Symmetries in four-dimensional multi-spin-two field theory: relations to Chern–Simons gravity and further applications

Nicolás Lorenzo González Albornoz

München 2019

Symmetries in four-dimensional multi-spin-two field theory: relations to Chern–Simons gravity and further applications

Nicolás Lorenzo González Albornoz

Dissertation
an der Fakultät für Physik
der Ludwig–Maximilians–Universität
München

vorgelegt von
Nicolás Lorenzo González Albornoz
aus Talca, Chile

München, den 7. März 2019

Erstgutachter:	Prof. Dr. Dieter Lüst
Zweitgutachter:	Prof. Dr. Ivo Sachs
Tag der mündlichen Prüfung:	27. Mai 2019

Zusammenfassung

Diese Dissertation widmet sich der Untersuchung der Symmetrie in Theorien für Spin-2-Felder. Insbesondere beschäftigen wir uns mit der Frage, ob die vollständig nichtlinearen Theorien für masselose und massive Spin-2-Teilchen, nämlich die Allgemeine Relativitäts- und die bimetrische Gravitationstheorie, aus der Formulierung der reinen Eichtheorie abgeleitet werden können. Darüber hinaus untersuchen wir die diskreten Symmetrien, die in einer Theorie mit mehr als einer Metrik auftreten können, wenn wir eine Metrik wählen, um die Dynamik der Raumzeit zu beschreiben, während alle anderen exakt auf gleicher Augenhöhe behandelt werden.

Zu diesem Zweck betrachten wir die Chern–Simons-Theorie in fünf Dimensionen, die mit der Anti-de-Sitter-Gruppe $\mathrm{AdS}_{4+1} = \mathrm{SO}(4,2)$ ausgestattet wird. Mit der Tatsache, dass diese Gruppe isomorph zur konformen Gruppe in vier Dimensionen, C_{3+1}, ist, drücken wir die Theorie in der Basis für die konforme Algebra aus. Die Eichfelder, die den Translationen $\boldsymbol{P}_a$ und speziellen konformen Transformationen $\boldsymbol{K}_a$ entsprechen, jeweils bezeichnet durch e^a und $\imath^a$, werden dann nach der Implementierung mehrerer dimensionaler Reduktionsschema als zwei Vierbeine interpretiert. Auf der vierdimensionalen Ebene finden wir für verschiedene Schemen, die die Chern–Simons-Theorie auf eine Generalisierung der Allgemeinen Relativitäts- und der konformen Gravitationstheorie erster Ordnung reduziert. Darüber hinaus führen wir eine doppelte Chern–Simons-Theorie in fünf Dimensionen mit der Symmetriegruppe $\mathrm{SO}(4,2)\times\mathrm{SO}(4,2)$ ein. Wir brechen die Symmetrie runter auf $\mathrm{SO}(3,1)\times\mathrm{SO}(2)$ und gelangen nach einer dimensionalen Reduktion zu einer Verallgemeinerung der vierdimensionalen bimetrischen Gravitationstheorie. In allen Fällen diskutieren wir die Restsymmetrie der Wirkungen auf der vierdimensionalen Ebene.

Wir betrachten auch eine geisterfreie Multigravitationstheorie mit der physikalischen Metrik $g_{\mu\nu}$ und den Satellitenmetriken $f^{(p)}_{\mu\nu}$, wobei $p = 1, \ldots, N$. Dazu erforschen wir die diskrete Symmetrie, die entsteht, wenn der Austausch zwischen den Satellitenmetriken die Wirkung invariant verlässt und auch wenn beide Quadratwurzeln $\pm\sqrt{g^{-1}f^{(p)}}$ in der Wirkung zum gleichen Interaktionspotenzial zwischen den physikalischen und den Satellitenmetriken führen. Die entstehende globale Symmetriegruppe ist isomorph zu $S_N \times (\mathbb{Z}_2)^N$. Darüber hinaus analysieren wir das Massenspektrum der Theorie mit der diskreten Symmetrie. Wir konzentrieren uns dann auf den trimetrischen Fall, da sich der multimetrische Fall ähnlich verhält und nicht zu einer neuen Phänomenologie führt. Das masselose Spin-2-Feld $G_{\mu\nu}$ vermittelt die weitreichende Gravitationskraft der Raumzeit, auf der die massiven Spin-2-Felder $M_{\mu\nu}$ und $\chi_{\mu\nu}$ propagieren. Mit störungstheoretischen Mitteln analysieren wir die Vertices der Theorie im Hinblick auf die Spin-2-Felder. Wir finden, dass das Spin-2-Teilchen mit der kleineren Masse, $\chi_{\mu\nu}$, stabil ist und insbesondere nicht in masselose Gravitons zerfallen kann. Wir postulieren, dass dieses Spin-2-Feld einen Bestandteil der Dunklen Materie darstellen kann.

Summary

This thesis is devoted to the investigation of the symmetry in theories for spin-2 fields. In particular, we address the question whether the fully non-linear theories for massless and massive spin-2 particles —namely standard general relativity and bimetric gravity, can be obtained from a pure gauge theory formulation. Furthermore, we explore the discrete symmetries that can arise in a theory for many metrics, when we choose one metric to describe the dynamics of spacetime while keeping all others exactly on an equal footing.

To this end, we consider the Chern–Simons gauge theory in five dimensions valued in the anti-de Sitter group $\mathrm{AdS}_{4+1} = \mathrm{SO}(4,2)$. Using the fact that this group is isomorphic to the conformal group in four dimensions, C_{3+1}, we express the theory in the basis for the conformal algebra. The gauge fields corresponding to the translations $\boldsymbol{P}_a$ and special conformal transformations $\boldsymbol{K}_a$, denoted by e^a and $\imath^a$ respectively, are then interpreted as two vierbeine after implementing several dimensional reduction schemes. At the four dimensional level we find for different schemes that the Chern–Simons theory reduces to a generalisation of standard general relativity and first-order conformal gravity. Moreover, we introduce a doubled Chern–Simons theory in five dimensions with symmetry group $\mathrm{SO}(4,2) \times \mathrm{SO}(4,2)$. We break down the symmetry to $\mathrm{SO}(3,1) \times \mathrm{SO}(2)$ and after a dimensional reduction, we recover a generalisation of 4-dimensional bimetric gravity. In all cases, we discuss the residual symmetry of the actions at the 4-dimensional level.

We also consider a ghost-free multigravity theory with the physical metric $g_{\mu\nu}$ and the satellite metrics $f^{(p)}_{\mu\nu}$ where $p = 1, \ldots, N$. For this, we explore the discrete symmetry that arises when the interchange between the satellite metrics leaves the action invariant and also when both square roots $\pm\sqrt{g^{-1}f^{(p)}}$ in the action lead to the same interaction potential between the physical and the satellite metrics. The global symmetry group that arises is isomorphic to $S_N \times (\mathbb{Z}_2)^N$. Moreover, we analyse the mass spectrum of the theory with the discrete symmetry. We focus then in the trimetric case since the multimetric case behaves similarly and does not lead to new phenomenology. The massless spin-2 mode $G_{\mu\nu}$ mediates the long-range gravitational force of the spacetime on which the massive spin-2 modes $M_{\mu\nu}$ and $\chi_{\mu\nu}$ propagate. By computing the perturbative expansion, we analyse the vertices of the theory in terms of the spin-2 modes. We find that the spin-2 particle with the smaller mass, $\chi_{\mu\nu}$, is stable and in particular, it cannot decay into massless gravitons. We postulate that this spin-2 field can be a component of dark matter.

Acknowledgements

Writing this thesis would not have been possible if I had not met such wonderful and interesting people in throughout my life.

For being a part of my life I thank to mi Mamá, mi Papá, Dani, Nata, Marti, Amparo, Dieguito, Jacinta, Estela, Álvaro, Diego, la Ita, tía Norma, tía María, Seba, Cami, my grandparents, my aunts, my uncles and my cousins.

I would like to express my sincere gratitude to my Doktorvater, Prof. Dieter Lüst, for the continuous support of my doctoral studies, for believing in me, for his motivation and immense knowledge both as a person and scientifically.

Besides my advisor, I would like to thank the rest of my thesis committee: Prof. Ivo Sachs, Prof. Stefan Hofmann and Prof. Otmar Biebel for accepting to participate in my examination and for their kindness.

I am specially thankful to Angnis Schmidt-May for her continuous support, her nice way to collaborate and her initiative.

I thank to the people at the Lehrstuhl Lüst of the Arnold Sommerfeld Center for Theoretical Physics, the Faculty of Physics and the Max Planck Institute for Physics for their kindness and support during my doctoral studies. Special thanks to Monika Kürzinger, Oleg Andreev, Andrea Orta, Livia Ferro, Matteo Parisi, Fabrizio Cordonier-Tello, Michael Haack, Erik Plauschinn, Valenti Vall Camell, Barbara Podolski, Felix Rudolph, Marvin Lüben, George Zoupanos, Enrico Andriolo, Marc Syväri, Brage Gording, Vero Errasti Diez and David Damgaard.

For their generosity and wonderful presence, I thank to the people I met since I arrived in Germany. Special thanks to Andrés Neumann, José Pedro Cornejo, Baraa Khamees, Valentina Decembrini, Salah Mabrouk, Klaus Alt, Wael Shehada, Dibya Chakraborty, Ignacio Arregui, Alejandra Armenta, Sarah Schkölziger, Camila Castro, Chu-Yi Chan, Stefan Jansky, Marco Paes, Ricardo Borrmann, María Hotes, Sibylle Wahl, María Andrea Leyton, Sebastián Paillán Störtzer, Friederike Merkel, Lavinia Picollo, Andrea Pisani, Luis Rosa, Eduardo Rodríguez, Daniel Carriazo, Fabian Montecinos, Jorge Olivos, Yoav Zagdon, the Mystery Bouldering Person and Slavko Kulli.

I thank to the people I met in Chile for their wonderful presence and for their encouragement when I decided to study in Germany. Thank you Yercko Stari, Francisco González, Bear Muñoz Pairikán, Nicolás Tordecilla, Silvia Gutiérrez, Felipe De La Cuadra, Eduardo Gajardo, Camila Reyes, Alfonso Gómez, Juan Pablo Terán, José Castro, Matias Lara, Juan Olivares, Nancy Moya, Raul Herrera, Darwin Nahuelpán, Miguel Campos, Cristofer Jiménez, José Vargas, Álvaro Cuevas, Diana Hidalgo, Patricio Muñoz, Eloisa Morales, Natalia Cisternas, Sebastián Casanueva, Angie Lagos, Marcelo Aguilar, Yeissy Jara, Cecilia Barrios and Theresa Buberl.

Special thanks to the Prof. Patricio Salgado Arias for his support and immense knowledge both as a person and scientifically. Thanks also to Gustavo Rubio and Sebastián Salgado with whom we spoke tremendously important things about life and physics.

I thank to my mentors for the kind support, for believing in me and their immense knowledge: Álvaro Rojas, Mario Quezada, Javier Frigerio, María Eugenia Marín, Vicente Sanhueza, Flavio Figueroa, Pablo Oliva, Myrna Wallace, Juan Crisóstomo, Guillermo Rubilar and Karin Richter.

Thanks to the people of the DAAD and CONICYT for their support. My doctoral studies have been funded by the bilateral DAAD-CONICYT grant 72150534.

Curriculum vitae

During my doctoral studies at the Arnold Sommerfeld Center for Theoretical Physics, I have fulfilled the following activities:

Teaching assistance

Nov 2017 - Mar 2018: Rechenmethoden der Theoretischen Physik
Lecture of Prof. Dr. Jan von Delft
Ludwig–Maximilians University of Munich

Publications

Dec 2018: *4D spin-2 fields from 5D Chern–Simons theory* [1]
N. L. Gonzalez Albornoz, D. Lüst, S. Salgado, A. Schmidt-May
Journal of High Energy Physics

Jan 2018: *Dark matter scenarios with multiple spin-2 fields* [2]
N. L. Gonzalez Albornoz, A. Schmidt-May, M. von Strauss
Journal of Cosmology and Astroparticle Physics

Apr 2016: *Generalized Galilean algebras and Newtonian gravity* [3]
N. L. González Albornoz, G. Rubio, P. Salgado, S. Salgado
Physics Letters B

Talks

Jan 2018: *4D Einstein–Hilbert from 5D Chern–Simons action*
Group Seminar
Arnold Sommerfeld Center for Theoretical Physics

Nov 2017: *Dark matter in multimetric gravity*
VI Postgraduate Meeting On Theoretical Physics
University of Valencia

Jul 2017: *Topological AdS black holes in massive gravity*
Lunch Seminar
Arnold Sommerfeld Center for Theoretical Physics

May 2016: *Expansion of Lie algebras and Einstein–Chern–Simons gravity*
Lunch Seminar
Arnold Sommerfeld Center for Theoretical Physics

Notation and conventions

In this thesis we repeatedly use the covariant index notation for tensors and connections defined in $(p+q)$-, $(p+q+1)$- and $(p+q+2)$-dimensional spaces. The following type of letters are used for indices of (pseudo)-tensors and connections under diffeomorphisms:

$$\begin{aligned} \mu, \nu, \ldots &= 1, \ldots, p+q \,, \\ m, n, \ldots &= 1, \ldots, p+q+1 \,, \\ M, N, \ldots &= 1, \ldots, p+q+2 \,, \end{aligned}$$

while for the indices of (pseudo)-tensors and connections under local Lorentz, (anti)-de Sitter and conformal transformations we use

$$\begin{aligned} a, b, \ldots &= 1, \ldots, p+q \,, \\ A, B, \ldots &= 1, \ldots, p+q+1 \,, \\ I, J, \ldots &= 1, \ldots, p+q+2 \,, \end{aligned}$$

respectively. The following table summarizes the symmetry groups mentioned in this manuscript:

Name	Symbol	Group	Killing metric ($p+q=3+1$)
Poincaré	P_{p+q}	$\mathrm{ISO}(p,q)$	
Lorentz	L_{p+q}	$\mathrm{SO}(p,q)$	$\eta_{ab} = \mathrm{diag}(+,+,+,-)$
translations	T_{p+q}	$\mathrm{T}(p,q)$ (abelian)	
anti-de Sitter	AdS_{p+q}	$\mathrm{SO}(p,q+1)$	$\eta_{AB} = \mathrm{diag}(+,+,+,-,-)$
de Sitter	dS_{p+q}	$\mathrm{SO}(p+1,q)$	$\eta_{AB} = \mathrm{diag}(+,+,+,-,+)$
Conformal	C_{p+q}	$\mathrm{SO}(p+1,q+1)$	$\eta_{IJ} = \mathrm{diag}(+,+,+,-,\mp,\pm)$

Contents

Chapter 1

Introduction

In this chapter we review the main aspects of the Lagrangian formulation of Einstein's general relativity and the actions for the first-order formalism of gravity. We then discuss the importance of symmetries in physical theories and motivate a gauge formulation for gravity. Next, some issues of general relativity are mentioned to motivate the generalisation of general relativity and the quest for modified gravity theories. We then present the generalisation of gravity to higher orders in the curvature and to arbitrary dimensions and we discuss its connection to Chern–Simons theory. The main aspects of the Lagrangian formulation of spin-2 field theory are presented afterwards and finally, we discuss how spin-2 field theory can be connected with other gauge formulations of gravity and whether the dark matter ingredient can be a massive spin-2 field. At the end of this chapter we present an item-list with the contents of this manuscript.

1.1 Standard general relativity

The main role of theoretical physics is to describe the motion of particles, fields and systems in terms of their energy and momentum. Standard general relativity is the theory for the motion of spacetime. The field equations for spacetime in absence of matter are known as the Einstein equations [4, 5] in vacuum

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 0 \,. \tag{1.1}$$

Here $g_{\mu\nu}(x)$ is the metric tensor of the manifold spacetime and it allows us to define distances and angles on it. We also have the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R, both expresable in terms of the Riemann tensor $R^{\mu}{}_{\nu\lambda\rho}$ (see § A.1 for the definitions), which describes the curvature of the manifold. The Riemann tensor depends directly on the Levi-Civita connection Γ and its first derivatives. Furthermore, Γ depends on the metric and on its first derivatives. Thus, the Einstein equations are second-order partial differential equations for the metric. The corresponding action for the Einstein equations is known as the Einstein–Hilbert action [6]. It has the metric as covariant dynamical field or the vielbein one-form $e^a(x)$ when we use the language of differential forms. In $D = p + q$ dimensions the action is given by

$$S_{\text{EH}}[g] = \int \mathrm{d}^D x \, \sqrt{|g|} \, g^{\mu\nu} R_{\mu\nu}(g) \,, \tag{1.2a}$$

$$S_{\text{EH}}[e] = \tfrac{1}{(D-2)!} \int \epsilon_{a_1 \ldots a_D} \, R^{a_1 a_2}(e) \wedge e^{a_3} \wedge \cdots \wedge e^{a_D} \,, \tag{1.2b}$$

respectively for the metric $g_{\mu\nu}$ or the vielbein form $e^a = e^a{}_\mu \mathrm{d}x^\mu$ where $g_{\mu\nu}(e) = e^a{}_\mu e^b{}_\nu \eta_{ab}$, with η_{ab} being a diagonal Minkowski metric with p entries $+1$ and q entries -1 and also where $R^{ab}(e)$ is defined in eq. (A.9). The actions above lead to the same dynamics, however, the tensor $g_{\mu\nu}$ has $D(D+1)/2$ independent fields while $e^a{}_\mu$ has D^2. The action (1.2b) has the extra symmetry of local Lorentz transformations $e'^a = \Lambda^a{}_b(\theta^{ab}) \, e^b$ where $\theta^{ab} = -\theta^{ba}$ are $D(D-1)/2$ parameters. The number of fields of $S_{\text{EH}}[e]$ minus the Lorentz symmetries gives the true numbers of fields as in $S_{\text{EH}}[g]$.

Besides distances and angles, parallelism is another geometrical concept that tells us how we transport a tangent vector along a curve on the manifold. To define this mathematically we need a connection, an object that in principle does not depend on the metric. The first-order formalism of standard general relativity consists in considering independently connection and metric. In the tensor language one considers an independent connection $\Gamma^\mu{}_{\nu\lambda}$ and in the differential forms language one considers the so-called spin connection $\omega^{\alpha\beta}{}_\mu$ defined through

$$\Gamma^\rho{}_{\mu\nu}(\omega, e) = e_a{}^\rho \partial_\mu e^a{}_\nu + e_a{}^\rho \omega^a{}_{b\mu} e^b{}_\nu \,. \tag{1.3}$$

This equality is the bridge between the tensors and forms language and it is known as the vielbein postulate because one cannot prove it but one has to include it as axiom. In the first order formalism the actions

$$S_{\mathrm{EP}}[\Gamma, g] = \int \mathrm{d}^D x \sqrt{|g|}\, g^{\mu\nu} R_{\mu\nu}(\Gamma) \,, \tag{1.4a}$$

$$S_{\mathrm{EC}}[\omega, e] = \tfrac{1}{(D-2)!} \int \epsilon_{a_1 \ldots a_D} R^{a_1 a_2}(\omega) \wedge e^{a_3} \wedge \cdots \wedge e^{a_D} \,, \tag{1.4b}$$

are known as the Einstein–Palatini [7] and Einstein–Cartan [8–13] respectively. For the Einstein–Palatini theory, the equation of motion for Γ implies that the connection Γ is the Levi-Civita connection, therefore by integrating Γ out one recovers Einstein–Hilbert theory. For the Einstein–Cartan theory, the equation of motion for ω and the vielbein postulate imply that the connection Γ is the Levi-Civita connection. A further generalisation of the Einstein–Cartan theory is given by the action introduced by Plebanski, for which the fundamental field that carries geometry is the set of two-forms $B^{ab} = e^a \wedge e^b$ [14].

All the theories above consider in addition that the metric is covariantly constant. This means that, given a connection Γ, then its related covariant derivative acting on the metric vanishes, i.e. $\nabla^{\Gamma}_{\mu} g_{\nu\lambda} = 0$, where the action of the covariant derivative ∇^{Γ}_{μ} on an arbitrary range-2 tensor $V_{\mu\nu}$ is defined in eq. (A.3a). This equation is called the compatibility condition and it demands that the metric has to be covariantly constant, i.e. from a point P, its components must not vary when transported along curves arriving to point Q. The compatibility condition is generally assumed since it implications are directly measurable: if we have two cubes of the same volume —notice that in order to measure their volumes we have to have used a metric— at P, they should have the same volume at Q.

As we explain in § 1.3, standard general relativity, although highly successfully supported by local tests and explaining numerous phenomena, fails when applied in quantum contexts or when compared to the predictions of quantum field theory. This motivates the search for modified or generalised gravity theories. As described in detail in § 1.5, in this dissertation we study both generalisations and modifications of standard general relativity.

1.2 Symmetries

One of the powerful aspects of physics is the capability to describe motion. Let us consider a sphere in a room, hanging by a thread from the ceiling. We can make the sphere rotate and that will be for sure a motion that we will be able to describe; in this case with classical mechanics. If we turn the lights off, then we make rotate the sphere and then we turn the lights on again, nobody will be able to say what was the motion suffered by the sphere, because throughout the motion the sphere is undistinguishable. The system has a symmetry and it prevents us to know information about its motion. As we already mentioned, to describe the motion of a system in terms of its momentum and energy is one of the main tasks in theoretical physics, but symmetry, apparently, prevents us to do it. We will see, however, that systems with symmetry are generically easier to describe.

In deterministic terms, one says that any motion in nature can be associated with a particular Lagrangian function $\mathcal{L}(\phi)$. This Lagrangian will eventually have some symmetries under the redefinition $\phi \to \phi'$ such that $\mathcal{L}(\phi')$ looks exactly as $\mathcal{L}(\phi)$. The Lagrangian that describes the geometry of the 2-sphere: $\mathcal{L}(x, y, z) = x^2 + y^2 + z^2$ has the symmetry

$$(x, y, z)^{\mathrm{T}} \longrightarrow (x', y', z')^{\mathrm{T}} = R(\theta^x, \theta^y, \theta^z)(x, y, z)^{\mathrm{T}} \,, \tag{1.5}$$

with $R(\theta^x, \theta^y, \theta^z)$ being a 3-parameter matrix in SO(3) and where T denotes the transpose. The geometric interpretation of this invariance is that, however we rotate the sphere, we will always see the same sphere.

Since along the direction of a symmetry it occurs always the same physics, we do not need to extend a coordinate in that direction to describe what happens. The equations of motion do not have to be integrated in that redundant coordinate and therefore, the calculations are significantly simpler. Identifying the symmetries of a physical system is thus relevant and being able to do it is connected with the mathematical structure of the system itself: when we have already considered all the sufficient coordinates to describe the system by implementing all the symmetries, the number of coordinates is precisily its number of degrees of freedom. A crucial insight in the mathematics of symmetries was made by Noether [15], who realized that for every symmetry that we can find in a physical system, there is then for a certain variable a conservation law. In a next level of abstraction: it is not only that we find symmetries in physical systems but also in physical theories themselves. For example, the Einstein–Hilbert Lagrangian is invariant under the general coordinate transformation $x^\mu \to x'^\mu$, as long as x'^μ depends smootly on the old set of coordinates x^μ and as this transformation is invertible. This invariance relates to the conservation of the stress-energy tensor. Another well-known examples are Newtonian and relativistic theory of mechanics. For those cases the homogeneity of space and time is related to the conservation of momentum and energy respectively.

A particular mathematical treatment for the symmetries of an action is the so-called gauge formulation. A gauge theory is a physical theory for which the action is invariant under some field transformations $A_\mu \to A'_\mu$, which can be derived from the symmetry rules of a certain Lie group[1]. For example, quantum electrodynamics is invariant under the field transformations induced by the Lie group $\mathrm{U}(1)$. The field content of a quantized gauge theory is made up by gauge bosons. The standard model of particle physics is invariant under the action of the Lie group $\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)$ and the gauge bosons correspond to the photon γ, the bosons $W^\pm$ and Z^0 and the gluons λ^i, $i = 1,\ldots,8$. Both mentioned theories have an action that can be expressed as a Yang–Mills action

$$S_{\mathrm{YM}}[\boldsymbol{A}] = \int \langle \boldsymbol{F}, \star \boldsymbol{F} \rangle \tag{1.6}$$

valued in the respective Lie groups. Here $\boldsymbol{F} = F^a{}_\mu\, \mathrm{d}x^\mu \otimes \boldsymbol{T}_a$ is the field strength associated with the gauge field $\boldsymbol{A}$ and $\langle \boldsymbol{T}_a, \boldsymbol{T}_b \rangle$ is the Killing metric of the Lie group. The usual 4-dimensional Einstein theory for gravity is, however, a theory that cannot be formulated as Yang–Mills gauge theory. We discuss more about this fact in § 1.3. Since gravity *à la* Einstein is not renormalizable and given that the gauge formulation of theory seems to be the key for its renormalization, we would like to formulate gravity as a gauge theory. It turns out though, that $(2n+1)$-dimensional gravity with cosmological constant plus

[1] In this manuscript, when there is no possible ambiguity, we refer indistinctly to the Lie algebra of a Lie group and to the Lie group itself. This we do in a context where the relevant meaning is the symmetry of a gauge theory. If we are studying the commutation relations we refer to the symmetry as the Lie algebra.

higher-order curvature terms can be formulated as the Chern–Simons action

$$S_{\mathrm{CS}}[\boldsymbol{A}] = \int Q^{(2n+1)}(\boldsymbol{A}) \tag{1.7}$$

for the group SO$(2n, 2)$. Here $Q^{(5)}$ is the Chern–Simons form of order 5 [16]. Against what one could have expected, 4-dimensional standard general relativity is not invariant under the gauge transformations induced by the Poincaré group (see § 4.1 for a detailed explanation as well as ref. [17]) and, therefore, we ask ourselves if it can be formulated a pure gauge theory in the same spirit as the Yang–Mills approach for the standard model. In this thesis, we deal with the formulation of 5-dimensional geometries as Chern–Simons actions and we study their connection with gravitational fields.

1.3 Issues with gravity

In the current state of theoretical and experimental physics research there are many problems that suggest that we should consider a more generalised theory of gravity. Some of those are: the cosmological constant problem, the nature of dark matter, why inflation happened, how to formulate quantum gravity. Also, given that the fundamental interaction of the standard model can be formulated as a Yang–Mills gauge theory, the pure gauge formulation of gravity has been object of study in the last decades, although without success. Up to the community knowledge, 4-dimensional standard general relativity cannot be formulated as the gauge theory for some Lie group (see e.g. refs. [17, 18]).

Furthermore, the search for modified gravity theories, i.e. generalised gravity theories satisfying the correspondence principle by leading to standard general relativity at some limit and that pass the small scale local tests of it has been an active subject of research in the last century in order to solve this queries: examples of modified gravity theories are massive gravity *à la* Fierz–Pauli [19], $f(R)$ gravity [20], Horndeski's theory (which generalises many other scalar-geometry interactions theories) [21], modified Newtonian dynamics [22], massive gravity *à la* de Rham–Gabadadze–Tolley [23, 24] and bimetric gravity [25], among others (see refs. [26–30] for detailed reviews).

To understand more: in the case of massive gravity it could be that the new parameter of the theory, i.e. the mass of the graviton m_{g}, adds desirable effects and that it solves the problem for dark matter. Other example; in the case of quantum gravity one main problem of the attempts to construct a quantum theory for the spacetime is that quantum Einstein gravity is not renormalizable when doing perturbation theory [31]. One might then have the hope that a quantum modified gravity was indeed renormalizable. As it occurs in the standard model, the gauge formulation *à la* Lie of the current and successful quantum theories of particle physics seems to be the key for their renormalizability and thus, formulating Einstein —or modified— gravity as a pure gauge theory has been subject of study since Witten formulated the 3-dimensional version of the Einstein–Cartan Lagrangian as a Chern–Simons 3-form valued in the 3-dimensional version of the Poincaré group [32].

A Chern–Simons form valued in Lie group is gauge invariant under the action of the group by definition [17]. This makes Chern–Simons theories attractive: their pure gauge invariance. The exterior derivative of a Chern–Simons form is by definition proportional to the trace of a polinomial of the two-form curvature $\boldsymbol{F}$ and therefore they are only Chern–Simons forms of odd order. One particular complication of this fact is that we cannot formulate the action of a 4-dimensional theory as the integral of some Chern–Simons form and, up to now, there is no consistent formulation of the 4-dimensional Einstein–Cartan theory with or without cosmological constant as a gauge theory —and that includes any attempt to formulate it as a Yang–Mills theory action. In other words, we do not know a set of gauge transformations $\delta_{\omega,e}$ induced by a Lie group, such that

$$\delta_{\omega,e} \int \epsilon_{abcd}\, R^{ab}(\omega) \wedge e^c \wedge e^d = 0\,. \tag{1.8}$$

Motivated by this, we can ask ourselves whether the solutions space of 4-dimensional gravity is a subset of a higher dimensional theory that can be formulated as a pure gauge theory. In this thesis we positively answer this question by showing that there is a 5-dimensional Chern–Simons theory on which we can restrict fields to get generalisations of standard general relativity as well as modified gravity theories.

1.4 Lanczos–Lovelock & Chern–Simons

Einstein–Hilbert (and Einstein–Cartan) theory in D-dimensions has an associated action which is linear in the Lorentz curvature R^{ab} and the rest is a factor of $D-2$ times the vielbein. Both ingredients are contracted with the invariant tensor of the Lorentz algebra SO(p,q), the Levi-Civita pseudo-tensor. Lanczos–Lovelock theory is the generalisation of the Einstein–Hilbert (and of Einstein–Cartan) theory formulated as the most general polynomial in R^{ab} and e^a contracted with the Levi-Civita pseudo-tensor [33–35]. For example, in $p+q=2+1$ dimensions one uses the invariant tensor ϵ_{abc} to construct the polynomial

$$S_{\text{LL}}[\omega,e]=\int \epsilon_{abc}\Big(a_{11}\,R^{ab}(\omega)\wedge e^c+a_{03}\,e^a\wedge e^b\wedge e^c\Big)\,. \tag{1.9}$$

Here a_{mn} are parameters going with the term of m order in R^{ab} and n order in e^a. The term corresponding to a_{03} is the action for the cosmological constant in 3 dimensions and, in D dimensions, it corresponds to the term of order D in the vielbein. In 4 dimensions the Lanczos–Lovelock action reads

$$\begin{aligned}S_{\text{LL}}[\omega,e]=\int \epsilon_{abcd}\Big(&a_{20}\,R^{ab}\wedge R^{cd}\\ &+a_{12}\,R^{ab}\wedge e^c\wedge e^d+a_{04}\,e^a\wedge e^b\wedge e^c\wedge e^d\Big)\,.\end{aligned} \tag{1.10}$$

This action contains also the cuadratic term in the curvature which is known as the Gauss–Bonnet term and it is topological by means of the Stokes' theorem, since it can be always expressed as a boundary term.

In odd dimensions $p+q=2n+1$ something highly remarkable happens: for a special choice of the coefficients a_{mn}, the Lanczos–Lovelock theory can be then formulated as a Chern–Simons action (see eq. (1.7)) valued in the AdS group in $2n+1$ dimensions, $\text{AdS}_{2n+1}\simeq \text{SO}(2n,2)$ [35]. The gauge connection is denoted as $\boldsymbol{A}=\frac{1}{2}\omega^{AB}\boldsymbol{J}_{AB}=\frac{1}{2}\omega^{ab}\boldsymbol{J}_{ab}+e^a\boldsymbol{P}_a$, respectively in the bases $\{\boldsymbol{J}_{AB}\}$ and $\{\boldsymbol{J}_{ab},\boldsymbol{P}_a\}$. The commutation rules of the AdS generators are specified in eq. (2.17). To make the Lanczos–Lovelock and the Chern–Simons theories coincide, one identifies $\omega^{a,p+q+1}$ with the vielbein e^a and ω^{ab} with the spin connection. For example, in three dimensions the action is the one obtained by Witten which is the action of eq. (1.9) with $a_{11} = 1$ and $a_{03} = \Lambda/3$. In 5 dimensions one has

$$S_{\rm CS}[\omega, e] = \int \epsilon_{abcde}\Big(R^{ab}(\omega)\wedge R^{cd}(\omega)\wedge e^e - \tfrac{2}{3\ell^2}R^{ab}(\omega)\wedge e^c\wedge e^d\wedge e^e + \tfrac{1}{5\ell^4}e^a\wedge e^b\wedge e^c\wedge e^d\wedge e^e\Big)\,. \tag{1.11}$$

The latter and the generalisation to any odd dimension was first obtained by Chamseddine [36]. The 5-dimensional Chern–Simons theory for the gauge group SO(4, 2) has many interesting properties. In refs. [37, 38] it was found that the number of degrees of freedom of the theory depends on the location in phase space and the same occurs when one goes to the generalisation using p-form gauge connections instead of the usual one-form [39]. In ref. [40] the holographic description of this Chern–Simons theory, as well as its Weyl anomaly, were derived: this is, by computing the vacuum expectation value of the trace of the stress-energy tensor for the conformal field theory by means of holography. They detected that only the type-A anomaly emerges and not the type-B[2], which is rather an unusual behavior.

In the AdS algebra (2.17) we have non-commutative translations $\boldsymbol{P}_a$ and therefore the gauge theory differs from a Poincaré gauge formulation. This kind of theories are attractive considering that the Poincaré invariance P_{2n+1} is obtained from the AdS invariance, at the level of the algebra and the action by means of an Inönü-Wigner contraction [42–44]. Using this fact, one can prove that the $(2n + 1)$-dimensional Chern–Simons Poincaré invariant gauge theory becomes [45]

$$S = a_{n1}\int \epsilon_{a_1\ldots a_{2n}}\, R^{a_1a_2}(\omega)\wedge\cdots\wedge R^{a_{2n-1}a_{2n}}(\omega)\wedge e^e\,. \tag{1.12}$$

Although this action is non-linear in the curvature it leads to first-order differential equations for the spin connection and the vielbein. It differs however from the Einstein–Cartan action and the question is, whether there is a possible way to formulate standard general relativity as a gauge theory for the Poincaré or a more general Lie group. Up to now, it is has been not possible to formulate 4-dimensional standard general relativity —or a modified 4-dimensional standard general relativity theory— as a gauge theory for the Poincaré group

[2]This classification of conformal anomalies was first suggested in ref. [41].

and, even worse, the Einstein–Cartan action itself is not invariant under the gauge transformations induced by the Poincaré group when the gauge connection is chosen as $\boldsymbol{A} = \frac{1}{2}\omega^{ab}\boldsymbol{J}_{ab} + e^a\boldsymbol{P}_a$, for commutative translations $\boldsymbol{P}_a$ (see § 4.1.2).

Furthermore, we can ask ourselves whether it is in general possible to formulate a 4-dimensional spin-2 field theory in a pure gauge formalism. In ref. [46] there was an insight to formulate 5-dimensional gravity for the natural generalisation of Chern–Simons actions, namely via transgressions actions [16] —basically the difference of two Chern–Simons actions which share the symmetry gauge group— for which they study solutions that consider a reference geometry associated with the presence of the extra Chern–Simons action.

In this thesis we gauge the 4-dimensional conformal group —with *to gauge* meaning to express a gauge connection and its associated gauge transformations in the basis of the Lie algebra associated with the Lie group associated with a gauge symmetry— using a certain parametrized conformal algebra

$$\mathrm{C}_{3+1}(M,\gamma) \simeq \mathrm{SO}(4,2)\ , \tag{1.13}$$

where M is a matrix in the two-dimensional general linear group over $\mathbb{C}$ and γ is a real or pure-imaginary parameter. In this particular basis we define a gauge connection containing two one-form vielbeine and we construct an invariant action under the gauge transformations induced by the group of eq. (1.13) and where the Lagrangian is formulated as a Chern–Simons form. We study the double vielbein field content of the theory and the relation between this theory to bimetric and further spin-2 field theories for gravity.

1.5 Spin-2 fields

In the last decades, physical theories including massive spin-2 fields have become subject of attention in particle physics and cosmology [26, 30]. Spin-2 particles with mass are a natural extension of the standard model-general relativity description: the standard model contains massless and massive particles from spin-0 to spin-1 and general relativity describes a massless spin-2 mode.

The first attempt for a linear massive spin-2 field theory was made by Fierz and Pauli [19] confronting, however, bad behaviors when taking the mass of the spin-2 mode to be zero and also ghost instabilities when going to the non-linear level. The problem was solved in the past two decades when a set of theories was proven to be free of ghost instabilities at any order in the perturbative expansion [23, 24, 47–51]. This set of theories are fully non-linear massive gravity and bimetric gravity. Both contain a massive mode that couples to standard model matter thus it mediates gravitational interactions. The mass of the spin-2 field is constrained to be small since the gravitational force is long-ranged.

The fully non-linear massive gravity action consists of a dynamical tensor field $g_{\mu\nu}$ with its own Einstein–Hilbert-like term. In addition, it has a second (fixed) reference metric $f_{\mu\nu}$, which is non-dynamical but it appears interacting with $g_{\mu\nu}$ in the potential (see § 3.4 for further details). This theory has, however, a lack of consistent cosmological solutions and one had to go further with spin-2 interactions theory. It was realized that if one adds the Einstein–Hilbert-like term for $f_{\mu\nu}$ to the action one does not reintroduces ghosts [25, 50] and that cosmological solutions are better behaved [52]. This results in a bimetric theory for gravity, describing nonlinear interactions of massless and massive spin-2 fields and their couplings to standard model matter [53].

Conformal gravity as pursued by Weyl [54, 55] and formulated by Bach [56] consists of an action for 4-dimensional gravity that is invariant under rescalings of the metric (see § (4.2) for further details). Since it is a theory for the metric of the spacetime, conformal gravity can be seen as spin-2 field theory. Connections between bimetric gravity have been studied e.g. in ref. [57]. Moreover, conformal gravity has the feature that it can be formulated through an auxiliary action containing a second dynamical vielbein field. By integrating out the extra vielbein one gets the original conformal gravity action. The auxiliary action is also invariant under Weyl rescalings. In ref. [1] we pointed out that conformal gravity can be obtained from first-order bimetric gravity by restricting both spin connections to be equal and related to a certain combination of the vielbeine through the vielbein postulate.

1.5.1 Chern–Simons and spin-2 fields

In this dissertation we study the well-known 5-dimensional Chern–Simons gauge theory for the group $SO(4,2)$. Its interpretation as a 5-dimensional gravity theory has been carried out, e.g., in refs. [18, 36, 58]. Inspired by the connections between Lanczos–Lovelock and Chern–Simons theory, we investigate the possibility of deriving a 4-dimensional spin-2 field theory from a pure gauge formulation in five dimensions. This setup would involve a dimensional reduction or truncation from the gauge theory. Kaluza–Klein-like reductions of Chern–Simons actions have been carried out in refs. [59, 60] and also, other types of dimensional reductions were proposed in refs. [61–66]. To our purpose, we present several unexplored dimensional reduction schemes that reveal relations between 5-dimensional Chern–Simons and 4-dimensional spin-2 theories, including general relativity and generalisations of conformal and bimetric gravity.

To be more precise, we explore these relations which have the following origin: the 5-dimensional AdS group $\mathrm{AdS}_{4+1} \simeq SO(4,2)$ is generated by the AdS rotations $\boldsymbol{J}_{AB}$ and AdS translations $\boldsymbol{T}_A$ with $A, B = 1, \ldots, 5$. This group is isomorphic to the 4-dimensional conformal group C_{3+1} which is generated by Lorentz rotations $\boldsymbol{J}_{ab}$, translations $\boldsymbol{P}_a$, conformal transformations $\boldsymbol{K}_a$ and dilations $\boldsymbol{D}$ where $a, b = 1, \ldots, 4$. Once we have performed the dimensional reduction, the generators $\boldsymbol{J}_{ab}$ will induce the appearence of the 4-dimensional spin connection and $\boldsymbol{P}_a$ together with $\boldsymbol{K}_a$ will give rise to two vierbein fields. Hence, in general we do not arrive at 4-dimensional general relativity but theories with two spin-2 fields.

For this aim we analyse the gauge algebra $SO(4,2)$ in different bases, all of them parametrized by the matrix M and the parameter γ, as mentioned at the end of § 1.4. In that way we identify the isomorphism to the algebra C_{3+1}. Although the 5-dimensional Chern–Simons actions written in different bases are all related by linear field transformations, the dimensional reduction schemes we use are basis-dependent and thus they lead to inequivalent 4-dimensional theories. As a general behavior, the truncation process breaks down the gauge symmetry to the Lorentz group $SO(3,1)$, which is the symmetry of the local Lorentz transformations. We recover with in this form the following 4-dimensional spin-2 field theories.

First, we carry out a simple dimensional reduction after having taken the Inönü–Wigner contraction that leads SO(4, 2) to ISO(3, 2) or ISO(4, 1). We obtain the Einstein–Cartan theory in four dimensions plus an action which involves a torsional Lorentz-breaking term. This term can be removed by restricting a field at the 5-dimensional Chern–Simons level or considering the case where the torsion is equal to zero. Second, without having taken Inönü–Wigner limit, a similar truncation of the Chern–Simons action gives first-order formulation of 4-dimensional conformal gravity. From a different setup, this action was first obtained by Kaku, Townsend and van Nieuwenhuizen in ref. [67] and, as well as conformal gravity *à la* Weyl, it has Weyl dilation invariance. Our procedure explains how this gauge symmetry originates from the SO(1, 1) symmetry present in the commutation relations of the gauge algebra (a rotation of the generators that leaves the commutation relations invariant). We refer to this symmetry as the Weyl rotation invariance (see § 4.3.2), because it acts on the vierbein fields as a continuous rotation in the same spirit of a Weyl dilation (see § 4.2 for more information about Weyl dilations).

Furthermore, we consider a dimensional reduction scheme that introduces one warp functions for each vierbein. After integrating along the warp direction the Chern–Simons action leads to an with a set of free parameters, distinctly to the previous case. This result corresponds to a generalisation of conformal gravity in the Cartan formalism. By last, we consider two copies of the Chern–Simons action which has a SO(4, 2) × SO(4, 2) gauge symmetry. By making the field content interdependent the gauge symmetry breaks to SO(3, 1) × SO(2). We then dimensionally reduce this theory to obtain a generalised bimetric theory with additional derivative terms. These novel kinetic terms can be removed through a fields restriction at the 5-dimensional level. In that case one obtains the bimetric gravity *à la* Hinterbichler and Rosen in the Cartan formalism. The latter procedure breaks the gauge symmetry to SO(3, 1). The generalised bimetric theory contains free parameters that can be chosen such that we recover another bimetric theory for which the Weyl rotation invariance typical of first-order conformal gravity is present.

1.5.2 Dark matter and spin-2 fields

By solving the mass eigenvalue problem for bimetric gravity one can calculate precisely what the field content is of such a theory. It turns out that there is one massless spin-2 mode $G_{\mu\nu}$ and one massive spin-2 mode $M_{\mu\nu}$, whose mass can be expressed in terms of the interaction parameters of the potential [25]. The massless mode can mediate a long-range gravitational force and this makes the constraints on the spin-2 mass less restrictive. The gravitational interactions of general relativity can be obtained in bimetric theory with the same precision for any value of the mass of the massive mode [68–71]. The way to achieve this is by decreasing the coupling between the $M_{\mu\nu}$ field and the stress-energy tensor $T^{\mu\nu}$ and, remarkably, this leaves the coupling between the massive and massless modes with the same strength. In this way the massive mode continues gravitating with the same strength as the fields of the standard model. The 3-metric case —so-called, trimetric gravity— contains one massless mode $G_{\mu\nu}$ and two massive modes: $M_{\mu\nu}$ and a lighter one, $\chi_{\mu\nu}$.

With the mentioned facts we could think that a vestige amount of spin-2 massive particles act as dark matter. The evidence of dark matter comes from astrophysical and cosmological measurements; the component of dark matter has only been observed through its interaction with gravity. If we assume that dark matter emerged as the result of a production mechanism one should then explain —at least at the classical level— how the massive modes are stable when interacting with the massless modes or whether the massive modes can decay in the massless modes, since the weak interactions of the dark matter particle shows its stability to the present time. Different approaches for dark matter that do not assume it to be made up of a certain particle are, among others, modified Newtonian dynamics [22, 72] and primordial black holes [73]. The attempts to produce or detect the dark matter particle have not been successful [74, 75]. This suggests that such a particle is heavy and therefore difficult to produce or that its interaction with baryonic matter particles is extremely weak.

In refs. [76, 77] a theoretical construction for massive spin-2 dark matter was suggested and later analysed in refs. [71, 78] (see also [79]). It was found that bimetric gravity gives a framework for dark matter made up of spin-2 particles with a mass of the order of `TeV`. The observed amount of dark matter in the Universe can be explained by through the production mechanism called "freeze-in" (see ref. [80] for a review on that production mechanism). In principle, the daughter products of the spin-2 particle could be detected indirectly in experiments.

In ref. [81] it was pointed out that strong spin-2 self-interactions could possibly affect the production mechanism when allowing thermalization of the dark sector and this would constraint the dark matter particle mass could be of the order of 1 `MeV`. Unfortunately, bimetric theory proposes a dark matter particle candidate in which the interactions with baryonic matter are too small to be detected. It has been then, up to now, not possible to produce massive spin-2 fields in particle accelerators.

In this thesis we explore multimetric interactions that generalise bimetric theory bringing the possibility to make have invariance under discrete symmetry groups. We suggest that the maximal global symmetry in multimetric theories with $(N+1)$ metrics is $S_N \times (\mathbb{Z}_2)^N$. Furthermore, we study the mass spectrum for this discrete invariance and we also calculate the cubic interaction vertices for the special case of trimetric theory, i.e. $N = 2$. It turns out that this case, in the same way of bimetric gravity, has a parameter α that regulates the deviations from general relativity. In particular, we analyse the parameter region $\alpha < 1$. In this case the deviations from general relativity are small with large mass range. The massive spin-2 mode with greater mass, $M_{\mu\nu}$, has a similar behavior as the massive mode of the bigravity case: it neither decays into $G_{\mu\nu}$ nor into the lighter mode $\chi_{\mu\nu}$. Moreover, it couples very weakly to the standard model fields. The lighter spin-2 mode has new features: it does not couples to matter and it does not decay into other spin-2 particles. Since it is a stable massive field we postulate it as a dark matter component. We discuss then the new parameter regions brought by the trimetric scenario.

1.6 Structure of this thesis

In this thesis we address the questions i) whether it is possible to obtain 4-dimensional spin-2 field theories from the 5-dimensional Chern–Simons theory for the $\mathrm{SO}(4,2)$ gauge group and ii) whether it is possible to describe the dark matter using maximal discrete symmetric multimetric models. To this end, this thesis is structured as follows:

- Chapter § 2 is devoted to a mathematical prelude. Here we study several topics of Lie algebras and their uses in gauge theory. In particular, we formally present the symmetry algebras used in this thesis and we review the Inönü–Wigner contractions. Also, we study the algebra $\mathrm{SO}(4,2)$ in different bases and we discuss the method of expansion of Lie algebras. By last, we discuss the machinery of Lie algebra-valued differential forms and Chern–Simons forms.

- Chapter § 3 constitutes a review of the ghost-free spin-2 field theories, namely massive and bimetric gravity. Also, we present some results of ref. [2]: the study of the mass spectrum for a multimetric theory for spin-2 fields, in the special case of maximal global discrete symmetry of the multimetric action.

- In chapter § 4 we discuss the fact that 4-dimensional gravity is not invariant under the gauge transformations induced by the Poincaré group, which motivates the gauge formulation of a modified gravity theory. Also, we study some aspects of conformal gravity in its first- and second-order formulation.

- In chapter § 5 we present the results obtained in ref. [1]. Here we compute the Chern–Simons theory for the gauge group $\mathrm{SO}(4,2)$ in convinient bases in order to apply several dimensional reduction schemes. We obtain generalisations of Einstein, conformal and bimetric gravity.

- Chapter § 6 is devoted to some results of ref. [2]. Here we analyse the perturbative expansion of trimetric gravity with maximal global discrete symmetry. The generalisation to N satellite metrics is also discussed.

We conclude that the massive mode $\chi_{\mu\nu}$ is completely stable and we suggest that it can be the component of dark matter.

- In chapter § 7 we present a conclusion for the aims that are pursued in this thesis.

Chapter 2

Symmetries

The concept of symmetry is something that we use over and over again in this manuscript. We devote this chapter –a mathematical prelude– to the explanation of different tools related to the description and use of symmetries in physics. In particular, we discuss some important symmetry groups in physics and we review the Inönü–Wigner contractions and the expansion of Lie algebras. Also, we present the algebra $\mathrm{SO}(4,2)$ in different bases and we review the properties of differential forms defined on a principal bundle together with the mathematical aspects of Chern–Simons theory.

2.1 Symmetry scenarios

2.1.1 Minkowski space

The $(p+q)$-dimensional Minkowski space $\mathcal{M}_{p+q}$ is defined through the metric tensor $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$, with $\mu, \nu = 1, \ldots, p+q$ and where η is a matrix with p times the entry $+1$ and q times the entry -1 in the diagonal and zeros out of it. If the diagonalized metric of a manifold has only positive entries, one says that the space is Euclidean if the entries of the metric are constant. If one or more of the entries are negative, then one says that the space is pseudo-Euclidean or Lorentzian.

We address, in the following, the problem of finding all the Killing vector fields of $\mathcal{M}_{p+q}$. For a given metric tensor, the equations that determine the Killing vectors of the manifold are

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \,, \tag{2.1}$$

which is the so-called Killing equation. Here $\vec{\xi} = \xi^\mu \partial_\mu$ is the Killing vector field. Since the Killing equation is symmetric in the lower indices, we can find at most $(p+q)(p+q+1)/2$ linearly independent Killing vector fields. Also, for a Levi-Civita connection it is possible to show that $\nabla_\mu \nabla_\nu \, \xi_\lambda = R^\rho{}_{\lambda\nu\mu} \xi_\rho$ [82]. Due to the fact that for the Minkowski space the metric is constant, the covariant derivative is simply the partial derivative and the Riemann tensor is equal to zero. The equations that determine the Killing vectors are then given by

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0 \,, \tag{2.2a}$$

$$\partial_\mu \partial_\nu \xi_\lambda = 0 \,. \tag{2.2b}$$

From eq. (2.2b) we see that $\xi_\mu = C_{\mu\nu} x^\nu + C_\mu$, where $C_{\mu\nu}$ and C_μ are constants. By plugging in this solution back in eq. (2.2a), we see that $C_{\mu\nu} = -C_{\nu\mu}$ and therefore $C_{\mu\nu}$ has $(p+q)(p+q-1)/2$ linearly independent entries. The $p+q$ quantities C_μ are linearly independent. In this way, the expression for a general Killing vector on M_{p+q} is

$$\vec{\xi} = \tfrac{1}{2}\, C^{\mu\nu} \left(x_\nu \partial_\mu - x_\mu \partial_\nu \right) + C^\mu \partial_\mu \,. \tag{2.3}$$

The tensor $x_\mu \partial_\nu - x_\nu \partial_\mu$ is anti-symmetric hence it corresponds to $(p+q)(p+q-1)/2$ linearly independent vector fields. Together with the fields ∂_μ they are

$$\frac{(p+q)(p+q-1)}{2} + (p+q) = \frac{(p+q)(p+q+1)}{2} \tag{2.4}$$

vectors, which coincide with the maximum number of possible independent Killing vectors that a $(p+q)$-dimensional manifold could have. We have found all the Killing vector fields of $\mathcal{M}_{p+q}$ and we denote them as

$$J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \,, \tag{2.5a}$$

$$P_\mu = \partial_\mu \,. \tag{2.5b}$$

2.1.2 Isometry group

It is well-known that all the linearly independent Killing vectors defined on a manifold make up a basis for a Lie algebra [16], which is called the Lie algebra of the isometry group of the manifold. In our case, the algebra of the isometry group for $\mathcal{M}_{p+q}$ is called the inhomogeneous special orthogonal algebra in $p+q$ dimensions $\mathrm{ISO}(p,q) = \mathrm{span}\,\{J_{\mu\nu}, P_\mu\}$, or simply the $(p+q)$-dimensional Poincaré algebra, also denoted by P_{p+q} .

At this point it is important to clarify that since we use the Poincaré algebra as well as its subalgebras or its extensions as gauge group in this thesis, we change the notation of its generators to emphasize that they are part of the basis of a principal bundle with elements $\boldsymbol{M} = M^a{}_\mu\, \mathrm{d}x^\mu \otimes \boldsymbol{T}_a$. Thus from now we denote the generators of the algebra as

$$J_{\mu\nu} \longrightarrow \boldsymbol{J}_{ab}\ , \tag{2.6a}$$

$$P_\mu \longrightarrow \boldsymbol{P}_a\ , \tag{2.6b}$$

with $a, b = 1, \ldots, p+q$.

2.1.2.1 Commutation relations

It is straightforward to compute the commutation relations. First, we find

$$[\boldsymbol{J}_{ab}, \boldsymbol{J}_{cd}] = f_{ab,cd}{}^{ef}\,\boldsymbol{J}_{ef}\ , \tag{2.7}$$

with structure constants

$$f_{ab,cd}{}^{ef} = -\tfrac{1}{2}\left(\eta_{ac}\delta^{ef}_{bd} + \eta_{bd}\delta^{ef}_{ac} - \eta_{bc}\delta^{ef}_{ad} - \eta_{ad}\delta^{ef}_{bc}\right)\ . \tag{2.8}$$

This shows that the generators $\boldsymbol{J}_{ab}$ make up a basis for a subalgebra by themselves. This algebra is called the special orthogonal algebra in $p+q$ dimensions $\mathrm{SO}(p,q) = \mathrm{span}\,\{J_{ab}\}$ or simply the $(p+q)$-dimensional Lorentz algebra, also denoted by L_{p+q}. The structure constants make up the components of a tensor

Lorentz transformations and they satisfy the anti-symmetry relations

$$f_{ab,cd}{}^{ef} = -f_{cd,ab}{}^{ef} = -f_{ba,cd}{}^{ef} = -f_{ab,dc}{}^{ef} = -f_{ab,cd}{}^{fe} \; . \tag{2.9}$$

Furthermore it holds that $[\boldsymbol{P}_a, \boldsymbol{P}_b] = 0$, which also shows that the generators $\boldsymbol{P}_a$ make up a basis for an abelian subalgebra. This is the algebra of translations in $p+q$ dimensions $\mathrm{T}(p,q)$, also denoted as T_{p+q}. It is then just left to compute how $\boldsymbol{J}_{ab}$ and $\boldsymbol{P}_a$ commute. One gets

$$[\boldsymbol{J}_{ab}, \boldsymbol{P}_c] = f_{ab,c}{}^{d}\, \boldsymbol{P}_d \; , \tag{2.10}$$

with

$$f_{ab,c}{}^{d} = -\Big(\eta_{ac}\delta^d_b - \eta_{bc}\delta^d_a\Big) \; . \tag{2.11}$$

satisfying that $f_{ab,c}{}^{d} = -f_{ba,c}{}^{d}$. Since the generators $\boldsymbol{J}_{ab}$ do not commute with $\boldsymbol{P}_a$, we conclude that the Lie algebra $\mathrm{ISO}(p,q)$ is the semidirect sum of the algebras $\mathrm{SO}(p,q)$ and $\mathrm{T}(p,q)$.

For purposes of computation it is relevant to note that the structure constants mentioned above satisfy the following identities.

1. Let G^{ab} and H^{ab} be anti-symmetric symbols. We have

$$f_{ab,cd}{}^{ef}\, G^{ab} H^{cd} = -2\eta_{ac}\Big(G^{ae}H^{cf} - G^{af}H^{ce}\Big) \; . \tag{2.12}$$

   Furthermore, for I^{ab} anti-symmetric one gets that

$$f_{ab,cd}{}^{ef}\, G^{ab} H^{cd} I_{ef} = 4\, G^{a}{}_{c} H^{cb} I_{ef} \; . \tag{2.13}$$

2. Let J^a be some arbitrary symbol. One gets that

$$f_{ab,c}{}^{d}\, G^{ab} J^c = 2\, G^{d}{}_{a} J^a \; . \tag{2.14}$$

2.1.3 Symmetry scenarios

Besides the Poincaré and Lorentz symmetries, other special orthogonal groups and similar structures appear in gravity and quantum field theory. The $(p+q)$-dimensional anti-de Sitter (de Sitter) spacetime is a solution of the Einstein equations with negative (positive) cosmological constant for which the Killing vectors satisfy the algebra $\mathrm{AdS}_{p+q} = \mathrm{SO}(p, q+1)$ ($\mathrm{dS}_{p+q} = \mathrm{SO}(p+1, q)$).

The conformal Killing vectors of the Minkowski space is the set of vectors $\vec{\xi} = \xi^\mu \partial_\mu$ that instead of satisfying eq. (2.1), satisfy $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \kappa\, g_{\mu\nu}$ for $g_{\mu\nu} = \eta_{\mu\nu}$, with κ being a constant. They satisfy the commutation relations of the algebra $\mathrm{SO}(p+1, q+1)$, also symbolized as C_{p+q}, which is called the conformal algebra in $p+q$ dimensions.

2.2 Inönü–Wigner contraction

Inönü–Wigner contractions [42] are a non-invertible process that abelianize some algebra sectors. To explain this, let us suppose we have an algebra expanded by the elements J_1 and J_2 and that they satisfy the commutation relations $[J_1, J_2] = J_1$. Let us then introduce a parameter by writing $\bar{J}_2 = \lambda J_2$. The commutation relations for J_1 and $\bar{J}_2$ are now $\left[J_1, \bar{J}_2\right] = \lambda J_1$. The algebra expanded by J_1 and $\bar{J}_2$ is then abelian in the contraction limit $\lambda \to 0$. Formally we are making the *dilation*-like change of basis of the form

$$\begin{pmatrix} \bar{J}_1 \\ \bar{J}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}, \tag{2.15}$$

in the sense that we are rescaling a generator. We observe that the determinant of the matrix of the change of basis is zero in the contraction limit, meaning that once we contract the algebra, we cannot go back. A well-known application of Inönü–Wigner contractions is that the contraction of the Poincaré algebra leads us to the Galileo algebra for the limit $1/c \to 0$ [83]. We discuss this in detail in § 2.2.2.

2.2.1 From (anti)-de Sitter to Poincaré

By means of an Inönü–Wigner contraction it is possible to exhibit a limit process through which we can find the Poincaré algebra $\mathrm{ISO}(p,q)$ from both the anti-de Sitter and de Sitter algebra $\mathrm{SO}(p,q+1)$ and $\mathrm{SO}(p+1,q)$ [43]. To see this, consider the Lorentz algebra $\mathrm{SO}(p,q)$ (eq. (2.7)) and write $p = P$, $q = Q+1$ and $p = P+1$, $q = Q$. The commutation relations read $[\boldsymbol{J}_{AB}, \boldsymbol{J}_{CD}] = f_{AB,CD}{}^{EF}\boldsymbol{J}_{EF}$, with $A, B = 1, \ldots, P+Q+1$, where now the η_{AB} inside $f_{AB,CD}{}^{EF}$ has P times the entry $+1$ or $(Q+1)$ times the entry -1 and $(P+1)$ times the entry $+1$ and Q times the entry -1 respectively. This means, those are the commutation relations for $\mathrm{SO}(P,Q+1)$ or $\mathrm{SO}(P+1,Q)$, respectively. A short way to distinguish whether we are talking about AdS_{P+Q} or dS_{P+Q} is to refer to the component $\eta_{(P+Q+1)(P+Q+1)} = \mp 1$, respectively.

Since $\boldsymbol{J}_{AB}$ is anti-symmetric, its number of linearly independent components is $(P+Q)(P+Q+1)/2$. We split the components as

$$\frac{(P+Q)(P+Q+1)}{2} = \frac{(P+Q)(P+Q-1)}{2} + (P+Q) \tag{2.16}$$

by making the change of basis $\boldsymbol{J}_{AB} = (\boldsymbol{J}_{ab}, \boldsymbol{J}_{a(P+Q+1)} = \boldsymbol{T}_a)$. Notice that this basis expressed the generators in a covariant way, now for tensors under $\mathrm{SO}(P,Q)$ transformations. The commutation relations for AdS_{P+Q} or dS_{P+Q} read

$$[\boldsymbol{J}_{ab}, \boldsymbol{J}_{cd}] = f_{ab,cd}{}^{ef}\, \boldsymbol{J}_{ef} \;, \tag{2.17a}$$

$$[\boldsymbol{J}_{ab}, \boldsymbol{T}_c] = f_{ab,c}{}^{d}\, \boldsymbol{T}_d \;, \tag{2.17b}$$

$$[\boldsymbol{T}_a, \boldsymbol{T}_b] = -\boldsymbol{J}_{ab} \;, \tag{2.17c}$$

for $\eta_{(P+Q+1)(P+Q+1)} = \mp 1$, respectively. The generators $\boldsymbol{T}_a$ are commonly called (anti)-de Sitter boosts. To carry out with the limit process we make the change of basis $\boldsymbol{T}_a \to \gamma\, \boldsymbol{T}_a$, where γ is a real parameter. The commutation relations affected by this redefinition are

$$[\boldsymbol{T}_a, \boldsymbol{T}_b] = -\tfrac{1}{\gamma^2}\boldsymbol{J}_{ab} \;. \tag{2.18}$$

In the limit $\gamma \to \infty$ the commutation relation of eq. (2.18) abelianizes and we recover exactly the commutation relations for the Poincaré algebra $\mathrm{P}_{P+Q} = \mathrm{ISO}(P,Q)$. Remarkably, this happens for both algebras $\mathrm{SO}(P,Q+1)$ and $\mathrm{SO}(P+1,Q)$.

2.2.2 From Poincaré to Galileo

The group of Galileo is defined as the set of transformations that leave Newtonian dynamics invariant i.e., Galilean transformations together with spatial rotations, spatial translations and time translations. Starting from the Poincaré algebra it is possible to obtain the algebra of the group of Galileo through an Inönü–Wigner contraction (see for example ref. [83]). Let us consider the Poincaré algebra P_{p+q}. One decomposes the basis as

$$\boldsymbol{J}_{ab} = (\boldsymbol{J}_{ij}, \boldsymbol{J}_{i(p+q)} = \boldsymbol{K}_i)\ , \tag{2.19a}$$
$$\boldsymbol{P}_a = (\boldsymbol{P}_i, \boldsymbol{P}_{(p+q)} = \boldsymbol{H})\ , \tag{2.19b}$$

and the commutation relations read

$$[\boldsymbol{J}_{ij}, \boldsymbol{J}_{k\imath}] = f_{ij,k\imath}{}^{mn}\, \boldsymbol{J}_{mn}\ , \tag{2.20a}$$
$$[\boldsymbol{J}_{ij}, \boldsymbol{K}_k] = f_{ij,k}{}^{\imath}\, \boldsymbol{K}_\imath\ , \tag{2.20b}$$
$$[\boldsymbol{J}_{ij}, \boldsymbol{P}_k] = f_{ij,k}{}^{\imath} \boldsymbol{P}_\imath\ , \tag{2.20c}$$
$$[\boldsymbol{J}_{ij}, \boldsymbol{H}] = 0\ , \tag{2.20d}$$
$$[\boldsymbol{K}_i, \boldsymbol{K}_j] = \boldsymbol{J}_{ij}\ , \tag{2.20e}$$
$$[\boldsymbol{K}_i, \boldsymbol{P}_j] = -\delta_{ij}\, \boldsymbol{H}\ , \tag{2.20f}$$
$$[\boldsymbol{K}_i, \boldsymbol{H}] = -\boldsymbol{P}_i\ , \tag{2.20g}$$
$$[\boldsymbol{P}_i, \boldsymbol{P}_j] = 0\ , \tag{2.20h}$$
$$[\boldsymbol{P}_i, \boldsymbol{H}] = 0\ . \tag{2.20i}$$

We introduce the parameter c through the rescalings $\boldsymbol{K}_i \to \mathrm{c}\,\boldsymbol{K}_i$ and $\boldsymbol{H} \to 1/\mathrm{c}\,\boldsymbol{H}$. The commutation relations that are affected by this redefinition are

$$[\boldsymbol{K}_i, \boldsymbol{K}_j] = \tfrac{1}{\mathrm{c}^2} \boldsymbol{J}_{ij}\ , \tag{2.21a}$$
$$[\boldsymbol{K}_i, \boldsymbol{P}_j] = -\tfrac{1}{\mathrm{c}^2} \delta_{ij}\, \boldsymbol{H}\ , \tag{2.21b}$$

and taking the limit $\mathrm{c} \to \infty$ we get

$$[\boldsymbol{K}_i, \boldsymbol{K}_j] = 0\,, \tag{2.22a}$$
$$[\boldsymbol{K}_i, \boldsymbol{P}_j] = 0\,. \tag{2.22b}$$

This is the algebra of the group of Galileo, where $\boldsymbol{K}_i$ is the generator of Galilean transformations and $\boldsymbol{J}_{ij}$, $\boldsymbol{P}_i$ and $\boldsymbol{H}$ are the generators of spatial rotations, spatial translations and time translations respectively. Using dimensional analysis we might be tempted interpret the parameter c as the speed of light, however, the previous analysis holds not only for c, but for any function $f(\mathrm{c})$ in $\boldsymbol{K}_i \to f(\mathrm{c})\,\boldsymbol{K}_i$ and $\boldsymbol{H} \to 1/f(\mathrm{c})\,\boldsymbol{H}$ that satisfies $f(\mathrm{c}) \to \infty$ as the parameter $\mathrm{c} \to \infty$.

The fact that we can get the Galilean symmetry from the Poincaré algebra by means of an Inönü–Wigner contraction limit can be used to study the newtonian limit of a relativistic theory. For example, in ref. [18] an expansion (see § 2.4) of the 5-dimensional Poincaré algebra was introduced and the corresponding Chern–Simons gravity theory was computed. Then in ref. [3] we computed the newtonian limit of the gravity theory by gauging the Inönü–Wigner contracted Lie algebra.

2.3 Parametrized conformal algebra

Conformal field theories in $(p-1)+q$ dimensions were shown to have the same dynamical content as a gravity theory in AdS in $p+q$ dimensions. One remarkable example for this is the AdS/CFT correspondence [84]. This can be understood at the level of symmetries because of the fact that

$$\mathrm{C}_{p+q} \simeq \mathrm{AdS}_{(p+1)+q} \simeq \mathrm{SO}(p+1, q+1)\,. \tag{2.23}$$

Further evidence that motivated AdS/CFT correspondence lies in the fact that the 3-dimensional Chern–Simons gravity gauge theory AdS group induces a Wess–Zumino model on the boundary [85]. Actions with this particular feature can be written as a Wess–Zumino–Witten action.

In this section, we start from the special orthogonal algebra of $\mathrm{SO}(4,2)$ and make a change to new bases introducing parameters in the same spirit as Weyl rotations (see § 4.3.2) and Inönü–Wigner contractions. This is to show an isomorphism to the conformal algebra C_{3+1}. The parameter-dependant algebra is symbolized as $\mathrm{C}_{3+1}(M,\gamma)$ as discussed at the end of § 1.4. As we see in § 5, when calculating the 5-dimensional Chern–Simons action in the basis of $\mathrm{C}_{3+1}(M,\gamma)$, the action exhibits curvature terms plus a potential involving all the possible interaction terms between the two vielbeine components e^a and $\imath^a$ living on a 5-dimensional manifold. This setup motivates a subsequent dimensional reduction in order to obtain 4-dimensional massive spin-2 field theories.

2.3.1 Bases of the algebra

In the following we decompose the generators $\boldsymbol{J}_{IJ}$ of $\mathrm{SO}(4,2)$ in many ways, with the aim to write the commutation relations, first with covariant indices in five dimensions, and then in four dimensions. We arrive at a basis with covariant Lorentz indices that exhibits the isomorphism between $\mathrm{SO}(4,2)$ and the four dimensional conformal group.

2.3.1.1 6-covariant basis

We begin from the basis with antisymmetric generators $\{\boldsymbol{J}_{IJ}\}$, with $I,J,\ldots = 1,\ldots,6$. Notice that the generators are expressed in a covariant manner, i.e., the make up the component of a tensor under $\mathrm{SO}(4,2)$ transformations. As discussed in § 2.1.2.1, in this basis the commutation relations are $[\boldsymbol{J}_{IJ},\boldsymbol{J}_{KL}] = f_{IJ,KL}{}^{MN}\boldsymbol{J}_{MN}$, with $\eta_{IJ} = \mathrm{diag}(+,+,+,-,-\eta,\eta)$ for $\eta = \pm 1$.

We refer to the basis $\{\boldsymbol{J}_{IJ}\}$ as the 6-covariant basis, since the generators $\boldsymbol{J}_{IJ}$ transform as a tensor under "Lorentz transformations" in $4+2$ dimensions. The invariant tensor of the Euler class reads

$$\langle \boldsymbol{J}_{IJ},\boldsymbol{J}_{KL},\boldsymbol{J}_{MN}\rangle = \epsilon_{IJKLMN}\ . \tag{2.24}$$

By knowing the commutation relations together with the invariant tensors of a certain Lie group, we have the essential components to compute Chern–Simons theory (see § 5).

2.3.1.2 5-covariant basis

In the following, we expand the indices to get the commutation relations of SO(4, 2) in a 5-covariant basis, i.e., a basis for which the generators since transform as a tensor under "Lorentz transformations" in 5 dimensions. For that end, we consider separately the generators $\boldsymbol{J}_{AB}$ and $\boldsymbol{J}_{A6} \equiv \gamma \boldsymbol{T}_A$ with $A, B, \ldots = 1, \ldots, 5$, with γ being a real or pure-imaginary parameter. According to our discussion of § 2.2, the parameter γ plays the role of an Inönü–Wigner limit controller: we use it to perform a contraction in the context of gravitational theories in § 5.

The commutation relations read in terms of the 5-covariant generators as follows

$$[\boldsymbol{J}_{AB}, \boldsymbol{J}_{CD}] = f_{AB,CD}{}^{EF} \boldsymbol{J}_{EF} , \tag{2.25a}$$

$$[\boldsymbol{J}_{AB}, \boldsymbol{T}_C] = f_{AB,C}{}^{D} \boldsymbol{T}_D , \tag{2.25b}$$

$$[\boldsymbol{T}_A, \boldsymbol{T}_B] = -\tfrac{\eta}{\gamma^2} \boldsymbol{J}_{AB} , \tag{2.25c}$$

with $\eta_{AB} = \mathrm{diag}(+,+,+,-,-\eta)$ with $\eta = \pm 1$. For this choice of basis the non-trivial invariant tensor of the Euler class read

$$\langle \boldsymbol{J}_{AB}, \boldsymbol{J}_{CD}, \boldsymbol{T}_E \rangle = \gamma^{-1} \, \epsilon_{ABCDE} \, . \tag{2.26}$$

In the same spirit of obtaining the Poincaré algebra from both (anti)-de Sitter algebras (see § 2.2.1), the commutation relations of eq. (2.25) show us that the contraction limit $\gamma \to \infty$ turns the algebra into ISO(3, 2) or ISO(4, 1) for $\eta = \pm 1$, respectively.

2.3.1.3 4-covariant basis

In order to go further with the decomposition, let us consider the generators $\boldsymbol{J}_{ab}$, $\boldsymbol{J}_{a5} \equiv \boldsymbol{B}_a$, $\boldsymbol{T}_a$, $\boldsymbol{T}_5 \equiv \boldsymbol{D}$. In the same spirit as above, they make up a 4-covariant basis with local Lorentz indices. The commutation relations of SO$(4,2)$ in this basis are

$$[\boldsymbol{J}_{ab}, \boldsymbol{J}_{cd}] = f_{ab,cd}{}^{ef}\, \boldsymbol{J}_{ef} \ , \tag{2.27a}$$
$$[\boldsymbol{J}_{ab}, \boldsymbol{B}_c] = f_{ab,c}{}^{d}\, \boldsymbol{B}_d \ , \tag{2.27b}$$
$$[\boldsymbol{J}_{ab}, \boldsymbol{T}_c] = f_{ab,c}{}^{d} \boldsymbol{T}_d \ , \tag{2.27c}$$
$$[\boldsymbol{J}_{ab}, \boldsymbol{D}] = 0 \ , \tag{2.27d}$$
$$[\boldsymbol{B}_a, \boldsymbol{B}_b] = \eta\, \boldsymbol{J}_{ab} \ , \tag{2.27e}$$
$$[\boldsymbol{B}_a, \boldsymbol{T}_b] = -\eta_{ab}\, \boldsymbol{D} \ , \tag{2.27f}$$
$$[\boldsymbol{B}_a, \boldsymbol{D}] = -\eta\, \boldsymbol{T}_a \ , \tag{2.27g}$$
$$[\boldsymbol{T}_a, \boldsymbol{T}_b] = -\eta\gamma^{-2} \boldsymbol{J}_{ab} \ , \tag{2.27h}$$
$$[\boldsymbol{T}_a, \boldsymbol{D}] = -\eta\gamma^{-2}\, \boldsymbol{B}_a \ . \tag{2.27i}$$

The gauge transformations associated with the SO$(4,2)$ symmetry in this particular basis are given in § 2.6.2. We observe in this basis that under SO$(1,1)$ rotations of the vector $(\boldsymbol{B}^a, \gamma\boldsymbol{T}^a)$, i.e.

$$\begin{pmatrix} \boldsymbol{B}^a \\ \gamma\boldsymbol{T}^a \end{pmatrix} \longrightarrow \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \boldsymbol{B}^a \\ \gamma\boldsymbol{T}^a \end{pmatrix} \ , \tag{2.28}$$

the algebra remains invariant. For $\gamma^2 < 0$, the invariance group is SO(2).

2.3.1.4 4-covariant canonical basis and parametrized conformal algebra

In the 4-covariant basis one still cannot see the isomorphism between SO$(4,2)$ and the conformal algebra. To this end, we perform the following linear transformation on the generators $\boldsymbol{B}_a$ and $\boldsymbol{T}_a$. Let us define the generators $\boldsymbol{P}_a$ and

$\boldsymbol{K}_a$ as

$$\begin{pmatrix} \boldsymbol{P}_a \\ \boldsymbol{K}_a \end{pmatrix} = M \begin{pmatrix} \boldsymbol{B}_a \\ \gamma \boldsymbol{T}_a \end{pmatrix} = \begin{pmatrix} a & b\gamma^{-1} \\ c & d\gamma^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_a \\ \gamma \boldsymbol{T}_a \end{pmatrix} , \tag{2.29}$$

where M is a matrix in $\mathrm{GL}(2,\mathbb{C})$ with a, b, c, and d in $\mathbb{C}$. The commutation relations for this new basis are then given by

$$[\boldsymbol{J}_{ab}, \boldsymbol{J}_{cd}] = f_{ab,cd}{}^{ef}\, \boldsymbol{J}_{ef} \;, \tag{2.30a}$$

$$[\boldsymbol{J}_{ab}, \boldsymbol{P}_c] = f_{ab,c}{}^{d}\, \boldsymbol{P}_d \;, \tag{2.30b}$$

$$[\boldsymbol{J}_{ab}, \boldsymbol{K}_c] = f_{ab,c}{}^{d}\, \boldsymbol{K}_d \;, \tag{2.30c}$$

$$[\boldsymbol{J}_{ab}, \boldsymbol{D}] = 0 \;, \tag{2.30d}$$

$$[\boldsymbol{P}_a, \boldsymbol{P}_b] = \eta \left(a^2 - b^2\gamma^{-2}\right) \boldsymbol{J}_{ab} \;, \tag{2.30e}$$

$$[\boldsymbol{P}_a, \boldsymbol{K}_b] = -\det M\, \eta_{ab}\, \boldsymbol{D} + \eta \left(ac - bd\gamma^{-2}\right) \boldsymbol{J}_{ab} \;, \tag{2.30f}$$

$$[\boldsymbol{P}_a, \boldsymbol{D}] = \tfrac{\eta}{\det M} \left[\left(ac - bd\gamma^{-2}\right) \boldsymbol{P}_a - \left(a^2 - b^2\gamma^{-2}\right) \boldsymbol{K}_a \right] , \tag{2.30g}$$

$$[\boldsymbol{K}_a, \boldsymbol{K}_b] = \eta \left(c^2 - d^2\gamma^{-2}\right) \boldsymbol{J}_{ab} \;, \tag{2.30h}$$

$$[\boldsymbol{K}_a, \boldsymbol{D}] = \tfrac{\eta}{\det M} \left[\left(c^2 - d^2\gamma^{-2}\right) \boldsymbol{P}_a - \left(ac - bd\gamma^{-2}\right) \boldsymbol{K}_a \right] . \tag{2.30i}$$

This basis represents the most general way to re-define translations and special conformal transformations. Therefore, we refer to it as the 4-covariant canonical basis. The gauge transformation induced by the group $\mathrm{SO}(4,2)$ are discussed in § 2.6.2. In the following we discuss special cases for the matrix M.

2.3.2 Parameter choices

In the following, we single out two particular cases for the parameters a, b, c, d and γ. This leads to different commutation relations, although corresponding to the same algebra $\mathrm{SO}(4,2)$. Of the set of all possible choices we exclude those that imply $\det M = 0$ because otherwise, we would not be able to find back the generator $\boldsymbol{T}_a$ and $\boldsymbol{B}_a$ from $\boldsymbol{P}_a$ and $\boldsymbol{K}_a$ through an inverse transformation.

2.3.2.1 Conformal basis

The purpose of introducing the 4-covariant canonical basis is to assign the vielbein components e^a and $\imath^a$ as gauge fields for the generators $\boldsymbol{P}_a$ and $\boldsymbol{K}_a$ (see § 5.1.4). To connect this construction with bimetric theory we should expect e^a and $\imath^a$ to appear with equal status. Therefore we treat $\boldsymbol{P}_a$ and $\boldsymbol{K}_a$ on the same footing as well. Both spaces span $\{\boldsymbol{P}_a\}$ and span $\{\boldsymbol{K}_a\}$ appear symmetrically in the subspaces structure SO(4, 2). Let us study the case when each one is an Abelian subalgebra. This occurs for

$$a^2 - b^2\gamma^{-2} = 0 \,, \tag{2.31a}$$
$$c^2 - d^2\gamma^{-2} = 0 \,. \tag{2.31b}$$

We should notice that the transpose matrix M^{T} can be constructed with the column vectors $\binom{a}{b/\gamma}$ and $\binom{c}{d/\gamma}$. In the special case of eq. (2.31), these two vectors have zero-norm with respect to the Killing metric of SO(1, 1), namely $\eta = \mathrm{diag}(1, -1)$.

The solutions for eq. (2.31) are given by i) $b(a) = \pm a\gamma$, $d(c) = \pm c\gamma$ or ii) $b(a) = \pm a\gamma$, $d(a) = \mp c\gamma$. By choosing the coefficients as in case i) we get $\det M = 0$, thus we neglect this case. For case ii) we get that $\det M = \mp 2ac$, which is, in general, different from zero. The commutators of the algebra of (2.30) that are affected by the parameters choice ii) are in this case

$$[\boldsymbol{P}_a, \boldsymbol{P}_b] = 0 \,, \tag{2.32a}$$
$$[\boldsymbol{P}_a, \boldsymbol{K}_b] = \pm 2ac\,(\gamma\eta_{ab}\,\boldsymbol{D} \pm \eta \boldsymbol{J}_{ab}) \,, \tag{2.32b}$$
$$[\boldsymbol{P}_a, \boldsymbol{D}] = \mp\eta\gamma^{-1}\boldsymbol{P}_a \,, \tag{2.32c}$$
$$[\boldsymbol{K}_a, \boldsymbol{K}_b] = 0 \,, \tag{2.32d}$$
$$[\boldsymbol{K}_a, \boldsymbol{D}] = \pm\eta\gamma^{-1}\boldsymbol{K}_a \,. \tag{2.32e}$$

We can see that in this case the commutation relations of SO(4, 2) take the usual form of the conformal algebra used in conformal field theory. This happens exactly when $\gamma = ac = 1$. Thus, we refer to the basis defined by this choice of parameters as the conformal basis. Note that the algebra of Poincaré in four dimensions makes up a subalgebra:

$$\mathrm{P}_{3+1} \simeq \mathrm{span}\left\{\boldsymbol{J}_{ab}, \boldsymbol{P}_a\right\} \simeq \mathrm{span}\left\{\boldsymbol{J}_{ab}, \boldsymbol{K}_a\right\} \ . \tag{2.33}$$

2.3.2.2 Orthogonal basis

There exists a second special case motivated by the geometric interpretation of the matrix M. If one demands the vectors $\binom{a}{b/\gamma}$ and $\binom{c}{d/\gamma}$ to be orthogonal with respect to the Killing metric of SO(1, 1), we are then requiring that

$$ac - bd\gamma^{-2} = 0 \ . \tag{2.34}$$

We call the basis defined by this choice the orthogonal basis. The solutions for eq. (2.31) are given by i) $b(a) = \pm \mathrm{i} a\gamma$, $d(c) = \pm \mathrm{i} c\gamma$ or ii) $b(a) = \pm a\gamma$, $d(a) = \mp c\gamma$. Again the first case implies that M is singular and we rule it out. For ii), we have $\det M = \mp 2\mathrm{i} ac$ so, in this case, the matrix M is, in general, non-singular. The commutation relations read

$$[\boldsymbol{P}_a, \boldsymbol{P}_b] = 2a^2\eta\, \boldsymbol{J}_{ab} \ , \tag{2.35a}$$
$$[\boldsymbol{P}_a, \boldsymbol{K}_b] = \pm 2\mathrm{i} ac\, \eta_{ab}\, \boldsymbol{D} \ , \tag{2.35b}$$
$$[\boldsymbol{P}_a, \boldsymbol{D}] = \mp \tfrac{\mathrm{i} a\eta}{c}\, \boldsymbol{K}_a \ , \tag{2.35c}$$
$$[\boldsymbol{K}_a, \boldsymbol{K}_b] = 2c^2\eta\, \boldsymbol{J}_{ab} \ , \tag{2.35d}$$
$$[\boldsymbol{K}_a, \boldsymbol{D}] = \pm \tfrac{\mathrm{i} c\eta}{a}\, \boldsymbol{P}_a \ , \tag{2.35e}$$

while the rest remain the same. The gauge transformations induced by SO(4, 2), written in the orthogonal basis, are discussed in § 2.6.2.

2.4 Expansion of Lie algebras

In ref. [86], a mathematical tool to obtain families of Lie algebras starting from a particular Lie algebra g was introduced. In this section we present that formalism and, before going to the mathematics, we motivate on how this procedure works.

2.4.1 Motivation: SO(4) from SO(3)

Consider the commutation relations of the algebra SO(3) (eq. (2.7) for $p = 3$ and $q = 0$). Making the change of basis $\boldsymbol{J}_a = \frac{\mathrm{i}}{2}\epsilon_{abc}\boldsymbol{J}_{bc}$ the commutation relations read $[\boldsymbol{J}_a, \boldsymbol{J}_b] = \epsilon_{abc}\boldsymbol{J}_c$. Let us also consider the cyclic group of order two, $\mathbb{Z}_2 = \{\lambda_0, \lambda_1\}$, whose multiplication rule is defined by the Cayley table

$\mathbb{Z}_2$	λ_0	λ_1
λ_0	λ_0	λ_1
λ_1	λ_1	λ_0

Let us define then (still in a heuristic way) the vector space spanned by the direct product between the sets $\{\lambda_0, \lambda_1\}$ and the basis of SO(3), $\{\boldsymbol{J}_a\}$. This is the space span $\{\lambda_\alpha \otimes \boldsymbol{J}_a\}$, with $\alpha = 0, 1$, and we denote it as $\mathbb{Z}_2 \times \mathrm{SO}(3)$. We symbolize the elements of its basis as $\boldsymbol{J}_{(a,\alpha)} \equiv \lambda_\alpha \otimes \boldsymbol{J}_a$. Moreover, let us define the internal operation

$$\left[\boldsymbol{J}_{(a,\alpha)}, \boldsymbol{J}_{(b,\beta)}\right] = \lambda_\alpha\lambda_\beta \otimes \epsilon_{abc}\boldsymbol{J}_c \,. \tag{2.36}$$

Using the multiplication rule of $\mathbb{Z}_2$, one finds the following commutation relations:

$$\left[\boldsymbol{J}_{(a,0)}, \boldsymbol{J}_{(b,0)}\right] = \epsilon_{abc}\boldsymbol{J}_{(c,0)} \,, \tag{2.37a}$$

$$\left[\boldsymbol{J}_{(a,0)}, \boldsymbol{J}_{(b,1)}\right] = \epsilon_{abc}\boldsymbol{J}_{(c,1)} \,, \tag{2.37b}$$

$$\left[\boldsymbol{J}_{(a,1)}, \boldsymbol{J}_{(b,1)}\right] = \epsilon_{abc}\boldsymbol{J}_{(c,0)} \,. \tag{2.37c}$$

We rename $\boldsymbol{J}_{(a,0)} = \boldsymbol{L}_a$ and $\boldsymbol{J}_{(a,1)} = \boldsymbol{K}_a$, and we find the commutation relations of the SO(4) algebra and, therefore, $\mathrm{SO}(4) \simeq \mathbb{Z}_2 \times \mathrm{SO}(3)$.

This procedure was shown to generalise into the following theorem: let $\mathrm{S} = \{\lambda_\alpha\}$, with $\alpha = 0, \ldots, N-1$, be a finite set equipped with an internal associative and commutative multiplication rule, i.e. a discrete abelian semigroup and let $\mathrm{g} = \mathrm{span}\,\{\boldsymbol{T}_a\}$ be a Lie algebra, with $a = 1, \ldots, \dim \mathrm{g}$ and commutation relations $[\boldsymbol{T}_a, \boldsymbol{T}_b] = C_{ab}{}^{c}\boldsymbol{T}_c$. The space $\mathrm{S} \times \mathrm{g} = \mathrm{span}\left\{\boldsymbol{T}_{(a,\alpha)} \equiv \lambda_\alpha \otimes \boldsymbol{T}_a\right\}$

together with the internal operation

$$\left[\boldsymbol{T}_{(a,\alpha)}, \boldsymbol{T}_{(b,\beta)}\right] = C_{ab}{}^{c}\, \boldsymbol{T}_{(c,\gamma)} \tag{2.38}$$

is a Lie algebra of dimension $N \dim \mathrm{g}$, with $\lambda_\gamma = \lambda_\alpha \lambda_\beta$ [86]. In the same reference, the authors proved further theorems for particular cases, depending on the finite semigroup internal structure and on the subspaces structure of the starting Lie algebra. We studied extensively the particular case when the semigroup is a cyclic group with an even number of elements in ref. [87].

As mentioned at the end of § 2.2, ref. [18] introduced an expansion of the Poincaré algebra with the aim of obtaining a 5-dimensional Chern–Simons theory. This algebra was called the $\mathfrak{B}$-algebra and it has the peculiarity that its associated Chern–Simons gravity in five dimensions satisfy the correspondence principle: one obtains the solely 5-dimensional Einstein–Cartan Lagrangian in a certain smooth limit of the parameters of the theory.

2.5 Differential forms

In Riemannian geometry, the rule for the commutativity between a k-form ψ and a $\imath$-form ζ is given by $\psi \wedge \zeta = (-1)^{k\imath} \zeta \wedge \psi$. If k is an odd number, then it follows from the commutativity rule that $\psi \wedge \psi = 0$. Since a differential form on a principal bundle is written in terms of the generators of the Lie algebra, i.e. $\boldsymbol{M} = M^a{}_\mu \, \mathrm{d}x^\mu \otimes \boldsymbol{T}_a$, the commutativity of such forms must also take into account the commutativity rule of the algebra. One defines then how differential forms on a principal bundle should commute through a Lie commutator in terms of the wedge product. Let M be a manifold and G be a Lie group. On the principal bundle (M, G) we define the commutator between forms as

$$[\boldsymbol{M}, \boldsymbol{N}] = \boldsymbol{M} \wedge \boldsymbol{N} - (-1)^{mn} \boldsymbol{N} \wedge \boldsymbol{M} \,, \tag{2.39}$$

where $\boldsymbol{M}$ and $\boldsymbol{N}$ are m- and n-forms respectively and $[\ ,\ ]$ stands for the Lie commutator of the Lie algebra g of the Lie group G, i.e., it acts on the generators $\boldsymbol{T}_a$. An important fact is that if $\boldsymbol{M}$ is an odd form, then the expression $\boldsymbol{M} \wedge \boldsymbol{M} \equiv \frac{1}{2} [\boldsymbol{M}, \boldsymbol{M}]$ is in general not equal to zero, because now the commutativity is subject to the commutation relations of the group. Furthermore, we also have a Leibniz rule for the exterior derivative d acting on forms defined on a principal bundle. In Riemannian geometry, we have $\mathrm{d}(\psi \wedge \zeta) = \mathrm{d}\psi \wedge \zeta + (-1)^k \psi \wedge \mathrm{d}\zeta$. Using this formula, one can prove that in the case of principal bundles we get

$$\mathrm{d}(\boldsymbol{M} \wedge \boldsymbol{N}) = \mathrm{d}\boldsymbol{M} \wedge \boldsymbol{N} + (-1)^m \boldsymbol{M} \wedge \mathrm{d}\boldsymbol{N} \ . \tag{2.40}$$

As in Riemannian geometry, once we have a connection we can define a covariant derivative with respect to that connection. The generalisation is done for the exterior derivative; now we need an exterior covariant derivative. The exterior covariant derivative of a form $\boldsymbol{M}$ with respect to a connection $\boldsymbol{A}$ is defined as

$$\mathrm{D}\boldsymbol{M} = \mathrm{d}\boldsymbol{M} + [\boldsymbol{A}, \boldsymbol{M}] \ . \tag{2.41}$$

To avoid confusion, we specify sometimes the connection $\boldsymbol{A}$ in the covariant derivative as $\mathrm{D} = \mathrm{D}_{\boldsymbol{A}}$. In the case of a local scalar, i.e. a differential form that does not have indices for the algebra: $\boldsymbol{\psi} = \psi$, we define in the same spirit as for a Riemaniann scalar that $\mathrm{D}\boldsymbol{\psi} = \mathrm{d}\boldsymbol{\psi}$.

The two-form curvature is the differential form associated with a particular field strength tensor, generally in the context of arbitrary Lie groups and not only in abelian theories like electrodynamics, where the field strength tensor is defined solely as derivatives of the fields. The two-form curvatures with respect to the connection $\boldsymbol{A}$ is defined as

$$\boldsymbol{F} = \mathrm{d}\boldsymbol{A} + \tfrac{1}{2}[\boldsymbol{A}, \boldsymbol{A}] \ . \tag{2.42}$$

Here we will also sometimes write the specific connection for a curvature as $\boldsymbol{F} = \boldsymbol{F_A}$ or for some remarkable curvatures, e.g. the one associated with the group $\mathrm{SO}(p,q)$ using the spin connection $\boldsymbol{\omega} = \frac{1}{2}\omega^{ab}\boldsymbol{J}_{ab}$, we use the notation $\boldsymbol{F_\omega} = \boldsymbol{R}(\boldsymbol{\omega})$. For an element of the Lie group $U(\boldsymbol{\epsilon})$, where $\boldsymbol{\epsilon} = \Lambda^a \boldsymbol{T}_a$ is the zero-form gauge parameter and Λ^a are the parameters of the transformation, the variation of a connection $\boldsymbol{A}$ under the gauge transformation $\boldsymbol{A}' = U^{-1}(\boldsymbol{\epsilon})\boldsymbol{A}U(\boldsymbol{\epsilon})$ can be written as

$$\delta\boldsymbol{A} = \mathrm{D}_{\boldsymbol{A}}\boldsymbol{\epsilon} \ . \tag{2.43}$$

In the following we present some useful differential forms identities on principal bundles. Let $\boldsymbol{A}$ be a connection on a principal bundle and let $\boldsymbol{M}$ and $\boldsymbol{N}$ be m- and n-forms respectively on the principal bundle.

1. One can find a kind of "antisymmetry" for the commutator

$$[\boldsymbol{M}, \boldsymbol{N}] = -(-1)^{mn} [\boldsymbol{N}, \boldsymbol{M}] \ . \tag{2.44}$$

2. The product of the algebra also satisfies a Leibniz rule

$$\mathrm{d}\, [\boldsymbol{M}, \boldsymbol{N}] = [\mathrm{d}\boldsymbol{M}, \boldsymbol{N}] + (-1)^{m} [\boldsymbol{M}, \mathrm{d}\boldsymbol{N}] \ . \tag{2.45}$$

   Then we find a Leibniz rule for the product of the algebra, which means that in general for every principal bundle, the map d is a derivation [83] of the Lie algebra g.

3. We have a Leibniz rule for the exterior covariant derivative

$$\mathrm{D}\,(\boldsymbol{M} \wedge \boldsymbol{N}) = \mathrm{D}\boldsymbol{M} \wedge \boldsymbol{N} + (-1)^{m} \boldsymbol{M} \wedge \mathrm{D}\boldsymbol{N} \ . \tag{2.46}$$

4. A significant property is the Bianchi identity in the context of principal bundles:

$$\mathrm{D}\boldsymbol{F} = 0 \ . \tag{2.47}$$

5. We have an expression for the second exterior covariant derivative in terms of the curvature and the commutator:

$$\mathrm{D}^2 \boldsymbol{M} = [\boldsymbol{F}, \boldsymbol{M}] \ . \tag{2.48}$$

6. The variation of the two-form curvature under gauge transformations can be expressed in two useful ways:

$$\delta\boldsymbol{F} = \mathrm{D}\,\delta\boldsymbol{A} \ , \tag{2.49}$$

   and since $\delta\boldsymbol{A} = \mathrm{D}\boldsymbol{\epsilon}$, using eq. (2.48) we find

$$\delta\boldsymbol{F} = [\boldsymbol{F}, \boldsymbol{\epsilon}] \ . \tag{2.50}$$

Eq. (2.49) is useful for the calculation of the equations of motion for an action written in terms of connection and curvature. On the other hand, eq. (2.50) is useful for the analysis of gauge invariance of such an action.

2.5.1 Connection separated in subspaces

A well-known result of differential geometry is that a connection Γ does not transform as a tensor under diffeomorphisms. This implies that there can be a frame of reference where $\bar{\Gamma}_{\mu\nu}{}^{\lambda} \neq 0$ while $\Gamma_{\mu\nu}{}^{\lambda} = 0$. This can be directly seen from the inhomogeneity of its transformation law

$$\bar{\Gamma}^{\mu}{}_{\nu\lambda} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial \bar{x}^{\nu}} \frac{\partial x^{\tau}}{\partial \bar{x}^{\lambda}} \Gamma^{\rho}{}_{\sigma\tau} + \frac{\partial \bar{x}^{\mu}}{\partial x^{\kappa}} \frac{\partial^2 x^{\kappa}}{\partial x^{\nu} \partial x^{\lambda}} \,. \tag{2.51}$$

This situation seems to be strange but in fact it has an analogue in nature: in the non-relativistic limit of general relativity one identifies the metric $g_{\mu\nu}$ with the gravitational potential ζ. The analogue of the gravitational force $\sim \vec{\nabla}\zeta$ are then the Christoffel symbols $\Gamma_{\mu\nu}{}^{\lambda} \sim \partial g_{\mu\nu}$. A frame of reference where the Christoffel symbol vanishes could be the classic example of the free-falling elevator, from where we cannot measure the effects of gravity. For a general connection we have the following properties: a connection plus a tensor transforms as a connection again, a connection minus another connection transform as a tensor, and a times a connection plus b times another connection transforms as a connection if $a + b = 1$. We can therefore define a new connection starting from an old one by adding a tensor, or starting from a times an old one and adding b times another one.

This kind of objects exist in gauge theory as well. One can construct gauge connections as the sum of a connection $\boldsymbol{A}$ and a one-form $\boldsymbol{G}$. Let us consider the case when the connection is $\boldsymbol{A} + \boldsymbol{G}$. In this case $\boldsymbol{A}$ is a connection valued in some subspace V_0 of the algebra $\mathrm{g} = V_0 \oplus V_1$ and $\boldsymbol{G}$ is a one-form valued in V_1. One example of this is the well-known gauge connection in gauge theories for gravity

$$\boldsymbol{\omega} + \boldsymbol{e} = \tfrac{1}{2}\omega^{ab}\boldsymbol{J}_{ab} + e^{a}\boldsymbol{P}_{a} \,, \tag{2.52}$$

where ω^{ab} is the one-form spin connection defined on V_0 and e^{a} is the one-form vielbein defined on V_1. Here $\boldsymbol{J}_{ab}$ and $\boldsymbol{P}_{a}$ are the generators of Poincaré or (A)dS algebras. V_0 corresponds to the Lorentz subalgebra and V_1 to the vectorial subspace of commutative and non-commutative translations when g is Poincaré or (A)dS respectively. It is important to notice that the connection $\boldsymbol{A}$ defines a gauge theory by itself. One example of this is that any gauge theory, for a gauge group containing the Lorentz group, will be Lorentz invariant.

In the general case, such a split of the total connection satisfies the following properties:

1. The exterior covariant derivative splits as

$$\mathrm{D}_{\boldsymbol{A}+\boldsymbol{G}}\boldsymbol{M} = \mathrm{D}_{\boldsymbol{A}}\boldsymbol{M} + [\boldsymbol{G}, \boldsymbol{M}] \ . \tag{2.53}$$

2. The two-form curvature splits as

$$\boldsymbol{F}_{\boldsymbol{A}+\boldsymbol{G}} = \boldsymbol{F}_{\boldsymbol{A}} + \tfrac{1}{2}[\boldsymbol{G}, \boldsymbol{G}] + \mathrm{D}_{\boldsymbol{A}}\boldsymbol{G} \ . \tag{2.54}$$

3. Since $\boldsymbol{A} + \boldsymbol{G}$ is a connection, the Bianchi identity is also satisfied for it, i.e.

$$\mathrm{D}_{\boldsymbol{A}+\boldsymbol{G}}\boldsymbol{F}_{\boldsymbol{A}+\boldsymbol{G}} = 0 \ . \tag{2.55}$$

   One can verify this explicitly by using eqs. (2.53) and (2.54).

4. For the total curvature $\boldsymbol{F}_{\boldsymbol{A}} + \mathrm{D}_{\boldsymbol{A}}\boldsymbol{G}$ one finds

$$\delta\boldsymbol{F}_{\boldsymbol{A}} + \delta\mathrm{D}_{\boldsymbol{A}}\boldsymbol{G} = \mathrm{D}_{\boldsymbol{A}}\delta\boldsymbol{A} + [\boldsymbol{G}, \delta\boldsymbol{A}] + \mathrm{D}_{\boldsymbol{A}}\delta\boldsymbol{G} \ . \tag{2.56}$$

2.6 Gauge theory for the conformal group

In the following we present the gauge theory formalism for a general base space M, and for the particular case of the group $\mathrm{G} = \mathrm{SO}(4, 2)$. This means that we show how the curvature and the gauge transformations look for a given connection in some basis of the algebra of the gauge group. To this end, we use all the bases of the special orthogonal group presented in § 2.3.1.

2.6.1 Connections and curvatures

The connection for the group SO(p,q) is usually chosen as a spin-like connection ω^{ab} in the basis $\boldsymbol{J}_{ab}$. For SO$(4,2)$ one writes $\boldsymbol{\omega}_6 = \frac{1}{2}\omega^{IJ}\boldsymbol{J}_{IJ}$ in the 6-covariant basis. We refer to this connection as the conformal spin connection. Its associated field strength $\boldsymbol{F}_{\boldsymbol{\omega}_6}$ coincides exactly with the Cartan curvature for the special orthogonal group, and in this basis it is given by

$$\boldsymbol{F}_{\boldsymbol{\omega}_6} = \boldsymbol{R}_6(\boldsymbol{\omega}_6) = \tfrac{1}{2}R_6^{IJ}(\omega_6)\boldsymbol{J}_{IJ} = \tfrac{1}{2}\Big(\mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}\Big)\boldsymbol{J}_{IJ} \,. \tag{2.57}$$

We call $R_6^{IJ}(\omega_6)$ the conformal curvature. In the 5-covariant basis the gauge connection ω_6 decomposes in the 5-covariant basis as

$$\boldsymbol{\omega}_6 = \tfrac{1}{2}\omega^{IJ}\boldsymbol{J}_{IJ} = \tfrac{1}{2}\omega^{AB}\boldsymbol{J}_{AB} + u^A\boldsymbol{T}_A = \boldsymbol{\omega}_5 + \boldsymbol{u}_5 \,, \tag{2.58}$$

where u^A is the gauge field for the gauge symmetry generators $\boldsymbol{T}_A$, and ω^{AB} is the so-called anti-de Sitter (de Sitter) spin connection for $\eta = \pm 1$ respectively. In § 2.5.1 we saw how the curvature of a gauge theory looks like when we split the connection as the sum of a connection plus a tensor. Using eq. (2.54) we see that, in the 5-covariant basis, the field strength reads

$$\begin{aligned}\boldsymbol{F}_{\boldsymbol{\omega}_5+\boldsymbol{u}_5} &= \boldsymbol{R}_5(\boldsymbol{\omega}_5) + \tfrac{1}{2}\left[\boldsymbol{u}_5, \boldsymbol{u}_5\right] + \mathrm{D}_{\boldsymbol{\omega}_5}\boldsymbol{u}_5 \,, \\ &= \tfrac{1}{2}\Big(R_5^{AB}(\omega_5) - \tfrac{\eta}{\gamma^2}u^A \wedge u^B\Big)\boldsymbol{J}_{AB} + \mathrm{D}_{\omega_5}u^A\,\boldsymbol{T}_A \,,\end{aligned} \tag{2.59}$$

where

$$\boldsymbol{R}_5(\boldsymbol{\omega}_5) = \tfrac{1}{2}R^{AB}(\omega_5)\boldsymbol{J}_{AB} = \tfrac{1}{2}\Big(\mathrm{d}\omega^{AB} + \omega^A{}_C \wedge \omega^{CB}\Big)\boldsymbol{J}_{AB} \tag{2.60}$$

is called the (anti)-de Sitter curvature and $\mathrm{D}_{\omega_5}u^A = \mathrm{d}u^A + \omega^A{}_B \wedge u^B$, which can be seen as the torsion of the fünfbein u^A in the Cartan formalism. Moreover, in the 4-covariant basis we denote the connection as

$$\boldsymbol{\omega}_6 = \boldsymbol{\omega}_4 + \boldsymbol{s} + \boldsymbol{u}_4 + \boldsymbol{\mu} = \tfrac{1}{2}\omega^{ab}\boldsymbol{J}_{ab} + s^a\boldsymbol{B}_a + u^a\boldsymbol{T}_a + \mu\boldsymbol{D} \,, \tag{2.61}$$

i.e., $\boldsymbol{\omega}_5 = \boldsymbol{\omega}_4 + \boldsymbol{s}$ and $\boldsymbol{u}_5 = \boldsymbol{u}_4 + \boldsymbol{\mu}$. Here $\boldsymbol{\omega}_4$ is the Lorentz spin connection and we simbolize it simply as $\boldsymbol{\omega}_4 = \boldsymbol{\omega}$. Also, we denote $\boldsymbol{u}_4 = \boldsymbol{u}$. The field strength in the 4-covariant basis is computed using eq. (2.54). We get

$$\begin{aligned} \boldsymbol{F}_{\omega+s+u+\mu} &= \boldsymbol{R}(\boldsymbol{\omega}) + \mathrm{D}\boldsymbol{s} + \mathrm{D}\boldsymbol{u} + \mathrm{D}\boldsymbol{\mu} \\ &\quad + \tfrac{1}{2}[\boldsymbol{s},\boldsymbol{s}] + [\boldsymbol{s},\boldsymbol{u}] + [\boldsymbol{s},\boldsymbol{\mu}] + \tfrac{1}{2}[\boldsymbol{u},\boldsymbol{u}] + [\boldsymbol{u},\boldsymbol{\mu}] + \tfrac{1}{2}[\boldsymbol{\mu},\boldsymbol{\mu}] \ , \end{aligned} \tag{2.62}$$

where

$$\boldsymbol{R}(\boldsymbol{\omega}) = \tfrac{1}{2}R^{ab}(\omega)\boldsymbol{J}_{ab} = \tfrac{1}{2}\left(\mathrm{d}\omega^{ab} + \omega^{a}{}_{c}\wedge\omega^{cb}\right)\boldsymbol{J}_{ab} \tag{2.63}$$

is the Lorentz curvature with respect to the Lorentz spin connection ω^{ab} and

$$\mathrm{D}u^{a} = \mathrm{d}u^{a} + \omega^{a}{}_{b}\wedge u^{b} \ , \tag{2.64a}$$

$$\mathrm{D}s^{a} = \mathrm{d}s^{a} + \omega^{a}{}_{b}\wedge s^{b} \ , \tag{2.64b}$$

$$\mathrm{D}\mu = \mathrm{d}\mu \ , \tag{2.64c}$$

where we have symbolized the covariant derivative with respect to ω as $\mathrm{D}_{\omega} = \mathrm{D}$.

The notation that we use to denote the gauge connection in the 4-covariant canonical basis is

$$\boldsymbol{\omega}_{6} = \boldsymbol{\omega} + \boldsymbol{e} + \boldsymbol{\imath} + \boldsymbol{\mu} = \tfrac{1}{2}\omega^{ab}\boldsymbol{J}_{ab} + e^{a}\boldsymbol{P}_{a} + \imath^{a}\boldsymbol{K}_{a} + \mu\boldsymbol{D} \ . \tag{2.65}$$

The components of the fields e^a and $\imath^a$ will be interpreted as vierbein after the dimensional reduction of a 5-dimensional Chern–Simons gravity theory in § 5. Comparing this with eq. (2.61), it is straightforward to derive the relations $s^a = s^a(e,\imath)$ and $u^a = u^a(e,\imath)$ using the change of basis from the 4-covariant to the 4-covariant canonical basis (eq. (2.29)). We obtain

$$s^{a} = a\, e^{a} + c\, i^{a} \ , \tag{2.66a}$$

$$u^{a} = b\, e^{a} + d\, i^{a} \ . \tag{2.66b}$$

2.6.2 Gauge transformations

In the following, we show the gauge transformations induced by the special orthogonal group $\mathrm{SO}(4,2)$ in different covariant bases that we use in § 5 with the aim of discussing the symmetries of gravitational theories. This gauge transformation emerge according to the infinitesimal law of transformation for the gauge connection $\delta\boldsymbol{A} = \mathrm{D}\boldsymbol{\epsilon}$. Here $\boldsymbol{\epsilon} = \frac{1}{2}\epsilon^{IJ}\boldsymbol{J}_{IJ}$ is the zero-form gauge parameter, so that $\exp(\boldsymbol{\epsilon})$ is an element of the group SO(4, 2).

2.6.2.1 4-covariant basis

For the 4-covariant basis we symbolize the parameters of the corresponding subspaces as

$$\boldsymbol{\epsilon} = \tfrac{1}{2}\theta^{ab}\boldsymbol{J}_{ab} + \beta^a\boldsymbol{B}_a + \tau^a\boldsymbol{T}_a + \lambda\boldsymbol{D}\,. \tag{2.67}$$

Using the commutation relations of eq. (2.27), one finds the gauge transformations

$$\tfrac{1}{2}\delta\omega^{ab} = \tfrac{1}{2}\mathrm{D}\theta^{ab} + \eta\, s^{[a}\wedge\beta^{b]} - \gamma^{-2}u^{[a}\wedge\tau^{b]}\,, \tag{2.68a}$$
$$\delta s^a = \mathrm{D}\beta^a - \theta^a{}_b\wedge s^b + \eta\gamma^{-2}\Big(\tau^a\wedge\mu - u^a\wedge\lambda\Big)\,, \tag{2.68b}$$
$$\delta u^a = \mathrm{D}\tau^a - \theta^a{}_b\wedge u^b + \eta\,\Big(\beta^a\wedge\mu - s^a\wedge\lambda\Big)\,, \tag{2.68c}$$
$$\delta\mu = \mathrm{d}\lambda - \eta_{ab}\Big(s^{(a}\wedge\tau^{b)} - \beta^{(a}\wedge u^{b)}\Big)\,. \tag{2.68d}$$

Since one recovers the Poincaré symmetries ISO(3, 2) or ISO(4, 1) (for $\eta = \pm1$, respectively) from SO(4, 2) by means of the Inönü–Wigner limit $\gamma\to\infty$, we can easily calculate the 5-dimensional Poincaré transformations in the 4-covariant basis from in eq. (2.68). We get

$$\tfrac{1}{2}\delta\omega^{ab} = \tfrac{1}{2}\mathrm{D}\theta^{ab} + \eta\, s^{[a}\wedge\beta^{b]}\,, \tag{2.69a}$$
$$\delta s^a = \mathrm{D}\beta^a - \theta^a{}_b\wedge s^b\,, \tag{2.69b}$$
$$\delta u^a = \mathrm{D}\tau^a - \theta^a{}_b\wedge u^b + \eta\,\Big(\beta^a\wedge\mu - s^a\wedge\lambda\Big)\,, \tag{2.69c}$$
$$\delta\mu = \mathrm{d}\lambda - \eta_{ab}\Big(s^{(a}\wedge\tau^{b)} - \beta^{(a}\wedge u^{b)}\Big)\,. \tag{2.69d}$$

2.6.2.2 4-covariant canonical basis

We denote the transformations parameters in the 4-covariant canonical basis as

$$\boldsymbol{\epsilon} = \boldsymbol{\theta} + \boldsymbol{\rho} + \boldsymbol{b} + \boldsymbol{\lambda} = \tfrac{1}{2}\theta^{ab}\boldsymbol{J}_{ab} + \rho^a\boldsymbol{P}_a + b^a\boldsymbol{K}_a + \lambda\boldsymbol{D}\,. \tag{2.70}$$

In this basis, the gauge transformations read

$$\begin{aligned}
\tfrac{1}{2}\delta\omega^{ab} &= \tfrac{1}{2}\mathrm{D}\theta^{ab} + \eta\left(a^2 - b^2\gamma^{-2}\right) e^{[a} \wedge \rho^{b]} \\
&\quad - \eta\left(ac - bd\gamma^{-2}\right)\Big(\rho^{[a} \wedge \imath^{b]} \\
&\quad - e^{[a} \wedge b^{b]}\Big) + \eta\left(c^2 - d^2\gamma^{-2}\right) \imath^{[a} \wedge b^{b]}\,, && (2.71a)\\
\delta e^a &= \mathrm{D}\rho^a - \theta^a{}_b \wedge e^b \\
&\quad - \tfrac{\eta}{\det M}\Big[\Big(ac - bd\gamma^{-2}\Big)\Big(\rho^a \wedge \mu - e^a \wedge \lambda\Big) \\
&\quad + \Big(c^2 - d^2\gamma^{-2}\Big)\Big(b^a \wedge \mu - \imath^a \wedge \lambda\Big)\Big]\,, && (2.71b)\\
\delta\imath^a &= \mathrm{D}b^a - \theta^a{}_b \wedge \imath^b \\
&\quad + \tfrac{\eta}{\det M}\Big[\Big(a^2 - b^2\gamma^{-2}\Big)\Big(\rho^a \wedge \mu - e^a \wedge \lambda\Big) \\
&\quad + \Big(ac - bd\gamma^{-2}\Big)\Big(b^a \wedge \mu - \imath^a \wedge \lambda\Big)\Big]\,, && (2.71c)\\
\delta\mu &= \mathrm{d}\lambda - \det M\, \eta_{ab}\Big(e^{(a} \wedge b^{b)} - \rho^{(a} \wedge \imath^{b)}\Big)\,. && (2.71d)
\end{aligned}$$

Note that if the linear transformation M is the identity, one recovers the gauge transformations (cf. eq. (2.69)). For M leading to the orthogonal basis (see § 2.3.2.2), the gauge transformations are given by

$$\begin{aligned}
\tfrac{1}{2}\delta\omega^{ab} &= \tfrac{1}{2}\mathrm{D}\theta^{ab} + 2a^2\eta\, e^{[a} \wedge \rho^{b]} + 2c^2\eta\, \imath^{[a} \wedge b^{b]}\,, && (2.72a)\\
\delta e^a &= \mathrm{D}\rho^a - \theta^a{}_b \wedge e^b \mp \mathrm{i}\tfrac{a\eta}{c}\Big(b^a \wedge \mu - \imath^a \wedge \lambda\Big)\,, && (2.72b)\\
\delta\ell^a &= \mathrm{D}b^a - \theta^a{}_b \wedge \imath^b \pm \mathrm{i}\tfrac{c\eta}{a}\Big(\rho^a \wedge \mu - e^a \wedge \lambda\Big)\,, && (2.72c)\\
\delta\mu &= \mathrm{d}\lambda \pm 2\mathrm{i}ac\,\eta_{ab}\Big(e^{(a} \wedge b^{b)} - \rho^{(a} \wedge \imath^{b)}\Big)\,. && (2.72d)
\end{aligned}$$

2.7 Chern–Simons forms

On principal bundles, one can define certain polynomials in $\boldsymbol{A}$ and $\boldsymbol{F}$ that satisfy the remarkable property that, no matter which symmetry group, they are invariant under the transformations of eq. (2.43). A well-known example are the Chern–Simons and transgression forms. Let us consider two connections

$\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$. A transgression form is defined as the $(2r-1)$-form

$$Q^{(2r-1)}(\boldsymbol{A}, \bar{\boldsymbol{A}}) = r \int_0^1 \mathrm{d}t \, \left\langle \boldsymbol{F}_t, \ldots, \boldsymbol{F}_t, \boldsymbol{A} - \bar{\boldsymbol{A}} \right\rangle_{\mathrm{g}} \,, \tag{2.73}$$

where $\boldsymbol{F}_t$ is the curvature associated with the interpolating connection $\boldsymbol{A}_t$ defined as $\boldsymbol{A}_t(\boldsymbol{A}, \bar{\boldsymbol{A}}) = \bar{\boldsymbol{A}} + t\left(\boldsymbol{A} - \bar{\boldsymbol{A}}\right)$ and $\langle\ , \ldots, \ \rangle_{\mathrm{g}}$ is an invariant tensors of order r of the algebra g of the group G. For the differential form in eq. (2.73), we have $\delta Q^{(2r-1)}(\boldsymbol{A}, \bar{\boldsymbol{A}}) = 0$ under the gauge transformation defined as $\delta \boldsymbol{A} = \delta \bar{\boldsymbol{A}} = \mathrm{D}\boldsymbol{\epsilon}$.

Since in physics we have theories with their actions and corresponding symmetries, we can ask ourselves whether a transgression form can be used as an action. The program for that would be i) to choose a gauge group, ii) to identify its algebra and invariant tensors, iii) to gauge it by constructing the connection and field strength valued in the algebra, iv) to calculate the transgression and v) to propose that

$$S[\boldsymbol{A}, \bar{\boldsymbol{A}}] = \int Q^{(2r-1)}(\boldsymbol{A}, \bar{\boldsymbol{A}}) \tag{2.74}$$

is an action. It seems to be a good idea because the action will be automatically invariant under the gauge group. Gravitational theories have been constructed as transgression actions in refs. [46, 88, 89]. When we choose that $\bar{\boldsymbol{A}} = 0$, we recover the Chern–Simons form of order r

$$Q^{(2r-1)}(\boldsymbol{A}, 0) = Q^{(2r-1)}(\boldsymbol{A}) = r \int_0^1 \mathrm{d}t \, \langle \boldsymbol{F}_t, \ldots, \boldsymbol{F}_t, \boldsymbol{A} \rangle_{\mathrm{g}} \,. \tag{2.75}$$

Given that $\bar{\boldsymbol{A}}$ is a connection, then the equation $\bar{\boldsymbol{A}} = 0$ is not invariant under a gauge transformation, and therefore the Chern–Simons form is not as well. However, one can prove that the variation of the Chern–Simons form under the gauge transformation of eq. (2.43) is equal to an exact form, i.e. to an exterior derivative of some differential $r-1$ form. Thus they would correspond to boundary terms under an integral by means of the Stokes theorem. This makes that the Chern–Simons actions

$$S[\boldsymbol{A}] = \int Q^{(2r-1)}(\boldsymbol{A}) \tag{2.76}$$

are invariant under gauge transformations $\delta \boldsymbol{A} = \mathrm{D}\boldsymbol{\epsilon}$ modulo boundary terms or *quasi-invariant*. Since in this dissertation we do not make an extensive analysis with boundary terms, we will not distinguish between the terms invariant and quasi-invariant unless there is ambiguity in the context.

As a consequence of the extended Cartan homotopy formula from ref. [90], we have a triangle identity for a transgression forms in the form

$$Q^{(2r-1)}(\boldsymbol{A}, \bar{\boldsymbol{A}}) = Q^{(2r-1)}(\boldsymbol{A}, \tilde{\boldsymbol{A}}) + Q^{(2r-1)}(\tilde{\boldsymbol{A}}, \bar{\boldsymbol{A}}) + \mathrm{d}B \,, \tag{2.77}$$

which means that we can write down a general transgression form in terms of two transgression forms with an intermediate connection $\tilde{\boldsymbol{A}}$, plus an exact term where B is a $2r-2$ form. We can explore in the following what happens to a transgression action when we use the triangle identity of eq. (2.77). For the case $\tilde{\boldsymbol{A}} = 0$, which can be true only locally but with the invariance not being spoiled, we find that

$$\int Q^{(2r-1)}(\boldsymbol{A}, \bar{\boldsymbol{A}}) = \int Q^{(2r-1)}(\boldsymbol{A}) - \int Q^{(2r-1)}(\bar{\boldsymbol{A}}) \,, \tag{2.78}$$

which means that the transgression action can be written as the difference of similar Chern–Simons actions but for different connections, modulo boundary terms.

Now that we know more about the machinery of transgression forms, we can to say something more about when they are used to construct gauge theories in physics. In general, there are two issues related with transgression theories. i) If the Chern–Simons action for $\boldsymbol{A}$ is well defined and with any luck it describes successfully something that we observe in nature, then the kinetic term of the Chern–Simons action for $\bar{\boldsymbol{A}}$ will have the opposite sign in front of it and this could be the origin for ghosts in the transgression theory. ii) One has moreover to deal with the fact that we have two connections and its interpretation. The latter does not sound so bad if we remember bimetric gravity in the first-order formalism. In that case we have two vielbeine and two corresponding spin connections.

In § 5.1.5 we will work with an analogue of transgression action avoiding the problem of the ghost in the theory, but paying the price that we have to break down the gauge symmetry of the action to a residual gauge symmetry. In particular, we will start with a double conformal symmetry $\mathrm{SO}(4,2)\times\mathrm{SO}(4,2)$ and then break it down to $\mathrm{SO}(3,1)\times\mathrm{SO}(2)$. We will dimensionally reduce this theory to study its relation with first-order bimetric gravity.

Chapter 3

Spin-2 fields

In this chapter we review the theories for spin-2 fields. Although the original works were presented in the language of tensors and in four dimensions, we present it in differential forms and for arbitrary dimensions. Their equivalence with the theories in the language of differential forms and their extension to higher dimensions was established by Hinterbichler and Rosen in ref. [91]. Moreover, in the last section of this chapter, we present some of the results of ref. [2], where the mass spectrum for tri- and multimetric gravity with maximal global discrete symmetry was obtained. Since it is more convenient when studying mass eigenstates, we present results in that section in the language of tensors rather than in differential forms.

3.1 Einstein

Let us consider Einstein's theory of gravitation. Shall there be a fundamental particle which deals with interactions at quantum level, namely the graviton, its propagation on spacetime itself should be governed by the linear Einstein equations at the classical limit. To linearize the Einstein equations (see eq. (1.1)) one says that the dynamical degrees of freedom of the spacetime, namely the metric $g_{\mu\nu}(x)$, are part of the perturbation metric $h_{\mu\nu}(x)$ propagating on a Minkowski background $\eta_{\mu\nu}$ such that

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \epsilon\, h_{\mu\nu}(x) \,, \tag{3.1}$$

where ϵ is the parameter of the perturbation expansion. At second order in ϵ the Einstein equations read $\Box_{\mu\nu}{}^{\lambda\rho}\, h_{\lambda\rho} = 0$, where $\Box_{\mu\nu}{}^{\lambda\rho}$ is the non-linear second order differential operator on Minkowski spacetime defined, for a general background, in eq. (A.11) and it is known as the Lichnerowicz operator. The Taylor expansion of Einstein–Hilbert Lagrangian in powers of ϵ takes the form

$$S_{\rm EH}[g] = \int \mathrm{d}^D x \Big(\mathcal{L}^{(0)}_{\rm EH}(h) + \epsilon\, \mathcal{L}^{(1)}_{\rm EH}(h) + \epsilon^2\, \mathcal{L}^{(2)}_{\rm EH}(h) + \mathcal{O}(\epsilon^3) \Big) \,. \tag{3.2}$$

From the structure of the Riemann tensor as well as from the Christoffel symbols we see that $\mathcal{L}^{(0)}_{\rm EH}(h) = 0$ and therefore, there is not an extra contributions to the cosmological constant coming from this expansion.

Any of the terms $\mathcal{L}^{(n)}_{\rm EH}(h)$, with $n = 1, 2, \ldots$, will always include spacetime derivatives of $h_{\mu\nu}$. As we can see indeed from the linear Einstein equations, there is no a mass term for the graviton. In § 3.2 we will discuss what happens to the linear Einstein theory if we add by hand a mass term. The understanding of linear Einstein theory is fundamental to describe its connection with Newton's theory for gravity [92] as well as for the description of gravitational waves [93]. See also [94] for a historical review on the mathematics of gravitational waves.

3.2 Fierz–Pauli

The first attempt for massive gravity was made by Fierz and Pauli [19], who added a self-interacting massive term to the Einstein linear theory. Fierz–Pauli theory is dictated by the Lagrangian defined as $\mathcal{L}^{(2)}_{\rm EH}(h) + \mathcal{L}_{\rm FP}(h)$ where

$$\mathcal{L}_{\rm FP}(h) = -m^2 \Big((\eta^{\mu\nu} h_{\mu\nu})^2 - h^{\mu\nu} h_{\mu\nu} \Big) \,, \tag{3.3}$$

where m is in this context supposed to be the mass of the graviton $m = m_{\rm g}$ and it is called the Fierz–Pauli mass, often symbolized as $m = m_{\rm FP}$. Notice that $\eta^{\mu\nu}h_{\mu\nu}$ is sometimes referred in the literature as the trace of the matrix $h_{\mu\nu}$, which has been then calculated by raising and lowering indices with the Minkowski metric. The equations of motion for Fierz–Pauli theory read

$$\Box_{\mu\nu}{}^{\lambda\rho}\, h_{\lambda\rho} + \tfrac{1}{2}m^2\Big(h^{\lambda}{}_{\lambda}\eta_{\mu\nu} - h_{\mu\nu}\Big) = 0\,. \tag{3.4}$$

It was noticed later by van Dam, Veltman and Zakharov [95,96] that the light bending predicted by general relativity [5] differs from the one predicted by the Fierz–Pauli theory in the limit $m \to 0$. This phenomenon is known as the van Dam–Veltman–Zakharov discontinuity in the literature. For the sake of clarity, notice that in Einstein's theory one does not use a perturbation of the Minkowski metric to calculate the light bending, but on a Schwarzschild background. Also, one considers that the light-like particle goes by the source to a much larger distance than the Schwarzschild radius $r_{\rm S} = 2GM/c^2$, where M is the mass of the star.

Vainshtein observed that the van Dam–Veltman–Zakharov discontinuity appears as a result of working at linear level, i.e., for non-linear generalisations of the Fierz–Pauli theory considering $\mathcal{O}(\epsilon^3)$ terms, it turns out that, below the so-called Vainshtein radius

$$r_{\rm V} = \ell_{\rm P}\left(\frac{m_{\rm P}^3 M}{m_{\rm g}^4}\right)^{\frac{1}{5}}\,, \tag{3.5}$$

non-linear corrections are indeed relevant and the theory predicts the same as general relativity. However, above this radius the linear approximation is valid and therefore the deviations produced by the graviton mass become relevant for a larger scale [97], which might yield detectable deviations from general relativity predictions .

The recent observations of gravitational waves set up the lower limit for the Compton length of the graviton to be $\lambda_{\rm C,g} > 10^{16}\,\texttt{m}$ (see refs. [98,99]), therefore there is an upper bound for the graviton mass given by $m_{\rm g} \approx 2\times10^{-59}\,\texttt{kg}$. Eq. (3.5) gives then for $M = M_\odot$ a Vainshtein radius of $r_{\rm V,\odot} \approx 4\times10^{13}\,\texttt{m}$, which is approximately where the termination shock beyond the solar system begins. Local tests of general relativity remain therefore valid even if the graviton had a non-zero mass.

Although the theory of a massive graviton seemed to be saved by the argument of Vainshtain, some time later it was observed that at higher orders in ϵ the theory propagates the so-called Boulware-Deser ghost [100]. More specifically, any higher-order extension of Fierz–Pauli theory would introduce a ghost scalar. At the linear level this ghost mode is removed by the term $-m^2(\eta^{\mu\nu}h_{\mu\nu})^2$ in the Lagrangian of eq. (3.3). This result was deeply accepted for the scientific community and it froze the research towards a theory for the massive graviton almost for forty years.

3.3 De Rham–Gabadadze–Tolley

To the rescue of the general idea of a massive graviton, in the year 2010 a major breakthrough was made by De Rham, Gabadadze and Tolley who constructed a massive gravity theory at any order, i.e. when ϵ in $g_{\mu\nu}$ is not necessarily small (see refs. [23, 24]). This was made by introducing a special self-interaction potential for the metric $g_{\mu\nu}$ which curiously can be seen as the most general scalar defined between contractions of $g_{\mu\nu}$ and a reference Minkowski metric $\eta_{\mu\nu}$.

The theory was successfully proven to be free of ghosts propagation in ref. [48]. To present their action we go to the vielbein formalism through $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu\, \eta_{ab}$ and $\eta_{\mu\nu} = \delta^a{}_\mu \delta^b{}_\nu\, \eta_{ab}$. In terms of the one-form vielbeine defined by $e^a = e^a{}_\mu \mathrm{d}x^\mu$ and $\mathrm{d}x^a = \delta^a{}_\mu \mathrm{d}x^\mu$ the de Rham–Gabadadze–Tolley action reads

$$
\begin{aligned}
S_{\mathrm{dRGT}}[e] = & \int \epsilon_{a_1\ldots a_D}\, R^{a_1 a_2}(e) \wedge e^{a_3} \wedge \ldots \wedge e^{a_D} \\
& + \textstyle\sum_{i=0}^{D} \beta_i \int \epsilon_{a_1\ldots a_D}\, e^{a_1} \wedge \ldots \wedge e^{a_i} \wedge \mathrm{d}x^{a_{i+1}} \wedge \ldots \wedge \mathrm{d}x^{a_D}\, . \qquad (3.6)
\end{aligned}
$$

Here the quantities β_i are arbitrary parameters and for any of their values the theory is ghost-free. Moreover, in terms of this parameters we can express the cosmological constant $\Lambda = \Lambda(\beta_i)$ and the mass of the graviton $m = m(\beta_i)$ at the linear level. The action above represents a non-linear completion of the Fierz–Pauli theory. At the linear level it describes the propagation of a massive graviton.

3.4 General reference metric

In § 3.2 we discussed that it is important to know which metric raises the indices of $h_{\mu\nu}$ in order to define its trace $\imath^{\mu}{}_{\mu}$. In that case the metric that raises the index is the Minkowski metric $\eta_{\mu\nu}$. If one goes further to a general reference metric $\eta_{\mu\nu} \to f_{\mu\nu}$, then we have two metrics and one has to indicate carefully which one is used to raise and lower indices.

Massive gravity *à la* de Rham–Gabadadze–Tolley for a general metric, i.e., the generalised theory where the Minkowski metric is now a non-dynamical general reference metric $f_{\mu\nu}$ was introduced in ref. [47]. One defines further a second dynamical vielbein as $\imath^a = \imath^a{}_\mu \mathrm{d}x^\mu$, which corresponds to do the generalisation $\delta^a{}_\mu \to \imath^a{}_\mu$. The massive gravity theory reads

$$\begin{aligned} S_{\mathrm{gdRGT}}[e] = &\int \epsilon_{a_1 \dots a_D} R^{a_1 a_2}(e) \wedge e^{a_3} \wedge \dots \wedge e^{a_D} \\ &+ \textstyle\sum_{i=0}^{D} \beta_i \int \epsilon_{a_1 \dots a_D} e^{a_1} \wedge \dots \wedge e^{a_i} \wedge \imath^{a_{i+1}} \wedge \dots \wedge \imath^{a_D} \, . \end{aligned} \tag{3.7}$$

The action above represents a non-linear completion of the Fierz–Pauli theory when $f_{\mu\nu} = \eta_{\mu\nu}$ and it was proven to be ghost-free in ref. [49]. The reader might wonder why the new field $\imath^a$ does not appear as a functional dependence of the action $S_{\mathrm{gdRGT}}[e]$. Notice that the general reference vielbein $\imath^a$ is not a dynamical field, which means that there is not an equation of motion associated with it. The action $S_{\mathrm{gdRGT}}[e, \imath]$, i.e., the action defined by the right-handed side of (3.7) when $\imath^a$ is considered as a dynamical field, is equivalent to the Einstein–Hilbert theory on-shell, that is, when one integrates $\imath^a$ out.

3.5 Hassan–Rosen

Some solutions needed for cosmology are absent in the framework of massive gravity *à la* de Rham–Gabadadze–Tolley. For instances, the open Friedmann–Lemaître–Robertson–Walker solution, describing an expanding Universe in the context of general relativity, is not present in massive gravity for a Minkowski reference metric [101]. When going to the generalised theory for an arbitrary reference metric, the solutions open, flat and closed Friedmann–Lemaître–Robertson–Walker exist, however, they are unstable and therefore they cannot describe the Universe as we observe it [102].

It was then necessary to go further with the massive spin-2 field interactions. It was realized that, around a generic background $g_{\mu\nu}$, the general reference metric can be expressed in function of the curvature of $g_{\mu\nu}$ [57, 103–106]. One can think about the complete theory of interaction between two dynamical metrics $g_{\mu\nu}$ and $f_{\mu\nu}$, whose kinetic terms lead second-order differential equations, taking into account not re-introducing the Boulware–Deser ghost. One completes the de Rham–Gabadadze–Tolley action with the Einstein–Hilbert like kinetic term as

$$\begin{aligned} S_{\text{HR}}[e,\imath] = &\int \epsilon_{a_1\ldots a_D}\, R^{a_1a_2}(e) \wedge e^{a_3} \wedge \ldots \wedge e^{a_D} \\ &+ \alpha^{D-2} \int \epsilon_{a_1\ldots a_D}\, R^{a_1a_2}(\imath) \wedge \imath^{a_3} \wedge \ldots \wedge \imath^{a_D} \\ &+ \textstyle\sum_{i=0}^{D} \beta_i \displaystyle\int \epsilon_{a_1\ldots a_D}\, e^{a_1} \wedge \ldots \wedge e^{a_i} \wedge \imath^{a_{i+1}} \wedge \ldots \wedge \imath^{a_D}\,, \end{aligned} \tag{3.8}$$

where again, unlike de Rham–Gabadadze–Tolley gravity, $\imath^a$ is now a dynamical field with its own corresponding Einstein–Hilbert-like kinetic term and its own equation of motion. Also, α is parameter of the theory that can be understood as the ratio of the Planck masses for both Einstein–Hilbert terms, after normalizing the action with a global constant. Hassan and Rosen first proposed such a bimetric theory in ref. [25] in the language of tensors and it was proven to be ghost-free at any order in ref. [50]. In the action (3.8) the abbreviation "HR" stands for Hinterbichler and Rosen though, the first who expressed it in differential forms [91]. In this case, cosmological solutions are better behaved [52] and emerging instabilities can be considered as belonging to the early Universe era [70].

The true equivalence between the Hassan–Rosen and the Hinterbichler–Rosen actions is established as follows, which by the way, is analogous to the equivalence between the Einstein–Hilbert action written in tensors and in differential forms as discussed in § 1.1: the Hinterbichler–Rosen theory under the symmetry condition

$$e^a{}_{[\mu}\imath^b{}_{\nu]}\eta_{ab} = 0 \tag{3.9}$$

becomes the Hassan–Rossen action for the fields $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}$ and $f_{\mu\nu} = \imath^a{}_\mu \imath^b{}_\nu \eta_{ab}$. Moreover, the equality (3.9) follows from the equations of motion (see refs. [91, 107, 108]) and therefore the Hassan–Rosen tensorial gravity can be seen as an integrated out version of Hinterbichler–Rosen tetrad gravity. If there is no possible ambiguity we will use then the abbreviation "HR" for both Hassan–Rosen and Hinterbichler–Rosen actions.

Up to now, we have discussed only the action for the behavior of the fields without introducing a Lagrangian for matter. In general relativity the Lagrangian matter is assumed to have the dependence $\mathcal{L}_{\text{matter}} = \mathcal{L}_{\text{matter}}(g, \phi)$ such that the stress-energy tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta(\sqrt{|g|}\mathcal{L}_{\text{matter}})}{\delta g^{\mu\nu}} , \tag{3.10}$$

and where ϕ represents collectively any standard model field. On the other hand, it turns out in bimetric gravity that the entire theory, i.e. including an action for matter, admits only a matter coupling to one of the metrics because, otherwise, dangerous ghost instabilities appear [109, 110]. Thus, the action for bimetric gravity coupled to matter takes the form

$$S[g, f, \phi] = S_{\text{HR}}[g, f] + S_{\text{matter}}[g, \phi] . \tag{3.11}$$

This result generalises as well for ghost-free multimetric gravity theories: the matter coupling is allowed only to one metric, in order that the action does not lead to ghost instabilities.

3.5.1 Metric formulation

In secs. § 3.5.2 and § 3.7.2 we show some features of spin-2 field theory at the level of solutions. At that point it becomes more natural to work in the language of tensors rather than in differential forms. In the following we present the action for bimetric gravity as it was formulated by Hassan and Rosen in ref. [25], although we write it here for arbitrary dimensions. The

Hassan–Rosen action is given by

$$
\begin{aligned}
S_{\mathrm{HR}}[g,f] = &\int \mathrm{d}^D x\,\sqrt{|g|}\,\Big(R(g) + \alpha^{D-2}\,\sqrt{|g^{-1}f|}\,R(f)\Big)\\
&+ \textstyle\sum_{i=0}^{D}\beta_i \displaystyle\int \mathrm{d}^D x\,\sqrt{|g|}\,P_i(\sqrt{g^{-1}f})\,,
\end{aligned}
\tag{3.12}
$$

where $|M|$ denotes the determinant of a matrix M and $P_i(M)$ is the symmetric polynomial of i-order for a matrix M, which is defined as

$$
P_i(M) = \delta^{\mu_1\ldots\mu_i}_{\nu_1\ldots\nu_i} M^{\nu_1}{}_{\mu_1}\cdots M^{\nu_i}{}_{\mu_i}\,,
\tag{3.13}
$$

and also where the matrix $\sqrt{g^{-1}f}$ should be understood as a matrix function such that

$$
\Big(\sqrt{g^{-1}f}\sqrt{g^{-1}f}\Big)^{\mu}{}_{\nu} = g^{\mu\lambda}f_{\lambda\nu}\,.
\tag{3.14}
$$

This particular form for the interaction potential ensures the absence of the Boulware–Deser instability. The β's, however, can assume any value and the theory will still be healthy. Different values for the β's lead to rather different phenomenological predictions. We emphasize that this potential, although cumbersome in the tensors formalism because of the appearance of the square-root matrix in calculations, is just the most general interaction that one can write between two vielbeine as it can be seen in a more natural way in eq. (3.8). A useful notation for the potential that we use later on is

$$
V(g,f;\beta_i) = \sum_{n=0}^{D}\beta_i P_i(\sqrt{g^{-1}f})\,.
\tag{3.15}
$$

The equations of motion for the action (3.12) are given by

$$
\mathrm{EOM}(g_{\mu\nu}):\qquad \mathcal{G}_{\mu\nu}(g) + M^2\,V_{\mu\nu}(g,f;\beta_i) = 0\,,
\tag{3.16a}
$$

$$
\mathrm{EOM}(f_{\mu\nu}):\qquad \mathcal{G}_{\mu\nu}(f) + \alpha^{2-D}M^2\,V_{\mu\nu}(f,g;\beta_{D-i}) = 0\,,
\tag{3.16b}
$$

where $\mathcal{G}_{\mu\nu}(g)$ is the Einstein tensor associated with the metric $g_{\mu\nu}$ and

$$
V_{\mu\nu}(g,f;\beta_i) = -\frac{2}{\sqrt{|g|}}\frac{\partial(\sqrt{|g|}V(g,f;\beta_i))}{\partial g^{\mu\nu}}\,.
\tag{3.17}
$$

3.5.2 Mass spectrum

In general, exact solutions in bimetric gravity are difficult to calculate. There is, however, one certain solution to the equations of motion of particular interest because it allows to calculate straightforwardly the mass spectrum of the theory. This solution, which we denote as $g_{\mu\nu} = \bar{g}_{\mu\nu}$ and $f_{\mu\nu} = \bar{f}_{\mu\nu}$, is when both metrics are Einstein metrics, i.e., $R_{\mu\nu}(\bar{g}) \propto \bar{g}_{\mu\nu}$ and $R_{\mu\nu}(\bar{f}) \propto \bar{f}_{\mu\nu}$. The equations of motion imply that $\bar{f}_{\mu\nu} \propto \bar{g}_{\mu\nu}$. We consider the case when that proportionality constant is positive and we denote it as

$$\bar{f}_{\mu\nu} = c^2 \bar{g}_{\mu\nu} \,. \tag{3.18}$$

This solution tells us that the manifold defined by $\bar{g}_{\mu\nu}$ is Einstein-like. On that manifold as a background we define perturbations as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\, h^{(g)}_{\mu\nu} \,, \tag{3.19a}$$

$$f_{\mu\nu} = c^2 \bar{g}_{\mu\nu} + \epsilon\, h^{(f)}_{\mu\nu} \,. \tag{3.19b}$$

At second order in ϵ the equations of motion read

$$\bar{\Box}_{\mu\nu}{}^{\lambda\rho}\, G_{\lambda\rho} = 0 \,, \tag{3.20a}$$

$$\bar{\Box}_{\mu\nu}{}^{\lambda\rho}\, M_{\lambda\rho} + \tfrac{1}{2} m^2 \Big(M^{\lambda}{}_{\lambda} \bar{g}_{\mu\nu} - M_{\mu\nu} \Big) = 0 \,, \tag{3.20b}$$

where $\bar{\Box}_{\mu\nu}{}^{\lambda\rho}$ is the Lichnerowicz operator on the background $\bar{g}_{\mu\nu}$ and where, from the old two metric perturbations we defined two linear combinations as

$$G_{\mu\nu} = h^{(g)}_{\mu\nu} + \alpha^2 h^{(f)}_{\mu\nu} \,, \tag{3.21a}$$

$$M_{\mu\nu} = -\Big(c^2 h^{(g)}_{\mu\nu} - h^{(f)}_{\mu\nu} \Big) \,. \tag{3.21b}$$

Also, $m^2 = (\beta_1 + 2\beta_2 + \beta_3)/\alpha^2$. From our discussion of Einstein linear theory and Fierz–Pauli theory (secs. (3.1) and (3.2) respectively) we see that eqs. (3.20a) and (3.20b) describe one massless and one massive mode denoted by $G_{\mu\nu}$ and $M_{\mu\nu}$ respectively, both propagating on the maximally symmetric background $\bar{g}_{\mu\nu}$. This mass spectrum was first derived in ref. [53].

3.5.3 First-order formulation

We want to go now to the first-order formalism of bimetric gravity, i.e., to introduce two connections that are independent of the metric or, similarly, two spin connections that are independent of the vielbeine. In the tetrad approach, the first-order formulation is straightforward: one has to introduce a spin connection for each vielbein and then to use two copies of the vielbein postulate. Doing this, the action for bimetric gravity[1] in the first-order formalism is given by

$$\begin{aligned} S_{\mathrm{HR}}[\omega, \sigma, e, \imath] = & \int \epsilon_{abcd}\, R^{ab}(\omega) \wedge e^c \wedge e^d \\ & + \alpha^2 \int \epsilon_{abcd}\, R^{ab}(\sigma) \wedge \imath^c \wedge \imath^d \\ & + \textstyle\sum_{i=0}^{4} \beta_i \int \epsilon_{a_1 \ldots a_4}\, e^{a_1} \wedge \ldots \wedge e^{a_i} \wedge \imath^{a_{i+1}} \wedge \ldots \wedge \imath^{a_4} \,. \end{aligned} \tag{3.22}$$

When using the equations of motion for the spin connections ω^{ab} and σ^{ab} and additionally the two vielbein postulates for the connections $\Gamma(\omega, e)$ and $\Gamma(\sigma, \imath)$, as we did in eq. (1.3), the latter become then Levi-Civita connections respectively for the metrics $g_{\mu\nu}(e)$ and $f_{\mu\nu}(\imath)$. Therefore, the Einstein–Cartan kinetic terms become then Einstein–Hilbert terms when integrating out the spin connections.

A special bimetric model, the so-called partial massless gravity, was studied in ref. [111]. The motivation for this model starts at the linear level, in Fierz–Pauli theory defined on de Sitter spacetime. In the particular case that the Fierz–Pauli mass is related to the cosmological constant as $m^2 = 2\Lambda/3$, a local gauge symmetry for the perturbation $h_{\mu\nu}(x)$ emerges. Here the origin of the mass of the graviton is due to the curvature of the de Sitter space and one refers to the graviton as something "partially massless". This relation between the mass of the graviton and the cosmological constant is known as the Higuchi bound [112]. A remarkable fact is that from bimetric gravity, for the de Sitter background solution one can also define a particular theory, i.e., set of particular β's, for which the same gauge symmetry arises for the perturbation $M_{\mu\nu}(x)$ as discussed in § 3.5.2. This happens when $\beta_1 = \beta_3 = 0$

[1] For simplicity, we will work in $D = 4$ until the end of this chapter.

for the parameters in eq. (3.22) and in addition when

$$\beta_0 = \tfrac{1}{2\alpha^2}\beta_2\,, \tag{3.23a}$$

$$\beta_4 = \tfrac{1}{2}\alpha^2\,\beta_2\,. \tag{3.23b}$$

The particular bimetric model for this choice of β's is known as partially massless gravity. The Cartan formalism for the action takes the form

$$\begin{aligned} S_{\rm PM}[\omega,\sigma,e,\imath] = &\int \epsilon_{abcd}\, R^{ab}(\omega)\wedge e^c\wedge e^d \\ &+\alpha^2 \int \epsilon_{abcd}\, R^{ab}(\sigma)\wedge \imath^c\wedge \imath^d \\ &-M^2 \int \epsilon_{abcd}\Big(e^a\wedge e^b + \alpha^2 \imath^a\wedge \imath^b\Big)\wedge\Big(e^c\wedge e^d+\alpha^2\imath^c\wedge\imath^d\Big)\,, \end{aligned} \tag{3.24}$$

where we defined the mass parameter M through $M^2 = -\beta_2/(2\alpha^2)$. This particular model shows a particular similitude with conformal gravity: by setting both spin connections to be equal $\sigma^{ab}{}_\mu(\omega) = \omega^{ab}{}_\mu$ and making the "Wick rotation" $\alpha \to i\alpha$, we then get the action for conformal gravity in the first-order formalism (see ref. [67] and § 4.3 for more details). Here we notice that the ghost of conformal gravity appears when making α pure imaginary, since this introduces the wrong minus sign multiplying the kinetic term.

In ref. [113] a 4-dimensional gauge theory was constructed for the Wick rotated conformal symmetry SO(5, 1). The gauge theory describes spin-2 fields interacting with a vector field. Interestingly, the interaction potential for the spin-2 fields has the same form as in eq. (3.24), although the different kinetic term and the potential for the vector field gives a residual symmetry SO(3, 1) × SO(2). Whether this SO(5, 1)-invariant theory has ghost instabilities or not is an open question.

As we will discuss in § 5, the 5-dimensional Chern–Simons construction for the conformal group SO(4, 2) can be related to first-order conformal gravity under some dimensional reduction scheme and, in a similar way, the doubled Chern–Simons construction for the SO(4, 2) × SO(4, 2) can be related to bimetric gravity.

3.6 Hinterbichler–Rosen

In this section we present the generalisation of bimetric gravity to a many-interacting-metrics theory; the so-called multimetric gravity. Also, we discuss the particular case of maximal discrete global symmetry of the multimetric model, as we proposed in ref. [2], together with its implications in the proportional background solutions and in the mass eigenvalue problem.

Whether the generalisation of general relativity to N non-interacting metrics is consistent was investigated in ref. [114], to find out indeed, that such a theory is inconsistent unless $N = 1$. Hinterbichler and Rosen first developed then a set of theories for N interacting vielbeine [91]. However, it was later realized that only those theories whose vielbeine interact pair-wisely through the bimetric potential are free of ghost instabilities [115]. Furthermore, loop interactions, i.e., when a vielbein interacts only with a next one and this latter in addition with a next one and so on until closing the circle, also lead to inconsistent theories (see refs. [91,116–118]). Those consistent theories were first formulated by Hinterbichler and Rosen using the language of differential forms for N vielbeine $e^a_{(p)}$, with $p = 1, \ldots, N$. Their equivalent theories in tensors is established when a generalised symmetry condition $(e^a_{(p)})_{[\mu|}(e^b_{(q)})_{|\nu]}\eta_{ab} = 0$ is considered as a constraint (see eq. (3.9) for the bimetric case).

The generalisation from two to N dynamical interacting vielbeine is not difficult to guess if we consider only pairwise interactions, i.e., separated interaction terms each one exclusively between two vielbeine. One writes down an action with all the possible pairwise combinations between the vielbeine, exactly as in the gdRGT action. It turns out, however, that only some combinations in the potential make up a theory that does not propagate ghosts, as we previously mentioned. For example, in the case $N = 4$, the only two possible potentials are depicted in figures 3.1 and 3.2. For the star graph the vielbeine interacting only with one vielbein are called the satellite vielbeine.

An example of a ghost-free theory of $N = 12$ vielbeine interacting pair-wisely is depicted in figure 3.3. A generalisation to non-pairwise interactions that lead to theories free of ghosts propagation has been studied in ref. [108]: this new theory emerges for a pairwise multimetric theory in the star layout when

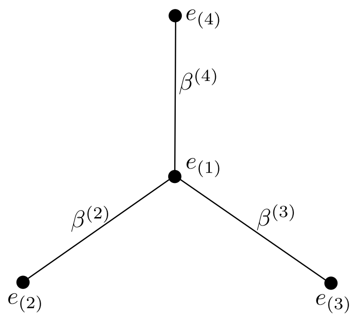

Figure 3.1: The ghost-free star graph for $N = 4$. The vielbeine $e^a_{(2)}$, $e^a_{(3)}$ and $e^a_{(4)}$ interact each one solely with $e^a_{(1)}$.

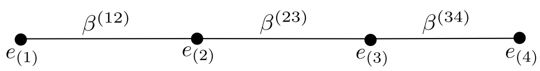

Figure 3.2: The ghost-free chain graph for $N = 4$. The vielbein $e^a_{(1)}$ interacts only with $e^a_{(2)}$, $e^a_{(2)}$ interacts only with $e^a_{(1)}$ and $e^a_{(3)}$, $e^a_{(3)}$ interacts only with $e^a_{(2)}$ and, by last, $e^a_{(4)}$ and $e^a_{(4)}$ interacts only with $e^a_{(3)}$.

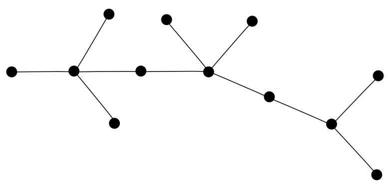

Figure 3.3: An example of a ghost-free 12-metric theory. Several combinations of the star and chain graphs are included, however no loop can be present otherwise one introduces a ghost.

integrating the central vielbein out. The interaction term is made up of the determinant of the sum of all satellite vielbeine.

3.7 Models with discrete symmetry

In this section we present our results of ref. [2] where we address the ♡ we develop in detail the multimetric theory in the case of maximal global discrete symmetry. We go then to the proportional backgrounds solution and then we perform the analysis of the mass eigenvalue problem. For simplicity we work out the latter only with three metrics, i.e., we study trimetric gravity case. The results, however, generalise to N spin-2 fields in a natural way, as discussed in section § 6.1.4.

3.7.1 Maximal discrete global symmetry

From the Hinterbichler–Rosen action we can see that 4-dimensional multi-metric gravity has $N-1$ α-parameters from the kinetic terms plus $5(N-1)$ β-parameters from the interaction potential. Moreover, we have a global constant in the action that eventually couples to a matter action. Different energy regimes can be then achieved by a multimetric gravity theory, therefore it would be useful to have a rule that relates parameters in a way we end up with a particular class of desired models. In this section we claim for maximal global discrete symmetries in the multimetric action. In particular, we look for theories with invariance under interchange of the highest number of metrics and, also, theories such that we have the same dynamics for both matrices $\pm\sqrt{g^{-1}f}$.

3.7.1.1 Interchange symmetry S_N

Invariance under the interchange of the highest possible number of metrics in a graph is present in the star graph. The action associated with the star graph of one central metric $g_{\mu\nu}$ and N satellite metrics $f^{(p)}$ has the form

$$S[g,f^{(p)}] = \int d^4x\,\sqrt{|g|}\left(R(g) + \textstyle\sum_{p=1}^{N}\alpha_{(p)}^2\sqrt{|g^{-1}f^{(p)}|}\,R(f^{(p)})\right) \\ - 2m^2\textstyle\sum_{p=1}^{N}\int d^4x\,\sqrt{|g|}\,V(g,f^{(p)};\beta_i^{(p)})\,. \tag{3.25}$$

From this point, we choose $g_{\mu\nu}$ to be the only (indeed the only allowed) metric that couples to matter.

We claim for invariance of the action under the S_N transformations:

$$f_{\mu\nu}^{(p)} \longleftrightarrow f_{\mu\nu}^{(q)}\,, \qquad \forall\ p,q = 1,\ldots,N\,, \tag{3.26}$$

which restricts the β-parameters in eq. (3.25). We see that, after doing the rescaling of the satellite metrics as $f_{\mu\nu}^{(p)} \to \tilde{f}_{\mu\nu}^{(p)} = \alpha_{(p)}^{-2} f_{\mu\nu}^{(i)}$, the sum of Einstein–Hilbert terms is symmetric under (3.26). From eq. (3.13) it can be easily seen that the symmetric polinomial in the bimetric potential satisfies $P_i(\lambda M) = \lambda^i P_i(M)$, where λ is a real function and, therefore, the potential becomes invariant when one demands that

$$\frac{\beta_i^{(p)}}{\alpha_{(p)}^i} = \frac{\beta_i^{(q)}}{\alpha_{(q)}^i}\,, \qquad \forall\ p,q = 1,\ldots,N\,. \tag{3.27}$$

Here, the letter i of the denominators represent a power and they should not be confused with labels, as they are for the β-parameters in the numerators. From the symmetry condition of eq. (3.27) we can write

$$\beta_i^{(p)} = \alpha_{(p)}^i\beta_i\,, \qquad \forall\ p = 1,\ldots,N\,, \tag{3.28}$$

for some set of parameters β_i. Thus, we are left now only with five free interaction parameters.

3.7.1.2 Reflection symmetry $(\mathbb{Z}_2)^N$

There is another discrete symmetry that we can claim for the N satellite metrics. We know that the potential depends on the matrices $\sqrt{g^{-1}f^{(p)}}$. From the bimetric case, eq. (3.14), we see that both $\pm\sqrt{g^{-1}f^{(p)}}$ satisfy the same definition. One can impose that

$$\sqrt{g^{-1}f^{(p)}} \longrightarrow -\sqrt{g^{-1}f^{(p)}} \qquad \forall\ p = 1, \dots, N\ , \tag{3.29}$$

leaves the potential invariant in order to get rid of such ambiguity. To see how the presence of the symmetry above constraints the parameters, it is easier to go to the vielbein language. The square-root matrix in terms of constrained vielbeine $e^a{}_\mu$ and $(\imath^a_{(p)})_\mu$ with $g_{\mu\nu} = \eta_{ab}e^a{}_\mu e^b{}_\nu$ and $f_{\mu\nu} = \eta_{ab}(\imath^a_{(p)})_\mu(\imath^b_{(p)})_\nu$ reads

$$\left(\sqrt{g^{-1}f^{(p)}}\,\right)^\mu{}_\nu = e_b{}^\mu(\imath^b_{(p)})_\nu\ . \tag{3.30}$$

For each one-form vielbein $\imath^a_{(p)}$ associated with $f^{(p)}_{\mu\nu}$, we consider then the transformations

$$\imath^a_{(p)} \longrightarrow -\imath^a_{(p)}\ , \qquad e^a \longrightarrow e^a\ . \tag{3.31}$$

This leaves all the metrics $g_{\mu\nu}$ and $f^{(p)}_{\mu\nu}$ invariant and changes the square-roots matrices as in eq. (3.29). The Einstein–Hilbert terms do not change since they all depend quadratically on the one-form vielbein. Moreover, the symmetric polinomials in the potential transform as

$$P_i(-\sqrt{g^{-1}f^{(p)}}) \longrightarrow (-1)^i P_i(\sqrt{g^{-1}f^{(p)}})\ . \tag{3.32}$$

From this we conclude that the potential will be invariant under (3.29) only if

$$\beta_1^{(p)} = \beta_3^{(p)} = 0 \qquad \forall\ p = 1, \dots, N\ . \tag{3.33}$$

3.7.1.3 Maximally symmetric action

Both conditions in eqs. (3.27) and (3.33) together lead us to an action for multimetric gravity with global $S_N \times (\mathbb{Z}_2)^N$ symmetry. The action results

$$
\begin{aligned}
S[g, f^{(p)}] = & \int d^4x \sqrt{|g|} \left(R(g) + \textstyle\sum_{p=1}^N \alpha_{(p)}^2 \sqrt{|g^{-1}f^{(p)}|}\, R(f^{(p)}) \right) \\
& - 2m^2 \textstyle\sum_{p=1}^N \int d^4x \sqrt{|g|}\, V(g, f^{(p)}; \alpha_{(p)}^i \beta_i) \,,
\end{aligned}
\tag{3.34}
$$

where $\beta_i \equiv \beta_i^{(p)}/\alpha_{(p)}^i$. The potentials have now the form

$$
\begin{aligned}
\sqrt{|g|}\, V(g, f^{(p)}; \alpha_{(p)}^i \beta_i) = & \ \beta_0 \sqrt{|g|} \\
& + \alpha_{(p)}^2 \beta_2 \sqrt{|g|}\, P_2(\sqrt{g^{-1}f^{(p)}}) + \alpha_{(p)}^4 \beta_4 \sqrt{|f^{(p)}|} \,,
\end{aligned}
\tag{3.35}
$$

with P_2 defined in eq. (3.13). To our knowledge there is no further global symmetries that can be imposed for the multimetric action (3.34). We observe that the parameters β_0 and β_4 are cosmological constant contributions while β_2 parametrizes the interaction between the center- and satellite-metrics.

3.7.2 Mass spectrum

In this section we review the mass spectrum of trimetric theory derived in [119], generalising the bimetric results of ref. [53] (see also § 3.5.2). At the end of the section we demand the maximal discrete global symmetry of the action, that we discussed in § 3.7.1.

The action for trimetric gravity is given by

$$
\begin{aligned}
S[g, f, k] = m_g^2 & \int \mathrm{d}^4x \sqrt{|g|} \Big(R(g) + \alpha_{(f)}^2 \sqrt{g^{-1}f}\, R(f) + \alpha_{(k)}^2 \sqrt{g^{-1}k}\, R(k) \Big) \\
- 2M^2 & \int \mathrm{d}^4x \sqrt{|g|} \Big(V(g, f; \beta_i^{(f)}) + V(g, k; \beta_i^{(k)}) \Big) \,,
\end{aligned}
\tag{3.36}
$$

where the interaction potentials V are defined in eq. (3.15). Here we have introduced the mass scale M for the entire potential as well as the global Planck mass m_g that normalizes the Einstein–Hilbert term for the metric $g_{\mu\nu}$.

The solution $g_{\mu\nu} = \bar{g}_{\mu\nu}$, $f_{\mu\nu} = \bar{f}_{\mu\nu}$ and $k_{\mu\nu} = \bar{k}_{\mu\nu}$ for proportional backgrounds is obtained through the ansatz

$$f_{\mu\nu} = c_{(f)}^2 g_{\mu\nu} \,, \tag{3.37a}$$
$$k_{\mu\nu} = c_{(k)}^2 g_{\mu\nu} \,, \tag{3.37b}$$

with $c_{(f)}$ and $c_{(k)}$ being constants, in complete analogy to eq. (3.18). For this ansatz, the equations of motion become

$$\mathrm{EOM}(g): \quad \mathcal{G}_{\mu\nu}(\bar{g}) + \left(\Lambda(\beta_i^{(f)}, c_{(f)}) + \Lambda(\beta_i^{(k)}, c_{(k)})\right) \bar{g}_{\mu\nu} = 0 \,, \tag{3.38a}$$
$$\mathrm{EOM}(f): \quad \mathcal{G}_{\mu\nu}(\bar{f}) + \tilde{\Lambda}(\beta_i^{(f)}, c_{(f)}, \alpha_{(f)})\, \bar{g}_{\mu\nu} = 0 \,, \tag{3.38b}$$
$$\mathrm{EOM}(k): \quad \mathcal{G}_{\mu\nu}(\bar{k}) + \tilde{\Lambda}(\beta_i^{(k)}, c_{(k)}, \alpha_{(k)})\, \bar{g}_{\mu\nu} = 0 \,, \tag{3.38c}$$

where

$$\Lambda(\beta_i^{(p)}, c_{(p)}) = m^2 \left(\beta_0^{(p)} + 3c_{(p)}\beta_1^{(p)} + 3c_{(p)}^2\beta_2^{(p)} + c_{(p)}^3\beta_3^{(p)}\right) \,, \tag{3.39a}$$
$$\tilde{\Lambda}(\beta_i^{(p)}, c_{(p)}, \alpha_{(p)}) = \frac{m^2}{\alpha_{(p)}^2 c_{(p)}^2} \left(c_{(p)}\beta_1^{(p)} + 3c_{(p)}^2\beta_2^{(p)} + 3c_{(p)}^3\beta_3^{(p)} + c_{(p)}^4\beta_4^{(p)}\right) \,, \tag{3.39b}$$

with $p = 1, 2$. Furthermore, since the Einstein tensor is invariant under constant rescalings of the metric, one finds the following conditions for the proportional background solution

$$\Lambda(\beta_i^{(f)}, c_{(f)}) + \Lambda(\beta_i^{(k)}, c_{(k)}) = \tilde{\Lambda}(\beta_i^{(f)}, c_{(f)}, \alpha_{(f)}) = \tilde{\Lambda}(\beta_i^{(k)}, c_{(k)}, \alpha_{(k)}) \,. \tag{3.40}$$

These determine the proportionality constants $c_{(f)}$ and $c_{(k)}$ in terms of the parameters of the theory.

3.7.2.1 Maximal global discrete symmetry

When the interchange symmetry S_2, as discussed in § 3.7.1.1, is present in the action then eq. (3.27) holds and

$$\alpha_{(f)}^2 c_{(f)}^2 = \alpha_{(k)}^2 c_{(k)}^2 \,, \tag{3.41}$$

solves the condition[2] $\tilde{\Lambda}(\beta_i^{(f)}, c_{(f)}, \alpha_{(f)}) = \tilde{\Lambda}(\beta_i^{(k)}, c_{(k)}, \alpha_{(k)})$. From now on, since all cosmological constant contributions in eq. (3.38) are equal, we will simply refer to them by the symbol Λ.

Following the standard procedure for the analysis of the mass spectrum, we derive the equations of motion for perturbations around the proportional backgrounds

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h^{(g)}_{\mu\nu} \,, \tag{3.42a}$$

$$f_{\mu\nu} = c^2_{(f)}\, \bar{g}_{\mu\nu} + h^{(f)}_{\mu\nu} \,, \tag{3.42b}$$

$$k_{\mu\nu} = c^2_{(k)}\, \bar{g}_{\mu\nu} + h^{(k)}_{\mu\nu} \,. \tag{3.42c}$$

The linearized equations of motion are not diagonal for the fluctuations above and thus they are not the mass eigenstates of the theory. After diagonalizing them, one finds that, for interaction parameters satisfying conditions of eqs. (3.27) and (3.33) —therefore we have $S_2 \times (\mathbb{Z}_2)^2 \simeq (\mathbb{Z}_2)^3$ invariance of the action—, the eigenstates of the mass matrix assume the form

$$G_{\mu\nu} = \frac{m_{\mathrm{P}} \left(h^{(g)}_{\mu\nu} + \alpha^2_{(f)}\, h^{(f)}_{\mu\nu} + \alpha^2_{(k)} h^{(g)}_{\mu\nu} \right)}{1 + \alpha^2_{(f)} c^2_{(f)} + \alpha^2_{(k)} c^2_{(k)}} \,, \tag{3.43a}$$

$$M_{\mu\nu} = -\frac{m_{\mathrm{P}} \left((\alpha^2_{(f)} c^2_{(f)} + \alpha^2_{(k)} c^2_{(k)})\, h^{(g)}_{\mu\nu} - \alpha^2_{(f)}\, h^{(f)}_{\mu\nu} - \alpha^2_{(k)}\, h^{(k)}_{\mu\nu} \right)}{(1 + \alpha^2_{(f)} c^2_{(f)} + \alpha^2_{(k)} c^2_{(k)}) \sqrt{\alpha^2_{(f)} c^2_{(f)} + \alpha^2_{(k)} c^2_{(k)}}} \,, \tag{3.43b}$$

$$\chi_{\mu\nu} = -\frac{m_{\mathrm{P}}\, \alpha_{(f)} \alpha_{(h)} \left(\frac{c_{(k)}}{c_{(f)}} h^{(f)}_{\mu\nu} - \frac{c_{(f)}}{c_{(k)}} h^{(k)}_{\mu\nu} \right)}{\sqrt{(1 + \alpha^2_{(f)} c^2_{(f)} + \alpha^2_{(k)} c^2_{(k)})(\alpha^2_{(f)} c^2_{(f)} + \alpha^2_{(k)} c^2_{(k)})}} \,, \tag{3.43c}$$

where we defined the Planck mass as

$$m^2_{\mathrm{P}} = m^2_g (1 + \alpha^2_{(f)} c^2_{(f)} + \alpha^2_{(k)} c^2_{(k)}) \,. \tag{3.44}$$

By calculating the linearized equations of motion in terms of the fluctuations defined in eq. (3.43), we see that the they have the following corresponding

[2]However, if we do not demand (3.33), there may exist other solutions.

squared Fierz–Pauli masses:

$$m_G^2 = 0\,, \tag{3.45a}$$
$$m_M^2 = \mathcal{A}(1 + \alpha_{(f)}^2 c_{(f)}^2 + \alpha_{(k)}^2 c_{(k)}^2) m^2\,, \tag{3.45b}$$
$$m_\chi^2 = \mathcal{A} m^2\,, \tag{3.45c}$$

with $\mathcal{A} = 2\beta_2^{(f)}/\alpha_{(f)}^2 = 2\beta_2^{(k)}/\alpha_{(k)}^2$. Those masses correspond to the eigenvalues of the matrix that has to be diagonalized, such that the linear equations of motion decouple in a way that exhibits the massless and the two mass eigenstates $G_{\mu\nu}$, $M_{\mu\nu}$ and $\chi_{\mu\nu}$ respectively. By using the background condition (3.41), the fluctuations, i.e., the mass eigenstates, can be expressed as

$$G_{\mu\nu} = \frac{m_{\rm P}}{1+\alpha^2}\Big(h_{\mu\nu}^{(g)} + \alpha_{(f)}^2\, h_{\mu\nu}^{(f)} + \alpha_{(k)}^2\, h_{\mu\nu}^{(k)}\Big)\,, \tag{3.46a}$$
$$M_{\mu\nu} = -\frac{m_{\rm P}}{\alpha(1+\alpha^2)}\Big(\alpha^2\, h_{\mu\nu}^{(g)} - \alpha_{(f)}^2\, h_{\mu\nu}^{(f)} - \alpha_{(k)}^2\, h_{\mu\nu}^{(k)}\Big)\,, \tag{3.46b}$$
$$\chi_{\mu\nu} = -\frac{m_{\rm P}}{\alpha\sqrt{1+\alpha^2}}\Big(\alpha_{(f)}^2\, h_{\mu\nu}^{(f)} - \alpha_{(k)}^2\, h_{\mu\nu}^{(k)}\Big)\,, \tag{3.46c}$$

where we defined the new parameter α through

$$\alpha^2 = \alpha_{(f)}^2 c_{(f)}^2 + \alpha_{(k)}^2 c_{(k)}^2 = 2\alpha_{(f)}^2 c_{(f)}^2 = 2\alpha_{(k)}^2 c_{(k)}^2\,. \tag{3.47}$$

The inverse relations are

$$h_{\mu\nu}^{(g)} = \frac{1}{m_{\rm P}}\Big(G_{\mu\nu} - \alpha\, M_{\mu\nu}\Big)\,, \tag{3.48a}$$
$$h_{\mu\nu}^{(f)} = \frac{\alpha}{2m_{\rm P}\alpha_{(f)}^2}\Big(\alpha\, G_{\mu\nu} + M_{\mu\nu} - \sqrt{1+\alpha^2}\,\chi_{\mu\nu}\Big)\,, \tag{3.48b}$$
$$h_{\mu\nu}^{(k)} = \frac{\alpha}{2m_{\rm P}\alpha_{(k)}^2}\Big(\alpha\, G_{\mu\nu} + M_{\mu\nu} + \sqrt{1+\alpha^2}\,\chi_{\mu\nu}\Big)\,. \tag{3.48c}$$

3.7.2.2 Enters dark matter

We present the action for trimetric gravity up to cubic terms in terms of the mass eigenstates in § B.3. There are some immediate implications for the phenomenology of trimetric gravity. The massive mode $\chi_{\mu\nu}$ is a spin-2 field that does not interact directly with the matter sector since, as discussed below eq. (3.10) for the bimetric case, the action for matter only includes couplings with $g_{\mu\nu}$ but $\chi_{\mu\nu}$ does not depend on the fluctuation $h^{(g)}_{\mu\nu}$, as seen from eq. (3.46c). This can be seen from the inverse relation (3.48a): adding the matter action $S[g,\phi]$ to the trimetric action of eq. (3.36) the coupling to matter in the quadratic action has the form

$$h^{(g)}_{\mu\nu} T^{\mu\nu} = \frac{1}{m_{\rm P}} \Big(G_{\mu\nu} - \alpha\, M_{\mu\nu} \Big) T^{\mu\nu}\,. \tag{3.49}$$

Here the stress-energy tensor is considered to be the source of small perturbations on the background $\bar{g}_{\mu\nu}$. Any matter coupling involving $\chi_{\mu\nu}$ is forbidden by the maximal global discrete symmetries. In addition, we see from eq. (3.45) that the mass of the massive mode $M_{\mu\nu}$ is larger than the mass of $\chi_{\mu\nu}$, with a factor of $\sqrt{1+\alpha^2}$. The massive graviton $\chi_{\mu\nu}$ is therefore prevented to decay into standard model particles and other massive spin-2 modes.

In § 6.1 we prove that $\chi_{\mu\nu}$ does not decay either into massless gravitons and therefore that the massive gravitons $\chi_{\mu\nu}$ are entirely stable. The heavier spin-2 particle $M_{\mu\nu}$ can decay into two lighter spin-2 particles ones $\chi_{\mu\nu}$, i.e. $M \to \chi\chi$ provided that $m_M > 2m_\chi$. Since $m_M^2 = (1+\alpha^2)m_\chi^2$ we must have $\alpha > \sqrt{3}$. The decay of the heavier mode into one massive particle $\chi_{\mu\nu}$ and a finite number of massless gravitons $G_{\mu\nu}$, i.e. $M \to G \cdots G\chi$, is not allowed again by the discrete symmetries, as explained in § 6.1.3. We therefore postulate the field $\chi_{\mu\nu}$ as the component of dark matter. In § 6.1.4 we carry out the generalisation of these results to N satellite-metric and maximal global discrete symmetry $S_N \times (\mathbb{Z}_2)^N$.

Chapter 4

Gauge theories for gravity

This chapter is devoted to the discussion of gauge formulations of gravity. In particular, we review the fact that the Einstein–Cartan action in four dimensions is not invariant under the gauge transformations induced by the Poincaré group. Moreover, we review the idea of Weyl on gauge theory, which led to the development of conformal gravity by Bach. By last, we review the theory from the first-order formulation point of view —namely, when the spin connection is independent of the vielbein.

4.1 Poincaré (non)-invariance of gravity

In this section we discuss the gauge invariance of standard general relativity under the Poincaré group. Namely, we study the invariance of the Einstein–Cartan Lagrangian under the local Poincaré transformations generated by $\boldsymbol{J}_{ab}$ and $\boldsymbol{P}_a$ when the gauge fields are ω^{ab} and e^a correspondingly. To show what happens with the invariance in different dimensions, we analyse the cases $D = 2+1$ and $D = 3+1$. In turns out that the (2+1)-dimensional action is indeed invariant but the (3+1)-dimensional one is not. This is due to the fact that the (3+1)-dimensional Einstein–Cartan Lagrangian is not linear in the vierbein, not as the $(2+1)$-dimensional Lagrangian which is linear in the dreibein.

4.1.1 Invariance of $(2+1)$-dimensional gravity

As discussed in § 1.4, according to the Lanczos–Lovelock generalisation of standard general relativity, the Einstein–Cartan action in 2+1 dimensions is given by

$$S_{\mathrm{EC}}[\omega, e] = \int \epsilon_{abc}\, R^{ab}(\omega) \wedge e^c \,, \tag{4.1}$$

where $a, b = 1, \ldots, p+q = 2+1$. The variation with respect to the fields reads

$$\delta_{\omega,e} S_{\mathrm{EH}}[\omega, e] = \int \epsilon_{abc} \Big(\delta R^{ab}(\omega) \wedge e^c + R^{ab}(\omega) \wedge \delta e^c \Big) \,. \tag{4.2}$$

Having started the analysis of § 2.3.1 from the $(2+1)$-dimensional special orthogonal algebra $\mathrm{SO}(2,2)$ instead of $\mathrm{SO}(4,2)$, we see that the $(2+1)$-dimensional Poincaré algebra span $\{\boldsymbol{J}_{ab}, \boldsymbol{P}_a\}$ is a subalgebra of $\mathrm{SO}(2,2)$ (see eq. (2.30) for $a^2 - b^2\gamma^{-2} = 0$).

Thus one can easily determine the Poincaré gauge transformation for the dreibein using the eq. (2.71b) for $b^a = \lambda = 0$. We get that $\delta e^a = \mathrm{D}\rho^a - \theta^a{}_b \wedge e^a$. Furthermore, from eq. (2.50) for the particular case of the Lorentz curvature $\boldsymbol{F} = \boldsymbol{R}(\boldsymbol{\omega})$ and zero-form parameter $\boldsymbol{\epsilon} = \boldsymbol{\theta}$ we have the following identity:

$$\delta R^{ab}(\omega) = -\theta^a{}_c \wedge R^{cb}(\omega) + \theta^b{}_c \wedge R^{ca}(\omega) \,. \tag{4.3}$$

Plugging in the variations for the vielbein and curvature into eq. (4.2) we get

$$\begin{aligned} \delta_{\omega,e} S_{\mathrm{EC}}[\omega, e] = \int \epsilon_{abc} \Big(& - \theta^a{}_d \wedge R^{db}(\omega) \wedge e^c \\ & + \theta^b{}_d \wedge R^{da}(\omega) \wedge e^c + R^{ab}(\omega) \wedge \mathrm{D}\rho^c - R^{ab}(\omega) \wedge \theta^c{}_d \wedge e^d \Big) \,. \end{aligned} \tag{4.4}$$

Since ϵ_{abc} is an invariant tensor for the $(2+1)$-dimensional Lorentz algebra $\mathrm{SO}(2,1)$ we have that $\mathrm{D}\epsilon_{abc} = 0$. Furthermore, from the Bianchi identity[1] $\mathrm{D}R^{ab}(\omega) = 0$, we note that

$$\int_M \epsilon_{abc}\, R^{ab}(\omega) \wedge \mathrm{D}\rho^c = \int_{\partial M} \epsilon_{abc}\, R^{ab}(\omega) \wedge \rho^c \,. \tag{4.5}$$

[1]Note that this identity arises in the context of gauge theory, as one can see from eq. (2.47) for the Lorentz group and Lorentz curvature $\boldsymbol{F} = \boldsymbol{R}(\boldsymbol{\omega})$.

Neglecting this boundary term, the variation of the action reads then

$$\delta_{\omega,e}S_{\rm EC}[\omega,e] = -\int \Big(\epsilon_{dbc}\theta^d{}_a + \epsilon_{adc}\theta^d{}_b + \epsilon_{abd}\theta^d{}_c\Big) \wedge R^{ab}(\omega) \wedge e^c \,, \tag{4.6}$$

thus using the identity of eq. (A.12) we obtain $\delta_{\omega,e}S_{\rm EH_{2+1}} = 0$. The 3-dimensional Einstein–Cartan theory is therefore Poincaré invariant.

4.1.2 Non-invariance of $(3+1)$-dimensional gravity

The Einstein–Cartan action in $3+1$ dimensions is given by

$$S_{\rm EC}[\omega,e] = \int \epsilon_{abcd}\, R^{ab}(\omega) \wedge e^c \wedge e^d \,. \tag{4.7}$$

Analogously as in § 4.1.1, the variation of this action with respect to the spin connection and the vierbein is

$$\delta_{\omega,e}S_{\rm EC}[\omega,e] = 2\int \epsilon_{abcd}\, R^{ab} \wedge e^c \wedge {\rm D}\rho^d \,. \tag{4.8}$$

With the aim of integrating by parts let us note that

$${\rm D}\Big(\epsilon_{abcd}\, R^{ab} \wedge e^c \wedge \rho^d\Big) = \epsilon_{abcd}\, R^{ab} \wedge T^c \wedge \rho^d - \epsilon_{abcd}\, R^{ab}(\omega) \wedge e^c \wedge {\rm D}\rho^d \,, \tag{4.9}$$

where we defined the torsion as $T^a(\omega,e) = {\rm D}_\omega e^a$; exactly as in the Cartan formalism. Neglecting the boundary term we find

$$\delta_{\omega,e}S_{\rm EC}[\omega,e] = 2\int \epsilon_{abcd}\, R^{ab}(\omega) \wedge T^c(\omega,e) \wedge \rho^d \,, \tag{4.10}$$

from where we see that the Einstein–Cartan action in four dimensions is explicitly non-invariant under the Poincaré group. This action is indeed locally Lorentz invariant because it is a contraction of (pseudo)-tensors, i.e. it is a Lorentz scalar. We see this fact explicitly from eq. (4.10) when $\rho^a = 0$, since in that case a Poincaré transformation becomes a Lorentz transformations.

The Cartan formalism considers independent spin connection and vielbein. A priori one does not assume a null torsion because the equation $T^a(\omega,e) = {\rm D}e^a = {\rm d}e^a + \omega^a{}_b \wedge e^c = 0$ would relate both independent fields, i.e. we would have an equation of the type $\omega^{ab} = \omega^{ab}(e)$. This would modify the degrees of freedom of the theory. Moreover, we cannot just impose $T^a(\omega, e) = 0$ as a constraint because this equation is not invariant under gauge transformations. To see this, consider eqs. (2.56) and (2.50) for the case of the $(3+1)$-dimensional Poincaré group with connection $\boldsymbol{A} + \boldsymbol{G} = \boldsymbol{\omega} + \boldsymbol{e}$ and zero-form gauge parameter $\boldsymbol{\epsilon} + \boldsymbol{\pi} = \boldsymbol{\theta} + \boldsymbol{\rho}$. Projecting out for the $\boldsymbol{P}_a$ generators we get that $\delta T^a = -R^{ab} \wedge \rho_b$ showing that, as long as we have Poincaré translations the torsion will generally change under gauge transformations.

By varying the Einstein–Cartan action with respect to ω^{ab} one finds that the equation of motion directly implies that $T^a(\omega, e) = 0$. Although the vierbein e^a is not a dynamical field as ω^{ab} —since the Lagrangian does not have derivatives of it and thus there is not a kinetic for it— the torsion being equals to zero makes it again dynamical. From eq. (4.10) we see that the action is invariant on-shell. This can be problematic when quantizing the theory because in the path integral formulation one wants to add every possible configuration and not only those for which the equation of motion are satisfied.

4.2 Conformal gravity

Gravity *à la* Einstein has a peculiar symmetry: its action is invariant under diffeomorphisms. Furthermore, written in the vielbein language, we find local Lorentz transformations (for more details see § 1.1). Local Lorentz transformations can be understood as a gauge symmetry because they are induced by the Lie group $\mathrm{SO}(3,1)$ and because this is not a change of coordinates but a change of the fields themselves in the form $e^a \to e'^a$. The concept of gauge symmetry was introduced by Weyl in ref. [54, 55] in an attempt of formulating a gravity theory that was invariant under the local transformation $g_{\mu\nu}(x) \to \phi^2(x) g_{\mu\nu}(x)$; the 4-dimensional Einstein–Hilbert action is not invariant under rescalings of the metric. This transformation has the name of Weyl dilation, Weyl transformation or conformal transformation in the Riemmanian sense —and not in the sense of the conformal transformations induced by the conformal group $\mathrm{SO}(4,2)$, which however, has spacetime dilations $x^\mu \to \lambda x^\mu$

as a subgroup of transformations[2].

Weyl invariance has become object of study given that string theory has to have scale invariance to be consistent: the Polyakov action has this symmetry. When quantizing the Polyakov action, the physical observables should not depend on the choice of the worldsheet metric $h_{\alpha\beta}$ and thus, the mentioned action should have enough local continuous symmetries such that one is able to get rid of those degrees of freedom. Weyl invariance allows us to do this. Further evidence relates Weyl invariance with the origin of mass [120].

An action for 4-dimensional gravity satisfying Weyl invariance was formulated by Bach in ref. [56]. This action is called the Weyl action and it is given by

$$S_{\mathrm{W}}[g] = \int \mathrm{d}^4x\, \sqrt{|g|}\, C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}\,, \tag{4.11}$$

where $C_{\mu\nu\lambda\rho}$ is the Weyl tensor defined in eq. (A.8). This theory is known as conformal or Weyl gravity and it is fully invariant under Weyl transformations. This 4-dimensional theory has a remarkable property: due to fact that, in $D = 4$, the integral of the Gauss–Bonnet term can be expressed as boundary term, the action of eq. (4.11) can be brought to the simple form

$$S_{\mathrm{CG}}[g] = \int \mathrm{d}^4x\, \sqrt{|g|}\, \left(R^{\mu\nu}(g)R_{\mu\nu}(g) - \tfrac{1}{3}R(g)^2\right)\,. \tag{4.12}$$

In this thesis we refer to this particular form of the action as conformal gravity in the second-order formalism. Written in this form, the action still has Weyl invariance.

4.2.1 Einstein-dilaton system

A usual redefinition of the Weyl rescaling is through the exponential function as $\phi(x) = e^{\Omega(x)}$. It is straightforward to compute that the transformation law of

[2]Here it is important to note that, Weyl dilations and spacetime dilations are transformations of different nature. However, when using the conformal group as the symmetry group for a gauge theory with two vielbeine, a Weyl rotation of the vielbeine $\delta e^a \sim \lambda \imath^a$ and $\delta \imath^a \sim \lambda e^a$ (cf. eqs. (2.71b) and (2.71c) and see § 4.3.2 for more information about Weyl rotations) is induced by the symmetry generator $\boldsymbol{D}$, which is the conformal Killing vector associated with the dilation $x^\mu \to \lambda x^\mu$.

the Ricci scalar under Weyl rescalings of the metric is given, in D dimensions, by

$$R(e^{2\Omega}g) = e^{-2\Omega}\Big(R(g) - 2(D-1)g^{\mu\nu}\nabla_\mu\nabla_\nu\Omega \\ - (D-1)(D-2)g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega\Big) . \tag{4.13}$$

This formula immediately motivates to write down an action for gravity that includes a scalar field counteracting the Weyl transformation such that the entire theory is Weyl invariant. By introducing the field Φ as

$$S_{\mathrm{ED}}[g,\Phi] = \int \mathrm{d}^D x\,\sqrt{|g|}\,\Phi^{-D}\left(\Phi^2 R(g) + (D-1)(D-2)g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi\right) , \tag{4.14}$$

we get the so-called Einstein-dilaton system, a scalar-tensor gravitational theory with invariance under the gauge transformations

$$g_{\mu\nu}(x) \longrightarrow \phi^2(x)\, g_{\mu\nu}(x) , \tag{4.15a}$$

$$\Phi(x) \longrightarrow \phi(x)\,\Phi(x) . \tag{4.15b}$$

The Einstein-dilaton system is an active subject of study since the classical models for inflation are mostly dilaton-based (see ref. [121] for a detailed review) and also since the dilaton is within the field content of (quantum) string theory (see e.g. ref. [122]).

4.3 First-order conformal gravity

In this section we study the action for conformal gravity in the first-order formalism. This action was first obtained in ref. [67] as a 4-dimensional gauge theory for the conformal group. Let us consider the following action for the two one-form vielbeine e^a and $\imath^a$ in four dimensions:

$$S[e,\imath] = \int \epsilon_{abcd}\Big(R^{ab}(e+\alpha\imath)\wedge\left[e^c\wedge e^d - \alpha^2\imath^c\wedge\imath^d\right] \\ + \tfrac{\eta}{2}\left[e^a\wedge e^b - \alpha^2\imath^a\wedge\imath^b\right]\wedge\left[e^c\wedge e^d - \alpha^2\imath^c\wedge\imath^d\right]\Big) , \tag{4.16}$$

where $\omega^{ab} = \omega^{ab}(e + \alpha\imath)$ is the spin connection related to the Levi-Civita connection as in eq. (1.3) but for the linear combination of two vierbeine $e^a + \alpha\,\imath^a$. The metric $g_{\mu\nu}$ related to the Levi-Civita connection $\Gamma(g)$ is obtained as $g = (e + \alpha\imath)^{\mathrm{T}}\eta(e + \alpha\imath)$. Let us define

$$E^a = e^a + \alpha\imath^a \,, \tag{4.17a}$$

$$I^a = e^a - \alpha\imath^a. \tag{4.17b}$$

The metric of the above mentioned Levi-Civita connection can be written as $g_{\mu\nu} = E^a{}_\mu E^b{}_\nu \eta_{ab}$. Let us also define the tensor $S_{\mu\nu} = E^a{}_\mu I^b{}_\nu \eta_{ab}$. In terms of those quantities the action of eq. (4.16) can then be written as

$$S[g,S] = \int \mathrm{d}^4x\,\sqrt{|g|}\,\Big(S^\mu{}_\mu R(g) - 2S^{\mu\nu}R_{\mu\nu}(g) - 2m^2\Big[(S^\mu{}_\mu)^2 - S^{\mu\nu}S_{\mu\nu}\Big]\Big)\,. \tag{4.18}$$

We aim in the following to integrate the field $S_{\mu\nu}$ out. Its equation of motion is given by

$$\tfrac{1}{4m^2}\Big(R(g)g_{\mu\nu} - 2R_{\mu\nu}(g)\Big) = S^\rho{}_\rho g_{\mu\nu} - S_{\mu\nu}\,, \tag{4.19}$$

which is an algebraic equation for $S_{\mu\nu}$. The general solution is

$$S_{\mu\nu} = \tfrac{1}{2m^2}\Big(R_{\mu\nu}(g) - \tfrac{1}{6}R(g)g_{\mu\nu}\Big)\,, \tag{4.20}$$

and by plugging in this into the action of eq. (4.18), we finally obtain the action for conformal gravity of eq. (4.12). This action is invariant under the rescaling $g_{\mu\nu}(x) \to \phi(x)^2 g_{\mu\nu}$ (or in the vielbein language, $e^a(x) \to \phi(x)\,e^a(x)$), i.e., Weyl transformations. We have seen then, that conformal gravity has an equivalent auxiliary action given by eq. (4.16).

4.3.1 First-order conformal gravity

Let us discuss, how it would be possible to go to the first-order formalism of conformal gravity. From the above discussion, one could think that doing

$$\omega^{ab} = \omega^{ab}(E) \longrightarrow \text{arbitrary } \omega^{ab}\,, \tag{4.21}$$

renders an equivalent first-order formulation. In this case, and also using the new vielbeine defined in eq. (4.17), we obtain

$$S[\omega, E, I] = \int \epsilon_{abcd}\Big(R^{ab}(\omega)\wedge E^b \wedge I^d - m^2\, E^a \wedge E^b \wedge I^c \wedge I^d\Big)\,, \tag{4.22}$$

However, it is straightforward to see that the spin connection $\omega^{ab}(E)$ is not a solution to the equations of motion of the action of eq. (4.22) and therefore, one cannot obtain the action of eq. (4.16) back from it. This occurs as well for $\omega^{ab}(I)$, because of the symmetry $E^a \leftrightarrow I^a$ of the action.

The action of eq. (4.22) was obtained in ref. [67] as a 4-dimensional gauge theory for the conformal group. It is invariant under conformal transformations only if the spin connection is related to the Levi-Civita connection through the vielbein postulate $\omega^{ab} = \omega^{ab}(E)$ and adding a correction term, which drops out from the final action when an auxiliary field is integrated out: the resulting action the conformal gravity of eq. (4.12). For completeness: the action of first-order conformal gravity in terms of the vielbeine e^a and $\imath^a$ is given by

$$\begin{aligned} S[\omega, e, \imath] = \int \epsilon_{abcd}\Big(& R^{ab}(\omega)\wedge\Big[e^c\wedge e^d - \alpha^2 \imath^c\wedge\imath^d\Big] \\ &+ \tfrac{\eta}{2}\Big[e^a\wedge e^b - \alpha^2\imath^a\wedge\imath^b\Big]\wedge\Big[e^c\wedge e^d - \alpha^2\imath^c\wedge\imath^d\Big]\Big)\,. \end{aligned} \tag{4.23}$$

4.3.2 Weyl rotations

A new symmetry emerges when going to an arbitrary spin connection: the action of eq. (4.23) is invariant under the SO(1,1) rotation of the vector defined as

$$\begin{pmatrix} e'^a(x) \\ \imath'^a(x)\end{pmatrix} = \begin{pmatrix} \cosh\phi(x) & \sinh\phi(x) \\ \sinh\phi(x) & \cosh\phi(x)\end{pmatrix}\begin{pmatrix} e^a(x) \\ \imath^a(x)\end{pmatrix}\,. \tag{4.24}$$

In the same spirit of a Weyl dilation $e^a(x) \to \phi(x)\, e^a(x)$ (see § 4.2), we refer to this rotation as a Weyl rotation. An infinitesimal Weyl rotation is given by

$$\delta e^a(x) = \phi(x)\, \imath^a(x)\,, \tag{4.25a}$$

$$\delta \imath^a(x) = \phi(x)\, e^a(x)\ . \tag{4.25b}$$

The set of Weyl rotations makes up a subgroup of the conformal group. This can be easily seen from eq. (2.72) with $\theta^{ab} = \rho^a = b^a = 0$ with the particular choice of parameters $c = \pm \mathrm{i} a\eta$. For this choice, a subset of conformal transformations coincide with the transformations in eq. (4.25).

Chapter 5

Conformal Chern–Simons theory

This chapter is devoted to present the results that we obtained in ref. [1]. We discuss the 5-dimensional Chern–Simons theory for the orthogonal group $SO(4,2)$ and we express the gauge fields in all the covariant bases discussed in § 2.3. One vierbein can be identified as a subset of the components of the fünfbein and a second vierbein can be identified as a subset of the components of the 5-dimensional spin connection. We show how one can perform a dimensional reduction in the 5-dimensional theory getting in the end 4-dimensional Einstein–Cartan theory. Also, in a similar setup, we perform a dimensional reduction to get 4-dimensional first-order conformal gravity. We construct then a doubled Chern–Simons theory and we break the gauge symmetry as $SO(4,2) \times SO(4,2) \to \text{Lorentz} \times \text{Dilation}$. In this case we have two vielbeine and two spin connections and the dimensional reduced theory coincides with the first-order version of bimetric gravity. In all the mentioned cases we discuss in which conditions the dimensional reduction can be seen as a gauge-fixing using the gauge redundancy of the Chern–Simons theory.

5.1 The action

We consider the Chern–Simons theory defined by a Chern–Simons form $Q^{(5)}$ integrated on a 5-dimensional manifold M_5. The mathematical details of a Chern–Simons theory were discussed in § 2.7. According to eq. (2.75) for a 5-order form ($r = 3$), after integrating the parameter t in along the interval $[0, 1]$, the Chern–Simons action takes the form

$$\begin{aligned} S_{\mathrm{CS}}[\boldsymbol{A}] = \int_{M_5} \Big\langle & \boldsymbol{F} \wedge \boldsymbol{F} \wedge \boldsymbol{A} \\ & - \tfrac{1}{2} \boldsymbol{F} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} + \tfrac{1}{10} \boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \Big\rangle \,, \end{aligned} \tag{5.1}$$

In the following we exhibit this action explicitly valued in the gauge algebra $\mathrm{G} = \mathrm{SO}(4, 2)$ choosing the different covariant bases discussed in § 2.3.1.

5.1.1 6-covariant basis

As discussed in the mathematical prelude (§ 2.6.1), in this basis the gauge connection takes the form $\boldsymbol{A} = \boldsymbol{\omega}_6 = \frac{1}{2} \omega^{IJ} \boldsymbol{J}_{IJ}$. The object ω^{IJ} can be seen as a spin connection since that one can associate it the curvature form $\boldsymbol{F} = \boldsymbol{R}_6(\boldsymbol{\omega}_6)$ with components $R_6^{IJ}(\omega_6) = \mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}$. This curvature satisfies

$$\mathrm{d}Q^{(5)}(\omega_6) = \langle \boldsymbol{R}_6(\boldsymbol{\omega}_6), \boldsymbol{R}_6(\boldsymbol{\omega}_6), \boldsymbol{R}_6(\boldsymbol{\omega}_6) \rangle \ , \tag{5.2a}$$
$$= \tfrac{1}{8}\, \epsilon_{IJKLMN} R_6^{IJ}(\omega_6) \wedge R_6^{KL}(\omega_6) \wedge R_6^{MN}(\omega_6) \,. \tag{5.2b}$$

where we have used the invariant tensor of $\mathrm{SO}(4, 2)$ of the Euler class (eq. (2.24)).

5.1.2 5-covariant basis

In the following we express the Chern–Simons action of eq. (5.1) in the 5-covariant of the orthogonal algebra $\mathrm{SO}(4, 2)$ discussed in § 2.3.1. In this case the gauge connection splits according to the subspaces decomposition of $\mathrm{SO}(4, 2) = V_0 \oplus V_1$ where V_0 is the AdS or dS subalgebra (generated by

$\boldsymbol{J}_{AB}$) and V_1 is the subspace of 5-dimensional non-commutative translations, $\boldsymbol{T}_A$. Namely the connection splits as in eq. (2.58).

The aim now is to write the action of eq. (5.1) in terms of 5-covariant objects. For this we use the subspace separation method of ref. [90]. The method consists in finding a subspace decomposition for the vectorial space SO(4, 2), for example $\mathrm{SO}(4,2) = V_0 \oplus V_1$ such that we can use the triangle formula of eq. (2.77) to write the Chern–Simons form. In our case $\boldsymbol{A}$ is the connection valued in the trivial subspace $V_0 \oplus V_1$, $\tilde{\boldsymbol{A}}$ is the connection valued in a non-trivial subspace V_0 or V_1 and $\bar{\boldsymbol{A}} = 0$ is the (local) connection valued in the trivial subspace which contain the neutral element. For the calculation it is convenient to choose V_0 as the non-trivial subspace, which means that the connections, with which we are working out, are

$$\boldsymbol{A} = \boldsymbol{\omega}_5 + \boldsymbol{u}_5 \,, \tag{5.3a}$$

$$\tilde{\boldsymbol{A}} = \boldsymbol{\omega}_5 \,, \tag{5.3b}$$

$$\bar{\boldsymbol{A}} = 0 \,. \tag{5.3c}$$

We put then this connections back into eq. (2.77) to calculate the Chern–Simons form $Q^{(5)}(\boldsymbol{A}, 0) = Q^{(5)}(\boldsymbol{A})$. Looking the invariant tensors of the algebra expressed in the 5-covariant basis (eq. (2.26)) we see that $Q^{(5)}(\tilde{\boldsymbol{A}}, \bar{\boldsymbol{A}})$ is zero, thus the only contribution to the Chern–Simons action is given by $Q^{(5)}(\boldsymbol{A}, \tilde{\boldsymbol{A}})$. A straightforward calculation of this transgression form results in the action

$$\begin{aligned} S_{\mathrm{CS}}[\omega_5, u_5] = \tfrac{1}{4} \int_{M_5} \epsilon_{ABCDE} \Big(& R_5^{AB}(\omega_5) \wedge R_5^{CD}(\omega_5) \wedge u^E \\ & - \tfrac{2\eta}{3\gamma^2} R_5^{AB}(\omega_5) \wedge u^C \wedge u^D \wedge u^E + \tfrac{1}{5\gamma^4} u^A \wedge u^B \wedge u^C \wedge u^D \wedge u^E \Big) \,. \end{aligned} \tag{5.4}$$

This action corresponds to the 5-dimensional Chern–Simons gravity for the gauge group $\mathrm{AdS}_{4+1} = \mathrm{SO}(4,2)$, where u^A is the fünfbein form. The first term is the dimensionally continued Gauss–Bonnet term, however it is not topological as its even-dimensional versions. The second and the third terms are the 5-dimensional Einstein–Cartan and cosmological constants actions written in differential forms respectively. The action of eq. (5.4) was proposed as gravity theory in five and any odd dimensions in ref. [36].

In ref. [35] it was proven that, imposing to the $(2n+1)$-dimensional Lanczos–Lovelock theory to have the maximum possible number of degrees of freedom, the coefficients of the Lanczos–Lovelock Lagrangian are such that the Lagrangian becomes the Chern–Simons form for the gauge group $\mathrm{AdS}_{2n+1} = \mathrm{SO}(2n+1,2)$. The action of eq. (5.4) is therefore a particular case of the 5-dimensional Lanczos–Lovelock theory.

5.1.3 4-covariant basis

In the following, we decompose the indices into the 4-covariant basis as discussed in § 2.3.1. The connection splits further up in the 4-covariant basis as described by eq. (2.61). Using this, we obtain the action

$$\begin{aligned} S_{\mathrm{CS}}[\omega,s,u,\mu] = \tfrac{1}{4}\int_{M_5} \epsilon_{abcd}\Big(& R^{ab}(\omega)\wedge R^{cd}(\omega) \\ & + 2\eta\, R^{ab}(\omega)\wedge s^c\wedge s^d - \tfrac{2\eta}{\gamma^2}\, R^{ab}(\omega)\wedge u^c\wedge u^d \\ & + s^a\wedge s^b\wedge s^c\wedge s^d - \tfrac{2}{\gamma^2}\, s^a\wedge s^b\wedge u^c\wedge u^d + \tfrac{1}{\gamma^4}\, u^a\wedge u^b\wedge u^c\wedge u^d\Big)\wedge\mu \\ & - \int_{M_5}\epsilon_{abcd}\Big(R^{ab}(\omega)\wedge T^c(\omega,s) + \eta\, T^a(\omega,s)\wedge s^b\wedge s^c \\ & - \tfrac{\eta}{3\gamma^2}\, T^a(\omega,s)\wedge u^b\wedge u^c\Big)\wedge u^d\,. \end{aligned} \tag{5.5}$$

We observe that neglecting μ and forgetting the coordinate dependence of the fields ω^{ab}, s^a and u^a then the first integral becomes similar to the first-order conformal gravity action (see § 4.3). This gives us an insight on how to perform a dimensional reduction to get such gravity action. Furthermore, doing $u^a = 0$ we get an action similar to the one of the Einstein–Cartan theory. Again, still with the task of integrating along the Σ to get the actual theory. In § 5.2.1 we will study two different dimensional reduction schemes for which the action of eq. (5.5) becomes related to Einstein–Cartan and conformal gravity, respectively.

5.1.4 4-covariant canonical basis

In this basis, the gauge connection $\boldsymbol{\omega}_6$ splits as described in § 2.6.1. Via the redefintion of eq. (2.66) we compute the Chern–Simons action in the 4-covariant canonical basis. It reads

$$\begin{aligned}
S_{\text{CS}}[\omega, e, \imath, \mu] = \tfrac{1}{4} \int_{M_5} \epsilon_{abcd}\, R^{ab}(\omega) \wedge R^{cd}(\omega) \wedge \mu \\
+ \tfrac{\eta}{2} \int_{M_5} \epsilon_{abcd} \Big(A\, R^{ab}(\omega) \wedge e^c \wedge e^d + 2B\, R^{ab}(\omega) \wedge e^c \wedge \imath^d \\
+ C\, R^{ab}(\omega) \wedge \imath^c \wedge \imath^d \Big) \wedge \mu \\
+ \tfrac{1}{4} \int_{M_5} \epsilon_{abcd} \Big(A^2\, e^a \wedge e^b \wedge e^c \wedge e^d + 4AB\, e^a \wedge e^b \wedge e^c \wedge \imath^d \\
+ 2(AC + 2B^2)\, e^a \wedge e^b \wedge \imath^c \wedge \imath^d + 4BC\, e^a \wedge \imath^b \wedge \imath^c \wedge \imath^d \\
+ C^2\, \imath^a \wedge \imath^b \wedge \imath^c \wedge \imath^d \Big) \wedge \mu \\
- \int_{M_5} \epsilon_{abcd}\, R^{ab}(\omega) \wedge \Big(ab\, T^c(\omega, e) \wedge e^d + ad\, T^c(\omega, e) \wedge \imath^d \\
+ bc\, T^c(\omega, \imath) \wedge e^d + cd\, T^c(\omega, \imath) \wedge \imath^d \Big) \\
- \eta \int_{M_5} \epsilon_{abcd} \left(a\, T^a(\omega, e) + c\, T^a(\omega, \imath) \right) \wedge \Big(b \left(A + \tfrac{2}{3} b^2 \gamma^{-2} \right) e^b \wedge e^c \wedge e^d \\
+ \left(d \left(A + \tfrac{2}{3} b^2 \gamma^{-2} \right) + 2b \left(B + \tfrac{2}{3} bd \gamma^{-2} \right) \right) e^b \wedge e^c \wedge \imath^d \\
+ \left(b \left(C + \tfrac{2}{3} d^2 \gamma^{-2} \right) + 2d \left(B + \tfrac{2}{3} bd \gamma^{-2} \right) \right) e^b \wedge \imath^c \wedge \imath^d \\
+ d \left(C + \tfrac{2}{3} d^2 \gamma^{-2} \right) \imath^b \wedge \imath^c \wedge \imath^d \Big) \,,
\end{aligned} \tag{5.6}$$

where we have defined the parameter combinations

$$A = a^2 - b^2 \gamma^{-2} \,, \tag{5.7a}$$

$$B = ac - bd \gamma^{-2} \,, \tag{5.7b}$$

$$C = c^2 - d^2 \gamma^{-2} \,. \tag{5.7c}$$

This is the 5-dimensional Chern–Simons action which is invariant under the parametrized conformal group $\mathrm{SO}(4,2) \simeq \mathrm{C}_{3+1}(M, \gamma)$, i.e., under the gauge transformations defined in eq. (2.71).

The first integral of the action in eq. (5.6) is similar to the Gauss–Bonnet boundary term (cf. eq. (A.10)), however the integrand is wedge μ and ω^{ab} has components and dependence in the fifth dimension, namely,

$$\omega^{ab} = \omega^{ab}{}_{m}(X)\,\mathrm{d}X^m\,, \tag{5.8}$$

with $X^m = (X^\mu = x^\mu, X^5 = w)$. This term is therefore not a boundary term as it happens with the pure 4-dimensional Gauss–Bonnet term. The same behavior is seen in the second and third integrals: the second integral has terms similar to the Einstein–Cartan actions for the vielbein e^a and $\imath^a$ plus and additional kinetic term. Remarkably, the third integral has all possible combinations between both vielbeine, similarly as in potential of 4-dimensional bimetric gravity (cf. eq. (3.22)). The action cannot be truly bimetric-like, though, because it has only a single spin connection. In § 5.1.5 we formulate a doubled Chern–Simons to solve this problem. The rest of the terms involve Lagrangian densities proportional to the torsions of the vielbein components e^a and $\imath^a$.

In the following we discuss the two special cases for the general action of eq. (5.6), which are given by the conformal and the orthogonal bases, i.e. when i) $A = C = 0$ and ii) $B = 0$ respectively.

5.1.4.1 Conformal basis

By choosing the parameters of the matrix M as in eq. (2.31), the action of eq. (5.6) takes the simpler form:

$$\begin{aligned} S_{\mathrm{CS}}[\omega, e, \imath, \mu] = \tfrac{1}{4}\int_{M_5} \epsilon_{abcd}\Big(&R^{ab}(\omega)\wedge R^{cd}(\omega) \\ &+ 8ac\eta\, R^{ab}(\omega)\wedge e^c \wedge \imath^d + 16a^2c^2\, e^a\wedge e^b\wedge \imath^c\wedge \imath^d\Big)\wedge\mu \\ &+ \gamma\, S_{\mathrm{torsion}}[\omega, e, \imath]\,. \end{aligned} \tag{5.9}$$

Here S_{torsion} is the action defined in eq. (B.11). These terms include the torsions of the vielbein components e^a and $\imath^a$. We note that non-torsional part of the action is similar to the action for first-order conformal gravity (cf. eq. (4.22)), including also the term that is similar to the Gauss–Bonnet boundary term. The torsional terms are required in order that the theory has the SO(4, 2) gauge invariance. Notice that that part of the action is proportional to the parameter γ, therefore we can get rid of the torsional terms by taking the limit $\gamma \to 0$. This limit, however, makes algebra ill-defined since the commutation relations (see eq. (2.32)) diverge and, therefore, they cannot lead to well-defined gauge transformations.

5.1.4.2 Orthogonal basis

Choosing the parameters of the matrix M that lead to the orthogonal basis, i.e., such that they satisfy eq. (2.34), the action of eq. (5.6) takes the form

$$\begin{aligned} S_{\text{CS}}[\omega, e, \imath, \mu] = \tfrac{1}{4} \int_{M_5} \epsilon_{abcd} \Big(& R^{ab}(\omega) \wedge R^{cd}(\omega) \\ & + 4a^2\eta\, R^{ab}(\omega) \wedge e^c \wedge e^d + 4c^2\eta\, R^{ab}(\omega) \wedge \imath^c \wedge \imath^d \\ & + 4a^4\, e^a \wedge e^b \wedge e^c \wedge e^d + 8a^2c^2\, e^a \wedge e^b \wedge \imath^c \wedge \imath^d \\ & + 4c^4\, \imath^a \wedge \imath^b \wedge \imath^c \wedge \imath^d \Big) \wedge \mu \\ & + \gamma\, S'_{\text{torsion}}[\omega, e, \imath] \, . \end{aligned} \tag{5.10}$$

The torsional part is S'_{torsion} is defined in eq. (B.12) and, again, it is proportional to γ. Here we have made the "Wick rotation" $\gamma \to \mathrm{i}\gamma$, such that the action remains real. This re-definition of γ changes the SO(1, 1) symmetry of the gauge algebra to SO(2), and now we have $\gamma^2 > 0$.

The family of algebras $\mathrm{C}_{3+1}(M, \gamma)$, for every value of the parameters, are isomorphic. Therefore, all the gauge theory actions defined by gauging[1] this algebras will be related by means of linear transformations M_1, M_2, and so on[2].

[1] Again, with "to gauge an algebra" meaning to construct the Chern–Simons gauge theory valued in the algebra.

[2] Even in the ill-behaved limit $\gamma \to 0$, it remarkably happens that the non-torsional parts of the actions of eqs. (5.9) and (5.10) are related by the following vielbein re-definition: making

$$\begin{aligned} e^a &\to \frac{1}{\sqrt{2}} \Big(e^a + \mathrm{i} \tfrac{c}{a}\, \imath^a \Big) \, , \\ \imath^a &\to \frac{1}{\sqrt{2}} \Big(\tfrac{a}{c}\, e^a - \mathrm{i} \imath^a \Big) \, , \end{aligned}$$

in eq. (5.9) we get eq. (5.10).

According to the discussion above, we have a family of equivalent actions written in different bases. This equivalance, however is broken by the dimensional reduction schemes that we present in § 5.2. Hence, the 4-dimensional theories that can be derived from the 5-dimensional Chern–Simons theory for different choices of M will not be indistinguishable.

5.1.5 Doubled Chern–Simons theory

Let us focus on the problem of introducing a second spin connection; this is to see how the Chern–Simons theory can describe a theory for two spin connections and two vielbeine to pursue a gauge formulation of bimetric gravity. As we already mentioned in § 2.7, by doing this by means of a transgression form, we are then introducing a ghost in the theory, given that the Einstein–Cartan kinetic terms for different spin connections and vielbein have opposite signs. Motivated by the fact that the potential of the action for conformal gravity, namely the action of eq. (4.23), has a similar form as partial massless gravity (see eq. (3.24)) and, given that one single Chern–Simons action has similar potential to conformal gravity, we construct a doubled conformal Chern–Simons geometry as the action

$$S_{\mathrm{DCS}}[\omega,\tilde{\omega},e,\tilde{e},\imath,\tilde{\imath},\mu,\tilde{\mu}] = S_{\mathrm{CS}}[\omega,e,\imath,\mu] + S_{\mathrm{CS}}[\tilde{\omega},\tilde{e},\tilde{\imath},\tilde{\mu}]\,. \tag{5.11}$$

Here each Chern–Simons action on the right-hand side is the action of eq. (5.6) for fields with and without tildes. Since each one has a gauge symmetry corresponding to the conformal group, i.e. eq. (2.71) with and without tildes on fields and gauge parameters, then the symmetry of the doubled Chern–Simons action is $\mathrm{SO}(4,2)\times\mathrm{SO}(4,2)$. We can visualize this doubled gauge theory as a two control knobs which can be gauged or calibrated separately.

In order to compare this theory to a bimetric construction we study the breaking-symmetry case when $\tilde{e}^a = e^a$ and $\tilde{\imath}^a = \imath^a$. Both Chern–Simons actions in eq. (5.11) have now interdependent fields, therefore gauging one of the Chern–Simons action will necessarily gauge the other action. We can visualize this double gauge theory with interdependent fields as the same control knobs of above but connected with a tape. The action for this special case is

$$S_{\rm DCS}[\omega,\tilde{\omega},e,\imath,\mu,\tilde{\mu}] = S_{\rm CS}[\omega,e,\imath,\mu] + S_{\rm CS}[\tilde{\omega},e,\imath,\tilde{\mu}]\ . \tag{5.12}$$

The gauge transformations of the group $\mathrm{SO}(4,2)\times\mathrm{SO}(4,2)$ break down to those eq. (2.71) with and without tildes and putting $\tilde{e}^a = e^a$ and $\tilde{\imath}^a = \imath^a$ everywhere. After imposing these last equalities, the mentioned field transformations become consistent only under the conditions

$$\begin{aligned}\mathrm{D}_\omega\rho^a - \theta^a{}_b\wedge e^b - \tfrac{\eta C}{\det M}(b^a\wedge\mu - \imath^a\wedge\lambda)&\\ = \mathrm{D}_{\tilde{\omega}}\tilde{\rho}^a - \tilde{\theta}^a{}_b\wedge e^b - \tfrac{\eta C}{\det M}(\tilde{b}^a\wedge\tilde{\mu} - \imath^a\wedge\tilde{\lambda})&\ , \end{aligned}\tag{5.13a}$$

$$\begin{aligned}\mathrm{D}_\omega b^a - \theta^a{}_b\wedge \imath^b + \tfrac{\eta A}{\det M}(\rho^a\wedge\mu - e^a\wedge\lambda)&\\ = \mathrm{D}_{\tilde{\omega}}\tilde{b}^a - \tilde{\theta}^a{}_b\wedge \imath^b + \tfrac{\eta A}{\det M}(\tilde{\rho}^a\wedge\tilde{\mu} - e^a\wedge\tilde{\lambda})&\ . \end{aligned}\tag{5.13b}$$

Now, we look for the general solution to the system of equations above, such that the gauge fields are not related by differential equations, since we do not want to modify the number of fields. The solution for the parameters of the system of equations above which neglects all cases that relate gauge fields in differential equations is given by

$$\tilde{\theta}^{ab} = \theta^{ab}\ , \tag{5.14a}$$

$$\tilde{\rho}^a = \rho^a = 0\ , \tag{5.14b}$$

$$\tilde{b}^a = b^a = 0\ , \tag{5.14c}$$

$$\tilde{\lambda} = \lambda\ , \tag{5.14d}$$

thus for the interdependent fields case the symmetry of the doubled Chern–Simons gauge theory becomes Lorentz $\times$ Dilation. Inserting the equalities (5.14a)-(5.14d) back into the gauge transformations of $\mathrm{SO}(4,2)\times\mathrm{SO}(4,2)$ for the interdependent fields case, namely eq. (2.71) with and without tildes and putting $\tilde{e}^a = e^a$ and $\tilde{\imath}^a = \imath^a$ everywhere, we get the following set of field transformations:

$$\delta\omega^{ab} = \mathrm{D}_\omega\theta^{ab}\ , \tag{5.15}$$

$$\delta\tilde{\omega}^{ab} = \mathrm{D}_{\tilde{\omega}}\theta^{ab}\ , \tag{5.16}$$

$$\delta e^a = -\theta^a{}_b\wedge e^b + \tfrac{\eta}{\det M}\big(B\,e^a + C\,\imath^a\big)\wedge\lambda\ , \tag{5.17}$$

$$\delta\imath^a = -\theta^a{}_b \wedge \imath^b - \tfrac{\eta}{\det M}\big(A\, e^a + B\, \imath^a\big) \wedge \lambda\,, \tag{5.18}$$

$$\delta\mu = \mathrm{d}\lambda\,, \tag{5.19}$$

$$\delta\tilde{\mu} = \mathrm{d}\lambda\,. \tag{5.20}$$

These are the Lorentz × Dilation transformations that leave the doubled Chern–Simons action of eq. (5.12) invariant. The explicit form of the doubled Chern–Simons action will not be discussed in this section since it is basically two copies of the SO(4, 2) Chern–Simons action. The relevant result comes after imposing a dimensional reduction scheme, which we present in § 5.2.3.

5.2 Dimensional reductions

In this section we analyse various different dimensional reduction schemes that we can impose to the Chern–Simons gauge theory. We decompose the 5-dimensional manifold as $M_5 = M_4 \ltimes \Sigma$, where M_4 is a 4-dimensional manifold, Σ is the 1-dimensional curve that symbolizes the dimensional reduction domain and $\ltimes$ denotes the semi-direct product of manifolds. Given that the theory enjoys a huge symmetry, i.e., SO(4, 2), in some cases the equations that define the dimensional reduction can be seen as a gauge-fixing.

5.2.1 4-covariant basis

5.2.1.1 Einstein–Cartan gravity

Let us consider the case of the 5-dimensional Chern–Simons theory written in the 4-covariant basis in the limit $\gamma \to \infty$. This breaks the symmetry SO(4, 2) down to the 5-dimensional Poincaré groups ISO(3, 2) or ISO(4, 1) (respectively for $\eta = \pm 1$), as discussed in § 2.2.1. The action of eq. (5.5) becomes

$$S_{\mathrm{CS}}[\omega, s, u, \mu] = \tfrac{1}{4} \int_{M_5} \epsilon_{abcd} \Big(R^{ab}(\omega) \wedge R^{cd}(\omega) \\ + 2\eta\, R^{ab}(\omega) \wedge s^c \wedge s^d + s^a \wedge s^b \wedge s^c \wedge s^d \Big) \wedge \mu$$

$$- \int_{M_5} \epsilon_{abcd} \Big(R^{ab}(\omega) + \eta\, s^a \wedge s^b \Big) \wedge T^c(\omega, s) \wedge u^d \,. \tag{5.21}$$

This is the Chern–Simons action that is invariant under the 5-dimensional Poincaré group, namely the under the gauge transformations of eq. (2.69).

Dimensional reduction scheme. We restrict the fields $\boldsymbol{A} = \boldsymbol{A}_m(X)\mathrm{d}X^m$ in the 5-dimensional action of eq. (5.21) as

$$\omega^{ab}{}_m(X)\,\mathrm{d}X^m = \omega^{ab}{}_\mu(x)\,\mathrm{d}x^\mu =: \bar{\omega}^{ab}(x) \,, \tag{5.22a}$$
$$s^a{}_m(X)\,\mathrm{d}X^m = s^a{}_\mu(x)\,\mathrm{d}x^\mu =: \bar{s}^a(x) \,, \tag{5.22b}$$
$$u^a{}_m(X)\,\mathrm{d}X^m = u^a{}_m(w)\,\mathrm{d}X^m =: \bar{u}^a(w) \,, \tag{5.22c}$$
$$\mu_m(X)\,\mathrm{d}X^m = \mu_m(w)\,\mathrm{d}X^m =: \bar{\mu}(w) \,. \tag{5.22d}$$

Here the bars are used to point out that this choice corresponds to a field setup that breaks the 5-dimensional Poincaré symmetry.

Interpretation as gauge-fixing. This field configuration can be seen as a gauge-fixing for a subset of all possible field configurations of the entire Chern–Simons theory. To see this, let us recall the field transformations of the 5-dimensional Poincaré group (eq. (2.69)). According to them, the most general gauge connection $\bar{\boldsymbol{A}}'$ that can be obtained from the gauge-fixed connection $\bar{\boldsymbol{A}}$ is

$$\tfrac{1}{2}\bar{\omega}'^{ab}{}_\mu(x,w) = \tfrac{1}{2}\bar{\omega}^{ab}{}_\mu(x) + \tfrac{1}{2}\bar{\mathrm{D}}_\mu\theta^{ab}(x,w) + \eta\,\bar{s}^{[a}{}_\mu(x)\beta^{b]}(x,w) \tag{5.23a}$$
$$\tfrac{1}{2}\bar{\omega}'^{ab}{}_5(x,w) = \tfrac{1}{2}\partial_w\theta^{ab}(x,w) \,, \tag{5.23b}$$
$$\bar{s}'^a{}_\mu(x,w) = \bar{s}^a{}_\mu(x) + \bar{\mathrm{D}}_\mu\beta^a(x,w) - \theta^a{}_b(x,w)\bar{s}^b{}_\mu(x) \tag{5.23c}$$
$$\bar{s}'^a{}_5(x,w) = \partial_w\beta^a(x,w) \,, \tag{5.23d}$$
$$\begin{aligned}\bar{u}'^a{}_\mu(x,w) &= \bar{u}^a{}_\mu(w) + \bar{\mathrm{D}}_\mu\tau^a(x,w) - \theta^a{}_b(x,w)\bar{u}^b{}_\mu(w) \\ &\quad + \eta\,\beta^a(x,w)\bar{\mu}_\mu(w) - \eta\,\bar{s}^a{}_\mu(x)\lambda(x,w) \,,\end{aligned} \tag{5.23e}$$
$$\begin{aligned}\bar{u}'^a{}_5(x,w) &= \bar{u}^a{}_5(w) + \partial_w\tau^a(x,w) \\ &\quad - \theta^a{}_b(x,w)\bar{u}^b{}_5(w) + \eta\,\beta^a(x,w)\bar{\mu}_5(w) \,,\end{aligned} \tag{5.23f}$$
$$\begin{aligned}\bar{\mu}'_\mu(x,w) &= \bar{\mu}_\mu(w) + \partial_\mu\lambda(x,w) - \eta_{ab}\,\bar{s}^{(a}{}_\mu(x)\tau^{b)}(x,w) \\ &\quad + \eta_{ab}\,\beta^{(a}(x,w) \wedge \bar{u}^{b)}{}_\mu(w) \,,\end{aligned} \tag{5.23g}$$

$$\bar{\mu}'_5(x,w) = \bar{\mu}_5(w) + \partial_5\lambda(x,w) + \eta_{ab}\,\beta^{(a}(x,w)\wedge\bar{u}^{b)}{}_5(w)\;. \tag{5.23h}$$

where $\bar{\mathrm{D}}_\mu$ is the 4-dimensional covariant derivative with respect to the 4-dimensional spin connection $\bar{\omega}^{ab}{}_\mu(x)$. At the 5-dimensional level, from all possible fields let us focus only on those that can split functionally as $F(x,w) = F(x)+G(w)$, $F(x,w) = F(x)G(w)$, $F(x,w) = \partial_5 G(x,w)$, etc., as suggested in eq. (5.23). This is a smaller subset of all possible functions delivered by the Chern–Simons theory, since after imposing the gauge-fixing we can get back at most the gauge connection $\bar{\boldsymbol{A}}'$. The most general gauge connection that we can obtain via a gauge transformation is such that it belongs to the space of functions defined by eq. (5.23). As we will see in the following, the 4-dimensional effective action contains a set of general fields that are not forced to have a special form.

To complete the analysis, we have to answer in which conditions it is possible to construct a gauge connection $\bar{\boldsymbol{A}}'$. Let us take look the field content of the reduced theory. Table 5.1 shows field content for the Chern–Simons theory and after the dimensional reduction. The number of fields of the general Chern–

Generator	Before DR	After DR	Parameter
$\boldsymbol{J}_{ab}$	$\omega^{ab}{}_m(\mathrm{X})$ (30)	$\bar{\omega}^{ab}{}_\mu(x)$ (24)	$\theta^{ab}(X)$ (6)
$\boldsymbol{B}_a$	$s^a{}_m(\mathrm{X})$ (20)	$\bar{s}^a{}_\mu(x)$ (16)	$\beta^a(X)$ (4)
$\boldsymbol{T}_a$	$u^a{}_m(\mathrm{X})$ (20)	$\bar{u}^a{}_m(w)$ (20)	$\tau^a(X)$ (4)
$\boldsymbol{D}$	$\mu_m(\mathrm{X})$ (5)	$\bar{\mu}_m(w)$ (5)	$\lambda(X)$ (1)
Total	75	65	15

Table 5.1: Field content for the Chern–Simons theory and after the dimensional reduction scheme of eq. (5.22). Here "DR" stands for dimensional reduction.

Simons theory is $30+20+20+5=75$ and the number of fields after fixing the gauge by means of eq. (5.22) is $24+16+20+5=65$. From eq. (5.23) we see that the $6+4+4+1=15$ group parameters restore the connection. Therefore, assuming that we can solve eq. (5.23) for θ^{ab}, β^a, τ^a and λ, we can find a new connection $\bar{\boldsymbol{A}}'$ from the gauge-fixed connection $\bar{\boldsymbol{A}}$.

By last, we analyse how the gauge symmetry is reduced by imposing the dimensional reduction scheme of eq. (5.22). For this, we analyse what are the conditions on the gauge parameters so that $\bar{\boldsymbol{A}}'$ satisfies the same gauge that the gauge-fixed connection $\bar{\boldsymbol{A}}$. That occurs when

$$\tfrac{1}{2}\bar{\omega}'^{ab}{}_{\mu}(x) = \tfrac{1}{2}\bar{\omega}^{ab}{}_{\mu}(x) + \tfrac{1}{2}\bar{\mathrm{D}}_{\mu}\theta^{ab}(x,w) + \eta\,\bar{s}^{[a}{}_{\mu}(x)\beta^{b]}(x,w)\,, \tag{5.24}$$

$$0 = \tfrac{1}{2}\partial_w\theta^{ab}(x,w)\,, \tag{5.25}$$

$$\bar{s}'^{a}{}_{\mu}(x) = \bar{s}^{a}{}_{\mu}(x) + \bar{\mathrm{D}}_{\mu}\beta^{a}(x,w) - \theta^{a}{}_{b}(x,w)\bar{s}^{b}{}_{\mu}(x)\,, \tag{5.26}$$

$$0 = \partial_w\beta^{a}(x,w)\,, \tag{5.27}$$

$$\begin{aligned}\bar{u}'^{a}{}_{\mu}(w) &= \bar{u}^{a}{}_{\mu}(w) + \bar{\mathrm{D}}_{\mu}\tau^{a}(x,w) - \theta^{a}{}_{b}(x,w)\bar{u}^{b}{}_{\mu}(w)\\ &\quad + \eta\left(\beta^{a}(x,w)\bar{\mu}_{\mu}(w) - \bar{s}^{a}{}_{\mu}(x)\lambda(x,w)\right)\,,\end{aligned} \tag{5.28}$$

$$\begin{aligned}\bar{u}'^{a}{}_{5}(w) &= \bar{u}^{a}{}_{5}(w) + \partial_w\tau^{a}(x,w)\\ &\quad - \theta^{a}{}_{b}(x,w)\bar{u}^{b}{}_{5}(w) + \eta\,\beta^{a}(x,w)\bar{\mu}_{5}(w)\,,\end{aligned} \tag{5.29}$$

$$\bar{\mu}'_{\mu}(w) = \bar{\mu}_{\mu}(w) + \partial_{\mu}\lambda(x,w) \tag{5.30}$$

$$\quad - \eta_{ab}\left(\bar{s}^{(a}{}_{\mu}(x)\tau^{b)}(x,w) - \beta^{(a}(x,w)\wedge\bar{u}^{b)}{}_{\mu}(w)\right)\,, \tag{5.31}$$

$$\bar{\mu}'_{5}(w) = \bar{\mu}_{5}(w) + \partial_{5}\lambda(x,w) + \eta_{ab}\,\beta^{(a}(x,w)\wedge\bar{u}^{b)}{}_{5}(w)\,. \tag{5.32}$$

This is fulfilled only if $\theta^{ab} = \beta^{a} = \tau^{a} = \lambda = 0$, which means that the gauge is completely fixed and we do not have any remaining gauge symmetry; we lost even local Lorentz invariance. In this sense, the dimensional reduction breaks the entire 5-dimensional Poincaré symmetry. As we see below, by plugging in the dimensional reduction scheme into the Chern–Simons action, the theory reduces to the 4-dimensional Lorentz-invariant Einstein–Cartan action plus as Lorentz-breaking term.

Reduced action. For the field components as in eq. (5.22), $R^{ab}(\bar{\omega})$ becomes the Lorentz curvature, $\bar{s}^{a}$ becomes the vierbein and $T^{a}(\bar{\omega},\bar{s}) = \mathrm{D}_{\bar{\omega}}\bar{s}$ becomes the torsion of the 4-dimensional Cartan formalism. The action of eq. (5.21) gives

$$\begin{aligned}\bar{S}[\bar{\omega},\bar{s},\bar{u},\bar{\mu}] &= \tfrac{\eta}{\ell^2}\int_{\Sigma}\bar{\mu}\int_{M_4}\epsilon_{abcd}\Big(\tfrac{\eta}{2}R^{ab}(\bar{\omega})\wedge R^{cd}(\bar{\omega})\\ &\quad + R^{ab}(\bar{\omega})\wedge\bar{s}^{c}\wedge\bar{s}^{d} + \tfrac{\eta}{\ell^2}\,\bar{s}^{a}\wedge\bar{s}^{b}\wedge\bar{s}^{c}\wedge\bar{s}^{d}\Big)\\ &\quad + \tfrac{\sqrt{2}}{\ell}\int_{\Sigma}\bar{u}^{a}\int_{M_4}\epsilon_{abcd}\left(R^{bc}(\bar{\omega}) + \tfrac{2\eta}{\ell^2}\,\bar{s}^{b}\wedge\bar{s}^{c}\right)\wedge T^{d}(\bar{\omega},\bar{s})\,.\end{aligned} \tag{5.33}$$

Here we rescaled the vierbein as $\bar{s}^a \to (\sqrt{2}/\ell)\bar{s}^a$, where ℓ is a constant. To integrate along Σ we define the constants $\kappa = \int_\Sigma \bar{\mu}$ and $\phi^a = \int_\Sigma \bar{u}^a$. We get then the following 4-dimensional action:

$$\bar{S}[\bar{\omega},\bar{s}] = \tfrac{\kappa\eta}{\ell^2}\int_{M_4} \epsilon_{abcd}\Big(R^{ab}(\bar{\omega}) \wedge \bar{s}^c \wedge \bar{s}^d + \tfrac{\eta}{\ell^2}\,\bar{s}^a \wedge \bar{s}^b \wedge \bar{s}^c \wedge \bar{s}^d\Big) + \tfrac{\sqrt{2}}{\ell}\phi^a \int_{M_4} \epsilon_{abcd}\left(R^{bc}(\bar{\omega}) + \tfrac{2\eta}{\ell^2}\,\bar{s}^b \wedge \bar{s}^c\right) \wedge \bar{T}^d(\bar{\omega},\bar{s}) \;, \tag{5.34}$$

where we have omitted the Gauss–Bonnet boundary term by means of the Stokes theorem. This action corresponds to the first-order 4-dimensional Einstein–Cartan action plus negative or positive cosmological constant respectively for $\eta = \pm 1$ plus terms involving torsion.

The presence of the constants ϕ^a breaks the local Lorentz symmetry, since the only constant vector that is local Lorentz invariant is $(0,0,0,0)^{\mathrm{T}}$. Therefore, the way to recover 4-dimensional local Lorentz invariance is by imposing $\phi^a = 0$, which further restricts the gauge fields to satisfy $\int_\Sigma \bar{u}^a = 0$. In this case we recover precisely the Einstein–Cartan action. Note, however, that defining a new Planck mass through $m_{\mathrm{P}}^2 = \kappa\eta/\ell^2$, we have in the large-$\ell$ limit that

$$\bar{S}[\bar{\omega},\bar{s}]\big|_{\ell\to\infty} = m_{\mathrm{P}}^2 \int_{M_4} \epsilon_{abcd}\, R^{ab}(\bar{\omega}) \wedge \bar{s}^c \wedge \bar{s}^d \;, \tag{5.35}$$

which is again the Einstein–Cartan action.

5.2.1.2 First-order conformal gravity

To go further with the analysis for different dimensional reductions, we consider again the SO(4, 2) invariant action of eq. (5.5), without having taken any limit for γ. Also, we consider the following slightly different scheme:

$$\omega^{ab}{}_m(X)\,\mathrm{d}X^m = \omega^{ab}{}_\mu(x)\,\mathrm{d}x^\mu =: \bar{\omega}^{ab}(x) \;, \tag{5.36a}$$

$$s^a{}_m(X)\,\mathrm{d}X^m = s^a{}_\mu(x)\,\mathrm{d}x^\mu =: \bar{s}^a(x) \;, \tag{5.36b}$$

$$u^a{}_m(X)\,\mathrm{d}X^m = u^a{}_\mu(x)\,\mathrm{d}x^\mu =: \bar{u}^a(x) \;, \tag{5.36c}$$

$$\mu_m(X)\,\mathrm{d}X^m = \mu_5(w)\,\mathrm{d}w =: \bar{\mu}(w) \;. \tag{5.36d}$$

The semi-direct product manifold $M_5 = M_4 \ltimes \Sigma$ becomes the direct product manifold $M_5 = M_4 \times \Sigma$ given that the components of the fünfbein $s^A{}_m(X)$ satisfy $s^a{}_5 = s^5{}_\mu = 0$. The same is valid for $u^A{}_m(X)$. Which one of those vielbeine shall correspond to the metric of the spacetime $g_{\mu\nu}$ in an effective action remains as matter of interpretation, as we do as well in bimetric theory: here one declares which metric is going to be the one that rises and lowers indices.

In the same way as the dimensional reduction scheme of § 5.2.1.1, the scheme of eq. (5.36) can be seen as a gauge-fixing under similar conditions. In this case, however, the gauge is not completely fixed by the scheme: one still has the free parameters $\theta^{ab} = \theta^{ab}(x)$ while necessarily $\beta^a = \tau^a = \lambda = 0$. Putting this values of parameters into the gauge transformations of eq. (2.68) one sees that residual gauge corresponds to 4-dimensional local Lorentz transformations $\mathrm{SO}(3,1)$.

The restriction of eq. (5.36d) makes that any other 5-component of the fields, e.g. $s^a{}_5$ drops out from the action since $\mathrm{d}w \wedge \mathrm{d}w = 0$. This also implies that all other appearing indices will be $D = 4$ spacetime indices. The action of eq. (5.5) becomes

$$\begin{aligned}\bar{S}[\bar{\omega}, \bar{s}, \bar{u}, \bar{\mu}] = \tfrac{\eta}{2} \int_\Sigma \bar{\mu} \int_{M_4} \epsilon_{abcd} \Big(&\tfrac{\eta}{2} R^{ab}(\bar{\omega}) \wedge R^{cd}(\bar{\omega}) \\ &+ R^{ab}(\bar{\omega}) \wedge \Big[\bar{s}^c \wedge \bar{s}^d - \tfrac{1}{\gamma^2} \bar{u}^c \wedge \bar{u}^d \Big] \\ &+ \tfrac{\eta}{2} \Big[\bar{s}^a \wedge \bar{s}^b - \tfrac{1}{\gamma^2} \bar{u}^a \wedge \bar{u}^b \Big] \wedge \Big[\bar{s}^c \wedge \bar{s}^d - \tfrac{1}{\gamma^2} \bar{u}^c \wedge \bar{u}^d \Big] \Big). \end{aligned} \tag{5.37}$$

Furthermore, defining the constant $\kappa = \int_\Sigma \bar{\mu}$ to integrate along Σ, the 4-dimensional action reads

$$\begin{aligned}\bar{S}[\bar{\omega}, \bar{s}, \bar{u}] = \tfrac{\kappa\eta}{2} \int_{M_4} \epsilon_{abcd} \Big(&R^{ab}(\bar{\omega}) \wedge \Big[\bar{s}^c \wedge \bar{s}^d - \tfrac{1}{\gamma^2} \bar{u}^c \wedge \bar{u}^d \Big] \\ &+ \tfrac{\eta}{2} \Big[\bar{s}^a \wedge \bar{s}^b - \tfrac{1}{\gamma^2} \bar{u}^a \wedge \bar{u}^b \Big] \wedge \Big[\bar{s}^c \wedge \bar{s}^d - \tfrac{1}{\gamma^2} \bar{u}^c \wedge \bar{u}^d \Big] \Big), \end{aligned} \tag{5.38}$$

where we omitted the Gauss–Bonnet boundary term. Remarkably, the Weyl rotation of the vector $\binom{u^a}{\gamma s^a}$ leaves the action invariant, which is a consequence of the $\mathrm{SO}(1,1)$ symmetry of the $\mathrm{SO}(4,2)$ algebra (cf. eq. (2.28)). The action of eq. (5.38) is the action for first-order conformal gravity (see eq. (4.23)) first obtained in ref. [67].

5.2.2 4-covariant canonical basis

5.2.2.1 Generalized first-order conformal gravity

As discussed in § 5.1.4.2 one can relate the 5-dimensional theories by linear field redefinitions, therefore they are all equivalent. In this section we present a dimensional reduction scheme that breaks this equivalence because the scheme basis-dependent. Namely, we restrict the fields as

$$\omega^{ab}{}_m(X)\,\mathrm{d}X^m = \omega^{ab}{}_\mu(x)\,\mathrm{d}x^\mu + \omega^{ab}{}_5(x,w)\,\mathrm{d}w =: \bar{\omega}^{ab}(x) + \omega^{ab}{}_5(x,w)\,\mathrm{d}w\,, \tag{5.39a}$$

$$e^a{}_m(X)\,\mathrm{d}X^m = e(w)\,e^a{}_\mu(x)\,\mathrm{d}x^\mu + e^a{}_5(x,w)\,\mathrm{d}w =: e(w)\,\bar{e}^a(x) + e^a{}_5(x,w)\,\mathrm{d}w\,, \tag{5.39b}$$

$$\imath^a{}_m(X)\,\mathrm{d}X^m = \imath(w)\,\imath^a{}_\mu(x)\,\mathrm{d}x^\mu + \imath^a{}_5(x,w)\,\mathrm{d}w =: \imath(w)\,\bar{\imath}^a(x) + \imath^a{}_5(x,w)\,\mathrm{d}w\,, \tag{5.39c}$$

$$\mu_m(X)\,\mathrm{d}X^m = \mu_5(w)\,\mathrm{d}w =: \bar{\mu}(w)\,. \tag{5.39d}$$

Here $e = e(w)$ and $\imath = \imath(w)$ are two arbitrary functions defined on the dimensional reduction domain Σ. They can be interpreted in the following way. A warped geometry has the general form for the metric

$$\mathrm{d}s^2 = f(w)g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu + g(w)\mathrm{d}w^2\,. \tag{5.40}$$

Thus our warped functions define a warped spacetime[3] in the direction $X^5 = w$. The scheme of eq. (5.39) is not the same for linear combinations of the 5-dimensional fields e^a and $\imath^a$. This means, a linear combination $q^a = ae^a + b\imath^a$ cannot be written in the form $q^a = q(w)\,\bar{q}^a(x) + q^a{}_5(x,w)\,\mathrm{d}w$ but it will read $q^a = ae(w)\,\bar{e}^a(x) + b\imath(w)\,\bar{\imath}^a(x) + q^a{}_5(x,w)\,\mathrm{d}w$. Our scheme requires then a particular choice of basis for the gauge algebra. We choose in the following the orthogonal basis, movivated by the feature that the potential in the action looks similar to the one of bimetric gravity and also by the fact that the

[3] The authors ref. [64] studied effective spacetimes starting from the theories with warped geometry in the context Chern–Simons theory.

torsional part goes away when putting $\gamma = 0$.

As in the previous cases, the dimensional reduction scheme can be also seen as a gauge-fixing. Two possibilities arise for residual gauge symmetry. Those are: i) a transformation with $\theta^{ab} = \theta^{ab}(x)$ which is 4-dimensional local Lorentz symmetry. In this case, the functions $e(w)$ and $\imath(w)$ are not constrained. We have also that ii) $\theta^{ab} = \theta^{ab}(x)$ and $\lambda = \lambda(w)$ corresponding to 4-dimensional local Lorentz symmetry together with Weyl dilations in the warping direction. For this case, the functions $e(w)$ and $\imath(w)$ are constrained to be proportional.

We study from now only the case i), since the integration along Σ of independent functions $e(w)$ and $\imath(w)$ will lead to a greater number of free parameters that allow us to study different sectors of the theory. In this scheme the action of. eq. (5.10) for $\gamma = 0$ becomes

$$\begin{aligned}\bar{S}[\bar{\omega}, \bar{e}, \bar{\imath}, \bar{\mu}] = a^2\eta \int_\Sigma e^2\bar{\mu} \int_{M_4} \epsilon_{abcd}\, R^{ab}(\bar{\omega}) \wedge \bar{e}^c \wedge \bar{e}^d \\ + c^2\eta \int_\Sigma \imath^2\bar{\mu} \int_{M_4} \epsilon_{abcd}\, R^{ab}(\bar{\omega}) \wedge \bar{\imath}^c \wedge \bar{\imath}^d \\ + a^4 \int_\Sigma e^4\bar{\mu} \int_{M_4} \epsilon_{abcd}\, \bar{e}^a \wedge \bar{e}^b \wedge \bar{e}^c \wedge \bar{e}^d \\ + 2a^2c^2 \int_\Sigma e^2\imath^2\bar{\mu} \int_{M_4} \epsilon_{abcd}\, \bar{e}^a \wedge \bar{e}^b \wedge \bar{\imath}^c \wedge \bar{\imath}^d \\ + c^4 \int_\Sigma \imath^4\bar{\mu} \int_{M_4} \epsilon_{abcd}\, \bar{\imath}^a \wedge \bar{\imath}^b \wedge \bar{\imath}^c \wedge \bar{\imath}^d\,. \end{aligned} \tag{5.41}$$

Here e^s and $\imath^s$, with $s = 1, 2, 4$, are powers of the warping function and they should not be confused with vielbein components.

It is necessary to emphasize that, since we are in the limit $\gamma \to 0$, the $\mathrm{C}_{3+1}(M, \gamma)$ algebra is not well-defined and formally, the theory is not a gauge-formulated anymore. Nevertheless, as one can see from the commutation relations of the algebra, the Lorentz subalgebra is unaffected by this limit and thus there should be still a residual local Lorentz invariance. To integrate along the domain Σ, we define the constants

$$p_{st} = \int_\Sigma \mathrm{d}w\, e^s(w)\imath^t(w)\mu_5(w)\,. \tag{5.42}$$

With this, the 4-dimensional theory becomes

$$\bar{S}[\bar{\omega}, \bar{e}, \bar{\imath}] = \int_{M_4} \epsilon_{abcd} \Big(a^2 \eta p_{20}\, R^{ab}(\bar{\omega}) \wedge \bar{e}^c \wedge \bar{e}^d \\ + c^2 \eta p_{02}\, R^{ab}(\bar{\omega}) \wedge \bar{\imath}^c \wedge \bar{\imath}^d + a^4 p_{40}\, \bar{e}^a \wedge \bar{e}^b \wedge \bar{e}^c \wedge \bar{e}^d \\ + 2a^2 c^2 p_{22}\, \bar{e}^a \wedge \bar{e}^b \wedge \bar{\imath}^c \wedge \bar{\imath}^d + c^4 p_{04}\, \bar{\imath}^a \wedge \bar{\imath}^b \wedge \bar{\imath}^c \wedge \bar{\imath}^d \Big) . \tag{5.43}$$

We observe that, as expected, the theory has residual local Lorentz invariance. Also, we see that this action contains all terms of the first-order conformal gravity action (cf. eq. (4.23)). The appendix B.1 is devoted to show that the constants p_{st} are arbitrary, meaning that they do not depend functionally on each other[4]. For the special choice

$$p_{40} = p_{20}^2 \, , \tag{5.44a}$$

$$p_{22} = p_{20} p_{02} \, , \tag{5.44b}$$

$$p_{04} = p_{02}^2 \, , \tag{5.44c}$$

by performing the field redefinition

$$a' E^a = a\sqrt{\eta p_{20}}\, \bar{e}^a + \mathrm{i}c\sqrt{\eta p_{02}}\, \bar{\imath}^a \, , \tag{5.45a}$$

$$c' I^a = a\sqrt{\eta p_{20}}\, \bar{e}^a - \mathrm{i}c\sqrt{\eta p_{02}}\, \bar{\imath}^a \, , \tag{5.45b}$$

the action takes the form of first-order conformal gravity, as in § 5.2.1.2. For general parameters p_{st}, the action of eq. (5.43) represents a generalisation of first-order conformal gravity. Notice that this generalisation does have the usual Weyl rotation invariance SO(1, 1).

5.2.3 Doubled Lorentz-Dilation Chern–Simons

5.2.3.1 Generalized first-order bimetric gravity

In the following we carry out a dimensional reduction of the doubled Chern–Simons theory introduced in § 5.1.5. To this end, we restrict the fields ω^{ab},

[4] This is, however, from the 4-dimensional point of view. At the 5-dimensional level, any restriction of the parameters p_{st} restricts fields in the 5-dimensional theory.

μ, e^a and i^a as in eq. (5.39) and also analogously for $\tilde{\omega}^{ab}$ and $\tilde{\mu}$. The action becomes

$$\begin{aligned}\overline{S+\tilde{S}} = \eta \int_{M_4} \epsilon_{abcd}\Big(& a^2 p_{20}\, R^{ab}(\bar{\omega}) \wedge \bar{e}^c \wedge \bar{e}^d + c^2 p_{02}\, R^{ab}(\bar{\omega}) \wedge \bar{\imath}^c \wedge \bar{\imath}^d \\ &+ a^2 \tilde{p}_{20}\, R^{ab}(\bar{\tilde{\omega}}) \wedge \bar{e}^c \wedge \bar{e}^d + c^2 \tilde{p}_{02}\, R^{ab}(\bar{\tilde{\omega}}) \wedge \bar{\imath}^c \wedge \bar{\imath}^d\Big) \\ &+ \int_{M_4} \epsilon_{abcd}\Big(a^4 (p_{40}+\tilde{p}_{40})\, \bar{e}^a \wedge \bar{e}^b \wedge \bar{e}^c \wedge \bar{e}^d + 2a^2c^2(p_{22}+\tilde{p}_{22})\, \bar{e}^a \wedge \bar{e}^b \wedge \bar{\imath}^c \wedge \bar{\imath}^d \\ &+ c^4(p_{04}+\tilde{p}_{04})\, \bar{\imath}^a \wedge \bar{\imath}^b \wedge \bar{\imath}^c \wedge \bar{\imath}^d\Big) + \gamma \bar{S}_{\text{torsion}} \, . \end{aligned} \tag{5.46}$$

Here the bar on the actions is to point out that we performed a dimensional reduction. Also, we have defined the constants

$$\tilde{p}_{st} = \int_\Sigma \mathrm{d}w\, e^s(w) \imath^t(w) \tilde{\mu}_5(w) \, . \tag{5.47}$$

Analogously as in § 5.2.2, the dimensional reduction scheme can be seen as gauge-fixing for which the entire symmetry breaks down to SO(3, 1). Since the parameters p_{st} and $\tilde{p}_{st}$ are arbitrary (see § B.1), the reduced action in eq. (5.46) represents a generalisation of 4-dimensional first-order bimetric gravity for $\beta_1 = \beta_3 = 0$ (cf. eq. (3.22)). For the case $\gamma \to 0$ we find the two following relevant cases:

First-order bimetric gravity. The potential in eq. (5.46) coincides with the one of bimetric theory for the special case of $\beta_1 = \beta_3 = 0$. By choosing

$$p_{20} = 1/a^2 \, , \tag{5.48a}$$

$$p_{02} = 0 \, , \tag{5.48b}$$

$$\tilde{p}_{20} = 0 \, , \tag{5.48c}$$

we eliminate the mixed kinetic term. In this case the action of (5.46) reads

$$\begin{aligned}\overline{S+\tilde{S}} = \eta \int_{M_4} \epsilon_{abcd}\Big(& R^{ab}(\bar{\omega}) \wedge \bar{e}^c \wedge \bar{e}^d + c^2 \tilde{p}_{02}\, R^{ab}(\bar{\tilde{\omega}}) \wedge \bar{\imath}^c \wedge \bar{\imath}^d\Big) \\ &+ \int_{M_4} \epsilon_{abcd}\Big(a^4 (p_{40}+\tilde{p}_{40})\, \bar{e}^a \wedge \bar{e}^b \wedge \bar{e}^c \wedge \bar{e}^d + 2a^2c^2(p_{22}+\tilde{p}_{22})\, \bar{e}^a \wedge \bar{e}^b \wedge \bar{\imath}^c \wedge \bar{\imath}^d \\ &+ c^4(p_{04}+\tilde{p}_{04})\, \bar{\imath}^a \wedge \bar{\imath}^b \wedge \bar{\imath}^c \wedge \bar{\imath}^d\Big) \, , \end{aligned} \tag{5.49}$$

which is ghost-free bimetric gravity in the first-order formulation with $\beta_1 = \beta_3 = 0$ (cf. eq. (3.22)). This action has the residual local Lorentz symmetry, as expected. Whether the more general action of eq. (5.46) propagates the Boulware-Deser ghost remains as an open question. Moreover, the special model of eq. (3.24), i.e., partial massless gravity, corresponds to the choice

$$p_{40} + \tilde{p}_{40} = -M^2/a^4 \,, \tag{5.50a}$$
$$p_{22} + \tilde{p}_{22} = -M^2\tilde{p}_{02}/a^2 \,, \tag{5.50b}$$
$$p_{04} + \tilde{p}_{04} = -M^2\tilde{p}_{02}^2 \,, \tag{5.50c}$$
$$\tilde{p}_{20} = 0 \,, \tag{5.50d}$$
$$p_{02} = 0 \,, \tag{5.50e}$$

where we also identify the constants $c = \alpha$ of both actions.

Weyl rotation invariant model. In § 4.3 we mentioned that the Weyl rotation symmetry of the action for first-order conformal gravity can be also realized as a subgroup of the conformal group SO(4, 2). Remarkably, there exists a choice for the parameters p_{st} that gives back this symmetry to the general case of eq. (5.46). Namely, by choosing the parameters as

$$p_{40} + \tilde{p}_{40} = p_{20}^2 \,, \tag{5.51a}$$
$$p_{22} + \tilde{p}_{22} = p_{20}\tilde{p}_{02} \,, \tag{5.51b}$$
$$p_{04} + \tilde{p}_{04} = \tilde{p}_{02}^2 \,, \tag{5.51c}$$
$$p_{02} = \tilde{p}_{02} \,, \tag{5.51d}$$
$$\tilde{p}_{20} = p_{20} \,, \tag{5.51e}$$

the action of eq. (5.46) becomes

$$\begin{aligned} S = \eta \int_{M_4} \epsilon_{abcd}\Big(\Big[R^{ab}(\bar{\omega}) + R^{ab}(\bar{\tilde{\omega}})\Big] \wedge \Big[a^2 p_{20}\, \bar{e}^c \wedge \bar{e}^d + c^2 \tilde{p}_{02}\, \bar{\imath}^c \wedge \bar{\imath}^d\Big] \\ + \Big[a^2 p_{20}\, \bar{e}^a \wedge \bar{e}^b + c^2 \tilde{p}_{02}\, \bar{\imath}^a \wedge \bar{\imath}^b\Big] \wedge \Big[a^2 p_{20}\, \bar{e}^c \wedge \bar{e}^d + c^2 \tilde{p}_{02}\, \bar{\imath}^c \wedge \bar{\imath}^d\Big]\Big) \,, \end{aligned} \tag{5.52}$$

which is invariant under SO(1, 1) (or SO(2), depending on the sign of $p_{20}\tilde{p}_{02}$) Weyl rotation. The action of eq. (5.52) differs, however, first-order conformal gravity since the curvatures depend on two independent spin connections. This model does not coincide with first-order ghost-free bimetric gravity either, due to the different structure of the kinetic terms.

Chapter 6

Dark matter in multimetric gravity

This chapter is devoted to some of our results of ref. [2]. We start with computing the full perturbative action of trimetric gravity with maximal discrete symmetry, up to cubic order in fluctuations. The generalisation to multimetric gravity in the star graph with N satellite metrics and maximal discrete symmetry is also discussed, showing the same behavior of the trimetric case and not leading to new phenomenology. We focus then on the trimetric case, where we show that certain features of the theory that occur up to cubic order hold to all orders. In particular, the heaviest massive spin-2 field $M_{\mu\nu}$ neither decays into massless gravitons $G_{\mu\nu}$ nor into lighter modes $\chi_{\mu\nu}$ and its coupling to matter is very small. The lighter mode does not interact with matter and it does not decay into $G_{\mu\nu}$ or $M_{\mu\nu}$. Given that the lighter mode does not have any decay channel, it is stable and we suggest that it can be the component of dark matter.

6.1 Perturbative expansion of the action

In this section we discuss the trimetric gravity action (eq. (3.36)) with maximal discrete symmetry in the perturbative expansion, in terms of the mass eigenstates $G_{\mu\nu}$, $M_{\mu\nu}$ and $\chi_{\mu\nu}$. The original metric fluctuations $h^{(g)}_{\mu\nu}$, $h^{(f)}_{\mu\nu}$ and $h^{(k)}_{\mu\nu}$ are expressed in terms of the mass eigenstates in eq. (3.48). By plugging in these expressions in the trimetric gravity action we are able to compute all possible interaction vertices between the mass eigenstates to all orders. We restrict the parameters of the theory to satisfy eqs. (3.27) and (3.33) in order that the theory has maximal discrete symmetry.

We see from eqs. (3.48) and (3.47) that, if $\alpha < 1$, the higher-order vertices of $M_{\mu\nu}$ and $\chi_{\mu\nu}$ are suppressed by factors of $1/(m_{\rm P}\alpha)$. This occurs as well in the bimetric case where α is defined as the ratio of the constants multiplying the Einstein–Hilbert terms in the bimetric action [71].

6.1.1 Nonlinear massless field

Let us consider the original metrics $g_{\mu\nu}$, $f_{\mu\nu}$ and $k_{\mu\nu}$ in terms of the original fluctuations (eq. (3.42)). Using the relations between the original fluctuations and the eigenstates, for the maximal discrete symmetric case (eq. (3.48)), we can write

$$g_{\mu\nu} = \gamma_{\mu\nu} - \frac{\alpha}{m_{\rm P}} M_{\mu\nu} \,, \tag{6.1a}$$

$$f_{\mu\nu} = \frac{\alpha^2}{2\alpha^2_{(f)}}\gamma_{\mu\nu} + \frac{\alpha}{2\alpha^2_{(f)} m_{\rm P}} \left(M_{\mu\nu} - \sqrt{1+\alpha^2}\,\chi_{\mu\nu} \right) \,, \tag{6.1b}$$

$$k_{\mu\nu} = \frac{\alpha^2}{2\alpha^2_{(k)}}\gamma_{\mu\nu} + \frac{\alpha}{2\alpha^2_{(k)} m_{\rm P}} \left(M_{\mu\nu} + \sqrt{1+\alpha^2}\,\chi_{\mu\nu} \right) \,, \tag{6.1c}$$

where we have defined $\gamma_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{m_{\rm P}} G_{\mu\nu}$. We see that the quantities $\gamma_{\mu\nu}$ take the role of a background metric for linear combinations of the fluctuations $M_{\mu\nu}$ and $\chi_{\mu\nu}$. Furthermore, for the case of no perturbations around the background $\gamma_{\mu\nu}$, i.e. $M_{\mu\nu} = \chi_{\mu\nu} = 0$, the trimetric gravity action for $g_{\mu\nu}$, $f_{\mu\nu}$ and $k_{\mu\nu}$ of eq. (6.1) reads

$$S[g,f,k]\Big|_{M=\chi=0} = m_{\rm P}^2 \int {\rm d}^4x\, \sqrt{|\gamma|}\, \left(R(\gamma) - 2\Lambda \right) \,, \tag{6.2}$$

which is the Einstein–Hilbert action for the background metric $\gamma_{\mu\nu}$ with the cosmological constant term[1]. The action above represents the dynamics for a general perturbation $G_{\mu\nu} \neq 0$ at any perturbative order. Therefore, we interpret the fluctuation $G_{\mu\nu}$ as a massless field that mediates the long-ranged gravitational force on the spacetime defined by metric $\gamma_{\mu\nu}$, on which the massive spin-2 modes $M_{\mu\nu}$ and $\chi_{\mu\nu}$ propagate.

6.1.2 Linear massive fluctuations

In the following we study the vertices that are linear in the massive modes. To the end, analyse the Lagrangian for trimetric gravity (eq. (3.36)) up to linear order in the massive modes $M_{\mu\nu}$ and $\chi_{\mu\nu}$ as perturbations around $\gamma_{\mu\nu}$, which keeps all orders of the massless eigenstate $G_{\mu\nu}$ (as discussed in § 6.1.1).

6.1.2.1 Contribution of the Einstein–Hilbert terms

The Taylor expansion of the the Einstein–Hilbert terms in the trimetric Lagrangian, up to first-order in metric perturbations is given by[2]

$$\begin{aligned}
& m_g^2\Big(\frac{\delta(\sqrt{|g|}\,R(g))}{\delta g_{\mu\nu}}\Big|_{g=\gamma} \delta g_{\mu\nu} + \alpha_{(f)}^2 \frac{\delta(\sqrt{|f|}\,R(f))}{\delta f_{\mu\nu}}\Big|_{f/c^2=\gamma} \delta f_{\mu\nu} + \alpha_{(k)}^2 \frac{\delta(\sqrt{|k|}\,R(k))}{\delta k_{\mu\nu}}\Big|_{k/c^2=\gamma} \delta k_{\mu\nu}\Big) \,, \\
& = m_g^2\Big(\frac{\delta(\sqrt{|g|}\,R(g))}{\delta g_{\mu\nu}}\Big|_{g=\gamma} h^{(g)}_{\mu\nu} + \alpha_{(f)}^2 \frac{\delta(\sqrt{|g|}\,R(g))}{\delta g_{\mu\nu}}\Big|_{g=\gamma} h^{(f)}_{\mu\nu} + \alpha_{(k)}^2 \frac{\delta(\sqrt{|g|}\,R(g))}{\delta g_{\mu\nu}}\Big|_{g=\gamma} h^{(k)}_{\mu\nu}\Big) \,, \\
& = m_g^2 \frac{\delta(\sqrt{|g|}\,R(g))}{\delta g_{\mu\nu}}\Big|_{g=\gamma} \left(h^{(g)}_{\mu\nu} + \alpha_{(f)}^2\, h^{(f)}_{\mu\nu} + \alpha_{(k)}^2\, h^{(k)}_{\mu\nu}\right) \,, \\
& = \frac{m_g^2}{m_{\mathrm{P}}} \frac{\delta\sqrt{|g|}\,R(g)}{\delta g_{\mu\nu}}\Big|_{g=\gamma} \Big[\Big(\frac{\alpha_{(f)}^2 c_{(f)}^2}{\alpha} + \frac{\alpha_{(h)}^2 c_{(h)}^2}{\alpha} - \alpha\Big) M_{\mu\nu} \\
& \qquad + \frac{\sqrt{1+\alpha^2}}{\alpha}\Big(\alpha_{(h)}^2 c_{(h)}^2 - \alpha_{(f)}^2 c_{(f)}^2\Big)\chi_{\mu\nu}\Big] \,, \\
& = 0 \,,
\end{aligned} \tag{6.3}$$

[1]The constant Λ is defined in the context of trimetric gravity in the first paragraph of § 3.7.2.1.

[2]We use the short notation $f_{\mu\nu}/c^2_{(f)} \to f/c^2$ and $k_{\mu\nu}/c^2_{(k)} \to k/c^2$.

where in the last line we have used the relations between α, $\alpha_{(h)}$ and $\alpha_{(f)}$ of eq. (3.47), provided by the discrete symmetries. This implies, that no vertices that are linear in the massive modes arise from the Einstein–Hilbert terms, in the case of maximal discrete symmetry.

6.1.2.2 Contribution of the trimetric potential

In the following we perform a Taylor expansion of the interaction potential up to first order around the proportional backgrounds:

$$
\begin{aligned}
&\left[\sqrt{|g|}\Big(V(g,f;\beta_i^{(f)})+V(g,k;\beta_i^{(k)})\Big)\right]_{f/c^2=k/c^2=g=\gamma}\\
&=\frac{\delta(\sqrt{|g|}\,V)}{\delta\chi_{\rho\sigma}}\bigg|_{f/c^2=k/c^2=g=\gamma}\delta\chi_{\rho\sigma}+\frac{\delta(\sqrt{|g|}\,V)}{\delta M_{\rho\sigma}}\bigg|_{f/c^2=k/c^2=g=\gamma}\delta M_{\rho\sigma}\ ,\\
&=\left[\frac{\delta(\sqrt{|g|}V(g,f;\beta_i^{(f)}))}{\delta f^{\mu\nu}}\frac{\delta f^{\mu\nu}}{\delta\chi_{\rho\sigma}}+\frac{\delta(\sqrt{|g|}V(g,k;\beta_i^{(k)}))}{\delta k^{\mu\nu}}\frac{\delta k^{\mu\nu}}{\delta\chi_{\rho\sigma}}\right]_{f/c^2=k/c^2=g=\gamma}\delta\chi_{\rho\sigma}\\
&+\left[\frac{\delta(\sqrt{|g|}V)}{\delta g^{\mu\nu}}\frac{\delta g^{\mu\nu}}{\delta M_{\rho\sigma}}+\frac{\delta(\sqrt{|g|}V(g,f;\beta_i^{(f)}))}{\delta f^{\mu\nu}}\frac{\delta f^{\mu\nu}}{\delta M_{\rho\sigma}}+\frac{\delta(\sqrt{|g|}V(g,k;\beta_i^{(k)}))}{\delta k^{\mu\nu}}\frac{\delta k^{\mu\nu}}{\delta M_{\rho\sigma}}\right]_{f/c^2=k/c^2=g=\gamma}\delta M_{\rho\sigma}\ ,\\
&=\frac{\sqrt{1+\alpha^2}}{2\alpha M^2}\tilde\Lambda(\beta_i^{(f)},c_{(f)},\alpha_{(f)})\Big(\alpha_{(k)}^2c_{(k)}^2-\alpha_{(f)}^2c_{(f)}^2\Big)\sqrt{|\gamma|}\,\gamma^{\rho\sigma}\delta\chi_{\rho\sigma}\\
&+\frac{1}{2M^2}\tilde\Lambda(\beta_i^{(k)},c_{(k)},\alpha_{(k)})\Big(\frac{\alpha_{(f)}^2c_{(f)}^2}{\alpha}+\frac{\alpha_{(k)}^2c_{(k)}^2}{\alpha}-\alpha\Big)\sqrt{|\gamma|}\,\gamma^{\rho\sigma}\delta M_{\rho\sigma}\ ,\\
&=0\ .
\end{aligned}
\tag{6.4}
$$

Here we used eqs. (3.40) and (6.1), as well as (3.47) to conclude the last line. Hence, also the potential does not contribute with linear term in the massive modes.

Considering also the results of § 6.1.2.1, we conclude that trimetric gravity with maximal discrete symmetry does not have any vertex that is linear in the massive modes $M_{\mu\nu}$ and $\chi_{\mu\nu}$ around the background $\gamma_{\mu\nu}$.

6.1.3 Cubic vertices

We continue then with the Taylor expansion of the trimetric gravity Lagrangian up to cubic order in the massive modes. Given that the calculation is rather

GGG	$1, \Lambda$	GMM	$1, \Lambda, m_M^2$	$MM\chi$	0
GGM	0	$G\chi\chi$	$1, \Lambda, m_\chi^2$	$M\chi\chi$	$\frac{1}{\alpha} \cdot (1, \Lambda, m_M^2, m_\chi^2)$
$GG\chi$	0	MMM	$\frac{1-\alpha^2}{\alpha} \cdot (1, \Lambda, m_M^2)$	$\chi\chi\chi$	0

Table 6.1: Factors of the cubic Lagrangian for trimetric gravity with maximal global discrete symmetry (cf. eq. (B.22)). Here we have omitted all numerical factors and any dimensionless constant but α. Also, all terms are divided by m_{P}.

long, the detailed procedure is summarized in § B.3. Table 6.1 shows the factors of the cubic interaction vertices.

We see from table 6.1 that the cubic self-interaction terms of the massless mode $G_{\mu\nu}$ are the same as in general relativity. This is in agreement the result of § 6.1.1, where we calculated the perturbative expansion to all orders. Also, we see that there are no cubic vertices that are linear in the massive modes $M_{\mu\nu}$ nor $\chi_{\mu\nu}$, which is in agreement with our discussion of § 6.1.2. The calculation of the cubic terms confirms that there are no vertices of the type GGM and $GG\chi$ and thus the massive gravitons cannot decay into massless gravitons.

The Lagrangian up to cubic order does not contain any term with odd powers of $\chi_{\mu\nu}$ which is due to the discrete symmetry. In fact, this extends to any order in the perturbative expansion. To see this, consider the interchange symmetry $\alpha_{(f)}^2 \, f_{\mu\nu} \leftrightarrow \alpha_{(k)}^2 \, k_{\mu\nu}$. This transformation leaves the entire trimetric action invariant, however the eigenstates transform as

$$G_{\mu\nu} \to G_{\mu\nu} \,, \tag{6.5a}$$

$$M_{\mu\nu} \to M_{\mu\nu} \,, \tag{6.5b}$$

$$\chi_{\mu\nu} \to -\chi_{\mu\nu} \,, \tag{6.5c}$$

as it can be seen directly from eq. (3.46). Thus, any term containing an odd power of $\chi_{\mu\nu}$ must be absent, otherwise it would spoil the invariance of the action. As a consequence of this, the decay $M \to G \cdots G\chi$, i.e., for an arbitrary number of massless gravitons, is not allowed, because that would require a vertex that is linear in $\chi_{\mu\nu}$. Said in other words, a decay of the heaviest mode into the lightest mode plus massless gravitons is not allowed.

Remarkably, the cubic vertices that are quadratic in the massive fields (GMM

and $G\chi\chi$) do not depend on the parameter α. This is important since, treating the massive fields as matter, they are contained in an expression for the gravitational stress-energy tensor, which is expected to be independent of α according to the Noether stress-energy tensor defined for the quadratic action in flat space [123].

6.1.4 Generalization to multiple fields

In the following we proceed with the generalisation of the trimetric model with maximal discrete symmetry for N satellite metrics. The mass spectrum for most general ghost-free multimetric theory is quite cumbersome due to the large amount of parameters. The case for the star graph (eq. (3.25)) was worked out in ref. [119]. Similarly as studied in § 3.7.2 and § 6.1, the mass eigenstates are linear combinations of the metric fluctuations $h^{(g)}_{\mu\nu}$ and $h^{(p)}_{\mu\nu}$ around a maximally symmetric background solution. Here $p = 1, \ldots, N$ labels the fluctuation for the satellite metric $f^{(p)}_{\mu\nu}$. Furthermore, the eigenmodes are always such that there is one single massless mode $G_{\mu\nu}$, one massive mode $M_{\mu\nu}$ with mass m_M and $N-1$ modes $\chi^{(r)}_{\mu\nu}$ with masses $m_{(r)} < m_M$, where $r = 1, \ldots, N-1$.

Following the results of ref. [119], it is straightforward to calculate the mass spectrum for the multimetric action with N satellite metrics (eq. (3.34)), now including maximal discrete symmetry. For the proportional background solution $f^{(p)}_{\mu\nu} = c^2_{(p)} g_{\mu\nu}$, the mass spectrum assumes the form

$$G_{\mu\nu} = \tfrac{m_{\mathrm{P}}}{1+\alpha^2}\left(h^{(g)}_{\mu\nu} + \textstyle\sum_{p=1}^{N} \alpha^2_{(p)} h^{(p)}_{\mu\nu}\right) , \tag{6.6a}$$

$$M_{\mu\nu} = -\tfrac{m_{\mathrm{P}}}{\alpha(1+\alpha^2)}\left(\alpha^2 h^{(g)}_{\mu\nu} - \sum_{p=1}^{N} \alpha^2_{(p)} f^{(p)}_{\mu\nu}\right) , \tag{6.6b}$$

$$\chi^{(r)}_{\mu\nu} = \tfrac{m_{\mathrm{P}}}{\alpha\sqrt{1+\alpha^2}}\left(\alpha^2_{(r)} h^{(r)}_{\mu\nu} - \alpha^2_{(r+1)} h^{(r+1)}_{\mu\nu}\right) , \tag{6.6c}$$

where $i = 1, \ldots, N-1$ and $\alpha^2 = \sum_{p=1}^{N} \alpha^2_{(p)} c^2_{(p)}$ and $m_{\mathrm{P}} = \sqrt{1+\alpha^2}\, m_g$. We recognize the same behavior as in the trimetric case: the massive states $\delta\chi^{(r)}_{\mu\nu}$ do not depend on the fluctuation of $h^{(g)}_{\mu\nu}$. Hence, this massive modes do not couple to matter (cf. eq. (3.11)). On the other hand, the massive mode $M_{\mu\nu}$

still depends on $h^{(g)}_{\mu\nu}$.

The masses of the eigenmodes satisfy

$$m_{(r)} = m_{(s)} = \frac{m_M}{\sqrt{1+\alpha^2}} \,, \tag{6.7}$$

for all values $r, s = 1, \ldots, N-1$. Exactly as in the trimetric case, solving eq. (6.6) for $h^{(g)}_{\mu\nu}$ gives

$$h^{(g)}_{\mu\nu} = \tfrac{1}{m_{\text{P}}} \left(G_{\mu\nu} - \alpha\, M_{\mu\nu}\right) \,. \tag{6.8}$$

The result in the trimetric theory, that $\chi_{\mu\nu}$ does not couple to matter generalises for the $(N-1)$ massive spin-2 particles corresponding to the modes $\chi^{(r)}_{\mu\nu}$. Moreover, as we can see from eq. (6.7) they have equal masses and are lighter than $M_{\mu\nu}$. This forbids $M_{\mu\nu}$ to decay into other massive spin-2 particles. Other channels for higher order vertices are again are not allowed by means of the discrete symmetries, which again forbid vertices that are linear in the modes $\chi^{(r)}_{\mu\nu}$. By last, generalising the discussion in § 6.1.2 shows that $\chi^{(r)}_{\mu\nu}$ cannot decay into massless modes $G_{\mu\nu}$. The novel structure of the trimetric theory lays on the fact that we have a new, completely stable, massive spin-2 field. The deviations from general relativity are then controlled by α and M. Now, in the multimetric case we have the same behavior as in the trimetric case: α and M control deviations and we have a bunch of lighter massive spin-2 fields with equal masses. Therefore, we do expect fundamentally new phenomena for $N > 2$ and hence we focus on the trimetric case in the following.

6.2 Lighter mode as component of dark matter

6.2.1 Assumptions

As we saw in § 6.1, due to the discrete symmetries, the massive mode $\chi_{\mu\nu}$ is stable because it does not couple to matter (standard model fields) and because it neither decays into massive spin-2 fields $M_{\mu\nu}$, nor massless gravitons[3] $G_{\mu\nu}$. We assume that the discrete symmetries of the trimetric Lagrangian are stable under quantum corrections. Note that there is no obvious symmetry giving rise to the vanishing coupling of $f_{\mu\nu}$ and $k_{\mu\nu}$ to the stress-energy tensor. This problem is still not solved in bi- and multi-metric gravity theory.

Moreover, we make an assumption on the parameter α. This is a dimensionless quantity that parameterizes the interaction strengths of massive spin-2 modes. Since α and M control deviations from general relativity, demanding that $\alpha \ll 1$ ensures that these deviations are small for a large range of spin-2 masses. This motivates us to focus on values $\alpha < 1$.

6.2.2 Dark matter

Taking the assumptions of § 6.2.1 into account, the fact that the lightest mode $\chi_{\mu\nu}$ is completely stable motivates us to consider it as a possible dark matter candidate.

For $\alpha < 1$, the masses of the spin-2 particles are of the same order, i.e., $m_M \simeq m_\chi$. Also, $M_{\mu\nu}$ cannot decay into $\chi_{\mu\nu}$ since $2\mu_\chi > \mu_M$. We take $\beta_2^f = \alpha_h^2 \beta_2^h / \alpha_f^2 \approx 1$. This assumption is without loss of generality since the scale of the β_2 can always be absorbed into M (cf. the potential of trimetric gravity, eq. (3.36)). In that case we have

$$m_M \simeq m_\chi \simeq M \, . \tag{6.9}$$

In this case, the heaviest mode can still decay into standard model fields and its matter coupling —as in bimetric theory— is controlled by the (weak) factor α/m_{P} (see eq. (6.8)).

In the context of bimetric gravity, ref. [71] argued that $M_{\mu\nu}$ makes up (part of) the observed dark matter density. The non-observation of dark matter particles in particle accelerator sets the constraint $10^{-15} \lesssim \alpha \lesssim 10^{-12}$ and $M \simeq 1 - 100\,\mathtt{TeV}$. The trimetric theory, however, enlarges the parameter scenario:

[3] This fact occurs even without the discrete symmetries.

we can require that $M_{\mu\nu}$ does not contribute to the observed amount of dark matter because it has decayed into standard model particles since the end of inflation. For that case, we recover reversed stability constraint

$$\alpha^{2/3} m_M > 0.13\,\mathtt{GeV}\,, \tag{6.10}$$

introduced in ref. [71]. Fixing the spin-2 mass m_M gives us then a further bound for α.

Chapter 7

Conclusions

In this dissertation we studied several mathematical tools that gave us an understanding on Lie algebras and Chern–Simons gauge theory. Furthermore, we analysed the current theories of massive spin-2 fields and the problems in the gauge formulation of gravitational theory. It was possible to conclude that, from the 5-dimensional Chern–Simons gauge theory for the group $\mathrm{SO}(4,2)$, we can obtain the following theories in the following senses:

- **Einstein–Cartan theory:** after taking the Inönü–Wigner contraction limit that reduces the symmetry from $\mathrm{SO}(4,2)$ to $\mathrm{ISO}(3,2)$ or $\mathrm{ISO}(4,1)$, we performed a simple dimensional reduction that led the Chern–Simons action to the Einstein–Cartan theory in four dimensions plus a Lorentz-breaking term involving torsion. This extra term could be removed by restricting a field in the 5-dimensional action or by taking a critical limit of the parameter ℓ.

- **First-order conformal gravity:** without having taken the Inönü–Wigner contraction limit, a similar dimensional reduction scheme as above led us to a first-order version of conformal gravity (studied by Kaku et al in ref. [67]). We concluded that the Weyl rotation symmetry of the 4-dimensional action does not arise from a subalgebra of the original gauge algebra $\mathrm{SO}(4,2)$, which is broken to $\mathrm{SO}(3,1)$ by the dimensional reduction. Instead, we observed that the Weyl rotation symmetry originates from the SO$(1,1)$ symmetry of the gauge algebra, i.e., a rotation of the generators that leaves the commutation relations invariant.

- **Generalized first-order conformal gravity:** a dimensional reduction scheme that introduces two warp functions was considered. This was made in a particular basis of the algebra. We could conclude that in a different basis, the same dimensional reduction is not possible, and therefore, we broke the similarity between all Chern–Simons theories written in different bases of SO$(4,2)$. Integrating along the warp direction led the Chern–Simons action to a generalisation of first-order conformal gravity. This theory contains standard first-order conformal gravity as in ref. [67] for a particular choice of parameters.

- **Generalized first-order bimetric theory:** we considered a doubled Chern–Simons action in $D = 5$ with symmetry group SO$(4,2)\times$SO$(4,2)$. We observed that making the fields interdependent in a certain way, breaks the gauge symmetry to SO$(3,1) \times$ SO(2). After a dimensional reduction we obtained an action that can be seen as a generalised bimetric theory involving a new type of kinetic interaction. For a certain choice of the parameters we obtained standard bimetric theory *à la* Hassan and Rosen in the first-order formalism. We also discussed another choice of parameters which recovers the Weyl rotation symmetry typical of first-order conformal gravity.

Furthermore, for the multimetric gravity theory:

- We concluded that the maximal global discrete symmetry in $(N+1)$-metric theories is $S_N \times (\mathbb{Z}_2)^N$. Also, we presented the corresponding action invariant under this extra symmetry. Moreover, we analysed the mass spectrum for the maximal global discrete symmetry. This showed that the multimetric theory does not bring new phenomenology for $N > 2$, thus we focused mainly in the trimetric case. The trimetric case turned out to contain one massless graviton $G_{\mu\nu}$, the massive graviton $M_{\mu\nu}$ and a lighter massive graviton $\chi_{\mu\nu}$, in agreement with the general results of ref. [119].

- Some features of the perturbative action with maximal global discrete symmetry were studied. For this, we computed all cubic interaction vertices in terms of the mass eigenstates. We found that the trimetric theory with maximal global discrete symmetry is a nontrivial generalisation of the bimetric case. This is mainly due to the existence of a lighter massive mode.

- We concluded that the heaviest spin-2 field $M_{\mu\nu}$ is neither allowed to decay into massless gravitons nor into lighter spin-2 fields. This is due to the discrete symmetries. Moreover, we saw that $M_{\mu\nu}$ couples to standard model matter very weakly. On the other hand, the lighter massive field does not interact with matter and it is not allowed to decay into other spin-2 particles either. Therefore, $\chi_{\mu\nu}$ was shown to be completely stable and we postulated it as the ingredient of dark matter by discussing the parameter regions for α and the mass scale M.

Appendix A

Formulae in geometry

A.1 Differential geometry and tensors

We start defining a derivative of the vector components V_μ as

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho{}_{\mu\nu} V_\rho \,, \tag{A.1}$$

where Γ is an arbitrary connection and ∇_μ is called the covariant derivative. Assuming that, for a scalar $\nabla_\mu \phi = \partial_\mu \phi$ and also that ∇_μ satisfy the Leibniz rule, one obtains

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\rho} V^\rho \,. \tag{A.2}$$

One can prove then that the covariant derivative on the tensor is

$$\nabla_\mu V_{\nu\rho} = \partial_\mu V_{\nu\rho} - \Gamma^\sigma{}_{\mu\nu} V_{\sigma\rho} - \Gamma^\sigma{}_{\mu\rho} V_{\nu\sigma} \,, \tag{A.3a}$$

$$\nabla_\mu V^{\nu\rho} = \partial_\mu V^{\nu\rho} + \Gamma^\nu{}_{\mu\sigma} V^{\sigma\rho} + \Gamma^\rho{}_{\mu\sigma} V^{\nu\sigma} \,. \tag{A.3b}$$

The commutator between two covariant derivatives is given by

$$[\nabla_\mu, \nabla_\nu]\, V_\rho = -R^\sigma{}_{\rho\mu\nu} V_\sigma - T^\sigma{}_{\mu\nu} \nabla_\sigma V_\rho \,, \tag{A.4}$$

where

$$R^{\sigma}{}_{\rho\mu\nu}(\Gamma) = \partial_\mu \Gamma^{\sigma}{}_{\nu\rho} - \partial_\nu \Gamma^{\sigma}{}_{\mu\rho} + \Gamma^{\sigma}{}_{\mu\tau}\Gamma^{\tau}{}_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\tau}\Gamma^{\tau}{}_{\mu\rho} \; , \tag{A.5a}$$

$$T^{\sigma}{}_{\mu\nu}(\Gamma) = \Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\sigma}{}_{\nu\mu} \; , \tag{A.5b}$$

are the Riemann and Torsion tensors respectively. They satisfy

$$R^{\sigma}{}_{\rho\mu\nu} = -R^{\sigma}{}_{\rho\nu\mu} \; , \tag{A.6a}$$

$$T^{\sigma}{}_{\mu\nu} = -T^{\sigma}{}_{\nu\mu} \; . \tag{A.6b}$$

Curvatures. In terms of the Riemann tensor one can define the following quantities, that we use repeatedly use in this thesis: we have the Ricci tensor $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$, the Ricci scalar $R = g^{\mu\nu}R^{\lambda}{}_{\mu\lambda\nu}$, the Einstein tensor $\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, the Gauss–Bonnet density

$$G = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \; , \tag{A.7}$$

the Weyl tensor

$$\begin{aligned} C_{\mu\nu\lambda\rho} = R_{\mu\nu\lambda\rho} &+ \tfrac{1}{D-2}\Big(R_{\mu\rho}g_{\nu\lambda} - R_{\mu\lambda}g_{\nu\rho} + R_{\nu\lambda}g_{\mu\rho} - R_{\nu\rho}g_{\mu\lambda}\Big) \\ &+ \tfrac{1}{(D-1)(D-2)}R\Big(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}\Big) \; , \end{aligned} \tag{A.8}$$

where $D = p + q$, and by last, we have the Lorentz curvature in terms of the vielbein matrix $e^{a}{}_{\mu}$

$$R^{ab}(e) = \tfrac{1}{2}\, e^{a}{}_{\mu} e^{b}{}_{\nu}\, R^{\mu\nu}{}_{\lambda\rho}(g(e))\, \mathrm{d}x^{\lambda} \wedge \mathrm{d}x^{\rho} \; , \tag{A.9}$$

with $g_{\mu\nu} = e^{a}{}_{\mu} e^{b}{}_{\nu}\, \eta_{ab}$.

Gauss–Bonnet density. It is straight forward to show that

$$\int_{M_4} \epsilon_{abcd}\, R^{ab}(\omega) \wedge R^{cd}(\omega) \propto \int_{M_4} \mathrm{d}^4x\, G \; , \tag{A.10}$$

where G is the Gauss–Bonnet density, provided that vielbein postulate (eq. (1.3)) holds.

Lichnerowicz operator. The expression for the Lichnerowicz operator defined in a curve spacetimes with metric $g_{\mu\nu}$ is given by

$$\Box_{\mu\nu}{}^{\lambda\rho} = -\tfrac{1}{2}\Big(\delta^\rho_\mu\delta^\sigma_\nu\nabla^2 + g^{\rho\sigma}\nabla_\mu\nabla_\nu - \delta^\rho_\mu\nabla^\sigma\nabla_\nu - \delta^\rho_\nu\nabla^\sigma\nabla_\mu - g_{\mu\nu}g^{\rho\sigma}\nabla^2 + g_{\mu\nu}\nabla^\rho\nabla^\sigma\Big)\,. \tag{A.11}$$

Here ∇_μ is the covariant derivative for the Levi-Civita connection of $g_{\mu\nu}$ and $\nabla^2 = g^{\mu\nu}\nabla_\mu\nabla_\nu$.

Levi-Civita symbol. Let A^{ab} be an anti-symmetric symbol, the generalised Levi-Civita symbol satisfies:

$$\sum_{\imath=1}^{p+q} \epsilon_{a_1\cdots a_{\imath-1}ba_{\imath+1}\cdots a_{p+q}}A^b{}_{a_\imath} = 0\,. \tag{A.12}$$

Appendix B

Further calculations and expressions

B.1 Proof for arbitrariness of the p's and $\tilde{p}$'s

In this section we show that the constants p_{st} and $\tilde{p}_{st}$ are arbitrary. For this we assume that the curve Σ is such that we can always find a chart of coordinate w and open interval (w_0, w_1) so that the integrals are not trivial. According to our discussion in § 5.1.1, the Chern–Simons theories that we constructed are such that Σ is homeomorphic to an interval.

Let us begin defining the functions

$$p_{st}(w) = \int_{w_0}^{w} \mathrm{d}w' \, e^s(w')\imath^t(w')\mu_5(w') \,, \tag{B.1a}$$

$$\tilde{p}_{st}(w) = \int_{w_0}^{w} \mathrm{d}w' \, e^s(w')\imath^t(w')\tilde{\mu}_5(w') \,. \tag{B.1b}$$

Clearly we have that

$$p_{st}(w_1) = p_{st} \,, \tag{B.2a}$$

$$\tilde{p}_{st}(w_1) = \tilde{p}_{st} \,, \tag{B.2b}$$

$$p_{st}(w_0) = 0 \,, \tag{B.2c}$$

$$\tilde{p}_{st}(w_0) = 0 \; . \tag{B.2d}$$

Using the fundamental theorem of calculus we observe that the derivatives of the functions p's are given by

$$\frac{\mathrm{d}p_{st}}{\mathrm{d}w}(w) = e^s(w)\imath^t(w)\mu_5(w) \; , \tag{B.3a}$$

$$\frac{\mathrm{d}\tilde{p}_{st}}{\mathrm{d}w}(w) = e^s(w)\imath^t(w)\tilde{\mu}_5(w) \; . \tag{B.3b}$$

From this we see that

$$\tilde{\mu}_5(w)\frac{\mathrm{d}p_{st}}{\mathrm{d}w}(w) = \mu_5(w)\frac{\mathrm{d}\tilde{p}_{st}}{\mathrm{d}w}(w) \; , \tag{B.4}$$

and using the product rule for derivatives we find

$$\frac{\mathrm{d}}{\mathrm{d}w}\Big(\tilde{\mu}_5(w)p_{st}(w) - \mu_5(w)\tilde{p}_{st}(w)\Big) = \frac{\mathrm{d}\tilde{\mu}_5}{\mathrm{d}w}(w)p_{st}(w) - \frac{\mathrm{d}\mu_5}{\mathrm{d}w}(w)\tilde{p}_{st}(w) \; . \tag{B.5}$$

Evaluating at $w = w_0$ and $w = w_1$ we find

$$\frac{\mathrm{d}}{\mathrm{d}w}\Big(\tilde{\mu}_5(w)p_{st}(w) - \mu_5(w)\tilde{p}_{st}(w)\Big)\Big|_{w=w_0} = 0 \; , \tag{B.6a}$$

$$\frac{\mathrm{d}}{\mathrm{d}w}\Big(\tilde{\mu}_5(w)(p_{st}(w) - p_{st}) - \mu_5(w)(\tilde{p}_{st}(w) - \tilde{p}_{st})\Big)\Big|_{w=w_1} = 0 \; , \tag{B.6b}$$

respectively. Here we can think about functions whose derivatives vanish at $w = w_0$ and $w = w_1$ respectively. The simplest function would be some power $q > 1$ of $w - w_0$ and $r > 1$ of $w - w_1$. We then have

$$\tilde{\mu}_5(w)p_{st}(w) - \mu_5(w)\tilde{p}_{st}(w) = a_{st}(w - w_0)^{q_{st}} \; , \tag{B.7a}$$

$$\tilde{\mu}_5(w)(p_{st}(w) - p_{st}) - \mu_5(w)(\tilde{p}_{st}(w) - \tilde{p}_{st}) = b_{st}(w - w_1)^{r_{st}} \; , \tag{B.7b}$$

where $q_{st}, r_{st} > 1$ and a_{st} and b_{st} are constants. One could also consider more sophisticated functions whose derivatives vanish at w_0 and w_1, however their Taylor expansion would contain $a_{st}(w - w_0)^{q_{st}}$ and $b_{st}(w - w_1)^{r_{st}}$ respectively as a term and therefore this latter function is just a special case with less free parameters (which are enough to make the proof). Eqs. (B.7a) and (B.7b) imply

$$\tilde{\mu}_5(w)p_{st} - \mu_5(w)\tilde{p}_{st} = a_{st}(w - w_0)^{q_{st}} - b_{st}(w - w_1)^{r_{st}} \; . \tag{B.8}$$

Evaluating this expression at $w = w_0$ and $w = w_1$ we get

$$\tilde{\mu}_5(w_0)p_{st} - \mu_5(w_0)\tilde{p}_{st} = -b_{st}(w_0 - w_1)^{r_{st}} \,, \tag{B.9a}$$

$$\tilde{\mu}_5(w_1)p_{st} - \mu_5(w_1)\tilde{p}_{st} = a_{st}(w_1 - w_0)^{q_{st}} \,, \tag{B.9b}$$

respectively. By last, we can solve for p's and $\tilde{p}$'s to get

$$p_{st} = \frac{\mu_5(w_0)a_{st}(w_1 - w_0)^{q_{st}} + \mu_5(w_1)b_{st}(w_0 - w_1)^{r_{st}}}{\mu_5(w_0)\tilde{\mu}_5(w_1) - \tilde{\mu}_5(w_0)\mu_5(w_1)} \,, \tag{B.10a}$$

$$\tilde{p}_{st} = \frac{\tilde{\mu}_5(w_0)a_{st}(w_1 - w_0)^{q_{st}} + \tilde{\mu}_5(w_1)b_{st}(w_0 - w_1)^{r_{st}}}{\mu_5(w_0)\tilde{\mu}_5(w_1) - \tilde{\mu}_5(w_0)\mu_5(w_1)} \,. \tag{B.10b}$$

From this expressions we see that the p's and the $\tilde{p}$'s are always proportional to the constants a_{st} and b_{st} respectively. Since a_{st} and b_{st} are arbitrary constants only subject to define the vanishing-first-derivative functions of eqs. (B.7a) and (B.7b), then the p's and the $\tilde{p}$'s are completely arbitrary and they can be used as free parameters of a theory.

B.2 Chern–Simons action in components

B.2.1 Conformal basis

The torsion terms in the conformal basis in eq. (5.9) are given by

$$\begin{aligned} S_{\text{torsion}}[\omega, e, h] = &\mp \int_{M_5} \epsilon_{abcd}\, R^{ab}(\omega) \wedge \Big(a^2\, T^c(\omega, e) \wedge e^d \\ &- ac\, T^c(\omega, e) \wedge h^d + ac\, T^c(\omega, h) \wedge e^d - c^2\, T^c(\omega, h) \wedge h^d \Big) \\ &\mp \frac{2\eta}{3} \int_{M_5} \epsilon_{abcd} \Big(a\, T^a(\omega, e) + c\, T^a(\omega, h) \Big) \wedge \Big(a^3\, e^b \wedge e^c \wedge e^d \\ &+ 3a^2 c\, e^b \wedge e^c \wedge h^d - 3ac^2\, e^b \wedge h^c \wedge h^d - c^3\, h^b \wedge h^c \wedge h^d \Big) \,, \end{aligned} \tag{B.11}$$

for the algebraic solutions of $A = C = 0$ given by $b = \pm a\gamma$ and $d = \mp c\gamma$ respectively.

B.2.2 Orthogonal basis

In the orthogonal basis, i.e. for $b = \pm ia\gamma$ and $d = \mp ic\gamma$, the torsion terms of eq. (5.10) are

$$
\begin{aligned}
S'_{\text{torsion}}[\omega, e, h] = \mp i \int_{M_5} \epsilon_{abcd}\, R^{ab}(\omega) \wedge \Big(a^2\, T^c(\omega, e) \wedge e^d \\
- ac\, T^c(\omega, e) \wedge h^d + ac\, T^c(\omega, h) \wedge e^d - c^2\, T^c(\omega, h) \wedge h^d \Big) \\
\pm i \frac{4\eta}{3} \int_{M_5} \epsilon_{abcd} \Big(a\, T^a(\omega, e) + c\, T^a(\omega, h) \Big) \wedge \Big(a^3\, e^b \wedge e^c \wedge e^d \\
- c^3\, h^b \wedge h^c \wedge h^d \Big) \, .
\end{aligned}
\tag{B.12}
$$

B.3 Quadratic action and cubic vertices

B.3.1 Useful definitions

In the following we introduce some useful definitions. Here we omit we bar on the background metric to simplify the notation.

Einstein gravity. Let us first define the bilinear operator

$$
\begin{aligned}
K^{(2)}_{\mu\nu}(h, \ell) = \nabla_\mu h_{\rho\sigma} \nabla_\nu \ell^{\rho\sigma} - \nabla_\mu h \nabla_\nu \ell + \nabla^\rho h_{\rho\mu} \nabla_\nu \ell \\
+ \nabla_\nu h_{\mu\rho} \nabla^\rho \ell - \nabla_\rho h_{\mu\nu} \nabla^\rho \ell + \nabla_\rho h^{\rho\sigma} \nabla_\sigma \ell_{\mu\nu} - 2\nabla_\mu h^{\rho\sigma} \nabla_\sigma \ell_{\nu\rho} \\
+ \nabla_\mu h \nabla^\rho \ell_{\rho\nu} + \nabla^\rho h_{\mu\nu} \nabla^\sigma \ell_{\rho\sigma} - 2\nabla_\rho h_{\mu\sigma} \nabla_\nu \ell^{\rho\sigma} - 2\nabla^\rho h_{\mu\sigma} \nabla^\sigma \ell_{\nu\rho} \\
+ 2\nabla^\rho h_{\mu\sigma} \nabla_\rho \ell_\nu{}^\sigma + \nabla^\rho h \nabla_\nu \ell_{\mu\rho} - \nabla^\rho h \nabla_\rho \ell_{\mu\nu} \, ,
\end{aligned}
\tag{B.13}
$$

where ∇_μ is the covariant derivative with respect to the Levi-Civita connection associated with $g_{\mu\nu}$. Moreover, let us define

$$
C^{(1)}_{\mu\nu}(h) = 2h_{\mu\nu} - g_{\mu\nu} h \, , \tag{B.14a}
$$

$$
P^{(1)}_{\mu\nu}(h) = h_{\mu\nu} - g_{\mu\nu} h \, , \tag{B.14b}
$$

and

$$C^{(2)}_{\mu\nu}(h) = 8h_{\mu\rho}h^{\rho}{}_{\nu} - 4hh_{\mu\nu} - 2g_{\mu\nu}h_{\rho\sigma}h^{\rho\sigma} + g_{\mu\nu}h^2 \,, \tag{B.15a}$$
$$P^{(2)}_{\mu\nu}(h) = 4h_{\mu\rho}h^{\rho}{}_{\nu} - 4hh_{\mu\nu} - g_{\mu\nu}h_{\rho\sigma}h^{\rho\sigma} + g_{\mu\nu}h^2 \,, \tag{B.15b}$$
$$Q^{(2)}_{\mu\nu}(h) = 4h_{\mu\rho}h^{\rho}{}_{\nu} - 3hh_{\mu\nu} - 2g_{\mu\nu}h_{\rho\sigma}h^{\rho\sigma} + g_{\mu\nu}h^2 \,. \tag{B.15c}$$

In terms of this operators, the Einstein–Hilbert Lagrangian with cosmological constant up to cubic order in perturbations reads

$$\mathcal{L}^{(3)}_{\mathrm{EH}}(h) = \sqrt{|g|}\Big(- \tfrac{1}{12}g^{\mu\nu}K^{(2)}_{\mu\nu}(h,h) + \tfrac{1}{4m_{\mathrm{P}}}\Big[h^{\mu\nu} - \tfrac{1}{6}h\,g^{\mu\nu}\Big]K^{(2)}_{\mu\nu}(h,h) + 2\Lambda m_{\mathrm{P}}^2 + \tfrac{\Lambda}{4}h^{\mu\nu}C^{(1)}_{\mu\nu}(h) - \tfrac{\Lambda}{12m_{\mathrm{P}}}h^{\mu\nu}C^{(2)}_{\mu\nu}[h]\Big) \,. \tag{B.16}$$

The first line corresponds to the kinetic terms and the second line to the self-interactions term up to cubic order.

Multimetric gravity. We use eq. (3.39) to express β_0 and β_4 in terms of α, Λ and β_2 as Λ using,

$$\beta_0 = \tfrac{1}{2}\Big(\tfrac{\Lambda}{M^2} - 3\alpha^2\beta_2\Big) \,, \tag{B.17a}$$
$$\beta_4 = \tfrac{2}{\alpha^2}\Big(\tfrac{\Lambda}{M^2} - 3\beta_2\Big) \,. \tag{B.17b}$$

Furthermore, we recall the expression for the masses of the massive modes. For maximal discrete symmetry they are

$$m_\chi^2 = 2\beta_2 M^2 \,, \tag{B.18a}$$
$$m_M^2 = 2(1+\alpha^2)\beta_2 M^2 \,. \tag{B.18b}$$

Those expressions are used in the following to replace dependence on β_2.

B.3.2 Trimetric action expanded to cubic order

In the following we calculate the cuadratic and cubic terms of the trimetric action of eq. (3.36) in terms of the mass eigenstates. We write the Lagrangian on the form,

$$\mathcal{L}_{\rm TM} = \mathcal{L}^{(0)}_{\rm TM} + \mathcal{L}^{(1)}_{\rm TM} + \mathcal{L}^{(2)}_{\rm TM} + \mathcal{L}^{(3)}_{\rm TM} + \dots \,, \tag{B.19}$$

First, one finds

$$\mathcal{L}^{(0)}_{\rm TM} = 2\Lambda m_{\rm P}^2 \sqrt{|g|}\,, \tag{B.20a}$$

$$\mathcal{L}^{(1)}_{\rm TM} = 0\,. \tag{B.20b}$$

Furthermore, the quadratic Lagrangian reads

$$\begin{aligned} \tfrac{1}{\sqrt{|g|}}\mathcal{L}^{(2)}_{\rm TM} = &-\tfrac{1}{12} g^{\mu\nu} K^{(2)}_{\mu\nu}(G,G) - \tfrac{1}{12} g^{\mu\nu} K^{(2)}_{\mu\nu}(M,M) - \tfrac{1}{12} g^{\mu\nu} K^{(2)}_{\mu\nu}(\chi,\chi) \\ &+ \tfrac{\Lambda}{4} G^{\mu\nu} C^{(1)}_{\mu\nu}(G) + \tfrac{\Lambda}{4} M^{\mu\nu} C^{(1)}_{\mu\nu}(M) + \tfrac{\Lambda}{4} \chi^{\mu\nu} C^{(1)}_{\mu\nu}(\chi) \\ &- \tfrac{m_M^2}{4} M^{\mu\nu} P^{(1)}_{\mu\nu}(M) - \tfrac{m_\chi^2}{4} \chi^{\mu\nu} P^{(1)}_{\mu\nu}(\chi)\,. \end{aligned} \tag{B.21}$$

We see that the first line contains the Fierz–Pauli kinetic terms for spin-2 fields. The second line contains the quadratic self-interaction terms that arise due to the interaction with the maximally symmetric background while the third line contains the self-interactions terms that give rise to masses of the eigenstates. The cubic interaction terms are given by

$$\begin{aligned} \tfrac{m_{\rm P}}{\sqrt{|g|}}\mathcal{L}^{(3)}_{\rm TM} = &\tfrac{1}{4}\Big(G^{\mu\nu} - \tfrac{1}{6} G^\rho{}_\rho g^{\mu\nu}\Big)\Big(K^{(2)}_{\mu\nu}(G,G) + K^{(2)}_{\mu\nu}(M,M) + K^{(2)}_{\mu\nu}(\chi,\chi)\Big) \\ &+ \tfrac{1}{2}\Big(M^{\mu\nu} - \tfrac{1}{6} M^\rho{}_\rho\, g^{\mu\nu}\Big)\Big(K^{(2)}_{\mu\nu}(G,M) + \tfrac{1}{2\alpha} K^{(2)}_{\mu\nu}(\chi,\chi) + \tfrac{1-\alpha^2}{2\alpha} K^{(2)}_{\mu\nu}(M,M)\Big) \\ &+ \tfrac{1}{2}\Big(\chi^{\mu\nu} - \tfrac{1}{6}\chi^\rho{}_\rho\, g^{\mu\nu}\Big)\Big(K^{(2)}_{\mu\nu}(G,\chi) + \tfrac{1}{\alpha} K^{(2)}_{\mu\nu}(M,\chi)\Big) \\ &- \tfrac{\Lambda}{12} G^{\mu\nu}\Big(C^{(2)}_{\mu\nu}(G) + 3C^{(2)}_{\mu\nu}(M) + 3C^{(2)}_{\mu\nu}(\chi)\Big) \\ &- \tfrac{\Lambda}{12\alpha} M^{\mu\nu}\Big((1-\alpha^2) C^{(2)}_{\mu\nu}(M) + 3C^{(2)}_{\mu\nu}(\chi)\Big) \\ &- \tfrac{1}{4} G^{\mu\nu}\Big(m_M^2 P^{(2)}_{\mu\nu}(M) + m_\chi^2 P^{(2)}_{\mu\nu}(\chi)\Big) - \tfrac{(1-\alpha^2)}{8\alpha} m_M^2\, M^{\mu\nu} P^{(2)}_{\mu\nu}(M) \\ &+ \tfrac{1}{4\alpha} M^{\mu\nu}\Big(m_\chi^2 P^{(2)}_{\mu\nu}(\chi) + \tfrac{m_M^2}{2} Q^{(2)}_{\mu\nu}(\chi)\Big)\,, \end{aligned} \tag{B.22}$$

where we introduced the trace with respect to the background metric as $A^\rho{}_\rho = g^{\mu\nu} A_{\mu\nu}$.

Bibliography

[1] N. L. González Albornoz, D. Lüst, S. Salgado, and A. Schmidt-May, "4D spin-2 fields from 5D Chern-Simons theory," *JHEP*, vol. 12, p. 026, 2018.

[2] N. L. González Albornoz, A. Schmidt-May, and M. von Strauss, "Dark matter scenarios with multiple spin-2 fields," *JCAP*, vol. 1801, no. 01, p. 014, 2018.

[3] N. González Albornoz, G. Rubio, P. Salgado, and S. Salgado, "Generalized Galilean algebras and Newtonian gravity," *Phys. Lett.*, vol. B755, pp. 433–438, 2016.

[4] A. Einstein, "Die Feldgleichungen der Gravitation," *Sitzung der physikalische-mathematischen Klasse*, vol. 25, pp. 844–847, 1915.

[5] A. Einstein, "Die Grundlage der allgemeinen Relativitätstheorie," *Ann. Phys.*, vol. 354, no. 7, pp. 769–822, 1916.

[6] D. Hilbert, "Die Grundlagen der Physik (erste Mitteilung)," *Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse*, vol. 1915, pp. 395–408, 1915.

[7] M. Ferraris, M. Francaviglia, and C. Reina, "Variational formulation of general relativity from 1915 to 1925 Palatini's method discovered by Einstein in 1925," *Gen. Rel. Gravit.*, vol. 14, no. 3, pp. 243–254, 1982.

[8] É. Cartan, "Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion," *Comptes Rendus, Ac. Sc. Paris*, vol. 174, pp. 593–595, 1922.

[9] É. Cartan, "Sur les variétés à connexion affine, et la théorie de la relativité généralisée (première partie)(suite)," in *Annales Scientifiques de l'Ecole Normale Supérieure*, vol. 41, pp. 1–25, Elsevier, 1924.

[10] E. Cartan, "Sur les variétés à connexion affine, et la théorie de la relativité généralisée (deuxième partie)," in *Annales scientifiques de l'École normale supérieure*, vol. 42, pp. 17–88, Elsevier, 1925.

[11] D. W. Sciama, "The Physical structure of general relativity," *Rev. Mod. Phys.*, vol. 36, pp. 463–469, 1964. [Erratum: Rev. Mod. Phys.36,1103(1964)].

[12] T. W. B. Kibble, "Lorentz invariance and the gravitational field," *J. Math. Phys.*, vol. 2, pp. 212–221, 1961. [,168(1961)].

[13] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick, and J. M. Nester, "General Relativity with Spin and Torsion: Foundations and Prospects," *Rev. Mod. Phys.*, vol. 48, pp. 393–416, 1976.

[14] J. F. Plebanski, "On the separation of Einsteinian substructures," *J. Math. Phys.*, vol. 18, pp. 2511–2520, 1977.

[15] E. Noether, "Invariante Variationsprobleme," *Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse*, vol. 57, pp. 235–257, 1918.

[16] M. Nakahara, *Geometry, topology and physics.* CRC Press, 2003.

[17] J. Zanelli, "Lecture notes on Chern–Simons (super-)gravities. Second edition (February 2008)," in *Proceedings, 7th Mexican Workshop on Particles and Fields (MWPF 1999): Merida, Mexico, November 10-17, 1999*, 2005.

[18] F. Izaurieta, E. Rodriguez, P. Minning, P. Salgado, and A. Perez, "Standard General Relativity from Chern–Simons Gravity," *Phys. Lett.*, vol. B678, pp. 213–217, 2009.

[19] M. Fierz and W. Pauli, "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field," *Proc. Roy. Soc. Lond.*, vol. A173, pp. 211–232, 1939.

[20] H. A. Buchdahl, "Non-linear Lagrangians and cosmological theory," *Mon. Not. Roy. Astron. Soc.*, vol. 150, p. 1, 1970.

[21] G. W. Horndeski, "Second-order scalar-tensor field equations in a four-dimensional space," *Int. J. Theor. Phys.*, vol. 10, pp. 363–384, 1974.

[22] M. Milgrom, "A Modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis," *Astrophys. J.*, vol. 270, pp. 365–370, 1983.

[23] C. de Rham and G. Gabadadze, "Generalization of the Fierz–Pauli Action," *Phys. Rev.*, vol. D82, p. 044020, 2010.

[24] C. de Rham, G. Gabadadze, and A. J. Tolley, "Resummation of Massive Gravity," *Phys. Rev. Lett.*, vol. 106, p. 231101, 2011.

[25] S. F. Hassan and R. A. Rosen, "Bimetric Gravity from Ghost-free Massive Gravity," *JHEP*, vol. 02, p. 126, 2012.

[26] C. de Rham, "Massive Gravity," *Living Rev. Rel.*, vol. 17, p. 7, 2014.

[27] T. P. Sotiriou and V. Faraoni, "$f(R)$ Theories Of Gravity," *Rev. Mod. Phys.*, vol. 82, pp. 451–497, 2010.

[28] C. Deffayet and D. A. Steer, "A formal introduction to Horndeski and Galileon theories and their generalizations," *Class. Quant. Grav.*, vol. 30, p. 214006, 2013.

[29] R. H. Sanders and S. S. McGaugh, "Modified Newtonian dynamics as an alternative to dark matter," *Ann. Rev. Astron. Astrophys.*, vol. 40, pp. 263–317, 2002.

[30] A. Schmidt-May and M. von Strauss, "Recent developments in bimetric theory," *J. Phys.*, vol. A49, no. 18, p. 183001, 2016.

[31] R. P. Feynman, B. Hatfield, F. B. Morinigo, and W. G. Wagner, *Feynman lectures on gravitation.* Addison-Wesley, 1995.

[32] E. Witten, "(2+1)-Dimensional Gravity as an Exactly Soluble System," *Nucl. Phys.*, vol. B311, p. 46, 1988.

[33] C. Lanczos, “A Remarkable property of the Riemann–Christoffel tensor in four dimensions,” *Annals Math.*, vol. 39, pp. 842–850, 1938.

[34] D. Lovelock, “The Einstein tensor and its generalizations,” *J. Math. Phys.*, vol. 12, pp. 498–501, 1971.

[35] R. Troncoso and J. Zanelli, “Higher dimensional gravity, propagating torsion and AdS gauge invariance,” *Class. Quant. Grav.*, vol. 17, pp. 4451–4466, 2000.

[36] A. H. Chamseddine, “Topological Gauge Theory of Gravity in Five-dimensions and All Odd Dimensions,” *Phys. Lett.*, vol. B233, pp. 291–294, 1989.

[37] M. Banados, L. J. Garay, and M. Henneaux, “The Local degrees of freedom of higher dimensional pure Chern–Simons theories,” *Phys. Rev.*, vol. D53, pp. 593–596, 1996.

[38] M. Banados, L. J. Garay, and M. Henneaux, “The Dynamical structure of higher dimensional Chern–Simons theory,” *Nucl. Phys.*, vol. B476, pp. 611–635, 1996.

[39] M. Banados, M. Henneaux, C. Iannuzzo, and C. M. Viallet, “A Note on the gauge symmetries of pure Chern–Simons theories with p-form gauge fields,” *Class. Quant. Grav.*, vol. 14, pp. 2455–2468, 1997.

[40] M. Banados, R. Olea, and S. Theisen, “Counterterms and dual holographic anomalies in CS gravity,” *JHEP*, vol. 10, p. 067, 2005.

[41] S. Deser and A. Schwimmer, “Geometric classification of conformal anomalies in arbitrary dimensions,” *Phys. Lett.*, vol. B309, pp. 279–284, 1993.

[42] E. Inonu and E. P. Wigner, “On the Contraction of groups and their represenations,” *Proc. Nat. Acad. Sci.*, vol. 39, pp. 510–524, 1953.

[43] E. Weimar-Woods, “The three-dimensional real lie algebras and their contractions,” *J. Math. Phys.*, vol. 32, no. 8, pp. 2028–2033, 1991.

[44] E. Weimar-Woods, “Contraction of Lie algebra representations,” *J. Math. Phys.*, vol. 32, no. 10, pp. 2660–2665, 1991.

[45] M. Banados, R. Troncoso, and J. Zanelli, "Higher dimensional Chern–Simons supergravity," *Phys. Rev.*, vol. D54, pp. 2605–2611, 1996.

[46] P. Mora, R. Olea, R. Troncoso, and J. Zanelli, "Transgression forms and extensions of Chern–Simons gauge theories," *JHEP*, vol. 02, p. 067, 2006.

[47] S. F. Hassan and R. A. Rosen, "On Non-Linear Actions for Massive Gravity," *JHEP*, vol. 07, p. 009, 2011.

[48] S. F. Hassan and R. A. Rosen, "Resolving the Ghost Problem in non-Linear Massive Gravity," *Phys. Rev. Lett.*, vol. 108, p. 041101, 2012.

[49] S. F. Hassan, R. A. Rosen, and A. Schmidt-May, "Ghost-free Massive Gravity with a General Reference Metric," *JHEP*, vol. 02, p. 026, 2012.

[50] S. F. Hassan and R. A. Rosen, "Confirmation of the Secondary Constraint and Absence of Ghost in Massive Gravity and Bimetric Gravity," *JHEP*, vol. 04, p. 123, 2012.

[51] S. F. Hassan, A. Schmidt-May, and M. von Strauss, "Proof of Consistency of Nonlinear Massive Gravity in the Stúckelberg Formulation," *Phys. Lett.*, vol. B715, pp. 335–339, 2012.

[52] F. Koennig, Y. Akrami, L. Amendola, M. Motta, and A. R. Solomon, "Stable and unstable cosmological models in bimetric massive gravity," *Phys. Rev.*, vol. D90, p. 124014, 2014.

[53] S. F. Hassan, A. Schmidt-May, and M. von Strauss, "On Consistent Theories of Massive Spin-2 Fields Coupled to Gravity," *JHEP*, vol. 05, p. 086, 2013.

[54] H. Weyl, "Gravitation and electricity," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, vol. 1918, p. 465, 1918. [,24(1918)].

[55] H. Weyl, "A New Extension of Relativity Theory," *Annalen Phys.*, vol. 59, pp. 101–133, 1919.

[56] R. Bach, "Zur weylschen relativitätstheorie und der weylschen erweiterung des krümmungstensorbegriffs," *Math. Z.*, vol. 9, no. 1, pp. 110–135, 1921.

[57] S. F. Hassan, A. Schmidt-May, and M. von Strauss, “Higher Derivative Gravity and Conformal Gravity From Bimetric and Partially Massless Bimetric Theory,” *Universe*, vol. 1, no. 2, pp. 92–122, 2015.

[58] J. D. Edelstein, M. Hassaine, R. Troncoso, and J. Zanelli, “Lie-algebra expansions, Chern–Simons theories and the Einstein–Hilbert Lagrangian,” *Phys. Lett.*, vol. B640, pp. 278–284, 2006.

[59] R. Aros, M. Romo, and N. Zamorano, “Compactification in first order gravity,” *J. Phys. Conf. Ser.*, vol. 134, p. 012013, 2008.

[60] I. Morales, B. Neves, Z. Oporto, and O. Piguet, “Chern–Simons gravity in four dimensions,” *Eur. Phys. J.*, vol. C77, no. 2, p. 87, 2017.

[61] A. H. Chamseddine, “Topological gravity and supergravity in various dimensions,” *Nucl. Phys.*, vol. B346, pp. 213–234, 1990.

[62] M. Banados, “Dilaton gravity (with a Gauss-Bonnet term) derived from five-dimensional Chern-Simons gravity,” *Phys. Rev.*, vol. D55, pp. 2051–2058, 1997.

[63] H. Nastase, “D=4 Einstein gravity from higher D CS and BI gravity and an alternative to dimensional reduction,” 2007.

[64] F. Izaurieta and E. Rodríguez, “Effectively four-dimensional spacetimes emerging from $d = 5$ Einstein–Gauss–Bonnet Gravity,” *Class. Quant. Grav.*, vol. 30, p. 155009, 2013.

[65] R. Aros and D. E. Diaz, “AdS Chern-Simons Gravity induces Conformal Gravity,” *Phys. Rev.*, vol. D89, no. 8, p. 084026, 2014.

[66] I. Morales, B. Neves, Z. Oporto, and O. Piguet, “Dimensionally compactified Chern–Simon theory in 5D as a gravitation theory in 4D,” *Int. J. Mod. Phys. Conf. Ser.*, vol. 45, p. 1760005, 2017.

[67] M. Kaku, P. K. Townsend, and P. van Nieuwenhuizen, “Gauge Theory of the Conformal and Superconformal Group,” *Phys. Lett.*, vol. 69B, pp. 304–308, 1977.

[68] V. Baccetti, P. Martin-Moruno, and M. Visser, “Massive gravity from bimetric gravity,” *Class. Quant. Grav.*, vol. 30, p. 015004, 2013.

[69] S. F. Hassan, A. Schmidt-May, and M. von Strauss, "Particular Solutions in Bimetric Theory and Their Implications," *Int. J. Mod. Phys.*, vol. D23, no. 13, p. 1443002, 2014.

[70] Y. Akrami, S. F. Hassan, F. Könnig, A. Schmidt-May, and A. R. Solomon, "Bimetric gravity is cosmologically viable," *Phys. Lett.*, vol. B748, pp. 37–44, 2015.

[71] E. Babichev, L. Marzola, M. Raidal, A. Schmidt-May, F. Urban, H. Veermae, and M. von Strauss, "Heavy spin-2 Dark Matter," *JCAP*, vol. 1609, no. 09, p. 016, 2016.

[72] M. Milgrom, "MOND theory," *Can. J. Phys.*, vol. 93, no. 2, pp. 107–118, 2015.

[73] B. Carr, F. Kuhnel, and M. Sandstad, "Primordial Black Holes as Dark Matter," *Phys. Rev.*, vol. D94, no. 8, p. 083504, 2016.

[74] K. A. Olive *et al.*, "Review of Particle Physics," *Chin. Phys.*, vol. C38, p. 090001, 2014.

[75] M. Ackermann *et al.*, "Updated search for spectral lines from Galactic dark matter interactions with pass 8 data from the Fermi Large Area Telescope," *Phys. Rev.*, vol. D91, no. 12, p. 122002, 2015.

[76] K.-i. Maeda and M. S. Volkov, "Anisotropic universes in the ghost-free bigravity," *Phys. Rev.*, vol. D87, p. 104009, 2013.

[77] A. Schmidt-May, "Nonlinear interactions for massive spin-2 fields," *PoS*, vol. CORFU2015, p. 157, 2016.

[78] E. Babichev, L. Marzola, M. Raidal, A. Schmidt-May, F. Urban, H. Veermäe, and M. von Strauss, "Bigravitational origin of dark matter," *Phys. Rev.*, vol. D94, no. 8, p. 084055, 2016.

[79] K. Aoki and S. Mukohyama, "Massive gravitons as dark matter and gravitational waves," *Phys. Rev.*, vol. D94, no. 2, p. 024001, 2016.

[80] N. Bernal, M. Heikinheimo, T. Tenkanen, K. Tuominen, and V. Vaskonen, "The Dawn of FIMP Dark Matter: A Review of Models and Constraints," *Int. J. Mod. Phys.*, vol. A32, no. 27, p. 1730023, 2017.

[81] X. Chu and C. Garcia-Cely, "Self-interacting Spin-2 Dark Matter," *Phys. Rev.*, vol. D96, no. 10, p. 103519, 2017.

[82] S. M. Carroll, *Spacetime and geometry: An introduction to general relativity*. 2004.

[83] A. O. Barut and R. Raczka, *Theory of group representations and applications*, vol. 2. World Scientific, 1986.

[84] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Int. J. Theor. Phys.*, vol. 38, pp. 1113–1133, 1999. [Adv. Theor. Math. Phys.2,231(1998)].

[85] E. Witten, "Global Aspects of Current Algebra," *Nucl. Phys.*, vol. B223, pp. 422–432, 1983.

[86] F. Izaurieta, E. Rodriguez, and P. Salgado, "Expanding Lie (super)algebras through Abelian semigroups," *J. Math. Phys.*, vol. 47, p. 123512, 2006.

[87] N. González, P. Salgado, G. Rubio, and S. Salgado, "EinsteinHilbert action with cosmological term from ChernSimons gravity," *J. Geom. Phys.*, vol. 86, pp. 339–358, 2014.

[88] F. Izaurieta, E. Rodriguez, and P. Salgado, "On transgression forms and Chern–Simons (super)gravity," 2005.

[89] P. Salgado, R. J. Szabo, and O. Valdivia, "Topological gravity and transgression holography," *Phys. Rev.*, vol. D89, no. 8, p. 084077, 2014.

[90] F. Izaurieta, E. Rodriguez, and P. Salgado, "The Extended Cartan homotopy formula and a subspace separation method for Chern-Simons supergravity," *Lett. Math. Phys.*, vol. 80, pp. 127–138, 2007.

[91] K. Hinterbichler and R. A. Rosen, "Interacting Spin-2 Fields," *JHEP*, vol. 07, p. 047, 2012.

[92] I. Newton, *Philosophiæ naturalis principia mathematica*, vol. 1. G. Brookman, 1833.

[93] A. Einstein, "Näherungsweise Integration der Feldgleichungen der Gravitation. Sitz.-ber. K," *Preuß. Akad. Wiss*, vol. 688, 1916.

[94] C. Denson Hill and P. Nurowski, “How the green light was given for gravitational wave search,” *arXiv preprint*, 2016.

[95] H. van Dam and M. J. G. Veltman, “Massive and massless Yang-Mills and gravitational fields,” *Nucl. Phys.*, vol. B22, pp. 397–411, 1970.

[96] V. I. Zakharov, “Linearized gravitation theory and the graviton mass,” *JETP Lett.*, vol. 12, p. 312, 1970. [Pisma Zh. Eksp. Teor. Fiz.12,447(1970)].

[97] A. I. Vainshtein, “To the problem of nonvanishing gravitation mass,” *Phys. Lett.*, vol. 39B, pp. 393–394, 1972.

[98] C. M. Will, “Bounding the mass of the graviton using gravitational wave observations of inspiralling compact binaries,” *Phys. Rev.*, vol. D57, pp. 2061–2068, 1998.

[99] B. P. Abbott *et al.*, “Observation of Gravitational Waves from a Binary Black Hole Merger,” *Phys. Rev. Lett.*, vol. 116, no. 6, p. 061102, 2016.

[100] D. G. Boulware and S. Deser, “Can gravitation have a finite range?,” *Phys. Rev.*, vol. D6, pp. 3368–3382, 1972.

[101] G. D’Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava, and A. J. Tolley, “Massive Cosmologies,” *Phys. Rev.*, vol. D84, p. 124046, 2011.

[102] A. De Felice, A. E. Gümrükcüoğlu, C. Lin, and S. Mukohyama, “On the cosmology of massive gravity,” *Class. Quant. Grav.*, vol. 30, p. 184004, 2013.

[103] L. Bernard, C. Deffayet, and M. von Strauss, “Consistent massive graviton on arbitrary backgrounds,” *Phys. Rev.*, vol. D91, no. 10, p. 104013, 2015.

[104] L. Bernard, C. Deffayet, and M. von Strauss, “Massive graviton on arbitrary background: derivation, syzygies, applications,” *JCAP*, vol. 1506, p. 038, 2015.

[105] L. Bernard, C. Deffayet, A. Schmidt-May, and M. von Strauss, “Linear spin-2 fields in most general backgrounds,” *Phys. Rev.*, vol. D93, no. 8, p. 084020, 2016.

[106] C. Mazuet and M. S. Volkov, "Massive gravitons in arbitrary spacetimes," *Phys. Rev.*, vol. D96, no. 12, p. 124023, 2017.

[107] S. Deser, M. Grisaru, and H. Pendleton, "Lectures on elementary particles and quantum field theory, volume 2," *Cambridge, Massachusetts, USA: The MIT Press*, p. 510, 1971.

[108] S. F. Hassan and A. Schmidt-May, "Interactions of multiple spin-2 fields beyond pairwise couplings," *arXiv preprint*, 2018.

[109] Y. Yamashita, A. De Felice, and T. Tanaka, "Appearance of Boulware–Deser ghost in bigravity with doubly coupled matter," *Int. J. Mod. Phys.*, vol. D23, p. 1443003, 2014.

[110] C. de Rham, L. Heisenberg, and R. H. Ribeiro, "On couplings to matter in massive (bi-)gravity," *Class. Quant. Grav.*, vol. 32, p. 035022, 2015.

[111] S. F. Hassan, A. Schmidt-May, and M. von Strauss, "On Partially Massless Bimetric Gravity," *Phys. Lett.*, vol. B726, pp. 834–838, 2013.

[112] A. Higuchi, "Forbidden Mass Range for Spin-2 Field Theory in De Sitter Space-time," *Nucl. Phys.*, vol. B282, pp. 397–436, 1987.

[113] L. Apolo and S. F. Hassan, "Non-linear partially massless symmetry in an $SO(1,5)$ continuation of conformal gravity," *Class. Quant. Grav.*, vol. 34, no. 10, p. 105005, 2017.

[114] N. Boulanger, T. Damour, L. Gualtieri, and M. Henneaux, "Inconsistency of interacting, multigraviton theories," *Nucl. Phys.*, vol. B597, pp. 127–171, 2001.

[115] C. de Rham and A. J. Tolley, "Vielbein to the rescue? Breaking the symmetric vielbein condition in massive gravity and multigravity," *Phys. Rev.*, vol. D92, no. 2, p. 024024, 2015.

[116] S. F. Hassan, A. Schmidt-May, and M. von Strauss, "Metric Formulation of Ghost-Free Multivielbein Theory," *arXiv preprint arXiv:1204.5202*, 2012.

[117] K. Nomura and J. Soda, "When is Multimetric Gravity Ghost-free?," *Phys. Rev.*, vol. D86, p. 084052, 2012.

[118] J. H. C. Scargill, J. Noller, and P. G. Ferreira, "Cycles of interactions in multi-gravity theories," *JHEP*, vol. 12, p. 160, 2014.

[119] O. Baldacchino and A. Schmidt-May, "Structures in multiple spin-2 interactions," *J. Phys.*, vol. A50, no. 17, p. 175401, 2017.

[120] A. R. Gover, A. Shaukat, and A. Waldron, "Weyl Invariance and the Origins of Mass," *Phys. Lett.*, vol. B675, pp. 93–97, 2009.

[121] D. Baumann, "Inflation," in *Physics of the large and the small, TASI 09, proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics, Boulder, Colorado, USA, 1-26 June 2009*, pp. 523–686, 2011.

[122] R. Blumenhagen, D. Lüst, and S. Theisen, *Basic concepts of string theory.* Theoretical and Mathematical Physics, Heidelberg, Germany: Springer, 2013.

[123] M. Leclerc, "Canonical and gravitational stress-energy tensors," *Int. J. Mod. Phys.*, vol. D15, pp. 959–990, 2006.