Geometric and non-geometric backgrounds from string dualities

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Zusammenfassung

In dieser Arbeit werden bestimmte Klassen von nicht-geometrischen String-Hintergründen, sowie weitere Aspekte im Wechselspiel zwischen Kompaktifizierungsmodellen und String-Dualitäten, analysiert. Konkreter ausgedrückt, werden die durchgeführten Analysen durch String-Zielraum-Dualitäten (T- S- und U- Dualitäten) motiviert, die Beziehungen zwischen verschiedenen Räumen sind, die, wenn sie als String-Hintergründe verwendet werden, zu den selben Quantentheorien führen.

Insbesondere beginnen wir mit der Konstruktion einer Klasse von Kodimension-zwei Lösungen der String-Hintergrundgleichungen, die aus nicht-trivialen Zwei-Torus-Faserungen über einer zweidimensionalen Basis $\mathcal{B}$ bestehen. Nachdem ein Degenerationspunkt auf $\mathcal{B}$ eingekreist wurde, werden die Fasern mit einem beliebigen Element der T-Dualitätsgruppe geklebt. Diese Gruppe schließt Transformationen ein, die weder Diffeomorphismen noch Eichtransformationen sind, die zu nicht-geometrischen Konfigurationen führen. Die entsprechenden Defekte, die wir T-fects nennen, können mit der T-Dualitätsmonodromie um sie herum identifiziert werden. Wir bestimmen alle möglichen derartigen Geometrien, indem wir meromorphe Funktionen finden, die sie charakterisieren.. Das bedeutet, dass die Konfigurationen einige Supersymmetrie bewahren. Mit Hilfe einer Hilfsfläche, die über $\mathcal{B}$ gefasert ist, wird ein geometrisches Bild entwickelt, in dem die Monodromie als ein Produkt von Dehn-Twists auf dieser Oberfläche beschrieben werden kann.

Die Zwei-Torus-Faserung Struktur bricht auf den T-fects zusammen, und um die vollständige String-Lösung zu erhalten, muss man eine mikroskopische Beschreibung der Degenerationen einkleben, die typischerweise einige Isometrien der Faser bricht. Wir analysieren die Physik in diesem Gebiet für parabolische T-fects durch Untersuchung der Dynamik von Strings mit nicht-verschwindenden Windungszahlen um den Defekt. Daraus ergibt sich, dass die Physik in der Nähe des Kerns im Allgemeinen Windungsmoden beinhaltet, die die Supergravitationsannäherung nicht erfasst. Aus diesem Grund werden auch T-fects und deren Physik in der Nähe ihres Kerns innerhalb Double Field Theory analysiert und es wird diskutiert, in welchem Ausmaß die gefundenen Windungsmoden in so einem Formalismus kodiert werden können. Eine der untersuchten Geometrien entspricht der kompaktifiziert NS5-Brane, für die auch T-Dualität entlang anderer Isometrierichtungen analysiert wird und die Ergebnisse werden mit der kompaktifiziert NS5-Brane verglichen.

Eine natürliche Generalisierung für T-fects Lösungen besteht darin, analoge Faserungen in M- und Type II-Theorien zu betrachten und U-Dualität-Monodromien zu erlauben. In diesem Fall schränkt jedoch Supersymmetrie die Elemente der Dualitätsgruppe ein, die als

Monodromien verwendet werden können. Insbesondere ergibt sich, dass nur Konfigurationen, die zu T-fects dual sind, konstruiert werden können. Wir untersuchen diese Konfigurationen anhand des neulich entwickelten Formalismus von generalisierten G-Strukturen in Exceptional Field Theory, dass die U-dualität Generalisierung von Double Field Theory ist.

Ein solcher Formalismus ist ein mächtiges Werkzeug für die Untersuchung von supersymmetrischen Flusskompaktifizierungen von Type II- und M- Theorien, da es Metrische- und Flussfreiheitsgrade vereint. Die entwickelten Techniken werden auch dazu verwendet, um Kompaktifizierungen von Type IIB zu AdS_6 Vakua zu untersuchen, wobei bekannte Ergebnisse in der Literatur durch eine einfachere Formulierung mit natürlichen geometrischen Objekten reproduziert werden. Eine solche Formulierung erlaubt es auch, notwendige Bedingungen für die allgemeinsten konsistenten Trunkierungen mit Vektormultiplets um diese Vakua zu finden. Diese konsistenten Trunkierungen sind wichtige Werkzeuge für die holographische Untersuchung dieser Vakua.

Abstract

In this thesis certain classes of non-geometric string backgrounds, as well as other aspects in the interplay between compactification models and string dualities, are analysed. Concretely, the performed analyses are motivated by string target space dualities (T- S- and U- dualities), which are relations between different spaces that, when used as backgrounds on which strings consistently propagate, give rise to the same quantum theory.

In particular, we begin by constructing a class of codimension-two solutions to the string background equations consisting of non-trivial two-torus fibrations over a two dimensional base $\mathcal{B}$. After encircling a degeneration point on $\mathcal{B}$, the fiber is glued with an arbitrary element of the T-duality group, including transformations that are neither diffeomorphisms nor gauge transformations, which lead to non-geometric configurations. The corresponding defects, which we call T-fects, can be identified with the T-duality monodromy around them. We determine all possible such geometries by finding meromorphic functions of the base characterising them, which implies that the configurations preserve some supersymmetry. Using an auxiliary surface fibered over $\mathcal{B}$, we develop a geometric picture where the monodromy can be described as a product of Dehn twists on this surface.

The two-torus fibration structure breaks down at the T-fects and to obtain the complete string solution, one needs to glue in a microscopic description of the degenerations, which typically breaks some of the isometries of the fiber. We analyse the physics in these regions for parabolic T-fects by studying dynamics of strings with non-zero winding numbers around the defect. We deduce that, in general, physics near the core involves winding modes that the supergravity approximation fails to capture. For this reason, we also analyse T-fects and its near-core physics within Double Field Theory, an extension of supergravity designed to capture momentum and winding modes in the same footing, and discuss to which extent the encountered physics can be encoded within this formalism. One of the geometries we study corresponds to the compactified NS5 brane, for which we also analyse T-duality along other isometry directions and compare the results with the compact NS5 brane.

A natural generalisation for T-fect solutions is to consider analogous fibrations in M- and type II theories and allow for U-duality monodromies. In this case, however, supersymmetry restricts the elements of the duality group that can be used as monodromies. In particular, we find that only configurations dual to T-fects can be constructed. We study these configurations using the recently developed formalism of generalised G-structures in Exceptional Field Theory, the U-duality generalisation of Double Field Theory.

This formalism is a powerful tool for the study of supersymmetric flux compactifications of type II and M- theories, since it unifies metric and fluxes degrees of freedom. We use the developed techniques to study compactifications of type IIB to AdS_6 vacua, rederiving known results in the literature using a simpler formulation in terms of natural geometric objects. This formulation also allows one to establish necessarily conditions for the most general consistent truncations with vector multiplets around these vacua. These consistent truncations are important tools for the holographic study of these vacua.

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Contents

Chapter 1

Introduction

1.1 Why string theory?

The ultimate goal of science would be to construct a *theory of everything*, that gives a complete and unique rational description of our entire universe[1]. An important issue to be taken into account in this context is that the universe happens to look very different at different scales. Typically, as we will discuss below, phenomena that look very different at some energy scale can become very closely related at higher energies, which is usually known as *unification* in theoretical physics. This is the reason why science has been historically developed as a set of apparently unrelated theories covering certain aspects of the reality at certain energies (typically close to the ones available in experiments in each period in history). Then, the current road towards a *theory of everything* is in fact a road towards the most fundamental theory at high energies that includes all lower energy partial theories and in this sense unifies them.

It is interesting to notice that, given the different appearance of reality at different scales and the complexity of a unified theory, it is still very convenient to consider the partial theories to study certain phenomena at certain energy scales. For instance, although we know that quantum mechanics or general relativity are more fundamental than Newtonian mechanics, the latter is a very good a approximation if one wants to study, for instance, the dynamics of a train; and corrections due to the more fundamental theories are irrelevant in this situation. In the same way, if one is interested in interactions between atoms, chemistry is a very good approximation though it does not take into account their internal structure, which are again irrelevant degrees of freedom. In this sense, the lower energy theories can be thought as "effective theories" in some limit of the unified fundamental theory.

Let us now briefly review how this unification occurred historically in physics and what

[1]Of course, a natural question to address is whether such theory indeed exists, namely if there is a rational explanation for everything, including for instance for the existence of the universe itself; or whether it is possible for us humans beings, which are part of the universe and therefore part of the theory, to completely reveal it. Though these are very interesting questions, they are out of the scope of this discussion.

its current status is. The first clear example of a unification of two theories in modern physics is probably Maxwell's theory of electromagnetism [1]. Elaborating on the work by Ørsted, Ampère and Faraday, Maxwell was able to construct a unified theory describing both magnetism and electricity. Furthermore, this new theory predicted the existence of electromagnetic waves propagating at a finite speed, which could be identified with light, turning also optics into electromagnetism. This theory was only experimentally proved about twenty years later, when Hertz succeeded in detecting the mentioned electromagnetic waves.

The success of the theory of electrodynamics had further implications. In particular, certain contradictions between this theory and galilean relativity, a well stablished principle of Newtonian mechanics, became soon apparent. The solution came with the theory of special relativity [2], by Lorentz, Poincaré and Einstein, which unified the concepts of space and time, established the speed of light as universal and the fastest speed allowed, and elliminated the idea of an absolut reference frame. Unifying newtonian gravity with this framework eventually culminated in Einstein's theory of general relativity [3].

General relativity and electrodynamics are two very successful theories to describe gravity and electromagnetism, two of the fundamental interactions of nature, at large distances and low energy. However, it was realised at the beginning of the 20th century that electromagnetism, the strongest of the two, failed to be consistent from a statistical mechanic point of view, as well as at atomic level. By imposing a quantisation for the energy of electromagnetic waves, Planck was able to successfully describe the spectrum of black body radiation at all frequencies, for which no consistent model could be found using the classical theory of electrodynamics. This idea was then further developed by Einstein, Heisenberg, Schrödinder and others leading to the theory of quantum mechanics. With this theory, the first successful models for atoms as electrons orbiting around a nucleus on stable orbits could be constructed.

The unification of quantum mechanics, electromagnetism and special relativity lead to the theory of quantum electrodynamics, constructed by Feynman, Dyson, Schwinger and Tomonaga, based on earlier work by Dirac and others. It was soon realised that this formalism could be generalised to include other interactions, leading to the framework of quantum field theory. In particular, this allowed for the first time to study the other two fundamental forces of nature, weak and strong nuclear interactions, the last one described by the theory of quantum chromodynamics. All this work culminated with the formulation of the "Standard Model for Particle Physics" a quantum field theory based on point-particle objects that successfully describes all interactions in nature except gravity, as well as their couplings to matter, at quantum level.

For example, within this model electromagnetism is also unified with the weak nuclear force into the electroweak interaction, with gauge group $SU(2) \times U(1)_Y$. This group is spontaneously broken to the quantum electrodynamics $U(1)$ at low energies through the Higgs mechanism, which gives mass to the weak interaction bosons. The success of this theory was confirmed on 2012, when the Higgs boson was detected at CERN with a mass of $m_H = 125\,\mathrm{GeV}$, establishing an energy scale for the unification of the electroweak force.

This success suggests that there could be a higher energy scale where also the strong nuclear interaction is unified with the electroweak force. A particular element that seems to play an important role in this unification is supersymmetry. In particular, models for a Grand Unified Theory has been constructed using a minimal supersymmetric extension of the Standard Model. Such models, predict a unification scale at energies $m_{GUT} = 10^{16}$ GeV, which are far away from the energies available in the current particle accelerators.

To complete a full unification of fundamental interactions, one should also include gravity. However, this interaction has properties that makes constructing theoretical models for a complete unification a very challenging task. Although the theory of general relativity successfully describes the effects of gravity at large distances, it generically include singularities, such as black holes, that indicate the need of a quantum version of the theory. However, constructing a quantum field theory for gravity following the same procedure as in the other interactions turns out to be impossible. The reason is that gravity as a quantum theory happens to be non-renormalisable, meaning that one cannot consistently remove the divergences that generically appear in a quantum field theory, and the theory cannot be used to make physical predictions.

It is very important to remark that having theory of quantum gravity is important even if it is not unified to the rest of interactions. Consider for instance the example of an electron moving through a double-slit experiment. Although very tiny, this electron generates a gravitational field due to its mass. However, quantum mechanics tells that the path that a single electron follows cannot be determined, and therefore its gravitational field cannot be constructed using classical gravity theories. Other more prominent examples where a theory of quantum gravity is needed include the resolution of general relativity singularities (i.e. black holes) and very early statges of the universe after the big bang. Furhtermore, for these last situations one also needs a theory for gauge interactions in highly curved backgrounds.

Nowadays, we have some candidates to provide a quantum theory for gravity. An example is, for instance, Loop Quantum Gravity. However, this theory has currently no consistent classical GR limit. Instead, the present thesis will deal with String Theory which has a natural classical limit and, unlike the former, can also include gauge interactions. This theory is based on changing the paradigm of point-like particle theories for one-dimensional objects (strings). Intuitively, the fact that the fundamental objects have a finite size implies the absence of ultra-violet divergences and, in fact, string theory is known to be divergence-free. Furthermore, as will be briefly reviewed below, string theory includes excitations that can be identified with gravitons and gauge bosons, becoming a very natural candidate for quantum gravity and for a framework where ultimate unification can occur.

In the last decades, string theory has been proven to have a very rich mathematical structure. In this thesis, some aspects related with non-geometry in string theory and string dualities will be analysed. The main concepts motivating this research will be introduced in the upcoming sections.

1.2 Basics of String Theory

The bosonic string

String theory is a theory of propagating one-dimensional objects. Analogous to a particle propagating in space-time, one can construct an action by integrating the area swept by the string while travelling between an initial and a final state. Such action is classically equivalent to the non-linear sigma model described by the Polyakov action

$$S_P = -\frac{T}{2}\int_\Sigma d^2\sigma\,\sqrt{-h}\,h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}\,, \tag{1.1}$$

where Σ is the two-dimensional surface swept by the string, called world-sheet, and $\sigma = (\sigma^1, \sigma^2)$ are a set of local coordinates on it. The fields X^μ can be interpreted as the embedding of the string in the target space, which is a manifold[2] with metric $g_{\mu\nu}$. $T = (2\pi\alpha')^{-1}$ is a parameter called string tension, related to the length of the fundamental string $l_s = 2\pi\sqrt{\alpha'}$, and the field $h_{\alpha\beta}$ is a metric tensor on the world-sheet. We note that gravity in two dimensions has no propagating degrees of freedom, which implies that $h_{\alpha\beta}$ is not dynamical. In fact, using the symmetries of the theory, one can always use a local frame where $h_{\alpha\beta}$ has the canonical form of a flat metric.

For the case where the target space is the empty Minkowski space, $g_{\mu\nu} = \eta_{\mu\nu}$, the theory can be quantised by promoting the fields X^μ to quantum operators. The corresponding Fourier modes, which can be seen as oscillations of the string, are interpreted as particles in the target space. Therefore, String Theory predicts a spectrum of infinitely many particles of different mass coming from a single fundamental object, the string. We note that the lower massive states of the spectrum will have a mass of order $M_s = (\alpha')^{-1/2}$, which is the string mass scale. Therefore, all massive states can decouple from the massless ones in situations when the characteristic energy of the system is much smaller than M_s, corresponding to situations where α' is small.

Another important aspect of the quantum spectrum of string theory is that it contains a spin-two state in its closed strings massless sector. Such state will be interpreted as a graviton when constructing effective actions. In particular, its space-time dynamics are controlled by Einstein gravity equations, as we will see in some more detail below. This fact is not only true for the bosonic string, but also for all other string theories, including Superstrings discussed below. For this reason, gravity is always naturally encoded in String Theory.

Furthermore, one can also consider open string states. In this case, their massless spectrum will contain spin-one excitations which, analogous to the discussion with the graviton, can be interpreted as gauge fields. Inuitively, one can see that a closed string can be obtained by colliding two open string states. Therefore, any theory of interacting open strings should also include closed string modes. In this sense, gravity is naturally included

[2] Along this thesis we will also consider strings propagating in spaces that globally fail to be a manifold. However, the current discussion is not affected by this consideration.

in any gauge theory constructed from String Theory, which therefore provides a natural framework for the unification of these two fundamental interactions.

While doing the quantisation procedure sketched above, one has to pay attention to possible anomalous behaviours of the symmetries of the classical theory. In particular, it turns out that the Weyl symmetry of the action (1.1), given by the rescaling of the world-sheet metric $h_{\alpha\beta} \to e^{2\Lambda} h_{\alpha\beta}$, becomes in general anomalous at quantum level. A surprising result is that, in Minkowski space-time, this anomaly disappears if and only if the dimension of the target space is fixed to a certain value, called critical dimension $D_{\text{crit.}}$, which for the bosonic string is $D_{\text{crit.}} = 26$.

Apart from a graviton state, the closed string massless spectrum of the bosonic string includes an anti-symmetric 2-tensor $B_{\mu\nu}$, called Kalb-Ramond field, and a scalar Φ, the dilaton. As we will discuss below, such string excitations are naturally related with fields in backgrounds where a string can consistently propagate. Therefore, one should consider a generalisation of the action (1.1) that includes the coupling with them. This is achieved with the action

$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h} \left(h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} + \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} + \alpha' \Phi \mathcal{R}(h) \right) , \qquad (1.2)$$

where $\mathcal{R}(h)$ is the Ricci scalar of the world-sheet metric. Since this metric do not encode any degree of freedom, the last term is topological and do not affect the classical equations of motion of the theory. However, it plays an important role when quantising the theory, organising the string perturbation theory. This can be seen as follows. Assuming that the dilaton changes on scales much larger than the string length l_s, the last term of the action (1.2) can be approximated by the Gauss-Bonnet integral

$$-\frac{\langle\Phi\rangle}{4\pi} \int_\Sigma \sqrt{-h}\ \mathcal{R}(h) = -\chi(\Sigma)\ \langle\Phi\rangle\,, \qquad (1.3)$$

where $\langle\Phi\rangle$ is the background expectation value of the dilaton and $\chi(\Sigma)$ the Euler number of the two-dimensional surface Σ, which is a topological invariant. For a Riemann surface[3] with g handles and b boundaries, $\chi = 2-2g-b$. In the context of path integral quantisation, one needs to consider exponentials of the action. The topological term then gives

$$e^{-\frac{\langle\Phi\rangle}{4\pi} \int_\Sigma \sqrt{-h}\ \mathcal{R}(h)} = g_s^{-\chi}, \qquad (1.4)$$

where $g_s = e^{\langle\Phi\rangle}$ can be defined to be the string coupling constant[4]. Since the path integral implies summing over all possible world-sheets, this factor organises them in powers of the string coupling constant. For instance, for closed strings (with $b = 0$) the expansion in powers of the string coupling corresponds to an expansion in the number of handles of the

[3] There is a generalisation of this topological invariant for non-oriented surfaces, which will not be discussed in this thesis.

[4] This can be deduced by looking at the simplest interacting diagram for closed strings, namely the one with two ingoing and one outgoing states.

world-sheet. This is the string theory analogous of the loop expansion in quantum field theories.

We conclude this part with a final remark about the string quantum theory. The fact that, unlike point-particle theories, the fundamental objects of string theory have a finite size implies that generally ultra-violet divergences are avoided in string theory. Together with the fact that gravity is naturally included in it, this observation leads to the conclusion that string theory is the most natural guess for a consistent theory of quantum gravity.

Superstrings

The bosonic string presented above has two important problems that prevent it to be useful in the description of our world: its spectrum contains a tachyon state, of negative mass squared, and it does not contain any fermionic states. It turns out that this two issues can be solved by generalising the world-sheet theory (1.1) and making it supersymmetric by including fermionic degrees of freedom on the world-sheet.

The resulting theories are generically called Superstring Theories and all of them can be consistently quantised when the critical dimension is $D_{\text{crit.}} = 10$. It turns out, however, that the procedures to obtain realistic theories with world-sheet supersymmetry are not unique. The different possibilities are the following:

- **Type II strings.** These are constructed by adding supersymmetry for both left- and right- moving sectors of the string and performing a GSO projection that eliminates the tachyon. The result are string theories with $\mathcal{N} = 2$ supersymmetry on the target space. There are two inequivalent ways of performing the projection leading to type IIA (non-chiral) and type IIB (chiral).

- **Heterotic strings.** These are constructed by adding supersymmetry on the left-moving sector while having only bosonic degrees of freedom in the right-moving. Since the critical dimensions of the bosonic string is bigger than the one for superstrings, one needs to compactify 16 of the bosonic directions using a self-dual even Euclidean lattice, which ensures modular invariance of the partition function. There are only two choices for such lattices, that lead to a string theory with either gauge group $SO(32)$ or gauge group $E_8 \times E_8$. In both cases the theory has $\mathcal{N} = 1$ supersymmetry.

- **Type I strings.** These is a theory of un-oriented strings. It can be obtained from type IIB by identifying the two orientations of the closed strings. Also, modular invariance of the partition function is only satisfied if one includes a so called twisted sector, that includes open strings. The theory has $\mathcal{N} = 1$ supersymmetry.

Similarly to the case of the bosonic string, all such superstring theories have a graviton and a dilaton in their massless spectrum. Furthermore the latter contains also different form fields (the exact number and their degrees are different in each theory) as well as the corresponding fermionic superpartners.

An astonishing fact about these theories is that all of them are related between themselves by a web of dualities, some of which will be reviewed in next section. Furthermore, they can be related to a unique theory defined in eleven dimensions: M-theory, whose fundamental objects are now two-dimensional (it follows from the above discussions that string theories cannot be consistently defined in eleven dimensions). A complete description of this theory is nowadays not known, but it should reduce to the unique 11-dimensional $\mathcal{N} = 1$ supergravity theory at low energies.

Low energy effective action and string background equations

As mentioned in the previous discussions, all string theories contain massless spin-two excitations that we interpret as gravitons. In order to prove that this interpretation is indeed correct, one needs to check that the dynamics of this states in the target space is indeed governed by a (super)gravity theory.

Furthermore, string theory predicts an infinite spectrum of massive states, corresponding to oscillations of the string, that have never been observed in the experiments available nowadays. A plausible reason for this is that the characteristic string mass scale M_s is significantly bigger that the energies currently available in experiments. As pointed out above, in this scenario the physics we observe is dominated by the massless modes of string theory. This corresponds to study the theory in a limit where the parameter α' is small, in which effects due to massive modes, which are proportional to powers of α', will be arbitrarily small.

For these reasons, it is convenient to study the effective theory of the massless modes of string theory, and check that this is indeed (super)gravity. Such checking is done by studying scattering amplitudes of the different string massless excitations and constructing a field theory that reproduce them. Undergoing this procedure, one obtains the expected result.

As an example, let us consider the case of the bosonic string, whose massless excitations are a graviton $g_{\mu\nu}$, a two-form field $B_{\mu\nu}$ and a scalar field Φ. The effective theory that reproduce the scatterings of such excitations is

$$S = \int d^D x \sqrt{-g} e^{-2\Phi} \left(\mathcal{R}(g) - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\nabla^\mu \Phi \nabla_\mu \Phi \right) , \tag{1.5}$$

with $H = dB$, which is indeed a gravity theory coupled to the two-form and the scalar fields. Analogous results are obtained for the superstring theories. Let us make the following observation about the theory(1.5): when the manifold is restricted to be a d-dimensional flat torus (or the product of a d-dimensional flat torus times an arbitrary manifold) the action (1.5) has a hidden $O(d, d, \mathbb{R})$ symmetry which non-trivially mixes the different background fields [4]. This is in general not present in arbitrary manifolds but, since any manifold looks flat in a small neighbourhood of any point (up to two derivatives terms like the curvature), one expects that some reminiscence of this symmetry can always be locally (i.e. close to each point in the manifold) found. We will reencounter these aspects when discussing string dualities and extended space formalisms in sections 1.3 and 1.5.

Let us now turn to an apparently unrelated question: above, we argued that quantising the Polyakov action (1.1) when the target space was Minkowski was only consistent in the case where its dimension was $D_{\text{crit.}}$. However, we also argued that it was natural to consider strings propagating on more general backgrounds, which include the fields obtained from the massless spectrum of the theory. The dynamics of the string in this situations is described by actions like (1.2). Now, one can wonder if the Weyl anomaly is still cancelled after doing this generalisation. It turns out that for arbitrary fields the symmetry is anomalous even in the critical dimension. However, by looking at the trace of the energy-momentum tensor at one loop, one can deduce under which conditions this anomaly cancels again. In particular, for the case of closed bosonic strings, the anomaly is given by [5–7]

$$2\alpha' T^{\alpha}_{\alpha} = \alpha' \beta^{(\Phi)} \mathcal{R}(h) + \beta^{(g)}_{\mu\nu} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu + \beta^{(B)}_{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \tag{1.6}$$

with

$$\begin{aligned} \beta^{(g)}_{\mu\nu} &= \alpha' \Big(\mathcal{R}_{\mu\nu} - \frac{1}{4} H_\mu{}^{\lambda\rho} H_{\nu\lambda\rho} + 2\nabla_\mu \nabla_\rho \Phi \Big) + \mathscr{O}(\alpha'^2)\,, \\ \beta^{(B)}_{\mu\nu} &= \alpha' \Big(-\frac{1}{2} \nabla_\lambda H^\lambda{}_{\mu\nu} + H^\lambda{}_{\mu\nu} \nabla_\lambda \Phi \Big) + \mathscr{O}(\alpha'^2)\,, \\ \beta^{(\Phi)} &= \frac{1}{4}(D - D_{\text{crit.}}) + \alpha' \Big(\mathcal{R}(g) + (\nabla\Phi)^2 - \frac{1}{2}\nabla^2\Phi - \frac{1}{24} H^2 \Big) + \mathscr{O}(\alpha'^2)\,, \end{aligned} \tag{1.7}$$

and all backgrounds (g, B, Φ) satisfying $\beta^{(g)} = \beta^{(B)} = \beta^{(\Phi)} = 0$ are anomaly-free. These set of differential equations for the background fields are called string background equations and, at first order on α', they coincide with the equations of motion of the effective supergravity theory (1.5)! This fact fits nicely in the idea of string theory being a theory of everything: the backgrounds along which a string can consistently propagate are naturally included within the theory itself.

String compactifications

Considering only massless string excitations is not enough to describe the physics we observe in experiments. In particular, we observe a four dimensional space, whereas superstring theories are only consistent in ten dimensions. This can be solved by considering that the extra dimensions are compact and small enough to escape from detection in the experiments we have nowadays. Therefore, the study of string theory on compact manifolds is a central topic within the field and it is generically called string compactifications. Generically, one considers string backgrounds of the form

$$\mathcal{M}_{D_{\text{crit.}}} = \tilde{\mathcal{M}}_{D_{\text{crit.}}-d} \times K_d\,, \tag{1.8}$$

where K_d is usually called internal manifold, and has dimension d, and $\tilde{\mathcal{M}}_{D_{\text{crit.}}-d}$ is the external manifold. Phenomenologically, one is interested in situations where $D_{\text{crit.}} - d = 4$, though other cases have also interest from a theoretical and mathematical point of view. As we will see, specially in the last chapters of this thesis, the effective physics on the external space (which is the one that should be eventually related with the physics observed in experiments) is highly influenced by the geometry and topology of the internal space in a non-trivial way. The consequence is that the same ten-dimensional theory compactified on different internal manifolds can lead to significantly different effective theories in lower dimensions and, out of the five ten dimensional theories described above, there is an enormous number of lower-dimensional effective backgrounds one can construct. Such set of constructions is generically known in the literature as string landscape.

The natural question that arises in this discussion is which of these internal manifolds lead to realistic lower-dimensional models. In general imposing some conditions for the lower dimensional physics implies important restrictions to the kind of internal manifolds we can have. For instance, demanding that the external space preserves some of the original supersymmetry (a reasonable assumption in view of a minimal supersymmetric extension of the Standard Model) imposes certain structures on the internal space or even restricts its holonomy group (see discussions in chapter 5 for details).

Another common fact of compactifications is that they always give rise to several scalar fields on the lower dimensional theory, which encode the moduli of the internal space. Since massless scalars are not observed in experiments, one is interested in models that naturally give a mass to such fields and stabilise them at some vacuum expectation value. These techniques are generically called moduli stabilisation. One way to do it at tree level is by turing on fluxes in the internal space [8–11]. By giving non-zero expectation values to fluxes, a scalar potential for the moduli arises which can contain minima that stabilise the moduli. In general, however, stabilising all moduli at the same time is not necessary possible, and one has to use other techniques.

From the above discussion it follows that the relation between the internal geometry and the effective lower dimensional theory is highly non-trivial. In fact, constructing an uplift for a given theory with a given field content in the lower dimensional space is an extremely hard task and there exist lots of cases where this uplift is still unknown. Actually, a related natural question to ask is whether all possible realistic effective field theories can be obtained from string theory. In the last years, there has been a lot of activity around this question, generically called the swampland conjectures [12], and the latest results seem to indicate that such theories could indeed exist.

1.3 String dualities

As already mentioned, string theory has a rich web of dualities which is not present in point-particle quantum field theories. Most of them are based on the fact that string theories deals with objects that extend in space (string are one-dimensional objects, but in general one could also consider branes which can be higher dimensional). This allows to have states winding or wrapping non-contractible cycles of our background. For this reason, dualities are closely related to the compact string backgrounds described above.

In general a duality is a relation between two apparently different theories that maps one to one all objects and observables from one to the other. In this sense, the two theories are in fact equivalent. Dualities can be useful both for a deeper conceptual understanding of the theory and for computational purposes, since it can be that sectors of one theory where we barely have access are mapped to a very well understood sectors in the dual theory. Dualities in string theory include gauge/gravity dualities, mirror symmetry, open/closed string dualities,. . . . For this thesis, we will be interested mainly in T-, S- and U- dualities, which will be reviewed next.

T-duality of the bosonic string

The simplest example where one encounters T-duality is the case of closed bosonic strings compactified on a circle of radius R. Such theory is described by the action (1.1), and the corresponding quantum spectrum is given by infinitely many states, labelled by (n, m), with $n, m \in \mathbb{Z}$, together with the excitation numbers of the different space-time oscillations. Such states are subject to the constraint

$$N - \bar{N} = n\, m\,, \tag{1.9}$$

where $N, \bar{N} \in \mathbb{Z}^+ \cup \{0\}$ are the sum of all excitation numbers for the left- and right- moving oscillations, and their masses M are given by (in units where $\alpha' = 1$)

$$M^2 = \frac{n^2}{R^2} + m^2 R^2 + 2(N + \bar{N} - 2)\,. \tag{1.10}$$

The integer n is related to the momentum of the state along the circle, which is quantised. We observe that its contribution to the total energy of the state is higher the smaller the radius is. On the other side, the m counts the number of times the string winds around the circle. In this case its contribution to the total energy is bigger the bigger the radius is.

A straight-forward observation is that, if one compactified the theory on a circle of radius $1/R$, the spectrum would be exactly the same as the one obtained by compactifying on a radius R. To relate the two situations, one needs to interpret the former winding modes as the new momentum modes and vice versa. After this identifications the two theories are indeed equivalent, obtaining

$$\text{String theory on } S^1 \text{ with radius } R \overset{\text{T-dual}}{\longleftrightarrow} \text{String theory on } S^1 \text{ with radius } 1/R\,.$$

This situation can be easily generalised to the case where the background is a flat d-dimensional torus T^d. In this case, the space is big enough to have a two-form field $B_{\mu\nu}$ and one needs to use the action (1.2). The spectrum is now labelled by d momentum numbers $\mathbf{n} \in \mathbb{Z}^d$ and d winding number $\mathbf{m} \in \mathbb{Z}^d$. Intuitively, the set of transformations that leave the spectrum invariant is now bigger since we have more ways to identify momentum and winding numbers. In fact, the set of all these transformations is the group $O(d, d, \mathbb{Z})$ and the $2d$-vector $(\mathbf{m}, \mathbf{n})$ transforms in its fundamental representation. More details about the action of the T-duality group on the background fields will be discussed in section 2.4.1.

It is interesting to remark that, as mentioned above, the low energy effective theory for the massless spectrum of the bosonic string, given by the action (1.5), has an $O(d, d, \mathbb{R})$

symmetry when toroidal backgrounds are considered. The way the background fields transform under such symmetry is the same as they transform under the T-duality group. In fact, what happens is that the $O(d,d,\mathbb{R})$ symmetry group of the lower dimensional theory is broken to the discrete group $O(d,d,\mathbb{Z})$ by non-perturbative effects when considering the full string theory. Note that the continuous group cannot be the duality group of string theory since fractional momentum and winding numbers are not allowed.

Finally, it is also interesting to ask whether this duality can be generalised to curved backgrounds. This is in fact possible in the cases where the background has a compact $U(1)$ isometry. In this case, starting from the sigma-model (1.2), one can gauge the isometry and introduce a Lagrange multiplier that ensures that the field strenght of the corresponding gauge field vanishes. Integrating out the bosonic field along the isometry direction in space-time $X^a(\sigma)$, one obtains another sigma model (where now the Lagrange multiplier plays the role of the boson along the isometry direction) where the new backround fields $(\tilde{g}, \tilde{B}, \tilde{\Phi})$ are given by the Busher rules [13–15]

$$\begin{aligned}
&\tilde{g}_{aa} = \frac{1}{g_{aa}}, \quad \tilde{g}_{a\mu} = \frac{B_{a\mu}}{g_{aa}}, \quad \tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{g_{a\mu}g_{a\nu} - B_{a\mu}B_{a\nu}}{g_{aa}}\,, \\
&\tilde{B}_{a\mu} = \frac{g_{a\mu}}{g_{aa}}, \quad \tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{g_{a\mu}B_{a\nu} - B_{a\nu}g_{a\nu}}{g_{aa}}\,,
\end{aligned} \tag{1.11}$$

togehter with the dilaton shift

$$\tilde{\Phi} = \Phi - \frac{1}{2}\log|g_{aa}|\,, \tag{1.12}$$

which is necessary at one loop. These set of transformations are indeed an $O(d,d,\mathbb{Z})$ transformation of the background fields, and one can check that the two theories are also equivalent as quantum theories [16].

In this section, T-duality was mainly discussed for the bosonic string. However, the arguments apply also for type II and heterotic superstring theories. In these cases, however, T-duality is not a self-duality of any of them, but instead relate type IIA with type IIB and the two heterotic string theories among each other.

S- and U- dualities of type II superstrings

The massless spectrum of type IIB superstring theory includes, apart from the dilaton Φ, another scalar field C_0. It also contains two 2-form fields, the usual B together with C_2. In type IIB supergravity, its low energy effective theory, these fields are related by an $SL(2,\mathbb{R})$ symmetry. In particular, given

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})\,, \tag{1.13}$$

the fields transform like

$$\tau'_{\text{a.d.}} = \frac{a\,\tau_{\text{a.d.}} + b}{c\,\tau_{\text{a.d.}} + d}\,, \qquad \begin{pmatrix} C'_2 \\ B' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} C_2 \\ B \end{pmatrix}\,, \tag{1.14}$$

D	9	8	7	6	5	4
$E_{d(d)}$	$SL(2) \times \mathbb{Z}_2$	$SL(3) \times SL(2)$	$SL(5)$	$Spin(5,5)$	$E_{6(6)}$	$E_{7(7)}$
H_d	$SO(2)$	$SO(3) \times SO(2)$	$SO(5)$	$USp(4) \times USp(4)$	$USp(8)$	$SU(8)$

Table 1.1: U-duality groups $E_{d(d)}$, and their maximal compact subgroups H_d, of M-theory compactified on a d-dimensional torus to D external dimensions.

where $\tau_{\text{a.d.}} = C_0 + i\, e^{-\Phi}$ is a complex field encoding the two scalar fields called axio-dilaton. Analogous to T-duality, such group cannot be the duality group of the full string theory. Instead, it is broken to $SL(2,\mathbb{Z})$. This can be seen as follows: the fundamental string is the object that couples electrically to the two-form B, and carries charge one. Similarly, the objects that couple to the two-form C_2 are $D1$ branes, which also carry charge one. When considering the full string theory, this symmetry transforms fundamental strings and $D1$ branes within each other. Since there cannot be fractional branes or strings, the group $SL(2,\mathbb{R})$ has to be broken to the discrete subgroup $SL(2,\mathbb{Z})$. This duality is called S-duality [17, 18]. We note that, when C_0 is turned off, it includes a transformation that relates $g_s \to 1/g_s$. Therefore, S-duality can be seen as a strong-weak non-perturbative duality.

As mentioned above, compactifications of type IIA and IIB supergravities on a d-dimensional torus have also an $SO(d,d,\mathbb{R})$ symmetry rotating the fields in the NS-NS sector (which are the fields that coincide with the field content of bosonic string), and are related between them by Buscher rules. Therefore, there should be a bigger group including both T- and S-dualities. In fact, it is known that the full symmetry group for type II supergravities on a d-dimensional torus is $E_{d+1(d+1)}(\mathbb{R})$, which is a non-compact version of the exceptional group E_{d+1} [19, 20] (see Table 1.1). Such group has an 11-dimensional origin, consistently with the close relation between type II and 11-dimensional supergravities: 11-dimensional supergravity compactified on a d-dimensional torus has a symmetry group $E_{d(d)}$.

Similarly to the case of T- and S- dualities, the group $E_{d+1(d+1)}(\mathbb{R})$ is conjectured to be brocken to its discrete subgroup $E_{d+1(d+1)}(\mathbb{Z})$ when considering the full string theory (similarly $E_{d(d)}(\mathbb{R})$ to $E_{d(d)}(\mathbb{Z})$ in M-theory). Such string (and M-theory) duality receive the name of U-duality (for reviews see [21–23]).

1.4 Non-geometric compactification models

An outcome of the above discussion on target space dualities is that geometry looks very different when probed with strings (or higher dimensional objects, such as branes) than when probed with point-like objects. They suggest then a natural generalisation to geometric backgrounds: one could construct spaces where the gluing between patches is done using string dualities rather than just diffeomorphisms. In general, such configurations will not be a manifold in the classical sense, but are smooth when interpreted as string backgrounds. Such spaces are generically called "non-geometric" [24].

There are several reasons why these configurations are worth being considered. The simplest one is that, from a world-sheet point of view, there is no indication that tells that the space where a string propagates should be geometric. Compactifications using non-geometric spaces are then a valid sector of the string landscape that can potentially include phenomenologically interesting models.

Non-geometric configurations and non-geometric fluxes

In fact, in the context of flux compactifications, such type of backgrounds need to be considered if one wants a fully T-duality invariant description of the four dimensional physics. As an example, let us consider the case of type II string theories compactified on a six-dimensional torus. If one turns on a non-zero H-flux in the internal space and applies T-duality transformations along the directions where the flux has legs, one obtains the following flux duality chain [25]

$$H_{abc} \xleftrightarrow{\text{T-dual.}} f_{ab}{}^{c} \xleftrightarrow{\text{T-dual.}} Q_a{}^{bc} \xleftrightarrow{\text{T-dual.}} R^{abc} . \tag{1.15}$$

As a simple example where such fluxes can be realised, one can consider the following toy model duality chain:

1. *Torus with H-flux.* We start from a flat three-torus equipped with a constant H-flux. Choosing a gauge for the corresponding B-field, this configurations is described by the fields

$$\begin{aligned} ds^2 &= d\theta^2 + dx^2 + dy^2 , \\ B &= \frac{N}{2\pi} \theta dx \wedge dy , \end{aligned} \tag{1.16}$$

   where N is a constant. Such background can be seen as a two-torus fibration over a circle, where θ is the base direction, with period 2π, and the periods of the coordinates on the fiber (x, y) are set to 1. The total H-flux is given by

$$H = \int_{T^3} dB = N , \tag{1.17}$$

   which for consistency at quantum level is quantised by $N \in \mathbb{Z}$.

2. *Twisted torus.* Applying Buscher rules procedure (1.11) along the direction y leads to the following configuration

$$\begin{aligned} ds^2 &= d\theta^2 + dx^2 + \left(dy - \frac{N}{2\pi}\theta dx\right)^2 , \\ B &= 0 , \end{aligned} \tag{1.18}$$

where, in order to make sense of this metric globally, one needs to identify $(x, y, \theta) \sim (x+1, y, \theta) \sim (x, y+1, \theta) \sim (x, y+Nx, \theta+2\pi)$. The resulting three-manifold is a Nil geometry and is usually called twisted torus, since it encodes a non-trivial circle fibration.

In fact, this background is known to have a geometric flux f related to the non-triviality of this fibration. In particular, following the same procedure we will follow in chapter 4 for more general situations, the geometric flux can be associated to the field strength of the globally defined form $\eta = dy - \frac{N}{2\pi}\theta dx$,

$$f = \int d\eta = \int \frac{N}{2\pi} dx \wedge d\theta = N \,, \tag{1.19}$$

and we observe that the original N units of H-flux have transformed into N units of geometric flux.

3. *T-fold.* A further application of Buscher rules (1.11) along direction x on the space (1.18) gives the configuration

$$\begin{aligned} ds^2 &= \frac{4\pi^2}{4\pi^2 + N^2\theta^2}(dx^2 + dy^2) + d\theta^2 \,, \\ B &= \frac{2\pi N\theta}{4\pi^2 + N^2\theta^2} dx \wedge dy \,. \end{aligned} \tag{1.20}$$

Now, after going around the base direction, the volume of the torus and its B-field get mixed in a non-trivial way, and the torus in the fiber cannot be glued using diffeomorphsism or gauge transformations. The gluing can only be done using T-duality transformations and the corresponding configuration is globally non-geometric. Such spaces are known in the literature as T-folds [26], and the corresponding flux controlling the non-geometry is called Q-flux. It can be obtained as the derivative of a bivector field β^{ij}, $Q_a{}^{bc} = \partial_a \beta^{bc}$, where β is constructed as the antisymmetric part of $(g+B)^{-1}$.

It is not difficult to see that neither the three torus with H-flux nor its duals are solutions to the string background equations, so the above example should only be understood as an illustrative toy model. Also, an important caveat of this example is the fact that the Nilmanifold is not a principal torus fibration, and one of the $U(1)$ isometries of the torus is in fact not globally defined, which makes the application of Buscher's procedure a priori not obvious (see for instance [27–30]). A way-out to this issue is to obtain the T-fold configuration directly from the initial flat torus with H-flux, where both $U(1)$ isometries are globally defined, either by performing a collective T-duality using the methods in [31,32] or by performing a fiberwise $SL(2,\mathbb{Z})_\rho$ rotation as explained in [33]. We will encounter again these backgrounds in chapter 2, embedded into a more general class of non-geometric configurations.

Finally, the R-flux in the right-hand side of the relation (1.15) is a bit more obscure. These sort of fluxes appear naturally in four-dimensional gauged supergravities as structure constants of the corresponding gauge algebras [25, 34–36]. However, it is very hard to construct configurations realising it. For instance, to obtain a background with such flux from (1.20), one would need to apply T-duality along the direction θ, which is not an isometry of the background. Furthermore, the conjectured configuration would not have a geometric description even locally, which make them very hard to study. During this thesis, this type of fluxes will not be further considered.

In the remaining of this section we will discuss some general ideas about how non-geometric fluxes can be used in the context of moduli stabilisation. Let us also mention that spaces with non-geometric fluxes have also very interesting mathematical properties. In particular, the Q-flux gives rise to non-associative geometries while R-flux to non-associative ones (see for instance [37–40]). Also, although in general CFT descripitions of such backgrounds are not available, in some particular cases these have been constructed from asymmetric orbifolds [40, 41].

Non-geometry and moduli stabilisation

So far we have introduced non-geometric backgrounds only through dualities. By definition of duality, if a certain non-geometric background is dual to a geometric one, the effective lower dimensional physics will be exactly that of the geometric compactification. However, by considering these cases one can learn about the couplings between the non-geometric fluxes and the moduli of the internal space, which is a valuable tool to engineer more general situations, using backgrounds which are not in any geometric orbit, that can help stabilising moduli that cannot be stabilised using only geometric fluxes.

As an example, let us consider the case the case of type IIB compactified on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, giving an $\mathcal{N} = 1$ theory in four-dimensions, discussed in [25, 42]. For simplicity, we consider a six-torus of the form $T^2 \times T^2 \times T^2$, where each T^2 factor will have the same complex structure τ and Kähler modulus ρ. In such compactifications, one obtains an effective scalar potential for the moduli of the form

$$V = e^K \left(K^{ij} D_i W \overline{D_j W} - 3|W|^2 \right) , \tag{1.21}$$

where K^{ij} is the inverse Kähler metric $K_{ij} = \partial_i \bar{\partial}_j K$, with K the Kähler potential

$$K = -3 \ln(-i(\tau - \bar{\tau})) - 3 \ln(-i(\rho - \bar{\rho})) - \ln(-i(\tau_{\text{a.d.}} - \bar{\tau}_{\text{a.d.}})) , \tag{1.22}$$

with $\tau_{\text{a.d}}$ the type IIB axio-dilaton. If one considers only geometric fluxes, the superpotential W is given by

$$W_{geo.} = P_1(\tau) + \tau_{\text{a.d.}} P_2(\tau) , \tag{1.23}$$

where P_1 is a polynomial whose coefficients are obtained from the RR three-form flux and P_2 another polynomial whose coefficients come from the NSNS flux H. We observe that, in this situation, the superpotential W does not depend on the Kähler moduli ρ, and it appears that such moduli cannot be stabilised (without considering other effects, such as non-perturvative ones). This is related to the fact that such IIB compactifications have a "no-scale" structure.

The situation changes completely if one considers also the non-geometric fluxes described above. In particular, if one wants to have an effective theory that is invariant under T-duality, one needs to generalise the superpotential to

$$W_{tot.} = P_1(\tau) + \tau_{\text{a.d.}} P_2(\tau) + \rho P_3(\tau)\,, \tag{1.24}$$

where now P_3 is a polynomial whose coefficients are obtained from the non-geometric flux Q. We now observe that such flux couples with the Kähler modulus ρ, and this can be stabilised. Intuitively, the fact that in the presence of Q-flux the volume of the torus changes non-trivially breaks the ”no-scale” structure.

From this example we learn that Q-flux can help stabilising the Kähler moduli of our background in situations where they cannot be stabilised using geometric fluxes. Therefore, non-geometric backgrounds are worth considering in the contexts of models for string phenomenology or stringy cosmology (see for instance [43–46]). In this context, it would be intersting to analyse to which extend they can be used to avoid no-go theorems of geometric compactifications. As an example, one could wonder whether non-geometry can avoid the Maldacena Nunez no-go theorem [47] for de Sitter spaces, and how this would fit into the recent discussion of de Sitter space within the swampland conjecture [48].

1.5 Generalised Geometry and extended space formalisms

In section 1.2, we have seen that the natural background where a bosonic string propagates contains a metric field g and a two-form field B, and are solutions of the theory (1.5). The local symmetries of such theory are diffeomorphisms and gauge transformations of the B-field. On the other hand, we have seen in section 1.3 that equivalent toroidal string backgrounds are related by $O(d, d, \mathbb{Z})$ transformations non-trivially mixing the metric and the B-field. Furthermore, when only zero modes of the string where considered, this group enhanced to $O(d, d, \mathbb{R})$, which was indeed the symmetry group of the action (1.5) when reduced on a d-dimensional torus. This symmetry is certainly no longer present when more general backgrounds are considered, but still one can find a reminiscence of it at a small neighbourhood around each point of the manifold, where any background looks flat (up to second derivatives terms like curvature).

One outcome of this discussion is that, from the string theory point of view, the metric and the two-form field are closely related. Such observation has motivation the appearance of formalisms that threat this fields in a same footing, covariantising the above mentioned symmetries and dualities.

$O(d, d)$ Generalised Geometry

The first of such formalisms is Generalised Geometry, which was initiated by Hitchin and Gualtieri [49–51] (in the context of string theory see [52, 53]). The starting point of this formalism is to construct a geometry generalising the tangent bundle TM of a given manifold M by formally substituting it to

$$TM \oplus T^*M \,. \tag{1.25}$$

A section $\mathbb{V}$ of such bundle can be written locally as $\mathbb{V} = v + \zeta$, where $v \in \Gamma(TM)$ is a tangent vector and $\zeta \in \Gamma(T^*M)$ a cotangent one. On the overlap of two patches of M, the bundles can be glued together by a $GL(d)$ transformation acting on both TM and T^*M (i.e. usual change of frame), as well as gauge transformations. The relevant fact of this construction is that one can naturally equip it with the $O(d,d)$ product

$$\langle \mathbb{V}_1, \mathbb{V}_2 \rangle = \langle v_1 + \zeta_1, v_2 + \zeta_2 \rangle = \iota_{v_1}\zeta_2 + \iota_{v_2}\zeta_1 = \eta_{MN}\mathbb{V}_1^M \mathbb{V}_2^N \,, \tag{1.26}$$

with

$$\eta_{MN} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{1.27}$$

The existence of this doubled bundle with this product defines an $SO(d,d)$ structure on the manifold in the same way a differential manifold has a $GL(d)$ structure. Also, any field over the manifold should organise into a representation of $SO(d,d)$. In particular, the metric and the B-field combine together into an object $\mathcal{H}$ called the generalised metric (we will encounter an explicit construction in chapter 2). In a Riemannian manifold, the metric breaks the structure group to $SO(d)$, which is the maximal compact subgroup of $GL(d)$, and inequivalent metrics parametrise the cosset space $\frac{GL(d)}{SO(d)}$. Similarly, having a generalised metric on the generalised tangent bundle breaks $SO(d,d)$ into its maximal subgroup, which is $SO(d) \times SO(d)$, and inequivalent generalised metrics parametrise the cosset $\frac{SO(d,d)}{SO(d)\times SO(d)}$.

Double Field Theory

The generalised geometry formalism is a natural framework for geometric bosonic string backgrounds with H-flux. Now, we want to present a generalisation where one could include also non-geometric configurations: Double Field Theory [54–56] (for reviews see also [57–59]). Such framework is based on doubling not only the tangent bundle but also the space-time dimensions, obtaining an extended manifold whose local coordinates on a patch are $\mathbb{X}^N = (x^n, \tilde{x}_n)$, being x^n the ones of the original d-dimensional manifold. In such construction, the above generalised bundle arises naturally as the tangent bundle of the $2d$-dimensional manifold.

In ordinary geometry, the infinitesimal transformation of any tensor field T along a diffeomorphism parameter v^n are given by the Lie derivative $\delta_v T = L_v T$. Now, on the doubled manifold, one can unify local diffeomorphisms and gauge transformations in a generalised Lie derivative along a generalised vector $\mathbb{V} = v + \zeta$, which acts on an arbitrary vector $\mathbb{U}$ as

$$\delta_{\mathbb{V}}\mathbb{U}^N = \mathcal{L}_{\mathbb{V}}\mathbb{U}^N = \mathbb{V}^M\partial_M\mathbb{U}^N + (\partial^N\mathbb{V}_M - \partial_M\mathbb{V}^N)\mathbb{U}^M \,, \tag{1.28}$$

where the indices $M, N = 1, \dots, 2d$ are raised and lowered with the $O(d,d)$ metric (1.27), and $\partial_N = (\partial_n, \tilde{\partial}^n)$. Analogously, one can construct actions of the generalised derivative to other generalised tensors, such as for instance the generalised metric. When restricted to $\mathbb{V} = v$ or $\mathbb{V} = \zeta$, one recovers the usual infinitesimal diffeomorphism and gauge transformations. In general, acting on an $O(d,d)$ tensor with the generalised derivative will produce another generalised tensor. In particular, the generalised derivative satisfy

$$\mathcal{L}_{\mathbb{V}} \eta_{MN} = 0 \,. \tag{1.29}$$

In ordinary geometry, the algebra of infinitesimal diffeomorphism transformations closes and the bracket of two Lie derivatives is again a Lie derivative. This is in general not true for the case of the generalised Lie derivative (1.28). In this case, the algebra closes only when the condition

$$\eta^{MN} \partial_M \bullet \partial_N \bullet = 0 \,, \tag{1.30}$$

is satisfied for any tensor field or gauge parameter in the theory inserted in $\bullet$. Such condition is known in the literature as strong or section constraint. Such condition is usually solved by restricting all fields to depend only on half of the coordinates. For instance, one can set $\tilde{\partial}^n \equiv 0$, in which case usual supergravity configurations are recovered. More concrete, if one solves the constraint by globally eliminating the dependence of any field on half of the coordinates, one can integrate out such directions, reducing the theory to the generalised geometry described above. However, one can also conceive spaces where the solution to the section is not globally defined. In this cases, such solution needs to be rotated by an $O(d,d)$ transformation on the overlap of two patches, and the configurations are non-geometric.

$E_{d(d)}$ Generalised Geometry and Exceptional Field Theory

The above discussion focused only on the case of $O(d,d)$ groups. There is however a natural way to generalise it to include $E_{d(d)}$ groups, motivated by U-duality of type II. It is interesting to notice that Exceptional Field Theory, the $E_{d(d)}$ version of DFT, will naturally unify both type II theories (as well as M-theory) into a single one in the way that will be explained below, which is consistent with the fact that type II/B are related by T-duality.

Starting from M-theory configurations, $E_{d(d)}$ generalised geometry [60–62] for $d \leq 7$ is based on extending the tangent bundle of a manifold M to

$$TM \oplus \Lambda^2 T^* M \oplus \Lambda^5 T^* M \oplus (T^* M \otimes \Lambda^7 T^* M) \,, \tag{1.31}$$

which accommodates diffeomorphisms and gauge transformations for the different form-fields of the theory. In dimensions $d < 7$, the space will be too small to accommodate some of these fields and, therefore, the corresponding term in (1.31) will be absent. Also analogous to the $O(d,d)$ case, the field content of M-theory will organise into a symmetric $E_{d(d)}$ matrix, the generalised metric, that parametrises the cosset

$$\frac{E_{d(d)}}{H_d} \,, \tag{1.32}$$

where H_d is the maximal compact subgroup of $E_{d(d)}$ (see Table 1.1).

Analogous to Double Field Theory, one can now construct a covariant theory for U-duality by enlarging the space-time dimensions in order to acommodate the bundle (1.31) as the tangent bundle of the extended manifold. Such theory recieves the name of Exceptional Field Theory [63,64], and within it one can unify infinitessimal diffeomorphism and gauge transformations into a generalised Lie derivative along a generalised infinitessimal parameter Λ^N. Such derivative acts on generalised vectors $\mathcal{V}^N$ as [65,66]

$$\mathcal{L}_\Lambda \mathcal{V}^N = \Lambda^M \partial_M \mathcal{V}^N + (\mathbb{P}_{\text{adj.}})^N{}_M{}^P{}_Q \mathcal{V}^M \partial_P \Lambda^Q + \lambda \mathcal{V}^N \partial_M \Lambda^M \,, \tag{1.33}$$

where $\mathbb{P}_{\text{adj.}}$ is a projector onto the adjoint of $E_{d(d)}$. In type II and M-theory, togehter with the $E_{d(d)}$ symmetry of the internal space, there is also an extra symmetry, called trombone symmetry [67], related to rescalings of the warp factor of the external fields. For this reason, it is convenient to formulate Exceptional Field Theory in terms of $E_{d(d)} \times \mathbb{R}^+$ tensors. Such objects will then have a weight under the $\mathbb{R}^+$, which corresponds to the factor λ in (1.33). For vectors of weight $\lambda = (D-2)^{-1}$, the derivative (1.33) can be rewritten as

$$\mathcal{L}_\Lambda \mathcal{V}^N = \Lambda^M \partial_M \mathcal{V}^N - \mathcal{V}^M \partial_M \Lambda^N + Y^{NM}_{PQ} \mathcal{V}^P \partial_M \Lambda^Q \,, \tag{1.34}$$

where Y^{NM}_{PQ} is an $E_{d(d)}$ invariant.

Finally, similar to the case of DFT, the algebra of infinitesimal generalised diffeomorphisms closes only if the section condition

$$Y^{NM}_{PQ} \partial_N \bullet \partial_M \bullet = 0 \,, \tag{1.35}$$

is satisfied. In this case, different solutions to this condition lead either to type IIB or to M-theory (and type IIA) configurations [63,64,68,69]. In this sense, EFT unifies all these theories. As in the DFT case, one can also imagine situations where the solution is not globally defined, leading to non-geometric configurations. In chapter 6 , we will discuss the case of $Spin(5,5)$ and give concrete expressions for the general discussion of this section.

1.6 Summary and overview

This thesis will be organised as follows:

- In chapter 2 we construct an analyse a class of configurations showing non-geometric features. These are two-torus fibrations with T-duality monodromies along non-contractible cycles. We will first analyse the case where the base is a circle, and construct configurations where the torus fiber is glued with any arbitrary element of the T-duality group. We then consider fibrations over a two-dimensional base and construct local solutions to the background equations around a degeneration with arbitrary monodromy in the T-duality group. We refer to such degenerations as T-fects. Some of these T-fects are identified with a semi-flat approximation of known brane solutions, while some of them are new exotic solutions.

- In chapter 3 we analyse the physics close to T-fects with parabolic monodromies. We argue that winding modes play an important role in such analysis, but supergravity approximation fails to encode them. For this reason, we also analyse this configurations within the formalism of Double Field Theory and discuss to which extend the physics we encounter can be encoded within it.

- One of the configurations that appears in the discussions on chapters 2 and 3 is the semi-flat approximation to the NS5 brane with two compact transverse directions. In chapter 4 we analyse T-duality transformations for the full un-compactified NS5 brane along angular isometries and compare the findings with the compact version.

- In chapter 5 we consider the T-fects configuration constructed in chapter 2 as solutions of type II and 11-dimensional supergravities and generalised them to allow also U-duality monodromies. After a detailed supersymmetry analysis, we conclude that one cannot construct supersymmetric solutions with any monodromy in the U-duality group. In fact, one can only construct configurations that are U-dual to T-fects. We analyse and classify all such configurations.

- In chapter 6 we analyse half-supersymmetric flux compactifications of type II and 11-dimensional supergravities to six dimensional Minkowski space using the formalism of generalised G-structure in Exceptional Field Theory. This is a very natural formalism to study these compactifications, since it unifies metric and fluxes degrees of freedom. After reviewing it, we construct the necessary tools to study compactifications to six external directions. We apply them to analyse the U-duality defects of chapter 5, reproducing the results obtained there.

- In chapter 7, we use the tools from chapter 6 to study half-supersymmetric flux compactifications of type IIB theory to AdS_6. In particular, we derive a classification for all such possible vacua, reproducing known results in terms of a simpler formulation using natural geometric objects. Furthermore, this formulation allows for studying the most general consistent truncations with vector multiplets around this vacua.

Chapter 2

Toroidal fibrations, T-folds and T-fects

In this chapter we construct and analyse a class of torus fibrations including certain configurations exhibiting non-geometric features. In particular, we will study two-torus fibrations with T-duality monodromies along non-contractible cycles.

In section 1.4 we have introduced non-geometric backgrounds as configurations where transition functions between patches are generalised to string dualities. Since, apart from simple realisations such as asymmetric orbifolds, little is known about string theory propagating in such backgrounds, it is interesting to construct and analyse explicit local geometries presenting such non-geometric features.

The approach we follow is to consider string theory compactified on a torus, and fiber the resulting T-duality group over a base $\mathcal{B}$. In particular, we will focus on the case of a two-torus T^2, whose T-duality group is $O(2,2,\mathbb{Z}) = SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where τ is the complex structure of T^2 and ρ is the complexified Kähler form. The $SL(2,\mathbb{Z})_\tau$ factor will be related with the large diffeomorphism group of the compactification torus $T^2_\tau = T^2$, while the other $SL(2,\mathbb{Z})$ factor corresponds to transformations that mix non-trivially the metric and the B-field.

On the first part of the chapter, we develop a geometric point of view obtained by interpreting the $SL(2,\mathbb{Z})_\rho$ factor as the large diffeomorphism group of an auxiliary torus T^2_ρ. We first consider the case in which the base $\mathcal{B}$ is a circle S^1, generating and classifying T-folds with any monodromy in the T-duality group. If only τ varies, the resulting 3-manifolds are geometric and can be understood in terms of a product of Dehn twists. For the cases with ρ-monodromies, one can think of them as Dehn twists of the auxiliary torus.

Second, we consider a fibration of the torus over a two dimensional base $\mathcal{B} = \mathbb{P}^1$, which is the familiar situation of stringy cosmic strings [70, 24]. In this situation the fibration degenerates at some point on the base around which we have a monodromy. In this sense the torus fibrations over a circle will be recovered at the boundary of a small disk encircling the defect. We will generically call this degenerations T-fects.

In order to construct approximate such geometries we use a semi-flat approximation [71–73] where the fields do not depend on the fiber coordinates. In our case this means to preserve the $U(1) \times U(1)$ isometry of the torus. On the degeneration this approximation breaks down and one needs to introduce by hand the exact geometry. This will be the main topic of chapter 3.

On the other hand, such T-fects solutions are co-dimension two objects and they have logarithmic divergences far away from the degeneration, which make them ill-defined as stand-alone objects. However, one can consider situations in which several of them are distributed along the same base, cancelling the divergences. The deficit angles of such objects will then bend the base and, by having enough of them, one can construct global models for compactification.

Finally, on the last section of the chapter, we will discuss how this configurations can be described within the formalism of Double Field Theory described in section 1.5, and compare this picture with the geometrisation of the $SL(2,\mathbb{Z})_\rho$ factor. This chapter closelly follows [74], as well as [33] in the final sections.

2.1 T-duality monodromies

Along this chapter we will consider two-tori fibered over a base $\mathcal{B}$. The moduli of a two-torus are encoded into its complex structure τ and complexified Kähler form ρ, defined as

$$\tau = \frac{g_{12}}{g_{22}} + i\frac{\sqrt{g}}{g_{22}}\,, \qquad \rho = B + i\sqrt{g}\,, \tag{2.1}$$

where g_{11}, g_{12}, g_{22} are the components of the metric on the torus, g its determinant, and B the unique component of an NS two-form field on the torus. Its T-duality group is $O(2,2,\mathbb{Z}) = SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where the $SL(2,\mathbb{Z})$ factors act on τ and ρ via Möbius transformations

$$\begin{aligned} \tau &\rightarrow M_\tau[\tau] \equiv \frac{a\tau + b}{c\tau + d}\,, \qquad M_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})_\tau\,, \\ \rho &\rightarrow M_\rho[\rho] \equiv \frac{\tilde{a}\rho + \tilde{b}}{\tilde{c}\tau + \tilde{d}}\,, \qquad M_\rho = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \in SL(2,\mathbb{Z})_\tau\,, \end{aligned} \tag{2.2}$$

The two $\mathbb{Z}_2$ factors are the mirror symmetry $(\tau, \rho) \rightarrow (\rho, \tau)$, which corresponds to factorised duality along one of the toroidal directions, and a reflection $(\tau, \rho) \rightarrow (-\bar{\tau}, -\bar{\rho})$. A partial geometrisation of the duality group is obtained by identifying the two $SL(2,\mathbb{Z})$ factors with the group of large diffeomorphisms of two tori, T^2_τ (the compactification torus) and T^2_ρ.

The first case we will consider, in section 2.2, will be torus bundles where the base $\mathcal{B}$ is a circle. This situation will be generalised in section 2.3 to fibrations over a two dimensional base, where the circle becomes contractible. Although in general configurations of the first kind do not satisfy string background equations, this situation has been analysed many times in the context of Scherk-Schwarz reductions and as toy model for non-geometric backgrounds [75, 76, 42, 77, 59]. Our discussion will focus in the role of the monodromy.

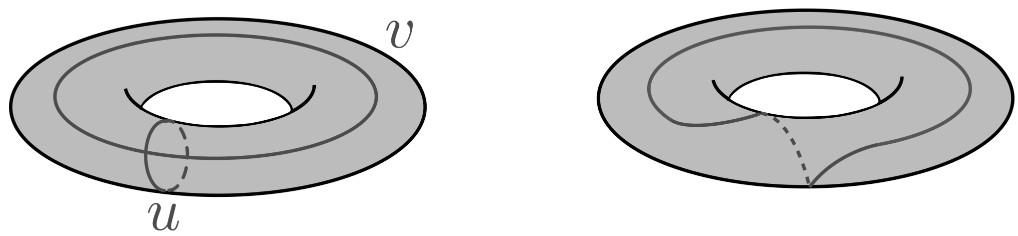

Figure 2.1: In the left, a torus with the cycles u, v. In the right, the action of a Dehn twist along the u-cycle, represented by the matrix U in (2.5), acting on the cycle v.

The case with $\mathcal{B} = \mathbb{P}^1$, where the configurations solve the string background equations, was introduced in [24]. Such situation is interesting since, in the case of heterotic theory, it can be understood from a duality with F-theory [78]. In particular, if one describes the auxiliary T^2_ρ fibration as an elliptic fibration, the corresponding line bundles on $\mathcal{B}$ that specify the fibration can be mapped explicitly to geometric compactifications on the F-theory side, and one can use this duality to obtain the conditions on the non-geomtric T^2_ρ fibration to obtain sensible string vacua. This analysis has been carried in [79–81].

2.1.1 Monodromy of mapping tori

Before turning into the study of such fibrations, we review some facts about monodromies that will be used along the discussion. We consider the fibration $T^2 \to \mathcal{N}_\phi \to S^1$ of a T^2 over a circle constructed as

$$\mathcal{N}_\phi = \frac{T^2 \times [0, 2\pi]}{(x^a, 0) \sim (\phi(x^a), 2\pi)} \,, \tag{2.3}$$

where ϕ is an element of the mapping class group $M(T^2)$, the group of large diffeomorphisms of a two torus. This group can be mapped one to one to $SL(2,\mathbb{Z})$ and, for this reason, we will simply write $\phi \in SL(2,\mathbb{Z})$. The geometry of $\mathcal{N}_\phi$ will be completely determined by the trace of ϕ or equivalently by the class of induced diffeomorphism of T^2.

The group $M(T^2)$ can be generated in terms of Dehn twists along two closed curves u and v with intersection number one. The corresponding Dehn twists, which we will denote by U and V, will satisfy the relations

$$\begin{aligned} UVU &= VUV \,, \\ (UV)^6 &= 1 \,, \end{aligned} \tag{2.4}$$

and the group will be generated by compositions of these two elements. A simple choice for the curves is the standard basis for the homology, see Figure 2.1. This gives the following matrix representation of the two Dehn twists:

$$U = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} , \qquad V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \tag{2.5}$$

which indeed satisfy (2.4).

Next we give a useful classification of the elements in $M(T^2)$, which will be later used to classify the different toroidal bundles in terms of their monodromy. This classification comes from the characterisation of elements in $SL(2,\mathbb{Z})$ inherited from the orientation-preserving isometries of the hyperbolic plane $\mathbb{H}^2$ through the identification $PSL(2,\mathbb{R}) \approx \text{Isom}^+(\mathbb{H}^2) \approx \text{Isom}^+(\text{Teich}(T^2))$ via Möbius transofrmations. Each isometry ϕ represented by a matrix $M \in SL(2,\mathbb{Z})$, is classified into one of the following types [82]:

1. *Elliptic type*: ϕ has one fixed point in $\mathbb{H}^2$, or equivalently $|tr(M)| < 2$.
2. *Parabolic type*: ϕ has no fixed point in $\mathbb{H}^2$ and exactly one fixed point on $\partial\mathbb{H}^2$, or equivalently $|tr(M)| = 2$.
3. *Hyperbolic type*: ϕ has no fixed point in $\mathbb{H}^2$ and exactly two fixed points in $\partial\mathbb{H}^2$, or equivalently $|tr(M)| > 2$.

Such classification for the elements of $SL(2,\mathbb{Z})$ is reflected in a thrichotomy of the corresponding torus diffeomorphisms, that are periodic, reducible and Anosov maps respectively. The geometry of the corresponding mapping torus $\mathcal{N}_\phi$ is determined by the class of such diffeomorphism and can be Euclidean, Nil or Solve. In the next section we will give concrete examples of this. In the rest of this section we study in more detail each of the three types of the above classification.

Ellyptic type

All elliptic elements are necessarily of finite order. There are exactly 6 conjugacy classes of elliptic type, given by the matrices

$$(UV)^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \qquad UVU = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad UV = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.6}$$

respectively of order 3, 4, 6; together with their inverses which, using the relations (2.4), are given by

$$\begin{aligned} &(UV)^{-2} = (UV)^4 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad (UVU)^{-1} = (UV)^4 U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ &(UV)^{-1} = (UV)^5 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}. \end{aligned} \tag{2.7}$$

The only other finite order element is $(UV)^3 = -\mathbb{1}$.

Parabolic type

For parabolic elements, which are of infinite order, there is an infinite number of conjugacy classes, which can be labelled by an integer N. Note that both the U and V Dehn twists are in the same conjugacy class labelled by $N = 1$. The general representative for the conjugacy class labelled by N is

$$\pm V^N = \pm \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} . \tag{2.8}$$

Elements in the same parabolic conjugacy class can be labelled by two coprime integers (p, q) as

$$L^{-1} V^N L = \begin{pmatrix} 1 + Npq & Np^2 \\ -Nq^2 & 1 - Npq \end{pmatrix} , \tag{2.9}$$

for any $L \in SL(2, \mathbb{Z})$.

Hyperbolic type

For hyperbolic elements there is a conjugacy class for each value of the trace, plus additional sporadic classes whose number can be computed with the methods of [83]. Two examples are given by

$$V^{1-N} U V = \begin{pmatrix} N & 1 \\ -1 & 0 \end{pmatrix} , \qquad U^{N+1} V U = \begin{pmatrix} 0 & 1 \\ -1 & -N \end{pmatrix} , \quad N \geq 3 . \tag{2.10}$$

The extra sporadic classes are listed in [83] up to $tr\, M = 15$. This are, in terms of Dehn twist decomposition (some of the matrices we use are conjugate to their representatives)

$$\begin{aligned} M_8 &= U^{-3} V^2 = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} , \quad M_{10} = U^{-4} V^2 = \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix} , \quad M_{12} = U^{-5} V^2 = \begin{pmatrix} 1 & 2 \\ 5 & 11 \end{pmatrix} , \\ M_{13} &= U^2 V^2 U^3 V^3 = \begin{pmatrix} -5 & -13 \\ 7 & 18 \end{pmatrix} , \quad M_{14} = U^{-6} V^2 = \begin{pmatrix} 1 & 2 \\ 6 & 13 \end{pmatrix} , \quad + \text{ inverses.} \end{aligned} \tag{2.11}$$

We note that the decomposition $M_{2N+2} = U^{-N} V^2$ seems not to be conjugate to the element (2.10) with the same trace nor to its inverse for $N > 1$. We therefore conjecture that this gives a list of sporadic conjugacy classes for hyperbolic elements of even trace $2N + 2$, $N \geq 2$.

The elements of hyperbolic conjugacy classes have always two real eigenvalues. For instance, the element $V^{1-N} U V$ in (2.10) has eigenvalues $(\lambda, 1/\lambda)$ with

$$\lambda = \frac{1}{2} (N + \sqrt{N^2 - 4}) , \qquad \text{for } N \geq 3 . \tag{2.12}$$

2.2 Three-manifolds and T-folds

We next apply the discussion of the previous section to string theory compactified on a two torus T^2_τ. Fibering this torus over an additional circle one generates a class of 3-manifolds that is commonly used to perform Scherk-Schwarz dimensional reduction and provides

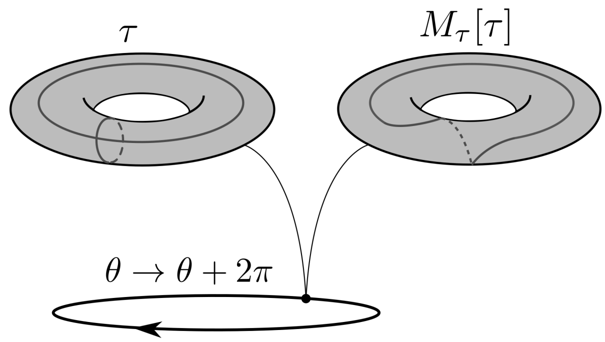

Figure 2.2: The mapping torus of a T^2_τ compactification. The monodromy of the fibration, which determine the geometry of the total space, is a product of Dehn twists. The simplest example is a twist along one cycle of the homology basis.

the simplest examples of Nil- and Sol-manifolds (see for instance [84]). If we also include monodromies in $SL(2,\mathbb{Z})_\rho$, which can be geometrised by an auxiliary T^2_ρ fibration over the circle, the resulting configurations will be in general non-geometric and are usually referred to as T-folds [27] (see also [85,86,30]). A usual approach in the literature to such spaces is through the chain of T-duality transformations (1.15), realised by the backgrounds (1.16), (1.18) and (1.20), which are simple toy models to introduce non-geometric fluxes.

The logic we will follow in the rest of the section will be different and will not rely on T-duality. Instead, our aim will be to give a classification of all possible geometric and non-geometric 3-spaces, coming from the corresponding classification of mapping class groups of the T^2_τ and T^2_ρ fibrations. We will construct explicit configurations for such T-folds that will be used in next section to classify all possible local geometries around defects arising from the fibration of the T-duality group on $\mathbb{P}^1$. Although, as mentioned above, the configurations considered in this section are not string vacua, after fibering the T-duality group over a two-dimensional base we will in fact obtain local solutions to the string background equations. The mapping tori presented in this section will arise at the boundary of a small disk encircling the defects.

We will now first review geometric torus bundles with $SL(2,\mathbb{Z})_\tau$ twists and then discuss spaces with non-geometric twists.

2.2.1 Geometric τ monodromies

We begin considering the situation where ρ is fixed to $\rho = i$ and we let $\tau(\theta)$ vary along the base $\mathcal{B} = S^1$ (see Figure 2.2). This is a well known situation, it has been discussed in some detail in [76], but will be useful to illustrate the role of the monodromy. The key point in the construction of such fibrations is to determine the function $\tau(\theta)$ encoding a given monodromy $M_\tau \in SL(2,\mathbb{Z})_\tau$. In terms of this function, the metric of the total space is given by

$$ds^2 = d\theta + g_{ab}dx^a dx^b\,, \tag{2.13}$$

where the metric on the toroidal fiber is, from (2.1),

$$g(\tau) = \frac{1}{\tau_2}\begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix}, \tag{2.14}$$

with $\tau = \tau_1 + i\tau_2$, and we fix the radius of the base circle to $R = 1$ and $(x^1, x^2) = (x, y)$. A given element M_τ acts on τ by Möbius transformation (2.2). We fix $\tau_0 = \tau(0)$ and demand that τ has the monodromy $\tau(2\pi) = M[\tau_0]$. To find such function we consider the element $\mathbf{m} = \log(M)$ of the Lie algebra $sl(2,\mathbb{R})$. Then, we construct the path $M(\theta)$ in $SL(2,\mathbb{R})$ given by the exponential map $M(\theta) = \exp(\mathbf{m}\,\theta/2\pi)$. By construction $M(2\pi) = M_\tau$, and we define

$$\tau(\theta) = M(\theta)[\tau_0], \tag{2.15}$$

which obviously has the desired monodromy properties: $\tau(0) = \tau_0$, $\tau(2\pi) = M[\tau_0]$. We will next discuss each of the three classes of $SL(2,\mathbb{Z})_\tau$ monodromies, and the corresponding torus diffeomorphism, separately.

Parabolic (reducible)

We begin considering monodromies of parabolic type, namely elements $M_\tau \in SL(2,\mathbb{Z})_\tau$ with $|tr(M_\tau)| = 2$. The simplest example is the shift $\tau \to \tau + N$, given by

$$M_\tau = V^N = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{m}_\tau = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}. \tag{2.16}$$

This monodromy implements an N-th power of a Dehn twist along the v cycle of the torus once encircling the base. Using the procedure described above, the complex structure of the fiber is given by

$$\tau(\theta) = \tau_0 + \frac{N}{2\pi}\theta, \tag{2.17}$$

where τ_0 is an arbitrary complex parameter. The corresponding total space is a Nilmanifold, as can be seen from the metric (2.13), which becomes:

$$ds_3^2 = d\theta^2 + dx^2 + \left(dy + \frac{N}{2\pi}\theta dx\right)^2, \tag{2.18}$$

with coordinate identifications described above, that implement the desired monodromy. Such metric can be written explicitly as a left-invariant metric on a nilpotent Lie group $\mathcal{G}$. In order to see this, we introduce the Maurer-Cartan forms $\eta^\theta = d\theta$ and $\eta^a = M(\theta)^a{}_b dx^b$, which satisfy

$$d\eta^\theta = 0, \qquad d\eta^a = (\mathbf{m}_\tau)^a{}_b \eta^\theta \wedge \eta^b. \tag{2.19}$$

Then, the generators (t_θ, t_1, t_2) of the corresponding Lie algebra $\mathfrak{g}$ satisfy

$$[t_\theta, t_a] = -(\mathbf{m}_\tau)_a{}^b t_b, \qquad [t_a, t_b] = 0. \tag{2.20}$$

From the precise form of $\mathbf{m}_\tau$, one can see that the lower central series, namely the sequences of ideals $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}^0]$, $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1]$,... terminates and the algebra is nilpotent. In fact, for $\mathbf{m}_\tau$ in (2.16), $\mathfrak{g}$ is the Heisenberg algebra, and the torus fibration can also be seen as a principal circle bundle over a torus. The global identification that makes the space compact is a quotient of $\mathcal{G}$ by a discrete subgroup. In the notation of [87] this space is referred to as $(0, 0, N \times 12)$.

The presented example was the simplest case of a Nil geometry and, as discussed above, it can be obtained by T-duality from a three torus with constant H-flux. One can also consider a monodromy which implements a Dehn twist around the u cycle of the fiber T^2, given by the element

$$M_\tau = U^N = \begin{pmatrix} 1 & 0 \\ -N & 1 \end{pmatrix} , \qquad \mathbf{m}_\tau = \begin{pmatrix} 0 & 0 \\ -N & 0 \end{pmatrix} . \tag{2.21}$$

This corresponds to the action on τ

$$\tau \to \frac{\tau}{1 - N\tau} . \tag{2.22}$$

which actually corresponds to a shift of τ^{-1}: $\tau^{-1} \to \tau^{-1} - N$. More generally, one can consider arbitrary elements in the N the conjugacy class, labelled by two coprime integers (p, q) as in (2.9),

$$M_\tau = \begin{pmatrix} 1 + Npq & Np^2 \\ -Nq^2 & 1 - Npq \end{pmatrix} , \qquad \mathbf{m}_\tau = \begin{pmatrix} Npq & Np^2 \\ -Nq^2 & -Npq \end{pmatrix} , \tag{2.23}$$

The corresponding solution for $\tau(\theta)$ is given by

$$\tau(\theta) = \exp(\mathbf{m}_\tau \theta / 2\pi)[\tau_0] = \frac{(2\pi + Npq\theta)\tau_0 + Np^2\theta}{-Nq^2\theta\, \tau_0 + (2\pi - Npq\theta)} , \tag{2.24}$$

and the corresponding metric on the total space is

$$ds_3^2 = d\theta^2 + d\tilde{x}^2 + \left(d\tilde{y} + \frac{N}{2\pi}(p^2 + q^2)\theta d\tilde{x} \right)^2 , \tag{2.25}$$

where

$$\tilde{x} + i\tilde{y} = e^{i\varphi}(x + iy) , \qquad \varphi = \arctan\left(\frac{q}{p}\right) , \tag{2.26}$$

and the identification now corresponds to gluing the T^2 after performing a N-th power of a Dehn twist along a cycle represented by $p[v] + q[u]$. Using (2.20) with $\mathbf{m}_\tau$ given in (2.23), one can see again that the corresponding algebra is nilpotent. In light of the result (2.25), one might think that studying different configurations with monodromies in the same conjugacy class is uninteresting. However, we will see that the same analysis with twists in $SL(2, \mathbb{Z})_\rho$ give rise to very different local configurations.

Elliptic (periodic)

We now turn to geometries with monodromies in elliptic conjugacy classes. We begin considering an elliptic conjugacy class represented by the order 4 monodromy:

$$M_\tau = UVU = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{m}_\tau = \begin{pmatrix} 0 & \frac{\pi}{2} \\ -\frac{\pi}{2} & 0 \end{pmatrix}, \tag{2.27}$$

for which the corresponding solution for $\tau(\theta)$ using (2.15) is given by (see also [76])

$$\tau(\theta) = \frac{\cos(\theta/4)\,\tau_0 + \sin(\theta/4)}{-\sin(\theta/4)\,\tau_0 + \cos(\theta/4)}. \tag{2.28}$$

The corresponding total space is a compactification of the Lie group ISO(2), the group of symmetries of the Euclidean plane. This can be seen from the local coordinate representation of the left-invariant forms:

$$\begin{aligned} \eta^1 &= d\theta\,, \\ \eta^2 &= \cos(\theta/4)\,dx + \sin(\theta/4)\,dy\,, \\ \eta^3 &= -\sin(\theta/4)\,dx + \cos(\theta/4)\,dy\,. \end{aligned} \tag{2.29}$$

which satisfy (2.19) with $\mathbf{m}_\tau$ in (2.27). In terms of this forms, the left invariant metric can be written as [76] $ds_3^2 = d\theta^2 + g_{a\,b}(\tau_0)\eta^a\eta^b$.

We can perform an analogous analysis for the rest of the remaining finite order conjugacy classes. For example, for the conjugacy classes of order 3 and 6, represented by $(UV)^k$, $k = 1, 2$, we have

$$\mathbf{m}_\tau^{UV} = \frac{\pi}{3\sqrt{3}} \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \qquad \mathbf{m}_\tau^{UVUV} = 2\mathbf{m}_\tau^{UV}, \tag{2.30}$$

and the corresponding complex structure is

$$\tau(\theta) = \frac{A^+\tau_0 + \frac{2}{\sqrt{3}}\sin(k\,\theta/6)}{-\frac{2}{\sqrt{3}}\sin(k\,\theta/6)\tau_0 + A^-}, \qquad A^\pm = \cos(k\,\theta/6) \pm \frac{1}{\sqrt{3}}\sin(k\,\theta/6)\,. \tag{2.31}$$

All 3-manifolds obtained in this way are compactifications of Lie groups of uni-modular solvable (but not nilpotent) algebra. Note that, if the parameter τ_0 is chosen to be the fixed points of the monodromy element we are considering, namely

$$UVU:\ \tau_0 = i\,, \qquad UV:\ \tau_0 = e^{i\pi/3}\,, \qquad UVUV:\ \tau_0 = e^{2i\pi/3}\,, \tag{2.32}$$

the complex structure is constant along the base and given by $\tau(\theta) = \tau_0$. These points are the minima for the potential for the moduli obtained by a reduction with duality twists. In them, one has a CFT description in terms of symmetric orbifolds (see for instance [75, 77, 85, 41]).

Hyperbolic (Anosov)

Finally, we consider hyperbolic monodromies, corresponding to Anosov diffeomorphisms of the fiber torus. To get some intuition about the kind of geometry we expect, we begin with a simple example with monodromy in $SL(2,\mathbb{R})$:

$$M_\tau = \begin{pmatrix} e^{\omega} & 0 \\ 0 & e^{-\omega} \end{pmatrix}, \qquad \mathbf{m}_\tau = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}. \tag{2.33}$$

The corresponding complex structure is given by $\tau(\theta) = e^{\omega\theta/\pi}\tau_0$, and corresponding metric for the total space (fixing $\tau_0 = i$)

$$ds_3^2 = e^{\omega\theta/\pi}dx^2 + e^{-\omega\theta/\pi}dy^2 + d\theta^2\,. \tag{2.34}$$

From the relations (2.20) we can deduce that the corresponding algebra is solvable, but not nilpotent, and therefore the configuration is Sol-manifold. The compact torus bundle can be obtained by a quotient of a Lie group Sol by a discrete quotient [84]. The background (2.34) was analysed in [88] in the context of a search of d Sitter vacua in string theory.

We now turn to genuine $SL(2,\mathbb{Z})$ hyperbolic monodromies. Some of the conjugacy classes can be labelled by integer N and are given by:

$$M_\tau = V^{1-N}UV = \begin{pmatrix} N & 1 \\ -1 & 0 \end{pmatrix}, \quad N \geq 3\,. \tag{2.35}$$

Note that the matrix

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \tag{2.36}$$

representing Arnold's cat map on the torus, is conjugate to the matrix above for $N = 3$. The monodromy matrix (2.35) can be diagonalized with two real eigenvalues (λ, $1/\lambda$), as given in (2.12). Powers of the Anosov diffeomorphism stretch and contract exponentially the two eigenspaces. For this reason, the repackaging of the torus image under the map into the fundamental domain results in a chaotic behaviour. In terms of λ, the algebra element is given by

$$\mathbf{m}_\tau = \frac{\log\lambda}{\lambda^2 - 1}\begin{pmatrix} \lambda^2+1 & 2\lambda \\ -2\lambda & -(\lambda^2+1) \end{pmatrix}, \tag{2.37}$$

and the corresponding torus complex structure by

$$\tau(\theta) = \frac{\tau_0 + \lambda - \lambda^{1+\theta/\pi}(1+\lambda\tau_0)}{-\lambda(\tau_0+\lambda) + \lambda^{\theta/\pi}(1+\lambda\tau_0)}\,. \tag{2.38}$$

From this expression one easily gets the metric on the total space. Analogous solutions can be found for the remaining conjugacy classes.

2.2.2 T-folds and ρ monodromies

We next turn to cases of a torus compactification where the complex structure is fixed to $\tau = i$ and we let $\rho(\theta)$ to vary along the base $\mathcal{B} = S^1$. Such spaces can be formally obtained by applying the results of an auxiliary T^2_ρ fibration on the circle, that geometrises the $SL(2,\mathbb{Z})_\rho$ factor of the T-duality group. A local expression for the fields characterising these configurations is given by

$$ds_3^2 = d\theta^2 + \rho_2(\theta)(dx^2 + dy^2)\,, \quad B = \rho_1(\theta) dx \wedge dy\,, \tag{2.39}$$

where $\rho = \rho_1 + i\rho_2$. The monodromy mixes the volume and the B-field and, in general, the resulting spaces are not manifold: after going around the circle, the torus fiber cannot be glued using diffeomorphisms or gauge transformations. Except for some particular limit in the moduli space in which we have a string description from asymmetric orbifold CFTs, there is in general no clear string description of such T-folds. However, as we will discuss in the next section, the non-geometric monodromies that we describe arise also in fibrations over $\mathbb{P}^1$, and in some cases one can obtain evidence for the existence of the corresponding string vacuum from string dualities.

Parabolic

We begin considering the parabolic monodromy

$$M_\rho = V^N = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}\,. \tag{2.40}$$

This corresponds to a shift of ρ, $\rho(\theta) = \rho_0 + N\theta$, and can be understood geometrically as a Dehn twist along the v cycle of the auxiliary T^2_ρ. The corresponding total space is just the three torus $T^3 = T^2_\tau \times S^1$ with N units of H-flux described by the fields (1.16), and can be obtained by T-duality from the Nilmanifold (2.18) as we discussed above. If we consider the conjugate monodromy

$$M_\rho = U^N = \begin{pmatrix} 1 & 0 \\ -N & 1 \end{pmatrix}\,, \tag{2.41}$$

we obtain the following solution for ρ:

$$\rho(\theta) = \frac{2\pi\rho_0}{-N\theta\rho_0 + 2\pi}\,. \tag{2.42}$$

The gluing condition now mixes the volume of the fiber torus with the B-field, as follow from the action on the volume,

$$\sqrt{g} \to \frac{\sqrt{g}}{N^2\, g + (NB - 1)^2}\,, \tag{2.43}$$

and we thus encounter an obstruction to glue the toroidal fiber to obtain a torus bundle. Note that for $\rho_0 = i$, from (2.42) we obtain the following field configuration:

$$ds_3^2 = d\theta^2 + \frac{1}{1 + N^2\theta^2}(dx^2 + dy^2)\,, \quad B = -\frac{N\theta}{1 + N^2\theta^2} dx \wedge dy\,, \tag{2.44}$$

which coincides with the T-fold metric we obtain from a T-duality on the Nilmanifold. We can also consider the general (p, q) parabolic conjugacy class as in (2.24), thus obtaining a class of T-fold metrics that interpolate between the two previous solutions.

Elliptic

We can now repeat the analysis of finite order elements done for the geometric τ monodromies. The corresponding solutions for $\rho(\theta)$, for monodromies of order 3, 4 and 6 are given by (2.28) and (2.31) by a fiberwise mirror symmetry $\tau \to \rho$. In this case, all configurations in this class present non-geometric features. For instance, the order 4 monodromy $M_\rho = UVU$, acting on ρ as

$$\rho \to -\frac{1}{\rho}\,, \tag{2.45}$$

has an action on the volume that transforms it like

$$\sqrt{g} \to \frac{\sqrt{g}}{B^2 + g}\,, \tag{2.46}$$

and the non-trivial mixing between the volume and the B-field becomes an obstruction to obtain a torus bundle. Analogous results can be also found in the cases of order 3 and 6.

These non-geometric T-folds have been discussed in the context of generalized Scherk-Schwarz reduction within double field theory in [89,44]. The corresponding potential $V(\rho_0)$ admits a minimum at the fixed point of the monodromy element. At such minimum, there exists a description in terms of an asymmetric orbifold CFT [40, 41]. We note that in the CFT description (as well as in double field theory) we see the presence of both H-flux and Q-flux. These fluxes should be identified by the corresponding monodromy, the perturbative shift $V_\rho \sim H$ and the non-geometric twist $U_\rho \sim Q$. The simultaneous presence of both fluxes fits well with the monodromy decomposition of the elliptic monodromy of order 4, $M_\rho^{ell} = U_\rho V_\rho U_\rho$. As we will discuss in the next section, such decompositions are defined up to some redundancy, which basically follows from a braid action on the Dehn twist decomposition. In this case we have for example, from (2.4), $U_\rho V_\rho U_\rho = V_\rho U_\rho V_\rho$, so it is not simple to match the monodromy decomposition with the corresponding flux parameters. We will come back to this point when we will study fibrations on a $\mathbb{P}^1$ base and we will identify sources for the fluxes.

Hyperbolic

We finally analyse the case of hyperbolic monodromy in ρ. As in the case of τ monodromies, one can construct a simple example by considering the $SL(2, \mathbb{R})$ monodromy (2.33). Choosing $\rho_0 = i$, this gives the metric

$$ds_3^2 = d\theta^2 + \rho_0 e^{2\theta\omega}(dx^2 + dy^2)\,, \tag{2.47}$$

where the volume of the toroidal fiber increases exponentially by going around the base.

We next discuss the case of the hyperbolic $SL(2,\mathbb{Z})$ representative element

$$M_\rho = \begin{pmatrix} N & 1 \\ -1 & 0 \end{pmatrix}, \quad N \geq 3\,. \tag{2.48}$$

The action on ρ is given by

$$\rho \to -\frac{1}{\rho} - N\,, \tag{2.49}$$

and the function $\rho(\theta)$ is given by (2.38) with $\tau \to \rho$. The action on the volume is the same as for the elliptic case,

$$\sqrt{g} \to \frac{\sqrt{g}}{B^2+g}\,. \tag{2.50}$$

but now iterations are not periodic and act in a complicated way. The action on the volume after n iterations, for instance, can be derived from (2.38) by setting $\tau \to \rho$ and $\theta = 2\pi n$. One obtains

$$\sqrt{g} \to \frac{\sqrt{g}\,(\lambda^2-1)^2\,\lambda^{2n}}{[\lambda(B+\lambda)-(B\lambda+1)\lambda^{2n}]^2 + g\lambda^2\,(\lambda^{2n}-1)^2} \tag{2.51}$$

2.2.3 General case

Finally, we also briefly analyse the case in which both $SL(2,\mathbb{Z})$ factors are non-trivially fibered over the same circle, letting $\tau(\theta)$ and $\rho(\theta)$ vary. These configurations are in general non-geometric and not dual to any geometric one, and they have been studied for instance in [41, 89]. An example is the order 4 solution

$$\begin{aligned} \tau(\theta) &= \frac{\cos(f\theta)\,\tau_0 + \sin(f\theta)}{-\sin(f\theta)\,\tau_0 + \cos(f\theta)}\,, \quad f \in \frac{1}{4} + \mathbb{Z}\,, \\ \rho(\theta) &= \frac{\cos(g\theta)\,\rho_0 + \sin(g\theta)}{-\sin(g\theta)\,\rho_0 + \cos(g\theta)}\,, \quad g \in \frac{1}{4} + \mathbb{Z}\,, \end{aligned} \tag{2.52}$$

where the parameters f and g should be identified with geometric and non-geometric fluxes. The potential for the moduli (τ_0, ρ_0) has a minimum at the fixed points of the monodromy $(\tau_0, \rho_0) = (i, i)$. At this minimum, an asymmetric orbifold description was studied in [41].

In the context of heterotic/F-theory duality, it is convinient to think of these configurations in terms of mapping class group of a genus surface of genus two obtained by gluing together both tori T_ρ and T_τ. If we then embed the representative matrices M_τ, M_ρ in 4×4 matrices acting on the homology of the genus-2 surface, we can think about the double elliptic monodromy as being decomposed into $M^{ell}_{\tau,\rho} = U_\tau V_\tau U_\tau U_\rho V_\rho U_\rho$. This would corresponds to having two kind of geometric fluxes, as well as H and Q-fluxes, in agreement with [41]. More details on this discussion can be found in [74, 79, 80].

2.3 Torus fibrations over a surface and T-fects

In this section we generalise the discussion of the previous section to the case where the duality group is fibered over a two-dimensional base $\mathcal{B} = \mathbb{P}^1$. This situation is well known from stringy cosmic strings [70] and was discussed in the context of non-geometric compactifications in [24]. In particular, it is known that in the case where the moduli τ and ρ are local holomorphic functions on the base, the configuration has local solutions to the killing spinor equations preserving half of the original supercharges, and it is therefore a local solution of the string background equations [70, 24, 90].

One can then imagine a situation where τ and ρ are meromorphic functions of the base, namely holomorphic everywhere except for a set of points, denoted by $\Delta = (x_1, \ldots, x_n)$, where the fibration degenerates and a defect sits (see Figure 2.3). These points are branch points for the functions τ and ρ and there is a non-trivial monodromy around them. A complementary point of view, described in [78, 91, 92], is to describe this fibrations as elliptic fibrations, and map the corresponding line bundles to an F-theory compactification via heterotic/F-theory duality, where even non-geometric ρ monodromies are mapped to geometric compactifications of F-theory. Possible degenerations of the elliptic fibrations are described by the Kodaira list as it is familiar in F-theory. Since such defects have T-duality monodromy around them, we will call them generically *T-fects*.

The approach we take here is slightly different and our goal is to understand the relation between the T-folds discussed in the previous section and the defects that arise in fibrations over $\mathbb{P}^1$. We construct local geometries in a region close to the defect for monodromies in all conjugacy classes of $SL(2, \mathbb{Z})$, naturally obtaining a classification for all such configurations arising in the neighbourhood of a given τ and ρ degeneration. In principle, the set of degenerations we consider is bigger than those appearing in the Kodaira classification, since not all $SL(2, \mathbb{Z})$ conjugacy classes arise from degenerations of elliptic curves (for an F-theory example see [83, 93]).

Within the geometric subclass of solutions, we encounter fibrations that can be interpreted as a semi-flat approximation to known 10-dimensional brane solutions, such as the NS5 or the KK monopole. However, it is important to emphasise that in general the existence of the solutions presented in this section cannot be taken as an automatic evidence that the corresponding degeneration exists in string theory. In fact the semi-flat torus fibration approximation breaks down on the defect, and one needs to complement it with a microscopic description of such degeneration, which in general it is not known. In some cases, part of this microscopic information can be inferred from T-duality arguments, as we will discuss in chapter 3. Also, there is the possibility that in some cases evidence of existence can be obtained from heterotic/F-theory duality.

2.3.1 Geometric τ-branes

We begin discussing local solutions in the neighbourhood of degenerations of the τ fibration. We fix the complexified Kähler parameter to $\rho = i$ and let $\tau(z)$ be a function of the two-dimensional base $\mathcal{B} = \mathbb{P}^1$, whose complex coordinate is denoted by $z = r\, e^{i\theta}$. We take a

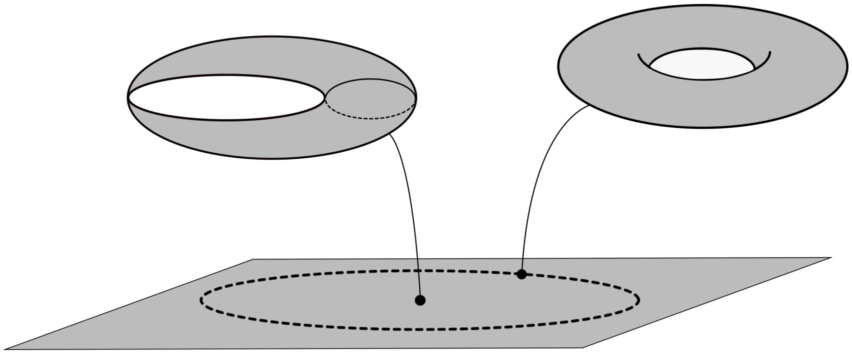

Figure 2.3: Torus fibration over $\mathbb{P}^1$.

semi-flat metric ansatz $\mathbb{R}^{1,5} \times \mathcal{B} \times_{\varphi} T^2$, where T^2 is fibered over $\mathcal{B}$ and the $U(1) \times U(1)$ isometries on the torus are preserved:

$$\begin{aligned} ds^2_{10} &= \eta_{\mu\nu} dx^{\mu} dx^{\nu} + e^{2\varphi_1} \tau_2 dz d\bar{z} + g_{ab}(z) d\xi^a d\xi^b \,, \\ e^{\Phi} &= \text{const}\,, \qquad B = 0\,, \qquad \mu, \nu = 1, \ldots, 5\,, \qquad a, b = 1, 2\,, \end{aligned} \tag{2.53}$$

where g_{ab} is obtained by inverting (2.1) and is given by

$$g(z) = \frac{1}{\tau_2} \begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix} . \tag{2.54}$$

One can check that, imposing $\tau(z) = \tau_1 + i\tau_2$ to be a meromorphic function on $\mathcal{B}$ and $\nabla^2 \varphi_1 = 0$, the metric is locally Ricci-flat and therefore a local solution to the string background equations. In fact, these conditions are necessary and suficient to preserve half of the supersymmetries (assuming that the corresponding spinor fields are globally defined on the manifold) [24, 90, 94]. For convenience, we will solve the condition $\nabla^2 \varphi_1 = 0$ by taking φ_1 to be the real part of a meromorphic function $\varphi(z) = \varphi_1 + i\varphi_2$.

In the present discussion, we will study the case where $\mathcal{B}$ is a disc. Such local solutions can in general be glued together to obtain a global fibration on $\mathbb{P}^1$, but we will not further elaborate on this point. We then take a small disc $D^2 : 0 < |z| < R_0$ and consider a function τ that has a branch point on its center, which means that τ will be multivalued around it. At the branch point the fibration will degenerates, the description break down and one has to resort to a string description of the degeneration. On the boundary of the disc, $S^1 = \partial D^2$, we will have a smooth torus fibration, which will be the mapping tori described in section 2.2.1 identified with the monodromy of the multivalued function τ.

Given an element $M_\tau \in SL(2, \mathbb{Z})_\tau$, one can construct the corresponding local geometry using the method described in section 2.2.1 together with solving Cauchy-Riemann equations for $\tau(z)$. In particular, we promote the free modulus τ_0 to be a function of $r = |z|$. The equations then reduce to an ordinary differential equation for $\tau_0(r)$ that, as we will see in the following for the three different types of the diffeomorphism in the fiber, has always solutions.

The resulting function $\tau(z)$ we obtain using this procedure should be understood as a function on an appropriate multi-sheeted Riemann surface, where the transition function between sheets is determined by a given $SL(2,\mathbb{Z})$ element, that we identify with an element of $M(T^2)$. For instance, given an element

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{2.55}$$

we obtain a function such that the analytic continuation along an arc that encircles the origin is

$$\tau(z) \to \frac{a\tau(z)+b}{c\tau(z)+d} . \tag{2.56}$$

Such function determines the semi-flat approximation to a given T-fect. For parabolic and elliptic conjugacy classes, we will recover in this way the known local solutions already appearing in the work by Kodaira [95], see table 2.1. Solutions in the hyperbolic conjugacy classes are more complicated and cannot be found by other methods.

Finally, to have the full description of the geometry, one needs to fix the warping factor φ_1. By various arguments [24, 90], one can show that in order for the metric on $\mathcal{B}$ to be single-valued the analytic continuation for the meromorphic function $\varphi = \varphi_1 + i\varphi_2$ should be

$$e^{\varphi(z)} \to e^{\varphi(z)}(c\tau(z)+d) . \tag{2.57}$$

Once $\tau(z)$ is known, one can use a similar logic as before and engineer a function φ satisfying (2.57). This can be done by

$$e^{\varphi(z)} = [c(\theta)\tau_0(r) + d(\theta)]e^{\varphi_0(r)} , \tag{2.58}$$

being $\tau_0(r)$ the r-dependent modulus from $\tau(z)$ and $c(\theta)$, $d(\theta)$ the elements in the bottom line of the matrix $M(\theta)$, constructed in section 2.2.1 for each given monodromy. The free modulus $\varphi_0(r)$ can be fixed again by Cauchy-Riemann conditions and the function $\varphi(z)$ has all the desired properties.

In the rest of the present section we construct and analyse explicit configurations for local geometries with monodromies in all three types of conjugacy classes.

Parabolic τ-branes and KK-monopoles

We consider the situation in which the tours fiber degenerate at z_0 by shrinking a cycle $p[v] + q[u]$. If $q = 0$ and $p = 1$ we have the monodromy (2.16) for $N = 1$, $M_\tau = V$. The corresponding action on τ is a shift:

$$\tau \to \tau + 1 . \tag{2.59}$$

The corresponding solution for $\tau(z)$ is well known [70]:

$$\tau(z) = \frac{i}{2\pi} \log\left(\frac{\mu}{z}\right) , \qquad e^{\varphi} = 1 , \tag{2.60}$$

Class	Type	Mondromy	Local model
Parabolic	I_n	$V^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$	$\frac{n}{2\pi i} \log z, \quad n > 0$
Elliptic order 6	II	$UV = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$\frac{\eta - \eta^2 z^{1/3}}{1 - z^{1/3}}, \quad \eta = e^{2\pi i/3}$
Elliptic order 4	III	$UVU = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\frac{i + i\sqrt{z}}{1 - \sqrt{z}}$
Elliptic order 3	IV	$UVUV = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\frac{\eta - \eta^2 z^{2/3}}{1 - z^{2/3}},$

Table 2.1: Locus solutions around degenerations of Kodaira type I_n, II, III, IV.

where μ is an integration constant. The corresponding metric in polar coordinates on $\mathbb{C}$ is thus:

$$\begin{aligned} ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{2\pi} \log\left(\frac{\mu}{r}\right) \left[d\theta^2 + r^2 dr^2 + (dx^8)^2\right] \\ + \frac{2\pi}{\log\left(\frac{\mu}{r}\right)} \left(d\xi^2 + \frac{\theta}{2\pi} d\xi^1\right)^2 . \end{aligned} \tag{2.61}$$

It is easy to check that this is a semi-flat approximation of a KK monopole with compact circle $\xi^2 \approx u$. Indeed, this is precisely the analysis done in [96]. To show this, let us start from the Taub-NUT metric:

$$ds^2_{KKM} = \eta_{\mu\nu} dx^\mu dx^\nu + h(\vec{x})\, d\vec{x}^2 + \frac{1}{h(\vec{x})} \left(d\xi^2 + A\right)^2 , \qquad h(\vec{x}) = 1 + \frac{R_2}{2|\vec{x}|} , \tag{2.62}$$

where $A = \vec{A} \cdot d\vec{x}$, $\vec{x} = (z, \bar{z}, \xi^2)$, and $dA = *_3 dh$. We will set the radius $R_2 = 1$. We now compactify on ξ^1. This corresponds to have an infinite array of sources on the covering space, resulting in the potential:

$$h = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[\frac{1}{\sqrt{(\xi^1 - 2\pi n)^2 + r^2}} - \frac{1}{|2\pi n|} \right] \approx \frac{1}{2\pi} \log(\mu/r) , \tag{2.63}$$

where the logarithm is a good approximation (up to exponentially supressed terms) in the region away from the origin, where the configuration reduces to the semi-flat metric (2.61). On the other side, if one takes the limit close to the origin in the semi-flat metric (2.61), the fibration breaks down and one needs to add corrections to the logarithm term in order to recover the full function (2.63). These will localised the shrinking cycle (NUT direction) along the perpendicular one, breaking the semi-flat torus ansatz. We will further elaborate on this discussion in chapter 3, where we will study the physics close to the degeneration.

The above analysis can be easily extended to the monodromy V^N, which corresponds to having a stack of N KK-monopoles at the degeneration point. We next consider the generic case corresponding with the (p, q) parabolic conjugacy class, with $q \neq 0$:

$$M_\tau = \begin{pmatrix} 1+pq & p^2 \\ -q^2 & 1-pq \end{pmatrix} . \tag{2.64}$$

We can construct the corresponding local geometry by using the method described above. We begin considering the ansatz for $\tau(r, \theta)$

$$\tau(r, \theta) = \exp(\mathbf{m}_\tau \theta / 2\pi)[\tau_0(r)] . \tag{2.65}$$

Plugging this ansatz into the Cauchy-Riemann equations

$$\frac{\partial \tau(r,\theta)}{\partial r} - \frac{1}{i\, r} \frac{\partial \tau(r,\theta)}{\partial \theta} = 0 , \tag{2.66}$$

they reduce to a single differential equation for $\tau_0(r)$, that can be written as

$$2\pi\, i\, r \frac{d\tau_0(r)}{dr} = [p + q\tau_0(r)]^2 . \tag{2.67}$$

The solution for $\tau(z)$ is then:

$$\tau(z) = \frac{2\pi i}{q^2 \log\left(\frac{\mu}{z}\right)} - \frac{p}{q} . \tag{2.68}$$

Following an analogous procedure for $e^{\varphi(z)}$ we obtain

$$e^{\varphi(z)} = i\kappa \log\left(\frac{\mu}{r}\right) , \tag{2.69}$$

where κ is an arbitrary integration constant. Since the monodromy corresponds now to a Dehn twist along the vanishing cycle $p[u] + q[v]$, we obtain a semi-flat approximation of a KK-monopole with special circle along an oblique direction in the (ξ^1, ξ^2) plane whose slope is given by q/p, as in (2.25). The semi-flat approximation arise after compactifying the orthogonal direction and taking the leading approximation for h as discussed before.

Elliptic τ-branes

We next consider the case in which the monodromy is of finite order, corresponding to periodic diffeomorphisms of the torus. Different degenerations can be classified in terms of elliptic conjugacy classes of $SL(2, \mathbb{Z})$, whose representatives are listed in (2.6),(2.7). We consider for illustration the monodromy of order 4 $M_\tau = UVU$, corresponding to the transformation

$$\tau \to -1/\tau . \tag{2.70}$$

Following the same procedure we used for the parabolic (p,q) monodromy (2.65), the Cauchy-Riemann equations coming for τ and e^{φ} reduce to

$$4r\frac{d\tau_0}{dr} + i(\tau_0^2+1) = 0\,, \qquad 4r\frac{d\varphi_0}{dr} = i\tau_0\,, \tag{2.71}$$

from which one obtains the solutions

$$\tau(z) = \tan\left[C - \frac{i}{4}\log(z)\right], \qquad e^{\varphi} = \kappa\cos\left[C - \frac{i}{4}\log(z)\right], \tag{2.72}$$

where C and κ are integration constants. Note that $\tau(z)$ can be written as

$$\tau(z) = \frac{i - i\,e^{2iC}\sqrt{z}}{1+\,e^{2iC}\sqrt{z}} = i - 2ie^{2iC}\sqrt{z} + 2ie^{4iC}z + \mathcal{O}(z^{3/2})\,, \tag{2.73}$$

and for $C = \pi/2$ the solution for τ reproduces the local model shown in Table 2.1. At the degeneration point the complex modulus is given, as expected, by the fixed point of the elliptic monodromy, $\tau_0 = i$. The corresponding metric can be found by plugging the functions τ and φ into the ansatz (2.53).

As we did for the parabolic case, we could also consider solutions associated to the general conjugacy class of the particular elliptic monodromy. This is specified by three parameters (p,q,w):

$$M_{p,q,w} = L^{-1}UVUL = \begin{pmatrix} pq + \dfrac{w(1+qw)}{p} & p^2+w^2 \\ -q^2 - \dfrac{(1+qw)^2}{p^2} & -\dfrac{w+q(p^2+w^2)}{p} \end{pmatrix}, \tag{2.74}$$

for any $L \in SL(2,\mathbb{Z})$. The corresponding solution to the Cauchy-Riemann equations for $\tau(z)$ and $e^{\varphi(z)}$ is

$$\tau(z) = w\left[\frac{1}{p + w\tan\left[p^2C - \frac{i}{4}\log(z)\right]} - \frac{1+wq}{p}\right]^{-1}, \tag{2.75}$$

$$e^{\varphi(z)} = \kappa\,p\,q\,\cos\left[p^2C - \frac{i}{4}\log(z)\right] + \kappa i(1+pw)\sin\left[p^2C - \frac{i}{4}\log(z)\right]. \tag{2.76}$$

Finally, a similar analysis can be done for all the remaining elliptic conjugacy classes. For example, the solution corresponding to the monodromy UV and $(UV)^2$ are found to be

$$UV\ :\qquad \tau(z) = \frac{\eta+\eta^2e^{iC}z^{1/3}}{1+e^{iC}z^{1/3}}\,, \qquad e^{\varphi} = \kappa(e^{-iC/2}z^{-1/6} + e^{iC/2}z^{1/6})\,, \tag{2.77}$$

$$UVUV\ :\qquad \tau(z) = \frac{\eta+\eta^2e^{iC}z^{2/3}}{1+e^{iC}z^{2/3}}\,, \qquad e^{\varphi} = \kappa(e^{-iC/2}z^{-1/3} + e^{iC/2}z^{1/3})\,, \tag{2.78}$$

which reproduce the solutions listed in table 2.1 when $C = \pi$.

Hyperbolic τ-branes

The last example case to consider are the defects with hyperbolic monodromy, corresponding to an Anosov map of the fiber torus. As we did for the maping class tori, it is useful to consider a simple example with monodormy in $SL(2,\mathbb{R})$ given by (2.33). In this case one obtains

$$\tau(z) = z^{-iw/\pi} = e^{\omega\theta/\pi}\left[\cos\left(\frac{w}{\pi}\log(r)\right) - i\sin\left(\frac{w}{\pi}\log(r)\right)\right], \qquad e^{\varphi(z)} = z^{iw/2\pi}. \tag{2.79}$$

We observe that both the imaginary and the real part of τ are highly oscillating close to the origin, and there are infinitely many intervals where $\tau_2 < 0$. The solutions for conjugacy classes of hyperbolic $SL(2,\mathbb{Z})$ are more involved, but share the same problem. As an example, for the monodromy(2.35), we obtain the following differential equations:

$$\pi r(\lambda^2-1)\frac{d\tau_0(r)}{dr} + i\log(\lambda)\left[\lambda+\tau_0(r)\right]\left[\lambda\tau_0(r)+1\right] = 0\,, \tag{2.80}$$

$$2\pi r(\lambda^2-1)\frac{d\phi_0(r)}{dr} - i\log(\lambda)\left[1+\lambda^2+2\lambda\tau_0(r)\right]\phi_0(r) = 0\,, \tag{2.81}$$

which lead to the solutions

$$\tau(z) = \frac{\sigma^{\lambda^2}(\lambda^2-1)}{\lambda(\sigma e^{i\tilde{z}}-\sigma^{\lambda^2})} - \frac{1}{\lambda}\,, \tag{2.82}$$

$$e^{\phi(z)} = \kappa\sqrt{\lambda}e^{\frac{i}{2}(\mu-\tilde{z}\lambda^2)}\left[\sigma^{\lambda^2}-\sigma e^{i\tilde{z}}\right], \tag{2.83}$$

with σ, κ and μ being integration constants and $\tilde{z}$ defined as

$$\tilde{z} = \mu(1-\lambda^2) + \frac{1}{\pi}\log\lambda\log\left[\pi z(\lambda^2-1)\right]. \tag{2.84}$$

Again, one can see that this solution has a bad behaviour close to the origin, having also infinite intervals where $\tau_2 < 0$, which makes it not obvious how to make sense of such solutions. This fits well with the fact that Anosov diffeomorphisms cannot be obtained as monodromies of a degenerating family of curves, and thus cannot be associated with a degeneration point. The F-theory analogous of such hyperbolic branes was studied in [93, 83]. In this context, there are separated 7-branes, and the associated massive states gives rise to infinite dimensional algebras.

A possible way out to the mentioned problems could be to think of this solutions as an approximation to a non-collapsed group of branes. In this case there is a size representing the brane distribution that can serve as a cutoff around the origin of the solution. To cure the problem at infinity one should add extra defects to cancel the total charge, as for the rest of T-fects. It would be interesting to understand better the nature of such hyperbolic solutions in the present context.

2.3.2 NS5 and exotic ρ-branes

In the previous section we discussed a classification of local geometries associated to monodromies filling the $SL(2,\mathbb{Z})_\tau$ mapping class group of the compactification torus. Global models of such τ fibrations will give rise to geometric 6d compactifications. We now consider the case of a ρ fibration, that can be geometrised as a mapping class group of an auxiliary torus T_ρ. The ρ fibrations, at constant $\tau = i$, are the fiberwise mirror-symmetric of the geometric τ fibrations discussed above. The ansatz in this case reads:

$$\begin{aligned} ds^2_{10} &= \eta_{\mu\nu}dx^\mu dx^\nu + e^{2\varphi_1}\rho_2 dz d\bar{z} + \rho_2(d\xi_1^2 + d\xi_2^2)\,, \\ H &= (\partial_z\rho_1 dz + \partial_{\bar{z}}\rho_1 d\bar{z}) \wedge d\xi_1 \wedge d\xi_2\,, \qquad e^\Phi = \rho_2\,, \end{aligned} \tag{2.85}$$

which, analogous to the τ case, it contains the configuration (2.39) on the boundary of a disc centered at the origin. Again, the fact the base $\mathcal{B}$ is two-dimensional and the circle contractible, allows to have configurations of the form (2.85) solving the equations of motion. As in the τ case these are solved by demanding that $\tau(z)$ and $\varphi(z)$ are locally holomorphic functions of the base, which also preserve half of the supersymmetry, as it follows from the careful analysis done in [24,73,90]. In appendix A we independently check that this ansatz satisfy the equations of motion. We stress that, also for the ρ fibrations, the semi-flat ansatz breaks down on the degenerations (branch points) of the fibration and one needs to resort to a string description of the degeneration. This aspect will be addressed in chapter 3 for some particular cases.

The meromorhic functions $\rho(z)$ describing the local geometries that will be discussed in this section can be obtained from the ones in the previous by the exchanging $\tau \to \rho$. We also note that in this case, unlike before, we have non-trivial dilaton field that varies around the defect. One should check that this transforms in the correct way under the T-duality monodromy, which is indeed the case for the ansatz (2.85).

Parabolic ρ branes

We begin again considering the simple case with monodromy of type V^N. This corresponds to gluing the torus with a gauge transformation for the B-field, so the corresponding solution is geometric. We have

$$\rho(z) = \frac{i}{2\pi}\log\left(\frac{\mu}{z}\right)\,, \qquad e^\varphi = 1\,, \tag{2.86}$$

and the background configuration is then given by the fields

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{2\pi}\log\left(\frac{\mu}{r}\right)\left[d\theta^2 + r^2 dr^2 + (d\xi^1)^2 + (d\xi^2)^2\right]\,. \tag{2.87}$$

$$B = \frac{\theta}{2\pi}d\xi^1 \wedge d\xi^2\,, \qquad e^\Phi = \frac{1}{2\pi}\log\left(\frac{\mu}{r}\right)\,. \tag{2.88}$$

As it is well known, this solution can be identified with the semi-flat approximation of a stack of NS5 branes [97] (see also [72, 90]). An indication of this can be found by integrating the flux over a three-dimensional surfaces surrounding the defect [24], which in this case we can take it to be the fiber torus times a constant radius circle γ on the complex plane. One obtains

$$\int_{T^2\times\gamma} H = \frac{N}{2\pi}\int_\gamma d\rho_1 = N\,, \tag{2.89}$$

which is an indication of the existence of a solitonic source for the H-flux in the interior of the surface. In fact, following a procedure analogous to the KK-monopole case we discussed in the previous section, this can be checked explicitly. The solution for a stack of NS5 branes localized on $\mathbb{R}^2\times T^2$ can be found by starting from the NS5 harmonic function h and compactify two directions, which is equivalent to consider an array of sources on the (ξ^1,ξ^2) plane. This gives

$$h(r) = \sum_{n,m}\frac{1}{(x^8-2\pi n)^2+(x^9-2\pi m)^2+r^2}\,. \tag{2.90}$$

At distances large compared to the distance between the sources, the result for the harmonic function is the same as the smeared KK monopole, and we obtain the metric (2.87). However, note that the corrections to the semi-flat approximation involve the breaking of both the $U(1)$ isometries of the torus. Since the degeneration is the same as for the KK-monopole, it seems that information about the breaking of the second $U(1)$ isometry is missing. We will further elaborate on this discussion when studying the physics close to the degeneration in chapter 3.

Next, we consider the solutions for the general conjugacy class of parabolic ρ monodromies, namely the ones associated to a (p,q) monodromy

$$M_\rho = \begin{pmatrix} 1+pq & p^2 \\ -q^2 & 1-pq\end{pmatrix}\,. \tag{2.91}$$

Note that for $p=0$ the monodromy is a β-transformation, while for $q=0$ this is just the B-transformation we just discussed. For general p,q both transformations are present at the same time. The NS5 brane corresponds to a $(1,0)$ brane, while the general solution for $q\neq 0$ is given by (2.68), (2.69). The $(0,1)$ solution, with monoromy U $(-1/\rho\to -1/\rho+1)$ is

$$\rho(z) = \frac{2\pi i}{\log\left(\frac{z}{\mu}\right)}\,,\qquad e^{\varphi} = i\sigma\log\left(\frac{\mu}{z}\right)\,, \tag{2.92}$$

and one obtains the following solution

$$\begin{aligned} ds^2_{10} &= \eta_{\mu\nu}dx^\mu dx^\nu + 2\pi\sigma^2 h(r)(dr^2+r^2d\theta^2) + \frac{2\pi h(r)}{h(r)^2+\theta^2}\left[(d\xi^1)^2+(d\xi^2)^2\right]\,, \\ B_2 &= \frac{2\pi\theta}{h(r)^2+\theta^2}d\xi^1\wedge d\xi^2\,,\qquad e^{2\Phi} = \frac{2\pi h(r)}{h(r)^2+\theta^2}\,, \end{aligned} \tag{2.93}$$

with

$$h(r) = \log\left(\frac{\mu}{r}\right) . \tag{2.94}$$

We see that the monodromy acts non-trivially on the volume, so this solution is non-geometric. This is the exotic brane solution discussed in length in [98, 90] (see also [99]) and usually called Q-brane or 5^2_2-brane in the notation of [23]. We have explicitly shown that the torus fibration on the boundary of a small neighbourhoods of such exotic brane reproduces the parabolic T-fold fibrations. In this case, the lack of a notion of integration in a globally non-geometric space such as the one described by the local geometry (2.93) does not allow to investigate the nature of the singularity by studying the flux on a hypersurface surrounding it.

Finally, we note that the geometry (2.93) can be obtained by applying Busher rules to the semi-flat KKM metric (2.60), in the same way the NS5 smeared on one circle is T-dual to the KKM solution (2.62). Corrections to this T-duality chain will be studied in chapter 3.

Elliptic ρ-branes

For the elliptic monodromies of finite orders, we can again obtain local solutions from the functions (2.72), (2.76), (2.77) and analogous solutions for other conjugacy classes. For example, the order 4 monodromy corresponds to the following background:

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}dx^\mu dx^\nu - \frac{1}{2}\sinh\left[\frac{1}{2}\log\left(\frac{r}{\mu}\right)\right](dr^2 + r^2 d\theta^2) \\ &\quad - \frac{\sinh\left[\frac{1}{2}\log\left(\frac{r}{\mu}\right)\right]}{\cos\left(\frac{\theta}{2}+\sigma\right) + \cosh\left[\frac{1}{2}\log\left(\frac{r}{\mu}\right)\right]}\left[(d\xi^1)^2 + (d\xi^2)^2\right] , \end{aligned} \tag{2.95}$$

$$\begin{aligned} B_2 &= -\frac{\sin(\theta/2+\sigma)}{\cos(\theta/2+\sigma) + \cosh\left[\frac{1}{2}\log\left(\frac{\mu}{r}\right)\right]} d\xi^1 \wedge d\xi^2 , \\ e^{2\Phi} &= -\frac{\sinh\left[\frac{1}{2}\log\left(\frac{r}{\mu}\right)\right]}{\cos\left(\frac{\theta}{2}+\sigma\right) + \cosh\left[\frac{1}{2}\log\left(\frac{r}{\mu}\right)\right]} , \end{aligned} \tag{2.96}$$

where, with respect to (2.72), we have redifined $C_1 = \sigma$ and $C_2 = 1/4 \log \mu$. Let us check explicitly that the solution has the desired monodromy. The action of the elliptic transformation is $\rho \to -1/\rho$. Recalling that $\rho = B + iV$, this corresponds to:

$$B(2\pi) = -\frac{B(0)}{B(0)^2 + g(0)} , \qquad \sqrt{g}(2\pi) = \frac{\sqrt{g}(0)}{B(0)^2 + g(0)} , \tag{2.97}$$

where the value within brackets is the value of the angle θ at which the fields are evaluated. It is not difficult to check that indeed the B-field and the fiber torus metric in the solution (2.95) satisfy the relations (2.97). Since the metric and B-field are mixed by the monodormy the solution is non-geometric, as in the case of the 5^2_2 parabolic brane.

Let us discuss the regime of validity of the solution (2.95). As for the other codimension two solutions, there is a scale at which the Ricci scalar blows up. This scale is determined by the parameter μ and can be taken to be large. In a global model this would be related to the scale at which the local solution breaks down and is glued to the global solution. Additionally, it would be very important to understand corrections to the semi-flat approximations near the degeneration.

We conclude by discussing the relation between the elliptic defect and the elliptic T-fold discussed in section 2.2.2. In the T-fold picture, which is related to an asymmetric $\mathbb{Z}_4$ orbifold construction [41], one can argue for the presence of both H and Q fluxes (this can also be inferred from a double field theory approach [89]). This seems to be compatible with the Dehn twist decomposition of the elliptic monodromy $M_\rho = UVU$, if we identify the source of H-flux with NS5 branes and the source of Q-flux with a 5^2_2 branes. The charge of such branes should indeed be identified with the parabolic monodromies V and U. There is some puzzle with this identification. The way to sum charges for codimension-2 defects is closely related to braids, and Dehn twists indeed satisfy the braid relation $UVU = VUV$. It is not clear how this fact could be seen in the corresponding gauged supergravity. Moreover, the elliptic asymmetric T-folds have fluxes quantized in fractional units, so that the relation with the corresponding brane is not obvious. It would be interesting to clarify these issues.

Hyperbolic ρ-branes

The last example in order to exhaust all possible conjugacy classes for the ρ fibration is a hyperbolic monodromy. As we discussed in 2.3.1, it is not possible to interpret such monodromy as coming from a degeneration of elliptic curves. Let us consider as an example the monodromy

$$M_\rho = \begin{pmatrix} N & 1 \\ -1 & 0 \end{pmatrix}, \quad N \geq 3\,. \tag{2.98}$$

The action on the B-field and the volume is then:

$$B \to -\frac{1}{\lambda} - \lambda - \frac{B}{B^2+g}\,, \qquad V \to \frac{\sqrt{g}}{B^2+g}\,. \tag{2.99}$$

We recall that λ is the bigger eigenvalue of M_ρ, and $\lambda^{-1} + \lambda = N$. The action (2.99) is similar to the elliptic case (2.97), but successive iterations of the hyperbolic transformation acts very differently on the fields. From the solution (2.82) it is possible to exhibit a metric and flux with such monodromy. For simplicity, we show only the solutions for the torus volume and the B-field:

$$\sqrt{g} = \frac{1}{\lambda}\frac{e^{\sigma}(\lambda^2-1)\left[e^{\sigma} - \lambda^{\theta/\pi}e^{\lambda^2\sigma}\cos\tilde{r}\right]}{e^{2\sigma} + \lambda^{2\theta/\pi}e^{2\sigma\lambda^2} - 2\lambda^{\theta/\pi}e^{\sigma(\lambda^2+1)}\cos\tilde{r}} - \lambda\,, \tag{2.100}$$

$$B = -\frac{1}{\lambda}\frac{e^{\sigma(\lambda^2+1)}(\lambda^2-1)\sin\tilde{r}}{\lambda^{-\theta/\pi}e^{2\sigma} + \lambda^{\theta/\pi}e^{2\sigma\lambda^2} - 2e^{\sigma(\lambda^2+1)}\cos\tilde{r}}\,, \tag{2.101}$$

where we defined

$$\tilde{r} = \mu(\lambda^2 - 1) + \frac{1}{\pi} \log \lambda \log \left[\pi r(\lambda^2 - 1)\right] \,. \tag{2.102}$$

It is easy to show that those solutions indeed satisfy (2.99). From the expression of $\sqrt{g}$ we see that close to the origin there are infinite points at which $\sqrt{g}$ turns negative. This solution can at best approximate the geometry of non-collapsed branes outside a small disk around the origin. Again, the solution cannot be trusted for large r, as for the other co-dimension two metrics.

2.3.3 Colliding degenerations

We finally discuss the most general case in which both τ and ρ vary along the base $\mathcal{B}$ and their degenerations collide at the same point z_0, that we again take to be the origin. The semi-flat metric ansatz is now:

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + e^{2\varphi_1} \tau_2 \rho_2 dz d\bar{z} + \rho_2 g_{ab} d\xi^a d\xi^b \,, \\ H_3 &= (\partial_z \rho_1 dz + \partial_{\bar{z}} \rho_1 d\bar{z}) \wedge d\xi^1 \wedge d\xi^2 \,, \qquad e^\Phi = \rho_2 \,, \end{aligned} \tag{2.103}$$

with g_{ab} given by (2.54). The conditions imposed by supersymmetry on this ansatz have been studied in [24, 73]. This fixes again ϕ, τ and ρ to be holomorphic functions on the punctured sphere. In appendix A we gives the expressions for the Ricci tensor and Ricci scalar for this ansatz, by explicitly checking that it solve the equations of motion.

In principle, by combining any of the solutions for $\rho(z)$ and $\tau(z)$ derived in the previous section we can obtain an explicit expression for the metric and the B-field that have an arbitrary monodromy (M_τ, M_ρ), namely any arbitrary element in $O(2, 2, \mathbb{Z})$ that can be connected to the identity with a path in $O(2, 2, \mathbb{R})$ (this excludes, for instance, $\tau \leftrightarrow \rho$). As in the case of the other local solutions described above, its existence cannot be taken as a direct proof of the existence of the corresponding defect in string theory. However, in this case this objects can be studied from heterotic/F-theory duality, where the monodromies of the double torus can be reinterpreted as monodromies on a genus 2 surface [74, 79].

Double parabolic T-fects

The simplest example one can consider is the case $M_\tau = V$, $M_\rho = V^N$ should represent a stack of N NS5 branes on top of a Taub-NUT space. This is trivial to check from(2.103), which by using previous results reads:

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + h_5(r) ds^2_{KKM} \,, \\ B_2 &= \frac{N\theta}{2\pi} d\xi^1 \wedge d\xi^2 \,, \qquad e^\Phi = h_5(r) \,, \end{aligned} \tag{2.104}$$

where

$$h_5(r) = \frac{N}{2\pi} \log\left(\frac{\mu}{r}\right) \,. \tag{2.105}$$

This is a smeared approximation of the solution for a NS5 (smeared along a single direction) on Taub-NUT space, which has precisely the same form (2.104) by harmonic superposition, with $h_5(r) \sim 1/r$. Analogous results can be found for arbitrary M_τ. If M_ρ is not a simple shift, we obtain non-geometric solutions that are not T-dual to any geometric background. Here we limit ourselves to discuss one example of this kind, obtained by considering an elliptic monodromy for both the τ and ρ modulus.

Double elliptic branes

As an illustrative example, we consider the monodromy

$$M_\tau = M_\rho = UVU = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{2.106}$$

which acts on τ and ρ by

$$\tau \to -\frac{1}{\tau} , \quad \rho \to -\frac{1}{\rho} . \tag{2.107}$$

From the solutions obtained for the corresponding τ and ρ one can construct a local geometry that has the T-fold described by (2.52). Here we show the explicit expressions for the metric of the torus that fiber over the two-dimensional base (for a particular choice of the integration constants)

$$G_{11} = \tau_1^2 + \tau_2^2 = \frac{r + \mu^2 - 2r\mu\cos(\theta + 2\sigma)}{\left[r + \mu + 2\sqrt{r\mu}\cos\left(\theta/2 + \sigma\right)\right]^2} , \tag{2.108}$$

$$G_{12} = \tau_1 = \frac{\sin(\theta/2 + \sigma)}{\cos(\theta/2 + \sigma) + \cosh\left[\frac{1}{2}\log\left(\frac{r}{\mu}\right)\right]} , \tag{2.109}$$

$$e^{2\Phi} = \tau_2 = -\frac{\sinh\left[\frac{1}{2}\log\left(\frac{r}{\mu}\right)\right]}{\cos\left(\frac{\theta}{2} + \sigma\right) + \cosh\left[\frac{1}{2}\log\left(\frac{r}{\mu}\right)\right]} , \tag{2.110}$$

$$B = \rho_1 = \tau_1 = G_{12} . \tag{2.111}$$

The action of the elliptic transformation on the metric components is:

$$G_{11}(2\pi) = \frac{1}{G_{11}(0)} + \frac{G_{12}(0)^2}{G_{11}(0)^2}\left[G_{12}(0)^2 - 1\right] , \qquad G_{12}(2\pi) = -\frac{G_{12}(0)^2}{G_{11}(0)^2} , \tag{2.112}$$

which reproduces the desired monodromy. We can now retake the discussion between the relation between the T-fect geometries described in this section and the T-folds of section 2.2 that we started for a single ρ-fect. Now, a T-fold with moduli satisfying the double elliptic monodromy (2.52) has been studied in [89, 41, 40], where it was argued that such backgrounds contain both geometric and non-geometric f, H and Q fluxes. It is interesting to ask if there is a relation between these fluxes and the charges of the T-fects considered here. A way to define the charge is via the monodromy that identifies a given T-fect. For a NS5 branes, the monodromy V_ρ indeed corresponds to measuring one unit of H-flux around the source. The exotic 5^2_2 brane is then identified by the parabolic monodromy U_ρ, and one can declare this to correspond to one unit of non-geometric Q-flux. For geometric fibrations, one could naively identify the parabolic monodromies V_τ and U_τ that correspond to KK monopoles with orthogonal special directions to the two kind of "geometric f-fluxes" that appear in the gauged supergravity. This is little more than terminology, and there are of course subtleties in making these relations concrete. For example, our discussion makes very clear that the monodromy is closely related to the topology of the space, so identifying a monodromy with a parameter of the effective theory requires some care [100]. However, it is interesting to see that the monodromy factorization in terms of Dehn twists, which would correspond to adding the charges of the parabolic T-fects, indeed contains the monodromies associated to NS5, 5^2_2 and KKM sources. It would be interesting to study further this relation.

2.3.4 Remarks on global constructions

Before closing this section, we want to give some final remarks about global issues. In general, all local geometries considered in this section have a bad behaviour far away from the origin. This is actually a characteristic fact of codimension-two objects, that typically have logarithmic divergences at infinity. However, in the present discussion we want to remark that such configurations can be thought as local geometries in a region of a global model.

In particular, retaking the discussion at the begining of this section, one can consider the situation where the toroidal moduli τ and ρ are meromorphic functions of $\mathbb{P}^1/\Delta$, where Δ is a discrete set of points where T-fects sit. One can then imagine that, by properly distributing the defects along the base and adjusting the integration constants of the local geometry, one can cancel the divergences and engineer global constructions.

The number of defects one can use for such constructions is however not arbitrary. One of the reasons for this can be phrased in terms of the monodromies. In particular, the monodromy along a path that encircles several defects is the product of the monodromies around each single defect. This can be seen by deforming the path, as explained in detail in [74]. Then, one expects that a path at infinity encircling all defects should have a trivial monodromy. From the second relation (2.4), one can conclude that one needs at least 12 fundamental degenerations (with monodromy U or V, which generate the $SL(2,\mathbb{Z})$ group) to construct any globlal model.

On the other hand, the codimension-two objects have typically a deficit angle, which bends the base surface and makes it eventually compact. This situation is well known from stringy cosmic strings [70], where it is known that one needs exactly 24 fundamental defects to construct a consistent compact string background, which is compatible with the above monodromy argument.

For the case where only τ varies and the fibration is geometric, the global model with

24 degenerations is a $K3$ surfaces. Models with non-trivial ρ fibrations are non-geometric and have been studied in [79, 80]. In both cases, the global model is not a torus fibrations, since the fibration degenerates on the defects. To obtain the full description of the model, one needs to smooth such degenerations by gluing in their exact geometric description, which tipically breaks some of the isometries of the fiber. Such corrections will be the topic of chapter 3.

2.4 T-fects in extended space theories

So far we have described two-torus fibrations over a base $\mathcal{B}$ with monodromies in the $SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho$ subgroup of the T-duality group. The $SL(2,\mathbb{Z})_\tau$ factor was identified with the mapping class group $M(T^2)$, the group of large diffeomorphisms of a two-torus. The factor $SL(2,\mathbb{Z})_\rho$, that includes the non-geometric monodromies, could be geometrised by thinking of it as a fibration of an auxiliary torus T_ρ. This picture is motivated by the heterotic/F-theory duality, where even non-trivial fibrations of the auxiliary T_ρ are mapped to geometric ones. Also, it has a natural generalisation to fibrations of genus two surfaces, corresponding to heterotic compactifications with a Wilson line, as discussed in [74].

In this chapter, we want to give a complementary picture for this configurations motivated by the doubled space formalisms such as the one described in [26] or Double Field Theory. Such picture will be useful in next chapter when discussing the physics next to the defects. The main idea is to describe the above configurations as four-torus fibration where the monodromies are restricted to be in $SO(2,2,\mathbb{Z}) \subset SL(4,\mathbb{Z})$, the last one being the group of large diffeomorphisms of the four-torus. As we will argue in the upcoming discussion, in the case where only one of the complex moduli (τ,ρ) is non-trivially fibered, the $\mathbb{T}^4$ fiber factorises into $\mathbb{T}^2 \times \tilde{\mathbb{T}}^2$. This is not the case for fibrations with non-trivial monodromies in both τ and ρ, where the gluing condition mixes both tori and one can only rely on the four-torus description. We want to stress that the extra torus $\tilde{\mathbb{T}}^2$ is not related with the auxiliary torus T^2_ρ in a trivial way. Instead, it is related to the extra winding coordinates of Double Field Theory. For the case of geometric τ-fects, the coordinates along the torus $\tilde{\mathbb{T}}^2$ indeed coincide with these extra coordinates.

In the rest of this section, we will first recall some facts about $O(2,2,\mathbb{Z})$ that will be then used to constructed the described picture for the T-fects configuration. When discussing the physics near the degenerations in chapter 3, we will discuss how and to which extend the physics we obtain can be encoded in this picture.

2.4.1 Actions of the T-duality group $O(2,2,\mathbb{Z})$

In section 2.1 we presented the T-duality group by its action on the complex moduli (τ,ρ) of a two-torus, which was the convenient language for the subsequent discussion. For the upcoming one, it will be more useful to describe it in a different way. For this reason, we first recall some well known facts about the group $O(2,2,\mathbb{Z})$ (for details see for instance [101]).

For general dimension d, an $O(d,d,\mathbb{Z})$ element can be written as a matrix

$$\Omega_{O(d,d)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,, \tag{2.113}$$

satisfying

$$A^T C + C^T A = B^T D + D^T B = 0 \quad A^T D + C^T B = I\,. \tag{2.114}$$

If one encodes the metric g and the 2-form field B on the torus into the background matrix

$$E = g + B \tag{2.115}$$

the element (2.113) acts on E as

$$E \to \frac{AE+B}{CE+D}\,. \tag{2.116}$$

Furthermore, one can also organise the background fields into an $O(d,d,\mathbb{Z})$ matrix constructed as

$$\mathcal{H} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}\,, \tag{2.117}$$

which is usually called generalised metric and is the natural way to parametrise the moduli space of a d-torus, which is the cosset $\frac{O(d,d)}{O(d)\times O(d)}$. The element (2.113) acts covariantly on the generalised metric,

$$\mathcal{H} \to \Omega\,\mathcal{H}\,\Omega^T\,. \tag{2.118}$$

In the context of Double Field Theory, $\mathcal{H}$ is interpreted as a symmetric bilinear form on a $2d$-dimensional space. This space is constructed by formally doubling the dimensions of the manifold, whose local coordinates are now denoted as $X^A = (\xi^a, \tilde{\xi}_a)$, where ξ^a are identified with the d-dimensional coordinates and $\tilde{\xi}_a$ are usually refered to as winding coordinates.

For the $d=2$ case, the $SL(2,\mathbb{Z})$ actions on τ and ρ, defined as

$$\tau \to \frac{a\tau+b}{c\tau+d}\,, \qquad \rho \to \frac{\tilde{a}\rho+\tilde{b}}{\tilde{c}\rho+\tilde{d}}\,, \quad ad-bc = \tilde{a}\tilde{d}-\tilde{b}\tilde{c} = 1\,, \tag{2.119}$$

can be embeded into an element $\Omega_{O(2,2)}$ which is determined as follows [59]

$$A = \tilde{a}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \quad B = \tilde{b}\begin{pmatrix} -b & a \\ -d & c \end{pmatrix}\,, \quad C = \tilde{c}\begin{pmatrix} -c & -d \\ a & b \end{pmatrix}\,, \quad D = \tilde{d}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\,. \tag{2.120}$$

Let us now briefly discuss this embedding for the cases where we have a non-trivial $SL(2,\mathbb{Z})$ action on one of the parameters, as well as the case of factorised dualities.

Möbius transformations on τ

The action of $SL(2,\mathbb{Z})_\tau$ is embeded into $O(2,2,\mathbb{Z})$ by matrices of the form

$$\Omega_{O(2,2),\tau} = \begin{pmatrix} M_\tau & 0 \\ 0 & (M_\tau^T)^{-1} \end{pmatrix}, \tag{2.121}$$

which acts on E as a basis change $E \to M_\tau E M_\tau^T$. Transformations of this type acting as $SL(4,\mathbb{Z})$ on the four-torus $\mathbb{T}^4$ factorise into two $SL(2,\mathbb{Z})$ transformation, one acting on the geometric torus along the coordinates (ξ^1, ξ^2) and the second acting on the winding toru along the directions $(\tilde{\xi}_1, \tilde{\xi}_2)$.

Möbius transformations on ρ

The action of $SL(2,\mathbb{Z})_\rho$ is embedded as

$$\Omega_{O(2,2),\rho} = \begin{pmatrix} \tilde{a} & 0 & 0 & \tilde{b} \\ 0 & \tilde{a} & -\tilde{b} & 0 \\ 0 & -\tilde{c} & \tilde{d} & 0 \\ \tilde{c} & 0 & 0 & \tilde{d} \end{pmatrix}. \tag{2.122}$$

Such transformations acting as $SL(4,\mathbb{Z})$ on the four-torus $\mathbb{T}^4$ factorise also into two $SL(2,\mathbb{Z})$. In this case, one of the two torus is constructed with the directions $(\xi^1, \tilde{\xi}_2)$, and the other with the directions $(\tilde{\xi}_1, \xi^2)$. If the $SL(2,\mathbb{Z})$ transformation acting on the first is M_ρ, the transormation on the second is given by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).M_\rho.\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Möbius transformations on τ and ρ

In the case where the $O(2,2,\mathbb{Z})$ element acts non-trivially on both τ and ρ, the action on the four-torus cannot be factorised. The only element acting non-trivially on ρ that has a geometric interpretation is the one given by $V^{\tilde{b}}$ in (2.5), which in the $O(2,2,\mathbb{Z})$ language can be identified with the upper right corner. We then see that upper triangular matrices represent the subgroup $G_{geom} \subset O(2,2,\mathbb{Z})$ of all geometric transformations, namely those that are product of diffeomorphisms and B-transformations. They are of the general form

$$\Omega_{geom} = \begin{pmatrix} M_\tau & \tilde{b}\, M_\tau \cdot \omega \\ 0 & (M_\tau^T)^{-1} \end{pmatrix}, \tag{2.123}$$

whith $\omega = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

A diffeomorphism combined with a $(0,1)$ element of the parabolic conjugacy classes in $SL(2,\mathbb{Z})_\rho$, namely by $U^{\tilde{c}}$ (2.5):

$$\rho \to \frac{\rho}{\tilde{c}\rho + 1}, \tag{2.124}$$

is represented by a lower triangular matrix

$$\Omega_\beta = \begin{pmatrix} M_\tau & 0 \\ \tilde{c}\,\omega \cdot M_\tau & (M_\tau^T)^{-1} \end{pmatrix}. \tag{2.125}$$

The action given by the transformation (2.124) forms an Abelian subgroup of $O(2,2,\mathbb{Z})$ usually referred to as β-transforms [50] since it acts naturally on bivectors $\beta \in \bigwedge^2 TM$. Such transformations implement TsT transformations and are very useful in holography to construct gravity duals of marginal deformations, such as β-deformations of $\mathcal{N}=4$ SYM [102, 103].

Other $SL(2,\mathbb{Z})_\rho$ transformations, such as the parabolic (p,q) monodromy, will result in $O(2,2,\mathbb{Z})$ elements in which both B- and β-transformations are present at the same time.

Factorised dualities

Finally, let us discuss the how the $\mathbb{Z}_2$ actions $\tau \leftrightarrow \rho$ and $\tau \leftrightarrow -1/\tau$ are encoded into $O(2,2,\mathbb{Z})$ matrices. These transformations have negative determinant, and therefore they do not belong to the group of large diffeomorphisms of the four-torus. They are embeded with the two matrices

$$\Omega_{\mathbb{Z}_2} = \begin{pmatrix} \mathbb{1} - e_a & e_a \\ e_a & \mathbb{1} - e_a \end{pmatrix}, \tag{2.126}$$

with $a = 1, 2$, where $\mathbb{1}$ is the identity matrix and e_a is a 2×2 matrix with all entries equal zero except for the a'th diagonal element. Their action reproduce the familiar Buscher rules, and in double field theory they correspond to $\xi^a \leftrightarrow \tilde{\xi}_a$.

2.4.2 Doubled torus fibrations

We now want to discuss to discuss how the configurations of chapter 2.3 can be described by a T-duality covariant formalism such as the doubled formalism of [26].

In the semi-flat approximation, the T-fect solutions are fully characterized by associating at each base point a string state $\Psi = \sum \Psi_{n,m} |\mathbf{n}, \mathbf{m}\rangle$ together with a monodromy, where the latter is an $O(2,2,\mathbb{Z})$ transformation acting on the momentum and winding numbers. We can then Fourier transform the basis $|\mathbf{n}, \mathbf{m}\rangle$ to position space as

$$|\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}\rangle = \sum_{n_i, m_i} e^{i n_1 \xi^1/R_1 + i n_2 \xi^2/R_2} e^{i m_1 \tilde{\xi}_1 \tilde{R}_1 + i m_2 \tilde{\xi}_2 \tilde{R}_2} |\mathbf{n}, \mathbf{m}\rangle\,, \tag{2.127}$$

where R_1 and R_2 are the radii of the two compact directions of the torus and we have introduced the coordinates $(\tilde{\xi}^1, \tilde{\xi}^2)$ as conjugate to the winding numbers. These additional coordinates can be thought of as defining an extended compact space, the four-torus $\mathbb{T}^4$ described above, which leads to the doubled formalism of [26].

In the case when monodromies are in the parabolic type, both the geometric and non-geometric ones act as generalised Dehn twists on such four-torus, defining a (non-principal) fibration over the two-dimensional base. As an example, let us consider the case where the defect has a parabolic geometric monodromy V_τ described in (2.16). The

$SO(2,2,\mathbb{Z}) \subset SL(4,\mathbb{Z})$ monodromy is given by

$$\Omega_{V_\tau} = \begin{pmatrix} M_\tau & 0 \\ 0 & (M_\tau)^{-T} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} , \tag{2.128}$$

from which one can see that the monodromy acts as a Dehn twist on both the geometric torus and on the torus formed by the winding directions. Similar analysis for general $(p,q)_\tau$ parabolic monodromies lead to analogous results.

One can similarly study parabolic monodromies in ρ. For instance, the cases represented by V_ρ and U_ρ are given respectively by the $SO(2,2,\mathbb{Z}) \subset SL(4,\mathbb{Z})$ elements

$$\Omega_{V_\rho} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad \Omega_{U_\rho} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} , \tag{2.129}$$

corresponding to a B-shift and β-transform as expected. In this case, the effect of the monodromy is two Dehn twists on the tori along directions $(\xi^1, \tilde{\xi}_2)$ and $(\tilde{\xi}_1, \xi^2)$, as it happens in general for all $(p,q)_\rho$-defects. The $O(2,2,\mathbb{Z})$ elements (2.129) can be obtained from (2.128) by exchanging $(\xi_2 \leftrightarrow \tilde{\xi}^2)$ or $(\xi_1 \leftrightarrow \tilde{\xi}^1)$, which is consistent with the fact that the corresponding local geometries can be obtained from the τ-fect with monodromy (2.128) by applying Buscher rules along ξ^2 and ξ^1 respectively.

Restricting these $\mathbb{T}^4$ fibrations over the boundary of a disk around the degeneration, $S^1 = \partial D^2$, one recovers the fibrations of [30]. In our case these fibrations extend over the punctured base and we need to ask if there exists a degeneration of the four-torus giving rise to such monodromies, aspect that will be further discussed in chapter 3, and if the local models can be glued together to form a global space.

Global construction

Before discussing how these configurations look like in Double Field Theory, we want to breafly adress the possibility of constructing global models with this configurations. This issue is in fact quite subtle, and we would like to make the following observation. A global, supersymmetric, non-geometric model can be obtained by pairing 12 non-local τ-degenerations with 12 non-local ρ-degenerations [24]. In the doubled space this is described by the factorization of the identity in terms of the (U,V) twists:

$$(U_\tau V_\tau)^6 (U_\rho V_\rho)^6 = \mathbb{1} \, . \tag{2.130}$$

However, as a $\mathbb{T}^4$ fibration (including monodromies in the full mapping class group $SL(4,\mathbb{Z})$), each degeneration can be seen as a collision of two elementary degenerations in which one cycle shrinks. Locally, these correspond to a singular fiber of type $I_1 \times \mathbb{T}^2$. We see then that the global doubled fibration is specified by 48 elementary degenerations, which appears to be incompatible with a holomorphic fibration of the $\mathbb{T}^4$ moduli.

This can be seen already in the geometric setting. Let us consider a smooth K3 surface, described by 24 mutually non-local I_1 degenerations, corresponding to the monodromy decomposition $(M_\tau^{(1,0)} M_\tau^{(0,1)})^{12} = \mathbb{1}$. The doubled torus fibration will then be described by the decomposition $(U_\tau V_\tau)^{12} = \mathbb{1}$. Now, there exists a global polarization that identifies the physical fiber with the (ξ_1, ξ_2) directions, and the fibration reduces to a $\mathbb{T}^2 \times \tilde{\mathbb{T}}^2$ fibration over $\mathbb{P}^1$. If we try to fiber the two complex structure moduli τ and $\tilde{\tau}$ of the two tori we can write the metric [104, 105]

$$ds^2 = e^{\varphi} \tau_2 \tilde{\tau}_2 dz d\bar{z} + \frac{1}{\tau_2} \left| d\xi^1 + \tau d\xi^2 \right|^2 + \frac{1}{\tilde{\tau}_2} \left| d\tilde{\xi}_1 + \tilde{\tau} d\tilde{\xi}_2 \right|^2 . \tag{2.131}$$

Each I_1 degeneration of τ or $\tilde{\tau}$ would give the same deficit angle as the physical I_1 singularity we started with, and a compact model seems to require a total of 24 degenerations, precisely half of the degenerations required to build the 24 $I_1 \times I_1$ degenerations of the doubled model. We leave this issue for future investigation.

2.4.3 T-fects in Double Field Theory

We finally discuss how the configurations in section 2.3 look like within Double Field Theory. As mentioned above, this theory interprets the generalised metric $\mathcal{H}$ in (2.117) as a bilinear form on a $2d$-dimensional space, with local coordinates $X^A = (\xi^a, \tilde{\xi}_a)$. Furhtermore the dilaton ϕ is encoded into the generalised dilaton $\tilde{\phi}$ defined by $e^{-2\tilde{\phi}} = \sqrt{g}\, e^{-2\phi}$. In order for the algebra of infinitesimal diffeomorphisms to close, one has to impose the so-called strong constraint

$$\eta^{MN} \partial_M \partial_N = 0 \, , \tag{2.132}$$

which implies that the fields only depend on half of the generalized coordinates. Taking this field content, one can write down a manifestly $O(d,d)$-covariant theory [56]

$$S \sim \int d^A X e^{-2\tilde{\phi}} \mathcal{R}(\mathcal{H}, \tilde{\phi}) \, , \tag{2.133}$$

where $\mathcal{R}(\mathcal{H}, \tilde{\phi})$ is the generalized curvature scalar (see for instance equation (4.24) in [56]). Solving the strong constraint by demanding no winding-coordinate dependence of the fields, the NS-NS supergravity action is recovered.

T-fect configurations

From the discussion above one can easily construct solutions of the equations of motion derived from the action (2.133) that correspond to the semi-flat limit of the doubled torus fibrations discussed in the previous section. As an example, the semi-flat NS5-brane solution is lifted to [1]

$$
\begin{aligned}
ds^2_{\mathrm{DFT}} = \quad & h(r)\Big[dr^2 + r^2 d\theta^2 + (d\xi^1)^2 + (d\xi^2)^2\Big] \\
& + \frac{1}{h(r)}\left[\left(d\tilde{\xi}_1 - \frac{\theta}{2\pi R_1 R_2} d\xi^2\right)^2 + \left(d\tilde{\xi}_2 + \frac{\theta}{2\pi R_1 R_2} d\xi^1\right)^2\right].
\end{aligned} \tag{2.134}
$$

This has a similar structure to the doubled torus fibration (2.131) with

$$
\tau = \frac{i}{2\pi R_1 R_2} \log(z^{-1}) \,, \qquad \tilde{\tau} = \frac{2\pi i R_1 R_2}{\log(z^{-1})} \,, \tag{2.135}
$$

where $z = r e^{i\theta}$, giving the expected monodromy Ω_{V_ρ} (2.129). By the simple basis change discussed above, one recovers the different semi-flat backgrounds discussed in the previous section with monodromies (2.128) and (2.129). In the next chapter we will use these solutions to discuss to which extend one can incorporate corrections to these solutions describing the local physics of the degeneration.

[1] We use ds^2_{DFT} as a short-hand notation to encode the form of the generalized metric as $ds^2_{\mathrm{DFT}} = \mathcal{H}_{MN} dX^M dX^N$.

Chapter 3

Physics of winding modes close to the T-fects

In the previous chapter, we have constructed a class of solutions consisting on flat two-tori whose moduli were meromorphically fibered over a two-dimensional base, with T-duality monodromies around a degeneration point. We also argued that one could construct global models by considering a finite number of them distributed over the same base. In the case where only the complex structure τ varies non-trivially, one can recover in this way K3 compact manifolds. In general, however, if one also allows the complexified Kähler parameter ρ to vary, one obtains non-geometric compactification models.

As it was also argued, close to a degeneration of the τ- and ρ-fibration this semi-flat approximation breaks down, and one should complement it by gluing in the exact local description of the degeneration. The latter, however, will in general break (some of) the isometries of the fiber. In this chapter, we will study this physics for the case of parabolic monodromies. As we will argue, in the geometric situation we have a good understanding of such a local description, while in the non-geometric case the situation is more delicate. In fact, we can rely on a dual description of the local non-geometric solution, but we will point out that in all known cases such duality is strictly valid only in the semi-flat approximation. Given that we lack a conformal field theory description of the degeneration,[1] and that supergravity is most certainly not valid for such stringy backgrounds, it is important to understand the physics of these degenerations.

By adapting the arguments of [106] to the torus case, we will argue that such physics is dominated by winding modes, and that the exotic brane solutions will receive stringy corrections that can be related to the modes correcting the semi-flat ansatz near geometric degenerations. In the last section, we will argue how and to which extend this physics can be encoded into the Double Field Theory picture described in section 2.4. The discussion on this chapter will follow [33, 107]

[1] Except particular cases such as asymmetric orbifold points. However, we will be interested in parabolic monodromies for which a CFT description is not available.

3.1 Exact metrics and T-duality

We will begin by analysing the geometric cases of monopoles with compact directions and study the dynamics of unwinding strings in the semi-flat limit. We will focus in the cases of the V monodromy degenerations both in τ and ρ and relate our findings using duality arguments.

3.1.1 I_1 degeneration

In the geometric class, the simplest semi-flat solution is the Kodaira type I_1 singularity, which corresponds to the semi-flat configuration (2.61). This is uniquely determined by the monodromy acting on the fiber torus when encircling the singularity: a Dehn twist around one of the cycles, which sends $\tau \to \tau + 1$. As argued in the previous chapter, we know that the exact metric is that of a Taub-NUT space with one transverse compact direction. Such metric breaks one of the $U(1) \times U(1)$ isometries of the semi-flat metric, and the modes that break such isometry localize the shrinking cycle (which coincides with the cycle along which the Dehn twist is performed) on the orthogonal cycle of the torus. We thus see that specifying the type of degeneration is enough to capture the symmetries of the exact solution.

The local metric can be derived by starting from the Euclidean Taub-NUT solution and compactifying one base direction. To do so, let us recall the background (2.62), whose transversal directions are

$$ds^2 = h(\vec{x})\, d\vec{x}^2 + \frac{1}{h(\vec{x})} \left(d\xi^2 + \omega\right)^2 , \qquad h(\vec{x}) = 1 + \frac{\tilde{R}_2}{2|\vec{x}|} , \tag{3.1}$$

where $\vec{x}$ denotes coordinates in $\mathbb{R}^3$ and ξ^2 denotes the coordinate on the S^1-fiber.[2] The background is regular if ξ^2 has a periodicity of $2\pi\tilde{R}_2$, where $\tilde{R}_2$ is the radius of the fiber at infinity[3]. Note furthermore that at the origin $\vec{x} = 0$ of the base, the cycle of the fiber shrinks to zero size. The one-form ω is not closed and encodes the non-triviality of the fibration. It is determined up to shifts by exact forms through the relation $d\omega = \star_3 dh$, where the latter ensures that the equations of motion with $H = 0$ and $e^{\phi} = g_s = \text{const.}$ are satisfied.

As mentioned in section 2.3.1, the compactification of this background is achieved by considering an infinite array of sources on one of the base directions. The harmonic function $h(\vec{x})$ becomes

$$h(r, \xi^1) = 1 + \sum_{n \in \mathbb{Z}} \frac{\tilde{R}_2}{2\sqrt{r^2 + (\xi^1 - 2\pi R_1 n)^2}} , \tag{3.2}$$

where we split the three-dimensional radial direction into $|\vec{x}|^2 = r^2 + (\xi^1)^2$. The sum in (3.2) does not converge but can be regularized. After Poisson resummation we obtain the

[2] Here and in the following we omit the additional six space-time directions that make the background into a full ten-dimensional solution of string theory.

[3] The tilde notation is introduced for convenience for the upcoming discussion.

Ooguri-Vafa metric [96] described by

$$h(r,\xi^1) = \frac{\tilde{R}_2}{2\pi R_1}\left[\log(\mu/r) + \sum_{n\neq 0} e^{in\xi^1/R_1}\, K_0\left(\frac{|n|\, r}{R_1}\right)\right], \tag{3.3}$$

with μ a constant that controls the regulator and absorbs also all other possible constants, for instance the first term in (3.2). K_0 is the zeroth-order modified Bessel function of the second kind, whose series expansion for large r reads

$$K_0\left(\frac{|n|\, r}{R_1}\right) = e^{-\frac{|n|r}{R_1}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\frac{1}{2}+k)^2}{\sqrt{\pi}k!}\left(\frac{R_1}{2|n|r}\right)^{k+\frac{1}{2}} . \tag{3.4}$$

Hence, the leading semi-flat term in (3.3) (i.e. the logarithm) is a good approximation of the exact metric far away from the degeneration point up to exponentially suppressed terms. In fact, the semi-flat approximation of a smooth K3 – repaired with the Ooguri-Vafa metric at the 24 I_1 points – gives a metric that is a good approximation of the exact Calabi-Yau metric [108]. The expression (3.3) can also be derived (in a simple way) field-theoretically [109,73] and (in a complicated way) by solving explicitly the Riemann-Hilbert problem with wall-crossing technology [110].

The semi-flat approximation of the above background is then the flat two-torus fibration (2.61). To obtain it from (3.1), we introduce polar coordinates (r,θ), bring the one-form ω mentioned above into the form $\omega^{\rm sf} = f\, d\xi^1$ with $df = \star_2 dh$ and take a gauge where

$$\omega^{\rm sf} = \frac{\tilde{R}_2}{2\pi R_1}\theta\, d\xi^1 . \tag{3.5}$$

We observe that, after encircling the defect in the base as $\theta \to \theta + 2\pi$, the shift $\omega^{\rm sf} \to \omega^{\rm sf} + \frac{\tilde{R}_2}{R_1} d\xi^1$ should be compensated by the shift $\xi^2 \to \xi^2 - \frac{\tilde{R}_2}{R_1}\xi^1$ which, as expected, corresponds to the action of a Dehn twist on the torus cycles.[4] The corrections to the semi-flat term in (3.3) explicitly break one of the $U(1)$ isometries of the torus fiber. This affects also the one-form (3.5), which is corrected (up to gauge transformations) by modified Bessel functions of the second kind as

$$\omega = \omega^{\rm sf} - \frac{\tilde{R}_2}{\pi R_1} r \sum_{k>0} K_1\left(\frac{k\, r}{R_1}\right) \sin\left(\frac{k\, \xi^1}{R_1}\right) d\theta . \tag{3.6}$$

The above analysis can be easily extended to a I_n degeneration. The solution is given by coalescing n Taub-NUT centers, and the Ooguri-Vafa corrections (3.3), which completely smooth out the semi-flat metric for $n = 1$, now replace the semi-flat singularity with an A_{n-1} singularity, as expected.

[4] We neglect a constant shift which is not captured by the action on the homology [110].

Monodromy and unwinding strings

An important point to investigate the near core physics is that even within the semi-flat approximation, some "remnant" of the corrections (3.3) survives. In fact, it is useful to look at the action of the monodromy on the momentum and winding of strings propagating on the torus fiber. From the discussion in section 2.4, it follows that for a monodromy $(M_\tau, M_\rho) \in SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho$, acting on the moduli as

$$\tau \to M_\tau[\tau] \equiv \frac{a\tau + b}{c\tau + d}, \qquad \rho \to M_\rho[\rho] \equiv \frac{\tilde{a}\rho + \tilde{b}}{\tilde{c}\rho + \tilde{d}}, \tag{3.7}$$

the corresponding $O(2,2,\mathbb{Z})$ transformation on the combined momentum($\mathbf{n}$)/winding($\mathbf{m}$) vector $(\mathbf{n}, \mathbf{m})$ is given by

$$\begin{aligned} \mathbf{n} &\to \tilde{a} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{n} + \tilde{b} \begin{pmatrix} -b & a \\ -d & c \end{pmatrix} \mathbf{m}, \\ \mathbf{m} &\to \tilde{c} \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \mathbf{n} + \tilde{d} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \mathbf{m}. \end{aligned} \tag{3.8}$$

In the simple case of constant ρ, that is say $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (+1, 0, 0, +1)$, the above transformation reduces to

$$\mathbf{n} \to M_\tau \, \mathbf{n}, \qquad \mathbf{m} \to \left(M_\tau^t\right)^{-1} \mathbf{m}. \tag{3.9}$$

In our case of interest, namely $\tau \to \tau + 1$, the monodromy is given by the matrix M_τ in (2.16), giving the transformation

$$(n_1, n_2) \to (n_1 + n_2, n_2), \qquad (m^1, m^2) \to (m^1, m^2 - m^1). \tag{3.10}$$

We see that momentum along the $(1,0)$-cycle (ξ^1-direction) is not conserved for a string that moves around a degeneration point at which the $(0,1)$-cycle (ξ^2-direction) shrinks. This is in contrast with the translation invariance of the semi-flat metric along both directions of the torus, which would make expect that momentum along both directions is preserved. The exact metric cures this problem by breaking the $U(1)$ isometry along the $(1,0)$-cycle, as we discussed above.

Note however that winding along the $(0,1)$-direction is also not conserved. This is easy to see by taking a string wrapped along the cycle $(1,1)$. Denoting the world-sheet coordinates by $(\hat{\tau}, \hat{\sigma})$, we consider the trajectory

$$\begin{aligned} \xi^1 &= 2\pi R_1 \hat{\sigma}, \qquad & \theta &= 2\pi \hat{\tau}, \\ \xi^2 &= 2\pi \tilde{R}_2 \hat{\sigma}, \qquad & r &= r_0, \end{aligned} \tag{3.11}$$

where (r, θ) are again polar coordinates on $\mathbb{R}^2$ and (ξ^1, ξ^2) are flat coordinates on $\mathbb{T}^2$ with periodicity $(\xi^1, \xi^2) = (\xi^1 + 2\pi R_1, \xi^2 + 2\pi \tilde{R}_2)$. The monodromy around the defect is a Dehn twist, which corresponds to cutting the torus along $(1,0)$, rotating by 2π and gluing it back. The process of unwinding the string along the $(1,0)$-cycle corresponds to the patching $\xi^2_{(2\pi)} = \xi^2_{(0)} - (\tilde{R}_2/R_1)\xi^1_{(0)}$, where $\xi^a_{(0)}$ and $\xi^a_{(2\pi)}$ are the torus coordinates at $\theta = 0$ and $\theta = 2\pi$, respectively. With this transformation the trajectory (3.11) unwinds the direction ξ^2 at $\theta = 2\pi$, which is the semi-flat version of the unwinding trajectory in the Taub-NUT space considered in [106]. In the latter case, the string can be unwound by taking it arbitrarily far-away from the core of the monopole because the S^1 circle is non-trivially fibered over S^2 at spatial infinity in the $\mathbb{R}^3$ base. Such a fibration is in fact the Hopf fibration of a three-sphere. The trajectory in this case takes a string wrapping the fiber and a $S^1 \subset S^2$ from the north pole to the south pole, where a rotation of the fiber effectively unwinds the string. Our case is a compactified version of this process. Although far away from the degeneration the space is locally $\mathbb{T}^2 \times S^1$, the global twist gives it the topology of a nilmanifold, which can be seen as a non-trivial fibration of the $(0,1)$-cycle over the remaining torus $\tilde{\mathbb{T}}^2 = (1,0) \times S^1$. The non-triviality of such fibration gives the unwinding in our case.

There is yet another way to understand equation (3.10). In the semi-flat limit we can quantize the string on the $\mathbb{T}^2$-fiber, and find for the left- and right-moving momenta the expressions

$$\left(p_{L,R}\right)_I = \pi_I \pm (G \mp B)_{IJ} L^J \,, \qquad\qquad I, J = 1, 2\,, \tag{3.12}$$

where π_I denotes the canonical momentum and L^I is the winding vector. In the present case, these are given by

$$\pi_I = \begin{pmatrix} n_1/R_1 \\ n_2/\tilde{R}_2 \end{pmatrix}, \qquad\qquad L^I = \begin{pmatrix} R_1 m^1 \\ \tilde{R}_2 m^2 \end{pmatrix}. \tag{3.13}$$

Furthermore, when encircling the defect as $\theta \to \theta + 2\pi$ the coordinates change as $(\hat{\xi}^1, \hat{\xi}^2) = (\xi^1, \xi^2 - \frac{\tilde{R}_2}{R_1}\xi^1)$, as discussed below equation (3.5). This gives rise to the diffeomorphism

$$\Omega^I{}_J = \frac{\partial \hat{\xi}^I}{\partial \xi^J} = \begin{pmatrix} 1 & 0 \\ -\frac{\tilde{R}_2}{R_1} & 1 \end{pmatrix}. \tag{3.14}$$

If we require the spectrum to be invariant under $\theta \to \theta + 2\pi$, we see that the momenta $p_{L,R}$ appearing in the mass formula have to be invariant. Recalling then that in the present situation $B_{IJ} = 0$ and $G(\theta + 2\pi) = \Omega^{-T} G(\theta)\, \Omega^{-1}$, we find

$$\begin{aligned} 0 &\stackrel{!}{=} \Delta\left(p_{L,R}\right)_I \\ &= \left(\Omega^T\right)_I{}^J \left(p_{L,R}(\theta + 2\pi)\right)_J - \left(p_{L,R}(\theta)\right)_I \\ &= \left[\left(\Omega^T\right)_I{}^J \pi_J(\theta + 2\pi) - \pi_I(\theta)\right] \pm G_{IJ}(\theta) \left[\left(\Omega^{-1}\right)^J{}_K L^K(\theta + 2\pi) - L^K(\theta)\right], \end{aligned} \tag{3.15}$$

which leads to the identifications shown in equation (3.10).

Charge inflow

As in [106], the non-conservation of the winding charge along the $(1,0)$-cycle is compensated by a radial inflow of charge towards the degeneration point. This arises from a coupling between the string and a collective coordinate excitation of the monopole.

For the Kaluza-Klein monopole this comes from a gauge transformation of the B-field in terms of the unique (up to a constant) self-dual two-form [111]

$$\mathcal{B} = \alpha\, d\Lambda\,, \qquad \Lambda = \frac{C}{h}\left(d\xi^2 + \omega\right), \tag{3.16}$$

where α is a parameter that becomes dynamical at the quantum level. The normalization constant C can be fixed by demanding that α has periodicity of $2\pi/\tilde{R}_2$. After compactification, it is possible to derive an exact expression for Λ [112]. Here, we will only need the semi-flat limit where Λ reduces to

$$\Lambda = \frac{C}{h}\left[d\xi^2 + \frac{\tilde{R}_2}{2\pi R_1}\theta\, d\xi^1\right], \tag{3.17}$$

which is indeed a self-dual form for the semi-flat I_1 degeneration. Let us now investigate the coupling of the string trajectory with α. For this, we embed the semi-flat supergravity configuration into $4+1$ dimensions and study the dynamics of a string moving in this background, described by the action

$$\begin{aligned} S = {} & S_{\text{sugra}} \\ & - \frac{1}{4\pi}\int d^5x \int d\hat{\rho}\, d\hat{\tau}\, \delta(x^a - X^a)\Big[\sqrt{\gamma}\,\gamma^{AB}\,\partial_A X^a \partial_B X^b\, G_{ab} + \epsilon^{AB}\,\partial_A X^a\,\partial_B X^b\,\mathcal{B}_{ab}\Big], \end{aligned} \tag{3.18}$$

where G and $\mathcal{B}$ are the five-dimensional background fields, S_{sugra} is the usual NS-NS supergravity action and we have set $\alpha' = 1$. Promoting α to $\alpha(t)$ and considering the unwinding trajectory (3.11), but leaving the motion along the base directions as arbitrary functions of $\hat{\tau}$, we find that the dynamics of $\alpha(t)$ can be described in terms of the Lagrangian density

$$\mathcal{L}_\alpha = \frac{1}{2}\,\dot{\alpha}^2 + K\alpha\left[h^{-1}\frac{d\theta}{dt} + (2\pi - \theta)\,\frac{h'}{h^2}\,\frac{dr}{dt}\right], \tag{3.19}$$

where K is a constant. The corresponding equations of motion are solved by

$$\dot{\alpha}(t) = K\,\frac{\theta - 2\pi}{h} + \alpha_0\,, \tag{3.20}$$

with α_0 an integration constant. For trajectories with $r = \text{const.}$, we see that after encircling the defect $\dot{\alpha}$ increases by $2\pi K/h$. We have therefore checked that a string configuration with initial winding charge $m_2 = 1$ following the unwinding trajectory (3.11), couples non-trivially with the background fields via the zero mode. Along this trajectory the string looses its winding charge but this is compensated by an increase of the kinetic energy of the zero mode. From the point of view of the theory reduced on the unwinding cycle, the winding charge is an electric-type charge associated to the gauge field obtained from the reduction of the B-field. With the discussed non-trivial coupling the unwinding trajectory generates an inflow of "winding" current which is eventually absorbed by the brane configuration [106]. We will come back to this point below when discussing the T-dual configurations.

3.1.2 NS5-branes on $\mathbb{R}^2 \times T^2$

The next case we would like to discuss is the parabolic degeneration of the auxiliary fiber ρ with monodromy $\rho \to \rho + 1$, described by the geometric configuration (2.87). As discussed in chapter 2, this configurations arises in the semi-flat limit of an NS5 brane and we will see that the exact solution on the degeneration breaks all the isometries of the fiber torus. We now discuss these corrections and, at the end of the section, we will relate them with those encountered in the case of a geometric I_1 degeneration discussed in the previous section which, at least at the semi-flat limit, is related with the NS5 by the factorised duality $\tau \leftrightarrow \rho$.

Compactification

To be more concrete, let us start from the uncompactified NS5-brane background, whose transversal space is described by the configuration

$$\begin{aligned} ds^2 &= h(\vec{x})\, d\vec{x}^2\,, \\ e^{\phi} &= g_s\, h(\vec{x})\,, \qquad\qquad h(\vec{x}) = 1 + \frac{1}{|\vec{x}|^2}\,, \\ H_3 &= \star_4 dh(\vec{x})\,, \end{aligned} \tag{3.21}$$

where $\vec{x} \in \mathbb{R}^4$. Next, we compactify two of the transversal directions on a two-torus. To this end, we split $\mathbb{R}^4 \to \mathbb{R}^2 \times \mathbb{T}^2$ and introduce polar coordinates (r, θ) on $\mathbb{R}^2$ and coordinates (ξ^1, ξ^2) on $\mathbb{T}^2$. The above solution can then be expressed in the following way

$$\begin{aligned} ds^2 &= h(r, \xi^1, \xi^2) \left[dr^2 + r^2 d\theta^2 + (d\xi^1)^2 + (d\xi^2)^2 \right], \\ e^{\phi} &= g_s\, h(r, \xi^1, \xi^2)\,, \\ H_3 &= \star_4 dh(r, \xi^1, \xi^2)\,, \end{aligned} \tag{3.22}$$

where, as in (2.90), the function h can be determined by considering a rectangular lattice of NS5-branes as

$$h(r, \xi^1, \xi^2) = 1 + \sum_{\vec{n} \in \mathbb{Z}^2} \frac{1}{r^2 + (\xi^1 - 2\pi R_1 n_1)^2 + (\xi^2 - 2\pi R_2 n_2)^2}\,. \tag{3.23}$$

This sum is not convergent, but can be regulated with a regulator of the form [73, 112]

$$\frac{1}{2\pi R_1 R_2} \sum_{n\in\mathbb{Z}^*} \frac{1}{|n|^2} \,. \tag{3.24}$$

By subtracting this term from the original function and performing a Poisson resummation we find

$$h(r,\xi^1,\xi^2) = \frac{1}{2\pi R_1 R_2}\left[\log\left(\frac{\mu}{r}\right) + \sum_{\vec{k}\in(\mathbb{Z}^2)^*} K_0\left(\lambda\, r\right) e^{-i\left(\frac{k_1\xi^1}{R_1}+\frac{k_2\xi^2}{R_2}\right)}\right], \tag{3.25}$$

where $(\mathbb{Z}^2)^* = \mathbb{Z}^2 - \{(0,0)\}$ and $\lambda = \sqrt{(k_1/R_1)^2 + (k_2/R_2)^2}$. The same result can be determined in purely field-theoretic terms [113, 73] from one-loop corrections to the gauge coupling of the non-linear sigma model, obtained by reducing a $\mathcal{N}=2$ theory on the torus. As in the case of the I_1 degeneration, the functions $K_0(\lambda\, r)$ vanish exponentially for large r and, at this region, the semi-flat term

$$h(r) = \frac{1}{2\pi R_1 R_2}\log\left(\frac{\mu}{r}\right)\,, \tag{3.26}$$

is a good approximation, leading to the configuration (2.87). In this case, however, the terms involving $K_0\left(\lambda\, r\right)$ break both $U(1)^2$ isometries of the background. Note also that, on the other side, taking the decompactification limit $r, \xi^1, \xi^2 \ll R_1, R_2$ in (3.23), one recovers the non-compact harmonic function shown in (3.21), that is

$$h(r,\xi^1,\xi^2) = 1 + \frac{1}{r^2 + (\xi^1)^2 + (\xi^2)^2}\,. \tag{3.27}$$

If we de-compactify only one of the cycles of the torus, say the one corresponding to ξ^1, and define as before $|\vec{x}|^2 = r^2 + (\xi^1)^2$, we obtain the familiar result for the H-monopole compactified along one direction [114, 106]

$$h\big(|\vec{x}|,\xi^2\big) = 1 + \frac{1}{2R_2|\vec{x}|}\,\frac{\sinh(|\vec{x}|/R_2)}{\cosh(|\vec{x}|/R_2) - \cos(\xi^2/R_2)}\,. \tag{3.28}$$

This solution encodes the breaking of the $U(1)$ isometry along the cycle which is dual to the shrinking one in the Taub-NUT space.

Monodromies

As in the previous case, we can understand the breaking of the $U(1)$ isometries from the action of the monodromies (3.8). Since now ρ is varying, there will be a mixing between momentum and winding states. Using the general expression shown in (3.8), we can deduce the action of $\rho \to \rho + 1$ on the momentum and winding modes as

$$(n_1, n_2) \to (n_1 + m^2, n_2 - m^1)\,, \qquad (m^1, m^2) \to (m^1, m^2)\,, \tag{3.29}$$

as expected from T-duality. Momentum can now unwind from the T-dual of the $(0,1)$-cycle, with a trajectory dual to (3.11)

$$\begin{aligned} \xi^1 &= 2\pi R_1 \hat{\sigma}\,, & \theta &= 2\pi\hat{\tau}\,, \\ \xi^2 &= \tfrac{2\pi}{R_2}\hat{\tau}\,, & r &= r_0\,, \end{aligned} \tag{3.30}$$

where we used that $\tilde{R}_2 = 1/R_2$. Similarly, for the $(1,0)$-cycle we can write

$$\begin{aligned} \xi^1 &= \tfrac{2\pi}{R_1}\hat{\tau}\,, & \theta &= 2\pi\hat{\tau}\,, \\ \xi^2 &= -2\pi R_2 \hat{\sigma}\,, & r &= r_0\,. \end{aligned} \tag{3.31}$$

The canonical momenta that generate translations along the fiber directions are

$$\pi_a = \int d\hat{\sigma}\,\left[g_{ab}\partial_{\hat{\tau}}X^b + B_{ab}\partial_{\hat{\sigma}}X^b\right]\,. \tag{3.32}$$

In order to compute such quantities we can set $h = 1$, effectively putting the brane in a asymptotically flat background. For the above trajectories we find, respectively,

$$\pi_2 = \frac{1}{R_2}(2\pi - \theta)\,, \qquad \pi_1 = \frac{1}{R_1}(2\pi - \theta)\,, \tag{3.33}$$

which indeed vanish after encircling the defect. Accordingly, the exact metric breaks translational invariance in both directions.

The non-conservation of momentum shown in (3.29) can also be understood in a fashion similar to the I_1-degeneration discussed in the previous section. Quantizing the string on the $\mathbb{T}^2$-fiber in the semi-flat limit, the left- and right-moving momenta are again given by the general expression (3.12), where

$$\pi_I = \begin{pmatrix} n_1/R_1 \\ n_2/R_2 \end{pmatrix}\,, \qquad L^I = \begin{pmatrix} R_1 m^1 \\ R_2 m^2 \end{pmatrix}\,. \tag{3.34}$$

When encircling the defect as $\theta \to \theta + 2\pi$, the coordinates change as $(\hat{\xi}^1, \hat{\xi}^2) = (\xi^1, \xi^2)$ and hence the diffeomorphism is trivial, however, now the B-field depends non-trivially on θ. Demanding again that the spectrum is invariant, we are led to requiring

$$\begin{aligned} 0 &\overset{!}{=} \Delta\big(p_{L,R}\big)_I \\ &= \big(p_{L,R}(\theta + 2\pi)\big)_I - \big(p_{L,R}(\theta)\big)_I \\ &= \Big[\pi_I(\theta + 2\pi) + \big(B_{IJ}L^J\big)(\theta + 2\pi) - \pi_I(\theta) - \big(B_{IJ}L^J\big)(\theta)\Big] \pm G_{IJ}\Big[L^J(\theta + 2\pi) - L^J(\theta)\Big]\,, \end{aligned} \tag{3.35}$$

which gives $L^I(\theta + 2\pi) = L^I(\theta)$ and $\pi_I(\theta + 2\pi) = \pi_I(\theta) + \frac{1}{R_1 R_2}\left(\begin{smallmatrix} 0 & +1 \\ -1 & 0 \end{smallmatrix}\right)_{IJ} L^J(\theta)$. Hence, we find the identifications of momentum and winding numbers shown in equation (3.29).

Charge inflow

As we did for the KK-monopole case, we can compute the coupling of the string with the background collective coordinates. In this case, the zero mode dual to (3.17) is a shift along the toroidal coordinate, $\xi^2 \to \xi^2 + \alpha$. In analogy to the above situation, we embed the semi-flat configuration into $4+1$ dimensions and study the dynamics of a string moving in this background using the action (3.18). Promoting α to $\alpha(t)$ and considering the trajectory (3.30) but letting the motion along the angular coordinate on the base be an arbitrary function $\theta(\hat{\tau})$, the dynamics of the zero modes is described by the effective Lagrangian density

$$\mathcal{L}_\alpha = \frac{1}{2}\dot{\alpha}^2 + \tilde{K}\dot{\alpha}\,(4\pi h - \theta)\,, \tag{3.36}$$

where $\tilde{K}$ is a constant. The corresponding equations of motion are solved by

$$\alpha = \tilde{K}\int^t dt\,\theta. \tag{3.37}$$

As for the KK-monopole, after going around the defect $\dot{\alpha}$ increases by $2\pi\tilde{K}$. In this case, the non-conserved charge along the trajectory is momentum, which couples to the background fields via the zero mode associated to the position of the brane along the fiber. Again, one can also perform an analysis from the point of view of the dimensionally reduced theory. In this case, momentum charge is associated to the KK gauge field coming from the reduction of the metric. The trajectory (3.30) will then produce an equivalent current inflow that is absorbed by the background via the discussed mechanism.

T-duality of exact metrics

It is interesting to ask what happens to the T-duality transformation between the NS5-brane and the I_1 degeneration, once corrections to the semi-flat approximation are taken into account and thus the Buscher rules cannot be applied.

Corrections to the $U(1)$ isometry of the $(1,0)$-cycle have a physical interpretation related to the non-conservation of momentum along that cycle. For the I_1 degeneration the corrections are captured by the Ooguri-Vafa metric related to (3.3) and have the form of a sum of non-perturbative terms

$$\mathcal{C}_n \sim e^{-\frac{|n|r}{R_1}}\, e^{-i\frac{n\,\xi^1}{R_1}}\,, \tag{3.38}$$

where each of these contains also the perturbative sum (3.4). The NS5-brane metric related to (3.25) on the other hand contains a double-sum of terms

$$\tilde{\mathcal{C}}_{n_1,n_2} \sim e^{-\lambda r}\, e^{-i\frac{n_1\xi^1}{R_1}}\, e^{-i\frac{n_2\xi^2}{R_2}} \qquad \text{with} \quad \lambda = \sqrt{(n_1/R_1)^2 + (n_2/R_2)^2}\,. \tag{3.39}$$

The corrections depending on ξ^2 break the isometry along the $(0,1)$-cycle, along which we dualize to arrive at the I_1 degeneration. The problem of dualizing these higher Fourier modes is in fact similar to the problem considered in [106], where it has been suggested (and to some extend checked in [115, 116], interpreting the corrections as world-sheet instanton effects) that the modes in $\tilde{\mathcal{C}}_{n_1,n_2}$ map to stringy modes of the Taub-NUT space. In our case we see that $\tilde{\mathcal{C}}_{n,0} = \mathcal{C}_n$ – including numerical factors – and it is plausible to conjecture that T-duality of the full NS5-background sends each mode $\tilde{\mathcal{C}}_{n,m}$ for $m \neq 0$ to a mixed momentum-winding mode (n, m) on the Taub-NUT side. Note that this is a very specific rule for the massive modes, and it might be valid only in the regime where the semi-flat approximation is broken only mildly (which is the regime in the analysis of [115, 116]).

A similar conclusion is found by considering elements of the T-duality group that are merely changes of basis, belonging to the geometric $SL(2,\mathbb{Z})_\tau$ subgroup. An example is the rotation that sends $\tau \to -1/\tau$, exchanging a $(1,0)$ I_1 degeneration with a $(0,1)$ one. The T-duality of exact metrics can now be derived by noticing that the two configurations are obtained by a compactification of Taub-NUT spaces along two orthogonal directions, and they are therefore related by a $\pi/2$ rotation of the toroidal coordinates. This results in a specific map for the massive modes (3.38) that sends $\mathcal{C}_n \to e^{-|n|r/R'_2}\, e^{-in\xi^2/R'_2}$, with $R'_2 = R_1$ and $R'_1 = R_2$.

3.2 Non-geometric parabolic T-fects

Next, we want to consider the degeneration with the non-geometric monodromy U_ρ, corresponding to β-transformations on the torus fiber, that sends

$$-1/\rho \to -1/\rho + 1\,. \tag{3.40}$$

As discussed in chapter 2, this configuration can be obtained either from the semi-flat I_1 degeneration discussed in section 3.1.1 by taking T-duality along the direction parametrised by ξ^1, though the Killing vector along this direction is not globally defined; or by a collective T-duality [32] from the semi-flat limit of the NS5 configuration in section 3.1.2, corresponding to the fiberwise $O(2,2,\mathbb{Z})$ transformation $(\rho \to -1/\rho, \tau \to -1/\tau)$. In both cases, one obtains

$$\begin{aligned} ds^2 &= h(r)\left(dr^2 + r^2 d\theta^2\right) + \frac{4\pi^2 h(r)}{4\pi^2 h(r)^2 + \tilde{R}_1^2\tilde{R}_2^2\theta^2}\left[(d\xi^1)^2 + (d\xi^2)^2\right], \\ B &= -\frac{2\pi\tilde{R}_1\tilde{R}_2\theta}{4\pi^2 h(r)^2 + \tilde{R}_1^2\tilde{R}_2^2\theta^2}\, d\xi^1 \wedge d\xi^2\,, \\ e^{2\phi} &= \frac{4\pi^2 h(r)}{4\pi^2 h(r)^2 + \tilde{R}_1^2\tilde{R}_2^2\theta^2}\,, \end{aligned} \tag{3.41}$$

where $\tilde{R}_a = 1/R_a$, $h(r)$ is the semi-flat harmonic function (3.26), and the solution coincides with the Q-brane configuration 2.93 after constant redefinitions, as expected.

As in the previous cases, we can now deduce from (3.8) the non-geometric monodromy action on the momentum and winding states, which is

$$(n_1, n_2) \to (n_1, n_2)\,, \qquad (m^1, m^2) \to (m^1 + n_2, m^2 - n_1)\,. \tag{3.42}$$

We note that this result can be also obtained from the NS5-monodromy (3.29) by the action of the transformation $\rho \to -1/\rho$ and $\tau \to -1/\tau$, which interchanges $n_a \leftrightarrow m^a$ for all a. The transformations (3.42) suggests that the $U(1)^2$ isometries of the metric (3.41) will not receive quantum corrections, as the momenta are conserved on both the torus directions. However, in analogy with the duality between the Taub-NUT space and the NS5-brane, there exist now trajectories along which strings initially wrapped along the $(1,0)$- and $(0,1)$-cycle of the torus unwind. For example, for the trajectory

$$\begin{aligned} \xi^1 &= 2\pi \tilde{R}_1 \hat{\sigma} \,, & \theta &= 2\pi \hat{\tau} \,, \\ \xi^2 &= -\tfrac{2\pi}{\tilde{R}_2} \hat{\tau} \,, & r &= r_0 \,, \end{aligned} \tag{3.43}$$

corresponding to a string with winding along the $(1,0)$-cycle and momentum along the $(0,1)$-cycle that will unwind after encircling the defect.

Let us again derive the change in momentum and winding numbers using the invariance of the left- and right-moving momenta (3.12) when encircling the defect. More concretely, we quantize the string on the $\mathbb{T}^2$-fiber in the semi-flat approximation, and find for the canonical momentum and winding vector the expressions

$$\pi_I = \begin{pmatrix} n_1/\tilde{R}_1 \\ n_2/\tilde{R}_2 \end{pmatrix} , \qquad L^I = \begin{pmatrix} \tilde{R}_1 m^1 \\ \tilde{R}_2 m^2 \end{pmatrix} . \tag{3.44}$$

Under $\theta \to \theta + 2\pi$, the background can be made globally-defined by identifying the tori $T^2(\theta + 2\pi)$ and $T^2(\theta)$, where the value within brackets indicates the position along the angular direction of the base, using an $O(2,2)$ transformation. In particular, we have

$$\big(G \mp B\big)(\theta + 2\pi) = \mathcal{O}_\beta^{-1} \left[\big(G \mp B\big)(\theta) \right] \,, \tag{3.45}$$

where $\mathcal{O}_\beta$ is a β-transformation. On the combined momentum-winding vector $(L^I, \pi_I)^T$ this transformation acts by matrix multiplication as

$$\mathcal{O}_\beta = \left(\begin{array}{c|c} \mathbb{1} & \begin{matrix} 0 & -\tilde{R}_1\tilde{R}_2 \\ +\tilde{R}_1\tilde{R}_2 & 0 \end{matrix} \\ \hline 0 & \mathbb{1} \end{array} \right) . \tag{3.46}$$

We now demand that the spectrum does not change when encircling the defect, which means that the left- and right-moving momenta have to be invariant under $\theta \to \theta + 2\pi$. We then compute

$$\begin{aligned} 0 &\overset{!}{=} \Delta\big(p_{L,R}\big)_I \\ &= \mathcal{O}_\beta \big(p_{L,R}(\theta + 2\pi)\big)_I - \big(p_{L,R}(\theta)\big)_I \\ &= \mathcal{O}_\beta \Big(\pi_I(\theta + 2\pi) \pm \mathcal{O}_\beta^{-1} \left[\big(G \mp B\big)(\theta) \right]_{IJ} L^J(\theta + 2\pi) \Big) - \Big(\pi_I(\theta) \pm \left[\big(G \mp B\big)(\theta) \right]_{IJ} L^J(\theta) \Big) \,, \end{aligned} \tag{3.47}$$

leading to the relation

$$0 \stackrel{!}{=} \mathcal{O}_\beta \cdot \begin{pmatrix} L^I(\theta+2\pi) \\ \pi_I(\theta+2\pi) \end{pmatrix} - \begin{pmatrix} L^I(\theta) \\ \pi_I(\theta) \end{pmatrix}, \tag{3.48}$$

which is solved by (3.42).

Additionally, the non-conservation of the winding charge should be compensated by an inflow current, and we expect the winding modes to couple to two dyonic coordinates arising from the flux. It is hard to make this concrete because of the non-geometric nature of the local metric (3.41), but we can make the following argument based on T-duality. Let us start from the solution of NS5-branes smeared on the $\mathbb{T}^2$, and consider the coordinate shifts $\xi^1 \to \xi^1 + f_1(r,\theta)$ and $\xi^2 \to \xi^2 + f_2(r,\theta)$.[5] If one applies T-duality along the ξ^2-direction to the transformed solution, we see that f_1 remains as a coordinate shift of the Taub-NUT solution, while f_2 is mapped to a gauge transformation of the B-field. This is consistent with the analysis of the monodromy action in section 3.1.1. On the other hand, if we start from the NS5-brane configuration and perform two T-dualities, both transformations become gauge transformations of the B-field and the metric is not affected. This suggests that the Q-brane has two dyonic zero modes, as expected from T-duality.

Beyond semi-flat approximation

As we did for the duality between the I_1 singularity and NS5-branes, we should ask what is the transformation of the modes (3.25) that localize the NS5-branes on the fiber torus under the T-duality that leads to the solution (3.41). The answer is roughly a T-dual version of the transformation between a $(1,0)$ and a $(0,1)$ type I_1 degeneration. A naive guess is that the NS5 Fourier modes are mapped to[6]

$$\tilde{\mathcal{C}}_{n_1,n_2} \sim e^{-\tilde{\lambda} r}\, e^{-in_1\tilde{\xi}_1\tilde{R}_1}\, e^{-in_2\tilde{\xi}_2\tilde{R}_2} \qquad \text{with} \quad \tilde{\lambda} = \sqrt{(n_1\tilde{R}_1)^2 + (n_2\tilde{R}_2)^2}\,, \tag{3.49}$$

where we define $\tilde{R}_i = \frac{1}{R_i}$. As in [106], the modes $\tilde{\xi}_i$ should be identified with dyonic degrees of freedom of the non-geometric solution, as follows from a particular effective action describing the type of couplings between winding and dyonic modes described above. The rationale for such transformations is that both the geometrical coordinates $\xi^i = \xi^i_L + \xi^i_R$ and the dual ones $\tilde{\xi}_i = \xi^i_L - \xi^i_R$ play a non-trivial role. The semi-flat solution for a NS5-brane is written in terms of a trivial fibration of the (ξ^1, ξ^2) fiber coordinates, and the excited Kaluza-Klein momentum states break both the U(1) symmetries associated to shifts in such coordinates. The dual stringy coordinates are instead exact. Note that from the previous discussion it seems that such stringy coordinates are associated with a non-geometric fibration structure, so that the present situation is substantially more

[5] These transformations are coordinate transformations, which result in another supergravity solution. The zero-mode is a particular case thereof.

[6] For the T-duality transformation between the I_1 degeneration and the Q-brane, this has been checked in [117, 118] in a regime where the isometries are mildly broken.

complicate than the usual duality between H-monopoles and Taub-NUT spaces. In this latter situation, the stringy coordinate is associated with a topologically non-trivial circle fibration, which is traded by T-duality with a B-field in the dual, trivially fibered solution. In fact there is a well-known geometrical construction that unifies both fibrations [119]. Starting from an oriented S^1 bundle over a compact connected manifold M: $S^1 \to E \xrightarrow{\pi} M$, one constructs the correspondence space $C = E \times_M \hat{E}$, where $\hat{E}$ is the T-dual fibration. C is both a circle bundle over E and a circle bundle over $\hat{E}$, and if $\hat{E}$ is a trivial fibration, as in the H-monopole case, we have that $C = E \times S^1$. For the present case of elliptic fibrations, this geometric construction cannot be easily generalized [28], in line with the above discussion. The breaking of both $U(1)^2$ isometries of the NS5-background poses in fact additional challenges for a geometric description in a extended space, as we will discuss in the next section.

3.3 Description in extended space

In the previous sections we have seen evidence for a "generalized T-duality" acting on higher Fourier modes of the string fields. We want now to discuss how and to what extend this new physics can be captured by the extended space picture described in section 2.4.

There, we described T-fects as $\mathbb{T}^4$ fibrations encoding the monodromy as $SO(2,2,\mathbb{Z}) \subset SL(4,\mathbb{Z})$, the last one being the group of large diffeomorphisms of the four-torus. In particular, the semi-flat approximations to the I_1 degeneration and the NS5-brane considered in this chapter, as well as the 5^2_2 brane, were characterised by the monodromies (2.128) and (2.129). In all of these cases the four-torus fiber factorised into $\mathbb{T}^2 \times \tilde{\mathbb{T}}^2$ and the monodromy could be geometrically described in terms of two Dehn twists, one along each two-torus. In this description, the different τ- and ρ-monodromies were related by a change of basis and one can conclude that all singular fibers should have the same topolgy.

Analogous to the two-torus fibrations, we then expect that the type of singular fiber in the $\mathbb{T}^4$ fibration should be determined by the conjugacy class of the monodromy around the boundary of a small disk encircling the degeneration. If we assume that monodromies of the type (2.129) arise as a Picard-Lefschetz type monodromy around a singular fiber, where two of the cycles of the $\mathbb{T}^4$ are pinched, we obtain a topology of type $I_1 \times I_1$. Higher Fourier modes should then correct the semi-flat approximation by localizing the shrinking cycles in a similar way as for the I_1 degenerations, and the change of $SL(4,\mathbb{Z})$ duality frames would give then a precise generalized T-duality between higher Fourier modes, of the kind discussed in the previous sections. Such degenerations can be described, within the semi-flat approximation, by the Ricci-flat metric (2.131). However, a way to include the corrections to such metric while keeping its Ricci tensor flat is not known. It would be interesting to study in more details this quantum corrected metric for the $I_1 \times I_1$ degeneration.

Finally, we will close this section by describing how the higher Fourier modes can be included in the DFT solutions constructed in section 2.4. In this case, we will be able to construct solutions to DFT equations of motion encoding these corrections. Furthermore, "generalised T-duality" will arise naturally as change of coordinate frames. However, with this picture we will lose the geometric intuition since the generalised metric is not a symmetric $GL(2d)$ matrix and cannot really be understood as a metric for the extended space. Also large diffeomorphisms is a delicate issue in DFT (see for instance [120], or [59] for a discussion in the context of T-folds), which makes it a complicated task to identify the Dehn twists.

3.3.1 Double field theory and generalized duality

In section 2.4, we constructed solutions of the equations of motion derived from the action (2.133) that corresponded to the semi-flat limit of the doubled torus fibrations (2.131). An interesting question is now to what extent we can construct solutions that incorporate the higher Fourier modes that localize the NS5-brane on the torus fiber. In fact, this is possible by applying the generalized dualization discussed in the previous subsection. This essentially corresponds to the particular T-duality transformation (2.126) on the massive fields.

We start by adding to the NS5 brane the Fourier modes that localised it on $\mathbb{R}^2 \times \mathbb{T}^2$. Such corrections enter in the harmonic function as described in (3.25). We solve $dB = \star dh$ by choosing a gauge where

$$B = \frac{\theta}{2\pi R_1 R_2}\, d\xi^1 \wedge d\xi^2 + \Pi_1\, d\theta \wedge d\xi^1 + \Pi_2\, d\theta \wedge d\xi^2 \ , \tag{3.50}$$

where $\Pi_{1,2}$ satisfy the equations

$$\partial_r \Pi_1 = r\partial_2 h \,, \qquad \partial_r \Pi_2 = -r\partial_1 h \,, \qquad (\partial_2\Pi_2 - \partial_1\Pi_1) = r\partial_r h + (2\pi R_1 R_2)^{-1} \,, \tag{3.51}$$

with h the localized harmonic function (3.25). These equations are solved by

$$\Pi_1(r,\xi^1,\xi^2) = + \sum_{k_1,k_2\geq 0} \frac{(2-\delta_{k_1,0}-\delta_{k_2,0})}{\pi R_1 R_2^2}\, \frac{k_2}{\lambda}\, r\, K_1(\lambda r) \cos\left(\frac{k_1\xi^1}{R_1}\right) \sin\left(\frac{k_2\xi^2}{R_2}\right) , \tag{3.52}$$

$$\Pi_2(r,\xi^1,\xi^2) = - \sum_{k_1,k_2\geq 0} \frac{(2-\delta_{k_1,0}-\delta_{k_2,0})}{\pi R_1^2 R_2}\, \frac{k_1}{\lambda}\, r\, K_1(\lambda r) \sin\left(\frac{k_1\xi^1}{R_1}\right) \cos\left(\frac{k_2\xi^2}{R_2}\right) , \tag{3.53}$$

with K_1 the first order modified Bessel function of second kind and $\lambda = \sqrt{(k_1/R_1)^2 + (k_2/R_2)^2}$.

Starting from this configuration and applying the "generalised T-duality" rules described above, we obtain the following configuration

$$\begin{aligned} ds^2 &= \tilde{h}\left[dr^2 + r^2 d\theta^2 + (d\xi^1)^2\right] + \frac{1}{\tilde{h}}\left[d\xi^2 + \frac{\tilde{R}_2}{2\pi R_1}\,\theta\, d\xi^1 + \tilde{\Pi}_2\, d\theta\right]^2 , \\ B &= \tilde{\Pi}_1\, d\theta \wedge d\xi^1 \,, \\ e^{2\Phi} &= \text{const.}\,, \end{aligned} \tag{3.54}$$

where $\tilde{\Pi}_{1,2} \equiv \Pi_{1,2}(r, \xi^1, \tilde{\xi}_2)$. Furthermore, $\tilde{h} \equiv h(r, \xi^1, \tilde{\xi}_2)$, with h the localized harmonic function (3.25), and we write the result in terms of $\tilde{R}_2 = 1/R_2$. This is a compactification of the solution presented in [115] and it is compatible with the naive T-duality discussed in the previous sections. A second dualization leads to the background described by

$$\begin{aligned} ds^2 &= \tilde{h}\left[dr^2 + r^2 d\theta^2\right] + \frac{4\pi^2 \tilde{h}}{4\pi^2 \tilde{h}^2 + \tilde{R}_1^2 \tilde{R}_2^2 \theta^2}\left[(d\xi^1 + \tilde{\Pi}_1 d\theta)^2 + (d\xi^2 + \tilde{\Pi}_2 d\theta)^2\right], \\ B &= -\frac{2\pi \tilde{R}_1 \tilde{R}_2 \theta}{4\pi^2 \tilde{h}^2 + \tilde{R}_1^2 \tilde{R}_2^2 \theta^2}\left(d\xi^1 + \tilde{\Pi}_1 d\theta\right) \wedge \left(d\xi^2 + \tilde{\Pi}_2 d\theta\right), \\ e^{2\Phi} &= \frac{4\pi^2 \tilde{h}}{4\pi^2 \tilde{h}^2 + \tilde{R}_1^2 \tilde{R}_2^2 \theta^2}, \end{aligned} \tag{3.55}$$

where now $\tilde{\Pi}_{1,2} \equiv \Pi_{1,2}(r, \tilde{\xi}_1, \tilde{\xi}_2)$ and $\tilde{h} \equiv h(r, \tilde{\xi}_1, \tilde{\xi}_2)$. We also substitute $R_{1,2} \to \tilde{R}_{1,2} = 1/R_{1,2}$. Configurations (3.54) and (3.55) depend explicitly on the winding coordinates, and should be understood as DFT configurations by inserting the fields into (2.117) to obtain the corresponding generalized metric. In the DFT language the above localized configurations can be obtained from the localized NS5-brane generalized metric by simple transformations of the type $\xi^a \leftrightarrow \tilde{\xi}_a$. All these configurations have vanishing generalized curvature $\mathcal{R}_{MN}$ and therefore are solutions of the equations of motion of the DFT action (2.133). DFT (and EFT) configurations describing the NS5 brane and its dual backgrounds have been also discussed from a different perspective in [121–123].

Before closing this section, let us mention that in principle one can consider defects with more general τ and ρ monodromies. The physics of such non-geometric defects has been recently studied in [79]. Analogous puzzles related to generalized T-duality will arise. However, in this situation both winding and momentum might not be conserved along both fiber directions, and a solution of the strong constraint is a priori not guaranteed.

3.4 Final remarks

In this chapter we have analysed the local physics of the parabolic T-fects degenerations constructed in the previous chapter. Although in some cases a full description of such physics might still be missing, we have discussed some physical effects that an exact description should include. In particular we argued that winding modes are crucial for understanding the near-core physics of T-duality defects.

This was already evident in the T-duality between I_1 singularities and NS5-branes, where our analysis becomes essentially a compactified version of [106]. We argued that a similar physics describes non-geometric Q-monopoles, where an essential role is played by two dyonic coordinates dual to the isometric directions of the fiber torus. In particular, the picture that emerges is that in both the Q- and the KK-monopole, winding modes correspond to localization along a dual circle, related with the dyonic coordinate, in the same way momentum modes localise the NS5 brane and the KK-monopole around a point

	NS5	I_1	Q-brane
Monodromy	$\rho \to \rho + 1$	$\tau \to \tau + 1$	$-1/\rho \to -1/\rho + 1$
Non-conservation along ξ^1	momentum	momentum	winding
Non-conservation along ξ^2	momentum	winding	winding
0-modes	Shift ξ^1	Shift ξ^1	Dyonic coordinate
	Shift ξ^2	Dyonic coordinate	Dyonic coordinate
Corrections dependence	ξ^1 (Geometric loc.)	ξ^1 (Geometric loc.)	$\tilde{\xi}_1$ (Stringy corr.)
	ξ^2 (Geometric loc.)	$\tilde{\xi}_2$ (Stringy corr.)	$\tilde{\xi}_2$ (Stringy corr.)

Table 3.1: Summary of the corrections for the different analised configurations. The first two rows indicate which charge fails to be conserved along the directions of the torus, and the third and the fourth the two zero modes of the background. The last two indicate the coordinate dependence of the corrections. Corrections that depend on the geometric coordiantes ξ^a correspond to localisations along the fiber directions while the ones depending on the dual coordinates ξ_a are stringy corrections.

on the geometric circles. The results obtained for each background are summarised in Table 3.1, where we observe the relation between winding modes and the dyonic coordinate.

Finally, in the last part of the chapter we observe that DFT can indeed reproduce the coordinate dependence suggested by the above analysis. It would be important, however, to search for an explicit CFT description of such winding mode physics, or a dual formulation in terms of more conventional dynamics.

Chapter 4

Spherical T-duality for the NS5-brane

Within the parabolic T-fects constructed in chapter 2, we found configurations corresponding to semiflat approximations to the NS5 brane and the KK-monopole. As described in chapter 3, such approximate geometries can be obtained from the exact configurations by taking some particular limit after compactifying some of the directions transversal to the brane. For the NS5, the process is as follows: the original configuration has four transversal directions. Out of them we make two of them compact, which is equivalent to place branes on a lattice along these directions. Finally, one takes a limit away from the brane in the uncompact directions, known as smearing limit, where one looses the information about the position of the brane along the compact constants and these directions become isometries.

By undergoing this procedure the topology and the geometry of the configuration change. For instance, the original $\mathfrak{so}(4)$ symmetry algebra of the transversal directions is brocken to $\mathfrak{so}(2)$ after the compactification. In the smearing limit, this isometry group is enhanced to $\mathfrak{so}(2)\times\mathfrak{u}(1)\times\mathfrak{u}(1)$, where the extra $\mathfrak{u}(1)$'s correspond to the new isometries along the compact directions that arise in this region. The resulting configuration can be seen as a trivial two-torus fibration, in the sense that the torus fiber has no geometric twistings, as described in chapter 2.

For the KK-monopole, the procedure is analogous, though in this case the background already has one compact direction in the transversal space. Then, by compactifying along one extra direction and smearing, we break the original isometry algebra $\mathfrak{so}(3)\times\mathfrak{u}(1)$ to $\mathfrak{so}(2)\times\mathfrak{u}(1)$ and then enhance it to $\mathfrak{so}(2)\times\mathfrak{u}(1)\times\mathfrak{u}(1)$. In this case, however, the circle corresponding to the original $\mathfrak{u}(1)$ is non-trivially fibered.

As it was discussed in chapter 2 and 3, the two semi-flat configurations are related by T-duality. Furthermore, since the original NS5 has two isometries, one can perform an extra T-duality obtaining the backgrounds [1]

$$\boxed{\text{smeared NS5}} \overset{\text{T-duality}}{\longleftrightarrow} \boxed{\text{smeared KK monopole}} \overset{\text{T-duality}}{\longleftrightarrow} \boxed{5^2_2\text{-brane}}$$

[1] For global issues on the last T-duality see discussions in chapters 2 and 3.

where the latter corresponds to a non-geometric background, as discussed in chapter 2. Such duality chain has been also discussed in the literature in the context of fluxes (see for instance [25]) where the smeared NS5 is interpreted as a source for H-flux; the KK-monopole as a source for a geometric flux f, encoding the non-triviality of the $U(1)$ fibration; and the 5^2_2-brane as a source for the non-geometric Q-flux.

A natural question that arises in the above discussion is the following: do the same features appear if one dualise the NS5 brane using the original isometries and without undergoing the smearing procedure? As mentioned above, the original configuration is topologically very different: the un-compactified NS5 brane is still a source of H-flux but the $U(1)$ isometries are not trivially fibered, which implies the presence of certain geometric flux.

In this chapter we will analyse such dualities and show that the outcome is significantly different. In particular, no non-geometric issues will be found after two T-dualities. Furhtermore, the duality procedure will break the original supersymmetries, leading to a non-supersymmetric configuration after two T-dualities for which, a priori, stability is not guaranteed. This result will be consistent with the general analysis in [124–128]. We will also reproduce the general result in [119], where it was argued that geometric and H- fluxes are interchanged under T-duality.

The isometries we will consider in this chapter are then those of a three-sphere surrounding the brane. These are convenient symmetries to consider since the corresponding vectors remain finite in the origin, which will lead to non-singular configurations after T-duality. In appendix B we review some important results about T-duality transfomations for the three-sphere, putting special emphasis to the global aspects, which are an appropriate warm up for the discussion in this chapter. The matherial presented in this chapter follows closely [129].

4.1 The NS5-brane and its orbifold projections

To start, let us briefly give some details for the NS5-brane and its orbifolds. In particular, we determine the geometric and NS-charges. For this discussion we will use a coordinate frame where the four-dimensional configuration in the space transverse to the NS5-brane is given by

$$\begin{aligned} ds^2 &= h(r)\, dr^2 + \frac{h(r)\, r^2}{4}\left(d\theta^2 + d\xi^2 + 4d\chi^2 - 4\cos\theta d\chi d\xi\right), \\ H &= \star_4 dh(r)\,, \\ e^{2\Phi} &= e^{2\phi_0} h(r)\,, \end{aligned} \tag{4.1}$$

where $\star_4$ denotes the Hodge-star operator in four Euclidean dimensions and the variables take values $r \in [0, \infty)$, $\theta \in [0, \pi]$ and $\chi, \xi \in [0, 2\pi)$ (see appendix B). The value of the dilaton at infinity ϕ_0 is constant, and we recall that the harmonic function $h(r)$ is given by

$$h(r) = 1 + \frac{k}{r^2}\,, \tag{4.2}$$

where $k \in \mathbb{Z}_+$ is interpreted as the number of coincident NS5-branes. Note that the solution (4.1) can be seen as a three-sphere fibered along the radial direction. This solution is asymptotically flat at $r \to \infty$ but, unlike empty Euclidean space, the angular directions do not shrink at the origin and the volume of the three-sphere at $r = 0$ remains finite.

NS5-orbifolds

In analogy to the case of the three-sphere discussed in appendix B, it is possible to construct a generalization of (4.1) by considering orbifold projections along the $U(1)$ fibers. The general form of this configuration is given by

$$\begin{aligned} ds^2 &= h(r)\,dr^2 + \frac{h(r)\,r^2}{4}\left(d\theta^2 + \frac{1}{k_2^2}d\xi^2 + \frac{4}{k_1^2}d\chi^2 - \frac{4}{k_1 k_2}\cos\theta\,d\chi d\xi\right), \\ H &= \frac{k_3}{2}\sin\theta\,d\theta\wedge d\xi\wedge d\chi\,, \\ e^{2\Phi} &= e^{2\phi_0}h(r)\,, \end{aligned} \tag{4.3}$$

where $|k_i| \in \mathbb{Z}_+$. The constraints on the possible values of k_1 and k_2 imposed by demanding a globally well-defined background coincide with those imposed on the three-sphere discussed in appendix B. The harmonic function is now given by

$$h(r) = 1 + \frac{|k_1 k_2 k_3|}{r^2}\,, \tag{4.4}$$

and this configuration is in general not asymptotically $\mathbb{R}^4$ anymore, but nevertheless a solution of the string equations-of-motion. The geometry at the origin $r = 0$ is the sphere-orbifold (B.41) with radius $R = \sqrt{|k_1 k_2 k_3|}$.

Geometric charges

Since for the above configurations the three-sphere formed by the angular directions does not shrink to zero at any point, we can describe (4.3) as a principal $U(1)$-bundle in the same way as it can be done for an ordinary three-sphere. We can then assign gauge fields to the fibration structures either along the coordinate χ or along ξ. In particular, for these two choices we have

$$\mathcal{A}_\chi = \frac{g_{\chi i}}{g_{\chi\chi}}\,dx^i\,, \qquad \mathcal{A}_\xi = \frac{g_{\xi i}}{g_{\xi\xi}}\,dx^i\,, \tag{4.5}$$

and by integrating the corresponding field strengths over a two-sphere surrounding the brane at fixed radius we determine the geometric charges as

$$n_\chi = \frac{k_1}{k_2}\,, \qquad n_\xi = 2\,\frac{k_2}{k_1}\,. \tag{4.6}$$

Note that $n_\chi \in \mathbb{Z}$ only when $k_2 = 1$ and $n_\xi \in 2\mathbb{Z}$ only when $k_1 = 1$, which are, respectively, the cases when the $U(1)_\chi$ and $U(1)_\xi$ fibers are globally-defined (for details on this discussion see appendix B).

NS charges

The NS charge h of the above configuration can be determined by integrating its Bianchi identity $dH = 4\pi^2 h\delta^{(4)}(x)$ over the transversal space which, as we did in (2.89) , can be expressed as in integral of the H-flux over a three-sphere S^3_∞ at $r = \infty$. The charge for the configuration (4.3) satisfies the quantization condition $h \in \mathbb{Z}$ and is given by

$$h = \frac{1}{4\pi^2}\int_{\mathbb{R}^4} dH = \frac{1}{4\pi^2}\int_{S^3_\infty} H = k_3\,. \tag{4.7}$$

4.2 T-duality

We are now going to perform T-duality transformations for the NS5-brane background and its orbifolds. Similarly to the case of the three-sphere discussed in appendix B, after dividing by a $\mathbb{Z}_p$ action some of the isometries of the original NS5-brane might be broken. Applying T-duality along these directions leads to globally ill-defined configurations. In a slightly different context, T-dualities along the Hopf fiber of the NS5 have been also considered in [130, 131].

T-duality along the direction χ

We begin by studying T-duality along the direction χ in (4.3). Applying the Buscher rules we find the following T-dual configuration

$$\begin{aligned} ds^2 &= h(r)dr^2 + \frac{h(r)\,r^2}{4}\left(d\theta^2 + \frac{1}{k_2^2}\sin^2\theta d\xi^2 + \frac{k_1^2}{h(r)^2\,r^4}(2d\chi - k_3\cos\theta\,d\xi)^2\right)\,,\\ B &= -\frac{k_1}{2k_2}\cos\theta\,d\xi\wedge d\chi\,,\\ e^{2\Phi} &= e^{2\phi_0}\,r^{-2}\,, \end{aligned} \tag{4.8}$$

which is again a solution of the supergravity equations of motion. It is not asymptotically-flat anymore, and close to the origin the geometry is locally a three-sphere. With the same prescriptions used above we can assign the following charges to this background

$$n'_\chi = k_3 = h\,, \qquad\qquad h' = \frac{k_1}{k_2} = n_\chi\,. \tag{4.9}$$

We observe that only in the case where $k_2 = 1$ the quantization condition $h' \in \mathbb{Z}$ will be satisfied. In fact, as it follows from the discussion in appendix B, this is the case where the $U(1)_\chi$ isometry of the original background is globally-defined. In this situation, the effect of T-duality along χ interchanges the geometric charge n_χ with the NS-charge h.

T-duality along the direction ξ

As can be seen from (4.3), we can equally-well perform a T-duality transformation along the direction ξ, and the corresponding dual background is

$$\begin{aligned} ds^2 &= h(r)dr^2 + \frac{h(r)\,r^2}{4}\left(d\theta^2 + \frac{4}{k_1^2}\sin^2\theta d\chi^2 + \frac{k_2^2}{h(r)^2\,r^4}(d\xi + 2k_3\cos\theta d\chi)^2\right), \\ B &= \frac{k_2}{2k_1}\cos\theta\, d\xi\wedge d\chi\,, \\ e^{2\Phi} &= e^{2\phi_0}\,r^{-2}\,, \end{aligned} \tag{4.10}$$

which is again a non-asymptotically-flat solution to the string equations-of-motion. As in the case obtained by T-duality along χ, one can compute the following charges

$$n'_\xi = -2k_3 = -2h\,, \qquad\qquad h' = -\frac{k_2}{k_1} = -\frac{1}{2}n_\xi\,. \tag{4.11}$$

The resulting NS charge is an integer only in the case when $k_1 = 1$, which is the situation where the $U(1)_\xi$ fiber of the original background is globally-defined. We observe that, in this case, T-duality exchanges $k_3 \leftrightarrow k_2$ as expected.

T-dualities along the directions χ and ξ

Finally, we discuss the background obtained after two T-duality transformations along the directions χ an ξ. The dual configuration can be expressed in the following way

$$\begin{aligned} ds^2 &= h(r)\left(dr^2 + \frac{r^2}{4}d\theta^2\right) + \frac{k_1^2k_2^2r^2h(r)}{4\Omega}\left(\frac{1}{k_1^2}d\xi^2 + \frac{4}{k_2^2}d\chi^2 + \frac{4}{k_1k_2}\cos\theta\, d\xi\, d\chi\right), \\ B &= \frac{k_1^2k_2^2k_3^2\cos\theta}{2k_3\Omega}d\xi\wedge d\chi\,, \\ e^{2\Phi} &= e^{2\phi_0}\frac{k_1^2k_2^2h(r)}{\Omega}\,, \end{aligned} \tag{4.12}$$

where we defined

$$\Omega = \left(r^2h(r)\sin\theta\right)^2 + \left(k_1k_2k_3\cos\theta\right)^2. \tag{4.13}$$

As expected, this background is again a solution to the string equations-of-motion. However, the geometry is somewhat peculiar: at $r = 0$ the directions θ, ξ and χ describe an S^3 orbifold, whereas at $r \to \infty$ the T^2-fiber corresponding to ξ and χ shrinks to zero size. The topology of the dual space is therefore different from the original NS5-brane topology. Furthermore, the NS-charge of the background (4.12) can in principle be computed in a similar ways as in (4.7), but without proper knowledge of the dual topology this is difficult in practice. Finally, as we will discuss in the next section, (4.12) does not preserve any supersymmetry and hence the stability of this solution is not guaranteed

On the other hand, we want to point-out that the configuration (4.12) obtained after two T-duality transformations does not show any non-geometric features – contrary to what one might have naively expected from [25]. This is in contrast to compactifying the NS5-brane solution, smearing along the compact directions and performing a T-duality along the latter, after which one obtains a non-geometric 5_2^2-brane [90, 99].

4.3 Supersymmetry

We now want to analyze the amount of supersymmetry preserved by the T-dual configurations determined in the last section. Even though the dual backgrounds are solutions to the string equations-of-motion, starting from the 1/2-BPS NS5-brane solution we will see that a single T-duality along the direction χ or ξ results in a 1/4-BPS configuration. Furthermore, after two T-dualities supersymmetry is completely broken. These results are in agreement with [124–127], where it was found that if a Killing spinor depends on the coordinate along which one T-dualizes then the corresponding supersymmetry will be broken.[2]

Conventions

In the rest of this section we want to give some details of our analysis. For type II supergravity theories in ten dimensions the supersymmetry variations of the dilatini and gravitini read as follows (for our conventions see [90])

$$\delta_\epsilon \lambda = \left(\frac{1}{2}\partial\!\!\!/\Phi - \frac{1}{4}H\!\!\!\!/\,\mathcal{P}\right)\epsilon, \qquad \delta_\epsilon \Psi_M = \left(\nabla_M - \frac{1}{4}H\!\!\!\!/_M\mathcal{P}\right)\epsilon, \tag{4.14}$$

where ϵ is a doublet of Majorana-Weyl spinors. The two components of ϵ have the same (IIB) or opposite (IIA) chiralities. The operator $\mathcal{P}$ acts on the spinor doublet as $\mathcal{P} = \sigma_3$ for type IIB and $\mathcal{P} = \Gamma_{[10]}\,\mathbb{I}_{2\times 2}$ for type IIA, where $\Gamma_{[10]}$ is the ten-dimensional chirality matrix. Note that if we choose a representation of eigenstates of $\Gamma_{[10]}$, $\mathcal{P}$ will act as $+\mathbb{I}_{32\times 32}$ on one component of the doublet and as $-\mathbb{I}_{32\times 32}$ on the other, both in type IIA/B. Therefore we will generically denote the two spinors as $\epsilon = (\epsilon_+, \epsilon_-)$.

The NS5-orbifold

We begin analyzing the amount of supersymmetry preserved by the configuration (4.3). This includes the NS5-brane for a particular choice of k_1 and k_2, which is known to be a

[2] A formulation of this condition independent of a particular coordinate frame was given in [128] using the Kosmann spinorial Lie-derivative.

1/2-BPS solution. The two Killing spinors for this background are[3]

$$\begin{aligned}
\epsilon_+ &= e^{\frac{\chi}{k_1}\Gamma_{\hat 7}\Gamma_{\hat 8}}\epsilon_{0,+} &&= \left(\cos\frac{\chi}{k_1} + \Gamma_{\hat 7}\Gamma_{\hat 8}\sin\frac{\chi}{k_1}\right)\epsilon_{0,+}\ ,\\
\epsilon_- &= e^{-\frac{\theta}{2}\Gamma_{\hat 8}\Gamma_{\hat 9}}e^{\frac{\xi}{2k_2}\Gamma_{\hat 7}\Gamma_{\hat 8}}\epsilon_{0,-} &&= \left(\cos\frac{\theta}{2} - \Gamma_{\hat 8}\Gamma_{\hat 9}\sin\frac{\theta}{2}\right)\left(\cos\frac{\xi}{2k_2} + \Gamma_{\hat 7}\Gamma_{\hat 8}\sin\frac{\xi}{2k_2}\right)\epsilon_{0,-}\ ,
\end{aligned} \tag{4.15}$$

where $\epsilon_{0,\pm}$ are constant Majorana-Weyl spinors satisfying $(1 \pm \Gamma_{\hat 6}\Gamma_{\hat 7}\Gamma_{\hat 8}\Gamma_{\hat 9})\epsilon_{0,\pm} = 0$, which projects-out half of their components.[4]

T-dual configurations

Let us discuss the amount of supersymmetry preserved by the various T-dual configurations discussed above. Here we only give the final results, but details of our computations can be found in appendix C.

- We start by considering the background (4.8) obtained after a T-duality transformation along the direction χ. Since ϵ_+ in (4.15) depends explicitly on χ, we expect that the corresponding supersymmetry will be broken. Indeed, for (4.8) we find only one Killing spinor given by (assuming $k_1k_2k_3 > 0$)

$$\begin{aligned}
\epsilon_- &= e^{-\frac{\theta}{2}\Gamma_{\hat 8}\Gamma_{\hat 9}}e^{\frac{\xi}{2k_2}\Gamma_{\hat 7}\Gamma_{\hat 8}}\epsilon_{0,-}\\
&= \left(\cos\frac{\theta}{2} - \Gamma_{\hat 8}\Gamma_{\hat 9}\sin\frac{\theta}{2}\right)\left(\cos\frac{\xi}{2k_2} + \Gamma_{\hat 7}\Gamma_{\hat 8}\sin\frac{\xi}{2k_2}\right)\epsilon_{0,-}\ ,
\end{aligned} \tag{4.16}$$

  where again $(1 - \Gamma_{\hat 6}\Gamma_{\hat 7}\Gamma_{\hat 8}\Gamma_{\hat 9})\epsilon_{0,-} = 0$. Note that this configuration preserves only half of the original supersymmetries.

- A similar analysis applies to a T-duality transformation along the direction ξ. Since in (4.15) the Killing spinor ϵ_- depends explicitly on ξ, we expect that the corresponding supersymmetry will be broken under T-duality. Indeed, for the background (4.10) we find only one Killing spinor given by (assuming $k_1k_2k_3 > 0$)

$$\begin{aligned}
\epsilon_+ &= e^{-\frac{\theta}{2}\Gamma_{\hat 8}\Gamma_{\hat 9}}e^{\frac{\chi}{k_1}\Gamma_{\hat 7}\Gamma_{\hat 9}}\epsilon_{0,-}\\
&= \left(\cos\frac{\theta}{2} - \Gamma_{\hat 8}\Gamma_{\hat 9}\sin\frac{\theta}{2}\right)\left(\cos\frac{\chi}{k_1} + \Gamma_{\hat 7}\Gamma_{\hat 9}\sin\frac{\chi}{k_1}\right)\epsilon_{0,+}\ ,
\end{aligned} \tag{4.17}$$

  with $(1 + \Gamma_{\hat 6}\Gamma_{\hat 7}\Gamma_{\hat 8}\Gamma_{\hat 9})\epsilon_{0,+} = 0$. This configuration again again preserves only half of the original supersymmetries.

[3] We note that ϵ_+ is $2\pi k_1$-periodic in χ and ϵ_- is $4\pi k_2$-periodic in ξ.

[4] The solution presented here correspond to the case where $k_1k_2k_3 > 0$. Details corresponding to the opposite case can be found in appendix C

- After applying two T-duality transformations along the directions χ and ξ, we expect that both of the supersymmetries corresponding to ϵ_+ and ϵ_- in (4.15) will be broken. For the background (4.12) the supersymmetry variations (4.14) can only be solved for vanishing spinors, and hence supersymmetry is completely broken. This means that stability is no longer guaranteed, and therefore we do not study this background in more detail.

4.4 T-duality along non-globally-defined $U(1)$ fibers

We finally want to generalize our previous discussion in the following way: if we interpret the three-sphere inside the transversal geometry of the NS5-brane solution as a two-torus fibered over a line-segment, we can in principle perform T-duality transformations also along an arbitrary direction of the two-torus. In general, such isometries are not globally well-defined and hence the dual background may show global problems. Nevertheless, locally this analysis is valid.

Let us rewrite the NS5-brane solution (4.1) in a different set of coordinates which make the $\mathbb{T}^2$-fibration structure explicit. Including orbifold projections along the two directions of the two-torus, we have

$$\begin{aligned} ds^2 &= h(r)\left(dr^2 + r^2 d\eta^2 + \frac{r^2}{\alpha_1^2}\cos^2\eta\, d\xi_1^2 + \frac{r^2}{\alpha_2^2}\sin^2\eta\, d\xi_2^2\right) , \\ H &= 2\alpha_3 \sin\eta\, \cos\eta\, d\eta\wedge d\xi_1 \wedge d\xi_2 , \\ e^{2\Phi} &= e^{2\phi_0} h(r) , \end{aligned} \tag{4.18}$$

where $\eta \in [0, \pi/2]$ and $\xi_{1,2} \in [0, 2\pi)$ and $|\alpha_i| \in \mathbb{Z}_+$, and where the harmonic function is given by

$$h(r) = 1 + \frac{|\alpha_1\alpha_2\alpha_3|}{r^2} . \tag{4.19}$$

After performing a T-duality transformation along the direction $\mathsf{v} = \beta_1\partial_{\xi_1} + \beta_2\partial_{\xi_2}$ we obtain, for a choice of local coordinates (ψ_1, ψ_2), the configuration

$$\begin{aligned} ds^2 &= h(r)\left(dr^2 + r^2 d\eta^2 + \frac{r^2}{4}\,\frac{\sin^2(2\eta)}{\Delta}\, d\psi_1^2\right) + \frac{1}{r^2 h(r)}\frac{\alpha_1^2\alpha_2^2}{\Delta}\left(d\psi_2 - \frac{\alpha_3}{2}\cos(2\eta)\, d\psi_1\right)^2 , \\ H &= \frac{\beta_1\beta_2\alpha_1^2\alpha_2^2\, \sin^2(2\eta)}{\Delta^2}\, d\eta\wedge d\psi_1\wedge d\psi_2 , \\ e^{2\Phi} &= e^{2\phi_0}\,\frac{\alpha_1^2\alpha_2^2}{r^2\Delta} , \end{aligned} \tag{4.20}$$

where we defined

$$\Delta = \big(\alpha_2\beta_1\sin\eta\big)^2 + \big(\alpha_1\beta_2\cos\eta\big)^2 . \tag{4.21}$$

As we mentioned in appendix B for the three-sphere, following Buscher's procedure in general does not give global information about the T-dual space, and hence we have not specified the range of the coordinates (ψ_1, ψ_2). Locally, however, (4.20) does solve the string equations-of-motion. Concerning the amount of supersymmetry preserved by (4.20), for arbitrary (β_1, β_2) all supersymmetries are broken – only for $(\beta_1, \beta_2) = (\alpha_1, \pm\alpha_2)$ the solution preserves half of the original supersymmetries. Note that the latter are precisely the examples (4.8) and (4.10) discussed above.

4.5 Comparison with similar configurations

Comparison with T-duality for $\mathbb{R}^n$

Let us note that performing a T-duality transformation along an angular direction for empty Euclidean space $\mathbb{R}^n$ results in a dual geometry which is singular at the origin [16]. The reason is that the norm of the corresponding Killing vector vanishes there. This is a puzzling observation, and it has been suggested that winding modes may play a role in resolving the singularity. On the other hand, we see for instance from (4.3) that the metric and H-flux of the NS5-brane at the origin $r = 0$ is finite. (We ignore the dilaton in the present discussion.) Performing a single T-duality transformation along an angular direction leads to a geometry which is again non-singular, as can be seen for instance from (4.8), and the reason for the non-singular behavior of the dual metric at $r = 0$ can be traced back to the non-vanishing H-flux $h = k_3 \neq 0$.

We do not have an answer to the question whether or how the singularity of the T-dual of $\mathbb{R}^n$ can be resolved. We want to stress, however, that for the NS5-brane the H-flux plays an important role in regard to this point, and one might suspect that it also can play a role in the $\mathbb{R}^n$ case.

Comparison with toroidal compactifications of the NS5

We also want to compare our results to T-duality transformations for the smeared NS5-brane solution. As mentioned above, these two cases are significantly different, since the smearing procedure change the geometry and the topology of the original background. In particular, by undergoing this compactification procedure, the transversal space is not anymore a three-sphere fibration along a radial direction and the new $U(1)$ isometries along which we perform T-duality transformations are trivial fibrations. This implies that, for this case, there is no geometric charge associated with this directions, in contrast with the Hopf fiber of the three-sphere. T-duality along the smeared directions leads to the KK-monopole and the 5^2_2-brane, which in both cases supersymmetry is preserved. The outcome of our analysis is then that dualising along this two configurations lead to very different results.

Comparison with the NS5–Taub-NUT configuration

Finally, we also want to point the existence of another configuration which containts a three-sphere orbifold at its near horison region. This is obtained by collapsing k_3 NS5 branes smeared along one direction and k_1 KK-monopoles. As mentioned above, the NS5 brane "loses" its geometric charge in the smearing procedure, which is restored by adding the KK-monopoles. The resulting configuration is

$$\begin{aligned} ds^2 &= \mathsf{h}_1(\rho)\mathsf{h}_3(\rho)\big(d\rho^2+\rho^2 d\theta^2+\rho^2\sin^2\theta d\varphi^2\big)+\frac{\mathsf{h}_3(\rho)}{\mathsf{h}_1(\rho)}\left(dx-\frac{k_1}{2}\cos\theta d\varphi\right)^2 , \\ B &= -\frac{k_3}{2}\cos\theta\, d\varphi\wedge dx\,, \\ e^{2\Phi} &= e^{2\phi_0}\,\mathsf{h}_3(\rho)\,, \end{aligned} \tag{4.22}$$

with $\rho\in[0,\infty)$, $\theta\in[0,\pi]$ and $\varphi, x\in[0,2\pi)$. The harmonic functions $\mathsf{h}_1(\rho)$ and $\mathsf{h}_3(\rho)$ correspond to the KK-monopole and the smeared NS5-brane, respectively, and read

$$\mathsf{h}_i(\rho)=1+\frac{k_i}{2\rho}\,. \tag{4.23}$$

A computation analogous to the ones carried before shows that this background has indeed k_1 units of geometric charge and k_3 units of NS-charge. Also, one can see that in the limit $\rho\to 0$ the metric in (4.22) remains finite, and the corresponding geometry is given by the three-sphere orbifold $SU(2)_{k_1k_3}/\mathbb{Z}_{k_1}$ [132]. However, even though the solution (4.22) close to the origin agrees with the $r\to 0$ behavior of the ones discussed in section 4.1, away from $r=0$ they are different. In particular, such configurations are 1/4-supersymmetric. After applying T-duality to the direction x, the resulting configuration is again a superposition of NS5-branes and KK-monopoles, where the numbers k_1 and k_3 have been interchanged

$$k_1 \quad \xleftrightarrow{\text{T-duality}} \quad k_3\,. \tag{4.24}$$

In this case, then, supersymmetry is preserved under T-duality.

Chapter 5

U-duality defects and supersymmetric geometric compactifications of M-theory

In chapter 2 we constructed a class of local geometries consisting in two-torus fibrations with T-duality monodromies around a degeneration point. Now, we generalise this constructions to two-torus fibrations in type II string theories with U-duality monodromies. The T-duality group of a two-torus, $SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})$, is now enlarged into the U-duality group $SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$. This group has a useful geometric interpretation in M-theory, where the $SL(3,\mathbb{Z})$ factor is the large diffeomorphisms group of the three-torus constructed from the original two-torus and the M-theory circle. The remaining $SL(2,\mathbb{Z})$ mixes the volume of this three-torus with the M-theory three-form in a way analogous to the $SL(2,\mathbb{Z})_\rho$ factor of the T-duality group in the heterotic case.

Because of this interpretation, we will actually begin our analysis by studying three-torus fibrations in M-theory. Since we already know from chapter 2 how the $SL(2,\mathbb{Z})$ factor can be fibered, we will focus in finding the most general supersymetric fibrations of the geometric $SL(3,\mathbb{Z})$ factor. This case will be studied by considering general supersymmetric compactifications of M-theory on a five-dimensional manifold without fluxes. The analysis of general compactification models fits into the picture that T-fects (and U-fects) solutions can be seen as local descriptions of a region of a global compactification model close to a degeneration, as argued in chapter 2.

After a meticulous supersymmetry analysis we will conclude that geometric M-theory compactifications on a five dimensional manifold can only preserve supersymmetry if the internal manifold is a trivial circle fibration. This will restrict the three-torus fibrations two be non-trivial two-torus fibrations times a trivial circle, which implies that all U-fects are dual to one of the T-fects configurations of chapter 2. We will close this chapter giving the complete classification of U-fects in type IIA and IIB supergravities, relating some of them to known configurations.

As in the case of T-fects, such configurations will include backgrounds with non-geometric features. In fact, these are a particular realisations of the spaces generically called U-manifolds or U-folds in the literature [133–135]. The discussion in this chapter is based on unpublished material. In slightly different contexts, three-torus fibrations in string theory have also been studied in [136, 137].

5.1 T-fects in M-theory

As a warm up before the full analysis, we will study T-duality defects as 11-dimensional supergravity solutions. Because they only involve fields in the NS sector, we can interpret τ-fects (2.53), ρ-fects (2.85) and $\tau\rho$-brane (2.103) solutions as solutions of type IIA supergravity and uplift them to 11-dimensional supergravity. By construction, these configurations will only have monodromies in a certain $SL(2,\mathbb{Z})$ subgroup of the $SL(3,\mathbb{Z})$ factor.

M-theory geometric τ-fects

We begin considering the pure geometric configuration (2.53). After an uplift we obtain

$$ds^2_{11} = dx^2_{||} + e^{2\phi}\tau_2 dz d\bar{z} + \tau_2 d\xi_1^2 + \frac{1}{\tau_2}(d\xi_2 + \tau_1 d\xi_1)^2 + d\xi_3^2, \tag{5.1}$$

and vanishing three-form field. The directions ξ_i are the compact directions of the 3-torus and $x_{||}$ world-volume of the original 10-dimensional five-brane. The function $\tau = \tau_1 + i\tau_2$ is again holomorphic in the $(z, \bar{z})$ plane. In the case where the monodromy of τ is in the parabollic class, these configurations correspond to smeared KK6 branes, which are Taub-NUT spaces embeded in 11-dimensional supergravity. As in the 10-dimensional case, different monodromies inside the same parabollic conjugacy class correspond to different orientations of the NUT direction (see discussion under (2.68)).

From the 11-dimensional point of view, the solution (5.1) preserves (locally) an $SO(1,6)$ subgroup of $SO(1,10)$ and, therefore, the configuration can be interpreted as a six-brane wrapping one of the compact directions. The space transverse to the brane is then four-dimensional, where only a two-torus fibration over a plane fits, consistently with the fact that we have only one complex modulus for the three-torus fiber.

M-theory non-geometric ρ-fects

Next, we consider configurations with monodromies in the $SL(2,\mathbb{Z})_\rho$ factor of the T-duality group, whose transversal space is described by the fields (2.85) . After an uplift, we obtain the 11-dimensional configuration

$$\begin{aligned} ds^2_{11} &= \frac{1}{\rho_2^{1/3}} dx^2_{||} + e^{2\phi}\rho_2^{2/3} dz d\bar{z} + \rho_2^{2/3}(d\xi_1^2 + d\xi_2^2 + d\xi_3^2)\,, \\ A &= \rho_1\, d\xi_1 \wedge d\xi_2 \wedge d\xi_3\,, \end{aligned} \tag{5.2}$$

where ρ is again an holomorphic function of the $(z, \bar{z})$ plane and the directions ξ_i are compact. In this case, from the 11-dimensional point of view, the configuration is a five-brane and the transverse space is a genuine three-torus fibration. When ρ has the parabolic monodromy V (2.16), the configuration is an M5 brane smeared along three directions of the transverse space. Instead, when we consider a solution with parabolic monodromy U (2.21) , the configuration is the non-geometric 5^3 brane [90]

$$\begin{aligned} ds^2_{11} &= \left(\frac{2\pi h(r)}{h(r)^2+\theta^2}\right)^{-1/3} dx^2_{||} + \frac{\Big(2\pi h(r)\Big)^{2/3}}{\Big(h(r)^2+\theta^2\Big)^{-1/3}} dz d\bar{z} + \left(\frac{2\pi h(r)}{h(r)^2+\theta^2}\right)^{2/3} (d\xi_1^2 + d\xi_2^2 + d\xi_3^2)\,, \\ A &= \frac{2\pi\theta}{h(r)^2+\theta^2}\, d\xi_1 \wedge d\xi_2 \wedge d\xi_3\,, \end{aligned} \tag{5.3}$$

with

$$h(r) = \log\left(\frac{\mu}{r}\right)\,. \tag{5.4}$$

As expected, this configuration can also be obtained by uplifting the Q-brane (2.93). The monodromy U of this configuration will act on the volume as

$$V_{T^3} \to \frac{V_{T^3}}{V_{T^3}^2 + (A-1)^2}\,, \tag{5.5}$$

where A is the unique component of the M-theory three-form on the three-torus, and the configuration will be non-geometric. Analogous to the case of heterotic ρ-fects in chapter 2, in general configurations (5.2) will show non-geometric features.

Colliding degenerations

As we argued in chapter 2, one can obtain general non-trivial fibrations for τ and ρ by colliding the corresponding individual fibrations. Here one can obtain analogous configurations in M-theory by uplifting the local geometry (2.103). The result will be configurations that are generically non-geometric and not dual to geometric configurations. From the 11-dimensional point of view, such configurations will look like five-branes lying inside six-branes that wrap one of the compact circles.

In M-theory, it seems a priori that there is another way of colliding $SL(2,\mathbb{Z})$ degenerations to obtain new configurations: collide two geometric $SL(2,\mathbb{Z})_\tau$ degenerations with different orientations inside the $SL(3,\mathbb{Z})$ factor. However, such collision actually breaks all supersymmetries of the background and the configuration is not a solution to the string background equations [136]. The intuitive reason is that the supersymmetry projectors of each individual τ-fect do not commute if they have different $SL(3,\mathbb{Z})$ orientations.

5.2 Supersymmetric U-duality defects in M-theory and geometric compactifications

As discussed, the solutions presented in the previous section only covered monodromies leaving in an $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ subgroup of the U-duality group. To generalise these results into the full duality group, we need to enlarge the geometric $SL(2,\mathbb{Z})$ factor into $SL(3,\mathbb{Z})$. This implies considering general geometric three-torus fibrations, turning on all possible geometric moduli of the three-torus.

In section 5.2.1 we will give a general ansatz for such construction. Then, in section 5.2.2 we will study the implications of compactifications of M-theory on five-dimensional geometric manifolds and apply the result to the ansatz for the three-torus fibrations. As we will see, turning on more than one complex modulus parameter at the same time will break all supersymmetry.

5.2.1 General ansatz for T^3 fibrations

The moduli space of a three-torus of unit volume[1] without fluxes can be parametrised by five real scalars, which we label by $(\tau_1, \tau_2, \sigma_1, \sigma_2, \sigma_3)$ and can be encoded into the three vielbein 1-forms as

$$\begin{aligned} \eta^1 &= \frac{1}{(\tau_2\sigma_3)^{1/3}}(d\xi_1 + \tau_1 d\xi^2 + \sigma_1 d\xi_3)\,, \\ \eta^2 &= \frac{1}{(\tau_2\sigma_3)^{1/3}}(\tau_2 d\xi_2 + \sigma_2 d\xi_3)\,, \\ \eta^3 &= \frac{1}{(\tau_2\sigma_3)^{1/3}}(\sigma_3 d\xi_3)\,, \end{aligned} \tag{5.6}$$

where the factor in front of each form is choosen in a way that the vielbein has unit determinant. The corresponding metric is given by

$$(g_{T^3})_{ij} = \eta^a_i \eta^b_j \delta_{ab} = \frac{1}{(\tau_2\sigma_3)^{2/3}} \begin{pmatrix} 1 & \tau_1 & \sigma_1 \\ \tau_1 & \tau_1^2 + \tau_2^2 & \tau_1\sigma_1 + \tau_2\sigma_2 \\ \sigma_1 & \tau_1\sigma_1 + \tau_2\sigma_2 & \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \end{pmatrix}, \tag{5.7}$$

which satisfies $\det g_{T^3} = 1$. The set of vectors dual to (5.6), satisfying $\iota_{e_i}\eta^j = \delta^j_i$, are

$$\begin{aligned} e_1 &= (\tau_2\sigma_3)^{1/3}\partial_{\xi^1}\,, \\ e_2 &= (\tau_2\sigma_3)^{1/3}\left(-\frac{\tau_1}{\tau_2}\partial_{\xi^1} + \frac{1}{\tau_2}\partial_{\xi^2}\right), \\ e_3 &= (\tau_2\sigma_3)^{1/3}\left(\left(\frac{\tau_1\sigma_2}{\tau_2\sigma_3} - \frac{\sigma_1}{\sigma_3}\right)\partial_{\xi^1} - \frac{\sigma_2}{\tau_2\sigma_3}\partial_{\xi^2} + \frac{1}{\sigma_3}\partial_{\xi^3}\right). \end{aligned} \tag{5.8}$$

[1] The volume is not affected by $SL(3,\mathbb{Z})$ transformations.

Next, we construct a flat torus fibration by letting the moduli parameters be functions of a 2 dimensional base: $\tau_1 = \tau_1(z,\bar{z})$, $\tau_2 = \tau_2(z,\bar{z})$,.... The local metric in a patch can be written as

$$ds^2 = f^2(z,\bar{z})dzd\bar{z} + (g_{T^3}(z,\bar{z}))_{ij}\, d\xi^i d\xi^j \,, \tag{5.9}$$

which is the most general ansatz preserving the $so(2)$ symmetry on the base. For further discussions we will consider the following 1-forms on the base

$$\begin{aligned} \Theta^1 &= \frac{1}{2} f(z,\bar{z})(dz + d\bar{z})\,, \\ \Theta^2 &= \frac{i}{2} f(z,\bar{z})(dz - d\bar{z})\,, \end{aligned} \tag{5.10}$$

which together with the 1-forms $\eta^a(z,\bar{z})$ form a set of five 1-forms defined in a local patch. We remark that these forms are not necessarily globally defined, and actually in general some of them are not. We will furhter comment on global issues during the supersymmetry analysis of next section.

5.2.2 Supersymmetry analysis

In previous chapters, the analysis of supersymmetry variations was already used as a technique to find stable solutions to the string background equations of motion. For instance, the T-fect configurations on chapter 2 preserved some supersymmetry only in the case where the moduli τ and ρ were holomorphically fibered. Here we apply this technique to the ansatz (5.9), in order to find the most general supersymmetric solutions to the string background equations. In this case, supersymmetry preservation will have strong implications to the ansatz, allowing only two out of the five real moduli of the three torus to be turned on simultaneously.

Before going into the concrete computation for the ansatz (5.9), we give some general considerations about compactifications of M-theory to six external directions to obtain some intuition to the kind of result we expect. In particular, we derive necessary conditions that any supersymmetric compactification needs to satisfy in terms of holonomy. In the last part of this section, these conditions will be applied to the ansatz (5.9) to obtain the results mentioned above.

General geometric compactifications to six-dimensional Minkowski vacua

As follows from the discussion in section 1.2, we are in general interested in considering solutions of 11-dimensional supergravites on manifolds of the form

$$\mathcal{M} = M_6 \times K\,, \tag{5.11}$$

where M_6 is the six-dimensional Minkowski space and K is a five-dimensional internal manifold. In section 5.1, we have seen that one possible way to construct geometric solutions to 11-dimensional supergravities is by taking solutions to the 10-dimensional theory and adding a trivial circle. Because of this close relation, we will also include in this general description the case of 10-dimensional supergravities compactified on a four-dimensional internal manifold. In this section, we will consider the simplest situation possible: the metric on $\mathcal{M}$ will be of the form

$$ds^2 = dx_\mu dx^\mu + ds_K^2 \,, \tag{5.12}$$

where ds_K^2 is a metric on the internal manifold K, which in our discussion will be eventually related to the metric of global models constructed out of the fibrations (5.9), and the rest of form fields (as well as fermionic fields) vanish. Generalisations to spaces with fluxes will be considered in chapter 6 and to the case where the external space is AdS_6 in chapter 7.

The main requirement we impose to our solutions is that they preserve some of the supersymmetries of the original theory. For the case without fluxes, the only non-trivial supersymetry variation is the one corresponding to the gravitino field, which will transform as

$$\delta_\varepsilon \Psi = \nabla \varepsilon \,, \tag{5.13}$$

where ∇ is a covariant derivative with respect to the spin connection of (5.12) and ε is the supersymmetry parameter. In the cases where (some amount of) supersymmetry is preserved, the right hand side of (5.13) will vanish for some spinor field ε and the background will then have a globally defined, nowhere vanishing covariantly constant spinor.

As we shall see, this statement has strong implications on the topology and the geometry of the configurations. On one side, it implies that the metric on K is Ricci-flat. Furthermore, it imposes strong restrictions to the holonomy group $\mathscr{H}$ of the internal manifold K, as we discuss in the following section.

Special holonomy for compactifications to six dimensions

The holonomy group $\mathscr{H}$ of a manifold K is the set of all possible transformations that any vector on any point of the manifold can experience after being parallel transported along any closed loop. In general, for a $\mathbb{d}$-dimensional Riemannian manifold, the holonomy group has to satisfy[2] $\mathscr{H} \subseteq SO(\mathbb{d})$. In the cases where the holonomy group is maximal, there cannot exist globally defined constant spinor fields, and compactifications using these manifolds will break all supersymmetry. On the other hand, if the holonomy group is trivial all supersymmetries of the original theory can be preserved. If K is compact, this last situation corresponds to flat tori.

A more interesting situation happens when $\mathscr{H}$ is strictly a non-trivial subgroup of the maximal one. In this case, one can have covariantly constant spinors if the decomposition of the spinor representation of the maximal group under the holonomy group contains singlets. Let us analyse the case where the external dimension is six. We will first discuss the case of 10-dimensional supergravities, corresponding to the geometric solutions discussed

[2] For simplicity we consider orientable manifolds which are simply-connected. For cases where the manifold is not simply-connected one has to use the concept of reduced holonomy group.

in chapter 2, and then we will compare the results with the situation in 11-dimensional gravity, which is the one we are eventually interested in this chapter.

10-dimensional supergravities

Under $SO(1,9) \to SO(1,5) \times SO(4)$, the 10-dimensional Weyl spinors (**16**) decompose as

$$\mathbf{16} \to (\mathbf{4}_{\rm L}, \bar{\mathbf{2}}) + (\mathbf{4}_{\rm R}, \mathbf{2}) \,, \tag{5.14}$$

where $\mathbf{4}_L$ and $\mathbf{4}_R$ are Weyl spinors of $SO(1,5)$ and $\mathbf{2}$, $\bar{\mathbf{2}}$ Weyl spinors of $SO(4)$. In ten dimensional Minkowski space, one can impose Majorana condition consistently with the Weyl condition and the 16 supercharges can be chosen to be real. If the internal space has trivial holonomy, the reduced theory has two real $\mathbf{4}_L$ and two real $\mathbf{4}_R$, which combines into one complex $\mathbf{4}_L$ and one $\mathbf{4}_R$. Therefore, starting from a theory with $\mathcal{N}=1/\mathcal{N}=2$, the reduced theory has $\mathcal{N}=2/\mathcal{N}=4$.

To study the cases with reduced holonomy, one needs to analyse how the representations $\mathbf{2}$ and $\bar{\mathbf{2}}$ transform under subgroups of $SO(4)$. At the level of Lie algebra, $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \times \mathfrak{su}(2)$ and we can rewrite $\mathbf{2} = (\mathbf{2},\mathbf{1})$ and $\bar{\mathbf{2}} = (\mathbf{1},\mathbf{2})$, where the numbers inside the brackets correspond to the representation of each $\mathfrak{su}(2)$. We then see that, if one chooses an internal manifold with $SU(2)$ holonomy, either the original $\mathbf{2}$ or $\bar{\mathbf{2}}$ transforms as a singlet under the holonomy group and can be used to construct a covariantly constant spinor. The reduced theory will then have half of the supersymmetries of the trivial holonomy case. One can easily check that $SU(2)$ (together with its Cartan subroup) is the only subgroup of $SO(4)$ that contains singlets, apart from the trivial one[3].

11-dimensional supergravities

We now reproduce the same kind of arguments of the case of 11-dimensional supergravity. In this case, there is no Weyl projection, and the minimal spinor representation is **32**. Under $SO(1,10) \to SO(1,5) \times SO(5)$, it decompose as

$$\mathbf{32} \to (\mathbf{4}_{\rm L}, \mathbf{4}) + (\mathbf{4}_{\rm R}, \mathbf{4}) \,, \tag{5.15}$$

where the $\mathbf{4}$ are spinors of $SO(5)$. In eleven dimensions one can also impose Majorana condition and choose the 32 supercharges to be real. Analogous to the ten-dimensional case, by compactifying on an internal five dimensional torus, the four real $\mathbf{4}_L$ and the four real $\mathbf{4}_R$ combine into two complex $\mathbf{4}_L$ and the two complex $\mathbf{4}_R$ and, starting with $\mathcal{N}=1$ in 11-dimensions, the reduced theory in six dimensions has $\mathcal{N}=4$.

Next, we analyse the the cases with reduced holonomy. Under $SO(5) \to SO(4)$, the $\mathbf{4}$ represenation of $SO(5)$ decomposes as $\mathbf{4} \to \mathbf{2} + \bar{\mathbf{2}}$. As discussed in the 10-dimensional case, the $\mathbf{2}$ and $\bar{\mathbf{2}}$ representations of $SO(4)$ contains one singlet each under the two different

[3]Another candidate that has singlets is the diagonal $SU(2)$ subgroup of $SO(4)$. However, in this case, the singlet is a spinor-bilinear and therefore cannot be used as a fermionic parameter for supersymmetry.

$SU(2)$ subgroups of $SO(4)$. Therefore, one can conclude that internal five-dimensional manifolds with $SU(2)$ holonomy also lead to supersymmetric compactifications. In this case, starting with $\mathcal{N}=1$ in eleven dimensions, the reduced theory will have $\mathcal{N}=2$. As we will see in the next section, when specifying to the torus fibration ansatz (5.9), these configurations with $SU(2)$ holonomy correspond to solutions of the form of (5.1).

Finally, by analysing all possible subgroups of $SO(5)$, one can conclude that $SU(2)$ and the trivial group are the only subgroups that preserve some supersymmetry when used as holonomy groups. For the ansatz (5.9), these will imply that the only supersymmetric solutions are the ones in (5.1), as we will encounter in the next sections by explicit calculation.

The above two cases above showed the strong connection between holonomy and supersymmetry preservation. Therefore, in general, it it useful for such analysis to know which subgroups of $SO(n)$ can in fact occur as holonomy groups for Riemannian manifolds. The cases where the manifold is simply connected, not locally a product space and not symmetric where classified by Berger in [138] . We observe that the only five-dimensional manifolds appearing in such classification have $SO(5)$ holonomy group, and they are therefore not supersymmetric. In fact, the only odd dimensional manifolds are G_2-manifolds, which appears in the context of compactifications of 11-dimensional supergravity to four external dimensions and allow constant spinors.

Next, we will analyse in more detail the precise implications of having covariant spinors on the five-dimensional internal manifold. This result will be then applied to the toroidal fibration ansatz (5.9).

Supersymmetric geometric solutions in five-dimensional euclidean spaces

In five euclidean dimensions the smallest spinor representation has 8 real dimensions, which can be combined into two complex four-dimensional spinors λ^a ($a=1,2$) satisfying the symplectic-Majorana condition

$$\overline{\lambda}_a = \epsilon_{ab}(\lambda^b)^T \mathcal{C}\,, \tag{5.16}$$

where $\overline{\lambda}_a$ is the Dirac conjugate and $\mathcal{C}$ is the charge conjugation matrix. If a five-dimensional geometric background without fluxes is supersymmetric, it has at least one doublet of covariantly constant symplectic-Majorana spinors, satisfying

$$\nabla\lambda^a = 0. \tag{5.17}$$

Finding all possible solutions for this equation is in general complicated. Instead, we will analyse what are the implications of the existence of such globally defined spinor field. For this, we will follow the analysis done in [139] for Minkowski signature, and adapt their arguments to the Euclidean space. Out of the spinor λ^a, one can construct the following bosonic fields:

$$\begin{aligned} f\epsilon^{ab} &= \bar{\lambda}^a\lambda^b\,, \\ V_{\bar{\imath}}\epsilon^{ab} &= \bar{\lambda}^a\gamma_{\bar{\imath}}\lambda^b\,, \\ \Phi^{ab}_{\bar{\imath}\bar{\jmath}} &= \bar{\lambda}^a\gamma_{\bar{\imath}\bar{\jmath}}\lambda^b\,, \end{aligned} \tag{5.18}$$

where $\bar{\imath}$ are flat five-dimensional indices, ϵ^{ab} is the Levi-Civita symbol, $\gamma_{\bar{\imath}}$ are five-dimensional euclidean gamma-matrices and $\gamma_{\bar{\imath}\bar{\jmath}} = \gamma_{[\bar{\imath}}\gamma_{\bar{\jmath}]}$. One can check that the scalar f and the 1-form V are real. Out of $\Phi^{ab}_{\bar{\imath}\bar{\jmath}}$, one can also construct three real two-forms as

$$X^{(1)} = \frac{1}{2}\left(\Phi^{11} + \Phi^{22}\right) , \qquad X^{(2)} = \frac{i}{2}\left(\Phi^{22} - \Phi^{11}\right) , \qquad X^{(3)} = i\Phi^{12} . \tag{5.19}$$

The above bosonic quantities are all constructed out of the same spinor field λ^a, which was globally defined by assumption, and they are therefore also globally defined objects. Furthermore, the components of these fields are not independent, but subject to some algebraic relations. Using Fierz identities, one can show that the following relations hold [139]

$$\begin{aligned} V_{\bar{\imath}}V^{\bar{\imath}} &= f^2 , \\ \iota_V X^{(u)} &= 0 , \\ \iota_V \star X^{(u)} &= f X^{(u)} , \\ X^{(u)} \wedge X^{(v)} &= 2\delta_{uv} f \star V , \\ \delta^{\bar{k}_1\bar{k}_2} X^{(u)}_{\bar{\imath}\bar{k}_1} X^{(v)}_{\bar{\jmath}\bar{k}_2} &= \delta_{uv}(f^2\delta_{\bar{\imath}\bar{\jmath}} - V_{\bar{\imath}}V_{\bar{\jmath}}) + \epsilon_{uvw} f X^{(w)}_{\bar{\imath}\bar{\jmath}} , \end{aligned} \tag{5.20}$$

where $u, v = 1, \ldots, 3$ and the vector $V^{\bar{\imath}}$ is constructed by rising the $\bar{\imath}$-index with $\delta^{\bar{\imath}\bar{\jmath}}$ and ι_V is the interior product. The $\star$ operation is defined on vectors as $(\star V)_{\bar{\imath}\bar{\jmath}\bar{k}\bar{l}} = \epsilon_{\bar{\imath}\bar{\jmath}\bar{k}\bar{l}\bar{m}} V^{\bar{m}}$, and on 2-forms as $(\star X)^{\bar{\imath}\bar{\jmath}\bar{k}} = \frac{1}{2}\epsilon^{\bar{\imath}\bar{\jmath}\bar{k}\bar{l}\bar{m}} X_{\bar{k}\bar{l}\bar{m}}$. The fact that all the bosonic quantities in (5.18) and (5.19) are constructed out of a covariantly constant spinor, satisfying (5.17), implies that all of them are closed [139]

$$df = 0 , \qquad dV = 0 , \qquad d(\star V) = 0 , \qquad dX^{(u)} = 0 . \tag{5.21}$$

Next, we choose a local coordinate frame where the vector V is along one of the coordinates, $V = \partial_{\mathsf{x}}$. The metric field in this local patch can be written as

$$ds^2 = f^2(d\mathsf{x} + \omega)^2 + f^{-1}h_{mn}dx^m dx^n , \tag{5.22}$$

where the metric $f^{-1}h_{mn}$ is obtained by projecting the full metric to the space perpendicular to the orbits of V, which we will call base manifold $\mathcal{B}$. The five coordinates on the patch are $x^i = (\mathsf{x}, x^n)$ and the i-indices are now lowered and raised using the metric (5.22) and its inverse.

The 1-form V dual to ∂_{x} is now $V = f^2(d\mathsf{x} + \omega)$. From (5.17) it follows that f has to be a constant and, therefore, the condition $dV = 0$ implies that $d\omega = 0$. Locally, this condition implies that ω can be absorbed into $d\mathsf{x}$, and the metric field then becomes

$$ds^2 = f^2 d\mathsf{x}^2 + f^{-1}h_{mn}dx^m dx^n . \tag{5.23}$$

We next analyse the implications of the other constraints in (5.20) and (5.21) on the two forms $X^{(u)}$. The second constraint in (5.20) implies that $X^{(u)}$ have legs only along the base manifold $\mathcal{B}$. Furthermore, the condition $dX^{(u)} = 0$ implies that they do not have dependence on the coordinate x and that they are also closed from the four-dimensional point of view. The four-form $*V$, where now the $*$ operation is the Hodge dual with respect to the metric field (5.23), can be regarded as a volume form for the base manifold $\mathcal{B}$ and is it follows from (5.21) that it is closed. The two last conditions of (5.20) become

$$\begin{aligned} X^{(u)} \wedge X^{(v)} &= 2\delta_{uv} Vol_{\mathcal{B}}\,, \\ X^{(u)}{}_m{}^p X^{(v)}{}_p{}^n &= -\delta_{uv}\delta_m{}^n + \epsilon_{uvw} X^{(w)}{}_m{}^n\,, \end{aligned} \tag{5.24}$$

which, together with the closure condition, indicate that $X^{(u)}$ define a hyperkähler structure on the four-dimensional manifold $\mathcal{B}$. We note that any hyperkähler manifold has dimension $4k$ and holonomy group $Sp(k)$. In our case, the dimension is four and the holonomy group is $Sp(1) \simeq SU(2)$, as expected from the discussion above.

Supersymmetric geometric U-fects

In the previous discussions, we have analysed in detail the implications of demanding that some supersymmetry is preserved in a general five-dimensional manifold. Now, we will apply them to the ansatz (5.9) in order to obtain all possible M-theory supersymmetric geometric three-torus fibrations.

As one can expect from the previous discussions, the result will be that the only cases where some supersymmetry is preserved are those where one of the circles of the torus is trivially constant along the base. We will find that this are in fact configurations (5.1), which were obtained by lifting T-fect solutions to M-theory. Therefore, only two out of the five geometric moduli can be turned on simultaneously. We remark that this situation can be improved by letting the base to be three-dimensional. This situation was studied for instance in [136, 137]

We will proceed systematically, analysing first the cases when V lives in the toroidal fiber and second the cases where it lives in the base.

V along the fiber

We start by considering the case where $V \sim \partial_{\xi^i}$. In particular, we will discuss the case where $V = e_1 \sim \partial_{\xi^1}$, where e_1 is defined in (5.8). The cases where $V \sim \partial_{\xi^2} \sim e_2 + \frac{\tau_1}{\tau_2} e_1$ and $V \sim \partial_{\xi^3} \sim e_3 + \left(\frac{\sigma_1}{\sigma_3} - \frac{\tau_1\sigma_2}{\tau_2\sigma_3}\right) e_1 + \frac{\sigma_2}{\tau_2\sigma_3} e_2$ can be discussed analogously.

The form dual to e_1 is

$$V = \eta_1 = \frac{1}{(\tau_2\sigma_3)^{1/3}} (d\xi^1 + \tau_1 d\xi^2 + \sigma_1 d\xi^3)\,, \tag{5.25}$$

where τ_1, τ_2, σ_1 and σ_3 are functions of the base coordinates. If one demands this form to be closed, one has to impose that

$$\begin{aligned} \sigma_1 &= \text{constant}\,, \\ \tau_1 &= \text{constant}\,, \\ \tau_2\sigma_3 &= \text{constant}\,. \end{aligned} \tag{5.26}$$

We will implement the last of this conditions by considering $\tau_2 = A^3/\sqrt{\rho_2}$ and $\sigma_3 = \sqrt{\rho_2}$, where A is a constant and ρ_2 a function of the base coordinates. With this conditions, the 1-forms on the torus (5.6) become

$$\begin{aligned} \eta^1 &= \frac{1}{A} d\bar{\xi}^1 \\ \eta^2 &= \frac{1}{A\sqrt{\rho_2}}(A^3 d\xi^2 + \rho_1\, d\xi^3) \\ \eta^3 &= \frac{1}{A\sqrt{\rho_2}}(\rho_2\, d\xi^3) \end{aligned} \tag{5.27}$$

where the coordinate $\bar{\xi}^1$ is defined via the $SL(3)$ transformation $\tilde{\xi}^1 = \xi^1 + \tau_1\xi^2 + \sigma_1\xi^3$. For convenience, we also introduced the function ρ_1 defined as $\rho_1 = \sigma_2\sqrt{\rho_2}$. The parameter A can be absorbed into the coordinates ξ^i by rescaling the radii of the circles. After this considerations, we are only left with two non-trivial moduli, ρ_1 and ρ_2, and we have recovered the T-fect solution (5.1).

V along the base

Next, we analyse the case where V is along the base. We take, for instance,

$$V = \Theta_1 = \frac{1}{2} f(z,\bar{z})(dz + d\bar{z})\,. \tag{5.28}$$

We automatically see that $dV = 0$ implies that f has to be $f = f(z + \bar{z})$. On the other side, we require that the 4-form

$$*V = \frac{1}{2} f(z,\bar{z})(dz - d\bar{z}) \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \tag{5.29}$$

is closed, which implies that $f = f(z - \bar{z})$. The only possibility we have to satisfy both requirements at the same time is that $f(z,\bar{z})$ is constant.

If $f(z,\bar{z})$ is a constant function, the base is flat and there is no back-reaction. Therefore, the only possible way that the metric in the total five-dimensional space is Ricci-flat is that the fibration is trivial.

5.3 Type II configurations

In the above section we found that all possible M-theory U-duality defects are in fact T-fects solutions uplifted to 11-dimensional supergravity. Reducing along the M-theory circle will bring us back to the T-fect configurations in type IIA. However, one can also consider reductions along one of the directions that originally belonged to the two-torus fiber. In this case, RR fields will appear and these configurations, together with T-fects, will constitute a full classification of the possible supersymmetric type IIA toroidal fibrations with $SL(2,\mathbb{Z})$ monodromies. Then, one can also investigate the type IIB case by using T-duality between type IIA and type IIB. This will complete the classification for type II defects with $SL(2,\mathbb{Z})$ monodromy.

In this section we will list all possible such configurations. Some of them will exhibit non-geometric behaviour, mixing the metric field with the diferent NS-NS and RR form fields. Furthermore, some of them will source non-geometric fluxes which are U-dual to RR fluxes (i.e. U-duality generalisations of the non-geometric Q-flux), which are necessary for a complete U-duality invariant description of the lower dimensional effective theory. Some of the solutions we present here, as well as their relations to fluxes, has been considered before in the literature (see for instance [140–142, 90, 143, 144]). With our formulation, we give a common framework to classify all them.

5.3.1 Type IIA configurations

τ- and ρ-fects

As just argued, reducing (5.1) along ξ_3 or (5.2) along any of the directions of the fiber one obtains again τ- and ρ- fects solutions in type IIA respectively.

6-brane U-duality defect

The only different configurations in type IIA one can obtain from M-theory T-fects is by reducing (5.1) along some of the directions of the non-trivial two-torus in the fiber. For example, if one reduces (5.1) along direction ξ_2, one obtains the configuration

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{\tau_2}}dx_{||}^2 + e^{2\phi}\sqrt{\tau_2}dzd\bar{z} + \sqrt{\tau_2}\left(d\xi_1^2 + \frac{1}{\tau_2}d\xi_3^2\right) , \\ C_1 &= \tau_1 d\xi^1 , \\ e^{2\Phi} &= \frac{1}{\tau_2^{3/2}} . \end{aligned} \tag{5.30}$$

Note that this is a 6-brane wrapping the compact direction ξ_3. If one takes τ to encode the monodromy V in the parabolic conjugacy class (see chapter 2), the solution (5.30) is a D6 brane wrapping the direction ξ_3 and smeared along the direction ξ_1 (in string frame). In this case, the effect of the monodromy can be expressed as gauge transformation of the 1-form C_1. Instead, if one considers τ to encode the parabolic monodromy U, the configuration will correspond to the non-geometric 6_3^1 brane (in the notation of [90]). For general elements in any conjugacy class, the monodromy will involve a mixing between the 1-form and the metric, and will lead to non-geometric configurations. Also, note that any other configuration obtained by reducing (5.1) along any other direction of the non-trivial two-torus will lead to a configuration related to (5.30) with an $SL(2)$ rotation acting on τ.

Colliding degenerations

Similarly to the M-theory case, one can now construct further supersymmetric degenerations by colliding a defect with monodromy in the $SL(3,\mathbb{Z})$ factor with one with monodromy in the $SL(2,\mathbb{Z})$ factor. In this context, this implies that, apart from colliding a τ-fect with a ρ-fect, one could also collide the last with a 6-brane U-duality defect.

The fact that defects with different $SL(2,\mathbb{Z})$ orientations within the $SL(3,\mathbb{Z})$ cannot be collide implies that one cannot collide a τ-fect with a 6-brane U-duality defect without breaking all supersymmetries.

5.3.2 Type IIB configurations

With the general classification of U-duality defects in type IIA presented in the previous section, one can now construct all possible type IIB configurations by factorised duality transformations of the former.

τ- and ρ-fects

As already discussed, T-duality acting on type IIA τ-fects give type IIB ρ-fects and vice versa.

RR 5-brane U-fects

For the case of 6-brane U-duality defects, the configurations obtained after factorised duality are different for each toroidal direction. For instance, if one applies Buscher rules along the compact direction of the brane world volume (along ξ_3 in (5.30)), one obtains the configuration becomes

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{\tau_2}}dx_{||}^2 + e^{2\phi}\sqrt{\tau_2}dzd\bar{z} + \sqrt{\tau_2}\left(d\xi_1^2 + d\xi_3^2\right) , \\ C_2 &= \tau_1 d\xi_1 \wedge d\xi_3 \, , \\ e^{2\Phi} &= \frac{1}{\tau_2} \, , \end{aligned} \tag{5.31}$$

which is the transverse space of a five-brane. When the monodromy on τ is the parabolic monodromy V, the configuration is a D5 brane with two smeared transverse directions. The effect of the monodromy can be interpreted as a gauge transformation of the 2-form C_2. Instead, if the monodromy is the parabolic element U, the configuration is the non-geometric 5_3^2 brane. In general, the monodromy will mix the 2-form with the volume of the torus, leading to non-geometric configurations. Also, we notice that this configurations are S-dual to ρ-fects in type IIB.

7-brane U-fects

If one performs a factorized duality along the transverse compact direction of the 6-brane U-duality defects (along ξ_1 in (5.30)), one obtains

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{\tau_2}}dx_{||}^2 + e^{2\phi}\sqrt{\tau_2}dzd\bar{z} + \frac{1}{\sqrt{\tau_2}}\left(d\xi_1^2 + d\xi_3^2\right) , \\ C_0 &= \tau_1 , \\ e^{2\Phi} &= \frac{1}{\tau_2^2} , \end{aligned} \tag{5.32}$$

which in Einstein frame is

$$\begin{aligned} ds^2 &= dx_{||}^2 + e^{2\phi}\sqrt{\tau_2}dzd\bar{z} + \left(d\xi_1^2 + d\xi_3^2\right) , \\ C_0 &= \tau_1 , \\ e^{2\Phi} &= \frac{1}{\tau_2^2} . \end{aligned} \tag{5.33}$$

These configurations are 7-branes wrapping 2 cycles (ξ_1 and ξ_3). The monodromy can now be express in terms of the axio-dilaton parameter $\tau_{\text{a.d.}} = C_0 + ie^{-\Phi}$ and the $SL(2,\mathbb{Z})$ is in fact the type IIB S-duality group. This are in fact local fields for the F-theory seven-branes. In particular, in the parabolic conjugacy classes one encounters the so-called seven-branes.

Colliding degenerations

Finally, following an analogous argumentation as in type IIA, we study supersymmetric collisions of U-duality defects. We conclude that it ρ-fects, RR 5-brane and 7-brae U-duality defects can be collided with τ-fects but not among themselves.

Chapter 6

Half-supersymmetric compactifications from $SO(5,5)$ Exceptional Field Theory

In the previous chapter, we have studied supersymmetric three-torus fibrations in M-theory, with special emphasis to the case without fluxes, as well as two-torus fibrations in type II theories. These where the natural generalisations of the heterotic two-torus fibrations constructed in chapter 2 and, in fact, all configurations we encountered in chapter 5 could be related to those in chapter 2 using some duality. As pointed out in chapter 2, such toroidal fibrations could be understood as local descriptions in a region of a global model close to a degeneration. For this reason, together with the fact that supersymmetry is closely related with global aspects of the configurations, the discussion in chapter 5 was more general, including any possible compactification to six dimensions without fluxes.

The next step will be to include fluxes in the internal space. As argued in the introduction, a natural framework to study flux compactifications of M- and type II theories is Exceptional Field Theory, since it unifies the metric and the form fields degrees of freedom into a covariant formulation. In this chapter, we will review and derive the necessary ingredients to describe half supersymmetric configurations in the context of $SO(5,5)$ EFT, which is the relevant duality group for our discussion. In the last section, we will apply the results to the three-torus fibrations of chapter 5, reproducing the results we found there. The techniques developed here will be also used in chapter 7 to study compactifications to AdS_6 vacua. Standard references for $SO(5,5)$ Exceptional Field Theory include [145,146] and for supersymmetric structures in EFT [147–151]. Some of the results in this section will appear in [152].

6.1 Review of $SO(5,5)$ EFT

In section 1.5, we introduced the general concepts for Exceptional Field Theories. Here we want to use this formalism to study compactifications of type II and M- theories to six external directions. We will begin reviewing the most important aspects of this EFT, which will be used in the rest of this chapter as well as in chapter 7.

The U-duality group for six external dimensions is $SO(5,5)$ (or rather its double cover $Spin(5,5)$), whose maximal compact subgroup is $USp(4) \times USp(4)$ (see Table 1.1). The diffeomorphism and gauge parameters organise into a generalised vector transforming in the fundamental representation of $Spin(5,5)$, corresponding to the **16** of $SO(5,5)$. In M-theory compactifications, the five-dimensional internal manifold K is extended to a 16-dimensional space, an analogous for type II compactifications. For the rest of this chapter, we use the conventions in Appendix D.

The infinitesimal parameters of the local symmetries (diffeomorphisms and gauge transformations) can be encoded into a **16** representation Λ^N. The action of the Lie derivative along Λ on a vector V^N in the **16** with weight $\lambda = 1/4$ is given by [151]

$$\mathcal{L}_\Lambda V^N = \Lambda^M \partial_M V^N - V^M \partial_M \Lambda^N + \frac{1}{2}(\gamma_I)^{NM}(\gamma^I)_{PQ} V^P \partial_N \Lambda^Q \,, \tag{6.1}$$

where $N = 1, \ldots, 16$ is the **16** index and $I = 1, \ldots, 10$ is an index in the **10**, which can be lowered and risen by the $O(5,5)$ metric η_{IJ}. From (6.1), one can read off the Y-tensor

$$Y^{NM}_{PQ} = \frac{1}{2}(\gamma_I)^{NM}(\gamma^I)_{PQ} \,, \tag{6.2}$$

where γ^I are the $SO(5,5)$ gamma matrices satisfying the Clifford algebra

$$\{\gamma_I, \gamma_J\} = 2\eta_{IJ}\delta^M_N \,. \tag{6.3}$$

For the upcoming discussion, it will be necessary to consider Lie derivatives acting on objects in the **10** of $SO(5)$ and with weight $\lambda = 1/2$. This will be given by [151]

$$\mathcal{L}_\Lambda V^I = \Lambda^M \partial_M V^I + \frac{1}{2}(\gamma_J \gamma^I)_M{}^N V^J \partial_N \Lambda^M \,. \tag{6.4}$$

Generalised metric

As mentioned in section 1.5, one of the central obects of EFT is the generalised metric, which encodes all the degrees of freedom on the internal manifold into a representation of the duality group[1]. In $SO(5,5)$ EFT, the generalised metric parametrises the cosset

$$\frac{SO(5,5)}{USp(4) \times USp(4)} \,. \tag{6.5}$$

Since all possible generalised metrics will be connected to the identity, they can be generically constructed by exponentiating elements of the $so(5,5)$ algebra. As reviewed in [157],

[1]Similarly, all components of the fields with legs both on the internal and external space will also organise in representations of the duality group. This fields appear in the tensor hierarchy of maximal gauged supergravity [153–156], but will remain turned off for the discussion in this chapter.

one can construct any element in the cosset (6.5) in any representation $\mathcal{R}$ by considering only the elements in the Cartan subalgebra together with the positive roots generators. In particular, one can construct the generalised vielbein $E_{\bar{\mathbb{N}}}{}^{\mathbb{N}}$, related to the generalised metric via $\mathcal{M}_{\mathbb{N}\mathbb{M}} = E_{\mathbb{N}}{}^{\bar{\mathbb{N}}}\, E_{\mathbb{M}}{}^{\bar{\mathbb{M}}}\, \delta_{\bar{\mathbb{N}}\bar{\mathbb{M}}}$, as

$$E_{\bar{\mathbb{N}}}{}^{\mathbb{N}} = \left[\exp\left(\sum_{\rho\in\mathfrak{h}} h_\rho K_\rho\right)\exp\left(\sum_{\alpha\in\Delta_+} h_\alpha K_\alpha\right)\right]_{\bar{\mathbb{N}}}{}^{\mathbb{N}}\,, \tag{6.6}$$

where $\mathfrak{h}$ is the Cartan subalgebra and Δ_+ the space of positive root generators. $K_\bullet$ denote generators of the algebra in the generic representation $\mathcal{R}$, $\mathbb{N} = 1, \ldots, \mathrm{dim}\mathcal{R}$, and the parameters h_ρ and h_α are eventually related to the vielbein and form fields of the internal manifold. In appendix D, we use this construction to obtain the generalised metric in the M-theory and type IIB solutions of the section constraint in both the**16** and the **10** representations.

Solutions to the section constraint

As already mentioned, the space of $SO(5,5)$ EFT is a product manifold with six external directions and 16 internal ones. The dynamics along the later are however restricted by the section condition

$$(\gamma^I)_{MN}(\gamma_I)^{PQ}\,\partial_P\, \bullet\; \partial_Q \bullet = 0 \tag{6.7}$$

for all M, N and for any field introduced in $\bullet$. When solving this constraint, one effectively breaks the $SO(5,5)$ symmetry of the extended space to some subgroup. As described in detail in [146], there are two inequivalent ways of solving it:

- One possibility is to break $SO(5,5)$ into $SL(5)\times\mathbb{R}^+$. In this case, one can still have dependence in five coordinates, which can be organised in the fundamental representation of $SL(5)$, and one can see that this restores the coordinate content of 11-dimensional supergravity.

- Another option is to break $SO(5,5)$ into $SL(4)\times SL(2)\times SL(2)$, in a way that the $SL(4)$ is not a subgroup of the $SL(5)$ used above. In this case, one can only keep dependence on four of the coordinates, and one restores the coordinate dependence of type IIB supergravity, where the S-duality group corresponds to one of the $SL(2)$ factors. The extra $SL(2)$ factor is accidental and is in fact broken when choosing the directions of the geometric four-dimensional coordinates.

We want to remark the fact that, from a single Exceptional Field Theory, one can recover both M- and type IIB theories, which is the reason why EFT provides a naturally unified framework for these two theories. More details on these solutions to the section constraint can be found in Appendix D.

6.2 Flux compactifications and EFT

In section 5.2 we analysed the implications of supersymmetry preservation on purely geometric manifolds. In this chapter we retake this discussion and generalise it by adding fluxes to the internal manifold. We will first give some considerations at the level of supergravity, and then discuss how EFT can be used to study such compactifications.

6.2.1 Compactifications with fluxes and G-structures

In this chapter we want to discuss flux compactifications of 11-dimensional and type II supergravities. The precise content flux content will depend on the theory we consider, but in general the gravitino variation (5.13) used in chapter 5 will be now modified to the general form

$$\delta_\varepsilon \Psi \quad = \quad (\nabla + \text{flux terms})\,\varepsilon\,, \tag{6.8}$$

where "flux terms" indicates contractions of the different form fields with gamma-matrices. In type II supergravities, the presence of fluxes also makes the variation of the dilatino non-trivial. As a concrete example, where only NS fields are turned on, see for instance (4.14). For the precise general form of the supergravity variations in type II and 11-dimensional supergravity we refer, for instance, to [158, 9].

One implication of the variations (6.8) is that the flux terms twist the spinor field and ε is no longer a covariantly constant spinor[2]. Therefore, the analysis of the holonomy group of the manifold is no further useful. However, there are still some necessary requirements that our manifold has to satisfy in order to be supersymmetric. In particular the spinor field ε needs to be globally defined, which is again a requirement that impose topological restrictions to the manifold. In this case, one can analyse them in terms of the structure group G of the internal manifold, which is the set of all linear transformations that can be used to glue sections of the tangent frame bundle (vielbeins) in the overlap of two patches. Manifolds with a structure group G are said to have a G-structure and all tensor fields will transform as representations of G. In general, for a d-dimensional riemannian manifold, its structure group will be $G \subseteq SO(d)$. If G is maximal, all spinor fields will twist non-trivially in the overlap of patches and we will have no globally defined sections of the spinor bundle. Therefore, a necessary requirement to preserve some supersymmetry is that the structure group G of the internal manifold is some subgroup in which the decomposition of the $SO(d)$ spinors contains singlets. Then, analogously to the analysis in section 5.2 for the holonomy groups, we can conclude that, in the cases of type II and 11-dimensional supergravities compactified down to six external directions, half of the supercharges can be preserved if the internal manifold has $G = SU(2)$ and all if the structure group of the internal manifold is trivial.

[2] The covariant derivative in (6.8) is defined with respect to the torsion-free Levi-Civita connection. On manifolds with G-structures considered below, one can always find a connection ∇' such that $\nabla'\varepsilon = 0$. Such connection will in general have torsion. Reduced holonomy as the one considered in the previous chapter is then equivalent to consider torsion-free G-structures.

The analysis of flux compactifications is important from the phenomenological point of view, since fluxes induce potentials in the lower dimensional effective theories that help stabilising the vevs of the scalar fields coming from objects living in the internal manifold (for reviews of flux compactifications and moduli stabilisation see for instance [9,10]). Here, we will now take a different path and reformulate the whole discussion in the context of Exceptional Field Theory. This will allow us to rewrite the Killing spinor conditions in a way where the symmetries of the theory become evident, which will give a valuable tool to find solutions, as we will see in section 6.4 and specially in chapter 7.

6.2.2 Exceptional G-structures in $SO(5,5)$ EFT

As discussed in sections 1.5 and 6.1, one key feature of Exceptional Field Theories and Exceptional Generalised Geometries is the fact that the bosonic fields of supergravity theories, that originally are $GL(d)$ tensors, are encoded into sections of generalised bundles, whose fibers are $E_{d(d)} \times \mathbb{R}^+$ tensors. In the same way that spinors in a d-dimensional manifolds are representations of $SO(d) \subset GL(d)$, one can now consider spinors in the extended spaces as representations of the maximal compact subgroup $H_d \subset E_{d(d)}$. In fact, it has been shown that the flux terms of (6.8) indeed generate an H_d action on spinors [148, 147, 149, 159].

One can therefore rephrase the relation between G-structures and globally defined spinors in the context of Exceptional Field Theory: a necessary condition to have globally defined spinors is that the generalised structure group, i.e. the structure group of the generalised tangent bundle, is a subgroup of H_d in which the decomposition of H_d spinors contains singlets [150]. Furthermore, demanding that the reduction yields to a Minkowski vacua impose certain differential constraints on the structure. In particular, we need that the internal manifold has vanishing generalised intrinsic torsion and the manifold is said to have "generalised special holonomy" [150], in analogy to the supergravity case.

This formalism can then be used to study flux compactifications using a logic analogous to geometric compactifications discussed in section 5.2: out of the globally defined generalised spinor on the internal manifold one can construct bosonic objects which will also transform as singlets and will define the (generalised) geometry of the internal manifold.

Alternatively, once one has fixed the generalised G-structure group that stabilise some spinor, one can directly identify the corresponding bosonic objects that will define the geometry by finding which representations contain singlets under such decomposition. In section 5.2, we found that the different bosonic fields characterising the structure were not independent, but related by a set of constraints (see (5.20)). Here, the analogous algebraic constraints will be those conditions that uniquely identify the singlets under the decomposition from the rest. They are constructed by taking products of the different representations and analysing the resulting field under the decomposition with respect to the generalised G-structure group.

$SU(2) \times SU(2)$ generalised structures

Let us now turn to the concrete case of half supersymmetric compactifications with six external directions and work out the fields that define their generalised structure, following the discussion in [151]. As already mentioned, the duality group of the internal space is $SO(5,5)$, and its maximal compact subgroup is $USp(4) \times USp(4)$. We will be interested in the case where the generalised G-structure group is $SU(2) \times SU(2) \subset USp(4) \times USp(4)$, which will lead to non-chiral $\mathcal{N} = (1,1)$ supergravities for the M-theory section and generalise the discussion in chapter 5 [3]. We consider then the branching rules of

$$SO(5,5) \to \Big(SU(2) \times SU(2)\Big) \times \Big(SU(2) \times SU(2)\Big)\,, \tag{6.9}$$

under which the lowest dimensional representations of $SO(5,5)$ decompose as

$$\begin{aligned} \mathbf{1} &\to (1,1,1,1)_0\,, \\ \mathbf{10} &\to (2,2,1,1)_0 \oplus (1,1,2,2)_0 \oplus (1,1,1,1)_2 \oplus (1,1,1,1)_{-2}\,, \\ \mathbf{16} &\to (2,1,2,1)_1 \oplus (2,1,1,2)_{-1} \oplus (1,2,2,1)_{-1} \oplus (1,2,1,2)_1\,. \end{aligned} \tag{6.10}$$

where the subscripts correspond to $U(1)$ charges. We now need to identify the singlets under the subgroup $SU(2) \times SU(2)$ obtained by taking one factor from each bracket in the right hand side of (6.9). We observe that such fields will organise into representations of the remaining $SU(2) \times SU(2) \simeq SO(4)$, which is the R-symmetry group of our compactification. We conclude that the generalised structure will be defined by [151]

- One scalar field κ, which is also a $\mathbf{1}$ of the $SO(4)_R$.
- One vector field K in the $\mathbf{10}$ of $SO(5,5)$, which is a $\mathbf{1}$ of $SO(4)_R$.
- Another vector field $\hat{K}$ in the $\mathbf{10}$ of $SO(5,5)$, again in the $\mathbf{1}$ of $SO(4)_R$.
- Four vector fields J_u, $u = 1, \ldots, 4$, in the $\mathbf{16}$ of $SO(5,5)$, organised into a $\mathbf{4}$ of $SO(4)_R$.

If one includes into the discussion the $\mathbb{R}^+$ group of the trombone symmetry mentioned in section 1.5, this objects naturally accomodate in bundles with weight $\lambda = 1/4$ for the case of κ and J_u, and $\lambda = 1/2$ for K and $\hat{K}$. Furthermore, as already mentioned, the fields defining the generalised structure cannot be independent, but they need to be related by some algebraic constraints. For the current case, these are [151]

$$\begin{aligned} K^I K_I = \hat{K}^I \hat{K}_I &= 0\,, \\ K^I \hat{K}_I &= \kappa^4\,, \\ \forall N: \quad J_u^M (\gamma^I)_{MN} K_I &= 0\,, \\ \frac{1}{2} J_u^M J_v^N (\gamma^I)_{MN} &= \delta_{uv} K^I\,. \end{aligned} \tag{6.11}$$

[3] There is still another subgroup of $USp(4) \times USp(4)$ that can be used as structure group for half-supersymmetric compactifications: $USp(4) \subset USp(4) \times USp(4)$. Compactifications of M-theory using this G-structure lead to chiral $\mathcal{N} = (2,0)$ supergravities in six dimensions, which will not be furhter discussed in this thesis. For its generalised structure we refer to [151].

The logic behind this constraints is the following: take for instance the two fields in the **10** of the G-structure group, K and $\hat{K}$. Contracting them among themselves or between each other using the $O(5,5)$ metric, one obtains objects that are singlets on $SU(2) \times SU(2)$. However, if we look at the $U(1)$ weights of the decomposition (6.10), one can easily see that the weight of the field κ can be only obtained by contracting K with $\hat{K}$ (the power 4 in the first equation of (6.11) is there to match the trombone symmetry weights). Then, the first two conditions of (6.11) impose that K and $\hat{K}$ are not just arbitrary fields in the **10** of $SO(5,5)$, but precisely those decomposing as $(1,1,1,1)_2$ and $(1,1,1,1)_{-2}$ respectively.

Analogous logic can be used to construct the rest of equations. In particular, in the last condition, two fields in the **4** of the R-symmetry group are contracted. The result is proportional to δ_{uv}, which is the only $SO(4)$-invariant with such index structure. Finally, one could also wonder why we did not analyse higher dimensional representations of $SO(5,5)$. In fact, by studying them and using the same logic we used for the algebraic constraints, one can convince himself that any other singlet in higher dimensional representations will be uniquely determined by constraints involving fields in the set $\{\kappa, K, \hat{K}, J_u\}$.

Finally, if one wants to obtain Minkowski vacua, one needs to impose that the internal manifold has vanishing intrinsic torsion. This condition can be phrased as a set of differential conditions on the structure fields, namely [151]

$$\begin{aligned} \forall M: \qquad (\gamma_I)^{MN}\partial_N K^I &= (\gamma_I)^{MN}\partial_N \hat{K}^I = 0\,, \\ \forall u: \qquad \mathcal{L}_{J_u}\hat{K}^I &= 0\,, \\ \forall u,v\ \forall M: \qquad \mathcal{L}_{J_u} J_v^M &= 0\,. \end{aligned} \tag{6.12}$$

Such set of conditions are usually referred to as integrability constraints.

6.3 The half supersymmetric generalised metric in $SO(5,5)$ EFT

In the analysis of section 5.2 the globally defined bosonic tensor fields (5.18) constructed out of the covariantly constant spinor completely determined the geometry of the background. This is also a general fact of Calabi-Yau manifolds, where the metric can be determined from the globally defined symplectic and complex structures. Here we expect a similar thing to happen, namely that the generalised metric of the internal manifold can be determined out of the generalised sections $\{\kappa,\ J_u,\ K,\ \hat{K}\}$. This is indeed possible. In particular, in the **10** representation the metric is given by

$$\mathcal{M}_{IJ} = \kappa^{-4}\left(\frac{\kappa^{-4}}{4!}\epsilon^{uvwx}(\gamma_{IK})_M{}^N(\gamma_J{}^K)_P{}^Q J_u{}^M \hat{J}_{v,N} J_w{}^P \hat{J}_{x,Q} + K_I K_J + \hat{K}_I \hat{K}_J\right)\,, \tag{6.13}$$

where $\gamma_{IK} = \gamma_{[I}\gamma_{K]}$ and we are using the conventions in Appendix D. For convenience, we have introduced the field $\hat{J}_u$, which is a section of a bundle whose fiber transforms in the $\overline{\mathbf{16}}$ representation and is constructed out of J_u and $\hat{K}$ as [151]

$$\hat{J}_{u,M} = \frac{1}{2}(\gamma^I)_{MN} J_u{}^N \hat{K}_I\,. \tag{6.14}$$

One can check that, by construction, $\mathcal{M}_{IJ}$ is symmetric and an $SO(5,5)$ matrix.

Similarly, one can also construct a generalised metric in the **16** representation in terms of $\{\kappa,\, J_u,\, K,\, \hat{K}\}$ as

$$\begin{aligned}\mathcal{M}_{MN} &= \frac{1}{\sqrt{2}}\Big(-\frac{\kappa^{-6}}{4!}\epsilon^{uvwx}(\gamma_I)_{MP}(\gamma_J)_{NQ}(\gamma^{IJ})^S{}_R J_u{}^P J_v{}^Q J_w{}^R \hat{J}_{x,S} \\ &\quad +4\kappa^{-6}\hat{J}^u{}_M \hat{J}_{u,N} - \kappa^{-2}(\gamma^I)_{MN}\hat{K}_I\Big)\,. \end{aligned} \tag{6.15}$$

The generalised metric in the **10** and **16** representations are compatible in the sense that they satisfy the relation

$$\mathcal{M}_{IJ}(\gamma^J)^{PQ}\mathcal{M}_{PM}\mathcal{M}_{QN} = (\gamma_I)_{MN}\,. \tag{6.16}$$

G-structures from the generalised vielbein in the M-theory solution of the section constraint

If the generalised metric for half-supersymmetric is determined by the generalised structures, it is also reasonable to think that one can construct the generalised structure out of the vielbein fields. Such expressions for the generalised structures give a way to prove that expressions (6.13) and (6.15) are correct.

Let us do this checking using fields in the M-theory solution to the section constraint. Using the conventions in Appendix D one can construct the following fields:

$$\begin{aligned} K^I &= \kappa^2\, E^{\bar{5}\,I}\,, \\ \hat{K}^I &= \kappa^2\, E_{\bar{5}}{}^I\,, \\ J_u{}^N &= \frac{\kappa}{2^{1/4}} E_{\bar{u}}{}^N + 2^{1/4}\kappa\, E^{\bar{u}\bar{5}\,N}\,, \qquad u = 1, \ldots 4 \end{aligned} \tag{6.17}$$

where $E_{\bar{I}}{}^I$ and $E_{\bar{N}}{}^N$ are the vielbeins in the **10** and **16** representations in terms of the M-theory fields (see (D.16) and (D.25) in Appendix D) and κ is here a density field of weight 1/4, for instance $\kappa^4 = \sqrt{g}$, with g the determinant of the normal metric on the five dimensional internal manifold.

One can check that structures defined in (6.17) indeed satisfy the algebraic constraint (6.11). Furthermore, plugging them into the expressions (6.13) and (6.15), one recovers the original generalised metrics, namely $\mathcal{M}_{NM} = E_N{}^{\bar{N}} E_M{}^{\bar{M}}\delta_{\bar{N}\bar{M}}$ and analogous for $\mathcal{M}_{IJ}$.

As final remark, let us mention that (6.17) is not necessarily the only way of constructing generalised structures out of the generalised vielbeins. Also, for (6.17) to be really structures, one has to be sure that they are globally defined, which a priori is not a requirement for the generalised vielbeins. These issues do not affect however our checking of the expressions (6.13) and (6.15).

6.4 An Example: Geometric half-supersymmetric U-duality defects in M-theory

We will now use the formalism described in the previous sections to study the problem of half supersymmetric toroidal fibrations in 11-dimensional supergravity analysed in chapter 5. We will encounter that, in the absence of fluxes, the EFT formalism will reproduce the results obtained from the analysis of special holonomy in supergravity, as expected. This example will also give us a first experience with the techniques that will be widely used in next chapter to discuss compactifications to AdS spaces.

We will then work using the M-theory section of $SO(5,5)$ EFT and using the conventions from Appendix D. The fields defining the generalised structure will then split into

$$\begin{aligned} \kappa^4 &= \star\kappa_{(5)} \\ K^I &= \omega_{(1)} + \omega_{(4)}\,, \\ \hat{K}^I &= \hat{\omega}_{(1)} + \hat{\omega}_{(4)}\,, \\ J_u &= v_u + \omega_{(2),u} + \omega_{(5),u} \end{aligned} \tag{6.18}$$

where the subscripts (n) indicate that the object transform as a space-time n-form, and the sums in the right hand side of the expressions have to be understood as a formal sums. For instance K^I has two type of components, (K^i, K_i), and K_i transforms under generalised derivatives as a 1-form and K^i as a dual of a 4-form; the forth power of κ transforms as a density, or as the unique component of a 5-form in five dimensions; and analogously for the rest. In the following, we will use this language to rewrite the constraints (6.11) and (6.12) in terms of the M-theory fields and solve them for the geometric three-torus fibration analysed in section 5.2.2, reproducing the results obtained there. We will first parametrise the fields in K and $\hat{K}$ and then solve the constraints for the generalised vectors J_u.

Parametrising K and $\hat{K}$

In terms of the M-theory fields, the first two equations of (6.11) imply that

$$\begin{aligned} \omega_{(1)} \wedge \omega_{(4)} = \hat{\omega}_{(1)} \wedge \hat{\omega}_{(4)} &= 0\,, \\ \omega_{(1)} \wedge \hat{\omega}_{(4)} + \hat{\omega}_{(1)} \wedge \omega_{(4)} &= \kappa_{(5)}\,. \end{aligned} \tag{6.19}$$

Any general solution to these equations has to fit in one of the following situations:

(a) $\omega_{(1)} = \hat{\omega}_{(4)} = 0$ and $\hat{\omega}_{(1)} \wedge \omega_{(4)} = \kappa_{(5)}$, or
$\hat{\omega}_{(1)} = \omega_{(4)} = 0$ and $\omega_{(1)} \wedge \hat{\omega}_{(4)} = \kappa_{(5)}$.

(b) $\omega_{(1)} = 0$, $\hat{\omega}_{(4)} = \hat{\omega}_{(1)} \wedge \hat{\Omega}_{(3)}$ and $\hat{\omega}_{(1)} \wedge \omega_{(4)} = \kappa_{(5)}$, or
$\hat{\omega}_{(1)} = 0$, $\omega_{(4)} = \omega_{(1)} \wedge \Omega_{(3)}$ and $\omega_{(1)} \wedge \hat{\omega}_{(4)} = \kappa_{(5)}$.

(c) $\omega_{(4)} = 0, \;\; \hat{\omega}_{(4)} = \hat{\omega}_{(1)} \wedge \hat{\Omega}_{(3)}$ and $\omega_{(1)} \wedge \hat{\omega}_{(1)} \wedge \hat{\Omega}_{(3)} = \kappa_{(5)}$, or

$\hat{\omega}_{(4)} = 0, \;\; \omega_{(4)} = \omega_{(1)} \wedge \Omega_{(3)}$ and $\hat{\omega}_{(1)} \wedge \omega_{(1)} \wedge \Omega_{(3)} = \kappa_{(5)}$.

(d) $\omega_{(4)} = \omega_{(1)} \wedge \Omega_{(3)}, \;\; \hat{\omega}_{(4)} = \hat{\omega}_{(1)} \wedge \hat{\Omega}_{(3)}$ and $\omega_{(1)} \wedge \hat{\omega}_{(1)} \wedge (\hat{\Omega}_{(3)} - \Omega_{(3)}) = \kappa_5$,

where $\Omega_{(3)}$ and $\hat{\Omega}_{(3)}$ are arbitrary non-zero three-forms. Furthermore, the first integrability condition in (6.12) implies that

$$d\omega_{(1)} = d\omega_{(4)} = d\hat{\omega}_{(1)} = d\hat{\omega}_{(4)} = 0\,. \tag{6.20}$$

The discussion was so far general. We will now turn to analyse geometric fibrations of the general ansatz (5.9). In section 5.2.2 we found that any such half-supersymmetric configuration had at least one closed one-form whose dual four-form was also closed. This condition, which is reproduced by the situation (a), already restricted our ansatz to be a geometric T-fect solution of the form (2.53), which were fully classified in chapter 2, times a trivial circle. Cases (b) and (c) add to this situation an extra closed one- or four-form, which are not proportional to the original one- and four-forms. These situations can only restrict further our ansatz, leading to trivial fibrations. A similar thing happens for situation (d), where we have two one-forms and two four-forms. These conclusions change completely if one allows that forms come also from background fluxes, but this will not be analysed in the present discussion.

Next, we will analyse what are the conditions imposed on J_u by the compatibility and integrability constraints in both of the situations described in (a). As expected, we will encounter that in one of the cases J_u includes three two-forms defined on the base manifold that define an hyperkähler structure on it.

Parametrising J_u for the case $\omega_{(1)} = \hat{\omega}_{(4)} = 0$

Fixing $\omega_{(1)} = \hat{\omega}_{(4)} = 0$ the algebraic constraints (6.11) on J_u can be written, in terms of the M-theory fields, as

$$\begin{aligned} \iota_{\star\omega_{(4)}}\omega_{(2),u} &= 0\,, \\ v_u{}^{[i}(\star\omega_{(4)})^{j]} &= 0\,, \\ \sqrt{2}\left(\iota_{v_{(u|}}\omega_{(5),|v)} + \omega_{(2),u}\wedge\omega_{(2),v}\right) &= \delta_{uv}\,\omega_{(4)}\,, \\ \iota_{v_{(u|}}\omega_{(2),|v)} &= 0\,, \end{aligned} \tag{6.21}$$

where $(u|\,|v)$ indicate symmetrisation in u and v and the equations hold $\forall u, v = 1, \ldots, 4$. The first two and the last equation are solved by taking v_u to be either zero or in the same direction of $\star\omega_{(4)}$, and with $\omega_{(2),u}$ having no legs along this direction. Using the coordinate frame from chapter 5.2, this implies that $v_u \sim \partial_\lambda$, and the two-forms $\omega_{(2),u}$ are defined on $\mathcal{B}$, whose volume form is proportional to $\omega_{(4)}$. The third equation in (6.21) can be solved by taking

$$\begin{aligned} \text{For } u=4: &\quad v_4 = \partial_\lambda\,, \quad \omega_{(5),4} = \frac{1}{\sqrt{2}} Vol_5\,, \quad \omega_{(2),4} = 0 \\ \text{For } u=1,2,3: &\quad v_u = 0\,, \quad \omega_{(5),u} = 0\,, \quad \omega_{(2),u} = \frac{1}{2^{3/4}} X^{(u)} \end{aligned} \tag{6.22}$$

where $X^{(u)}$ are the two-forms defined in (5.19) and the constraints (6.21) with such parametrisation imply the conditions (5.24), defining an (almost) hyperkähler structure on the base manifold $\mathcal{B}$. Note that the choice of encoding ∂_λ in $J_{u=4}$ is arbitrary, but any other choice can be related to this with an R-symmetry rotation.

Finally, we analyse the last two differential conditions from (6.12), which in terms of the M-theory fields read

$$\begin{aligned} L_{v_u} v_v &= 0 \\ L_{v_u}\omega_{(2),v} - \iota_{v_v} d\omega_{(2),u} &= 0\,, \\ L_{v_u}\omega_{(5),v} + \omega_{(2),v}\wedge d\omega_{(2),u} &= 0\,, \\ \hat{\omega}_{(1)}\wedge d\omega_{(2),u} &= 0\,, \\ L_{v_u}\hat{\omega}_{(1)} &= 0\,, \end{aligned} \tag{6.23}$$

where L_v refers to the usual Lie derivative with respect to a vector v. Because v_u is either ∂_λ or zero, all Lie derivatives in (6.23) vanish if non of the forms has coordinate dependence on λ. The rest are solved by taking

$$d\omega_{(2),u} = 0\,, \tag{6.24}$$

for all u, which implies that the hyperkähler structure is integrable. With this parametrisation we therefore reproduce the results found in section 5.2.2.

Furhtermore, once we know the generalised structures describing the manifold, one can obtain the background fields by constructing the generalised metric (6.13) or (6.15) and then read off the corresponding supergravity fields using the dictionaries derived in Appendix D. The result is that the metric on the five dimensional internal manifold is given by

$$g_{ij} = \frac{\sqrt{2}}{3!\,\kappa}\epsilon^{uvw}\epsilon^{klmn}\,\omega_{(2),u,ik}\,\omega_{(2),u,jl}\,\omega_{(2),w,mn}\,, \tag{6.25}$$

and the three-form flux vanishes.

Parametrising J_u for the case $\hat{\omega}_{(1)} = \omega_{(4)} = 0$

We now reproduce the same analysis for the situation where $\hat{\omega}_{(1)} = \omega_{(4)} = 0$. In this case, the algebraic constraints (6.11) for the components of J_u are

$$\begin{aligned} \omega_{(5),u} &= 0\,, \\ \omega_{(2),u} \wedge \omega_{(1)} &= 0\,, \\ \iota_{v_u}\omega_{(1)} &= 0\,, \\ \omega_{(2),u} \wedge \omega_{(2),v} &= 0\,, \\ -2\iota_{v_{(u|}}\omega_{(2),|v)} &= \delta_{uv}\,\omega_{(1)}\,. \end{aligned} \tag{6.26}$$

The second condition can be solved by taking $\omega_{(2),u} = \omega_{(1)} \wedge \Omega_{(1),u}$ for any one-form $\Omega_{(1),u}$, which automatically satisfies the fifth. The last condition becomes

$$\begin{aligned} \iota_{v_u}\Omega_{(1),u} &= 1\,, \qquad \forall u\,, \\ \iota_{v_u}\Omega_{(1),v} &= -\iota_{v_v}\Omega_{(1),u}\,, \qquad \text{for } u \neq v\,, \end{aligned} \tag{6.27}$$

which can only be solved if the four vectors v_u are non-zero and linear independent. To see this, consider that $v_4 = \sum_{i=1}^{3} a_i v_i$ for some functions a_i. Then, using the relations in (6.27),

$$\iota_{v_4}\Omega_{(1),1} = \sum_{i=1}^{3} a_i \iota_{v_i}\Omega_{(1),1} = a_1 - \iota_{v_1}(a_2\,\Omega_{(1),2} + a_3\,\Omega_{(1),3})\,, \tag{6.28}$$

$$\iota_{v_1}\Omega_{(1),4} = \frac{1}{a_1} + \frac{1}{a_1}\iota_{v_4}(a_2\,\Omega_{(1),2} + a_3\,\Omega_{(1),3}) = \frac{1}{a_1} + \iota_{v_1}(a_2\,\Omega_{(1),2} + a_3\,\Omega_{(1),3}) + \frac{a_2^2}{a_1} + \frac{a_3^2}{a_1}\,,$$

and, since $\iota_{v_4}\Omega_{(1),1} = -\iota_{v_1}\Omega_{(1),4}$, one would need that a_i satisfy

$$a_1^2 + a_2^2 + a_3^2 = -1\,, \tag{6.29}$$

which cannot be solved. The third condition of (6.26) indicates that all four vectors v_u lie in the kernel of the closed one-form $\omega_{(1)}$, which implies that they are a base of the tangent space of the submanifold $\mathcal{B}$ defined in section 5.2.2. Since v_u cannot have zeros, otherwise at some point the first relation in (6.27) would not be satisfied, the submanifold $\mathcal{B}$ is parallelisable.

Finally, we look at the differential conditions, which in terms of the current parametrisation read

$$\begin{aligned} L_{v_u}v_v &= 0 \\ L_{v_u}\omega_{(2),v} - \iota_{v_v}d\omega_{(2),u} &= 0\,, \\ \omega_{(2),v} \wedge d\omega_{(2),u} &= 0\,, \\ L_{v_u}\hat{\omega}_{(4)} &= 0\,. \end{aligned} \tag{6.30}$$

The first of this conditions implies that the the globally defined vector fields on $\mathcal{B}$ form an abelian algebra. In our ansatz (5.9), this automatically implies that the fibration is trivial, and all other conditions are automatically satisfied.

Chapter 7

Half-supersymmetric type IIB AdS_6 vacua from $SO(5,5)$ Exceptional Field Theory

Exceptional field theory is a very natural formalism to study type II and M-theory compactifications, since it unifies metric and flux degrees of freedom into the same representation. Therefore, the analysis of generalised structures in EFT in chapter 6 provides a powerful tool to study supersymmetric flux compactifications. In section 6.4 we could already taste it by applying the formalism to the case of toroidal fibrations of chapter 5. In this chapter, we will use these techniques to study more involved compactifications. In particular we will construct and classify all possible flux compactifications of type IIB leading the AdS_6 vacua. Such classification is actually known from the work in [160–162] (for previous work, see also references within), but our formulation will be in terms of natural geometric objects, making it much simpler. Furhtermore, such a geometric formulation will allow us to establish necesarily conditions for the most general consistent truncations with vector multiplets around the previously found vacua. This are important tools for the holographic study of these vacua. The results in this chapter will appear in [152], where also a successful analysis for the case of AdS_7 vacua in massive type IIA theory is performed.

7.1 Algebraic and differential constraints

In the last two chapters, we have studied compactifications to Minkowski space. Since this space is flat and its topology is trivial, the globally defined spinors we used for supersymmetry were just constant along the external directions. This situation changes if one demands the external space to be AdS, which has a constant curvature related with a non-zero cosmological constant Λ. In particular, the right hand side of equation (6.8) (where ∇ is the covariant derivative along the internal space) is not zero any more, but proportional to Λ.

Let us now discuss how this modification affects the algebraic and topological constraints (6.11) and (6.12) of the EFT formalism for half supersymmetric compactifications. We recall that the algebraic conditions (6.11) came from the analysis of G-structures, and were necessary conditions to have globally defined spinor bundles in the internal space, which is in turn necessary for supersymmetry. Then, such conditions cannot be modified by the introduction of a cosmological constant, since the requirement of having global spinors in the internal space remains.

The cosmological constant do instead modify the differential constraints (6.11). Intuitively, the fact that the right hand side of (6.8) is not zero implies that the internal space has no vanishing intrinsic torsion any more. The differential constraints are then modified to [151]

$$\begin{aligned} \mathcal{L}_{J_u} J_v &= -\tilde{c}_1 \, \epsilon_{uvwx} \, J^w \Lambda^x \,, \\ \mathcal{L}_{J_u} \hat{K} &= 0 \,, \\ d\hat{K} &= \tilde{c}_2 \, \Lambda^u \, J_u \,, \\ dK &= 0 \,, \end{aligned} \tag{7.1}$$

where Λ_u encodes the cosmological constant and we can use an $SO(4)$ rotation to write

$$\Lambda_u = (0,0,0,\Lambda) \ . \tag{7.2}$$

This choice breaks the original $SO(4)$ symmetry to $SO(3)$, which is the R-symmetry group of AdS$_6$ vacua. In the following, we will then split $u = (A,4)$ with $A = 1,2,3$. The non-vanishing coefficients $\tilde{c}_1$ and $\tilde{c}_2$ cannot be fixed within the present formalism. This can be done by careful comparison with six-dimensional half-maximal supergravity [163], or by comparison with a known AdS$_6$ vacuum. With respect to $(A,4)$ the differential conditions become

$$\begin{aligned} \mathcal{L}_{J_A} J_B &= -c_1 \, \epsilon_{ABC} J^C \,, \\ \mathcal{L}_{J_A} J_4 &= 0 \,, \\ \mathcal{L}_{J_A} \hat{K} &= 0 \,, \\ d\hat{K} &= c_2 \, J_4 \,, \\ dK &= 0 \,, \end{aligned} \tag{7.3}$$

where we have introduced $c_1 = \tilde{c}_1 \Lambda$ and analogous for c_2. Note that the conditions $\mathcal{L}_{J_4} J_u = 0$ and $\mathcal{L}_{J_4} \hat{K} = 0$ are automatically satisfied by $d\hat{K} = c_1 \Lambda J_4$ (for non-zero c_1).

Generalised sections in terms of the type IIB fields

Analogous to the discussion in section 6.4, it is convenient to parametrise the content of the generalised tensors defining the G-structure in terms of type IIB fields. In this case, following the conventions in Appendix D, this is done by

$$\begin{aligned} J_u &= v_u + \lambda^\alpha_{(1),u} + \sigma_{(3),u} \,, \\ K &= \omega^\alpha_{(0)} + \omega_{(2)} + \omega^\alpha_{(4)} \,, \\ \hat{K} &= \hat{\omega}^\alpha_{(0)} + \hat{\omega}_{(2)} + \hat{\omega}^\alpha_{(4)} \,, \end{aligned} \tag{7.4}$$

where α is the S-duality $SL(2)$-index and again the subscript (n) indicates that the object transforms as a space-time n-form. The components v_u are space-time vectors. As in the case in section 6.4, needs to write the constraints in terms of these fields to perform the upcoming calculation.

7.2 Parametrising generalised structures

As argued above, the fields defining the generalised structure organise themselves into representations of the R-symmetry group $SO(3)$. The only assumption we will consider in order to construct the complete classification of AdS_6 half-supersymmetric vacua is that this R-symmetry group is inherited from the symmetry group of a two-sphere in the internal space. We will therefore assume that all such vacua can be obtained from 10-dimensional type IIB string theory after compactifying on an internal space which is a non-trivial fribration of a two-sphere over a Riemann surface Σ. In this case, the S-duality $SL(2)$ will be obtained from the $SL(2)$ symmetry in Σ. As we will see a posteriori, these assumptions are enough to obtain all vacua in [160–162].

The systematics we will follow to construct all possible generalised structures satisfying the above mentioned conditions is the following:

1. We begin constructing the generalised vectors J_A. In terms of the parametrisation (7.4), each of these objects contains a vector, an $SL(2)$-doublet of 1-forms, and a three form. We construct, in terms of the natural tensorial objects on the two-sphere and on Σ, the most general ansatz for them that organise into the **3** representation of $SO(3)$. We then check how these ansätze are constraint by the algebraic conditions (6.11).

2. The last condition in (6.11) uniquely determine the field K out of any of the fields J_A. Then, following the same logic as before, we construct the most general ansatz for $\hat{K}$ and constraint it by demanding that it satisfies the conditions $\hat{K}^I\hat{K}_I = 0$ and $\hat{K}^I K_I = \kappa^4$. The field κ is in fact determined by this last constraint, and the only condition one needs to further impose is that it is defined everywhere and no-where vanishing (which we implement by assuming, without loss of generality, that it is strictly positive everywhere).

3. Next, the field J_4 is constructed out of $\hat{K}$ using the forth condition in (7.3), $J_4 = c_2^{-1}\Lambda^{-1}d\hat{K}$. This has to be plugged again into the last condition of (6.11), which further restricts the components in $\hat{K}$. We notice that the third condition in (6.11) is in fact implied by the rest.

4. With the three steps above, we have constructed the most general solution to the algebraic constraints (6.11), as well as to the forth condition in (7.3). The last step is to check the rest of differential constraints in (7.3) which translate into simple conditions for the fields parametrising the generalised structures.

Before moving towards the explicit construction of these fields, let us make the following clarification: as explained in chapter (6), generalised fields defining the generalised structure are in fact sections of generalised bundles, and therefore they need to be globally defined. However, in this formulation one does not have to impose any further global constraint on the supergravity fields parametrising them. In particular, the latter could vanish at some point if, at the same time, some other field parametrising the same generalised section is not vanishing at this point.

We next give the explicit solution to the described construction. We will first state our conventions for the fields on the two-sphere and on the Riemann surface Σ. We will then give some of the details of the calculation and state the final result. In next section, this result will be used to read off the corresponding type IIB supergravity fields.

7.2.1 Conventions for the sphere and the Riemann surface

We begin with the S^2, for which we use conventions similar to [164]. We define the $SO(3)$ triplet of functions y^A, $A = 1,2,3$ satisfying

$$y^A y_A = 1\,, \tag{7.5}$$

which are in fact the coordinates on the embedding space $\mathbb{R}^3 \supset S^2$, and we use them as *constrained* coordinates on the S^2. Furthermore, we consider the Killing vectors v_A, as well as the 1-forms

$$dy_A\,, \qquad \text{and} \qquad \theta_A = \epsilon_{ABC} y^B dy^C\,, \tag{7.6}$$

where ϵ_{ABC} is the three-dimensional Levi-Civita symbol. Together with the vectors v_A they satisfy the following relations

$$\begin{aligned} y^A dy_A &= 0\,, \\ \imath_{v_A} dy_B &= -\epsilon_{ABC} y^C\,, \\ \imath_{v_A} \theta_B &= \delta_{AB} - y_A y_B\,. \end{aligned} \tag{7.7}$$

The only 2-form on the S^2 is the volume form given by

$$vol_{S^2} = \frac{1}{2}\epsilon_{ABC} y^A dy^B \wedge dy^C\,, \tag{7.8}$$

which satisfies the relations

$$\begin{aligned} \frac{1}{2}\epsilon_{ABC} dy^B \wedge dy^C &= y_A vol_{S^2}\,, \\ \imath_{v_A} vol_{S^2} &= dy_A\,. \end{aligned} \tag{7.9}$$

On the Riemann surface Σ, we use the local coordinates χ^α, $\alpha = 1,2$. One can then construct any 1-form as

$$h^\alpha d\chi_\alpha\,, \tag{7.10}$$

for arbitrary h^α. The only two-form is the volume form

$$vol_\Sigma = \frac{1}{2}\epsilon_{\alpha\beta}d\chi^\alpha \wedge d\chi^\beta\,, \tag{7.11}$$

which satisfies

$$d\chi_\alpha \wedge d\chi_\beta = \epsilon_{\alpha\beta} vol_\Sigma\,, \tag{7.12}$$

with $\epsilon_{\alpha\beta}$ the Levi-Civita symbol. This object can also be used to rise and lower $SL(2)$ indices, for which we use the conventions

$$V^\alpha = \epsilon^{\alpha\beta}V_\beta\,, \qquad V_\alpha = V^\beta\epsilon_{\beta\alpha}\,. \tag{7.13}$$

7.2.2 Parametrising J_A

In this part, we want to give some details of the computation described above. In particular, we will give some details on the parametrisation of the generalised vectors J_A, as an example for the logic followed during the whole calculation.

As mentioned, these objects are parametrised by a vector, an $SL(2)$-doublet of 1-forms, and a three form. The most general ansatz for J_A constructed from the objects on the sphere and the Riemann surface Σ listed above, and that is also compatible with the $SO(3)$ R-symmetry, is

$$\begin{aligned} J_A = {}& c_1 v_A + h_\alpha y_A vol_{S^2} \wedge d\chi^\alpha + t\,\epsilon_{ABC}y^B dy^C \wedge vol_\Sigma + l^\alpha dy_A \\ & + y_A m^{\alpha\beta}d\chi_\beta + n^\alpha \epsilon_{ABC}y^B dy^C + f dy_A \wedge vol_\Sigma\,, \end{aligned} \tag{7.14}$$

where the $SL(2)$ matrix $m^{\alpha\beta}$, as well as the functions f, h_α, t, l_α, n^α only depend on the coordinates on Σ, χ^α. Note that the vector part of this ansatz has already been fixed to be one of the vectors on the sphere v_A times the constant c_1. This is in fact a consequence of the first differential condition in (7.3), but it is convenient to already introduce it here since it simplifies the computations. Also, the last term in (7.14) can be removed by the generalised diffeomorphism parametrised by $\mathcal{V} = \tilde{f}\chi_\alpha vol \wedge d\chi^\beta$, with $d\tilde{f}\chi_\alpha = f d\chi_\alpha$, and we will therefore set $f = 0$.

Taking into account this considerations, the algebraic constraints in (6.11) restrict the ansatz to

$$J_A = c_1\, v_A + m^{\alpha\beta}y_A d\chi_\beta + \rho\, k^\alpha dy_A - k^\beta m_{\beta\alpha}y_A vol_{S^2}\wedge d\chi^\alpha + \frac{|m|}{c_1}\theta_A \wedge vol_\Sigma\,, \tag{7.15}$$

where $|m|$ is the determinant of $m^{\alpha\beta}$, and we have introduced

$$k^\alpha = -\frac{1}{|m|}m^{\alpha\beta}h_\beta\,. \tag{7.16}$$

The fields (7.15) are the most general solution to the algebraic constraints. These are furthermore restricted by the differential constraints (7.3), which impose that

$$m_{\alpha\beta} = -c_1\partial_\beta k_\alpha\,. \tag{7.17}$$

7.2.3 Final result

Following the procedure described at the beginning of this section, and using the same logic that we used for the fields J_A, one obtains that the most general solution satisfying all algebraic and differential constraints is

$$\begin{aligned} J_A &= c_1\, v_A + c_1 d\,(k^\alpha\, y_A) + \frac{1}{2} c_1 d\,(k^\alpha\, \theta_A \wedge dk_\alpha)\ , \\ J_4 &= dp^\alpha - k_\alpha\, dp^\alpha \wedge vol_{S^2}\ , \\ K &= -\frac{c_1^2}{\sqrt{2}}\left(\partial_\beta k_\gamma \partial^\beta k^\gamma\right)(vol_\Sigma + k^\alpha vol_{S^2} \wedge vol_\Sigma)\ , \\ \hat{K} &= \frac{c_2}{\sqrt{2}}\,(p_\alpha - (r + p_\alpha k^\alpha)\, vol_{S^2})\ , \\ \kappa^4 &= \frac{c_2\, c_1^2}{2}\left(\partial_\alpha k_\beta \partial^\alpha k^\beta\right)\, r\, vol_{S^2} \wedge vol_\Sigma\,, \end{aligned} \tag{7.18}$$

which is given in terms of the functions on the Riemann surface r, k^α and p^α, which are subject to the differential conditions

$$\begin{aligned} \partial_\alpha r &= -p_\beta \partial_\alpha k^\beta\,, \\ \partial_\alpha p_\beta \partial^\alpha p^\beta &= c_2^2\, \partial_\alpha k_\beta \partial^\alpha k^\beta > 0\,. \end{aligned} \tag{7.19}$$

These conditions are implied if one consider the functions

$$f^\alpha = p^\alpha + i c_2\, k^\alpha\,, \tag{7.20}$$

to be holmorphic functions of Σ. In fact, one can prove that this is the most general solution to (7.19) up to diffeomorphisms [152]. We can then conclude that the space of all AdS$_6$ half-supersymmetric vacua can be parametrised by an $SL(2)$-doublet of holomorphic functions f^α (and the integration constant of the function r).

7.3 Background fields

Given the fields (7.18), we can now obtain the corresponding 10-dimensional configurations by plugging them into the generalised metric (6.13) or (6.15), and reading off the corresponding supergravity fields using the dictionary in Appendix D for the type IIB solution to the section constraint. We obtain the following fields on the internal four-dimensional space

$$\begin{aligned} ds^2 &= \frac{c_1^{5/4} r^{5/4} c_2^{1/2}\,(dkdk)^{3/2}}{2^{5/4}} \Delta^{1/4}\left(\Delta^{-1} ds^2_{S^2} + \frac{4}{c_1^2\, c_2\, r^2\,(dkdk)^2} dk^\alpha \otimes dp_\alpha\right)\,, \\ C_{(2)}{}^\alpha &= -\left(k^\alpha + \frac{c_2\, r\, p_\gamma\, \partial_\beta k^\gamma\, \partial^\beta p^\alpha}{2\Delta}(dkdk)\right) vol_{S^2}\,, \\ H^{\alpha\beta} &= \frac{1}{2\sqrt{2}\,\Delta}\left(\frac{c_2\,(dkdk)}{\sqrt{c_1\, r}} p^\alpha p^\beta + 4\sqrt{c_1\, r}\, \partial_\gamma k^\alpha \partial^\gamma p^\beta\right)\,, \end{aligned} \tag{7.21}$$

with $dkdk = \partial_\alpha k_\beta \partial^\alpha k^\beta$, and

$$\Delta = \frac{1}{2}\rho^3\, r\, \left(dkdk\right)^2 + \frac{1}{2}\tau\, (dkdk)\, p_\gamma\, p_\delta\, \partial_\sigma k^\gamma\, \partial^\sigma p^\delta\,. \tag{7.22}$$

After some redefinitions, these fields can be related to the results obtained in [160–162] (for details see [152]).

7.4 Consistent truncations around the AdS$_6$ vacua

The geometric language used by the EFT formalism allows us, not only to construct the most general AdS$_6$ half-supersymmetric vacua, but also to study general consistent truncations around them. In particular, following the discussion in [151], one can study consistent truncations with vector multiplets around a vacua by adding to the set of well defined sections defining the generalised structure extra generalised vectors in the **16** representation. Concretely, the consistent truncation is then identified with the set of fields $\{\omega_{\mathbb{u}}, K, \hat{K}, \kappa\}$, where $\omega_{\mathbb{u}} = (J_u, \omega_{\bar{u}})$, where we split $\mathbb{u} = (u, \bar{u})$, with $u = 1, \ldots, 4$, $\bar{u} = 1, \ldots, N$, and $N \leq 4$. The fields $\omega_{\bar{u}}$ will be related with the extra vector multiplets when reduced to six dimensions.

Such extra new fields are constraint by the algebraic relations

$$\begin{aligned} \omega_{\mathbb{u}}^M (\gamma^I)_{MN} K_I &= 0\,, \\ \frac{1}{2}\omega_{\mathbb{u}}^M \omega_{\mathbb{v}}^N (\gamma^I)_{MN} &= \eta_{\mathbb{u}\mathbb{v}} K^I\,, \end{aligned}$$

where $\eta_{\mathbb{u}\mathbb{v}}$ is the invariant metric on $SO(4, N)$, with u the directions with positive signature and $\bar{u}$ those with negative signature. Also, in general, the fields $\omega_{\mathbb{u}}$ satisfy

$$\mathcal{L}_{\omega_{\mathbb{u}}} \omega_{\mathbb{v}} = f_{\mathbb{u}\mathbb{v}}{}^{\mathbb{w}} \omega_{\mathbb{w}}\,, \qquad \mathcal{L}_{\omega_{\mathbb{u}}} \hat{n} = 0\,, \tag{7.23}$$

where $f_{\mathbb{u}\mathbb{v}}{}^{\mathbb{w}}$ are structure constants that appear in the embedding tensor.

With this fields one can construct the objects

$$\begin{aligned} \mathcal{J}_u(x, Y) &= \Sigma^{-1}(x)\, b_u{}^{\mathbb{u}}(x)\, \omega_{\mathbb{u}}(Y)\,, \\ \hat{\mathcal{K}}(x, Y) &= \Sigma^2(x)\, \hat{K}(Y)\,, \end{aligned} \tag{7.24}$$

where x are coordinates on the six-dimensional dimensional external space and Y on the internal. The functions $b_u{}^{\mathbb{u}}(x)$ are subject to the constraint

$$b_u{}^{\mathbb{u}} b_v{}^{\mathbb{v}} \eta_{\mathbb{u}\mathbb{v}} = \delta_{uv}\,, \tag{7.25}$$

and therefore parametrise the cosset $\frac{SO(4,N)}{SO(4)}$; and $\Sigma(x)$ are scalar fields leading to scalar multiplets in six dimensions. The objects $\mathcal{J}_u$ and $\hat{\mathcal{K}}$ are the analogous generalised fields to the J_u and $\hat{K}$ fields in the vacua, and therefore define the geometry.

We will close this chapter by discussing the situation in which one vectormultiplet transforming as an R-symmetry singlet is added to the vacuum solutions of section 7.2.1. This can be taken as an example for the analyses that can be performed with these techniques, and can actually be generalised to situations with more R-symmetry singlet vectormultiplets, as well as for the case with three vectormultiplets in the **3** of $SO(3)$. The results for all these cases will appear in [152].

7.4.1 Consistent truncations with one vector multiplet around AdS$_6$ vacua

For the case with one vector multiplet $\omega_{\bar{1}}$ around the AdS$_6$ vacua discussed in section 7.2.1, the conditions (7.23) imply that

$$\begin{aligned} \frac{1}{2} J_A^M \omega_{\bar{1}}^N (\gamma^I)_{MN} &= 0\,, \\ \frac{1}{2} J_4^M \omega_{\bar{1}}^N (\gamma^I)_{MN} &= 0\,, \\ \frac{1}{2} \omega_{\bar{1}}^M \omega_{\bar{1}}^N (\gamma^I)_{MN} &= -K^I\,, \end{aligned}$$

with K, J_A and J_4 given by (7.18). Furthermore, the differential conditions

$$\mathcal{L}_{\omega_{\bar{1}}} J_A = \mathcal{L}_{\omega_{\bar{1}}} J_4 = 0\,, \qquad \mathcal{L}_{\omega_{\bar{1}}} \hat{K} = 0\,, \tag{7.26}$$

have to be satisfied. Following a procedure similar to the one used in section 7.2.1, one can conclude that the most general field satisfying these conditions is

$$J_{\bar{1}} = \rho \left(\epsilon_{\alpha\beta} M^{\sigma\beta} \epsilon_{\sigma\rho} k^\rho d\chi^\alpha \wedge Vol_{S^2} + M^{\alpha\beta} \epsilon_{\beta\gamma} d\chi^\gamma \right)\,, \tag{7.27}$$

where $M^{\alpha\beta}$ is an $SL(2)$ matrix, depending on the coordinates on the Riemann surface, and subject to the constraints

$$\begin{aligned} M^{\alpha\beta} \partial_\beta k_\alpha &= 0\,, \\ M^{\alpha\beta} \partial_\beta p_\alpha &= 0\,, \\ M^{\alpha\beta} M_{\alpha\beta} &= -\partial_\alpha k_\beta \partial^\alpha k^\beta\,, \\ \partial_\beta M^{\alpha\beta} &= 0\,. \end{aligned} \tag{7.28}$$

To solve these constraints it is useful to use the holomorphic functions f^α (7.20) and encode the matrix $M^{\alpha\beta}$ into two real one-forms

$$M^\alpha = M^{\alpha\beta} \epsilon_{\beta\gamma} d\chi^\gamma\,, \tag{7.29}$$

in terms of which the conditions (7.28) become

$$\begin{aligned} df_\alpha \wedge M^\alpha &= 0\,, \\ 2c_2^2 M^\alpha \wedge M_\alpha &= df^\alpha \wedge \overline{df}_\alpha\,, \\ dM^\alpha &= 0\,. \end{aligned} \tag{7.30}$$

The first two of them can be generically solved for any f^α with

$$M^\alpha = g\,\partial_w f^\alpha d\bar{w} + \text{c.c.}\,, \tag{7.31}$$

where $w = \chi^1 + i\chi^2$ is an holomorphic coordinate on the Riemann surface and g is a complex function constrained by

$$4\,c_2^2\,|g|^2 = 1\,. \tag{7.32}$$

Finally, the closure condition $dM^\alpha = 0$ further restricts g to be defined via the implicit relation

$$\partial_w(g\partial_w f^\alpha) \in \text{Real functions of } \Sigma\,. \tag{7.33}$$

To our knowledge, there exist no method to generically solve this condition, not even to tell under which conditions this solution exist. However, this condition gives a very strong necessarily condition that any consistent truncation around the AdS_6 vacua should satisfy. Furthermore, by plugging the obtained fields into the generalised metric, one can obtain the expressions for the fields in the 10-dimensional configuration [152], which shows the potential of the presented formalism.

Chapter 8

Conclusions and discussion

In this thesis we have studied certain aspects in the interplay between geometric and non-geometric backgrounds, string compactifications and string dualities. We now summarise our findings, discuss the results and point out some ideas for future research motivated by them.

Chapters 2, 3 and 5 were dedicated to the construction and analysis of classes of non-geometric string backgrounds. These are in general motivated by string target space dualities such as T- S- and U- dualities. Such dualities, which are not present in point-particle theories, indicate that strings (and in general extended objects) probe geometry differently from zero-dimensional objects, and suggest that the classical concept of differentiable manifold should be generalised in string theory. In particular, one can construct spaces where strings can smoothly propagate by using dualities as transition functions between patches, in a way that would not be allowed in classical geometry. It is also important to remark that, from the world-sheet perspective, there is no reason that indicates that string propagating backgrounds should be geometric. The only reason why non-geometric backgrounds are less used as string compactification models is because we lack the powerful tools of classical geometry.

Non-geometric backgrounds are then valid configurations in string landscape that can lead to interesting phenomenological models and therefore a better understanding of them is necessary. With this motivation, in chapter 2 we constructed and classified a general class of two-torus fibrations including configurations with non-geometric features. In particular, we began considering the case of a two-torus over a base $\mathcal{B} = S^1$. By going around $\mathcal{B}$ the fiber does not come to itself but is glued with a T-duality transformation in $SO(2,2,\mathbb{Z}) \simeq SL(2)_\tau \times SL(2)_\rho$. In the cases where such transformation is in the geometric subgroup, $\phi \in SL(2,\mathbb{Z})_\tau$, the configurations we obtained are the mapping tori characterised by the fibration $T^2 \to \mathcal{N}_\phi \to S^1$. We constructed explicit metrics for them and observed that the trichotomy of conjugacy classes is in correspondence with the possible geometries of the total space: Euclidean, Nil, and Solv.

On the other hand, when monodromies are in the $SL(2,\mathbb{Z})_\rho$ subgroup, the T-duality transformation mixes non-trivially the volume of the torus with its B-field and the total space becomes non-geometric. We developed a picture where such monodromies can be understood as Dehn twists on an auxiliary torus T^2_ρ.

Next, we considered fibrations over a two dimensional base $\mathcal{B} = \mathbb{P}^1$, where solutions to the string background equations can be constructed by letting the moduli of the torus fiber be meromorphic functions of the base, which also implies the preservation of some of the original supersymmetries. In particular, we considered situations where the base was a punctured disk D^2/x_0, at whose boundary the previous torus fibrations over a circle were recovered. The defect in x_0, which we call T-fect, is then identified with the monodromy at the boundary of the disk. Again, when only the complex structure τ is allowed to vary, one obtains geometric configurations. By solving Cauchy-Riemann equations, we argued that one could determine the function τ for each monodromy in the T-duality group, thus obtaining all possible local geometries for the mentioned configurations with arbitrary monodromy. The simplest example is the semi-flat approximation to the Taub-NUT space, obtained in the parabolic conjugacy class.

For the case where only ρ varies, the local solutions could be obtained by a fiberwise mirror symmetry from the geometric fibrations. By applying this to all the conjugacy classes of $SL(2,Z)$, we obtained a classification of the corresponding local solutions. Analogous to fibrations over a circle, the monodromies in this situation act non-trivially on the volume and the corresponding solutions are non-geometric. In the $\mathbb{P}^1$ case, the geometric picture in terms of an auxiliary fibration was motivated by the heterotic/F-theory duality.

The two-torus fibration structure breaks down at the degeneration point. In order to have a complete understanding of these solutions, one needs to supplement the encountered local geometries with a string description of the T-fect, which typically breaks (some of) the isometries of the background. In chapter 3, we analysed the physics close to T-fects with parabolic monodromies. We concluded that winding modes play a crucial role for the understanding of such physics. In particular, it is known that the degeneration with monodromy $\rho \to \rho + 1$ corresponds to an smeared NS5 brane, and the exact geometry is obtained by gluing in corrections that localise the brane along both directions of the torus, breaking both isometries. Using duality arguments, we argued that one of these set of corrections is related, in the geometric degeneration $\tau \to \tau + 1$, to winding modes localising the solution on a dyonic coordinate dual to the isometry direction. This analysis was extended to degenerations with monodromy $-1/\rho \to -1/\rho + 1$, corresponding to non-geometric Q-branes, where we concluded that corrections to the semi-flat approximation involve two dyonic coordinates.

These effects due to winding modes cannot be captured by the supergravity approximation, which is a point-like theory. For this reason, we also constructed solutions in Double Field Theory, a theory conceived to capture momentum and winding modes in the same footing, encoding such effects. We furthermore argued that, at least in a region where the isometries are mildly broken, these corrections could be understood in terms of instanton effects. To have a complete understanding of such degenerations, however, it would be necessary to find an explicit CFT descriptions encoding such winding physics. Also, it would be interesting to know how these corrections are encoded within the heterotic/F-theory duality mentioned above.

As mentioned, one of the geometries we encountered in the discussions summarised above is the semi-flat approximation to the NS5 brane. This configuration is obtained by taking a limit away from the brane after making two of the transverse directions compact. After a T-duality transformation along both compact directions, one obtains the non-geometric Q-brane. The full NS5 configuration, however, has also other isometries that can be used for T-duality. In chapter (4), we analysed T-duality transformations along angular directions in the space transverse to the NS5-brane and concluded that the situation is significantly different from the T-duality along the flat toroidal directions of the semi-flat approximation. In particular, the angular isometries are non-trivially fibered and one can assign them a geometric charge, which is interchange with the characteristic NS charge of the NS5 after T-duality, consistent with general results in the literature. Moreover, the configuration obtained after two T-dualities shows no non-geometric feature, unlike the Q-brane, and the original supersymmetry of the NS5 brane is completely broken by the dualisation procedure.

Furthermore, we compared the case of T-duality along angular directions of the NS5 brane with the case of angular T-duality on empty space. Unlike in the latter case, the angular isometries of the NS5 brane do not shrink at the origin and the dual geometry is non-singular. In empty space, instead, one obtains dual geometries with a singularity at the origin. It would be interesting to further analyse this situation and find a way to resolve this singularity.

A natural generalisation for the T-fect constructions of chapter 2 is to consider analogous two-torus fibrations in type II and M-theory and allow for U-duality monodromies, whose group is $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$. These fibrations were considered in chapter 5, where we argued that the $SL(3,\mathbb{Z})$ factor could be interpreted as the large diffeormorphisms group of a three-torus in M-theory. This interpretation motivated the study of general supersymmetric geometric compactifications of M-theory on five-dimensional internal manifolds. After a detailed analysis, we concluded that in this number of dimensions supersymmetry preservation impose severe restrictions to the background. In particular, for the U-duality defects, the only possible supersymmetric configurations where uplifts of the geometric τ-fects of chapter 2, and the elements in the $SL(3,\mathbb{Z})$ factor that could be used as monodromy were restricted to be in an $SL(2,\mathbb{Z})$ subrgoup. We also briefly discussed all possible local geometries of type II with U-duality monodromies, which are all U-dual to some of the T-fect configurations from chapter 2.

In chapter 6, we studied half-supersymmetric flux compactifications of type II and M-theory using the formalism of generalised G-structures in Exceptional Field Theory. The last is in fact a natural theory for the study of flux compactification, since it threats metric and flux degrees of freedom in the same footing. In that chapter, we reviewed generalised G-structures and constructed the necessary tools for the study of half-supersymmetric flux compactification, such as an expression for the generalised metric in terms of the fields defining the G-structure or a dictionary for the generalised metric in terms of the type IIB and M-theory fields, summarised in Appendix D. As a first example, we applied the formalism to the case of U-duality defects of chapter 5, reproducing the results.

Finally, this formalism was applied to a much more involved situation in chapter 7, namely half-supersymmetric compactifications to AdS_6 spaces. In particular, we were able to obtain a full classification for such vacua in terms of two holomorphic functions f^α, $\alpha = 1, 2$, which reproduced known results in the literature in a much simpler formulation. Furthermore, this geometric language also allowed us to stablish necessary conditions for the most general consistent truncations with vector multiplets around these vacua, as well as to construct the explicit 10-dimensional configurations. These consistent truncations are very usefool tools for the holographic study of the vacua.

The great success of this method in the considered situations makes generalisations to other cases to look very promissing. In particular, one could next study AdS_5 vacua, which have a richer structure and less results are known, and continue to lower dimensional AdS spaces. Furthermore, it would be interesting to investigate how non-geometric compactifications, such as the ones considered in the first chapters of this thesis, can be studied within this formalism. An important issue in this context is that the generalised G-structure formalism is based on objects that are globally defined on the manifold. In the non-geometric configurations, however, the notion of "globally defined" is missing, and one should find an analogous concept. It is reasonable to think that geometric pictures in terms of auxiliary fibrations, such as the one described in chapter 2 for T-fect solutions, could help in this direction. Altogether we conclude that the formalism of generalised G-structures in Exceptional Field Theory has a promissing future in the study of supersymmetric flux compactifications.

Appendix A

Geometry of torus fibration

We summarize some details about the geometry of the ansatz (2.103). The relevant equations of motion are:

$$R - 4(d\Phi)^2 + 4\nabla^2\Phi - \frac{1}{2}H^2 = 0\,, \tag{A.1}$$

$$R_{MN} + 2\nabla_M\nabla_N\Phi - \frac{1}{2}\iota_M H \iota_N H = 0\,, \tag{A.2}$$

$$d(e^{-2\Phi} \star_{10} H) = 0\,. \tag{A.3}$$

It is easy to check that the Cauchy-Riemann equations for the functions τ, ρ and ϕ imply that the ansatz (2.103) solve the equations of motion. The Ricci scalar (in the string frame) for such ansatz is:

$$R = \frac{3e^{-2\varphi_1(r,\theta)}}{2r^2[\rho_2(r,\theta)]^3\tau_2(r,\theta)}\left[r^2\left(\frac{\partial\rho_2(r,\theta)}{\partial r}\right)^2 + \left(\frac{\partial\rho_2(r,\theta)}{\partial\theta}\right)^2\right]\,. \tag{A.4}$$

For completeness we also give the components of the Ricci tensor along the torus and the base directions:

$$\begin{aligned}
R_{r,r} &= \frac{(1+\partial_\theta^2\phi_2)\partial_r\rho_2 + \partial_\theta\rho_2\partial_r\phi_2}{r\rho_2} + \frac{r^2\partial_r\rho_2\partial_r\tau_2 - \partial_\theta\rho_2\partial_\theta\tau_2}{2r^2\rho_2\tau_2} + \frac{\partial_\theta^2\rho_2}{r^2\rho_2} + \frac{3(\partial_r\rho_2)^2}{2\rho_2^2}\,,\\
R_{r,\theta} &= \frac{\partial_\theta\rho_2(1+\partial_\theta\phi_2)}{r\rho_2} + \frac{3\partial_\theta\rho_2\partial_r\rho_2}{2\rho_2^2} + \frac{\partial_\theta\rho_2\partial_r\tau_2 + \partial_\theta\tau_2\partial_r\rho_2}{2\rho_2\tau_2} - \frac{r\partial_r\rho_2\partial_r\phi_2 + \partial_r\partial_\theta\rho_2}{\rho_2}\,,\\
R_{\theta,\theta} &= \frac{3(\partial_\theta\rho_2)^2}{2\rho_2^2} - \frac{\partial_\theta^2\rho_2 + r(1+\partial_\theta\phi_2)\partial_r\rho_2}{\rho_2} - \frac{\partial_\theta\partial_\theta\tau_2 + r^2\partial_r\rho_2\partial_r\tau_2}{2\rho_2\tau_2} - \frac{r\partial_\theta\rho_2\partial_r\phi_2}{\rho_2}\,,\\
R_{8,8} &= \frac{e^{-2\phi_1}\tau_1}{r\rho_2\tau_2^2}\left[\partial_\theta\rho_2\partial_r\tau_2 - \partial_r\rho_2\partial_\theta\tau_2\right] + \frac{e^{-2\phi_1}(\tau_1^2-\tau_2^2)}{2r^2\rho_2\tau_2^3}\left[\partial_\theta\rho_2\partial_\theta\tau_2 + r^2\partial_r\rho_2\partial_r\tau_2\right]\,,\\
R_{8,9} &= \frac{e^{-2\phi_1}}{2r\rho_2\tau_2^2}\left[\partial_\theta\rho_2\partial_r\tau_2 - \partial_r\rho_2\partial_\theta\tau_2\right] + \frac{e^{-2\phi_1}\tau_1}{2r^2\rho_2\tau_2^3}\left[\partial_\theta\rho_2\partial_\theta\tau_2 + r^2\partial_r\rho_2\partial_r\tau_2\right]\,,\\
R_{9,9} &= \frac{e^{-2\phi_1}}{2r^2\rho_2\tau_2^3}\left[\partial_\theta\rho_2\partial_\theta\tau_2 + r^2\partial_r\rho_2\partial_r\tau_2\right]\,.
\end{aligned} \tag{A.5}$$

Appendix B

Spheres, Lens spaces and T-duality

In this appendix we want to review T-duality transformations on the three-sphere with H-flux, which is useful for the discussion on spherical T-duality on the NS5 brane in chapter 4. Some of the results presented in this section can already be found in the literature (see for instance [165–167, 32]), but we will put some special focus on the global aspects. This discussion will closely follow [129].

B.1 Hopf fibration and $U(1)$ actions

The three-sphere S^3 can be described as a non-trivial S^1 fibration as

$$S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2 \,, \tag{B.1}$$

and the standard way is by using the projection

$$\pi : \begin{cases} S^3 & \longrightarrow \quad S^2 \\ (z_0, z_1) & \longmapsto \quad (2z_0 z_1^*,\, |z_0|^2 - |z_1|^2) \end{cases} \tag{B.2}$$

It is easy to check that $\pi(z_0, z_1) \in S^2$ and that $\pi(z_0, z_1) = \pi(w_0, w_1)$ if an only if $(z_0, z_1) = (\lambda w_0, \lambda w_1)$ for $\lambda \in U(1)$. Therefore, at each point of S^2 there is a $U(1)$ fiber. Note that the action of $U(1)$ on each fiber is transitive and free. The fibration is then a principal $U(1)$ bundle. To define the standard Hopf fibration, we use the first expression in (B.13) to construct the $U(1)$ action on the fiber. However, we can also employ the second action in (B.13) as

$$\bar{\pi} : \begin{cases} S^3 & \longrightarrow \quad S^2 \\ (z_0, z_1) & \longmapsto \quad (2z_0 z_1,\, |z_0|^2 - |z_1|^2) \end{cases} \tag{B.3}$$

Again, one can show that $\bar{\pi}(z_0, z_1) \in S^2$ and that, now, $\bar{\pi}(z_0, z_1) = \bar{\pi}(w_0, w_1)$ if an only if $(z_0, z_1) = (\lambda w_0, \lambda^* w_1)$ for $\lambda \in U(1)$. As before, the action acts transitively and freely on all fibers, and the fibration is again a principal $U(1)$ bundle.

Let us also address the question whether one can construct fibrations using actions infinitesimally characterized by a linear combination of the vectors v and $\bar{v}$ in (B.11). If we take the linear combination $av + b\bar{v}$ with where $a, b \in \mathbb{R}$, the corresponding group action is

$$g(z_0, z_1) = (e^{i(a+b)\alpha} z_0,\, e^{i(a-b)\alpha} z_1)\,, \tag{B.4}$$

where $\alpha \in [0, 2\pi)$. Consider now a $U(1)$ orbit through the point $(z_0, z_1) = (0, 1)$ defined as all points satisfying $|z_1| = 1$ and $z_0 = 0$. The action (B.4) acts freely and transitively on it if and only if $a + b = \pm 1$. In other words, only when this condition is satisfied the orbit through the point $(z_0, z_1) = (0, 1)$ constructed with the action (B.4) is isomorphic to $U(1)$. Similarly, the action is free and transitive on the $U(1)$ orbit through the point $(z_0, z_1) = (0, 1)$ if and only if $a - b = \pm 1$. Then, we can conclude that the only cases that can be used to construct principal bundles are $(a, b) = (\pm 1, 0)$, $(0, \pm 1)$, which are the ones discussed above.

Finally, we note that although the three-sphere S^3 can be described as a principal $U(1)$ bundle in two different ways, it cannot be described as a principal $U(1) \times U(1)$ bundle. The reason is that the orbits constructed using the $U(1) \times U(1)$ action

$$g(z_0, z_1) = (e^{i(\alpha+\beta)} z_0,\, e^{i(\alpha-\beta)} z_1)\,, \tag{B.5}$$

with $\alpha, \beta \in [0, 2\pi)$, are not isomorphic to $U(1) \times U(1)$ everywhere. In particular, the orbits through the points $(z_0, z_1) = (0, 1)$ and $(z_0, z_1) = (1, 0)$ are two $U(1)$ one on top of each other.

B.2 The three-sphere as string background

A three-sphere of radius $R^2 = k$ can be used as a consistent string background if it also containts k units of H-flux. The background can be then described by the $SU(2)$ Wess-Zumino-Witten (WZW) model at level k. We will now review the main geometrical aspects of this background.

The setting

Apart from the presentation of section B.1, a round three-sphere of radius R can be also defined by its embedding into $\mathbb{C}^2$ through the equation

$$|z_0|^2 + |z_1|^2 = R^2\,, \tag{B.6}$$

where $(z_0, z_1) \in \mathbb{C}^2$. Using Hopf coordinates

$$z_0 = R e^{i\xi_1} \cos\eta\,, \qquad\qquad z_1 = R e^{i\xi_2} \sin\eta\,, \tag{B.7}$$

with $\eta \in [0, \pi/2]$ and $\xi_1, \xi_2 \in [0, 2\pi)$, the metric and H-flux of the $SU(2)_k$ WZW model take the following form

$$\begin{aligned} ds^2 &= R^2 \left(d\eta^2 + \cos^2\eta\, d\xi_1^2 + \sin^2\eta\, d\xi_2^2 \right), \\ H &= 2k \sin\eta\, \cos\eta\, d\eta \wedge d\xi_1 \wedge d\xi_2\,. \end{aligned} \tag{B.8}$$

The dilaton is taken to be $\Phi = \phi_0 =$ const. Note that in order for the theory to be conformal one has to impose a relation between the radius and the level k which reads $R^2 = |k|$. However, for practical purposes it will be convenient for us to keep the dependence on R^2 explicit. In our subsequent analysis we will also make use of the following coordinate system with $\theta \in [0, \pi]$ and $\chi, \xi \in [0, 2\pi)$

$$\chi = \frac{1}{2}(\xi_1 + \xi_2)\,, \qquad \xi = \xi_1 - \xi_2\,, \qquad \theta = 2\eta\,. \tag{B.9}$$

The metric and H-flux (B.8) in these coordinates take the form

$$\begin{aligned} ds^2 &= \frac{R^2}{4}\left(d\theta^2 + 4d\chi^2 + d\xi^2 - 4\cos\theta\, d\chi\, d\xi\right), \\ H &= \frac{k}{2}\sin\theta\, d\theta \wedge d\xi \wedge d\chi\,. \end{aligned} \tag{B.10}$$

Isometries

The isometry group of the round three-sphere is $O(4)$, and therefore the isometry algebra is $\mathfrak{so}(4) \cong \mathfrak{su}(2)\times\mathfrak{su}(2)$. Note that this algebra contains $\mathfrak{u}(1)\times\mathfrak{u}(1)$ as an abelian subalgebra. Using Hopf coordinates, the corresponding Killing vector-fields for these isometries are

$$v = \partial_{\xi_1} + \partial_{\xi_2} = \partial_\chi\,, \qquad \bar{v} = \partial_{\xi_1} - \partial_{\xi_2} = 2\,\partial_\xi\,, \tag{B.11}$$

which have a nowhere vanishing norm

$$|v|^2 = |\bar{v}|^2 = R^2\,. \tag{B.12}$$

These vector-fields can be integrated to $U(1)$ group-actions on the three-sphere, and for $\lambda \in U(1)$ the group acts on the embedding coordinates (z_0, z_1) in the following way

$$\begin{aligned} v:&\quad g_\lambda(z_0, z_1) = (\lambda z_0, \lambda z_1)\,, \\ \bar{v}:&\quad \bar{g}_\lambda(z_0, z_1) = (\lambda z_0, \lambda^* z_1)\,, \end{aligned} \tag{B.13}$$

which are precisely the $U(1)$ actions described in section B.1 and their orbits are therefore $U(1)$ fibers everywhere in S^3.

Geometric charges

As follows from the above discussion, a three-sphere can be described as a principal $U(1)$ bundle whose fiber is along the direction defined by one of the vectors in (B.11). There is a natural procedure to assign a $U(1)$ connection to such fibrations [168]. In particular, for $U(1)$ fiber along some direction $\star$, we determine the corresponding connection as

$$\mathcal{A}_\star = \frac{g_{\star i}}{g_{\star\star}}\, dx^i\,, \tag{B.14}$$

where $\{dx^i\}$ is a local basis on the co-tangent space. For the three-sphere we then obtain the following:

- Using the coordinates (B.9), the metric in (B.10) can be rewritten as

$$ds^2 = \frac{R^2}{4}\left(d\theta^2 + \sin^2\theta\, d\xi^2\right) + R^2\left(d\chi - \frac{1}{2}\cos\theta\, d\xi\right)^2 . \tag{B.15}$$

  The $U(1)$ gauge connection associated to the direction χ reads

$$\mathcal{A}_\chi = -\frac{1}{2}\cos\theta d\xi\,, \tag{B.16}$$

  and the corresponding field strength is computed as

$$\mathcal{F}_\chi = d\mathcal{A}_\chi = \frac{1}{2}\sin\theta\, d\theta\wedge d\xi\,. \tag{B.17}$$

  We can then define a geometric charge n_χ associated to the fibration by integrating $\mathcal{F}_\chi$ over the base manifold $\mathcal{B}$. The latter is a two-sphere of radius $R/2$, and the charge is computed as

$$n_\chi = \frac{1}{2\pi}\int_{\mathcal{B}}\mathcal{F}_\chi = 1\,. \tag{B.18}$$

  This is precisely the first Chern class of the fibration, which in general has to be an integer for principal $U(1)$ bundles.

- Since also the direction ξ in (B.11) corresponds to a $U(1)$ fiber, we can compute the charge with respect to such a fibration structure. To do so, we note that the metric (B.10) can be rewritten as

$$ds^2 = \frac{R^2}{4}\left(d\theta^2 + 4\sin^2\theta\, d\chi^2\right) + \frac{R^2}{4}\left(d\xi - 2\cos\theta\, d\chi\right)^2 , \tag{B.19}$$

  from which we can determine the gauge field associated to this fiber-bundle structure as

$$\mathcal{A}_\xi = -2\cos\theta\, d\chi\,. \tag{B.20}$$

  This gauge field has field strength

$$\mathcal{F}_\xi = 2\sin\theta\, d\theta\wedge d\chi\,, \tag{B.21}$$

  and the geometric charge we associate to this fibration has a different normalization as compared to (B.18). It is given by[1]

$$n_\xi = \frac{1}{4\pi}\int_{\mathcal{B}}\mathcal{F}_\xi = 2\,. \tag{B.22}$$

  The fact that we obtain two units of geometric charge is because there is an effective orbifolding in our coordinate ξ (see the upcoming discussion in section B.3). Due to this effect the charge along the coordinate ξ in this coordinate frame is always even.

[1] Using the definition (B.18), the charge associated with the field strength (B.21) would be $n = 4$. However, the base-manifold $\mathcal{B}$ in (B.19) is not a two-sphere but rather a double-cover. This has to be taken into account when computing the charge, and it is therefore the reason for the extra factor of two in the normalisation.

Remark

Let us briefly note that in addition to the coordinates (B.9) there is one other consistent choice, namely

$$\tilde{\chi} = \xi_1 + \xi_2 \,, \qquad \tilde{\xi} = \frac{1}{2}(\xi_1 - \xi_2) \,, \qquad \tilde{\theta} = 2\eta \,, \tag{B.23}$$

again with periods $\tilde{\chi}, \tilde{\xi} \in [0, 2\pi)$ and $\tilde{\theta} \in [0, \pi]$. This choice is one the same footing as (B.9), and is more suitable to describe the background as a $U(1)_\xi$ fibration. The natural definitions for the geometric charges are now

$$\tilde{n}_\chi = \frac{1}{4\pi} \int_{\mathcal{B}} \tilde{\mathcal{F}}_\chi \,, \qquad \tilde{n}_\xi = \frac{1}{2\pi} \int_{\mathcal{B}} \tilde{\mathcal{F}}_\xi \,, \tag{B.24}$$

where $\tilde{\mathcal{F}}_\xi$ and $\tilde{\mathcal{F}}_\chi$ are the field strengths computed in the $(\tilde{\chi}, \tilde{\xi})$ coordinates. In this frame we obtain $\tilde{n}_\chi = 2$ and $\tilde{n}_\xi = 1$ for the three-sphere. However, the coordinates (B.23) will not play a major role in our subsequent discussion.

NS charge

Apart from geometric charges, the background (B.10) also has a non-trivial H-flux. Its associated charge is defined as usual as

$$h = \frac{1}{4\pi^2} \int_{S^3} H \,. \tag{B.25}$$

In the case of the three-sphere (B.10), we find $h = k$ independent of the coordinate frame we choose. Furthermore, note that the H-flux is quantized as $h \in \mathbb{Z}$.

B.3 Lens spaces

In section B.1 we defined the three sphere as a Hopf fibration. We want now to construct new manifolds by performing orbifold projections on it. An orbifold constructed using a finite symmetry group of the original space is again a manifold if and only if the action of any element of the group on the original space is a homeomorphism and the symmetry group acts freely on the space. These conditions are satisfied if one consider a $\mathbb{Z}_k$ group acting on the $U(1)$ fiber of the three-sphere.

In particular, given two natural numbers p and q with $p > q$ and relatively prime, one can construct the Lens space $L(p, q)$ defined as the orbifold $S^3/\mathbb{Z}_p$ with $\mathbb{Z}_p$ acting on the sphere as

$$(z_0, z_1) \to \left(e^{\frac{2\pi i m}{p}} z_0,\, e^{\frac{2\pi i\, q\, m}{p}} z_1 \right) , \tag{B.26}$$

for any $m \in \mathbb{Z}_p$. This action is free if and only if the condition of p and q being relatively prime is satisfied. Note that, in fact, q is defined modulo p. In terms of Hopf coordinates

(B.7), the action is

$$\xi_1 \to \xi_1 + \frac{2\pi m}{p}\,, \qquad\qquad \xi_2 \to \xi_2 + \frac{2\pi q m}{p}\,. \tag{B.27}$$

We will next analyse these orbifolds using the coordinate frame (B.9).

The $\mathbb{Z}_{k_1}^{(\chi)}$ orbifold

Taking $q = 1$, the action (B.27) leave the coordinate ξ invariant and acts on χ as

$$\chi \to \chi + \frac{2\pi m}{p}\,. \tag{B.28}$$

Then, the configuration constructed with such an orbifold projection is the space $L(k_1, 1)$.

The $\mathbb{Z}_{k_2}^{(\xi)}$ orbifold

The action that leaves χ invariant and acts only on ξ corresponds to the case $q = p - 1$ (equivalent to $q = -1$). However, in this case the situation is more subtle. The reason is that, as discussed before (B.22), the coordinates (B.9) are not the most appropriate to describe the three-sphere as a $U(1)_\xi$ fibration. In particular, if one considers the inverse transformation of (B.9),

$$\xi_1 = \chi + \frac{1}{2}\xi\,, \qquad\qquad \xi_2 = \chi - \frac{1}{2}\xi\,, \tag{B.29}$$

one can easily see that, by sending $\xi \to \xi + 2\pi$, the coordinates ξ_1 and ξ_2 do not come back to themselves. The intuition behind this fact is that, when assigning a period of 2π to $\xi_1 - \xi_2$, we are effectively orbifolding this direction. In fact, the points related by the $\mathbb{Z}_2$ action

$$(z_0, z_1) \to \left(e^{i\pi m} z_0, e^{-i\pi m} z_1\right)\,, \tag{B.30}$$

with $m \in \{0, 1\}$, are identified in the coordinates (ξ, χ). We stress, however, that this is an artifact of the coordinate frame we use, and not an orbifolding of the actual background. Nevertheless, this effect arises when computing the charges, since we compute them in a particular frame. This is then the reason for the factor 4π in the normalization of (B.22) and the even charge along this direction. Changing to the coordinate frame (B.23) will move this effect from the direction ξ to the direction $\tilde{\chi}$.

With this considerations, we next want to find an action that leaves χ invariant. The natural guess is

$$(z_0, z_1) \to \left(e^{\frac{2\pi i m_2}{k_2}} z_0, e^{-\frac{2\pi i m_2}{k_2}} z_1\right)\,, \tag{B.31}$$

for $m_2 \in \mathbb{Z}_{k_2}$. The space constructed using this action is the $L(k_2, k_2 - 1)$ Lens space. In terms of the coordinate system (B.9), the action (B.31) acts on the coordinate ξ as

$$\xi \to \xi + \frac{4\pi m_2}{k_2}\,. \tag{B.32}$$

Note that, for even k_2, the action looks rather as a $\mathbb{Z}_{k_2/2}$-orbifold. However, since $m_2 \in \mathbb{Z}_{k_2}$, the action (B.32) has then fixed points and the corresponding configuration fails to be an appropriate description for the orbifold constructed with the action (B.31). Again this is just a coordinate dependent effect. In fact, Lens spaces $L(k_2, k_2 - 1)$ with even k_2 can be described using the coordinates (B.23). In this frame, $L(k_1, 1)$ spaces can only be described in the cases where k_1 is odd.

The $\mathbb{Z}_{k_1}^{(\chi)} \times \mathbb{Z}_{k_2}^{(\xi)}$ orbifold

Finally, under certain assumptions it is possible to construct orbifolds with an action along both χ and ξ directions. The natural generalization to the orbifold actions discussed above is

$$(z_0, z_1) \to \left(e^{2\pi i \frac{k_2 m_1 + k_1 m_2}{k_1 k_2}} z_0, e^{2\pi i \frac{k_2 m_1 - k_1 m_2}{k_1 k_2}} z_1\right), \tag{B.33}$$

with $m_i \in \mathbb{Z}_{k_i}$. In order to have a globally well-defined orbifold, one has to check whether this action is free. As already discussed, the two $U(1)$ isometries of the three-sphere collapse into a single $U(1)$ in two points of the base. However, if the integers k_1 and k_2 are relatively prime, the action (B.33) becomes free at these points. In particular, one can convince oneself that the sets $\{\tilde{k}_2 m_1 + k_1 m_2 \,|\, m_1 \in [0, k_1 - 1], m_2 \in [0, \tilde{k}_2 - 1]\}$ and $\{\tilde{k}_2 m_1 - k_1 m_2 \,|\, m_1 \in [0, k_1 - 1], m_2 \in [0, \tilde{k}_2 - 1]\}$ contain exactly the same elements as $\mathbb{Z}_{k_1 k_2}$ but in a different order. Therefore, the total space will be $L(k_1 k_2, q)$ for some $q \neq 1$, and the resulting space is in fact a $\mathbb{Z}_p$-orbifold along an oblique direction.

In terms of the coordinates (B.9), the action (B.33) is

$$\chi \to \chi + \frac{2\pi m_1}{k_1}, \qquad \xi \to \xi + \frac{4\pi m_2}{k_2}, \tag{B.34}$$

where again the frame fails to describe the cases with even k_2. These cases can be described using the coordinates (B.23), which fail to describe cases with even k_1. Note that k_1 and k_2 can never be even at the same time.

B.4 Orbifolds of the $SU(2)_k$ WZW models

We next use the discussion in last section to construct orbifolds of the $SU(2)_k$ WZW models presented in section B.2. We will construct local metrics on them and compute the corresponding charges. We will see that the configurations presented here are the natural generalisations of the three-sphere to spaces with higher geometric charges.

The $\mathbb{Z}_{k_1}^{(\chi)}$ orbifold

We begin by considering the case where a discrete symmetry $\mathbb{Z}_{k_1}$ acts along the $U(1)_\chi$ fiber. The resulting space is locally the same as the original three-sphere, but is globally different. In particular, the orbifold background can be described by the same fields as in (B.10) but with the coordinate χ having the period

$$\chi \in \left[0, \frac{2\pi}{k_1}\right), \qquad k_1 \in \mathbb{Z}_+ . \tag{B.35}$$

To restore the 2π-periodicity we rescale $\chi \to k_1 \chi$, after which the resulting background fields read

$$\begin{aligned} ds^2 &= \frac{R^2}{4}\left(d\theta^2 + \frac{4}{k_1^2}d\chi^2 + d\xi^2 - \frac{4}{k_1}\cos\theta\, d\chi\, d\xi\right), \\ H &= \frac{k_3}{2}\sin\theta\, d\theta \wedge d\xi \wedge d\chi , \end{aligned} \tag{B.36}$$

where now $\chi \in [0, 2\pi)$ and where we defined $k_3 = k/k_1$. In order for this model to be conformal, the radius has to satisfy $R = \sqrt{|k_1 k_3|}$. As described in section B.3, this manifold is the Lens space $L(k_1, 1)$ and the full configuration corresponds to a $SU(2)_{k_1 k_3}/\mathbb{Z}_{k_1}$ WZW model. Next, we compute the charges following the same procedure as in section B.2. With respect to the $U(1)_\chi$ fiber, the associated gauge field is

$$\mathcal{A}_\chi^{k_1} = -\frac{k_1}{2}\cos\theta\, d\xi , \tag{B.37}$$

from which we determine the geometric charge as $n_\chi = k_1$. The charge corresponding to the H-flux is determined as before and is given by $h = k_3$.

Let us now observe that similarly as in (B.10), the orbifolded background (B.36) has in addition a local isometry along the direction ξ. However, if one naively tried to calculate the geometric charge with respect to this directions, following the method above, one would obtain $n_\xi = 2/k_1$, which fails to be in $2\mathbb{Z}$. The reason is that after the orbifolding procedure, the ξ-fiber is not a $U(1)$ bundle anymore. Intuitively, the fiber along this direction remains a $U(1)$ everywhere except at the points $(z_0, z_1) = (0, 1)$ and $(z_0, z_1) = (1, 0)$, where the two fibers collide. There, the ξ-fiber becomes $U(1)/\mathbb{Z}_{k_1}$, and one cannot define a $U(1)$ connection along this direction.

The $\mathbb{Z}_{k_2}^{(\xi)}$ orbifold

A very similar discussion applies if one constructs an orbifold by acting with $\mathbb{Z}_{k_2}$ on the $U(1)_\xi$ fiber of the original S^3. In this case, the resulting configuration is

$$\begin{aligned} ds^2 &= \frac{R^2}{4}\left(d\theta^2 + 4d\chi^2 + \frac{1}{k_2^2}d\xi^2 - \frac{4}{k_2}\cos\theta\, d\chi\, d\xi\right), \\ H &= \frac{k_3}{2}\sin\theta\, d\theta \wedge d\xi \wedge d\chi , \end{aligned} \tag{B.38}$$

with $\chi, \xi \in [0, 2\pi)$ and the radius has to satisfy $R^2 = |k_2 k_3|$. One can now compute the charge with respect to the ξ direction, for which the corresponding gauge field reads

$$\mathcal{A}_\xi^{k_2} = -2k_2 \cos\theta\, d\chi . \tag{B.39}$$

Using the conventions (B.22) one obtains for the charge $n_\xi = 2k_2$, which satisfies $n_\xi \in 2\mathbb{Z}$. Analogous to what happened in the previous case, the configuration is not a $U(1)_\chi$ bundle anymore, therefore the charge is not well-defined. Furthermore, as argued in section B.3 the configuration (B.38) is only well-defined for odd k_2, which are the only cases compatible with the artificial orbifolding along the direction ξ encoded in the coordinate frame. If one wants to construct $\mathbb{Z}^{(\xi)}_{k_2}$ orbifolds with even k_2, one needs to use the frame (B.23), where the configuration would read

$$\begin{aligned} ds^2 &= \frac{R^2}{4}\left(d\theta^2 + d\tilde{\chi}^2 + \frac{4}{k_2^2}d\tilde{\xi}^2 - \frac{4}{k_2}\cos\theta\, d\tilde{\chi}\, d\tilde{\xi}\right), \\ H &= \frac{k_3}{2}\sin\theta\; d\theta\wedge d\tilde{\xi}\wedge d\tilde{\chi}\,. \end{aligned} \tag{B.40}$$

The $\mathbb{Z}^{(\chi)}_{k_1}\times\mathbb{Z}^{(\xi)}_{k_2}$ orbifold

Finally, let us comment on $\mathbb{Z}_{k_1}\times\mathbb{Z}_{k_2}$ orbifolds. As discussed in section B.4, the only possible cases are those where k_1 and k_2 are relatively prime, which are the only cases where the action acts freely in the points where the two $U(1)$'s collapse. However, since the group acts as $\mathbb{Z}_{k_1k_2}$ on these $U(1)$'s but as $\mathbb{Z}_{k_1}\times\mathbb{Z}_{k_2}$ elsewhere, none of the directions χ or ξ will be global isometries.

Following the discussions above, we write the most general orbifold configuration using the coordinates (B.9) as

$$\begin{aligned} ds^2 &= \frac{R^2}{4}\left(d\theta^2 + \frac{4}{k_1^2}d\chi^2 + \frac{1}{k_2^2}d\xi^2 - \frac{4}{k_1k_2}\cos\theta\, d\chi\, d\xi\right), \\ H &= \frac{k_3}{2}\sin\theta\, d\theta\wedge d\xi\wedge d\chi\,, \end{aligned} \tag{B.41}$$

or using the frame (B.23) as

$$\begin{aligned} ds^2 &= \frac{R^2}{4}\left(d\theta^2 + \frac{1}{k_1^2}d\tilde{\chi}^2 + \frac{4}{k_2^2}d\tilde{\xi}^2 - \frac{4}{k_1k_2}\cos\theta\, d\tilde{\chi}\, d\tilde{\xi}\right), \\ H &= \frac{k_3}{2}\sin\theta\, d\theta\wedge d\tilde{\xi}\wedge d\tilde{\chi}\,. \end{aligned} \tag{B.42}$$

Note that the first configuration fails to describe the cases with even k_2 and the second the cases with even k_1. In both situations the radius is $R=\sqrt{|k_1k_2k_3|}$. A direct computation shows that the geometric charges are $n_\chi = \frac{k_1}{k_2}$ and $n_\xi = \frac{2k_2}{k_1}$ for the first case and $\tilde{n}_\chi = \frac{2k_1}{k_2}$ and $\tilde{n}_\xi = \frac{k_2}{k_1}$ for the second. Except for $k_1 = 1$ or $k_2 = 1$, all of them are non-integers since k_1 and k_2 are relatively prime.

B.5 Clifford tori

As it can be inferred for instance from the metric in (B.8), the three-sphere can locally be seen as a two-torus – constructed with the two $U(1)$ directions – fibered over a line segment. This is not a globally-defined fibration structure, since there are two points on the base where the torus degenerates to a circle. However, this picture is nevertheless useful since we can interpret T-duality transformations as $O(2,2;\mathbb{Z})$ acting on the $\mathbb{T}^2$.

Kähler and complex structure of $\mathbb{T}^2$

Let us therefore parametrize the two-torus in terms of its complex and Kähler structure τ and ρ as

$$\tau = \frac{g_{\chi\xi}}{g_{\chi\chi}} + i\frac{\sqrt{\det g_{\mathbb{T}^2}}}{g_{\chi\chi}}\,, \qquad \rho = B_{\mathbb{T}^2} + i\sqrt{\det g_{\mathbb{T}^2}}\,. \tag{B.43}$$

In terms of these parameters the general configuration (B.41) is described by

$$\tau = -\frac{1}{2}\,\frac{k_1}{k_2}\,e^{-i\theta}\,, \qquad \rho = -\frac{1}{2}\,k_3\,e^{-i\theta}\,, \tag{B.44}$$

where $\theta \in [0,\pi]$ is the coordinate along the line segment. Note that at the end-points of the segment the imaginary parts of τ and ρ vanish, which are the points where one of the two cycles of the torus collapses. For the following analysis, we will also consider the coordinate system (B.23), in which the parameters (B.43) for (B.42) are

$$\tilde{\tau} = -2\,\frac{k_1}{k_2}\,e^{-i\theta} = 4\tau\,, \qquad \tilde{\rho} = -\frac{1}{2}\,k_3\,e^{-i\theta} = \rho\,. \tag{B.45}$$

Finally, for all three-spheres the component of the metric on the base is required to be $R^2 = |k_1k_2k_3|$. This is not affected by any transformation of the toroidal coordinates, and hence a transformation that preserves R^2 will preserve the three-sphere structure.

$SL(2,\mathbb{Z})_\tau$ transformations

To get some understanding of the three-sphere from the point of view of a torus fibration, we analyze how $SL(2,\mathbb{Z})$ transformations act on the complex structure τ. These transformations are large diffeomorphisms, and therefore we can also understand them as transformations acting on the coordinates (ξ,χ). However, as already discussed, this coordinate frame has some orbifold structure intrinsically encoded in it, which also needs to be taken into account.

In particular, let us analyze the transformation of the form $\tau \to -1/\tau$, which corresponds to a $\pi/2$-rotation of the coordinates. To do this, we will rely on the original Hopf coordinates (B.7), where the two angular directions form a torus with no additional structure. Under this transformation, the coordinates (ξ,χ) transform as

$$\xi \to -2\chi = -\tilde{\chi}\,, \qquad \chi \to \frac{1}{2}\xi = \tilde{\xi}\,, \tag{B.46}$$

and the orbifolding structure is also rotated. Therefore, the natural transformation to study is

$$\tau \to -\frac{1}{\tilde{\tau}} = -\frac{1}{4\tau}, \tag{B.47}$$

which is not an $SL(2,\mathbb{Z})$ transformation anymore, but is indeed the transformation obtained by conjugating $\tau \to -1/\tau$ with the coordinate transformation (B.9). Applying this transformation to (B.44), the resulting configuration is described by

$$\tau' = \frac{1}{2}\,\frac{k_2}{k_1}\,e^{i\theta}\,, \tag{B.48}$$

which acts on the integer numbers k_1 and k_2 by

$$k_1' = k_2\,, \qquad\qquad k_2' = -k_1\,, \tag{B.49}$$

as one would have naturally guessed. The correct sign for $\operatorname{Im}\tau'$ is obtained by taking absolute values of k_i'. This transformation preserves the radius R of the three-sphere and, therefore, the resulting configuration is again a three-sphere orbifold.

Finally, we point-out that the same result can also be obtained by an explicit rotation of the metric, after which the geometric charges in the conventions (B.18) and (B.22) are $n_\chi = -\frac{k_2}{k_1}$ and $n_\xi = -\frac{2k_1}{k_2}$, which is in agreement with (B.49).

B.6 T-duality

We now want to investigate the configurations obtained after applying a factorized T-duality on the sphere and its orbifolds. Buscher's approach to T-duality involves gauging isometries of the background [14, 15, 13], and we therefore expect to find globally well-defined T-dual spaces when the isometry of the original background is globally defined.

T-duality along χ: $\tau \leftrightarrow \rho$

We start by considering T-duality transformations along the direction χ, which in the conventions (B.43) with coordinate frame (B.9) corresponds to the interchange $\tau \leftrightarrow \rho$. The configuration dual to (B.44) is characterized by the parameters

$$\tau' = -\frac{1}{2}\,k_3\,e^{-i\theta}, \qquad\qquad \rho' = -\frac{1}{2}\,\frac{k_1}{k_2}\,e^{-i\theta}\,, \tag{B.50}$$

with corresponding charges

$$n_\chi' = k_3\,, \qquad\qquad n_\xi' = \frac{2}{k_3}\,, \qquad\qquad h' = \frac{k_1}{k_2}\,. \tag{B.51}$$

The condition $h' \in \mathbb{Z}$ can only be satisfied if $k_2 = 1$, since the original background is globally well-defined if and only if k_1 and k_2 are coprime. In fact, as discussed in section B.4, this is the only case when the isometry along χ of the original configuration is globally-defined. Therefore, dualizing along it when $k_2 \neq 1$ leads to a background globally ill-defined. If the condition $k_2 = 1$ is satisfied, the radius R of the three-sphere remains invariant and the resulting configuration is a three-sphere orbifold with charges $(n_\chi', h') = (h, n_\chi)$ [167].

The same results can be also obtained by direct application of Buscher rules to the configuration (B.41) with $k_2 = 1$, obtaining

$$\begin{aligned} ds^2 &= \frac{k_1 k_3}{4}\left(d\theta^2 + \frac{4}{k_3^2}d\chi^2 + d\xi^2 - \frac{4}{k_3}\cos\theta\, d\chi\, d\xi\right), \\ H &= \frac{k_1}{2}\sin\theta\, d\theta\wedge d\xi\wedge d\chi\,, \end{aligned} \tag{B.52}$$

and we can then conclude that the effect of T-duality along the direction χ is the interchange $k_1 \leftrightarrow k_3$, provided $k_2 = 1$. Therefore, in this case, T-duality relates the following two conformal theories [132]

$$\frac{SU(2)_{k_1k_3}}{\mathbb{Z}_{k_1}} \quad\longleftrightarrow\quad \frac{SU(2)_{k_1k_3}}{\mathbb{Z}_{k_3}}\,. \tag{B.53}$$

T-duality along $\tilde{\xi}$: $\tilde{\tau} \leftrightarrow -1/\tilde{\rho}$

Next, we consider the T-duality transformation along direction ξ. The natural coordinate-frame to describe this duality is (B.23), where it corresponds to the interchange $\tilde{\tau} \leftrightarrow -1/\tilde{\rho}$ in (B.45). The dual background is then characterized by

$$\tilde{\tau}' = 2\,\frac{1}{k_3}e^{i\theta}\,, \qquad\qquad \tilde{\rho}' = \frac{1}{2}\,\frac{k_2}{k_1}e^{i\theta}\,, \tag{B.54}$$

with charges

$$\tilde{n}'_\chi = -\frac{2}{k_3}\,, \qquad\qquad \tilde{n}'_\xi = -k_3\,, \qquad\qquad \tilde{h}' = -\frac{k_2}{k_1}\,, \tag{B.55}$$

and again the condition $\tilde{h} \in \mathbb{Z}$ is only satisfied when $k_1 = 1$, which is the case where the isometry in the original configuration is globally well-defined.

As in the case of T-duality along χ, the same results can be obtained by direct application of Buscher rules to the configuration (B.42). More interestingly, it is possible to obtain an equivalent result within the coordinate-frame of (B.41). In this frame, the Killing vector is $\bar{v} = 2\partial_\xi$ (see (B.11)), and applying the Buscher rules gives the T-dual background[2]

$$\begin{aligned} ds^2 &= \frac{k_2 k_3}{4}\left(d\theta^2 + 4d\chi^2 + \frac{1}{k_3^2}d\xi^2 + \frac{4}{k_3}\cos\theta\, d\xi\, d\chi\right), \\ H &= -\frac{k_2}{2}\sin\theta\, d\theta\wedge d\xi\wedge d\chi\,, \end{aligned} \tag{B.56}$$

and one can use the previously-mentioned procedure to compute the charges to obtain

$$n'_\xi = -2k_3\,, \qquad\qquad h' = -k_2\,, \tag{B.57}$$

[2] For T-duality transformations along Killing vector-fields which are not normalized to one, see for instance [31, 32].

which are consistent with the results found using the other coordinate-frame. Furthermore, these results could have been obtained also by considering the transformation $\tau \leftrightarrow -1/4\rho$. We then conclude that the effect of T-duality along the direction ξ, with $k_1 = 1$, is the interchange $k_2 \leftrightarrow k_3$ and relates the conformal theories

$$\frac{SU(2)_{k_2k_3}}{\mathbb{Z}_{k_2}} \quad \longleftrightarrow \quad \frac{SU(2)_{k_2k_3}}{\mathbb{Z}_{k_3}} . \tag{B.58}$$

T-duality along χ and ξ: $\tau \leftrightarrow -1/4\tau$, $\rho \leftrightarrow -1/4\rho$

Finally, we consider factorized dualities along both directions of the torus simultaneously. Consecutively applying T-duality along the directions χ and ξ corresponds to the transformation (in terms of the coordinate frame(B.9))

$$\tau \to -\frac{1}{4\tau} , \qquad \rho \to -\frac{1}{4\rho} . \tag{B.59}$$

Applying these transformations to (B.41) one obtains a configuration where the fibered torus is described by

$$\tau' = \frac{1}{2}\,\frac{k_2}{k_1}e^{i\theta} , \qquad \rho' = \frac{1}{2}\,\frac{1}{k_3}e^{i\theta} , \tag{B.60}$$

and corresponding charges read

$$n'_\chi = -\frac{k_2}{k_1}, \qquad n'_\xi = -\frac{2k_1}{k_2}, \qquad h' = -\frac{1}{k_3}. \tag{B.61}$$

The NS charge is properly quantized only for $k_3 = 1$. This is in fact the self-dual point for the ρ-transformation and also the case where the radius R of the three-sphere remains invariant. We emphasize that it is enough to have only one of the geometric charges correctly quantized in order to have at least one integer geometric charge in the dual background. In fact, the dual background for the case $k_3 = 1$ is the original background after a $\pi/2$-rotation.

Summary

The various cases of T-duality transformations discussed in this section are summarized in table B.1. From there one can see how geometric charges and the NS-charge are interchanged, and we highlight the cases of a globally-defined $U(1)_\chi$ and $U(1)_\xi$ isometry as well as the case $h = 1$. We observe that, after a T-duality transformation along a globally-defined isometry of the three-sphere orbifold, the dual background is again a three-sphere orbifold.

B.6.1 T-duality along non-globally-defined $U(1)$ fibers

So far we have discussed duality transformations for three-sphere orbifolds along the directions of the vector fields (B.11). In this section, we analyze the local fields obtained

model				global $U(1)_\chi$			global $U(1)_\xi$			$h=1$		
	n_χ	n_ξ	h	n_χ	n_ξ	h	n_χ	n_ξ	h	n_χ	n_ξ	h
S^3	$\frac{k_1}{k_2}$	$2\,\frac{k_2}{k_1}$	k_3	k_1		k_3		$2\,k_2$	k_3	$\frac{k_1}{k_2}$	$2\,\frac{k_2}{k_1}$	1
$T_\chi(S^3)$	k_3	$2\,\frac{1}{k_3}$	$\frac{k_1}{k_2}$	k_3		k_1				1	2	$\frac{k_1}{k_2}$
$T_\xi(S^3)$	$\frac{1}{k_3}$	$2\,k_3$	$\frac{k_2}{k_1}$					$2\,k_3$	k_2	1	2	$\frac{k_2}{k_1}$
$T_{(\chi,\xi)}(S^3)$	$-\frac{k_2}{k_1}$	$-2\,\frac{k_1}{k_2}$	$-\frac{1}{k_3}$							$-\frac{k_2}{k_1}$	$-2\,\frac{k_1}{k_2}$	-1

Table B.1: Summary of geometric and NS-charges for the three-sphere orbifold and its T-dual configurations. In the first column, $T_\star(S^3)$ denotes the T-dual of S^3 orbifold along the direction(s) $\star$. For a globally-defined $U(1)_\chi$ isometry $k_2 = 1$ is needed, and we have displayed only the integer charges. Similarly, for a globally-defined $U(1)_\xi$ isometry $k_1 = 1$ has to be required and we again only showed the integer charges.

by T-duality transformations along an arbitrary direction of the Clifford torus. In general, these directions will not be globally-defined $U(1)$ fibers, and consequently the dual backgrounds may not be globally-defined or may become non-compact.

For the present discussion, we will go back to the original Hopf coordinates to avoid any choice of frame that singles out a particular direction. In this frame, a general sphere-orbifold takes the form

$$\begin{aligned} ds^2 &= R^2 \left(d\eta^2 + \frac{1}{\alpha_1^2} \cos^2\eta\, d\xi_1^2 + \frac{1}{\alpha_2^2} \sin^2\eta\, d\xi_2^2 \right) , \\ H &= 2\,\alpha_3 \sin\eta\, \cos\eta\, d\eta \wedge d\xi_1 \wedge d\xi_2 , \end{aligned} \tag{B.62}$$

with $R^2 = |\alpha_1\alpha_2\alpha_3|$ and $|\alpha_i| \in \mathbb{Z}_+$. As follows from our previous discussion, not all values of (α_1, α_2) are possible in order to have a globally-defined background. Next, we T-dualize along the isometry $\mathsf{v} = \beta_1\partial_{\xi_1} + \beta_2\partial_{\xi_2}$ for arbitrary (β_1, β_2) using the methods described in [31, 32]. After a choice of local coordinates (ψ_1, ψ_2) we obtain the dual configuration

$$\begin{aligned} ds^2 &= R^2 \left(d\eta^2 + \frac{\cos^2\eta}{\Delta}\, d\psi_1^2 + \frac{\sin^2\eta}{\Delta}\, d\psi_2^2 \right) , \\ H &= \frac{\alpha_1^2\,\alpha_2^2\,\alpha_3\,\beta_1\,\beta_2 \sin(2\eta)}{\Delta^2}\, d\eta \wedge d\psi_1 \wedge d\psi_2 , \\ e^{2\Phi} &= e^{2\Phi_0}\, \frac{\alpha_1\alpha_2}{\alpha_3}\, \frac{1}{\Delta} , \end{aligned} \tag{B.63}$$

where we have defined

$$\Delta = \big(\alpha_1 \beta_2 \sin\eta\big)^2 + \big(\alpha_2 \beta_1 \cos\eta\big)^2 . \tag{B.64}$$

Since this is a local procedure, it does not give information about the global properties of the coordinates (ψ_1, ψ_2), which could be even non-compact. We however observe that by choosing $(\beta_1, \beta_2) = (\alpha_1, \pm\alpha_2)$ the dual configuration is locally a three-sphere, and for $\beta_1, \beta_2 \neq 0$ the space is non-singular.

Appendix C

Supersymmetry analysis of the NS5-branes and its duals

In this appendix we give details of the analysis of the supersymmetry variations (4.14) for some of the configurations appearing in section 4.1. During the calculation the following notation will be used:

- Capital letters $M, N, ...$ correspond to curved space-time indices and take values $M \in \{0, \dots 9\}$.
- Space-time indices are separated into those along the brane, $\mu \in \{0, \dots, 5\}$, and those perpendicular to it, $i \in \{r, \theta, \chi, \xi\}$.
- Flat indices are denoted by a hat. For the transverse space we have $\hat{\imath} \in \{\hat{6}, \hat{7}, \hat{8}, \hat{9}\}$.

The ten-dimensional fields are constructed by trivially adding the brane-volume directions to the fields describing the transverse space. Note that both the spin-connection and the NS field strength will have non-zero components only along the directions transversal to the brane. Furthermore, we use conventions where $\not{H} = \frac{1}{3!}\Gamma^{\hat{M}\hat{N}\hat{P}} H_{\hat{M}\hat{N}\hat{P}}$, $\not{H}_M = \frac{1}{2!}\Gamma^{\hat{N}\hat{P}} H_{\hat{N}\hat{P}M}$ and $\Gamma^{\hat{M}\dots\hat{N}} = \frac{1}{p!}\Gamma^{[\hat{M}} \dots \Gamma^{\hat{N}]}$.

The NS5-orbifold

We will first analyze the supersymmetry preserved by the NS5-orbifold solution (4.3). This includes the NS5-brane as a particular case, which is well-known that it preserves half of the supersymmetries.

Dilatino variation We begin by analyzing the first of the variations in (4.14). A direct calculation shows that, in the present case,

$$\delta_\epsilon \lambda = -\frac{|k_1 k_2 k_3|}{2r^3 h(r)^{3/2}}\,\Gamma_{\hat{6}}\,(1 \pm \text{sgn}(k_1 k_2 k_3)\Gamma_{\hat{6}}\Gamma_{\hat{7}}\Gamma_{\hat{8}}\Gamma_{\hat{9}})\,\epsilon\,, \tag{C.1}$$

where $\mathrm{sgn}(x)$ is the sign function and we pick the plus sign when acting on ϵ_+ and the minus when acting on ϵ_-. The matrix $\Gamma_{[4]} = \Gamma_{\hat{6}}\Gamma_{\hat{7}}\Gamma_{\hat{8}}\Gamma_{\hat{9}}$ is the chirality operator of a representation of the four dimensional euclidean Clifford algebra and, therefore, $\frac{1}{2}\left(1 \pm \Gamma_{[4]}\right)$ are projectors. Also, note that the two chirality operators $\Gamma_{[4]}$ and $\Gamma_{[10]}$ commute and one can construct spinors which are chiral with respect to both of them. We conclude then, that supersymmetry variations of the dilatino vanishes for spinors satisfying the condition.

$$\left(1 \pm \mathrm{sgn}(k_1k_2k_3)\Gamma_{\hat{6}}\Gamma_{\hat{7}}\Gamma_{\hat{8}}\Gamma_{\hat{9}}\right)\epsilon_{\pm} = 0\,, \tag{C.2}$$

which reduces the degrees of freedom of the original Majorana-Weyl spinors by one half. For the following we will assume that without loss of generality $\mathrm{sgn}(k_1k_2k_3) = 1$. In the case where $\mathrm{sgn}(k_1k_2k_3) = -1$, the two spinors of the doublet ϵ are interchanged.

Gravitino variation for ϵ_+ We next analyze the second condition in (4.14) for the component ϵ_+. For the components $M = \mu$, the equations $\delta_\epsilon\Psi_M = 0$ reduce to $\partial_\mu\epsilon = 0$, and the Killing spinors have to be constant along the brane directions. For $M = i$ the variations are

$$\begin{aligned}
\delta_{\epsilon_+}\Psi_r &= \partial_r\epsilon_+,\\
\delta_{\epsilon_+}\Psi_\theta &= \partial_\theta\epsilon_+ + \tfrac{1}{4h(r)}\left(\Gamma_{\hat{8}}\Gamma_{\hat{9}} - \Gamma_{\hat{6}}\Gamma_{\hat{7}}\right)\epsilon_+\,,\\
\delta_{\epsilon_+}\Psi_\xi &= \partial_\xi\epsilon_+ - \tfrac{1}{4k_2h(r)}\Big(\sin\theta\left(\Gamma_{\hat{7}}\Gamma_{\hat{9}} + \Gamma_{\hat{6}}\Gamma_{\hat{8}}\right) + \cos\theta\left(\Gamma_{\hat{7}}\Gamma_{\hat{8}} - \Gamma_{\hat{6}}\Gamma_{\hat{9}}\right)\Big)\epsilon_+\,,\\
\delta_{\epsilon_+}\Psi_\chi &= \partial_\chi\epsilon_+ - \tfrac{1}{2r^2k_1h(r)}\Big(2k_1k_2k_3\,\Gamma_{\hat{7}}\Gamma_{\hat{8}} + r^2\left(\Gamma_{\hat{7}}\Gamma_{\hat{8}} + \Gamma_{\hat{6}}\Gamma_{\hat{9}}\right)\Big)\epsilon_+\,,
\end{aligned} \tag{C.3}$$

and applying them to a spinor ϵ_+ satisfying (C.2) they reduce to

$$\begin{aligned}
\delta_{\epsilon_+}\Psi_r &= \partial_r\epsilon_+\,, & \delta_{\epsilon_+}\Psi_\xi &= \partial_\xi\epsilon_+\,,\\
\delta_{\epsilon_+}\Psi_\theta &= \partial_\theta\epsilon_+\,, & \delta_{\epsilon_+}\Psi_\chi &= \left(\partial_\chi - \tfrac{1}{k_1}\Gamma_{\hat{7}}\Gamma_{\hat{8}}\right)\epsilon_+\,.
\end{aligned} \tag{C.4}$$

A general solution to the equations $\delta_{\epsilon_+}\lambda = 0$ and $\delta_{\epsilon_+}\Psi_M = 0$ is

$$\epsilon_+ = e^{\left(\frac{\chi}{k_1}\Gamma_{\hat{7}}\Gamma_{\hat{8}}\right)}\epsilon_{0,+} \qquad \text{with } \left(1 + \Gamma_{\hat{6}}\Gamma_{\hat{7}}\Gamma_{\hat{8}}\Gamma_{\hat{9}}\right)\epsilon_{0,+} = 0\,, \tag{C.5}$$

where $\epsilon_{0,+}$ is a Majorana-Weyl spinor with constant entries.

Gravitino variation for ϵ_- We perform now the same analysis for ϵ_-. The supersymmetry variations for the gravitino along the $M = i$ directions are

$$\begin{aligned}
\delta_{\epsilon_-}\Psi_r &= \partial_r\epsilon_-\,,\\
\delta_{\epsilon_-}\Psi_\theta &= \partial_\theta\epsilon_- - \tfrac{1}{4r^2h(r)}\Big(2N\,\Gamma_{\hat{8}}\Gamma_{\hat{9}} + r^2\left(\Gamma_{\hat{8}}\Gamma_{\hat{9}} - \Gamma_{\hat{6}}\Gamma_{\hat{7}}\right)\Big)\epsilon_-\,,\\
\delta_{\epsilon_-}\Psi_\xi &= \partial_\xi\epsilon_- - \tfrac{1}{4k_2r^2h(r)}\Big(\left(2k_1k_2k_3 + r^2\right)\Gamma_7\left(\sin\theta\,\Gamma_9 + \cos\theta\,\Gamma_8\right)\\
&\qquad\qquad - r^2\Gamma_6\left(\cos\theta\,\Gamma_9 - \sin\theta\,\Gamma_8\right)\Big)\epsilon_-\,,\\
\delta_{\epsilon_-}\Psi_\chi &= \partial_\chi\epsilon_- - \tfrac{1}{2k_1h(r)}\left(\Gamma_{\hat{7}}\Gamma_{\hat{8}} + \Gamma_{\hat{6}}\Gamma_{\hat{9}}\right)\epsilon_-\,,
\end{aligned} \tag{C.6}$$

and applying them to a spinor ϵ_- satisfying (C.2) they reduce to

$$\begin{aligned} \delta_{\epsilon_-}\Psi_r &= \partial_r\epsilon_- \,, & \delta_{\epsilon_-}\Psi_\xi &= \big(\partial_\xi - \tfrac{1}{2k_2}\sin\theta\,\Gamma_{\hat 7}\Gamma_{\hat 9} - \tfrac{1}{2k_2}\cos\theta\,\Gamma_{\hat 7}\Gamma_{\hat 8}\big)\epsilon_- \,, \\ \delta_{\epsilon_-}\Psi_\theta &= \big(\partial_\theta + \tfrac{1}{2}\Gamma_{\hat 8}\Gamma_{\hat 9}\big)\epsilon_- \,, & \delta_{\epsilon_-}\Psi_\chi &= \partial_\chi\epsilon_- \,. \end{aligned} \tag{C.7}$$

The general solution to the supersymmetry equations is

$$\epsilon_- = e^{-\frac{\theta}{2}\Gamma_{\hat 8}\Gamma_{\hat 9}} e^{\frac{\xi}{2k_2}\Gamma_{\hat 7}\Gamma_{\hat 8}}\epsilon_{0,-} \qquad \text{with } (1 - \Gamma_{\hat 6}\Gamma_{\hat 7}\Gamma_{\hat 8}\Gamma_{\hat 9})\,\epsilon_{0,-} = 0\,, \tag{C.8}$$

where $\epsilon_{0,-}$ is again a Majorana-Weyl spinor with constant entries.

T-dual configuration along χ

Next, we analyze the amount of supersymmetry preserved by the configuration obtained after performing a T-duality transformation along the direction χ to the NS5-orbifold, described by the fields (4.8). We will find that part of the supersymmetry of the original background is broken by the T-duality transformation.

Dilatino variation The dilatino variation for the present background is now

$$\delta_\epsilon\lambda = -\frac{1}{2r\sqrt{h(r)}}\,\Gamma_{\hat 6}\,(1 \pm \Gamma_{\hat 6}\Gamma_{\hat 7}\Gamma_{\hat 8}\Gamma_{\hat 9})\,\epsilon, \tag{C.9}$$

which is again solved by a doublet of spinors satisfying

$$(1 \pm \Gamma_{\hat 6}\Gamma_{\hat 7}\Gamma_{\hat 8}\Gamma_{\hat 9})\,\epsilon_\pm = 0\,. \tag{C.10}$$

Note that in this case this condition is independent of the sign of $k_1k_2k_3$. As in the case before, this condition projects out half of the components of each Majorana-Weyl spinor.

Gravitino variation for ϵ_+ Although the dilatino variations do not depend on the sign of $k_1k_2k_3$, the gravitino variations do depend on it. For simplicity, we will only discuss the case where $k_1k_2k_3 > 0$. The case where $k_1k_2k_3 < 0$ can be discussed analogously, and the results are interchanged between the two components of the doublet. With the mentioned sign assumption, the variations $\delta_{\epsilon_+}\Psi_i$ for a spinor ϵ_+ satisfying (C.10) are

$$\begin{aligned} \delta_{\epsilon_+}\Psi_r &= \partial_r\epsilon_+ \,, \\ \delta_{\epsilon_+}\Psi_\theta &= \Big(\partial_\theta + \tfrac{1}{2h(r)}\Gamma_{\hat 6}\Gamma_{\hat 7}\Big)\,\epsilon_+ \,, \\ \delta_{\epsilon_+}\Psi_\xi &= \Big(\partial_\xi + \tfrac{k_1k_2k_3}{2k_2r^2h(r)^2}\cos\theta\Gamma_{\hat 7}\Gamma_{\hat 8} + \tfrac{1}{2k_2h(r)}(\sin\theta\Gamma_{\hat 6}\Gamma_{\hat 8} + \cos\theta\Gamma_{\hat 7}\Gamma_{\hat 8})\Big)\,\epsilon_+ \,, \\ \delta_{\epsilon_+}\Psi_\chi &= \Big(\partial_\chi - \tfrac{k_1^2k_2k_3}{r^4h(r)^2}\Gamma_{\hat 7}\Gamma_{\hat 8}\Big)\,\epsilon_+ \,. \end{aligned} \tag{C.11}$$

The first equation, $\delta_{\epsilon_+}\Psi_r = 0$, is solved by spinors which are constant along the direction r. However, assuming this condition, the other equations $\delta_{\epsilon_+}\Psi_i = 0$ cannot be solved. Therefore, for the present configuration there is no spinor ϵ_+ satisfying the condition (C.10) with plus sign.

Gravitino variation for ϵ_- For the case of ϵ_-, the variations $\delta_{\epsilon_-}\Psi_i$ for a spinor ϵ_- satisfying (C.10) are

$$\begin{aligned} \delta_{\epsilon_-}\Psi_r &= \partial_r\epsilon_- \,, \\ \delta_{\epsilon_-}\Psi_\xi &= \left(\partial_\xi - \tfrac{1}{2k_2}\sin\theta\,\Gamma_{\hat{7}}\Gamma_{\hat{9}} - \tfrac{1}{2k_2}\cos\theta\,\Gamma_{\hat{7}}\Gamma_{\hat{8}}\right)\epsilon_- \,, \\ \delta_{\epsilon_-}\Psi_\theta &= \left(\partial_\theta + \tfrac{1}{2}\Gamma_{\hat{8}}\Gamma_{\hat{9}}\right)\epsilon_- \,, \\ \delta_{\epsilon_-}\Psi_\chi &= \partial_\chi\epsilon_- \,, \end{aligned} \tag{C.12}$$

which are the same as in the original NS5-orbifold. Therefore, the equations $\delta_{\epsilon_-}\Psi_M = 0$ are solved again by

$$\epsilon_- = e^{-\frac{\theta}{2}\Gamma_{\hat{8}}\Gamma_{\hat{9}}}e^{\frac{\xi}{2k_2}\Gamma_{\hat{7}}\Gamma_{\hat{8}}}\epsilon_{0,-} \qquad \text{with } \left(1+\Gamma_{\hat{6}}\Gamma_{\hat{7}}\Gamma_{\hat{8}}\Gamma_{\hat{9}}\right)\epsilon_{0,-} = 0 \,. \tag{C.13}$$

This solution corresponds to the case where $k_1k_2k_3 > 0$. In the case where $k_1k_2k_3 < 0$, the ϵ_- component has no solution whereas the ϵ_+ has a solution of the form (C.13). In both cases, only half of the Killing spinors of the original background are present after the T-duality transformation. The configuration is then 1/4-supersymmetric.

Appendix D

Sections of $SO(5,5)$ Exceptional Field Theory

In this Appendix we summarise the most relevant facts about M-theory and type IIB sections of $SO(5,5)$ Exceptional Field Theory.

D.1 M-theory section

D.1.1 Conventions

By choosing the solution to the section condition that breaks $SO(5,5) \to SL(5) \times \mathbb{R}^+$ the fundamental ($\mathbf{10}$) and spinor ($\mathbf{16}$) representations decompose into

$$\begin{aligned} \mathbf{10} &\to \mathbf{5}_{-2} \oplus \bar{\mathbf{5}}_2 \,, \\ \mathbf{16} &\to \bar{\mathbf{5}}_{-3} \oplus \mathbf{10}_1 \oplus \mathbf{1}_5 \,, \end{aligned} \tag{D.1}$$

where the subscript corresponds to the weight with respect to the $\mathbb{R}^+$ factor. We will then write a vector V^N in the $\mathbf{16}$ and a vector V^I in the $\mathbf{10}$ representations as

$$\begin{aligned} V^I &= (V^i, V_i) \,, \\ V^N &= (V^n, V_{n_1 n_2}, V_{(z)}) \,, \end{aligned} \tag{D.2}$$

where $N \in \{1, \ldots, 16\}$, $I \in \{1, \ldots, 10\}$ and $i, n, \cdots \in \{1, \ldots, 5\}$. The indices $n_1 n_2$ are anti-symmetrised and the subscript (z) denotes a singlet. For the M-theory section we use the multiplication convention

$$\begin{aligned} V^I W_I &= V^i W_i + V_i W^i \,, \\ V^N W_N &= V^n W_n + V_{n_1 n_2} W^{n_1 n_2} + V_{(z)} W_{(z)} \,, \end{aligned} \tag{D.3}$$

Following these conventions the identity matrices are

$$\delta^I_J = \begin{pmatrix} \delta^i_j & 0 \\ 0 & \delta_i^j \end{pmatrix} \,, \qquad \delta^M_N = \begin{pmatrix} \delta^m_n & 0 & 0 \\ 0 & \delta^{n_1 n_2}_{m_1 m_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \,, \tag{D.4}$$

where we define $\delta^{n_1n_2}_{m_1m_2} = \frac{1}{2}(\delta^{m_1}_{n_1}\delta^{m_2}_{n_2} - \delta^{m_2}_{n_1}\delta^{m_1}_{n_2})$. The $SO(5,5)$ metric and its inverse are

$$\eta_{IJ} = \begin{pmatrix} 0 & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix}, \qquad \eta^{IJ} = \begin{pmatrix} 0 & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix}. \tag{D.5}$$

D.1.2 M-theory γ-matrices

We construct a set of matrices $(\gamma^I)^{MN}$ and $(\gamma^I)_{MN}$ satisfying the Clifford algebra relations

$$\left\{\gamma^I, \gamma^J\right\}^M{}_N = 2\eta^{IJ}\delta^M{}_N\,. \tag{D.6}$$

A possible choice is

$$\begin{aligned}
\left(\gamma^i\right)^{MN} &= \begin{pmatrix} (\gamma^i)^{m\,n} & (\gamma^i)^m{}_{n_1n_2} & (\gamma^i)^m{}_{(z)} \\ (\gamma^i)_{m_1m_2}{}^n & (\gamma^i)_{m_1m_2\,n_1n_2} & (\gamma^i)_{m_1m_2\,(z)} \\ (\gamma^i)^n_{(z)} & (\gamma^i)_{(z)\,n_1n_2} & (\gamma^i)_{(z)(z)} \end{pmatrix} = \begin{pmatrix} 0 & 2\delta^{i\ m}_{n_1n_2} & 0 \\ 2\delta^{i\ n}_{m_1m_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\left(\gamma_i\right)^{MN} &= \begin{pmatrix} (\gamma_i)^{m\,n} & (\gamma_i)^m{}_{n_1n_2} & (\gamma_i)^m{}_{(z)} \\ (\gamma_i)_{m_1m_2}{}^n & (\gamma_i)_{m_1m_2\,n_1n_2} & (\gamma_i)_{m_1m_2\,(z)} \\ (\gamma_i)^n_{(z)} & (\gamma_i)_{(z)\,n_1n_2} & (\gamma_i)_{(z)(z)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sqrt{2}\delta^m_i \\ 0 & \frac{1}{\sqrt{2}}\epsilon_{im_1m_2n_1n_2} & 0 \\ \sqrt{2}\delta^n_i & 0 & 0 \end{pmatrix}, \\
\left(\gamma^i\right)_{MN} &= \begin{pmatrix} (\gamma^i)_{m\,n} & (\gamma^i)_m{}^{n_1n_2} & (\gamma^i)_m{}^{(z)} \\ (\gamma^i)^{m_1m_2}{}_n & (\gamma^i)^{m_1m_2\,n_1n_2} & (\gamma^i)^{m_1m_2\,(z)} \\ (\gamma^i)^{(z)}_n & (\gamma^i)^{(z)\,n_1n_2} & (\gamma^i)^{(z)(z)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sqrt{2}\delta^i_m \\ 0 & \frac{1}{\sqrt{2}}\epsilon^{im_1m_2n_1n_2} & 0 \\ \sqrt{2}\delta^i_n & 0 & 0 \end{pmatrix}, \\
\left(\gamma_i\right)_{MN} &= \begin{pmatrix} (\gamma_i)_{m\,n} & (\gamma_i)_m{}^{n_1n_2} & (\gamma_i)_m{}^{(z)} \\ (\gamma_i)^{m_1m_2}{}_n & (\gamma_i)^{m_1m_2\,n_1n_2} & (\gamma_i)^{m_1m_2\,(z)} \\ (\gamma_i)^{(z)}_n & (\gamma_i)^{(z)\,n_1n_2} & (\gamma_i)^{(z)(z)} \end{pmatrix} = \begin{pmatrix} 0 & 2\delta^{n_1n_2}_{i\ m} & 0 \\ 2\delta^{m_1m_2}_{i\ n} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{D.7}$$

where ϵ_{ijklm} is the Levi-Civita symbol and we use the conventions where $\epsilon_{12345} = \epsilon^{12345} = 1$.

D.1.3 10 representation

We construct the generalised metric in the **10** representation $\mathcal{M}_{IJ}$ using the method described in (6.6). We also write down the generalised derivatives in terms of the M-theory fields. These will be used to compare the supergravity fields with the coefficients in the exponential when constructing the generalised vielbein.

Generalised derivatives

The generalised metric for vielbein in the **10** representation is

$$
\begin{aligned}
\mathcal{L}_\Lambda E_{\bar{I}}{}^J &= \Lambda^M \partial_M E_{\bar{I}}{}^J + \frac{1}{2}\left(\gamma_K{}^J\right)_M{}^N E_{\bar{I}}{}^K \partial_N \Lambda^M \\
&= \Lambda^M \partial_M E_{\bar{I}}{}^J + \frac{1}{2}\left(\gamma_K \gamma^J\right)_M{}^N E_{\bar{I}}{}^K \partial_N \Lambda^M - \frac{1}{2} E_{\bar{I}}{}^J \partial_M \Lambda^M .
\end{aligned} \tag{D.8}
$$

The generalised vector Λ^N encodes diffeomorphisms and gauge transformations for the different supergravity fields as $\Lambda^N = (\xi^n, \lambda_{n_1 n_2}, f)$, where ξ^n is a vector field corresponding to diffeomorphisms, $\lambda_{n_1 n_2}$ a two-form corresponding to gauge transformations of the M-theory 3-form and f is a scalar field. We will see that the generalised vielbein does not transform under this scalar field (neither in the **16** representation).

After imposing the section condition (all derivatives are zero except for ∂_m), the generalized derivatives for the M-theory parameters read:

- Diffeomorphisms: $\Lambda^M = (\xi^m, 0, 0)$

$$
\begin{aligned}
\mathcal{L}_\xi E_{\bar{I}}{}^j &= \xi^n \partial_n E_{\bar{I}}{}^j - E_{\bar{I}}{}^n \partial_n \xi^j + \frac{1}{2} E_{\bar{I}}{}^j \partial_n \xi^n , \\
\mathcal{L}_\xi E_{\bar{I} j} &= \xi^n \partial_n E_{\bar{I} j} + E_{\bar{I} n} \partial_j \xi^n - \frac{1}{2} E_{\bar{I} j} \partial_n \xi^n ,
\end{aligned} \tag{D.9}
$$

- 3-form gauge transformations: $\Lambda^M = (0, \lambda_{m_1 m_2}, 0)$

$$
\begin{aligned}
\mathcal{L}_\lambda E_{\bar{I}}{}^j &= \frac{1}{\sqrt{2}} E_{\bar{I} k}\, \epsilon^{kjnm_1 m_2} \partial_n \lambda_{m_1 m_2} , \\
\mathcal{L}_\lambda E_{\bar{I} j} &= 0 ,
\end{aligned} \tag{D.10}
$$

- Scalar parameter: $\Lambda^M = (0, 0, f)$

$$
\begin{aligned}
\mathcal{L}_f E_{\bar{I}}{}^j &= 0 , \\
\mathcal{L}_f E_{\bar{I} j} &= 0 .
\end{aligned} \tag{D.11}
$$

Algebra and generators

In the fundamental representation, the elements of the algebra $K_I{}^J$ satisfy the relation

$$
\left(\eta^{IK} K_K{}^J\right)^T = -\eta^{IK} K_K{}^J . \tag{D.12}
$$

They can be labelled by $(K^{AB})_I{}^J$, where A and B are indices in the **10**, and the generators satisfy $K^{AB} = -K^{BA}$. They are

$$
(K^a{}_b)_I{}^J = \begin{pmatrix} \delta^a_i \delta^j_b & 0 \\ 0 & -\delta^i_b \delta^a_j \end{pmatrix} , \quad (K^{ab})_I{}^J = \begin{pmatrix} 0 & 2\delta^{ab}_{ij} \\ 0 & 0 \end{pmatrix} , \quad (K_{ab})_I{}^J = \begin{pmatrix} 0 & 0 \\ 2\delta^{ij}_{ab} & 0 \end{pmatrix} , \tag{D.13}
$$

and they satisfy the algebra

$$\begin{aligned}
\left[K^{a_1}{}_{b_1}, K^{a_2}{}_{b_2}\right] &= \delta^{a_2}_{b_1} K^{a_1}{}_{b_2} - \delta^{a_1}_{b_2} K^{a_2}{}_{b_1}\,, & \left[K^{a_1}{}_{b_1}, K_{a_2 b_2}\right] = -\delta^{a_1}_{a_2} K_{b_1 b_2} - \delta^{a_1}_{b_2} K_{a_2 b_1}\,,\\
\left[K^{a_1}{}_{b_1}, K^{a_2 b_2}\right] &= \delta^{a_2}_{b_1} K^{a_1 b_2} + \delta^{b_2}_{b_1} K^{a_2 a_1}\,, & \left[K^{a_1 b_1}, K^{a_2 b_2}\right] = \left[K_{a_1 b_1}, K_{a_2 b_2}\right] = 0\,,\\
\left[K^{a_1 b_1}, K_{a_2 b_2}\right] &= -\delta^{a_1}_{a_2} K^{b_1}{}_{b_2} + \delta^{a_1}_{b_2} K^{b_1}{}_{a_2} + \delta^{b_1}_{a_2} K^{a_1}{}_{b_2} - \delta^{b_1}_{b_2} K^{a_1}{}_{a_2}\,. & \qquad (D.14)
\end{aligned}$$

One can check that a Cartan-subalgebra is given by the generators $\{K^a{}_a\}$ and the positive positive root generators are $\{K_{a_1 a_2},\, K^{b_1}{}_{b_2}\,|b_1 < b_2\}$.

Generalised vielbein

Following the construction (6.6), we can write the generalised vielbein in the M-theory section as

$$E_{\bar{\mathbb{N}}}{}^{\mathbb{N}} = \left[\exp\left(\sum_{a<b} h_a{}^b K^a{}_b\right) \exp\left(\sum_a h_a K^a{}_a\right) \exp\left(\sum_{a,b} M^{ab} K_{ab}\right)\right]_{\bar{\mathbb{N}}}{}^{\mathbb{N}}\,, \qquad (D.15)$$

where h_a, $h_a{}^b$ and M^{ab} are arbitrary fields. Using the generators in the **10** representation (D.13), we construct the generalized vielbein

$$E_{\bar{I}}{}^{J} = \begin{pmatrix} e^{1/2} e_{\bar{\imath}}{}^{j} & 0 \\ \frac{1}{\sqrt{2}} e^{-1/2} e^{\bar{\imath}}{}_k \beta^{kj} & e^{-1/2} e^{\bar{\imath}}{}_j \end{pmatrix}\,, \qquad (D.16)$$

where now $e_{\bar{\imath}}{}^j$ is the (inverse) vielbein of the 5-dimensional internal space, $e = \det(e^{\bar{\imath}}{}_j)$ and $\beta^{ij} = \frac{1}{3!}\epsilon^{ijklm} A^{(3)}_{klm}$, with $A^{(3)}$ the M-theory 3-form flux. To construct this vielbein from (D.15), we have expressed the fields h_a, $h_a{}^b$ and M^{ab} in terms of the supergravity fields by demanding that, when acting on $E_{\bar{I}}{}^J$ with generalised derivatives, all fields transform correctly under five-dimensional diffeomorphsims and 3-form gauge transformations.

The invers vielbein is given by

$$E_I{}^{\bar{J}} = \begin{pmatrix} e^{-1/2} e_i{}^{\bar{\jmath}} & 0 \\ -\frac{1}{\sqrt{2}} e^{-1/2} \beta^{ik} e_k{}^{\bar{\jmath}} & e^{1/2} e^i{}_{\bar{\jmath}} \end{pmatrix}\,. \qquad (D.17)$$

Generalised metric

Finally, one can construct the generalised metric $\mathcal{M}_{IJ} = E_I{}^{\bar{I}} E_J{}^{\bar{J}} \delta_{\bar{I}\bar{J}}$, obtaining

$$\mathcal{M}_{IJ} = \begin{pmatrix} \frac{1}{e}\, g_{ij} & \frac{1}{\sqrt{2}\, e}\, g_{ik}\beta^{kj} \\ -\frac{1}{\sqrt{2}\, e}\, \beta^{ik} g_{kj} & e\, g^{ij} - \frac{1}{2e}\, \beta^{ik} g_{kl} \beta^{lj} \end{pmatrix}\,, \qquad (D.18)$$

where $g_{ij} = e_i{}^{\bar{\imath}} e_j{}^{\bar{\jmath}} \delta_{\bar{\imath}\bar{\jmath}}$ and $e = \sqrt{\det g}$.

D.1.4 16 representation

We construct the generalised metric in the **16**, $\mathcal{M}_{MN}$, using the same procedure as in the previous section. In particular, we construct the generalised vielbein from (D.15) using the same coeficients h_a, $h_a{}^b$ and M^{ab} and changing only the representation of the generators. By construction, the resulting vielbein will transform consistently under generalised Lie derivatives, which we will check *a posteriori*.

Generalised derivatives

The generalized vielbein in the **16** transforms under generalized diffeomorphisms Λ^N as

$$\mathcal{L}_\Lambda E_{\bar{M}}{}^N = \Lambda^P \partial_P E_{\bar{M}}{}^N - E_{\bar{M}}{}^P \partial_P \Lambda^N + \frac{1}{2} (\gamma_I)^{NS} \left(\gamma^I\right)_{PQ} E_{\bar{M}}{}^P \partial_S \Lambda^Q - \frac{1}{4} E_{\bar{M}}{}^N \partial_P \Lambda^P . \quad \text{(D.19)}$$

As we did in the **10** representation, we write this derivative in terms of the supergravity diffeomorphisms and gauge parameters for the different components of the vielbein

- Diffeomorphisms $\Lambda^M = (\xi^m, 0, 0)$

$$\begin{aligned}
\mathcal{L}_\xi E_{\bar{M}}{}^n &= \xi^p \partial_p E_{\bar{M}}{}^n - E_{\bar{M}}{}^p \partial_p \xi^n - \frac{1}{4} E_{\bar{M}}{}^n \partial_p \xi^p \,, \\
\mathcal{L}_\xi E_{\bar{M}\, n_1 n_2} &= \xi^p \partial_p E_{\bar{M}\, n_1 n_2} + E_{\bar{M}\, n_1 q} \partial_{n_2} \xi^q + E_{\bar{M}\, q n_2} \partial_{n_1} \xi^q - \frac{1}{4} E_{\bar{M}\, n_1 n_2} \partial_p \xi^p \,, \\
\mathcal{L}_\xi E_{\bar{M}\,(z)} &= \xi^p \partial_p E_{\bar{M}\,(z)} + \frac{3}{4} E_{\bar{M}\,(z)} \partial_p \xi^p \,,
\end{aligned} \quad \text{(D.20)}$$

- 3-form gauge transformation $\Lambda^M = (0, \lambda_{m_1 m_2}, 0)$

$$\begin{aligned}
\mathcal{L}_\lambda E_{\bar{M}}{}^n &= 0 \,, \\
\mathcal{L}_\lambda E_{\bar{M}\, n_1 n_2} &= -E_{\bar{M}}{}^p \left(\partial_p \lambda_{n_1 n_2} + \partial_{n_1} \lambda_{n_2 p} + \partial_{n_2} \lambda_{p n_1}\right) \,, \\
\mathcal{L}_\lambda E_{\bar{M}\,(z)} &= \frac{1}{2} \epsilon^{p_1 p_2 s q_1 q_2} E_{\bar{M}\, p_1 p_2} \partial_s \lambda_{q_1 q_2} \,,
\end{aligned} \quad \text{(D.21)}$$

- Diffeomorphisms $\Lambda^M = (0, 0, f)$

$$\begin{aligned}
\mathcal{L}_f E_{\bar{M}}{}^n &= 0 \,, \\
\mathcal{L}_f E_{\bar{M}\, n_1 n_2} &= 0 \,, \\
\mathcal{L}_f E_{\bar{M}\,(z)} &= 0 \,.
\end{aligned} \quad \text{(D.22)}$$

Algebra

To construct generators of the algebra in the spinor representation we use

$$K^{AB} = \frac{1}{4} \left[\gamma^A, \gamma^B\right] \,. \quad \text{(D.23)}$$

Using the representation (D.7), these generators are

$$\begin{aligned}
(K^a{}_b)_M{}^N &= \frac{1}{4}\left[\gamma^a,\gamma_b\right] = \begin{pmatrix} \delta^a_m\delta^n_b - \frac{1}{2}\delta^a_b\delta^n_m & 0 & 0 \\ 0 & -2\delta_b^{[m_1|}\delta^{a|m_2]}_{n_1n_2} + \frac{1}{2}\delta^a_b\delta^{m_1m_2}_{n_1n_2} & 0 \\ 0 & 0 & \frac{1}{2}\delta^a_b \end{pmatrix}, \\
(K^{ab})_M{}^N &= \frac{1}{4}\left[\gamma^a,\gamma^b\right] = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}\epsilon^{abm_1m_2n} & 0 & 0 \\ 0 & -\sqrt{2}\delta^{ab}_{n_1n_2} & 0 \end{pmatrix} \\
(K_{ab})_M{}^N &= \frac{1}{4}\left[\gamma_a,\gamma_b\right] = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}}\epsilon_{abmn_1n_2} & 0 \\ 0 & 0 & \sqrt{2}\delta^{m_1m_2}_{ab} \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{D.24}$$

and a straight-forward calculation shows that they satisfy the algebra relations (D.14).

Vielbein

Using (D.15) with the generators in the spinor representation and using the same fields $h_a{}^b$, h_a and M^{ab} we used for the generalised vielbein in the fundamental representation one obtains

$$E_{\bar{M}}{}^N = \begin{pmatrix} e^{-1/4}e_{\bar{m}}{}^n & -\frac{e^{-1/4}}{2}e_{\bar{m}}{}^l A^{(3)}_{ln_1n_2} & -\frac{e^{-1/4}}{8}e_{\bar{m}}{}^l A^{(3)}_{lpq}\beta^{pq} \\ 0 & e^{-1/4}e^{\bar{m}_1\bar{m}_2}{}_{n_1n_2} & \frac{e^{-1/4}}{2}e^{\bar{m}_1\bar{m}_2}{}_{l_1l_2}\beta^{l_1l_2} \\ 0 & 0 & e^{3/4} \end{pmatrix}, \tag{D.25}$$

where $e^{\bar{m}_1\bar{m}_2}{}_{n_1n_2} = \frac{1}{2}(e^{\bar{m}_1}{}_{n_1}e^{\bar{m}_2}{}_{n_2} - n_1 \leftrightarrow n_2)$. One can check that all the fields transform correctly under generalized derivatives as expected.

The inverse generalised vielbein is

$$E_M{}^{\bar{N}} = \begin{pmatrix} e^{1/4}e_m{}^{\bar{n}} & \frac{e^{1/4}}{2}A^{(3)}_{mp_1p_2}e^{p_1p_2}{}_{\bar{n}_1\bar{n}_2} & -\frac{e^{-3/4}}{8}A^{(3)}_{mpq}\beta^{pq} \\ 0 & e^{1/4}e^{m_1m_2}{}_{\bar{n}_1\bar{n}_2} & -\frac{e^{-3/4}}{2}\beta^{m_1m_2} \\ 0 & 0 & e^{-3/4} \end{pmatrix}, \tag{D.26}$$

Generalised metric

The generalised metric in the **16** representation, $\mathcal{M}_{MN} = E_M{}^{\bar{M}}E_N{}^{\bar{N}}\delta_{\bar{M}\bar{N}}$, is

$$\mathcal{M}_{MN} = \tag{D.27}$$
$$\begin{pmatrix} e^{1/2}g_{mn} + \frac{e^{1/2}}{4}A^{(3)}_{mpq}A^{(3)\,pq}_n + \frac{e^{-3/2}}{64}X_mX_n & \frac{e^{1/2}}{2}A^{(3)\,n_1n_2}_m + \frac{e^{-3/2}}{16}X_m\beta^{n_1n_2} & -\frac{e^{-3/2}}{8}X_m \\ \frac{e^{1/2}}{2}A^{(3)\,m_1m_2}_n + \frac{e^{-3/2}}{16}\beta^{m_1m_2}X_n & e^{1/2}g^{m_1m_2\,n_1n_2} + \frac{e^{-3/2}}{4}\beta^{m_1m_2}\beta^{n_1n_2} & -\frac{e^{-3/2}}{2}\beta^{m_1m_2} \\ -\frac{e^{-3/2}}{8}X_n & -\frac{e^{-3/2}}{2}\beta^{n_1n_2} & e^{-3/2} \end{pmatrix},$$

which, up to prefactors, coincides with the results in [145]. Again $\beta^{mn} = \frac{1}{3!}\epsilon^{mnpqr}A^{(3)}_{pqr}$ and $X_m = A^{(3)}_{mpq}\beta^{pq}$. The metric with two pairs of anti-symmetric indices is defined as $g^{m_1m_2\,n_1n_2} = \frac{1}{2}(g^{m_1n_1}g^{m_2n_2} - g^{m_1n_2}g^{m_2n_1})$ and is used to raise and lower a pair of anti-symmetric indices. The determinant of the vielbein is related to the metric as $e = \sqrt{\det g}$. One can check that the generalised metrics in the **10** and **16** representations are related by

$$\left(\gamma^I\right)^{PQ} \mathcal{M}_{MP}\,\mathcal{M}_{NQ}\,\mathcal{M}_{IJ} = \left(\gamma_I\right)_{MN} . \tag{D.28}$$

D.2 Type IIB section

D.2.1 Conventions

We next choose a solution of the section condition that breaks $SO(5,5) \to SL(4)\times SL(2)\times SL(2)$. The **10** and the **16** representations decompose into

$$\begin{aligned} \mathbf{10} &\longrightarrow (\mathbf{2},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{6}),\\ \mathbf{16} &\longrightarrow (\mathbf{2},\mathbf{1},\mathbf{4}) \oplus (\mathbf{1},\mathbf{2},\bar{\mathbf{4}}), \end{aligned} \tag{D.29}$$

and we write vectors V^I in the **10** and V^N in the **16** as

$$\begin{aligned} V^I &= (V^{\alpha_i}_{u_i}, V_{i_1i_2}) = (V^{\alpha_i}_{+}, V^{\alpha_i}_{-}, V_{i_1i_2}),\\ V^N &= (V^n_{u_n}, V^{\alpha_n}_{n}) = (V^n_+, V^n_-, V^{\alpha_n}_n), \end{aligned} \tag{D.30}$$

where again $I \in \{1,\ldots,10\}$, $N \in \{1,\ldots,16\}$ and now $\alpha_i \in \{1,2\}$, $u_i \in \{+,-\}$ and $i,n,\dots \in \{1,2,3,4\}$. Since the generalised coordinates are in the **16** representation, the supergravity coordinates x^n should be included in X^N. We take the convention where $X^n_+ = x^n$ and $X^n_- = \frac{1}{3!}\epsilon^{nmpq}\tilde{x}_{mpq}$. This choice breaks one of the $SL(2)$ factors, which was accidental.

For the type IIB section, we will use the multiplication conventions (note they are slightly different from the ones used in the M-theory case)

$$\begin{aligned} V^IW_I &= V^{\alpha_i}_{u_i}W^{u_i}_{\alpha_i} + \frac{1}{2}V_{i_1i_2}W^{i_1i_2}\,,\\ V^NW_N &= V^n_{u_n}W^{u_n}_n + V^{\alpha_n}_nW^n_{\alpha_n}\,, \end{aligned} \tag{D.31}$$

for which the identity matrices are

$$\delta^I_J = \begin{pmatrix} \delta^{\alpha_i}_{\alpha_j}\delta^{u_j}_{u_i} & 0\\ 0 & 2\delta^{j_1j_2}_{i_1i_2} \end{pmatrix}, \qquad \delta^N_M = \begin{pmatrix} \delta^n_m\delta^{u_m}_{u_n} & 0\\ 0 & \delta^m_n\delta^{\alpha_n}_{\alpha_m} \end{pmatrix}. \tag{D.32}$$

The $SO(5,5)$ metric and its inverse are

$$\eta_{IJ} = \begin{pmatrix} \epsilon_{\alpha_i\alpha_j}\epsilon^{u_iu_j} & 0\\ 0 & \epsilon^{i_1i_2j_1j_2} \end{pmatrix}, \qquad \eta^{IJ} = \begin{pmatrix} \epsilon^{\alpha_i\alpha_j}\epsilon_{u_iu_j} & 0\\ 0 & \epsilon_{i_1i_2j_1j_2} \end{pmatrix}, \tag{D.33}$$

and we use the conventions $\epsilon^{12} = \epsilon_{12} = \epsilon^{+-} = \epsilon_{+-} = \epsilon^{1234} = \epsilon_{1234} = 1$.

D.2.2 γ-matrices

A set of matrices $(\gamma^I)^{MN}$ and $(\gamma^I)_{MN}$ satisfying the Clifford algebra relation (D.6) is

$$\begin{aligned}
\left(\gamma^{\alpha_i}_{u_i}\right)^{MN} &= \begin{pmatrix} \left(\gamma^{\alpha_i}_{u_i}\right)^{m\ n}_{u_m u_n} & \left(\gamma^{\alpha_i}_{u_i}\right)^{m\ \alpha_n}_{u_m n} \\ \left(\gamma^{\alpha_i}_{u_i}\right)^{\alpha_m n}_{m\ u_n} & \left(\gamma^{\alpha_i}_{u_i}\right)^{\alpha_m \alpha_n}_{m\ n} \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2}\delta^m_n \epsilon^{\alpha_i\alpha_n}\epsilon_{u_i u_m} \\ \sqrt{2}\delta^n_m \epsilon^{\alpha_i\alpha_m}\epsilon_{u_i u_n} & 0 \end{pmatrix}, \\
\left(\gamma_{i_1 i_2}\right)^{MN} &= \begin{pmatrix} \left(\gamma_{i_1i_2}\right)^{m\ n}_{u_m u_n} & \left(\gamma_{i_1i_2}\right)^{m\ \alpha_n}_{u_m n} \\ \left(\gamma_{i_1i_2}\right)^{\alpha_m n}_{m\ u_n} & \left(\gamma_{i_1i_2}\right)^{\alpha_m \alpha_n}_{m\ n} \end{pmatrix} = \begin{pmatrix} 2\sqrt{2}\delta^{mn}_{i_1i_2}\epsilon_{u_m u_n} & 0 \\ 0 & -\sqrt{2}\epsilon_{i_1 i_2 mn}\epsilon^{\alpha_m\alpha_n} \end{pmatrix}, \\
\left(\gamma^{\alpha_i}_{u_i}\right)_{MN} &= \begin{pmatrix} \left(\gamma^{\alpha_i}_{u_i}\right)^{u_m u_n}_{m\ n} & \left(\gamma^{\alpha_i}_{u_i}\right)^{u_m n}_{m\ \alpha_n} \\ \left(\gamma^{\alpha_i}_{u_i}\right)^{m\ u_n}_{\alpha_m n} & \left(\gamma^{\alpha_i}_{u_i}\right)^{m\ n}_{\alpha_m \alpha_n} \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2}\delta^n_m \delta^{\alpha_i}_{\alpha_n}\delta^{u_m}_{ui} \\ \sqrt{2}\delta^m_n \delta^{\alpha_i}_{\alpha_m}\delta^{u_n}_{u_i} & 0 \end{pmatrix}, \quad \text{(D.34)} \\
\left(\gamma_{i_1 i_2}\right)_{MN} &= \begin{pmatrix} \left(\gamma_{i_1i_2}\right)^{u_m u_n}_{m\ n} & \left(\gamma_{i_1i_2}\right)^{u_m n}_{m\ \alpha_n} \\ \left(\gamma_{i_1i_2}\right)^{m\ u_n}_{\alpha_m n} & \left(\gamma_{i_1i_2}\right)^{m\ n}_{\alpha_m \alpha_n} \end{pmatrix} = \begin{pmatrix} \sqrt{2}\epsilon_{i_1 i_2 mn}\epsilon^{u_m u_n} & 0 \\ 0 & -2\sqrt{2}\delta^{mn}_{i_1i_2}\epsilon_{\alpha_m\alpha_n} \end{pmatrix}.
\end{aligned}$$

Similarly, one can construct $(\gamma_I)^{MN}$ and $(\gamma_I)_{MN}$ by

$$\begin{aligned}
\left(\gamma^{u_i}_{\alpha_i}\right) &= \epsilon_{\alpha_i\alpha_k}\epsilon^{u_i u_k}\left(\gamma^{\alpha_k}_{u_k}\right), \\
\left(\gamma^{i_1 i_2}\right) &= \frac{1}{2}\epsilon^{i_1 i_2 k_1 k_2}\left(\gamma_{k_1k_2}\right). \quad \text{(D.35)}
\end{aligned}$$

D.2.3 10 representation

As we did for the M-theory section, we construct the generalised metric in terms of the type IIB from (6.6).

Generalised derivatives

The generalised derivative in the **10** representation is given in (D.8). Now, the generalised vector Λ^N encodes diffeomorphisms, 4-forms and 2-forms gauge transformations as $\Lambda^M = (\xi^m, \chi^m, \Omega^\alpha_m)$, where $\chi^m = \frac{1}{3!}\epsilon^{mpqr}\chi_{pqr}$. After imposing the section condition (all derivatives are zero except for $\partial^+_m = \partial_m$), the generalized derivatives for the different components of the generalised vielbein in terms of the type IIB parameters are:

- Diffeomorphisms: $\Lambda^M = (\xi^m, 0, 0)$

$$\begin{aligned}
\mathcal{L}_\xi E_{\bar{I}\ +}^{\ \alpha_j} &= \xi^m\partial_m E_{\bar{I}\ +}^{\ \alpha_j} - \frac{1}{2}E_{\bar{I}\ +}^{\ \alpha_j}\partial_m\xi^m, \\
\mathcal{L}_\xi E_{\bar{I}\ -}^{\ \alpha_j} &= \xi^m\partial_m E_{\bar{I}\ -}^{\ \alpha_j} + \frac{1}{2}E_{\bar{I}\ -}^{\ \alpha_j}\partial_m\xi^m, \quad \text{(D.36)} \\
\mathcal{L}_\xi E_{\bar{I}\, j_1 j_2} &= \xi^m\partial_m E_{\bar{I}\, j_1j_2} + E_{\bar{I}\, j_1 m}\partial_{j_2}\xi^m + E_{\bar{I}\, m j_2}\partial_{j_1}\xi^m - \frac{1}{2}E_{\bar{I}\, j_1 j_2}\partial_m\xi^m,
\end{aligned}$$

- $C^{(4)}$ gauge transformations: $\Lambda^M = (0, \chi^m, 0)$

$$\begin{aligned} \mathcal{L}_\chi E_{\bar{I}\,+}{}^{\alpha_j} &= 0\,, \\ \mathcal{L}_\chi E_{\bar{I}\,-}{}^{\alpha_j} &= -E_{\bar{I}\,+}{}^{\alpha_j}\partial_m \chi^m\,, \\ \mathcal{L}_\chi E_{\bar{I}\,j_1 j_2} &= 0\,, \end{aligned} \tag{D.37}$$

- $C^{(2)}$ gauge transformations: $\Lambda^M = (0, 0, \Omega^\alpha_n)$

$$\begin{aligned} \mathcal{L}_\Omega E_{\bar{I}\,+}{}^{\alpha_j} &= 0\,, \\ \mathcal{L}_\Omega E_{\bar{I}\,-}{}^{\alpha_j} &= -\frac{1}{2}\epsilon^{k_1 k_2 m n}\partial_m \Omega^{\alpha_j}_n E_{\bar{I}\,k_1 k_2}\,, \\ \mathcal{L}_\Omega E_{\bar{I}\,j_1 j_2} &= \epsilon_{\alpha_k \alpha_s} E_{\bar{I}\,+}{}^{\alpha_k}(\partial_{j_1}\Omega^{\alpha_s}_{j_2} - \partial_{j_2}\Omega^{\alpha_s}_{j_1})\,. \end{aligned} \tag{D.38}$$

Algebra

As in the M-theory case, the generators in the fundamental representation satisfy the condition (D.12) and can be labeled by $(K^{AB})_I{}^J$, where A and B are indices in the **10** and the generators satisfy $K^{AB} = -K^{BA}$. In the type IIB section we choose the generators

$$\begin{aligned} \left(K^{\alpha_a \alpha_b}_{u_a u_b}\right)_I{}^J &= \begin{pmatrix} \delta^{\alpha_a}_{\alpha_i}\delta^{u_i}_{u_a}\epsilon^{\alpha_b \alpha_j}\epsilon_{u_b u_j} - \delta^{\alpha_b}_{\alpha_i}\delta^{u_i}_{u_b}\epsilon\alpha_a\alpha_j\epsilon_{u_a u_j} & 0 \\ 0 & 0 \end{pmatrix}\,, \\ \left(K_{a_1 a_2 u_b}{}^{\alpha_b}\right)_I{}^J &= \begin{pmatrix} 0 & -\epsilon_{a_1 a_2 j_1 j_2}\delta^{\alpha_b}_{\alpha_i}\delta^{u_i}_{u_b} \\ 2\delta^{i_1 i_2}_{a_1 a_2}\epsilon^{\alpha_b \alpha_j}\epsilon_{u_b u_j} & 0 \end{pmatrix}\,, \\ \left(K_{a_1 a_2 b_1 b_2}\right)_I{}^J &= \begin{pmatrix} 0 & 0 \\ 0 & 2\delta^{i_1 i_2}_{a_1 a_2}\epsilon_{b_1 b_2 j_1 j_2} - 2\delta^{i_1 i_2}_{b_1 b_2}\epsilon_{a_1 a_2 j_1 j_2} \end{pmatrix}\,, \end{aligned} \tag{D.39}$$

which satisfy the algebra

$$\begin{aligned} \left[K^{\alpha_a \alpha_b}_{u_a u_b}, K^{\alpha_c \alpha_d}_{u_c u_d}\right] &= \epsilon^{\alpha_b \alpha_c}\epsilon_{u_b u_c}K^{\alpha_a \alpha_d}_{u_a u_d} - \epsilon^{\alpha_b \alpha_d}\epsilon_{u_b u_d}K^{\alpha_a \alpha_c}_{u_a u_c} + \epsilon^{\alpha_a \alpha_d}\epsilon_{u_a u_d}K^{\alpha_b \alpha_c}_{u_b u_c} - \epsilon^{\alpha_a \alpha_c}\epsilon_{u_a u_c}K^{\alpha_b \alpha_d}_{u_b u_d}\,, \\ \left[K^{\alpha_a \alpha_b}_{u_a u_b}, K_{c_1 c_2 u_d}{}^{\alpha_d}\right] &= -\epsilon^{\alpha_a \alpha_d}\epsilon_{u_a u_d}K_{c_1 c_2 u_b}{}^{\alpha_b} + \epsilon^{\alpha_b \alpha_d}\epsilon_{u_b u_d}K_{c_1 c_2 u_a}{}^{\alpha_a}\,, \\ \left[K^{\alpha_a \alpha_b}_{u_a u_b}, K_{c_1 c_2, d_1 d_2}\right] &= 0 \\ \left[K_{a_1 a_2 u_b}{}^{\alpha_b}, K_{c_1 c_2 u_d}{}^{\alpha_d}\right] &= -\epsilon_{a_1 a_2 c_1 c_2}K^{\alpha_b \alpha_d}_{u_b u_d} - \epsilon^{\alpha_b \alpha_d}\epsilon_{u_b u_d}K_{a_1 a_2 c_1 c_2}\,, \\ \left[K_{a_1 a_2 u_b}{}^{\alpha_b}, K_{c_1 c_2 d_1 d_2}\right] &= -\epsilon_{a_1 a_2 d_1 d_2}K_{c_1 c_2 u_b}{}^{\alpha_b} + \epsilon_{a_1 a_2 c_1 c_2}K_{d_1 d_2 u_b}{}^{\alpha_b}\,, \\ [K_{a_1 a_2 b_1 b_2}, K_{c_1 c_2 d_1 d_2}] &= \epsilon_{b_1 b_2 c_1 c_2}K_{a_1 a_2 d_1 d_2} + \epsilon_{a_1 a_2 d_1 d_2}K_{b_1 b_2 c_1 c_2} - \epsilon_{b_1 b_2 d_1 d_2}K_{a_1 a_2 c_1 c_2} - \epsilon_{a_1 a_2 c_1 c_2}K_{b_1 b_2 d_1 d_2}\,. \end{aligned} \tag{D.40}$$

We identify as Cartan subalgebra the one spanned by the generators $\{K^{1\,2}_{-+},\ K^{1\,2}_{+-},\ K_{1234}\, K_{1324},\ K_{1423}\}$ and the positive root generators are $\{K_{1213}, K_{1214}, K_{1223}, K_{1224}, K_{1314}, K_{1323}, K^{1\,1}_{-+}, K^{1\,2}_{++}, K_{a_1 a_2\,+}{}^{\alpha_b}\}$.

Vielbein

Analogously to (D.15), we construct the generalised vielbein as

$$E_{\bar{\mathbb{N}}}{}^{\mathbb{N}} = \left[E_{(geo.)}\cdot \exp\left[M_1 K^{1\,1}_{-+}\right]\cdot \exp\left[M_2 K^{1\,2}_{++}\right]\cdot \exp\left[M^{a_1 a_2}{}_{\alpha_b}K_{a_1 a_2\,+}{}^{\alpha_b}\right]\right]_{\bar{\mathbb{N}}}{}^{\mathbb{N}}\,, \tag{D.41}$$

where

$$E_{(geo.)\,\bar{\mathbb{N}}}{}^{\mathbb{N}} = \left[\exp\left[\sum_{\rho\in\mathfrak{h}} h_\rho K_\rho\right]\cdot\exp\left[\sum_{(a_1a_2a_3a_4)\in\Delta} h_{a_1a_2a_3a_4}K_{a_1a_2a_3a_4}\right]\right]_{\bar{\mathbb{N}}}{}^{\mathbb{N}}\,, \tag{D.42}$$

with K_ρ the generators in the Cartan subalgebra and $\Delta = \{1213, 1214, 1223, 1224, 1314, 1323\}$, is the vielbein with zero fluxes.

Using the generators in the **10** representation, one obtains the vielbein

$$E_{\bar{I}}{}^{J} = \begin{pmatrix} E^{\bar{u}_i\alpha_j}_{\bar{\alpha}_iu_j} & E^{\bar{u}_i}_{\bar{\alpha}_ij_1j_2} \\ E^{\bar{\imath}_1\bar{\imath}_2\alpha_j}{}_{u_j} & E^{\bar{\imath}_1\bar{\imath}_2}{}_{j_1j_2} \end{pmatrix} \tag{D.43}$$

with

$$\begin{aligned}
E^{\bar{u}_i\alpha_j}_{\bar{\alpha}_iu_j} &= \left\{\begin{array}{lll} E^{\bar{+}\alpha_j}_{\bar{\alpha}_i+} = e^{-1/2}\,h(\tau)_{\bar{\alpha}_i}{}^{\alpha_j} & E^{\bar{-}\,\alpha_j}_{\bar{\alpha}_i+} = 0 & E^{\bar{-}\,\alpha_j}_{\bar{\alpha}_i-} = e^{1/2}h(\tau)_{\bar{\alpha}_i}{}^{\alpha_j} \\ E^{\bar{+}\alpha_j}_{\bar{\alpha}_i-} = -e^{-1/2}C_{(4)}\,h(\tau)_{\bar{\alpha}_i}{}^{\alpha_j} + \frac{1}{8}e^{-1/2}\epsilon^{k_1k_2k_3k_4}\,h(\tau)_{\bar{\alpha}_i}{}^{\alpha_k}\,C^{(2)}_{k_1k_2\alpha_k}C^{(2)}_{k_3k_4\alpha_s}\epsilon^{\alpha_j\alpha_s} & & \end{array}\right\} \\
E^{\bar{u}_i}_{\bar{\alpha}_ij_1j_2} &= -E^{\bar{u}_i\alpha_k}_{\bar{\alpha}_i+}\,C^{(2)}_{j_1j_2\alpha_k} = \left\{\begin{array}{c} E^{\bar{+}}_{\bar{\alpha}_ij_1j_2} = -e^{-1/2}\,h(\tau)_{\bar{\alpha}_i}{}^{\alpha_k}\,C^{(2)}_{j_1j_2\alpha_k} \\ E^{\bar{-}}_{\bar{\alpha}_ij_1j_2} = 0 \end{array}\right\} \\
E^{\bar{\imath}_1\bar{\imath}_2\alpha_j}{}_{u_j} &= \frac{1}{2}E^{\bar{\imath}_1\bar{\imath}_2}{}_{k_1k_2}\beta^{k_1k_2}{}_{\alpha_k}\epsilon_{+u_j}\epsilon^{\alpha_k\alpha_j} = \left\{\begin{array}{c} E^{\bar{\imath}_1\bar{\imath}_2\alpha_j}{}_{+} = 0 \\ E^{\bar{\imath}_1\bar{\imath}_2\alpha_j}{}_{-} = \frac{1}{2e^{1/2}}e^{\bar{\imath}_1\bar{\imath}_2}{}_{k_1k_2}\beta^{k_1k_2}{}_{\alpha_k}\epsilon^{\alpha_k\alpha_j} \end{array}\right\} \\
E^{\bar{\imath}_1\bar{\imath}_2}{}_{j_1j_2} &= \frac{1}{e^{1/2}}e^{\bar{\imath}_1\bar{\imath}_2}{}_{j_1j_2}
\end{aligned} \tag{D.44}$$

where the axio-dilaton $\tau = \tau_1 + i\tau_2$ is encoded into the $SL(2)$ matrix

$$h(\tau)_{\bar{\alpha}_i}{}^{\alpha_k} = \begin{pmatrix} \frac{1}{\sqrt{\tau_2}} & -\frac{\tau_1}{\sqrt{\tau_2}} \\ 0 & \sqrt{\tau_2} \end{pmatrix}\,, \tag{D.45}$$

and $C_{(4)} = \frac{1}{4!}\epsilon^{i_1i_2i_3i_4}C_{(4),i_1i_2i_3i_4} = C_{(4),1234}$ and $\beta^{i_1i_2}{}_{\alpha} = \frac{1}{2}\epsilon^{i_1i_2k_1k_2}C^{(2)}_{k_1k_2\alpha}$. The fields $C_{(4)}$ and $C^{(2)}_\alpha$ fields transform under gauge transformations as

$$\begin{aligned}
\delta C^{(2)}_{i_1i_2\alpha} &= (d\Omega^\beta)_{i_1i_2}\epsilon_{\beta\alpha} = (\partial_{i_1}\Omega^\beta_{i_2} - \partial_{i_2}\Omega^\beta_{i_1})\epsilon_{\beta\alpha}\,, \\
\delta C_{(4)} &= \partial_m\chi^m - \frac{1}{8}\epsilon^{i_1i_2i_3i_4}(d\Omega^\alpha)_{i_1i_2}C^{(2)}_{i_3i_4\alpha} = \partial_m\chi^m - \frac{1}{2}\star(d\Omega\wedge C^{(2)}_\alpha)\,.
\end{aligned} \tag{D.46}$$

To obtain this vielbein from (D.41), we have adjusted the fields in the exponential by demanding that, under generalised derivatives of the vielbein, all supergravity fields transform according to the gauge transformations (D.46) and space-time diffeomorphisms.

Inverse generalized vielbein $E_I{}^{\bar{I}}$

The components of $E_I{}^{\bar{I}}$ are:

$$\begin{aligned}
E^{+\,\bar{\alpha}_i}_{\alpha_i\,\bar{+}} &= e^{1/2}\, h_{\alpha_i}{}^{\bar{\alpha}_i}\,, \qquad\qquad E^{-\,\bar{\alpha}_i}_{\alpha_i\,\bar{-}} = e^{-1/2}\, h_{\alpha_i}{}^{\bar{\alpha}_i}\,, \qquad\qquad E^{-\,\bar{\alpha}_i}_{\alpha_i\,\bar{+}} = 0 \\
E^{-\,\bar{\alpha}_i}_{\alpha_i\,\bar{-}} &= \frac{C_{(4)}}{e^{1/2}}\, h_{\alpha_i}{}^{\bar{\alpha}_i} - \frac{1}{2e^{1/2}} \star (C^{(2)}_{\alpha_i} \wedge C^{(2)}_{\alpha_k}) \epsilon^{\alpha_k \alpha_t}\, h_{\alpha_t}{}^{\bar{\alpha}_i} \\
E^{+}_{\alpha_i\,\bar{\imath}_1\bar{\imath}_2} &= \frac{e^{1/2}}{2} C^{(2)}_{k_1k_2\,\alpha_i}\, e^{k_1k_2}{}_{\bar{\imath}_1\bar{\imath}_2}\,, \qquad\qquad E^{-}_{\alpha_i\,\bar{\imath}_1\bar{\imath}_2} = 0 \\
E^{i_1i_2\,\bar{\alpha}_i}{}_{\bar{+}} &= 0\,, \qquad\qquad\qquad\qquad E^{i_1i_2\,\bar{\alpha}_i}{}_{\bar{-}} = -e^{-1/2} \beta^{i_1i_2}{}_{\alpha_k} \epsilon^{\alpha_k\alpha_t}\, h_{\alpha_t}{}^{\bar{\alpha}_i} \\
E^{i_1i_2}{}_{\bar{\imath}_1\bar{\imath}_2} &= e^{1/2} e^{i_1i_2}{}_{\bar{\imath}_1\bar{\imath}_2}
\end{aligned} \tag{D.47}$$

Generalised metric

The components of the generalized metric $\mathcal{M}_{IJ} = E_I{}^{\bar{I}} E_J{}^{\bar{J}} \delta_{\bar{I}\bar{J}}$ are

$$\begin{aligned}
\mathcal{M}^{++}_{\alpha_i\alpha_j} &= \frac{1}{e}\left(e^2 + C^2_{(4)}\right) H_{\alpha_i\alpha_j} + \frac{1}{4e} \star \left(C^{(2)}_{\alpha_i} \wedge C^{(2)}_{\alpha_p}\right) \star \left(C^{(2)}_{\alpha_j} \wedge C^{(2)}_{\alpha_q}\right) H^{\alpha_p\alpha_q} \\
&\quad + \frac{C_{(4)}}{2e}\left(H_{\alpha_i\alpha_p} \star \left(C^{(2)}_{\alpha_j} \wedge C^{(2)}_{\alpha_q}\right) \epsilon^{\alpha_p\alpha_q} + (\alpha_i \leftrightarrow \alpha_j)\right) + \frac{e}{4} C^{(2)}_{k_1k_2\,\alpha_i} C^{(2)}_{k_3k_4\,\alpha_j} g^{k_1k_2\;k_3k_4} \\
\mathcal{M}^{+-}_{\alpha_i\alpha_j} &= \frac{C_{(4)}}{e} H_{\alpha_i\alpha_j} - \frac{1}{2e} \star \left(C^{(2)}_{\alpha_i} \wedge C^{(2)}_{\alpha_k}\right) \epsilon^{\alpha_k\alpha_p} H_{\alpha_p\alpha_j} \\
\mathcal{M}^{--}_{\alpha_i\alpha_j} &= \frac{1}{e} H_{\alpha_i\alpha_j} \\
\mathcal{M}^{+\,j_1j_2}_{\alpha_i} &= \frac{1}{e}\left(C_{(4)} H_{\alpha_i\alpha_k} - \frac{1}{2} \star \left(C^{(2)}_{\alpha_i} \wedge C^{(2)}_{\alpha_p}\right) \epsilon^{\alpha_p\alpha_q} H_{\alpha_q\alpha_k}\right) \epsilon^{\alpha_k\alpha_l} \beta^{j_1j_2}{}_{\alpha_l} + \frac{e}{2} g^{j_1j_2\;k_1k_2} C^{(2)}_{k_1k_2\alpha_i} \\
\mathcal{M}^{-\,j_1j_2}_{\alpha_i} &= \frac{1}{e} H_{\alpha_i\alpha_k} \epsilon^{\alpha_k\alpha_p} \beta^{j_1j_2}{}_{\alpha_p} \\
\mathcal{M}^{i_1i_2\;j_1j_2} &= e\, g^{i_1i_2\;j_1j_2} + \frac{1}{e} \beta^{i_1i_2}{}_{\alpha_p} \beta^{j_1j_2}{}_{\alpha_q} H^{\alpha_p\alpha_q}
\end{aligned} \tag{D.48}$$

where $e = \sqrt{|g|}$ and $g^{i_1i_2\;j_1j_2} = \frac{1}{2}(g^{i_1j_1}g^{i_2j_2} - g^{i_1j_2}g^{i_2j_1})$.

D.2.4 16 representation

We next reproduce the analogous computations for the **16** representation

Generalised derivatives

The generalized vielbein in the 16 transforms under generalized diffeomorphisms Λ^N as

$$\mathcal{L}_\Lambda E_{\bar{M}}{}^N = \Lambda^P \partial_P E_{\bar{M}}{}^N - E_{\bar{M}}{}^P \partial_P \Lambda^N + \frac{1}{2} (\gamma_I)^{NS} \left(\gamma^I\right)_{PQ} E_{\bar{M}}{}^P \partial_S \Lambda^Q - \frac{1}{4} E_{\bar{M}}{}^N \partial_P \Lambda^P \tag{D.49}$$

After imposing the section condition (all derivatives are zero except for $\partial^+_m = \partial_m$), the generalized derivatives for the type IIB parameters read:

- Diffeomorphisms: $\Lambda^M = (\xi^m, 0, 0)$

$$\begin{aligned}
\mathcal{L}_\xi E_{\bar{M}+}{}^{n} &= \xi^p\partial_p E_{\bar{M}+}{}^{n} - E_{\bar{M}+}{}^{p}\partial_p\xi^n - \frac{1}{4}E_{\bar{M}+}{}^{n}\partial_p\xi^p \\
\mathcal{L}_\xi E_{\bar{M}-}{}^{n} &= \xi^p\partial_p E_{\bar{M}-}{}^{n} - E_{\bar{M}-}{}^{p}\partial_p\xi^n + \frac{3}{4}E_{\bar{M}-}{}^{n}\partial_p\xi^p \\
\mathcal{L}_\xi E_{\bar{M}n}{}^{\alpha_n} &= \xi^p\partial_p E_{\bar{M}n}{}^{\alpha_n} + E_{\bar{M}p}{}^{\alpha_n}\partial_n\xi^p - \frac{1}{4}E_{\bar{M}n}{}^{\alpha_n}\partial_p\xi^p
\end{aligned} \tag{D.50}$$

- $C^{(4)}$ gauge transformations: $\Lambda^M = (0, \chi^m, 0)$

$$\begin{aligned}
\mathcal{L}_\chi E_{\bar{M}+}{}^{n} &= 0 \\
\mathcal{L}_\chi E_{\bar{M}-}{}^{n} &= -E_{\bar{M}+}{}^{n}\partial_p\chi^p \\
\mathcal{L}_\chi E_{\bar{M}n}{}^{\alpha_n} &= 0
\end{aligned} \tag{D.51}$$

- $C^{(2)}$ gauge transformations: $\Lambda^M = (0, 0, \Omega^\alpha_n)$

$$\begin{aligned}
\mathcal{L}_\Omega E_{\bar{M}+}{}^{n} &= 0 \\
\mathcal{L}_\Omega E_{\bar{M}-}{}^{n} &= -\epsilon^{mkpq}\epsilon_{\alpha_k\alpha_q}E_{\bar{M}k}{}^{\alpha_k}\partial_p\Omega^{\alpha_q}_q \\
\mathcal{L}_\Omega E_{\bar{M}n}{}^{\alpha_n} &= -E_{\bar{M}+}{}^{k}(\partial_k\Omega^{\alpha_n}_n - \partial_m\Omega^{\alpha_n}_k)
\end{aligned} \tag{D.52}$$

Algebra

We reproduce the same same calculation we did for the 10 but now using the generators in the 16, which are

$$(K^{AB})_M{}^N = \frac{1}{4}[\gamma^A, \gamma^B]_M{}^N \tag{D.53}$$

This generators satisfy the algebra (D.40).

Vielbein

Finally, we generate the rest of the fluxes again with

$$\begin{aligned}
E_{\bar{M}}{}^{N} &= \left[E_{(geo.)}\cdot\exp\left[-\tau_1 K^{1\,1}_{-+}\right]\cdot\exp\left[C_{(4)}K^{1\,2}_{++}\right]\cdot\exp\left[\frac{1}{2}\beta^{a_1a_2}{}_{\alpha_b}K_{a_1a_2+}{}^{\alpha_b}\right]\right]_{\bar{M}}{}^{N} \\
&= \begin{pmatrix} E^{\bar{u}_m n}_{\bar{m}\ u_n} & E^{\bar{u}_m\alpha_n}_{\bar{m}\ n} \\ E^{\bar{m}\ n}_{\bar{\alpha}_m u_n} & E^{\bar{m}\ \alpha_n}_{\bar{\alpha}_m n} \end{pmatrix}
\end{aligned} \tag{D.54}$$

with

$$
\begin{aligned}
E_{\bar{m}\ u_n}^{\bar{u}_m n} &= \left\{ \begin{array}{cc} E_{\bar{m}+}^{\bar{+}\ n} = e^{-1/4} e_{\bar{m}}{}^{n} & E_{\bar{m}-}^{\bar{+}\ n} = -e^{-1/4} C_{(4)} e_{\bar{m}}{}^{n} - \frac{1}{2} e^{-1/4} e_{\bar{m}}{}^{k} C^{(2)}_{kr\,\alpha_r} \beta^{rn}{}_{\alpha_k} \epsilon^{\alpha_r \alpha_k} \\ E_{\bar{m}+}^{\bar{-}\ n} = 0 & E_{\bar{m}-}^{\bar{-}\ n} = e^{3/4} e_{\bar{m}}{}^{n} \end{array} \right\} \\
E_{\bar{m}\ n}^{\bar{u}_m \alpha_n} &= e^{-1/4} e_{\bar{m}}{}^{k} C^{(2)}_{kn\,\alpha_k} \epsilon^{\alpha_k \alpha_n} \delta_{+}^{\bar{u}_m} = \left\{ \begin{array}{c} E_{\bar{m}\ n}^{\bar{+}\alpha_n} = e^{-1/4} e_{\bar{m}}{}^{k} C^{(2)}_{kn\,\alpha_k} \epsilon^{\alpha_k \alpha_n} \\ E_{\bar{m}\ n}^{\bar{-}\alpha_n} = 0 \end{array} \right\} \\
E_{\bar{\alpha}_m u_n}^{\bar{m}\ n} &= -e^{-1/4} e^{\bar{m}}{}_{k} h_{\bar{\alpha}_m}{}^{\alpha_k} \beta^{kn}{}_{\alpha_k} \delta_{u_n}^{-} = \left\{ \begin{array}{c} E_{\bar{\alpha}_m +}^{\bar{m}\ n} = 0 \\ E_{\bar{\alpha}_m -}^{\bar{m}\ n} = -e^{-1/4} e^{\bar{m}}{}_{k} h_{\bar{\alpha}_m}{}^{\alpha_k} \beta^{kn}{}_{\alpha_k} \end{array} \right\} \\
E_{\bar{\alpha}_m\, n}^{\bar{m}\ \alpha_n} &= e^{-1/4} e^{\bar{m}}{}_{n} h_{\bar{\alpha}_m}{}^{\alpha_n}
\end{aligned}
\tag{D.55}
$$

where

$$
h_{\bar{\alpha}_m}{}^{\alpha_n} = \begin{pmatrix} \frac{1}{\sqrt{\tau_2}} & -\frac{\tau_1}{\sqrt{\tau_2}} \\ 0 & \sqrt{\tau_2} \end{pmatrix}
\tag{D.56}
$$

Again, one can check that, under generalized derivatives of the generalized vielbein, all fields transform consistenly with their transformations under diffeomorphisms and gauge transformations.

Inverse generalized vielbein $E_M{}^{\bar{M}}$

The components of $E_M{}^{\bar{M}}$ are:

$$
\begin{aligned}
E_{m\,\bar{+}}^{+\,\bar{m}} &= e^{1/4}\, e_m{}^{\bar{m}}\,, \qquad E_{m\,\bar{-}}^{-\,\bar{m}} = e^{-3/4}\, e_m{}^{\bar{m}}\,, \qquad E_{m\,\bar{+}}^{-\,\bar{m}} = 0 \\
E_{m\,\bar{-}}^{+\,\bar{m}} &= C_{(4)}\, e^{-3/4}\, e_m{}^{\bar{m}} - \frac{1}{2} e^{-3/4}\, C^{(2)}_{mk\,\alpha_k}\, \beta^{kr}{}_{\alpha_r} \epsilon^{\alpha_k \alpha_r}\, e_r{}^{\bar{m}} \\
E_{m\,\bar{m}}^{+\bar{\alpha}_m} &= -e^{1/4} C^{(2)}_{mk\,\alpha_r} e^{k}{}_{\bar{m}} \epsilon^{\alpha_r \alpha_k} h_{\alpha_k}{}^{\bar{\alpha}_m}\,, \qquad E_{m\,\bar{m}}^{-\bar{\alpha}_m} = 0 \\
E_{\alpha_m \bar{+}}^{m\ \bar{m}} &= 0\,, \qquad E_{\alpha_m \bar{-}}^{m\ \bar{m}} = e^{-3/4} \beta^{mr}{}_{\alpha_m}\, e_r{}^{\bar{m}} \\
E_{\alpha_m \bar{m}}^{m\ \bar{\alpha}_m} &= e^{1/4} e^{m}{}_{\bar{m}} h_{\alpha_m}{}^{\bar{\alpha}_m}
\end{aligned}
\tag{D.57}
$$

Generalised metric

The components of the generalized metric $\mathcal{M}_{MN} = E_M{}^{\bar{M}} E_N{}^{\bar{N}} \delta_{\bar{M}\bar{N}}$ are

$$
\begin{aligned}
\mathcal{M}^{++}_{m\,n} &= e^{1/2} g_{mn} + e^{-3/2} \left(C^2_{(4)} g_{mn} + \frac{1}{4} C_{mk\,\alpha_k} \beta^{kr}{}_{\alpha_r} C_{ns\,\alpha_s} \beta^{st}{}_{\alpha_t} g_{rt} \epsilon^{\alpha_k \alpha_r} \epsilon^{\alpha_s \alpha_t} \right) \\
&\quad - \frac{1}{2} e^{-3/2} C_{(4)} \left(g_{mr} C^{(2)}_{nk\,\alpha_k} \beta^{kr}{}_{\alpha_t} \epsilon^{\alpha_k \alpha_t} + (m \leftrightarrow n) \right) + e^{1/2} C^{(2)}_{mk\,\alpha_k} C^{(2)}_{ns\,\alpha_s} g^{ks} H^{\alpha_k \alpha_s} \\
\mathcal{M}^{+-}_{m\,n} &= e^{-3/2} \left(C_{(4)} g_{mn} - \frac{1}{2} C_{mk\,\alpha_k} \beta^{kr}{}_{\alpha_r} \epsilon^{\alpha_k \alpha_r} g_{rn} \right) \\
\mathcal{M}^{--}_{m\,n} &= e^{-3/2} g_{mn} \\
\mathcal{M}^{+\ n}_{m\,\alpha_n} &= e^{-3/2} \left(C_{(4)} g_{mr} \beta^{nr}{}_{\alpha_n} - \frac{1}{2} C^{(2)}_{mk\,\alpha_k} \beta^{kr}{}_{\alpha_r} g_{rs} \beta^{ns}{}_{\alpha_n} \epsilon^{\alpha_k \alpha_r} \right) - e^{1/2} C^{(2)}_{mk\,\alpha_r} g^{kn} \epsilon^{\alpha_r \alpha_k} H_{\alpha_k \alpha_n} \\
\mathcal{M}^{-\ n}_{m\,\alpha_n} &= e^{-3/2} g_{mp} \beta^{np}{}_{\alpha_n} \\
\mathcal{M}^{m\ n}_{\alpha_m \alpha_n} &= e^{1/2} g^{mn} H_{\alpha_m \alpha_n} + e^{-3/2} \beta^{mp}{}_{\alpha_m} \beta^{nq}{}_{\alpha_n} g_{pq}
\end{aligned}
\qquad \text{(D.58)}
$$

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