Selected Topics in Complex Geometry

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Abstract

This thesis consists of three parts.

In the first part the problem of lifting holomorphic projective bundles to holomorphic vector bundles is considered. Namely, we give a criterion to ensure the existence of a holomorphic lift in terms of the existence of a smooth lift and a holomorphic structure on the determinant bundle of the lift. The solution of the lifting problem is then applied in two settings: to obtain a classification of Kähler contact 3-folds and certain Kähler string manifolds. The fact that both types of spaces are spin allows us to make use of a consequence (due to Schreieder-Tasin) of the minimal model program for Kähler 3-folds.

In the second part, we study the geometry of holomorphic Engel structures on the complex space $\mathbb{C}^4$. Applying certain methods due to Forstneric, we are able to produce uncountably infinitely many non-isomorphic Engel structures.

In the third part some partial results concerning holomorphic domination by a product manifold are discussed. We obtain non-domination results from certain negativity properties of the tangent bundle. This is subsequently applied to the case of a Fano surface of lines.

Zusammenfassung

Diese Arbeit besteht aus drei Teilen.

Im ersten Teil wird die Hochhebung eines holomorphen projektiven Bündels zu einem holomorphen Vektorbündel betrachet. Es wird im Besonderen ein Kriterium angegeben, das die Existenz einer holomorphen Hochhebung sichert, wenn zugleich eine differenzierbare Hochhebung und eine holomorphe Struktur auf dem Determinantenbündel des hochgehobenen Bündels existieren. Die Chararakterisierung der Existenz einer Hochhebung wird auf zwei Arten verwendet: um Kähler 3-Mannigfaltigkeiten mit holomorphen Kontaktstrukturen zu klassifizieren und um Kähler String-Mannigfaltigkeiten zu untersuchen. In beiden Fälle lassen die Räume Spinstrukturen zu, was die Verwendung einer Konsequenz des Mori Programs für Kähler 3-Faltigkeiten zulässt, die von Schreider-Tasin angegeben wurde.

Im zweiten Teil wird die Geometrie von holomorphen Engel Strukturen auf $\mathbb{C}^4$ untersucht. Mit Methoden, die auf Forstneric zurück gehen, wird die Konstruktion von überabzählbar vielen unterschiedlichen Engel Strukturen ausgeführt.

Im dritten Teil werden gewisse Resultate über dominante Abbildungen von kartesischen Produkten auf komplexe Mannigfaltigkeiten bewiesen. Es wird untersucht, wie gewisse Negativitätseigenschaften des Tangentialbündels das Dominieren durch ein kartesiches Produkt ausschließen. Nachfolgend wird damit bewiesen, dass Fano Flächen sich nicht durch ein Produkt dominieren lassen.

Contents

Chapter 1

Introduction

"... l'élément ne préexiste pas à l'ensemble, il n'est ni plus immédiat ni plus ancien, ce ne sont pas les éléments qui déterminent l'ensemble, mais l'ensemble qui détermine les éléments: la connaissance du tout et de ses lois, de l'ensemble et de sa structure, ne saurait être déduite de la connaissance séparée des parties qui le composent: cela veut dire qu'on peut regarder une pièce d'un puzzle pendant trois jours et croire tout savoir de sa configuration et de sa couleur sans avoir le moins du monde avancé: seule compte la possibilité de relier cette pièce à d'autres pièces ..."

– Georges Perec. *La Vie. Mode d'emploi.*

This thesis discusses three rather unrelated topics. First, the problem of lifting a holomorphic projective bundle to a holomorphic vector bundle, with a view to certain applications for 3-dimensional string and contact manifolds. Second, the study and construction of non-standard holomorphic Engel structures on $\mathbb{C}^4$. Third, obstructions to domination by products by means of negativity conditions on the tangent bundle. We start with an overview of the main ideas and results for each of these topics.

1.1 Lifts of projective bundles

The work described in this section, and included in Chapters 2, 3 and 4, represents previously unpublished joint work with D. Kotschick.

Given a vector bundle, one can projectivize each fiber to obtain a bundle of projective spaces, the *projectivization.* The *lifting problem for projective bundles* is the problem of writing a *projective bundle*, i.e. a bundle of projective spaces[1], as the projectivization of a vector bundle. This cannot be done in general. Indeed, in algebraic geometry this observation leads one to define the Brauer group of a variety, which endows the projective bundles modulo projectivizations with a group structure. The Brauer group is the subject of active research and generally a delicate object to study.

We pursue here the more practical question of deciding for a given holomorphic projective bundle whether the lifting problem can be solved. To obtain a usable criterion, the idea is to separate the topological and the holomorphic sides of the problem. One attempts first to lift the projective bundle topologically to a complex vector bundle. If this is possible, one then applies a differential geometric criterion to decide whether the complex vector bundle can be endowed with a holomorphic structure. The main result is the following theorem.

Theorem 2.1. *Let $X \longrightarrow B$ be a holomorphic $\mathbb{CP}^{k-1}$-bundle over a complex manifold B. Then there is a holomorphic rank k vector bundle $\mathcal{E} \longrightarrow B$ with $X = \mathbb{P}(\mathcal{E})$ if and only if the smooth projective space bundle underlying X lifts to a smooth rank k vector bundle $E \longrightarrow B$ for which the determinant bundle $\det(E)$ admits the structure of a holomorphic line bundle.*

In the special case where the total space is a spin manifold and the fibers of the projective bundle are one-dimensional, Theorem 2.1 has the following corollary:

Corollary 2.7. *Let $X \longrightarrow B$ be a holomorphic $\mathbb{CP}^1$-bundle over a complex manifold B. If X is a spin manifold, there exists a holomorphic rank 2 vector bundle $\mathcal{E} \longrightarrow B$ with $X = \mathbb{P}(\mathcal{E})$.*

In other words, the lifting problem can always be solved in this instance.

We now explain our interest in the specific assumptions employed in the corollary. The minimal model program was recently extended to Kähler threefolds by Höring and Peternell [36, 37]. In analogy to the 2-dimensional phenomenon that a spin complex surface is automatically minimal, Schreieder-Tasin observed in [69] that for spin Kähler threefolds, the steps necessary to get to the minimal model are greatly simplified. More concretely, a spin Kähler threefold is, up to a sequence of blow-downs to smooth points, either a Mori fibre space or a minimal model (that is, it has a pseudo-effective canonical bundle). Moreover, in the former case, it is one of the following three: a Fano manifold, a $\mathbb{CP}^1$ bundle over a Kähler surface, or a quadric bundle over a complex curve. Corollary 2.7 allows us to obtain a slight refinement of the second possibility in this list. We use this fact to investigate string Kähler threefolds and contact Kähler threefolds.

String Kähler threefolds

A spin manifold is called string if its first Pontryagin class vanishes. The condition $p_1(M) = 0$ arose first in string theory, as the condition for having a spin structure on the loop space of the manifold. It has since been studied extensively in the context of elliptic

[1] In the smooth case, one requires also that the structure group be the projective general linear group.

cohomology, see for example [71]. We apply the methods and results of [69], together with some additional arguments, to analyze Kähler structures on threefolds which are not just spin, but string.

While a spin structure rules out blowdowns to smooth points in even complex dimensions (since $\mathbb{CP}^{2n}$ is not spin), in odd complex dimensions $\mathbb{CP}^{2n+1}$ is spin, but is not string. Therefore, for string manifolds there can be no blowdowns to smooth points in odd dimensions, although the spin condition alone is not sufficient for this conclusion. We will prove the following result.

Theorem 3.2. *Let X be a string Kähler threefold. If $\chi(\mathcal{O}_X) > 0$, then $b_2(X) \geq 2$. If furthermore $b_2(X) > 2$, then X is the projectivisation of a holomorphic rank 2 vector bundle over a projective surface S with $\chi(\mathcal{O}_S) > 0$.*

Conversely, every projective surface S with $\chi(\mathcal{O}_S) > 0$ arises in this way from a string threefold.

The fact that non-projective Kähler surfaces cannot occur for S in this theorem came as a surprise. This part of the conclusion arises from non-existence results for certain holomorphic rank two vector bundles on non-projective surfaces initiated by Elencwajg and Forster [22] and developed further by Bănică and Le Potier [54, 7].

The proof of this theorem has the following further consequence, the last part of which relates to the scarcity of spin threefolds of general type discussed in [69, Subsection 1.2]. That many Kähler threefolds are in fact (close to) projective is one of the major themes in extending Mori theory to the Kähler setting [36, 37, 16].

Corollary 3.3. *If X is a string Kähler threefold with $\chi(\mathcal{O}_X) > 0$, then X is projective, with negative Kodaira dimension.*

The assumption $\chi(\mathcal{O}_X) > 0$ is trivially satisfied whenever X has no odd-degree cohomology. In contrast to this corollary, there are string projective manifolds of general type with no odd-degree cohomology in all even complex dimensions. Without the assumption $\chi(\mathcal{O}_X) > 0$ Kähler but non-projective string threefolds do exist, as shown by the product $T \times C$, where T is a non-projective complex torus of complex dimension two, and C is a curve.

Contact Kähler threefolds

A holomorphic contact structure on a complex manifold X of dimension $2n+1$ is a rank $2n$ holomorphic subbundle F of the tangent bundle TX which is maximally non-integrable. Putting $L = TX/F$ and denoting by $\alpha : TX \longrightarrow L$ the quotient projection, one can restate this condition as saying that the L^{n+1}-valued form $\alpha \wedge (d\alpha)^n$ is everywhere non-zero, that is, it defines an isomorphism $\det(TX) \simeq L^{n+1}$. If X is 3-dimensional, this implies that its first Chern class is even, thus X is a spin manifold.

The canonical examples of contact manifolds are $\mathbb{CP}^{2n+1}$ with $L = \mathcal{O}_{\mathbb{CP}^{2n+1}}(2)$ and the projectivization $\mathbb{P}(T^*S)$ of the cotangent bundle of a complex surface S, with $L = \mathcal{O}_{\mathbb{P}(T^*S)}(1)$.

We will give a quick proof of the classification of compact Kähler threefolds admitting holomorphic contact structures, stating that these two examples are exhaustive.

Theorem 4.1. *Let X be a compact Kähler threefold with a holomorphic contact structure. Then X is $\mathbb{CP}^3$ or the projectivization of the cotangent bundle of a Kähler surface. Conversely, both $\mathbb{CP}^3$ and the projectivizations of cotangent bundles have standard contact structures.*

For X projective, Theorem 4.1 was proved by Ye [79]; see also [42]. However, in his proof, Ye does not explain why a certain $\mathbb{CP}^1$-bundle over a projective surface is the projectivization of a vector bundle[2], which is something that does not hold in general.

As explained above, a threefold with a contact structure is spin, so we are in a position to use the results of Schreieder-Tasin [69], which together with Corollary 2.7 lead to a proof of Theorem 4.1. Modulo [69], this gives both a self-contained proof of Ye's original result, and its extension from projective to Kähler threefolds. This extension was also carried out by Frantzen and Peternell [28], completing an earlier result of Peternell [64]. However, those papers appeal to Ye's [79] arguments, without addressing the issue we found with the latter.

1.2 Holomorphic Engel structures

The content of chapter 5 is joint work with N. Pia and was published in the *Journal of Geometric Analysis*, Volume 28(3) (2018) [18].

An Engel structure is a 2-distribution D on a smooth 4-dimensional manifold M with the property that $E := [D, D]$ has constant rank equal to 3 and $[E, E] = TM$ (so E is an even contact structure). Associated to an Engel structure there exists also a canonical line field $W \subset D$, the *characteristic foliation*, so that D comes with an associated canonical flag $W \subset D \subset E \subset TM$.

These structures sit in the rather restrictive class of topologically stable distributions[3], which besides Engel structures, contains only line fields, contact structures and even contact structures. This very same classification holds true in the holomorphic case as well (see [65]), which makes Engel structures natural objects to study in complex geometry. Nevertheless, unlike contact structures, this study has only relatively recently been started by Presas-Solá Conde in [65], who studied Engel structures in compact projective manifolds. They show among other things, that if a certain weak positivity condition holds, then an Engel structure on a projective manifold is a so-called *prolongation* of a holomorphic contact structure on a 3-fold.

By contrast, we will be interested here in the study of holomorphic Engel structures on $\mathbb{C}^4$, specifically with the following: are there holomorphic Engel structures on $\mathbb{C}^4$ which are not isomorphic to the standard one? This question is essentially motivated by the

[2]See the middle of p. 313 in [79].

[3]This means that the set of Engel structures on M is open within the space of distributions in TM, and that there is a universal model to which any Engel structure is locally isomorphic.

analogous one for contact structures on $\mathbb{C}^{2n+1}$, which was studied by Forstnerič in [25]. There he constructs non-standard contact structures which are distinguishable from the standard one because they enjoy a certain type of *directed hyperbolicity.* More specifically, the contact structures of Forstnerič admit no *Legendrian complex lines*, that is, non-constant holomorphic maps $\mathbb{C} \longrightarrow \mathbb{C}^{2n+1}$ which are everywhere tangent to the contact distribution.

Forstnerič's construction can be described as follows: one starts by proving certain estimates that Legendrian complex lines with respect to the standard contact structure must fulfill. One then constructs a certain subset K of $\mathbb{C}^{2n+1}$ which is shown, by means of the previously computed estimates, to intersect every Legendrian complex line. Finally, a proper subset $\Omega \subset \mathbb{C}^{2n+1}$ biholomorphic to $\mathbb{C}^{2n+1}$ and disjoint from K is shown to exist. Therefore a non-standard contact structure is obtained by restricting the standard one to Ω. It is not clear at present whether one can obtain different contact structures by varying the parameters in the construction.

We will use similar methods to Forstnerič to construct Engel structures on $\mathbb{C}^4$ which are non-isomorphic to the standard model. A stark difference between the Engel and the contact settings is the flag that comes with an Engel structure. This will allow us to use the configuration of complex lines tangent to each of the distributions in the flag as an invariant:

Theorem 5.1. *On $\mathbb{C}^4$ there are Engel structures $\mathcal{D}_\mathcal{E}$, $\mathcal{D}_\mathcal{D}$ and $\mathcal{D}_\mathcal{W}$ with the following properties*

1. *$\mathcal{D}_\mathcal{E}$ admits no lines tangent to its associated even contact structure;*
2. *$\mathcal{D}_\mathcal{D}$ admits no $\mathcal{D}_\mathcal{D}$-lines but does admit lines tangent to its associated even contact structure;*
3. *$\mathcal{D}_\mathcal{W}$ admits no lines tangent to its characteristic foliation but does admit $\mathcal{D}_\mathcal{W}$-lines.*

In particular these Engel structures are pairwise non-isomorphic and not isomorphic to the standard Engel structure $(\mathbb{C}^4, \mathcal{D}_{st})$.

Moreover a theorem of Fornæss-Buzzard will provide fine control on the complex lines tangent to the characteristic foliation of the so-called *Cartan prolongation* of contact structures on $\mathbb{C}^3$ without Legendrian complex lines as constructed by Forstnerič, and lead to the construction of two infinite families of holomorphic Engel structures.

Theorem 5.2. *For every $n \in \mathbb{N} \cup \{\infty\}$ there exists an Engel structure $\mathcal{D}_n$ on $\mathbb{C}^4$ for which the only $\mathcal{D}_n$-lines are tangent to the characteristic foliation $\mathcal{W}_n$, and such that*

$$L_n := \{p \in \mathbb{C}^4 : \exists f : \mathbb{C} \to \mathbb{C}^4 \ \mathcal{D}_n\text{-line with } f(0) = p\}$$

is a proper subset of $\mathbb{C}^4$ which has exactly n connected components for $n \in \mathbb{N}$, and $L_\infty = \mathbb{C}^4$.

In the above, we use the number of connected components of L_n as an invariant; it turns out one can also use a modulus given by the relative position of such components and obtain a family of uncountably many pairwise non-isomorphic Engel structures on $\mathbb{C}^4$.

Theorem 5.3. *For every $R \in \mathbb{R} \setminus \{0\}$ there exists an Engel structure $\mathcal{D}_R$ for which the only $\mathcal{D}_R$-lines are tangent to the characteristic foliation $\mathcal{W}_R$, and such that the set of points which admit such $\mathcal{W}_R$-lines is exactly $\mathbb{C} \times \{0, 1, R\sqrt{-1}\} \times \mathbb{C}^2 \subset \mathbb{C}^4_{(w,x,y,z)}$. Moreover $\mathcal{D}_R$ is isomorphic to $\mathcal{D}_{R'}$ if and only if $R = R'$.*

1.3 Domination by products in the holomorphic setting

In the last chapter, we will be concerned with the following question: given Y a complex manifold, does there exist a dominant map $X \longrightarrow Y$ from a product $X = X_1 \times \cdots \times X_k$, where $\dim(X_i) < \dim(Y)$?

This problem is treated systematically by Schoen [67] in the context of algebraic varieties and rational maps. There, a birational invariant τ coming from Hodge theory is defined, which is monotonic under the rational dominance relation: if X rationally dominates Y, then $\tau(X) \geq \tau(Y)$. Schoen gives numerous examples where τ obstructs rational domination by a product.

We take a (complex) geometric perspective and seek negativity conditions on the tangent bundle of complex manifolds that obstruct holomorphic domination by products. There are two basic heuristics that suggest that a negativity condition is a fruitful one to consider: the first is that products of complex manifolds cannot be too negative. The second is that in order to dominate a negative space, the domain tends to need some negativity as well. The tension between these two sides is where we look for our obstruction.

One elementary but instructive observation is that if a manifold Y is dominated by a product, there are non-trivial continuous families of holomorphic maps into Y, simply by viewing factors in the product as parameters. Conditions that ensure the discreteness of spaces of holomorphic maps are known, and have been studied among others by Noguchi-Sunada [61] and Kalka-Shiffman-Wong [41]. One such condition is *p–negativity.* A holomorphic vector bundle $E \longrightarrow X$ is p–negative if it admits a non-negative continuous real valued function $f : E \longrightarrow \mathbb{R}$ such that f vanishes exactly along the zero section of E, and elsewhere it is of class C^2 with a Hessian $\partial\bar{\partial} f$ that has $\dim(X) + p$ positive eigenvalues at each point. Kalka-Shiffman-Wong [41] prove that, if X is a compact space and Y a complex manifold with p–negative tangent bundle, then the space of holomorphic maps $X \longrightarrow Y$ of rank greater than $\dim(Y) - p$ is finite. For instance, if Y admits a Hermitian metric with strictly negative bisectional curvature, then TY is n-negative, and therefore Y not dominated by a product.

To obtain stronger results, we localize the notion of p–negativity to a condition we call *quasi p–negativity.* As an example, the tangent bundle of a Hermitian manifold with holomorphic bisectional curvature nonpositive everywhere and negative at one point is quasi n-negative. The main result in this chapter is the following theorem:

Theorem 6.25. *Let Y be a complex manifold of dimension n. Let $l > 1$ be an integer and let $X = X_1 \times \cdots \times X_l$, where X_i are compact complex manifolds of dimension $m_i < n$. Further let $k = \min_i(m_i)$ and p an integer satisfying $p > k$. Assume that TY is quasi*

p–negative, and let $\varphi : X \longrightarrow Y$ be a holomorphic map such that φ does not factor through $X_1 \times \cdots \times \widehat{X_j} \times \cdots \times X_l \subset X$ for any j. Then φ is not dominant.

As a corollary, we obtain:

Corollary 6.27. *Let Y be a projective variety which admits a metric with quasi-negative bisectional curvature. Then Y is not rationally dominated by a product.*

Natural candidates to which the corollary applies are submanifolds of complex tori: since the bisectional curvature is non-increasing on submanifolds, non-positivity of the bisectional curvature is automatic, and one need only find a point where it is negative. We show in this way that the *Fano surface of lines in a cubic* is not rationally dominated by a product of curves. This was proved via different methods by Schreieder in [68].

Chapter 2

Lifts of Projective Bundles

This chapter deals with the lifting problem for projective bundles, that is, the question of deciding for a given $PGL(k, \mathbb{C})$–bundle whether it comes from the projectivization of a vector bundle.

Although we are especially interested in the holomorphic setting, our approach is to consider the topological side first. This will allow us to split the holomorphic problem into two steps: after deciding whether a certain projective bundle admits – topologically – a lift to a complex vector bundle, we can study the obstruction to endowing the lift with a holomorphic structure. The topological lifting problem is treated in Subsection 2.2.1. While the discussion is at first carried out in full generality, we then focus in the special case where the fiber is a 2-sphere, where a particularly simple criterion arises in terms of the Euler class of the sphere bundle.

In the holomorphic setting, we will prove the following theorem:

Theorem 2.1. *Let $X \longrightarrow B$ be a holomorphic $\mathbb{CP}^{k-1}$–bundle over a complex manifold B. Then there is a holomorphic rank k vector bundle $\mathcal{E} \longrightarrow B$ with $X = \mathbb{P}(\mathcal{E})$ if and only if the smooth projective space bundle underlying X lifts to a smooth rank k vector bundle $E \longrightarrow B$ for which the determinant bundle* $\det(E)$ *admits the structure of a holomorphic line bundle.*

Thus if a holomorphic $\mathbb{CP}^{k-1}$–bundle lifts smoothly to a complex vector bundle, the condition that it lifts holomorphically is simply that the determinant of some smooth lift admits a holomorphic structure. The proof of Theorem 2.1 is carried out through somewhat classical methods: the condition that a complex bundle projectivizes to a holomorphic one will be seen in Theorem 2.13 to be equivalent to the existence of a certain type of connection, which we name projective. When the determinant bundle has a holomorphic structure – and therefore admits a connection whose curvature form has vanishing $(0, 2)$ part – it is possible to modify the projective connection to one whose curvature has vanishing $(0,2)$ part, thus endowing the vector bundle with a holomorphic structure.

2.1 Projective bundles

We start with the notion of a projective bundle and recall the standard construction of the simplest projective bundles: projectivizations of vector bundles.

Definition 2.2. Let M be a smooth manifold. A *smooth projective bundle* on M is a smooth fiber bundle with fiber $\mathbb{CP}^k$ and structure group $PGL(k+1,\mathbb{C})$.

Remark 2.3. In the special case where $k=1$, the fiber is $\mathbb{CP}^1 \cong S^2$. By a result of Smale (see [73]), the orientation preserving diffeomorphism group of S^2 deformation retracts to $SO(3)$. Thus a smooth orientable projective bundle with fiber $\mathbb{CP}^1$ is in fact the sphere bundle of a rank 3 vector bundle.

Since the group of holomorphic automorphisms of $\mathbb{CP}^k$ is $PGL(k+1,\mathbb{C})$, it is not necessary to assume that a holomorphic $\mathbb{CP}^k$-bundle has structure group $PGL(k+1,\mathbb{C})$. Rather, this is an immediate consequence of having holomorphic trivializations, and we omit the structure group in the holomorphic counterpart to Definition 2.2.

Definition 2.4. Let X be a complex manifold. A *holomorphic projective bundle* on X is a holomorphic fiber bundle with fiber a complex projective space $\mathbb{CP}^k$.

The first example of a projective bundle is the *projectivization* of a vector bundle, the bundle whose fiber over a point consists of the lines in the corresponding fiber of the vector bundle. We review the construction in the subsequent examples.

Example 2.5 (Smooth projectivizations)**.** Let M be a smooth manifold and let $\pi : E \longrightarrow M$ be a complex vector bundle of rank $k+1$ on M. Denote by $E^\times$ the total space of E with its zero section removed. The group $\mathbb{C}^*$ acts freely on $E^\times$ by multiplication on the fibers yielding a quotient space $\mathbb{P}(E) = E^\times/\mathbb{C}^*$ called the *projectivization* of the vector bundle E. Denote by $p : E^\times \longrightarrow \mathbb{P}(E)$ the quotient map. With the projection induced by π, $\mathbb{P}(E) \longrightarrow M$ is a fiber bundle over M with fiber $\mathbb{CP}^k$. Concretely, the trivializations of E restrict to $E^\times$ to fit into a diagram

$$\begin{tikzcd} E_U^\times \arrow[r, "\psi"] \arrow[d, "p"] & U \times \mathbb{C}^{k+1}\backslash\{0\} \arrow[d] \\ \mathbb{P}(E)_U \arrow[r] & U \times \mathbb{CP}^k, \end{tikzcd}$$

where for $x \in M$ the map on the bottom sends a line in E_x to the corresponding line in the trivialization, and the map on the right is the obvious one making the diagram commute. The cocycle of $\mathbb{P}(E)$ is therefore given by the image of the cocycle of E under the quotient map $GL(k+1,\mathbb{C}) \longrightarrow PGL(k+1,\mathbb{C})$. This is the first example of a projective bundle. ◻

It is a matter of convention that we defined $\mathbb{P}(E)$ as the bundle of *lines* in E. The projectivization of E is sometimes defined instead as the bundle $\mathbb{P}_{\mathrm{quot}}(E)$ of rank 1 quotients of E; the resulting projective bundle is not in general isomorphic to $\mathbb{P}(E)$, although the total spaces are diffeomorphic. The two are related by $\mathbb{P}(E) = \mathbb{P}_{\mathrm{quot}}(E^*)$.

Example 2.6 (Holomorphic projectivizations)**.** The preceding example works *mutatis mutandis* in the holomorphic setting, yielding in this case a holomorphic projective bundle. □

A construction of a smooth projective bundle which does not come from a vector bundle is given below in Example 2.8.

2.2 The lifting problem

We now begin the study of the lifting problem for projective bundles. This will lead in particular to Theorem 2.1 stated in the introduction and to the following corollary:

Corollary 2.7. *Let $X \longrightarrow B$ be a holomorphic $\mathbb{CP}^1$-bundle over a complex manifold B. If X is a spin manifold, then there is a holomorphic rank 2 vector bundle $\mathcal{E} \longrightarrow B$ with $X = \mathbb{P}(\mathcal{E})$.*

2.2.1 Smooth projective bundles

Let B be a smooth manifold and $M \longrightarrow B$ a smooth projective bundle. Since $PU(k)$ sits in $PGL(k, \mathbb{C})$ as a maximal compact subgroup, the structure group of a $PGL(k, \mathbb{C})$-bundle can be reduced to $PU(k, \mathbb{C})$, and isomorphism classes of $PGL(k, \mathbb{C})$-bundles are in bijection with isomorphism classes of $PU(k)$-bundles. We will consider the latter.

One has the following commutative diagram of Lie groups with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] & U(k) \arrow[r] & PU(k) \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{Z}_k \arrow[r] \arrow[u, hook] & SU(k) \arrow[r] \arrow[u, hook] & PU(k) \arrow[r] \arrow[u, equal] & 1.
\end{tikzcd}$$

The induced long exact sequences in Čech cohomology give rise to the following commutative diagram of pointed sets:

$$\begin{tikzcd}
\check{H}^1(B; U(1)) \arrow[r] & \check{H}^1(B; U(k)) \arrow[r] & \check{H}^1(B; PU(k)) \arrow[r, "\delta"] & \check{H}^2(B; U(1)) \\
\check{H}^1(B, \mathbb{Z}_k) \arrow[r] \arrow[u] & \check{H}^1(B; SU(k)) \arrow[r] \arrow[u] & \check{H}^1(B; PU(k)) \arrow[r, "\tilde{\delta}"] \arrow[u, equal] & \check{H}^2(B; \mathbb{Z}_k). \arrow[u, "\beta"']
\end{tikzcd}$$

Here we denote by the same symbol a Lie group G and the corresponding sheaf of smooth functions on B with values in G.

For a given isomorphism class $[M] \in \check{H}^1(B; PU(k))$, the obstruction to lifting $[M]$ to a $U(k)$ bundle is $\delta([M])$. Since the sheaf of smooth functions is fine, the exponential sequence identifies $\check{H}^i(B; U(1))$ with $H^{i+1}(B;\mathbb{Z})$, so $\delta([M])$ can be seen as an element in $H^3(M;\mathbb{Z})$. Moreover, under this identification β is the Bockstein homomorphism, as can be seen by considering the sequence

$$1 \longrightarrow \mathbb{Z}_k \longrightarrow U(1) \xrightarrow{\times k} U(1) \longrightarrow 1$$

where the group $\mathbb{Z}_k$ is viewed as the subgroup of k^{th} roots of unity.

By naturality, the map δ factors through $H^2(B;\mathbb{Z}_k)$, which implies that the obstruction class $\delta([M])$ is an element of order (at most) k in $H^3(B;\mathbb{Z})$. Moreover, the $PU(k)$-bundle M lifts to a $U(k)$-bundle if and only if the obstruction to lifting M to an $SU(k)$-bundle lies in the kernel of the Bockstein homomorphism β.

When $k = 2$, the obstruction class has a particularly simple interpretation which we now explain. First we have that $PU(2) = SU(2)/\pm 1 = SO(3)$, which means that an element $[M] \in \check{H}^1(B; PU(2))$ can be viewed as the sphere bundle of an orientable rank 3 real vector bundle V over B. The element $\delta([M]) \in H^3(B;\mathbb{Z})$ is a characteristic class of V in degree 3, which implies already that it must be the Euler class of V. This can also be seen directly. Lifting M to an $SU(2)$-bundle is the same as endowing V with a spin structure, the obstruction to which is the second Stiefel-Whitney class $w_2(V)$. By the previous paragraph, M lifts to a $U(2)$-bundle if and only if this obstruction is killed by the Bockstein homomorphism. But $\beta(w_2(V)) = W_3(V)$ is exactly the Euler class of V, since V has rank 3.

We conclude that a smooth $\mathbb{CP}^1$-bundle with structure group $PU(2)$ is the projectivization of a complex rank 2 vector bundle if and only if $e(V) = 0$. In fact, this holds for *any* orientable smooth $\mathbb{CP}^1$-bundle, without any assumption on the structure group, due to remark 2.3.

Example 2.8 (A projective bundle which is not a projectivization)**.** Let M be a 4 dimensional manifold on which there exists a class $x \in H^2(M,\mathbb{Z}_2)$ which does not lift to $H^2(M,\mathbb{Z})$. Let f be a map to an Eilenberg-MacLane space $K(\mathbb{Z}_2, 2)$

$$f : M \longrightarrow K(\mathbb{Z}_2, 2)$$

representing x under the correspondence between $H^2(M;\mathbb{Z}_2)$ and homotopy classes of maps $M \longrightarrow K(\mathbb{Z}_2, 2)$. In the same manner, represent the second Stiefel-Whitney class w_2 of the universal bundle on the classifying space $BSO(3)$ of $SO(3)$ bundles by $h : BSO(3) \longrightarrow K(\mathbb{Z}_2, 2)$. Since $\pi_i(BSO(3)) = \pi_i(K(\mathbb{Z}_2, 2))$ for $0 \leq i \leq 3$ and $\pi_4(K(\mathbb{Z}_2, 2)) = 0$, h is a 4-equivalence. It follows that (see for example Theorem 11.12, p. 485 in [9]) the map f factors through $BSO(3)$ for some map g:

$$f : M \xrightarrow{g} BSO(3) \xrightarrow{h} K(\mathbb{Z}_2, 2).$$

Denoting by $\alpha \in H^2(K(\mathbb{Z}_2, 2), \mathbb{Z}_2)$ the fundamental class of $K(\mathbb{Z}_2, 2)$, we have

$$x = f^*(\alpha) = g^*h^*(\alpha) = g^*(w_2),$$

so the pull-back of the universal bundle on $BSO(3)$ to M, yields a rank 3 real oriented vector bundle E on M with $w_2(E) = x$. Since x does not lift to $H^2(M, \mathbb{Z})$, it has non-trivial image under the Bockstein homomorphism $\beta : H^2(M, \mathbb{Z}_2) \longrightarrow H^3(M, \mathbb{Z})$. But as mentioned above, this is exactly the Euler class $e(E) = \beta(w_2(E))$.

For instance, we may consider the 4-dimensional manifold $M = \mathbb{RP}^3 \times S^1$, whose cohomology with integer coefficients is given in degree 2 by $H^2(M; \mathbb{Z}) = \mathbb{Z}_2$. On the other hand, we compute from the Künneth formula $H^2(M; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Thus there is a class in $H^2(M; \mathbb{Z}_2)$ which does not lift to $H^2(M; \mathbb{Z})$ and hence a $\mathbb{CP}^1$-bundle on M, namely the sphere bundle of a vector bundle E of rank 3, which does not lift to a complex rank 2 bundle. In this case one can also explicitly write down such an E. Let L_1 and L_2 be (the) non-orientable real line bundles on $\mathbb{RP}^3$ and S^1, respectively. Let L_3 be the line bundle on M with first Stiefel-Whitney class given by $w_1(L_3) = w_1(L_1) + w_1(L_2)$ and put $E = (L_1 \oplus L_2 \oplus \underline{\mathbb{R}}) \otimes L_3$. Then

$$\begin{aligned} w_1(E) =& w_1(L_1) + w_1(L_2) + 3w_1(L_3) = 0 \\ w_2(E) =& w_1(L_1) \cup w_1(L_2). \end{aligned}$$

Since the class $w_1(L_1) \cup w_1(L_2)$ does not lift to an integral class, we conclude that E is orientable with non-trivial Euler class. □

One instance where the lift exists is given in the following lemma, which will be useful later.

Lemma 2.9. *Let $\pi \colon M \longrightarrow B$ be a smooth orientable 2-sphere bundle over an orientable manifold B. If M is a spin manifold and B admits a Spin^c-structure, then there exists a complex vector bundle $E \longrightarrow B$ such that $M = \mathbb{P}(E)$ as an oriented 2-sphere bundle.*

The assumption about the existence of a *Spin*^{c}-structure on B is always satisfied if $\dim(B) \leq 4$, or if B admits an almost complex structure.

Proof. Let $\pi \colon V \longrightarrow B$ be an orientable real rank 3 vector bundle whose sphere bundle is isomorphic to M. Notice that

$$TM \oplus \underline{\mathbb{R}} = \pi^*(V) \oplus \pi^*(TB) \ ,$$

so $w_2(TM) = \pi^*(w_2(V) + w_2(TB))$. This must vanish since M is assumed to be spin. Since π^* is injective in degree 2, we conclude $w_2(V) = w_2(TB)$. Now $\beta(w_2(TB)) = 0$ is exactly the condition that B admits a *Spin*^{c}-structure. ∎

One has the following well known (see [35]) relation between the characteristic classes of V and E.

Lemma 2.10. *Let $V \longrightarrow B$ be an orientable rank 3 real vector bundle. Suppose there exists a complex vector bundle $E \longrightarrow B$ of rank 2 such that $\mathbb{P}(E)$ and $S(V)$ are isomorphic as $SO(3)$ bundles. Then $p_1(V) = c_1(E)^2 - 4c_2(E)$.*

Proof. Note first that the expression for $p_1(V)$ remains unchanged under tensoring E with a complex line bundle L:

$$\begin{aligned} c_1(E \otimes L)^2 - 4c_2(E \otimes L) &= (c_1(E) + 2c_1(L))^2 - 4\left(c_2(E) + c_1(E)c_1(L) + c_1(L)^2\right) \\ &= c_1(E)^2 - 4c_2(E). \end{aligned}$$

By the splitting principle, we may assume that

$$E = L_1 \oplus L_2 = L_1 \otimes \left(\underline{\mathbb{C}} \oplus (L_1^{-1} \otimes L_2)\right).$$

Putting $L := L_1^{-1} \otimes L_2$ we now have $\mathbb{P}(E) = \mathbb{P}\left(\underline{\mathbb{C}} \oplus L\right)$. The transition maps of V are given by composing the transition maps of $\underline{\mathbb{C}} \oplus L$ with the morphism

$$\mu : U(2) \longrightarrow SU(2) \xrightarrow{\phi} SO(3),$$

where ϕ is defined by identifying an element $\alpha \in SU(2)$ with a unit quaternion q_α and taking $\phi(\alpha)$ to be the element of $SO(3)$ defined by letting q_α act by conjugation on the imaginary quaternions. Restricting μ to $1 \times U(1) \subset U(2)$ we see that

$$\mu \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix},$$

so $V \cong \underline{\mathbb{R}} \oplus L$. Since L is a complex line bundle we have $p_1(V) = p_1(\underline{\mathbb{R}}) + p_1(L) = 0 + c_1(L)^2$. Carrying on the computation we obtain

$$\begin{aligned} p_1(V) &= (c_1(L_1) - c_1(L_2))^2 \\ &= (c_1(L_1) + c_1(L_2))^2 - 4c_1(L_1)c_1(L_2) \\ &= c_1(E)^2 - 4c_2(E), \end{aligned}$$

which is the claim of the lemma. ∎

Returning to the general case, if a projective bundle can be lifted to a vector bundle, then it is natural to ask about the (non-)uniqueness of the lift. Two elements in $\check{H}^1(B; U(k))$ whose projectivizations are isomorphic differ by the tensor product with a complex line bundle.

2.2.2 Holomorphic projective bundles

We now discuss the problem of lifting a holomorphic projective bundle $X \longrightarrow B$ over a complex manifold B to a holomorphic vector bundle. This is completely analogous to the previous discussion, but considering instead the sequence of algebraic groups

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{C}^* \arrow[r] & GL(k,\mathbb{C}) \arrow[r] & PGL(k,\mathbb{C}) \arrow[r] & 1 \\ 1 \arrow[r] & \mathbb{Z}_k \arrow[r] \arrow[u, hook] & SL(k,\mathbb{C}) \arrow[r] \arrow[u, hook] & PSL(k,\mathbb{C}) \arrow[r] \arrow[u, equal] & 1 \end{tikzcd}$$

and replacing sheaves of smooth functions by sheaves of holomorphic functions with values in these groups. The corresponding exact sequences in cohomology give rise to the following commutative diagram:

$$\begin{tikzcd}
\check{H}^1(B;\mathcal{O}^*) \arrow[r] & \check{H}^1(B;GL(k,\mathcal{O})) \arrow[r] & \check{H}^1(B;PGL(k,\mathcal{O})) \arrow[r] & \check{H}^2(B;\mathcal{O}^*) \\
\check{H}^1(B;\mathbb{Z}_k) \arrow[r] \arrow[u, hook] & \check{H}^1(B;SL(k,\mathcal{O})) \arrow[r] \arrow[u, hook] & \check{H}^1(B;PGL(k,\mathcal{O})) \arrow[r] \arrow[u, equal] & \check{H}^2(B;\mathbb{Z}_k)\,. \arrow[u, hook]
\end{tikzcd}$$

For a given holomorphic projective bundle $X \longrightarrow B$ we again find that the obstruction class $c \in H^2(B;\mathcal{O}^*)$ to lifting X to a holomorphic vector bundle $\mathcal{E}$ is an element of order (at most) k.

We can now isolate the holomorphic part of the problem from its topological part. Namely, we assume that the obstruction to lifting X to a smooth complex vector bundle vanishes, and ask what obstruction remains to lifting X to a holomorphic vector bundle. Requiring that a smooth lift exists implies that the holomorphic obstruction is in the kernel of the map

$$\varphi\colon H^2(B;\mathcal{O}^*) \longrightarrow \check{H}^2(B;\underline{\mathbb{C}}^*) = \check{H}^2(B,U(1)) = H^3(B;\mathbb{Z})$$

induced by the forgetful sheaf homomorphism $\mathcal{O}^* \longrightarrow \underline{\mathbb{C}}^*$. Here $\underline{\mathbb{C}}^*$ denotes the sheaf of nowhere vanishing smooth $\mathbb{C}$-valued functions on B. But φ is in fact, under the above identification of $\check{H}^2(B;\underline{\mathbb{C}}^*)$ with $H^3(B;\mathbb{Z})$, the connecting homomorphism in the long exact sequence coming from the exponential sequence, as can be seen by considering the commutative diagram of sheaves

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & \mathcal{O}^* \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \underline{\mathbb{C}} \arrow[r] & \underline{\mathbb{C}}^* \arrow[r] & 1
\end{tikzcd}$$

where the vertical maps are forgetful morphisms.

Considering now

$$\cdots \longrightarrow H^2(B;\mathbb{Z}) \longrightarrow H^2(B,\mathcal{O}) \longrightarrow H^2(B;\mathcal{O}^*) \xrightarrow{\varphi} H^3(B;\mathbb{Z}) \longrightarrow \cdots$$

we see that if the smooth obstruction vanishes, i.e. $\varphi(c) = 0$, then there exists a lift $\tilde{c} \in H^2(B;\mathcal{O})$ of c. Moreover, c itself vanishes if and only if there exists an element in $L \in H^2(B;\mathbb{Z})$ which maps to $\tilde{c} \in H^2(B;\mathcal{O}) \cong H^{0,2}(B)$, by the Dolbeault theorem.

Proposition 2.11. *Let $E \longrightarrow B$ be a complex vector bundle of rank k which is a smooth lift of the holomorphic projective bundle $X \longrightarrow B$. Let α be the curvature form of a connection on* $\det(E)$*. Then $[\alpha^{0,2}/k] \in H^{0,2}(B)$ is in the image of $H^2(B,\mathbb{Z}) \longrightarrow H^2(B,\mathcal{O})$ if and only if X lifts to a holomorphic vector bundle.*

We postpone the proof of this proposition to Subsection 2.2.4 below.

2.2.3 Projective connections

In this section we will think of the lifting problem in a more differential-geometric way. It is well known that connections whose curvature has no $(0,2)$-component correspond to holomorphic structures on complex vector bundles, cf. [44, Ch. 1]. We will, in parallel with this standard fact, establish a correspondence between certain projective connections on vector bundles and holomorphic structures on their projectivizations.

Definition 2.12. Let B be a complex manifold and E a smooth complex vector bundle over B. Suppose that there is an isomorphism of smooth projective bundles $\mathbb{P}(E) \cong X$, where X is a holomorphic projective bundle. Then E is said to be *projectively holomorphic*.

Our aim is to prove the following:

Theorem 2.13. *A projectively holomorphic vector bundle E admits a holomorphic structure if and only if* $\det(E)$ *admits a holomorphic structure.*

The theorem will follow from the Lemmata 2.16 and 2.17 below. In order to state them conveniently, we introduce the following terminology:

Definition 2.14. Suppose that E is projectively holomorphic. A (local) non-vanishing section of E is called a *projectively holomorphic section* if it projectivizes to a holomorphic (local) section of $\mathbb{P}(E)$.

Definition 2.15. Let $E \longrightarrow B$ be a smooth complex vector bundle over a complex manifold B. A connection ∇ on E is called *projective* if its curvature F_∇ satisfies $F_\nabla^{0,2} = \alpha^{0,2} \otimes Id_E \in \Omega^{0,2}(\mathrm{End}(E))$ for some smooth 2–form α.

Lemma 2.16. *If E is projectively holomorphic, then it admits a projective connection.*

Proof. Suppose that E is projectively holomorphic of rank k. Then $\mathbb{P}(E)$ has the structure of a holomorphic projective bundle. We fix a local holomorphic trivialization for $\mathbb{P}(E)$ over some trivializing open set $U \subset B$, which we lift to obtain a local trivialization of E. In such trivializations the projectivization map is given by

$$\begin{aligned} U \times \mathbb{C}^k \setminus 0 &\longrightarrow U \times \mathbb{CP}^{k-1} \\ (x, v) &\longmapsto (x, [v]) \ . \end{aligned}$$

Fix the frame $s_i(x) = (x, (\ldots, 0, 1, 0, \ldots))$ on $E|_U$. The s_i are obviously projectively holomorphic. Let s be a local smooth non-vanishing section of E over U, $s = \sum_i \lambda_i s_i$. Then s is projectively holomorphic if and only if there exists a smooth, non-vanishing $\mathbb{C}$-valued f such that $f\lambda_i$ is holomorphic for all i.

Denote by ∇_U the flat connection on $E|_U$ defined by declaring the frame $\{s_i\}$ to be parallel. Let $s = f \sum_i \lambda_i s_i$ with λ_i holomorphic and f smooth. Then

$$\nabla_U^{0,1}(s) = \frac{\bar{\partial} f}{f} \otimes s = \bar{\partial} \log(f) \otimes s.$$

Gluing such locally defined connections ∇_U by a partition of unity yields a projective connection on E, as we will now see.

Put $\nabla = \sum_U \rho_U \nabla_U$ for a partition of unity $\{\rho_U\}$ subordinate to an open covering of B by trivializing open sets for the holomorphic structure of $\mathbb{P}(E)$. Let s be a projectively holomorphic section defined over an open set V. We have $F_\nabla(s) = \bar{\nabla} \circ \nabla(s)$, where $\bar{\nabla}$ denotes the covariant derivative on E-valued 1-forms determined by ∇, i.e.

$$\bar{\nabla}(\eta \otimes s) = d\eta \otimes s - \eta \wedge \nabla s$$

for any 1-form η. Since s is projectively holomorphic, we have $\nabla_U^{0,1}(s) = \bar{\partial}(\log(f_U)) \otimes s$. Now we can compute the curvature applied to the projectively holomorphic section s via

$$\begin{aligned} F_\nabla^{0,2}(s) &= \big(\bar{\nabla} \circ \nabla(s)\big)^{0,2} = \big(\bar{\nabla} \circ \nabla^{0,1}(s)\big)^{0,2} \\ &= \left(\bar{\nabla}\left(\sum_U \rho_U \, \bar{\partial} \log(f_U) \otimes s\right)\right)^{0,2} \\ &= \sum_U \big(\bar{\partial}\rho_U \wedge \bar{\partial} \log(f_U)\big) \otimes s \\ &= \alpha \otimes s \ , \end{aligned}$$

where in the third line, the second term in the definition of $\bar{\nabla}$ vanishes since we obtain the square of a one-form. We have thus $F_\nabla^{0,2} = \alpha \otimes Id_E$ with $\alpha = \sum_U \bar{\partial}\rho_U \wedge \bar{\partial} \log(f_U)$ a $(0,2)$-form as required. ∎

Lemma 2.17. *Let E be a projectively holomorphic vector bundle. Suppose that* $\det(E)$ *is holomorphic. Then there exists a projective connection ∇ on E which induces a connection on* $\det(E)$ *compatible with its holomorphic structure.*

Proof. We proceed as in the proof of Lemma 2.16, obtaining a projectively holomorphic frame $\{s_i\}_i$. Since $\det(E)$ is holomorphic, there exists a non-vanishing $f \in C^\infty_{\mathbb{C}}(U)$ such that $f s_1 \wedge \cdots \wedge s_k$ is a local holomorphic section of $\det(E)$. Then consider $\tilde{s}_i = \sqrt[n]{f} s_i$ (after possibly restricting U to a smaller set). This is a projectively holomorphic frame which induces a holomorphic frame in the determinant bundle. Let ∇_U denote the flat connection on E_U for which the frame $\{\tilde{s}_i\}_i$ is parallel. Fix a partition of unity $\{\rho_U\}$ subordinate to an open covering of X by open sets as above. Then the connection $\nabla = \sum_U \rho_U \nabla_U$ is projective and induces on $\det(E)$ a connection compatible with its holomorphic structure. ∎

The following lemma is not necessary to prove Theorem 2.13, but it is the converse of Lemma 2.16 and we include the statement here for completeness.

Lemma 2.18. *Let E be a smooth complex vector bundle over a complex manifold B. If E admits a projective connection, then it is projectively holomorphic.*

Remark 2.19. Recall that the total space of a complex vector bundle $\pi : E \longrightarrow X$ always admits an almost complex structure, but it is not canonical. One would like to use the fact $TE \cong V \oplus \pi^*(TB)$, where $V = \ker(D\pi) \subset TE$, and take the direct sum of the almost complex structures on the summands. However, there is a choice allowed for the isomorphism between TE and $V \oplus \pi^*(TB)$, upon which the almost complex structure on E does depend. Specifying a connection ∇ on E determines a horizontal *subbundle* H on TE and this determines the isomorphism.

Proof. We use a projective connection on E to obtain an almost complex structure on the total space of E, as mentioned in the above remark (and hence on $E^\times$, the total space of E without the zero section). This almost complex structure will pass to the quotient $\mathbb{P}(E)$, and we will show directly that it is integrable.

Step 1: construction of the almost complex structure on E. The vertical bundle $V = \ker(D\pi)$ is canonically isomorphic to $\pi^*(E)$. Namely, for $e \in E$, an element in V_e is represented by a path $\gamma(t) = e + te'$, which can be identified with $(e, e') \in \pi^*(E)$. In this way, V is naturally a complex bundle, and its complex structure restricts to the (integrable, standard) almost complex structure on the fibers of E.

Let ∇ be a projective connection on E. Recall that the horizontal subbundle $H \subset TE$ associated to ∇ is given as follows: for $e \in E$, the fiber H_e is the image of the differential $Ds\colon TB \longrightarrow TE$ of a section $s\colon B \longrightarrow E$ with $s(\pi(e)) = e$ and $(\nabla s)(\pi(e)) = 0$. On the other hand, the connection ∇ is recovered via

$$\nabla s : TB \xrightarrow{Ds} s^*(TE) \xrightarrow{p_V} s^*(V) = s^*\pi^*(E) = E,$$

where p_V denotes the projection to the vertical bundle. The map Ds can be extended i-linearly to the complexifications $TB \otimes \mathbb{C} = T^{1,0}B \oplus T^{0,1}B$ and $TE \otimes \mathbb{C}$. Set $H^{1,0} = Ds(TB^{1,0})$, and similarly for $H^{0,1}$.

A choice of ∇ on E thus defines an almost complex structure on E via the splitting

$$TE \otimes \mathbb{C} = (V \oplus H) \otimes \mathbb{C} = (V^{1,0} \oplus H^{1,0}) \oplus (V^{0,1} \oplus H^{0,1}).$$

Step 2: computing Lie brackets of complex valued vector fields on E. The Lie bracket of $(0,1)$ vector-fields X, Y in E with respect to the almost complex structure defined above will now be computed. For this purpose, we will take the vector fields X, Y to lie completely along one of the subbundles $V \otimes \mathbb{C}$, $H \otimes \mathbb{C}$ of $TE \otimes \mathbb{C}$, and do the computation of the bracket for each of the 3 possible configurations separately.

Note that for every vector field $\hat{X} \in \Gamma(TB)$ on B there exists a unique horizontal lift $X \in \Gamma(TE)$. This remains true on the respective complexifications of TB and TE. Moreover, $\Gamma(H \otimes \mathbb{C})$ is locally generated as a $C^\infty_{\mathbb{C}}(E)$-module by such lifts. This fact will be useful in what follows.

Consider first the case where $X = X_V$ and $Y = Y_V$ are both vertical, i.e. $X_V, Y_V \in \Gamma(V^{0,1})$. Then the bracket can be computed on a fiber E_x of E, where the almost complex structure is known to restrict to an integrable one. Thus $[X_V, Y_V] \in \Gamma(V^{0,1})$.

Suppose now that $X = X_H$ is horizontal and $Y = Y_V$ is vertical. We may without loss of generality assume that X_H is the horizontal lift of a vector field $\hat{X}$ in B. Then

$$D\pi\left([X_H, Y_V]\right) = [D\pi(X_H), D\pi(Y_V)] = 0,$$

so $[X_H, Y_V] \in V \otimes \mathbb{C}$. It follows that $[X_H, Y_V]$ is of type $(0,1)$ if and only if it evaluates to zero on smooth functions $E \longrightarrow \mathbb{C}$ which are holomorphic along the fibers of E. Let f be such a function. We have $X_H(Y_V(f)) = 0$, since Y_V is of type $(0,1)$. It is now enough to show a function f on a neighbourhood of $e \in E_x$ which is holomorphic along E_x can be extended so that the restriction of $X_H(f)$ to E_x is holomorphic; if that is the case, then $Y_V(X_H(f)) = 0$ and hence $[X_H, Y_V]$ is of type $(0,1)$. To show that such an extension exists, we can work on a trivialization $U \times \mathbb{C}^k$ of E. Given $f_0 : \mathbb{C}^k \longrightarrow \mathbb{C}$ a holomorphic map, consider $f : U \times \mathbb{C}^k \longrightarrow \mathbb{C}$ given by $f(x, v) = f_0(v)$. If X_H is the horizontal lift of $\hat{X} \in \Gamma(TU)$, we will have

$$X_H(x, v) = F_I(x)\frac{\partial}{\partial x_I} + F_{\bar{I}}(x)\frac{\partial}{\partial \overline{x_I}} + G_J(x)\frac{\partial}{\partial v_J} + G_{\bar{J}}(x)\frac{\partial}{\partial \overline{v_J}},$$

where the components of X_H only depend on $x \in U$. Therefore the restriction of the function $X_H(f)$ to any fiber E_x remains holomorphic.

Finally, let $X = X_H$ and $Y = Y_H$ both be horizontal. We again assume that they are horizontal lifts of $\hat{X}$ and $\hat{Y}$. The horizontal part of $[X_H, Y_H]$ is of type $(0,1)$ since $D\pi([X_H, Y_H])$ has type $(0,1)$ in B. For the vertical part we have, from the usual relation between the curvature of the Ehresmann connection and that of its corresponding covariant derivative, that

$$p_V([X_H, Y_H])(e) = F^{\nabla}(\hat{X}, \hat{Y})e,$$

where F^{∇} denotes the curvature of ∇. On the other hand, the assumption that ∇ is projectively holomorphic implies

$$p_V([X_H, Y_H])(e) = \alpha(\hat{X}, \hat{Y})e,$$

where α is as in Definition 2.15.

Denote by $\nu \in \Gamma(TE)$ the vertical vector field on E given tautologically by $e \longmapsto e$ (this is sometimes called the *Euler vector field*). Since $p_V([X_H, Y_H])(e)$ lies along the line spanned by ν, the arguments laid out so far prove that

$$[T^{0,1}E, T^{0,1}E] \subset T^{0,1}E \oplus \langle \nu \rangle.$$

Step 3: almost complex structure on $\mathbb{P}(E)$ and integrability. We now consider $E^{\times}$, i.e. the vector bundle E with its 0 section removed. The almost complex structure on E restricts to $E^{\times}$ and since it is compatible with the $\mathbb{C}^*$ action on the fibers, it passes to the quotient $\mathbb{P}(E)$.

Let $p : E^{\times} \longrightarrow \mathbb{P}(E)$ denote the quotient map. By definition, this map is compatible with the almost complex structures on both spaces. Given a point $e \in E^{\times}$ we have the exact sequence of vector spaces

$$0 \longrightarrow \ker(D_e p) = \langle \nu_e \rangle \longrightarrow T_e E \longrightarrow T_{p(e)}\mathbb{P}(E) \longrightarrow 0.$$

Let now $\tilde{X}$ and $\tilde{Y}$ be $(0,1)$ vector fields on $\mathbb{P}(E)$. If X and Y are their respective lifts to $E^{\times}$, we have

$$Dp([X,Y]) = Dp(\alpha(\hat{X},\hat{Y})\nu) = 0,$$

and therefore $[\tilde{X},\tilde{Y}] = 0$. ∎

2.2.4 The conclusions about the lifting problem

We now use the tools developed above to prove the advertised solution to the lifting problem for holomorphic projective bundles.

Proof of Theorem 2.13. If E has a holomorphic structure, then it is projectively holomorphic and its determinant is holomorphic. Conversely, suppose that E is projectively holomorphic and that its determinant bundle is holomorphic. Let ∇ be the connection obtained in Lemma 2.17. Recall that the curvature of the connection induced on the determinant is given by the trace.

Let k be the rank of E. Since $F_{\nabla}^{0,2} = \alpha^{0,2} \otimes Id_E$ and $\mathrm{tr}(F) = F_{\det(E)}$, we have

$$\alpha^{0,2} = \frac{1}{k}\mathrm{tr}(F)^{0,2} = \frac{1}{k}F_{\det(E)}^{0,2} = 0 \ ,$$

where the last equality comes from the holomorphicity of $\det(E)$. Thus ∇ induces a holomorphic structure on E, since the $(0,2)$ part of its curvature vanishes. ∎

Theorem 2.1 is just an elaboration on Theorem 2.13.

Proof of Theorem 2.1. If a given holomorphic projective bundle $X \longrightarrow B$ lifts to a holomorphic vector bundle $\mathcal{E} \longrightarrow B$, then the smooth vector bundle underlying $\mathcal{E}$ provides a lift of X as a smooth projective bundle to a vector bundle whose determinant has a holomorphic structure given by $\mathcal{E}$.

Conversely, suppose that X admits a smooth lift E for which $\det(E)$ admits a holomorphic structure. By Lemma 2.17 there is a projective connection on E which induces a connection on $\det(E)$ compatible with its holomorphic structure. By the proof of Theorem 2.13 this connection induces a holomorphic structure $\mathcal{E}$ on E, which, by construction, has the property that $\mathbb{P}(\mathcal{E})$ is isomorphic to X as a holomorphic projective bundle. ∎

Proof of Corollary 2.7. If $X \longrightarrow B$ is a holomorphic $\mathbb{CP}^1$-bundle whose total space is spin, then Lemma 2.9 shows that X lifts smoothly to a rank 2 vector bundle E. Moreover, by the proof of that lemma, the first Chern class of E is an integral lift of $w_2(B)$. By twisting with a line bundle any such lift can be realised as $c_1(E)$. Thus we may assume $c_1(E) = -c_1(B)$. In this case the determinant bundle of E has a holomorphic structure given by the canonical line bundle of B. The conclusion now follows from Theorem 2.1. ∎

Proof of Proposition 2.11. There exists a smooth complex vector bundle E to which X lifts, i.e. E is projectively holomorphic. By Theorem 2.13, E admits a holomorphic structure if and only if $\det(E)$ does. If α is the curvature form of a connection ∇ on

$\det(E)$, then ∇ induces a holomorphic structure if and only if $\alpha^{0,2} = 0$. We can change the connection and curvature forms by twisting E by a complex line bundle L and a connection on it[1]. This twisting replaces $\det(E)$ by $\det(E) \otimes L^k$, and $\alpha^{0,2}$ by $\alpha^{0,2} + k\beta^{0,2}$, where β is the curvature of L. This can be made to vanish, equivalently the determinant of the twisted vector bundle can be given a holomorphic structure, if and only if $\frac{1}{k}\alpha^{0,2}$ is the image of some $L \in H^2(B;\mathbb{Z})$ under the map $H^2(B;\mathbb{Z}) \longrightarrow H^{0,2}(B)$ in the exact cohomology sequence of the exponential sequence. ∎

[1] This includes the possibility of twisting by a connection on the trivial line bundle.

Chapter 3

String Manifolds

In this and the subsequent chapter, we shall give two applications of Theorem 2.1, specifically of Corollary 2.7. Here we will focus on string Kähler threefolds.

Definition 3.1. A smooth manifold M is called *string* if it satisfies $w_2(M) = 0$ and $p_1(M) = 0$.

Some authors use the potentially stronger condition $\frac{1}{2}p_1(M) = 0$, but we will not do this here. The condition $p_1(M) = 0$ arose first in string theory, as the condition for having a spin structure on the loop space of the manifold. It has since been studied extensively in the context of elliptic cohomology, see for example [71]. We apply the methods and results of [69], together with some additional arguments, to analyze Kähler structures on threefolds which are not just spin, but string.

After some preliminaries and examples of string manifolds, we will give a proof of the following characterization theorem for string Kähler threefolds with positive holomorphic Euler characteristic.

Theorem 3.2. *Let X be a string Kähler threefold. If $\chi(\mathcal{O}_X) > 0$, then $b_2(X) \geq 2$. If furthermore $b_2(X) > 2$, then X is the projectivization of a holomorphic rank 2 vector bundle over a projective surface S with $\chi(\mathcal{O}_S) > 0$.*

Conversely, every projective surface S with $\chi(\mathcal{O}_S) > 0$ arises in this way from a string threefold.

The proof appears in Section 3.3. As a corollary, we obtain:

Corollary 3.3. *If X is a string Kähler threefold with $\chi(\mathcal{O}_X) > 0$, then X is projective, with negative Kodaira dimension.*

3.1 Basics on string manifolds

To begin with, we note that orientable manifolds of dimension strictly smaller than 4 are spin and therefore string. Moreover, it is obvious that Cartesian products of string manifolds are string.

In dimension 4 a closed oriented manifold is string if and only if it is spin with vanishing signature. The standard arguments showing that every finitely presentable group is the fundamental group of a smooth closed oriented four-manifold can be carried out in this category, showing that the string condition does not impose restrictions on the fundamental group.

The following lemma follows from the additivity of w_2 and of p_1 in connected sums.

Lemma 3.4. *If a manifold M of dimension > 4 decomposes as a smooth connected sum $M = M_1 \# M_2$, then M is string if and only if both M_i are string.*

While a connected sum of string manifolds is string in any dimension, the converse does not hold in dimension 4, as shown by the connected sum of two copies of the $K3$ surface with opposite orientations.

We will use the lemma in order to rule out splittings of string manifolds M with $M_2 = \mathbb{CP}^3$, using the fact that although $\mathbb{CP}^3$ is spin, it is not string.

Recall that due to Remark 2.3 every oriented smooth S^2-bundle can be thought of as the unit sphere bundle of a rank 3 oriented vector bundle. Continuing the discussion in Lemma 2.9, we have the following.

Lemma 3.5. *Let $X \longrightarrow B$ be the unit sphere bundle of the oriented rank 3 vector bundle $V \longrightarrow B$. Then X is string if and only if $w_2(V) = w_2(B)$ and $p_1(V) = -p_1(B)$.*

Proof. Note that

$$TX \oplus \underline{\mathbb{R}} = \pi^*(V) \oplus \pi^*(TB) \ ,$$

so $w_2(TX) = \pi^*(w_2(V) + w_2(TB))$ and $p_1(X) = \pi^*(p_1(V) + p_1(TB))$. Therefore the claim follows from the injectivity of π^* in cohomology. ∎

3.2 Kähler examples

As observed in the previous section, every complex curve is string, and a compact complex surface is string if and only if it is spin and its signature vanishes, equivalently if the canonical class is even and $c_1^2 = 2c_2$.

Example 3.6 (String surfaces of general type)**.** There exist simply connected string surfaces of general type. For example, Moishezon and Teicher [58] constructed simply connected surfaces of general type of both positive and zero signature. It was observed in [48] that these surfaces are spin. Therefore, the ones of zero signature are string. □

Example 3.7 (Even-dimensional examples of general type)**.** Taking products of the surfaces from the previous example, we see that in every even complex dimension there are projective string manifolds of general type with no odd-degree cohomology. ∎

This is in sharp contrast with what happens for threefolds, see Corollary 3.3. We now check some other potential sources of examples, showing that it is hard to find string projective manifolds from these constructions.

Example 3.8 (String hypersurfaces)**.** A hypersurface X_d of degree d in $\mathbb{CP}^{n+1}$ has $p_1(X_d) = 0$ if and only if $d^2 = n + 2$. Therefore, for every $d \geq 2$ there is one example. By adjunction such a hypersurface has $c_1(X_d) = (n+2-d)x = d(d-1)x$, where x is the positive generator of $H^2(\mathbb{CP}^{n+1}; \mathbb{Z})$. Since $c_1(X_d)$ is even, X_d is automatically string. ∎

These examples are all Fano, and for $d = 2, 3, 4, \ldots$ they are quadrics in $\mathbb{CP}^3$, cubics in $\mathbb{CP}^8$, quartics in $\mathbb{CP}^{15}$,... In particular there are no such threefolds.

Example 3.9 (String complete intersections)**.** A smooth complete intersection $X \subset \mathbb{CP}^{n+r}$ of codimension r and multidegree $(d_1, \ldots, d_r)$ has $p_1(X) = 0$ if and only if

$$\sum_{i=1}^{r} d_i^2 = n + 1 + r \ .$$

Again there are no solutions for $n = 3$, so we do not find threefold examples this way. By adjunction such a complete intersection has $c_1(X) = (n+1+r-\sum_i d_i)x = \sum_i d_i(d_i-1)x$, where x is the positive generator of $H^2(\mathbb{CP}^{n+1}; \mathbb{Z})$. Since $c_1(X)$ is even, X is automatically string, and since $c_1(X)$ is positive, X is Fano. ∎

Example 3.10 (Projectivised vector bundles)**.** Let S be a compact complex manifold and $\mathcal{E} \longrightarrow S$ a holomorphic rank 2 vector bundle. By Lemma 3.5 the projectivization $X = \mathbb{P}(\mathcal{E})$ is string if $c_1(\mathcal{E}) = c_1(S) \pmod 2$ and $c_1^2(\mathcal{E}) - 4c_2(\mathcal{E}) = 2c_2(S) - c_1^2(S)$. In the special case $c_1(\mathcal{E}) = c_1(S)$ the first condition is automatic, and the second one reduces to $c_2(\mathcal{E}) = \frac{1}{2}(c_1^2(S) - c_2(S))$. ∎

Example 3.11 (Projectivised tangent bundles of surfaces)**.** Let S be a compact complex surface satisfying $c_1^2(S) = 3c_2(S)$. Then either the two sides of the equation vanish, or they are positive and S is $\mathbb{CP}^2$ or a ball quotient. Let $X = \mathbb{P}(TS)$ be the projectivization of the holomorphic tangent bundle. This is a complex threefold, which is Kähler if and only if S is. Moreover, X is string by the previous example. For $S = \mathbb{CP}^2$ we find $X = \mathbb{P}(T\mathbb{CP}^2) = F(1, 2)$, the (complete) flag manifold of $\mathbb{C}^3$. ∎

Proposition 3.12. *Let S be any projective surface. Then there is a holomorphic rank* 2 *bundle $\mathcal{E} \longrightarrow S$ such that $X = \mathbb{P}(\mathcal{E})$ is a string projective threefold.*

Proof. By a result of Wu [76, p.68], the total Chern class defines a bijective correspondence between the isomorphism classes of complex vector bundle of rank 2 over a smooth 4-manifold M and the set $H^2(M, \mathbb{Z}) \times H^4(M, \mathbb{Z})$. Then let $E \longrightarrow S$ be the smooth complex vector bundle of rank 2 with[1] $\det(E) = \det(TS)$ and $c_2(E) = \frac{1}{2}(c_1^2(S) - c_2(S))$. The determinant bundle $\det(E)$ carries the holomorphic structure given by the dual of the canonical bundle of S. Therefore, by the result of Schwarzenberger [70, Theorem 9], E has a holomorphic and hence algebraic structure $\mathcal{E}$. Its projectivization $\mathbb{P}(\mathcal{E})$ is a projective threefold, which is string by Lemma 3.5. ∎

The results of [22, 54, 7] show that the proposition becomes false if one replaces the projective surface S by an arbitrary Kähler surface. The reason is that Schwarzenberger's theorem fails for non-projective surfaces.

Note that S and X have the same fundamental group. Therefore, we have the following conclusion for the possible fundamental groups of Kähler string threefolds, showing that certain aspects of the topology of threefolds are not constrained by the string condition.

Corollary 3.13. *If Γ is the fundamental group of any smooth Kähler threefold, then it is also the fundamental group of a string Kähler, and, in fact, projective, threefold.*

Proof. A recent theorem of Claudon, Höring and Lin [16, Theorem 1.2], shows that fundamental groups of Kähler threefolds are projective. So Γ is a projective fundamental group, and, by the Lefschetz hyperplane theorem, the fundamental group of a smooth projective surface S, to which the proposition applies. ∎

3.3 The main theorem on string manifolds

We now show that many string threefolds must be ruled, and so in particular are not of general type.

Proof of Theorem 3.2. Since $\mathbb{CP}^3$ is not string, Lemma 3.4 tells us that X does not split off a copy of $\mathbb{CP}^3$ in a connected sum. In particular, X does not allow blowdowns to smooth points. Therefore, by [69, Theorem 10] we know that X has K_X nef, or is Fano, or is a smooth $\mathbb{CP}^1$-bundle or a quadric bundle over a curve. In the case of quadric bundles one also knows that the relative Picard number must be one, which implies $b_2(X) = 2$. So there is nothing to prove in that case.

The string assumption implies that $c_1^2(X) = 2c_2(X)$, and so by the Riemann–Roch theorem

$$\chi(\mathcal{O}_X) = \frac{1}{24} c_1(X) c_2(X) = \frac{1}{48} c_1^3(X) \ .$$

Since this is assumed to be positive, we conclude $c_1^3(X) \geq 48$, equivalently $K_X^3 \leq -48$. This means in particular that K_X cannot be nef.

If X is Fano, we write $c_1(X) = d \cdot c_1(L)$, with L ample and $c_1(L)$ indivisible in integral cohomology. Since X is string and therefore spin, the divisibility d must be even. By the work of Kobayashi–Ochiai [47] we know that $d \leq 4$, with equality only if X is $\mathbb{CP}^3$.

[1] Such a vector bundle can also be constructed without appealing to Wu's result by taking the fiber connected sum of TS with sufficiently many copies of the complex rank 2 vector bundles $F_{\pm}$ over S^4 with $c_2(F_{\pm}) = \pm 1$.

This is impossible, since $\mathbb{CP}^3$ is not string. Thus we conclude that $d = 2$. In this case the inequality $c_1^3(X) \geq 48$ becomes $c_1^3(L) \geq 6$. The classification of del Pezzo threefolds, see for example [30, Theorem 8.11], together with the fact that $\mathbb{CP}^3$ and its blowups are not string, tell us that X must be $\mathbb{P}(T\mathbb{CP}^2) = F(1,2)$ from Example 3.11 or $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, which is the projectivization of the trivial rank 2 holomorphic vector bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Finally, if X is a smooth $\mathbb{CP}^1$-bundle, then because the total space is spin, it is in fact the projectivization of a holomorphic rank 2 vector bundle $\mathcal{E} \longrightarrow S$ by Corollary 2.7. It remains to show that S is not just Kähler, but must in fact be projective.

Note first that we assumed that $X = \mathbb{P}(\mathcal{E})$ has $\chi(\mathcal{O}_X) > 0$. The multiplicativity of χ then implies that $\chi(\mathcal{O}_S) > 0$. Since χ is a birational invariant, it is also positive on the minimal model of S. Thus S is a Kähler surface with positive Todd genus, and so by the Enriques–Kodaira classification is a blowup of one of the following: a rational surface, an Enriques surface, a $K3$ surface, a minimal elliptic surface of Kodaira dimension one, or a surface if general type. These surfaces are all projective, except in the cases of $K3$ surfaces and of elliptic surfaces, so we assume from now on that S is a blowup of a non-projective elliptic or $K3$ surface.

For elliptic surfaces we have $c_1^2 \leq 0$, and so by Riemann–Roch the condition $\chi(\mathcal{O}_X) > 0$ becomes $c_2 > 0$. This condition is of course also valid for $K3$ surfaces, and for all the blowups.

So suppose now that we have a holomorphic rank two bundle $\mathcal{E}$ over such a surface S, so that $X = \mathbb{P}(\mathcal{E})$ is string. By Lemma 3.5 this gives the equation

$$c_1^2(\mathcal{E}) - 4c_2(\mathcal{E}) = 2c_2(S) - c_1^2(S),$$

where the left hand side is from Lemma 2.10. By the above discussion of the numerical constraints, the right hand side is in fact positive. However, for $\mathcal{E}$ to exist as a holomorphic bundle over a non-projective S, the left hand side must be non-positive by a result of Bănică and Le Potier [7, Théorème (0.3)]. This contradiction rules out non-projective surfaces S in the conclusion of Theorem 3.2. That all projective surfaces with $\chi(\mathcal{O}_S) > 0$ do arise follows from Proposition 3.12. This completes the proof of Theorem 3.2. ∎

Proof of Corollary 3.3. The proof of Theorem 3.2 has shown that any Kähler threefold with $\chi(\mathcal{O}_X) > 0$ is $\mathbb{P}(\mathcal{E})$ for a holomorphic bundle over a projective surface S, or else has $b_2(X) = 2$. In the first case X is projective because S is projective and $\mathcal{E}$ is algebraic by GAGA. In the second case we have $h^{0,2}(X) = 0$, again giving the conclusion that X is projective. Both the projective bundles and the additional quadric bundles appearing in the second case are uniruled and hence of negative Kodaira dimension. ∎

Chapter 4

Contact Structures on Threefolds

We present here the second application of Theorem 2.1. The main goal is to give the following classification of compact contact Kähler threefolds:

Theorem 4.1. *Let X be a compact Kähler threefold with a holomorphic contact structure. Then X is $\mathbb{CP}^3$ or the projectivization of the cotangent bundle of a Kähler surface. Conversely, both $\mathbb{CP}^3$ and the projectivizations of cotangent bundles have standard contact structures.*

This theorem was proved initially by Ye in [79] for X a projective threefold, and extended to the Kähler case by Peternell [64] and Frantzen-Peternell [28].

After an exposition of the basic theory of contact complex threefolds, we will give a proof of the theorem in section 4.3. Our proof relies on the simplification of the minimal model program for spin Kähler threefolds obtained by Schreieder-Tasin in [69], and addresses a point that is omitted in previously existing proofs, namely why a certain $\mathbb{CP}^1$–bundle over a surface can be lifted to a holomorphic vector bundle. This is done by means of Corollary 2.7 of Theorem 2.1.

In contrast, Theorem 4.1 does not remain true for complex threefolds. To get a grasp on the extent to which the theorem breaks down, we will present a number of examples in Section 4.4.

4.1 Preliminaries on holomorphic contact structures

Although the emphasis is on contact Kähler threefolds, we will start with a general overview of holomorphic contact structures on complex manifolds of arbitrary dimension.

In this section, X will be a complex manifold and $F \subset TX$ will be a holomorphic distribution of codimension one. We will also denote by L the holomorphic line bundle TX/F, and write $\alpha : TX \longrightarrow L$ for the quotient projection.

Lemma 4.2. *There is a well-defined vector bundle morphism (not necessarily of constant rank) $\phi : \Lambda^{2n}F \longrightarrow L^{\otimes n}$ given by*

$$\phi(X_1 \wedge \cdots \wedge X_{2n}) = \sum_{\sigma \in S_{2n}} (-1)^{|\sigma|} \bigotimes_{i=1}^{n} \alpha([X_{\sigma_{2i-1}}, X_{\sigma_{2i}}]),$$

where the Lie brackets are computed by taking vector field extensions along F.

Proof. Fix $p \in X$. Suppose that ϕ has been computed using extensions $\tilde{X}_i$ of $X_i \in T_pX$. Since the vector fields $\tilde{X}_i$ span F around p, it is enough to show that the right-hand side remains the same after replacing X_1 by $X_1 + fX_i$ where f is a holomorphic function with $f(p) = 0$. Then the general case will follow. But

$$[\tilde{X}_1 + f\tilde{X}_i, \tilde{X}_j] = [X_1, X_j] + f[X_i, X_j] - X_j(f)X_i,$$

so at the point p we have the equality

$$\alpha_p([X_1 + fX_i, X_j]) = \alpha_p([X_1, X_j] + f[X_i, X_j] - X_j(f)X_i) = \alpha_p([X_1, X_j]),$$

because $f(p) = 0$ and $X_i(p) \in F_p = \ker(\alpha_p)$. It is clear that a permutation σ of the X_i on the left-hand side changes the right-hand side by $(-1)^{|\sigma|}$. ∎

We can now introduce the notion of a holomorphic contact structure.

Definition 4.3. Let X be a complex manifold of dimension $2n + 1$. A distribution F of codimension 1 on X is a *holomorphic contact structure* if ϕ is an isomorphism of line bundles. A complex manifold X equipped with a holomorphic contact structure is called a (complex) *contact manifold*. The induced exact sequence of vector bundles

$$0 \longrightarrow F \longrightarrow TX \longrightarrow L := TX/F \longrightarrow 0$$

is called the *contact sequence*.

As an immediate consequence of Lemma 4.2 we have the following topological constraint on the canonical bundle of a contact manifold, originally due to Kobayashi [43].

Corollary 4.4. *Let X be a complex contact manifold. Then the canonical bundle K_X is divisible by $n + 1$. In particular, a complex contact manifold of dimension 3 is spin.*

Proof. From the contact sequence it follows immediately that $K_X^{-1} = \det(TX) = \det(F) \otimes L$. On the other hand, $\det(F) \simeq L^{\otimes n}$ since F is a contact structure. It follows that $K_X^{-1} \simeq L^{\otimes n+1}$. ∎

To clarify the semantics, the following definition draws the parallel between the notions of real and complex contact manifolds, while establishing also the relation between Definition 4.3 and Kobayashi's original definition in [43].

Proposition 4.5. *Let X be a $(2n+1)$–dimensional complex manifold and let F be a holorphic codimension 1 distribution on X. The distribution F is a contact structure if and only if for every point $p \in X$ there exists an open neighbourhood U of p and a holomorphic 1-form θ on U with $\theta \wedge (d\theta)^n \neq 0$ such that $F|_U = \ker\theta$.*

Proof. Suppose F is a contact structure and let $\alpha : TX \longrightarrow L := TX/F$ be the quotient map. Let $\{U_i\}_i$ be a trivializing open covering for L. Using the local trivializations of L we obtain $\alpha_i : TU_i \longrightarrow \mathbb{C}$, that is, for each i a holomorphic 1-form on U_i. Let $X_0, \dots, X_{2n}$ be pointwise linearly independent local holomorphic vector fields around $p \in U_i$ such that $X_j \in \Gamma(F|_{U_i})$ for $j > 0$. Then

$$\begin{aligned}
\alpha_i \wedge (d\alpha_i)^n(X_0, \dots, X_{2n}) =& \alpha_i(X_0)(d\alpha_i)^n(X_1, \dots, X_{2n}) \\
=& \alpha_i(X_0)\left(\sum_{\sigma \in S_{2n}} (-1)^{|\sigma|} \prod_{j=1}^{n} d\alpha_i(X_{\sigma_{2j-1}}, X_{\sigma_{2j}})\right) \\
=& \alpha_i(X_0)\left(\sum_{\sigma \in S_{2n}} (-1)^{|\sigma|} \prod_{j=1}^{n} \alpha_i([X_{\sigma_{2j-1}}, X_{\sigma_{2j}}])\right) \\
\neq& 0,
\end{aligned}$$

where the first and third inequalities are due to $X_j \in \ker(\alpha_j)$ for $j > 0$, the second is the Cartan formula, and the inequality is the contact condition written on U_i.

Conversely, let $\{U_i\}_i$ be an open covering of X and α_i holomorphic 1-forms on U_i such that $F|_{U_i} = \ker(\alpha_i)$. Then on $U_i \cap U_j$ we also have $\ker(\alpha_i) = \ker(\alpha_j)$, whence it follows that $\alpha_i = f_{ij}\alpha_j$ for some $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. The f_{ij} define a 1-cocycle with values on $\mathcal{O}^*$, hence a holomorphic line bundle L on F. Then the α_i glue to a vector bundle morphism $\alpha : TX \longrightarrow L$ with $\ker(\alpha) = F$. The same above computation shows that the vector bundle morphism ϕ defined in Lemma 4.2 is therefore an isomorphism. ∎

Recall that in the real case, one defines a contact structure ξ on a smooth $(2n+1)$–manifold as a codimension 1 distribution which is locally given as the kernel of a local 1-form with $\theta \wedge (d\theta)^n \neq 0$. Proposition 4.5 remains valid, and when ξ is co-orientable (i.e. TM/ξ is orientable, hence trivial), the TM/ξ-valued 1-form α becomes simply a 1-form on M. In general, the line bundle TX/ξ is defined by a homomorphism $\pi_1(M) \longrightarrow \mathbb{Z}_2$, so one may always pass to a 2-covering of M where the pulled-back contact structure is defined by a global contact form.

In summary, while in the real case no generality is lost by assuming that the contact structure is the kernel of a global 1-form, in the complex setting the situation is starkly different. For instance, the kernel of a global holomorphic 1-form on a compact Kähler manifold defines a foliation outside of the zero set.

Nevertheless, locally one has the same Darboux theorem as in the real case. This has been known since the beginnings of the holomorphic theory; a proof can be found in [2, Appendix A].

Theorem 4.6 (Darboux theorem for holomorphic contact structures)**.** *Let X be a complex contact manifold of dimension $2n+1$. Then around every $p \in X$ there exist coordinates $z_0, \dots, z_{2n}$ such that the contact structure F is given locally as the kernel of the 1-form*

$$dz_0 + \sum_{i=1}^{n} z_{2i-1} dz_{2i}.$$

4.2 Complex contact threefolds

In this section we make some simple remarks particular to the three-dimensional case, with which we will be concerned for the remainder of this chapter. We start with the following elementary lemma relating the characteristic classes of the bundles TX, F and L.

Lemma 4.7. *Let X be a contact threefold with contact distribution F given as the kernel of a homomorphism $TX \longrightarrow L$. Then the following relations between the Chern classes of TX, F and L hold:*

$$\begin{aligned} c_1(X) &= c_1(F) + c_1(L) = 2c_1(L) \ , \\ c_2(X) &= c_2(F) + c_1(L)^2 \ , \\ c_3(X) &= c_2(F) c_1(L) \ . \end{aligned}$$

Proof. This is a direct consequence of the isomorphism as complex bundles $TX \simeq F \oplus L$ implied by the contact sequence, together with the fact that $\det(F)$ is isomorphic to L, which holds by definition for a contact structure. ∎

We will use the following remark:

Lemma 4.8. *Let X be a complex manifold with a contact structure F given as the kernel of a homomorphism $\phi\colon TX \longrightarrow L$. Let $C \subset X$ be a smooth rational curve with $C \cdot c_1(L) \leq 1$. Then C is tangent to the contact distribution.*

Proof. The restriction of ϕ to $TC \subset TX|_C$ gives a morphism $TC \longrightarrow L|_C$. Since C is a rational curve, we have $TC = \mathcal{O}_C(2)$. But $\mathcal{O}_C(2) \longrightarrow L|_C \cong \mathcal{O}_C(k)$ is trivial for $k < 2$. ∎

The two standard examples of contact threefolds are the following:

Example 4.9. Consider the complex projective space $\mathbb{CP}^3$ with homogeneous coordinates $[z_0 \colon \cdots : z_3]$. Then the kernel of the 1-form

$$z_0 dz_1 - z_1 dz_0 + z_2 dz_3 - z_3 dz_2$$

defines a contact structure. In this case the contact line bundle L is $\mathcal{O}_{\mathbb{CP}^3}(2)$. □

Example 4.10. Let S be any complex surface. Then $X = \mathbb{P}(T^*S)$ has a tautological contact structure F. At a point $(x, [\alpha]) \in \mathbb{P}(T^*S)$ it is defined as the kernel of $\pi^*(\alpha)$. Note that this means that the fibers of the projection $\pi \colon X \longrightarrow S$ are tangent to F. In this case the contact line bundle L is $\mathcal{O}_X(1)$. □

4.3 Proof of Theorem 4.1

We are now ready to prove the classification of contact Kähler threefolds. Example 4.9 shows that $\mathbb{CP}^3$ has a contact structure. Example 4.10 shows that the projectivized cotangent bundle X of a surface S has a contact structure. Clearly X is Kähler if and only if S is.

For the proof of Theorem 4.1 we need to show that no other Kähler threefold can have a contact structure. So let X be a compact Kähler threefold with a holomorphic contact structure. Then X is spin by Lemma 4.7, and so we can apply the classification result of Schreieder and Tasin [69, Theorem 10]. The conclusion is that X admits a finite sequence of blowdowns to smooth points $f \colon X \longrightarrow Y$ onto a manifold Y which is either a Mori fiber space or has K_Y nef.

Let E be an exceptional divisor contracted by f. Since $K_X = K_Y + 2E$, any contracted rational curve $C \subset E$ must have $K_X \cdot C = -2$ (and hence $L \cdot C = 1$ by Lemma 4.7). By Lemma 4.8 it follows that C is contact. Since E is swept out by such rational curves C, it is tangent to the contact distribution, which is a contradiction to the non-integrability. We conclude that X itself is a Mori fiber-space or has nef canonical bundle. However, the latter cannot occur, by a result of Demailly [20]. It then follows from [69, Theorem 10] that X is a quadric bundle over a smooth curve, a Fano threefold, or a $\mathbb{CP}^1$-bundle over a Kähler surface S.

If X is a quadric bundle $q \colon X \longrightarrow C$ over a curve C, then the generic fiber is a smooth quadric in $\mathbb{CP}^3$, which is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. If $Q = q^{-1}(x)$ is any such fiber, then $c_1(X|_Q) = c_1(Q)$. Then for any leaf C of the foliations corresponding to the factors of Q, we have $K_X \cdot C = -2$. As above, this implies that Q is tangent to the contact distribution, which is again a contradiction.

If X is Fano, then $\chi(\mathcal{O}_X) = 1$. By Hirzebruch-Riemann-Roch, it follows that $24 = c_1(X)c_2(X)$, while by Lemma 4.7 we have

$$c_1(X)c_2(X) = 2c_1(L)\left(c_2(F) + c_1^2(L)\right) = 2c_1^3(L) + 2c_3(X).$$

Since in this case $b_1(X) = 0$, we have $c_3(X) = \chi_{top}(X) = 2 + 2b_2(X) - b_3(X)$, and it follows that

$$b_3(X) = 2b_2(X) + c_1^3(L) - 10\ ,$$

in particular $c_1^3(L) = d$ is even. An argument due to Fujita (see [29]) shows that $d \leq 8$, with equality if and only if $(X, L) = (\mathbb{CP}^3, \mathcal{O}(2))$. Therefore, if $b_2 = 1$, it follows that $d = 8$ and hence $X = \mathbb{CP}^3$. Moreover, if $b_2 \geq 2$, then

$$c_1^3(X) = b^3(X) - 2b_2(X) + 10 \leq b^3(X) + 6.$$

It follows that X is isomorphic to either $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ or to the projectivization of the cotangent bundle of the complex projective plane, see the table in [29, p. 710]. However, $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ is not contact, by the argument for quadric bundles given above.

We may then assume that X is a holomorphic $\mathbb{CP}^1$-bundle $\pi\colon X \longrightarrow S$ over a Kähler surface. It remains to show that X is isomorphic to the projectivization of the cotangent bundle of S. By Corollary 2.7 there is a holomorphic rank 2 bundle E such that $X = \mathbb{P}(E)$. Moreover, we may assume that $\det(E)$ is the canonical bundle K_S of S. That $E \cong T^*S$ follows by the same argument as in [79], which we now summarise.

We have $K_X = \pi^*(K_S) \otimes T\pi^*$, where $T\pi$ is the tangent bundle along the fibers of π. The Euler sequence of the projectivization

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \pi^*E \longrightarrow T\pi \otimes \mathcal{O}_X(-1) \longrightarrow 0 \tag{4.1}$$

implies that $\mathcal{O}_X(-2) \otimes T\pi \cong \pi^*(\det(E)) \cong \pi^*(K_S)$. Then $K_X = \mathcal{O}_X(-2)$, so $L \cong \mathcal{O}_X(1)$. Lemma 4.8 implies that the fibers of π are tangent to the contact distribution. This implies that the contact sequence descends to an exact sequence of vector bundles

$$0 \longrightarrow F/T\pi \longrightarrow TX/T\pi \longrightarrow L \longrightarrow 0 \ , \tag{4.2}$$

and using that $TX/T\pi \cong \pi^*(TS)$ and that $L \cong \mathcal{O}_X(1)$ we get by taking the determinant that

$$F/T\pi \cong \pi^*(K_S^*) \otimes \mathcal{O}_X(-1) \cong T\pi^* \otimes \mathcal{O}_X(1) \ .$$

Using this and passing to the dual of (4.2), we obtain the exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \pi^*(T^*S) \longrightarrow T\pi \otimes \mathcal{O}_X(-1) \longrightarrow 0 \ . \tag{4.3}$$

Comparing (4.1) and (4.3), we see that both π^*E and $\pi^*(T^*S)$ are extensions of $T\pi\otimes\mathcal{O}_X(-1)$ by $\mathcal{O}_X(-1)$. Now we have

$$\mathrm{Ext}^1_{\mathcal{O}_X}(T\pi \otimes \mathcal{O}_X(-1), \mathcal{O}_X(-1)) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, T\pi^*) \cong H^1(X; T\pi^*)$$

and the group $H^1(X; T\pi^*)$ is isomorphic to $H^2(X; \pi^*(K_S))$, which is just a copy of $\mathbb{C}$ by the Leray spectral sequence $\mathbb{E}$ of $X \to S$ and the sheaf $\pi^*(K_S)$, as we now explain. Recall that $\mathbb{E}$ converges to $H^{p+q}(X; \pi^*(K_S))$ and has as its second page $\mathbb{E}_2$

$$\mathbb{E}_2^{p,q} = H^p(S, R^q\pi_*(\pi^*(K_S))).$$

By the projection formula, $R^q\pi_*(\pi^*(K_S)) = R^q\pi_*(\mathcal{O}_X) \otimes K_S$. Moreover, the higher direct images $R^q\pi_*(\mathcal{O}_X)$ are given by the sheafification of the presheaf $U \longmapsto H^q(\pi^{-1}(U), \mathcal{O}_X)$. Since $X \longrightarrow S$ is a $\mathbb{CP}^1$–bundle, we have $H^q(\pi^{-1}(U), \mathcal{O}_X) = 0$ for $q > 0$ and U in trivializing good cover; it follows that $R^q(\pi_*(\mathcal{O}_X)) = 0$ for $q > 0$. The Leray spectral sequence degenerates at $\mathbb{E}_2$ and we have

$$H^2(X, \pi^*(K_S)) = H^2(S, K_S) \cong \mathbb{C}.$$

Since neither sequence splits, we conclude that $\pi^*E \cong \pi^*(T^*S)$, and then $E \cong T^*S$ as well. This completes the proof of Theorem 4.1.

4.4 Non-Kähler examples

There are many constructions of compact complex threefolds with holomorphic contact structures, which are non-Kähler and also have other properties which are very different from those in Examples 4.9 and 4.10.

The existence of these other constructions is often overlooked in the algebraic geometry literature. For example, Beauville [4, pp. 60,61] explicitly states that the only known compact contact manifolds are either homogeneous (and hence projective and even Fano), or projectivised cotangent bundles. Similar statements appear implicitly in other places, e.g. [28]. On the other hand, these constructions are more familiar to differential geometers, as exemplified by the survey in Blair's book [6, Chapter 12]. They include (compact quotients of) the complex Heisenberg group and of $SL(2, \mathbb{C})$, certain holomorphic fiber bundles with fiber an elliptic curve over holomorphic symplectic manifolds (the holomorphic analogs of the so-called Boothby–Wang construction in real contact geometry), and twistor spaces of self-dual Einstein four-manifolds with non-zero scalar curvature.

In this section, several of these complex non-Kähler examples will be considered, some of which illustrate the failure of Theorem 4.1 in the complex case. We start with the simple remark that the construction from Example 4.10 already yields several non-Kähler contact manifolds.

Example 4.11 (Non-Kähler projectivizations)**.** As explained in example 4.10, we may take the projectivization of the cotangent bundle of a non-Kähler surface to obtain a complex threefold X with a contact structure. Recall that Kähler surfaces are characterized among complex surfaces, as those whose first Betti number is even (see e.g. [11] or [52]). Since X is the projectivization of a vector bundle over a surface with odd first Betti number, it follows e.g. by the Leray-Hirsch theorem that $b_1(X)$ is also odd. Therefore X is not a Kähler manifold. □

Another important example is the Iwasawa manifold, a compact quotient of $\mathbb{C}^3$, where the contact structure is induced by the standard one in $\mathbb{C}^3$.

Example 4.12 (Iwasawa manifold)**.** Let $H_{\mathbb{C}}$ denote the Lie group

$$H_{\mathbb{C}} = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{C} \right\}.$$

The form 1-form $\alpha = dy - xdz$ (which satisfies $\alpha \wedge d\alpha \neq 0$) is left invariant in $H_{\mathbb{C}}$, hence it passes to the quotient $Z = H_{\mathbb{C}}/H_{\mathbb{Z}[i]}$, where $H_{\mathbb{Z}[i]}$ denotes the lattice

$$H_{\mathbb{Z}[i]} = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{Z}[i] \right\} \subset H_{\mathbb{C}}.$$

The threefold Z thus admits a contact structure defined by the kernel α.

Notice the contact structure on Z is given as the kernel of a globally defined holomorphic 1-form, which means that Z is not Kähler, as remarked in Section 4.1. □

4.4.1 Twistor Spaces

Let (M, g) be a 4-dimensional oriented Riemannian manifold. The twistor space Z of M is, as a smooth manifold, the total space of the 2-sphere bundle $\pi : S(\bigwedge^{-}(M)) \longrightarrow M$, where $\bigwedge^{-}(M)$ denotes the bundle of anti-selfdual 2-forms on M. One may as well view Z as the fiber bundle on M whose fiber at each point $p \in M$ consists of those complex structures on T_pM which are compatible with the metric g on M and induce the opposite orientation. The Levi-Civita connection on M induces a connection on the bundle of 2-forms, which then defines a splitting of the tangent bundle of Z, $TZ = V \oplus H$, where V is the kernel of $D\pi$ and H is isomorphic to $\pi^*(TM)$. There is a canonical almost-complex structure J on TZ, which restricts to the usual complex structure on the 2-sphere along the fibers, and which acts tautologically on the horizontal distribution H.

Recall that in dimension 4 the curvature tensor splits into four pieces, viz. the selfdual and anti-selfdual Weyl tensors, the traceless Ricci tensor and the scalar curvature. These pieces are given by the Ricci decomposition into irreducible representations for the orthogonal group. The integrability of the almost complex structure J turns out to be equivalent to the vanishing of the anti-selfdual Weyl tensor, by a result of Atiyah-Hitchin-Singer (see e.g. [5] chapter 13 or the original [3]). In particular, since the Weyl tensor is a conformal invariant, a locally conformally flat manifold will have vanishing Weyl tensor. The corresponding twistor space therefore has a complex structure.

By definition of J, both V and H are complex subbundles of TZ. Since V restricts on each fiber of $Z \longrightarrow M$ to the tangent bundle, the fibers become rational curves with respect to J. While V is not a holomorphic subbundle of TZ, H turns out to be holomorphic and if (M, g) be a self-dual Einstein manifold of dimension 4 with non-zero scalar curvature, H becomes a holomorphic contact structure on Z (see e.g. [5, chapter 14] or [23]) [1]. Twistor spaces have in common with the projective bundles from Example 4.10 the feature that they are both covered by a family of rational curves. However, while in projective bundles the rational curves are tangent to the contact distribution, in twistor spaces they are transverse.

Proposition 4.13. *Let X be the twistor space of a hyperbolic manifold M. Then X admits a holomorphic contact structure, but X is not homotopy equivalent to the projectivization of the cotangent bundle of a complex surface.*

Proof. It is enough to consider the fundamental group of M. The exact sequence in homotopy induced by the fibration $X \longrightarrow M$ yields $\pi_1(X) \cong \pi_1(M)$. Such a group cannot arise as the fundamental group of a complex surface by a result of Carlson-Toledo (see [12] for the Kähler case and [13] for the general statement; see also Subsection 3.2 of Chapter 1 in [1] for a simpler argument in dimension ≤ 4, due to Kotschick). ∎

In particular, there exist infinitely many homotopy classes of complex contact threefolds which are not of the form $\mathbb{P}(T^*S)$ for a complex surface S.

[1] This is also a special case of a result of Salamon stating that any quaternionic twistor space admits a holomorphic contact structure [66], since self-dual Einstein 4-manifolds are the 4-dimensional analogues of quaternionic Kähler manifolds.

4.4.2 Holomorphic Boothby-Wang

The following construction is due to Foreman [24]. Let Y be a compact complex manifold with a holomorphic symplectic structure $\omega \in H^0(X, \Omega_X^2)$. Denote by α, resp. β, the real, resp. imaginary, part of ω. An easy computation shows that α and β are both symplectic forms on Y. Suppose additionally that α and β both determine integral classes in cohomology. Then one can construct (principal) $U(1)$–bundles $A \longrightarrow S$ and $B \longrightarrow Y$ with $c_1(A) = [\alpha]$ and $c_1(B) = [\beta]$ with connection forms η_A and η_B fulfilling $d\eta_A = \pi_A^*(\alpha)$ and $d\eta_B = \pi_B^*(\beta)$. Then η_A and η_B are (real) contact forms on A and B which yield Reeb vector fields R_A and R_B, i.e. $\eta_A(R_A) = 1$ and $R_A \in \ker(d\eta_A)$, and similarly for B. Note moreover that the connections η_A and η_B define a connection on the fiber product $X = A \times_Y B$, with corresponding vertical and horizontal bundles V and H. One can define on X an almost complex structure as follows. On the vertical bundle V of X we put $J(R_A) = -R_B$ and $J(R_B) = R_A$. On the horizontal bundle H we use the isomorphism $H \cong \pi^*(TY)$ given by the connection to pull-back the almost complex structure on TY.

This almost complex structure is in fact integrable (see [24]), so X is a complex manifold. Moreover, the projection $\pi : X \longrightarrow Y$ is a holomorphic map, and the form $\eta_A + i\eta_B$ on X is holomorphic and therefore defines a global holomorphic contact form on X.

One can use the above construction to obtain non-Kähler simply connected contact manifolds. We will need the following lemma.

Lemma 4.14. *Let $M \longrightarrow B$ be a principal $U(1)$–bundle. Then the connecting homomorphism $\delta : \pi_2(B) \longrightarrow \pi_1(S^1)$ in the long exact sequence in homotopy of the fibration $S^1 \longrightarrow M \longrightarrow B$ factors through $H_2(M, \mathbb{Z})$ as*

$$\pi_2(B) \longrightarrow H_2(B, \mathbb{Z}) \longrightarrow \pi_1(S^1) \cong \mathbb{Z}$$

the map on the left is the Hurewicz morphism and the map on the right evaluates the Euler class of $M \longrightarrow B$ in homology.

Proof. Consider S^∞ as the universal $U(1)$–bundle $S^\infty \longrightarrow \mathbb{CP}^\infty$ over the classifying space $BU(1) \simeq \mathbb{CP}^\infty$. Then M is isomorphic to $f^*(S^\infty)$ for some $f : B \longrightarrow \mathbb{CP}^\infty$. We have the natural maps

$$\begin{tikzcd}
S^1 \arrow[r] & S^\infty \arrow[r] & \mathbb{CP}^\infty \\
S^1 \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & B \arrow[u]
\end{tikzcd}$$

between the fibrations, and thus obtain corresponding maps in the long exact sequences in homotopy:

$$\begin{tikzcd}
0 = \pi_2(S^\infty) \arrow[r] & \pi_2(\mathbb{CP}^\infty) \arrow[rr] \arrow[dr, "h"'] & & \pi_1(S^1) \arrow[r] & \pi_1(S^\infty) = 0 \\
& & H_2(\mathbb{CP}^\infty, \mathbb{Z}) \arrow[ur, "\langle u,-\rangle"'] & & \\
\pi_2(M) \arrow[r] \arrow[uu] & \pi_2(B) \arrow[rr] \arrow[uu, "f_*"] \arrow[dr, "h"'] & & \pi_1(S^1) \arrow[r] \arrow[uu, "\cong"'] & \pi_1(M) \arrow[uu] \\
& & H_2(B, \mathbb{Z}) \arrow[ur] \arrow[uu, crossing over, "f_*"'] & &
\end{tikzcd}$$

Here $u \in H^2(\mathbb{CP}^\infty, \mathbb{Z})$ denotes the Euler class of the universal bundle. Note also that the isomorphism $\pi_1(S^1) \cong \mathbb{Z}$ is natural, since we have a $U(1)$-principal bundle.

We need to show that the lower triangle in the diagram commutes. This follows by naturality if the upper triangle commutes. To show that this is the case, notice that since $\mathbb{CP}^\infty$ is simply connected, the Hurewicz map is an isomorphism. On the other hand, the Euler class u of the universal bundle evaluates to 1 on the generator of $H_2(\mathbb{CP}^\infty, \mathbb{Z})$, because u generates the cohomology ring $H^*(\mathbb{CP}^\infty, \mathbb{Z})$ and $\mathbb{CP}^\infty$ is simply connected, so there is no torsion in degree 1. ∎

The lemma implies in particular that if the Euler class is spherical (i.e. it evaluates non-trivially on the image of the Hurewicz map) and is sent to the generator in $\pi_1(S^1)$, the fundamental group of the total space of the fiber bundle is isomorphic to that of the base.

Example 4.15 (Holomorphic Boothby-Wang on K3–surfaces)**.** Let S be a K3–surface. Such surfaces are by definition simply connected and have trivial canonical bundle. That they are simply connected implies by the Hurewicz theorem that every class in $H_2(S, \mathbb{Z})$ can be represented by an element in $\pi_2(S)$. On the other hand, the triviality of the canonical bundle says exactly that S admits a holomorphic symplectic form.

In order to carry out the holomorphic Boothby-Wang construction explained above, it is necessary to produce K3–surfaces which admit holomorphic symplectic forms whose real and imaginary parts determine integral cohomology classes. While one can produce concrete examples where this is the case (for example, the standard quartic $\sum_i x_i^4 = 0$ in $\mathbb{CP}^3$), one can give a more systematic account using the known theory on K3–surfaces (a detailed reference is [40]).

Recall first that $b_2(S) = 22$ and since S is simply connected $H^2(S) \cong \mathbb{Z}^{22}$. Moreover, the lattice $(H^2(S, \mathbb{Z}), Q_S)$, where $Q_S : H^2(S, \mathbb{Z}) \times H^2(S, \mathbb{Z}) \longrightarrow \mathbb{Z}$ denotes the intersection form of S, is isomorphic to $\Lambda := (\mathbb{Z}^{22}, Q)$. Here Q is the unimodular form $2E_8 \oplus 3H$, where E_8 denotes the negative of the Cartan matrix of the E_8 root system and H denotes the matrix of the standard non-trivial involution on $\mathbb{Z}^2$ fixing the diagonal. The *period domain* $\mathcal{D}$ is defined by

$$\mathcal{D} = \{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) \,|\, x^2 = 0 \text{ and } x \cdot \overline{x} > 0\}.$$

This space parametrizes Hodge structures with underlying $\mathbb{Z}$-module Λ such that for all type $(2,0)$ elements $0 \neq \alpha \in \Lambda^{2,0} \subset \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ we have $\alpha^2 = 0$, $\alpha \cdot \overline{\alpha}$ and $\Lambda^{1,1} \perp \alpha$.

On the other hand, one has the moduli space $\mathcal{M}$ of marked[2] $K3$ surfaces and the *period map* $P : \mathcal{M} \longrightarrow \mathcal{D}$ given by

$$(S, \varphi) \longmapsto [\varphi_{\mathbb{C}}(H^{2,0}(S))].$$

A far-reaching result in the theory of K3–surfaces states that the period map is surjective. In other words, given an admissible (in the above sense) Hodge structure on Λ, there exists a K3–surface realizing it; namely, if the Hodge structure is given by $x \in \Lambda_{\mathbb{C}}$, there exists an S and an isomorphism $\varphi : H^2(S, \mathbb{Z})\Lambda$ such that $\varphi_{\mathbb{C}}^{-1}(x)$ spans $H^{2,0}(X)$.

We now conclude the following lemma from the above discussion.

Lemma 4.16. *Let S be a smooth manifold diffeomorphic to a K3–surface. For every $x \in (\mathbb{Z}[i])^{22}$ satisfying the relations*

$$\overline{x}^T \cdot Q \cdot x > 0 \quad \textit{and} \quad x^T \cdot Q \cdot x = 0,$$

there exists a complex structure on S and an isometry $\varphi : H^2(S, \mathbb{Z}) \longrightarrow \mathbb{Z}^{22}$ such that $\varphi_{\mathbb{C}}^{-1}(x) \in H^{2,0}(X)$ can be represented by a non-vanishing holomorphic 2-form whose real and imaginary parts are integral in cohomology.

Let $\omega = \alpha + i\beta$ be a holomorphic symplectic form as obtained in the lemma on some K3–surface S. From inspection of the conditions in the lemma, it is clear we may take $[\alpha]$ and $[\beta]$ to be indivisible classes in integral cohomology. Let $\pi_A : M_A \longrightarrow S$ and $\pi_B : M_B \longrightarrow B$ be the S^1-bundles corresponding to α and β. We may view X as $\pi_B^*(M_A) = \pi_A^*(M_B)$. The manifold M_A is simply connected by Lemma 4.14, and applying the same lemma again shows that X is also simply-connected.

We have thus obtained a simply connected complex contact threefold X which fibers holomorphically over a K3–surface with elliptic fibers. Notice that the fibers of X are null-homologous, and therefore X cannot be Kähler.

While there seems to be a lot of freedom in picking α and β, we remark here that for indivisible α and β the resulting manifold X will have the same diffeomorphism type. This can be seen by computing the cohomology ring and Chern classes of X and verifying that the invariants are independent of α and β. By virtue of X being spin and simply connected, these invariants determine the diffeomorphism type by a theorem of Wall (see [75] or [62] for more detailed results). In fact, we have

$$X \simeq \sharp_{k-1}(S^3 \times S^3)\sharp_{k-2}(S^2 \times S^4)$$

as a smooth manifold, where $k = b_2(S) = 22$. It is evident from this description that X does not have the cohomology ring of a Kähler manifold, so it is not even homotopy equivalent to a Kähler manifold. $\square$

[2] A marking is an isomorphism $\varphi : (H^2(S, \mathbb{Z}), Q_S) \longrightarrow (\Lambda, Q)$.

Chapter 5

Holomorphic Engel Structures

The content of this chapter is joint work with Nicola Pia and was published in the *Journal of Geometric Analysis*, Volume 28(3) (2018) [18]. The contribution of both authors to this work is equal. In comparison with the published article, section 5.2 explaining the proof of a theorem of Forstnerič was added to make the exposition more self-contained.

A *holomorphic Engel structure* on a complex manifold M of complex dimension 4 is a holomorphic subbundle $\mathcal{D} \hookrightarrow TM$ of complex rank 2 which is maximally non-integrable. More precisely $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$ has constant rank 3 and satisfies $[\mathcal{E}, \mathcal{E}] = TM$ (a 3-distribution satisfying this condition is called a *holomorphic even contact structure*). Every holomorphic even contact structure $\mathcal{E}$ admits a unique holomorphic line field $\mathcal{W} \subset \mathcal{E}$ such that $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$. This line field $\mathcal{W}$ is called *characteristic foliation.* If $\mathcal{D}$ is an Engel structure and $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is its associated even contact structure then the characteristic foliation $\mathcal{W}$ satisfies $\mathcal{W} \subset \mathcal{D}$. Hence an Engel structure $\mathcal{D}$ determines a flag of distributions $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$.

Every holomorphic Engel structure $(M, \mathcal{D})$ is locally isomorphic to the complex Euclidean space $\mathbb{C}^4$ with coordinates (w, x, y, z) and the Engel structure given by

$$\mathcal{D}_{st} = \ker(dy - zdx) \cap \ker(dz - wdx).$$

The associated even contact structure is $\mathcal{E}_{st} = \ker(dy - zdx)$ and the characteristic foliation is $\mathcal{W}_{st} = \ker(dxdydz)$.

These structures are the holomorphic analogues of the usual Engel structures. Together with line fields, contact structures and even contact structures, these are the only topologically stable distributions (see [14]). The existence of an orientable Engel structure on a closed orientable (real) 4-manifold M implies that M is parallelizable. Conversely the existence of Engel structures on parallelizable 4-manifolds was established in [74]. The geometry of these structures is closely related to even contact structures, which are known to satisfy a complete h-principle (see [57]). An existence h-principle has been established for Engel structures in [15].

Holomorphic Engel structures on closed complex 4-manifolds have been studied in [65]. The only known constructions are the Cartan prolongation of a holomorphic contact structure and the Lorentz tube of a holomorphic conformal structure on a 3-manifold. These two families of structures are classified in the projective case, and the main result in [65] is a partial classification of Engel structures on closed projective manifolds. The existence of a holomorphic Engel structure on a closed complex manifold which is not a Cartan prolongation or a Lorentz tube remains an open problem.

We are interested in constructing non-standard holomorphic Engel structures on $\mathbb{C}^4$. Forstnerič constructed non-standard holomorphic contact structures on $\mathbb{C}^{2n+1}$ in [25]. There the idea is to find a Fatou-Bieberbach domain where the standard holomorphic contact structure is *hyperbolic* in a directed sense, as explained below. One of the aims of this note is to use the same method to prove the analogous statement for holomorphic Engel structures. In what follows, given a distribution $\mathcal{H} \longrightarrow T\mathbb{C}^4$, we will use the terms *$\mathcal{H}$-line* or *line tangent to $\mathcal{H}$* to designate a non-constant holomorphic map $f : \mathbb{C} \longrightarrow \mathbb{C}^4$ such that $f'(\zeta) \in \mathcal{H}_{f(\zeta)}$ for all $\zeta \in \mathbb{C}$. If no ambiguity concerning the distribution may arise, we also use *horizontal line* as a synonym for $\mathcal{H}$-line.

Theorem 5.1. *On $\mathbb{C}^4$ there are Engel structures $\mathcal{D}_\mathcal{E}$, $\mathcal{D}_\mathcal{D}$ and $\mathcal{D}_\mathcal{W}$ with the following properties*

1. *$\mathcal{D}_\mathcal{E}$ admits no lines tangent to its associated even contact structure;*
2. *$\mathcal{D}_\mathcal{D}$ admits no $\mathcal{D}_\mathcal{D}$-lines but does admit lines tangent to its associated even contact structure;*
3. *$\mathcal{D}_\mathcal{W}$ admits no lines tangent to its characteristic foliation but does admit $\mathcal{D}_\mathcal{W}$-lines.*

In particular these Engel structures are pairwise non-isomorphic and not isomorphic to the standard Engel structure $(\mathbb{C}^4, \mathcal{D}_{st})$.

As we verify below, the standard Engel structure admits many $\mathcal{D}_{st}$-lines, including many tangent to the characteristic foliation.

Controlling the geometry of the characteristic foliation, we are able to construct infinite families of non-isomorphic holomorphic Engel structures.

Theorem 5.2. *For every $n \in \mathbb{N} \cup \{\infty\}$ there exists an Engel structure $\mathcal{D}_n$ on $\mathbb{C}^4$ for which the only $\mathcal{D}_n$-lines are tangent to the characteristic foliation $\mathcal{W}_n$, and such that*

$$L_n := \{p \in \mathbb{C}^4 : \exists f : \mathbb{C} \longrightarrow \mathbb{C}^4 \ \mathcal{D}_n\text{-line with } f(0) = p\}$$

is a proper subset of $\mathbb{C}^4$ which has exactly n connected components for $n \in \mathbb{N}$, and $L_\infty = \mathbb{C}^4$.

We first construct $\mathcal{D}_\infty$ using an open set in the Cartan prolongation of a Kobayashi hyperbolic contact structure in $\mathbb{C}^3$. This will admit very few $\mathcal{D}_\infty$-lines by construction. Then we use a result, due to Buzzard and Fornæss (Theorem 5.10, for a proof see [10]) that allows one to control the set of points in $\mathbb{C}^4$ which admit such horizontal lines. A more careful analysis leads to

Theorem 5.3. *For every $R \in \mathbb{R} \setminus \{0\}$ there exists an Engel structure $\mathcal{D}_R$ for which the only $\mathcal{D}_R$-lines are tangent to the characteristic foliation $\mathcal{W}_R$, and such that the set of points which admit such $\mathcal{W}_R$-lines is exactly $\mathbb{C} \times \{0, 1, R\sqrt{-1}\} \times \mathbb{C}^2 \subset \mathbb{C}^4_{(w,x,y,z)}$. Moreover $\mathcal{D}_R$ is isomorphic to $\mathcal{D}_{R'}$ if and only if $R = R'$.*

5.1 Hyperbolicity and holomorphic Engel structures

For the proof of Theorem 5.1 we will need the notion of hyperbolicity on a complex directed manifold. Recall that the Kobayashi pseudo-distance d_M on a complex manifold M may be written in terms of the Finsler pseudo-metric

$$F(v_p) = \inf\left\{\frac{1}{|\lambda|} : \exists \text{ a holomorphic } f : D \longrightarrow M \text{ s.t. } f(0) = p,\ f'(0) = \lambda v\right\}, \tag{5.1}$$

by integration. Explicitly,

$$d(p, q) = \inf\left\{\int_0^1 F(\gamma'(t))dt : \gamma \text{ piecewise smooth, } \gamma(0) = p \text{ and } \gamma(1) = q\right\}. \tag{5.2}$$

Given a holomorphic subbundle $\mathcal{H} \subset TM$, a disc $D \longrightarrow M$ is called *horizontal* if it is tangent to $\mathcal{H}$. The Finsler pseudo-metric $F_{\mathcal{H}}$ directed by $\mathcal{H}$ is defined by requiring that the infimum in (5.1) be taken only over horizontal discs. Likewise, the Kobayashi pseudo-distance $d_{\mathcal{H}}$ on the directed manifold $(M, \mathcal{H})$ is defined by requiring that the infimum in (5.2) be taken only over paths γ that are tangent to $\mathcal{H}$. This is finite because, by Chow's theorem, these paths always exist if the distribution is bracket generating. This is the case, by definition, for an Engel structure. The directed manifold $(M, \mathcal{H})$ is said to be *Kobayashi hyperbolic* if $d_{\mathcal{H}}$ is a genuine distance. Note that if $(M, \mathcal{H})$ is Kobayashi hyperbolic, there can be no $\mathcal{H}$-line.

Remark 5.4. Notice that the standard Engel structure is not hyperbolic, since it admits many horizontal lines $f : \mathbb{C} \longrightarrow \mathbb{C}^4$. For instance, one can take the leaves of the characteristic foliation $\mathcal{W}$ of $\mathcal{D}_{st}$. In fact, given a point $p = (w_0, x_0, y_0, z_0) \in \mathbb{C}^4$ and a vector $v = (v_w, v_x, v_y, v_z) \in \mathcal{D}_p$ (hence $v_z = w_0 v_x$ and $v_w = z_0 v_x$) the map

$$f(\zeta) = \left(w_0 + v_w\zeta, x_0 + v_x\zeta, y_0 + v_y\zeta + v_x v_z\frac{\zeta^2}{2} + v_x^2 v_w\frac{\zeta^3}{6}, z_0 + v_z\zeta + v_x v_w\frac{\zeta^2}{2}\right)$$

is a horizontal line with $f(0) = p$ and $f'(0) = v$.

The idea for proving Theorem 5.1 is to construct certain (directed) hyperbolic subsets of $\mathbb{C}^4$ and look for biholomorphic copies of $\mathbb{C}^4$ inside these domains.

Definition 5.5. A Fatou-Bieberbach domain is a proper subset $\Omega \subset \mathbb{C}^n$ such that Ω is biholomorphic to $\mathbb{C}^n$.

Following [25] we let $\{c_n\}_{n\in\mathbb{N}}$, $\{d_n\}_{n\in\mathbb{N}}$ and $\{e_n\}_{n\in\mathbb{N}}$ be positive diverging monotonic sequences. Denote with D_y (resp. D_z) the unit disc in the y (resp. z) direction, with $\partial D^2_{(w,x)}$ the boundary of the unit polydisc in the (w,x)-plane in $\mathbb{C}^4$ and with $\partial D^3_{(w,x,z)}$ the boundary of the unit polydisc in the (w,x,z)-plane in $\mathbb{C}^4$. Let

$$A = \bigcup_{i=1}^{\infty} 2^{i-1}\partial D^3_{(w,x,z)} \times c_i\overline{D}_y. \tag{5.3}$$

$$B = \bigcup_{i=1}^{\infty} 2^{i-1}\partial D^2_{(w,x)} \times d_i\overline{D}_y \times e_i\overline{D}_z. \tag{5.4}$$

By a direct adaptation of lemma 2.1 in [25], we can prove the following:

Lemma 5.6. *Assume $d_n \geq 2^{5n+2}$ and $e_n \geq 2^{3n+1}$ for every $n \in \mathbb{N}$. Let $N_0 \in \mathbb{N}$ and $f : D \longrightarrow \mathbb{C}^4 \setminus B$ be a $\mathcal{D}_{st}$-horizontal embedding of a disc with $f(0) \in 2^{N_0}D^4$. Then we have the estimates*

$$|w'(0)| < 2^{N_0+1},\ |x'(0)| < 2^{N_0+1},\ |y'(0)| < 2^{3N_0+2},\ |z'(0)| < 2^{2N_0+1}.$$

Proof. We may assume without loss of generality that f is holomorphic on $\overline{D}$ (replace f by $\zeta \longmapsto f(r\zeta)$ for some $r < 1$). This gives $N \in \mathbb{N}$ such that $|x(\zeta)| < 2^N$ and $|w(\zeta)| < 2^N$ for all $\zeta \in \overline{D}$. The Cauchy integral formula for a circle centered at $\zeta = 0$ of ray $r = 1 - 2^{-N}$ gives

$$|x'(\zeta)| < 2^{2N} \quad \text{and} \quad |w(\zeta)x'(\zeta)| < 2^{3N}$$

for $|\zeta| \leq r$. Since f is horizontal, we have the conditions

$$y'(\zeta) = z(\zeta)x'(\zeta) \quad \text{and} \quad z'(\zeta) = w(\zeta)x'(\zeta) \tag{5.5}$$

which in turn give

$$|z(\zeta)| \leq |z(0)| + \left|\int_0^\zeta w dx\right| < 2^{N_0} + 2^{3N} < 2^{3N+1} \leq d_N$$
$$|y(\zeta)| \leq |y(0)| + \left|\int_0^\zeta z dx\right| < 2^{N_0} + 2^{5N+1} < 2^{5N+2} \leq c_N$$

for $|\zeta| \leq r$. From these estimates, the definition of B, and the fact that $f(D)$ does not intersect B, it follows that $(w(\zeta), x(\zeta))$ does not intersect $2^{N-1}\partial D^2$ for $|\zeta| \leq r$. Since $2^{N-1}\partial D^2$ disconnects $2^N D^2$ and $(w(0), x(0)) \in 2^{N_0}D^2 \subset 2^{N-1}D^2$, we conclude that

$$(w(\zeta), x(\zeta)) \in 2^{N-1}D^2 \quad \text{for} \quad |\zeta| \leq 1 - 2^{-N}.$$

If $N - 1 > N_0$, we can repeat the same argument to get

$$(w(\zeta), x(\zeta)) \in 2^{N-2}D^2 \quad \text{for} \quad |\zeta| \leq 1 - 2^{-N} - 2^{-(N-1)},$$

and after finitely many repetitions

$$(w(\zeta), x(\zeta)) \in 2^{N_0} D^2 \quad \text{for} \quad |\zeta| \leq 1 - 2^{-N} - \ldots - 2^{-(N_0+1)} \leq \frac{1}{2}.$$

Applying the Cauchy estimate now gives $|x'(0)| \leq 2^{N_0+1}$ and $|w'(0)| \leq 2^{N_0+1}$, while using equation (5.5) we get

$$|z'(0)| = |w(0)x'(0)| \leq 2^{2N_0+1} \quad \text{and} \quad |y'(0)| = |z(0)x'(0)| \leq 2^{3N_0+2},$$

completing the proof of the lemma. ∎

The following lemma has a completely analogous proof.

Lemma 5.7. *Assume $c_n \geq 2^{3n+1}$ for every $n \in \mathbb{N}$. Let $N_0 \in \mathbb{N}$ and $f : D \longrightarrow \mathbb{C}^4 \setminus A$ be a $\mathcal{D}_{st}$-horizontal embedding of a disc with $f(0) \in 2^{N_0} D^4$. Then we have the estimates*

$$|w'(0)| < 2^{N_0+1},\ |x'(0)| < 2^{N_0+1},\ |y'(0)| < 2^{2N_0+1},\ |z'(0)| < 2^{N_0+1}.$$

5.2 A theorem of Forstnerič

The following theorem was proved by Forstnerič in [25]. Due to its central importance in our constructions, we will now sketch its proof.

Theorem 5.8 (Forstnerič)**.** *Let $0 < a_1 < b_1 < a_2 < b_2 < \ldots$ and $c_i > 0$ be sequences of real numbers such that $\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} b_n = +\infty$. Let $n > 1$ be an integer and*

$$K = \bigcup_{i=1}^{\infty} \left(b_i \overline{D}^{n-1} \setminus a_i D^{n-1} \right) \times c_i \overline{D} \subset \mathbb{C}^n. \tag{5.6}$$

Then there exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^n \setminus K$.

The proof of the theorem relies on the *pushout method*. Loosely, one constructs a sequence of automorphisms of $\mathbb{C}^2$ which progressively push the set K further away to infinity; at the same time, one ensures that each successive automorphism is close to the identity on increasingly larger polydisks: this will imply that the composition of the infinite sequence of automorphisms converges to a bijective map defined on the whole $\mathbb{C}^2$ onto a Fatou-Bieberbach domain Ω with $K \cap \Omega = \emptyset$.

Lemma 5.9. *Let $l > 0$ and $0 < l < s_1 < t_1 < s_2 < t_2 < \ldots$ be a sequence of real numbers diverging to ∞. Moreover let $\{u_i\}_{i\in\mathbb{N}}$ be a sequence of positive reals and $\delta > 0$. Then there exists an entire holomorphic map $f : \mathbb{C} \longrightarrow \mathbb{C}$ and a sequence $0 < l + 1 < \mu_1 < \nu_1 < \mu_2 < \nu_2 < \ldots$ of real numbers diverging to ∞ such that if $s_i \leq |z| \leq t_i$ and $|w| \leq u_i$ for some $i \in \mathbb{N}$, then*

$$\mu_i < |f(z)| - u_i \leq |w + f(z)| \leq u_i + |f(z)| < \nu_i.$$

Furthermore, for all $z \in \mathbb{C}$ with $|z| \leq l$, $|z - f(z)| < \delta$.

Sketch of proof. Let $t_0 = l$ and pick for each $i \in \mathbb{N}$ a real number r_i such that $t_{i-1} < r_i < s_i$. We make the ansatz

$$f(z) = \sum_{i=1}^{\infty} \left(\frac{z}{r_i}\right)^{N_i}.$$

For sufficiently large N_i, the corresponding summand of f is arbitrarily small in the disc of radius t_{i-1} and arbitrarily large on the annulus $\{z \in \mathbb{C} \mid s_i \leq |z| \leq t_i\}$. One uses this fact to inductively pick N_i, μ_i and ν_i. We refer to [25] for details. ∎

Sketch of proof of Theorem 5.8. For conciseness, we give the argument for $n = 2$; the generalization is straighforward. For each $i \in \mathbb{N}$ pick $\epsilon_i \in (0,1)$ such that $\sum_{i=1}^{\infty} \epsilon_i < \infty$. In the course of the proof we will construct families of sequences $\{a_{i,k}\}_i$, $\{b_{i,k}\}_i$, $\{c_{i,k}\}_i$, $\{\alpha_{i,k}\}_i$, $\{\beta_{i,k}\}_i$ and $\{\gamma_{i,k}\}_i$ indexed in k, given which we construct the sets

$$K_j = \bigcup_{i=1}^{\infty} \left(b_{i,k}\overline{D} \setminus a_{i,k}D\right) \times c_{i,k}\overline{D} \quad \text{and}$$
$$L_j = \bigcup_{i=1}^{\infty} \gamma_{i,k}\overline{D} \times \left(\beta_{i,k}\overline{D} \setminus \alpha_{i,k}D\right).$$

We start with $k = 1$. Put $a_{i,1} = a_i$, $b_{i,1} = b_i$ and $c_{i,1} = c_i$. Let f_1 be the holomorphic map obtained from Lemma 5.9 with the data $l = 1$, $s_i = a_{i,1}$, $t_i = b_{i,1}$, $u_i = c_{i,1}$, $\delta = \epsilon_1$ and let $\alpha_{i,1} = \mu_i$, $\beta_{i,1} = \nu_i$ and $\gamma_{i,1} = b_{i,1}$. In the same way, let g_1 be the holomorphic map obtained from the lemma with the data $l = 1$, $s_i = \alpha_{i,1}$, $t_i = \beta_{i,1}$, $u_i = \gamma_{i,1}$, $\delta = \epsilon_1$ and let $a_{i,2} = \mu_i$, $b_{i,2} = \nu_i$ and $c_{i,2} = \beta_{i,1}$.

We define two shear-like automorphisms φ_1 and ψ_1 of $\mathbb{C}^2$ via

$$\varphi_1(z,w) = (z, w + f_1(z)) \text{ and } \psi_1(z,w) = (z + g_1(w), w).$$

Now $\varphi_1(K_1) \subset L_1$ and $L_1 \cap (\mathbb{C} \times 2\overline{D}) = \emptyset$ by the Lemma. In the same way, $\psi_1(L_1) \subset K_2$ and $K_2 \cap (2\overline{D} \times \mathbb{C}) = \emptyset$. Putting $\theta_1 = \psi_1 \circ \varphi_1$ we have

$$\theta_1(K_1) \subset K_2, \;\; K_2 \cap 2D^2 = \emptyset \text{ and } \sup_{z \in \overline{D}^2} |\theta_1(z) - z| < \epsilon_1.$$

Working inductively on k, we obtain a sequence of sets K_k and automorphisms θ_k with

$$\theta_k(K_k) \subset K_{k+1}, \;\; K_{k+1} \cap (k+1)D^2 = \emptyset \text{ and } \sup_{z \in k\overline{D}^2} |\theta_k(z) - z| < \epsilon_k.$$

By standard arguments [26, pp. 114–115], the sequence of automorphisms $\Theta_k := \theta_k \circ \cdots \circ \theta_1$ converges to an injective holomorphic map $\Theta : \mathbb{C}^2 \to \mathbb{C}^2$ which is onto a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^2$. ∎

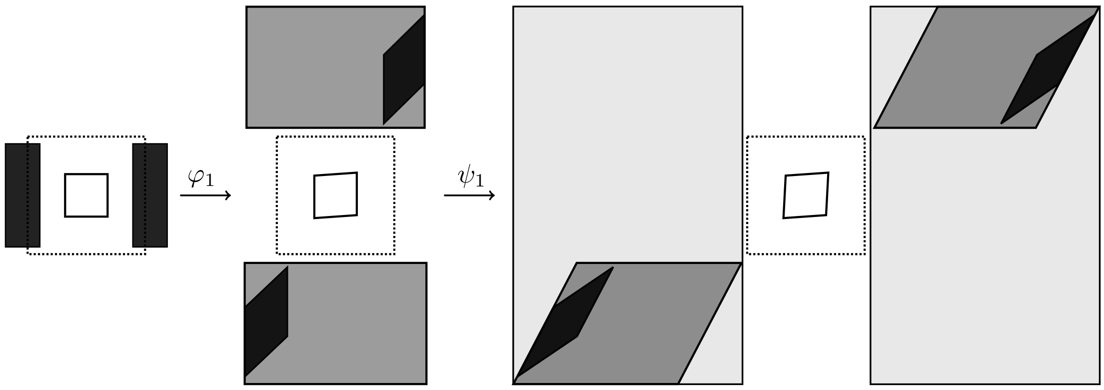

Figure 5.1: The action of the automorphism θ_1 on the component $(b_1\overline{D} \setminus a_1 D) \times c_1\overline{D}$ (in black) of the set $K_1 = K$ is depicted above. The dark, respectively light, gray area is the corresponding component of L_1, respectively K_2.

5.3 Proof of Theorem 5.1

In what follows fix $0 < \varepsilon < 1$ and consider the real sequences

$$a_i = 2^{i-1} - \varepsilon,\ b_i = 2^{i-1} + \varepsilon.$$

To construct $\mathcal{D}_{\mathcal{E}}$ we fix $c_i = 2^{3i+1}$ and let A be the set determined by c_i according to (5.3). Lemma 5.7 ensures that $(\mathbb{C}^4 \setminus A, \mathcal{E}_{st})$ is hyperbolic, moreover Theorem 5.8 gives a Fatou-Bieberbach map $\Phi : \mathbb{C}^4 \longrightarrow \Omega \subset \mathbb{C}^4 \setminus A$. We set $\mathcal{D}_{\mathcal{E}} := \Phi^*\mathcal{D}_{st}$ so that its associated even contact structure is $\Phi^*\mathcal{E}_{st}$. Lemma 5.7 furnishes a lower bound for the Finsler metric, whence it follows that the $\Phi^*\mathcal{E}_{st}$-directed Kobayashi pseudo-distance on Ω is a genuine distance, i.e. the restriction of the standard even contact structure to Ω is hyperbolic.

To construct $\mathcal{D}_{\mathcal{D}}$ we fix $d_i = 2^{5i+2}$ and $e_i = 2^{3i+1}$ and let K be the set determined by $n = 3, a_i, b_i$ and $c_i = d_i$ according to (5.6). Let B be the set determined by d_i and e_i according to (5.4), and notice that $B \subset K \times \mathbb{C}$. By Theorem 5.8 there exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^3$ with $\Omega \cap K = \emptyset$. Define $\Xi = \Omega \times \mathbb{C}$. The subset $\Xi \subset \mathbb{C}^4$ is a Fatou-Bieberbach domain in $\mathbb{C}^4$ which fulfills $\Xi \cap (K \times \mathbb{C}) = \emptyset$; in particular, $\Xi \cap B = \emptyset$. Let $\Phi : \mathbb{C}^4 \longrightarrow \Xi$ be the Fatou-Bieberbach map. We define $\mathcal{D}_{\mathcal{D}} = \Phi^*(\mathcal{D}_{st})$. Lemma 5.6 furnishes a lower bound for the Finsler metric, whence it follows that the $\mathcal{D}_{st}$-directed Kobayashi pseudo-distance on Ξ is a genuine distance, i.e. the restriction of the standard Engel structure to Ξ is hyperbolic. Notice that in this construction the associated even contact structure $\mathcal{E}$ is not hyperbolic. Indeed we have many $\mathcal{E}_{st}$-lines $f : \mathbb{C} \longrightarrow \Xi$ of the form

$$f(\zeta) = (w_0, x_0, y_0, \zeta)$$

where (w_0, x_0, y_0) is not contained in A, which can be pulled-back.

To construct $\mathcal{D}_{\mathcal{W}}$ consider the set

$$K = \bigcup_{i=1}^{\infty} 2^{i-1}\partial D^2_{(w,y)} \times 2^i \overline{D}_z$$

contained in the (w, y, z)-plane in $\mathbb{C}^4$. All $\mathcal{W}$-horizontal holomorphic copies of $\mathbb{C}$ are of the form $f(\zeta) = (w(\zeta), x_0, y_0, z_0)$ for some w holomorphic and hence they will intersect K for some ζ. Indeed if $N_0 \in \mathbb{N}$ is such that $|z_0| < 2^{N_0}$ then f does not intersect K only if $|w(\zeta)| < 2^{N_0-1}$ for all $\zeta \in \mathbb{C}$, which is not true. Theorem 5.8 ensures the existence of a Fatou-Bieberbach map $\tilde{\Phi} : \mathbb{C}^3 \longrightarrow \Omega \subset \mathbb{C}^3 \setminus K$ so that also $\Phi = \tilde{\Phi} \times id : \mathbb{C}^4 \longrightarrow \Omega \times \mathbb{C} \subset \mathbb{C}^4$ is a Fatou-Bieberbach map. By the above discussion there are no copies of $\mathbb{C}$ tangent to the characteristic foliation of the standard Engel structure restricted to Ω. We then define $\mathcal{D}_{\mathcal{W}} := \Phi^*\mathcal{D}_{st}$, this structure does not have lines tangent to the characteristic foliation, nevertheless $\mathbb{C}^4$ is not $\mathcal{D}_{\mathcal{W}}$-hyperbolic, since the pull-back of the $\mathcal{D}_st$-line

$$f : \mathbb{C} \hookrightarrow \mathbb{C}^4 \qquad f(\zeta) = (0, \zeta, 0, 0).$$

is a $\mathcal{D}_{\mathcal{W}}$-line.

5.4 Construction of the infinite families

In this section we will prove theorems 5.2 and 5.3.

5.4.1 Proof of Theorem 5.2

We use Forstnerič's hyperbolic contact structure on $\mathbb{C}^3$, which is the pull-back $\alpha = \Phi^*\alpha_{st}$ of the restriction of the standard contact structure on a hyperbolic Fatou-Bieberbach domain in $\mathbb{C}^3 \setminus K$ (see [25]). Consider the Cartan prolongation[1] $M = \mathbb{P}(\xi_h)$ of $\xi_h = \ker \alpha$ with its Engel structure $\mathcal{D}(\xi_h)$. Since $\ker \alpha_{st}$ is trivial as a holomorphic bundle, M is biholomorphic to $\mathbb{C}^3 \times \mathbb{CP}^1$. Given $p \in \mathbb{CP}^1$, consider in M the open set $\mathbb{C}^4 = \mathbb{C}^3 \times \mathbb{CP}^1 \setminus (\mathbb{C}^3 \times \{p\})$ and the restriction of the Engel structure $\mathcal{D}_\infty = \mathcal{D}(\xi_h)|_{\mathbb{C}^4}$. We claim that this structure has the properties stated in Theorem 5.2.

Indeed suppose that $f : \mathbb{C} \longrightarrow \mathbb{C}^4$ is a $\mathcal{D}_\infty$-line. Then if we denote by $\pi : M \longrightarrow \mathbb{C}^3$ the canonical projection of the projectivisation, the composition $\pi \circ f$ is tangent to ξ_h in $\mathbb{C}^3$. Since $(\mathbb{C}^3, \xi_h)$ is hyperbolic, $\pi \circ f$ must be constant, so f is tangent to the fibers. This proves that the only $\mathcal{D}_\infty$-lines are tangent to the characteristic foliation $\mathcal{W}_\infty$.

Fix $n \in \mathbb{N}$. In order to construct $\mathcal{D}_n$, we use the following result

Theorem 5.10 (Buzzard and Fornæss, [10]). *Let L be a closed, 1-dimensional, complex subvariety of $\mathbb{C}^2$, and B_0 a ball with $\overline{B}_0 \cap L = \emptyset$. Then there exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^2 \setminus \overline{B}_0$ with $L \subset \Omega$ and a biholomorphic map Ψ from Ω onto $\mathbb{C}^2$ such that $\mathbb{C}^2 \setminus \Psi(L)$ is Kobayashi hyperbolic. Moreover, all nonconstant images of $\mathbb{C}$ in $\mathbb{C}^2$ intersect $\Psi(L)$ in infinitely many points.*

[1] Explicitly, after fixing an affine chart $w \longmapsto [(1, w)]$ on the fibers of the projectivization, we have for (x, y, z) in the Fatou-Bieberbach domain and $\alpha_{st} = dz + xdy$, $\mathcal{D}(\xi_h)_{(w,x,y,z)} = \langle \frac{\partial}{\partial w}, w(x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}) + \frac{\partial}{\partial x} \rangle$.

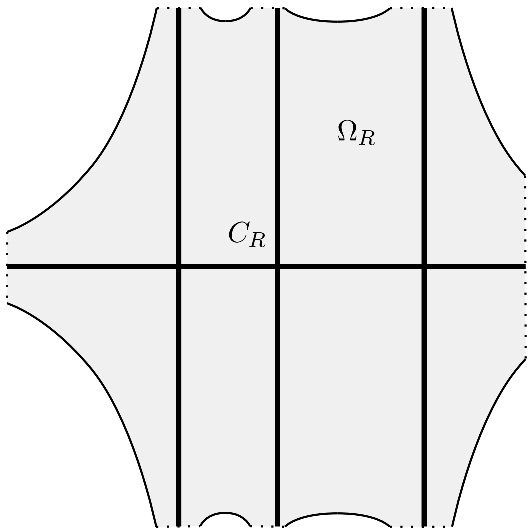

Figure 5.2: The Fatou-Bieberbach domain Ω_R in Theorem 5.3.

Now we choose

$$\tilde{L}_n = \bigcup_{k=1}^{n} \mathbb{C} \times \{k\} \subset \mathbb{C}^2_{(w,x)} \ .$$

Then Theorem 5.10 gives a Fatou-Bieberbach map $\Phi_n : \mathbb{C}^2 \longrightarrow \Omega_n \subset \mathbb{C}^2$ such that $\Omega_n \setminus \tilde{L}_n$ is Kobayashi hyperbolic and the w-curves $f_i : \mathbb{C} \longrightarrow \mathbb{C}^2$ s.t. $\zeta \longmapsto (\zeta, i)$ are still contained in Ω_n. Now take the Fatou-Bieberbach map $\Psi_n = \Phi_n \times id : \mathbb{C}^4 \longrightarrow \Omega_n \times \mathbb{C}^2 \subset \mathbb{C}^4$ and the Engel structure $\mathcal{D}_n = \Psi_n^* \mathcal{D}_\infty$. By construction $\mathcal{D}_n$ only admits $\mathcal{D}_n$-lines on the points

$$L_n = \tilde{L}_n \times \mathbb{C}^2 = \big\{(w,x,y,z) \in \mathbb{C}^4 : x \in \{1,...,n\}\big\}$$

hence completing the proof of Theorem 5.2.

5.4.2 Proof of Theorem 5.3

For some $R \in \mathbf{R} \setminus \{0\}$, we will consider the subvariety $C_R = \big(\mathbb{C} \times \{0, 1, R\sqrt{-1}\}\big) \cup (\{0\} \times \mathbb{C}) \subset \mathbb{C}^2$. By Theorem 5.10, there exists a Fatou-Bieberbach domain $\Omega_R \subset \mathbb{C}^2$ which contains C_R, and such that the complement $\Omega_R \setminus C_R$ is Kobayashi hyperbolic. Moreover, any curve $\mathbb{C} \longrightarrow \Omega_R$ intersects C_R an infinite number of times. Denote by $\mathcal{W}_R$, resp. $\mathcal{W}_{R'}$, the 1-foliation on $\Omega_R \times \mathbb{C}^2$, resp. $\Omega'_R \times \mathbb{C}^2$, determined by the projections $p : \Omega_R \times \mathbb{C}^2 \longrightarrow \mathbb{C}^3$, resp. $p' : \Omega_{R'} \times \mathbb{C}^2 \longrightarrow \mathbb{C}^3$, given by $(w,x,y,z) \longmapsto (x,y,z)$. We introduce also the projections $\pi : \Omega_R \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ and $\pi' : \Omega_{R'} \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ given by $(w,x,y,z) \longmapsto x$ and the notation $V_R = \pi^{-1}\{0, 1, R\sqrt{-1}\}$ and $V'_{R'} = {\pi'}^{-1}\{0, 1, R'\sqrt{-1}\}$. Notice that V_R, resp. $V'_{R'}$, consists exactly of the points of Ω_R, resp. $\Omega_{R'}$ through which a $\mathcal{W}_R$-line, resp. $\mathcal{W}_{R'}$-line, passes.

Lemma 5.11. *Suppose that $R, R' \in \mathbf{R}\backslash\{0\}$ and $R \neq R'$. Then there exists no biholomorphic map $\Phi : \Omega_R \times \mathbb{C}^2 \longrightarrow \Omega_{R'} \times \mathbb{C}^2$ such that $\Phi_*(\mathcal{W}_R) = \mathcal{W}_{R'}$.*

Proof. Suppose such a Φ exists and consider the map $h : \mathbb{C} \longrightarrow \mathbb{C}$ given by $h = \pi' \circ \Phi \circ \iota$, where ι is the inclusion $\iota(\zeta) = (0, \zeta, 0, 0) \in \Omega_R \times \mathbb{C}^2$. Notice that horizontal curves in $\mathcal{W}_R$ must map to horizontal curves in $\mathcal{W}_{R'}$. Moreover, we have $h^{-1}\{0, 1, R'\sqrt{-1}\} = \{0, 1, R\sqrt{-1}\}$. It follows that we have a biholomorphic map $\Phi|_{V_R} : V_R \longrightarrow V'_{R'}$. This implies in particular that $h : \{0, 1, R\sqrt{-1}\} \longrightarrow \{0, 1, R'\sqrt{-1}\}$ is bijective. Since h is non-constant, it either has an essential singularity or a pole at infinity.

If h has an essential singularity at infinity, then by the big Picard theorem h takes every value in $\mathbb{C}$ infinitely many times, with one possible exception. This contradicts the fact that $h : \{0, 1, R\sqrt{-1}\} \longrightarrow \{0, 1, R'\sqrt{-1}\}$ is bijective.

Otherwise, h is a polynomial with exactly one zero, so it must be linear. On the other hand, $h(\{0, 1, R\sqrt{-1}\}) = \{0, 1, R'\sqrt{-1}\}$, which is impossible for $R \neq R'$. ∎

Now given the Fatou-Bieberbach map $\Phi_R : \mathbb{C}^4 \longrightarrow \Omega_R \times \mathbb{C}^2 \subset \mathbb{C}^4$ we define $\mathcal{D}_R := \Phi_R^* \mathcal{D}_{st}$ and Theorem 5.3 is a direct consequence of Lemma 5.11.

Chapter 6

Domination by Products

Let X and Y be complex manifolds. A holomorphic or meromorphic map $X \longrightarrow Y$ is called *dominant* if it has dense image in Y. We write $X \geq Y$ to indicate that a dominant map from X to Y exists, and say that X *dominates* Y. In this chapter, we consider a geometrical approach to the problem of determining whether a complex manifold is *dominated by a product*.

Definition 6.1. Let Y be a complex manifold. Y is said to be *(holomorphically) dominated by a product* if there exist compact complex manifolds $X_1, \ldots, X_k$ such that $\dim(X_i) < \dim(Y)$ and a dominant holomorphic map $X_1 \times \cdots \times X_k \longrightarrow Y$.

Similarly, if a dominant meromorphic map from a product exists, we say that Y is meromorphically dominated by a product. In the algebraic setting, we say also that Y is rationally dominated by a product.

Developing an idea of Deligne, Schoen introduced in [67] a Hodge theoretic integer valued birational invariant τ for algebraic varieties which is monotonic under the rational dominance relation, that is, if $X \geq Y$ rationally, then $\tau(X) \geq \tau(Y)$. Very roughly, τ is a measure of how complicated the irreducible Hodge substructures in the cohomology of a given variety are. It has good properties under taking products, so it provides obstructions to being rationally dominated by a product. Schoen proves in [67] among other things that, for varieties X over $\mathbb{C}$:

- if $X = X_1 \times \cdots \times X_k$, then $\tau(X) = \max\{\tau(X_i) \mid 1 \leq i \leq k\}$;
- if X is an Abelian variety, $\tau(X) = 0$ if X is of CM type, and $\tau(X) = 1$ otherwise;
- if X is a smooth projective curve, $\tau(X) = \tau(\mathrm{Jac}(X))$;
- if $X \subset \mathbb{CP}^N$ is a sufficiently general complete intersection of degree $> N + 1$, $\tau(X) = \dim(X)$;

- if X is a sufficiently ample, sufficiently general smooth divisor on a smooth projective variety of dimension $n+1$, then $\tau(X) = n$.

On the other hand, for τ to be non-trivial, the Hodge structure of the underlying variety cannot be too simple. For instance, τ vanishes when there are no holomorphic forms, which occurs for example in certain complex ball quotients. This shortcoming is noted in [67], where it is also remarked that a result due to Zheng precludes the rational domination of a complex variety with ample cotangent bundle by a product. That is indeed the case on a compact ball quotient.

Our point of view is that of complex (differential) geometry: we obtain obstructions from certain types of negativity of the tangent bundle of a manifold, which are geometric (or directly motivated by geometric notions) in the sense that they come from properties of the curvature of the Chern connection with respect to a Hermitian structure on the tangent bundle. We review known notions of negativity in Section 6.1; the results obstructing domination by products are discussed in Section 6.2. In Section 6.3, we apply the results of Section 6.2 to show that that the Fano surface of lines in a cubic is not rationally dominated by a product. This was previously proved by Schreieder in [68].

In the topological world, domination (meaning the existence of a continuous map of non-zero degree) by a product manifold has been studied in [49], where certain obstructions coming from the fundamental group can be found. Any topological obstruction obviously applies to holomorphic domination by products in the same dimension, but it turns out that the results in [49] can sometimes be applied to the rational case, without the assumption on dimension [50].

6.1 Negativity Notions

We start by recalling notions of negativity for the tangent bundle of a complex manifold, as well as some of their consequences. The content in this section is standard. We start in 6.1.1 with some basic curvature properties of Hermitian vector bundles, as can be found in any introductory textbook (e.g. [39] or [32]). In 6.1.2 we specialize to the case of the tangent bundle, recalling the notion of bisecional curvature and collecting a number of consequences of its negativity. Finally, 6.1.3 contains an analytic counterpart to the notion of negative bisectional curvature. The material therein can be found e.g. in [45, Ch. 6].

6.1.1 Curvature on Hermitian bundles

Let X be a complex manifold and $E \longrightarrow X$ a holomorphic vector bundle with a Hermitian metric h. The pair (E, h) is called a Hermitian vector bundle. Denote by ∇ the Chern connection of h, that is, the unique connection on E which is compatible with the holomorphic structure $\bar{\partial}_E$ and the metric h. Let d_∇ be the exterior derivative associated to ∇ and $F^\nabla = d_\nabla^2 \in A^{1,1}(\mathrm{End}(E))$ the curvature. When no confusion may arise, we will drop the reference to the connection and the bundle in our notation.

Definition 6.2. A Hermitian vector bundle (E, h) over X is called *Griffiths-positive* if for every $p \in X$ and every non-zero $v \in T_pX$ and $e \in E_p$ we have $h(F(v, \bar{v})e, e) > 0$.

We define Griffiths negative (semi-positive, etc.) Hermitian vector bundles in the same manner. When E is a line bundle, positivity of E is the same as positivity of its real first Chern class $c_1(E) = \frac{i}{2\pi}[F_E] \in H^2(X; \mathbb{R})$ in the usual sense. As for line bundles, it makes sense to define positivity for holomorphic vector bundles. Namely, we say a holomorphic vector bundle is Griffiths positive when it admits a metric which makes it a positive Hermitian vector bundle.

If $S \subset E$ is a holomorphic subbundle, the quotient bundle $Q = E/S$ is also holomorphic. Furthermore, we have (smoothly, but not necessarily holomorphically) $E = S \oplus Q$. In what follows, we implicitly identify Q with $S^\perp$. Let ∇_S, resp. ∇_Q, be the Chern connection of the induced metric on S, resp. Q, and set

$$\begin{aligned} \lambda_S &= \nabla_E|_S - \nabla_S \in A^{1,0}(\mathrm{Hom}(S, Q)) \\ \lambda_Q &= \nabla_E|_Q - \nabla_Q \in A^{0,1}(\mathrm{Hom}(Q, S)). \end{aligned}$$

We call λ_S the second fundamental form of S. Let $\{s_i\}$ be a local unitary frame for S, which we complete to a local unitary frame for E. Denote by W_E, W_S and W_Q the corresponding connection matrices of ∇_E, ∇_S and ∇_Q, all skew-hermitian, since the frame was picked unitary. Similarly, let L_S, L_Q be the matrices of λ_S, λ_Q. Then

$$W_E = \begin{pmatrix} W_S & L_S \\ L_Q & W_Q \end{pmatrix}.$$

We have $L_S = -L_Q^\dagger$, where $\dagger$ denotes the conjugate transpose. The curvature matrix Ω_E of ∇_E is given in terms of the Cartan formula by $\Omega_E = dW_E - W_E \wedge W_E$, so

$$\begin{aligned} \Omega_E|_S =&\Omega_S - L_S \wedge L_Q = \Omega_S + L_S \wedge L_S^\dagger \\ \Omega_E|_Q =&\Omega_Q - L_Q \wedge L_S = \Omega_Q + L_Q \wedge L_Q^\dagger. \end{aligned}$$

On the other hand, $L_S \wedge L_S^\dagger \geq 0$, since L_S is a matrix of $(1,0)$-forms. We conclude that $F_S \leq F_E|_S$ and $F_Q \geq F_E|_Q$, i.e. *curvature decreases on subbundles and increases on quotient bundles.*

Remark 6.3. Let E, E_1 and E_2 be Hermitian vector bundles on X with Chern connections ∇, ∇_1 and ∇_2. Recall that the curvature of a connection has the following naturality properties:

(i) the curvature of induced connection ∇^* on the dual E^* is given by $F^{\nabla^*} = -(F^\nabla)^T$;

(ii) the curvature of induced connection $\nabla^\otimes$ on the tensor product $E_1 \otimes E_2$ is given by $F^{\nabla^\otimes} = F^{\nabla_1} \otimes \mathrm{Id}_{E_2} + \mathrm{Id}_{E_1} \otimes F^{\nabla_2}$;

(iii) the curvature of induced connection $\nabla^\oplus$ on the direct sum $E_1 \oplus E_2$ is given by $F^{\nabla^\oplus} = F^{\nabla_1} \oplus F^{\nabla_2}$.

In all the above, the induced connections are the Chern connections of the respective induced Hermitian metrics. It follows that Griffiths positivity (semi-positivity, negativity, semi-negativity) is preserved under tensor products and direct sums. On the other hand, E is Griffiths (semi-)positive if and only if its dual is (semi-)negative. Moreover, since $\Lambda^k E$ is a direct summand of $E^{\otimes k}$ (hence both a quotient and a subbundle), it is Griffiths (semi-)positive, respectively (semi-)negative, if E is Griffiths (semi-)positive, respectively (semi-)negative.

We prove now the following well known Bochner type formula for later use:

Lemma 6.4. *Let X be a complex manifold and (E,h) a Hermitian vector bundle on X with associated Chern connection ∇. Let s be a section of E, $f : X \longrightarrow \mathbb{R}$ the function $f(x) = h(s,s)(x)$ and $\alpha = \partial\overline{\partial} f$. Then we have*

$$\alpha(w,\overline{w}) = \|\nabla_w s\|^2 - h(F^\nabla(w,\overline{w})s,s).$$

Proof. Recall that the exterior derivative d_∇ induced on $A^*(E)$, the exterior algebra of the smooth forms with values in E fulfills $dh(\psi_1,\psi_2) = h(d_\nabla\psi_1,\psi_2) + (-1)^k h(\psi_1, d_\nabla\psi_2)$ for $\psi_i \in A^k(E)$, and that on decomposable elements $\psi_i = \beta_i \otimes e_i \in A^{k_i}(E)$ we have $h(\psi_1,\psi_2) = (\beta_1 \wedge \overline{\beta_2})h(e_1,e_2)$. Then

$$\begin{aligned} \alpha =& -\overline{\partial}\partial h(s,s) = -d\partial h(s,s) = -d(dh(s,s))^{1,0} \\ =& -d(h(\nabla s,s) + h(s,\nabla s))^{1,0} \\ =& -d(h(\nabla^{1,0}s,s) + h(s,\nabla^{0,1}s)). \end{aligned}$$

Using the fact that ∇ is the Chern connection, so that $\nabla^{0,1} = \overline{\partial}_E$, we see that

$$\begin{aligned} \alpha =& -dh(\nabla^{1,0}s,s) = -dh(\nabla s,s) \\ =& -(h(F^\nabla s,s) - h(\nabla s,\nabla s)) \\ =& h(\nabla s,\nabla s) - h(F^\nabla s,s). \end{aligned}$$

Therefore, for $w \in TX$, we have

$$\alpha(w,\overline{w}) = h(\nabla_w s,\nabla_w s) - h(F^\nabla(w,\overline{w})s,s)$$

as claimed. ∎

This lemma has the following consequence for the sections of a Griffiths semi-negative vector bundle:

Proposition 6.5. *Let X be a compact complex manifold and $E \longrightarrow X$ a vector bundle with a Griffiths semi-negative Hermitian metric h. Then every section $s \in H^0(X,E)$ of E is parallel with respect to the Chern connection of h. If moreover E is Griffiths negative around a point, then $H^0(X,E) = 0$.*

Proof. Let $s \in H^0(X,E)$. The form α from Lemma 6.4 is then semi-positive, so f is plurisubharmonic on X and therefore constant, so α vanishes identically. This implies that $\nabla s = 0$. Suppose that E is also Griffiths negative around a point. Were s not to vanish identically, then α would be non-zero. Therefore $s = 0$. ∎

6.1.2 Holomorphic bisectional curvature

We now consider the case where $E = TX$, keeping the notation as in the previous section.

Definition 6.6. Let X be a Hermitian manifold, $p \in X$ and $0 \neq v, w \in T_pX$. Denote by F the curvature of the Chern connection on TX. The *holomorphic bisectional curvature* B between v and w is

$$B(v,w) = \frac{h(F(v,\bar{v})w,w)}{\|v\|^2\|w\|^2}.$$

The *holomorphic sectional curvature* H of v is $H(v) = B(v,v)$.

The holomorphic bisectional curvature is thus a curvature between complex lines in each complex tangent space. Due to the principle of decreasing curvature on subbundles, the holomorphic bisectional curvature B_Y of a submanifold $Y \subset X$ fulfills $B_Y \leq B|_{TY}$.

Lemma 6.7. *Let X be a complex manifold, viewed as an (integrable) almost complex manifold (X, J) with a Kähler metric g. Let $p \in X$ and let $A, B \in T_pX$ be two vectors with unit length in the (real) tangent bundle of X. Then the holomorphic bisectional curvature between the J-invariant planes defined by A and B is $B(A, B) = g(R(A, JA)JB, B)$, where R is the Riemann tensor of the metric g.*

Proof. The Riemannian metric g in TX extends sesquilinearly to $T_{\mathbb{C}}X$. Denote this extension also by g. We can identify (TX, h, J) with $(T^{1,0}X, g|_{T^{1,0}X}, i) \subset T_{\mathbb{C}}X$. Moreover, a unit vector $v \in T^{1,0}X$ is written uniquely in terms of a unit real vector $A \in TX$ via $v = 2^{-\frac{1}{2}}(A - iJA)$. Since X is a Kähler manifold, the Chern connection (on $T^{1,0}X$) and the Levi-Civita connection (on TX) are identified, and the curvature of the Chern connection is given by the restriction to $T^{1,0}X$ of the complex linear extension of the Riemann tensor R. Setting $w = 2^{-\frac{1}{2}}(B - iJB)$ we have

$$\begin{aligned}
B(A,B) := B(v,w) =& h(F(v,\bar{v})w,w) \\
=& \tfrac{1}{4} g(R(A - iJA, A + iJA)(B - iJB), B - iJB)) \\
=& \tfrac{1}{4} g((R(A, iJA) - R(iJA, A))(B - iJB), B - iJB) \\
=& \tfrac{i}{2} g(R(A, JA)(B - iJB), B - iJB) \\
=& \tfrac{i}{2} \left(g(R(A, JA)B, -iJB) - g(R(A, JA)iJB, B)) \right) \\
=& \tfrac{i}{2} \left(ig(R(A, JA)B, JB) + ig(R(A, JA)B, JB)) \right) \\
=& - g(R(A, JA)B, JB) = g(R(A, JA)JB, B),
\end{aligned}$$

as claimed. ∎

Remark 6.8. This relation between B and R suggests a different way to define a curvature between complex lines, which makes sense even in the almost complex case – in fact, this is the definition of Goldberg and Kobayashi in [31]. When the metric is not Kähler, there several distinct connections (and curvatures) one can consider, all natural in a certain sense, which capture different aspects of the underlying Hermitian structure. Consequently, the curvature should be chosen in terms of the properties one wants to study. The way we defined the holomorphic bisectional curvature makes it especially suitable to study questions pertaining to complex objects. Losing the Riemannian picture is the price we have to pay when we don't have the Kähler condition to connect the Riemannian and the complex world.

Due to the symmetries of a Kähler curvature tensor, the holomorphic bisectional curvature of a Kähler metric is symmetric in its entries, and it can be related to the sectional curvature by the following lemma, in which we keep the notation of Lemma 6.7.

Lemma 6.9. *The holomorphic bisectional curvature of a Kähler metric is given in terms of the sectional curvature by* $B(A,B) = K(A,B)(1-g(A,B)^2) + K(A,JB)(1-g(A,JB)^2)$.

Proof. By Lemma 6.7, for A,B unit vectors in the real tangent bundle of X, we have

$$\begin{aligned} B(A,B) =& g(R(A,JA)JB,B) \\ =& -g(R(JA,JB)A,B) - g(R(JB,A)JA,B) \\ =& -g(R(A,B)A,B) - g(R(JB,A)JJA,JB) \\ =& g(R(A,B)B,A) + g(R(JB,A)A,JB) \\ =& K(A,B)(1-g(A,B)^2) + K(A,B)(1-g(A,JB)^2), \end{aligned}$$

where we used the Bianchi identity in the second equality, and a symmetry of the Kähler curvature tensor in the third. In particular, if A and B are orthonormal, $B(A,B) = K(A,B) + K(A,JB)$. ∎

Remark 6.10. On a Kähler manifold, the holomorphic sectional curvature determines completely the curvature tensor (see e.g. [46]).

As in the Riemannian setting, it is interesting to study conditions on the sign of B; of course due to Lemma 6.9, given any Kähler metric, a sign condition on K implies the same sign condition on B.

For a compact complex manifold X admitting a Hermitian metric with $B > 0$, Mori [60] proved using algebraic methods that X is isomorphic to the complex projective space (the *Frankel conjecture*). Yau-Siu also gave a differential geometric argument proving the same result in [72]. A generalization to non-negative bisectional curvature in the form of a uniformization theorem (the *generalized Frankel conjecture*) was later given by Mok in [59].

In the case $B < 0$, note that, because of Remark 6.3, X has an ample canonical bundle (and is in particular projective of general type), since K_X^* is an exterior power of the tangent bundle, and therefore negative. When X is Kähler, one has the following theorem, proved by Wu-Zheng for analytic metrics, and extended by Liu to an arbitrary Kähler metric.

Theorem 6.11 (Wu-Zheng [77], Liu [55])**.** *Let X be a compact Kähler manifold of dimension n with non-positive bisectional curvature. Then there exists a finite covering X' of X such that X' is a metric and holomorphic fiber bundle over a compact Kähler manifold N of dimension k with non-positive bisectional curvature, and fiber a complex torus. Moreover, N has $c_1(N) < 0$ (in particular, X has Kodaira dimension k).*

The following Lemma 6.12 computes the holomorphic bisectional curvature of a convex linear combination of metrics. This kind of control of the curvature of a sum of metrics allows us to deduce the well known Proposition 6.13. We provide a detailed proof of the lemma, since we will need it in the subsequent section.

Lemma 6.12. *Let g, h be Hermitian (not necessarily Kähler) metrics on a complex manifold X and denote by k the Hermitian metric $k = g + h$. Let $\{s_i\}_i$ be a local holomorphic frame on the tangent bundle of X, and let G, H, K denote the matrices of g, h, k, respectively, in this frame. Let W_g and W_h denote the connection matrices of the Chern connections of g and h, and let B_g, B_h, B_k denote the matrices $B_g = \Omega_g G$, $B_h = \Omega_h H$ and $B_k = \Omega_k K$, where Ω_g, Ω_h, Ω_k denote the curvature matrices of g, h, k on the frame $\{s_i\}_i$. Then*

$$B_k = B_g + B_h - (W_h - W_g)(G^{-1} + H^{-1})^{-1}(W_h - W_g)^\dagger. \tag{6.1}$$

Proof. Let g, h be as above. We compute the holomorphic bisectional curvature of $k = g+h$. We have $G_{ij} = g(s_i, s_j)$, $H_{ij} = h(s_i, s_j)$ and $K = G + H$. Since our frame is holomorphic, $W_g G = \partial G$ and $W_h H = \partial H$, where W_g and W_h denote the connection matrices of g and h. Taking the conjugate transpose of these equations and noting that G and H are Hermitian, we see that $GW_g^\dagger = \bar\partial G$ and $HW_h^\dagger = \bar\partial H$. Then the matrix of the Chern connection of k is $W_k = \partial(G + H)(G + H)^{-1}$. On the other hand, we have

$$\begin{aligned}
\bar\partial(W_k K) = \bar\partial W_k K - W_k \wedge \bar\partial K =& \bar\partial\partial(G + H)\\
=& \bar\partial\partial G + \bar\partial\partial H\\
=& \bar\partial W_g G - W_g \wedge \bar\partial G + \bar\partial W_h H - W_h \wedge \bar\partial H.
\end{aligned}$$

Note now that the holomorphic bisectional curvature is given by the matrix $B_k = \Omega_k K$, where $\Omega_k = \bar\partial W_k$ is the curvature matrix of the Chern connection. By this we mean that if $v, w \in T_pX$ are a unit vectors, with v expressed in terms of our frame by $v = \sum_i \lambda_i s_i(p)$, then $(\lambda^\dagger \cdot \Omega_k K \cdot \lambda)(w, \overline{w}) = k(\Omega_k(w, \overline{w})v, v)$. So we find

$$B_k = B_g + B_h - W_g \wedge GW_g^\dagger - W_h \wedge HW_h^\dagger + W_k \wedge \bar\partial K. \tag{6.2}$$

For the last term, omitting the wedge for a cleaner notation,

$$\begin{aligned}
W_k\bar\partial K =& \partial K K^{-1}\bar\partial K\\
=& (W_g G + W_h H)K^{-1}(GW_g^\dagger + HW_h^\dagger)\\
=& (W_g K + W_h H)K^{-1}(GW_g^\dagger + HW_h^\dagger) - W_g H K^{-1}(GW_g^\dagger + HW_h^\dagger)\\
=& (W_g + (W_h - W_g)HK^{-1})(GW_g^\dagger + HW_h^\dagger)\\
=& W_g(GW_g^\dagger + HW_h^\dagger) + (W_h - W_g)HK^{-1}(GW_g^\dagger + HW_h^\dagger)\\
=& W_g(GW_g^\dagger + HW_h^\dagger) + (W_h - W_g)HK^{-1}(GW_g^\dagger + KW_h^\dagger)+\\
& - (W_h - W_g)HK^{-1}GW_h^\dagger\\
=& W_g GW_g^\dagger + (W_h - W_g)HK^{-1}GW_g^\dagger + W_h HW_h^\dagger - (W_h - W_g)HK^{-1}GW_h^\dagger\\
=& W_g GW_g^\dagger + W_h HW_h^\dagger + (W_h - W_g)HK^{-1}G(W_g^\dagger - W_h^\dagger).
\end{aligned}$$

Replacing this term in (6.2), and using the identity $H(G+H)^{-1}G = (G^{-1}+H^{-1})^{-1}$, we finally get

$$B_k = B_g + B_h - (W_h - W_g)(G^{-1}+H^{-1})^{-1}(W_h - W_g)^\dagger,$$

which is the expression in (6.1). ∎

Since the inverse of a positive definite matrix is positive definite, Proposition 6.13 follows from (6.1),

Proposition 6.13. *Let g, h be Hermitian (not necessarily Kähler) metrics with nonpositive holomorphic bisectional curvature on a complex manifold. Then $g+h$ is a Hermitian metric with nonpositive holomorphic bisectional curvature. If in addition g has strictly negative holomorphic bisectional curvature, then $g+h$ is a Hermitian metric with strictly negative holomorphic bisectional curvature.*

Corollary 6.14. *The set $\mathcal{F}(X)$ of Hermitian metrics on X with nonpositive holomorphic bisectional curvature forms a convex cone in the space of sections of $TX \otimes \overline{T}X$.*

6.1.3 Analytic Negativity

We now come to a more flexible notion of negativity for vector bundles, which we will see generalizes Griffiths negativity.

Definition 6.15. Let r, p be integers with $0 \leq p \leq n$ and let X be a complex manifold of dimension n. A holomorphic vector bundle $E \longrightarrow X$ of rank r is said to be *p–negative* if there exists a continuous function $f : E \longrightarrow \mathbb{R}$, $f \geq 0$, such that:

(i) $f^{-1}(0)$ is the zero section of E,

(ii) f is of class C^2 on $E^\times$, the complement of the zero section of E, and

(iii) the complex Hessian $i\partial\bar\partial f$ of the restriction of f to $E^\times$ has *at least* $r+p$ positive eigenvalues.

We say that a complex manifold is p–negative if its tangent bundle is p–negative.

In particular, if E is p–negative, then it is l-negative for all $0 \leq l \leq p$. The following examples show that p–negativity encompasses both algebraic and differential geometric notions of negativity.

Example 6.16 (Griffiths negative bundles)**.** Let E be a Hermitian bundle of rank r. Define on E a function $f : E \longrightarrow \mathbb{R}$ by $f(e) := h(e,e)$. On a trivialization $(x,u) \longmapsto \sum_i u_i s_i$ given by r pointwise linearly independent local holomorphic sections s_i, f is given by $f(x,u) = \sum_{i,j} h(s_i,s_j)u_i\overline{u_j}$ so that, by Lemma 6.4,

$$\partial\bar\partial f = \sum_{i,j} h(s_i,s_j)du_i \wedge d\overline{u_j} + \sum_i \left(h(u_i\nabla s_i, u_i\nabla s_i) - h(F^\nabla(u_i s_i), u_i s_i)\right).$$

The number of positive eigenvalues of $i\partial\bar{\partial}f$ is invariant under a biholomorphism. Therefore, if E admits a metric h such for each $x \in X$ and $e \in E_x$ the Hermitian form $v \longmapsto h(F^{\nabla}(v,\bar{v})e,e)$ on T_xX has p negative eigenvalues, then E is p–negative.

Therefore a Griffiths negative vector bundle E of rank r is n-negative. In particular, a complex manifold with negative holomorphic bisectional curvature is n-negative. ∎

Example 6.17 (The dual of an ample vector bundle)**.** Let $E \longrightarrow X$ be a holomorphic vector bundle on a compact complex manifold X, and let $\pi : \mathbb{P}(E) \longrightarrow X$ be its projectivization. On $\mathbb{P}(E)$ one can consider the tautological bundle line bundle

$$\mathcal{O}_{\mathbb{P}(E)}(-1) := \{([e],e) \in \pi^*(E) \,|\, e \in E\},$$

as well as its dual $\mathcal{O}_{\mathbb{P}(E)}(1)$. Recall E is called *ample*[1] if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is ample as a line bundle over $\mathbb{P}(E^*)$.

Assume then that E^* is ample. Then $\mathcal{O}_{\mathbb{P}(E)}(-1)$ admits a Hermitian metric h with strictly negative curvature. The metric h defines a function $f : E^{\times} \longrightarrow \mathbb{R}$

$$f(e) := h_{[e]}(e,e).$$

Note that f extends, by setting $f \equiv 0$ on the zero section of E, to a continuous function $E \longrightarrow \mathbb{R}$, which we continue to denote by f.

Note that $\mathcal{O}_{\mathbb{P}(E)}(-1)$ is the total space of the blow-up of E along its zero section, with exceptional divisor the zero section of $\mathcal{O}_{\mathbb{P}(E)}(-1)$. Thus the natural map $\mathcal{O}_{\mathbb{P}(E)}(-1)^{\times} \longrightarrow E^{\times}$ is a biholomorphism. Now on a local trivialization $([e],\lambda) \longmapsto \lambda s([e])$ for $\mathcal{O}_{\mathbb{P}(E)}(-1)$ given by a local non-vanishing holomorphic section s, the metric h is given by $\|\lambda\|^2 h(s,s)$. Moreover, due to Lemma 6.4,

$$\partial\bar{\partial}(|\lambda|^2 h(s,s)) = h(s,s)d\lambda \wedge d\bar{\lambda} + |\lambda|^2 \left(h(\nabla s, \nabla s) - h(s,s)\Omega\right),$$

where ∇ denotes the Chern connection of h and Ω its curvature, which is strictly negative if E is ample. We conclude that $i\partial\bar{\partial}f$ is positive definite on $E^{\times}$ so that E is n-negative. More generally, whenever $\mathcal{O}_{\mathbb{P}(E)}(-1)$ admits a metric whose curvature has k negative eigenvalues, E is $(k+1)$-negative. ∎

The concept of p–negativity appears in connection with discreteness and finiteness results for spaces of holomorphic (and meromorphic) maps between complex manifolds, for instance in [41] and [61]. This is relevant to us due to the following reason: suppose that a complex manifold Y is given, and that one knows a priori that the space of holomorphic maps

$$\mathrm{Hol}_k(X,Y) := \{\varphi : X \longrightarrow Y \,|\, \varphi \text{ holomorphic and } \dim(\varphi(X)) = k\}$$

is discrete for arbitrary X with $\dim(X) < \dim(Y)$ and arbitrary $k < \dim(Y)$. Then Y is not holomorphically dominated by a product. Indeed, if Y is dominated by a product $X_1 \times X_2$

[1] This definition of ampleness for vector bundles may seem strange; this is due to our convention for the projectivization.

with $\dim(X_i) < \dim(Y)$, take a dominant map $f : X_1 \times X_2 \longrightarrow Y$ and pick a regular point (x_1, x_2) of f. The restrictions $f_x := f|_{\{x\}\times X_2}$, where x lies in a sufficiently small neighbourhood of x_1, therefore form a non-trivial continuous family of maps $X_2 \longrightarrow Y$.

Information about the finiteness of $\mathrm{Hol}_k(X, Y)$ is provided by the following:

Theorem 6.18 (Kalka-Schiffman-Wong [41])**.** *Let X be a compact connected complex space and let Y be a connected n-dimensional manifold, not necessarily compact. If TY is p–negative, then $\mathrm{Hol}_{n-p+1}(X, Y)$ is discrete.*

A similar result due to Noguchi and Sunada is the following.

Theorem 6.19 (Noguchi-Sunada [61])**.** *Let X and Y be algebraic varieties, with Y smooth and complete of dimension n. If the k-th exterior power $\Lambda^k(TY)$ of TY is n-negative, then the space $\mathrm{Mer}_k(X, Y)$ of meromorphic maps of rank k is finite.*

By the remarks above, these theorems have in particular the following corollaries:

Corollary 6.20. *Let Y be a complex manifold.*

(i) If Y admits a metric with negative bisectional curvature, then Y is not holomorphically dominated by a product.

(ii) Suppose that Y is compact and projective with ample cotangent bundle. Then Y is not rationally dominated by a product.

6.2 Quasi-negativity and domination by products

In order to get more general domination results than Corollary 6.20, we wish to consider weaker notions of negativity on the tangent bundle. The idea is that, in contrast to imposing conditions on the tangent bundle which prescribe negativity *everywhere*, like Griffiths negativity, p–negativity or ampleness of the dual, it should suffice to prescribe it at a single point, provided one has non-positivity everywhere[2]. This is partly motivated by the Bochner type formula in Lemma 6.4 and the subsequent Proposition 6.5.

Going in this direction, the following is a natural geometric condition to consider on the holomorphic bisectional curvature:

Definition 6.21. A Hermitian manifold X has quasi-negative holomorphic bisectional curvature if $B \leq 0$ and there exists a point $p \in X$ such that $B_p < 0$.

For the reasons explained in section 6.1.1, quasi-negativity of the holomorphic bisectional curvature is inherited by submanifolds. A direct application of Proposition 6.5 yields:

Corollary 6.22. *Let X be a Hermitian manifold and $Y \subset X$ a compact complex submanifold. If the holomorphic bisectional curvature of X is quasi-negative along Y, a holomorphic vector field on X must vanish along Y.*

[2]It is manifestly not enough to impose negativity at a point without global non-positivity: due to Lemma 6.12, it is always possible to make the bisectional curvature arbitrarily negative at a point.

Proof. This follows from Proposition 6.5 with $E = TX|_Y$. ∎

Therefore a complex manifold $X = X_1 \times X_2$ where one of the factors, say X_1, is compact admits no metric with quasi-negative bisectional curvature, as one can apply the above corollary to $X_1 \times U$ for a sufficiently small open set $U \subset X_2$. A modification of this argument can be used to show that if $\dim(X) = \dim(Y)$, and Y has a metric with quasi-negative bisectional curvature, then no dominant holomorphic map $X \longrightarrow Y$ exists:

Proposition 6.23. *Let Y be a compact complex manifold with a Hermitian metric of quasi-negative holomorphic bisectional curvature. Then Y is not holomorphically dominated by a product $X = X_1 \times X_2$, where X_1 is compact and* $\dim(X) = \dim(Y)$.

Proof. Suppose such a map $\varphi : X_1 \times X_2 \longrightarrow Y$ exists. Since φ is surjective, the regular point set R of φ is open and dense in $X_1 \times X_2$.

We consider on X an arbitrary product metric g and on Y a Hermitian metric h with quasi-negative holomorphic bisectional curvature. Then we can define a new metric $k := g + \varphi^* h$ on X. Notice that $\varphi^* h$ defines a metric on R with quasi-negative holomorphic bisectional curvature.

Let then $(x_1, x_2) \in R$ be a point where $\varphi^* h$ has negative holomorphic bisectional curvature. Let v_2 be a non-vanishing vector field defined on a neighbourhood $U_2 \subset X_2$ of x_2, and consider $f : X_1 \longrightarrow \mathbb{R}$ defined by $f(x) = h(v_2, v_2)(x, x_2)$. By Lemma 6.4 we have for $\alpha = -\bar\partial\partial f$

$$\alpha(w, \overline{w}) = \|\nabla_w v_2\|^2 - B(v_2, w)\|v_2\|^2\|w\|^2,$$

where $w \in T_x X_1$.

On the other hand, by virtue of g being a product metric, we have $B_g(v_2, w) = 0$. Therefore we can use Lemma 6.12 to conclude that $B_k(v_2, w) \leq 0$, and moreover, for all $w \in T_{x_1} X_1$, $B_k(v_2, w) < 0$. Then $i\alpha$ is an exact semi-positive 2-form on the compact manifold X_1 which is strictly positive at a point, which is a contradiction. ∎

The proof of the above proposition gives some direct geometric insight concerning the role the quasi-negativity condition plays in obstructing domination by products. We aim now to prove a more general result, where however this role will become in some ways less apparent. We begin by weakening the notion of n–negativity to *quasi-p*–negativity, in such a way that the notion of quasi-n-negativity becomes a generalization of the notion of quasi-negative holomorphic bisectional curvature.

Definition 6.24. Let $0 \leq p \leq n$ and let X be a complex manifold of dimension n. A holomorphic vector bundle $E \longrightarrow X$ of rank r is said to be quasi p–negative if there exists a continuous function $f : E \longrightarrow \mathbb{R}$, $f \geq 0$ such that:

(i) $f^{-1}(0)$ is the zero section of E,

(ii) f is of class C^2 on $E^\times$, the complement of the zero section of E,

(iii) there exists a point $x \in X$ and a neighbourhood $U \subset X$ of x such that the restriction of the complex Hessian $i\partial\bar\partial f$ to $E^\times_U$ has *at least* $r + p$ positive eigenvalues and

(iv) $i\partial\bar\partial f$ is non-negative on $E^\times$ (i.e. f is plurisubharmonic).

Then we can adapt the methods in [41] to derive the following theorem (see also [45, Ch. 7]).

Theorem 6.25. *Let Y be a complex manifold of dimension n. Let $l > 1$ be an integer and let $X = X_1 \times \cdots \times X_l$, where X_i are compact complex manifolds of dimension $m_i < n$. Further let $k = \min_i(m_i)$ and p an integer satisfying $p > k$. Assume that TY is quasi p–negative, and let $\varphi : X \longrightarrow Y$ be a holomorphic map such that φ does not factor through $X_1 \times \cdots \times \widehat{X_j} \times \cdots \times X_l \subset X$ for any j. Then φ is not dominant.*

Proof. For a contradiction, suppose that such a φ exists. Without loss of generality, assume $m_1 = k$ and define $Z = X_2 \times \cdots \times X_l$.

Let f be as in Definition 6.24, let $W \subset Y$ be such that $i\partial\bar\partial h$ has $n+p$ positive eigenvalues on $TY|_W$, and put $V = \varphi^{-1}(W)$. There exists a regular point $x = (x_1, \dots, x_l) \in V \subset X$ of φ and a vector $v_1 \in T_{x_1}X_1$ such that $v = (v_1, 0, \dots, 0) \in T_xX$ has $D\varphi(v) \neq 0$ (otherwise, φ would not depend on X_1, and thus factor through Z contradicting our assumption).

Now consider the embedding $\psi : Z \hookrightarrow TX$ given by taking Z to the product of $v_1 \in T_{x_1}X_1$ with the zero section of TZ, that is

$$(z_2, \dots, z_l) \overset{\psi}{\longmapsto} (v_1, 0, \dots, 0) \in T_{x_1}X_1 \times T_{z_2}X_2 \times \cdots \times T_{z_l}X_l.$$

Note that the image of $\tilde Z := \psi(Z)$ under $D\varphi$ is not contained in the zero section of TY, and the dimension of $D\varphi(\tilde Z)$ is at least $n - m_1$. Now $\tilde f := f \circ \varphi$ is continuous and plurisubharmonic, therefore constant on $\tilde Z$. On the other hand, consider the point $w = D_x\varphi(v) \in \varphi(\tilde Z)$. Since $w \in TY|_W$, $i\partial\bar\partial h$ has $n+p$ positive eigenvalues at w. Therefore there exists through w a local $n+p$ complex submanifold S where h is strictly plurisubharmonic. Comparing dimensions, we see that $\dim(S \cap \varphi(\tilde Z)) \geq (n+p) + (n - m_1) - 2n = p - k > 0$. It follows that f cannot be constant on $D\varphi(\tilde Z)$, so $\tilde f$ is not constant on $\tilde Z$, which is a contradiction. ∎

Our notion of quasi p–negativity has obvious minimality implications for projective varieties:

Lemma 6.26. *Let Y be quasi p–negative for some $p \geq 0$. Then every map $\varphi : \mathbb{CP}^m \longrightarrow Y$ is constant.*

Proof. The tangent bundle of $\mathbb{CP}^m$ is globally generated, so for every $x \in \mathbb{CP}^m$ and $v \in T\mathbb{CP}^m$ there exists a holomorphic vector field V with $V(x) = v$. Let $f : TY \longrightarrow \mathbb{R}$ be plurisubharmonic with $f^{-1}(0)$ equal to the zero section of TY. The map $f \circ D\varphi \circ V : X \longrightarrow \mathbb{R}$ takes the value 0 on a zero of V, and since f is plurisubharmonic it must be constant equal to zero. Hence $D\varphi(v)$ vanishes for every $v \in T\mathbb{CP}^m$. ∎

Thus Theorem 6.25 yields also the following corollary.

Corollary 6.27. *Let Y be a projective variety which admits a metric with quasi-negative bisectional curvature. Then Y is not rationally dominated by a product.*

Proof. Suppose there exists a rational map $\varphi : X \longrightarrow Y$, where $X = X_1 \times \cdots \times X_k$ is compact with $\dim(X_i) < Y$. After a finite sequence of blow-ups

$$\tilde{X}^N \longrightarrow \ldots \longrightarrow \tilde{X}^1 \longrightarrow X$$

along subvarieties, one obtains from X a smooth projective subvariety $\tilde{X} = \tilde{X}^N$ and a holomorphic map $\tilde{\varphi} : \tilde{X} \longrightarrow Y$ which factorizes through φ and the blowdown to X. At each stage $\tilde{X}^i \longrightarrow \tilde{X}^{i-1}$, the exceptional divisor fibers over the blow-up locus with fiber a projective space of dimension at least one. On the other hand, by Lemma 6.26, the composition of the blowdown to $\tilde{X}^N \longrightarrow X$ with φ has to be constant along each such fiber. Proceeding inductively from N to 1, we conclude that the rational map φ was holomorphic after all.

Since Y admits a metric with quasi-negative bisectional curvature, its tangent bundle is quasi n-negative, and we obtain a contradiction to Theorem 6.25. ∎

6.3 Examples

In this section we give two examples that illustrate our results.

6.3.1 Immersions in the Albanese torus

Given a Kähler manifold which is immersed by its Albanese map, we want to understand under what conditions the induced metric has quasi-negative holomorphic bisectional curvature.

Recall that the *Albanese torus* $\mathrm{Alb}(X)$ of a Kähler manifold X is given by

$$\mathrm{Alb}(X) = \frac{H^0(X;\Omega^1)^*}{H_1(X;\mathbb{Z})},$$

where a class $l \in H_1(X;\mathbb{Z})$ is identified with $\int_l \in H^0(X;\Omega^1)^*$. Fix $x \in X$. Then

$$p \longmapsto (\alpha \longmapsto \int_x^p \alpha) \in H^0(X;\Omega^1)^*$$

is defined up to a translation $l \in H_1(X;\mathbb{Z})$, so it defines a map

$$\alpha_X : X \longrightarrow \mathrm{Alb}(X),$$

the *Albanese map* of X with basepoint x. Its differential at p is

$$\begin{aligned} D_p\mathrm{Alb} : T_pX &\longrightarrow T_p\mathrm{Alb}(X) \cong H^0(X;\Omega^1)^* \\ v &\longmapsto (\alpha \longmapsto \alpha_p(v)) = \mathrm{Ev}_v. \end{aligned}$$

Clearly, when Alb is an immersion, $\Omega^1(X)$ is globally generated. The converse is also true, as can be seen by taking the dual of the exact sequence

$$0 \longrightarrow TX \longrightarrow T\mathrm{Alb}|_X \longrightarrow \mathcal{N}_X \longrightarrow 0.$$

Assume now that α_X is an immersion. Let $q = h^{1,0}$ and $n = \dim(X) \leq q$. Let $p \in X$ and define

$$E_p = \{V \in H^0(X,\Omega^1)^* \,|\, \exists\, v \in T_pX \text{ s.t. } V(\alpha) = \alpha(v),\ \forall \alpha \in H^0(X,\Omega^1)\},$$

so that E_p is the image of T_pX under the Albanese map α_X. Define in addition

$$\begin{aligned} F_p =& H^0(X,\Omega^1)^*/E_p, \text{ and} \\ Z_p =& \{\alpha \in H^0(X,\Omega^1) \,|\, \alpha_p = 0\}. \end{aligned}$$

Then we have $F_p \cong Z_p^*$ via the non-degenerate natural pairing $V \otimes \alpha \longmapsto V(\alpha)$. Now recall from section 6.1.1 that the holomorphic bisectional curvature of a submanifold of a flat manifold is (up to normalization) $-L^\dagger \wedge L$, where L is the matrix of the second fundamental form λ. Therefore, the holomorphic bisectional curvature is negative at p if and only if for every $v \in T_pX$,

$$\left(L^\dagger \wedge L\right)(v,\bar{v}) > 0,$$

which is to say that the matrix $L(v)$ has maximal rank.

In the present case, we can write λ as

$$\begin{aligned} \lambda_p :\ T_pX &\longrightarrow \mathrm{Hom}(T_pX, Z_p^*) \\ v &\longmapsto (w \longmapsto (\alpha \longmapsto (\mathcal{L}_v\alpha)(w))), \end{aligned}$$

where $\mathcal{L}_v\alpha$ denotes the Lie derivative of α in the direction of v. Note that since holomorphic 1-forms on Kähler manifolds are closed, we have that $(\mathcal{L}_v\alpha)(w) = w(\alpha(v))$ by the Cartan formula, and $w(\alpha(v)))_p$ does not depend on the local extension one takes for v, because α vanishes at p. Hence the holomorphic bisectional curvature is negative at p if, and only if, for every nonzero $v \in T_pX$ the map

$$\begin{aligned} \lambda_p(v) : T_pX &\longrightarrow Z_p^* \\ w &\longmapsto (\mathcal{L}_v \cdot)(w) \end{aligned}$$

is injective. Note that $w \in \ker(\lambda_p(v))$ if and only if $(\mathcal{L}_v\alpha)(w) = 0$ for every $\alpha \in Z_p$. We have proved

Proposition 6.28. *Let X be a compact Kähler manifold with globally generated cotangent bundle. Then the metric induced by the immersion $\alpha_X : X \hookrightarrow \mathrm{Alb}(X)$ has negative holomorphic bisectional curvature at p if, and only if, for every $0 \neq v, w \in T_pX$ there exists a holomorphic 1-form α which vanishes at p with $0 \neq (\mathcal{L}_v\alpha)(w)$.*

6.3.2 Fano Surfaces

Now we will apply Corollary 6.27 to *Fano surfaces of lines in a cubic*, namely we will show that such surfaces are not rationally dominated by a product by showing that they admit a metric with quasi-negative holomorphic bisectional curvature.

Let V be a cubic threefold in 4-dimensional projective space, with at most one nodal singularity. We consider the surface S of lines in V, concretely

$$S = \{L \subset V \mid L \text{ is a line.}\}.$$

These surfaces are extensively studied in [17], where they are used in the proof of the non-rationality of V. A crucial ingredient in that proof is the following:

Theorem 6.29 (Tangent bundle theorem, [17])**.** *There is a commutative diagram*

$$\begin{array}{ccc} S & \xrightarrow{\alpha_S} & \alpha_S(S) \subset \mathrm{Alb}(S) \\ \Big\downarrow{\scriptstyle \iota} & & \Big\downarrow{\scriptstyle g} \\ \mathrm{Gr}(2,5) & \xrightarrow{\tilde{\rho}} & \mathrm{Gr}(2, T_0\mathrm{Alb}(S)) \end{array}$$

where $\iota : S \longrightarrow \mathrm{Gr}(2,5)$ *is the natural embedding one gets from sending a line* $L \subset V \subset \mathbb{CP}^4$ *to the* 2*-plane it defines in* $\mathbb{C}^5$*,* α *is the Albanese map (which for Fano surfaces is an embedding),* g *is the Gauss map, and* $\tilde{\rho}$ *is the isomorphism induced on the Grassmannians by an isomorphism* $\rho : \mathbb{C}^5 \longrightarrow T_0\mathrm{Alb}(S)$*.*

We make use of the fact that Fano surfaces embed in their Albanese tori, whose flat metric then restricts to a metric of non-positive holomorphic bisectional curvature on the Fano surface.[3] Our task is thus to prove the existence of a point where the holomorphic bisectional curvature is strictly negative. The idea is the following. To find a point where the holomorphic bisectional curvature is negative is to find a point where the second fundamental form is non-degenerate. But the second fundamental form is given by the derivative of the Gauss map $g \circ \alpha_S$. The commutativity of the diagram in the tangent bundle theorem says that the second fundamental form of the embedding in the Albanese variety is given by the differential of the natural embedding of the Fano surface in $Gr(2,5)$.

The holomorphic bisectional curvature is therefore negative (with respect to the metric induced by the Albanese) at a point $s \in S$ if, and only if, for every $v \in T_sS \subset T_s\mathrm{Gr}(2,5)$ the map $\lambda_v : L_s \longrightarrow \mathbb{C}^5/L_s$ defined by the identification $T_K\mathrm{Gr}(k,n) \equiv \mathrm{Hom}(K, \mathbb{C}^n/K)$ is injective. We will now determine the maps λ_v for $v \in T_sS$ when L_s is a *line of first type* (see [17, Def. 6.6]). We may assume that L_s is given by $[x_0 : x_1] \longmapsto [x_0 : x_1 : 0 : 0 : 0]$. After a linear change of coordinates in x_2, x_3, x_4, we can, as in [17, p. 308 (6.9)] write the defining polynomial F for V as

$$F(x_0, \ldots, x_4) = x_2x_0^2 + x_3x_0x_1 + x_4x_1^2 + \sum_{i>1} O(x_i^2).$$

[3] One should bear in mind that, while our argument is differential-geometric, the metric we chose is canonical and depends only on algebraic-geometric properties of Fano surfaces.

Now we consider the chart around L_s for the Grassmannian $\mathrm{Gr}(2,5)$

$$(u_2, u_3, u_4, z_2, z_3, z_4) \longmapsto \mathrm{span}\{(1, 0, u_2, u_3, u_4), (0, 1, z_2, z_3, z_4)\}.$$

In this chart, S is given by

$$\begin{aligned} u_2 + O(u_i^2, z_i^2) &= 0 \\ u_3 + z_2 + O(u_i^2, z_i^2) &= 0 \\ u_4 + z_3 + O(u_i^2, z_i^2) &= 0 \\ z_4 + O(u_i^2, z_i^2) &= 0. \end{aligned}$$

Let $G : \mathbb{C}^6 \longrightarrow \mathbb{C}^4$ be the function determined by the expression above. Then

$$\ker(D_0G) = \ker \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

so in terms of the coordinate basis, the tangent space to S at s is given by

$$T_sS = \mathrm{span}\{\underbrace{(0, 1, 0, -1, 0, 0)}_{e_1}, \underbrace{(0, 0, 1, 0, -1, 0)}_{e_2}\}.$$

In terms of the coordinate basis, the identification $T_{L_s}\mathrm{Gr}(2,5) \equiv \mathrm{Hom}(L_s, \mathbb{C}^5/L_s)$ is

$$(v_1, v_2, v_3, w_1, w_2, w_3) \longmapsto \begin{pmatrix} * & * & v_1 & v_2 & v_3 \\ * & * & w_1 & w_2 & w_3 \end{pmatrix}^t,$$

and it follows that for $v = a_1e_1 + a_2e_2 \in T_sS$,

$$\lambda_v = \begin{pmatrix} * & * & 0 & a_1 & a_2 \\ * & * & -a_1 & -a_2 & 0 \end{pmatrix}^t.$$

It is then clear that λ_v is injective for any $v \neq 0 \in T_sS$, so the holomorphic bisectional curvature at s is negative.

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, § 8, Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

Coelho, Rui

Name, Vorname

München, 09.09.2019	Rui Miguel Serrão Coelho
Ort, Datum	Unterschrift Doktorand/in

Formular 3.2