The Generalized Vaserstein Symbol

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Abstract

Let R be a commutative ring. An important question in the study of projective modules is under which circumstances a projective R-module P is cancellative, i.e. under which circumstances any isomorphism $P \oplus R^k \cong Q \oplus R^k$ for some projective R-module Q and $k > 0$ already implies $P \cong Q$.

If R is an affine algebra of dimension d over an algebraically closed field k, then it is known that projective R-modules of rank $r \geq d$ are cancellative. While it is known that projective modules of rank $r = d - 2$ are not cancellative in general, it remains an open question whether projective modules of rank $r = d - 1$ are cancellative or not. By substantially using a map called the Vaserstein symbol, Fasel-Rao-Swan could prove that at least R^{d-1} is cancellative if $d \geq 4$, $(d-1)! \in k^{\times}$ and R is normal.

Motivated by the cancellation problem of projective modules, the aim of this work is to construct a generalized Vaserstein symbol associated to any projective R-module P_0 of rank 2 with a trivialization of its determinant: The generalized Vaserstein symbol is defined on the orbit space $Um(P_0 \oplus R)/E(P_0 \oplus R)$ of the set $Um(P_0 \oplus R)$ of epimorphisms $P_0 \oplus R \rightarrow R$ under the right action of the subgroup $E(P_0 \oplus R)$ of the group $Aut(P_0 \oplus R)$ of automorphisms of $P_0 \oplus R$ generated by elementary automorphisms and maps into the abelian group $\tilde{V}(R)$, which can be identified with the so-called elementary symplectic Witt group $W_E(R)$.

We prove that the generalized Vaserstein symbol is a bijection if R is a regular Noetherian ring of dimension 2 or a regular affine algebra of dimension 3 over a perfect field k with $c.d.(k) \leq 1$ and $6 \in k^{\times}$. This enables us to generalize a result of Fasel-Rao-Swan on transformations of unimodular rows via elementary matrices. Furthermore, by means of the generalized Vaserstein symbol, we can give a necessary and sufficient condition for the triviality of the orbit space $Um(P_0 \oplus R)/SL(P_0 \oplus R)$ over affine algebras of dimension 4 over an algebraically closed field k. We can also classify stably isomorphic oriented projective modules of rank 2 with a trivial determinant over affine algebras of dimension 3 over finite fields.

Zusammenfassung

Sei R ein kommutativer Ring. Eine wichtige Frage im Studium projektiver Moduln ist, unter welchen Bedingungen ein projektiver R-Modul P kürzbar ist, d.h. unter welchen Bedingungen jeder Isomorphismus $P \oplus R^k \cong Q \oplus R^k$ für einen projektiven R-Modul Q und $k > 0$ bereits $P \cong Q$ impliziert.
Ist R eine affine Algebra von Dimension d über einem algebraisch abgeschlossenen Körper k, dann sind projektive R-Moduln von Rang $r \geq d$ kürzbar. Während projektive R-Moduln von Rang $r = d - 2$ nicht immer kürzbar sind, ist es immer noch eine offene Frage, ob projektive Moduln von Rang $r = d - 1$ kürzbar sind oder nicht. Fasel-Rao-Swan konnten unter Verwendung des sogenannten Vaserstein-Symbols beweisen, dass zumindest R^{d-1} kürzbar ist, falls $d \geq 4$, $(d-1)! \in k^{\times}$ und R normal ist.
Motiviert vom Studium projektiver Moduln ist es das Ziel dieser Arbeit, ein verallgemeinertes Vaserstein-Symbol, das jedem projektiven R-Modul P_0 von Rang 2 mit einer Trivialisierung seiner Determinante zugeordnet wird, zu definieren: Diese Abbildung ist auf dem Orbitraum $Um(P_0 \oplus R)/E(P_0 \oplus R)$ der Menge $Um(P_0 \oplus R)$ der Epimorphismen $P_0 \oplus R \rightarrow R$ unter der Rechtswirkung der von den elementaren Automorphismen erzeugten Untergruppe $E(P_0 \oplus R)$ der Automorphismengruppe $Aut(P_0 \oplus R)$ von $P_0 \oplus R$ definiert und bildet in die abelsche Gruppe $\tilde{V}(R)$ ab, die sich mit der elementaren symplektischen Witt-Gruppe $W_E(R)$ identifizieren lässt.
Wir beweisen, dass das verallgemeinerte Vaserstein-Symbol eine Bijektion ist, falls R ein regulärer noetherscher Ring von Dimension 2 oder eine reguläre affine Algebra von Dimension 3 über einem perfekten Körper k mit $c.d.(k) \leq 1$ und $6 \in k^{\times}$ ist. Dies ermöglicht es uns, ein Resultat von Fasel-Rao-Swan über die Umformbarkeit unimodularer Reihen mittels elementarer Matrizen zu verallgemeinern. Außerdem können wir anhand des verallgemeinerten Vaserstein-Symbols eine notwendige und hinreichende Bedingung für die Trivialität des Orbitraums $Um(P_0 \oplus R)/SL(P_0 \oplus R)$ über einer affinen Algebra von Dimension 4 über einem algebraisch abgeschlossenen Körper k finden. Wir können ebenfalls stabil isomorphe orientierte projektive Moduln von Rang 2 mit trivialer Determinante über affinen Algebren von Dimension 3 über endlichen Körpern klassifizieren.

Contents

Introduction

In this thesis, we construct a generalized Vaserstein symbol map and explore its applications to the classification of finitely generated projective modules. Projective modules were first introduced in 1956 by Henri Cartan and Samuel Eilenberg (cp. [CE]) and have since then been widely studied by many mathematicians. As a matter of fact, if R is a commutative ring and $X = Spec(R)$, then finitely generated projective R-modules correspond to locally free coherent sheaves of $\mathcal{O}_X$-modules. The study of finitely generated projective modules can hence be interpreted as the study of algebraic vector bundles over affine schemes.
First of all, note that every finitely generated projective R-module P gives rise to a map

$$rank_P : Spec(R) \to \mathbb{Z}$$

which assigns to every $\mathfrak{p} \in Spec(R)$ the rank of the finitely generated projective and hence free module $P_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$. This map is locally constant and hence continuous if we equip $\mathbb{Z}$ with the discrete topology. Since $Spec(R)$ is quasi-compact, the map $rank_P$ takes only finitely many values r_i, $i = 1, ..., n$. The decomposition $Spec(R) = \bigcup_{i=1}^{n} rank_P^{-1}(r_i)$ induces decompositions $R = R_1 \times ... \times R_n$ and $P = P_1 \oplus ... \oplus P_n$, where $P_i = P \otimes_R R_i$ for $i = 1, ..., n$. Since any P_i is a finitely generated projective R_i-module such that $rank_{P_i} : Spec(R_i) \to \mathbb{Z}$ is a constant map, we can restrict our study of finitely generated projective modules to projective modules of constant rank. For any commutative ring R and $r \geq 0$, we let $\mathcal{V}_r(R)$ denote the set of isomorphism classes of finitely generated projective R-modules of constant rank r. Then we consider the stabilization maps

$$\phi_r : \mathcal{V}_r(R) \to \mathcal{V}_{r+1}(R), [P] \mapsto [P \oplus R].$$

The main goal in the classification of finitely generated projective modules is to give cohomological descriptions of the sets $\mathcal{V}_r(R)$ for all $r \geq 0$ and of the images and the fibers of the stabilization maps above. Of course, any class of a projective R-module P' of rank $r + 1$ lies in the image of the map ϕ_r if and only if $P' \cong P \oplus R$ for some P of rank r. In general,

the fiber $\phi_r^{-1}([P\oplus R])$ can be identified with the orbit space $Um(P\oplus R)/Aut(P\oplus R)$ of the set $Um(P\oplus R)$ of R-linear epimorphisms $P\oplus R\to R$ under the right action of the group $Aut(P\oplus R)$ of R-linear automorphisms of $P\oplus R$. If P is free, we can identify $Um(P\oplus R)$ with the set $Um_{r+1}(R)$ of unimodular rows of length $r+1$, $Aut(P\oplus R)$ with $GL_{r+1}(R)$ and hence $\phi_r^{-1}([R^{r+1}])$ with $Um_{r+1}(R)/GL_{r+1}(R)$.
In the end, the goal is to describe the set of isomorphism classes of finitely generated projective R-modules. The direct sum of projective R-modules endows this set with the structure of an abelian monoid. Its group completion is just the group $K_0(R)$; in other words, $K_0(R)$ is the quotient of the free abelian group generated by the isomorphism classes $[P]$ of finitely generated projective R-modules P modulo the subgroup generated by $[P]+[Q]-[P\oplus Q]$ for all such P and Q. By abuse of notation, we also denote by $[P]$ the class of a finitely generated projective R-module in $K_0(R)$. General properties of group completions imply that the classes of finitely generated projective R-modules P and Q coincide in $K_0(R)$ if and only if $P\oplus R^n\cong Q\oplus R^n$ for some $n\geq 0$. Furthermore, if we denote by $[Spec(R),\mathbb{Z}]$ the group of continuous maps from $Spec(R)$ to $\mathbb{Z}$ (with the discrete topology), then there is a natural group homomorphism

$$rank : K_0(R)\to[Spec(R),\mathbb{Z}]$$

induced by the rank maps for all finitely generated projective modules over R. In fact, this rank map is a split surjection; a right-inverse of $rank$ can be defined as follows: First of all, note that $[Spec(R),\mathbb{Z}]$ is the group completion of the abelian monoid $[Spec(R),\mathbb{N}]$ of continuous maps from $Spec(R)$ to $\mathbb{N}$ (with the discrete topology). If $f: Spec(R)\to\mathbb{N}$ is continuous, we let r_i, $i=1,...,n$, denote the finitely many values of f and $R=R_1\times...\times R_n$ be the corresponding decomposition of R. Then we let R^f be the direct sum of the R-modules $R_i^{r_i}$, $i=1,...,n$. The assignment $f\mapsto R^f$ induces a homomorphism $[Spec(R),\mathbb{Z}]\to K_0(R)$, which clearly defines a right-inverse of $rank$. Hence we obtain a split short exact sequence

$$0\to\tilde{K}_0(R)\to K_0(R)\to[Spec(R),\mathbb{Z}]\to 0,$$

where we let $\tilde{K}_0(R)$ denote the kernel of $rank$. It is also called the reduced K_0-group of R and yields a stable classification of finitely generated projective modules over R as follows: As above, we let $\mathcal{V}_r(R)$ denote the set of isomorphism classes of projective R-modules of constant rank r. We can equip these sets with basepoints by taking the free R-modules of each particular rank. Then we form the direct limit $\mathcal{V}(R):=lim_{r\geq 0}\mathcal{V}_r(R)$ with respect to the stabilization maps. By abuse of notation, we also denote by $[P]$ the class of a finitely generated projective module P of constant rank in $\mathcal{V}(R)$. We obtain maps

$$\pi_r : \mathcal{V}_r(R) \to \mathcal{V}(R)$$

for all $r \geq 0$. Since the stabilization maps are all pointed, $\mathcal{V}(R)$ is also pointed by $\pi_0([0])$. We then define maps $f_r : \mathcal{V}_r(R) \to \tilde{K}_0(R)$ for all $r \geq 0$ by $f_r([P]) = [P] - [R^r]$. Clearly, these maps are compatible with the stabilization maps and hence induce a map

$$f : \mathcal{V}(R) \to \tilde{K}_0(R),$$

which can easily be checked to be a pointed bijection. It follows that the pointed set $\mathcal{V}(R)$ can be endowed with the structure of an abelian group via the bijection $f : \mathcal{V}(R) \xrightarrow{\cong} \tilde{K}_0(R)$. Thus, the stable classification of finitely generated projective modules already becomes part of a theory which behaves in many aspects like a cohomology theory. The pointed set $\mathcal{V}(R)$ can consequently be studied and computed via cohomological methods. For this reason and furthermore by analogy with the study of topological vector bundles, it is reasonable to try to extend these cohomological methods to the unstable classification of finitely generated projective modules, i.e. to the study of the pointed sets $\mathcal{V}_r(R)$ and the stabilization maps ϕ_r for $r \geq 0$.

Let us now review the major results in the classification of finitely generated projective modules over commutative rings. In this thesis, we will mainly be interested in affine algebras over fields or, more generally, Noetherian rings. As explained above, one can restrict oneself to projective modules of constant rank; in fact, since the spectrum of a Noetherian ring is a Noetherian topological space with only finitely many connected components, it is in this case sufficient to study projective modules over Noetherian rings R such that $Spec(R)$ is connected. The first important classification results were proven by Jean-Pierre Serre and by Hyman Bass for Noetherian commutative rings:

Theorem (Serre). Let R be a commutative Noetherian ring of dimension d. Then any finitely generated projective R-module P of constant rank $r > d$ is of the form $P \cong P' \oplus R^{r-d}$ for some projective R-module P' of constant rank d.

In his original paper, Jean-Pierre Serre proved this result under the assumption that $Spec(R)$ is a connected topological space (cp. [JPS1, Théorème 1]). The theorem above then follows by applying his result connected component by connected component.

Theorem (Bass). Let R be a commutative Noetherian ring of dimension d. If P and Q are finitely generated projective R-modules of constant rank $r > d$, then $P \oplus R^k \cong Q \oplus R^k$

for some $k \geq 0$ implies $P \cong Q$.

As a matter of fact, Bass could prove that in the situation of the theorem the subgroup $E(P \oplus R)$ of $Aut(P \oplus R)$ generated by elementary automorphisms of $P \oplus R$ acts transitively on the right on $Um(P \oplus R)$; in particular, the orbit spaces $Um(P \oplus R)/E(P \oplus R)$ and hence $Um(P \oplus R)/Aut(P \oplus R)$ are trivial (cp. [HB, Chapter IV, Theorem 3.4 and Corollary 3.5]). As a special case, one obtains that $Um_{r+1}(R)/E_{r+1}(R)$ is trivial for such rings.
The theorems by Jean-Pierre Serre and Hyman Bass in particular show that we may restrict to projective modules of constant rank $r \leq \dim(R) + 1$: It follows immediately from the theorems that, for a Noetherian ring R of dimension d, the map $\phi_r : \mathcal{V}_r(R) \to \mathcal{V}_{r+1}(R)$ is injective if $r \geq d+1$ and surjective if $r \geq d$; in particular, $\tilde{K}_0(R) \cong \mathcal{V}(R) = \mathcal{V}_{d+1}(R)$.
Furthermore, the map $\phi_1 : \mathcal{V}_1(R) \to \mathcal{V}_2(R)$ is always injective for an arbitrary commutative ring R as $P \cong \det(P \oplus R) \cong \det(Q \oplus R) \cong Q$ for projective modules P and Q of rank 1 such that $P \oplus R \cong Q \oplus R$. In particular, if R is a Noetherian ring of dimension 1, then $\tilde{K}_0(R) \cong \mathcal{V}(R) = \mathcal{V}_1(R) = Pic(R)$.
For general Noetherian rings or affine algebras over arbitrary fields, the result by Bass is the best possible: Indeed, let $A = \mathbb{R}[x,y,z]/\langle x^2+y^2+z^2-1\rangle$ be the real algebraic 2-sphere, which is an affine algebra over $\mathbb{R}$ of dimension 2. Then the unimodular row (x,y,z) of length 3 over A cannot be completed to an invertible matrix. In particular, the kernel P of the homomorphism $(x,y,z) : A^3 \to A$ is a non-free stably free A-module of rank 2. In order to see this, note that any triple $(a,b,c) \in A^3$ induces a vector field $S^2 \to \mathbb{R}^3$; the row (x,y,z) then corresponds to the vector field which is pointing radially outward. Hence any element of P gives a vector field on S^2 which is tangent to the 2-sphere. Now if P was a free A-module of rank 2, then any free A-basis $\{f,g\}$ would give two vector fields $f,g : S^2 \to \mathbb{R}^3$ such that $f(p)$ and $g(p)$ are linearly independent for every point $p \in S^2$. But it follows from a well-known theorem by Brouwer (cp. [Br, Satz 2]) that this is impossible. Consequently, P cannot be free and the map $\phi_2 : \mathcal{V}_2(A) \to \mathcal{V}_3(A)$ cannot be injective.
Since the result by Bass cannot be improved for Noetherian rings or affine algebras over arbitrary fields, we will henceforth consider affine algebras over algebraically closed fields. In [S1], Andrei Suslin proved the following cancellation theorem for affine algebras over algebraically closed fields:

Theorem (Suslin). Let R be an affine algebra of dimension d over an algebraically closed field k. Then any finitely generated projective R-module P of rank d is cancellative, i.e.

any isomorphism $P \oplus R^k \cong Q \oplus R^k$ for some Q and $k \geq 0$ implies $P \cong Q$.

For any given dimension d, the cancellation theorem above holds for all affine algebras of dimension d over an algebraically closed field k if and only if it holds for all reduced affine algebras of dimension d over k (cp. [HB, Chapter III, Proposition 2.12]). For a reduced affine algebra R of dimension d over an algebraically closed field k and a projective module P of rank d, Andrei Suslin then proves that $Um(P \oplus R)/Aut(P \oplus R)$ is trivial by first studying the orbit space $Um(P \oplus R)/E(P \oplus R)$, where $E(P \oplus R)$ is the subgroup of $Aut(P \oplus R)$ generated by elementary automorphisms; in fact, he does this in the language of unimodular elements by using a version of Swan's Bertini theorem (cp. [Sw, Theorem 1.3]). Writing any $a \in Um(P \oplus R)$ as (a_P, a_R) (where a_P is the restriction of a to P and a_R is the element of R corresponding to the restriction of a to R respectively), he proves that any $a \in Um(P \oplus R)$ can be transformed via elementary automorphisms to an element $b \in Um(P \oplus R)$ of the form $b = (b_P, b_R^d)$ such that $P \otimes_R R/b_R R$ is free and $\dim(R/b_R R) \leq d-1$. For elements of this form, he then proves that they are equivalent to the projection $P \oplus R \to R$ with respect to the action of $Aut(P \oplus R)$ on $Um(P \oplus R)$ as soon as $Aut(P \otimes_R R/b_R R)$ acts transitively on $Um(P \otimes_R R/b_R R)$ (cp. [S1, Lemma 2]). This enables him to prove his theorem by induction on d.
Again using a version of Swan's Bertini theorem (cp. [Sw, Theorem 1.5]), Andrei Suslin could also prove in [S5] that if R is a normal affine algebra of dimension d over a field k such that $c.d.(k) \leq 1$ and $d! \in k^{\times}$, then stably free R-modules of rank d are free. Using similar methods, Shrikant Bhatwadekar could prove that any projective R-module of rank d is cancellative whenever R is an affine algebra of dimension d over an infinite perfect field k such that $c.d.(k) \leq 1$ and $d! \in k^{\times}$ (cp. [B, Theorem 4.1 and Remark 4.2]).
Henceforth, let R be a smooth affine algebra of dimension d over an algebraically closed field k and let $X = Spec(R)$. By analogy with the situation in algebraic topology, there are Chern classes $c_i(P) \in CH^i(X)$, $i \geq 0$, associated to any finitely generated projective R-module P of rank r, which satisfy the expected properties: First of all, one has $c_0(P) = 1$ and $c_i(P) = 0$ for $i > r$. Furthermore, one has a Whitney sum formula, i.e. for any short exact sequence

$$0 \to P_1 \to P_2 \to P_3 \to 0$$

of finitely generated projective R-modules, one has $c(P_2) = c(P_1) \cdot c(P_3) \in CH^*(X)$, where $c(P_k) = \sum_{i \geq 0} c_i(P_k)$ denotes the total Chern class of P_k, $k = 1,2,3$. The Chern classes induce maps

$$(c_1, ..., c_r) : \mathcal{V}_r(R) \to \prod_{i=1}^r CH^i(X)$$

for all $r \geq 1$. It is in general very difficult to determine whether these maps are injective or surjective, but some results have been proven in lower dimensions:
We let $G_0(R)$ be the quotient of the free abelian group generated by isomorphism classes $[M]$ of finitely generated R-modules M modulo the subgroup generated by the elements of the form $[M_1] + [M_3] - [M_2]$ for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of finitely generated R-modules. There is an obvious map

$$K_0(R) \to G_0(R), [P] \mapsto [P],$$

which is an isomorphism because R is assumed to be smooth; this basically follows from the fact that any finitely generated R-module has a finite projective resolution.
The group $G_0(R)$ (and hence also $K_0(R)$) has an obvious filtration $(F^i G_0(R))_{i \geq 0}$, where $F^i G_0(R)$ is defined as the subgroup of $G_0(R)$ generated by the classes of finitely generated R-modules whose support has codimension $\geq i$. We denote by $(F^i K_0(R))_{i \geq 0}$ the induced filtration of $K_0(R)$. For all $i \geq 1$, the ith Chern class induces a group homomorphism

$$c_i : F^i K_0(R)/F^{i+1} K_0(R) \to CH^i(X).$$

Moreover, there is an isomorphism $K_0(R)/F^1 K_0(R) \to CH^0(X)$ induced by the 0th Chern class and *rank*. Furthermore, the groups $F^i G_0(R)/F^{i+1} G_0(R)$ are in fact generated by the classes $[R/\mathfrak{p}]$ for prime ideals of height i. The assignment $Spec(R/\mathfrak{p}) \mapsto [R/\mathfrak{p}]$ factors through rational equivalence and hence induces a natural surjective homomorphism

$$\varphi_i : CH^i(X) \to F^i K_0(R)/F^{i+1} K_0(R).$$

It is well-known that both composites $\varphi_i \circ c_i$ and $c_i \circ \varphi_i$ are multiplication by $(-1)^{i-1}(i-1)!$ for $i \geq 1$. In particular, the maps φ_i and c_i are automatically isomorphisms if $i \leq 2$. This leads to a classification of projective modules of rank 2 on smooth affine surfaces:

Theorem. Let R be a smooth affine algebra of dimension 2 over an algebraically closed field k and let $X = Spec(R)$ be connected. Then the Chern classes induce a bijection $(c_1, c_2) : \mathcal{V}_2(R) \to CH^1(X) \times CH^2(X)$.

The theorem was basically a consequence of the fact that projective modules of rank 2 over smooth affine algebras of dimension 2 over algebraically closed fields are cancellative. This was first proven in [MS] and generalized by Suslin's cancellation theorem above. The rough idea of the proof is the following: Since projective modules of rank 2 over R are cancellative, we know that $\tilde{K}_0(R) \cong \mathcal{V}_2(R)$. Furthermore, we have two short exact sequences

$$0 \to F^1K_0(R) \to K_0(R) \to CH^0(X) \to 0$$

and

$$0 \to F^2K_0(R) \to F^1K_0(R) \to CH^1(X) \to 0.$$

Since the homomorphism $K_0(R) \to CH^0(X) \cong \mathbb{Z}$ corresponds to the rank map, it follows that $\tilde{K}_0(R) = F^1K_0(R)$. Moreover, the second Chern class induces an isomorphism $F^2K_0(R) \cong CH^2(X)$ as $F^iK_0(R) = 0$ for $i \geq 3$. The theorem above can be deduced from these observations.
If R is a smooth affine algebra of dimension 3 over an algebraically closed field, then one can also use the filtration $(F^iK_0(R))_{i\geq 0}$ in order to study the corresponding maps on $\mathcal{V}_2(R)$ and $\mathcal{V}_3(R)$ induced by Chern classes. Indeed, the following results were proven by N. Mohan Kumar and M. Pavaman Murthy in [KM]:

Theorem (Kumar-Murthy). Let R be a smooth affine algebra of dimension 3 over an algebraically closed field k with $char(k) \neq 2$ and let $X = Spec(R)$ be connected. Then the map $(c_1, c_2) : \mathcal{V}_2(R) \to CH^1(X) \times CH^2(X)$ is surjective and, moreover, the map $(c_1, c_2, c_3) : \mathcal{V}_3(R) \to CH^1(X) \times CH^2(X) \times CH^3(X)$ is bijective.

One of the main ingredients in their proof was the fact that $CH^3(X)$ is uniquely 2-divisible and hence isomorphic to $F^3K_0(R)$ in the situation of the theorem (cp. [Sr]). Nevertheless, they were still unable to prove with the filtration $(F^iK_0(R))_{i\geq 0}$ that the map $(c_1, c_2) : \mathcal{V}_2(R) \to CH^1(X) \times CH^2(X)$ is injective. This was established by Aravind Asok and Jean Fasel in [AF2] using $\mathbb{A}^1$-homotopy theory.
The $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ and its pointed version $\mathcal{H}_{\bullet}(k)$ over a base field k were introduced by Fabien Morel and Vladimir Voevodsky in [MV] and provide a framework to apply methods used in the classical homotopy theory of topological spaces to questions in algebraic geometry. We denote by Sm_k the category of smooth separated schemes of finite type over k. Note that we can interpret any smooth k-scheme as a Nisnevich sheaf of sets

$Sm_k^{op} \to Sets$; since any set can be interpreted as a constant simplicial set, we can interpret any smooth k-scheme as a simplicial Nisnevich sheaf of sets. Furthermore, any simplicial set $\mathcal{S}$ can also be considered a simplicial Nisnevich sheaf of sets by setting $\mathcal{S}(Y) = \mathcal{S}$ for any $Y \in Sm_k$ and $\mathcal{S}(f) = id_{\mathcal{S}}$ for any morphism in Sm_k. Hence the category $\Delta^{op}Shv_{Nis}(Sm_k)$ of simplicial Nisnevich sheaves of sets over Sm_k contains both the category Sm_k and the category of simplicial sets. As a first step, one equips this category with a model structure called the simplicial model structure. A morphism between simplicial Nisnevich sheaves is a weak equivalence with respect to the simplicial model structure if it induces weak equivalences of simplicial sets on the stalks at all points of the Nisnevich topology. One then obtains the $\mathbb{A}^1$-model structure from the simplicial model structure by formally inverting the projections $\mathcal{X} \times \mathbb{A}_k^1 \to \mathcal{X}$ for all $\mathcal{X} \in \Delta^{op}Shv_{Nis}(Sm_k)$. The weak equivalences of this model structure are called $\mathbb{A}^1$-weak equivalences. The $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ is then defined as the homotopy category of $\Delta^{op}Shv_{Nis}(Sm_k)$ with respect to the $\mathbb{A}^1$-model structure and is thus obtained from $\Delta^{op}Shv_{Nis}(Sm_k)$ by inverting the $\mathbb{A}^1$-weak equivalences. Analogously, one defines the pointed $\mathbb{A}^1$-model structure on the category of pointed Nisnevich sheaves of sets; its weak equivalences are called pointed $\mathbb{A}^1$-weak equivalences. The pointed $\mathbb{A}^1$-homotopy category $\mathcal{H}_\bullet(k)$ is then defined as its homotopy category and hence obtained from the category of pointed simplicial Nisnevich sheaves by inverting pointed $\mathbb{A}^1$-weak equivalences.

We refer to objects of $\mathcal{H}(k)$ (or of $\mathcal{H}_\bullet(k)$) as (pointed) spaces. For two spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $[\mathcal{X},\mathcal{Y}]_{\mathbb{A}^1}$ the set of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ in $\mathcal{H}(k)$; similarly, for two pointed spaces $(\mathcal{X},x)$ and $(\mathcal{Y},y)$, we denote by $[(\mathcal{X},x),(\mathcal{Y},y)]_{\mathbb{A}^1,\bullet}$ the set of morphisms from $(\mathcal{X},x)$ to $(\mathcal{Y},y)$ in $\mathcal{H}_\bullet(k)$. For any pointed space $(\mathcal{X},x)$ and $i \geq 0$, one can define $\mathbb{A}^1$-homotopy sheaves $\pi_i^{\mathbb{A}^1}(\mathcal{X},x)$, which are Nisnevich sheaves of sets on Sm_k if $i \geq 0$, Nisnevich sheaves of groups on Sm_k if $i \geq 1$ and Nisnevich sheaves of abelian groups on Sm_k if $i \geq 2$.

One nice feature of $\mathbb{A}^1$-homotopy theory is that there exists a representability result for algebraic vector bundles over affine schemes which is the algebro-geometric analogue of Steenrod's homotopy classification of topological vector bundles (cp. [Ste, §19.3]): For any $r \geq 0$, there is a natural bijection

$$\mathcal{V}_r(R) \cong [X, BGL_r]_{\mathbb{A}^1}$$

for any smooth affine scheme $X = Spec(R)$ over k, where BGL_r is the simplicial classifying space of the scheme GL_r of invertible $r \times r$-matrices. This result was proven in its greatest generality in [AHW] and is due to Fabien Morel, Marco Schlichting, Aravind Asok, Marc

Hoyois and Matthias Wendt.
By analogy with the situation in topology, one can use a version of a Postnikov tower in $\mathbb{A}^1$-homotopy theory in order to compute the set $[X, BGL_r]_{\mathbb{A}^1}$: For any pointed space $(\mathcal{Y}, y)$ such that $\pi_0^{\mathbb{A}^1}(\mathcal{Y}, y) = 0$, there exist pointed spaces $(\mathcal{Y}^{(i)}, y)$, pointed morphisms $p_i : (\mathcal{Y}, y) \to (\mathcal{Y}^{(i)}, y)$ and pointed morphisms $f_i : (\mathcal{Y}^{(i+1)}, y) \to (\mathcal{Y}^{(i)}, y)$ such that

1) $\pi_j^{\mathbb{A}^1}(\mathcal{Y}^{(i)}, y) = 0$ for $j > i$,

2) the morphism p_i induces an isomorphism on $\mathbb{A}^1$-homotopy sheaves in degree $\leq i$,

3) the morphism f_i is an $\mathbb{A}^1$-fibration whose homotopy fiber is an Eilenberg-MacLane space $K(\pi_{i+1}^{\mathbb{A}^1}(\mathcal{Y}, y), i+1)$,

4) the induced morphism $(\mathcal{Y}, y) \to holim_i(\mathcal{Y}^i, y)$ is a pointed $\mathbb{A}^1$-weak equivalence.

In addition, there exists a homotopy cartesian square of the form

$$\begin{tikzcd} \mathcal{Y}^{(i+1)} \arrow[r] \arrow[d, "f_i"'] & B\pi_1^{\mathbb{A}^1}(\mathcal{Y}, y) \arrow[d] \\ \mathcal{Y}^{(i)} \arrow[r] & K^{\pi_1^{\mathbb{A}^1}(\mathcal{Y},y)}(\pi_{i+1}^{\mathbb{A}^1}(\mathcal{Y}, y), i+2), \end{tikzcd}$$

where $B\pi_1^{\mathbb{A}^1}(\mathcal{Y}, y)$ is the classifying space of $\pi_1^{\mathbb{A}^1}(\mathcal{Y}, y)$ and $K^{\pi_1^{\mathbb{A}^1}(\mathcal{Y},y)}(\pi_{i+1}^{\mathbb{A}^1}(\mathcal{Y}, y), i+2)$ is a twisted Eilenberg-MacLane space.
For any smooth k-scheme X, we let $X_+ = X \sqcup *$ be the disjoint union of X and an artificially added basepoint. One can compute the set $[X, \mathcal{Y}]_{\mathbb{A}^1} = [X_+, (\mathcal{Y}, y)]_{\mathbb{A}^1,\bullet}$ by means of the spaces $(\mathcal{Y}^{(i)}, y)$. Indeed, because of property 4), a morphism $X_+ \to (\mathcal{Y}, y)$ in $\mathcal{H}_\bullet(k)$ is given by a sequence of compatible morphisms from X_+ to the spaces $(\mathcal{Y}^{(i)}, y)$ in $\mathcal{H}_\bullet(k)$.
A morphism $X_+ \to (\mathcal{Y}^{(i)}, y)$ lifts to a morphism $X_+ \to (\mathcal{Y}^{(i+1)}, y)$ if and only if the composite $X_+ \to (\mathcal{Y}^{(i)}, y) \to K^{\pi_1^{\mathbb{A}^1}(\mathcal{Y},y)}(\pi_{i+1}^{\mathbb{A}^1}(\mathcal{Y}, y), i+2)$ lifts to a map $B\pi_1^{\mathbb{A}^1}(\mathcal{Y}, y)$. The set of morphisms from X_+ to $K^{\pi_1^{\mathbb{A}^1}(\mathcal{Y},y)}(\pi_{i+1}^{\mathbb{A}^1}(\mathcal{Y}, y), i+2)$ as well as the set of lifts of a morphism $X_+ \to \mathcal{Y}^{(i)}$ to $\mathcal{Y}^{(i+1)}$ have cohomological descriptions. Very roughly speaking, the $\mathbb{A}^1$-Postnikov tower above translates the computation of $[X, \mathcal{Y}]_{\mathbb{A}^1}$ into cohomological terms.
Now let the base field k be algebraically closed such that $char(k) \neq 2$, let $(\mathcal{Y}, y)$ be $(BGL_2, *)$ with its canonical basepoint and let $X = Spec(R)$ be a smooth affine threefold over k. First of all, one has $\pi_0^{\mathbb{A}^1}(BGL_2, *) = 0$ and one can use the $\mathbb{A}^1$-Postnikov tower in order to compute $\mathcal{V}_2(R)$. In [AF2], Aravind Asok and Jean Fasel develop a sufficient understanding of the higher $\mathbb{A}^1$-homotopy sheaves of BGL_2 and of their Nisnevich

cohomology in order to compute $[X, BGL_2]_{\mathbb{A}^1} = [X_+, (BGL_2, *)]_{\mathbb{A}^1,\bullet}$. In particular, their computations show that $[X_+, (BGL_2, *)]_{\mathbb{A}^1,\bullet} \cong [X_+, (BGL_2^{(2)}, *)]_{\mathbb{A}^1,\bullet}$.
Again using the $\mathbb{A}^1$-Postnikov tower, one can see that any morphism $X_+ \to (BGL_2^{(2)}, *)$ in $\mathcal{H}_\bullet(k)$ is uniquely determined by pairs of cohomology classes (ξ, α), where ξ corresponds to a class in $[X_+, (BGL_2^{(1)}, *)]_{\mathbb{A}^1,\bullet} = [X_+, (B\mathbb{G}_m, *)]_{\mathbb{A}^1,\bullet} \cong CH^1(X)$ and α corresponds to a class in $[X_+, (K(\mathbf{K}_2^M, 2), *)]_{\mathbb{A}^1,\bullet} \cong H^2(X, \mathbf{K}_2^M) \cong CH^2(X)$. Aravind Asok and Jean Fasel then verify that these classes are in fact the first and second Chern classes of the associated finitely generated projective R-module. This yields:

Theorem (Asok-Fasel). Let R be a smooth affine algebra of dimension 3 over an algebraically closed field k such that $char(k) \neq 2$ and let $X = Spec(R)$. Then the map $(c_1, c_2) : \mathcal{V}_2(R) \to CH^1(X) \times CH^2(X)$ is bijective.

In particular, this completes the classification of finitely generated projective modules over smooth affine threefolds. As an immediate corollary, one obtains the following cancellation theorem:

Theorem (Asok-Fasel). Let R be a smooth affine algebra of dimension 3 over an algebraically closed field k with $char(k) \neq 2$. Then any finitely generated projective R-module P of rank 2 is cancellative, i.e. any isomorphism $P \oplus R^k \cong Q \oplus R^k$ for some Q and some $k > 0$ implies $P \cong Q$.

This raises the question whether a projective module P of rank $d-1$ over a smooth affine algebra R of dimension d over an algebraically closed field is cancellative in general. This is an open question, but the special case $P = R^{d-1}$ has been settled in [FRS, Theorem 7.5]:

Theorem (Fasel-Rao-Swan). Let R be a normal affine algebra of dimension $d \geq 3$ over an algebraically closed field k with $(d-1)! \in k^\times$; if $d = 3$, furthermore assume that R is smooth. Then stably free modules of rank $d-1$ are free, i.e. R^{d-1} is cancellative.

In order to prove the theorem, one only has to show that any unimodular row $a = (a_1, ..., a_d)$ of length d is equivalent to $(1, 0, ..., 0)$ with respect to the right action of $GL_d(R)$ on $Um_d(R)$. In fact, it follows from a theorem by Suslin that any unimodular row of the form $(b_1^{(d-1)!}, b_2, ..., b_d)$ is completable to an invertible matrix. In particular, it is sufficient to prove that a is equivalent to a row of this form with respect to the action of $GL_d(R)$. For this purpose, we set $I = \langle a_4, ..., a_d \rangle$ and let $B = R/I$. Then we consider the map

$$Um_3(B)/E_3(B) \to Um_d(R)/E_d(R),\ (\bar{b}_1, \bar{b}_2, \bar{b}_3) \mapsto (b_1, b_2, b_3, a_4, ..., a_d),$$

which is easily seen to be well-defined. It follows that it suffices to show that $(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ is equivalent to a row $(\bar{b}_1^{(d-1)!}, \bar{b}_2, \bar{b}_3)$ with respect to the action of $E_3(B)$. As a consequence of Swan's Bertini theorem (cp. [Sw, Theorem 1.5]), we can actually assume that B is a smooth threefold over k. Furthermore, there is a map

$$V : Um_3(B)/E_3(B) \to W_E(B)$$

called the Vaserstein symbol, which maps into the so-called elementary symplectic Witt group (cp. [SV, §3]). In case of a smooth threefold over a field with the properties of k, it is known that this map is a bijection (cp. [RvdK, Corollary 3.5]) and hence induces a group structure on $Um_3(B)/E_3(B)$. Furthermore, one has $nV(\bar{b}_1, \bar{b}_2, \bar{b}_3) = V(\bar{b}_1^n, \bar{b}_2, \bar{b}_3)$ and hence $n(\bar{b}_1, \bar{b}_2, \bar{b}_3) = (\bar{b}_1^n, \bar{b}_2, \bar{b}_3)$ for all $(\bar{b}_1, \bar{b}_2, \bar{b}_3)$ with respect to the group structure induced by the Vaserstein symbol.

The group $W_E(B)$ is actually a reduced higher Grothendieck-Witt group; using the Gersten-Grothendieck-Witt spectral sequence, one can prove that it is divisible prime to $char(k)$. This implies in particular that there exists a unimodular row $(\bar{b}_1, \bar{b}_2, \bar{b}_3)$ of length 3 over B such that $(\bar{a}_1, \bar{a}_2, \bar{a}_3) = (d-1)!(\bar{b}_1, \bar{b}_2, \bar{b}_3) = (\bar{b}_1^{(d-1)!}, \bar{b}_2, \bar{b}_3)$ in $Um_3(B)/E_3(B)$, which concludes the proof given in [FRS].

As already mentioned, the general case of a projective module P of rank $d-1$ remains an open problem. Of course, one can also ask whether projective modules of rank $\leq d-2$ are cancellative, but this is not true in general: For any prime number p, N. Mohan Kumar constructed in [NMK] examples of non-free stably free modules of rank p over a smooth affine algebra of dimension $p+2$ over an algebraically closed field.

Given any polynomial $f(X)$ of degree p over a field K with $f(0) = a \in K^{\times}$, N. Mohan Kumar recursively defines polynomials by

$$F_1(X_0, X_1) = X_1^p f(\tfrac{X_0}{X_1}) \text{ and}$$
$$F_{i+1}(X_0, ..., X_{i+1}) = F_1(F_i(X_0, ..., X_i), a^{\frac{p^i-1}{p-1}} X_{i+1}^{p^i}).$$

Clearly, the polynomial F_n is homogeneous of degree p^n. Then he proves that F_n is irreducible if $f(X^{p^{n-1}})$ is irreducible. In particular, if $f(X^{p^p})$ is irreducible, then F_{p+1} is irreducible. Under this assumption, he then considers the smooth affine scheme over K defined by $X = \mathbb{P}_K^{p+1} \smallsetminus V(F_{p+1})$.

As a next step, he constructs a Zariski covering of X given by $Y = (\mathbb{P}_K^{p+1} \smallsetminus V(F_p)) \cap X$ (where F_p is naturally viewed as a polynomial in $p+2$ variables) and $Z = (\mathbb{P}_K^{p+1} \smallsetminus V(G)) \cap X$, where

$$G(X_0, ..., X_{p+1}) = F_p(X_0, ..., X_p) - a^{\frac{p^p-1}{p-1}} X_{p+1}^{p^p}.$$

Since $F_{p+1} \in \langle F_p, G \rangle$, one clearly has $X = Y \cup Z$. Then he considers the smooth affine scheme $Y \cap Z$ of dimension $p+1$ over K.

He further shows that the point $y = [0 : 0 : ... : 0 : 1 : 1]$ is a complete intersection in Y and hence corresponds to a maximal ideal $\mathfrak{m}_y$ in $\mathcal{O}_Y(Y)$ generated by a regular sequence $(b_1, ..., b_{p+1})$ of elements in $\mathcal{O}_Y(Y)$. Since $y \notin Z$, the sequence defines a unimodular row of length $p+1$ over $\mathcal{O}_{Y \cap Z}(Y \cap Z)$. Using intersection theory, N. Mohan Kumar then proves that this unimodular row cannot be completed to an invertible matrix over $\mathcal{O}_{Y \cap Z}(Y \cap Z)$ and hence defines a non-free stably free module of rank p over $\mathcal{O}_{Y \cap Z}(Y \cap Z)$.

If we let $K = k(T)$ be the function field in one variable over an algebraically closed field k and $f(X) = X^p + T$, then the construction gives an example of a smooth affine algebra of dimension $p+1$ over $k(T)$ which admits a non-free stably free module of rank p. Clearing denominators, one obtains a smooth affine scheme $X_{\mathcal{K}} = Spec(R_{\mathcal{K}})$ of dimension $p+2$ over k together with a non-free stably free $R_{\mathcal{K}}$-module of rank p.

For $p = 2$, the construction gives in particular an example of a smooth affine algebra $R_{\mathcal{K}}$ of dimension 4 which admits a non-free stably free module of rank 2. As a consequence, the maps $\phi_2 : \mathcal{V}_2(R_{\mathcal{K}}) \to \mathcal{V}_3(R_{\mathcal{K}})$ and $(c_1, c_2) : \mathcal{V}_2(R_{\mathcal{K}}) \to CH^1(X_{\mathcal{K}}) \times CH^2(X_{\mathcal{K}})$ cannot be injective.

If R is a smooth affine algebra of dimension 4 over an algebraically closed field, the classification of finitely generated projective R-modules can therefore not be completely determined by the intersection theory of the underlying affine scheme $X = Spec(R)$; the classification of projective modules of rank 2 seems to be a particularly subtle problem. In view of N. Mohan Kumar's examples, it is natural to ask whether there is a cohomological criterion for a projective R-module of rank 2 to be cancellative.

In this thesis, we investigate the cancellation problem of finitely generated projective modules, i.e. the question whether an isomorphism $P \oplus R^k \cong Q \oplus R^k$ for projective modules P and Q over a commutative ring R and $k > 0$ implies that $P \cong Q$. As we have seen above, the usual approach to this problem is to study the orbit spaces $Um(P \oplus R)/E(P \oplus R)$. Motivated by the methods used by Fasel-Rao-Swan in the proof of their results on stably free modules, we construct a generalized Vaserstein symbol map

$$V_{\theta_0} : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$$

associated to any projective R-module P_0 of rank 2 with a fixed trivialization θ_0 of its determinant, where R is a commutative ring and $\tilde{V}(R)$ is a group which is canonically isomorphic to the elementary symplectic Witt group $W_E(R)$.
By means of this map, we generalize the approach of Fasel-Rao-Swan to the cancellation problem of projective modules of rank $d-1$ over smooth affine algebras of dimension d over algebraically closed fields and, moreover, we study the cohomological obstructions for the cancellation of projective modules of rank 2 with trivial determinant over smooth affine algebras of dimension 4 over algebraically closed fields. Our applications of the generalized Vaserstein symbol in this thesis are representative of methods and techniques that, we think, might be useful for future developments in the study of projective modules.

Overview of the main results. Let R be a commutative ring and, furthermore, let P_0 be a finitely generated projective R-module of constant rank 2 with a fixed trivialization $\theta_0 : R \xrightarrow{\cong} \det(P_0)$ of its determinant. In order to explain our results, let us fix some notation first: For all $n \geq 3$, we let $P_n = P_0 \oplus Re_3 \oplus ... \oplus Re_n$ be the direct sum of P_0 and free R-modules of rank 1 with explicit generators e_i, $i = 3, ..., n$. Furthermore, we let $\pi_{k,n} : P_n \to R$ be the projections onto the free direct summands of rank 1 with index $k = 3, ..., n$. Any $a \in Um(P_n)$ can be written as $(a_0, a_3, ..., a_n)$, where a_0 is the restriction of a to P_0 and any $a_i = a(e_i)$, $i = 3, ..., n$, corresponds to the restriction of a to Re_i. We let $E(P_n)$ denote the subgroup of the group $Aut(P_n)$ of automorphisms of P_n generated by elementary automorphisms. Note that there are embeddings $E(P_n) \to E(P_{n+1})$ for all $n \geq 3$; we let $E_\infty(P_0)$ denote the direct limit of the groups $E(P_n)$ via these embeddings. Moreover, we let $Um(P_n)$ denote the set of epimorphisms $P_n \to R$ and we let $Unim.El.(P_n)$ denote the set of unimodular elements of P_n. The group $Aut(P_n)$ acts on the right on $Um(P_n)$ and on the left on $Unim.El.(P_n)$. Evidently, the same also holds for any subgroup of $Aut(P_n)$. As already mentioned, we construct a generalized Vaserstein symbol

$$V_{\theta_0} : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$$

associated to P_0 and the fixed trivialization θ_0 of its determinant (cp. Theorem 4.6). The terminology is justified by the following observation: If we take $P_0 = R^2$ and let $e_1 = (1, 0)$

and $e_2 = (0,1)$, then it is well-known that there is a canonical isomorphism $\theta_0 : R \xrightarrow{\cong} \det(R^2)$ given by $1 \mapsto e_1 \wedge e_2$. As we will see, the generalized Vaserstein symbol associated to $-\theta_0$ then coincides with the usual Vaserstein symbol via the identification $\tilde{V}(R) \cong W_E(R)$. Of course, any two trivializations of $\det(P_0)$ are equal up to multiplication by a unit of R. We will actually make precise how the generalized Vaserstein symbol depends on the choice of a trivialization of $\det(P_0)$ by means of a canonical $R^\times$-action on $\tilde{V}(R)$.
We then generalize criteria found by Andrei Suslin and Leonid Vaserstein in [SV, §5] for the injectivity and surjectivity of the usual Vaserstein symbol (cp. Theorems 4.8 and 4.17):

Theorem. The generalized Vaserstein symbol $V_{\theta_0} : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$ is surjective if $Um(P_{2n+1}) = \pi_{2n+1,2n+1}E(P_{2n+1})$ for all $n \geq 2$. Furthermore, it is injective if $E(P_{2n})e_{2n} = (E_\infty(P_0) \cap Aut(P_{2n}))e_{2n}$ for all $n \geq 3$ and $E_\infty(P_0) \cap Aut(P_4) = E(P_4)$.

If R is a Noetherian ring of dimension $d \leq 3$, then it follows from [HB, Chapter IV, Theorem 3.4] that $\pi_{n,n}E(P_n) = Um(P_n)$ and $E(P_n)e_n = Unim.El.(P_n)$ for $n \geq 5$. In particular, the generalized Vaserstein symbol is a bijection if moreover $E_\infty(P_0) \cap Aut(P_4) = E(P_4)$. Using local-global principles for transvection groups (cp. [BBR]), we may then prove the following result (Theorems 1.21, 1.22, 4.18 and 4.19 in the text):

Theorem. The equality $E_\infty(P_0) \cap Aut(P_4) = E(P_4)$ holds if R is a 2-dimensional regular Noetherian ring or if R is a 3-dimensional regular affine algebra over a perfect field k such that $c.d.(k) \leq 1$ and $6 \in k^\times$. In particular, it follows that the generalized Vaserstein symbol $V_{\theta_0} : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$ is a bijection in these cases.

Now recall that one of the main ingredients in the proof of [FRS, Theorem 7.5] was the formula $nV(a_1, a_2, a_3) = V(a_1^n, a_2, a_3)$ for all unimodular rows (a_1, a_2, a_3) of length 3 whenever R is a smooth affine algebra over an algebraically closed field. It is therefore natural to ask whether an analogous formula holds for the generalized Vaserstein symbol. By reinterpreting the generalized Vaserstein symbol in the language of motivic homotopy theory, we can indeed prove (cp. Theorem 4.23):

Theorem. Let R be a smooth affine algebra over a perfect field k with $char(k) \neq 2$ such that $-1 \in {k^\times}^2$ and $n \in \mathbb{N}$. If $n \equiv 0, 1 \ mod\ 4$, then the sum formula $V_{\theta_0}(a_0, a_R^n) = n \cdot V_{\theta_0}(a_0, a_R)$ holds for all $(a_0, a_R) \in Um(P_0 \oplus R)$.

This theorem enables us to generalize the approach of Fasel-Rao-Swan to stably free modules of rank $d-1$ over normal affine algebras of dimension d over algebraically closed fields. Using Swan's Bertini theorem, we prove (cp. Theorem 4.24):

Theorem. Let R be a normal affine algebra of dimension $d \geq 3$ over an algebraically closed field k with $char(k) \neq 2$; if $d = 3$, furthermore assume that R is smooth. Then, for any $a \in Um(P_d)$ and $j \in \mathbb{N}$ with $gcd(char(k), j) = 1$, there is an automorphism $\varphi \in E(P_d)$ such that $a\varphi$ has the form $b = (b_0, b_3^j, ..., b_d)$.

In particular, if there exists $j \in \mathbb{N}$ with $gcd(char(k), j) = 1$ such that any epimorphism of the form $b = (b_0, b_3^j, ..., b_d)$ is completable to an automorphism $\psi \in Aut(P_d)$ (i.e. $b = \pi_{d,d}\psi$), then $P_{d-1} = P_0 \oplus R^{d-3}$ is cancellative. If $d = 3$ and $j = 2$, then we can explicitly construct such an automorphism with determinant 1 by generalizing a construction given by Krusemeyer in [Kr] (cp. Section 1.4). This immediately proves the following cancellation theorem (cp. Corollary 4.25):

Theorem. Let R be a smooth affine algebra of dimension 3 over an algebraically closed field k with $char(k) \neq 2$. Then $Um(P_0 \oplus R)/SL(P_0 \oplus R)$ is trivial; in particular, P_0 is cancellative.

In the sequel, we also prove that the generalized Vaserstein symbol descends to a map

$$V_{\theta_0} : Um(P_0 \oplus R)/SL(P_0 \oplus R) \to \tilde{V}_{SL}(R),$$

which we call the generalized Vaserstein symbol modulo SL. The group $\tilde{V}_{SL}(R)$ is the cokernel of a hyperbolic map $SK_1(R) \to \tilde{V}(R)$. Focusing on Noetherian rings of dimension ≤ 4, we then study the generalized Vaserstein symbol modulo SL and give again criteria for its surjectivity and injectivity. The criterion for the surjectivity is the following (cp. Theorem 4.27):

Theorem. Let R be a Noetherian ring of Krull dimension ≤ 4. Furthermore, assume that $SL(P_5)$ acts transitively on the set $Um(P_5)$. Then the generalized Vaserstein symbol $V_{\theta_0} : Um(P_0 \oplus R)/SL(P_0 \oplus R) \to \tilde{V}_{SL}(R)$ modulo SL is surjective.

The group $\tilde{V}(R)$ is a subgroup of a group usually denoted $V(R)$ (cp. [FRS, Section 4.2]), which is generated by isometry classes of the form $[P,\chi_1,\chi_2]$ for non-degenerate alternating forms χ_1,χ_2 on a finitely generated projective R-module P. For any non-degenerate alternating form χ on P_{2n}, we define $Sp(\chi) = \{\varphi \in Aut(P_{2n}) | \varphi^t \chi \varphi = \chi\}$. There is a non-degenerate alternating form χ_0 on P_0 given by $(p,q) \mapsto \theta_0^{-1}(p \wedge q)$. Furthermore, there is a canonical non-degenerate alternating form ψ_2 on R^2 given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Our criterion for the injectivity of $V_{\theta_0} : Um(P_0 \oplus R)/SL(P_0 \oplus R) \to \tilde{V}_{SL}(R)$ is the following (cp. Theorem 4.32):

Theorem. Let R be a Noetherian ring of dimension ≤ 4. Then the generalized Vaserstein symbol $V_{\theta_0} : Um(P_0 \oplus R)/SL(P_0 \oplus R) \to \tilde{V}_{SL}(R)$ modulo SL is injective if and only if $SL(P_4)e_4 = Sp(\chi)e_4$ for all non-degenerate alternating forms χ on P_4 such that $[P_4, \chi_0 \perp \psi_2, \chi] \in \tilde{V}(R)$.

As an immediate consequence, we obtain the following criterion for the triviality of the orbit space $Um(P_0 \oplus R)/SL(P_0 \oplus R)$ (cp. Corollary 4.33):

Theorem. Let R be a Noetherian ring of dimension ≤ 4. Assume that $SL(P_5)$ acts transitively on the set $Um(P_5)$. Then the orbit space $Um(P_0 \oplus R)/SL(P_0 \oplus R)$ is trivial if and only if $\tilde{V}_{SL}(R)$ is trivial and $SL(P_4)e_4 = Sp(\chi_0 \perp \psi_2)e_4$.

If $P_0 = R^2$, we can take the trivialization $R \to \det(R^2), 1 \mapsto e_1 \wedge e_2$, mentioned above. The non-degenerate alternating form χ_0 then corresponds to ψ_2. In particular, we can identify $Sp(\chi_0 \perp \psi_2)$ with $Sp_4(R)$. Moreover, the sets $Um(P_n)$ and $Unim.El.(P_n)$ can be identified with the sets $Um_n(R)$ of unimodular rows of length n over R and $Um_n^t(R)$ of unimodular columns of length n over R in this case. Motivated by the previous theorem, we then study symplectic orbits of unimodular columns. Using motivic homotopy theory and Suslin matrices, we prove (cp. Corollary 4.45):

Theorem. Let R be a smooth affine algebra of dimension $d \geq 4$ over an algebraically closed field k with $d! \in k^\times$. Assume that d is divisible by 4. Then $Sp_d(R)$ acts transitively on $Um_d^t(R)$; in particular, $Sp_d(R)e_d = SL_d(R)e_d$.

As a direct consequence of the previous two theorems, we obtain the following criterion for the triviality of $Um_3(R)/SL_3(R)$ (cp. Theorem 4.46):

Theorem. Let R be a 4-dimensional smooth affine algebra over an algebraically closed field k with $6 \in k^\times$. Then $Um_3(R)/SL_3(R)$ is trivial if and only if $\tilde{V}_{SL}(R) = 0$.

In the situation of the theorem, the group $\tilde{V}_{SL}(R)$ is actually a 2-torsion group: Indeed, the usual Vaserstein symbol surjects on the group $W_{SL}(R) \cong \tilde{V}_{SL}(R)$ and, moreover, one has $2V(a_1, a_2, a_3) = V(a_1^2, a_2, a_3) = V(1, 0, 0) = 0$ because any row of the form (a_1^2, a_2, a_3) can be completed to an invertible 3×3-matrix with determinant 1 (cp. [SwT] or [Kr]). In particular, $\tilde{V}_{SL}(R) = 0$ if and only if $\tilde{V}_{SL}(R)$ is 2-divisible. Motivated by this, we use the Gersten-Grothendieck-Witt spectral sequence in order to find cohomological criteria for the 2-divisibility of the groups $\tilde{V}(R)$ and $\tilde{V}_{SL}(R)$. These criteria enable us to prove (cp. Corollary 4.47):

Theorem. Let R be a 4-dimensional smooth affine algebra over an algebraically closed field k with $6 \in k^\times$ and let $X = Spec(R)$. Then $Um_3(R)/SL_3(R)$ is trivial if $CH^3(X)$ and $H^2(X, \mathbf{K}_3^{MW})$ are 2-divisible. Furthermore, $Um_3(R)/SL_3(R)$ is trivial if $H^2(X, \mathbf{I}^3)$ is 2-divisible and $CH^3(X) = CH^4(X) = 0$.

As a corollary of this, it follows that any finitely generated projective R-module over a smooth affine algebra R of dimension 4 over an algebraically closed field k with $6 \in k^\times$ is free if $CH^i(X) = 0$ for $i = 1, 2, 3, 4$ and $H^2(X, \mathbf{I}^3) = 0$, where $X = Spec(R)$ (cp. Corollary 4.48 in the text).
Finally, let us remark that our methods do not only apply to smooth affine algebras over algebraically closed fields. For example, we can also classify stably isomorphic oriented projective modules of rank 2 with a trivial determinant over affine algebras of dimension 3 over finite fields (cp. Theorem 4.34):

Theorem. Assume that R is an affine algebra of dimension $d = 3$ over a finite field $\mathbb{F}_q$. Then $Sp(\chi)e_4 = Unim.El.(P_4)$ for any non-degenerate alternating form χ on P_4. In particular, it follows that the generalized Vaserstein symbol descends to a bijection $V_{\theta_0} : Um(P_0 \oplus R)/SL(P_0 \oplus R) \xrightarrow{\cong} \tilde{V}_{SL}(R)$.

Structure of the thesis. The first chapter of this thesis is dedicated to the study of finitely generated projective modules over commutative rings. In particular, we study non-degenerate alternating forms on projective modules, the group of automorphisms of projective modules and its subgroup generated by transvections as well as the actions of these groups on the set of unimodular elements. Moreover, we study the stabilization maps for projective modules and oriented projective modules and we use the local-global principle for transvection groups in order to prove stability results on automorphisms of projective modules. The results proven in this chapter provide the technical groundwork for the proofs of some of the main results in this thesis.

The second chapter gives a brief introduction to motivic homotopy theory. First of all, we outline the construction of the unstable $\mathbb{A}^1$-homotopy category $\mathcal{H}(S)$ and of its pointed version $\mathcal{H}_{\bullet}(S)$ over a base scheme S. Then we study the endomorphisms of $\mathbb{P}^1_S$ in the pointed $\mathbb{A}^1$-homotopy category $\mathcal{H}_{\bullet}(S)$ over the spectrum $S = Spec(R)$ of a smooth affine algebra R over a perfect field k with $char(k) \neq 2$. Furthermore, we shortly discuss $\mathbb{A}^1$-fiber sequences and Suslin matrices at the end of the second chapter.

In the third chapter of this thesis, we introduce higher Grothendieck-Witt groups, which are a modern version of Hermitian K-theory. In this context, we also define and study the groups $\tilde{V}(R)$ and $\tilde{V}_{SL}(R)$ mentioned above. Moreover, we define Grothendieck-Witt sheaves and use the Gersten-Grothendieck-Witt spectral sequence in order to give cohomological criteria for the 2-divisibility of $\tilde{V}(R)$ and $\tilde{V}_{SL}(R)$ whenever R is a smooth affine algebra of dimension 4 over an algebraically closed field k of characteristic $\neq 2$.

In the last chapter of this thesis, we first review the usual Vaserstein symbol for unimodular rows and then reinterpret it by means of the isomorphism $W_E(R) \cong \tilde{V}(R)$ for any commutative ring R. We then construct the generalized Vaserstein symbol associated to any projective R-module P_0 of rank 2 with a fixed trivialization $\theta_0 : R \xrightarrow{\cong} \det(P_0)$ of its determinant and finally prove the main results in this thesis. In the last section of this thesis, we relate our results to some open questions in the study of projective modules.

Remark. Parts of this thesis appear in similar form in [Sy1] and [Sy2]: This concerns Sections 1.1, 1.3, 1.4, 1.5, 2.3, 3.1, 3.2, 3.3, 3.4 and 4.3 as well as parts of Sections 1.2, 2.1, 4.1 and 4.2. The main results in [Sy1] are Theorems 4.8, 4.17, 4.18 and 4.19 in this thesis; the main results in [Sy2] are Theorems 4.27 and 4.32, Corollaries 4.33 and 4.45, Theorem 4.46 and Corollary 4.47 in this thesis.

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The Study Of Projective Modules

In the first chapter of this thesis, we recall some basic definitions and facts on projective modules over commutative rings and prove some technical lemmas which will be largely used later in the proofs of the main results of this thesis. In particular, we study the group of elementary automorphisms of projective modules and prove some results on transformations of unimodular elements via elementary automorphisms. In this context, we briefly recall in Section 1.3 how projective modules which are stably isomorphic to a given projective module P can be classified in terms of the orbit space of the set of epimorphisms $P \oplus R \to R$ under the action of the group of automorphisms of $P \oplus R$. Furthermore, we construct explicit completions of some specific epimorphisms $P \oplus R \to R$, which generalizes a construction given by Krusemeyer in [Kr]. At the end of this chapter, we also recall the local-global principle for transvection groups from [BBR] in order to prove stability results on automorphisms of projective modules.

1.1 Projective modules

Let R be a commutative ring. An R-module P is projective if it is the direct summand of a free R-module; if P is finitely generated, it is projective if and only if there is an R-module Q such that $R^n \cong P \oplus Q$ for some $n \in \mathbb{N}$. For any projective R-module P and for any prime ideal $\mathfrak{p}$ of R, the localized $R_\mathfrak{p}$-module $P_\mathfrak{p}$ is again projective and therefore free (because projective modules over local rings are free). In this weak sense, projective modules are locally free. If the rank of $P_\mathfrak{p}$ as an $R_\mathfrak{p}$-module is finite for every prime $\mathfrak{p}$, then we say that P is a projective module of finite rank. In this case, there is a well-defined map

$$rank_P : Spec(R) \to \mathbb{Z}$$

which sends a prime ideal $\mathfrak{p}$ of R to the rank of $P_\mathfrak{p}$ as an $R_\mathfrak{p}$-module. In general, it is not true that projective modules of finite rank are finitely generated; nevertheless, this is known to hold if $rank_P$ is a constant map (cp. [W, Chapter I, Ex. 2.14]). We will say that P is locally free of finite rank (in the strong sense) if it admits elements $f_1, ..., f_n \in R$ generating the unit ideal such that the localizations P_{f_k} are free R_{f_k}-modules of finite rank for $k = 1, ..., n$. In fact, it is well-known that this is true if and only if P is a finitely generated projective R-module. The following lemma follows from [W, Chapter I, Lemma 2.4] and [W, Chapter I, Ex. 2.11]:

Lemma 1.1. *Let R be a commutative ring and M be an R-module. Then the following statements are equivalent:*

a) M is a finitely generated projective R-module;

b) M is locally free of finite rank (in the strong sense);

c) M is a finitely presented R-module and $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for every prime ideal $\mathfrak{p}$ of R;

d) M is a finitely generated R-module, $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for every prime ideal $\mathfrak{p}$ of R and the induced map $rank_M : Spec(R) \to \mathbb{Z}$ is continuous.

In this thesis, we will study projective modules which satisfy the equivalent conditions of Lemma 1.1 by primarily focusing on projective modules of finite constant rank.

1.2 Alternating forms, elementary automorphisms and unimodular elements

Now let R be a commutative ring. For any projective R-module P of finite rank, we let $P^\vee = Hom_{R\text{-}\mathfrak{mod}}(P, R)$ be its dual. There is a canonical isomorphism

$$can : P \to P^{\vee\vee},\ p \mapsto (ev_p : P^\vee \to R, a \mapsto a(p)),$$

induced by evaluation. A symmetric isomorphism on P is an isomorphism $f : P \to P^\vee$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "can"'] & P^\vee \arrow[d, equal, "id"] \\ P^{\vee\vee} \arrow[r, "f^\vee"'] & P^\vee \end{tikzcd}$$

is commutative. Furthermore, a skew-symmetric isomorphism on P is an isomorphism $f : P \to P^\vee$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "-can"'] & P^\vee \arrow[d, equal, "id"] \\ P^{\vee\vee} \arrow[r, "f^\vee"'] & P^\vee \end{tikzcd}$$

is commutative. Finally, an alternating isomorphism on P is an isomorphism $f : P \to P^\vee$ such that $f(p)(p) = 0$ for all $p \in P$.

Analogously, a symmetric form on a projective R-module P of finite rank is an R-bilinear map $\chi : P \times P \to R$ such that $\chi(p,q) = \chi(q,p)$ for all $p,q \in P$. Similarly, a skew-symmetric form on a projective R-module P of finite rank is an R-bilinear map $\chi : P \times P \to R$ such that $\chi(p,q) = -\chi(q,p)$ for all $p,q \in P$. Moreover, an alternating form on a projective R-module P of finite rank is an R-bilinear map $\chi : P \times P \to R$ such that $\chi(p,p) = 0$ for all $p \in P$. Note that any alternating form on P is automatically skew-symmetric. If $2 \in R^\times$, any skew-symmetric form is alternating as well. A (skew-)symmetric or alternating form χ is non-degenerate if the induced map $P \to P^\vee, q \mapsto (p \mapsto \chi(p,q))$ is an isomorphism. Obviously, the data of a non-degenerate (skew-)symmetric form is equivalent to the data of a (skew-)symmetric isomorphism. Analogously, the data of a non-degenerate alternating form is equivalent to the data of an alternating isomorphism.

Now let $\chi : M \times M \to R$ be any R-bilinear form on M. This form induces a homomorphism $M \otimes_R M \to R$. Moreover, for any prime $\mathfrak{p}$ of R, there is an induced homomorphism $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} M_\mathfrak{p} \cong (M \otimes_R M)_\mathfrak{p} \to R_\mathfrak{p}$. This gives an R-bilinear form $\chi_\mathfrak{p} : M_\mathfrak{p} \times M_\mathfrak{p} \to R_\mathfrak{p}$ on $M_\mathfrak{p}$. The following lemma shows that these localized forms completely determine χ:

Lemma 1.2. *If χ_1 and χ_2 are R-bilinear forms on an R-module M. Then $\chi_1 = \chi_2$ if and only if ${\chi_1}_\mathfrak{p} = {\chi_2}_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R.*

Proof. The forms χ_1 and χ_2 agree if and only if $\chi_1(p,q) - \chi_2(p,q) = 0$ for all $p,q \in M$. Therefore the lemma follows immediately from the fact that being 0 is a local property for elements of any R-module. □

Now let $M = \bigoplus_{i=1}^n M_i$ be an R-module which admits a decomposition into a direct sum of R-modules M_i, $i = 1, ..., n$. An elementary automorphism φ of M with respect to the given decomposition is an endomorphism of the form $\varphi_{s_{ij}} = id_M + s_{ij}$, where $s_{ij} : M_j \to M_i$ is an R-linear homomorphism for some $i \neq j$ (cp. [HB, Chapter IV, §3]). Any such

homomorphism automatically is an isomorphism with inverse given by $\varphi_{s_{ij}}^{-1} = id_M - s_{ij}$. In the special case $M = R^n \cong \bigoplus_{i=1}^n R$ one just obtains the automorphisms given by elementary matrices. We denote by $Aut(M)$ the group of automorphisms of M and by $E(M_1, ..., M_n)$ (or simply $E(M)$ if the decomposition is understood) the subgroup of $Aut(M)$ generated by elementary automorphisms.
The following lemma gives a list of some useful formulas, which can be checked easily by direct computation:

Lemma 1.3. *Let $M = \bigoplus_{i=1}^n M_i$ be a direct sum of R-modules. Then we have*

a) $\varphi_{s_{ij}}\varphi_{t_{ij}} = \varphi_{(s_{ij}+t_{ij})}$ *for all* $s_{ij} : M_j \to M_i$, $t_{ij} : M_j \to M_i$ *and* $i \neq j$*;*

b) $\varphi_{s_{ij}}\varphi_{s_{kl}} = \varphi_{s_{kl}}\varphi_{s_{ij}}$ *for all* $s_{ij} : M_j \to M_i$, $s_{kl} : M_l \to M_k$, $i \neq j$, $k \neq l$, $j \neq k$, $i \neq l$*;*

c) $\varphi_{s_{ij}}\varphi_{s_{jk}}\varphi_{-s_{ij}}\varphi_{-s_{jk}} = \varphi_{(s_{ij}s_{jk})}$ *for all* $s_{ij} : M_j \to M_i$, $s_{jk} : M_k \to M_j$ *and distinct* i, j, k*;*

d) $\varphi_{s_{ij}}\varphi_{s_{ki}}\varphi_{-s_{ij}}\varphi_{-s_{ki}} = \varphi_{(-s_{ki}s_{ij})}$ *for all* $s_{ij} : M_j \to M_i$, $s_{ki} : M_i \to M_k$ *and distinct* i, j, k.

If we restrict to the case $M_i = M_n$ for $i \geq 2$, we obtain the following result on $E(M)$:

Corollary 1.4. *If $M_i = M_n$ for $i \geq 2$, then the group $E(M)$ is generated by the elementary automorphisms of the form $\varphi_s = id_M + s$, where s is an R-linear map $M_i \to M_n$ or $M_n \to M_i$ for some $i \neq n$. The same statement holds if one replaces n by any other $k \geq 2$.*

Proof. Since $M_i = M_n$ for all $i \geq 2$, we have identities $id_{in} : M_n \to M_i$ and $id_{ni} : M_i \to M_n$ for all $i \geq 2$. Let $s_{ij} : M_j \to M_i$ be a morphism with $i \neq j$ and therefore either $i \geq 2$ or $j \geq 2$. We may assume that i, j, n are distinct. If $i \geq 2$, then

$$\varphi_{s_{ij}} = \varphi_{id_{in}}\varphi_{id_{ni}s_{ij}}\varphi_{-id_{in}}\varphi_{(-id_{ni}s_{ij})}$$

by the third formula in Lemma 1.3. If $j \geq 2$, then

$$\varphi_{s_{ij}} = \varphi_{(s_{ij}id_{jn})}\varphi_{id_{nj}}\varphi_{(-s_{ij}id_{jn})}\varphi_{-id_{nj}}.$$

by the third formula in Lemma 1.3. This proves the first part of the corollary. The last part follows in the same way if n is replaced by $k \geq 2$. □

The proof of Corollary 1.4 also shows:

Corollary 1.5. *Let $M = \bigoplus_{i=1}^n M_i$ be a direct sum of R-modules and also let $s : M_j \to M_i$, $i \neq j$, be an R-linear map. Assume that there is $k \neq i$ with $M_k = M_i$ or $k \neq j$ with $M_k = M_j$. Then the induced elementary automorphism φ_s is a commutator.*

The following lemma is a version of Whitehead's lemma (cp. [SV, Lemma 2.2]) in our general setting:

Lemma 1.6. *Let $M = M_1 \oplus M_2$ and let $f : M_1 \to M_2$, $g : M_2 \to M_1$ be morphisms. Assume that $id_{M_1} + gf$ is an automorphism of M_1. Then:*

a) $id_{M_2} + fg$ is an automorphism of M_2;

b) $(id_{M_1} + gf) \oplus (id_{M_2} + fg)^{-1}$ is an element of $E(M_1 \oplus M_2)$.

Proof. We have $id_{M_1} \oplus (id_{M_2} + fg) = \varphi_{-f}\varphi_{-g}((id_{M_1} + gf) \oplus id_{M_2})\varphi_f\varphi_g$. This shows the first statement. For the second statement one checks that

$$(id_{M_1} + gf) \oplus (id_{M_2} + fg)^{-1} = \varphi_{-g}\varphi_{-f}\varphi_g\varphi_{(id_{M_1}+gf)^{-1}g-g}\varphi_{fgf+f}.$$

So $(id_{M_1} + gf) \oplus (id_{M_2} + fg)^{-1}$ lies in $E(M_1 \oplus M_2)$. □

Now let P be a finitely generated projective R-module. We denote by $Um(P)$ the set of epimorphisms $P \to R$. The group $Aut(P)$ of automorphisms of P then acts on the right on $Um(P)$. Consequently, the same holds for any subgroup of $Aut(P)$; in particular, it holds for the subgroup $SL(P)$ of automorphisms of determinant 1. If we fix a decomposition $P \cong \bigoplus_{i=1}^n P_i$, the group $E(P) = E(P_1, ..., P_n)$ acts on $Um(P)$ as well. In this case, we may write any $a \in Um(P)$ as $(a_1, .., a_n)$, where any a_i, $i = 1, ..., n$, is the restriction of a to the direct summand P_i respectively.
An element $p \in P$ is called unimodular if there is an $a \in Um(P)$ such that $a(p) = 1$; this means that the morphism $R \to P, 1 \mapsto p$ defines a section for the epimorphism a. We denote by $Unim.El.(P)$ the set of unimodular elements of P. Note that the group $Aut(P)$ and hence also $SL(P)$ and $E(P)$ with respect to any decomposition act on the left on P; these actions restrict to actions on $Unim.El.(P)$. Again, if we fix a decomposition $P \cong \bigoplus_{i=1}^n P_i$, we can write any $a \in Unim.El.(P)$ as $(a_1, .., a_n)$, where any a_i, $i = 1, ..., n$, is the coordinate of a in the direct summand P_i respectively.
The canonical isomorphism $can : P \to P^{\vee\vee}$ identifies the set of unimodular elements $Unim.El.(P)$ of P with the set $Um(P^\vee)$ of epimorphisms $P^\vee \to R$, i.e. an element $p \in P$ is unimodular if and only if $ev_p : P^\vee \to R$ is an epimorphism. Furthermore, if p and q are unimodular elements of P and $\varphi \in Aut(P)$ with $\varphi(p) = q$, then $ev_p\varphi^\vee = ev_q : P^\vee \to R$.
We therefore obtain a well-defined map

$$Unim.El.(P)/Aut(P) \to Um(P^\vee)/Aut(P^\vee).$$

Let us show that this map is actually a bijection. Since the map is automatically surjective, it only remains to show that it is injective. So let $\psi \in Aut(P^\vee)$ such that $ev_p\psi = ev_q$. One can easily check that the map $Aut(P) \to Aut(P^\vee)$, $\varphi \mapsto \varphi^\vee$, is bijective; hence $\psi = \varphi^\vee$ for some $\varphi \in Aut(P)$. Thus, we obtain $ev_q = ev_p\varphi^\vee = ev_{\varphi(p)}$ and therefore $\varphi(p) = q$ because $can : P \to P^{\vee\vee}$ is injective. Altogether, we obtain a bijection

$$Unim.El.(P)/Aut(P) \xrightarrow{\cong} Um(P^\vee)/Aut(P^\vee).$$

In particular, if $P \cong P^\vee$, then $Unim.El.(P)/Aut(P) \cong Um(P)/Aut(P)$.
If $P \cong \bigoplus_{i=1}^n P_i$ is a direct sum, then obviously $P^\vee \cong \bigoplus_{i=1}^n P_i^\vee$ and we have the identification

$$Unim.El.(P)/E(P) \xrightarrow{\cong} Um(P^\vee)/E(P^\vee).$$

In this thesis, we will study these orbit spaces and will use both interpretations as orbit spaces of the set of epimorphisms or unimodular elements of a projective module.
If $P = R^n$, we naturally identify $Um(P)$ with the set $Um_n(R)$ of unimodular rows of length n and $Unim.El.(P)$ with the set $Um_n^t(R)$ of unimodular columns of length n. We also identify $Aut(P)$, $SL(P)$ and $E(P)$ with $GL_n(R)$, $SL_n(R)$ and $E_n(R)$ in this case.

We now introduce some notation: Let P_0 be a finitely generated projective R-module of rank 2. For any $n \geq 3$, let $P_n = P_0 \oplus Re_3 \oplus ... \oplus Re_n$ be the direct sum of P_0 and free R-modules Re_i, $3 \leq i \leq n$, of rank 1 with explicit generators e_i. Note that we can write any $a \in Um(P_n)$ as $(a_0, a_3, ..., a_n)$, where a_0 is the restriction of a to P_0 and any a_i, $i = 3, ..., n$, is the element of R corresponding to the restriction of a to Re_i respectively, i.e. $a_i = a(e_i)$. We denote by $\pi_{k,n} : P_n \to R$ the projections onto the free direct summands of rank 1 with index $k = 3, ..., n$. For any non-degenerate alternating form χ on P_{2n}, $n \geq 2$, we set $Sp(\chi) = \{\varphi \in Aut(P_{2n}) | \varphi^t\chi\varphi = \chi\}$.
For $n \geq 3$, we have embeddings $Aut(P_n) \to Aut(P_{n+1})$ and $E(P_n) \to E(P_{n+1})$. We denote by $Aut_\infty(P_0)$ (resp. $E_\infty(P_0)$) the direct limits of the groups $Aut(P_n)$ (resp. $E(P_n)$) via these embeddings.
In the following lemmas, we denote by ψ_2 the non-degenerate alternating form on R^2 given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus, for any non-degenerate alternating form χ on P_{2n} for some $n \geq 2$, we obtain a non-degenerate alternating form on P_{2n+2} given by the orthogonal sum $\chi \perp \psi_2$.
With this notation in mind, we may now prove a few lemmas which provide the technical groundwork for the proofs of some of the main results in this thesis:

Lemma 1.7. *Let χ be a non-degenerate alternating form on P_{2n} for some $n \geq 2$. Let $p \in P_{2n-1}$ and $a : P_{2n-1} \to R$. Then there are $\varphi, \psi \in Aut(P_{2n-1})$ such that*

a) the morphism $(\varphi \oplus 1)(id_{P_{2n}} + p\pi_{2n,2n})$ is an element of $E(P_{2n}) \cap Sp(\chi)$;

b) the morphism $(\psi \oplus 1)(id_{P_{2n}} + ae_{2n})$ is an element of $E(P_{2n}) \cap Sp(\chi)$.

Proof. We let $\Phi : P_{2n} \to P_{2n}^{\vee}$ be the alternating isomorphism induced by χ and Φ^{-1} be its inverse.
For the first part, we introduce the following homomorphisms: Let d denote the morphism $R \to P_{2n-1}$ which sends 1 to $\Phi^{-1}(\pi_{2n,2n})$; note that it can be considered an element of P_{2n-1} because of $\pi_{2n,2n}(\Phi^{-1}(\pi_{2n,2n})) = \chi(\Phi^{-1}(\pi_{2n,2n}), \Phi^{-1}(\pi_{2n,2n})) = 0$. Furthermore, let $\nu = \chi(p, -) : P_{2n-1} \to R$. We observe that $\nu d = 0$. By Lemma 1.6, the homomorphism $\varphi = id_{P_{2n-1}} - d\nu$ is an automorphism and $\varphi \oplus 1$ is an elementary automorphism. In particular, $(\varphi \oplus 1)(id_{P_{2n}} + p\pi_{2n,2n})$ is an elementary automorphism. In light of the proof of [SV, Lemma 5.4] and Lemma 1.2, one can check locally that it also lies in $Sp(\chi)$.
For the second part, we introduce the following homomorphisms: We let c denote the homomorphism $\chi(-, e_{2n}) : P_{2n-1} \to R$. Furthermore, we let $a \oplus 0 : P_{2n} \to R$ be the extension of a to P_{2n} which sends e_{2n} to 0; then we denote by ϑ the homomorphism $R \to P_{2n-1}$ which sends 1 to $\pi\Phi^{-1}(a \oplus 0)$, where $\pi : P_{2n} \to P_{2n-1}$ is the projection. Note that $c\vartheta = 0$. Again by Lemma 1.6, the morphism $\psi = id_{P_{2n-1}} - \vartheta c$ is an automorphism and $\psi \oplus 1$ is an elementary automorphism. In particular, $(\psi \oplus 1)(id_{P_{2n}} + ae_{2n})$ is an elementary automorphism as well. Again, in light of the proof of [SV, Lemma 5.4] and Lemma 1.2, one can check locally that it also lies in $Sp(\chi)$. □

Lemma 1.8. *Let χ be a non-degenerate alternating form on the module P_{2n} for some $n \geq 2$. Then $E(P_{2n})e_{2n} = (E(P_{2n}) \cap Sp(\chi))e_{2n}$.*

Proof. Let $p \in E(P_{2n})e_{2n}$. By Corollary 1.4, the group $E(P_{2n})$ is generated by automorphisms of the form $id_{P_{2n}} + s$, where s is a morphism $P_{2n-1} \to Re_{2n}$ or $Re_{2n} \to P_{2n-1}$. Hence we can write $(\alpha_1 ... \alpha_r)(p) = e_{2n}$, where each α_i is one of these generators. We show by induction on r that $p \in (E(P_{2n}) \cap Sp(\chi))e_{2n}$. If $r = 0$, there is nothing to show. So let $r \geq 1$. Lemma 1.7 shows that there is $\gamma \in Aut(P_{2n-1})$ such that $(\gamma \oplus 1)\alpha_r$ lies in $E(P_{2n}) \cap Sp(\chi)$.

We set $\beta_i = (\gamma \oplus 1)\alpha_i(\gamma^{-1} \oplus 1)$ for each $i < r$. Each of the β_i lies in $E(P_{2n})$ and is again one of the generators of $E(P_{2n})$ given above. Furthermore, $(\beta_1 ... \beta_{r-1}(\gamma \oplus 1)\alpha_r)(p) = e_{2n}$. This enables us to conclude by induction. $\square$

Lemma 1.9. *Let χ_1 and χ_2 be non-degenerate alternating forms on the module P_{2n} such that $\varphi^t(\chi_1 \perp \psi_2)\varphi = \chi_2 \perp \psi_2$ for some $\varphi \in E_\infty(P_0) \cap Aut(P_{2n+2})$. Now let $\chi = \chi_1 \perp \psi_2$. If $(E_\infty(P_0) \cap Aut(P_{2n+2}))e_{2n+2} = (E_\infty(P_0) \cap Sp(\chi))e_{2n+2}$ holds, then one has $\psi^t \chi_1 \psi = \chi_2$ for some $\psi \in E_\infty(P_0) \cap Aut(P_{2n})$.*

Proof. Let $\psi'' e_{2n+2} = \varphi e_{2n+2}$ for some $\psi'' \in E_\infty(P_0) \cap Sp(\chi)$. Then we set $\psi' = (\psi'')^{-1}\varphi$. Since $(\psi')^t(\chi_1 \perp \psi_2)\psi' = \chi_2 \perp \psi_2$, the composite $\psi : P_{2n} \xrightarrow{\psi'} P_{2n+2} \to P_{2n}$ and ψ' satisfy the following conditions:

- $\psi'(e_{2n+2}) = e_{2n+2}$;
- $\pi_{2n+1,2n+2}\psi' = \pi_{2n+1,2n+2}$;
- $\psi^t \chi_1 \psi = \chi_2$.

The first two conditions imply that ψ equals ψ' up to elementary morphisms and also that $\psi \in E_\infty(P_0) \cap Aut(P_{2n})$, which finishes the proof. $\square$

Lemma 1.10. *Assume that $\pi_{2n+1,2n+1}(E_\infty(P_0) \cap Aut(P_{2n+1})) = Um(P_{2n+1})$ holds for some $n \in \mathbb{N}$. Then, for any non-degenerate alternating form χ on P_{2n+2}, there exists an automorphism $\varphi \in E_\infty(P_0) \cap Aut(P_{2n+2})$ such that $\varphi^t \chi \varphi = \psi \perp \psi_2$ for some non-degenerate alternating form ψ on P_{2n}.*

Proof. Let $d = \chi(-, e_{2n+2}) : P_{2n+1} \to R$. Since d can be locally checked to be an epimorphism, there is an automorphism $\varphi' \in E_\infty(P_0) \cap Aut(P_{2n+1})$ such that $d\varphi' = \pi_{2n+1,2n+1}$. Then the alternating form $\chi' = (\varphi' \oplus 1)^t \chi(\varphi' \oplus 1)$ satisfies that $\chi'(-, e_{2n+2}) : P_{2n+1} \to R$ is just $\pi_{2n+1,2n+1}$. Now we simply define $c = \chi'(-, e_{2n+1}) : P_{2n+1} \to R$ and let $\varphi_c = id_{P_{2n+2}} + ce_{2n+2}$ be the elementary automorphism on P_{2n+2} induced by c; then ${\varphi_c}^t \chi' \varphi_c = \psi \perp \psi_2$ for some non-degenerate alternating form ψ on P_{2n}, as desired. $\square$

Lemma 1.11. *We have $E(P_0 \oplus R) \subset SL(P_0 \oplus R)$. Furthermore, if $\varphi \in SL(P_0 \oplus R)$, then the induced morphism $\varphi_* : \det(P_0 \oplus R) \to \det(P_0 \oplus R)$ is the identity on $\det(P_0 \oplus R)$.*

Proof. If $\varphi \in E(P_0 \oplus R)$ and $\mathfrak{p}$ is any prime ideal of R, then $\varphi_\mathfrak{p}$ will obviously correspond to an elementary automorphism of $(P_0)_\mathfrak{p} \oplus R_\mathfrak{p}$. Choosing any isomorphism $(P_0)_\mathfrak{p} \cong R_\mathfrak{p}^2$, it will

therefore correspond to an element of $E_3(R_{\mathfrak{p}}) \subset SL_3(R_{\mathfrak{p}})$. Thus, $E(P_0 \oplus R) \subset SL(P_0 \oplus R)$, as desired.
Since being 0 is a local property, the second statement can also be checked locally. Again, choosing any isomorphism $(P_0)_{\mathfrak{p}} \cong R^2_{\mathfrak{p}}$, the automorphism $\varphi_{\mathfrak{p}}$ will by assumption correspond to an element of $SL_3(R_{\mathfrak{p}})$. But since for any automorphism of $R^3_{\mathfrak{p}}$ the induced automorphism on $\det(R^3_{\mathfrak{p}})$ is just multiplication by its determinant, the second statement follows immediately. □

We will now introduce useful maps which allow us to some degree to restrict our study of the orbit spaces $Um(P_n)/E(P_n)$ to the orbit spaces of the form $Um(P_3)/E(P_3)$. For this, let $n \geq 4$ and $a \in Um(P_n)$. As usual, we can write a as $(a_0, a_3, ..., a_n)$, where a_0 is the restriction of a to P_0 and any a_i, $i = 3, ..., n$, is the element of R corresponding to the restriction of a to Re_i respectively, i.e. $a_i = a(e_i)$. We denote by I the image of the homomorphism $\tilde{a} = (a_4, ..., a_n) : \bigoplus_{i=4}^n Re_i \to R$; in other words, $I = \langle a_4, ..., a_d \rangle$. From now on, we write by abuse of notation π for the canonical projection $Q \to Q/I$ for any R-module Q. We consider the R/I-module P_3/IP_3 and naturally identify it with $(P_0/IP_0) \oplus (R/I)$. Furthermore, we let $Um(P_3/IP_3)$ be the set of R/I-linear epimorphisms onto R/I. As usual, we may write any $\bar{b} \in Um(P_3/IP_3)$ as $(\bar{b}_0, \bar{b}_3)$. For any such $\bar{b} \in Um(P_3/IP_3)$, there exists an R-linear map $b = (b_0, b_3)$ such that the diagram

$$\begin{tikzcd} & P_3 \arrow[dl, dotted, "b"'] \arrow[d, "\bar{b}\pi"] \\ R \arrow[r, two heads, "\pi"] & R/I \end{tikzcd}$$

commutes because P_3 is projective and $R \to R/I$ is an R-linear surjective map. Clearly, the homomorphism $(b_0, b_3, a_4, ..., a_n)$ is then an element of $Um(P_n)$.
Now assume that $b' = (b'_0, b'_3)$ is another R-linear map such that the diagram above is commutative. Then the R-linear map $b - b'$ maps P_3 into I. Thus, as P_3 is projective, there exists an R-linear map $s : P_3 \to \bigoplus_{i=4}^n Re_i$ such that the diagram

$$\begin{tikzcd} & P_3 \arrow[dl, dotted, "s"'] \arrow[d, "b-b'"] \\ \bigoplus_{i=4}^n Re_i \arrow[r, two heads, "\tilde{a}"] & I \end{tikzcd}$$

is commutative. In particular, if we let $\varphi_s = id_{P_n} + s$ be the elementary automorphism of P_n induced by s, then $(b'_0, b'_3, a_4, ..., a_n)\varphi_s = (b_0, b_3, a_4, ..., a_n)$.
It follows that the assignment $(\bar{b}_0, \bar{b}_3) \mapsto (b_0, b_3, a_4, ..., a_n)$ induces a well-defined map

$$Um(P_3/IP_3) \to Um(P_n)/E(P_n).$$

Finally, let $\bar{b} \in Um(P_3/IP_3)$ and $\bar{s} : P_0/IP_0 \to R/I$ and $\bar{t} : R/I \to P_0/IP_0$ be R/I-linear maps. Then, again since P_0 and R are projective R-modules, there exist R-linear lifts $s : P_0 \to R$ and $t : R \to P_0$ of $\bar{s}$ and $\bar{t}$ respectively. In particular, $\varphi_s = id_{P_3} + s$ and $\varphi_t = id_{P_3} + t$ are lifts of the elementary automorphisms $\varphi_{\bar{s}}, \varphi_{\bar{t}}$ of P_3/IP_3 induced by $\bar{s}$ and $\bar{t}$ respectively. If $\bar{b}' = \bar{b}\varphi_{\bar{s}}$ and $\bar{b}'' = \bar{b}\varphi_{\bar{t}}$ and $b : P_3 \to R$ is an R-linear map which lifts $\bar{b}$, then $b\varphi_s$ and $b\varphi_t$ are R-linear lifts of $\bar{b}'$ and $\bar{b}''$ to $P_3 \to R$ respectively. In particular, if we let $b = (b_0, b_3)$, $b' = b\varphi_s = (b_0', b_3')$ and $b'' = b\varphi_t = (b_0'', b_3'')$, then the classes of $(b_0, b_3, a_4, ..., a_n)$, $(b_0', b_3', a_4, ..., a_n)$ and $(b_0'', b_3'', a_4, ..., a_n)$ in $Um(P_n)/E(P_n)$ coincide.
Altogether, it follows from this that the map above descends to a well-defined map

$$\Phi(a) : Um(P_3/IP_3)/E(P_3/IP_3) \to Um(P_n)/E(P_n).$$

More generally, let $I_i = \langle a_{i+1}, ..., a_n \rangle$ for $3 \leq i \leq n-1$. By repeating the reasoning above, we can prove that there is a well-defined map

$$\Phi_i(a) : Um(P_i/I_iP_i)/E(P_i/I_iP_i) \to Um(P_n)/E(P_n)$$

which sends the class of $(\bar{b}_0, \bar{b}_3, ..., \bar{b}_i) \in Um(P_i/I_iP_i)$ to the class represented by the homomorphism $(b_0, ..., b_i, a_{i+1}, ..., a_n) \in Um(P_n)$, where $(b_0, b_3, ..., b_i) : P_i \to R$ is any R-linear lift of $(\bar{b}_0, \bar{b}_3, ..., \bar{b}_i)$. In particular, $\Phi_3(a) = \Phi(a)$.
By dualizing the reasoning above, one can also prove that, for any unimodular element $a = (a_0, ..., a_n) \in P_n$, there are analogously defined maps

$$\Phi_i(a) : Unim.El.(P_i/I_iP_i)/E(P_i/I_iP_i) \to Unim.El.(P_n)/E(P_n),$$

where again $I_i = \langle a_{i+1}, ..., a_n \rangle$ for $3 \leq i \leq n-1$.

The following two lemmas are generalizations to our situations of the corresponding well-known statements when $P_0 = R^2$ (cp. [Va] and [SV, Lemma 2.7(c)]):

Lemma 1.12. *Let $n \geq 5$, $a = (a_0, a_3, ..., a_n) \in Um(P_n)$ and let $k \in \mathbb{N}$ and $3 \leq i, j \leq n$. Then there exists $\varphi \in E(P_n)$ such that $(a_0, ..., a_i^k, ..., a_n)\varphi = (a_0, ..., a_j^k, ..., a_n)$.*

Proof. Let J denote the image of a_0. We consider the ring R/J and the unimodular rows $(\bar{a}_3, ..., \bar{a}_i^k, ..., \bar{a}_n)$ and $(\bar{a}_3, ..., \bar{a}_j^k, ..., \bar{a}_n)$. Then it is well-known that there is $\bar{\varphi}' \in E_{n-2}(R/J)$ such that $(\bar{a}_3, ..., \bar{a}_i^k, ..., \bar{a}_n)\bar{\varphi}' = (\bar{a}_3, ..., \bar{a}_j^k, ..., \bar{a}_n)$.

We now lift $\bar{\varphi}'$ to an element φ' of $E(P_d)$ which is the identity on P_0. If we then set $(b_0, b_3, ..., b_n) = (a_0, a_3, ..., a_i^k, ..., a_n)\varphi'$, there exist $p_l \in P_0$, $3 \le l \le n$, such that $a_l - b_l = a_0(p_l)$ for $l \neq j$, $3 \le l \le n$, and $a_j^k - b_j = a_0(p_j)$. Furthermore, $b_0 = a_0$.
Then we let $p : \bigoplus_{l=3}^n Re_l \to P_0$ be the homomorphism which sends e_l to p_l for $3 \le l \le n$. If we then let φ_p be the induced element of $E(P_n)$, the automorphism $\varphi = \varphi'\varphi_p$ lies in $E(P_n)$ and transforms $(a_0, ..., a_i^k, ..., a_n)$ to $(a_0, ..., a_j^k, ..., a_n)$. □

Lemma 1.13. *Let $a = (a_0, a_3, ..., a_n) \in Um(P_n)$ such that $(a_4, ..., a_n) \in Um_{n-3}(R)$. Then there exists $\varphi \in E(P_n)$ such that $a\varphi = \pi_{n,n}$.*

Proof. As in the previous proof, we let J denote the image of a_0. We consider the ring R/J and the unimodular row $(\bar{a}_3, ..., \bar{a}_n)$. Then there is $\bar{\varphi}' \in E_{n-2}(R/J)$ such that $(\bar{a}_3, ..., \bar{a}_i, ..., \bar{a}_n)\bar{\varphi}' = (0, ..., 1)$ (cp. [SV, Lemma 2.7(c)]).
We can then lift $\bar{\varphi}'$ to an element φ' of $E(P_n)$ which is the identity on P_0. If we set $(b_0, b_3, ..., b_n) = (a_0, a_3, ..., a_n)\varphi'$, there exist $p_l \in P_0$, $3 \le l \le n$, such that $a_l - b_l = a_0(p_l)$ for $3 \le l \le n$. Moreover, $b_0 = a_0$.
Then we let $p : \bigoplus_{l=3}^n Re_l \to P_0$ be the homomorphism which sends e_l to p_l for $3 \le l \le n$. If we then let φ_p be the induced element of $E(P_n)$, the automorphism $\varphi = \varphi'\varphi_p$ lies in $E(P_n)$ and transforms $(a_0, ..., a_n)$ to $\pi_{n,n}$. □

In the remainder of this section, we will prove some statements which allow us to restrict our study of orbit spaces of unimodular elements over affine algebras to algebras of lower dimensions:

Lemma 1.14. *Assume that R is a normal affine algebra of dimension $d \ge 4$ over an algebraically closed field k with $char(k) \neq 2$; furthermore, let $a = (a_0, ..., a_d) \in Um(P_d)$. Then there exists $\varphi \in E(P_d)$ such that if we let $a\varphi = (b_0, ..., b_d)$ and $I = \langle b_4, ..., b_d \rangle$, then R/I is either 0 or a smooth affine algebra of dimension 3 over k.*

Proof. Since R is normal, the ideal J of the singular locus of R has height at least 2 and therefore $\dim(R/J) \le d - 2$. Hence it follows from [HB, Chapter IV, Theorem 3.4] that $Um(P_d/JP_d) = \pi_{d,d}E(P_d/JP_d)$ and therefore we can assume that the image of a_0 and any a_i for $3 \le i \le d-1$ lie in J and that $a_d - 1 \in J$.
Now let $p = (p_0, c_3, ..., c_d) \in P_d$ be a section of a, i.e. $a(p) = 1$. Then we consider the unimodular row $\tilde{a} = (a_0(p_0), a_3, ..., a_d)$. By Swan's Bertini theorem (cp. [Sw, Theorem 1.5]), there is an upper triangular matrix $B = (\beta_{i,j})_{2 \le i,j \le d}$ (notice the indexing!) of rank $d-1$ over R such that $\tilde{a}B = (a_0(p_0), a_3', ..., a_d')$ has the property that if $I = \langle a_4', ..., a_d' \rangle$, then

R/I is either 0 or a smooth threefold outside the singular locus of R. But by the previous paragraph, it follows that we still have $a'_d - 1 \in J$, which means that R/I is either 0 or a smooth affine threefold over k.

We now define a homomorphism $s_0 : \bigoplus_{i=3}^d Re_i \to P_0$ by $s_0(e_i) = \beta_{2,i}p_0$. Furthermore, we define homomorphisms $s_l : \bigoplus_{i=l+1}^d Re_i \to Re_l$ for each l, $3 \le l \le d-1$, by $s_l(e_i) = \beta_{l,i}e_l$. Then we let φ_0 and φ_l, $3 \le l \le d-1$, be the elementary automorphisms of P_d induced by s_0 and the s_l respectively and we define $\varphi = \varphi_{d-1} \circ ... \circ \varphi_3 \circ \varphi_0$. By construction, we have $a\varphi = (a_0, a'_3, ..., a'_d)$, which finishes the proof. □

We introduce some further notation: For any commutative ring R, any finitely generated projective R-module P and any element $p \in P$, we let $o(p) = \{f(p) | f \in P^\vee\} \subset R$, which is clearly an ideal of R. Note that p is unimodular if and only if $o(p) = R$.

Proposition 1.15. *Assume that R is an affine algebra of dimension $d \ge 3$ over a finite field $\mathbb{F}_q$ or its algebraic closure $\bar{\mathbb{F}}_q$. Then the group $E(P_{d+1})$ acts transitively on $Unim.El.(P_{d+1})$.*

Proof. First of all, we note that we can assume that R is reduced: If R is not reduced and the proposition is proven for reduced algebras, we may consider R/N, where N is the nilradical of R. Then it follows that any unimodular element can be transformed via elementary automorphisms to an element of the form $a = (a_0, a_3, ..., a_{d+1})$, where $a_{d+1} - 1$ is nilpotent. But this means that a_{d+1} is a unit in R. Hence Lemma 1.13 shows that there is $\varphi \in E(P_{d+1})$ such that $\varphi(a) = e_{d+1}$. So let us henceforth assume that R is reduced.

Following the proof of [S1, Theorem 1], we pick a point on each irreducible component of the maximal spectrum $Max(R)$ of R, denote the resulting finite set by V and set $\nu = \prod_{\mu \in V} \mu$. Then, for each $\mu \in V$, we pick $p_{1,\mu}, ..., p_{2,\mu} \in P_0$ such that their classes in $P_0/\mu P_0$ form a basis of $P_0/\mu P_0$ as an $R/\mu R$-module. Then we find $p_1, p_2 \in P_0$ such that $p_i - p_{i,\mu} \in \mu P_0$ for $i = 1, 2$. Note that if we let $p_i = e_i$ for $3 \le i \le d+1$ and denote their classes in $P_{d+1}/\mu P_{d+1}$ by $p_{i,\mu}$, then $p_{1,\mu}, ..., p_{d+1,\mu}$ form a basis of $P_{d+1}/\mu P_{d+1}$.

Now let $\mathfrak{A} = o(p_1 \wedge p_2)$. By construction, $\mathfrak{A}$ is not contained in any $\mu \in V$. Since every minimal prime ideal of R is contained in some $\mu \in V$, this implies that $\dim(R/\mathfrak{A}) \le d-1$; in particular, $\dim(R/\nu\mathfrak{A}) \le d-1$. Therefore it follows from [HB, Chapter IV, Theorem 3.4] that any unimodular element $a = (a_0, a_3, ..., a_{d+1})$ can be transformed via elementary automorphisms of P_{d+1} to an element $(b_0, b_3, ..., b_{d+1})$ with $b_{d+1} - 1 \in \nu\mathfrak{A}$. If we let $\bar{p}_i$ denote the classes of the p_i, $i = 1, ..., d+1$, modulo b_{d+1}, then $o(\bar{p}_1 \wedge \bar{p}_2) = R/b_{d+1}R$; this implies that $P_0/b_{d+1}P_0$ is free with a basis given by the classes $\bar{p}_i$, $i = 1, 2$, and, in particular, that $P_d/b_{d+1}P_d$ is free with a basis given by the classes $\bar{p}_i$, $i = 1, ..., d$. As $b_{d+1} - 1 \in \nu\mathfrak{A}$, $R/b_{d+1}R$

has dimension $\leq d-1$; hence the group $E_d(R/b_{d+1}R)$ acts transitively on $Um_d^t(R/b_{d+1}R)$ (cp. [SV, Corollary 17.3]).

It follows that $(b_0,...,b_{d+1})$ and hence $(a_0,...,a_{d+1})$ can be transformed via elementary automorphisms to an element of the form $a' = (0,...,1+cb_{d+1},b_{d+1})$ for some $c \in R$. We then let $s : Re_{d+1} \to Re_d, 1 \mapsto -c$ and $t : Re_d \to Re_{d+1}, 1 \mapsto -b_{d+1}$ and let φ_s, φ_t be the induced elementary automorphisms on P_{d+1}; furthermore, we let ψ_2 be the automorphism of $Re_d \oplus Re_{d+1}$ given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in E_2(R)$$

and let $\psi \in E(P_{d+1})$ be the automorphism of P_{d+1} which is ψ_2^{-1} on $Re_d \oplus Re_{d+1}$ and the identity on the other direct summands. Then $\psi\varphi_t\varphi_s(a') = (0,...,0,1)$, which finishes the proof. □

Proposition 1.16. *Assume that R is a normal affine algebra of dimension $d \geq 4$ over the algebraic closure $k = \bar{\mathbb{F}}_q$ of a finite field $\mathbb{F}_q$ of characteristic $\neq 2$. Then, for any unimodular element $a \in P_d$, there exists an automorphism $\varphi \in E(P_d)$ such that $\varphi(a) = (b_0, b_3, ..., b_d)$, where R/b_dR is a smooth k-algebra of dimension $d-1$ and P_0/b_dP_0 is a free R/b_dR-module.*

Proof. Since R is normal, the ideal J defining the singular locus of $Spec(R)$ has height ≥ 2. Following the proof of [B, Theorem 1], we let t be a non-zero-divisor such that $(P_0)_t$ is free of rank 2. We can assume that $t \in J$ (as $ht(J) \geq 2$). Note that if we pick two elements of P_0 which form a basis of $(P_0)_t$, then the induced map $R^2 \to P_0$ is injective. Hence we obtain a free submodule $F = R^2$ of P_0 such that $F_t = (P_0)_t$. Furthermore, we let $s = t^l$ such that $sP_0 \subset F$. We denote by (e_1, e_2) the standard basis of F; in particular, for $n \geq 3$, $(e_1, e_2, e_3, ..., e_n)$ is a basis of $F_n = F \oplus Re_3 \oplus ... \oplus Re_n \subset P_n$ and $sP_n \subset F_n$.

Since s is a non-zero-divisor, we have $\dim(R/sR) \leq d-1$. Using Proposition 1.15, we can then conclude that there exists $\varphi_1 \in E(P_d)$ such that $\varphi_1(a) = (b_0, b_3, ..., b_{d-1}, b'_d)$ with $(b_0, b_3, ..., b_{d-1}) \in sP_{d-1} \subset F_{d-1}$ and with $1 - b'_d \in sR$.

If we let $b_0 = b_1e_1 + b_2e_2$, then $(b_1, b_2, b_3, ..., b_{d-1}, b'_d)$ and, furthermore, as $1 - b'_d \in sR$, $(sb_1, sb_2, sb_3, ..., sb_{d-1}, b'_d)$ are unimodular rows over R. Therefore Swan's Bertini theorem (cp. [Sw, Theorem 1.5]) implies that there exist $f_i \in R$, $1 \leq i \leq d-1$, such that for $b_d = b'_d + \sum_{i=1}^{d-1} f_i s b_i$ the ring R/b_dR is a smooth k-algebra of dimension $d-1$.

Now let $(e_1^\vee, ..., e_{d-1}^\vee)$ be the (dual) basis of $F_{d-1}^\vee$ and let $\alpha = \sum_{i=1}^{d-1} sf_ie_i^\vee$. Note that, since $sP_{d-1} \subset F_{d-1}$, we have $sF_{d-1}^\vee \subset P_{d-1}^\vee$ and we can interpret α as a homomorphism $P_{d-1} \to Re_d$. We let φ_2 be the elementary automorphism on P_d induced by α.

By construction, one has $\varphi_2(b_0, b_3, ..., b_{d-1}, b'_d) = (b_0, b_3, ..., b_{d-1}, b_d)$. Note that $1 - b_d \in sR$. Since $F_s = (P_0)_s$, the inclusion $F \subset P_0$ induces an equality $F/b_dF = P_0/b_dP_0$. Thus, P_0/b_dP_0 is a free R/b_dR-module of rank 2, as desired. □

1.3 The stabilization maps

Let R be a commutative ring. We consider the map

$$\phi_r : \mathcal{V}_r(R) \to \mathcal{V}_{r+1}(R), [P] \mapsto [P \oplus R],$$

from the set of isomorphism classes of projective modules of rank r to the set of isomorphism classes of projective modules of rank $r+1$ and fix a projective module $P \oplus R$ representing an element of $\mathcal{V}_{r+1}(R)$ in the image of this map. An element $[P']$ of $\mathcal{V}_r(R)$ lies in the fiber over $[P \oplus R]$ if and only if there is an isomorphism $i : P' \oplus R \xrightarrow{\cong} P \oplus R$. Any such isomorphism yields an element of $Um(P \oplus R)$ given by the composite

$$a(i) : P \oplus R \xrightarrow{i^{-1}} P' \oplus R \xrightarrow{\pi_R} R.$$

Note that if one chooses another module P'' representing the isomorphism class of P' and any isomorphism $j : P'' \oplus R \xrightarrow{\cong} P \oplus R$, the resulting element $a(j)$ of $Um(P \oplus R)$ still lies in the same orbit of $Um(P \oplus R)/Aut(P \oplus R)$: For if we choose an isomorphism $k : P' \xrightarrow{\cong} P''$, then we have an equality

$$a(i) = a(j) \circ (j(k \oplus id_R)i^{-1}).$$

Thus, we obtain a well-defined map

$$\phi_r^{-1}([P \oplus R]) \to Um(P \oplus R)/Aut(P \oplus R).$$

Conversely, any element $a \in Um(P \oplus R)$ gives an element of $\mathcal{V}_r(R)$ lying over $[P \oplus R]$, namely $[P'] = [\ker(a)]$. Note that the kernels of two epimorphisms $P \oplus R \to R$ are isomorphic if these epimorphisms are in the same orbit in $Um(P \oplus R)/Aut(P \oplus R)$. Thus, we also obtain a well-defined map

$$Um(P \oplus R)/Aut(P \oplus R) \to \phi_r^{-1}([P \oplus R]).$$

One can then easily check that the maps $\phi_r^{-1}([P \oplus R]) \to Um(P \oplus R)/Aut(P \oplus R)$ and $Um(P \oplus R)/Aut(P \oplus R) \to \phi_r^{-1}([P \oplus R])$ are inverse to each other. Note that $[P]$ corresponds to the class represented by the canonical projection $\pi_R : P \oplus R \to R$ under these bijections. In conclusion, we have a pointed bijection between the sets $Um(P \oplus R)/Aut(P \oplus R)$ and $\phi_r^{-1}([P \oplus R])$ equipped with $[\pi_R]$ and $[P]$ as basepoints respectively. Moreover, we also obtain a (pointed) surjection $Um(P \oplus R)/E(P \oplus R) \to \phi_r^{-1}([P \oplus R])$.
Furthermore, we denote by $\mathcal{V}_r^o(R)$ the set of isomorphism classes of oriented projective modules of rank r, i.e. isomorphism classes of pairs (P, θ), where P is projective of constant rank r and $\theta : \det(P) \xrightarrow{\cong} R$ is an isomorphism. An isomorphism between two such pairs (P, θ) and (P', θ') is an isomorphism $k : P \xrightarrow{\cong} P'$ such that $\theta = \theta' \circ \det(k)$. Note that if (P, θ) is an oriented projective module of rank r, then there is an induced orientation on $P \oplus R$ given by the composite $\theta^+ : \det(P \oplus R) \cong \det(P) \xrightarrow{\theta} R$.
We now consider the stabilization maps

$$\phi_r^o : \mathcal{V}_r^o(R) \to \mathcal{V}_{r+1}^o(R), [(P, \theta)] \mapsto [(P \oplus R, \theta^+)]$$

from the set of isomorphism classes of oriented projective modules of rank r to the set of isomorphism classes of oriented projective modules of rank $r+1$. We fix an oriented projective module $(P \oplus R, \theta^+)$ representing an element of $\mathcal{V}_{n+1}^o(R)$ in the image of this map. An element $[(P', \theta')]$ of $\mathcal{V}_n^o(R)$ lies in the fiber over $[(P \oplus R, \theta^+)]$ if and only if there is an isomorphism $i : P' \oplus R \xrightarrow{\cong} P \oplus R$ such that $\theta^+ \circ \det(i) = \theta'^+$. Any such isomorphism yields an element of $Um(P \oplus R)$ given by the composite

$$a(i) : P \oplus R \xrightarrow{i^{-1}} P' \oplus R \xrightarrow{\pi_R} R.$$

If one chooses another module (P'', θ'') representing the isomorphism class of (P', θ') and any isomorphism $j : P'' \oplus R \xrightarrow{\cong} P \oplus R$ with $\theta''^+ = \theta^+ \circ \det(j)$, the resulting element $a(j)$ of $Um(P \oplus R)$ still lies in the same orbit of $Um(P \oplus R)/SL(P \oplus R)$: For if we choose an isomorphism $k : P' \xrightarrow{\cong} P''$ with $\theta' = \theta'' \circ \det(k)$, then $j(k \oplus id_R)i^{-1} \in SL(P \oplus R)$ and we have an equality

$$a(i) = a(j) \circ (j(k \oplus id_R)i^{-1}).$$

Thus, we obtain a well-defined map

$$\phi_r^{o\,-1}([(P \oplus R, \theta^+)]) \to Um(P \oplus R)/SL(P \oplus R).$$

Conversely, any element $a \in Um(P \oplus R)$ gives an element of $\mathcal{V}_r^o(R)$ lying over $[(P \oplus R, \theta^+)]$: If we let $P' = \ker(a)$, then the short exact sequence

$$0 \to P' \to P \oplus R \xrightarrow{a} R \to 0$$

is split and any section s of a induces an isomorphism $i : P' \oplus R \xrightarrow{\cong} P \oplus R$. The induced isomorphism $\det(i) : \det(P' \oplus R) \xrightarrow{\cong} \det(P \oplus R)$ does not depend on the section s; hence we can canonically define an orientation θ' on P' given by the composite

$$\det(P') \cong \det(P' \oplus R) \xrightarrow{\det(i)} \det(P \oplus R) \xrightarrow{\theta^+} R.$$

Then $[(P', \theta')] \in \phi_r^{o-1}([(P \oplus R, \theta^+)])$. Note that this assignment only depends on the class of a in $Um(P \oplus R)/SL(P \oplus R)$.
Thus, we also obtain a well-defined map

$$Um(P \oplus R)/SL(P \oplus R) \to \phi_r^{o-1}([(P \oplus R, \theta^+)]).$$

Again, one can check that the maps $\phi_r^{o-1}([(P \oplus R, \theta^+)]) \to Um(P \oplus R)/SL(P \oplus R)$ and $Um(P \oplus R)/SL(P \oplus R) \to \phi_r^{o-1}([(P \oplus R, \theta^+)])$ are inverse to each other. Note that $[(P, \theta)]$ corresponds to the class represented by the canonical projection $\pi_R : P \oplus R \to R$ under these bijections. Altogether, we have a pointed bijection between the sets $Um(P \oplus R)/SL(P \oplus R)$ and $\phi_r^{o-1}([(P \oplus R, \theta^+)])$ equipped with $[\pi_R]$ and $[(P, \theta)]$ as basepoints respectively.
Finally, if (P, θ) is an oriented projective module of rank r as above, the canonical projection $Um(P \oplus R)/SL(P \oplus R) \to Um(P \oplus R)/Aut(P \oplus R)$ then corresponds to the map $\phi_r^{o-1}([(P \oplus R, \theta^+)]) \to \phi_r^{-1}([P \oplus R])$ forgetting the orientation of P.

1.4 Explicit completions of unimodular elements

Let R be a commutative ring and let P_0 be a projective R-module of rank 2 with a trivialization $\theta_0 : R \xrightarrow{\cong} \det(P_0)$ of its determinant. We use the notation of Section 1.2 and let $P_3 = P_0 \oplus Re_3$. In particular, any $a \in Um(P_3)$ can be written as (a_0, a_R), where a_0 is the restriction of a to P_0 and $a_R = a(e_3) \in R$. For any $a \in Um(P_3)$ of the form (a_0, a_R^2), there exists $\varphi \in SL(P_3)$ such that $\pi_{3,3}\varphi = a$ (cp. [B, Proposition 2.7] or [S1, Lemma 2]). We now construct an explicit completion φ of $a = (a_0, a_R^2)$. For this, let us first look at the case $P_0 = R^2$: If (b, c, a) is a unimodular row and $qb + rc + pa = 1$, then it follows from a construction given by Krusemeyer in [Kr] that the matrix

$$\begin{pmatrix} -p-qr & q^2 & -c+2aq \\ -r^2 & -p+qr & b+2ar \\ b & c & a^2 \end{pmatrix}$$

is a completion of (b, c, a^2) with determinant 1. We observe that

$$\begin{pmatrix} -qr & q^2 \\ -r^2 & qr \end{pmatrix} = \begin{pmatrix} q \\ r \end{pmatrix} \begin{pmatrix} -r & q \end{pmatrix}$$

and also

$$\begin{pmatrix} -c \\ b \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}.$$

This shows how to generalize the construction of this explicit completion. We denote by $\chi_0 : P_0 \to P_0^\vee$ the alternating isomorphism given by $q \mapsto (\chi_0(q) : P_0 \to R, p \mapsto \theta_0^{-1}(p \wedge q))$. If $a = (a_0, a_R)$ is an element of $Um(P_0 \oplus R)$ with a section s uniquely given by the element $s(1) = (q, p) \in P_0 \oplus R$, we consider the following morphisms: We define an endomorphism of P_0 by

$$\varphi_0 = -(\pi_{P_0} s) \circ \chi_0(q) - p \cdot id_{P_0} : P_0 \to P_0$$

and we also define a morphism $R \to P_0$ by

$$\varphi_R : R \to P_0,\ 1 \mapsto 2a_R \cdot q + \chi_0^{-1}(a_0).$$

Then we consider the endomorphism of $\varphi : P_0 \oplus R$ given by

$$\begin{pmatrix} \varphi_0 & \varphi_R \\ a_0 & a_R^2 \end{pmatrix}.$$

Essentially by construction, φ is a completion of (a_0, a_R^2):

Proposition 1.17. *The endomorphism φ of $P_0 \oplus R$ defined above is an automorphism of $P_0 \oplus R$ of determinant 1 such that $\pi_{3,3}\varphi = (a_0, a_R^2)$.*

Proof. Choosing locally a free basis $(e_1^{\mathfrak{p}}, e_2^{\mathfrak{p}})$ of $(P_0)_{\mathfrak{p}}$ at any prime $\mathfrak{p}$ with $(\theta_0^{-1})_{\mathfrak{p}}(e_1^{\mathfrak{p}} \wedge e_2^{\mathfrak{p}}) = 1$, we can check locally that this endomorphism is an automorphism of determinant 1 (because locally it coincides with the completion given in [Kr]); by definition, we obviously have $\pi_{3,3}\varphi = (a_0, a_R^2)$. Thus, φ has the desired properties and generalizes the explicit completion given in [Kr]. $\square$

1.5 Local-global principles

We will now briefly review the local-global principle for transvection groups proven in [BBR] and use it in order to deduce stability results for stably elementary automorphisms of $P_0 \oplus R^2$. For this, we only have to assume that R is an arbitrary commutative ring with unit.

First of all, let P be a finitely generated projective R-module and $q \in P$, $\varphi \in P^\vee$ such that $\varphi(q) = 0$. This data naturally determines a homomorphism $\varphi_q : P \to P$ by $\varphi_q(p) = \varphi(p)q$ for all $p \in P$. An automorphism of the form $id_P + \varphi_q$ is called a transvection if either $q \in Unim.El.(P)$ or $\varphi \in Um(P)$. We denote by $T(P)$ the subgroup of $Aut(P)$ generated by transvections.

Now let $Q = P \oplus R$ be a direct sum of a finitely generated projective R-module P of rank ≥ 2 and the free R-module of rank 1. Then the elementary automorphisms of $P \oplus R$ are easily seen to be transvections and are also called elementary transvections. Consequently, we have $E(Q) \subset T(Q) \subset Aut(Q)$.

In the theorem stated below, we denote by $R[X]$ the polynomial ring in one variable over R and let $Q[X] = Q \otimes_R R[X]$. The evaluation homomorphisms $ev_0, ev_1 : R[X] \to R$ induce maps $Aut(Q[X]) \to Aut(Q)$. If $\alpha(X) \in Aut(Q[X])$, then we denote its images under these maps by $\alpha(0)$ and $\alpha(1)$ respectively. Similarly, the localization homomorphism $R \to R_{\mathfrak{m}}$ at any maximal ideal $\mathfrak{m}$ of R induces a map $Aut(Q[X]) \to Aut(Q_{\mathfrak{m}}[X])$, where $Q_{\mathfrak{m}}[X] = Q[X] \otimes_{R[X]} R_{\mathfrak{m}}[X]$; if $\alpha(X) \in Aut(Q[X])$, we denote its image under this map by $\alpha_{\mathfrak{m}}(X)$.

We will use the following results proven by Bak, Basu and Rao (cp. [BBR, Theorems 3.6 and 3.10]):

Theorem 1.18. *The inclusion $E(Q) \subset T(Q)$ is actually an equality. Furthermore, if $\alpha(X) \in Aut(Q[X])$ satisfies $\alpha(0) = id_Q \in Aut(Q)$ and $\alpha_{\mathfrak{m}}(X) \in E(Q_{\mathfrak{m}}[X])$ for all maximal ideals $\mathfrak{m}$ of R, then $\alpha(X) \in E(Q[X])$.*

In order to prove the desired stability results, we introduce the following property: Let $\mathfrak{C}$ be either the class of Noetherian rings or the class of affine k-algebras over a fixed field k. Furthermore, let $d \geq 1$ be an integer and $m \in \mathbb{N}$. We say that $\mathfrak{C}$ has the property $\mathcal{P}(d,m)$ if, for R in $\mathfrak{C}$ of dimension d and for any finitely generated projective R-module P of rank m, the group $SL(P \oplus R^n)$ acts transitively on $Um(P \oplus R^n)$ for all $n \geq 2$. If k is a field, we simply say that k has the property $\mathcal{P}(d,m)$ if the class of affine k-algebras has the property $\mathcal{P}(d,m)$.

Of course, if the class of Noetherian rings has the property $\mathcal{P}(d,m)$, then the same holds for every field. The class of Noetherian rings has the property $\mathcal{P}(d,m)$ for $m \geq d$. Furthermore, it follows from [B, Remark 4.2] that any infinite perfect field k of cohomological dimension ≤ 1 satisfies property $\mathcal{P}(d,d-1)$ if $d! \in k^{\times}$.
In the remainder of this section, we will denote by π the canonical projection $P \oplus R^n \to R$ onto the "last" free direct summand of R^n.

Lemma 1.19. *Let $\mathfrak{C}$ be the class of Noetherian rings or affine k-algebras over a fixed field k. Assume that $\mathfrak{C}$ has the property $\mathcal{P}(d,m)$. Let R be a d-dimensional ring in $\mathfrak{C}$, P a projective R-module of rank m and $a \in Um(P \oplus R^n)$ for some $n \geq 2$. Moreover, assume that there is an element $t \in R$ and a homomorphism $w : P \oplus R^n \to R$ such that $a - \pi = tw$. Then there is $\varphi \in SL(P \oplus R^n)$ such that $a = \pi\varphi$ and $\varphi(x) \equiv id_{P\oplus R^n}(x)$ modulo $\langle t \rangle$ for all x.*

Proof. We set $B = R[X]/\langle X^2 - tX \rangle$. By assumption, we have $a = \pi + tw$. Then we lift it to $a(X) = \pi + Xw : (P \oplus R^n) \otimes_R B \to B$, which can be checked to be an epimorphism (as in the proof of [RvdK, Proposition 3.3]). Hence we have $a(X) \in Um((P \oplus R^n) \otimes_R B)$. Since B is a ring in $\mathfrak{C}$ of dimension d, property $\mathcal{P}(d,m)$ now gives an element $\varphi(X) \in SL((P \oplus R^n) \otimes_R B)$ with $a(X) = \pi\varphi(X)$. Then $\varphi = \varphi(0)^{-1}\varphi(t)$ is the desired automorphism. □

For any $n \geq 2$, we say that two automorphisms $\varphi, \psi \in SL(P \oplus R^n)$ are isotopic if there is an automorphism $\tau(X)$ of $(P \oplus R^n) \otimes_R R[X]$ with determinant 1 such that $\tau(0) = \varphi$ and $\tau(1) = \psi$.

Theorem 1.20. *Let $\mathfrak{C}$ be the class of Noetherian rings or affine k-algebras over a fixed field k. Assume that $\mathfrak{C}$ has the property $\mathcal{P}(d+1,m+1)$. Let R be a d-dimensional ring in $\mathfrak{C}$, P a projective R-module of rank m and $\sigma \in Aut(P \oplus R^n)$ for some $n \geq 2$. Assume that $\sigma \oplus 1 \in E(P \oplus R^{n+1})$. Then σ is isotopic to $id_{P\oplus R^n}$.*

Proof. Since $\sigma \oplus 1 \in E(P \oplus R^{n+1})$, there is a natural isotopy $\tau(X) \in E((P \oplus R^{n+1}) \otimes_R R[X])$ with $\tau(0) = id_{P\oplus R^{n+1}}$ and $\tau(1) = \sigma \oplus 1$. Now apply the previous lemma to $R[X]$, $X^2 - X$ and $a = \pi\tau(X)$ in order to obtain an automorphism $\chi(X) \in SL((P \oplus R^{n+1}) \otimes_R R[X])$ with $\pi\chi(X) = a$ such that $\chi(X)(x) \equiv x$ modulo $\langle X^2 - X \rangle$. Thus, $\pi\tau(X)\chi(X)^{-1} = \pi$. Therefore $\tau(X)\chi(X)^{-1}$ equals $\rho(X) \oplus 1$ for some $\rho(X) \in SL((P \oplus R^n) \otimes_R R[X])$ up to elementary automorphisms. But then $\rho(X)$ is an isotopy from $id_{P\oplus R^n}$ to σ. □

We can now use Theorem 1.20 in order to deduce the following stability results:

Theorem 1.21. *With the notation of Section 1.2, we assume that P_0 has rank 2. If R is a regular Noetherian ring of dimension 2, then there is an equality $E_\infty(P_0) \cap Aut(P_4) = E(P_4)$.*

Proof. If $\sigma \in SL(P_4)$ is stably elementary, then $\sigma \in E(P_{n+1})$ for some $n \geq 4$. We can now apply Theorem 1.20 to $P = P_0$ and deduce that there is an isotopy $\rho(X) \in SL(P_n[X])$ from id_{P_n} to σ. But since R is regular, we know that $\rho_{\mathfrak{m}}(X)$ is stably elementary (for any maximal ideal $\mathfrak{m}$ of R); in fact, we can deduce that $\rho_{\mathfrak{m}}(X)$ is elementary because $\dim(R) = 2$. Therefore Theorem 1.18 implies that $\rho(X) \in E(P_n[X])$ and consequently $\sigma = \rho(1) \in E(P_n)$. The theorem now follows by inductively repeating this argument and deducing that $\sigma \in E(P_4)$. □

Theorem 1.22. *With the notation of Section 1.2, we further assume that P_0 has rank 2. Let k be a field with $\mathcal{P}(4,3)$. If R is a regular affine k-algebra of dimension 3, then $E_{\infty}(P_0) \cap Aut(P_4) = E(P_4)$.*

Proof. We know that there is an equality $SL_N(R_{\mathfrak{p}}[X]) = E_N(R_{\mathfrak{p}}[X])$ for any prime $\mathfrak{p}$ of R and $N \geq 4$ by a famous theorem of Vorst (cp. [V]). We can thus argue as in the proof of Theorem 1.21. □

Motivic Homotopy Theory

In this chapter, we give a brief introduction to motivic homotopy theory as developed by Fabien Morel and Vladimir Voevodsky in [MV]. At first, we outline the construction of the unstable $\mathbb{A}^1$-homotopy category over a regular Noetherian base scheme S of finite Krull dimension and of its pointed version. In the subsequent section, we study the endomorphisms of $\mathbb{P}^1_S$ in the pointed $\mathbb{A}^1$-homotopy category over the spectrum $S = Spec(R)$ of a smooth affine algebra R over a perfect field of characteristic $\neq 2$. In fact, we will extend some computations which are known over a perfect field k with $char(k) \neq 2$ as a base scheme to the case of a smooth affine algebra over k. Finally, we briefly discuss $\mathbb{A}^1$-fiber sequences and Suslin matrices in the last section of this chapter.

2.1 The unstable $\mathbb{A}^1$-homotopy category

Let S be a regular Noetherian base scheme of finite Krull dimension and let Sm_S be the category of smooth separated schemes of finite type over S. Furthermore, we denote by $Spc_S = \Delta^{op}Shv_{Nis}(Sm_S)$ (resp. $Spc_{S,\bullet}$) the category of (pointed) simplicial Nisnevich sheaves of sets over Sm_S. We refer to objects of Spc_S ($Spc_{S,\bullet}$) as (pointed) spaces. Note that any (pointed) simplicial set or any (pointed) Nisnevich sheaf of sets can be considered (pointed) spaces. In particular, any (pointed) scheme $X \in Sm_S$ defines an object of Spc_S (resp. $Spc_{S,\bullet}$).

One can define a model structure on Spc_S as follows: A (simplicial) cofibration in Spc_S is just defined to be a monomorphism of simplicial sheaves. A morphism $f : \mathcal{X} \to \mathcal{Y}$ is called a simplicial weak equivalence if $f(x) : \mathcal{X}(x) \to \mathcal{Y}(x)$ is a weak equivalence of simplicial sets for any point x in the Nisnevich site on Sm_S. The (simplicial) fibrations are then defined to be the morphisms with the right lifting property with respect to trivial cofibrations, i.e. (simplicial) cofibrations which are also simplicial weak equivalences; this means that a morphism $p : \mathcal{E} \to \mathcal{B}$ is a simplicial fibration if, for any diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "i"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{X} \arrow[r] \arrow[ur, dotted, "f"] & \mathcal{B} \end{tikzcd}$$

with $i : \mathcal{A} \to \mathcal{X}$ a trivial cofibration, there is a morphism $f : \mathcal{X} \to \mathcal{E}$ making the diagram commutative.

We denote by C_s the class of (simplicial) cofibrations, by W_s the class of simplicial weak equivalences and by F_s the class of (simplicial) fibrations defined as above. The triple (C_s, W_s, F_s) determines a model structure on Spc_S called the simplicial model structure or the local injective model structure. One also obtains a simplicial model structure on $Spc_{S,\bullet}$ by defining a morphism $f : (\mathcal{X}, x) \to (\mathcal{Y}, y)$ of pointed simplicial Nisnevich sheaves to be a pointed (simplicial) cofibration, weak equivalence or fibration if it is a cofibration, weak equivalence or fibration of simplicial Nisnevich sheaves with respect to the simplicial model structure on Spc_S just described. We write $\mathcal{H}_s(S)$ (resp. $\mathcal{H}_{s,\bullet}(S)$) for the (pointed) simplicial homotopy category, which can be obtained from Spc_S (resp. $Spc_{S,\bullet}$) by inverting (pointed) simplicial weak equivalences.

The $\mathbb{A}^1_S$-model structure can be obtained as a Bousfield localization of the simplicial model structure described above: A space $\mathcal{Z} \in Spc_S$ is called $\mathbb{A}^1_S$-local if the map

$$Hom_{H_s(S)}(\mathcal{X}, \mathcal{Z}) \to Hom_{H_s(S)}(\mathcal{X} \times \mathbb{A}^1_S, \mathcal{Z})$$

induced by the projection $\mathcal{X} \times \mathbb{A}^1_S \to \mathcal{X}$ is a bijection for any $\mathcal{X} \in Spc_S$. Furthermore, a morphism $f : \mathcal{X} \to \mathcal{Y}$ of simplicial Nisnevich sheaves is called an $\mathbb{A}^1_S$-weak equivalence if the map

$$f^* : Hom_{H_s(S)}(\mathcal{Y}, \mathcal{Z}) \to Hom_{H_s(S)}(\mathcal{X}, \mathcal{Z})$$

is a bijection for any $\mathbb{A}^1_S$-local space $\mathcal{Z}$. We denote by $W_{\mathbb{A}^1_S}$ the class of $\mathbb{A}^1_S$-weak equivalences. Note that all simplicial weak equivalences and all the projections $\mathcal{X} \times \mathbb{A}^1_S \to \mathcal{X}$ are automatically $\mathbb{A}^1_S$-weak equivalences.

We define the class $C_{\mathbb{A}^1_S}$ of ($\mathbb{A}^1_S$-)cofibrations again as the class of monomorphisms, i.e. $C_{\mathbb{A}^1_S} = C_s$. Moreover, we define the class $F_{\mathbb{A}^1_S}$ of $\mathbb{A}^1_S$-fibrations to be class of morphisms with the right lifting property with respect to trivial $\mathbb{A}^1_S$-cofibrations, i.e. monomorphisms which are also $\mathbb{A}^1_S$-weak equivalences. Then the triple $(C_{\mathbb{A}^1_S}, W_{\mathbb{A}^1_S}, F_{\mathbb{A}^1_S})$ determines the $\mathbb{A}^1_S$-model structure on Spc_S. Again, one obtains an $\mathbb{A}^1_S$-model structure on $Spc_{S,\bullet}$ by defining a morphism $f : (\mathcal{X}, x) \to (\mathcal{Y}, y)$ of pointed simplicial Nisnevich sheaves to be a pointed

$\mathbb{A}^1_S$-cofibration, $\mathbb{A}^1_S$-weak equivalence or $\mathbb{A}^1_S$-fibration if it is a cofibration, weak equivalence or fibration of simplicial Nisnevich sheaves with respect to the $\mathbb{A}^1_S$-model structure on Spc_S just described.

The (pointed) unstable $\mathbb{A}^1_S$-homotopy category $\mathcal{H}(S)$ (resp. $\mathcal{H}_\bullet(S)$) is the homotopy category associated to the (pointed) $\mathbb{A}^1_S$-model structure and is obtained from Spc_S (resp. $Spc_{S,\bullet}$) by inverting (pointed) $\mathbb{A}^1_S$-weak equivalences. In case of an affine base scheme $S = Spec(R)$, we simply write Spc_R, $Spc_{R,\bullet}$, $\mathcal{H}(R)$ or $\mathcal{H}_\bullet(R)$ for the respective categories. Objects of $\mathcal{H}(S)$ (resp. $\mathcal{H}_\bullet(S)$) will be referred to as (pointed) spaces. For two spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $[\mathcal{X},\mathcal{Y}]_{\mathbb{A}^1_S} = Hom_{\mathcal{H}(S)}(\mathcal{X},\mathcal{Y})$ the set of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ in $\mathcal{H}(S)$; similarly, we denote by $[(\mathcal{X},x),(\mathcal{Y},y)]_{\mathbb{A}^1_S,\bullet} = Hom_{\mathcal{H}_\bullet(S)}((\mathcal{X},x),(\mathcal{Y},y))$ the set of morphisms from a pointed space $(\mathcal{X},x)$ to another pointed space $(\mathcal{Y},y)$ in $\mathcal{H}_\bullet(S)$. Sometimes we will omit the basepoints from the notation or write R instead of $Spec(R)$ if $S = Spec(R)$ is an affine scheme.

If $\mathcal{X},\mathcal{Y} \in Spc_S$, we say that two morphisms $f,g : \mathcal{X} \to \mathcal{Y}$ are naively $\mathbb{A}^1_S$-homotopic if there is a morphism $H : \mathcal{X} \times \mathbb{A}^1_S \to \mathcal{Y}$ such that $H(-,0) = f$ and $H(-,1) = g$. We denote by $[\mathcal{X},\mathcal{Y}]_N$ the set of equivalence classes of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ under the equivalence relation generated by the relation of naive $\mathbb{A}^1_S$-homotopies. A (pointed) space $\mathcal{Y}$ is called $\mathbb{A}^1_S$-fibrant if the unique morphism $\mathcal{Y} \to * = S$ is an $\mathbb{A}^1_S$-fibration; in fact, for any (pointed) space $\mathcal{Y}$, there is a (pointed) $\mathbb{A}^1_S$-fibrant space $\mathcal{Y}'$ together with a (pointed) $\mathbb{A}^1_S$-weak equivalence $\mathcal{Y} \to \mathcal{Y}'$. If $\mathcal{Y}$ is an $\mathbb{A}^1_S$-fibrant space and $\mathcal{X}$ is any space, then the relation of naive $\mathbb{A}^1_S$-homotopies on the set of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ is an equivalence relation and the natural map $[\mathcal{X},\mathcal{Y}]_N \to [\mathcal{X},\mathcal{Y}]_{\mathbb{A}^1_S}$ is a bijection.

For any space $\mathcal{X}$, the product functor $\mathcal{X} \times - : Spc_S \to Spc_S$ admits a right adjoint $\underline{Hom}(\mathcal{X},-) : Spc_S \to Spc_S$; the adjoint pair forms a Quillen pair and therefore induces an adjunction

$$\mathcal{X} \times - : \mathcal{H}(S) \rightleftarrows \mathcal{H}(S) : \mathcal{R}\underline{Hom}(\mathcal{X},-)$$

on the $\mathbb{A}^1_S$-homotopy category; here, $\mathcal{R}\underline{Hom}(\mathcal{X},-)$ denotes a right derived functor of $\underline{Hom}(\mathcal{X},-)$.

Like in classical topology, one can define a wedge product $(\mathcal{X},x) \vee (\mathcal{Y},y)$ and a smash product $(\mathcal{X},x) \wedge (\mathcal{Y},y)$ of two pointed spaces $(\mathcal{X},x)$ and $(\mathcal{Y},y)$: The wedge product is defined by the pushout square

$$\begin{tikzcd} * \arrow[r, "y"] \arrow[d, "x"'] & (\mathcal{Y}, y) \arrow[d] \\ (\mathcal{X}, x) \arrow[r] & (\mathcal{X}, x) \vee (\mathcal{Y}, y) \end{tikzcd}$$

and the smash product by the pushout square

$$\begin{tikzcd} (\mathcal{X}, x) \vee (\mathcal{Y}, y) \arrow[r] \arrow[d, "(id \times y) \vee (x \times id)"] & * \arrow[d] \\ (\mathcal{X} \times \mathcal{Y}, x \times y) \arrow[r] & (\mathcal{X}, x) \wedge (\mathcal{Y}, y). \end{tikzcd}$$

For any pointed space $(\mathcal{X}, x)$, the functor $(\mathcal{X}, x) \wedge - : Spc_{S,\bullet} \to Spc_{S,\bullet}$ admits a right adjoint $\underline{Hom}_{\bullet}((\mathcal{X}, x), -) : Spc_{S,\bullet} \to Spc_{S,\bullet}$; the adjoint pair forms a Quillen pair and hence descends to an adjunction

$$(\mathcal{X}, x) \wedge - : \mathcal{H}_{\bullet}(S) \rightleftarrows \mathcal{H}_{\bullet}(S) : \mathcal{R}\underline{Hom}_{\bullet}((\mathcal{X}, x), -)$$

on the level of the pointed $\mathbb{A}^1_S$-homotopy category; here, $\mathcal{R}\underline{Hom}_{\bullet}((\mathcal{X}, x), -)$ is a right derived functor of $\underline{Hom}_{\bullet}((\mathcal{X}, x), -)$. As a particularly interesting special case, one obtains the functor $\Sigma_s = S^1 \wedge - : Spc_{S,\bullet} \to Spc_{S,\bullet}$, which is called the simplicial suspension functor; its right adjoint $\Omega_s = \underline{Hom}_{\bullet}(S^1, -) : Spc_{S,\bullet} \to Spc_{S,\bullet}$ is called the simplicial loop space functor. We denote by Σ_s^n and Ω_s^n the iterated suspension and loop space functors for any $n \in \mathbb{N}$. For any pointed space $(\mathcal{X}, x)$, its simplicial suspension $\Sigma_s(\mathcal{X}, x) = S^1 \wedge (\mathcal{X}, x)$ has the structure of an h-cogroup in $\mathcal{H}_{\bullet}(S)$ (cp. [A, Definition 2.2.7] or [Ho, Section 6.1]); in particular, for any pointed space $(\mathcal{Y}, y)$, there is a natural group structure on the set $[\Sigma_s(\mathcal{X}, x), (\mathcal{Y}, y)]_{\mathbb{A}^1_S,\bullet}$ induced by the h-cogroup structure of $\Sigma_s(\mathcal{X}, x)$. For any pointed space $(\mathcal{Y}, y)$, the space $\mathcal{R}\Omega_s(\mathcal{Y}, y)$ has the structure of an h-group in $\mathcal{H}_{\bullet}(S)$ and hence the set $[(\mathcal{X}, x), \mathcal{R}\Omega_s(\mathcal{Y}, y)]_{\mathbb{A}^1_S,\bullet}$ has a natural group structure for any pointed space $(\mathcal{X}, x)$ induced by the h-group structure of $\mathcal{R}\Omega_s(\mathcal{Y}, y)$.
Furthermore, the functor $()_+ : Spc_S \to Spc_{S,\bullet}, \mathcal{X} \mapsto \mathcal{X}_+ = \mathcal{X} \sqcup *$ and the forgetful functor $\mathcal{F} : Spc_{S,\bullet} \to Spc_S, (\mathcal{X}, x) \mapsto \mathcal{X}$ form a Quillen pair and hence descend to an adjunction

$$()_+ : \mathcal{H}(S) \rightleftarrows \mathcal{H}_{\bullet}(S) : \mathcal{R}\mathcal{F},$$

where $\mathcal{R}\mathcal{F}$ is a right derived functor of $\mathcal{F}$. We will tacitly use this in some proofs in order to force some spaces to have a basepoint.

2.2 Endomorphisms of $\mathbb{P}^1$

For any base scheme S as above, we let $\mathbb{P}^1_S = \mathbb{P}^1 \times_{\mathbb{Z}} S$ and $\mathbb{G}_{m,S} = \mathbb{G}_m \times_{\mathbb{Z}} S$, where $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{Z}} = Proj(\mathbb{Z}[T_0, T_1])$ and $\mathbb{G}_m = Spec(\mathbb{Z}[T]_T)$. If $S = Spec(R)$ is an affine scheme, we simply write $\mathbb{P}^1_R$ and $\mathbb{G}_{m,R}$ instead of $\mathbb{P}^1_{Spec(R)}$ and $\mathbb{G}_{m,Spec(R)}$. The scheme $\mathbb{P}^1_S$ is canonically pointed by ∞, the scheme $\mathbb{G}_{m,S}$ by 1. It is well-known that there is a pointed $\mathbb{A}^1_S$-weak equivalence between $\mathbb{P}^1_S$ and $S^1 \wedge \mathbb{G}_{m,S}$. Via this identification of $\mathbb{P}^1_S$ and $S^1 \wedge \mathbb{G}_{m,S}$ in $\mathcal{H}_\bullet(S)$, the space $\mathbb{P}^1_S$ obtains the structure of an h-cogroup (cp. [A, Definition 2.2.7] or [Ho, Section 6.1]). In particular, for any pointed space $(\mathcal{X}, x)$, the set $[\mathbb{P}^1_S, (\mathcal{X}, x)]_{\mathbb{A}^1_S,\bullet}$ has a natural group structure.

Now let $S = Spec(k)$ be the spectrum of a perfect field k with $char(k) \neq 2$. The group $[\mathbb{P}^1_k, \mathbb{P}^1_k]_{\mathbb{A}^1_k,\bullet}$ has been computed in [C] as follows: We say that two pointed morphisms $f, g : \mathbb{P}^1_k \to \mathbb{P}^1_k$ are naively $\mathbb{A}^1_k$-homotopic if there is a morphism $H : \mathbb{P}^1_k \times_k \mathbb{A}^1_k \to \mathbb{P}^1_k$ with $H(-,0) = f$, $H(-,1) = g$ and such that $H(\infty, -) = \infty$. We then denote by $[\mathbb{P}^1_k, \mathbb{P}^1_k]_{N,\bullet}$ the set of equivalence classes of pointed morphisms under the equivalence relation generated by the relation of naive $\mathbb{A}^1_k$-homotopies.

As it was proven in [C, Theorem 3.24], the set $[\mathbb{P}^1_k, \mathbb{P}^1_k]_{N,\bullet}$ can be endowed with a structure of an abelian monoid such that the map $[\mathbb{P}^1_k, \mathbb{P}^1_k]_{N,\bullet} \to [\mathbb{P}^1_k, \mathbb{P}^1_k]_{\mathbb{A}^1_k,\bullet}$ is a group completion.

Any pointed morphism $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ has an associated non-degenerate symmetric bilinear form $Bez(f)$ called the Bézout form of f. We let $MW(k)$ be the Witt monoid of isomorphism classes of non-degenerate symmetric bilinear forms over k. The Grothendieck group of $MW(k)$ is the Grothendieck-Witt ring $GW(k)$ of non-degenerate symmetric bilinear forms over k. The discriminant induces a well-defined monoid homomorphism $MW(k) \to k^\times / k^{\times 2}$.

It is proven in [C, Corollary 3.11] that the assignment

$$(f : \mathbb{P}^1_k \to \mathbb{P}^1_k) \mapsto (Bez(f), \det(Bez(f)))$$

induces a monoid isomorphism

$$[\mathbb{P}^1_k, \mathbb{P}^1_k]_{N,\bullet} \xrightarrow{\cong} MW(k) \times_{k^\times / k^{\times 2}} k^\times,$$

where the right-hand term is the fiber product with respect to the discriminant map $MW(k) \to k^\times / k^{\times 2}$ and the projection $k^\times \to k^\times / k^{\times 2}$.

It follows that we have a group isomorphism

$$[\mathbb{P}^1_k, \mathbb{P}^1_k]_{\mathbb{A}^1_k,\bullet} \xrightarrow{\cong} GW(k) \times_{k^\times / k^{\times 2}} k^\times.$$

For any $n \in \mathbb{N}$, there is a natural pointed morphism of schemes $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ induced by the k-algebra homomorphism $k[T]_T \to k[T]_T, T \mapsto T^n$. Taking the smash product with S^1, we obtain a morphism $\psi_k^n : S^1 \wedge \mathbb{G}_{m,k} \to S^1 \wedge \mathbb{G}_{m,k}$, which corresponds up to canonical pointed $\mathbb{A}_k^1$-weak equivalence to a morphism $\mathbb{P}_k^1 \to \mathbb{P}_k^1$. The Bézout form $Bez(\psi_k^n)$ is given by the $n \times n$-matrix with only 1's on the anti-diagonal and 0's elsewhere. Its class in $GW(k)$ equals n_ϵ, which is given by the formula

$$n_\epsilon = \sum_{i=1}^n < (-1)^{(i-1)} > \in GW(k).$$

As $\det(Bez(\psi_k^n)) = (-1)^{n(n-1)/2}$, it follows that ψ_k^n corresponds to the pair $(n_\epsilon, (-1)^{n(n-1)/2})$ under the isomorphism $[\mathbb{P}_k^1, \mathbb{P}_k^1]_{\mathbb{A}_k^1,\bullet} \xrightarrow{\cong} GW(k) \times_{k^\times / k^{\times 2}} k^\times$. In particular, if $-1 \in k^{\times 2}$, i.e. if -1 is a square in k, and if furthermore $n \equiv 0, 1 \ mod\ 4$, then the morphism ψ_k^n corresponds to $n \cdot id_{S^1 \wedge \mathbb{G}_{m,k}}$ in $[S^1 \wedge \mathbb{G}_{m,k}, S^1 \wedge \mathbb{G}_{m,k}]_{\mathbb{A}_k^1,\bullet}$.
We now want to prove the latter computation for a more general base scheme. For this, we let k be a perfect base field with $char(k) \neq 2$ as in the computation above and we let $f : X = Spec(R) \to Spec(k)$ be a smooth affine scheme of finite type over k.
If we take X as a base scheme, we again consider the morphism $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ given by $k[T]_T \to k[T]_T, T \mapsto T^n$, for all $n \in \mathbb{N}$. Its pullback along the morphism $f : X \to Spec(k)$ gives a morphism $\mathbb{G}_{m,R} \to \mathbb{G}_{m,R}$. Taking the smash product with S^1, we obtain a morphism $\psi_R^n : S^1 \wedge \mathbb{G}_{m,R} \to S^1 \wedge \mathbb{G}_{m,R}$ in $\mathcal{H}_\bullet(R)$.

Lemma 2.1. *The morphism $f : X \to Spec(k)$ induces a well-defined group homomorphism $[S^1 \wedge \mathbb{G}_{m,k}, S^1 \wedge \mathbb{G}_{m,k}]_{\mathbb{A}_k^1,\bullet} \to [S^1 \wedge \mathbb{G}_{m,R}, S^1 \wedge \mathbb{G}_{m,R}]_{\mathbb{A}_R^1,\bullet}$.*

Proof. There is a restriction functor $f^* : Spc_{k,\bullet} \to Spc_{R,\bullet}$ induced by f. It follows from [MV, Proposition 3.2.8] that f^* commutes with the smash product of pointed spaces, sends $\mathbb{A}_k^1$-weak equivalences to $\mathbb{A}_R^1$-weak equivalences and hence descends to a functor $f^* : \mathcal{H}_\bullet(k) \to \mathcal{H}_\bullet(R)$. The functor f^* sends any smooth k-scheme U to its pullback $U \times_k R$ along f and similarly sends a morphism $g : U \to V$ between two k-schemes to its pullback $g \times_k R : U \times_k R \to V \times_k R$; furthermore, it fixes simplicial sets and morphisms between them. Hence we obtain a map $[S^1 \wedge \mathbb{G}_{m,k}, S^1 \wedge \mathbb{G}_{m,k}]_{\mathbb{A}_k^1,\bullet} \to [S^1 \wedge \mathbb{G}_{m,R}, S^1 \wedge \mathbb{G}_{m,R}]_{\mathbb{A}_R^1,\bullet}$. As the group structure of both sets is induced by the structure of S^1 as an h-cogroup, the map is clearly a group homomorphism. □

As an immediate consequence of the previous lemma, we obtain:

Corollary 2.2. *If $-1 \in k^{\times 2}$, $n \equiv 0, 1 \ mod\ 4$ and $\mathcal{X} \in Spc_{R,\bullet}$, then the class of $\psi_R^n \wedge \mathcal{X}$ in $[S^1 \wedge \mathbb{G}_{m,R} \wedge \mathcal{X}, S^1 \wedge \mathbb{G}_{m,R} \wedge \mathcal{X}]_{\mathbb{A}_R^1,\bullet}$ equals the class of $n \cdot id_{S^1 \wedge \mathbb{G}_{m,R} \wedge \mathcal{X}}$.*

2.3 $\mathbb{A}^1$-fiber sequences and Suslin matrices

In any pointed model category, i.e. in any model category whose initial and terminal objects are isomorphic, there exists the notion of fiber sequences $(\mathcal{F}, f) \hookrightarrow (\mathcal{E}, e) \to (\mathcal{B}, b)$ (cp. [Ho, Section 6.2]). Since $Spc_{S,\bullet}$ is a pointed model category with its $\mathbb{A}^1_S$-model structure, this notion in particular exists in motivic homotopy theory. Analogous to the situation in classical topology, such fiber sequences give rise to long exact sequences of the form

$$... \to [\mathcal{X}, \mathcal{R}\Omega_s(\mathcal{B}, b)]_{\mathbb{A}^1,\bullet} \to [\mathcal{X}, (\mathcal{F}, f)]_{\mathbb{A}^1,\bullet} \to [\mathcal{X}, (\mathcal{E}, e)]_{\mathbb{A}^1,\bullet} \to [\mathcal{X}, (\mathcal{B}, b)]_{\mathbb{A}^1,\bullet}$$

for any pointed space $\mathcal{X}$.

Now let us fix a perfect field k with $char(k) \neq 2$. For any pointed space $(\mathcal{X}, x) \in Spc_{k,\bullet}$ and $i \geq 0$, we define the ith $\mathbb{A}^1_k$-homotopy sheaf $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$ as the Nisnevich sheaf on Sm_k associated to the presheaf $U \mapsto [\Sigma_s^i U_+, (\mathcal{X}, x)]_{\mathbb{A}^1_k,\bullet}$. In general, the $\mathbb{A}^1_k$-homotopy sheaves $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$ are Nisnevich sheaves of sets on Sm_k for $i \geq 0$, Nisnevich sheaves of groups for $i \geq 1$ and Nisnevich sheaves of abelian groups for $i \geq 2$. Since sheafification is exact, any $\mathbb{A}^1_k$-fiber sequence $(\mathcal{F}, f) \hookrightarrow (\mathcal{E}, e) \to (\mathcal{B}, b)$ yields a long exact sequence

$$... \to \pi_{i+1}^{\mathbb{A}^1}(\mathcal{B}, b) \to \pi_i^{\mathbb{A}^1}(\mathcal{F}, f) \to \pi_i^{\mathbb{A}^1}(\mathcal{E}, e) \to \pi_i^{\mathbb{A}^1}(\mathcal{B}, b) \to ...$$

of $\mathbb{A}^1_k$-homotopy sheaves. For the purpose of this thesis, we simply state the existence of the following $\mathbb{A}^1_k$-fiber sequences, which follows from [W, Section 5] and [AHW2, Section 2]:

Theorem 2.3. *Let (X, x) be a pointed k-scheme. If $G = Sp_{2n}, SL_n, GL_n$ and $P \to X$ is a G-torsor, then there is an $\mathbb{A}^1_k$-fiber sequence of the form*

$$G \hookrightarrow P \to X.$$

As special cases of this theorem, we obtain $\mathbb{A}^1_k$-fiber sequences of the form

$$SL_n \hookrightarrow SL_{n+1} \to SL_{n+1}/SL_n,$$
$$Sp_{2n} \hookrightarrow SL_{2n} \to SL_{2n}/Sp_{2n},$$
$$Sp_{2n} \hookrightarrow GL_{2n} \to GL_{2n}/Sp_{2n}.$$

Let us describe the quotients SL_n/SL_{n-1}: For $n \geq 1$, the projection on the first column induces a morphism $SL_n/SL_{n-1} \to \mathbb{A}^n_k \setminus 0$, which is Zariski locally trivial with fibers isomorphic to $\mathbb{A}^{n-1}_k$ and hence an $\mathbb{A}^1_k$-weak equivalence.

For all $n \geq 1$, let $Q_{2n-1}^k = Spec(k[x_1,...,x_n,y_1,...,y_n]/\langle\sum_{i=1}^n x_iy_i - 1\rangle)$ the smooth affine quadric hypersurfaces in $\mathbb{A}_k^{2n}$. The projection on the coefficients $x_1,...,x_n$ induces a morphism $p_{2n-1}^k : Q_{2n-1}^k \to \mathbb{A}_k^n \setminus 0$, which is locally trivial with fibers isomorphic to $\mathbb{A}_k^{n-1}$ and hence an $\mathbb{A}_k^1$-weak equivalence. Thus, we have $\mathbb{A}_k^1$-weak equivalences

$$SL_n/SL_{n-1} \simeq_{\mathbb{A}_k^1} \mathbb{A}_k^n \setminus 0 \simeq_{\mathbb{A}_k^1} Q_{2n-1}^k$$

for all $n \geq 1$. Note that these $\mathbb{A}_k^1$-weak equivalences are all pointed if we equip SL_n/SL_{n-1} with the identity matrix, $\mathbb{A}_k^n \setminus 0$ with $(1,0,..,0)$ and Q_{2n-1}^k with $(1,0,..,0,1,0,..,0)$ as basepoints.

If R is a smooth affine k-algebra and $X = Spec(R)$, then it is well-known that

$$Um_n(R) \cong Hom_{Sm_k}(X, \mathbb{A}_k^n \setminus 0)$$

and

$$\{(a,b)|a,b \in Um_n(R), ab^t = 1\} = Hom_{Sm_k}(X, Q_{2n-1}^k).$$

If $n \geq 3$, it follows from [Mo, Remark 8.10] and [F, Theorem 2.1] that

$$Um_n(R)/E_n(R) \cong [X, \mathbb{A}_k^n \setminus 0]_{\mathbb{A}_k^1}.$$

In particular, if we let $S_{2m-1}^k = k[x_1,...,x_m,y_1,...,y_m]/\langle\sum_{i=1}^m x_iy_i - 1\rangle$ for $m \geq 1$, then the orbit space $Um_n(S_{2m-1}^k)/E_n(S_{2m-1}^k)$ is just given by

$$[Q_{2m-1}^k, \mathbb{A}_k^n \setminus 0]_{\mathbb{A}_k^1} \cong [\mathbb{A}_k^m \setminus 0, \mathbb{A}_k^n \setminus 0]_{\mathbb{A}_k^1}.$$

It is well-known that $\mathbb{A}_k^m \setminus 0$ is isomorphic to $\Sigma_s^{m-1}\mathbb{G}_{m,k}^{\wedge m}$ in $\mathcal{H}_\bullet(k)$ for all $m \geq 1$; therefore $\mathbb{A}_k^m \setminus 0$ inherits the structure of an h-cogroup in $\mathcal{H}_\bullet(k)$ for $m \geq 2$ (cp. [A, Definition 2.2.7] or [Ho, Section 6.1]). In particular, the orbit space $Um_n(S_{2m-1}^k)/E_n(S_{2m-1}^k)$ has a natural group structure for $m \geq 2$, $n \geq 3$.

Now let R be a commutative ring, $n \geq 1$ and $a = (a_1,...,a_n), b = (b_1,...,b_n)$ be row vectors of length n. In [S2], Suslin inductively constructed matrices $\alpha_n(a,b)$ of size 2^{n-1} called Suslin matrices as follows: For $n = 1$, one simply sets $\alpha_1(a,b) = (a_1)$; for $n \geq 2$, one sets $a' = (a_2,...,a_n), b' = (b_2,...,b_n)$ and defines

$$\alpha_n(a,b) = \begin{pmatrix} a_1 Id_{2^{n-2}} & \alpha_{n-1}(a',b') \\ -\alpha_{n-1}(b',a')^t & b_1 Id_{2^{n-2}} \end{pmatrix}.$$

In [S2, Lemma 5.1], Suslin proved that $\det(\alpha_n(a,b)) = (ab^t)^{2^{n-2}}$ if $n \geq 2$; in particular, if $a = (a_1, ..., a_n)$ is a unimodular row of length n and $b = (b_1, ..., b_n)$ defines a section of a, i.e. $ab^t = \sum_{i=1}^n a_i b_i = 1$, then $\alpha_n(a,b) \in SL_{2^{n-1}}(R)$.
Suslin originally introduced these matrices in order to show that if $a = (a_1, a_2, a_3, ..., a_n)$ is a unimodular row of length $n \geq 3$, then the row of the form $a' = (a_1, a_2, a_3, ..., a_n^{(n-1)!})$ is completable to an invertible matrix. In fact, he proved that, for any a with section b, there exists an invertible $n \times n$-matrix $\beta(a,b)$ whose first row is a' such that the classes of $\beta(a,b)$ and $\alpha_n(a,b)$ in $K_1(R)$ coincide (cp. [S4, Proposition 2.2 and Corollary 2.5]).
As explained in [AF4], one can in fact interpret Suslin's construction as a morphism of schemes: We let $Q_{2n-1}^k = Spec(k[x_1, ..., x_n, y_1, ..., y_n]/\langle \sum_{i=1}^n x_i y_i - 1 \rangle)$ as above. Then there exists a morphism $\alpha_n : Q_{2n-1}^k \to SL_{2^{n-1}}$ induced by $\alpha_n(x,y)$, where $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$; if we equip Q_{2n-1}^k with $(1,0,..,0,1,0,...,0)$ and $SL_{2^{n-1}}$ with the identity as basepoints, this morphism is pointed. Composing with the canonical map $SL_{2^{n-1}} \to SL$, we obtain a morphism $Q_{2n-1}^k \to SL$, which we also denote by α_n. If R is a smooth affine algebra over k and $n \geq 3$, then the induced morphism

$$Um_n(R)/E_n(R) \cong [Spec(R), Q_{2n-1}^k]_{\mathbb{A}_k^1} \xrightarrow{\alpha_{n*}} [Spec(R), SL]_{\mathbb{A}_k^1} \cong SK_1(R)$$

takes the class of any $a \in Um_n(R)$ to the class of $\alpha_n(a,b)$ in $SK_1(R)$, where b is any section of a.

Hermitian K-Theory

In this chapter, we give a brief introduction to Hermitian K-theory and we give several presentations of the higher Grothendieck-Witt group $GW_1^3(R)$ for any smooth affine algebra R over a perfect field of characteristic $\neq 2$. For any commutative ring R, we introduce the group $W_E'(R)$ and its subgroup $W_E(R)$ called the elementary symplectic Witt group in the second section of this chapter; moreover, we define the group $W_{SL}'(R)$, which is the cokernel of a hyperbolic map $SK_1(R) \to W_E'(R)$, and its subgroup $W_{SL}(R)$. In the following section, we define the groups $V(R)$ and $V_{SL}(R)$; in fact, we will see that there are canonical isomorphisms $W_E'(R) \cong V(R)$ and $W_{SL}'(R) \cong V_{SL}(R)$. Furthermore, we will see that there is a canonical isomorphism $GW_1^3(R) \cong W_E'(R)$ for any smooth affine algebra R over a perfect field k of characteristic $\neq 2$. Finally, we introduce Grothendieck-Witt sheaves and study their Nisnevich cohomology in order to give criteria for the 2-divisibility of $W_E(R)$ and the group $W_{SL}(R)$.

3.1 Grothendieck-Witt groups

In this section, we recall some basics about higher Grothendieck-Witt groups, which are a modern version of Hermitian K-theory. The general references of the modern theory are [MS1], [MS2] and [MS3]. Let X be a scheme with $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$ and let $\mathcal{L}$ be a line bundle on X. Then we consider the category $C^b(X)$ of bounded complexes of locally free coherent $\mathcal{O}_X$-modules. The category $C^b(X)$ inherits a natural structure of an exact category from the category of locally free coherent $\mathcal{O}_X$-modules by declaring $C_\bullet' \to C_\bullet \to C_\bullet''$ to be exact if and only if $C_n' \to C_n \to C_n''$ is exact for all n. The duality $Hom_{\mathcal{O}_X}(-, \mathcal{L})$ induces a duality $\#_{\mathcal{L}}$ on $C^b(X)$ and the isomorphism $id \to Hom_{\mathcal{O}_X}(Hom_{\mathcal{O}_X}(-, \mathcal{L}), \mathcal{L})$ for locally free coherent $\mathcal{O}_X$-modules induces a natural isomorphism of functors $\varpi_{\mathcal{L}} : id \xrightarrow{\sim} \#_{\mathcal{L}}\#_{\mathcal{L}}$ on $C^b(X)$. Moreover, the translation functor $T : C^b(X) \to C^b(X)$ yields new dualities $\#_{\mathcal{L}}^j = T^j \#_{\mathcal{L}}$ and natural isomorphisms $\varpi_{\mathcal{L}}^j = (-1)^{j(j+1)/2} \varpi_{\mathcal{L}}$. We say that a morphism in

$C^b(X)$ is a weak equivalence if and only if it is a quasi-isomorphism and we denote by qis the class of quasi-isomorphisms. For all j, the quadruple $(C^b(X), qis, \#^j_{\mathcal{L}}, \varpi^j_{\mathcal{L}})$ is an exact category with weak equivalences and strong duality (cp. [MS2, §2.3]).
Following [MS2], one can associate a Grothendieck-Witt space $\mathcal{GW}$ to any exact category with weak equivalences and strong duality. The (higher) Grothendieck-Witt groups are then defined to be its homotopy groups:

Definition 3.1. *For all j, let $\mathcal{GW}(C^b(X), qis, \#^j_{\mathcal{L}}, \varpi^j_{\mathcal{L}})$ denote the Grothendieck-Witt space associated to the quadruple $(C^b(X), qis, \#^j_{\mathcal{L}}, \varpi^j_{\mathcal{L}})$ as above. Then, for any $i \geq 0$, we define $GW^j_i(X, \mathcal{L}) = \pi_i \mathcal{GW}(C^b(X), qis, \#^j_{\mathcal{L}}, \varpi^j_{\mathcal{L}})$. If $\mathcal{L} = \mathcal{O}_X$, we also denote $GW^j_i(X, \mathcal{O}_X)$ by $GW^j_i(X)$. Furthermore, if $X = Spec(R)$, we simply denote $GW^j_i(X, \mathcal{L})$ or $GW^j_i(X)$ by $GW^j_i(R, \mathcal{L})$ or $GW^j_i(R)$ respectively.*

The groups $GW^j_i(X, \mathcal{L})$ are 4-periodic in j. If we let $X = Spec(R)$ be an affine scheme, the groups $GW^j_i(X)$ coincide with Hermitian K-theory and U-theory as defined by Karoubi (cp. [MK1] and [MK2]) because $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$ by our assumption (cp. [MS1, Remark 4.13] and [MS3, Theorems 6.1-2]). In particular, there are isomorphisms $K_iO(R) \cong GW^0_i(R)$, ${}_{-1}U_i(R) \cong GW^1_i(R)$, $K_iSp(R) \cong GW^2_i(R)$ and $U_i(R) \cong GW^3_i(R)$.
The Grothendieck-Witt groups defined as above carry a multiplicative structure. Indeed, the tensor product of complexes induces product maps

$$GW^j_i(X, \mathcal{L}_1) \times GW^s_r(X, \mathcal{L}_2) \to GW^{j+s}_{i+r}(X, \mathcal{L}_1 \otimes \mathcal{L}_2)$$

for all i, j, r, s and line bundles $\mathcal{L}_1, \mathcal{L}_2$ on a scheme X with $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$ (cp. [MS3, §9.2]). For all $i, j \geq 0$, there exist forgetful homomorphisms $f_{i,j} : GW^j_i(X, \mathcal{L}) \to K_i(X)$, hyperbolic homomorphisms $H_{i,j} : K_i(X) \to GW^j_i(X, \mathcal{L})$ and also boundary homomorphisms $\eta : GW^{j+1}_{i+1}(X, \mathcal{L}) \to GW^j_i(X, \mathcal{L})$, which are connected by means of the exact sequence called Karoubi periodicity sequence of the form

$$K_{i+1}(X) \xrightarrow{H_{i+1,j+1}} GW^{j+1}_{i+1}(X, \mathcal{L}) \xrightarrow{\eta} GW^j_i(X, \mathcal{L}) \xrightarrow{f_{i,j}} K_i(X) \xrightarrow{H_{i,j+1}} GW^{j+1}_i(X, \mathcal{L}).$$

In this thesis, the group of our interest is $GW^3_1(R) = U_1(R)$ for a smooth affine algebra R over a perfect field k with $char(k) \neq 2$. As a matter of fact, it is argued in [FRS] and [AF4] that there is a natural isomorphism between $GW^3_1(R)$ and the group $W'_E(R)$, the latter of which will be introduced in the next section of this chapter.
Now let S be a regular Noetherian affine scheme of finite Krull dimension with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. Then it is known (cp. [JH, Theorem 3.1]) that higher Grothendieck-Witt groups of smooth

separated schemes of finite type over S are representable in the pointed $\mathbb{A}^1_S$-homotopy category $\mathcal{H}_\bullet(S)$ as defined by Morel and Voevodsky. More precisely, if we let X be a smooth separated scheme of finite type over S, it is shown that there are pointed spaces $\mathcal{GW}^j$ and natural isomorphisms

$$[\Sigma^i_s X_+, \mathcal{GW}^j]_{\mathbb{A}^1_S,\bullet} \cong GW^j_i(X).$$

In particular, we have identifications

$$[X, \mathcal{R}\Omega^i_s \mathcal{GW}^j]_{\mathbb{A}^1_S} \cong GW^j_i(X).$$

It follows also from this that any morphism $f : Y \to X$ of smooth separated schemes of finite type over S induces a pullback morphism $f^* : GW^j_i(X) \to GW^j_i(Y)$.
Following [ST], we are now going to make these spaces more explicit: For $n \in \mathbb{N}$, we let GL_n, O_{2n}, Sp_{2n} be the schemes (defined over S) of invertible $n \times n$-matrices, of orthogonal $2n \times 2n$-matrices and of symplectic $2n \times 2n$-matrices. Then we consider for all $n \in \mathbb{N}$ the closed embeddings $GL_n \to O_{2n}$ and $GL_n \to Sp_{2n}$ induced by

$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & (M^{-1})^t \end{pmatrix}.$$

For any $n \in \mathbb{N}$, these embeddings are compatible with the standard stabilization embeddings $GL_n \to GL_{n+1}$, $O_{2n} \to O_{2n+2}$ and $Sp_{2n} \to Sp_{2n+2}$. Taking direct limits over all n with respect to the induced maps $O_{2n}/GL_n \to O_{2n+2}/GL_{n+1}$ and $Sp_{2n}/GL_n \to Sp_{2n+2}/GL_{n+1}$, we obtain spaces O/GL and Sp/GL. Similarly, the natural inclusions $Sp_{2n} \to GL_{2n}$ are compatible with the standard stabilization embeddings and we analogously obtain a space $GL/Sp = colim_n\, GL_{2n}/Sp_{2n}$. As proven in [ST, Theorems 8.2 and 8.4], there are canonical pointed $\mathbb{A}^1_S$-weak equivalences

$$\mathcal{GW}^j \simeq_{\mathbb{A}^1_S} \begin{cases} \mathbb{Z} \times OGr & \text{if } j \equiv 0 \bmod 4 \\ Sp/GL & \text{if } j \equiv 1 \bmod 4 \\ \mathbb{Z} \times HGr & \text{if } j \equiv 2 \bmod 4 \\ O/GL & \text{if } j \equiv 3 \bmod 4 \end{cases}$$

and

$$\mathcal{R}\Omega_s^1 O/GL \simeq_{\mathbb{A}^1_S} GL/Sp,$$

where OGr is an "infinite orthogonal Grassmannian" and HGr is an "infinite symplectic Grassmannian". As a consequence of all the previous paragraphs, there is a natural isomorphism $[X, GL/Sp]_{\mathbb{A}^1_S} \cong GW_1^3(X)$.
To conclude this section, we let the base scheme $S = Spec(k)$ be the spectrum of a perfect field k with $char(k) \neq 2$ and describe two actions of $R^\times$ on $GW_1^3(R)$ for any smooth affine k-algebra R. We consider the product map

$$GW_0^0(R) \times GW_1^3(R) \to GW_1^3(R)$$

for a smooth affine algebra R over k induced by the multiplicative structure on the higher Grothendieck-Witt groups mentioned above. As described above, there is a canonical isomorphism $GW_0^0(R) \cong K_0O(R)$ and the latter group can be identified with the Grothendieck-Witt ring $GW(R)$ of non-degenerate symmetric bilinear forms, i.e. the Grothendieck completion of the abelian monoid of non-degenerate symmetric bilinear forms over R. Furthermore, there is a canonical map

$$R^\times \to GW(R),\ u \mapsto (R \times R \to R, (x,y) \mapsto uxy),$$

which induces an action of $R^\times = \mathbb{G}_{m,k}(R)$ on $GW_1^3(R)$ via the product map mentioned above. Following [AF3, Section 3.5], we refer to this action as the multiplicative action of $R^\times = \mathbb{G}_{m,k}(R)$ on $GW_1^3(R)$.
We now describe an action of $\mathbb{G}_{m,k}$ on GL/Sp. For any smooth affine k-algebra R and any unit $u \in R^\times$, we denote by $\gamma_{2n,u}$ the invertible $2n \times 2n$-matrix inductively defined by

$$\gamma_{2,u} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

and $\gamma_{2n+2,u} = \gamma_{2n,u} \perp \gamma_{2,u}$. Conjugation by $\gamma_{2n,u}^{-1}$ induces an action of $\mathbb{G}_{m,k}$ on GL_{2n} for all n. As Sp_{2n} is preserved by this action, there is an induced action on GL_{2n}/Sp_{2n}. Since all the morphisms $GL_{2n}/Sp_{2n} \to GL_{2n+2}/Sp_{2n+2}$ are equivariant for this action, we obtain an action of $\mathbb{G}_{m,k}$ on GL/Sp. In particular, there is an induced action of $R^\times = \mathbb{G}_{m,k}(R)$ on $GW_1^3(R) \cong [Spec(R), GL/Sp]_{\mathbb{A}^1_k}$ for any smooth affine k-algebra R by taking the $\mathbb{A}^1_k$-homotopy classes of morphisms. Again following [AF3, Section 3.5], we refer to this action as the conjugation action of $R^\times$ on $GW_1^3(R)$. It follows from the proof of [AF3, Proposition 3.5.1] that the conjugation action coincides with the multiplicative action.

3.2 The elementary symplectic Witt group

Let R be a commutative ring and let G be any group such that $E(R) \subset G \subset SL(R)$. For any $n \in \mathbb{N}$, we denote by $A_{2n}(R)$ the set of alternating invertible matrices of rank $2n$. We inductively define an element $\psi_{2n} \in A_{2n}(R)$ by setting

$$\psi_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $\psi_{2n+2} = \psi_{2n} \perp \psi_2$. For any $m < n$, there is an embedding of $A_{2m}(R)$ into $A_{2n}(R)$ given by $M \mapsto M \perp \psi_{2n-2m}$. We denote by $A(R)$ the direct limit of the sets $A_{2n}(R)$ under these embeddings. Two alternating invertible matrices $M \in A_{2m}(R)$ and $N \in A_{2n}(R)$ are called G-equivalent, $M \sim_G N$, if there is an integer $s \in \mathbb{N}$ and a matrix $E \in SL_{2n+2m+2s}(R) \cap G$ such that

$$M \perp \psi_{2n+2s} = E^t (N \perp \psi_{2m+2s}) E.$$

This defines an equivalence relation on $A(R)$ and the set of equivalence classes $A(R)/{\sim_G}$ is denoted $W'_G(R)$. Since

$$\begin{pmatrix} 0 & id_s \\ id_r & 0 \end{pmatrix} \in E_{r+s}(R)$$

for even rs, it follows that the orthogonal sum equips $W'_G(R)$ with the structure of an abelian monoid. As it is shown in [SV], this abelian monoid is actually an abelian group. An inverse for an element of $W'_G(R)$ represented by a matrix $N \in A_{2n}(R)$ is given by the element represented by the matrix $\sigma_{2n} N^{-1} \sigma_{2n}$, where the matrices σ_{2n} are inductively defined by

$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\sigma_{2n+2} = \sigma_{2n} \perp \sigma_2$. In particular, for $G = E(R)$ or $SL(R)$, we obtain abelian groups $W'_E(R)$ and $W'_{SL}(R)$.

Now recall that one can assign to any alternating invertible matrix M an element $Pf(M)$ of $R^\times$ called the Pfaffian of M. The Pfaffian satisfies the following formulas:

- $Pf(M \perp N) = Pf(M)Pf(N)$ for all $M \in A_{2m}(R)$ and $N \in A_{2n}(R)$;

- $Pf(G^tNG) = \det(G)Pf(N)$ for all $G \in GL_{2n}(R)$ and $N \in A_{2n}(R)$;
- $Pf(N)^2 = \det(N)$ for all $N \in A_{2n}(R)$;
- $Pf(\psi_{2n}) = 1$ for all $n \in \mathbb{N}$.

Therefore the Pfaffian determines a group homomorphism $Pf : W'_G(R) \to R^\times$; its kernel is denoted $W_G(R)$. If $G = E(R)$, the group $W_G(R)$ is simply denoted $W_E(R)$ and is called the elementary symplectic Witt group of R. Furthermore, if $G = SL(R)$, we denote $W_G(R)$ simply by $W_{SL}(R)$.
As mentioned in the previous section, it is argued in [FRS] and [AF4] that there is a natural isomorphism between $GW_1^3(R)$ and the group $W'_E(R)$ for any smooth affine algebra R over a perfect field of characteristic $\neq 2$. One of the main tools to compute the group $GW_1^3(R)$ is the Karoubi periodicity sequence also mentioned in the previous section. By means of the identification $GW_1^3(X) \cong W'_E(R)$, this yields an exact sequence of the form

$$K_1Sp(R) \xrightarrow{f_{1,2}} K_1(R) \xrightarrow{H_{1,3}} W'_E(R) \xrightarrow{\eta} K_0Sp(R) \xrightarrow{f_{0,2}} K_0(R).$$

The homomorphisms in this sequence can be explicitly described as follows: The forgetful homomorphisms $K_1Sp(R) \xrightarrow{f_{1,2}} K_1(R)$ and $K_0Sp(R) \xrightarrow{f_{0,2}} K_0(R)$ are induced by the obvious inclusions $Sp_{2n}(R) \to GL_{2n}(R)$ and the assignment $(P, \varphi) \mapsto P$ for any skew-symmetric space (P, φ), i.e. for any finitely generated projective R-module P with a non-degenerate skew-symmetric form φ on P, respectively. Moreover, the hyperbolic map $K_1(R) \xrightarrow{H_{1,3}} W'_E(R)$ is induced by the assignment $M \mapsto M^t\psi_{2n}M$ for all $M \in GL_{2n}(R)$. Finally, the boundary homomorphism $W'_E(R) \xrightarrow{\eta} K_0Sp(R)$ is induced by the assignment $M \mapsto [R^{2n}, M] - [R^{2n}, \psi_{2n}]$ for all $M \in A_{2n}(R)$.
As the image of $K_1Sp(R)$ under $f_{1,2}$ in $K_1(R)$ lies in $SK_1(R)$, one can rewrite the sequence above as

$$K_1Sp(R) \xrightarrow{f_{1,2}} SK_1(R) \xrightarrow{H_{1,3}} W_E(R) \xrightarrow{\eta} K_0Sp(R) \xrightarrow{f_{0,2}} K_0(R).$$

We are now going to explain the identification $GW_1^3(R) \cong W'_E(R)$ in terms of the representability results for higher Grothendieck-Witt groups in motivic homotopy theory. For this, we fix a base scheme $S = Spec(R)$, where R is a smooth affine algebra over any perfect field k with $char(k) \neq 2$. As explained in the previous section, there is an $\mathbb{A}^1_R$-weak equivalence $\mathcal{R}\Omega^1_s\mathcal{GW}^3 \simeq_{\mathbb{A}^1_R} GL/Sp$, where GL and Sp denote the infinite linear and symplectic groups (over R) respectively. We let A_{2n} denote the scheme (over R) of skew-symmetric

invertible $2n \times 2n$-matrices. For any $n \in \mathbb{N}$, one can define a morphism $GL_{2n}/Sp_{2n} \to A_{2n}$ by $M \mapsto M^t \psi_{2n} M$. By the same reasoning as in [AF4, Section 2.3.2], these morphisms are isomorphisms and hence induce an isomorphism between GL/Sp and $A = colim_n A_{2n}$ (transition maps are defined by adding ψ_2). If B is a smooth affine R-algebra and $Y = Spec(B)$, the obvious map $A(B) \to [Y, A]_{\mathbb{A}^1_R}$ induces the identification $W'_E(B) \cong GW^3_1(B)$. Analogously, there is an isomorphism between SL/Sp and $A' = colim_n A'_{2n}$, where A'_{2n} is the scheme (over R) of skew-symmetric invertible $2n \times 2n$-matrices of Pfaffian 1. Again, if B is a smooth affine R-algebra and $Y = Spec(B)$, there is an analogous indentification $W_E(B) \cong [Y, A']_{\mathbb{A}^1_R}$.
In fact, if $R = k$, the $\mathbb{A}^1_k$-fiber sequences

$$Sp \to GL \to GL/Sp$$
$$Sp \to SL \to SL/Sp$$

induce the homomorphisms

$$K_1Sp(R) \xrightarrow{f} K_1(R) \xrightarrow{H} W'_E(R)$$
$$K_1Sp(R) \xrightarrow{f} SK_1(R) \xrightarrow{H} W_E(R)$$

in the Karoubi periodicity sequence above.

3.3 The groups $V_G(R)$

Again, let R be a commutative ring. Consider the set of triples (P, g, f), where P is a finitely generated projective R-module and f, g are alternating isomorphisms on P. Two such triples (P, f_0, f_1) and (P', f'_0, f'_1) are called isometric if there exists an isomorphism $h : P \to P'$ such that $f_i = h^\vee f'_i h$ for $i = 0, 1$. We denote by $[P, g, f]$ the isometry class of the triple (P, g, f).
Let $V(R)$ be the quotient of the free abelian group on isometry classes of triples as above modulo the subgroup generated by the relations

- $[P \oplus P', g \perp g', f \perp f'] = [P, g, f] + [P', g', f']$ for alternating isomorphisms f, g on P and f', g' on P';
- $[P, f_0, f_1] + [P, f_1, f_2] = [P, f_0, f_2]$ for alternating isomorphisms f_0, f_1, f_2 on P.

Note that these relations yield the useful identities

- $[P, f, f] = 0$ in $V(R)$ for any alternating isomorphism f on P;
- $[P, g, f] = -[P, f, g]$ in $V(R)$ for alternating isomorphisms f, g on P;
- $[P, g, \beta^\vee \alpha^\vee f \alpha \beta] = [P, f, \alpha^\vee f \alpha] + [P, g, \beta^\vee f \beta]$ in $V(R)$ for all automorphisms α, β of P and alternating isomorphisms f, g on P.

We may also restrict this construction to free R-modules of finite rank. The corresponding group will be denoted $V_{free}(R)$. Note that there is an obvious group homomorphism $V_{free}(R) \to V(R)$. This homomorphism can be seen to be an isomorphism as follows:
For any finitely generated projective R-module P, we call

$$H(P) = \begin{pmatrix} 0 & id_{P^\vee} \\ -can & 0 \end{pmatrix} : P \oplus P^\vee \to P^\vee \oplus P^{\vee\vee}$$

the hyperbolic isomorphism on P.
Now let (P, g, f) be a triple as above. Since P is a finitely generated projective R-module, there is another R-module Q such that $P \oplus Q \cong R^n$ for some $n \in \mathbb{N}$. In particular, it follows that $P \oplus P^\vee \oplus Q \oplus Q^\vee$ is free of rank $2n$. Therefore the triple

$$(P \oplus P^\vee \oplus Q \oplus Q^\vee, g \perp can\ g^{-1} \perp H(Q), f \perp can\ g^{-1} \perp H(Q))$$

represents an element of $V_{free}(R)$; this element is independent of the choice of Q. It follows that the assignment

$$(P, g, f) \mapsto (P \oplus P^\vee \oplus Q \oplus Q^\vee, g \perp can\ g^{-1} \perp H(Q), f \perp can\ g^{-1} \perp H(Q))$$

induces a well-defined group homomorphism

$$V(R) \to V_{free}(R).$$

Since

$$[P, g, f] = [P \oplus P^\vee \oplus Q \oplus Q^\vee, g \perp can\ g^{-1} \perp H(Q), f \perp can\ g^{-1} \perp H(Q)]$$

in $V(R)$ by the first of the useful identities listed above, this homomorphism is actually an inverse to the canonical homomorphism $V_{free}(R) \to V(R)$. Thus, $V_{free}(R) \cong V(R)$.
There is a canonical isomorphism between $V(R)$ and the group $W'_E(R)$ defined in the previous section. In order to discuss this identification, we first need to prove Lemma 3.2 and Corollaries 3.3 and 3.4 below. They will also be used in the proofs of some later results in this thesis.

Lemma 3.2. *Let $P = \bigoplus_{i=1}^n P_i$ be a finitely generated projective R-module and f_i alternating isomorphisms on P_i, $i = 1, ..., n$. Let $f = f_1 \perp ... \perp f_n$. Then $[P, f, \varphi^\vee f \varphi] = 0$ in $V(R)$ for any element φ of the commutator subgroup of $Aut(P)$. In particular, the same holds for every element of $E(P)$ with respect to the given decomposition.*

Proof. By the third of the useful identities listed above, we have

$$[P, f, \varphi_2^\vee \varphi_1^\vee f \varphi_1 \varphi_2] = [P, f, \varphi_1^\vee f \varphi_1] + [P, f, \varphi_2^\vee f \varphi_2].$$

Therefore we only have to prove the first statement for commutators. If $\varphi = \varphi_1 \varphi_2 \varphi_1^{-1} \varphi_2^{-1}$ is a commutator, then the formula above yields

$$[P, f, \varphi^\vee f \varphi] = [P, f, \varphi_1^\vee f \varphi_1] + [P, f, \varphi_2^\vee f \varphi_2] + [P, f, (\varphi_1^{-1})^\vee f \varphi_1^{-1}] + [P, f, (\varphi_2^{-1})^\vee f \varphi_2^{-1}] = 0,$$

which proves first part of the lemma.
For the second part, observe that by the formula above we only need to prove the statement for elementary automorphisms. So let φ_s be the elementary automorphism induced by $s : P_j \to P_i$. Since we can add the summand $[P_i, f_i, f_i] = 0$, we may assume that we are in the situation of Corollary 1.5. Therefore we may assume that φ_s is a commutator and the second statement then follows from the first part of the lemma. □

Corollary 3.3. *Let P be a finitely generated projective R-module and χ be an alternating isomorphism on P. Then $[P \oplus R^{2n}, \chi \perp \psi_{2n}, \varphi^\vee(\chi \perp \psi_{2n})\varphi] = 0$ in $V(R)$ for any elementary automorphism φ of $P \oplus R^{2n}$. In particular, if f is any alternating isomorphism on $P \oplus R^{2n}$, it follows that there is an equality $[P \oplus R^{2n}, \chi \perp \psi_{2n}, \varphi^\vee f \varphi] = [P \oplus R^{2n}, \chi \perp \psi_{2n}, f]$ in $V(R)$.*

Proof. The first part follows directly from the previous lemma. The second part is then a direct consequence of the second relation given in the definition of the group $V(R)$. □

Corollary 3.4. *For any matrix $E \in E_{2n}(R)$, we have $[R^{2n}, \psi_{2n}, E^t \psi_{2n} E] = 0$ in $V(R)$. In particular, we have $[R^{2n}, \psi_{2n}, N] = [R^{2n}, \psi_{2n}, E^t N E]$ in $V(R)$ for any alternating invertible matrix $N \in A_{2n}(R)$.*

Using the previous corollary, the group $V_{free}(R)$ can be identified with $W_E'(R)$ as follows: If $M \in A_{2m}(R)$ represents an element of $W_E'(R)$, then we assign to it the class in $V_{free}(R)$ represented by $[R^{2m}, \psi_{2m}, M]$. By Corollary 3.4, this assignment descends to a well-defined homomorphism $\nu : W_E'(R) \to V_{free}(R)$.
Now let us describe the inverse $\xi : V_{free}(R) \to W_E'(R)$ to this homomorphism. Let (L, g, f) be a triple with L free and g, f alternating isomorphisms on L. We can choose an isomorphism $\alpha : R^{2n} \xrightarrow{\cong} L$ and consider the alternating isomorphism

$$(\alpha^\vee f\alpha) \perp \sigma_{2n}(\alpha^\vee g\alpha)^{-1}\sigma_{2n}^\vee : R^{2n} \oplus (R^{2n})^\vee \to (R^{2n})^\vee \oplus R^{2n}.$$

With respect to the standard basis of R^{2n} and its dual basis of $(R^{2n})^\vee$, we may interpret this alternating isomorphism as an element of $A_{4n}(R)$ and then consider its class $\xi([L,g,f])$ in $W'_E(R)$. In fact, this class is independent of the choice of the isomorphism $\alpha : R^{2n} \xrightarrow{\cong} L$. If $\beta : R^{2n} \xrightarrow{\cong} L$ is another isomorphism, then it suffices to show that the alternating matrix M corresponding to $\alpha^\vee f\alpha \perp \beta^\vee g\beta$ is equivalent in $W'_E(R)$ to the alternating matrix corresponding to $\beta^\vee f\beta \perp \alpha^\vee g\alpha$. But there is an isometry $\gamma = (\alpha^{-1}\beta) \perp (\beta^{-1}\alpha)$ from $\alpha^\vee f\alpha \perp \beta^\vee g\beta$ to $\beta^\vee f\beta \perp \alpha^\vee g\alpha$, which is an elementary automorphism by Whitehead's lemma. One then also checks easily that the defining relations of $V_{free}(R)$ are also satisfied by the assignment above. Hence it follows that this assignment induces a well-defined homomorphism $\xi : V_{free}(R) \to W'_E(R)$. By construction, ν and ξ are obviously inverse to each other and therefore identify $W'_E(R)$ with $V_{free}(R)$. From now on, we denote by $\tilde{V}(R)$ the subgroup of $V(R)$ corresponding to the elementary symplectic Witt group $W_E(R)$ under the isomorphisms $V(R) \cong V_{free}(R) \cong W'_E(R)$.

In view of the previous paragraph, we obtain the following new presentation of the group $W'_{SL}(R)$: Let $V_{SL}(R)$ be the quotient of the free abelian group on isometry classes of triples (P,g,f) modulo the subgroup generated by the relations

- $[P \oplus P', g \perp g', f \perp f'] = [P,g,f] + [P',g',f']$ for alternating isomorphisms f,g on P and f',g' on P';
- $[P,f_0,f_1] + [P,f_1,f_2] = [P,f_0,f_2]$ for alternating isomorphisms f_0,f_1,f_2 on P;
- $[P,g,f] = [P,g,\varphi^\vee f\varphi]$ for alternating isomorphisms g,f on P and $\varphi \in SL(P)$.

Then $V_{SL}(R) \cong W'_{SL}(R)$. We denote by $\tilde{V}_{SL}(R)$ the subgroup of $V_{SL}(R)$ corresponding to $W_{SL}(R)$. Automatically, there is a canonical epimorphism $\tilde{V}(R) \to \tilde{V}_{SL}(R)$ corresponding to the map $W_E(R) \to W_{SL}(R)$.

Since we have isomorphisms $GW_1^3(R) \cong W'_E(R) \cong V(R)$ for any smooth affine algebra R over a perfect field k with $char(k) \neq 2$, we can now make the functoriality of the Grothendieck-Witt group $GW_1^3(R)$ more explicit in terms of the group $V(R)$. For this, we let $S = Spec(R)$ be a base scheme, where R is a smooth affine algebra over a perfect field k with $char(k) \neq 2$. Assume that $Y = Spec(B)$ and $Z = Spec(C)$ are all smooth affine schemes over $S = Spec(R)$. Any morphism $Z \to Y$ over $Spec(R)$ then corresponds to an R-algebra homomorphism $f : B \to C$. If P is a finitely generated projective B-module with alternating isomorphisms χ_1 and χ_2, then the class of the triple $[P,\chi_2,\chi_1] \in V(B)$ is sent under the pullback morphism f^* to $[P \otimes_B C, \chi_2 \otimes_B C, \chi_1 \otimes_B C] \in V(C)$.

Now let R be a commutative ring. In order to conclude this section, we describe some group actions on $V(R)$: For any finitely generated projective R-module P, alternating isomorphism $\chi : P \to P^\vee$ and $u \in R^\times$, the morphism $u \cdot \chi : P \to P^\vee$ is again an alternating isomorphism on P. Note that the alternating isomorphism $u \cdot \chi$ is canonically isometric to the alternating isomorphism $u \otimes_R \chi : R \otimes_R P \xrightarrow{u \otimes_R \chi} R \otimes_R P^\vee \cong (R \otimes_R P)^\vee$ and we therefore have an equality

$$[P, u \cdot \chi_2, u \cdot \chi_1] = [R \otimes_R P, u \otimes_R \chi_2, u \otimes_R \chi_1] \text{ in } V(R)$$

for all χ_1, χ_2. One can check easily that the assignment

$$(u, (P, \chi_2, \chi_1)) \mapsto (P, u \cdot \chi_2, u \cdot \chi_1)$$

descends to a well-defined action of $R^\times$ on $V(R)$.
Now let us assume that $2 \in R^\times$, let $\varphi : Q \to Q^\vee$ be a symmetric isomorphism on a finitely generated projective R-module Q. Then, for any skew-symmetric isomorphism $\chi : P \to P^\vee$ as above, there is an induced homomorphism $\varphi \otimes_R \chi : Q \otimes_R P \to Q^\vee \otimes_R P^\vee \cong (Q \otimes_R P)^\vee$, which is a skew-symmetric isomorphism on $Q \otimes_R P$. One can check easily that the assignment

$$((Q, \varphi), (P, \chi_2, \chi_1)) \mapsto (Q \otimes_R P, \varphi \otimes_R \chi_2, \varphi \otimes_R \chi_1)$$

induces a well-defined action of the Grothendieck-Witt group $GW(R) = GW_0^0(R)$ of R on $V(R)$.
For any smooth affine algebra R over a perfect field k with $char(k) \neq 2$, recall that we have defined also an action of $R^\times$ on $GW_1^3(R)$ in the previous section called the conjugation action, which coincides with another action called the multiplicative action. By means of the identifications $GW_1^3(R) \cong W'_E(R) \cong V(R)$, we have many equivalent ways to describe this action: If $M \in GL_{2n}(R)$ represents a morphism $Spec(R) \to GL_{2n}$ and u is a unit of R, note that the conjugation of M by $\gamma_{2n,u}^{-1}$ is sent via the morphism $GL_{2n} \to GL_{2n}/Sp_{2n} \xrightarrow{\cong} A_{2n}$ to

$$\gamma_{2n,u}^{-1} M^t \gamma_{2n,u} \psi_{2n} \gamma_{2n,u} M \gamma_{2n,u}^{-1} = \gamma_{2n,u}^{-1} M^t (u \cdot \psi_{2n}) M \gamma_{2n,u}^{-1}.$$

Furthermore, note that the isometry induced by the matrix $\gamma_{2n,u}$ yields an equality

$$[R^{2n}, \psi_{2n}, \gamma_{2n,u}^{-1} M^t (u \cdot \psi_{2n}) M \gamma_{2n,u}^{-1}] = [R^{2n}, u \cdot \psi_{2n}, M^t (u \cdot \psi_{2n}) M]$$

in $V(R)$. As a consequence, the conjugation action of $R^\times$ on $GW_1^3(R)$ can be described via the isomorphism $GW_1^3(R) \cong V(R)$ as follows: If (P, χ_2, χ_1) is a triple as in the definition of the group $V(R)$ and $u \in R^\times$, then the action is given by

$$(u, (P, \chi_2, \chi_1)) \mapsto (P, u \cdot \chi_2, u \cdot \chi_1).$$

Hence in this case the conjugation action is just given by the action of $R^\times$ on $V(R)$ which we defined above. The conjugation action is thus a homotopy-theoretic interpretation of the action defined above in case of a smooth affine algebra over a perfect field of characteristic $\neq 2$. Since the conjugation action coincides with the multiplicative action, we therefore also obtain another interpretation of the $R^\times$-action on $V(R)$ defined above via the multiplicative structure of higher Grothendieck-Witt groups.

3.4 The Gersten-Grothendieck-Witt spectral sequence

In the last section of this chapter, we introduce Grothendieck-Witt sheaves and study their cohomology. This will give cohomological obstructions to the 2-divisibility of $W_E(R)$ and $W_{SL}(R)$ for any smooth affine fourfold over an algebraically closed field k of characteristic $\neq 2$.

First of all, we fix a perfect base field k with $char(k) \neq 2$. Recall that we have defined $\mathbb{A}_k^1$-homotopy sheaves $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$ for any pointed space $(\mathcal{X}, x) \in Spc_{k,\bullet}$. As a special case, we define Grothendieck-Witt sheaves as follows:

Definition 3.5. *For any $i, j \geq 0$, we set $\boldsymbol{GW}_i^j = \pi_i^{\mathbb{A}^1}(\mathcal{GW}^j)$.*

Now let $X = Spec(R)$ be a smooth affine k-scheme. The Karoubi periodicity sequence induces an exact sequence of sheaves

$$\mathbf{K}_4^Q \xrightarrow{H_{4,3}} \mathbf{GW}_4^3 \xrightarrow{\eta} \mathbf{GW}_3^2 \xrightarrow{f_{3,2}} \mathbf{K}_3^Q,$$

where $\mathbf{K}_i^Q$ denotes the ith Quillen K-theory sheaf for $i \geq 0$. We denote by $\mathbf{A}$ the image of $H_{4,3}$ and by $\mathbf{B}$ the image of η and obtain a short exact sequence

$$0 \to \mathbf{A} \to \mathbf{GW}_4^3 \to \mathbf{B} \to 0$$

of sheaves. It follows from [AF2, Lemma 4.11] and from the computations in [AF3, Section 3.6] that the associated exact sequence of cohomology groups yields an exact sequence of the form

$$H^3(X,\mathbf{K}_4^Q/2) \to H^3(X,\mathbf{GW}_4^3) \to Ch^3(X) \to Ch^4(X) \to H^4(X,\mathbf{GW}_4^3) \to 0,$$

where $Ch^i(X) = CH^i(X)/2$ for $i = 3,4$. Since $CH^4(X)$ is 2-divisible for any smooth affine fourfold X over an algebraically closed field, we obtain:

Proposition 3.6. *If R is a smooth affine algebra of dimension 4 over an algebraically closed field k with $char(k) \neq 2$ and $X = Spec(R)$, then there is an exact sequence of the form $H^3(X,\boldsymbol{K}_4^Q/2) \to H^3(X,\boldsymbol{GW}_4^3) \to Ch^3(X) \to 0$.*

In particular, if $H^3(X,\mathbf{K}_4^Q/2)$ and $Ch^3(X)$ are trivial, then also $H^3(X,\mathbf{GW}_4^3)$ is trivial. In fact, one can prove the following statement:

Proposition 3.7. *If R is a smooth affine algebra of dimension 4 over an algebraically closed field k with $char(k) \neq 2$ and $X = Spec(R)$, then $H^3(X,\boldsymbol{K}_4^Q)$ is 2-divisible and $H^3(X,\boldsymbol{K}_4^Q/2) = 0$. In particular, $H^3(X,\boldsymbol{GW}_4^3)$ is 2-divisible if and only if $CH^3(X)$ is 2-divisible.*

Proof. We let $2\mathbf{K}_4^Q$ be the image and $\{2\}\mathbf{K}_4^Q$ be the kernel of the morphism $\mathbf{K}_4^Q \to \mathbf{K}_4^Q$ induced by multiplication by 2. Then we consider the two short exact sequences of sheaves

$$0 \to \{2\}\mathbf{K}_4^Q \to \mathbf{K}_4^Q \to 2\mathbf{K}_4^Q \to 0$$

and

$$0 \to 2\mathbf{K}_4^Q \to \mathbf{K}_4^Q \to \mathbf{K}_4^Q/2 \to 0.$$

The Gersten resolutions of $\{2\}\mathbf{K}_4^Q$ and $\mathbf{K}_4^Q/2$ are flasque resolutions of these sheaves and can therefore be used in order to compute their cohomology.
Since $K_0(F) = \mathbb{Z}$ for any field F, we have $H^4(X,\{2\}\mathbf{K}_4^Q) = 0$. It follows that the map $H^3(X,\mathbf{K}_4^Q) \to H^3(X,2\mathbf{K}_4^Q)$ is surjective. As the composite

$$H^3(X,\mathbf{K}_4^Q) \to H^3(X,2\mathbf{K}_4^Q) \to H^3(X,\mathbf{K}_4^Q)$$

is multiplication by 2, it thus suffices to prove that $H^3(X,\mathbf{K}_4^Q/2) = 0$.
For any $q,m \in \mathbb{N}$, we let $\mathcal{H}^q(m)$ be the sheaf associated to the presheaf

$$U \mapsto H^q_{et}(U,\mu_2^{\otimes m}).$$

Recall that the Bloch-Ogus spectral sequence (cp. [BO]) converges to the étale cohomology groups $H^*_{et}(X, \mu_2^{\otimes m})$ and its terms on the second page are $H^p_{Zar}(X, \mathcal{H}^q(m))$. These groups can be computed via the Gersten complex

$$H^q(k(X), \mu_2^{\otimes m}) \xrightarrow{d_0} \oplus_{x_1 \in X^{(1)}} H^{q-1}(k(x_1), \mu_2^{\otimes m-1}) \xrightarrow{d_1} ....$$

By [JPS2, §4.2, Proposition 11], one has $cd(k(x_p)) \leq 4-p$ for any $x_p \in X^{(p)}$. Hence it follows that $H^p_{Zar}(X, \mathcal{H}^q(m)) = 0$ for all $q \geq 5$; consequently, $H^3(X, \mathcal{H}^4(m)) = H^7_{et}(X, \mu_2^{\otimes m}) = 0$ because X is affine.
Since $H^3(X, \mathbf{K}_4^M/2) = H^3(X, \mathbf{K}_4^Q/2)$ and $H^3(X, \mathcal{H}^4(4)) = H^3(X, \mathbf{K}_4^M/2)$ because of the proof of the Bloch-Kato conjecture, this proves the result. □

In the remainder of this section, we will use the Gersten-Grothendieck-Witt spectral sequence in order to compute $W_E(S^k_{2n-1})$ for all n divisible by 4 and in order to find cohomological obstructions for the 2-divisibility of $W_E(R)$ and $W_{SL}(R)$ when R is a smooth affine algebra of dimension 4 over an algebraically closed field k with $char(k) \neq 2$.
Recall that if X is a smooth k-scheme of dimension d, then the Gersten-Grothendieck-Witt spectral sequence $E(3)$ associated to X has terms of the form

$$E(3)_1^{p,q} \cong \begin{cases} \bigoplus_{x_p \in X^{(p)}} GW^{3-p}_{3-p-q}(k(x_p), \omega_{x_p}) & \text{if } 0 \leq p \leq d \text{ and } 3 \geq p+q \\ 0 & \text{else} \end{cases}$$

on the first page and converges to $GW^3_{3-*}(X)$. There is a filtration

$$0 = F_{d+1} \subset F_d \subset ... \subset F_1 \subset GW^3_1(X) = F_0$$

with $F_p/F_{p+1} \cong E(3)^{p,2-p}_\infty$ for all p. Furthermore, the terms $E(3)^{p,q}_2$ on the second page are isomorphic to $H^p(X, \mathbf{GW}^3_{3-q})$ for $0 \leq p \leq d$ and $p+q \leq 3$. We define $GW^3_{1,red}(X) = F_1$. In general, the group $GW^3_{1,red}(X)$ coincides with $[X, SL/Sp]_{\mathbb{A}^1_k}$. In particular, if $X = Spec(R)$ is affine, then it coincides with $W_E(R)$. Hence we can compute the group $W_E(R)$ via the limit terms $E(3)^{p,2-p}_\infty$.

Proposition 3.8. *Let $n \in \mathbb{N}$ be divisible by 4 and k be a perfect field with $char(k) \neq 2$. Then $W_E(S^k_{2n-1}) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. We have identifications

$$W_E(S^k_{2n-1}) \cong [Q^k_{2n-1}, SL/Sp]_{\mathbb{A}^1_k} \cong [\mathbb{A}^n_k \setminus 0, SL/Sp]_{\mathbb{A}^1_k} \cong GW^3_{1,red}(\mathbb{A}^n_k \setminus 0).$$

We use the Gersten-Grothendieck-Witt spectral sequence $E(3)$ associated to $X = \mathbb{A}^n_k \setminus 0$ in order to compute $GW^3_{1,red}(\mathbb{A}^n_k \setminus 0)$. As indicated above, we have a filtration

$$0 = F_{n+1} \subset F_n \subset ... \subset GW^3_{1,red}(X) = F_1 \subset GW^3_1(X) = F_0$$

with $F_p/F_{p+1} \cong E(3)^{p,2-p}_\infty$ for all p.
Let us compute the limit terms $E(3)^{p,q}_\infty$. It is known that the terms $E(3)^{p,q}_2$ on the second page are precisely isomorphic to $H^p(X, \mathbf{GW}^3_{3-q})$. Since n is divisible by 4, it follows from [AF1, Lemma 4.5] that

$$E(3)^{p,q}_2 \cong \begin{cases} GW^3_{3-q}(k) & \text{if } p = 0 \\ GW^3_{3-n-q}(k) & \text{if } p = n-1 \\ 0 & \text{else.} \end{cases}$$

It follows immediately from this that $GW^3_{1,red}(X) = F_1 = GW^3_0(k)$. But $GW^3_0(k) \cong \mathbb{Z}/2\mathbb{Z}$ by [FS, Lemma 4.1]. This proves the proposition. $\square$

Finally, we can give the following cohomological criteria for the 2-divisibility of $W_E(R)$ and $W_{SL}(R)$:

Proposition 3.9. *Let R be a smooth affine algebra of dimension 4 over an algebraically closed field k with $char(k) \neq 2$ and $X = Spec(R)$. Then $W_E(R)$ is 2-divisible if $H^2(X, \boldsymbol{K}^{MW}_3)$ and $H^3(X, \boldsymbol{GW}^3_4)$ are 2-divisible. Furthermore, $W_{SL}(R)$ is 2-divisible if $H^2(X, \boldsymbol{I}^3)$ is 2-divisible and $CH^3(X) = CH^4(X) = 0$.*

Proof. We use the Gersten-Grothendieck-Witt spectral sequence $E(3)$ associated to X. We have a filtration

$$0 = F_5 \subset F_4 \subset ... \subset GW^3_{1,red}(R) = F_1 \subset GW^3_1(R) = F_0$$

with $F_p/F_{p+1} \cong E(3)^{p,2-p}_\infty$ for all p. The terms $E(3)^{p,q}_2$ on the second page are $H^p(X, \mathbf{GW}^3_{3-q})$ for $0 \leq p \leq 4$ and $p + q \leq 3$ and 0 elsewhere.
First of all, [FRS, Lemma 2.2] implies that $E(3)^{p,1}_1 = 0$ for all p. Therefore $E(3)^{1,1}_\infty = 0$ and hence $F_2 = W_E(R)$. Moreover, since k is algebraically closed, the limit term $F_4 = E(3)^{4,-2}_\infty$ is a quotient of $\oplus_{x \in X^{(4)}} k^\times$ and therefore 2-divisible. Altogether, we have two short exact sequences

$$0 \to F_3 \to W_E(R) \to E(3)_\infty^{2,0} \to 0,$$
$$0 \to F_4 \to F_3 \to E(3)_\infty^{3,-1} \to 0,$$

where F_4 is 2-divisible. In particular, $W_E(R)$ is 2-divisible as soon as $E(3)_\infty^{2,0}$ and $E(3)_\infty^{3,-1}$ are 2-divisible.
However, $E(3)_\infty^{3,-1}$ is a quotient of $H^3(X, \mathbf{GW}_4^3)$. Furthermore, we know that $E(3)_2^{2,0}$ is precisely $H^2(X, \mathbf{GW}_3^3) \cong H^2(X, \mathbf{K}_3^{MW})$. Hence $E(3)_\infty^{2,0}$ is precisely the kernel of the differential mapping into $E(3)_2^{4,-1} \cong H^4(X, \mathbf{GW}_4^3)$. But by the fact that $CH^4(X)$ is 2-divisible and by [AF3, Proposition 3.6.4], we can conclude that $H^4(X, \mathbf{GW}_4^3) = 0$. Thus, the limit term $E(3)_\infty^{2,0}$ is precisely $H^2(X, \mathbf{K}_3^{MW})$ and the first statement follows.
For the second statement, we will use the Brown-Gersten-Quillen spectral sequence $E'(3)$ associated to X, which has terms of the form

$$E'(3)_1^{p,q} \cong \begin{cases} \bigoplus_{x_p \in X^{(p)}} K_{3-p-q}^Q(k(x_p)) & \text{if } 0 \le p \le d \text{ and } 3 \ge p+q \\ 0 & \text{else} \end{cases}$$

on the first page and converges to $K_{3-*}^Q(X)$. The group $SK_1(R)$ can be computed via the limit terms $E'(3)_\infty^{p,2-p}$: There is a filtration

$$0 = F_5' \subset F_4' \subset ... \subset SK_1(R) = F_1' \subset K_1(R) = F_0'$$

with $F_p'/F_{p+1}' \cong E'(3)_\infty^{p,2-p}$ for all p. Moreover, the terms $E'(3)_2^{p,q}$ on the second page are isomorphic to $H^p(X, \mathbf{K}_{3-q}^Q)$ for $0 \le p \le d$ and $p+q \le 3$.
By construction of both the Brown-Gersten-Quillen and the Gersten-Grothendieck-Witt spectral sequences, the hyperbolic morphism induces a morphism of spectral sequences. Hence we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & F_3' \arrow[r] \arrow[d, "H_{1,3}"] & SK_1(X) \arrow[r] \arrow[d, "H_{1,3}"] & F_1'/F_3' \arrow[r] \arrow[d, "H_{1,3}"] & 0 \arrow[d] \\
0 \arrow[r] & F_3 \arrow[r] & W_E(X) \arrow[r] & H^2(X, \mathbf{K}_3^{MW}) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. If $H^3(X, \mathbf{GW}_4^3)$ is 2-divisible (in particular, if $CH^3(X)$ is 2-divisible), then we have seen above that F_3 is 2-divisible. Since $W_{SL}(R)$ is 2-torsion, the snake lemma induces an isomorphism $W_{SL}(R) \xrightarrow{\cong} H^2(X, \mathbf{K}_3^{MW})/H_{1,3}(F_1'/F_3')$. In particular, there is a surjection $H^2(X, \mathbf{K}_3^{MW})/H_{1,3}(F_2'/F_3') \to W_{SL}(R)$.

Since $CH^4(X) = 0$, the group $H^2(X, \mathbf{K}_3^Q)$ surjects onto $F_2'/F_3' \cong E'(3)_\infty^{2,0}$ and it follows that there is an equality $H^2(X, \mathbf{K}_3^{MW})/H_{1,3}(F_2'/F_3') = H^2(X, \mathbf{K}_3^{MW})/H_{3,3}(H^2(X, \mathbf{K}_3^Q))$. Finally, as the homomorphism $H^2(X, \mathbf{K}_3^Q) \to H^2(X, 2\mathbf{K}_3^Q)$ is surjective, the long exact sequence of cohomology groups associated to the short exact sequence

$$0 \to 2\mathbf{K}_3^M \to \mathbf{K}_3^{MW} \to \mathbf{I}^3 \to 0$$

shows that $H^2(X, \mathbf{K}_3^{MW})/H_{3,3}(X, \mathbf{K}_3^Q) \cong H^2(X, \mathbf{I}^3)$. This yields the second statement. $\square$

The Generalized Vaserstein Symbol

In the last chapter of this thesis, we finally define the generalized Vaserstein symbol and prove our main results. First of all, we start by reviewing the definition and basic properties of the usual Vaserstein symbol as defined by Suslin and Vaserstein in [SV, §5]. In the subsequent section, this leads to the construction of the generalized Vaserstein symbol $V_{\theta_0} : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$ associated to a finitely generated projective module P_0 of rank 2 over a commutative ring R together with a fixed trivialization of its determinant $\theta_0 : R \xrightarrow{\cong} \det(P_0)$. We will then study its basic properties and prove in particular that it is a bijection if R is either a regular Noetherian ring of dimension 2 or a regular affine algebra of dimension 3 over a perfect field k with $c.d.(k) \leq 1$ and $6 \in k^{\times}$. Furthermore, we prove a sum formula $V_{\theta_0}(a_0, a_R^n) = n \cdot V_{\theta_0}(a_0, a_R)$ for $n \equiv 0, 1 \ mod\ 4$ over smooth affine algebras over perfect fields with characteristic $\neq 2$ such that $-1 \in k^{\times 2}$ and explore its applications. Finally, we will see that the generalized Vaserstein symbol descends to a map $V_{\theta_0} : Um(P_0 \oplus R)/SL(P_0 \oplus R) \to \tilde{V}_{SL}(R)$, which we will study in the third section of this chapter. As an application of this map, we will give a criterion for the triviality of the orbit space $Um(P_0 \oplus R)/SL(P_0 \oplus R)$ for Noetherian rings of dimension ≤ 4. Motivated by this criterion, we study symplectic orbits of unimodular rows and prove in particular that $Sp_d(R)$ acts transitively on $Um_d(R)$ whenever $d \geq 4$ is divisible by 4 and R is a smooth affine algebra of dimension d over an algebraically closed field k with $d! \in k^{\times}$. As an immediate consequence of this, we will prove that $Um_3(R)/SL_3(R)$ is trivial if and only if $\tilde{V}_{SL}(R)$ is trivial whenever R is a smooth affine algebra of dimension 4 over an algebraically closed field k with $6 \in k^{\times}$. Furthermore, we will prove that the generalized Vaserstein symbol induces a bijection $V_{\theta_0} : Um(P_0 \oplus R)/SL(P_0 \oplus R) \to \tilde{V}_{SL}(R)$ if R is an affine algebra of dimension 3 over a finite field. In order to conclude this thesis, we contextualize our results in the last section of this chapter by relating them to some open questions in the study of projective modules.

4.1 The Vaserstein symbol for unimodular rows

In this section, we review the Vaserstein symbol map as introduced by Suslin and Vaserstein in [SV, §5]. Furthermore, we prove a cancellation theorem for finitely generated projective modules of a specific form over normal affine algebras over the algebraic closure of a finite field. We conclude this section by reinterpreting the Vaserstein symbol by means of the isomorphism $W_E'(R) \cong V_{free}(R)$.

First of all, we let R be a commutative ring and we let $Um_3(R)$ be its set of unimodular rows of length 3, i.e. triples $a = (a_1, a_2, a_3)$ of elements in R such that there are elements $b_1, b_2, b_3 \in R$ with $\sum_{i=1}^{3} a_i b_i = 1$. This data determines an exact sequence of the form

$$0 \to P(a) \to R^3 \xrightarrow{a} R \to 0,$$

where $P(a) = \ker(a)$. The triple $b = (b_1, b_2, b_3) \in R^3$ gives a section to the epimorphism $a : R^3 \to R$ and induces a retraction $r : R^3 \to P(a), e_i \mapsto e_i - a_i b$, where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. One then obtains an isomorphism $i = r + a : R^3 \to P(a) \oplus R$, which induces an isomorphism $\det(R^3) \to \det(P(a) \oplus R)$. Finally, by composing with the canonical isomorphisms $\det(P(a) \oplus R) \cong \det(P(a))$ and $R \to \det(R^3), 1 \mapsto e_1 \wedge e_2 \wedge e_3$, one obtains an isomorphism $\theta : R \to \det(P(a))$.

The element of $W_E'(R)$ defined by the matrix

$$V(a,b) = \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -b_3 & b_2 \\ a_2 & b_3 & 0 & -b_1 \\ a_3 & -b_2 & b_1 & 0 \end{pmatrix}$$

has Pfaffian 1 and does not depend on the choice of the section b (cp. [SV, Lemma 5.1]). We therefore obtain a well-defined map $V : Um_3(R) \to W_E(R)$. In particular, if we let G be any group such that $E(R) \subset G \subset SL(R)$, we obtain a well-defined map $V_G : Um_3(R) \to W_G(R)$; in the case $G = E(R)$, we just recover the map V. These maps were introduced and studied by Suslin and Vaserstein in [SV, §5]. They proved (cp. [SV, Theorem 5.2]):

Theorem 4.1. *Let R be a commutative ring and, moreover, let G be any group such that $E(R) \subset G \subset SL(R)$. For all $n \geq 1$, we let $\pi_{1,n} = (1,0,...,0)$ be the standard unimodular row of length n. The map $V_G : Um_3 \to W_G(R)$ has the following properties:*

a) $V_G(a) = V_G(a\varphi)$ for all $a \in Um_3(R)$ and $\varphi \in G \cap SL_3(R)$.

b) If $\pi_{1,2n+1}(G \cap SL_{2n+1}(R)) = Um_{2n+1}(R)$ for all $n \geq 2$, then V_G is surjective.

c) If $\pi_{1,2n}E_{2n}(R) = \pi_{1,2n}(G^t \cap SL_{2n}(R))$ for all $n \geq 2$, then $V_G(a) = V_G(a')$ for some $a, a' \in Um_3(R)$ implies that $a = a'\varphi$ for some $\varphi \in G \cap SL_3(R)$.

In particular, the theorem implies that the maps $V_G : Um_3(R) \to W_G(R)$ descend to maps

$$V_G : Um_3(R)/(G \cap SL_3(R)) \to W_G(R).$$

Moreover, one obtains a map

$$V : Um_3(R)/E_3(R) \to W_E(R)$$

called the Vaserstein symbol. Of course, this is just the composite

$$V : Um_3(R)/E_3(R) \to Um_3(R)/(E(R) \cap SL_3(R)) \xrightarrow{V_{E(R)}} W_E(R).$$

Suslin and Vaserstein studied the injectivity and surjectivity of the Vaserstein symbol by means of the criteria given by the theorem above. In [SV, Corollary 7.4], they proved that the Vaserstein symbol is a bijection if $\dim(R) \leq 2$. In fact, if we let $\pi_{1,n} = (1, 0, ..., 0)$ be the standard unimodular row of length n, their proof showed that the Vaserstein symbol is surjective if $\pi_{1,2n+1}E_{2n+1}(R) = Um_{2n+1}(R)$ for $n \geq 2$ and injective if $\pi_{1,2n}E_{2n}(R) = \pi_{1,2n}SL_{2n}(R)$ for $n \geq 3$ and $E(R) \cap SL_4(R) = E_4(R)$. Since $\pi_{1,n}E_n(R) = Um_n(R)$ if $n \geq 5$ for any Noetherian ring of dimension 3 (cp. [HB, Chapter IV, Theorem 3.4]), the only remaining criterion which needs to be proven for a Noetherian ring of dimension 3 is $E(R) \cap SL_4(R) = E_4(R)$. Using local-global principles, Rao and van der Kallen could prove in [RvdK, Theorem 3.4 and Corollary 3.5]:

Theorem 4.2. *Assume that R is a regular affine algebra of dimension 3 over a field k with $c.d.(k) \leq 1$ which is perfect if $char(k) = 2, 3$. Then $E(R) \cap SL_4(R) = E_4(R)$. In particular, the Vaserstein symbol $V : Um_3(R)/E_3(R) \to W_E(R)$ is a bijection.*

In particular, the orbit space $Um_3(R)/E_3(R)$ can be endowed with an abelian group structure in the situation of the theorem. In [FRS], this abelian group structure was substantially used in order to show that stably free modules of rank $d-1$ over normal affine algebras of dimension $d \geq 4$ or smooth affine algebras of dimension $d = 3$ over an algebraically closed field k with $(d-1)! \in k^{\times}$ are free. In their proof, they implicitly showed that if $j \in \mathbb{N}$ such that $gcd(char(k), j) = 1$, then any unimodular row of length d can be transformed via

elementary matrices to a row of the form $(a_1, \ldots, a_d^j)$; in particular, if one takes $j = (d-1)!$, then Suslin's $n!$-factorial theorem (cp. [S1, Remark after Lemma 2]) enabled them to conclude their proof.
We are now going to use their implicit result in order to prove a cancellation theorem for projective modules over normal affine algebras over the algebraic closure of a finite field:

Theorem 4.3. *Let R be a normal affine algebra of dimension d over the algebraic closure $\bar{\mathbb{F}}_q$ of a finite field $\mathbb{F}_q$. Furthermore, let $d \geq 5$ and P_0 be a projective R-module of rank 2 and assume that $(d-1)! \in \mathbb{F}_q^\times$. Then the projective R-module $P_{d-1} = P_0 \oplus R^{d-3}$ of rank $d-1$ is cancellative.*

Proof. By Proposition 1.16, we know that any unimodular element in P_d can be transformed via elementary automorphisms to a unimodular element of the form $a = (a_0, a_3, \ldots, a_d)$ such that R/a_dR is a smooth k-algebra of dimension $d-1$ and P_0/a_dP_0 is a free R/a_dR-module of rank 2.
The proof of [FRS, Theorem 7.5] now shows that the unimodular element $(\bar{a}_0, \bar{a}_3, \ldots, \bar{a}_{d-1})$ of P_{d-1}/a_dP_{d-1} can be transformed via elementary automorphisms to an element of the form $\bar{b} = (\bar{b}_0, \bar{b}_3^{(d-1)!}, \bar{b}_4, \ldots, \bar{b}_{d-1})$. Using the map $\Phi_{d-1}(a)$ associated to a and using Lemma 1.12, it follows that a can be transformed via elementary automorphisms to a unimodular element of the form $b = (b_0, b_3, \ldots, b_{d-1}^{(d-2)!}, a_d^{d-1})$. By [S1, Lemma 2], there is an automorphism φ of P_d such that $\varphi(e_d) = b$, which proves the theorem. □

In fact, it was proven in [DK] that projective modules of rank $d-1$ over affine algebras of dimension $d \geq 4$ over the algebraic closure of a finite field $\mathbb{F}_q$ with $(d-1)! \in \mathbb{F}_q^\times$ are cancellative in general. Hence our theorem above illustrates how the implicit result of Fasel-Rao-Swan on transformations of unimodular rows via elementary matrices immediately implies this cancellation theorem in some special cases.
In order to conclude this section, let us now reinterpret the Vaserstein symbol map in light of the isomorphism $W_E'(R) \cong V(R)_{free}$ discussed in the previous chapter. The symbol $V(a)$ is sent to the element of $V_{free}(R)$ represented by the isometry class $[R^4, \psi_4, V(a,b)]$. If we denote by χ_a the alternating form $P(a) \times P(a) \to R, (p,q) \mapsto \theta^{-1}(p \wedge q)$, we obtain an alternating form on R^4 given by $(i \oplus 1)^t(\chi_a \perp \psi_2)(i \oplus 1)$. Moreover, if we set

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in E_4(R),$$

then one can check that the form $(i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)$ is given by the matrix $\sigma^t V(a,b)^t \sigma$. In particular, if we let $M : Um_3(R) \to Um_3(R)$ be the map which sends a unimodular row $a = (a_1, a_2, a_3)$ to the row $M(a) = (-a_1, -a_2, -a_3)$, then the composite $\nu \circ V \circ M$ is given by $a \mapsto [R^4, \psi_4, (i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)]$. Since both M and ν are bijections, these considerations lead to a generalization of the Vaserstein symbol.

4.2 The generalized Vaserstein symbol

In this section, we will define the generalized Vaserstein symbol and prove criteria for its surjectivity and injectivity which are the analogues of Suslin's and Vaserstein's criteria for the Vaserstein symbol mentioned in the previous section. Our criteria will enable us to prove that the generalized Vaserstein symbol is a bijection if R is a regular Noetherian ring of dimension 2 or a regular affine algebra of dimension 3 over a perfect field k such that $c.d.(k) \leq 1$ and $6 \in k^{\times}$. Finally, we give an alternative definition of the generalized Vaserstein symbol for smooth affine algebras over perfect fields of characteristic $\neq 2$ and use this definition in order to prove a sum formula for the generalized Vaserstein symbol. As an application, we can generalize the implicit result in [FRS, Theorem 7.5] that any unimodular row of length d over a normal affine algebra of dimension $d \geq 4$ or a smooth affine algebra of dimension $d = 3$ over an algebraically closed field k with $(d-1)! \in k^{\times}$ can be transformed via elementary matrices to a row of the form $(a_1, ..., a_d^{(d-1)!})$. Moreover, this will enable us to re-prove a cancellation theorem for projective modules of rank 2 with a trivial determinant over smooth affine threefolds (cp. [AF2, Corollary 3.8]).
Now let R be a commutative ring and P_0 be a projective R-module of rank 2. We will use the notation of Section 1.2: For all $n \geq 3$, we let $P_n = P_0 \oplus Re_3 \oplus ... \oplus Re_n$ be the direct sum of P_0 and free R-modules Re_i, $3 \leq i \leq n$, of rank 1 with explicit generators e_i. We will sometimes omit these explicit generators in the notation. Moreover, we denote by $\pi_{k,n} : P_n \to R$ the projections onto the free direct summands of rank 1 with index $k = 3, ..., n$. Recall that any $a \in Um(P_n)$ can be written as $(a_0, a_3, ..., a_n)$, where a_0 is the restriction of a to P_0 and $a_i = a(e_i)$, $i = 3, ..., n$, is the element of R corresponding to the restriction of a to Re_i. We assume that P_0 admits a fixed trivialization $\theta_0 : R \xrightarrow{\cong} \det(P_0)$ of its determinant. Furthermore, we denote by χ_0 the non-degenerate alternating form on P_0 given by $P_0 \times P_0 \to R, (p,q) \mapsto \theta_0^{-1}(p \wedge q)$.
Any element a of $Um(P_0 \oplus R)$ gives rise to an exact sequence of the form

$$0 \to P(a) \to P_0 \oplus R \xrightarrow{a} R \to 0,$$

where $P(a) = \ker(a)$. Any section $s : R \to P_0 \oplus R$ of a determines a canonical retraction $r : P_0 \oplus R \to P(a)$ given by $r(p) = p - sa(p)$ and an isomorphism $i : P_0 \oplus R \to P(a) \oplus R$ given by $i(p) = a(p) + r(p)$.
The exact sequence above yields an isomorphism $\det(P_0) \cong \det(P(a))$ and therefore an isomorphism $\theta : R \to \det(P(a))$ obtained by composing with θ_0. We denote by χ_a the non-degenerate alternating form on $P(a)$ given by $P(a) \times P(a) \to R, (p,q) \mapsto \theta^{-1}(p \wedge q)$.
We now want to define the generalized Vaserstein symbol

$$V_{\theta_0} : Um(P_0 \oplus R) \to V(R)$$

associated to P_0 and the fixed trivialization θ_0 of $\det(P_0)$ by

$$V_{\theta_0}(a) = [P_0 \oplus R^2, \chi_0 \perp \psi_2, (i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)].$$

If there is no ambiguity, we will usually suppress the fixed trivialization θ_0 and denote V_{θ_0} simply by V in order to simplify our notation. In order to prove that this generalized symbol is well-defined, one has to show that our definition is independent of a section of a:

Theorem 4.4. *The generalized Vaserstein symbol is well-defined, i.e. the element $V(a)$ defined as above is independent of the choice of a section of a.*

Proof. Let $a \in Um(P_0 \oplus R)$ with two sections $s, t : R \to P_0 \oplus R$. We denote by i_s and i_t the isomorphisms $P_0 \oplus R \cong P(a) \oplus R$ induced by the sections s and t respectively. Since the isomorphism $\det(P(a)) \cong \det(P_0)$ does not depend on the choice of a section (because the difference of two sections maps R into $P(a)$), the form χ_a is independent of the choice of a section as well. Therefore it suffices to show that the elements

$$V(a,s) = [P_0 \oplus R^2, \chi_0 \perp \psi_2, (i_s \oplus 1)^t (\chi_a \perp \psi_2)(i_s \oplus 1)] \text{ and}$$
$$V(a,t) = [P_0 \oplus R^2, \chi_0 \perp \psi_2, (i_t \oplus 1)^t (\chi_a \perp \psi_2)(i_t \oplus 1)]$$

are equal in $V(R)$.
We do this in the following three steps:

- We define a map $d : P_0 \oplus R \to R$. We get a corresponding automorphism $\varphi \in E(P_0 \oplus R^2)$ defined by $\varphi = id_{P_0 \oplus R^2} - de_4$.
- We show that $\varphi^t (i_s \oplus 1)^t (\chi_a \perp \psi_2)(i_s \oplus 1)\varphi = (i_t \oplus 1)^t (\chi_a \perp \psi_2)(i_t \oplus 1)$.
- Using Corollary 3.3, we conclude that $V(a,s) = V(a,t)$.

Now let us carry out the first step: First of all, we define a map $d' : P_0 \oplus R \to \det(P_0 \oplus R)$ by $p \mapsto s(1) \wedge t(1) \wedge p \in \det(P_0 \oplus R)$. Then $d : P_0 \oplus R \to R$ is the map obtained from d' by composing with the isomorphisms $\det(P_0 \oplus R) \cong \det(P_0) \cong R$. Let d_0 and d_R be its restrictions to P_0 and R respectively. Furthermore, we let $\varphi_0 = id_{P_0 \oplus R^2} - d_0 e_4$ and $\varphi_R = id_{P_0 \oplus R^2} - d_R e_4$ be the elementary automorphisms of $P_0 \oplus R^2$ defined by $-d_0$ and $-d_R$ respectively. Moreover, we let $\varphi = id_{P_0 \oplus R^2} - de_4$. Note that $\varphi = \varphi_0 \varphi_R = \varphi_R \varphi_0 \in E(P_0 \oplus R^2)$.
Now let us conduct the second step. By Lemma 1.2, we can check the desired equality locally. So let $\mathfrak{p}$ be a prime ideal of R and $(e_1^{\mathfrak{p}}, e_2^{\mathfrak{p}})$ be a basis of the free $R_{\mathfrak{p}}$-module $(P_0)_{\mathfrak{p}}$ of rank 2. We may further assume that $(\theta_0^{-1})_{\mathfrak{p}}(e_1^{\mathfrak{p}} \wedge e_2^{\mathfrak{p}}) = 1$. With respect to the basis $(e_1^{\mathfrak{p}}, e_2^{\mathfrak{p}}, e_3)$ of $(P_0)_{\mathfrak{p}} \oplus R_{\mathfrak{p}}$, the epimorphism $a_{\mathfrak{p}}$ can be represented by the unimodular row $(a_1^{\mathfrak{p}}, a_2^{\mathfrak{p}}, a_3^{\mathfrak{p}})$ and both sections $s_{\mathfrak{p}}$ and $t_{\mathfrak{p}}$ can be represented by the columns $(s_1^{\mathfrak{p}}, s_2^{\mathfrak{p}}, s_3^{\mathfrak{p}})^t$ and $(t_1^{\mathfrak{p}}, t_2^{\mathfrak{p}}, t_3^{\mathfrak{p}})^t$. Using the basis $(e_1^{\mathfrak{p}}, e_2^{\mathfrak{p}}, e_3, e_4)$ of $(P_0)_{\mathfrak{p}} \oplus R_{\mathfrak{p}}^2$, we can check the desired equality locally: If we let $d_1^{\mathfrak{p}} = t_3^{\mathfrak{p}} s_2^{\mathfrak{p}} - t_2^{\mathfrak{p}} s_3^{\mathfrak{p}}$, $d_2^{\mathfrak{p}} = t_1^{\mathfrak{p}} s_3^{\mathfrak{p}} - t_3^{\mathfrak{p}} s_1^{\mathfrak{p}}$ and $d_3^{\mathfrak{p}} = t_2^{\mathfrak{p}} s_1^{\mathfrak{p}} - t_1^{\mathfrak{p}} s_2^{\mathfrak{p}}$ and

$$M_{\mathfrak{p}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -d_1^{\mathfrak{p}} & -d_2^{\mathfrak{p}} & -d_3^{\mathfrak{p}} & 1 \end{pmatrix},$$

this amounts to verifying the equality

$$M_{\mathfrak{p}}^t \begin{pmatrix} 0 & s_3^{\mathfrak{p}} & -s_2^{\mathfrak{p}} & a_1^{\mathfrak{p}} \\ -s_3^{\mathfrak{p}} & 0 & s_1^{\mathfrak{p}} & a_2^{\mathfrak{p}} \\ s_2^{\mathfrak{p}} & -s_1^{\mathfrak{p}} & 0 & a_3^{\mathfrak{p}} \\ -a_1^{\mathfrak{p}} & -a_2^{\mathfrak{p}} & -a_3^{\mathfrak{p}} & 0 \end{pmatrix} M_{\mathfrak{p}} = \begin{pmatrix} 0 & t_3^{\mathfrak{p}} & -t_2^{\mathfrak{p}} & a_1^{\mathfrak{p}} \\ -t_3^{\mathfrak{p}} & 0 & t_1^{\mathfrak{p}} & a_2^{\mathfrak{p}} \\ t_2^{\mathfrak{p}} & -t_1^{\mathfrak{p}} & 0 & a_3^{\mathfrak{p}} \\ -a_1^{\mathfrak{p}} & -a_2^{\mathfrak{p}} & -a_3^{\mathfrak{p}} & 0 \end{pmatrix}$$

But this follows from the proof of [SV, Lemma 5.1].
Finally, we conclude by Corollary 3.3: Since φ_0 and φ_R are elementary automorphisms of $P_0 \oplus R^2$, the automorphism $\varphi = \varphi_0 \varphi_R$ is an element of $E(P_0 \oplus R^2)$. By Corollary 3.3, we deduce that

$$\begin{aligned} V(a,s) &= [P_0 \oplus R^2, \chi_0 \perp \psi_2, (i_s \oplus 1)^t (\chi_a \perp \psi_2)(i_s \oplus 1)] \\ &= [P_0 \oplus R^2, \chi_0 \perp \psi_2, \varphi^t (i_s \oplus 1)^t (\chi_a \perp \psi_2)(i_s \oplus 1)\varphi]. \end{aligned}$$

But by the second step, we also know that

$$\begin{aligned} &[P_0 \oplus R^2, \chi_0 \perp \psi_2, \varphi^t (i_s \oplus 1)^t (\chi_a \perp \psi_2)(i_s \oplus 1)\varphi] \\ &= [P_0 \oplus R^2, \chi_0 \perp \psi_2, (i_t \oplus 1)^t (\chi_a \perp \psi_2)(i_t \oplus 1)] = V(a,t). \end{aligned}$$

This finishes the proof. $\square$

We note that there is a natural homomorphism $\overline{Pf} : V(R) \to R^\times$ obtained as the composite $V(R) \xrightarrow{\cong} V_{free}(R) \xrightarrow{\xi} W'_E(R) \xrightarrow{Pf} R^\times$. We denote its kernel by $\tilde{V}(R)$. Of course, the isomorphism $V(R) \cong W'_E(R)$ induces an isomorphism $\tilde{V}(R) \cong W_E(R)$.
As stated in the previous section, the usual Vaserstein symbol of a unimodular row is an element of $W_E(R)$ and is invariant under elementary transformations. We will now prove that the analogous statements also hold for the generalized Vaserstein symbol:

Lemma 4.5. *The generalized Vaserstein symbol $V : Um(P_0 \oplus R) \to V(R)$ maps $Um(P_0 \oplus R)$ into $\tilde{V}(R)$.*

Proof. For this, we note that the Pfaffian of an element of $V(R)$ is completely determined by the Pfaffians of all its images under the maps $V(R) \to V(R_\mathfrak{p})$ induced by localization at any prime ideal $\mathfrak{p}$. But the localization $(P_0)_\mathfrak{p}$ at any prime $\mathfrak{p}$ is a free $R_\mathfrak{p}$-module of rank 2; choosing a basis $(e_1^\mathfrak{p}, e_2^\mathfrak{p})$ of $(P_0)_\mathfrak{p}$ such that $(\theta_0^{-1})_\mathfrak{p}(e_1^\mathfrak{p} \wedge e_2^\mathfrak{p}) = 1$ as in the proof of Theorem 4.4, we may calculate the Pfaffian of any Vaserstein symbol by the usual formula for the Pfaffian of an alternating 4×4-matrix. The lemma then follows immediately. $\square$

Theorem 4.6. *Let φ be an elementary automorphism of $P_0 \oplus R$. Then we have an equality $V(a) = V(a\varphi)$ for any $a \in Um(P_0 \oplus R)$. In particular, we obtain a well-defined map $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$.*

Proof. Let φ be an elementary automorphism of $P_0 \oplus R$ and let $s : R \to P_0 \oplus R$ be a section of $a \in Um(P_0 \oplus R)$. Then clearly $\varphi^{-1}s$ is a section of $a\varphi$. We let $i : P_0 \oplus R \to P(a) \oplus R$ and $j : P_0 \oplus R \to P(a\varphi) \oplus R$ be the isomorphisms induced by the sections s and $\varphi^{-1}s$. We will show that

$$(\varphi \oplus 1)^t (i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)(\varphi \oplus 1) = (j \oplus 1)^t (\chi_{a\varphi} \perp \psi_2)(j \oplus 1).$$

The theorem then follows from Corollary 3.3.
So let us show the equality above. Directly from the definitions, one immediately checks that $(i \oplus 1)(\varphi \oplus 1) = ((\varphi \oplus 1) \oplus 1)(j \oplus 1)$, where by abuse of notation we understand φ as the induced isomorphism $P(a\varphi) \to P(a)$. Altogether, it only remains to show that $\varphi^t \chi_a \varphi = \chi_{a\varphi}$.
For this, let (p, q) a pair of elements in $P(a\varphi)$; by definition, $\chi_{a\varphi}$ sends these elements to the image of $p \wedge q$ under the isomorphism $\det(P(a\varphi)) \cong R$. This element can also be described as the image of $p \wedge q \wedge \varphi^{-1}s(1)$ under the isomorphism $\det(P_0 \oplus R) \cong R$.

Analogously, the alternating form $\varphi^t \chi_a \varphi$ sends the pair (p,q) to the image of the element $\varphi(p) \wedge \varphi(q) \wedge s(1)$ under the isomorphism $\det(P_0 \oplus R) \cong R$. Therefore Lemma 1.11 allows us to conclude as desired, which finishes the proof of the theorem. $\square$

Note that if we equip the set $Um(P_0 \oplus R)$ with the projection $\pi_R = \pi_{3,3} : P_0 \oplus R \to R$ onto R as a basepoint, then the generalized Vaserstein symbol is a map of pointed sets because $V(\pi_R) = [P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi_0 \perp \psi_2] = 0$.

Let us briefly discuss how the generalized Vaserstein symbol depends on the choice of the trivialization θ_0 of the determinant of P_0. For this, recall that we have defined an action of $R^\times$ on $V(R)$ in Section 3.3. In case of a smooth affine algebra over a perfect field of characteristic $\neq 2$, we saw also in Section 3.3 that this action can be identified with the multiplicative action induced by a product map in the theory of higher Grothendieck-Witt groups.

Now let P_0 be a projective R-module of rank 2 which admits a trivialization θ_0 of its determinant. Furthermore, let $a \in Um(P_0 \oplus R)$ with section s and let i, χ_0, χ_a as in the definition of the generalized Vaserstein symbol. We consider another trivialization θ_0' of $\det(P_0)$ and we let χ_0' and χ_a' be the corresponding alternating forms on P_0 and $P(a)$. Obviously, there is a unit $u \in R^\times$ such that $\theta_0 = u \cdot \theta_0'$; in particular, we have $u \cdot \chi_0 = \chi_0'$ and $u \cdot \chi_a = \chi_a'$. Thus, if we denote the Vaserstein symbol associated to θ_0' by $V_{\theta_0'}$, then

$$V_{\theta_0'} = [P_0 \oplus R^2, (u \cdot \chi_0) \perp \psi_2, (i \oplus 1)^t ((u \cdot \chi_a) \perp \psi_2)(i \oplus 1)].$$

Finally, the isometry given by $P_0 \oplus R^2 \xrightarrow{id_{P_0} \oplus 1 \oplus u} P_0 \oplus R^2$ yields an equality

$$\begin{aligned} &[P_0 \oplus R^2, (u \cdot \chi_0) \perp \psi_2, (i \oplus 1)^t ((u \cdot \chi_a) \perp \psi_2)(i \oplus 1)] \\ &= [P_0 \oplus R^2, u \cdot (\chi_0 \perp \psi_2), u \cdot (i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)]. \end{aligned}$$

Thus, if we denote the Vaserstein symbol associated to θ_0 by V_{θ_0}, then

$$V_{\theta_0'} = u \cdot V_{\theta_0}.$$

In particular, the property of the generalized Vaserstein symbol to be injective, surjective or bijective onto $\tilde{V}(R)$ does not depend on the choice of θ_0.

There is another immediate consequence of this: If we let $P_0 = R^2$ be the free R-module of rank 2 and let $e_1 = (1,0), e_2 = (0,1) \in R^2$ be the obvious elements, then there is a canonical isomorphism $\theta_0 : R \xrightarrow{\cong} \det(R^2)$ given by $1 \mapsto e_1 \wedge e_2$. Then recall that the usual Vaserstein symbol can be described as $V_{\theta_0} \circ M$ (up to the identification $W_E(R) \cong \tilde{V}(R)$). But by the formula above, it immediately follows that the generalized Vaserstein symbol associated to $-\theta_0$ coincides with the usual Vaserstein symbol via the identification $\tilde{V}(R) \cong W_E(R)$ mentioned above.

We will now study the generalized Vaserstein symbol $V_{\theta_0} : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$ and give some criteria for its surjectivity and injectivity. As we have already seen, these properties are independent of the choice of a trivialization of $\det(P_0)$. So let us again fix such a trivialization $\theta_0 : R \xrightarrow{\cong} \det(P_0)$ and let us denote V_{θ_0} simply by V.
Recall that a unimodular row of length n is an n-tuple $a = (a_1, ..., a_n)$ of elements in R such that there are elements $b_1, ..., b_n \in R$ with $\sum_{i=1}^n a_i b_i = 1$. We denote by $Um_n(R)$ the set of unimodular rows of length n. For any $n \geq 3$, there are obvious maps $U_n : Um_{n-2}(R) \to Um(P_n)$. As a first step towards our criterion for the surjectivity of the generalized Vaserstein symbol (cp. Theorem 4.8 below), we prove the following statement:

Lemma 4.7. *Any element of the form $[P_4, \chi_0 \perp \psi_2, \chi] \in \tilde{V}(R)$ for a non-degenerate alternating form χ on P_4 is in the image of the generalized Vaserstein symbol.*

Proof. First of all, we set $a = \chi(-, e_4) : P_0 \oplus Re_3 \to R$. Since χ is non-degenerate, there is an element $p \in P_4$ such that $\chi(-, p) : P_4 \to R$ is just $-\pi_{4,4}$. In fact, since $\chi(p, p) = 0$, it immediately follows that $p \in P_3$. But then $a(p) = \chi(p, e_4) = -\chi(e_4, p) = 1$. Hence p defines a section $s : R \to P_3$, $1 \mapsto p$, of $a : P_0 \oplus Re_3 \to R$.
The generalized Vaserstein symbol $V(a)$ of a may therefore be computed by means of this section: As in the definition of the generalized Vaserstein symbol, we obtain an isomorphism $i : P_0 \oplus R \to P(a) \oplus R$ and an alternating form χ_a on $P(a) = \ker(a)$ induced by a and its section s. The generalized Vaserstein symbol $V(a)$ of a is then just given by the element $[P_0 \oplus R^2, \chi_0 \perp \psi_2, (i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)]$ of $\tilde{V}(R)$. But one can check easily that the form $(i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)$ locally coincides with χ by construction. By Lemma 1.2, it thus follows that it also coincides with χ globally. Therefore we obtain the desired equality $V(a) = [P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi]$. □

Using Lemma 4.7 and the technical lemmas proven in previous chapters, we may now prove the following criteria for the surjectivity of the generalized Vaserstein symbol:

Theorem 4.8. *Let $N \in \mathbb{N}$. Assume that $\beta \in \tilde{V}(R)$ is of the form $[P_{2N+2}, \chi_0 \perp \psi_{2N}, \chi]$ for some non-degenerate alternating form χ on P_{2N+2}. Furthermore, assume that we have an equality $\pi_{2n+1,2n+1}(E_\infty(P_0) \cap Aut(P_{2n+1})) = Um(P_{2n+1})$ for any $n \in \mathbb{N}$ such that $1 < n \leq N$.*

Then β lies in the image of the generalized Vaserstein symbol. As a consequence, the generalized Vaserstein symbol $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$ is surjective if the equality $\pi_{2n+1,2n+1}(E_\infty(P_0) \cap Aut(P_{2n+1})) = Um(P_{2n+1})$ holds for all $n \geq 2$.

Proof. By assumption, $\beta \in \tilde{V}(R)$ has the form $\beta = [P_{2N+2}, \chi_0 \perp \psi_{2N}, \chi]$ for some non-degenerate alternating form χ on P_{2N+2}. Furthermore, we may inductively apply Lemma 1.10 (because of the second assumption) in order to deduce that there is a stably elementary automorphism φ on P_{2N+2} such that $\varphi^t \chi \varphi = \psi \perp \psi_{2N-2}$ for some non-degenerate alternating form ψ on P_4. In particular, $\beta = [P_4, \chi_0 \perp \psi_2, \psi]$ by Corollary 3.3. Finally, any element of this form is in the image of the generalized Vaserstein symbol by Lemma 4.7. So β is in the image of the generalized Vaserstein symbol.
For the last statement, note that any element of $\tilde{V}(R)$ is of the form $[R^{2n}, \psi_{2n}, \chi]$ for some non-degenerate alternating form χ on R^{2n} (because of the isomorphism $\tilde{V}(R) \cong W_E(R)$). We may then artificially add a trivial summand $[P_0, \chi_0, \chi_0]$; hence any element of $\tilde{V}(R)$ is of the form $[P_{2n+2}, \chi_0 \perp \psi_{2n}, \chi_0 \perp \chi]$ for some non-degenerate alternating form χ on R^{2n}. We can then conclude by the previous paragraph. □

Theorem 4.9. *Let $N \in \mathbb{N}$. Assume that the following conditions are satisfied:*

- *Every element of $\tilde{V}(R)$ is of the form $[R^{2N}, \psi_{2N}, \chi]$ for some non-degenerate alternating form χ on R^{2N}.*
- *One has $\pi_{2n+1,2n+1}(E_\infty(P_0) \cap Aut(P_{2n+1})) = Um(P_{2n+1})$ for any $n \in \mathbb{N}$ with $1 < n < N$ and $U_{2N+1}(Um_{2N-1}(R)) \subset \pi_{2N+1,2N+1}E(P_{2N+1})$.*

Then the generalized Vaserstein symbol $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$ is surjective.

Proof. We proceed as in the proof of Theorem 4.8: By the first assumption, any element of $\tilde{V}(R)$ is of the form $[R^{2N}, \psi_{2N}, \chi]$ for some non-degenerate alternating form χ on R^{2N}. Again adding a trivial summand $[P_0, \chi_0, \chi_0]$, we see that any element of $\tilde{V}(R)$ is of the form $[P_{2N+2}, \chi_0 \perp \psi_{2N}, \chi_0 \perp \chi]$ for some non-degenerate alternating form χ on R^{2N}. As in the proof of Theorem 4.8, it then follows inductively from Lemma 1.10 that any element of $\tilde{V}(R)$ is of the form $[P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi]$ for some non-degenerate alternating form χ on $P_0 \oplus R^2$. The generalized Vaserstein symbol is then surjective by Lemma 4.7. Note that the condition $\pi_{2N+1,2N+1}E(P_{2N+1}) = Um(P_{2N+1})$ can be replaced by the weaker condition $U_{2N+1}(Um_{2N-1}) \subset \pi_{2N+1,2N+1}E(P_{2N+1})$ in our situation. □

Corollary 4.10. *Assume that the following conditions are satisfied:*

- *The usual Vaserstein symbol* $V : Um_3(R)/E_3(R) \to W_E(R)$ *is surjective;*

- $U_5(Um_3(R)) \subset \pi_{5,5}(E_\infty(P_0) \cap Aut(P_5))$.

Then the generalized Vaserstein symbol $V_{\theta_0} : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$ *is surjective.*

Proof. The surjectivity of the usual Vaserstein symbol means that any element of $\tilde{V}(R)$ is of the form $[R^4, \psi_4, \chi]$ for some non-degenerate alternating form χ on R^4. Now the corollary follows from the proof of Theorem 4.9. □

In order to prove our criterion for the injectivity of the generalized Vaserstein symbol, we first introduce the following condition: We will say that P_0 satisfies condition $(*)$ if $[P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi_1] = [P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi_2] \in \tilde{V}(R)$ for alternating forms χ_1, χ_2 on $P_0 \oplus R^2$ implies $\alpha^t(\chi_1 \perp \psi_{2n})\alpha = \chi_2 \perp \psi_{2n}$ for some automorphism $\alpha \in E_\infty(P_0) \cap Aut(P_{2n+4})$.
If P_0 is a free R-module, condition $(*)$ is satisfied, which basically follows from the isomorphism $V_{free}(R) \cong W_E'(R)$. Moreover, using the isomorphisms $V(R) \cong V_{free}(R) \cong W_E'(R)$, we will see that it is possible to prove that condition $(*)$ is always satisfied (cp. Lemma 4.12). As a first step towards Lemma 4.12, we observe:

Lemma 4.11. *Let* χ *be a non-degenerate alternating form on a finitely generated projective* R*-module* P*. Then there exists a finitely generated projective* R*-module* P' *with a non-degenerate alternating form* χ' *on* P' *and an isomorphism* $\tau : R^{2n} \xrightarrow{\cong} P \oplus P'$ *such that* $\tau^t(\chi \perp \chi')\tau = \psi_{2n}$.

Proof. Let Q be a finitely generated projective R-module such that $P \oplus Q$ is free. Then, for $Q_1 = P^\vee \oplus Q \oplus Q^\vee$, one has $P \oplus Q_1 \cong R^{2m}$ for some $m \geq 0$. Moreover, for $\phi_1 = can\chi^{-1} \perp H_Q$, the form $\chi \perp \phi_1$ is hence isometric to a form ϕ_2 on R^{2m}. Now let ϕ_3 be a form on R^{2s} for some $s \geq 0$ which represents the inverse of ϕ_2 in $W_E'(R)$. Then $\phi_2 \perp \phi_3 \perp \psi_{2t}$ is isometric to $\psi_{2m+2s+2t}$ for some $t \geq 0$. We set $P' = Q_1 \oplus R^{2s+2t}$ and $\chi' = \phi_1 \perp \phi_3 \perp \psi_{2t}$. Then there is an isometry $\tau : R^{2m+2s+2n} \to P'$ between from $\psi_{2m+2s+2t}$ and $\chi \perp \chi'$, as desired. □

Using Lemma 4.11, we may prove:

Lemma 4.12. *Any* P_0 *satisfies condition* $(*)$.

Proof. We use the explicit description of the inverse of $V_{free}(R) \to V(R)$ to prove Lemma 4.13 below, which obviously implies Lemma 4.12 for $P = P_0 \oplus R^2$ and $\chi = \chi_0 \perp \psi_2$: □

Lemma 4.13. *If $[P,\chi,\chi_1] = [P,\chi,\chi_2] \in V(R)$ for non-degenerate alternating forms χ, χ_1 and χ_2 on a finitely generated projective R-module P, then $\alpha^t(\chi_1 \perp \psi_{2n})\alpha = \chi_2 \perp \psi_{2n}$ for some $n \in \mathbb{N}$ and some automorphism $\alpha \in E(P \oplus R^{2n})$.*

Proof. The equality $[P,\chi,\chi_1] = [P,\chi,\chi_2]$ means that $[P,\chi_1,\chi_2] = 0$. By Lemma 4.11, it follows that there is a finitely generated projective R-module P_1 with a non-degenerate alternating form χ' on P_1 and, moreover, with an isomorphism $\tau : R^{2m} \xrightarrow{\cong} P \oplus P_1$ such that $\tau^t(\chi_1 \perp \chi')\tau = \psi_{2m}$. In particular, one has $0 = [P,\chi_1,\chi_2] = [R^{2m},\psi_{2m},\tau^t(\chi_2 \perp \chi')\tau]$ in $V(R)$. Therefore the class of $\tau^t(\chi_2 \perp \chi')\tau$ in $W'_E(R)$ is trivial and there exist $u \geq 1$ and $\zeta \in E(R^{2m+2u})$ such that $\zeta^t((\tau^t(\chi_2 \perp \chi')\tau) \perp \psi_{2u})\zeta = \psi_{2m+2u}$. Note that ζ lies in the commutator subgroup of $Aut(R^{2m+2u})$.
Again by Lemma 4.11, there exists a finitely generated projective R-module P_2 with a non-degenerate alternating form χ'' on P_2 and with an isomorphism $\beta : R^{2v} \xrightarrow{\cong} P_1 \oplus R^{2u} \oplus P_2$ such that $\beta^t(\chi' \perp \psi_{2u} \perp \chi'')\beta = \psi_{2v}$. But then the composite

$$\xi = (id_P \oplus \beta^{-1})(\tau \oplus id_{R^{2u}} \oplus id_{P_2})(\zeta^{-1} \oplus id_{P_2})(\tau^{-1} \oplus id_{R^{2u}} \oplus id_{P_2})(id_P \oplus \beta)$$

is an isometry from $\chi_1 \perp \psi_{2v}$ to $\chi_2 \perp \psi_{2v}$ and also lies in the commutator subgroup of $Aut(P \oplus R^{2v})$ because it is a conjugate of $\zeta^{-1} \perp id_{P_2}$. In particular, it follows that $\xi \perp id_{R^{2w}} \in E(P \oplus R^{2v+2w})$ for some $w \geq 0$. Finally, if we then set $\alpha = \xi \perp id_{R^{2w}}$ and $n = v + w$, the lemma is proven. □

Now that we have proven that condition $(*)$ is always satisfied, we can find conditions which imply that two elements $a, b \in Um(P_0 \oplus R)$ with the same Vaserstein symbol are equal up to a stably elementary automorphism of $P_0 \oplus R$. More precisely:

Theorem 4.14. *Assume that $E(P_{2n})e_{2n} = (E_\infty(P_0) \cap Aut(P_{2n}))e_{2n}$ for $n \geq 2$. Then $V(a) = V(b)$ for $a, b \in Um(P_0 \oplus R)$ implies that $b = a\varphi$ for some $\varphi \in E_\infty(P_0) \cap Aut(P_3)$.*

Proof. Let $a, b \in Um(P_0 \oplus R)$ with sections s, t respectively and let $i : P_0 \oplus R \to P(a) \oplus R$ and $j : P_0 \oplus R \to P(a) \oplus R$ be the isomorphisms induced by these sections. Furthermore, we let $V(a,s) = (i \oplus 1)^t(\chi_a \perp \psi_2)(i \oplus 1)$ and $V(b,t) = (j \oplus 1)^t(\chi_b \perp \psi_2)(j \oplus 1)$ be the alternating forms on $P_0 \oplus R^2$ appearing in the definition of the generalized Vaserstein symbols of a and b respectively. Now let us assume that $V(a) = V(b)$. Since P_0 satisfies condition $(*)$, there exist $n \in \mathbb{N}$ and an automorphism $\alpha \in E_\infty(P_0) \cap Aut(P_{2n+4})$ such that $\alpha^t(V(a,s) \perp \psi_{2n})\alpha = V(b,t) \perp \psi_{2n}$. Using Lemma 1.9, we may inductively deduce that $\beta^t V(a,s)\beta = V(b,t)$ for some $\beta \in E_\infty(P_0) \cap Aut(P_0 \oplus R^2)$. Now by Lemma 1.8 and the assumption in the theorem, there exists an automorphism $\gamma \in E(P_0 \oplus R^2) \cap Sp(V(a,s))$

such that $\beta e_4 = \gamma e_4$.
We now define $\delta : P_0 \oplus R \to P_0 \oplus R$ as the composite

$$P_0 \oplus Re_3 \to P_0 \oplus Re_3 \oplus Re_4 \xrightarrow{\gamma^{-1}\beta} P_0 \oplus Re_3 \oplus Re_4 \to P_0 \oplus Re_3.$$

One can then check that δ is an element of $E_\infty(P_0) \cap Aut(P_0 \oplus R)$. Moreover, we have

$$\beta^t (\gamma^{-1})^t V(a,s) \gamma^{-1} \beta = V(b,t)$$

and in particular $a\delta = b$, as desired. □

Corollary 4.15. *Under the hypotheses of Theorem 4.14, furthermore assume that the equality $a(E_\infty(P_0) \cap Aut(P_0 \oplus R)) = aE(P_0 \oplus R)$ holds for all $a \in Um(P_0 \oplus R)$. Then the generalized Vaserstein symbol $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$ is injective.*

Proof. By Theorem 4.14, we already know that $V(a) = V(b)$ implies $b = a\varphi'$ for some $\varphi' \in E_\infty(P_0) \cap Aut(P_0 \oplus R)$. Now by the additional assumption, there also exists an elementary automorphism φ of $P_0 \oplus R$ such that $b = a\varphi$. So the generalized Vaserstein symbol is injective. □

Regarding the additional assumption in Corollary 4.15, it is actually possible to adapt the arguments given in the proof of [SV, Corollary 7.4] in order to prove that the desired equality $a(E_\infty(P_0) \cap Aut(P_0 \oplus R)) = aE(P_0 \oplus R)$ holds for all $a \in Um(P_0 \oplus R)$ if one has $E_\infty(P_0) \cap Aut(P_0 \oplus R^2) = E(P_4)$:

Lemma 4.16. *If the equality $E_\infty(P_0) \cap Aut(P_0 \oplus R^2) = E(P_4)$ holds, then also the equality $a(E_\infty(P_0) \cap Aut(P_0 \oplus R)) = aE(P_0 \oplus R)$ holds for all $a \in Um(P_0 \oplus R)$.*

Proof. Let $a \in Um(P_0 \oplus R)$ with section s and let $\varphi \in E_\infty(P_0) \cap Aut(P_0 \oplus R)$. If we let $V(a,s)$ be the alternating form from the definition of the generalized Vaserstein symbol, then it follows from the proof of Lemma 4.7 that

$$(\varphi \oplus 1)^t V(a,s)(\varphi \oplus 1) = V(a', s')$$

for some $a' \in Um(P_0 \oplus R)$ with section s'. By assumption, the automorphism $\varphi \oplus 1$ of P_4 is an element of the group $E(P_4)$. Moreover, by Corollary 1.4, the group $E(P_4)$ is generated by elementary automorphisms $\varphi_g = id_{P_4} + g$, where g is a homomorphism

1) $g : Re_3 \to P_0$,

2) $g : P_0 \to Re_3$,

3) $g : Re_3 \to Re_4$ or

4) $g : Re_4 \to Re_3$.

It therefore suffices to show the following: If $\varphi_g^t V(a,s)\varphi_g = V(a',s')$ for some g as above, then $a' = a\psi$ for some $\psi \in E(P_0 \oplus R)$. The only non-trivial case is the last one, i.e. if g is a homomorphism $Re_4 \to Re_3$.
So let $g : Re_4 \to Re_3$ and let φ_g be the induced elementary automorphism of P_4. As explained above, we assume that

$$\varphi_g^t V(a,s)\varphi_g = V(a',s')$$

for some epimorphism $a' : P_0 \oplus Re_3 \to R$ with section s'. Now write $a = (a_0, a_R)$, where a_0 is the restriction of a to P_0 and $a_R = a(e_3)$ respectively. Furthermore, let $p = \pi_{P_0}(s(1))$. From now on, we interpret the alternating form χ_0 in the definition of the generalized Vaserstein symbol as an alternating isomorphism $\chi_0 : P \to P^\vee$. Then one can check locally that

$$a' = (a_0 - g(1) \cdot \chi_0(p), a_R).$$

Then let us define an automorphism ψ of P_3 as follows: We first define an endomorphism of P_0 by

$$\psi_0 = id_{P_0} - g(1) \cdot \pi_{P_0} \circ s \circ \chi_0(p) : P_0 \to P_0$$

and we also define a morphism $P_0 \to Re_3$ by

$$\psi_R = -g(1) \cdot \pi_R \circ s \circ \chi_0(p) : P_0 \to R.$$

Then we consider the endomorphism of $P_0 \oplus R$ given by

$$\psi = \begin{pmatrix} \psi_0 & 0 \\ \psi_R & id_R \end{pmatrix}.$$

First of all, this endomorphism coincides up to an elementary automorphism with

$$\begin{pmatrix} \psi_0 & 0 \\ 0 & id_R \end{pmatrix}.$$

Since $\chi_0(p) \circ \pi_{P_0} \circ s = 0$, this endomorphism is an element of $E(P_0 \oplus R)$ by Lemma 1.6. Hence the same holds for ψ. Finally, one can check easily that $a\psi = a'$ by construction. $\square$

As an immediate consequence, we can finally deduce our criterion for the injectivity of the generalized Vaserstein symbol:

Theorem 4.17. *Assume that* $E(P_{2n})e_{2n} = (E_\infty(P_0) \cap Aut(P_{2n}))e_{2n}$ *for all* $n \geq 3$ *and furthermore that* $E_\infty(P_0) \cap Aut(P_4) = E(P_4)$. *Then the generalized Vaserstein symbol* $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$ *is injective.*

Proof. Combine Corollary 4.15 and Lemma 4.16. □

Let us now study the criteria for the surjectivity and injectivity of the generalized Vaserstein symbol found in this section. In [HB], the conditions of Theorem 4.8 and Theorem 4.17 are studied in a very general framework. If R is a Noetherian ring of Krull dimension d, it follows from [HB, Chapter IV, Theorem 3.4] that actually $Unim.El.(P_n) = E(P_n)e_n$ for all $n \geq d+2$ (or $Um(P_n) = \pi_{n,n}E(P_n)$ for all $n \geq d+2$). In particular, if $\dim(R) \leq 4$, then the generalized Vaserstein symbol is injective as soon as $E_\infty(P_0) \cap Aut(P_4) = E(P_4)$; if $\dim(R) \leq 3$, it is surjective. Hence the following results are immediate consequences of our stability results in Section 1.5:

Theorem 4.18. *Assume that* R *is a regular Noetherian ring of dimension* $d = 2$. *Then the generalized Vaserstein symbol* $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$ *is a bijection.*

Proof. This follows directly from Theorem 1.21. □

Theorem 4.19. *Assume that* R *is a regular affine algebra of dimension* $d = 3$ *over an algebraically closed field* k *or over a perfect field* k *such that* $c.d.(k) \leq 1$ *and* $6 \in k^\times$. *Then the generalized Vaserstein symbol* $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$ *is a bijection.*

Proof. It follows from [S1] and [B, Remark 4.2] that k satisfies property $\mathcal{P}(4,3)$ if k is algebraically closed or if k is infinite perfect with $c.d.(k) \leq 1$ and $6 \in k^\times$. If k is finite, this follows from Proposition 1.15. Hence the theorem follows directly from Theorem 1.22. □

Because of the pointed surjection $Um(P_0 \oplus R)/E(P_0 \oplus R) \to \phi_2^{-1}([P_0 \oplus R])$, the bijectivity of the generalized Vaserstein symbol always gives rise to a surjection $W_E(R) \to \phi_2^{-1}([P_0 \oplus R])$; in this case, it seems that the group structure of $W_E(R) \cong Um(P_0 \oplus R)/E(P_0 \oplus R)$ essentially governs the structure of the fiber $\phi_2^{-1}([P_0 \oplus R])$.
The following application follows - to some degree - the pattern of the proof of [FRS, Theorem 7.5] and illustrates the previous paragraph:

Theorem 4.20. *Let* R *be a commutative ring and* P_0 *be a projective* R*-module of rank* 2 *which admits a trivialization* θ_0 *of its determinant. Assume that the following conditions are satisfied:*

a) *The generalized Vaserstein symbol* $V_{\theta_0} : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}(R)$ *induced by* θ_0 *is a bijection.*

b) $2V_{\theta_0}(a_0, a_R) = V_{\theta_0}(a_0, a_R^2)$ *for* $(a_0, a_R) \in Um(P_0 \oplus R)$.

c) *The group* $W_E(R)$ *is* 2*-divisible.*

Then $\phi_2^{-1}([P_0 \oplus R])$ *is trivial.*

Proof. Assume $P' \oplus R \cong P_0 \oplus R$. As we have seen in Section 1.3, P' has an associated element in the orbit space $Um(P_0 \oplus R)/Aut(P_0 \oplus R)$. We lift this element to an element $[b]$ of $Um(P_0 \oplus R)/E(P_0 \oplus R)$ ($[b]$ denotes the class of $b \in Um(P_0 \oplus R)$). Since the generalized Vaserstein symbol is a bijection and $W_E(R)$ is a 2-divisible group by assumption, we get that $[b] = 2[a]$, where $[a]$ denotes the class of an element $a = (a_0, a_R)$ of $Um(P_0 \oplus R)$ in the orbit space $Um(P_0 \oplus R)/E(P_0 \oplus R)$. But then the second assumption shows that $2[a] = [(a_0, a_R^2)]$. It follows from [B, Proposition 2.7] or [S1, Lemma 2] that any element of $Um(P_0 \oplus R)$ of the form (a_0, a_R^2) is completable to an automorphism of $P_0 \oplus R$, i.e. $\pi_R \varphi = (a_0, a_R^2)$ for some automorphism φ of $P_0 \oplus R$, where $\pi_R = \pi_{3,3} : P_0 \oplus R \to R$ is the projection. Altogether, π_R and b therefore lie in the same orbit under the action of $Aut(P_0 \oplus R)$ and hence $P' \cong P$. Thus, $\phi_2^{-1}([P_0 \oplus R])$ is trivial. $\square$

In the remainder of this section, we are going to address the second condition in the theorem above. In fact, we are going to prove that the sum formula $nV(a_0, a_R) = V(a_0, a_R^n)$ holds for a smooth affine algebra R over a perfect field k with $char(k) \neq 2$ such that $-1 \in k^{\times 2}$ if $n \equiv 0, 1 \ mod \ 4$. This formula will be proven by using the $\mathbb{A}^1_R$-homotopy category over the base scheme $Spec(R)$.

For the first lemma below, we let R be any commutative ring and P_0 be as usual a projective R-module of rank 2 with a fixed trivialization θ_0 of its determinant. Furthermore, we let $f : R \to B$ and $g : B \to C$ be ring homomorphisms. Then we have canonical maps

$$f^*_{Um} : Um(P_0 \oplus R) \to Um((P_0 \otimes_R B) \oplus B)$$

and

$$g^*_{Um} : Um((P_0 \otimes_R B) \oplus B) \to Um((P_0 \otimes_R C) \oplus C).$$

Furthermore, the B-module $P_0 \otimes_R B$ and the C-module $P_0 \otimes_R C$ have trivial determinants; their trivializations are given by $\theta_0 \otimes_R B$ and $\theta_0 \otimes_R C$ respectively. Finally, note that there is a group homomorphism $g^* : V(B) \to V(C)$ which sends any class $[P, \chi_1, \chi_2]$ in $V(B)$ to the class $[P \otimes_B C, \chi_1 \otimes_B C, \chi_2 \otimes_B C]$ in $V(C)$.

Lemma 4.21. *We have $V_{\theta_0 \otimes_R C}(g^*_{Um}(a)) = g^*(V_{\theta \otimes_R B}(a))$ for any $a \in Um((P_0 \otimes_R B) \oplus B)$.*

Proof. If $s : B \to (P_0 \otimes_R B) \oplus B$ is a section of a, then s clearly induces a section $s \otimes_B C$ of $a \otimes_B C \in Um((P_0 \otimes_R C) \oplus C)$. We let $P(a) = \ker(a)$, $P(a \otimes_B C) = \ker(a \otimes_B C)$ and we let $i_s : (P_0 \otimes_R B) \oplus B \to P(a) \oplus B$ and $i_{s \otimes_B C} : (P_0 \otimes_R C) \oplus C \to P(a \otimes_B C) \oplus C$ be the isomorphisms induced by s and $s \otimes_B C$. Furthermore, we let $V(a, s)$ and $V(a \otimes_B C, s \otimes_B C)$ be the alternating forms $(i_s \oplus 1)^t (\chi_a \perp \psi_2)(i_s \oplus 1)$ and $(i_{s \otimes_B C} \oplus 1)^t (\chi_{a \otimes_B C} \perp \psi_2)(i_{s \otimes_B C} \oplus 1)$ from the definition of the generalized Vaserstein symbol. As usual, we let $P_4 = P_0 \oplus R^2$. Then, under the isomorphism $(P_4 \otimes_R B) \otimes_B C \xrightarrow{\cong} P_4 \otimes_R C$, it is routine to check that the alternating form $V(a, s) \otimes_B C$ corresponds to $V(a \otimes_B C, s \otimes_B C)$. This proves the lemma. □

We now fix a smooth affine algebra R over a perfect field k with $char(k) \neq 2$ as a base ring and give an alternative description of the generalized Vaserstein symbol for smooth affine algebras over the base ring R.

For this, we start with a few general remarks: We fix finitely generated projective R-modules P and Q such that $P \oplus Q = R^n$ for some $n \in \mathbb{N}$. Furthermore, we denote by $Sym(P)$, $Sym(Q)$ and $Sym(R^n) = R[X_1, ..., X_n]$ the symmetric R-algebras of P, Q and R^n respectively. Next we set $E(P) = Spec(Sym(P))$, $E(Q) = Spec(Sym(Q))$ and identify $\mathbb{A}^n_R$ with $Spec(Sym(R^n))$. Note that the inclusions i_P, i_Q of P and Q into R^n and the projections π_P, π_Q of R^n onto P and Q respectively induce R-algebra homomorphisms between the corresponding symmetric algebras.

We denote by $\langle P \rangle$ and $\langle Q \rangle$ the ideals in $Sym(P)$ and $Sym(Q)$ generated by the homogeneous elements of degree ≥ 1 and denote by 0 their corresponding closed subschemes of $E(P)$ and $E(Q)$. By abuse of notation, we also denote by $\langle P \rangle$ and $\langle Q \rangle$ the ideals generated by their images in $Sym(R^n)$. Note that $Sym(R^n)/\langle Q \rangle \cong Sym(P)$.

Now let $S^R_{2n-1} = R[X_1, ..., X_n, Y_1, ..., Y_n]/\langle \sum_{i=1}^n X_i Y_i - 1 \rangle$ and let $Q^R_{2n-1} = Spec(S^R_{2n-1})$. Then the R-algebra homomorphism

$$i_n : Sym(R^n) = R[X_1, ..., X_n] \to S^R_{2n-1}, X_i \mapsto \bar{X}_i$$

induces a Zariski-locally trivial morphism of schemes

$$pr_n : Q^R_{2n-1} \to \mathbb{A}^n_R \setminus 0$$

with fibers isomorphic to $\mathbb{A}^{n-1}_R$.

Again by abuse of notation, we will denote by $\langle Q \rangle$ the ideal generated by the image of $\langle Q \rangle \subset Sym(Q)$ under the map $Sym(Q) \to Sym(R^n) \xrightarrow{i_n} S^R_{2n-1}$; furthermore, we define

$\bar{S}^R_{2n-1} = S^R_{2n-1}/\langle Q\rangle$ and $\bar{Q}^R_{2n-1} = Spec(\bar{S}^R_{2n-1})$. One can check easily that the composite of R-algebra homomorphisms

$$\bar{i}_n : Sym(P) \to Sym(R^n) \xrightarrow{i_n} S^R_{2n-1} \to \bar{S}^R_{2n-1}$$

induces a Zariski-locally trivial morphism of schemes

$$\bar{pr}_n : \bar{Q}^R_{2n-1} \to E(P) \smallsetminus 0$$

with fibers isomorphic to $\mathbb{A}^{n-1}_R$. It follows that $\bar{Q}^R_{2n-1}$ is a smooth scheme over $Spec(R)$ and $\bar{pr}_n$ is an $\mathbb{A}^1_R$-weak equivalence.
Now let B be a smooth affine algebra over R. Then one can check easily that there are natural bijections

$$Hom_{R\text{-}\mathfrak{Alg}}(Sym(P), B) \xrightarrow{\cong} Hom_{B\text{-}\mathfrak{Mod}}(P \otimes_R B, B)$$

and

$$Hom_{R\text{-}\mathfrak{Alg}}(\bar{S}^R_{2n-1}, B) \xrightarrow{\cong} \{(a,s) \in Um(P \otimes_R B) \times B^n | a(\pi_{P\otimes_R B}(s)) = 1\}.$$

We apply the previous paragraphs now to the case $P = P_3 = P_0 \oplus R$ (with P_0 a projective R-module of rank 2 with a fixed trivialization θ_0 of its determinant as usual). The epimorphism $\pi_R : P_0 \oplus R \to R$ with section $(0,1) \in P_0 \oplus R$ induces basepoints $Spec(R) \to \bar{Q}_{2n-1}$ and $Spec(R) \to E(P) \smallsetminus 0$. The morphism $\bar{pr}_n : \bar{Q}^R_{2n-1} \to E(P) \smallsetminus 0$ is then a pointed morphism; in particular, it has an inverse $\bar{pr}_n^{-1}$ in $\mathcal{H}_\bullet(R)$. Forgetting the basepoints, we may also interpret this morphism as a morphism in $\mathcal{H}(R)$.
The identity of $\bar{S}^R_{2n-1}$ corresponds to an epimorphism $a : P_3 \otimes_R \bar{S}^R_{2n-1} \to \bar{S}^R_{2n-1}$ with a section $s \in P_3 \otimes_R \bar{S}^R_{2n-1}$ and an element $t \in Q \otimes_R \bar{S}^R_{2n-1}$. Therefore the identity on $\bar{Q}^R_{2n-1}$ determines a well-defined generalized Vaserstein symbol (with respect to the fixed trivialization $\theta_0 \otimes_R \bar{S}^R_{2n-1}$ of $\det(P_0) \otimes_R \bar{S}^R_{2n-1}$)

$$V(a) \in V(\bar{S}^R_{2n-1}) \cong [\bar{Q}^R_{2n-1}, \mathcal{R}\Omega^1_s \mathcal{GW}^3]_{\mathbb{A}^1_R},$$

which corresponds to a morphism $\bar{Q}^R_{2n-1} \to \mathcal{R}\Omega^1_s \mathcal{GW}^3$ in $\mathcal{H}(R)$; we will denote this morphism by $\mathcal{V}$. Furthermore, the composite

$$E(P) \smallsetminus 0 \xrightarrow{\bar{pr}_n^{-1}} \bar{Q}^R_{2n-1} \xrightarrow{\mathcal{V}} \mathcal{R}\Omega^1_s \mathcal{GW}^3$$

defines a morphism $E(P) \smallsetminus 0 \to \mathcal{R}\Omega^1_s \mathcal{GW}^3$ in $\mathcal{H}(R)$, which we again denote by $\mathcal{V}$.

Lemma 4.22. *Assume that B is a smooth affine algebra over R, $a \in Um(P_3 \otimes_R B)$ and $f_a : Spec(B) \to E(P_3) \setminus 0$ the morphism of schemes corresponding to a. Then $\mathcal{V} \circ f_a$ corresponds to the generalized Vaserstein symbol of a associated to the trivialization $\theta_0 \otimes_R B$.*

Proof. This follows directly from Lemma 4.21. □

As a matter of fact, there is a formal way to prove that we can assume that the composite $E(P) \setminus 0 \xrightarrow{\bar{pr}_n^{-1}} \bar{Q}_{2n-1}^R \xrightarrow{\mathcal{V}} \mathcal{R}\Omega_s^1 \mathcal{GW}^3$ can be represented by an actual morphism of pointed spaces $E(P) \setminus 0 \to \mathcal{R}\Omega_s^1 \mathcal{GW}^3$: Since $\mathcal{R}\Omega_s^1 \mathcal{GW}^3$ is $\mathbb{A}_R^1$-fibrant, we already know that the composite $E(P) \setminus 0 \xrightarrow{\bar{pr}_n^{-1}} \bar{Q}_{2n-1}^R \xrightarrow{\mathcal{V}} \mathcal{R}\Omega_s^1 \mathcal{GW}^3$ is given by an actual morphism of spaces. Moreover, since the composite $Spec(R) \to E(P) \setminus 0 \xrightarrow{\mathcal{V}\bar{pr}_n^{-1}} \mathcal{R}\Omega_s^1 \mathcal{GW}^3$ computes the generalized Vaserstein symbol of the projection $\pi_R : P_0 \oplus R \to R$, it is null-homotopic. As $\mathcal{R}\Omega_s^1 \mathcal{GW}^3$ is $\mathbb{A}_R^1$-fibrant, there is a naive $\mathbb{A}_R^1$-homotopy from the basepoint $Spec(R) \to \mathcal{R}\Omega_s^1 \mathcal{GW}^3$ of $\mathcal{R}\Omega_s^1 \mathcal{GW}^3$ to the composite $Spec(R) \to E(P) \setminus 0 \xrightarrow{\mathcal{V}\bar{pr}_n^{-1}} \mathcal{R}\Omega_s^1 \mathcal{GW}^3$. By adjuntion, this naive $\mathbb{A}_R^1$-homotopy is represented by a morphism $H : Spec(R) \to \underline{Hom}(\mathbb{A}_R^1, \mathcal{R}\Omega_s^1 \mathcal{GW}^3)$. As $\underline{Hom}(Spec(R), \mathcal{R}\Omega_s^1 \mathcal{GW}^3) = \mathcal{R}\Omega_s^1 \mathcal{GW}^3$, we obtain a commutative diagram

$$\begin{tikzcd}
Spec(R) \arrow[r, "H"] \arrow[d] & \underline{Hom}(\mathbb{A}_R^1, \mathcal{R}\Omega_s^1 \mathcal{GW}^3) \arrow[d, "ev_1"] \\
E(P) \setminus 0 \arrow[r, "\mathcal{V}\bar{pr}_n^{-1}"'] & \mathcal{R}\Omega_s^1 \mathcal{GW}^3,
\end{tikzcd}$$

where the right-hand vertical morphism is induced by evaluation at 1. By [MV, Lemma 2.2.9], this morphism is a simplicial fibration and weak equivalence; since furthermore the morphism $Spec(R) \to E(P) \setminus 0$ is a cofibration, there automatically exists a morphism $F : E(P) \setminus 0 \to \underline{Hom}(\mathbb{A}_R^1, \mathcal{R}\Omega_s^1 \mathcal{GW}^3)$ making the two resulting triangles commute. If we let $ev_0 : \underline{Hom}(\mathbb{A}_R^1, \mathcal{R}\Omega_s^1 \mathcal{GW}^3) \to \mathcal{R}\Omega_s^1 \mathcal{GW}^3$ be the morphism induced by evaluation at 0, then the composite $ev_0 F$ is a pointed morphism $E(P) \setminus 0 \to \mathcal{R}\Omega_s^1 \mathcal{GW}^3$ which is naively $\mathbb{A}_R^1$-homotopic to $\mathcal{V}\bar{pr}_n^{-1}$. Hence it follows that we can assume that the composite $E(P) \setminus 0 \xrightarrow{\bar{pr}_n^{-1}} \bar{Q}_{2n-1}^R \xrightarrow{\mathcal{V}} \mathcal{R}\Omega_s^1 \mathcal{GW}^3$ can be represented by an actual morphism of pointed spaces $E(P) \setminus 0 \to \mathcal{R}\Omega_s^1 \mathcal{GW}^3$.

The previous paragraph finally enables us to prove the desired sum formula:

Theorem 4.23. *Let R be a smooth affine algebra over a perfect field k with $char(k) \neq 2$ such that $-1 \in k^{\times 2}$ and $n \in \mathbb{N}$. Furthermore, let P_0 be a projective R-module of rank 2 with a fixed trivialization $\theta_0 : R \xrightarrow{\cong} \det(P_0)$ of its determinant. If $n \equiv 0, 1$ mod 4, then the sum formula $V_{\theta_0}(a_0, a_R^n) = n \cdot V_{\theta_0}(a_0, a_R)$ holds for all $(a_0, a_R) \in Um(P_0 \oplus R)$.*

Proof. As we have just seen, the generalized Vaserstein symbol can be defined by means of a pointed morphism $\mathcal{V} : E(P_3) \setminus 0 \to \mathcal{R}\Omega^1_s \mathcal{GW}^3$ in $Spc_{R,\bullet}$.
Setting $P = P_3$, we now consider the pushout square

$$\begin{tikzcd}
((E(P_0) \setminus 0) \times \mathbb{G}_{m,R})_+ \arrow[r] \arrow[d] & (E(P_0) \times \mathbb{G}_{m,R})_+ \arrow[d] \\
((E(P_0) \setminus 0) \times \mathbb{A}^1_R)_+ \arrow[r] & (E(P) \setminus 0)_+
\end{tikzcd}$$

in $Spc_{R,\bullet}$ given by the Zariski covering of $(E(P) \setminus 0)_+$, which is also a homotopy pushout square. Furthermore, we also consider the square

$$\begin{tikzcd}
(E(P_0) \setminus 0)_+ \times \mathbb{G}_{m,R} \arrow[r] \arrow[d] & E(P_0) \times \mathbb{G}_{m,R} \\
(E(P_0) \setminus 0)_+ \times \mathbb{A}^1_R &
\end{tikzcd}$$

and let $\mathcal{Y}$ be its homotopy pushout. Clearly, the obvious morphism from the first to the second diagram induces a morphism $i : (E(P) \setminus 0)_+ \to \mathcal{Y}$. Furthermore, the n-fold power map $\mathbb{A}^1_R \to \mathbb{A}^1_R$ can be used to define power operations $\psi_n : (E(P) \setminus 0)_+ \to (E(P) \setminus 0)_+$ and $\bar{\psi}_n : \mathcal{Y} \to \mathcal{Y}$ respectively. By sending $* \times \mathbb{G}_{m,R}$ and $* \times \mathbb{A}^1_R$ to the basepoint of $\mathcal{R}\Omega^1_s \mathcal{GW}^3$, we can extend the morphism $\mathcal{V}_+$ obtained from the morphism $\mathcal{V}$ defining the generalized Vaserstein symbol to a morphism $\bar{\mathcal{V}} : \mathcal{Y} \to \mathcal{R}\Omega^1_s \mathcal{GW}^3$.
Now the commutative diagram

$$\begin{tikzcd}
(E(P) \setminus 0)_+ \arrow[r, "\psi_n"] \arrow[d, "i"] & (E(P) \setminus 0)_+ \arrow[r, "\mathcal{V}_+"] \arrow[d, "i"] & \mathcal{R}\Omega^1_s \mathcal{GW}^3 \arrow[d, equal] \\
\mathcal{Y} \arrow[r, "\bar{\psi}_n"] & \mathcal{Y} \arrow[r, "\bar{\mathcal{V}}"] & \mathcal{R}\Omega^1_s \mathcal{GW}^3
\end{tikzcd}$$

shows that it suffices to show that the composition $\bar{\mathcal{V}} \circ \bar{\psi}_n$ is equal to $n \cdot \bar{\mathcal{V}}$ in $\mathcal{H}_\bullet(R)$ with respect to the group structure on $[\mathcal{Y}, \mathcal{R}\Omega^1_s \mathcal{GW}^3]_{\mathbb{A}^1_R,\bullet}$ induced by $\mathcal{R}\Omega^1_s \mathcal{GW}^3$.
But since $\mathcal{Y}$ is the homotopy pushout of the diagram

$$\begin{tikzcd}
(E(P_0) \setminus 0)_+ \times \mathbb{G}_{m,R} \arrow[r] \arrow[d] & \mathbb{G}_{m,R}\ , \\
(E(P_0) \setminus 0)_+ &
\end{tikzcd}$$

it is weakly equivalent to $S^1 \wedge \mathbb{G}_{m,R} \wedge (E(P_0) \setminus 0)_+$ (cp. [Mo, p.219]) and therefore has the structure of an h-cogroup. Under this weak equivalence, the power operation $\bar{\psi}_n$ then corresponds to the smash product of the n-fold power map on $\mathbb{G}_{m,R}$ with $S^1 \wedge (E(P_0) \setminus 0)_+$. By Corollary 2.2, this implies that $\bar{\psi}_n$ is equal to $n \cdot id_{\mathcal{Y}}$ in $[\mathcal{Y}, \mathcal{Y}]_{\mathbb{A}^1_R, \bullet}$ and also that $\bar{\mathcal{V}} \circ \bar{\psi}_n$ is equal to $n \cdot \bar{\mathcal{V}}$ in $[\mathcal{Y}, \mathcal{R}\Omega^1_s \mathcal{GW}^3]_{\mathbb{A}^1_R, \bullet}$ with respect to the group structures induced by $\mathcal{Y}$ as an h-cogroup. By the usual Eckmann-Hilton argument, it follows that $\bar{\mathcal{V}} \circ \bar{\psi}_n$ is equal to the n-fold sum of $\bar{\mathcal{V}}$ with respect to the group structure on $[\mathcal{Y}, \mathcal{R}\Omega^1_s \mathcal{GW}^3]_{\mathbb{A}^1_R, \bullet}$ induced by $\mathcal{R}\Omega^1_s \mathcal{GW}^3$ as an h-group. This proves the theorem. □

We conclude this section with two applications of the previous theorem:

Theorem 4.24. *Let R be a normal affine algebra of dimension $d \geq 3$ over an algebraically closed field k with $char(k) \neq 2$; if $d = 3$, furthermore assume that R is smooth. Let P_0 be a projective R-module of rank 2 with a trivial determinant and let $P_n = P_0 \oplus R^{n-2}$ for $n \geq 3$. Then, for any $a \in Um(P_d)$ and $j \in \mathbb{N}$ with $gcd(char(k), j) = 1$, there is an automorphism $\varphi \in E(P_d)$ such that $a\varphi$ has the form $b = (b_0, b_3^j, ..., b_d)$.*

Proof. Let $a = (a_0, a_3, ..., a_d) \in Um(P_d)$ and $I = \langle a_4, ..., a_d \rangle$. By Lemma 1.14, we know that we can assume that R/I is either 0 or a smooth affine algebra of dimension 3 over k. If $R/I = 0$, then Lemma 1.13 proves the statement of the theorem. So let us assume that R/I is a smooth affine algebra of dimension 3 over k. In this case, we know that the generalized Vaserstein symbol associated to P_0/IP_0 and any fixed trivialization of its determinant gives a pointed bijection between $Um(P_3/IP_3)/E(P_3/IP_3)$ and $\tilde{V}(R/I)$; this bijection induces a group structure on $Um(P_3/IP_3)/E(P_3/IP_3)$ (cp. Theorem 4.19). Since the latter group is divisible prime to $char(k) \neq 2$ (cp. [FRS, Propositions 5.1 and 6.1]), there is $(\bar{b}_0, \bar{b}_3) \in Um(P_3/IP_3)$ with $4j \cdot (\bar{b}_0, \bar{b}_3) = (\bar{a}_0, \bar{a}_3)$ in $Um(P_3/IP_3)/E(P_3/IP_3)$. Then the previous theorem implies that in fact $(\bar{b}_0, \bar{b}_3^{4j}) = (\bar{a}_0, \bar{a}_3)$ in $Um(P_3/IP_3)/E(P_3/IP_3)$. Applying the map $\Phi_3(a)$ now yields the theorem. □

Corollary 4.25. *Let R be a smooth affine algebra of dimension 3 over an algebraically closed field k with $char(k) \neq 2$ and let P_0 a projective R-module of rank 2 with trivial determinant. Then $Um(P_0 \oplus R)/SL(P_0 \oplus R)$ is trivial; in particular, P_0 is cancellative.*

Proof. Let $a = (a_0, a_3) \in Um(P_0 \oplus R)$. By the previous theorem, there is $\varphi \in E(P_0 \oplus R)$ such that $a\varphi$ is of the form $b = (b_0, b_3^2)$. By Proposition 1.17, there is $\psi \in SL(P_0 \oplus R)$ such that $b\psi$ is the projection onto R. This proves that $Um(P_0 \oplus R)/SL(P_0 \oplus R)$ is trivial. In particular, this implies that the orbit space $Um(P_0 \oplus R)/Aut(P_0 \oplus R)$ is trivial and hence that P_0 is cancellative.

$\square$

4.3 The generalized Vaserstein symbol modulo SL

In this section, we compose the generalized Vaserstein symbol V_{θ_0} with the canonical epimorphism $\tilde{V}(R) \to \tilde{V}_{SL}(R)$. In fact, we will see immediately that this map descends to a map $V_{\theta_0} : Um(P_0 \oplus R)/SL(P_0 \oplus R) \to \tilde{V}_{SL}(R)$. We will study this map under some suitable assumptions and deduce a criterion for the triviality of the orbit space $Um(P_0 \oplus R)/SL(P_0 \oplus R)$. In particular, by studying symplectic orbits of unimodular rows, we will prove that $Um_3(R)/SL_3(R)$ is trivial if and only if $\tilde{V}_{SL}(R)$ is trivial whenever R is a smooth affine algebra of dimension 4 over an algebraically closed field k with $6 \in k^\times$.
We will use the notation of Sections 1.2 and 4.2: Throughout this section, we let R be a commutative ring and we let P_0 be a projective R-module of rank 2 with a fixed trivialization $\theta_0 : R \xrightarrow{\cong} \det(P_0)$ of its determinant. For all $n \geq 3$, we let $P_n = P_0 \oplus Re_3 \oplus ... \oplus Re_n$ and we will sometimes omit the explicit generators e_i, $i = 3, ..., n$, of the free direct summands of rank 1 in the notation. Again, we denote by $\pi_{k,n} : P_n \to R$ the projections onto the free direct summands of rank 1 with index $k = 3, ..., n$.
As usual, we will mostly omit the trivialization θ_0 in our notation and denote V_{θ_0} simply by V if there is no ambiguity. As a first step, we prove:

Theorem 4.26. *Let $\varphi \in SL(P_0 \oplus R)$ and $a \in Um(P_0 \oplus R)$. Then $V(a) = V(a\varphi)$ in $\tilde{V}_{SL}(R)$. In particular, we obtain a well-defined map $V : Um(P_0 \oplus R)/SL(P_0 \oplus R) \to \tilde{V}_{SL}(R)$, which we call the generalized Vaserstein symbol modulo SL.*

Proof. Let $\varphi \in SL(P_0 \oplus R)$ and let $s : R \to P_0 \oplus R$ be a section of $a \in Um(P_0 \oplus R)$. Then $\varphi^{-1}s$ is a section of $a\varphi$. We let $i : P_0 \oplus R \to P(a) \oplus R$ and $j : P_0 \oplus R \to P(a\varphi) \oplus R$ be the isomorphisms induced by the sections s and $\varphi^{-1}s$. Obviously, it suffices to show that

$$(\varphi \oplus 1)^t (i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)(\varphi \oplus 1) = (j \oplus 1)^t (\chi_{(a\varphi)} \perp \psi_2)(j \oplus 1).$$

As in the proof of Theorem 4.6, one checks that $(i \oplus 1)(\varphi \oplus 1) = ((\varphi \oplus 1) \oplus 1)(j \oplus 1)$, where by abuse of notation we understand φ as the induced isomorphism $P(a\varphi) \to P(a)$. Hence it suffices to show that $\varphi^t \chi_a \varphi = \chi_{a\varphi}$.
For this, we let (p, q) a pair of elements in $P(a\varphi)$; by definition, $\chi_{a\varphi}$ sends these elements to the image of $p \wedge q$ under the isomorphism $\det(P(a\varphi)) \cong R$. This element can also be described as the image of $p \wedge q \wedge \varphi^{-1}s(1)$ under the isomorphism $\det(P_0 \oplus R) \cong R$.

Analogously, the alternating form $\varphi^t \chi_a \varphi$ sends the pair (p,q) to the image of the element $\varphi(p) \wedge \varphi(q) \wedge s(1)$ under the isomorphism $\det(P_0 \oplus R) \cong R$. Since φ has determinant 1, the automorphism of $\det(P_0 \oplus R)$ induced by φ is the identity (cp. Lemma 1.11). This proves the desired equality $\varphi^t \chi_a \varphi = \chi_{a\varphi}$. □

By abuse of notation, we denote by $V = V_\theta : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}_{SL}(R)$ the composite of the generalized Vaserstein symbol associated to θ_0 and the canonical epimorphism $\tilde{V}(R) \to \tilde{V}_{SL}(R)$.

Theorem 4.27. *Let R be a Noetherian ring of Krull dimension ≤ 4. Assume that $SL(P_5)$ acts transitively on $Um(P_5)$. Then the map $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}_{SL}(R)$ is surjective.*

Proof. Let $\beta \in \tilde{V}_{SL}(R)$. Since $\dim(R) \leq 4$, we know that $Um(P_n) = \pi_{n,n}E(P_n)$ for all $n \geq 6$. Therefore every element in $\tilde{V}(R)$ is of the form $[P_6, \chi_0 \perp \psi_4, \chi]$ for some non-degenerate alternating form χ on P_6 by Lemma 1.10; hence the same holds for any element in $\tilde{V}_{SL}(R)$. Consequently, we can write $\beta = [P_6, \chi_0 \perp \psi_4, \chi]$.
Now let $d = \chi(-, e_6) : P_5 \to R$. Since d can be locally checked to be an epimorphism, there is an automorphism $\varphi \in SL(P_5)$ such that $d\varphi = \pi_{5,5}$. Then the alternating form $\chi' = (\varphi \oplus 1)^t \chi (\varphi \oplus 1)$ satisfies that $\chi'(-, e_6) : P_5 \to R$ is just $\pi_{5,5}$. Now we simply define $c = \chi'(-, e_5) : P_5 \to R$ and let $\varphi_c = id_{P_6} + ce_6$ be the elementary automorphism on P_6 induced by c; then ${\varphi_c}^t \chi' \varphi_c = \psi \perp \psi_2$ for some non-degenerate alternating form ψ on P_4. Since all the isometries we used have determinant 1, we conclude that $\beta = [P_4, \chi_0 \perp \psi_2, \psi]$. As any element of this form lies in the image of the generalized Vaserstein symbol by Lemma 4.7, this proves the theorem. □

We remark that the assumption in the last theorem is satisfied if R is an affine algebra of dimension ≤ 4 over an infinite perfect field k of cohomological dimension ≤ 1 with $6 \in k^\times$ (cp. [S1], [S5] and [B]) or if R is a Noetherian ring of dimension ≤ 3 ([HB, Chapter IV, Corollary 3.5]).
In order to study the fibers of the map $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}_{SL}(R)$, we prove the analogue of Lemma 4.13 for the group $\tilde{V}_{SL}(R)$:

Lemma 4.28. *If two elements $[P, \chi, \chi_1], [P, \chi, \chi_2] \in \tilde{V}_{SL}(R)$ are equal, then there is an automorphism φ of $SL(P \oplus R^{2n})$ for some $n > 0$ such that $\chi_1 \perp \psi_{2n} = \varphi^\vee (\chi_2 \perp \psi_{2n})\varphi$.*

Proof. The equality $[P, \chi, \chi_1] = [P, \chi, \chi_2]$ means that $[P, \chi_1, \chi_2] = 0$. By Lemma 4.11, it follows that there is a finitely generated projective R-module P_1 with a non-degenerate

alternating form χ' on P_1 and, moreover, with an isomorphism $\tau : R^{2m} \xrightarrow{\cong} P \oplus P_1$ such that $\tau^t(\chi_1 \perp \chi')\tau = \psi_{2m}$. In particular, one has $0 = [P, \chi_1, \chi_2] = [R^{2m}, \psi_{2m}, \tau^t(\chi_2 \perp \chi')\tau]$ in $\tilde{V}_{SL}(R)$. Therefore the class of $\tau^t(\chi_2 \perp \chi')\tau$ in $W'_{SL}(R)$ is trivial and hence there exist $u \geq 1$ and $\zeta \in SL(R^{2m+2u})$ such that $\zeta^t((\tau^t(\chi_2 \perp \chi')\tau) \perp \psi_{2u})\zeta = \psi_{2m+2u}$.
Again by Lemma 4.11, there exists a finitely generated projective R-module P_2 with a non-degenerate alternating form χ'' on P_2 and with an isomorphism $\beta : R^{2n} \xrightarrow{\cong} P_1 \oplus R^{2u} \oplus P_2$ such that $\beta^t(\chi' \perp \psi_{2u} \perp \chi'')\beta = \psi_{2n}$. But then the composite

$$\alpha = (id_P \oplus \beta^{-1})(\tau \oplus id_{R^{2u}} \oplus id_{P_2})(\zeta^{-1} \oplus id_{P_2})(\tau^{-1} \oplus id_{R^{2u}} \oplus id_{P_2})(id_P \oplus \beta)$$

is an isometry from $\chi_1 \perp \psi_{2n}$ to $\chi_2 \perp \psi_{2n}$ and clearly has determinant 1. This proves the lemma. $\square$

Now let us study the fibers of the map $V : Um(P_0 \oplus R)/E(P_0 \oplus R) \to \tilde{V}_{SL}(R)$. For this, we describe an action of $SL(P_4)$ on $Um(P_0 \oplus R)/E(P_0 \oplus R)$ as follows:
First of all, note that $E(P_4)$ is a normal subgroup of $SL(P_4)$: If we let $\varphi \in SL(P_4)$ and $\varphi' \in E(P_4)$, then there is a natural isotopy from id_{P_4} to $\varphi^{-1}\varphi'\varphi$. By Theorem 1.18 and Suslin's normality theorem (cp. [S3]), it follows that $\varphi^{-1}\varphi'\varphi \in E(P_4)$.
Now let $\varphi \in SL(P_4)$ and $a \in Um(P_0 \oplus R)$. We choose a section $s : R \to P_0 \oplus R$ of a and obtain a non-degenerate alternating form

$$V(a,s) = (i_s \oplus 1)^t(\chi_a \perp \psi_2)(i_s \oplus 1)$$

as in the definition of the generalized Vaserstein symbol. Then we consider the alternating form $\varphi^t V(a,s)\varphi$. By abuse of notation, we also denote by a the class of a in $Um(P_0 \oplus R)/E(P_0 \oplus R)$ and define $a \cdot \varphi$ to be the class in $Um(P_0 \oplus R)/E(P_0 \oplus R)$ represented by $\varphi^t V(a,s)\varphi(-, e_4) : P_0 \oplus R \to R$.
Now let us show that this assignment gives a well-defined right action of $SL(P_4)$ on $Um(P_0 \oplus R)/E(P_0 \oplus R)$: If we choose another section s' of a, then there is $\varphi' \in E(P_4)$ such that $\varphi' V(a,s')\varphi' = V(a,s)$ (cp. the proof of Theorem 4.4). Since $E(P_4)$ is a normal subgroup of $SL(P_4)$, it follows that

$$(\varphi)^t V(a,s)\varphi = (\varphi'')^t(\varphi)^t V(a,s')\varphi\varphi''$$

for some $\varphi'' \in E(P_4)$. The lemma below will hence imply that our assignment does not depend on the choice of the section s of a.
Similarly, if $a' = a\varphi'$ for $\varphi' \in E(P_0 \oplus R)$, then $V(a',s') = (\varphi' \oplus 1)^t V(a,s)(\varphi' \oplus 1)$, where $s' = (\varphi')^{-1}s$ (this follows from the proof of Theorem 4.6). Again, since $E(P_4)$ is normal in $SL(P_4)$, it follows that

$$(\varphi)^t V(a,s)\varphi = (\varphi'')^t (\varphi)^t V(a',s')\varphi\varphi''$$

for some $\varphi'' \in E(P_4)$. The following lemma then also implies that our assignment does only depend on the class of a in $Um(P_0 \oplus R)/E(P_0 \oplus R)$.

Lemma 4.29. *Let χ and χ' be non-degenerate alternating forms on the module P_4 such that $[P_4, \chi_0 \perp \psi_2, \chi], [P_4, \chi_0 \perp \psi_2, \chi'] \in \tilde{V}(R)$ and let $a = \chi(-, e_4), a' = \chi'(-, e_4) \in Um(P_0 \oplus R)$. If $\varphi^t \chi \varphi = \chi'$ for some $\varphi \in E(P_4)$, then the classes of a and a' coincide in the orbit space $Um(P_0 \oplus R)/E(P_0 \oplus R)$.*

Proof. First of all, the group $E(P_4)$ is generated by elementary automorphisms $\varphi_g = id_{P_4} + g$, where g is a homomorphism

1) $g : Re_3 \to P_0$,

2) $g : P_0 \to Re_3$,

3) $g : Re_3 \to Re_4$ or

4) $g : Re_4 \to Re_3$.

Furthermore, we can write $\chi = V(a,s)$ and $\chi' = V(a',s')$ for sections s and s' of a and a' respectively (cp. the proof of Lemma 4.7). Hence it suffices to show the following:
If $\varphi_g^t V(a,s)\varphi_g = V(a',s')$ for some g as above, then $a' = a\psi$ for some $\psi \in E(P_0 \oplus R)$. The only non-trivial case is the last one, i.e. if g is a homomorphism $Re_4 \to Re_3$.
As in the proof of Lemma 4.16, we let $g : Re_4 \to Re_3$ and let φ_g be the induced elementary automorphism of P_4 and we assume that

$$\varphi_g^t V(a,s)\varphi_g = V(a',s')$$

for some epimorphism $a' : P_0 \oplus Re_3 \to R$ with section s'. We then write a as $a = (a_0, a_R)$, where a_0 is the restriction of a to P_0 and $a_R = a(e_3)$. Moreover, we define $p = \pi_{P_0}(s(1))$. From now on, we interpret the alternating form χ_0 in the definition of the generalized Vaserstein symbol as an alternating isomorphism $\chi_0 : P \to P^\vee$. One can verify locally that

$$a' = (a_0 - g(1) \cdot \chi_0(p), a_R).$$

Then let us define an automorphism ψ of P_3 as follows: We first define an endomorphism of P_0 by

$$\psi_0 = id_{P_0} - g(1) \cdot \pi_{P_0} \circ s \circ \chi_0(p) : P_0 \to P_0$$

and we also define a morphism $P_0 \to Re_3$ by

$$\psi_R = -g(1) \cdot \pi_R \circ s \circ \chi_0(p) : P_0 \to R.$$

Then we consider the endomorphism of $P_0 \oplus R$ given by

$$\psi = \begin{pmatrix} \psi_0 & 0 \\ \psi_R & id_R \end{pmatrix}.$$

First of all, this endomorphism coincides up to an elementary automorphism with

$$\begin{pmatrix} \psi_0 & 0 \\ 0 & id_R \end{pmatrix}.$$

Since $\chi_0(p) \circ \pi_{P_0} \circ s = 0$, this endomorphism is an element of $E(P_0 \oplus R)$ by Lemma 1.6. Hence the same holds for ψ. Finally, one can check easily that $a\psi = a'$ by construction. $\square$

As indicated above, the previous lemma shows that our previous assignment gives a well-defined map

$$Um(P_0 \oplus R)/E(P_0 \oplus R) \times SL(P_4) \xrightarrow{-\cdot-} Um(P_0 \oplus R)/E(P_0 \oplus R).$$

Note that if $a \in Um(P_0 \oplus R)$ with section s and $\varphi \in SL(P_4)$, then it follows from the proof of Lemma 4.7 that the alternating form $\varphi^t V(a,s)\varphi$ equals $V(a\cdot\varphi, s')$ for some section s' of $a\cdot\varphi$. It follows that the map above is indeed a right action of $SL(P_4)$ on $Um(P_0 \oplus R)/E(P_0 \oplus R)$. In fact, the previous lemma shows that this action descends to an action of $SL(P_4)/E(P_4)$ on $Um(P_0 \oplus R)/E(P_0 \oplus R)$.

Lemma 4.30. *Let χ_1 and χ_2 be non-degenerate alternating forms on the module P_{2n} such that $\varphi^t(\chi_1 \perp \psi_2)\varphi = \chi_2 \perp \psi_2$ for some $\varphi \in SL(P_{2n+2})$. Furthermore, let $\chi = \chi_1 \perp \psi_2$. If $SL(P_{2n+2})e_{2n+2} = Sp(\chi)e_{2n+2}$ holds, then one has $\psi^t\chi_1\psi = \chi_2$ for some $\psi \in SL(P_{2n})$.*

Proof. Let $\psi'' e_{2n+2} = \varphi e_{2n+2}$ for some $\psi'' \in Sp(\chi)$. Then we set $\psi' = (\psi'')^{-1}\varphi$. Since $(\psi')^t(\chi_1 \perp \psi_2)\psi' = \chi_2 \perp \psi_2$, the composite $\psi : P_{2n} \xrightarrow{\psi'} P_{2n+2} \to P_{2n}$ and ψ' satisfy the following conditions:

- $\psi'(e_{2n+2}) = e_{2n+2}$;
- $\pi_{2n+1,2n+2}\psi' = \pi_{2n+1,2n+2}$;
- $\psi^t\chi_1\psi = \chi_2$.

These conditions imply that ψ equals ψ' up to elementary morphisms of P_{2n+2} and hence has determinant 1 as well. This finishes the proof. $\square$

Theorem 4.31. *Let R be a Noetherian ring of dimension ≤ 4. Let $a, a' \in Um(P_0 \oplus R)$. Then $V(a) = V(a')$ in $\tilde{V}_{SL}(R)$ if and only if $a \cdot \varphi = a'$ in $Um(P_0 \oplus R)/E(P_0 \oplus R)$ for some $\varphi \in SL(P_4)$.*

Proof. We let $s, s' : R \to P_0 \oplus R$ be sections of a and a' and $V(a,s)$ and $V(a',s')$ be the alternating forms induced by s and s' which appear in the definition of the generalized Vaserstein symbol. Now assume that $V(a) = V(a')$. Since $\dim(R) \leq 4$, we know that $E(P_n)e_n = Um(P_n)$ for all $n \geq 6$. In particular, one has $(E(P_{2n}) \cap Sp(\chi))e_{2n} = Um(P_{2n})$ for all $n \geq 3$ and all non-degenerate alternating forms on P_{2n} (cp. Lemma 1.8). Hence we can apply Lemma 4.28 and Lemma 4.30 in order to deduce that $\varphi^t V(a,s)\varphi = V(a',s')$ for some $\varphi \in SL(P_4)$. By definition of the action of $SL(P_4)$ on $Um(P_0 \oplus R)/E(P_0 \oplus R)$, this means that $a \cdot \varphi = a'$.
Conversely, assume that $a \cdot \varphi = a'$ for some $\varphi \in SL(P_4)$. By definition, this means that $\varphi^t V(a,s)\varphi = V(a'',s'')$, where the class of $a'' \in Um(P_0 \oplus R)$ coincides with the class of a' in $Um(P_0 \oplus R)/E(P_0 \oplus R)$ and s'' is a section of a''. In particular, it follows from the proofs of Theorem 4.4 and Theorem 4.6 that there exists $\psi \in E(P_4)$ with $\psi^t \varphi^t V(a,s)\varphi\psi = V(a',s')$. This clearly implies that $V(a) = V(a')$ in $\tilde{V}_{SL}(R)$. $\square$

For any Noetherian ring R of dimension ≤ 4, we have established the following exact sequence of groups and pointed sets whenever $SL(P_5)$ acts transitively on $Um(P_5)$:

$$SL(P_4) \Rightarrow Um(P_0 \oplus R)/E(P_0 \oplus R) \xrightarrow{V} \tilde{V}_{SL}(R) \to 0.$$

In this situation, we mean by exactness at $Um(P_0 \oplus R)/E(P_0 \oplus R)$ that two classes in $Um(P_0 \oplus R)/E(P_0 \oplus R)$ represented by $a, a' \in Um(P_0 \oplus R)$ satisfy $V(a) = V(a')$ in $\tilde{V}_{SL}(R)$ if and only if $a\varphi = a'$ for some $\varphi \in SL(P_4)$.
Furthermore, there is a well-defined right action of $SK_1(R)$ on $W_E(R) \cong \tilde{V}(R)$ given by the following assignment: If $\varphi \in SL_{2n}(R)$ and $\theta \in A_{2n}(R)$ represent elements of $SK_1(R)$ and $W_E(R)$, then $\theta \cdot \varphi$ is represented by the class of $\varphi^t \theta \varphi$ in $W_E(R)$. This action is compatible with the right action introduced above: Following [We, Chapter III, Lemma 1.6], any finitely generated projective R-module Q such that $P_0 \oplus Q \cong R^n$ for some $n > 0$ induces a well-defined group homomorphism $SL(P_4) \to SL_{n+2}(R)$. This induces a well-defined map $SL(P_4) \to SK_1(R)$ independent of the choice of Q. In fact, the map descends to a well-defined group homomorphism $St : SL(P_4)/E(P_4) \to SK_1(R)$. One can then check easily that the diagram

$$\begin{tikzcd}
Um(P_3)/E(P_3) \times SL(P_4)/E(P_4) \arrow[r] \arrow[d, "V\times St"] & Um(P_3)/E(P_3) \arrow[r] \arrow[d, "V"] & Um(P_3)/SL(P_3) \arrow[d, "V"] \\
\tilde{V}(R) \times SK_1(R) \arrow[r] & \tilde{V}(R) \arrow[r] & \tilde{V}_{SL}(R)
\end{tikzcd}$$

is commutative.

As a consequence of the previous theorem, we obtain the following criterion for the injectivity of the map $V: Um(P_0 \oplus R)/SL(P_0 \oplus R) \to \tilde{V}_{SL}(R)$:

Theorem 4.32. *Let R be a Noetherian ring of dimension ≤ 4. Then the induced map $V: Um(P_0 \oplus R)/SL(P_0 \oplus R) \to \tilde{V}_{SL}(R)$ is injective if and only $SL(P_4)e_4 = Sp(\chi)e_4$ for all non-degenerate alternating forms χ on P_4 such that $[P_4, \chi_0 \perp \psi_2, \chi] \in \tilde{V}(R)$.*

Proof. First of all, assume that $SL(P_4)e_4 = Sp(\chi)e_4$ for all non-degenerate alternating forms χ on P_4 such that $[P_4, \chi_0 \perp \psi_2, \chi] \in \tilde{V}(R)$. Now let $a, a' \in Um(P_0 \oplus R)$ such that $V(a) = V(a')$. Then $\varphi^t V(a,s)\varphi = V(a', s')$ for some $\varphi \in SL(P_4)$ and sections s, s' of a and a' by the previous theorem. By assumption there is $\varphi' \in Sp(V(a,s))$ with $\varphi e_4 = \varphi' e_4$. If we let $\varphi'' = \varphi'^{-1}\varphi$, then $\varphi'' e_4 = e_4$ and $\varphi''^t V(a,s)\varphi'' = V(a', s')$. Thus, if we write

$$\varphi'' = \begin{pmatrix} \varphi''_0 & 0 \\ \varphi''_R & 1 \end{pmatrix} \in Aut(P_3 \oplus R),$$

then φ''_0 has determinant 1 and satisfies $a' = a\varphi''_0$. In particular, the classes of a and a' in $Um(P_0 \oplus R)/SL(P_0 \oplus R)$ coincide and V is injective.

Conversely, assume that V is injective. Let χ be an arbitrary non-degenerate alternating form on P_4 such that $[P_4, \chi_0 \perp \psi_2, \chi] \in \tilde{V}(R)$ and also let $\varphi \in SL(P_4)$. We write $\chi = V(a,s)$ and $\varphi^t \chi \varphi = V(a', s')$ for $a, a' \in Um(P_0 \oplus R)$ with sections s and s'. Then obviously $V(a) = V(a')$. By assumption, there is $\varphi' \in SL(P_0 \oplus R)$ with $a' = a\varphi'$ and hence $(\varphi' \oplus 1)^t V(a,s)(\varphi' \oplus 1) = V(a', s'')$, where s'' is a section of a'. Furthermore, there exists $\varphi'' \in E(P_4)$ with $\varphi'' e_4 = e_4$ such that $\varphi''^t V(a', s'')\varphi'' = V(a', s')$ (cp. the proof of Theorem 4.4). The automorphism $\beta = \varphi\varphi''^{-1}(\varphi' \oplus 1)^{-1}$ lies in $Sp\chi$ and satisfies $\beta e_4 = \varphi e_4$, which proves the theorem. $\square$

The proof of Theorem 4.32 shows in particular the following statement:

Corollary 4.33. *Let R be a Noetherian ring of dimension ≤ 4. Assume that $SL(P_5)$ acts transitively on $Um(P_5)$. Then the orbit space $Um(P_0 \oplus R)/SL(P_0 \oplus R)$ is trivial if and only if $W_{SL}(R)$ is trivial and $SL(P_4)e_4 = Sp(\chi_0 \perp \psi_2)e_4$.*

As an immediate consequence, we can classify stably isomorphic oriented projective modules of rank 2 over affine algebras of dimension 3 over finite fields:

Theorem 4.34. *Assume that R is an affine algebra of dimension $d = 3$ over a finite field $\mathbb{F}_q$. Then $Sp(\chi)e_4 = Unim.El.(P_4)$ for any non-degenerate alternating form χ on P_4. In particular, the generalized Vaserstein symbol associated to any trivialization θ_0 of $\det(P_0)$ gives a bijection $V_{\theta_0} : Um(P_0 \oplus R)/SL(P_0 \oplus R) \xrightarrow{\cong} \tilde{V}_{SL}(R)$.*

Proof. Proposition 1.15 and Lemma 1.8 imply the first statement. The second statement follows from the first statement and Theorem 4.32. □

We remark that the group $\tilde{V}_{SL}(R)$ is not trivial in general for an affine algebra of dimension 3 over a finite field: Let $\mathbb{F}_p$ be the field with p elements for a prime number p with $p \equiv 1 \ mod\ 8$. We consider the polynomial $X^8 - a$ for some element $a \in \mathbb{F}_p^\times$ which is not a square; furthermore, we let $\sqrt[8]{a}$ be a root of this polynomial in an algebraic closure of $\mathbb{F}_p$. Since $p \equiv 1 \ mod\ 8$, the field $\mathbb{F}_p$ contains all 8th roots of unity, i.e. all zeros of the polynomial $X^8 - 1$ over $\mathbb{F}_p$. In particular, by Kummer's theorem on cyclic field extensions, we see that $\mathbb{F}_p(\sqrt[8]{a})$ is Galois over $\mathbb{F}_p$ and $[\mathbb{F}_p(\sqrt[8]{a}) : \mathbb{F}_p] = r$ such that r divides 8. Therefore the minimal polynomial $\mathcal{M}(\sqrt[8]{a})$ of $\sqrt[8]{a}$ over $\mathbb{F}_p$ has degree $1, 2, 4$ or 8. But the coefficient of $\mathcal{M}(\sqrt[8]{a})$ in degree zero is a product of 8th roots of unity (which are all in $\mathbb{F}_p$) and $\sqrt[8]{a}^r$. Since a is not a square in $\mathbb{F}_p$, it follows that $\sqrt[8]{a}^i \notin \mathbb{F}_p$ for $i = 1, 2, 4$ and r has to be 8. Hence $X^8 - a$ is irreducible over $\mathbb{F}_p$. If we take the polynomial $X^2 - a$ for N. Mohan Kumar's construction of stably free modules in [NMK], then we produce a smooth affine algebra $R_{\mathcal{K}}$ of dimension 3 over $\mathbb{F}_p$ which admits a non-free stably free module of rank 2. It follows from the previous theorem that $\tilde{V}_{SL}(R_{\mathcal{K}}) \neq 0$.
Recall that one of the basic tools to study the groups $W_E(R)$ and $W_{SL}(R)$ is the Karoubi periodicity sequence

$$K_1Sp(R) \to SK_1(R) \to W_E(R) \to K_0Sp(R) \to K_0(R).$$

We let $W_E(R)/SL_3(R)$ be the cokernel of the composite $SL_3(R) \to SK_1(R) \to W_E(R)$. Then we can deduce the following result from Corollary 4.33:

Corollary 4.35. *Assume that R is a smooth 4-dimensional algebra over the algebraic closure $k = \bar{\mathbb{F}}_q$ of a finite field such that $6 \in k^\times$. Then the orbit space $Um_3(R)/SL_3(R)$ is trivial if and only if $W_E(R)/SL_3(R)$ is trivial.*

Proof. As a matter of fact, it was proven in [FRS, Corollary 7.8] that the homomorphism $SL_4(R)/E_4(R) \xrightarrow{\cong} SK_1(R)$ is an isomorphism.
Now assume that $Um_3(R)/SL_3(R)$ is trivial. By Corollary 4.33, this means that the map $SK_1(R) \to W_E(R)$ is surjective and $Sp_4(R)e_4 = SL_4(R)e_4$. The second condition and the isomorphism $SL_4(R)/E_4(R) \cong SK_1(R)$ easily imply that any matrix in $SL_4(R)$ lies in $SL_3(R)$ up to a matrix in $Sp_4(R)E_4(R)$. Since elements in $Sp_4(R)E_4(R)$ are sent to 0 in $W_E(R)$ under the hyperbolic map $SK_1(R) \to W_E(R)$, this immediately implies that $W_E(R)/SL_3(R) = W_{SL}(R) = 0$.
Conversely, assume that $W_E(R)/SL_3(R)$ is trivial. Then $W_{SL}(R)$ is obviously trivial. Now let $\varphi \in SL_4(R)$. Then the class of the matrix $\varphi^t\psi_4\varphi$ is trivial in $W_E(R)/SL_3(R)$. By the Karoubi periodicity sequence, this means that there exists a matrix $\varphi' \in SL_3(R)$ such that $\varphi(\varphi' \oplus 1)^{-1}$ is in the image of the map $K_1Sp(R) \to SK_1(R)$. Since $\dim(R) = 4$, $K_1Sp(R)$ is generated by $Sp_4(R)$; the isomorphism $SL_4(R)/E_4(R) \cong SK_1(R)$ then implies that $\varphi''^{-1}\varphi(\varphi' \oplus 1)^{-1}$ lies in $E_4(R)$ for some $\varphi'' \in Sp_4(R)$. Since for any $v \in Um_4^t(R)$ one has $E_4(R)v = (E_4(R) \cap Sp_4(R))v$, it follows that there is an element $\psi \in E_4(R) \cap Sp_4(R)$ with $\varphi''^{-1}\varphi(\varphi' \oplus 1)^{-1}e_4 = \psi e_4$. Since $\varphi(\varphi' \oplus 1)^{-1}e_4 = \varphi e_4$, it follows that $\varphi e_4 = \varphi''\psi e_4$ and $\varphi''\psi \in Sp_4(R)$. This proves the corollary. □

Corollary 4.36. *Assume that R is a smooth affine algebra of dimension 3 over an algebraically closed field k with $char(k) \neq 2$. Then $Sp(\chi_0 \perp \psi_2)e_4 = Unim.El.(P_4)$.*

Proof. By Corollary 4.25 we know that $Um(P_0 \oplus R)/SL(P_0 \oplus R)$ is trivial. Since moreover $SL(P_4)e_4 = Unim.El.(P_4)$ and $SL(P_5)$ acts transitively on $Um(P_5)$, the result follows by Corollary 4.33. □

Recall that the Bass-Quillen conjecture $BQ(R)$ asserts that all finitely generated projective modules over $R[X_1, ...X_n]$ are extended from R whenever R is a regular Noetherian ring; in particular, all finitely generated projective $R[X]$-modules are free if R is a regular local ring such that $BQ(R)$ holds. The Bass-Quillen conjecture is known to hold in many cases, e.g. if R is a regular k-algebra essentially of finite type over a field k (cp. [Li]). Furthermore, it follows from the Quillen-Suslin theorem that all finitely generated projective $R[X]$-modules are free if R is a regular local ring of dimension ≤ 1. Moreover, M. P. Murthy proved in [M] that all finitely generated projective $R[X]$-modules are free if R is a regular local ring of dimension 2 and later R. A. Rao proved in [R] that the same statement holds if R is a regular local ring of dimension 3 with $6 \in R^\times$. Note that if R is a regular local ring, the assumption on regularity implies that all finitely generated projective modules over $R[X]$

are stably free and hence the conjecture holds if and only if $GL_r(R[X])$ acts transitively on $Um_r(R[X])$ (or, equivalently, on $Um_r^t(R[X])$) for all $r \geq 3$. We may thus deduce the following statement from the previous results:

Proposition 4.37. *Let R be a regular local ring of dimension 4 with $6 \in R^\times$. Then all finitely generated projective $R[X]$-modules are free if and only if $Sp_4(R[X])$ acts transitively on $Um_4^t(R[X])$.*

Proof. Since $R[X]$ is essentially of dimension 4, we know that $E_r(R[X])$ acts transitively on $Um_r(R[X])$ for $r \geq 6$. Moreover, it was proven in [R, Corollary 2.7] that $E_5(R[X])$ acts transitively on $Um_5(R[X])$ as well.
Of course, if we let $P_0 = R^2$, then there exists a canonical trivialization θ_0 of $\det(R^2)$ given by $1 \mapsto e_1 \wedge e_2$, where $e_1 = (1,0), e_2 = (0,1) \in R^2$. Consequently, there is a generalized Vaserstein symbol $V_{\theta_0} : Um_3(R[X])/SL_3(R[X]) \to \tilde{V}_{SL}(R[X])$ associated to θ_0. Although $\dim(R[X]) = 5$, the proofs of Theorems 4.31, 4.32 and Corollary 4.33 work for $R[X]$ because $E_r(R[X])$ acts transitively on $Um_r^t(R[X])$ for $r \geq 5$.
Now let us first assume that all finitely generated projective $R[X]$-modules are free. Then $SL_r(R[X])$ acts transitively on $Um_r^t(R[X])$ for $r = 3, 4$. In particular, the orbit space $Um_3(R[X])/SL_3(R[X])$ is trivial. Then it follows directly from Corollary 4.33 that $Sp_4(R[X])$ acts transitively on $Um_4^t(R[X])$.
Conversely, assume only that $Sp_4(R[X])$ acts transitively on $Um_4^t(R[X])$. The proofs of [R, Proposition 2.2 and Proposition 2.9] show that the usual Vaserstein symbol $V_{-\theta_0}$ and hence also $V_{\theta_0} : Um_3(R[X])/SL_3(R[X]) \to \tilde{V}_{SL}(R[X])$ is a constant map. But the proof of Theorem 4.32 then shows that it is also injective because $Sp_4(R[X])$ acts transitively on $Um_4^t(R[X])$. Consequently, all finitely generated projective $R[X]$-modules are free. □

Let R be a Noetherian commutative ring of dimension ≤ 4 such that $SL(P_5)$ acts transitively on $Um(P_5)$. We now try to use the previous results in order to give descriptions of the orbit spaces $Um(P_0 \oplus R)/E(P_0 \oplus R)$ and $Um(P_0 \oplus R)/SL(P_0 \oplus R)$.
For any map $F : M \to N$ between sets M and N, one obviously has $M = \cup_{x \in N} F^{-1}(x)$. Therefore we also have $Um(P_0 \oplus R)/E(P_0 \oplus R) = \cup_{\beta \in \tilde{V}_{SL}(R)} V^{-1}(\beta)$. Now let us fix an element $a \in Um(P_0 \oplus R)$ together with a section s and give a description of the preimage $V^{-1}(V(a)) \subset Um(P_0 \oplus R)/E(P_0 \oplus R)$. We set $\chi = V(a, s)$. We have an obvious map

$$i_a : SL(P_4) \to V^{-1}(V(a)), \varphi \mapsto a \cdot \varphi,$$

induced by the right action of $SL(P_4)$ on $Um(P_0 \oplus R)/E(P_0 \oplus R)$. By our observations above, this map is immediately surjective.

Now assume that there are two elements φ_1 and φ_2 of $SL(P_4)$ with $\varphi_1{\varphi_2}^{-1} \in Sp(\chi)E(P_4)$. Then obviously $i_a(\varphi_1) = i_a(\varphi_2)$. Conversely, let $\varphi_1, \varphi_2 \in SL(P_4)$ such that $i_a(\varphi_1) = i_a(\varphi_2)$. Then it follows from the proofs of Theorems 4.4 and 4.6 that there is an element $\varphi \in E(P_4)$ such that

$$\varphi_1{}^t \chi \varphi_1 = \varphi^t \varphi_2{}^t \chi \varphi_2 \varphi.$$

In particular, since $E(P_4)$ is a normal subgroup of $SL(P_4)$, it follows that $\varphi_1{\varphi_2}^{-1}$ lies in $Sp(\chi)E(P_4)$. Thus, it follows that i_a induces a bijection

$$i_a : Sp(\chi)E(P_4)\backslash SL(P_4) \xrightarrow{\cong} V^{-1}(V(a))$$

between the set of right cosets of $Sp(\chi)E(P_4)$ in $SL(P_4)$ and the preimage $V^{-1}(V(a))$. Altogether, we have just established the following description of $Um(P_0 \oplus R)/E(P_0 \oplus R)$:

Theorem 4.38. *Let R be a Noetherian commutative ring of dimension ≤ 4 such that $SL(P_5)$ acts transitively on $Um(P_5)$. Let $\{\chi_i\}_{i\in I}$ be a set of non-degenerate alternating forms on P_4 such that $I \to \tilde{V}_{SL}(R), i \mapsto [P_4, \chi_0 \perp \psi_2, \chi_i]$, is a bijection. Then there is a bijection $Um(P_0 \oplus R)/E(P_0 \oplus R) \cong \cup_{i\in I} Sp(\chi_i)E(P_4)\backslash SL(P_4)$.*

Remark 4.39. *We remark that $SL(P_4)/E(P_4)$ is abelian if R is a smooth affine algebra of dimension 4 over an algebraically closed field k such that $6 \in k^\times$ and P_0 is free: This follows from the fact that the map $SL_4(R)/E_4(R) \to SK_1(R)$ is injective in this situation (cp. [FRS, Corollary 7.7]). Hence the subgroup $Sp(\chi)E_4(R)$ of $SL_4(R)$ is normal and $Sp(\chi)E_4(R)\backslash SL_4(R) = SL_4/Sp(\chi)E_4(R)$.*

Let us now describe the orbit space $Um(P_0 \oplus R)/SL(P_0 \oplus R)$. Analogously, we consider the surjective map $V : Um(P_0 \oplus R)/SL(P_0 \oplus R) \to \tilde{V}_{SL}(R)$ and describe the preimages $V^{-1}(V(a))$ for $a \in Um(P_0 \oplus R)$. Henceforth we assume that $SL(P_4)/E(P_4)$ is an abelian group. By repeating the arguments above appropriately, we obtain a bijection

$$i_a : SL(P_4)/Sp(\chi)SL(P_3)E(P_4) \xrightarrow{\cong} V^{-1}(V(a)).$$

Theorem 4.40. *Let R be a Noetherian commutative ring of dimension ≤ 4 such that $SL(P_5)$ acts transitively on $Um(P_5)$. Let $\{\chi_i\}_{i\in I}$ be a set of non-degenerate alternating forms on P_4 such that $I \to \tilde{V}_{SL}(R), i \mapsto [P_4, \chi_0 \perp \psi_2, \chi_i]$, is a bijection. Furthermore, assume in addition that $SL(P_4)/E(P_4)$ is an abelian group. Then there is a bijection $Um(P_0 \oplus R)/SL(P_0 \oplus R) \cong \cup_{i\in I} SL(P_4)/Sp(\chi_i)SL(P_3)E(P_4)$.*

Because of Remark 4.39, we obtain the following description of $Um_3(R)/SL_3(R)$:

Corollary 4.41. *Let R be a smooth affine algebra of dimension ≤ 4 over an algebraically closed field k of characteristic $\neq 2,3$. Furthermore, let $\{\chi_i\}_{i\in I}$ be a set of non-degenerate alternating forms on R^4 such that the map $I \to \tilde{V}_{SL}(R), i \mapsto [R^4, \psi_4, \chi_i]$, is a bijection. Then there is a bijection $Um_3(R)/SL_3(R) \cong \cup_{i\in I} SL_4(R)/Sp(\chi_i)SL_3(R)E_4(R)$.*

Now let R be a smooth affine algebra of even dimension d over an algebraically closed field k with $d! \in k^\times$. Motivated by the previous results, we study the orbits of unimodular rows of length d under the right actions of $SL_d(R)$ and $Sp_d(R)$. We will use this to prove the equality $SL_d(R)e_d = Sp_d(R)e_d$. Since we have $SL_d(R)e_d = Um_d^t(R)$ in this case (cp. [FRS, Theorem 7.5]), this means that one has to prove that $Sp_d(R)$ acts transitively on the left on $Um_d^t(R)$.
As already indicated, we will approach this problem in terms of the right actions of $SL_d(R)$ and $Sp_d(R)$ on $Um_d(R)$. For the remainder of this section, we let $\pi_{1,d} = (1,0,...,0)$ and $\pi_{d,d} = (0,...,0,1)$ be the standard unimodular rows of length d and $e_{1,d} = \pi_{1,d}^t$ and $e_{d,d} = \pi_{d,d}^t$ the corresponding unimodular columns. As a first step, let us recall some basic facts about symplectic and elementary symplectic orbits. The following result by Gupta is a special case of [G, Theorem 3.9] and extends [CR, Theorem 5.5]:

Theorem 4.42. *Let R be a commutative ring. For any $n \in \mathbb{N}$ and any unimodular row $v \in Um_{2n}(R)$, the equality $vE_{2n}(R) = vESp_{2n}(R)$ holds.*

Corollary 4.43. *Let R be a commutative ring. If $v, v' \in Um_{2n}(R)$ for some $n \in \mathbb{N}$ and $vE_{2n}(R) = v'E_{2n}(R)$, then $vSp_{2n}(R) = v'Sp_{2n}(R)$.*

We can then give a partial answer to a question raised by Gupta (cp. [G, Question 5.5]):

Theorem 4.44. *Let R be a smooth affine algebra of dimension $d \geq 4$ over an algebraically closed field k with $d! \in k^\times$. Assume that d is divisible by 4. Then $Sp_d(R)$ acts transivitely on $Um_d(R)$.*

Proof. It follows from the proof of [FRS, Theorem 7.5] that any unimodular row of length d can be transformed via elementary matrices to a row of the form $(a_1, ..., a_{d-1}, a_d^{(d-1)!^2})$. By the previous corollary, it thus suffices to show that any such row of length d is the first row of a symplectic matrix.
So let $a = (a_1, ..., a_{d-1}, a_d^{(d-1)!})$ and let $b = (b_1, ..., b_{d-1}, b_d)$ be a unimodular row such that $ab^t = 1$. Furthermore, let $a' = (a_1, ..., a_{d-1}, a_d^{(d-1)!^2})$. It follows from [S4, Proposition 2.2,

Corollary 2.5] that there exists a matrix $\beta(a,b) \in SL_d(R)$ whose first row is a' such that $[\beta(a,b)] = [\alpha_d(a,b)]$ in $SK_1(R)$.

Now let us first assume that the class of $\alpha_d(a,b)$ in $K_1(R)$ lies in the image of the forgetful map $K_1Sp(R) \xrightarrow{f} K_1(R)$. Then it is well-known that $Sp(R) = ESp(R)Sp_d(R)$ (cp. [SV, Theorem 7.3(b)]). Therefore the class of $\alpha_d(a,b)$ actually lies in the image of the composite $Sp_d(R) \to K_1Sp(R) \xrightarrow{f} K_1(R)$. In other words, there exists a matrix $\varphi \in Sp_d(R)$ such that $[\varphi] = [\alpha_d(a,b)] = [\beta(a,b)]$ in $K_1(R)$. As the homomorphism $SL_d(R)/E_d(R) \to SK_1(R)$ is injective (cp. [FRS, Corollary 7.7]), it follows that $\beta(a,b)\varphi^{-1} \in E_d(R)$. Since the equality $\pi_{1,d}E_d(R) = \pi_{1,d}ESp_d(R)$ holds, there is $\psi \in ESp_d(R)$ such that $\pi_{1,d}\beta(a,b)\varphi^{-1} = \pi_{1,d}\psi$. In particular, $a' = \pi_{1,d}\beta(a,b) = \pi_{1,d}\psi\varphi$ lies in the orbit of $\pi_{1,d}$ under the action of $Sp_d(R)$.

Thus, it suffices to show that the class of $\alpha_d(a,b)$ in $K_1(R)$ indeed lies in the image of $K_1Sp(R) \xrightarrow{f} K_1(R)$. For this, recall that a unimodular row of length d over R corresponds to a morphism $X = Spec(R) \to \mathbb{A}^d_k \setminus 0$ and there is a canonical pointed $\mathbb{A}^1_k$-weak equivalence $p_{2d-1} = p^k_{2d-1} : Q^k_{2d-1} \to \mathbb{A}^d_k \setminus 0$. As a matter of fact, a morphism $X \to Q^k_{2d-1}$ corresponds to a unimodular row of length d with the choice of an explicit section. Furthermore, there is an $\mathbb{A}^1_k$-fiber sequence $Sp \to GL \to GL/Sp$, which induces the Karoubi periodicity sequence by taking the sets of morphisms in $H(k)$. There is a pointed morphism $\alpha_d : Q^k_{2d-1} \to SL \hookrightarrow GL$ induced by $\alpha_d(x,y)$.

Let $a'' = (a_1, ..., a_{d-1}, a_d) \in Um_d(R)$. We now interpret this unimodular row as a morphism $a'' : X \to \mathbb{A}^d_k \setminus 0$ of spaces. If we let $\Psi^{(d-1)!} : \mathbb{A}^d_k \setminus 0 \to \mathbb{A}^d_k \setminus 0$ be the morphism induced by $(x_1, ..., x_{d-1}, x_d) \mapsto (x_1, ..., x_{d-1}, x_d^{(d-1)!})$, then we obviously have $a = \Psi^{(d-1)!}a'' : X \to \mathbb{A}^d_k \setminus 0$. It thus suffices to prove the existence of a morphism $\mathbb{A}^d_k \setminus 0 \to Sp$ in $H(k)$ that makes the diagram

$$\begin{tikzcd} \mathbb{A}^d_k \setminus 0 \arrow[r, "p^{-1}_{2d-1}\Psi^{(d-1)!}"] \arrow[d, dotted] & Q^k_{2d-1} \arrow[d, "\alpha_d"] & \\ Sp \arrow[r] & GL \arrow[r] & GL/Sp \end{tikzcd}$$

commutative. For this purpose, we first of all note that the motivic Brouwer degree of $\Psi^{(d-1)!} \in [\mathbb{A}^d_k \setminus 0, \mathbb{A}^d_k \setminus 0]_{\mathbb{A}^1_k,\bullet} = GW(k)$ is $(d-1)!_\epsilon$. Since k is algebraically closed, it follows that $\alpha_d p^{-1}_{2d-1}\Psi^{(d-1)!}$ equals $(d-1)! \cdot \alpha_d p^{-1}_{2d-1} \in [\mathbb{A}^d_k \setminus 0, GL]_{\mathbb{A}^1_k,\bullet}$, where the group structure is understood with respect to the structure of $\mathbb{A}^d_k \setminus 0$ as an h-cogroup in $H_\bullet(k)$. The usual Eckmann-Hilton argument then implies that also $\alpha_d p^{-1}_{2d-1}\Psi^{(d-1)!} = (d-1)! \cdot \alpha_d p^{-1}_{2d-1}$ in $[\mathbb{A}^d_k \setminus 0, GL]_{\mathbb{A}^1_k,\bullet}$, where the group structure is understood with respect to the structure of an h-group of $GL \simeq_{\mathbb{A}^1_k} \mathcal{R}\Omega_s BGL$ in $H_\bullet(k)$. As $[\mathbb{A}^d_k \setminus 0, GL/Sp]_{\mathbb{A}^1_k} \cong W_E(S^k_{2n-1}) \cong \mathbb{Z}/2\mathbb{Z}$ and

$(d-1)!$ is even, it follows that

$$(\mathbb{A}^d_k \smallsetminus 0) \xrightarrow{p_{2d-1}^{-1}\Psi^{(d-1)!}} Q^k_{2d-1} \xrightarrow{\alpha_d} GL \to GL/Sp$$

is trivial and hence the factorization exists, as desired. □

As a consequence, we can prove a corresponding statement for the left action of $Sp_d(R)$ on $Um^t_d(R)$:

Corollary 4.45. *Let R be a smooth affine algebra of dimension $d \geq 4$ over an algebraically closed field k with $d! \in k^\times$. Assume that d is divisible by 4. Then $Sp_d(R)$ acts transitively on $Um^t_d(R)$; in particular, $Sp_d(R)e_d = SL_d(R)e_d$.*

Proof. First of all, let

$$\varphi_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(R).$$

We can then inductively define $\varphi_{2n+2} = \varphi_{2n} \perp \varphi_2 \in GL_{2n+2}(R)$ for all $n \in \mathbb{N}$. Furthermore, we have $\varphi^t_d \psi_d \varphi_d = \psi^t_d$, $\varphi^t_d = \varphi_d$ and $\varphi^{-1}_d = \varphi_d$.
Now let $v \in Um_d(R)$ and v^t the corresponding unimodular column. By the previous theorem, there is $\varphi \in Sp_d(R)$ with $\pi_{d,d}\varphi = v\varphi_d$. Then it follows that $\varphi_d \varphi^t \varphi_d \in Sp_d(R)$. Finally, one has $\varphi_d \varphi^t \varphi_d e_{d,d} = \varphi_d \varphi_d v^t = v^t$, which proves the corollary. □

Theorem 4.46. *Let R be a 4-dimensional smooth affine algebra over an algebraically closed field k with $6 \in k^\times$. Then $Um_3(R)/SL_3(R)$ is trivial if and only if $W_{SL}(R) = 0$.*

Proof. This follows immediately from Corollary 4.33 and Corollary 4.45. □

Corollary 4.47. *Let R be a 4-dimensional smooth affine algebra over an algebraically closed field k with $6 \in k^\times$ and let $X = Spec(R)$. Then $Um_3(R)/SL_3(R)$ is trivial if $CH^3(X)$ and $H^2(X, \boldsymbol{K}^{MW}_3)$ are 2-divisible. Furthermore, $Um_3(R)/SL_3(R)$ is trivial if $H^2(X, \boldsymbol{I}^3)$ is 2-divisible and $CH^3(X) = CH^4(X) = 0$.*

Proof. By Theorem 4.46, we have to show that $W_{SL}(R) = 0$ if $CH^3(X)$ and $H^2(X, \mathbf{K}^{MW}_3)$ are 2-divisible or if $H^2(X, \mathbf{I}^3)$ is 2-divisible and $CH^3(X) = CH^4(X) = 0$. But since the Vaserstein symbol surjects onto $W_{SL}(R)$ and k is algebraically closed, it follows from [FRS, Lemma 7.4] and the Swan-Towber theorem [SwT, Theorem 2.1] that $W_{SL}(R)$ is 2-torsion. Hence it suffices to show that $W_E(R)$ or $W_{SL}(R)$ is 2-divisible. So the first statement follows from Propositions 3.7 and 3.9. The second statement follows directly from Proposition 3.9. □

Corollary 4.48. *Let R be a 4-dimensional smooth affine algebra over an algebraically closed field k with $6 \in k^{\times}$ and let $X = Spec(R)$. Moreover, assume that $CH^i(X) = 0$ for $i = 1,2,3,4$ and that $H^2(X, \mathbf{I}^3) = 0$. Then all finitely generated projective R-modules are free.*

Proof. We may assume that $X = Spec(R)$ is connected; in particular $[X, \mathbb{Z}] \cong \mathbb{Z}$. The fact that $CH^i(X) = 0$ for $i = 1,2,3,4$ immediately implies that $F^i K_0(R) = 0$ for $i = 1,2,3,4$. Hence $rank : K_0(R) \xrightarrow{\cong} [X, \mathbb{Z}] \cong \mathbb{Z}$ is an isomorphism and all finitely generated projective R-modules are stably free. Since stably free R-modules of rank ≥ 3 are free (cp. [S1] and [FRS]), it suffices to prove that stably free modules of rank 2 are free. But this follows from Corollary 4.47. □

4.4 Further thoughts

In the last section of this thesis, we discuss some open questions in the study of projective modules and relate them to our results. For this, we let R be a smooth affine algebra of dimension $d \geq 3$ over an algebraically closed field k such that $(d-1)! \in k^{\times}$.

In general, it is an open question whether any finitely generated projective R-module P of rank $d-1$ is cancellative. If $d = 3$, then P is cancellative by results of Asok-Fasel (cp. [AF2, Corollary 6.8]); if P has a trivial determinant, we are already able to re-prove their cancellation theorem by means of the generalized Vaserstein symbol (cp. Corollary 4.25). Moreover, if $P = R^{d-1}$, then P is also cancellative by results of Fasel-Rao-Swan (cp. [FRS, Theorem 7.5]).

Our results in Section 4.2 suggest that any projective R-module of the form $P_{d-1} = P_0 \oplus R^{d-3}$, where P_0 has rank 2 and a trivial determinant, is cancellative. Indeed, because of Theorem 4.24, it suffices to prove that for some j prime to $char(k)$ any epimorphism of the form $b = (b_0, b_3^j, b_4, ..., b_d) \in Um(P_{d-1} \oplus R)$ can be completed to an automorphism of $P_{d-1} \oplus R$, i.e. there is an automorphism $\varphi \in Aut(P_{d-1} \oplus R)$ such that $b = \pi\varphi$, where π is just the projection $P_{d-1} \oplus R \to R$ onto the last free direct summand.

By analogy with Andrei Suslin's proof of his cancellation theorem in [S1], one could try to prove this by induction on d. In fact, we have already settled the base case by constructing an explicit completion of an epimorphism of the form (b_0, b_3^2) in Section 1.4.

Another approach would be to reformulate the above problem in the language of $\mathbb{A}^1$-homotopy theory; for example, the special case of unimodular rows and their completability is discussed in [AF1, Section 5]. Since the free R-module is always extended from the base field k, one can use the $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ over the base field in this case. In general, one possibly has to use the $\mathbb{A}^1$-homotopy category $\mathcal{H}(R)$ with base scheme $X = Spec(R)$. Nevertheless, we leave the investigation of this remaining problem to future work.

Throughout this thesis, we have considered projective modules with a trivial determinant. An affine scheme $X = Spec(R)$ is called a topologically contractible smooth affine variety over $\mathbb{C}$ if it is an irreducible smooth affine scheme over $\mathbb{C}$ such that its associated complex manifold $X(\mathbb{C})$ is a contractible topological space. It is known that $CH^1(X) = 0$ for such a variety. Hence it follows in particular that our results apply to topologically contractible smooth affine varieties over $\mathbb{C}$. The generalized Serre conjecture on algebraic vector bundles asserts that algebraic vector bundles over topologically contractible smooth affine varieties over $\mathbb{C}$ are trivial. The conjecture is known to hold in dimensions ≤ 2, but is open in higher dimensions. Under the assumption that the generalized Serre conjecture holds in higher dimensions and in view of Theorem 4.46, it might be expected that $W_{SL}(R) = 0$ for a smooth affine algebra R of dimension 4 over $\mathbb{C}$ such that $X = Spec(R)$ is a topologically contractible variety.

Fabien Morel and Vladimir Voevodsky have defined complex realization functors

$$\mathfrak{R}_{\mathbb{C}} : \mathcal{H}(\mathbb{C}) \to \mathcal{H},$$
$$\mathfrak{R}_{\mathbb{C},\bullet} : \mathcal{H}_{\bullet}(\mathbb{C}) \to \mathcal{H}_{\bullet},$$

where $\mathcal{H}$ and $\mathcal{H}_{\bullet}$ are the homotopy categories of topological spaces and pointed topological spaces respectively. These functors extend the assignment which sends any smooth affine scheme X to the topological space $X(\mathbb{C})$. For any base scheme S, a space $\mathcal{X} \in Spc_S$ is called $\mathbb{A}^1_S$-contractible if $\mathcal{X}$ is isomorphic to S in $\mathcal{H}(S)$. It follows from the existence of the complex realization functors that $\mathbb{A}^1_{\mathbb{C}}$-contractible smooth affine varieties over $\mathbb{C}$ are topologically contractible. By the algebro-geometric analogue of Steenrod's homotopy classification of topological vector bundles (cp. [AHW]), it follows that all algebraic vector bundles on $\mathbb{A}^1_{\mathbb{C}}$-contractible smooth affine varieties over $\mathbb{C}$ are trivial. Thus, the subtle question behind the generalized Serre conjecture is under which circumstances topologically contractible smooth affine varieties are in fact $\mathbb{A}^1_{\mathbb{C}}$-contractible.

In fact, there exist examples of topologically contractible smooth affine varieties of dimension 3 over $\mathbb{C}$ which are not isomorphic to $\mathbb{A}^3_{\mathbb{C}}$ called the Koras-Russell threefolds of the first and second kind. In [HKØ], it was proven that all algebraic vector bundles over the Koras-Russell threefolds of the first and second kind are trivial. As a matter of fact, it was later proven in [DF] that the Koras-Russell threefolds of the first kind are in fact $\mathbb{A}^1_{\mathbb{C}}$-contractible, which trivialized the result on their vector bundles. Nonetheless, it still remains an open question whether the Koras-Russell threefolds of the second kind are $\mathbb{A}^1_{\mathbb{C}}$-contractible as well.

Now let us return to the case of a general commutative ring R. Our results raise the question whether one can define a generalized Vaserstein symbol for any projective R-module of rank 2 (not necessarily with a trivial determinant). For any projective R-module L of rank 1, we set $P^{\vee_L} = Hom_{R\text{-}\mathfrak{mod}}(P, L)$. For any projective R-module P of finite rank, one has a natural isomorphism

$$can_L : P \to P^{\vee_L \vee_L}, p \mapsto (ev_p : P^{\vee_L} \to L, a \mapsto a(p)),$$

induced by evaluation. Then an L-oriented alternating morphism on P is a morphism $f : P \to P^{\vee_L}$ such that $f(p)(p) = 0$ for all $p \in P$. An L-oriented alternating isomorphism on P is an L-oriented alternating morphism on P which is an isomorphism. Replacing alternating isomorphisms by L-oriented alternating isomorphisms, we can then mimic our definition of the group $V(R)$ (cp. Section 3.3) in order to define a group $V(R, L)$. Note that for any finitely generated projective R-module P there is a hyperbolic L-oriented alternating isomorphism $H_L(P) : P \oplus P^{\vee_L} \to P^{\vee_L} \oplus P^{\vee_L \vee_L}$ given by

$$\begin{pmatrix} 0 & id \\ -can_L & 0 \end{pmatrix}.$$

Now let P_0 be a projective R-module of rank 2 and let $L = \det(P_0)$. Then the form $P_0 \times P_0 \to \det(P_0), (p, q) \mapsto p \wedge q$ induces an L-oriented alternating isomorphism on P_0, which we denote by χ_0.

If $a \in Um(P_0 \oplus R)$ with section $s : R \to P_0 \oplus R$ and $P(a) = \ker(a)$, then we obtain as usual isomorphisms $i_s : P_0 \oplus R \xrightarrow{\cong} P(a) \oplus R$ and $\theta : \det(P_0) \xrightarrow{\cong} \det(P(a))$. Then the form $P(a) \times P(a) \to L, (p, q) \mapsto \theta^{-1}(p \wedge q)$ induces an L-oriented alternating isomorphism on $P(a)$, which we denote by χ_a. We can then associate to a the element

$$V(a) = [P_0 \oplus R \oplus R^{\vee_L}, \chi_0 \perp H_L(R), (i_s \oplus 1)^{\vee_L}(\chi_a \perp H_L(R))(i_s \oplus 1)]$$

in $V(R, L)$. In order to define a Vaserstein symbol for P_0, it then remains to prove that the element $V(a)$ does not depend on the choice of the section s above. But we can mimic the proof of Theorem 4.4 for this: If t is another section of a and $i_t : P_0 \oplus R \xrightarrow{\cong} P(a) \oplus R$ is the isomorphism induced by t, then one has to show that the elements

$$[P_0 \oplus R \oplus R^{\vee_L}, \chi_0 \perp H_L(R), (i_s \oplus 1)^{\vee_L}(\chi_a \perp H_L(R))(i_s \oplus 1)] \text{ and}$$
$$[P_0 \oplus R \oplus R^{\vee_L}, \chi_0 \perp H_L(R), (i_t \oplus 1)^{\vee_L}(\chi_a \perp H_L(R))(i_t \oplus 1)]$$

are equal in $V(R, L)$. For this, we define a homomorphism $d' : P_0 \oplus R \to \det(P_0 \oplus R)$ by $p \mapsto s(1) \wedge t(1) \wedge p \in \det(P_0 \oplus R)$. Then we let $d : P_0 \oplus R \to R^{\vee_L}$ be the map obtained from d' by composing with the canonical isomorphisms $\det(P_0 \oplus R) \cong \det(P_0) \cong R^{\vee_L}$. Furthermore, we let φ be the elementary automorphism on $P_0 \oplus R \oplus R^{\vee_L}$ induced by $-d$. As in the proof of Theorem 4.4, one can then check locally that

$$\varphi^{\vee_L}(i_s \oplus 1)^{\vee_L}(\chi_a \perp H_L(R))(i_s \oplus 1)\varphi = (i_t \oplus 1)^{\vee_L}(\chi_a \perp H_L(R))(i_t \oplus 1)$$

and conclude that our assignment does not depend on the choice of s. Of course, this raises the question whether one can then prove results without the assumption of a trivial determinant which are analogous to our results in this thesis. We leave the investigation of this to future work.

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