with the exceptions of Sp(M) × SO($N>2$) and spin(7) × SU(2) [15, 16], since the necessary reality condition is not available for the group spin(6) = SU(4). The solutions can be written as

$$H^{I,k} = -A(\bar{\Sigma}^I)^{kl}(\{Q_l\bar{Q}^iQ_i - Q_i\bar{Q}^iQ_l) + 2(\bar{\Sigma}^I)^{ij}Q_i\bar{Q}^kQ_j\} \tag{3.4.41}$$

and correspond to the above solutions for $\mathcal{N} = 5$ after obvious replacements.

3.4.3. $\mathcal{N} = 6$ (Clifford), $\mathcal{N} = 7$ and $\mathcal{N} = 8$ (chiral)

The transition between the representations $\mathcal{N} = 6$ (chiral) and $\mathcal{N} = 6$ (Clifford) is performed by formally defining the Clifford spinor

$$q_a \doteq \begin{pmatrix} Q_i \\ P^i \end{pmatrix} \tag{3.4.42}$$

and the Clifford matrices

$$\gamma^{1,\dots,6} \doteq \begin{pmatrix} 0 & \Sigma^{1,\dots,6} \\ \bar{\Sigma}^{1,\dots,6} & 0 \end{pmatrix} \quad , \quad \gamma^* = \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1}, \tag{3.4.43}$$

which are equal to the SO(7) spin matrices with the identification $\gamma^* = \gamma^7$. Conveniently, these have an imaginary and antisymmetric representation, which is constructed via the basis transformation

$$\gamma^I \longrightarrow U\gamma^I U^\dagger \quad , \quad U = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathrm{i}\mathbb{1} & \mathrm{i}\mathbb{1} \end{pmatrix}. \tag{3.4.44}$$

The chiral blocks $(\Sigma^I)_{i\bar{i}}$ and $(\bar{\Sigma}^I)_{\bar{i}i}$ for $\mathcal{N} = 8$ are given by

$$\begin{aligned} \Sigma^1 &= \bar{\Sigma}^1 = \mathbb{1} \\ \Sigma^{2,\dots,8} &= -\bar{\Sigma}^{2,\dots,8} = \mathrm{i}\tilde{\gamma}^{1,\dots,7}, \end{aligned} \tag{3.4.45}$$

so that $(\Sigma^I)^{\mathrm{T}} = \bar{\Sigma}^I$.

The spin matrices fulfil

$$(\gamma^I)_{i(j}(\gamma^I)_{l)k} = \delta_{ik}\delta_{jl} - \delta_{i(j}\delta_{l)k} \tag{3.4.46}$$

and are subject to the Fierz lemma (see Appendix A)

$$8\delta_{ik}\delta_{jl} = \delta_{il}\delta_{jk} + \gamma^I_{[il]}\gamma^I_{jk} - \tfrac{1}{2}\gamma^{IJ}_{[il]}\gamma^{IJ}_{jk} - \tfrac{1}{6}\gamma^{IJK}_{(il)}\gamma^{IJK}_{jk}. \tag{3.4.47}$$

Some relevant consequences are

$$\begin{aligned}
\gamma^{I[K}_{ij}\gamma^{L]I}_{kl} &= 4\delta_{[i[k}\gamma^{KL}_{l]j]} - \gamma^{[K}_{ij}\gamma^{L]}_{kl} && (3.4.48a)\\
\gamma^{IJ[K}_{ij}\gamma^{L]IJ}_{kl} &= -8\delta_{(k(i}\gamma^{KL}_{j)l)} && (3.4.48b)\\
-8\delta_{(k[i}\gamma^{K}_{j]l)} &= \delta_{kl}\gamma^K_{ij} - \tfrac{1}{2}\gamma^{KIJ}_{kl}\gamma^{IJ}_{ij} && (3.4.48c)\\
-8\delta_{[k[i}\gamma^{K}_{j]l]} &= -\gamma^{KI}_{kl}\gamma^I_{ij} + \gamma^{KI}_{ij}\gamma^I_{kl}. && (3.4.48d)
\end{aligned}$$

In terms of γ^* instead of γ^7, the Fierz lemma is

$$\begin{aligned}
8\delta_{ab}\delta_{cd} =& \delta_{ad}\delta_{cb} + \gamma^I_{[ad]}\gamma^I_{cb} + \gamma^*_{[ad]}\gamma^*_{cb} - \tfrac{1}{2}\gamma^{IJ}_{[ad]}\gamma^{IJ}_{cb} - \gamma^{*I}_{[ad]}\gamma^{*I}_{cb}\\
&- \tfrac{1}{6}\gamma^{IJK}_{(ad)}\gamma^{IJK}_{cb} - \tfrac{1}{2}\gamma^{*IJ}_{(ad)}\gamma^{*IJ}_{cb}
\end{aligned} \tag{3.4.49}$$

and provides the identities

$$\begin{aligned}
\gamma^{I[K}_{ab}\gamma^{L]I}_{cd} + \gamma^{*[K}_{ab}\gamma^{L]*}_{cd} &= 4\delta_{[a[c}\gamma^{KL}_{d]b]} - \gamma^{[K}_{ab}\gamma^{L]}_{cd} && (3.4.50a)\\
8\delta_{(c[b}\gamma^{K}_{a]d)} &= \delta_{cd}\gamma^K_{ab} - \gamma^{K*J}_{cd}\gamma^{*J}_{ab} - \tfrac{1}{2}\gamma^{KIJ}_{cd}\gamma^{IJ}_{ab} && (3.4.50b)\\
8\delta_{[c[b}\gamma^{K}_{a]d]} &= -\gamma^{KI}_{cd}\gamma^I_{ab} + \gamma^{KI}_{ab}\gamma^I_{cd} - \gamma^{K*}_{cd}\gamma^*_{ab} + \gamma^{K*}_{ab}\gamma^*_{cd}. && (3.4.50c)
\end{aligned}$$

In terms of the chiral matrices follows the so-called "triality relation"[1]

$$(\Sigma^I)_{i(\bar{i}}(\Sigma^I)_{j\bar{j})} = \delta_{ij}\delta_{\bar{i}\bar{j}} \tag{3.4.51}$$

and

$$(\Sigma^{K[I})_{ij}(\Sigma^{J]K})_{kl} = 4\delta_{[i[k}(\Sigma^{IJ})_{l]j]}. \tag{3.4.52}$$

As suggested by the Fierz identities, a sufficiently general expression for H^I in terms

[1] It indicates that interchanging the role of the SO(8) indices with that of one of the spin(8) matrix indices specifies new spin matrices solving the Clifford algebra. For the superspace it is then formally possible to let the spinor coordinates transform under one of the spin(8) representations while the scalar multiplet carries an SO(8) vector index.

of an $\mathcal{N}=7$ spinor is

$$\begin{aligned} H^I_i =&\gamma^I_{ij}\left[A\{q^j\bar{q}^k q_k\} + B\{q^k\bar{q}^j q_k\} + C\{q^k\bar{q}_k q^j\}\right] \\ &+\gamma^I_{kl}\left[D\{q^k\bar{q}^l q_i\} + E\{q^k\bar{q}_i q^l\} + F\{q_i\bar{q}^k q^l\}\right] \\ &+\gamma^{IK}_{kl}\gamma^K_{ij}\left[G\{q^k\bar{q}^l q^j\} + H\{q^j\bar{q}^k q^l\}\right]. \end{aligned} \tag{3.4.53}$$

Without loss of generality one can choose $H=0$. In order to produce only field-strength terms, the necessary relations are $G=-D$, $E=0$ and $A=-B=-G$, leading to

$$(\gamma^{[J}H^{I]})_m =G\{(q\gamma^{IJ}\bar{q})q_m + q_m(\bar{q}\gamma^{IJ}q) - \gamma^{IJ}_{kj}q_k\bar{q}_m q_j\}. \tag{3.4.54}$$

The third term in this expression can only be converted into a field-strength term if the gauge group is real SU(2) × SU(2) [21], where

$$\bar{q}_{\bar{v}}{}^{v} = \varepsilon^{vw} q_w{}^{\bar{w}}\varepsilon_{\bar{w}\bar{v}}. \tag{3.4.55}$$

In this case, the relation

$$q_{[k}\bar{q}_{|i|}q_{m]} = -\tfrac{1}{2}q_{[k}\bar{q}_{m]}q_i - \tfrac{1}{2}q_i\bar{q}_{[k}q_{m]}, \tag{3.4.56}$$

following from

$$q_l \mathrm{tr}(\bar{q}_k q_m) = q_k\bar{q}_m q_l + q_m\bar{q}_k q_l = q_l\bar{q}_k q_m + q_l\bar{q}_m q_k, \tag{3.4.57}$$

applies and leads to

$$(\gamma^{[J}H^{I]})_k =\tfrac{3}{2}Gc_3\left[(q\gamma^{IJ}\bar{q})q_k + q_k(\bar{q}\gamma^{IJ}q)\right]. \tag{3.4.58}$$

With $Gc_3 = -\frac{2}{3}a$, the resulting solution is

$$H^I_i =\tfrac{2}{3}a\left[(\gamma^I q)_i\bar{q}^k q_k - q^k(\gamma^I\bar{q})_i q_k + (q\gamma^I\bar{q})q_i - (q\gamma^{IK}\bar{q})(\gamma^K q)_i\right]. \tag{3.4.59}$$

For fundamental representations one finds no solution.

The analysis in terms of an $\mathcal{N}=6$ Clifford spinor leads to the same conclusion as for the $\mathcal{N}=7$ spinor, with the condition

$$q_a \doteq \begin{pmatrix} Q_i \\ \bar{Q}^i \end{pmatrix}, \tag{3.4.60}$$

where $\bar{Q}^i = (Q_i)^*$. In the real basis introduced above, this spinor becomes real in the form

$$\begin{pmatrix} Q_i \\ \bar{Q}^i \end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathrm{i}\mathbb{1} & \mathrm{i}\mathbb{1} \end{pmatrix} \cdot \begin{pmatrix} Q_i \\ \bar{Q}^i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} Q_i + \bar{Q}^i \\ -\mathrm{i}(Q_i - \bar{Q}^i) \end{pmatrix} . \tag{3.4.61}$$

This formulation is therefore equivalent to $\mathcal{N} = 7$, if the replacement of γ^7 by γ^* takes place. Correspondingliy, the solution is written as

$$H_i^I = \tfrac{2}{3} a \left[(\gamma^I q)_i \bar{q}^k q_k - q^k (\gamma^I \bar{q})_i q_k + (q\gamma^I \bar{q}) q_i - (q\gamma^{IK} \bar{q})(\gamma^K q)_i - (q\gamma^{I*} \bar{q})(\gamma^* q)_i \right] . \tag{3.4.62}$$

The most general form for H^I in terms of a left-handed chiral $\mathcal{N} = 8$ spinor is

$$H_{\bar{k}}^I = (\bar{\Sigma}^I)_{\bar{k}k} \left[A\{Q_k \bar{Q}_i Q_i\} + B\{Q_i \bar{Q}_k Q_i\} \right] + C(\Sigma^{IK})_{ij} (\bar{\Sigma}^K)_{\bar{k}k} \{Q_i \bar{Q}_j Q_k\}. \tag{3.4.63}$$

Choosing $C = -A = B$ leads to

$$(\Sigma^{[J} H^{I]})_m = C\{(Q\Sigma^{IJ}\bar{Q})Q_m + Q_m(\bar{Q}\Sigma^{IJ}Q) - \Sigma_{kl}^{IJ} Q_k \bar{Q}_m Q_l\}. \tag{3.4.64}$$

As is known from the analysis for $\mathcal{N} = 7$ the only solution is for real SU(2) $\times$ SU(2) and reads

$$H_{\bar{k}}^I = \tfrac{2}{3} a (\bar{\Sigma}^I)_{\bar{k}k} \left[Q_k \bar{Q}_i Q_i - Q_i \bar{Q}_k Q_i \right] - \tfrac{2}{3} a (\Sigma^{IK})_{ij} (\bar{\Sigma}^K)_{\bar{k}k} Q_i \bar{Q}_j Q_k. \tag{3.4.65}$$

This concludes the determination of allowed gauge groups in flat superspace. An overview will be given in Section (3.6). In the next section, the above analysis is adapted to the presence of superconformal gravity.

3.5. All gauge groups for the gravitationally coupled spin($\mathcal{N}$) scalar multiplet

The coupling of the spin($\mathcal{N}$) scalar multiplet to both a gauge sector and supergravity demands a solution to the equation

$$\gamma^{[J}H^{I]} = F^{IJ}q + qG^{IJ} + \tfrac{1}{4}(W^{IJKL}\gamma_{KL} + 4K\gamma^{IJ})q. \tag{3.5.1}$$

As mentioned in Section (3.1), this solution is not always given by

$$H^I = H^I_{\text{SG}} + H^I_{\text{CS}} \tag{3.5.2}$$

or can in some cases be more general. In the following, each case $\mathcal{N}$ with matter fields transforming under irreducible chiral representations of spin($\mathcal{N}$) is considered separately. The on-shell super-Cotton tensor is specified in terms of its corresponding matter current implied by the coupling to a gauge-coupled scalar and the implications of the coupling condition are analysed.

3.5.1. $\mathcal{N} = 4$

On shell, the super-Cotton tensor $W^{IJKL} \equiv W\varepsilon^{IJKL}$ is, in terms of a gauge-coupled scalar, given by

$$W = \tfrac{\lambda}{16}|\bar{Q}^i Q_i|, \tag{3.5.3}$$

where $|.|$ indicates the trace over gauge-group indices. The complete solution to the coupling condition is then

$$H^I = H^I_{\text{SG}} + H^I_{\text{CS}}, \tag{3.5.4}$$

where

$$H^I_{\text{SG}} = \tfrac{1}{2}(\tfrac{\lambda}{16}|\bar{Q}^i Q_i| - 2K)\bar{\Sigma}^I Q \tag{3.5.5}$$

and H^I_{CS} is the contribution from the gauge sector of the desired gauge group. There is no other solution both representing the supergravity sector and generalising the gauge groups possible in flat superspace than the above sum.

3.5.2. $\mathcal{N} = 5$

The on-shell super-Cotton tensor is given by

$$W^I = -\tfrac{\lambda}{16}|q\gamma^I\bar{q}|, \tag{3.5.6}$$

in terms of which H^I_{SG} becomes

$$\begin{aligned} H^I_{\text{SG},k} =& -\tfrac{1}{2}W_K(\gamma^{IK}q)_k + \tfrac{1}{2}W^I q_k - K(\gamma^I q)_k \\ =& -\tfrac{\lambda}{32}\left[(\gamma^I|q)_k\bar{q}^j|q_j - |q^j(\gamma^I\bar{q}|)_k q_j - |q_k(\bar{q}|\gamma^I q) + |q^m\bar{q}_k|(\gamma^I q)_m + |q\gamma^I\bar{q}|q_k\right] \\ & - K(\gamma^I q)_k. \end{aligned} \tag{3.5.7}$$

Also here, the gauge groups found in flat superspace cannot be generalised.

3.5.3. $\mathcal{N} = 6$

The on-shell super-Cotton tensor is

$$W^{IJ} = -\tfrac{\lambda}{16}\mathrm{i}|Q\bar{\Sigma}^{IJ}\bar{Q}|, \tag{3.5.8}$$

so that

$$\begin{aligned} H^{I,k}_{\text{SG}} =& \tfrac{\mathrm{i}}{4}W_{KL}(\bar{\Sigma}^{KLI}Q)^k - \tfrac{\mathrm{i}}{2}W^{IK}(\bar{\Sigma}_K Q)^k - K(\bar{\Sigma}^I Q)^k \\ =& -\tfrac{\lambda}{8}|Q_i\bar{Q}^k|(\bar{\Sigma}^I Q)^i + \tfrac{\lambda}{32}|Q_i\bar{Q}^i|(\bar{\Sigma}^I Q)^k - K(\bar{\Sigma}^I Q)^k. \end{aligned} \tag{3.5.9}$$

As opposed to the cases $\mathcal{N} = 4$ and $\mathcal{N} = 5$, the presence of the supergravity coupling admits more general gauge groups compared to flat superspace [30]. As can be seen in the expression

$$\gamma^{[J}H^{I]} = \tfrac{1}{4}(-\mathrm{i}W^{KL}\gamma_{KLIJ} + 4\tilde{q}W_{IJ} + 4K\gamma_{IJ})q + f^{IJ}Q + Qg^{IJ}, \tag{3.5.10}$$

the R-symmetry U(1) charge can be used to liberate the U(1) factors of the special-unitary-group factors in order to generalise $\mathrm{SU}(N) \times \mathrm{SU}(N)$ to $\mathrm{SU}(N) \times \mathrm{SU}(M)$.

One way to do this is to shift the charge $\tilde{q}$ to $\tilde{q} - af_{NM}$ in order to cancel the parts of the gauge field strength corresponding to the U(1) factors in the algebra, leaving the algebra and the solution H^I formally invariant, i.e.

$$\begin{aligned} H^I = &-\tfrac{\lambda}{8}|Q_i\bar{Q}^k|(\bar{\Sigma}^I Q)^i + \tfrac{\lambda}{32}|Q_i\bar{Q}^i|(\bar{\Sigma}^I Q)^k - K(\bar{\Sigma}^I Q)^k \\ &- 2a(\bar{\Sigma}^I)^{ij} Q_i\bar{Q}^k Q_j - a(\bar{\Sigma}^I)^{kl}(Q_l\bar{Q}^i Q_i - Q_i\bar{Q}^i Q_l). \end{aligned} \tag{3.5.11}$$

This bifundamental representation corresponds to matter in the fundamental representation of $\mathrm{SU}(M)\times\mathrm{U}(1)$ with arbitrary $\mathrm{U}(1)$ charge if the replacement $\frac{a}{N}\to\frac{q^2}{M}+a$ takes place. The solution can be written as

$$\begin{aligned} H^I = &-\tfrac{\lambda}{8}Q_i^\beta\bar{Q}_\beta^k(\bar{\Sigma}^I Q^\alpha)^i + \tfrac{\lambda}{32}Q_i^\beta\bar{Q}_\beta^i(\bar{\Sigma}^I Q^\alpha)^k - K(\bar{\Sigma}^I Q^\alpha)^k \\ &- 2a(\bar{\Sigma}^I)^{ij} Q_i^\alpha\bar{Q}_\beta^k Q_j^\beta - a(\bar{\Sigma}^I)^{kl}(Q_l^\alpha\bar{Q}_\beta^i Q_i^\beta - Q_i^\alpha\bar{Q}_\beta^i Q_l^\beta). \end{aligned} \tag{3.5.12}$$

Another possibility is to formally gauge $\mathrm{U}(N)\times\mathrm{U}(M)$ and then shift $\tilde{q}\to\tilde{q}+af_{NM}$ in order to produce field-strength terms in the algebra completing the $\mathrm{U}(N)\times\mathrm{U}(M)$ to $\mathrm{SU}(N)\times\mathrm{SU}(M)$. In this case, it is required that $\tilde{q}+af_{NM}=-\frac{\mathrm{i}}{2}$, thus fixing the gauge coupling for given $\tilde{q}$, N and M.

3.5.4. $\mathcal{N}=7$

The super-Cotton tensor is $W^{IJKL}=\frac{1}{6}\varepsilon^{IJKLSPQ}W_{SPQ}$ and

$$\gamma^{[J}H_{\mathrm{SG}}^{I]} = \tfrac{1}{4}\left(\tfrac{\mathrm{i}}{3}W_{SPQ}\gamma^{IJSPQ} + 4K\gamma^{IJ}\right)q. \tag{3.5.13}$$

As mentioned in Section (3.1) and opposed to the cases $\mathcal{N}=4$, $\mathcal{N}=5$ and $\mathcal{N}=6$, H_{SG}^I cannot be expressed in terms of W^{IJK}, since the general ansatz

$$H_{\mathrm{SG}}^I = XW_{KLM}\gamma^{IKLM}q + YW^{IKL}\gamma_{KL}q - K\gamma^I q \tag{3.5.14}$$

admits no adequate choice of X, Y.

In terms of a matter current, the on-shell super-Cotton tensor is

$$W^{IJK} = \mathrm{i}\tfrac{\lambda}{16}|q\gamma^{IJK}\bar{q}|, \tag{3.5.15}$$

so that

$$(\gamma^{[J}H^{I]}_{\text{SG}})_i = \tfrac{\lambda}{32}\left[|q_i(\bar{q}|\gamma^{IJ}q) + |q_k(\gamma^{IJ}\bar{q}|)_i q_k + (\gamma^{IJ}|q)_i\bar{q}_k|q_k + |q_k\bar{q}_i|(\gamma^{IJ}q)_k\right] - \tfrac{\lambda}{16}\left[(\gamma^{[I}|q)_i(\bar{q}|\gamma^{J]}q) + |q_k(\gamma^{[I}\bar{q}|)_i(\gamma^{J]}q)_k\right] + K(\gamma^{IJ}q)_i. \tag{3.5.16}$$

The general ansatz becomes

$$\begin{aligned} H^I_{k,\text{SG}} = &- 4\tfrac{\mathrm{i}\lambda}{16}(Y-3X)\left[|q_i(\bar{q}|\gamma^I q) + |q_k\bar{q}_i|(\gamma^I q)_k\right] \\ &- 4\tfrac{\mathrm{i}\lambda}{16}(Y+3X)\left[|q_k(\gamma^I\bar{q}|)_i q_k + (\gamma^I|q)_i\bar{q}_k|q_k\right] \\ &+ 2Y\tfrac{\mathrm{i}\lambda}{16}|q_k\bar{q}_k|(\gamma^I q)_i - K(\gamma^I q)_i, \end{aligned} \tag{3.5.17}$$

implying

$$\begin{aligned} (\gamma^{[J}H^{I]}_{\text{SG}})_i = &- 4\tfrac{\mathrm{i}\lambda}{16}(Y-3X)\left[(\gamma^{[J}|q)_i(\bar{q}|\gamma^{I]}q) + |q_k(\gamma^{[J}\bar{q}|)_i(\gamma^{I]}q)_k\right] \\ &+ 4\tfrac{\mathrm{i}\lambda}{16}(Y+3X)\left[|q_k(\gamma^{IJ}\bar{q}|)_i q_k + (\gamma^{IJ}|q)_i\bar{q}_k|q_k\right] \\ &- 2Y\tfrac{\mathrm{i}\lambda}{16}|q_k\bar{q}_k|(\gamma^{IJ}q)_i + K(\gamma^{IJ}q)_i. \end{aligned} \tag{3.5.18}$$

Still, the coefficients cannot in general be chosen to reproduce the supergravity term, except in the absence of gauge group indices, where $X = 0$, $Y = -\frac{\mathrm{i}}{6}$ and

$$H^I_{\text{SG}} = -\tfrac{\lambda}{16}q_k q_k(\gamma^I q) - K\gamma^I q. \tag{3.5.19}$$

Even though the supergravity sector cannot exist separately in presence of a gauge group, there exist solutions for matter which is coupled to both supergravity and gauge fields. For the gauge-coupled scalar transforming in the bifundamental representation of SU(2) × SU(2), the traces over gauge indices can be turned into matrix products, so that

$$(\gamma^{[J}H^{I]}_{\text{SG}})_i = \tfrac{\lambda}{16}\left[\tfrac{1}{2}q_i(\bar{q}\gamma^{IJ}q) + q_k(\gamma^{IJ}\bar{q})_i q_k + (\gamma^{IJ}q)_i\bar{q}_k q_k - \tfrac{1}{2}(q\gamma^{IJ}\bar{q})q_i\right] - \tfrac{\lambda}{8}\left[(\gamma^{[I}q)_i(\bar{q}\gamma^{J]}q) + q_k(\gamma^{[I}\bar{q})_i(\gamma^{J]}q)_k\right] + K(\gamma^{IJ}q)_i. \tag{3.5.20}$$

Conveniently one can add the respective ansatzes for H^I from the gauge and supergravity sectors, i.e.

$$\begin{aligned}H^I_{i,\text{SG}} =& -8\tfrac{\mathrm{i}\lambda}{16}(Y-3X)\left[q_i(\bar{q}\gamma^I q)+q_k\bar{q}_i(\gamma^I q)_k\right]\\ &-8\tfrac{\mathrm{i}\lambda}{16}(Y+3X)\left[q_k(\gamma^I\bar{q})_i q_k+(\gamma^I q)_i\bar{q}_k q_k\right]\\ &+4Y\tfrac{\mathrm{i}\lambda}{16}q_k\bar{q}_k(\gamma^I q)_i - K(\gamma^I q)_i \end{aligned} \tag{3.5.21}$$

and

$$\begin{aligned}H^I_{i,\text{CS}} =& \gamma^I_{ij}\left[A\{q^j\bar{q}^k q_k\}+B\{q^k\bar{q}^j q_k\}+C\{q^k\bar{q}_k q^j\}\right]\\ &+\gamma^I_{kl}\left[D\{q^k\bar{q}^l q_i\}+E\{q^k\bar{q}_i q^l\}+F\{q_i\bar{q}^k q^l\}\right]\\ &+\gamma^{IK}_{kl}\gamma^K_{ij}\left[G\{q^k\bar{q}^l q^j\}+H\{q^j\bar{q}^k q^l\}\right], \end{aligned} \tag{3.5.22}$$

and solve

$$\begin{aligned}(\gamma^{[J}H^{I]})_i =& -8\tfrac{\mathrm{i}\lambda}{16}(Y-3X)\left[(\gamma^{[J}q)_i(\bar{q}\gamma^{I]}q)+q_k(\gamma^{[J}\bar{q})_i(\gamma^{I]}q)_k\right]\\ &+8\tfrac{\mathrm{i}\lambda}{16}(Y+3X)\left[q_k(\gamma^{IJ}\bar{q})_i q_k+(\gamma^{IJ}q)_i\bar{q}_k q_k\right]-4Y\tfrac{\mathrm{i}\lambda}{16}q_k\bar{q}_k(\gamma^{IJ}q)_i\\ &-[A(\gamma^{IJ}q)_i\bar{q}_k q_k+Bq_k(\gamma^{IJ}\bar{q})_i q_k+Cq_k\bar{q}_k(\gamma^{IJ}q)_i]\\ &+\gamma^{[J}_{ij}\gamma^{I]}_{kl}[Dq_k\bar{q}_l q_j+Eq_k\bar{q}_j q_l+Fq_j\bar{q}_k q_l]\\ &+(\gamma^{IJ}_{kl}\delta_{ij}-4\delta_{[k[i}\gamma^{IJ}_{j]l]}+\gamma^{[J}_{ij}\gamma^{I]}_{kl})[Gq_k\bar{q}_l q_j+Hq_j\bar{q}_k q_l]\\ &+K(\gamma^{IJ}q)_i. \end{aligned}$$

One can choose the coefficients $H=E=F=0$ and $G=-A=B=-D$ such that

$$\begin{aligned}(\gamma^{[J}H^{I]})_i =& -8\tfrac{\mathrm{i}\lambda}{16}(Y-3X)\left[(\gamma^{[J}q)_i(\bar{q}\gamma^{I]}q)+q_k(\gamma^{[J}\bar{q})_i(\gamma^{I]}q)_k\right]\\ &+8\tfrac{\mathrm{i}\lambda}{16}(Y+3X)\left[q_k(\gamma^{IJ}\bar{q})_i q_k+(\gamma^{IJ}q)_i\bar{q}_k q_k\right]-4Y\tfrac{\mathrm{i}\lambda}{16}q_k\bar{q}_k(\gamma^{IJ}q)_i\\ &-Cq_k\bar{q}_k(\gamma^{IJ}q)_i\\ &+\tfrac{3}{2}G(\gamma^{IJ}_{kl}\delta_{ij}+\delta_{ki}\gamma^{IJ}_{lj})q_k\bar{q}_l q_j\\ &+K(\gamma^{IJ}q)_i. \end{aligned}$$

The couplings are then related by $\frac{3}{2}G=-a-\frac{\lambda}{32}=-b+\frac{\lambda}{32}$, since the field-strength terms of the gauge sector also have to contribute to the supergravity sector. Fixing the remaining constants $X = -Y = -\frac{\mathrm{i}}{16}$ and $C = \frac{\lambda}{64}$ leads to the solution

$$\begin{aligned} H_i^I =& \tfrac{\lambda}{8}\left[q_i(\bar{q}\gamma^I q) + q_k\bar{q}_i(\gamma^I q)_k\right] - \tfrac{\lambda}{16}\left[q_k(\gamma^I\bar{q})_i q_k + (\gamma^I q)_i\bar{q}_k q_k\right] - \tfrac{\lambda}{64}q_k\bar{q}_k(\gamma^I q)_i - K(\gamma^I q)_i \\ &+ \tfrac{2}{3}(-a - \tfrac{\lambda}{32})\left[-(\gamma^I q)_i\bar{q}_k q_k + q_k(\gamma^I\bar{q})_i q_k - (q\gamma^I\bar{q})q_i + (q\gamma^{IK}\bar{q})(\gamma^K q)_i\right] + \tfrac{\lambda}{64}q_k\bar{q}_k(\gamma^I q)_i, \end{aligned} \tag{3.5.23}$$

which can be rewritten in terms of traces as

$$\begin{aligned} H_i^I =& \tfrac{\lambda}{8}|q_i(\bar{q}|\gamma^I q) - \tfrac{\lambda}{16}|q_k(\gamma^I\bar{q}|)q_k - K(\gamma^I q)_i \\ &+ \tfrac{2}{3}(-a - \tfrac{\lambda}{32})\left[-(\gamma^I q)_i|\bar{q}_k q_k| + |q_k(\gamma^I\bar{q}|)_i q_k - (q\gamma^I\bar{q})q_i + (q\gamma^{IK}\bar{q})(\gamma^K q)_i\right]. \end{aligned} \tag{3.5.24}$$

Other gauge groups with a bifundamental representation are not possible.

Regarding fundamental representations, one case derived from the bifundamental representation is SU(2) with one of the couplings a, b set to zero. As opposed to the previous cases, there are more solutions which are not special cases of bifundamental representations. The supergravity sector in terms of scalars transforming under a fundamental representation,

$$\begin{aligned} (\gamma^{[J}H_{\mathrm{SG}}^{I]})_i^\alpha =& \tfrac{\lambda}{32}\left[q_i^\beta(\bar{q}_\beta\gamma^{IJ}q^\alpha) + q_k^\beta(\gamma^{IJ}\bar{q}_\beta)_i q_k^\alpha + (\gamma^{IJ}q^\beta)_i\bar{q}_\beta^k q_k^\alpha + q_k^\beta\bar{q}_{\beta,i}(\gamma^{IJ}q^\alpha)_k\right] \\ &- \tfrac{\lambda}{16}\left[(\gamma^{[I}q^\beta)_i(\bar{q}_\beta\gamma^{J]}q^\alpha) + q_k^\beta(\gamma^{[I}\bar{q}_\beta)_i(\gamma^{J]}q^\alpha)_k\right] + K(\gamma^{IJ}q^\alpha)_i, \end{aligned} \tag{3.5.25}$$

has to be compared to the ansatz (choosing $H = E = F = 0$ and $G = -A = B = -D$)

$$\begin{aligned} (\gamma^{[J}H^{I]})_i^\alpha =& -4\tfrac{\mathrm{i}\lambda}{16}(Y - 3X)\left[(\gamma^{[J}q^\beta)_i(\bar{q}_\beta\gamma^{I]}q^\alpha) + q_k^\beta(\gamma^{[J}\bar{q}_\beta)_i(\gamma^{I]}q^\alpha)_k\right] \\ &+ 4\tfrac{\mathrm{i}\lambda}{16}(Y + 3X)\left[q_k^\beta(\gamma^{IJ}\bar{q}_\beta)_i q_k^\alpha + (\gamma^{IJ}q^\beta)_i\bar{q}_\beta^k q_k^\alpha\right] \\ &- 2Y\tfrac{\mathrm{i}\lambda}{16}q_k^\beta\bar{q}_\beta^k(\gamma^{IJ}q^\alpha)_i + K(\gamma^{IJ}q^\alpha)_i \\ &- C_1 q_k^\beta\bar{q}_\beta^k(\gamma^{IJ}q^\alpha)_i \\ &+ G_3(\gamma_{kl}^{IJ}\delta_{ij} - \delta_{ki}\gamma_{jl}^{IJ} + \delta_{li}\gamma_{jk}^{IJ})q_k^\alpha\bar{q}_\beta^l q_j^\beta \\ &+ G_4(\gamma_{kl}^{IJ}\delta_{ij} - \delta_{ki}\gamma_{jl}^{IJ} + \delta_{li}\gamma_{jk}^{IJ})C^{\alpha\beta\gamma\delta}q_k^\beta\bar{q}_\gamma^l q_j^\delta. \end{aligned}$$

As before, $X = -Y = -\frac{\mathrm{i}}{16}$, and $C_1 = \frac{\lambda}{128}$ cancels the superfluous term in the supergravity ansatz. G_3 provides both the missing terms in the supergravity ansatz and the gauge field strength. A solution is then possible for $\mathrm{SU}(N) \times \mathrm{U}(1)$ with the specification

$G_3 = \frac{\lambda}{32} = a - \frac{\lambda}{32} = \frac{a}{N} - \frac{q^2}{N}$ and is given by

$$H_i^I = \tfrac{\lambda}{16}\left[q_i^\beta(\bar{q}_\beta\gamma^I q^\alpha) + q_k^\beta\bar{q}_{\beta,i}(\gamma^I q^\alpha)_k\right] - \tfrac{\lambda}{32}\left[q_k^\beta(\gamma^I\bar{q}_\beta)_i q_k^\alpha + (\gamma^I q^\beta)_i\bar{q}_\beta^k q_k^\alpha\right] - K(\gamma^I q)_i$$
$$- \tfrac{\lambda}{32}(\gamma^I q_\alpha)_i\bar{q}_\beta^k q_k^\beta + \tfrac{\lambda}{32}q_k^\alpha(\gamma^I\bar{q}_\beta)_i q_k^\beta - \tfrac{\lambda}{32}(q^\alpha\gamma^I\bar{q}_\beta)q_i^\beta + \tfrac{\lambda}{32}(q^\alpha\gamma^{IK}\bar{q}_\beta)(\gamma^K q^\beta)_i. \quad (3.5.26)$$

For SO(N), the ansatz for H^I_{CS} cannot provide field-strength terms; however, the missing terms in the supergravity ansatz can be explained by an SO(N) field strength cancelling them. Concretely, in the supergravity sector in terms of a scalar transforming under SO(N),

$$(\gamma^{[J}H^{I]}_{\text{SG}})_i = \tfrac{\lambda}{32}\left[2(q_{[\beta}\gamma^{IJ}q_{\alpha]})q_{i,\beta} + 2(\gamma^{IJ}q_\beta)_i q_\beta^k q_\alpha^k\right] - \tfrac{\lambda}{16}2(\gamma^{[I}q_\beta)_i(q_\beta\gamma^{J]}q_\alpha) + K(\gamma^{IJ}q_\alpha)_i, \quad (3.5.27)$$

the first term can be cancelled by choosing $a = -\frac{\lambda}{32}$. The solution depends therefore only on λ and is given by

$$H^I_\alpha = \tfrac{\lambda}{8}q_\beta(q_\beta\gamma^I q_\alpha) - \tfrac{\lambda}{16}(\gamma^I q_\beta)q_\beta^k q_\alpha^k - K\gamma^I q_\alpha. \quad (3.5.28)$$

This SO(N) gauge group can further be promoted to spin(7) by taking $3G_4 = -\frac{3}{2}\frac{\lambda}{32}$. The corresponding solution reads

$$H^I_\alpha = \tfrac{\lambda}{8}q_\beta(q_\beta\gamma^I q_\alpha) - \tfrac{\lambda}{16}(\gamma^I q_\beta)q_\beta^k q_\alpha^k - \tfrac{\lambda}{64}C_{\alpha\beta\gamma\delta}[(q^\beta\gamma^{IK}q^\gamma)(\gamma^K q^\delta) - (q^\beta\gamma^I q^\gamma)q^\delta] - K\gamma^I q_\alpha. \quad (3.5.29)$$

3.5.5. $\mathcal{N} = 8$

The super-Cotton tensor is self-dual and

$$\Sigma^{[J}H^{I]}_{\text{SG}} = \tfrac{1}{4}(W^{IJKL}\Sigma_{KL}Q + 4K\Sigma^{IJ}Q). \quad (3.5.30)$$

The ansatz

$$H^I_{\text{SG}} = XW^{IKLM}\bar{\Sigma}_{KLM}Q + 2K\bar{\Sigma}^I Q \quad (3.5.31)$$

provides no solution off shell since

$$\Sigma^{[J}H^{I]}_{\text{SG}} = XW^{[I|KLM|}\Sigma^{J]KLM}Q + 3XW^{IJLM}\Sigma_{LM}Q - 2K\Sigma^{IJ}Q. \quad (3.5.32)$$

The on-shell super-Cotton tensor is

$$W^{IJKL} = \tfrac{\lambda}{16}|Q\Sigma^{IJKL}\bar{Q}|, \tag{3.5.33}$$

so that

$$\begin{aligned}\Sigma^{[J}H^{I]}_{\text{SG}} =& \tfrac{\lambda}{32}|Q_i\bar{Q}_i|(\Sigma^{IJ}Q) - \tfrac{\lambda}{16}|Q_i(\Sigma^{IJ}\bar{Q}|)Q_i - \tfrac{\lambda}{16}(\Sigma^{IJ}|Q)\bar{Q}_i|Q_i \\ &- \tfrac{\lambda}{16}|Q(\bar{Q}|\Sigma^{IJ}Q) - \tfrac{\lambda}{16}|Q_i\bar{Q}|(\Sigma^{IJ}Q)_i + K\Sigma^{IJ}Q.\end{aligned} \tag{3.5.34}$$

The ansatz becomes

$$H^I_{\text{SG}} = X\tfrac{\lambda}{16}\left[24|Q_i(\bar{\Sigma}^I\bar{Q}|)Q_i + 24(\bar{\Sigma}^I|Q)\bar{Q}_i|Q_i - 6|Q_i\bar{Q}_i|(\bar{\Sigma}^I Q)\right] - K\bar{\Sigma}^I Q, \tag{3.5.35}$$

implying

$$\begin{aligned}\Sigma^{[J}H^{I]}_{\text{SG}} =& - X\tfrac{\lambda}{16}\left[24|Q_i(\Sigma^{IJ}\bar{Q}|)Q_i + 24(\Sigma^{IJ}|Q)\bar{Q}_i|Q_i - 6|Q_i\bar{Q}_i|(\Sigma^{IJ}Q)\right] \\ &+ K\Sigma^{IJ}Q.\end{aligned} \tag{3.5.36}$$

Similarly to the case $\mathcal{N} = 7$, the coefficients cannot in general be chosen to reproduce the supergravity term, except in the absence of gauge-group indices, where $X = \frac{1}{28}$ and

$$H^I_{\text{SG}} = \tfrac{3}{2}\tfrac{\lambda}{16}Q^2\Sigma^I Q - K\Sigma^I Q. \tag{3.5.37}$$

However, a solution with a gauge-coupled scalar field exists for the combined gauge and supergravity sector. Conveniently, the on-shell ansatz of the gauge sector to the one for the supergravity sector can be added. For a bifundamental representation, the only possibility remains real SU(2) × SU(2) [21]. In this case, it can be written

$$\begin{aligned}\Sigma^{[J}H^{I]}_{\text{SG}} =& - \tfrac{\lambda}{8}Q_i(\Sigma^{IJ}\bar{Q})Q_i - \tfrac{\lambda}{16}(\Sigma^{IJ}Q)\bar{Q}_iQ_i \\ &- \tfrac{\lambda}{16}Q(\bar{Q}\Sigma^{IJ}Q) + \tfrac{\lambda}{16}(Q\Sigma^{IJ}\bar{Q})Q + K\Sigma^{IJ}Q\end{aligned} \tag{3.5.38}$$

and for the combined ansatz

$$\begin{aligned}\Sigma^{[J}H^{I]} =& - X\tfrac{\lambda}{16}\left[48Q_i(\Sigma^{IJ}\bar{Q})Q_i + 36(\Sigma^{IJ}Q)\bar{Q}_iQ_i\right] + K\Sigma^{IJ}Q \\ &- \left[A(\Sigma^{IJ}Q)\bar{Q}_iQ_i + BQ_i(\Sigma^{IJ}\bar{Q})Q_i\right] \\ &+ C\left[\tfrac{3}{2}(Q\Sigma^{IJ}\bar{Q})Q + \tfrac{3}{2}Q(\bar{Q}\Sigma^{IJ}Q) + Q_i(\Sigma^{IJ}\bar{Q})Q_i - (\Sigma^{IJ}Q)\bar{Q}_iQ_i\right].\end{aligned} \tag{3.5.39}$$

The couplings must fulfil $a+b=-3C$ and $a-b=\frac{\lambda}{8}$ [21], while e.g. $X=\frac{1}{24}$, $C=B$ and $A=-C-\frac{\lambda}{32}$. The solution is

$$\begin{aligned} H^I =& \tfrac{\lambda}{16}\left[2Q_i(\bar{\Sigma}^I\bar{Q})Q_i+\tfrac{3}{2}(\bar{\Sigma}^I Q)\bar{Q}_iQ_i\right]-K\bar{\Sigma}^I Q \\ &+\tfrac{1}{3}(2a-\tfrac{\lambda}{8})[(\bar{\Sigma}^I Q)\bar{Q}_iQ_i-Q_i(\bar{\Sigma}^I\bar{Q})Q_i-(Q\Sigma^{IK}\bar{Q})(\bar{\Sigma}^K Q)] \\ &-\tfrac{\lambda}{32}(\bar{\Sigma}^I Q)\bar{Q}_iQ_i. \end{aligned} \tag{3.5.40}$$

As for gauge groups with fundamental representations, one example is given by SU(2), realised by setting a or b to zero in the above gauging of SU(2) $\times$ SU(2). In general, the supergravity sector in terms of scalars transforming under a fundamental representation reads

$$\begin{aligned} \Sigma^{[J}H^{I]}_{\text{SG}} =& \tfrac{\lambda}{32}Q^\beta_i\bar{Q}^i_\beta(\Sigma^{IJ}Q^\alpha)-\tfrac{\lambda}{16}Q^\beta_i(\Sigma^{IJ}\bar{Q}_\beta)Q^\alpha_i-\tfrac{\lambda}{16}(\Sigma^{IJ}Q^\beta)\bar{Q}^i_\beta Q^\alpha_i \\ &-\tfrac{\lambda}{16}Q^\beta(\bar{Q}_\beta\Sigma^{IJ}Q^\alpha)-\tfrac{\lambda}{16}Q^\beta_i\bar{Q}_\beta(\Sigma^{IJ}Q^\alpha)_i+K\Sigma^{IJ}Q^\alpha. \end{aligned} \tag{3.5.41}$$

Comparing with the combined ansatz

$$\begin{aligned} \Sigma^{[J}H^{I]} =& -X\tfrac{\lambda}{16}\left[24Q^\beta_i(\Sigma^{IJ}\bar{Q}_\beta)Q^\alpha_i+24(\Sigma^{IJ}Q^\beta)\bar{Q}^i_\beta Q^\alpha_i-6Q^\beta_i\bar{Q}^i_\beta(\Sigma^{IJ}Q^\alpha)\right]+K\Sigma^{IJ}Q^\alpha \\ &-\left[A(\Sigma^{IJ}Q)\bar{Q}_iQ_i+BQ_i(\Sigma^{IJ}\bar{Q})Q_i\right] \\ &+C_3[(Q^\alpha\Sigma^{IJ}\bar{Q}_\beta)Q^\beta+Q^\alpha(\bar{Q}_\beta\Sigma^{IJ}Q^\beta)-Q^\alpha_i\bar{Q}_{k,\beta}(\Sigma^{IJ}Q^\beta)_i \\ &\quad+Q^\alpha_i(\Sigma^{IJ}\bar{Q}_\beta)Q^\beta_i-(\Sigma^{IJ}Q^\alpha)\bar{Q}_{i,\beta}Q^\beta_i], \end{aligned} \tag{3.5.42}$$

a solution exists for SU(N) $\times$ U(1)$^\circ$ with $a=-\frac{\lambda}{8}$ and $q^2=-\frac{\lambda}{8}(\frac{1}{N}-\frac{1}{2})$, reading

$$\begin{aligned} H^{I,\alpha} =& \tfrac{\lambda}{32}[Q^\beta_i\bar{Q}^i_\beta(\bar{\Sigma}^I Q^\alpha)+2Q^\alpha_i\bar{Q}^i_\beta(\bar{\Sigma}^I Q^\alpha)]-\tfrac{\lambda}{16}(Q^\alpha\Sigma^{IK}\bar{Q}_\beta)(\bar{\Sigma}^K Q^\beta) \\ &+2K\bar{\Sigma}^I Q^\alpha. \end{aligned} \tag{3.5.43}$$

For a scalar transforming under SO(N), the supergravity sector becomes

$$\begin{aligned} \Sigma^{[J}H^{I]}_{\text{SG}} =& -\tfrac{\lambda}{8}Q^\beta(Q_\beta\Sigma^{IJ}Q^\alpha)+\tfrac{\lambda}{32}Q^\beta_iQ^i_\beta(\Sigma^{IJ}Q^\alpha)-\tfrac{\lambda}{8}(\Sigma^{IJ}Q^\beta)Q^i_\beta Q^\alpha_i \\ &+K\Sigma^{IJ}Q^\alpha. \end{aligned} \tag{3.5.44}$$

If the SO(N) coupling is chosen to be $a=\frac{\lambda}{16}$ [21], the first term is cancelled and a solution can be given as

$$H^I = \tfrac{\lambda}{32}\left[4(\bar{\Sigma}^I Q^\beta)Q^i_\beta Q^\alpha_i - Q^\beta_i Q^i_\beta(\bar{\Sigma}^I Q^\alpha)\right] - K\bar{\Sigma}^I Q^\alpha. \tag{3.5.45}$$

This solution can be modified for spin(7) to

$$H^I = \tfrac{\lambda}{32}\left[4(\bar{\Sigma}^I Q^\beta)Q^i_\beta Q^\alpha_i - Q^\beta_i Q^i_\beta(\bar{\Sigma}^I Q^\alpha)\right] + \tfrac{\lambda}{32}C_{\alpha\beta\gamma\delta}(Q^\beta\Sigma^{IK}Q^\gamma)(\bar{\Sigma}^K Q^\delta) - K\bar{\Sigma}^I Q^\alpha. \tag{3.5.46}$$

This concludes the determination of allowed gauge groups in curved superspace, where the scalar multiplet is additionally coupled to superconformal gravity. An overview of the results from the last two sections is given in the next section.

3.6. Overview of gauge groups

The following table summarises the findings from the previous two sections. The columns "fundamental" and "bifundamental" list the allowed gauge groups for which the scalar fields are in the fundamental or bifundamental representation, respectively. Together with the values $\mathcal{N}$ it is implied that the scalar multiplet transforms under an irreducible (chiral) representation of spin($\mathcal{N}$). The lines denoted by "+SG" contain the allowed gauge groups in curved superspace, i.e. in the presence of conformal supergravity. For $\mathcal{N}=4$ and $\mathcal{N}=5$ they are the same as in flat space. The subscripts of the group factors indicate the relative coupling constants.

	fundamental	bifundamental
$\boldsymbol{\mathcal{N}=4}$	$\mathrm{SU}(N)_a \times \mathrm{U}(1)_{a-a/N}$	$\mathrm{U}(M)_a \times \mathrm{U}(N)_{-a}$
	$\mathrm{Sp}(N)_a \times \mathrm{U}(1)_{-a}$	$\mathrm{SU}(M)_a \times \mathrm{SU}(N)_{-a} \times \mathrm{U}(1)_{a/N-a/M}$
		$\mathrm{Sp}(M)_a \times \mathrm{SO}(N)_{-a}$
		$\mathrm{spin}(7)_a \times \mathrm{SU}(2)_{-a}$
$\boldsymbol{\mathcal{N}=5}$	$\mathrm{SU}(N)_a \times \mathrm{U}(1)_{a-a/N}$	$\mathrm{U}(M)_a \times \mathrm{U}(N)_{-a}$
	$\mathrm{Sp}(N)_a \times \mathrm{U}(1)_{-a}$	$\mathrm{SU}(M)_a \times \mathrm{SU}(N)_{-a} \times \mathrm{U}(1)_{a/N-a/M}$
		$\mathrm{Sp}(M)_a \times \mathrm{SO}(N)_{-a}$
		$\mathrm{spin}(7)_a \times \mathrm{SU}(2)_{-a}$
$\boldsymbol{\mathcal{N}=6}$	$\mathrm{SU}(N)_a \times \mathrm{U}(1)_{a-a/N}$	$\mathrm{U}(M)_a \times \mathrm{U}(N)_a$
	$\mathrm{Sp}(N)_a \times \mathrm{U}(1)_{-a}$	$\mathrm{SU}(M)_a \times \mathrm{SU}(N)_a \times \mathrm{U}(1)_{a/N-a/M}$
		$\mathrm{Sp}(M)_a \times \mathrm{SO}(2)_a$
+SG	$\mathrm{SU}(N) \times \mathrm{U}(1)$	$\mathrm{SU}(M)_a \times \mathrm{SU}(N)_a$
$\boldsymbol{\mathcal{N}=7}$		$\mathrm{SU}(2)_a \times \mathrm{SU}(2)_a$
+SG	$\mathrm{SU}(N)_{-\lambda/8} \times \mathrm{U}(1)_{(2-N)\lambda/16}$	$\mathrm{SU}(2)_a \times \mathrm{SU}(2)_{a-\lambda/8}$
	$\mathrm{SO}(N)_{-\lambda/16}$	
	$\mathrm{spin}(7)_{-\lambda/16}$	
$\boldsymbol{\mathcal{N}=8}$		$\mathrm{SU}(2)_a \times \mathrm{SU}(2)_a$
+SG	$\mathrm{SU}(N)_{-\lambda/4} \times \mathrm{U}(1)_{(2-N)\lambda/8}$	$\mathrm{SU}(2)_a \times \mathrm{SU}(2)_{a-\lambda/4}$
	$\mathrm{SO}(N)_{-\lambda/8}$	
	$\mathrm{spin}(7)_{-\lambda/8}$	

As mentioned during the above analysis, the groups with fundamental representation in flat space are equivalent to corresponding groups with bifundamental representation for a certain case of N, M. The same is true in curved superspace, except for SO(N) and spin(7) in the cases $\mathcal{N} = 7, 8$.

It is worth to note that for the cases $\mathcal{N} = 6, 7, 8$ the coupling constants of the two group factors with bifundamental representation appear with equal sign, which is due to the conventional ordering of the conjugated scalar fields in the on-shell expressions for the left- and right-acting field strengths, as defined in Section (3.4). This however correspond to opposite signs of the matter currents, which contain the conjugate fields in the usual order. Concretely, since for $\mathcal{N} = 6, 7, 8$ the matrices γ^{IJ} have the property

$$Q\gamma^{IJ}\bar{Q} = -\,\bar{Q}\gamma^{IJ}Q \tag{3.6.1a}$$

$$Q\Sigma^{IJ}\bar{Q} = -\,\bar{Q}\bar{\Sigma}^{IJ}Q, \tag{3.6.1b}$$

the sign of the right-acting field strength gets reverted in the usual ordering.

This sign difference is reflected in opposite signs of the two Chern-Simons terms for the two group factors. It is essential for describing the world-volume theory of M2-branes. Since one Chern-Simons term is parity-odd due to the cubic vector fields, the interchange of left- and right-acting gauge fields under parity can be imposed in order to leave the combined Chern-Simons terms invariant [10]. The requirement of parity invariance comes from the principle that M2-branes correspond to the strong-coupling limit of D2-branes, which are parity-even.

The results in the table have been partially present in various places in the literature. Regarding flat space, the groups for $\mathcal{N} = 4$, except for $\text{spin}(7)_a \times \text{SU}(2)_{-a}$, had been initially discovered in [13], and extended for $\mathcal{N} = 5$ in [15]. Classifications for $\mathcal{N} = 6$ were given in [14], [12] and [16]. The unique gauge group for $\mathcal{N} = 8$ appeared in the BLG model [9, 8] and subsequent comments, e.g. [10]. The group $\text{spin}(7)_a \times \text{SU}(2)_{-a}$ was found in [17], where also the case $\mathcal{N} = 7$ had been mentioned. As for the presence of supergravity, the group SO(N) for $\mathcal{N} = 8$ was found in [21] and the generalisation to SU(M) $\times$ SU(N) of $\mathcal{N} = 6$ was mentioned in [30]. The groups with fundamental representation SU(N) $\times$ U(1) and spin(7) for $\mathcal{N} = 7, 8$ [32] are new results of the present analysis.

4. Topologically massive supergravity

In this chapter, the gravitationally coupled spin($\mathcal{N}$) scalar multiplet is interpreted as a conformal compensator in order to realise extended topologically massive supergravities and to specify the corresponding values of $\mu\ell$.

In Section (4.1), basic properties of three-dimensional Einstein-Hilbert gravity with a negative cosmological constant, conformal gravity, and topologically massive gravity are reviewed.

In Section (4.2), the relation between the cosmological constant of anti-de Sitter space and the geometry of anti-de Sitter superspace is established. Further, the action for the scalar conformal compensator is given.

In Section (4.3), the spin($\mathcal{N}$) superfield describing the compensator multiplet is introduced into anti-de Sitter superspace. Consistency with this background together with previous results directly leads to a formula for $\mu\ell$ in the cases $4 \leq \mathcal{N} \leq 8$, which is evaluated. Subsequently, it is analysed how the values of $\mu\ell$ can change if the compensator transforms under additional gauge groups.

4.1. Einstein gravity and topologically massive gravity

In three space-time dimensions, the Riemann tensor $R_{[mn][kl]} = R_{[kl][mn]}$ is dual to a symmetric tensor of rank two

$$R_{mnkl} \equiv \varepsilon_{mnr}\varepsilon_{kls}\tilde{R}^{rs}. \tag{4.1.1}$$

It is completely determined by the Ricci tensor and curvature scalar, since

$$\tilde{R}_{mn} = R_{mn} + \tilde{R}g_{mn} = R_{mn} - \tfrac{1}{2}Rg_{mn}, \tag{4.1.2a}$$

which translates back to

$$R_{rsmn} = 4R_{[m[r}g_{s]n]} - Rg_{m[r}g_{s]n}. \tag{4.1.3}$$

In consequence, the Einstein equation with negative cosmological constant $\Lambda = -\ell^{-2}$

$$0 = R_{mn} - \tfrac{1}{2} g_{mn} R - g_{mn} \frac{1}{\ell^2} \tag{4.1.4}$$

completely fixes the geometry to describe space-time with constant negative curvature, or anti-de Sitter space, specified by

$$R = -\frac{6}{\ell^2} \tag{4.1.5a}$$

$$R_{mn} = -2 g_{mn} \frac{1}{\ell^2}. \tag{4.1.5b}$$

This leaves no freedom for a locally propagating graviton. However, the solution of the Einstein equation with negative cosmological constant can be generalised to a metric which describes the analogue of a black hole in four dimensions, known as the BTZ black hole [23]. Its parameter corresponding to the mass is positive when describing a black hole and can take a separated negative value to recover the anti-de Sitter metric. Asymptotically, the BTZ metric always becomes the anti-de Sitter metric. Gravity on this asymptotic anti-de Sitter space can be described by a two dimensional conformal field theory on its boundary [24] with the total central charge

$$c_{\mathrm{R}} + c_{\mathrm{L}} = \frac{3\ell}{2G}. \tag{4.1.6}$$

The left- and right-moving modes correspond to massless gravitons propagating on this boundary.

The model of cosmological topologically massive gravity is given by the action [25]

$$\begin{aligned} S =& \frac{1}{\kappa^2} \int \mathrm{d}^3 x \; e \; \left(R + 2\ell^{-2} \right) \\ &+ \frac{1}{4\mu\kappa^2} \int \mathrm{d}^3 x \; e \; \varepsilon^{mnl} \left(\omega_m^{ab} R_{nl,ab} - \tfrac{2}{3} \omega_m^{ab} \omega_{n,b}{}^{c} \omega_{l,ca} \right), \end{aligned} \tag{4.1.7}$$

where $\kappa^2 = 16\pi G$. It is the sum of the Einstein-Hilbert action and the conformal gravity action for the spin connection $\omega_m(e)$. The resulting equations of motion are

$$G_{mn} + \tfrac{1}{\mu} C_{mn} = 0 \tag{4.1.8}$$

with

$$G_{mn} = R_{mn} - \tfrac{1}{2} g_{mn} R - g_{mn} \frac{1}{\ell^2} \tag{4.1.9a}$$

$$C_{mn} = \varepsilon_m{}^{kl} \nabla_k (R_{ln} - \tfrac{1}{4} g_{ln} R). \tag{4.1.9b}$$

Vanishing of the Cotton tensor C_{mn} is equivalent to a conformally flat space-time. Therefore, conformal gravity alone also has no propagating degrees of freedom. While the anti-de Sitter solution for pure cosmological Einstein-Hilbert gravity, including black holes and massless boundary gravitons, is still valid, the presence of the conformal-gravity term gives rise to a massive propagating graviton with mass μ. In absence of the cosmological constant, the Einstein-Hilbert term is usually taken with the sign opposite to the one above, in order to ensure a positive energy for the graviton. This, however, would cause the mass of the BTZ black hole to become negative [40].

The central charges of the boundary conformal field theory are modified due to the conformal-gravity term, becoming [27]

$$c_{\mathrm{R}} = \frac{3\ell}{2G}(1 + \tfrac{1}{\mu\ell}) \tag{4.1.10a}$$

$$c_{\mathrm{L}} = \frac{3\ell}{2G}(1 - \tfrac{1}{\mu\ell}). \tag{4.1.10b}$$

Choosing the appropriate sign for positive-mass black holes, the analysis of the graviton modes in presence of the negative cosmological constant [26] then shows that for general values of $\mu\ell$ the theory suffers from negative energies of the gravitons. Only in the special case $\mu\ell = 1$, the massive and left-moving gravitons become degenerate and non-propagating, leaving a chiral conformal field theory with $(c_{\mathrm{R}}, c_{\mathrm{L}}) = (3\ell/G, 0)$ [26, 41] and a right-moving degree of freedom with positive energy. This model is called chiral gravity or topologically massive gravity at the chiral point.

The parameter $\mu\ell$ can freely be chosen to have this most sensible value of $\mu\ell = 1$. However, in the following it will be shown that $\mu\ell$ is fixed by the superconformal geometry for $\mathcal{N} \geq 4$ supersymmetric topologically massive gravity, where the Einstein-Hilbert action is realised via a superconformal compensator [31, 32]. The resulting topologically massive gravities and values of $\mu\ell$ will be classified corresponding to all possible gauged compensators.

4.2. Conformal supergravity and conformal compensators

In the supersymmetric case, the Chern-Simons action for conformal gravity is replaced by the action for extended superconformal gravity [39]. Most importantly, besides the terms for fermionic and auxiliary fields, it is supplemented by a Chern-Simons term for the gauge field B_m^{IJ} of the SO($\mathcal{N}$) local structure group

$$\frac{1}{4\mu\kappa^2}\int \mathrm{d}^3x\ e\ \varepsilon^{mnl}\left(-2B_m^{IJ}F_{nl,IJ}-\tfrac{4}{3}B_m^{IJ}B_n^{JK}B_l^{KI}\right). \tag{4.2.1}$$

In view of the anti-de Sitter solution of cosmological Einstein gravity, the geometries of anti-de Sitter space and the anti-de Sitter background of superconformal gravity can be compared using the algebra of covariant derivatives. The Riemann tensor is defined by the commutator

$$[\nabla_m,\nabla_n]V^k=\left(2\partial_{[m}\Gamma^k_{n]l}+2\Gamma^k_{[m|p|}\Gamma^p_{n]l}\right)V^l\equiv R_{mn}{}^k{}_lV^l=-\frac{2}{\ell^2}\delta^k_{[m}g_{n]l}V^l, \tag{4.2.2}$$

which can be translated to the tangent space as

$$[\nabla_a,\nabla_b]V^c=-\frac{2}{\ell^2}\delta^c_{[a}\eta_{b]d}V^d. \tag{4.2.3}$$

Comparison with the commutator from superconformal geometry in the background of anti-de Sitter superspace [38]

$$[\mathscr{D}_a,\mathscr{D}_b]V^c=-4K^2\mathscr{M}_{ab}V^c=-8K^2\delta^c_{[a}\eta_{b]d}V^d \tag{4.2.4}$$

shows that the cosmological constant is generated by the value of the field K with [31]

$$\ell^{-1}=2|K|. \tag{4.2.5}$$

The Einstein-Hilbert term can be realised by a supersymmetric conformal compensator q subject to the bosonic part of the conformally invariant action [42]

$$S_{\text{bos.}}=\int \mathrm{d}^3x\ e\ \left(-\tfrac{1}{2}(\overline{\mathscr{D}^a q})(\mathscr{D}_a q)-\tfrac{1}{16}R|q|^2+\lambda(|q|^2)^3\right), \tag{4.2.6}$$

where λ is a constant. The Einstein-Hilbert term is recovered in the super-Weyl gauge for q, where

$$|q|^2 = 16\kappa^{-2}. \tag{4.2.7}$$

This action is naturally part of the action for the gravitationally coupled spin($\mathcal{N}$) scalar multiplet [29, 21]. It would be desirable to obtain it from the superfield action (3.2.22) presented in Chapter 3; however, the action would have to be formulated in a locally scale invariant way, and the relation of the Riemann scalar R in terms of the geometry of conventional curved superspace to the fields appearing in the component action would have to be established, possibly accounting for the differential relations appearing in the expression for the dimension-two Lorentz curvature derived in Chapter 2.

4.3. Extended topologically massive supergravity

4.3.1. Fixation of $\mu\ell$

In order to agree with the background defining anti-de Sitter superspace, the superfield describing the spin($\mathcal{N}$) scalar compensator multiplet has to be covariantly constant

$$\mathscr{D}_A \boldsymbol{Q} = 0. \tag{4.3.1}$$

In consequence, the anticommutator of spinor covariant derivatives acting on $\boldsymbol{Q}$

$$\{\mathscr{D}^I_\alpha, \mathscr{D}^J_\beta\}\boldsymbol{Q} = 0 \tag{4.3.2}$$

implies the algebraic condition [31, 32]

$$4\boldsymbol{K}\gamma^{IJ}\boldsymbol{Q} = -\boldsymbol{W}^{IJKL}\gamma_{KL}\boldsymbol{Q}. \tag{4.3.3}$$

Conveniently, this condition is equivalent to $\check{q}_{[\alpha,\beta]}^{\;J} = 0$ or $H^J = 0$ [32]. Using the relations

$$K = \frac{1}{2\ell} \tag{4.3.4}$$

and

$$W^{IJKL} = \frac{\mu\kappa^2}{16}|\bar{q}\gamma^{IJKL}q| \tag{4.3.5}$$

leads to

$$\frac{2}{\ell}\gamma^{IJ}q = -\frac{\mu\kappa^2}{16}|\bar{q}\gamma^{IJKL}q|\gamma_{KL}q. \tag{4.3.6}$$

Replacing the gravitational coupling with

$$\kappa^2 = 16|q|^{-2} \tag{4.3.7}$$

and multiplying by γ^{IJ} leads to the formula

$$(\mu l)^{-1} q = \tfrac{1}{2\mathcal{N}(\mathcal{N}-1)} |q|^{-2} |\bar{q}\gamma^{IJKL} q| \gamma_{IJKL} q. \tag{4.3.8}$$

Its evaluation yields [32]

$\mathcal{N} =$	4	5	6	7	8
$(\mu\ell)^{-1} =$	1	3/5	1	2	3

where for $\mathcal{N} = 6$ the formula was adjusted due to the additional U(1) field strength associated with the local structure group, contributing to the super-Cotton tensor.

4.3.2. Gauged compensators

For a gauge-coupled compensator, the adapted condition for anti-de Sitter background

$$H^I = H^I_{\text{SG}} + H^I_{\text{CS}} = 0 \tag{4.3.9}$$

generally leads to deformations of μ, which can involve the gauge coupling, special properties of the gauge group and the number of compensator components with non-vanishing values. If the compensator transforms under a bifundamental representation, the deformation typically vanishes.

Usually, if H^I_{CS} can be written in the form

$$(H^I_{\text{CS}})^{\alpha}_{a} = X q^{\gamma}_{c} \bar{q}^{c}_{\gamma} (\gamma^I)_a{}^b q^{\alpha}_{b} \tag{4.3.10}$$

for some subset of the range of spin($\mathcal{N}$) indices a and gauge indices α, then there is a shift

$$\mu \longrightarrow \mu + 32 X \kappa^{-2}. \tag{4.3.11}$$

Special effects can appear, if the gauge and supergravity sectors depend on each other.

In the following, features of each case $\mathcal{N}$ are commented.

$\mathcal{N}=4$ For $\mathcal{N}=4$ the deformation vanishes for compensators in bifundamental representations due to the opposite signs of the coupling constants of the factor groups and to the antisymmetry of the spin(7) tensor C_{rsvw}. For the groups with fundamental representation, since one can choose only one gauge component to be non-vanishing, the deformation likewise cancels.

$\mathcal{N}=5$ For $\mathcal{N}=5$, the same statements as for $\mathcal{N}=4$ hold.

$\mathcal{N}=6$ For $\mathcal{N}=6$, the statements made for $\mathcal{N}=4$ and $\mathcal{N}=5$ still apply; however, the gauge group $\mathrm{SU}(N)\times\mathrm{U}(1)$ in the presence of supergravity causes a deformation of μ. In this case, the AdS background requires

$$\begin{aligned} K(\bar{\Sigma}^I Q^\alpha)^k =& -\tfrac{\lambda}{8} Q_i^\beta \bar{Q}_\beta^k (\bar{\Sigma}^I Q^\alpha)^i + \tfrac{\lambda}{32} Q_i^\beta \bar{Q}_\beta^i (\bar{\Sigma}^I Q^\alpha)^k \\ & - 2a(\bar{\Sigma}^I)^{ij} Q_i^\alpha \bar{Q}_\beta^k Q_j^\beta - a(\bar{\Sigma}^I)^{kl}(Q_l^\alpha \bar{Q}_\beta^i Q_i^\beta - Q_i^\alpha \bar{Q}_\beta^i Q_l^\beta). \end{aligned} \tag{4.3.12}$$

In general, only one component of the matrix Q_i^α can be non-vanishing, leaving $\mu\ell=1$. However, in the special case where $a=\frac{\lambda}{16}$, or

$$K(\bar{\Sigma}^I Q^\alpha)^k = -\tfrac{\lambda}{32} Q_i^\beta \bar{Q}_\beta^i (\bar{\Sigma}^I Q^\alpha)^k + \tfrac{\lambda}{16} Q_i^\alpha \bar{Q}_\beta^i (\bar{\Sigma}^I Q^\beta)^k, \tag{4.3.13}$$

one can choose $Q_\alpha^i = 4\kappa^{-1}p^{-1}\delta_\alpha^i$, where $\alpha=1,...,p\leq 4$. This leads to the relation

$$\frac{1}{2\ell} = -\frac{\mu\kappa^2}{32}16\kappa^{-2} + \frac{\mu\kappa^2}{16}16\kappa^{-2}\frac{1}{p} \tag{4.3.14}$$

and to the formula [32]

$$(\mu\ell)^{-1} = \frac{2}{p} - 1. \tag{4.3.15}$$

$\mathcal{N}=7$ For the compensator coupled to supergravity and the gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)$, the condition

$$\begin{aligned} K\gamma_{ab}^I q_b =& \tfrac{\lambda}{8}|q_a\bar{q}_b|\gamma_{bc}^I q_c - \tfrac{\lambda}{16}|q_c\bar{q}_b|\gamma_{ab}^I q_c \\ & + \tfrac{2}{3}(a-\tfrac{\lambda}{32})\left[-\gamma_{ab}^I q_b|\bar{q}_c q_c| + \gamma_{ab}^I|q_c\bar{q}_b|q_c - (q\gamma^I\bar{q})q + (q\gamma^{IK}\bar{q})(\gamma^K q)\right] \end{aligned} \tag{4.3.16}$$

is realised by one non-vanishing spin(7) component, leaving $(\mu\ell)^{-1}=2$. Similarly, compensators transforming under a possible fundamental representation do not realise any deformation of $(\mu\ell)^{-1}$ [32].

$\mathcal{N} = 8$ For the compensator coupled to supergravity and the gauge group SU(2) $\times$ SU(2), the AdS condition yields $(\mu\ell)^{-1} = 3$ [21].

The gauge group SO(N) allows a deformation of $\mu\ell$. The condition is

$$K(\bar{\Sigma}^I Q^\alpha)^k = \tfrac{\lambda}{32}\left[4(\bar{\Sigma}^I Q^\beta)^k Q^i_\beta Q^\alpha_i - Q^\beta_i Q^i_\beta (\bar{\Sigma}^I Q^\alpha)^k\right]. \qquad (4.3.17)$$

One can choose $Q^i_\alpha = 4\kappa^{-1}p^{-1}\delta^i_\alpha$, where $\alpha = 1, ..., p \leq 8$. This leads to

$$\frac{1}{2\ell} = -\frac{\mu\kappa^2}{32}16\kappa^{-2} + \frac{\mu\kappa^2}{8}16\kappa^{-2}\frac{1}{p} \qquad (4.3.18)$$

or [33]

$$(\mu\ell)^{-1} = \frac{4}{p} - 1. \qquad (4.3.19)$$

In summary, the gravitationally coupled scalar multiplet was used as a conformal compensator, which naturally realises topologically massive supergravities with fixed value of $\mu\ell$ for $\mathcal{N} \geq 4$. This fixation is mainly owed to the fact that the super-Cotton tensor appearing for $\mathcal{N} \geq 4$ contains the SO($\mathcal{N}$) field strength whose corresponding coupling constant involves μ. The relation between μ and the anti-de Sitter radius ℓ is implied by the background of anti-de Sitter superspace. Using an ungauged compensator field, the values $\mu\ell$ are given by a generic formula which can be evaluated for the specific cases $\mathcal{N}$. For a gauged compensator, a number of gauge components can be given an expectation value generating the gravitational coupling constant. This has no effect for $\mathcal{N} = 4, 5, 7$, but leads to specific diversifications for $\mathcal{N} = 6$ with gauge group SU(N) and $\mathcal{N} = 8$ with gauge group SO(N) given by formulas involving the number of chosen gauge components.

5. Conclusion

In this thesis, the coupling of scalar supermultiplets to superconformal Chern-Simons gauge theories and superconformal gravity has been described in the framework of $\mathcal{N}$-extended superspace.

As for the gauge coupling, a list of all possible gauge groups for theories with $4 \leq \mathcal{N} \leq 8$ supersymmetries has been produced. It agrees with the classifications found in various places in the literature. However, the superfield formalism employed here proves more powerful in several regards: First of all, it does not require any knowledge about the component actions of the theories, but rather provides the superfield action principle readily producing such component actions. Furthermore, these actions are delivered for all $\mathcal{N}$, only requiring the gauge-group-dependent field H^I appearing in the transformation law of the fermionic matter field.

Most importantly for the main goal of this thesis, the superfield analysis is equally suited for superconformal gravity-coupled matter. The thorough investigation of modifications or new emergence of gauge groups in curved superspace, as compared to flat superspace, has produced some new results, completing the existing knowledge in this regard.

For the topologically massive gravities realised by using the coupled matter field as a conformal compensator, the resulting fixed values $\mu\ell$ have been determined and its possible modifications due to the presence of gauge groups have been discussed. These values are important concerning the stability, or positive energies, of topologically massive gravity. The value $\mu\ell = 1$ preferred by the model of chiral gravity is indeed implied by $\mathcal{N} = 4$ and $\mathcal{N} = 6$ supersymmetry, and can also be achieved for $\mathcal{N} = 8$ with gauge group SO(N).

However, as was mentioned in the introduction, the conformal compensator generates the opposite sign of the Einstein-Hilbert action. This corresponds to a negative mass of the BTZ black hole. This problem has often been noted in the literature [29], with no satisfying conclusion so far. One promising discussions is given in [43]. There, it is conjectured that the negative-mass black holes may be excluded from interaction with the physical matter fields or, in other words, could never be formed by the collapse of the positive-energy matter and therefore would live in a different superselection sector of the theory.

Such a solution would be necessary in view of a positive total mass of topologically massive gravity. General relativity in four space-time dimensions possesses various positive-energy theorems, stating that space-times with appropriate asymptotic symmetries allowing for conserved asymptotic charges have non-negative mass with unique zero-energy states. Classic examples are asymptotically-Minkowski and -anti-de Sitter spaces. The most simple proofs for these theorems can be derived from supergravity [44], whose positivity of energy is implied by the supersymmetry algebra [45]. Also for three-dimensional cosmological topologically massive gravity, positive energy has been established by its embeddability in simple supergravity [46]. The value $\mu\ell$ relevant for perturbative stability is however free to choose. The feature of the new extended topologically massive supergravities is the fixation of this parameter, in some cases to the value $\mu\ell = 1$ required by perturbative stability. As is known to be possible in four dimensions [47], it could therefore be beneficial to generalise the analysis of mass-positivity to these extended supergravities.

A. Symmetry groups

A.1. Fundamental representations

If the elements of a Lie group are represented by matrices M, a vector v transforms under the standard-fundamental representation as

$$\tilde{v} \longrightarrow M \cdot v, \tag{A.1.1}$$

a dual vector transforms under the dual representation as

$$\tilde{v}^{(*)} \longrightarrow (M^{-1})^{\mathrm{T}} \cdot v^{(*)}, \tag{A.1.2}$$

a complex conjugate vector transforms under the complex conjugate representation as

$$\tilde{v}^{*} \longrightarrow M^{*} \cdot v^{*}, \tag{A.1.3}$$

and a dual complex conjugate vector transforms under the dual complex conjugate representation as

$$\tilde{v}^{*(*)} \longrightarrow (M^{-1})^{\dagger} \cdot v^{*(*)}. \tag{A.1.4}$$

Dual vectors map vectors to numbers according to

$$v^{(*)}(w) \equiv (v^{(*)})^{\mathrm{T}} \cdot w \in \mathbb{C}. \tag{A.1.5}$$

If there is an isomorphism between a vector w and a dual vector $w^{(*)}$ via a scalar product

$$\langle w, v \rangle = (w^{(*)})^{\mathrm{T}} \cdot v \tag{A.1.6}$$

(holding for all v) being invariant under the group transformations, the two representations are equivalent. If the scalar product is described by a metric g

$$\langle w, v \rangle = w^{\mathrm{T}} \cdot g^{\mathrm{T}} \cdot v, \tag{A.1.7}$$

the metric is a linear map between w and $w^{(*)}$

$$w^{(*)} = g \cdot w \tag{A.1.8}$$

If the elements of a group are defined to leave a certain matrix g invariant, i.e.

$$g^{-1} \cdot M^{\mathrm{T}} \cdot g = M^{-1}, \tag{A.1.9}$$

then g is such a metric.

A.2. Index notation

Components of vectors are denoted by a lower index, components of dual vectors are denoted by an upper index. Same named upper and lower indices are summed over. The scalar product reads

$$\langle w, v\rangle = (w^{(*)})^i v_i. \tag{A.2.1}$$

A metric mapping vectors to dual vectors has two upper indices (for real groups the $(*)$ superscript is usually omitted)

$$(w^{(*)})^i = g^{ij} w_j \tag{A.2.2}$$

and the inverse metric has two lower indices

$$w_i = (g^{-1})_{ij}(w^{(*)})^j = (w^{(*)})^j((g^{-1})^{\mathrm{T}})_{ji} \equiv (w^{(*)})^j g_{ji}. \tag{A.2.3}$$

The indices of the transformation matrices can be deduced and are summarised in the relation

$$g \cdot M \cdot g^{-1} =(M^{-1})^{\mathrm{T}} \tag{A.2.4a}$$

$$g^{ij} M_j{}^k (g^{-1})_{kl} =((M^{-1})^{\mathrm{T}})^i{}_l. \tag{A.2.4b}$$

Examples. *Elements of the* ***pseudo-orthogonal group*** $\mathrm{SO}(M,N)$ *are real and have* $g = \eta$*, where*

$$\eta = \mathrm{diag}(-1, .., -1, 1, .., 1), \tag{A.2.5}$$

and elements of the ***symplectic group*** $\mathrm{Sp}(N)$ *have* $g = \varepsilon$ *where* ε *is antisymmetric.*

Elements of the ***unitary group*** $\mathrm{SU}(N)$ *fulfil*

$$M^\dagger = M^{-1}, \tag{A.2.6}$$

identifying the dual and complex conjugate representations. The isomorphism between the fundamental and dual representations is given by complex conjugation and the scalar product is

$$\langle w, v\rangle = (w^*)^{\mathrm{T}} \cdot v. \tag{A.2.7}$$

A.3. Generators

A Lie-group element is given by the exponential

$$M = \mathrm{e}^X = \mathrm{e}^{X^A T_A}, \tag{A.3.1}$$

where X is the generator of this group element and is expanded in the basis elements T_A depending on the group, which are the generators of the group. Due to the formula

$$\mathrm{e}^X \cdot \mathrm{e}^Y = \mathrm{e}^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X[X,Y]]+\dots} \tag{A.3.2}$$

the generators must fulfil a Lie algebra

$$[T_A, T_B] = f_{AB}^C T_C \tag{A.3.3}$$

where f_{AB}^C are constants.

For the fundamental representation of the **special unitary group** acting on vectors with components x_i, the matrix $X_i{}^j$ is anti-hermitian,

$$X^* = -X. \tag{A.3.4}$$

It is expanded in a basis T_A

$$X_i{}^j = X^A (T_A)_i{}^j \tag{A.3.5}$$

with the normalisation

$$\mathrm{tr}(T_A, T_B) = \delta_{AB}. \tag{A.3.6}$$

This implies

$$X_A = X_i^{\ j} (T_A)_j^{\ i} \tag{A.3.7}$$

and, together with the tracelessness,

$$(T^A)_i^{\ j} (T_A)_k^{\ l} = \delta_i^l \delta_k^j - \tfrac{1}{N} \delta_i^j \delta_k^l. \tag{A.3.8}$$

The generators can thus be relabelled as

$$X^A T_A = X_i^{\ j} T_j^{\ i}, \tag{A.3.9}$$

in terms of which the Lie algebra becomes

$$[T_i^{\ j}, T_k^{\ l}] = \delta_k^j T_i^{\ l} - \delta_i^l T_k^{\ j}. \tag{A.3.10}$$

For the fundamental representation of the **special pseudo-orthogonal group** $\mathrm{SO}(m,n)$ acting on vectors with components x_i, the matrix $X_i^{\ j}$ fulfils

$$g^{-1} \cdot X^{\mathrm{T}} \cdot g = -X. \tag{A.3.11}$$

The space of X is spanned by $D = \frac{1}{2}(d^2 - d)$ matrices of dimension d

$$(T_A)_i^{\ j} = (\tilde{T}_A)_{ik} (g^{\mathrm{T}})^{kj}, \tag{A.3.12}$$

where $\tilde{T}_A$ span the space of antisymmetric matrices. If the positions in a matrix are numbered like

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ & 0 & 8 & 9 & 5 \\ & & 0 & 10 & 6 \\ & & & 0 & 7 \\ & & & & 0 \end{pmatrix}, \tag{A.3.13}$$

$\tilde{T}_A$ is chosen to have only the entries 1 at position A and -1 at the mirror position. Matrices $\tilde{\tilde{T}}_A$ with two upper indices (i.e. multiplied from both sides by g)

$$(\tilde{\tilde{T}}_A)^{ij} = g^{ik} g^{jl} (\tilde{T}_A)_{kl} \tag{A.3.14}$$

are related to $\tilde{T}_A$ by

$$\tilde{\tilde{T}}_A = \left(-\tilde{T}_1, ..., -\tilde{T}_S, \tilde{T}_{S+1}, ..., \tilde{T}_D\right), \tag{A.3.15}$$

where $S = 2mn - n$. Therefore,

$$\mathrm{tr}(\tilde{T}_A \tilde{\tilde{T}}_B) = -2\eta_{AB} \tag{A.3.16}$$

where η is the metric of the group $\mathrm{SO}(T, S)$, where $T = D - S$. It follows that

$$\mathrm{tr}(T_A T_B) = -2\eta_{AB}, \tag{A.3.17}$$

because

$$(T_A)_i^{\ j}(T_A)_j^{\ i} = (\tilde{T}_A)_{ik} g^{kj} (\tilde{T}_A)_{jl} g^{li} = (\tilde{T}_A)_{ik} (\tilde{\tilde{T}}_A)^{ki}. \tag{A.3.18}$$

Then

$$X_A = -\tfrac{1}{2} X_i^{\ j} (T_A)_j^{\ i} \tag{A.3.19}$$

and

$$(T^A)_i^{\ j} (T_A)^{kl} = (T^{kl})_i^{\ j} = 2\delta_i^{[k} g^{l]j}. \tag{A.3.20}$$

This suggests the relabelling

$$X^A T_A = \tfrac{1}{2} X^{ij} T_{ij} \tag{A.3.21}$$

and the Lie algebra becomes

$$[T^{ab}, T^{cd}] = -4 g^{[c[a} T^{b]d]}. \tag{A.3.22}$$

A.4. Spin groups

The N-dimensional Clifford algebra is

$$\{\gamma_I, \gamma_J\} = 2\delta_{IJ}. \tag{A.4.1}$$

The matrices

$$\mathscr{N}^{IJ} = \tfrac{1}{4}[\gamma^I, \gamma^J] = \tfrac{1}{2}\gamma^{IJ} \tag{A.4.2}$$

solve the SO(N) Lie algebra and are the generators defining the group spin(N). One solution for the Clifford algebras of dimension $N = 2m$ and $N = 2m + 1$ is

$$\begin{aligned} \gamma^1 &= \sigma_1 \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} \\ \gamma^{2,...,2m} &= \mathrm{i}\sigma_2 \otimes \mathrm{i}\hat{\gamma}^{1,...,N-1} \\ \gamma^* &= \gamma^{2m+1} \equiv -\mathrm{i}^m \gamma^1 \cdot ... \cdot \gamma^{2m} = \sigma_3 \otimes \mathbb{1} \otimes ... \otimes \mathbb{1}, \end{aligned} \tag{A.4.3}$$

where $\hat{\gamma}$ solve the $N = 2m - 1$ dimensional Clifford algebra and each line consists of m Kronecker-factors of 2×2 matrices. For $N = 2m$, the generators of spin(N) commute with the matrices

$$P_{\mathrm{L/R}} = \tfrac{1}{2}(\mathbb{1} \pm \gamma^*), \tag{A.4.4}$$

which project on the generators of irreducible representations. These are called left- and right-handed and transform under the generators defined by the chiral Clifford algebra

$$\Sigma^I \bar{\Sigma}^J = \delta^{IJ} + \Sigma^{IJ} \tag{A.4.5a}$$

$$\bar{\Sigma}^I \Sigma^J = \delta^{IJ} + \bar{\Sigma}^{IJ}, \tag{A.4.5b}$$

where

$$\gamma^I = \begin{pmatrix} 0 & \Sigma^I \\ \bar{\Sigma}^I & 0 \end{pmatrix}. \tag{A.4.6}$$

The map to the corresponding SO(N) representation is given by

$$v^I = \psi^a (\gamma^I)_a{}^b \chi_b, \tag{A.4.7}$$

which transforms as an SO(N) vector since for infinitesimal parameters ω^{IJ},

$$\begin{aligned} (A^{-1})_c{}^a (\gamma^I)_a{}^b A_b{}^d &= (1 - \tfrac{1}{4}\omega^{KL}\gamma_{KL})_c{}^a (\gamma^I)_a{}^b (1 + \tfrac{1}{4}\omega^{PQ}\gamma_{PQ})_b{}^d \\ &= (\gamma^I)_c{}^d - \tfrac{1}{4}\omega^{KL}[\gamma_{KL}, \gamma^I]_c{}^d \\ &= (\gamma^I)_c{}^d + \omega^{IL}(\gamma^L)_c{}^d. \end{aligned} \tag{A.4.8}$$

An SO(N) vector is therefore equivalent to a matrix in the space spanned by γ^I according to

$$v_i{}^j = v^I (\gamma_I)_i{}^j \Longleftrightarrow v^I = \tfrac{1}{2^m}(\gamma_I)_i{}^j v_j{}^i. \tag{A.4.9}$$

This implies an (in-)completeness relation

$$(\gamma_I)_i^{\ j}(\gamma_I)_k^{\ l} = 2^m \delta_i^l \delta_k^j + C_{i\ k}^{\ j\ l} \tag{A.4.10}$$

with

$$(\gamma_I)_j^{\ i} C_{i\ k}^{\ j\ l} = (\gamma_I)_l^{\ k} C_{i\ k}^{\ j\ l} = 0. \tag{A.4.11}$$

There are completeness relations, called Fierz-indentities or -lemmas, for odd $N = 2m+1$ with the basis

$$\Gamma = \{\mathbb{1}, \gamma^{I_1}, \gamma^{I_1 I_2}, ..., \gamma^{I_1 \dots I_{\lfloor N/2 \rfloor}}\} \tag{A.4.12}$$

and even $N = 2m$ with the basis

$$\Gamma = \{\mathbb{1}, \gamma^{I_1}, \gamma^{I_1 I_2}, ..., \gamma^{I_1 \dots I_N}\}. \tag{A.4.13}$$

They are given by [48]

$$2^m \delta_i^j \delta_k^l = \delta_i^l \delta_k^j + (\gamma^I)_i^{\ l}(\gamma_I)_k^{\ j} + ... + \tfrac{1}{\lfloor N/2 \rfloor !}(\gamma^{I_1 \dots I_{\lfloor N/2 \rfloor}})_i^{\ l}(\gamma_{I_{\lfloor N/2 \rfloor} \dots I_1})_k^{\ j} \tag{A.4.14}$$

and

$$2^m \delta_i^j \delta_k^l = \delta_i^l \delta_k^j + (\gamma^I)_i^{\ l}(\gamma_I)_k^{\ j} + ... + \tfrac{1}{N!}(\gamma^{I_1 \dots I_N})_i^{\ l}(\gamma_{I_N .. I_1})_k^{\ j}, \tag{A.4.15}$$

respectively.

B. Systematic supercovariant projection

The component fields

$$\partial^{I_1}_{\alpha_1}...\partial^{I_k}_{\alpha_k}\boldsymbol{A}| \equiv a^{I_1...I_k}_{\alpha_1...\alpha_k} \tag{B.0.1}$$

and supercovariant projections

$$D^{I_1}_{\alpha_1}...D^{I_k}_{\alpha_k}\boldsymbol{A}| \equiv A^{I_1...I_k}_{\alpha_1...\alpha_k} \tag{B.0.2}$$

are related by

$$A^{I_1...I_k}_{\alpha_1...\alpha_k} = a^{I_1...I_k}_{\alpha_1...\alpha_k} + \mathfrak{A}^{I_1...I_k}_{\alpha_1...\alpha_k}. \tag{B.0.3}$$

The fields $\mathfrak{A}^{I_1...I_k}_{\alpha_1...\alpha_k}$ can be constructed after the pattern

$$\begin{aligned}\mathfrak{A}^{I_1...I_k}_{\alpha_1...\alpha_k} =& \mathrm{i}\sum_{b>a}^{k} \tilde{\partial}^{I_aI_b}_{\alpha_a\alpha_b} a^{I_1...I_k\setminus I_{a,b}}_{\alpha_1...\alpha_k\setminus\alpha_{a,b}} \\ &+ \mathrm{i}^2 \sum_{a<b,c<d,a<c}^{k} r^{ab,cd}\tilde{\partial}^{I_aI_b}_{\alpha_a\alpha_b}\tilde{\partial}^{I_cI_d}_{\alpha_c\alpha_d} a^{I_1...I_k\setminus I_{a,b,c,d}}_{\alpha_1...\alpha_k\setminus\alpha_{a,b,c,d}} \\ &+ \mathrm{i}^3 \sum_{a<b,c<d,e<f,a<c<e}^{k} r^{ab,cd,ef}\tilde{\partial}^{I_aI_b}_{\alpha_a\alpha_b}\tilde{\partial}^{I_cI_d}_{\alpha_c\alpha_d}\tilde{\partial}^{I_eI_f}_{\alpha_e\alpha_f} a^{I_1...I_k\setminus I_{a,b,c,d,e,f}}_{\alpha_1...\alpha_k\setminus\alpha_{a,b,c,d,e,f}} \\ &+ ... .\end{aligned} \tag{B.0.4}$$

with

$$\tilde{\partial}^{I_aI_b}_{\alpha_a\alpha_b} = (-1)^{b-a+1}\delta^{I_aI_b}\partial_{\alpha_a\alpha_b}. \tag{B.0.5}$$

The symbol $r^{ab,cd,ef,...}$ carries several pairs of indices. In each pair, the value of the second index is higher than the value of the first index. Two neighbouring pairs are said to be crossing if the value of the second index of the first pair lies between the values of the two indices of the second pair (eg. $15, 28$). They are said to be embracing, if the value of the second index of the first pair is higher than the values of the indices of the second pair (e.g. 18, 35). They are said to be paired, if the value of the second index of the first pair is lower than the values of the indices of the second pair (e.g. 13, 67). If the number of crossing neighbouring pairs is C, then the value of the symbol is given by

$$r^{ab,cd,ef,\dots} = (-1)^C. \tag{B.0.6}$$

In the following, the indices I_a are not displayed, and the summation constraints are implied, i.e. the indices of α_a carried by a ∂ are increasing from left to right, as are the first indices of the different ∂.

An iterative formula is

$$\begin{aligned}
\mathfrak{A}_{\alpha_1\dots\alpha_k\alpha_{k+1}} =&\mathfrak{A}_{\alpha_1\dots\alpha_k}(\partial a_{(k-2)k+1}, \partial\partial a_{(k-4)k+1}, \dots) \\
&+ \mathrm{i}\sum_a^k \tilde{\partial}_{\alpha_a\alpha_{k+1}} a_{\alpha_1\dots\alpha_k\backslash\alpha_a} \\
&+ \mathrm{i}^2\sum_{a,c,d}^k r^{a(k+1),cd}\tilde{\partial}_{\alpha_a\alpha_{k+1}}\tilde{\partial}_{\alpha_c\alpha_d} a_{\alpha_1\dots\alpha_k\backslash\alpha_{a,c,d}} \\
&+ \mathrm{i}^2\sum_{a,b,c}^k r^{ab,c(k+1)}\tilde{\partial}_{\alpha_a\alpha_b}\tilde{\partial}_{\alpha_c\alpha_{k+1}} a_{\alpha_1\dots\alpha_k\backslash\alpha_{a,b,c}} \\
&+ \mathrm{i}^3\sum_{a,c,d,e,f}^k \tilde{\partial}_{\alpha_a\alpha_{k+1}} r^{a(k+1),cd,ef}\tilde{\partial}_{\alpha_c\alpha_d}\tilde{\partial}_{\alpha_e\alpha_f} a_{\alpha_1\dots\alpha_k\backslash\alpha_{a,c,d,e,f}} \\
&+ \mathrm{i}^3\sum_{a,b,c,e,f}^k r^{ab,c(k+1),ef}\tilde{\partial}_{\alpha_a\alpha_b}\tilde{\partial}_{\alpha_c\alpha_{k+1}}\tilde{\partial}_{\alpha_e\alpha_f} a_{\alpha_1\dots\alpha_k\backslash\alpha_{a,b,c,e,f}} \\
&+ \mathrm{i}^3\sum_{a,b,c,d,e}^k r^{ab,cd,e(k+1)}\tilde{\partial}_{\alpha_a\alpha_b}\tilde{\partial}_{\alpha_c\alpha_d}\tilde{\partial}_{\alpha_e\alpha_{k+1}} a_{\alpha_1\dots\alpha_k\backslash\alpha_{a,b,c,d,e}} \\
&+ \dots ,
\end{aligned} \tag{B.0.7}$$

where, symbolically, $a_{(k-2)k+1}$ has the indices carried by ∂ removed and the indices α_{k+1} and I_{k+1} attached.

The above formulas can be verified by convincing, direct calculations. A more convenient method for calculating supercovariant projections is to perform successive supersymmetry transformations, according to the relation

$$-\varepsilon^\beta A_{\beta\alpha_1..\alpha_k} = -\varepsilon^\beta D_\beta \boldsymbol{A}_{\alpha_1..\alpha_k}| = \mathrm{i}\varepsilon^\beta Q_\beta \boldsymbol{A}_{\alpha_1..\alpha_k}| = \delta_\varepsilon A_{\alpha_1..\alpha_k}. \tag{B.0.8}$$

A resulting list with the pattern $\overset{(n)}{a}_{\alpha_k..\alpha_1} = -\overset{(n)}{\mathfrak{A}}_{\alpha_k..\alpha_1}$ $(k \leq 2)$, which may prove useful for some applications is

$$
\begin{aligned}
\overset{(n)}{a}_{\alpha_k..\alpha_1} =& -\overset{(n)}{\mathfrak{A}}_{\alpha_k..\alpha_1} \\
\tilde{a} =& 0 \\
\tilde{a}^I_\alpha =& \mathrm{i}\partial_\alpha{}^\beta a^I_\beta \\
\tilde{a}^{JI}_{\beta\alpha} =& \varepsilon^{JI}_{\beta\alpha}\Box a - 2\mathrm{i}\partial_{(\beta}{}^\mu a^{[IJ]}_{\alpha)\mu} \\
\tilde{\tilde{a}} =& \mathcal{N}\Box a \\
\tilde{\tilde{a}}_\alpha =& 2\mathrm{i}\partial_\alpha{}^\mu \tilde{a}_\mu + \mathcal{N}\Box a_\alpha \\
\tilde{\tilde{a}}_{\beta\alpha} =& -4\mathrm{i}\partial_{(\beta}{}^\mu \tilde{a}_{\alpha)\mu} + 2\varepsilon_{\beta\alpha}\Box\tilde{a} - 2\partial_\alpha{}^\mu\partial_\beta{}^\nu a_{\mu\nu} + \mathcal{N}\Box a_{\beta\alpha} \\
\tilde{\tilde{\tilde{a}}} =& (3\mathcal{N} - 2)\Box\tilde{a} \\
\tilde{\tilde{\tilde{a}}}_\alpha =& 3\mathrm{i}\partial_\alpha{}^\mu \tilde{\tilde{a}}_\mu + (3\mathcal{N} - 2)\Box(\tilde{a}_\alpha - \mathrm{i}\partial_\alpha{}^\beta a_\beta) \\
\tilde{\tilde{\tilde{a}}}_{\beta\alpha} =& 6\mathrm{i}\partial_{(\alpha}{}^\mu \tilde{\tilde{a}}_{\beta)\mu} + 3\varepsilon_{\beta\alpha}\Box\tilde{\tilde{a}} - 6\partial_\alpha{}^\mu\partial_\beta{}^\nu \tilde{a}_{\mu\nu} + (3\mathcal{N} - 2)\Box(\tilde{a}_{\beta\alpha} - \varepsilon_{\beta\alpha}\Box a + 2\mathrm{i}\partial_{(\beta}{}^\mu a_{\alpha)\mu}) \\
\tilde{\tilde{\tilde{\tilde{a}}}} =& (6\mathcal{N} - 8)\Box\tilde{\tilde{a}} - \mathcal{N}(3\mathcal{N} - 2)\Box\Box a \\
\tilde{\tilde{\tilde{\tilde{a}}}}_\alpha =& 4\mathrm{i}\partial_\alpha{}^\mu \tilde{\tilde{\tilde{a}}}_\mu + (6\mathcal{N} - 8)\Box(\tilde{\tilde{a}}_\alpha - 2\mathrm{i}\partial_\alpha{}^\mu \tilde{a}_\mu) - \mathcal{N}(3\mathcal{N} - 2)\Box\Box a_\alpha \\
\tilde{\tilde{\tilde{\tilde{a}}}}_{\beta\alpha} =& -8\mathrm{i}\partial_{(\beta}{}^\mu \tilde{\tilde{\tilde{a}}}_{\alpha)\mu} + 4\varepsilon_{\beta\alpha}\Box\tilde{\tilde{\tilde{a}}} - 12\partial_\alpha{}^\mu\partial_\beta{}^\nu \tilde{\tilde{a}}_{\mu\nu} \\
&+ (6\mathcal{N} - 8)\Box(\tilde{\tilde{a}}_{\beta\alpha} - 4\mathrm{i}\partial_{(\alpha}{}^\mu \tilde{a}_{\beta)\mu} - 2\varepsilon_{\beta\alpha}\Box\tilde{a} + 2\partial_\alpha{}^\mu\partial_\beta{}^\nu a_{\mu\nu}) - \mathcal{N}(3\mathcal{N} - 2)\Box\Box a_{\beta\alpha} \\
\tilde{\tilde{\tilde{\tilde{\tilde{a}}}}} =& (10\mathcal{N} - 20)\Box\tilde{\tilde{\tilde{a}}} + (-15\mathcal{N}^2 + 26\mathcal{N} - 16)\Box\Box\tilde{a}.
\end{aligned}
$$

Bibliography

[1] W. Nahm. Supersymmetries and their Representations. *Nucl. Phys.*, B135:149, 1978. [,7(1977)].

[2] M. J. Duff and K. S. Stelle. Multimembrane solutions of D = 11 supergravity. *Phys. Lett.*, B253:113–118, 1991. [,110(1990)].

[3] E. Bergshoeff, E. Sezgin, and P. K. Townsend. Supermembranes and Eleven-Dimensional Supergravity. *Phys. Lett.*, B189:75–78, 1987. [,69(1987)].

[4] E. Bergshoeff, E. Sezgin, and P. K. Townsend. Properties of the Eleven-Dimensional Super Membrane Theory. *Annals Phys.*, 185:330, 1988.

[5] G. W. Gibbons and P. K. Townsend. Vacuum interpolation in supergravity via super p-branes. *Phys. Rev. Lett.*, 71:3754–3757, 1993.

[6] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Int. J. Theor. Phys.*, 38:1113–1133, 1999. [Adv. Theor. Math. Phys.2,231(1998)].

[7] John H. Schwarz. Superconformal Chern-Simons theories. *JHEP*, 11:078, 2004.

[8] Andreas Gustavsson. Algebraic structures on parallel M2-branes. *Nucl. Phys.*, B811:66–76, 2009.

[9] Jonathan Bagger and Neil Lambert. Gauge symmetry and supersymmetry of multiple M2-branes. *Phys. Rev.*, D77:065008, 2008.

[10] Mark Van Raamsdonk. Comments on the Bagger-Lambert theory and multiple M2-branes. *JHEP*, 05:105, 2008.

[11] Miguel A. Bandres, Arthur E. Lipstein, and John H. Schwarz. N = 8 Superconformal Chern-Simons Theories. *JHEP*, 05:025, 2008.

[12] Ofer Aharony, Oren Bergman, Daniel Louis Jafferis, and Juan Maldacena. N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals. *JHEP*, 10:091, 2008.

[13] Davide Gaiotto and Edward Witten. Janus Configurations, Chern-Simons Couplings, And The theta-Angle in N=4 Super Yang-Mills Theory. *JHEP*, 06:097, 2010.

[14] Ofer Aharony, Oren Bergman, and Daniel Louis Jafferis. Fractional M2-branes. *JHEP*, 11:043, 2008.

[15] Kazuo Hosomichi, Ki-Myeong Lee, Sangmin Lee, Sungjay Lee, and Jaemo Park. N=5,6 Superconformal Chern-Simons Theories and M2-branes on Orbifolds. *JHEP*, 09:002, 2008.

[16] Martin Schnabl and Yuji Tachikawa. Classification of N=6 superconformal theories of ABJM type. *JHEP*, 09:103, 2010.

[17] Eric A. Bergshoeff, Olaf Hohm, Diederik Roest, Henning Samtleben, and Ergin Sezgin. The Superconformal Gaugings in Three Dimensions. *JHEP*, 09:101, 2008.

[18] Evgeny I. Buchbinder, Sergei M. Kuzenko, and Igor B. Samsonov. Implications of $\mathcal{N} = 4$ superconformal symmetry in three spacetime dimensions. *JHEP*, 08:125, 2015.

[19] Sergei M. Kuzenko and Igor B. Samsonov. Implications of $\mathcal{N} = 5, 6$ superconformal symmetry in three spacetime dimensions. *JHEP*, 08:084, 2016.

[20] Henning Samtleben and Robert Wimmer. N=6 Superspace Constraints, SUSY Enhancement and Monopole Operators. *JHEP*, 10:080, 2010.

[21] Ulf Gran, Jesper Greitz, Paul S. Howe, and Bengt E. W. Nilsson. Topologically gauged superconformal Chern-Simons matter theories. *JHEP*, 12:046, 2012.

[22] Henning Samtleben and Robert Wimmer. N=8 Superspace Constraints for Three-dimensional Gauge Theories. *JHEP*, 02:070, 2010.

[23] Maximo Banados, Claudio Teitelboim, and Jorge Zanelli. The Black hole in three-dimensional space-time. *Phys. Rev. Lett.*, 69:1849–1851, 1992.

[24] J. David Brown and M. Henneaux. Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity. *Commun. Math. Phys.*, 104:207–226, 1986.

[25] Stanley Deser, R. Jackiw, and S. Templeton. Topologically Massive Gauge Theories. *Annals Phys.*, 140:372–411, 1982. [Annals Phys.281,409(2000)].

[26] Wei Li, Wei Song, and Andrew Strominger. Chiral Gravity in Three Dimensions. *JHEP*, 04:082, 2008.

[27] Per Kraus and Finn Larsen. Holographic gravitational anomalies. *JHEP*, 01:022, 2006.

[28] Ulf Gran and Bengt E. W. Nilsson. Three-dimensional $N = 8$ superconformal gravity and its coupling to BLG M2-branes. *JHEP*, 03:074, 2009.

[29] Xiaoyong Chu and Bengt E. W. Nilsson. Three-dimensional topologically gauged N=6 ABJM type theories. *JHEP*, 06:057, 2010.

[30] Bengt E. W. Nilsson. Aspects of topologically gauged M2-branes with six supersymmetries: towards a 'sequential AdS/CFT'? In *Proceedings, 7th International Conference on Quantum Theory and Symmetries (QTS7): Prague, Czech Republic, August 7-13, 2011*, 2012.

[31] Frederik Lauf and Ivo Sachs. Topologically massive gravity with extended supersymmetry. *Phys. Rev.*, D94(6):065028, 2016.

[32] Frederik Lauf and Ivo Sachs. Complete superspace classification of three-dimensional Chern-Simons-matter theories coupled to supergravity. *JHEP*, 02:154, 2018.

[33] Bengt E. W. Nilsson. Critical solutions of topologically gauged N = 8 CFTs in three dimensions. *JHEP*, 04:107, 2014.

[34] Sergei M. Kuzenko, Ulf Lindstrom, and Gabriele Tartaglino-Mazzucchelli. Off-shell supergravity-matter couplings in three dimensions. *JHEP*, 03:120, 2011.

[35] S. J. Gates, Marcus T. Grisaru, M. Rocek, and W. Siegel. Superspace Or One Thousand and One Lessons in Supersymmetry. *Front. Phys.*, 58:1–548, 1983.

[36] Paul S. Howe, J. M. Izquierdo, G. Papadopoulos, and P. K. Townsend. New supergravities with central charges and Killing spinors in (2+1)-dimensions. *Nucl. Phys.*, B467:183–214, 1996.

[37] Daniel Butter, Sergei M. Kuzenko, Joseph Novak, and Gabriele Tartaglino-Mazzucchelli. Conformal supergravity in three dimensions: New off-shell formulation. *JHEP*, 09:072, 2013.

[38] Sergei M. Kuzenko, Ulf Lindstrom, and Gabriele Tartaglino-Mazzucchelli. Three-dimensional (p,q) AdS superspaces and matter couplings. *JHEP*, 08:024, 2012.

[39] Daniel Butter, Sergei M. Kuzenko, Joseph Novak, and Gabriele Tartaglino-Mazzucchelli. Conformal supergravity in three dimensions: Off-shell actions. *JHEP*, 10:073, 2013.

[40] Karim Ait Moussa, Gerard Clement, and Cedric Leygnac. The Black holes of topologically massive gravity. *Class. Quant. Grav.*, 20:L277–L283, 2003.

[41] Andrew Strominger. A Simple Proof of the Chiral Gravity Conjecture. 2008.

[42] N. D. Birrell and P. C. W. Davies. *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, Cambridge, UK, 1984.

[43] S. Deser and J. Franklin. Is BTZ a separate superselection sector of CTMG? *Phys. Lett.*, B693:609–611, 2010.

[44] C. M. Hull. The Positivity of Gravitational Energy and Global Supersymmetry. *Commun. Math. Phys.*, 90:545, 1983.

[45] Stanley Deser and Claudio Teitelboim. Supergravity Has Positive Energy. *Phys. Rev. Lett.*, 39:249, 1977.

[46] S. Deser. Positive energy of topologically massive gravity. *Class. Quant. Grav.*, 26(19):192001, 2009.

[47] G. W. Gibbons, C. M. Hull, and N. P. Warner. The Stability of Gauged Supergravity. *Nucl. Phys.*, B218:173, 1983.

[48] Daniel Z. Freedman and Antoine Van Proeyen. *Supergravity*. Cambridge Univ. Press, Cambridge, UK, 2012.