Black holes are quantum complete

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München 2018

Black holes are quantum complete

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Dissertation
an der Fakultät für Physik
der Ludwig–Maximilians–Universität
München

vorgelegt von
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aus Trier

München, den 6. August 2018

Erstgutachter: Prof. Dr. Stefan Hofmann

Zweitgutachter: Prof. Dr. Peter Mayr

Tag der mündlichen Prüfung: 14. September 2018

Only those who will risk going too far can possibly find out how far one can go.

– Thomas Stearns Eliot –
(Preface to Transit of Venus: Poems)

Contents

List of figures

Zusammenfassung

Vollständigkeit ist ein äußerst wichtiges Konzept in der theoretischen Physik. Die Hauptidee besagt, dass eine Bewegung oder ein Freiheitsgrad eindeutig und für alle Zeiten definiert ist. Die Kriterien, die Vollständigkeit in klassischen und quantenmechanischen Theorien beschreiben, sind je nach Theorie unterschiedlich. Sie sind eng mit dem spezifischen Messprozess der jeweiligen Theorie verknüpft. Unvollständigkeit geht oft mit der Präsenz einer Singularität einher. Die Existenz einer Singularität ist untrennbar gegeben durch die Freiheitsgrade der Theorie.

In Allgemeiner Relativitätstheorie sind Raumzeiten charakterisiert durch die Länge geodätischer Kurven. Dieses Kriterium fußt auf der klassischen Punktteilchenbeschreibung mittels Differentialgeometrie und führte zur Entwicklung der Singularitätentheoreme nach Hawking und Penrose.

Ein Quantendetektor in einer dynamischen Raumzeit kann nicht durch Quantenmechanik im engeren Sinne beschrieben werden, denn es ist nicht möglich eine konsistente relativistische Einteilcheninterpretation einer Quantentheorie zu formulieren. Infolgedessen übertragen wir den Terminus der Vollständigkeit auf Situationen deren einzig adäquate Beschreibung durch Quantenfeldtheorie auf gekrümmten Raumzeiten erfolgen kann. Um Unitaritätsverletzungen, welche sich in endlicher Zeit ereignen, auflösen zu können, nutzen wir die Schrödingerdarstellung der Quantenfeldtheorie, da diese eine Zeitauflösung ermöglicht. Sinnhaftigkeit der Persistenzamplitude des Wellenfunktionales, d.h. Wahrscheinlichkeitsverlust oder Stabilität werden mit Vollständigkeit in Verbindung gebracht.

Anhand eines schwarzen Loches der Schwarzschildgattung wenden wir unser Kriterium an und testen Vereinbarkeit mit freien, massiven Skalarfeldern. Abweichungen vom

Gauß'schen Ansatz für das Wellenfunktional, angeregte Zustände und Selbstwechselwirkung der Testfelder, werden betrachtet und deren gutartige Entwicklung wird gezeigt.

Die Analyse wird auf eine weitere Klasse von Raumzeiten, den Kasner-Raumzeiten angewandt, die hohe Relevanz durch die Vermutung von Belinskii, Khalatnikov und Lifshitz haben.

Abstract

Completeness is a very important concept in theoretical physics. The main idea is that the motion or the degree of freedom is uniquely defined for all times. The criterion for completeness is different for classical and quantum theories. This corresponds to a specific measurement process in the corresponding theories. Incompleteness is often related to the occurrence of a singularity. The notion of a singularity is closely related to the corresponding degree of freedom.

In general relativity space-times are characterised by the extendibility of geodesic curves. This criterion founded on the point particle description through differential geometry has given rise to the singularity theorems of Hawking and Penrose.

A quantum probing of dynamical space-times can not be described with quantum mechanics because there is no consistent relativistic one particle interpretation of a quantum theory. Hence, we extend the notion for completeness to situations where the only adequate description is in terms of quantum field theory on curved space-times. In order to analyse unitarity violations occurring during a finite time, we use the Schrödinger representation of quantum field theory which allow for time resolution. Consistency of the wave-functional's persistency amplitude, i.e. probability loss or stability, will be connected to completeness.

For a Schwarzschild type back hole we apply the criterion and probe with free massive scalar fields for consistency. Furthermore, deviations from Gaussianity, i.e. excited states and self-interaction of the probing fields are derived and consistency is showed for those deformations.

The analysis is furthermore applied to Kasner space-times, which have high relevance due to the conjecture of Belinskii, Khalatnikov, and Lifshitz.

1 Concerning singularities

Considering the case that an random person on the street asks you about black holes and the big bang, most people which are educated in science would inevitably come to the point where they have to talk about singularities, but this will trigger the next question: „What is a singularity?“. The concept is known to every physicist, although the notion is not very familiar to non-physicists or non-mathematicians because it has no equivalence in their daily life, at least, however, it seems to be important in order to understand what a black hole or the big bang is. During a day without thoughts about physics or mathematics the concept of singularities is not needed. From a heuristic point of view one would deny that a system in nature becomes singular, e.g. reaches an infinite value of energy.

Amongst physicists, it is clear that the occurrence of a singularity is equivalent of having a severe problem our theory. The black hole and the big bang singularities are a direct outcome of Einstein's equations, hence, they are predicted to occur by general relativity. It is, however, not clear that these solutions can be reached dynamically. Hence, they could also be a mathematical artefact.

Usually the term „singularity“ is connected to the situation where at least one observable could be measured to grow unbounded. Experimentalists have never reported an infinite value of a measurable quantity. This fact motivates to question the physical justification for the concept.

Let us recall the definition of singularities in mathematics. The definition of a singular point is [Simon, 2015a]:

Definition 1. *If Ω is a region, $f \in \mathfrak{A}(\Omega)$ (all analytic functions on Ω) and $z_0 \in \partial\Omega$, we say that f is regular at z_0 if and only if there is $\delta > 0$, g analytic in $D_\delta(z_0)$ (disk of radius δ about z_0) so that $f(z) = g(z)$ for all $z \in \Omega \cap D_\delta(z_0)$. If f is not regular at z_0, we say that* f *is singular at* z_0.

The above definition says that there exists a point z_0 for which the function acquires an infinite value: $f(z_0) > M, \quad \forall M \in \mathbb{R}$.

The language of physics is mathematics, therefore, it is very natural that mathematical concepts are also present in physics. Singularities appear in different theories; in some they signal a breakdown of the description. Before we specify the notion in physical examples we want to elucidate what is generally meant by „breakdown of the description“. Physical theories are devoted to specific energy (or length) scales, below this energy scales the theory is effectively describing the system while beyond the scale the theory loses its predictability.

In general, the pure presence of a singularity starts to become a problem only in conjunction with the possibility to dynamically reach the singularity in a finite amount of time. Considering the mathematical definition, the first singularity, ever deduced in physics, occurred in the theory of gravitation. Newton's law of the attraction between two massive bodies admits a singular value at the origin. The formula $F = -G_N \frac{mM}{r^2}$ suggests that the force at $r = 0$ will be infinitely strong. This is clearly a pathology of the theory signalling its breakdown close to this point. If we assume two point-like particles with masses m_1 and m_2 at $r \equiv 0$ it will not be possible to separate both, no matter how small the masses are. The motion generated by the gravitational potential ends for both particles at the origin. This is the first singularity ever known in physics.

In 1864 James Clerk Maxwell formulated his very successful classical field theory of electromagnetism. This theory was a benchmark for the development of modern theoretical physics. Its success was overwhelming, but when applied to atomic physics, it was plagued by the singularity of the atomic potential and the stability problems due to synchrotron emission.

For a hydrogen atom the attractive Coulomb potential serves as a suitable description for the potential of the nucleus. The potential has a similar form as the gravitational potential when considering a point charge: $V_{\text{Coulomb}} \propto -\frac{1}{r}$. Here again, we have a mathematical singularity at $r = 0$ (this analysis holds as long as the charges are opposite, instead when we have equal charges the potential becomes repulsive $V_{\text{Coulomb}} \propto \frac{1}{r}$ and it is impossible to reach $r = 0$ because the potential is unbounded from above). If this has been the whole story, a hydrogen atom could not be stable. The energy loss due to radiation of the electron orbiting the proton would result in a life-time of $\tau \approx 10^{-11}$s. From experiments (or birthday parties) we know it is different. This problem was unsolvable in classical physics because the predictability of the theory has broken down.

Quantum mechanics has provided a resolution of this singularity. The electron is now described as a state of a Hilbert space given by a wave function and the system by a Hamilton operator with Coulomb potential. In this notion, the wave-function of the non-relativistic bound-state electron yields a different result than the classical point-particle description, because the wave-function shows no support at the origin, therefore, never reaches the pathological point $r \equiv 0$. The ultimate reason is the probabilistic interpretation of quantum mechanics which prevents the electron from reaching $r = 0$ although the classical point-particle motion would predict this point to be realised in a finite amount of time.

This rather intuitive example illustrates that the term „singularity" needs some specification in physics. The mere existence of a singular point is not significant as long as the degrees of freedom are excluded from this point. Observables constructed from the degrees of freedom, e.g. energy, will not diverge and hence, the theory would be regular. In other words, if theory A detects a singular point, theory B does not need to agree.

The result concerning the hydrogen atom is a hint that the quantum description is more fundamental, because it is backed-up by experimental data.

Catching up the example of the hydrogen atom, one could think about gravity in the same way such that the singularity in the gravitational force might be a relic of the point particle description. The more potent theory describing gravitation is general relativity, but this theory predicts singularities in the case of black holes and big bang. One question, which is the key question of this thesis, arises: „How can we find out, whether these singularities are a mathematical relic or not?"

The presence of a singularity is connected to the concept of completeness. This is closer to our experience than a singularity which might be the reason for incompleteness. The singularity acts as a sink where degrees of freedom are absorbed, e.g. the classical motion of the electron in the Coulomb potential stops abruptly at $r = 0$. Completeness can be illustrated by a pool billiard game. Blocking the pockets makes the game complete, since no ball is able to leave the table. The walls serve as an infinitely high potential. No matter what initial conditions we set for the kinetic energy of the balls, they will be unable to leave. Bringing the pockets into the game means that the ball can disappear from the table, therefore the system describing only the table is not complete, while the system covering the room in which the table stands is still complete. Reasons for incompleteness can be various; a singularity is just one of them.

Completeness is a powerful tool which roughly tells us whether degrees of freedom could leave the system or not, and is closely connected to the underlying theory and the measurement process. The example of the hydrogen atom was classically incomplete but quantum mechanically complete when measured with the bound state electron. In classical physics it is predicted that the orbiting electron will lose energy through synchrotron radiation and hits the proton in a very short time, but on the quantum level, the probability for the electron wave-function is zero at $r = 0$. Therefore, the system is quantum-mechanically complete, although the classical potential between proton and electron is singular in the mathematical sense and suggests classical incompleteness. What counts for physics is the relevance of the singularity for the measurement processes that involve the appropriate degrees of freedom.

In general relativity methods have been developed to predict the occurrence of singularities. A characterisation of the structure of a manifold under rather generic assumptions has condensed in the famous singularity theorems [Hawking and Ellis, 1973]. They show rigorously under what conditions a singularity is inevitable. Along with this comes a notion of completeness which is measured by free-falling point particles serving as probes. Free-falling observers measure in proper time whether the end-point of a manifold can be reached in a finite amount of time. Black holes are shown to be incomplete by the singularity theorems because all radial geodesics have finite length and end at the singularity.

Famous examples for other singular space-times are Friedmann space-times, or the de Sitter space-time, admitting the cosmological singularity at the beginning of time (big bang). Although the big bang scenario does not meet the energy conditions of the theorems they fit perfectly in their picture. The black hole solutions, such as the Schwarzschild, Kerr, Reißner-Nordström, etc., and also Kasner are fully covered by Hawking and Penrose's theorems. However, all those mentioned solutions of Einstein's equation contain a space-like singularity. This means there is a singular hypersurface bordering on the physical space-time.

Horowitz and Marolf developed a criterion for a quantum probing of space-times which analyses quantum-mechanical probes on static space-times. Their criterion led to various research; a vast variety of static space-times have been investigated with respect to quantum probes.

Horowitz and Marolf's notion is very limited since it is restricted to static space-times, this means especially it will not be applicable to big bang and the gravitational singularity of a Schwarzschild black hole, since they occur in dynamical, i.e. explicitly time-dependent space-times. While these singularities are space-like, Horowitz and Marolf investigated time-like singularities. A spacelike singularity calls for a different treatment than a time- or lightlike, since in dynamical set-ups quantum mechanics is not applicable. The reason is that there is no consistent relativistic version of quantum mechanics. Dynamical space-times support emission and absorption processes, i.e. the particle number is not conserved, and the only adequate quantum description is via quantum field theory. The question we want to investigate in this thesis concerns exactly the physical significance of spacelike singularities when probed with quantum fields.

The significance of spacelike singularities can therefore only be detected with quantum field theory on curved space-time. In this thesis we develop a probing criterion for quantum field theory and apply it to the generalised Kasner space-time as well as to the interior of a black hole. The thesis is structured as follows:

In Chapter 2 we provide a brief introduction into the concept of completeness in several theories. We define and motivate these notions from the general idea and show how completeness is realised in classical physics, quantum mechanics and general relativity. We will explicitly compare the classical and the quantum-mechanical criteria and work out the cases where there is a tension. Afterwards, we explain the quantum probing of static space-times invented by Horowitz and Marolf. This introduction to the concept of completeness should motivate the notion we are proposing for quantum field theory on curved spaces.

With the concept of completeness in mind, we will proceed in Chapter 3 with a thorough investigation of the geometrical properties of the two space-times we intend to probe, that is, the Schwarzschild space-time and the generalised Kasner space-time. We will analyse in great detail what their geometrical properties are. We start with the black hole space-time. As an example we consider the Schwarzschild space-time for the reasons of simplicity, since we are only interested in the implications of the gravitational singularity without incorporating charged or rotating black hole solutions. We will discuss both the exterior and interior solution of the black hole. The explicit form of the metric allows - after some slight approximations - to transform the Schwarzschild solution into a Kasner space-time close to the singularity. This feature will then be used to draw a connexion to the conjecture of Belinskii, Khalatnikov and Lifshitz as well as to the following analysis of Kasner space-times.

The second space-time we investigate is the generalised Kasner space-time which is a very important class of space-times in physics because it has various applications especially due to the conjecture of Belinskii, Khalatnikov, and Lifshitz which states that the behaviour of fields close to spacelike singularities is generically described by a Kasner space-time.

In the fourth chapter we give a brief introduction into the required tools of functional calculus which we need in order to explain the Schrödinger representation of quantum field theory (first on Minkowski space-time, afterwards on curved spaces). After the preliminaries we state the definition of quantum completeness, i.e. completeness where the only adequate description is in terms of quantum field theory. Our criterion is due to the ground-states of the Schrödinger representation but we will show that this is so far sufficient. Then we apply the criterion to the Schwarzschild space-time, and afterwards to Kasner space-time. Moreover we analyse non-Gaussian deformations of the ground-state wave-functionals such as excitation with respect to the ground state and self-interaction of the quantum probes. For the latter we give an argument why they cannot change the result, whatsoever. In the last two sections of Chapter 4 we calculate the energy density which is in full accordance with the result of quantum completeness and show that charges are conserved inside Schwarzschild black hole. In the end we will draw a link to the black hole final-state proposed by Horowitz and Maldacena. The link between Heisenberg and Schrödinger representation is presented in the appendix.

The last chapter gives a short summary and discusses probable future directions.

2 The fellowship of completeness

We start with a warm-up introducing into different notions of completeness. A pedagogical treatment as well as the mathematical preliminaries are presented in the books about mathematical physics of Reed and Simon [Reed and Simon, 1980, Reed and Simon, 1975, Reed and Simon, 1979, Reed and Simon, 1978]. We will follow basically their conceptual punchline and provide examples which strengthen the intuition, additionally the presented definitions, theorems, and some of the examples are taken from these books.

Starting with classical (non-relativistic) completeness we proceed with quantum mechanical completeness and outline differences between both notions. Afterwards, we discuss completeness for general relativity which is connected to the singularity theorems of Hawking and Penrose.

Finally, we will conclude this section with Horowitz and Marolf's criterion for quantum-mechanical completeness on static space-times. For all presented notions of completeness we come up with brief examples and comparisons in order to supplement the intuition for the concept of completeness and its realisation in various theories.

2.1. Classical completeness

In this section we discuss the basic concept of completeness and explain in short how we can apply it to classical motions. The basic idea of completeness is the following: A motion generated by a potential is complete, if it is uniquely defined for all times and under arbitrary initial conditions. In other words, a degree of freedom can not disappear or appear out of nowhere, moreover its evolution must not be ambiguous. We will see in the following what consequences this idea implies.

For classical (non-relativistic) completeness [Reed and Simon, 1980] we investigate a motion of a point particle $x(t)$ generated by a potential $V(x)$. Let us furthermore call the velocity of the particle $v(t) \in \mathbb{R}$, where t is time. We restrict ourself to the half-line $(0, \infty) \ni x$ and we say the potential has a continuous derivative which is Lipshitz[1] on every compact subset of the half-line. This ensures that the potential and its derivative is continuous, smooth, and does not vary too fast.

Classical completeness is potential completeness, in order to introduce this notion we will follow essentially [Reed and Simon, 1980] and write down the Hamilton function of the system, given by $H(x,p) = \frac{1}{2m}p^2 + V(x)$. In general the potential can have an explicit time dependence, but we will restrict ourself to static potentials for simplicity. The solution $x(t)$ is specified through the equations of motion:

$$m\frac{dx}{dt}(t) = p(t), \qquad \frac{dp}{dt}(t) = -\frac{dV}{dx}(x(t)) \tag{2.2}$$

Since we have a differential equation of second order in $x(t)$ we will need a pair of initial conditions which fully determine the solution. The definition of classical completeness is given by [Reed and Simon, 1980]:

Definition 2. *A classical motion generated by a potential* V *is* ***complete*** *at* 0*, or* ∞*, if there is no pair of initial conditions* $\langle x_0, v_0 \rangle \in (0, \infty) \times \mathbb{R}$ *so that the solution* $x(t)$ *runs off to* 0*, or* ∞ *in a finite time.*

Completeness in classical non-relativistic physics says, no matter what initial conditions we assume, the trajectory will not reach the end-point in a finite amount of time; the mere existence of a singularity (at one end-point) in the potential, or the possibility for a singular value of $x(t)$, is not significant for the system in order to be incomplete. It should be mentioned that the initial conditions are formulated in x and v although it would have been more suitable to state the initial conditions in x and p.

An illustrative example can be constructed by looking at a queue game. Without pockets, the game is totally complete, i.e. the balls cannot leave the table because the potential describing the boundary of the table is infinite. If there are pockets, the motion on the table will be incomplete since the balls can leave the table.

[1] Let (X, ρ) be a metric space. $f : X \to V$, a normed linear space, is called **Lipshitz continuous** if and only if for some $C > 0$ and all $x, y \in X$ with $\rho(x,y)$, we have that

$$\|f(x) - f(y)\| \leq C\rho(x,y). \tag{2.1}$$

Classical completeness puts some restriction at the potential and the analysis can therefore be reduced to investigations of the potential V.

Theorem 1. *Let* $V(x)$ *have a continuous derivative which is uniformly Lipshitz on each compact subset of* $(0, \infty)$*. Then the classical motion generated by* $V(x)$*:*

(a) is not complete at 0 if and only if $V(x)$ *is bounded from above near zero.*

(b) is not complete at ∞ *if and only if* $V(x)$ *is bounded from above for* $x \geq 1$ *and*

$$\int_1^\infty \frac{dx}{\sqrt{K - V(x)}} < \infty \qquad \textit{for some} \quad K > \sup_{x \geq 1} V(x).$$

In other words, bounded (from above) potentials generate incomplete motions. This is intuitively clear, we can find initial conditions such that we can reach the end-point in a finite amount of time, for example a high initial velocity ν_0.

Nevertheless, classical completeness should only be considered as an introducing example.

2.2. Quantum mechanical completeness

Quantum mechanical completeness is technically different to the notion of classical physics. We will give a brief mathematical introduction which is based on the books by Reed and Simon. The notion for quantum-mechanical completeness on a half-line is given by

Theorem 2. *The potential* $V(x)$ *is called* ***quantum-mechanically complete*** *if* $H = -\frac{d^2}{dx^2} + V(x)$ *is essentially self-adjoint on* $C_0^\infty(0, \infty)$ *(continuous functions on* $(0, \infty)$*)* . $V(x)$ *is said to be complete at* ∞ *(respectively at* 0*) if at least one solution of* $\varphi''(x) = V(x)\varphi$ *is not in* L^2 *near* ∞ *(respectively near* 0*).*

The key requirement on the Hamilton operator is given by essential self-adjointness. The criterion stated above is connected to Weyl's limit point/limit circle criterion (Definition 6) for self-adjointness on a half-line. In [Simon, 2015c] the relation to the usual (not on a half-line) definition of self-adjoint operators can be found which we will provide for the sake of completeness in Definition 4. Before we go into details what quantum-mechanical completeness implies, we will first explain why self-adjointness is similar to a complete motion for a quantum state.

Our first objective is to get some intuition for self-adjointness and how we can see that an operator has this property. We will focus on the basic criterion stated by von Neumann [Neumann, 1930]. A basic knowledge of functional analysis is assumed, the focus lays on the definitions and theorems which are important for us.

Definition 3. *A densely defined operator* T *on a Hilbert space* $\mathcal{H}$ *is called* ***symmetric****, or* ***Hermitian****, if* $\mathsf{T} \subset \mathsf{T}^*$*, that is, if* $\mathsf{D}(\mathsf{T}) \subset \mathsf{D}(\mathsf{T}^*)$ *and* $\mathsf{T}\varphi = \mathsf{T}^*\varphi$ *for all* $\varphi \in \mathsf{D}(\mathsf{T})$*. Equivalently,* T *is symmetric if and only if*

$$(\mathsf{T}\varphi, \psi) = (\varphi, \mathsf{T}\psi) \qquad \textit{for all} \qquad \varphi, \psi \in \mathsf{D}(\mathsf{T}).$$

Here, the brackets denote the L^2 bilinear product and the star the Hilbert space adjoint of the operator which is conjugate linear. The domain of the operator is given by $\mathsf{D}(\mathsf{T})$ or $\mathsf{D}(\mathsf{T}^*)$ for the adjoint operator. Symmetric operators are always closable, since $\mathsf{D}(\mathsf{T}^*) \supset \mathsf{D}(\mathsf{T})$ is dense in $\mathcal{H}$. A symmetric operator is the basis for self-adjointness which is defined by

Definition 4. *A densely defined operator* T *is called* ***self-adjoint*** *if* $\mathsf{T} = \mathsf{T}^*$*, that is, if and only if* T *is symmetric and* $\mathsf{D}(\mathsf{T}) = \mathsf{D}(\mathsf{T}^*)$*.*

The main difference between symmetric and self-adjoint is given by the domain of the operator; the distinction between both properties is very important. Self-adjointness is, for example, the essential hypothesis for the spectral theorem, which is the decomposition of operators into eigenvalues and eigenbasis. Additionally, only self-adjoint operators act as generators for a one-parameter unitary group. Note, the domain of a symmetric operator is adjustable by boundary conditions such that the operator becomes self-adjoint. In a sense, self-adjointness can be seen as a compromise such that the domain is small enough for the operator to be symmetric and big enough to equal the domain of the adjoint.

Since the criterion for completeness states that essentially self-adjointness is sufficient, we will give the basic definition here:

Definition 5. *A symmetric operator* T *is called* ***essentially self-adjoint*** *if its closure* $\bar{\mathsf{T}}$ *is self-adjoint. If* T *is closed, a subset* $\mathsf{D} \subset \mathsf{D}(\mathsf{T})$ *is called* ***core*** *for* T *if* $\overline{\mathsf{T} \restriction \mathsf{D}} = \mathsf{T}$*.*

Essentially self-adjointness implies that the operator has a unique self-adjoint extension[2]. Existence of such extensions is often sufficient, therefore essential self-adjointness is enough

[2]Imposition of different boundary conditions may induce different self-adjoint extensions which are mutually incomparable.

in order to formulate the completeness criterion for quantum mechanics. The standard procedure for finding self-adjoint extensions is to construct the Friedrichs's extension or by exploiting Green's identity which has been developed by von Neumann [Neumann, 1930]. The addition of suitable boundary conditions which depend on the eigenfunctions of the adjoint operator ensures that the domains coincide. It may be difficult to determine the domain of an operator, to give some core is much easier. This is the reason why essential self-adjointness is important in mathematics, too. We will now present the basic criterion for self-adjointness.

Theorem 3. *Let T be a symmetric operator on a Hilbert space $\mathcal{H}$. Then the following three statements are equivalent:*

(a) T *is self-adjoint*

(b) T *is closed and* $Ker(T^* \pm i) = \{0\}$

(c) $Ran(T \pm i) = \mathcal{H}$.

The proof for this theorem can be found in Reed and Simon [Reed and Simon, 1980]. For essentially self-adjointness we can find a corollary

Corollary 1. *Let T be a symmetric operator on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(a) T *is essentially self-adjoint*

(b) $Ker(T^* \pm i) = \{0\}$

(c) $Ran(T \pm i)$ *is dense.*

These so called von Neumann criteria can be condensed to one important statement. Point (b) of the above theorem says that the equation $T\varphi = \mp i\varphi$ has no solution except from $\varphi = 0$ which is similar to say that the spectrum of the operator T consists of only real eigenvalues. Looking at Stone's theorem [Stone, 1932, Stone, 1929b, Stone, 1930, Stone, 1929a] unveils why this is important:

Theorem 4 (Stone's Theorem)**.** *Let $U(t)$ be a strongly continuous one-parameter unitary group on a Hilbert space $\mathcal{H}$. Then, there is a self-adjoint operator A on $\mathcal{H}$ so that $U(t) = e^{iAt}$.*

Note, Stone's theorem is only applicable in the above form when A is time-independent. $U(t)$ is called the infinitesimal generator of the unitary group. This $U(t)$ can become well known objects for example the time-evolution operator in quantum mechanics. Suppose having the self-adjoint Hamilton operator H, by Stone's theorem we know that it admits $U(t) = e^{iHt}$ as one-parameter unitary group. H serves as the infinitesimal generator of the time-evolution operator.

Time-evolution is crucial in order to decide whether or not a motion is complete. It is sometimes good to begin with heuristic considerations. We start with the time-dependent Schrödinger equation

$$i\frac{d\psi}{dt} = H\psi. \tag{2.3}$$

The solution for the time-evolution can be found by integrating the above equation

$$\psi(t) = \exp(iHt)\psi(0). \tag{2.4}$$

If the Hamilton operator were not essentially self-adjoint, we would get complex eigenvalues as consequence. In the time evolution (2.4) complex eigenvalues yield exponentially decreasing and increasing real parts $e^{\pm|\Im(H)|t}$ of the solutions. With these one might get amplification or damping terms depending on the sign of $\Im(H)$.

In the quantum-mechanical case the von Neumann criterion can be shown to coincide with the limit point-limit circle criterion of Weyl [Weyl, 1910, Reed and Simon, 1975]

Theorem 5. *Let $V(x)$ be a continuous real-valued function $(0, \infty)$. Then $H = -\frac{d^2}{dx^2} + V(x)$ is essentially self-adjoint on $C_0^\infty(0, \infty)$ if and only if $V(x)$ is in the limit point case at both zero and infinity.*

A few lines below we will explain what is meant with limit point and limit circle, and the etymology of the criterion for the interested reader.

Definition 6. *A potential $V(x)$ is in the **limit circle case** at infinity (respectively zero) if for some, and therefore all, λ, all solutions of*

$$-\varphi''(x) + V(x)\varphi(x) = \lambda\varphi(x)$$

*are L^2-functions (square-integrable) at infinity (respectively zero). If $V(x)$ is not in the limit limit circle case at infinity (respectively zero), it is said to be in the **limit point case**.*

The origin of this terminology [Reed and Simon, 1975] is due to the idea of considering self-adjointness problems of $H = -\frac{d^2}{dx^2} + V(x)$ on (a, ∞) as a limit of problems on (a, b) in the limit of $b \to \infty$. Suppose φ and ψ are solutions of $-\varphi''(x) + V(x)\varphi(x) = i\varphi(x)$ on (a, ∞) which obey the boundary conditions $\varphi(a) = \psi'(a) = 0$ and $-\varphi'(a) = \psi(a) = 1$. For a fixed b we could take a set $z \in \mathbb{C}$ and with angle $\alpha \in [0, 2\pi)$ we define $\eta = \varphi + z\psi$ which obeys $\cos(\alpha)\eta(b) + \sin(\alpha)\eta'(b) = 0$ which form for some α a circle C_b. By sending $b \to \infty$ this circle has two options, it either converges to a limiting circle of finite radius or it shrinks to a point. In case of a limiting circle both solutions to the above equation are in L^2, in the other case one fails to be square integrable. Therefore, completeness, and self-adjointness, is related to the limit point. The connection is given by failing of one solution to be square-integrable and therefore the initial conditions do not need to be specified. If the above equation has only one square integrable solution at 0 then we lose the dependence on the boundary at ∞ [Weyl, 1910].
In case that the Hamilton operator is not essentially self-adjoint, one could hope to find a self-adjoint extension by fixing the boundary conditions at the end-points.

For a spherically symmetric set-up, the criterion can be adapted [Reed and Simon, 1975]:

Theorem 6. *Let $V(r)$ be a continuous symmetric potential on $\mathbb{R}^n \backslash \{0\}$ with r being the distance from the origin. If the potential satisfies*

$$V(r) + \frac{(n-1)(n-3)}{4r^2} \geq \frac{3}{4r^2} \tag{2.5}$$

then the Schrödinger operator $-\Delta + V(r)$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^n \backslash \{0\})$. If in contrast $V(r)$ satisfies

$$0 \leq V(r) + \frac{(n-1)(n-3)}{4r^2} \leq \frac{c}{r^2}, \quad \textit{with} \quad c < \frac{3}{4} \tag{2.6}$$

then the operator is not essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^n \backslash \{0\})$.

This theorem shows explicitly that the Laplace-Beltrami operator is essentially self-adjoint for all dimensions $n \geq 4$ [Reed and Simon, 1975]. For the dimensions $n > 4$ they claim that the proof is straightforward while it is more subtle for $n = 4$. The proof of the above theorem can be found in [Reed and Simon, 1975].

2.3. Quantum-mechanical versus classical completeness

The quantum-mechanical criterion looks very similar to the criterion of classical completeness. Roughly speaking, an infinite travel time to the endpoint of a system for arbitrary initial conditions in classical mechanics finds its analogue in the fact that no boundary conditions for the quantum-mechanical state need to be specified at the endpoint. When looking at the right endpoint there are sufficient conditions for the potential to be complete [Reed and Simon, 1975]:

Theorem 7. *Let* $V(x)$ *be a continuous real-valued function on the half-line* $(0, \infty)$ *and suppose there exists a positive differentiable function* $M(x)$ *so that:*

(i) $V(x) \geq -M(x)$

(ii) $\int_1^\infty \frac{dx}{\sqrt{M(x)}} = \infty$

(iii) $\frac{M'(x)}{(M(x))^{3/2}}$ *is bounded near* ∞.

Then $V(x)$ *is in the limit point circle (complete) at infinity.*

Before we proceed, we take some time to understand the theorem above. The potential is bounded from below by $-M(x)$ while the function $M(x)$ has the property that its derivative is also bounded. Theorem 7 can be restated such that when the potential $V(x)$ fulfils the classical completeness criterion it suffices for $V(x)$ to be such that $V'|V|^{-\frac{3}{2}}$ is bounded near infinity in order to be quantum-mechanically complete at ∞. This condition says, that the derivative of the potential should not be too large compared to the potential itself. In fact, if the derivative of the potential is too large, the two notions are independent from each other at the end-point. In this case, the classical and the quantum mechanical completeness criteria coincide and the theorem can be reformulated

Theorem 8. *Let* $V(x)$ *be a twice continuously differentiable function on* $(0, \infty)$ *which satisfies* $V(x) \to -\infty$ *as* $x \to \infty$*, and suppose that*

$$\int_c^\infty \left| \left[\frac{(\sqrt{-V})'}{(-V)^{3/4}} \right]' \sqrt[4]{-V} \right| dx < \infty \tag{2.7}$$

for some $c > 0$*. Then* $V(x)$ *is quantum-mechanical complete at* ∞ *if and only if* $V(x)$ *is classically complete at infinity.*

The relation between the two formulations is explained in [Reed and Simon, 1975]. However, there can be examples constructed which show that both notion do not need to give the same result. This is not just a mathematical gimmick, the explanation will include physics and will show the overwhelming evidence for the power of quantum probing.

We present two pedagogical examples following Rauch and Reed [Rauch and Reed, 1973] and also [Reed and Simon, 1975] where quantum-mechanical and classical incompleteness do not agree.

Example: classically incomplete, quantum-mechanically complete at ∞: The potential be a series of steps at height $-\pi^2 k^4$ with $k \in \mathbb{N}$, two plateaus are smoothly connected by steep cliffs in a very short interval (α_k, β_k). A sketch of the potential can be seen in Figure 2.1. When we calculate $\int_0^\infty \frac{dx}{(-V(x))^{1/2}}$ we see it acquires a finite value, hence, the potential is classically complete. If the steepness of the connecting lines is high enough,

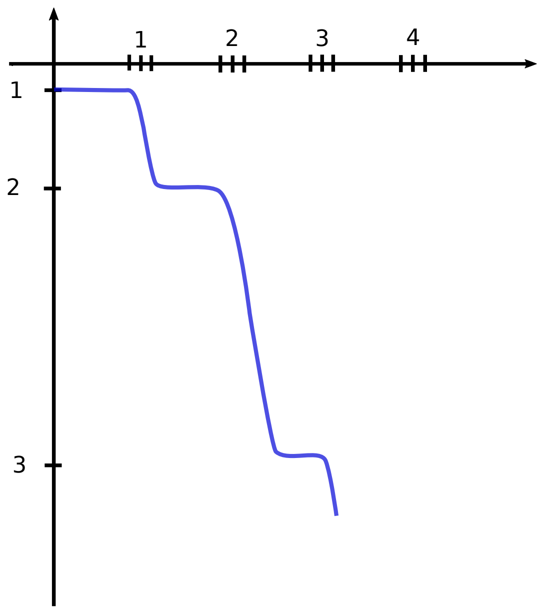

Figure 2.1.: Potential $V(x)$ with steps.

the interval around the integer number k (α_k, β_k) should be very small, then the potential will be quantum-mechanically complete. The idea is to construct $V(x)$ such that it is at least $\mathcal{C}^2(0, \infty)$ and that we can find solutions of the Schrödinger equation which are not in L^2. The potential is monotonically decreasing. Take $\alpha_1 = 1$ and let $\varphi(x) = -\cos(\pi x)$ on $(0, 1]$. At $x = 1$, we see $\varphi(1) = 1$ and $\varphi'(1) = 0$. Now, we can choose β_1 such that the solution has not much descended at this point. The solution $\varphi(x)$ is concave downward until the next zero which we call r_1. We can deduce for $\varphi(x)$

$$\varphi(x) - 1 = \int_1^x \left(\int_1^s V(t)\varphi(t)dt \right) ds. \tag{2.8}$$

The norm of the above mentioned relation allows us to estimate the upper bound on the interval $(1, \min\{r_1, \alpha_1\})$ by

$$|\varphi(x) - 1| \leq \frac{(x-1)^2}{2} 2^4 \pi^2. \tag{2.9}$$

The potential can be put on the interval (α_1, β_1). We choose β_1 such that $\varphi(\beta_1) \geq 1 - \frac{1}{4}$ is guaranteed. On the next interval, i.e. (β_1, α_2) the solution is given by $\varphi_2(x) = A\cos(4\pi x - \gamma_2)$ where A obeys the same estimate as $\varphi(\beta_1)$. We choose α_2 such that it is the closest point to 2 where $\varphi_2(x)$ has a maximum. Following the above steps we find another estimate $\varphi(\beta_2) \geq 1 - \frac{1}{4} - \frac{1}{8}$ by an appropriate choice of β_2. Repeating the procedure lead to a solution

$$\varphi(x) = A_n \cos(n^2 \pi x - \gamma_n) \tag{2.10}$$

on (β_{n-1}, α_n) with $|A_n| \geq \frac{1}{2}$. Thus $\varphi(x) \notin L^2(0, \infty)$ and therefore it is in the limit point case at infinity and quantum-mechanically complete.

This result calls for a physical interpretation. The quantum mechanical waves are reflected by the steps which are chosen such that the reflected waves are coherent and infinity can not be reached because of destructive interference. The system is complete.

This looks like the quantum-mechanical completeness is superior in the sense that quantum mechanics make classical systems complete but this is a fallacy. It is in another sense superior; this is presented in an example where a system is classically complete but quantum-mechanically incomplete. The observed incompleteness will give rise to a physical phenomenon which is not present in classical mechanics.

Example: classically complete, quantum-mechanically incomplete at ∞: We consider the following potential

$$V(x) = \frac{1}{x^2} - x^4 + \sum_{k=1}^{\infty} \sigma_k(x) \tag{2.11}$$

where $\sigma_k(x)$ are very narrow spikes with increasing height such that $V(k) = k$. The potential is classically complete because it is unbounded from above at ∞. With Theorem 7 we can show that for the potential depicted in Figure 2.2. without the spikes the Hamilton operator is not essentially self-adjoint on C_0^∞. It can be shown [Reed and Simon, 1975] when

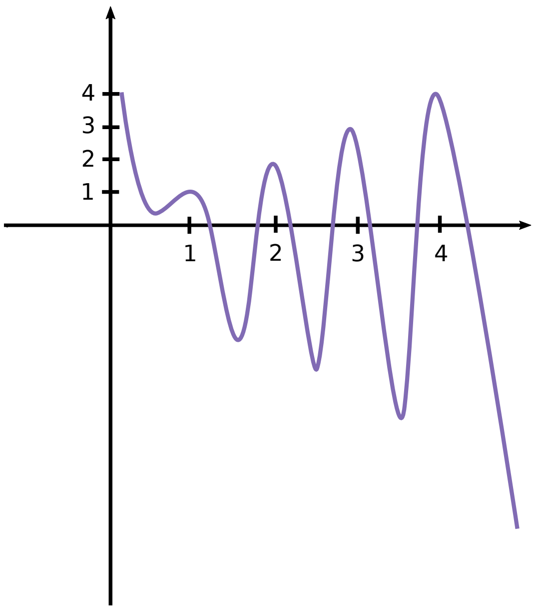

Figure 2.2.: Potential $V(x)$ with spikes.

the spikes are narrow enough that the Hamilton operator for the whole potential is not essentially self-adjoint and the motion generated by $V(x)$ is not complete. The particles, if the spikes are narrow enough, can tunnel through the potential barrier and reach infinity. The mathematical analysis gives rise to a phenomenon we can experimentally observe which is not present in classical physics.

There are many other examples where classical completeness is contrasted to the quantum-mechanical e.g. [Shubin, 1998]. In a publication by Simon the reader can find a lot of applications of eigenvalue problems [Simon, 1991]. The power of the quantum-mechanical completeness concept can be used in order to probe space-times. Before we evaluate on this, we will explain the relativistic classical criterion based on geodesic completeness.

2.4. Geodesic completeness

In general relativity the motion of test bodies is governed by the background geometry which is described by a connected four-dimensional differentiable Hausdorff C^{∞} manifold $\mathcal{M}$ and a bilinear form defined on $\mathcal{M}$, the Lorentz metric g (for mathematical definitions cf. [Kobayashi and Nomizu, 1963, Kobayashi and Nomizu, 1969]). We use as mathematical model for a space-time the pair $(\mathcal{M}, g)$ as a collections of events [Hawking and Ellis, 1973]. The space-time curvature from the point of view of a point-particle sitting on the manifold is interpreted as a potential which shape dictates its path in case it is freely falling. In the non-relativistic limit of Newtonian mechanics this can be identified with the trajectory of the test body.

It should be noticed that this description is only appropriate in the case of point masses. For extended bodies, the motion in arbitrary geometries turns out to be more complicated [Dixon, 1970a, Dixon, 1970b, Dixon, 1974, Ehlers and Rudolph, 1977]. By sending the size of the object to zero, the point particle case can be reproduced.
A space-time singularity corresponds to a point or a region where the metric tensor degenerates (at least one component of g goes to zero) or diverges (at least one component of g goes to infinity). If the rest of the space-time is differentiable and in the above sense non-pathologic, one could cut out the point by ensuring that no regular point is omitted from $\mathcal{M}$. In other words, singular space-times in the framework of general relativity mean that there are points or regions which are cut out in order to preserve Lorentzian signature and the differentiability of the metric everywhere.

In this context, the question occur whether a space-time could be extended with the required differentiability or not. Geodesic curves γ serve as a diagnostic tool; they describe the trajectory a point mass would follow in the absence of forces. The mathematical definition is [Kobayashi and Nomizu, 1963]:

Definition 7. *A curve $\gamma = x_t$, with $a < t < b$, where $-\infty \leq a < b \leq \infty$, of class $C^1(\mathcal{M})$ with a linear connection Γ is called a **geodesic** if the vector field $X = \dot{x}_t$ defined along γ, that is, if the transport along X: $\nabla_X X$ exists and equals 0 for all t, with $\dot{x}_t$ being the tangent vector to the curve γ at the point x_t.*

The connection of the metric g along the vector field X is given by ∇_X, a dot denotes a differentiation with respect to the parameter t. If a curve γ is parametrised by a unique affine parameter then γ is turned into a geodesic. Its equation of motion can either be deduced from the equivalence principle, or calculated through the action principle or through parallel displacement along a curve. The curvature of the background encoded in the connection form shapes the geodesic. The equation of motion, the so called geodesic equation is written as:

$$\nabla_X X = 0 \quad , \text{if} \quad X = \dot{x}_t. \tag{2.12}$$

This equation can be expressed in a local coordinate neighbourhood. Let $x^i(t)$ be the equations of a curve $\gamma = x_t$, then γ is a geodesic if and only if

$$\frac{d^2x^i}{dt^2} + \sum_{j,k} \Gamma^i_{jk} \frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \quad i \in \mathbb{N} \setminus \{0\}. \tag{2.13}$$

With the notion of geodesics, we are able to state the criterion for completeness which is given by [Hawking and Ellis, 1973]

Definition 8 (geodesic completeness)**.** *A semi-Riemannian (Lorentzian) manifold is complete if every geodesic can be extended to arbitrary values of its affine parameter.*

The affine parameter has the physical interpretation of proper time. A geodesic with affine parameter range from minus until plus infinity corresponds to a motion which is uniquely defined for all (proper) time. When we have a singularity at one end-point the affine parameter ends at a finite value, for example in Schwarzschild at $r = 0$ which corresponds to a finite affine parameter depending on the initial conditions. It is impossible to extend the geodesic through this point, the motions abruptly ends in a configuration where the curvature diverges. However, the completeness can also be seen from the metric itself [Hawking and Ellis, 1973]

Definition 9 (metric completeness)**.** *The pair* $(\mathcal{M}, g)$ *is metrically complete if every Cauchy sequence with respect to the distance function converges to a point in* $\mathcal{M}$*.*

This means that for a small parameter $\varepsilon > 0$ and for all $n \geq N$ the metric acting as a difference function $g(x_n, x) < \varepsilon$. The above definitions, metric and geodesic completeness, can be shown to coincide generically for Riemannian manifolds by the Hopf-Rinow theorem [Kobayashi and Nomizu, 1963] but for Lorentzian manifolds there are counterexamples, e.g. Clifton-Pohl torus.

Turning to the physical implications of the mathematical terms, we first want to say something about singularities in the context of general relativity. There are three possibilities of singularities: space-like, time-like, and light-like (and combinations thereof).

Space-like singularities occur at a specific time. Either a whole spatial hypersurface becomes singular or the whole metric collapses to one point (this can only happen for non-vacuum solutions). These types occur for example for Schwarzschild black holes (vacuum) and for Friedmann universes (non-vanishing energy-momentum tensor).

Time-like singularities are points in space where matter enters an infinite curvature regime and are located in space. As example serves the negative mass Schwarzschild solution which we present in the following or charged black holes.

Light-like singularities are somehow exotic and can be found in compactified supergravity [van Baal and Bais, 1983].
How can we find out that a space-time admits a singularity? In 1965 Penrose [Penrose, 1965] has formulated the first of a series of singularity theorems [Hawking, 1965, Hawking, 1966, Hawking, 1976b]. The most powerful and most general is the singularity theorem of Hawking and Penrose [Hawking and Penrose, 1970]. They show that a manifold admits a singularity under rather general properties. We state the version of the most popular theorem without giving a proof [Hawking and Ellis, 1973]:

Theorem 9. *A space-time* $(\mathcal{M}, g)$ *is not timelike and null geodesically complete if:*

(1) $R_{\mu\nu}v^{\mu}v^{\nu} \geq 0$ *holds for every non-spacelike vector* $\mathbf{v}$

(2) The generic condition is satisfied, that is, every geodesic contains a point at which $t_{[\mu}R_{\nu]\alpha\beta[\rho}t_{\sigma}t_{\lambda]}t^{\alpha}t^{\beta} \neq 0$*, where* $\mathbf{t}$ *is the tangent vector of the geodesic.*

(3) The chronology condition holds on the manifold

(4) There exists at least either a compact achronal set without edge or a closed trapped surface or a point p *such that on every past (or future) null geodesic from* p *the divergence* θ *of the null geodesic from* p *becomes negative.*

In this theorem $R_{\mu\nu}$ denotes the Ricci curvature tensor. These rather mathematical statements can be brought into a more physical language. The first condition can be paraphrased into the statement that gravity acts always attractive, the second says that every geodesic feels the influence of the curvature, and the chronology condition is equal to the statement that there are no closed timelike curves, i.e. we have a notion of the light cone and causality. The last statement is a bit more involved, but what it actually means is that at some point geodesics tend to approach each other and the light cone is reconverging. The strength of reconvergence is given by the parameter θ, called expansion. In other words under gravity (which is purely attractive) generic conditions predict a singularity which can be seen in the metric tensor by diverging or vanishing components.

However, we have to make sure, that the singularity is not just an artefact of the coordinate choice. Diffeomorphism invariant quantities such as the Kretschmann scalar extract the relevant singularities because the coordinate dependence is gone. Another way could be to find a coordinate neighbourhood which is regular at the specific point, i.e. the Kruskal-Szekeres for the black-hole manifold [Kruskal, 1960, Szekeres, 1960].

The parameter θ measures how the distance of two neighbouring geodesics changes with respect to the affine parameter. Raychaudhuri's equation [Raychaudhuri, 1957] describes the movement of two particles on neighbouring geodesics with affine parameter λ:

$$\frac{d\theta}{d\lambda} = \omega^2 - \sigma^2 - \frac{\theta}{3} - R(X,X) - \frac{d(dX)}{d\lambda}. \tag{2.14}$$

The different tensors θ, σ, and ω have physical meanings: θ is the expansion tensor, σ the shear tensor, and ω the vorticity tensor. This equation is essential in Hawking and Penrose's proof of the singularity theorems.

Singularity theorems only make statements about the occurrence but say little about the nature of the singularity, like the dimensionality or the orientation, nor do they explain their physical impact. While for example a singular region in space could be avoided by staying far away, it is not so intuitive how a spacelike singularity can be omitted. In the first case the geodesics which do not hit the singularity can be extended uniquely to infinite affine parameter, in the latter case it seems pretty hopeless to extend the affine parameter to infinity length.

Geodesic completeness is close to the basic idea of completeness, because the geodesic should be uniquely defined for all affine parameter (which can be interpreted as proper time) and this has to hold for all geodesics which corresponds to the demand of having arbitrary initial conditions. No matter how much we appreciate differential geometry in the confrontation of non-relativistic classical and quantum-mechanical completeness, the latter scores. Before we toss in the towel in the light of spacelike singularities, we shall consider quantum theory; first for timelike singularities and then we deploy the full power of quantum field theory and investigate spacelike singularities.

2.5. Quantum-mechanical probes of space-time singularities

Completeness in the theory of quantum mechanics and general relativity are rather different because we have two quite different criteria for probing the existence of a singularity within each theory. On the quantum-mechanical side essential self-adjointness of the Hamilton operator, which corresponds directly to a unique time-evolution, guarantees completeness. On the general relativity's side we saw that a geodesic of infinite length corresponds to a complete manifold.

In section 2.3 we compared the classical with quantum-mechanical completeness and uncovered that there exists a tension. We could ask, does this tension occurs when we probe a space-time with quantum probes, which is exactly the question we address in this section, additionally we want to point out the interdependence between both concepts.

A very intriguing example is the already mentioned hydrogen atom where the quantum-mechanical states are ignorant of the classical singularity of the Coulomb potential. Let us mention that quantum mechanics can only be considered on a static, globally hyperbolic space-time. In dynamical space-times there is no consistent relativistic one particle description [Ashtekar and Magnon, 1975]. Probing of dynamical space-times is hence only appropriate when performed via quantum field theory. Therefore, we first recap the probing of timelike singularities which has been developed by Horowitz and Marolf [Horowitz and Marolf, 1995].

Their argument is basically the same as for quantum mechanical completeness. We call the space-time quantum-mechanically complete if the quantum probes have a unitary time evolution generated by a self-adjoint Hamilton operator. Quantum-mechanical as well as classical completeness is basically a notion of potential completeness, i.e. the shape of the potential decides whether the system is complete or not complete. The geometry of a space-time can be transformed into an effective potential. Compared to flat space-time the equation of motion is different because the differential operator $\Box = g^{-1}(\nabla, \nabla)$ depends explicitly on the metric components of g which are functions of the coordinates, and form effectively a potential. Albeit there are several ways to define completeness, e.g. [Traschen and Brandenberger, 1990], we want to explain the so-called Horowitz-Marolf criterion which is an accepted proposal.

When we start with a static space-time which admits a timelike Killing vector field ξ^μ, no matter whether it is regular or singular, we can write down the wave function $(\nabla^\mu \nabla_\mu - m^2)\psi = 0$ with aid of the Killing parameter t

$$\frac{\partial^2 \psi}{\partial t^2} = V D^i (V D_i \psi) - V^2 m^2 \psi \tag{2.15}$$

with $V^2 = -\xi^\mu \xi_\mu$, and D_a is the spatial covariant derivative on the spatial hypersurface Σ. The idea of a quantum probing of singular space-times has been developed by Wald [Wald, 1979], while [Blau et al., 2006] characterised a lot more solutions to (2.15) with respect to their completeness. Horowitz and Marolf's approach is based on Wald's analysis. We will briefly review their idea which presents the inspiration to the quantum probing with respect to quantum field theory.

In equation (2.15) we define the operator acting on ψ as $A = -VD^i(VD_i) + V^2m^2$. The underlying Hilbert space is the space of square integrable functions. This is an essential feature in the construction of quantum mechanics, because it ensures the validity of a probabilistic interpretation. Without this we would not be able to interpret a quantum theory since the solutions of the Schrödinger equations are complex, hence, not observable. The domain of A will be the smooth functions of compact support on the spatial hypersurface $D(A) = \mathcal{C}_0^\infty(\Sigma)$ and A is positive and symmetric/Hermitian and its eigenvalues are real which is to say that the deficiency indices are equal [Horowitz and Marolf, 1995] and a self-adjoint extension A_E always exists. Nevertheless, we shall mention the crucial question is whether this extension is unique or not. In case of a unique extension, A_E will be positive definite and we can take the positive root of (2.15)

$$i\frac{d\psi}{dt} = \sqrt{A_E}\psi. \tag{2.16}$$

The solution can be found just by integration with respect to time

$$\psi(t) = \exp\left(-i\sqrt{A_E}t\right)\psi(0). \tag{2.17}$$

Assume the extension of A is unique, then the quantum theory on the space-time is called regular. If in contrast there are more than one extension, we face some ambiguity in our theory which means a loss of predictability.

A self-adjoint operator generates a unique unitary time evolution and preserves the norm of the state ψ.

Horowitz and Marolf picked a very easy example to illustrate what happens during a quantum probing of static space-times. We recall this here in order to give an intuition for their completeness criterion.

Consider the metric of a general static, spherically symmetric space-time in four dimensions

$$g = -V^2 dt \otimes dt + \frac{1}{V^2} dr \otimes dr + R^2 d\Omega_2 \tag{2.18}$$

with $V(r)$ and $R(r)$ functions only of the radial component r and $d\Omega_2$ is the line element of the solid angle. Our task is to test for the self-adjointness of the operator A. It is important to note that the completeness criterion is the same as for quantum mechanics because we will extract the impact of the metric and rewrite it as an effective potential for ψ. From Theorem 3 we get that self-adjointness corresponds to having no solutions besides $\psi = 0$

to the equation

$$(A \pm i)\psi = 0. \tag{2.19}$$

The solution for the above equation can be found by a separation of variables and expansion in spherical harmonics $\psi = f(r)Y(\vartheta, \varphi)$. Recall, the singularity of the metric is purely timelike; it only depends on the radial component. The resulting radial equation is

$$\frac{d^2f}{dr^2} + \frac{1}{V^2R^2}\frac{d(V^2R^2)}{dr}\frac{df}{dr} - \frac{c}{V^2R^2}f - \frac{m^2}{V^2}f \pm i\frac{f}{V^4} = 0. \tag{2.20}$$

with arbitrary constant $c \in \mathbb{C}$. Self-adjoint A implies at least one of the solutions to (2.20) fails to be in L^2 with respect to the correct measure, where the measure function denotes R^2V^{-2} near the origin. If one solution fails to be square integrable at the endpoint then we are left with only one unique solution. Note, this has to hold for all c, either negative or positive.

At $r = 0$ it turns out only one solution meets the condition to be square integrable from which follows that the potential term $-\frac{m^2}{V^2}f$ acts as a barrier which prevents the wave function from reaching the end-point at the origin in a finite amount of time. In fact, it drives the smaller solution faster to zero while the larger solution diverges even more at $r = 0$. The self-adjoint extension is unique. This example of a classically singular and quantum-mechanically regular metric nevertheless shows that there is a tension between both notions. Another example is a charged dilatonic black hole in four dimensions [Gibbons et al., 1995]

$$S = \int d^4x\sqrt{-\det(g)}\left[R - 2\left(\nabla\phi\right)^2 - e^{-2a\phi}F^2\right]. \tag{2.21}$$

Here, ϕ is the dilaton and F the Maxwell field, a is the coupling constant of the dilaton field. It is an example of a quantum-mechanically non-singular space-time. For $a = \sqrt{3}$ it becomes Kaluza-Klein theory [Horowitz and Marolf, 1995] and it can be shown that this theory is quantum mechanically complete because of the occurrence of an infinitely high potential barrier [Holzhey and Wilczek, 1992].

In this situation the opposite outcome can occur, classically regular but singular with respect to the quantum probing. The negative mass black hole [Ishibashi and Hosoya, 1999] is a Schwarzschild solution with $m < 0$, the corresponding Penrose diagram can be seen in Figure 2.3. This modification of the Schwarzschild space-time implies that the whole configuration is static and spherically symmetric and the horizon vanishes; we have a naked singularity at the origin (wiggly line on the left of Figure 2.3) where the curvature scalar

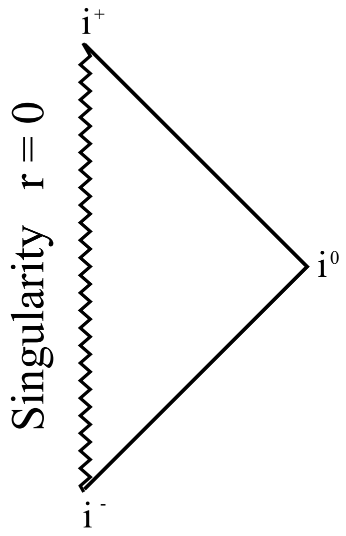

Figure 2.3.: Penrose diagram of the negative mass Schwarzschild solution

diverges. However, this point cannot be reached in a finite amount of (proper) time because the effective potential grows unbounded and general relativity predicts completeness for the end-point.

The potential effectively generated from the geometry is $V \propto \frac{1}{r}$; at the origin it diverges to positive infinity and becomes infinite in height and steepness. Quantum mechanics show [Horowitz and Myers, 1995] for this background that both solution of the Schrödinger equation are locally normalisable near $r = 0$ which implies that we have two solutions and the Hamilton operator fails to meet the criterion for essential self-adjointness, consequently, we have lost predictability and time evolution is not unique. The Horowitz and Marolf criterion classifies this system as quantum mechanically singular. This does of course no harm to the notion of quantum mechanical completeness, moreover it points out that something pathologic is happening. To the best knowledge of todays physics, negative mass is not realised in nature, whatsoever. Quantum mechanics, in contrast to general relativity, points out that there is a serious problem with the construction by running into a singularity. The interpretation would be that it is not possible to do any quantum physics on this background consistently, another reasonable conclusion would be that there is no negative mass. Additionally, when we consider metric perturbations around this background, there will be only one initial configuration which leaves the fluctuations finite [Ishibashi and Wald, 2003]. In fact, a negative mass black hole is a highly unstable configuration.

The completeness criterion of Horowitz and Marolf inspired a lot of research. Konkowski et al.[Konkowski and Helliwell, 2001, Konkowski et al., 2003, Konkowski et al., 1985, Konkowski and Helliwell, 1985] investigated cosmologically relevant space-times which are known to be quasiregular, the so-called Taub-NUT solution found by [Taub, 1951] and later generalised by [Newman et al., 1963]. These are generalisations of the Schwarzschild metric which combine a dynamical patch of space-time with a static one. Although the name suggests regularity, the quasiregular spacetimes also admit a singularity which sometimes is called a mild singularity [Helliwell et al., 2003]. The classification as mild suggests they could be harmless, however, they are quantum mechanically singular. Furthermore they extended the quantum probing by using different types of degrees of freedom, for example Klein-Gordon, Dirac, and Maxwell fields.

Ishibashi and Hosoya [Ishibashi and Hosoya, 1999] set up quantum probings for naked singularities which are forbidden in general relativity. The cosmic censorship hypothesis claims that naked singularities are hidden behind a horizon. However, the big bang represents the most famous naked singularity. Most articles which were built upon the research in [Horowitz and Marolf, 1995], show a tension between geodesical and quantum-mechanical completeness.

Blau, Frank, and Weiss [Blau et al., 2006] specify the result to cases where a singular space-time with timelike singularity fulfils the dominant energy condition[3]. If so, the space-time is also singular in the sense of Horowitz and Marolf [Horowitz and Marolf, 1995].

The whole approach is strictly limited to static space-times with a timelike singularity. The limitation is set by the probing theory. In dynamical space-times we observe emission and absorption processes, hence the number of degrees os freedom is no longer a conserved quantity. Quantum mechanical completeness will not be appropriate in a dynamical space-time, and has to be amended by quantum field theory.

[3]The dominant energy condition: For every W_a, $T^{ab}W_aW_b \geq 0$, and $T^{ab}W_a$ is a non-spacelike vector [Hawking and Ellis, 1973]. This can be interpreted that no-one can observe a local negative energy density and the energy flow is non-spacelike.

3 The two singularities

Singularities are a widely discussed concept in various scientific disciplines. Originally they were defined in mathematics but when adopted to physics they became the sign for the breakdown of the chosen description. For example the Coulomb potential is fine in order to describe the nuclear potential of an atom unless the electron is not too close to the nucleus, that is, the structure of the nuclear constituents is negligible. General relativity boosted the popularity of singularities a lot after the discovery of the black hole solution to Einstein's equation.

In this regard arises the important question whether a singularity is an artefact of the used language of mathematics or a real object appearing in nature. We have seen in the last chapter, it strongly depends on the chosen degree of freedom whether the system is measured singular or not. While a geodesic observer would conclude there is a singularity, the quantum-mechanical observer might not agree. In this chapter we will take the mathematical point of view and analyse two distinct space-times with respect to its geometrical properties.

The idea of the thesis is to understand the evolution of a quantum field in a singular dynamical background geometry. Before we could reach this goal, we need to understand the geometry which affects this evolution. Our analysis covers two space-times - Schwarzschild and Kasner space-time - which are both singular in the sense of the singularity theorems. Their singular structure coincides; both admit a spacelike singularity at one endpoint and both space-times are globally hyperbolic. Similar singular structure means in this example a whole hypersurface becomes singular - in case the space-time is foliated into spacelike hypersurfaces with respect a global timelike Killing vector field. Let us now analyse the two space-times and discuss their huge importance for physics.

3.1. The gravitational singularity

Gravitational singularities describe all kinds of black hole space-times which are spherically symmetric solutions of the Einstein equation. The most prominent examples are the Schwarzschild [Schwarzschild, 1916], Kerr [Kerr, 1963], Reißner-Nordström [Reissner, 1916, Nordström, 1918] and Kerr-Newman [Newman and Janis, 1965] black holes. For an anti-de Sitter Schwarzschild metric can the singularity be resolved in principle [Maldacena, 2003]. The Kerr-Newman is the most general form of a black hole. By appropriately choosing charge and angular momentum this solution reduces to more special black holes. Classical no hair theorems [Israel, 1967] characterise black holes by only three quantities: charge Q, mass M, and angular momentum J. Whilst in the Kerr-Newman none are zero, we can reduce to Kerr by setting $Q = 0$, to Reißner-Nordström with $J = 0$, and when both $Q = 0$ and $J = 0$ we get the Schwarzschild solution.

Schwarzschild's intention was to solve Einstein's equation for a spherically symmetric configuration [Schwarzschild, 1916]. Birkhoff's theorem [Birkhoff and Langer, 1923]

Theorem 10 (Birkhoff's Theorem). *Any C^2 solution of Einstein's empty space equations which is spherically symmetric in an open set $\mathcal{V}$, is locally equivalent to part of the maximally extended Schwarzschild solution in $\mathcal{V}$.*

tells us that all spherically symmetric objects admit a Schwarzschild geometry when the observer is located far enough from the central object. In this limit the volume and the shape are negligible and the description collapses to one of a point particle which deforms the background to the Schwarzschild solution. As a side remark it has be shown that this is also true for C^0 and piecewise C^1 solutions [Bergmann et al., 1965]. We will see what the term maximally extended means in the remainder of this chapter.

General relativity predicts that all collapsing objects end in a Kerr phase [Penrose, 1965]. Nevertheless, let us focus on the most simple but also the most important case, the Schwarzschild black hole. Although Kerr black holes are more likely to occur, the Schwarzschild black hole is sufficient for our analysis. The metric in the Schwarzschild coordinate neighbourhood is

$$g = -\left(1 - \frac{2M}{r}\right) dt \otimes dt + \frac{dr \otimes dr}{1 - \frac{2M}{r}} + r^2 d^2\Omega \tag{3.1}$$

with $d^2\Omega = d\vartheta \otimes d\vartheta + \sin^2(\vartheta) d\varphi \otimes d\varphi$ the line-element for the solid angle. This metric describes the solution around a pointlike object of mass M. Taking the limit of $M \to 0$

or $r \to \infty$ we recover approximately the Minkowski space-time. Therefore, (3.1) is called asymptotically flat. This can be seen by taking the Weyl tensor, its only non-zero component is $\Psi_2 = -\frac{M}{r^3}$. The metric belongs algebraically to the type D (two double principal null directions) in Petrov's classification [Petrov, 1954] which is best in order to describe the geometry around massive objects. When we analyse the singular structure we will discover two pathological points: $r = 2M$ and $r = 0$. In both cases one of the metric coefficients blow up and at least one goes to zero.

The first singular point has a remarkable property, it divides the space-time into two separate patches. For $r > 2M$ the space-time patch $\mathcal{E}$ is described as in (3.1) by a static spherically symmetric solution. If $r < 2M$ the g_{00} and the g_{rr} component both change their sign. The time signature is then in front of the $dr \otimes dr$ term which means t becomes spatial while r becomes temporal. The metric in this coordinate patch $\mathcal{B}$ is dynamical and has a different topology compared to $\mathcal{E}$. The Penrose diagram is shown in Figure 3.1. We

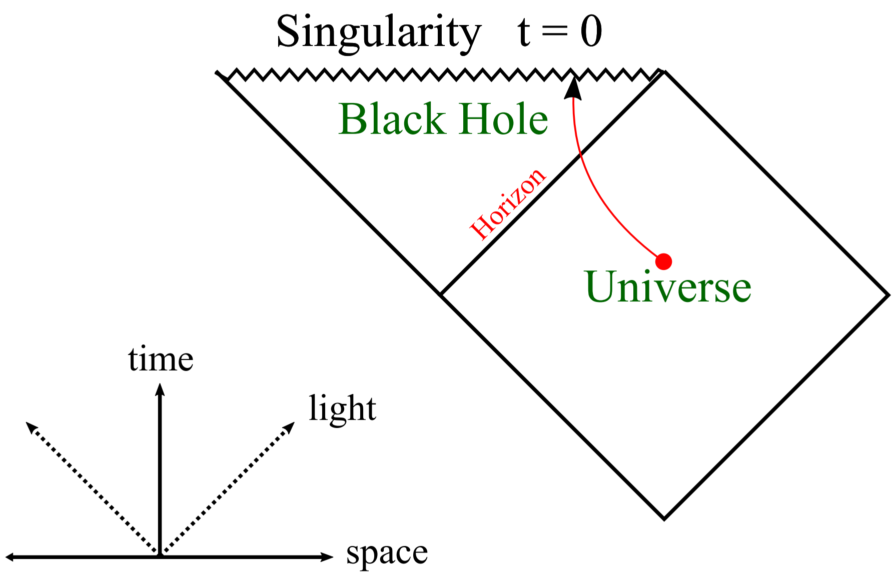

Figure 3.1.: Penrose diagram of the Schwarzschild solution

see the two different patches the outside region $\mathcal{E}$ described by (3.1) and the interior region $\mathcal{B}$ by (3.4). Angular coordinates are suppressed as usual. Both regions are separated by a S^2 null surface, the event horizon[1] located at $r = 2M$. The name event horizon is because nothing which has crossed can ever reach the horizon again; it separates events. Of course

[1]Event horizons can only be defined when the observer is infinitely far away, the more general concept are apparent horizons which can be defined at arbitrary distance from the black hole.

this includes also photons, hence, we cannot see inside the black hole, therefore is it black. Outside is a static spherically symmetric vacuum solution of Einstein's equation, inside is a dynamical spacetime. Both patches admit a timelike Killing vector field ∂_t and are globally hyperbolic on its own. In order to see this we can use Geroch's theorem [Geroch, 1970]

Theorem 11. *If an open set $\mathcal{N}$ is globally hyperbolic, then $\mathcal{N}$, regarded as a manifold, is homeomorphic to $\mathbb{R} \times \Sigma$, with Σ the three-dimensional manifold, and for each $a \in \mathbb{R}$, $\{a\} \times \Sigma$ is a Cauchy surface for $\mathcal{N}$.*

and choose a foliation along the timelike Killing vector with the spacelike hypersurfaces Σ normal to ∂_t. This foliation will become important for our quantum completeness criterion but we will refer to this later on again. It has to be mentioned that the patches when glued smoothly together will not stay globally hyperbolic near the horizon, because then the Killing vector becomes a superposition of a space- and a time-like vector field $\xi = \alpha(r)\partial_t + \beta(r)\partial_r$; for $r > 2M$ the function β decreases fast, which does α for $r < 2M$. The naming of the coordinates is due to (3.1). We see at the horizon that the light cone flips which accompanies the change of the Killing vector.

The true singular structure is given by a spacelike singularity, at the hypersurface $r = 0$ where all geodesics end without being extended. This fulfils the criterion of incompleteness mentioned in the previous chapter. However, the components of the metric tensor are not diffeomorphism invariant, especially not invariant under a coordinate transformation. We need an invariant quantity in order to decide whether we have chosen an awkward coordinate system or not. Therefore we consider the Kretschmann scalar which is a contraction of the Riemann tensor R with itself

$$K = R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} = \frac{48M^2}{r^6}. \tag{3.2}$$

For the metric (3.1) is the Kretschmann scalar $\propto r^{-6}$ which is a coordinate invariant statement. K blows up in the limit $r = 0$ which in general relativity is the real, physical singularity and it cannot be removed by a diffeomorphisms. The divergence at the Schwarzschild radius $2M$ has disappeared in K, which indicates that a new coordinate system could be chosen without the singularity at the horizon. Such singularities are called coordinate singularities.

There is a bunch of regular[2] coordinate neighbourhoods for black holes like for example Painlevé-Gullstrand [Painlevé, 1921, Gullstrand, 1922]. However, we have to pay a price

[2] Regular at the Schwarzschild coordinate $r = 2M$.

for this, those type of coordinates come with off-diagonal elements in the metric tensor. Kruskal [Kruskal, 1960] and Székeres [Szekeres, 1960] found independently a coordinate system where the metric components have a finite value at the horizon and are given in light-cone coordinates u and v. The black hole metric in these coordinates can be deduced from Schwarzschild coordinates by the following transformations:

$$v = t + r_*, \qquad u = t - r_* \quad \text{with} \quad r_* = r + 2M \ln\left|\frac{r}{2M} - 1\right| . \tag{3.3}$$

The space-time coordinate r_* is called tortoise coordinate from the antique paradox of Zeno: achilles and the tortoise. We can perform the transformation and end with the fully extended solution of the Schwarzschild metric

$$g = -\frac{32M^3}{r} dv \otimes dv + \frac{32M^3}{r} du \otimes du + r^2(d\vartheta \otimes d\vartheta + \sin^2(\vartheta) d\varphi \otimes d\varphi). \tag{3.4}$$

Note that the coordinate r is now a function of the light-cone coordinates u and v where u denotes the outgoing and v the ingoing null coordinates. The explicit dependence of $r(u, v)$ is given through the Lambert W-function which is defined by the functional equation $z = W(z) \exp(W(z))$ such that

$$r(u, v) = 2M \left[1 + W\left(e^{\frac{u+v}{2M} - 1} \right) \right] . \tag{3.5}$$

This function has two branches, here the upper one is used. It can be seen that the metric components remain singular at the origin but stays regular around $r = 2M$. Obviously, nothing pathological happens at the horizon. The same result can be seen by changing to proper time; for the free falling particle nothing special occurs at this point whereas for an observer, which is located infinitely far away, the infalling object slows down and becomes more and more redshifted[3]. Kruskal-Székeres coordinates agree to the result in proper time that at the horizon nothing special happens for a geodesic observer.

For the sake of completeness we just want to mention why solution (3.4) is called fully extended (cf. figure 3.2.). In this coordinate neighbourhood, we get the two Schwarzschild patches (Black Hole and Universe) as before and two additional ones. One is a second outside region (parallel Universe) which is not causally connected to the other outside region but geometrically the Universe and parallel Universe are equal. There is as well

[3] Nevertheless, this could also be the physical effect which describes the photons, sent by the infallen object, spiralling outwards

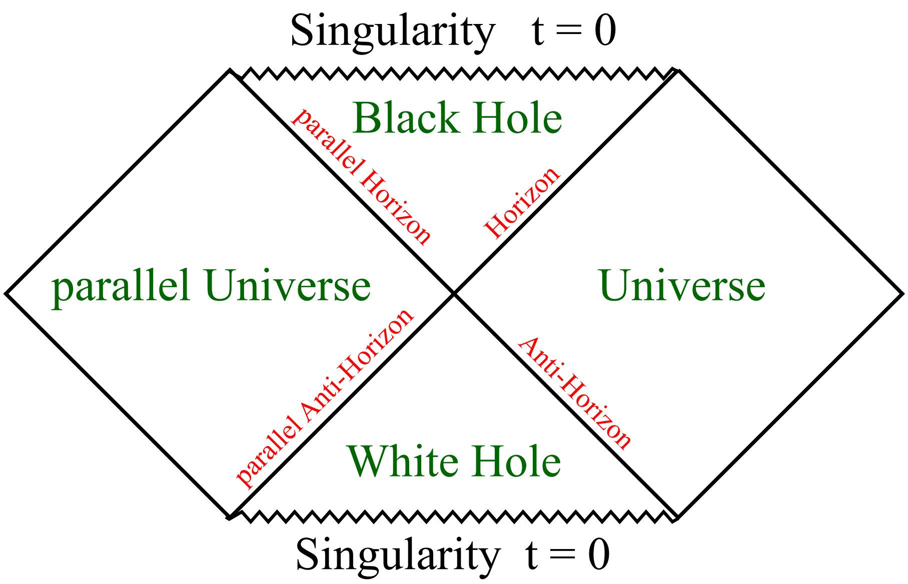

Figure 3.2.: Penrose diagram of the fully extended Schwarzschild solution.

something in the past similar to the black hole, with the directions of geodesics is reversed, such that all point towards the anti-horizon. It is a reversed time black hole, called a white hole [Hawking, 1976a]. The full setup are two universes sharing one white and one black hole. There is the theoretical possibility to go from one universe to the other by a wormhole acting as Einstein-Rosen bridge [Maldacena and Susskind, 2013].

Our aim in this thesis is to investigate the physical relevance of geometrical singularities. Therefore we will focus from now on the interior of a black hole, where the geometrical singularity is located. Note, the probing of the white hole could be similar to the case of the interior metric but the singularity lies in the past.

As we mentioned above, the metric can be decomposed into an interior $r \geq 2M$ and an exterior region. Inside the black hole the (outside) time component becomes spacelike and the radial component timelike. By renaming the components t for the timelike and r for the spacelike[4] we get the following metric for the interior

$$g = -\frac{dt \otimes dt}{\left|1 - \frac{2M}{t}\right|} + \left|1 - \frac{2M}{t}\right| dr \otimes dr + t^2 d^2\Omega. \tag{3.6}$$

The solid angle remains unchanged when entering the horizon but we want to emphasise that in front of the angular part is now a time dependent function, instead of a radius

[4]From now on the coordinate referring to the time coordinate will be called t.

dependent function like in $\mathcal{E}$, hence the metric becomes anisotropic. Such a metric is of the Kantowski-Sachs type with topology $\mathbb{R} \times \mathbb{R} \times S^2$

$$g = -A(t)dt \otimes dt + B(t)dr \otimes dr + F(t)d^2\Omega. \tag{3.7}$$

The functions A, B, F are distinct scale functions which can be read off in the Schwarzschild interior to be $B(t) = A(t)^{-1} = \left|1 - \frac{2M}{t}\right|$ and $F(t) = t^2$. Outside, the Schwarzschild metric describes a sphere, while inside the space has the shape of a cylinder or a cigar. This leads to a spherically homogeneous but anisotropic cosmological model. If space-times are spatially homogeneous and isotropic then they belong to the Friedmann-Lemaître-Robertson-Walker case.

Metric (3.6) admits a four-parameter group with three-parameter subgroup acting on two-dimensional surfaces of constant curvature. The Kantowski-Sachs metric [Collins, 1977] contains a spacelike singularity at $t = 0$ and is globally hyperbolic, foliation along a timelike Killing vector field is possible. Under the assumption of a perfect fluid inside the black hole, the black hole interior metric becomes conformally flat after a sophisticated coordinate transformation [Buchdahl, 1971]. In general, the Schwarzschild solution is not isomorphic to Minkowski. Nevertheless, the spatial hypersurfaces are conformally flat, which is tested by the Cotton tensor, a 3-tensor defined as

$$\mathsf{Cot} = \star\nabla \left(\mathfrak{Ric} - \frac{\mathcal{R} g_\Sigma}{4} \right), \tag{3.8}$$

with $\mathfrak{Ric}$ the Ricci tensor and $\mathcal{R}$ the Ricci scalar curvature, and g_Σ the metric tensor of the hypersurface. Necessary and sufficient for a three dimensional manifold to be conformally flat is a vanishing Cot. In four dimensions the analogue is the Weyl tensor. Note, the hypersurfaces in its conformally flat form do not need to be orthogonal to the timelike Killing vector field admitted by the full metric. A rescaling of the coordinates by a time-dependent function might be appropriate in order to see that the interior Schwarzschild admits conformally flat hypersurfaces. For completeness reasons we stress that assuming a perfect fluid inside the black hole, the interior transforms such that its Weyl tensor vanishes and it becomes conformally flat [Raychaudhuri and Maiti, 1979].

For the the whole Kantowski-Sachs class we can show that $\mathsf{Cot} \equiv 0$ therefore the time slices are isometric to flat spacetime

$$g = -f(t)dt \otimes dt + K(t)d^3\Sigma \tag{3.9}$$

where $d^3\Sigma$ can be expressed as a three dimensional Euclidean space-time. This allows for an expansion of the harmonic functions (corresponding to the spatial Laplace operator) in exponential functions. By re-scaling $r' \to \sqrt{t^3/(t-2M)}r$ we can bring the metric into the form of (3.9) with $K(t) = t^2$. Note that here occurs the effect that r' depends on t which produces a tilt of the spatial hypersurface with respect to the Killing vector.

Leaving this intermezzi behind, we come back to (3.6). The interior Schwarzschild solution has a very remarkable feature when asymptotically expanded for small times, namely,

$$\bar{g} = -\frac{t}{2M} dt \otimes dt + \frac{2M}{t} dr \otimes dr + t^2 d^2\Omega. \tag{3.10}$$

which is a so-called A-metric first developed by Ehlers and Kundt [Ehlers and Kundt, 1962]. This class of space-times is related to the Schwarzschild metric. For example the gravitational potential of a point particle with mass M or a planet, etc. (Schwarzschild solution) or the gravitational field of a tachyon (AIII-metric) is covered by this class. Generically, they are given by

$$g_A = -\left(\varepsilon - \frac{2M}{r}\right) dt \otimes dt + \frac{dr \otimes dr}{\varepsilon - \frac{2M}{r}} + r^2 \frac{d\zeta \otimes d\zeta^*}{(1 + \frac{\varepsilon}{2}|\zeta|^2)^2}. \tag{3.11}$$

The $\zeta \in \mathbb{C}$ are coordinates on the Argand-Wessel-Gauß plane and yield the angular coordinates through a specific transformation; ε is a parameter reflecting the Gaußian curvature of the manifold.

For $\varepsilon = 1$ and $\zeta = \sqrt{2}\tan\left(\frac{\vartheta}{2}\right) e^{i\varphi}$ we get the usual Schwarzschild solution which in this context is called AI solution.

A particularly interesting A-metric comes with the parameters $\varepsilon = 0$ and the transformation $\zeta = \frac{\vartheta}{\sqrt{2}} e^{i\varphi}$. Plugging this in (3.11) leads to

$$g_0 = -\frac{t}{2M} dt \otimes dt + \frac{2M}{t} dr \otimes dr + t^2(d\vartheta \otimes d\vartheta + \vartheta^2 d\varphi \otimes d\varphi) \tag{3.12}$$

which is called AIII and is very similar to the asymptotically expanded interior Schwarzschild solution $\bar{g}$ - after we Taylor expand $\sin^2(\vartheta)$ we get in first order ϑ^2 (small-angle approximation) and receive an AIII metric which describes perfectly the two singularities presented in this thesis. The following change of coordinates

$$r \to \sqrt[3]{\frac{3}{4M}} z, \quad t \to \sqrt[3]{\frac{9M}{2}} \tau^{2/3}, \quad \vartheta e^{i\varphi} \to \sqrt[3]{\frac{2}{9M}} (x + iy) \tag{3.13}$$

allows to re-state the AIII metric accordingly

$$g = -d\tau \otimes d\tau + \tau^{4/3}(dx \otimes dx + dy \otimes dy) + \tau^{-2/3} dz \otimes dz. \tag{3.14}$$

that it becomes a type-D Kasner solution with exponents $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$.

From the Belinsky-Khalatnikov-Lifshitz conjecture (we will in the next subsection introduce Kasner space-times more detailed) we know the important coordinate dependence close to a spacelike singularity is due to time; its behaviour is given by Kasner metrics. In our Kantowski-Sachs model, the universe is described via an anisotropic, homogeneous space-time and it is shown that the time dependence can be factorised. Close to the singularity (asymptotically expansion in small time is meaningful), assuming a small angle ϑ (which is not too restrictive since we could have restricted ourself to the equatorial plane) we find that the interior Schwarzschild metric corresponds to a Kasner type-D metric. Kasner space-times belong to the Bianchi I type of symmetries but additionally need to obey two conditions, the Kasner plane ($\sum p_i = 1$) and the Kasner sphere ($\sum p_i^2 = 1$) simultaneously. The Kasner spacetime, therefore, is the intersection of the sphere and the plane. With rescaling of z we can bring (3.14) into spatially conformally flat, or Bianchi I, form with a purely time dependent conformal weight $\Omega(x) \equiv \Omega(\tau)$.

Those geometric properties will become important for quantum completeness. In Minkowski space-time is not much worry about conceptual things like the definition of a Fourier transform since the harmonics are exponential functions. For Schwarzschild there might appear some obstacles coming along with the more complicated topology and geometry. However, the reduction to a Kasner type metric, which is Bianchi type-I, show that the important information about the singularity is encoded in the time component which stays untouched by just restricting to the Kasner type-D solution. Nevertheless, we will do the calculation for the full Schwarzschild metric (3.4) and show consistency with the Kasner metric (3.14), and we draw a connection between the black hole singularity and the BKL conjecture.

Both singularities, the Kasner as well as the Schwarzschild case, lead to incompleteness with respect to the singularity theorems, because both singularities can be reached within a finite affine parameter. Since the gravitational singularity is governed by a Kasner-like behaviour, it is natural to analyse this class of space-time, too.

3.2. The generalised Kasner singularity

The second singular space-time which we investigate is the generalised Kasner metric. For the first time this solution was proposed by Edward Kasner in 1921 [Kasner, 1921] as an anisotropic[5] vacuum solution of Einstein's equation which means, the Ricci curvature vanishes; Kasner space-times belong to the Bianchi class metrics which are generically given by the following from

$$g = -dt \otimes dt + g_{ab}(t)(e^a_i dx^i)(e^b_j dx^j) \tag{3.15}$$

The vierbeine are denoted by e^i_j and the components of the spatial metric by $g_{ab}(t)$. Note, $g_{ab}(t)$ is a purely time-dependent function; this is an essential feature of this space-time class. Bianchi I metrics have numerous applications in cosmological models, they are generally given by

$$g = -dt \otimes dt + \sum_{i=1}^{3} a_i^2(t)(dx^i \otimes dx^i). \tag{3.16}$$

A special case of this class are for example the Kasner solutions [Kasner, 1921] or the Friedmann space-time. They have been widely analysed by physicists and mathematicians. Besides their belonging to the Bianchi I class the time dependent scale factors of Kasner space-times obey two additional conditions we will show below. These conditions reflect that the space-times are vacuum solutions.

Generalised Kasner space-times are described through a multiply warped product of a base manifold $\mathcal{B}$ and fibres $\mathcal{F}_i$ with warping factor $f : \mathcal{B} \to \mathcal{F}$. The warped product manifold is [Dobarro and Ünal, 2005]

$$\mathcal{M} = \mathcal{B} \times_{f^{p_1}} \mathcal{F}_1 \times_{f^{p_2}} \mathcal{F}_2 \times_{f^{p_3}} \mathcal{F}_3. \tag{3.17}$$

with metric tensor

$$g = -dt \otimes dt + \sum_i (f^{p_i} \circ \pi) g_{\mathcal{F}_i} \tag{3.18}$$

and projection onto the base space π and $g_{\mathcal{F}_i}$ the metric part corresponding to the fibre $\mathcal{F}_i$. The warping factor is given by the function f which is identical for all fibres and the exponents fulfil so-called generalised Kasner conditions (which are only valid for vacuum

[5]In contrast, the Friedmann solution expands isotropically. But this is due to the presence of matter, usually a cosmological constant or a dilaton degree of freedom.

solutions)

$$\sum_i \zeta_i p_i = 1, \quad \sum_i \zeta_i p_i^2 = 1. \tag{3.19}$$

First we see the generalisation of the Kasner-plane condition and the second is the Kasner-sphere, the generalisation occurs through the number ζ. Note, the definition also applies to space-times which are conformally isomorphic to Kasner space-times as long as the characterising behaviour is still preserved. We will see that the Kasner conditions and the form of the metric (3.17) simplifies if $f(t) = t^2$. This will in the following be called Kasner space-time. A Kasner metric is then given by

$$g = -dt \otimes dt + \sum_{i=1}^{3} t^{2p_i}(dx^i \otimes dx^i), \tag{3.20}$$

where the exponents p_i form the Kasner plane $\sum p_i = 1$ and the (non-generalised) Kasner sphere $\sum p_i^2 = 1$. However, through the Kasner conditions the three exponents become linearly dependent and could be expressed by one single parameter λ via

$$p_1(\lambda) = \frac{-\lambda}{1+\lambda+\lambda^2} \tag{3.21}$$

$$p_2(\lambda) = \frac{1+\lambda}{1+\lambda+\lambda^2} \tag{3.22}$$

$$p_3(\lambda) = \frac{\lambda(1+\lambda)}{1+\lambda+\lambda^2}. \tag{3.23}$$

Note, this does not hold for all λ, instead if $\lambda < 1$ then we have to do the identifications

$$p_1\left(\frac{1}{\lambda}\right) = p_1(\lambda) \tag{3.24}$$

$$p_2\left(\frac{1}{\lambda}\right) = p_3(\lambda) \tag{3.25}$$

$$p_3\left(\frac{1}{\lambda}\right) = p_2(\lambda). \tag{3.26}$$

In the Schwarzschild analysis we have already referred to the conjecture of Belinskii, Khalatnikov, and Lifshitz (BKL) about the singularity of a Kasner metric; here, we want to continue the discussion: BKL proposed that close to space-like singularities all space-times behave as Kasner space-times because time gradients dominate over spacial gradients [Ashtekar et al., 2011]. This behaviour seems to be rather general and we have already seen that the black hole singularity fits exactly in the picture of the BKL conjecture. Belinskii et.

al. observed complicated (chaotic) oscillations around the singularity [Belinskii et al., 1970] and that in the vicinity of a spacelike singularity the variation of observables in space becomes irrelevant compared to the change in time.

Kasner metrics are related to gravitational chaos [Damour et al., 2001] which is also called cosmological billiards. Consider a pool billiard game, we have to identify the size of the table and the reflection at the walls. Pockets are by now excluded from the description, this billiard game will focus on the bouncing behaviour. A group theory point of view relates the billiard description to the Weyl chamber of a Kac-Moody algebra and the reflections at the walls to the Weyl reflections. The size of the table corresponds to the size of the Weyl chamber while the walls could be different things: in the gravitational context they are connected to the spatial curvature but they could also come from off-diagonal components of the spatial metric (symmetry walls) or electric and magnetic walls from p-forms.

When we approach the singularity, it turns out that Kasner space-times are not stable under the influence of fluctuations, they evolve into other Kasner space-times with different exponents [Damour et al., 2001]. These fluctuations bring the metric into this oscillatory, chaotic behaviour, because one can show that at least one Kasner exponent must come with a negative exponent. Consequently, in this direction the fluctuation gets boosted and amplified, causing the change in the exponents. The rotation of the Kasner axes (spatial vectors) induces a motion which is as chaotic as the motion in a billiard game.

The Kasner space-time can be found not only by field theoretic motivation, it also follows from geometrical observations combined with the BKL conjecture. Let us recall the metric and its special properties which can be written as

$$g = -dt^2 + t^{2p_1} dx_1^2 + t^{2p_2} dx_2^2 + t^{2p_3} dx_3^2 \tag{3.27}$$

with the three Kasner exponents p_1, p_2, and p_3 fulfilling the two relations:

(i) Kasner plane: $\sum_i p_i = 1$,

(ii) Kasner sphere: $\sum_i p_i^2 = 1$.

Our idea is to show that Kasner solutions arise naturally when the manifold has a singular spatial hypersurface close to which time gradients dominate over spatial gradients (i.e. we assume the BKL conjecture to hold).

First of all we want to start with the defining equation for a general vacuum solution g to Einstein's equation

$$\mathfrak{Ric} \equiv 0. \tag{3.28}$$

In order to understand this equation from geometrical reasons we have to understand the interpretation of the Ricci tensor. Mathematically the Ricci tensor gives something like an average over the curvature of all planes involving specific vectors which are given by the indices of the components. The more concrete picture is the volume deviation; the Ricci tensor is a measure of the volume change along a geodesic of a curved space-time compared to the volume change along a flat space-time's geodesic given by

$$\frac{D^2}{d\tau^2}\delta V - \frac{D^2_{\text{flat}}}{d\tau^2}\delta V = -\delta V \mathfrak{Ric}(T,T) \tag{3.29}$$

with volume change δV and $\frac{D^2}{d\tau^2}$ Fermi-Walker derivation with respect to an affine parameter τ. The direction in which the volume moves is given by the vector T. Vacuum solutions do not change the volume compared to the flat case because the right hand side of (3.29) would be zero.

The goal is to find the form of the space-time which describes the curvature close to a spacelike singularity. From differential geometry the criterion for a complete manifold is the existence of a complete connection [Kobayashi and Nomizu, 1969]. Complete connection means that all geodesic curves $\gamma(\lambda)$ can always be parametrised by an affine parameter ranging $-\infty \leq \lambda \leq \infty$ on this manifold. This implies the singular structure is encoded in the connection Γ which is the Levi-Civita (or Riemannian) connection since we have pseudo-Riemannian (or Riemannian) manifolds. Hence, we could on the level of connections perform a gradient analysis where we neglect spatial gradients when compared to time gradients.

The Levi-Civita connection $\Gamma = g^{-1}dg$ is given in a local normal coordinate neighbourhood by

$$\Gamma^l_{ij} = \frac{1}{2} g^{lk} \left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right). \tag{3.30}$$

Note, we adopted the notation of Kobayashi and Nomizu's book in order to ensure consistency of the formulae. In particular, the latin indices here refer to the index set of the four dimensional manifold.

One important step in the calculation of BKL is the choice of the coordinate frame. We choose similarly to work in Gauß normal coordinates (which is in physics often called synchronous gauge). A normal coordinate system at a point x is a coordinate neighbourhood where the $\frac{\partial}{\partial x^i}$ form an orthonormal frame. Parallel displacement along geodesics allows to attach the normal coordinates to every point in a neighbourhood U of x.

By the choice of a normal coordinate neighbourhood we can re-express the metric in the following form

$$g = -dt^2 + h_{ij} dx^i dx^j, \tag{3.31}$$

with the metric of the three dimensional hypersurface h. Now, we see why this coordinates are called normal, the time direction has been rotated such that ∂_t is normal to the spatial part (no $dtdx^i$ terms). Therefore, we have a good notion of the vertical and horizontal subspace. We actually have split the space-time such that we have a purely spacelike submanifold $\mathcal{S}$ described by h which is a function of all coordinates and a notion of time by t.

Differential geometry has a huge apparatus for treating submanifolds. One special set of equations are the Gauß-Codazzi-Mainardi equations which are considered in order to rewrite the Riemann tensor $R(X, Y)Z$ on this manifold [Kobayashi and Nomizu, 1969]

$$R(W, Z, X, Y) = \mathcal{R}(W, Z, X, Y) + g(\alpha(X, Z), \alpha(Y, W)) - g(\alpha(Y, Z), \alpha(X, W)) \tag{3.32}$$

where we have the Riemann tensor of the hypersurface $\mathcal{R}(X, Y)Z$ lifted to the four dimensional space-time and the second fundamental form $\alpha(X, Y) = h(X, Y)\xi$, often called the extrinsic curvature, which is the lifted metric of the submanifold $\mathcal{S}$. In local coordinates:

$$R_{ijkl} = \mathcal{R}_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk} \tag{3.33}$$

with extrinsic curvature $K_{ij} = \frac{1}{2}\partial_t h_{ij}$. Concerning the spatial metric h the indices only run over the spatial components, i.e. h is a 3×3 matrix and the 0-component is excluded, in contrast to g.

We see in a normal coordinate neighbourhood we could split the manifold into a spatial submanifold and the subspace normal to it. For the Riemann tensor we can do the same and distinguish the parts contributing to the vertical and horizontal subspace.

We expect to get from the split into vertical and horizontal subspace the normal component of the Riemann tensor $R(X, Y)Z$

$$^{\perp}R(X, Y)Z = \left(\tilde{\nabla}_X \alpha\right)(Y, Z) - \left(\tilde{\nabla}_Y \alpha\right)(X, Z) \tag{3.34}$$

with $\tilde{\nabla}_X \alpha$ being the covariant derivative of α with respect to the connection in the tangent space $T(M) + T(M)^\perp$. The corresponding connections are ∇_X for $T(M)$ and D_X for $T(M)^\perp$. In general

$$\tilde{\nabla}_X \alpha(Y, Z) = D_X \left(\sum_i h^i(Y, Z)\xi_i \right) - \sum_i \{h^i(\nabla_X Y, Z) + h^i(Y, \nabla_X Z)\}\xi_i. \tag{3.35}$$

it is to say that the normal component is given exclusively by the second fundamental form.

We are almost at the point where we can evaluate equation (3.28). Of course we could contract the Riemann tensor but we want to advertise a more elegant way using the structure equation for a connection form ω and its curvature form Ω

$$d\omega(X, Y) = -\frac{1}{2}[\omega(X), \omega(Y)] + \Omega(X, Y). \tag{3.36}$$

On the bundle of orthonormal frames we can state by calling Ω now Ψ and ω by ψ the above equation as

$$\Psi_A^B = d\psi_A^B + \sum_k \psi_A^k \wedge \psi_k^B + \sum_r \psi_A^r \wedge \psi_r^B \tag{3.37}$$

The indices denoted by capital letters range over the whole four manifold where k, and r over the subspace normal to $\mathcal{S}$ which is in fact only the 0-component. We could use this structure equation in order to derive the equation of Gauß and Codazzi

$$\Psi_j^i = \Omega_j^i + \sum_r \psi_r^i \psi_i^r, \tag{3.38}$$

where Ω denotes the curvature form of the orthonormal bundle of the spatial metric to the orthonormal bundle of the full four dimensional metric. Note, for a tangent bundle of a (pseudo-)Riemannian manifold the curvature form corresponds to the Ricci tensor when we use the canonical form θ^i

$$\Omega_j^i = \frac{1}{2} \sum_{k,l} R^i_{kjl} \theta^k \wedge \theta^l. \tag{3.39}$$

From this analysis we see that in a normal coordinate neighbourhood we get two contributions to the four dimensional Ricci tensor, one coming from a lift of the three dimensional curvature form and the second from the squared connection form. Expressing these quantities by the second fundamental form and the extrinsic curvature we get for the components of the Ricci tensor the three relations:

$$(\mathfrak{Ric})^0_0 = -\frac{\partial}{\partial t}K^k_k - K^l_k K^k_l, \tag{3.40}$$

$$(\mathfrak{Ric})^0_i = \frac{\partial}{\partial x^k}K^k_i - \frac{\partial}{\partial x^i}K^k_k, \tag{3.41}$$

$$(\mathfrak{Ric})^j_i = -^{(3)}(\mathfrak{Ric})^j_i - \frac{1}{\sqrt{\det(h)}}\frac{\partial}{\partial t}\left(\sqrt{\det(h)}K^j_i\right). \tag{3.42}$$

All three equations have to fulfil (3.28). When we have a look at the 00-component we can identify the two different contributions, one comes from the lifted curvature form on the spatial submanifold and the second from the squared connection forms. Choosing a normal coordinate neighbourhood will be already generating a system of differential equations which have Kasner space-times as solutions. Until now we did not made any assumptions concerning the gradients.

Since we are interested in the behaviour close to a singular hypersurface, time has come to assume that spatial gradients are irrelevant when compared to time gradients. Next, we will perform the gradient expansion explicitly starting from the connections. In normal coordinates (i.e. $g_{00} = -1$, $g_{0i} = 0$ and $g_{ij} = h_{ij}$) after exploiting that $\partial_x h \ll \partial_t h$ we have only two types of non-vanishing Christoffel symbols

$$\Gamma^t_{ij} = \frac{1}{2}\frac{\partial h_{ij}}{\partial t} \tag{3.43}$$

$$\Gamma^j_{it} = \frac{1}{2}g^{jk}\frac{\partial h_{ik}}{\partial t}. \tag{3.44}$$

These Christoffel symbols allow to calculate the Ricci tensor with the general formula

$$(\mathfrak{Ric})_{ij} = \frac{\partial \Gamma^l_{ij}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^j} + \Gamma^m_{ij}\Gamma^l_{lm} - \Gamma^m_{il}\Gamma^l_{jm}. \tag{3.45}$$

Here, it becomes clear why an imposition of the condition $\partial_x h \ll \partial_t h$ is much less work when it is imposed on the level of the connection. By the way if the gradient expansion had been performed in the Ricci tensor we would have gotten the same result. With the Christoffel symbols we have already derived above we deduce the following three equations for the different types of components of the Ricci tensor

$$(\mathfrak{Ric})_{00} = -\frac{1}{2}\frac{\partial}{\partial t}\left(h^{ik}\frac{\partial h_{ki}}{\partial t}\right) - \frac{1}{4}h^{lk}\frac{\partial h_{ki}}{\partial t}h^{im}\frac{\partial h_{ml}}{\partial t}, \tag{3.46}$$

$$(\mathfrak{Ric})_{0i} = \frac{1}{2}\frac{\partial}{\partial x^l}\left(h^{lk}\frac{\partial h_{ki}}{\partial t}\right) - \frac{1}{2}\frac{\partial}{\partial x^i}\left(h^{jk}\frac{\partial h_{kj}}{\partial t}\right), \tag{3.47}$$

$$(\mathfrak{Ric})_{ij} = \frac{1}{2}\frac{\partial^2 h_{ij}}{\partial t^2} - \frac{1}{4}\frac{\partial h_{ki}}{\partial t}h^{km}\frac{\partial h_{mj}}{\partial t}. \tag{3.48}$$

The 00 and the 0i-component are easily comparable to the form derived above by the structure equation, the third will be a bit more involved. Nevertheless, what is immediately visible is that the 00- and the 0i- components are similar to the equations above.
Now comes the point where we need that every real symmetric matrix is diagonalisable. Let us, hence use this fact and consider a diagonal metric. It is easy to see that the 0i-components then trivially obey (3.28). The 00-component reduces drastically to

$$(\mathfrak{Ric})_{00} = -\frac{1}{2}\left[\frac{\partial}{\partial t}\left(h^{ii}\frac{\partial h_{ii}}{\partial t}\right) + \frac{1}{2}\left(h^{ii}\frac{\partial h_{ii}}{\partial t}\right)^2\right] = 0. \tag{3.49}$$

Since $(\mathfrak{Ric})_{0i}$ is trivially zero we can find a solution to (3.49) by taking a polynomial ansatz for h

$$h = t^{k_1}dx_1^2 + t^{k_2}dx_2^2 + t^{k_3}dx_3^2. \tag{3.50}$$

This is justified since no spatial derivatives occur and we get a system of coupled differential equation for the three metric components only consisting of time derivatives. Using (3.50) and plugging it into (3.49) we get from the part coming from the lifted Ricci tensor

$$\Omega \to \frac{k_1 + k_2 + k_3}{2t^2} \tag{3.51}$$

and similarly from the squared connection form

$$\omega \wedge \omega \to -\frac{k_1^2 + k_2^2 + k_3^2}{4t^2}. \tag{3.52}$$

When we now take into account (3.49) and additionally propose the following identification $k_i = 2p_i$ then we see immediately that we get the defining conditions for Kasner space-times which have to be fulfilled simultaneously

$$(\mathfrak{Ric})_{00} = -\frac{1}{2t^2}\left[p_1 + p_2 + p_3 - (p_1^2 + p_2^2 + p_3^2)\right] = 0. \tag{3.53}$$

From the lifted curvature form Ω we obtain the Kasner plane

$$\Omega : p_1 + p_2 + p_3 = 1 \tag{3.54}$$

and from the squared connection forms ω we get the Kasner sphere

$$\omega^2 : p_1^2 + p_2^2 + p_3^2 = 1. \tag{3.55}$$

Only if those relations are fulfilled together then this metric describes the behaviour close to a spacelike singularity properly. This behaviour we found also close to the singularity of the Schwarzschild black hole (cf. (3.13)) which supports the BKL conjecture.

Bianchi's classification scheme for example the Bianchi I space-time, which characterises three surfaces upon their symmetries [Bianchi, 1898] also covers Kasner space-time. In general it has been developed in order to classify three dimensional Lie algebras and distinguishes eleven classes. In cosmology this scheme is used for homogeneous four dimensional space-times with the trivial foliation.

Although not covered by the Bianchi classification the Schwarzschild space-time is related to the conjecture of BKL because asymptotically it resembles a Kasner type-D metric. This underlines the generality of the conjecture and shows their importance for physics.

4

The return of regularity

In the previous chapter we put a lot of effort in understanding the geometry of the background in terms of general relativity. Both space-times, Schwarzschild and Kasner, admit a spacelike singularity such that for $t = 0$ the spatial hypersurface becomes degenerate. We are almost ready to start with the quantum probing of these space-times. Probing with quantum mechanics cannot be afforded since there is no consistent relativistic one-particle description on a dynamical space-time. The reason is that the theory will run into inconsistencies unless one of the principles of quantum mechanics is violated [Feynman et al., 2005]. An intuitive reason is that a dynamical background breaches against the conservation of particle number [DeWitt, 1975].

Only quantum field theory in curved spaces provide an adequate description; instead of a one particle description we use quantum fields and work in a Fock space rather than in Hilbert space.

Chapter 3 showed that singularities occur at a specific time, therefore, as opposed to an asymptotic framework pertinent to a scattering description, the functional Schrödinger approach allows us to analyse unitarity violations occurring during a finite amount of time, and, in particular, during the time interval $(0, t_0]$. The other advantage is that we could set up our criterion in the style of quantum-mechanical completeness because the Schrödinger representation could be seen as a functional generalisation of quantum mechanics.

In this chapter we start with a very brief excursus into the techniques of functional calculus (only what is needed to understand the calculations performed in this thesis). Afterwards, we introduce the Schrödinger representation of quantum field theory on flat and on curved backgrounds and explain explicitly the construction of the states and their interpretation. As a first example and as a consistency check we show that Minkowski space-time is complete, then we state our criterion for curved space-times.

Both the Kasner and the Schwarzschild manifold are incomplete under application of the Hawking and Penrose criterion (cf. Theorem 9). For dynamical space-times the Horowitz and Marolf's analysis (self-adjointness of the Hamilton operator) is not applicable, because Stone's theorem is not valid for time-dependent Hamilton operators. Therefore, those space-times were filed as incomplete. Nevertheless we present in this thesis that these backgrounds are complete when probed with quantum fields.

4.1. The Schrödinger representation of quantum field theory

Quantum field theory is usually represented as an $\mathbb{S}$-matrix theory. In this picture, we look at initially prepared states which have to be free at past infinity [Reed and Simon, 1979]. Then we evolve the state, scatter, and look at the outgoing products at future infinity where they are free again. This Heisenberg picture is very powerful for calculating scattering amplitudes but due to the construction of the scattering operator, the $\mathbb{S}$-matrix, there is no possibility to do a time resolution for the scattering process.

A remedy is provided by the Schrödinger representation which allow for a time-resolution and which will be used throughout the thesis. The idea is that dynamical quantities are expressed in canonical variables at a fixed time and the quantisation is imposed by a commutation relation.

Similar to quantum mechanics, Heisenberg and Schrödinger formulation are equivalent but we have to emphasise that the Stone-von Neumann theorem will not work in this regard because the configuration space is infinitely dimensional and a unitary mapping between the two representations can not be assumed.

In the Schrödinger representation the completeness analysis appears to be very analogous to Horowitz and Marolf's analysis. We will find a criterion for the (functional) Hamilton operator to fulfil in order to have a complete set-up. Before we are turning our attention to physics we recap a few mathematical tools, such as the functional calculus. The preliminaries will be based on the introduction of the book [Hatfield, 1992]

4.1.1. Functional calculus

In this subsection explain the technical and mathematical preliminaries concerning calculations in function space. A function space[1] is an infinitely dimensional space where the points are functions of space-time points. In other words, each point is a mapping of space-time points into real or complex numbers. Those functions can be scalar functions, vectors, tensors in general, or spinors.

If a point in function space is mapped to a number, this is called a functional. In mathematical terms it is a mapping F which takes elements of a vector space V into a field e.g. $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, in short $\mathsf{F} : \mathsf{V} \to \mathbb{K}$. This can be illustrated by figure 4.1.

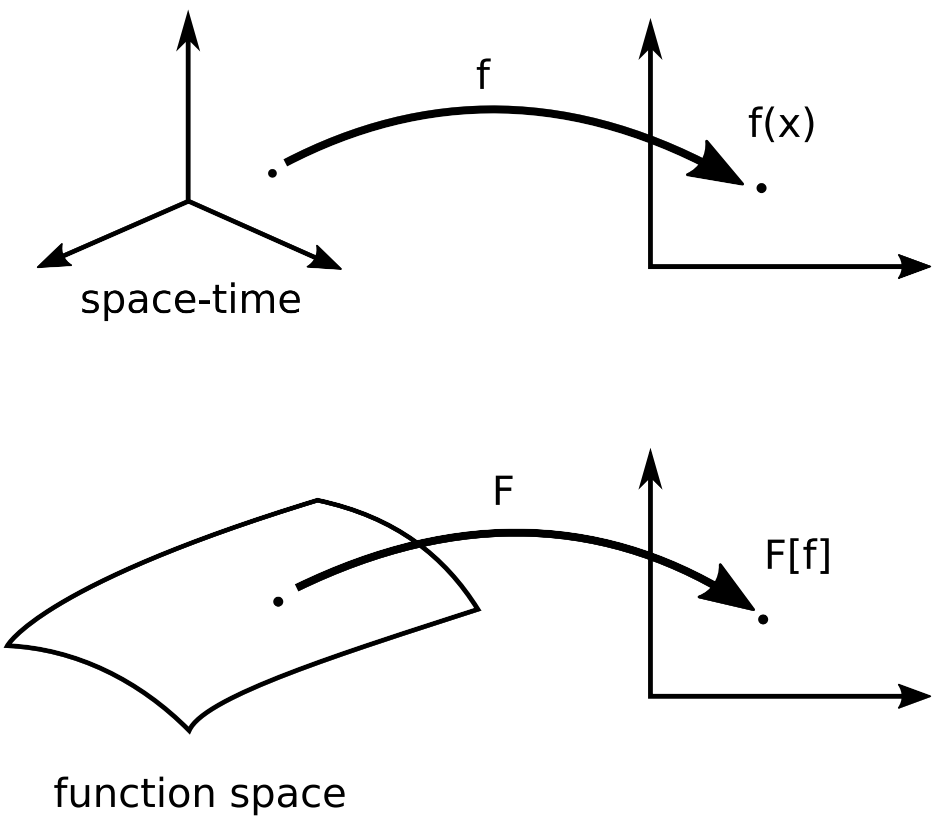

Figure 4.1.: Relation between functions $f(x)$ and functionals $F[f]$.

In figure 4.1 we see a comparison between a normal function and a functional. A function $f(x)$ maps a point in the space-time to a number in some field $\mathbb{K}$. In the figure, the field $\mathbb{K}$ is depicted by the two axis, the mapped point is called $f(x)$. This kind of map could have

[1]For most of these spaces existence is not clearly proven but recent developments have been made (cf. [Amour et al., 2015]).

been a diffeomorphism for example. A functional F turns a function f into a number in $\mathbb{K}$; we see that there are similarities between both objects.

We can formulate rules for functional integration and differentiation which are comparable to the calculus of functions. In the following we will evaluate the most important features of the functional calculus which we will need in order to understand and perform calculations in the Schrödinger representation. First of all, we define the functional derivative in analogy to the total derivative as a limit of a difference quotient. Consider a functional $F[a]$ with function $a(x)$. Moving in function space implies a change of the function which is done by a Dirac δ-distribution. The functional derivative is defined as follows:

$$\frac{\delta F[a]}{\delta a} = \lim_{\epsilon \to 0} \frac{F[a + \epsilon\delta] - F[a]}{\epsilon}. \tag{4.1}$$

One can of course define a functional derivative which is directional to another function. In our case it is directional to the δ-distribution. From this we directly get

$$\frac{\delta a(y)}{\delta a(x)} = \delta(x - y). \tag{4.2}$$

Similarly, a directional functional derivative along a function f gives $f(x - y)$ as result. With the above figure 4.1 we see why we have to use a function in the definition of the derivative: in functional space functions play the role of a local bookkeeping device (similar to x for functions). Consider the usual difference quotient of functions, here, we move a tiny instant further in the coordinate, shrink this to zero, and receive the first derivative.

A functional derivative is motivated from this picture, because we move a tiny instant further as well. Since we are in function space, we must do this shift by a function or a distribution.

If we take an action we could derive the equation of motion by applying the functional derivative. Hence, we can formulate functional differential equations. The solution techniques are less powerful - or less developed - than for partial differential equations but a few methods can be borrowed.

It is possible to define an integral of functionals over function space. In general the measure is not well defined, such as the Feynman measure while the Wiener measure is well-defined [Simon, 1979]. Note the latter is the Euclidean and finite dimensional version of the Feynman measure mostly used in probability theory when describing random walks (also called Wiener process). Nevertheless, the problems arising from the infinite dimensionality in the Feynman integral are cured by imposing a cut-off scale in the function space. Nevertheless, let us come up with a natural way to define an integration. One problem is that to present knowledge there are only a few integrable situations, one example is the Dirac measure,

$$\int \mathcal{D}a\delta[a-\xi] = 1. \tag{4.3}$$

This Dirac measure is very similar to the one in Lebesgue or Riemann integration, however, here the distribution is a functional. Integration over all paths is denoted by

$$\mathcal{D}a = \bigotimes_x da(x). \tag{4.4}$$

This gives us an idea how to write down the functional delta distribution which is consistent with the measure

$$\delta[a-\xi] = \prod_x \delta(a(x) - \xi(x)). \tag{4.5}$$

A functional integral is an infinite product of independent integrals. It can also be motivated by starting with discrete objects and going to a continuum limit [Kleinert, 2009].

The second integrable function is the Gauß wave-functional which is of huge importance for physics

$$\int \mathcal{D}a e^{\int dx a^2(x)}. \tag{4.6}$$

Note, that such a functional is also called a Wiener measure [Simon, 1979] if the function space is finite and Euclidean. Similar to the δ-functional the Gaussian can be factorised into separate integrations over usual Gaussian integrals

$$\int \mathcal{D}a e^{\int dx a^2(x)} = \prod_x \int da(x) e^{-a^2(x)} = \prod_x \sqrt{\pi}. \tag{4.7}$$

If there were a function $f(x)$ in the exponent, i.e. $\exp(\int dx f(x) a^2(x))$ the result would be a quotient of π and the functional determinant Det of $f(x)$ as long as it can be interpreted as infinite-dimensional diagonal matrix $f(x) = f(y)\delta(x-y)$.

$$\int \mathcal{D}a \exp\left(-\int dx dy a(x) k(x,y) a(y)\right) = \lim_{d\to\infty} \frac{\sqrt{\pi}^d}{\sqrt{\mathrm{Det}(k)}}. \tag{4.8}$$

In order to be mathematically as exact as possible we put the limit over the infinite dimensions of the Fock space in the above equation, and formally give the result coming from the finite dimensional case.

One side remark on fermionic variables, the Fock space structure is much less complicated than in the bosonic case because there we have to introduce anti-commuting numbers, i.e. Graßmann variables which square to zero, and our fields are described by spinors. Anyone who is interested in this topic can have a look into [Hatfield, 1992]. This thesis will restrict only to bosonic variables which are commuting and lying in the symmetric Fock space.

Since we have all relevant tools at hand, we start with the explanation of the Schrödinger representation of quantum field theory.

4.1.2. Flat space-time formulation

Flat space-time serves usually as the background where our intuition works best, we will exploit this here as well and choose Minkowski space-time to be our starting point where we introduce all relevant features of the Schrödinger representation without having the subtleties from curved space-time. Minkowski space is blessed with Poincaré invariance of the vacuum state, therefore the vacuum is unique; this is also one Gårding-Wightman axiom [Simon, 2015d].

We present a basic introduction which is based on [Hatfield, 1992]. The main description of the Schrödinger representation is very similar to the Hamilton formulation in classical physics: Dynamical quantities are expressed in terms of canonical variables at fixed time. In other words, all tensors, especially the degrees of freedom, are pulled-back to the spatial hypersurface. The elements of the Fock space are the field operators, or more explicitly, the physical degrees of freedom which are expressed as the classical eigenvalues of the corresponding operator.

For a better understanding we will discuss the basic issues for scalar field theory in Schrödinger representation and discuss afterwards the implications for fermions and for spin-1 Maxwell fields.

Scalar fields

The theory of a free real scalar field $\Phi(x)$ with mass m is described by the following action

$$S[\Phi] = -\frac{1}{2}\int d^4x \left(\partial_\mu\Phi\partial^\mu\Phi + m^2\Phi^2\right). \tag{4.9}$$

In the Schrödinger representation of quantum field theory we are working with the Hamilton operator which is given through a Legendre transformation of the Lagrangian

$$H[\pi, \Phi] = \mathcal{L}^* = \pi\partial_0\Phi - \mathcal{L} \tag{4.10}$$

with $\mathcal{L}^*$ being the convex conjugate to $\mathcal{L}$. The momentum π is the conjugate variable to the degree of freedom Φ. Note, both π and Φ are now time-independent. In principle we could work with the one-particle description of quantum mechanics, however in order to introduce the formalism we will stick to this cumbersomely appearing method. In general space-times we would start with an ADM split [Arnowitt et al., 1960]; or similarly we could use Gauß-Codazzi (e.g. [Kobayashi and Nomizu, 1969]); another possibility to combine general relativity and quantum field theory would be to express the Hamilton operator like in the Wheeler-deWitt [DeWitt, 1967] formalism[2].

Our task is to write down a functional Schrödinger equation for a scalar field on Minkowski space-time. Therefore, we need some ingredients: the functional state, the Hamilton operator, and the quantisation prescription. The momentum conjugate to Φ is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\Phi)} = \partial_0\Phi(x). \tag{4.11}$$

Note, π is likewise real. With the conjugate variable to Φ we construct the Hamilton operator in terms of the field and its conjugate momentum

$$H[\pi, \Phi] = \frac{1}{2}\int d^3x : \pi^2 : + : |\boldsymbol{\partial}\Phi|^2 : + m^2 : \Phi^2 : \tag{4.12}$$

Here, $\boldsymbol{\partial}$ denotes the purely spatial derivatives and the double dot the normal ordering. The operators π and Φ fulfil a canonical equal-time commutation relation

$$[\Phi(x), \pi(y)]|_t = i\delta(\boldsymbol{x} - \boldsymbol{y}). \tag{4.13}$$

Commutation of Φ and π with itself are zero. Let us now have a deeper look at the underlying Fock space. The construction of the Fock space is very important because it can make calculations simple. We choose to span the Fock space by the Φ fields such that $\Phi(x)$ is diagonal. In other words we work in the basis of eigenstates of $\Phi(x)$ which we call

[2] However, we have to be careful since for example in the de Sitter space we must take care of the Higuchi bound [Higuchi, 1987] when assuming a massive field.

$|\phi\rangle$ such that

$$\Phi(x)|\phi\rangle = \phi(x)|\phi\rangle \tag{4.14}$$

and the eigenvalues $\phi(x)$ represent the classical fields. In this basis $\Phi(x)$ acts as multiplicative operator and with (4.13) the conjugate momentum as functional derivative

$$\pi(x) = -i\frac{\delta}{\delta\phi(x)} \tag{4.15}$$

on $\Psi[\phi] = \langle\phi|\Psi\rangle$ due the definition of the functional derivative (4.2). The differential expression of $\pi(x)$ turns the Hamilton operator into the form of a functionally generalised Schrödinger operator

$$H\left[\frac{\delta}{\delta\phi(x)},\Phi\right] = \frac{1}{2}\int d^3x\left(-\frac{\delta^2}{\delta\phi(x)^2} + |\partial\Phi|^2 + m^2\Phi^2\right). \tag{4.16}$$

Note, that we have two parts of H, one containing derivative operators and the other one multiplication operators. They do not mix unless the metric has non-diagonal components. We want to mention that all Hamilton operators are normal ordered. With (4.16) we can formulate the Schrödinger equation

$$i\frac{\partial\Psi}{\partial t}[\phi] = H\left[\frac{\delta}{\delta\phi(x)},\Phi\right]\Psi[\phi] = \frac{1}{2}\int d^3x\left(-\frac{\delta^2\Psi}{\delta\phi(x)^2}[\phi] + |\partial\Phi|^2\Psi[\phi] + m^2\Phi^2\Psi[\phi]\right). \tag{4.17}$$

Since in our example the Hamilton operator as well as the wave-functional have no time dependence, (4.17) reduces to an eigenvalue problem, the time-independent Schrödinger equation

$$H[\Phi]\Psi[\phi] = E\Psi[\phi] \tag{4.18}$$

with energy eigenvalue E. This ansatz implies the applicability of a generalised version Stone's theorem, which subsequently implies that a unitary time evolution operator can be factorised out of the wave functional

$$\Psi[\phi](t) = U(t,t_0)\Psi[\phi](t_0). \tag{4.19}$$

Note, we could always define a time-evolution operator, which is given by formally integrating (4.17)

$$\Psi[\phi](t) = \exp\left(-i\int_{t_0}^{t} d\tau H[\Phi](t)\right)\Psi[\phi](t_0), \tag{4.20}$$

however, time-independent and self-adjoint Hamilton operators allow for a unitary time-evolution in (4.19). We solve (4.18) for the ground state wave functional. Further, we assume, the ground state functional has no nodes and is positive [Hatfield, 1992]. Our ansatz

$$\Psi_0[\phi] = N_0 \exp\left(-\iint d^3x d^3y \phi(x) K(x,y) \phi(y)\right) \tag{4.21}$$

is a Gaussian wave packet with integral kernel $K(x,y)$. Let us introduce a more compact notation for the functional which will be used in the later parts of the thesis: $\mathcal{K}_2[\phi,\phi] = \iint d^3x d^3y \phi(x) K(x,y) \phi(y)$. We add here that in time-dependent cases, $\mathcal{K}_2[\phi,\phi](t)$ is a function of time, so is $K(t,x,y)$ as well as the measure function of the integral. This ansatz is legitimate because power counting in the action suggests that the exponent should be at least quadratic in the fields $\phi(x)$. Moreover, the two functional derivatives applied on $\Psi_0[\phi]$ yield one field independent term which is matched with the right hand side of (4.18): the constant E is identified with E_0 the ground state energy eigenvalue.

The Schrödinger formulation of quantum field theory allows in principle for a third quantisation because we can find a quantisation relation for the wave-functional itself. Whether this turns out to be useful or not has to be tested. A first attempt has been made by Giddings and Strominger [Giddings and Strominger, 1989].

Plugging (4.21) into the Schrödinger equation (4.18) yields two relations for the kernel $K(x,y)$ (a field independent and a field dependent equation)

$$\int d^3x K(x,x) = E_0, \tag{4.22}$$

$$\iiint d^3x d^3y d^3z \phi(x) K(x,z) K(z,y) \phi(y) = \frac{1}{4}\int d^3x \phi(x)(\Delta - m^2)\phi(x). \tag{4.23}$$

The second equation can be reformulated into another condition

$$\int d^3x K(x,z) K(z,y) = \frac{1}{4}\int d^3x (\Delta - m^2)\delta(y-z). \tag{4.24}$$

Both sides only depend on the spatial difference $y - z$. The whole kernel is translationally invariant with consequence that the left hand side is a convolution of two kernels. With the convolution formula (A.4) applied after Fourier transformation (4.24) reduces to a simple algebraic equation

$$\widehat{K^2(k)} = -\frac{1}{4}(k^2 + m^2). \tag{4.25}$$

The Fourier transform of the Laplacian gives the dispersion relation for the field $\Phi(x)$

which is used to construct the explicit form of the kernel:

$$K(x,y) = \frac{1}{2}\int d^3k\sqrt{k^2+m^2}e^{i2\pi k(x-y)}. \tag{4.26}$$

It seems from the above equation that the kernel and the propagator are closely related: each obey Huygens's principle. The ground state energy can easily determined once knowing $K(x,x)$

$$E_0 = \frac{1}{2}\int d^3x\int d^3k\sqrt{k^2+m^2}e^{i2\pi k(x-x)} = \frac{1}{2}\int d^3k\sqrt{k^2+m^2}\delta^3(0). \tag{4.27}$$

We get a divergence due to the sum over infinite zero point energies of all oscillators which can be renormalised and should do no harm to the theory [Hatfield, 1992], since it agrees with the result in the operator representation. The states are normalised via the Fock space integral

$$1 = \langle 0|0\rangle = \int \mathcal{D}\phi\langle 0|\phi\rangle\langle\phi|0\rangle = \int \mathcal{D}\phi\Psi_0^*[\phi]\Psi_0[\phi]. \tag{4.28}$$

The probabilistic interpretation of quantum field theory allows to set the norm of the wave-functional to one. Therefore, we end up with

$$|N_0|^2 = \left[\int \mathcal{D}\phi \exp\left(-2\iint d^3x d^3y \phi(x)K(x,y)\phi(y)\right)\right]^{-1} = \frac{\sqrt{\mathrm{Det}\left(\sqrt{\Delta+m^2}\right)}}{\sqrt{\pi}^{\infty}}. \tag{4.29}$$

Note, that π is here the irrational number and not the conjugate momentum. We see (4.29) is the infinite product of ground state harmonic oscillators. Therefore, we get infinity in the normalisation which we cancel by de L'Hôpitale's rule[3]. The wave-functional can be written as

$$\Psi_0[\hat{\phi}] = \prod_k \left(\frac{\sqrt{-k^2+m^2}}{\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{1}{2}\frac{1}{(2\pi)^3}\sqrt{-k^2+m^2}\hat{\phi}^2(k)\right) \tag{4.30}$$

such that we can see its structure as infinite copies of one-dimensional harmonic oscillators. We used that the harmonic functions of the Laplace operator are exponential functions.

Now, we saw how to calculate the ground state in the Schrödinger picture which is a

[3]This rule was actually invented by Johann Bernoulli. He had sent de l'Hôpital his verbatim copies which he included in his textbook about calculus. The well known de l'Hôpital rule should actually be named Bernoulli's rule.

wave-functional describing field configurations of ϕ.

Excited states

When we think of quantum mechnanics we know that there are more states than just the ground state; we can have infinitely many excitations of Ψ_0. How do they look in the Schrödinger picture? The answer is given through the definition of creation and annihilation operators. Their functional versions can be interpreted as including a field into the Fock space, or destroying a field in the Fock space, with some momentum, say k_1 which obeys the dispersion relation and consequently the on-shell condition. The physical reason is that they are produced by the space-time dynamics and, hence, they must be harmonic functions of the d'Alembert operator of the background.

Construction of arbitrary excitations can be realised when we define the creation and annihilation operators. First, we start with

$$a^-[f]\Psi^{(0)}[\phi] = 0 \tag{4.31}$$

that is the functional annihilation operator $a^-[f]$ containing the on-shell fields $f(t, x)$. How does the creation operator $(a^-[f])^*$ acts? Similar to a quantum mechanical creation operator! On-shell fields $f(t, x)$ are included into the field configuration space by $(a^-[f])^*$ acting on the state Ψ. Their representation depends on the specific system. For a free scalar field theory in Minkowski space they are given by

$$a^-[f] = \int d^3x e^{-ikx} \left(\sqrt{-k^2 + m^2} f^*(x) + \frac{\delta}{\delta\phi(x)} \right), \tag{4.32}$$

$$(a^-[f])^* = \int d^3x e^{ikx} \left(\sqrt{-k^2 + m^2} f(x) - \frac{\delta}{\delta\phi(x)} \right). \tag{4.33}$$

Multiple application of the creation operator produces higher and higher excitations with respect to the ground state $\Psi^{(0)}[\phi]$. The wave part $\mathcal{G}^{(0)}[\phi]$ will not change, but the normalisation $N^{(0)}$ will. Note that in this picture excitations correspond to a functional „renormalisation“[4] with respect to the on-shell fields, i.e the normalisation will be f dependent $N^{(0)}[f]$. Excitations will be discussed in more detail in the later sections where we specify the background geometry; we explicitly construct a representation of the operators and apply them to the ground state.

[4]The quotation marks shall symbolise that we do not refer to the physical process of renormalisation.

Interacting theory

In the last two sections we have only considered free field theory. The most physically interesting systems contain interaction terms. The question is what changes when we include a polynomial self-interaction term in the action functional. In the functional Schrödinger representation we do not use a reduction formalism nor a Green's function. Interacting theories are best investigated perturbatively (the coupling is weak by construction) since unfortunately, the full equation is not soluble by current knowledge. Like in quantum mechanics we use Rayleigh-Schrödinger perturbation theory but a functionally generalised version which is presented in [Hatfield, 1992].

Consider as example a four-vertex interacting theory of a scalar field $\Phi(x)$ (we will evaluate the example close to the treatment of Hatfield)

$$S = -\frac{1}{2}\int d^4x \left(\partial_\mu \Phi \partial^\mu \Phi + m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4(x)\right). \tag{4.34}$$

The coupling strength λ is chosen such that perturbation theory is applicable, that is $\lambda < 1$. Let us be extreme and extreme smart and prepare the system such that $\lambda \ll 1$. The mass term gives the renormalised mass m and therefore the interaction Hamiltonian is given by $H_{int} = \int d^3x \frac{\lambda}{4!} : \Phi^4(x) :$. The equal time commutation relation as well as the field momentum are not affected and are the same as in the free field theory.

Let $\Psi_N[\phi]$ be the eigenfunctional of the Hamilton operator with energy eigenvalue E_N, obeying the time-independent Schrödinger equation

$$H[\phi]\Psi_N[\phi] = E_N \Psi_N[\phi]. \tag{4.35}$$

The energy eigenvalues E_N correspond to the N^{th} iteration of the perturbation theory, that is, the N^{th} correction to the energy eigenvalue.

We can decompose the Hamilton operator in its free and its interaction part with parameter χ which can acquire values from zero to one, such that we can control the strength of the interaction term,

$$H[\phi] = H_0[\phi] + \chi H_{int}[\phi]. \tag{4.36}$$

For the Rayleigh-Schrödinger perturbation theory we assume the interaction strength to be much smaller than one which allows us to expand, analogous to quantum mechanics, both the wave functional and the energy eigenvalue as a power series in χ around the N^{th}

free state and the free N^{th} energy eigenvalue,

$$\Psi_N[\phi] = \sum_{k=0}^{\infty} \chi^k \Psi_N^{(k)}[\phi] \tag{4.37}$$

$$E_N = \sum_{k=0}^{\infty} \chi^k E_N^{(k)} \tag{4.38}$$

and plug this into the Schrödinger equation (4.35). Equating both sides of this eigenequation order by order in χ, and taking the inner product with $\Psi_N^{(0)}[\phi]$, we obtain,

$$\Psi_N^{(1)}[\phi] = \sum_{M \neq N} \frac{\langle \Psi_M^{(0)}[\phi] | H_{int} | \Psi_N^{(0)}[\phi] \rangle}{E_N^{(0)} - E_M^{(0)}} \Psi_M^{(0)}[\phi]. \tag{4.39}$$

Similarly, we get the energy eigenvalues. Here, we present the first two corrections to the N^{th} free state energy

$$E_N^{(1)} = \langle \Psi_N^{(0)}[\phi] | H_{int} | \Psi_N^{(0)}[\phi] \rangle, \qquad E_N^{(2)} = \sum_{M \neq N} \frac{\langle \Psi_M^{(0)}[\phi] | H_{int} | \Psi_N^{(0)}[\phi] \rangle}{E_N^{(0)} - E_M^{(0)}}. \tag{4.40}$$

To get the first order of perturbation with respect to the previously derived ground state energy means we have to derive $E_0^{(1)}$. The interaction Hamilton operator is a polynomial in the field coordinates $\phi(\vec{x})$. We conclude from this that all expectation values will be functional integrals that are moments of the Gauß function $\left(\Psi_0^{(0)}[\phi]\right)^* \Psi_0^{(0)}[\phi]$.[5]

Spinor fields

The Schrödinger representation applies to other spin particles as well. Nevertheless, we will see that we have to cope with some problems which we will see in the following subsections. For fermionic fields we have the Dirac Hamiltonian

$$H = \int d^3x \Psi^\dagger(x)(-i\gamma^\mu \nabla_\mu + m)\Psi(x). \tag{4.41}$$

with γ matrices from the Clifford algebra. Note, the explicit form of the spinors depends on their representation. Instead of commutation relations the fields Ψ and $\Psi^\dagger$ obey equal-time

[5] The calculation can be simplified by construction of a generating functional. This is achieved with addition of a source term. The interested reader is referred to [Hatfield, 1992].

anti-commutation relations

$$\{\Psi_a(x), \Psi_b^\dagger(y)\} = \delta_{ab}\delta^{(3)}(x-y). \tag{4.42}$$

We call the spinor indices a and b. In the same fashion as for the scalar field we choose the eigenbasis similar and identify $\Psi^\dagger$ with the functional derivative $\frac{\delta}{\delta\psi(x)}$. The $\psi(x)$ are spinors of anti-commuting variables of Graßmann functions and it follows $\psi^2(x) = 0$. The phase space of the spinor theory is pretty much simpler than the bosonic phase space, due to the Grassmann property.
We see that in general the construction goes very similar no matter what field content we impose. In the end we receive a wave-functional obeying the Schrödinger equation. This makes it possible to use the formulation for all types of spin. For a detailed treatment of fermionic fields we refer to [Jackiw, 1990].

Photon field

In Maxwell theory, the photon field $A_\mu(x)$ propagates two degrees of freedom although the tensor has four components. Therefore, we usually choose a gauge, which reduces the theory to the physically relevant degrees of freedom. This is crucial also in the Schrödinger because we need to perform the quantisation prescription in the physical degrees of freedom. For electrodynamics we have three convenient gauges: Coulomb ($\partial^i A_i = 0$), Lorentz ($\partial^\mu A_\mu = 0$), and temporal gauge ($A^0 = 0$). After we fix the gauge and extract the physical degrees of freedom, we get a Hamilton operator of the form

$$H = \frac{1}{2}\int d^3x(E^2 + B^2) \tag{4.43}$$

where the physical degrees of freedom are given by E the electric and B the magnetic field. Without going into details we just state here, that the commutation relations result in an identification of the electric field with the functional derivative [Hatfield, 1992]. In other words, one degree of freedom will correspond to a functional derivative. The spin causes some complications both on a conceptional and on the technical level. The reason is that the wave-functional which satisfies the corresponding Schrödinger equation must also satisfy a functional version of Gauß's law ($\nabla E = 0$) which is here nothing but an additional constraint

$$\nabla \frac{\delta}{\delta a(x)}\Psi[a] = 0 \tag{4.44}$$

with three-vector degree of freedom $a(x)$. The cumbersome fact is to keep in mind the gauge freedom and its implications. For those who intend to use the Schrödinger formulation for spin-1 particles can find an introduction in [Hatfield, 1992]. We close this short remarks on other spin fields and come to the purpose of this thesis, the quantum field probing of singular space-times.

Note, it has been shown [Stewart and Hájíček, 1973] that spin can not prevent from hitting the singularity. Therefore, a scalar field quantum probing seems enough for our purpose.

4.1.3. Curved space-time formulation

Quantum field theory in curved space-times has several subtleties which are not present in Minkowski space. For example in the Heisenberg picture it is complicated to formulate a scattering operator. The states at infinite past and infinite future do not necessarily lie in the same Hilbert space, $\mathcal{H}_{\text{in}} \neq \mathcal{H}_{\text{out}}$, i.e. a global vacuum can not be uniquely defined for in- and out states. This makes it cumbersome to calculate a scattering process because one is obliged to perform a Bogolubov transformation [Birrell and Davies, 1984] defined in [Reed and Simon, 1975] because Poincaré invariance will not hold in curved spaces. The Heisenberg formulation can be visualised like in figure 4.2: We start with a state in $\mathcal{H}_{\text{in}}$ which is free and obeys an on-shell relation. $(\Omega^-)^*$ evolve this state in time to the point where the scattering occurs, say in $\mathcal{H}_0$, now the state is not free. From the scattering point (the explosion in figure 4.2) the state evolves further to the infinite future which is described by Ω^+ where we end with a state in $\mathcal{H}_{\text{out}}$. The combination of both Ω is called the Jauch scattering matrix, or S-matrix, $S = \Omega^+(\Omega^-)^*$. If both Hilbert spaces are the same one speaks about asymptotic completeness [Reed and Simon, 1979], to emphasise again, curved space-times are in general not asymptotically complete.

The Schrödinger representation circumvents this problem because it does not use the scattering operator S. In this description the fields in the functional are not specified further, they could be on-shell fields, but they do not need to. We could see them more or less as local bookkeeping devices. Those can be all types of fields obeying the assumed spin statistics. A scalar field $\Phi(x)$ in the wave functional describes field configurations of scalar fields in the symmetric Fock space; their evolution is described by the functional Schrödinger equation which for static space-times reduces to the time-independent Schrödinger equation like we showed for Minkowski space-time.

However, we are interested to probe how a space-time singularity affects quantum field

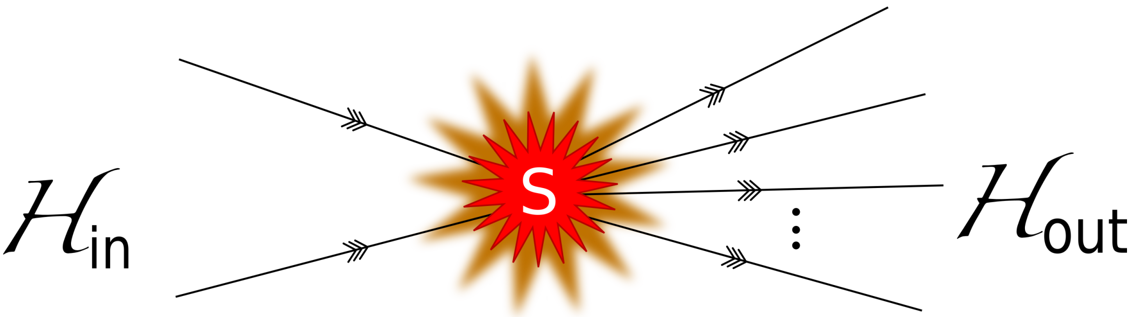

Figure 4.2.: Schematical illustration of the scattering operator from two to n particles.

theory. In other words, only geometrically singular space-times are relevant for our analysis. The Minkowski background is of course interesting since our concepts and formalism should be consistent on flat space-time as well. Static space-times allow for the application of the Horowitz-Marolf criterion, however, we could solve the time-independent functional Schrödinger equation as well, but this would turn out to be an overkill.

Our task is to investigate the time-evolution of the wave-functional $\Psi[\phi](t)$ describing field configurations of scalar fields on a dynamical curved background. Because of the explicit time-dependence of $\Psi[\phi](t)$, the Schrödinger representation is favoured because this formalism allows for an explicit time-resolution.

Note, that throughout the analysis we will ignore backreaction of the probing fields with the background geometry. The reason is that we intend to probe the considered space-time with respect to quantum completeness (we will define this term later on). If backreaction became important the previously given background geometry would be deformed and the quantum probing of the initial set-up would become obsolete. To this extent, we were not more restrictive than Hawking and Penrose in their formulation of the singularity theorems.

Essentially, we follow the same steps as in flat space-time, starting with the construction of the Hamilton operator, we define the explicit form of the wave functional and its normalisation. However, it is at present knowledge not designed for generic backgrounds. The space-time has to fulfil mild conditions like being connected and differentiable but these conditions are not exclusive for the Schrödinger representation. We assume additionally that the space-time is globally hyperbolic, which is, at least, not too restrictive. Due to Geroch's theorem a globally hyperbolic space-time foliates into $\mathbb{R} \times \Sigma$. The spatial hypersurfaces are denoted by Σ and are foliated along the time direction with $t \in \mathbb{R}$. Usually, one foliates the space-time along the time-like Killing vector field, if possible. This is called the (1+3)-split or ADM-split[6].

We start foliating the space-time $(\mathcal{M}, g)$ by Cauchy hypersurfaces Σ_t with normal n defined through $g(n, n) = -1$, the vector x^μ gives the coordinates and the time will be chosen along the Killing vector. Let $T \in V(\mathcal{M})$ a vector in the vectorspace with $\nabla_T t = 1$, that is, $T = (\partial_t X)(t, x)$. A basic statement in differential geometry is that the manifold can be decomposed into a horizontal and an vertical space [Kobayashi and Nomizu, 1963] with the projectors of these subspaces given by

$$P^{(\perp)} = n \otimes n, \quad P^{(\|)} = g + g(V, n)n, \tag{4.45}$$

for $V \in V(\mathcal{M})$. We introduce the lapse function N and the shift vector N_a by

$$N = g(T, n), \qquad N_a = P^{(\|)}(T, \cdot). \tag{4.46}$$

Hence, the vector T can be written as $T = N_\perp n + N_\|$ and the metric tensor in a similar way. Consider first the differentials dX^μ which are

$$dX^\mu = \partial_t X^\mu + \partial_a X^\mu = (N_\perp n^\mu + N_\|^a \partial_a X^\mu)dt + \partial_a X^\mu dx^a. \tag{4.47}$$

Now we turn our attention to the metric $g_{\mu\nu} dX^\mu \otimes dX^\nu$, with the results from above we find the three kinds of metric components

$$g(\partial_t, \partial_t) = -N_\perp^2 + g_{ab} N_\|^a N_\|^b, \tag{4.48}$$

$$g(\partial_t, \partial_a) = g_{ab} N_\|^a, \tag{4.49}$$

$$g(\partial_a, \partial_b) = g_{ab}, \tag{4.50}$$

[6]A covariant (2+2)-split has been performed by [d'Inverno and Smallwood, 1980].

with latin indices ranging over spatial coordinates. For the inverse metric we find similar expressions

$$
\begin{aligned}
g^{-1}(t,t) &= -\frac{1}{N_\perp^2} + g_{ab} N_\parallel^a N_\parallel^b, && (4.51)\\
g^{-1}(t,x^a) &= \frac{N_\parallel^a}{N_\perp^2}, && (4.52)\\
g^{-1}(x^a,x^b) &= g^{ab} + \frac{N_\parallel^a N_\parallel^b}{N_\perp^2}, && (4.53)
\end{aligned}
$$

Going through all this geometrical things helps to formulate the Hamilton operator. Let us consider a free massive scalar field $\Phi(x)$ as quantum probe, the Hamilton operator for a space-time g is then given by [Hofmann and Schneider, 2015]

$$H[\pi,\Phi] = \int_{\Sigma_t} d\mu(x)(N_\perp \mathcal{H}^\perp + N_i^\parallel \mathcal{H}_\parallel^i). \tag{4.54}$$

Here, all tensors are pulled back to the spatial hypersurface Σ_t. We have defined the measure $d\mu(x) = d^3x\mu(x)$ with measure function $\mu(x) = \sqrt{\det(g_\Sigma)}$. The tensor g_Σ is the induced metric on the hypersurface Σ_t. Additionally, we identified the lapse function $N_\perp = \sqrt{-g_{00}}$ and the shift vector $N_i^\parallel = g_{0i}$. We should mention that we imposed a normal ordering to the Hamilton operator. The parallel Hamiltonian density is given by

$$\mathcal{H}_\parallel^i = \frac{\pi(x)\partial^i\Phi(x)}{\sqrt{\det(g_\Sigma)}}. \tag{4.55}$$

A quantity constructed from a Legendre transformation possesses a term which mixes the field $\Phi(x)$ and its conjugate variable, so does the Hamilton operator (in general)

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\Phi(x))} = \frac{\sqrt{\det(g_\Sigma)}}{\sqrt{-g^{00}}}(\partial_0\Phi(x) - g_{0i}\partial^i\Phi(x)). \tag{4.56}$$

We see, we will get various contributions from the metric in this set-up. If the metric tensor depends explicitly on time this dependence will yield an explicit time-dependence of the Hamilton operator as well as for the wave-functional which we will see later. For a diagonal space-time the only contribution will be

$$\mathcal{H}^\perp = \frac{1}{2}\left[\frac{1}{\det(g_\Sigma)}\pi^2(x) + g_\Sigma^{ij}\partial_i\Phi(x)\partial_j\Phi(x) + (m^2 + \zeta\mathcal{R})\Phi^2(x)\right]. \tag{4.57}$$

The Ricci scalar curvature defined as contraction of the Ricci tensor is denoted by $\mathcal{R}$ and is coupled through ζ. In four dimensions, for example $\zeta = 0$ is known as minimal coupling and $\zeta = \frac{1}{6}$ as conformal coupling. For all vacuum solutions of Einstein's equation this contribution vanishes because of the vanishing Ricci tensor.

Similar to the Minkonwski case, we could construct the basis of the Fock space such that it is spanned by the quantum fields $\Phi(x)$. Let us just repeat, we choose the basis such that the field operator fulfils the eigenvalue equation $\Phi(x)|\Phi\rangle = \phi(x)|\Phi\rangle$, with eigenvalue $\phi(x)$ being the classical fields. The ϕ-representation of an arbitrary state $|\Psi\rangle$ is a non-linear wave functional $\Psi[\phi](t)$. In curved space-time we can formulate the canonical quantisation prescriptions for the field and its conjugate momentum

$$[\pi(t,x), \Phi(t,y)]|\phi\rangle = i\delta^{(3)}(x-y)|\phi\rangle. \tag{4.58}$$

We evaluated the operators π and Φ in the eigenbasis $|\phi\rangle$. This implies the identification of $\pi(x)$ with the functional derivative with respect to the field $\phi(x)$

$$\pi(x) = -i\frac{\delta}{\delta\phi(x)}. \tag{4.59}$$

Now, we are able to deal with the conjugate momentum, because we know how it operates on the state functional $\Psi[\phi](t)$. With all required ingredients we are able to give the Schrödinger equation for time-dependent states

$$i\partial_t\Psi[\phi](t) = H[\phi](t)\Psi[\phi](t), \tag{4.60}$$

in its explicit form which with (4.57) and (4.59) can be written as

$$i\partial_t\Psi[\phi](t) = \frac{1}{2}\sqrt{-\det(g)}\left[\frac{1}{\det(g_\Sigma)}\frac{\delta^2\Psi}{\delta\phi^2}[\phi](t) + \phi(x)(\Delta - m^2 - \zeta\mathcal{R})\phi(x)\Psi[\phi](t)\right]. \tag{4.61}$$

Now we need to have a look at the wave-functional Ψ; power counting arguments motivate a Gaussian ansatz for the ground state wave-functional $\Psi^{(0)}[\phi](t)$. We cover the higher excitations in a later chapter where we explicitly construct the creation and annihilation operators for the Kasner and Schwarzschild space-time. The ground state can be factorised into two parts

$$\Psi^{(0)}[\phi](t) = N^{(0)}(t)\mathcal{G}^{(0)}[\phi](t). \tag{4.62}$$

One part is the wave part $\mathcal{G}^{(0}[\phi](t)$ depending on $\phi(x)$ and time and the second is the normalisation $N^{(0)}(t)$ which contains all contributions that are independent of the quantum fields. The time dependence in $\Psi^{(0}[\phi](t)$ stems from the explicit time-dependence of the metric.
Consistency of the field theory is closely related to the norm of the functional states $\|\Psi^{(0)}\|^2$

$$\|\Psi^{(0)}\|^2(t) = \int \mathcal{D}\phi \left(\Psi^{(0)}[\phi](t)\right)^* \Psi^{(0)}[\phi](t). \tag{4.63}$$

which is a functional generalisation of the quadratic norm of quantum mechanics. This norm is in principle an observable quantity. A meaningful theory should at least be normalisable, i.e. the norm should not diverge. The origin of the time-dependence lays in the construction of the Schrödinger states, because the fields are defined on the spatial hypersurface while the time-dependence comes from the metric. The wave-part $\mathcal{G}^{(0)}[\phi](t)$ in curved space-times is

$$\mathcal{G}^{(0)}[\phi](t) = \exp\left(-\frac{1}{2}\iint d\mu(x,y)\phi(x)K(t,x,y)\phi(y)\right). \tag{4.64}$$

The integral kernel in the exponent is now explicitly time-dependent. Plugging this into the Schrödinger equation we get a relation for the normalisation

$$i\partial_t \ln(N^{(0)}(t))\mathcal{G}^{(0)}[\phi](t) + i\partial\mathcal{G}^{(0)}[\phi](t) = H[\phi](t)\mathcal{G}^{(0)}[\phi](t). \tag{4.65}$$

Substituting the ansatz for (4.64) into (4.65), all contributions without ϕ-dependence must be contained in $N^{(0)}(t)$. The Hamilton operator applied on $\mathcal{G}^{(0)}[\phi](t)$ gives a defining equation for the normalisation

$$i\partial_t \ln(N^{(0)}(t)) = \frac{1}{2}\int_{\Sigma_t} d^3z\sqrt{\det(g)}(t,z)K(t,z,z). \tag{4.66}$$

We integrate over the whole spatial hypersurface Σ_t where z denotes a spatial coordinate. Solving (4.66) through integration with respect to t we get

$$N^{(0)}(t) = \mathcal{N}_0 \exp\left[-\frac{i}{2}\int_{t_0}^{t} d\tau \int_{\Sigma_t} d^3z\sqrt{\det(g)}(t,z)K(t,z,z)\right]. \tag{4.67}$$

The kernel in the above equation is evaluated at position z while the time dependence can be seen as a normalisation on each hypersurface Σ_t. The second equation for $K(t,x,y)$ is a Riccati differential equation given by the ϕ-dependent part. After differentiating with respect to the fields and performing the integrations of the resulting Dirac measures we receive

$$\begin{aligned}&\frac{i\partial_t\left[\sqrt{\det(g_\Sigma)}(t,x)\sqrt{\det(g_\Sigma)}(t,y)K(t,x,y)\right]}{\sqrt{\det(g_\Sigma)}(t,x)\sqrt{\det(g_\Sigma)}(t,y)}=\\&\int_{\Sigma_t}\sqrt{-g_{tt}}(t,z)d\mu(z)\ K(t,x,z)K(t,z,y)+\sqrt{-g_{tt}}(t,x)\left(\Delta-m^2-\zeta R\right)\delta^{(3)}(x,y).\end{aligned}\tag{4.68}$$

This equation is a nonlinear integro-differential equation. The spatial part of the Laplace-Beltrami operator is defined by

$$\Delta=\frac{1}{\sqrt{\det(g_\Sigma)}}\partial_i\left[\sqrt{\det(g_\Sigma)}g^{ij}_\Sigma\partial_j\right].\tag{4.69}$$

We should note that the Dirac δ-distribution in curved space-times is defined with respect to the measure

$$\delta^{(3)}(x,y)=\frac{\delta^{(3)}(x-y)}{\sqrt{\det(g_\Sigma)}}.\tag{4.70}$$

Solving equation (4.68) is highly non-trivial; we can use the spatial Fourier transform in order to transform (4.68) into a form which is easier to handle. The Fourier transform we used is

$$f(t,z)=\int d^3k e^{i2\pi kz}\widehat{f(t,k)}.\tag{4.71}$$

Note, the spatial measure must be absorbed in the δ-distribution, otherwise, the Fourier inversion theorem will not hold. The coordinate vector is spanned by the specific coordinates, e.g. in the Schwarzschild case (r,ϑ,φ) where the angles are interpreted as radian which is in this example

$$f(t,z)=\int dk_r dk_\vartheta dk_\varphi e^{i2\pi(rk_r+\vartheta k_\vartheta+\varphi k_\varphi)}\widehat{f(t,k)}.\tag{4.72}$$

Fortunately, the whole equation for the kernel (4.68) can be Fourier transformed. All hypersurfaces Σ_t are conformally flat and translational invariance in the spatial coordinates is a consequence. The Fourier transformed version of (4.68) is then given by

$$i\partial_t\left[\widehat{\det(g_\Sigma)}(t,k)\widehat{K(t,k)}\right]=\widehat{\sqrt{g_{tt}}}(t,k)\widehat{\det(g_\Sigma)}^{3/2}(t,k)\widehat{K(t,k)}^2-\widehat{\sqrt{\det(g_\Sigma)}}(t,k)\widehat{\Omega^2(t,k)}\tag{4.73}$$

with Fourier transformed Laplace-Beltrami operator

$$\widehat{\Omega^2(t,k)} = \widehat{g^{-1}}(t,k) - \hat{V} \tag{4.74}$$

where V encodes some other terms like mass m^2 and the Ricci scalar $\zeta\mathcal{R}$.
Equation (4.73) is a Riccati type differential equation which is difficult to treat in general. Finding a solution is possible if we know one special solution [Gradshteyn and Ryzhik, 2014]. In appendix B we will give some further information about this type of equation. It can appear as algebraic and differential equation [Lasiecka and Triggiani, 1991]. Both are non-trivial to solve. However, in mathematics is a lot of research on this type of equations.

For our purposes it should be enough to say that we can perform a transformation (unless the transformation is singular) given by

$$\widehat{K(t,k)} = -\frac{1}{2\sigma(t)}\partial_t \ln(\sigma(t) f^2(k,t)) \tag{4.75}$$

in order to rewrite the Riccati equation in terms of an ordinary second order linear differential equation. Here, we defined the function

$$\sigma(t) = -i\frac{\sqrt{\det(g)}}{\det(g_\Sigma)}. \tag{4.76}$$

4.2. The quantum completeness criterion

We are now at the point where we can state the completeness criterion for quantum field theory. In chapter 2 we have seen different concepts of completeness depending on the theory we have employed. Albeit there are different notions, the main principle stays the same.

In quantum field theory Lorentz geometry separates completeness by its causal character into null, time- and spacelike completeness. The singularity theorems of Hawking and Penrose use timelike and null geodesics as diagnostic tool. Chapter 2 shows that geodesic incompleteness does not need to coincide with unitarity violation.

Horowitz and Marolf [Horowitz and Marolf, 1995] took the example of the hydrogen atom for pedagogical reasons because there the classical instability coming from the Coulomb potential is cured by quantum mechanics. The crucial property was self-adjointness of the Hamilton operator, since self-adjointness leads to a unitary time-evolution, in other words the bound-state electron wave function is uniquely defined for all times under arbitrary initial conditions and hence complete. The applicability of Horowitz and Marolf's criterion is restricted to static space-times because a probing with quantum mechanics can only be appropriate for systems with conserved particle number, i.e. systems with timelike singularities. However, there is no simple extension of this formalism to spacelike singularities.

The articles [Balcerzak and Dabrowski, 2006] and [Horowitz and Polchinski, 2002] proposed some classical strings within a singular set-up for future null singularities while some other attempts were made: Horowitz and Steif [Horowitz and Steif, 1990] showed that in string theory a passing through a singular region is not well-behaved. Although string theory has some candidate for black holes [Horowitz and Strominger, 1991], it is beyond the scope of this thesis to investigate how the dynamical resolution of the black hole geometry can be described, nevertheless it is a compelling task to dynamically resolve the space-time and the singularity; by the way, there are several techniques to cure singular points with techniques from algebraic geometry (e.g. blow-up techniques). String theory might be a viable candidate as parent theory from this point of view. Loop quantum gravity for example allows as well for a more mighty dealing with singularities [Modesto, 2004]. In each case the price we have to pay is we must introduce new physics.

Our attempt is to take the geometry as a given set-up which means the classical space-time is our test object. Let us now build our detector for the space-time. Our standpoint is we choose quantum field theory as a well established and well understood theory. We propose to use quantum theories which are constructed such that they are regular and without any intrinsic pathology. Quantum fields are usually given by operator valued distributions, so either a regularisation scheme has to be proposed, such as minimal subtraction or dimensional regularisation [Collins, 1984], or distribution theory has to be considered [Hörmander, 1990] in order to avoid divergencies arising from an inappropriate handling of distributions. These divergencies which arise from the distributional character of the fields are excluded from this quality management because there exists a treatment by Epstein and Glaser [Epstein and Glaser, 1973] which avoids these type of divergencies just from the beginning, called finite quantum field theory.

Nevertheless, a renormalisation prescription is also of need in order to distinguish between divergencies arising from quantum field theory, which arise independently from the geometry and divergencies stemming from the space-time, although it is not easy to separate background contributions from field theoretic contributions. Hence, we probe a singular background with a regularised and renormalised quantum theory such that all arising divergencies can be traced back to the influence of the geometry.

Before we reveal the criterion for quantum completeness we want to present the scientific embedding. Static space-times with timelike singularities can be probed by quantum mechanics. For null singularities, Wald [Wald, 1980] has shown that the operator giving the dynamics is always self-adjoint. If not, there would be a contradiction due to a non-distinct evolution of initial conditions. Spacelike singularities occur in dynamical space-times, and describe singular spatial hypersurfaces. Their characterising property is, that a whole hypersurface Σ_t becomes singular and for all times beyond either space-time has ended or one has to pass through the singularity which is in general not possible. There are a few examples like in a bouncing cosmology it has been shown to be possible and regular to pass through a singularity [Gielen and Turok, 2016].
Concerning interpretations, a space-time in a quantum-mechanical probing could be regarded as an effective potential. If this is bounded from above the result is an incomplete space-time. In other words, only an infinitely repulsive barrier or a depleting measure function (like for the $1S$ orbital of the hydrogen atom) prevents quantum states from populating the singular region.

In our case, it would be possible to interpret the space-time geometry as external source. External because the internal degrees of freedom are not resolved dynamically. Field theories influenced by external sources have been investigated by Schwinger. He identified that the vacuum persistency (probability that the vacuum stays the vacuum) qualifies best for describing the effect of an external source on the system. Schwinger [Schwinger, 1951] has shown that fields coupled to external sources suffer from a dynamical principle $\mathcal{W}$ changing the vacuum persistency amplitude

$$\langle 0|0\rangle = e^{i\mathcal{W}}. \tag{4.77}$$

The imaginary part of $\mathcal{W}$, which is describing the influence of the source on the fields, bears a positive sign in a consistent theory. Even in Minkowski we observe a depletion of the persistency amplitude in presence of a source. Hence, stability of the persistency amplitude may be an invalid criterion. In general, a meaningful quantum field theory shows a depleting or normalisable persistency amplitude when coupled to an external source. The idea is that an interaction with the source excites states and cause particle production.

The equation of motion for a field ϕ on a time-dependent background consists of friction terms $\propto \partial_t\phi$ which produce a non-trivial dispersion which results in a transfer of probability to the background (external source).

Consistent evolutions are not harmed by a probability loss, for example in open systems it is expected to see a loss, however, a gain of probability is problematic. The reason is that the probabilistic interpretation is an intrinsic feature of quantum theory and state normalisability must be guaranteed.
Quantum completeness means: unitarity is replaced by state normalisability. On the level of groups it is equal to admitting a contraction semi-group [Reed and Simon, 1975]:

Definition 10. *A family of bounded operators* $\{T(t)|0 \leq t < \infty\}$ *on a Banach space* X *is called a* ***strongly continuous semi-group*** *if:*

(a) $T(0) = \mathrm{id}$

(b) $T(s)T(t) = T(s+t)$ *for all* $s, t \in \mathbb{R}^+$

(c) For each $\varphi \in X, t \to T(t)\varphi$ *is continuous.*

Semi-groups are more general than the unitary one-parameter group we know from standard quantum mechanics, however, their importance is on equal footing. For open systems, such as quantum field theory in curved spaces, they are expected to occur. They also turn out to be useful for the description of ground state energies [Simon, 1982]. Contraction semi-groups:

Definition 11. *A family of bounded operators* $\{T(t)|0 \leq t < \infty\}$ *on a Banach space* X *is called* ***contraction semi-group*** *if it is a strongly continuous semi-group and moreover* $\|T(t)\| \leq 1$ *for all* $t \in [0, \infty)$.

are of our special interest because the contraction could ensure that the state normalisability is preserved. Then no observable will diverge since time-evolution is governed by the semi-group. Such groups are well-known solutions of partial differential equations.

In open systems, the unitary group will be replaced by a contraction semi-group where the dissipation depletes the probability amplitude. To our understanding of completeness the contraction should happen towards the singularity because this feature would correspond to the basic idea of completeness that the end-point cannot be reached in a finite amount of time.

The Hamilton operator has to have a special property in order to generate the contraction, while self-adjointness implies unitarity, accretiveness implies normalisability:

Definition 12. *A densely defined operator* A *on a Banach space* X *is called* ***accretive*** *if for each* $\varphi \in D(A)$, *Re*$(\ell(A\varphi)) \geq 0$ *for some normalised tangent functional to* φ. A

is called maximal accretive (or m-accretive) if A *is accretive and has no proper accretive extension.*

The direct consequence of this definition is that the corresponding operator acts as a generator for a contraction semi-group if it is accretive[7] and $\mathrm{Ran}(\lambda_0 + A) = X$. Note, we could as well work in Hilbert spaces.

For the sake of understanding we should state the definition if the normalised tangent functional:

Definition 13. *Let* X *be a Banach space,* $\varphi \in X$*. An element* $\ell \in X^*$ *that satisfies* $\|\ell\| = \|\varphi\|$*, and* $\ell = \|\varphi\|^2$ *is called a normalised tangent functional to* φ*. By the Hahn-Banach theorem, each* φ *has at least one normalised tangent functional.*

Similar to self-adjoint operators we find a core theorem (equivalent to Stone's theorem) about the generators of contraction semi-groups [Reed and Simon, 1975]:

Theorem 12. *Let* A *be the generator of a contraction semi-group on a Banach space* X*. Let* D *be a dense set,* $D \subset D(A)$*, so that* $e^{-tA} : D \to D$*. Then* D *is a core for* A *(i.e.,* $\overline{A \restriction D} = A$*).*

Taking all this as prelude, we approach our statement defining quantum completeness. The state functional $\Psi[\phi](t)$ is constructed in the eigenbasis of the field. Furthermore, we assume Ψ to be the ground state and the field theory to be free. Throughout our analysis we will neglect back-reaction. This set-up enables us to state the consistency criterion [Hofmann and Schneider, 2015]:

Definition 14 (Quantum completeness)**.** *Let* $\mathcal{M}$ *be a globally hyperbolic manifold and* g *a metric, we call a space-time* $(\mathcal{M}, g)$ *with spacelike singularity at* 0 ***quantum complete*** *(to the left) with respect to a free field theory if the* L^2 *norm of the Schrödinger wave-functional* $\Psi[\phi](t)$ *of free test fields* ϕ *can be normalised at initial time* t_0*, the normalisation is bounded from above by its initial value for any* $t \in (0, t_0)$*, and the probability* $\|\Psi\|^2(t) = 0$ *at* $t = 0$*.*

Note, we have assumed the singularity to be at the specific point $t = 0$ which coincides with the Kasner and Schwarzschild singularity but we could have also defined it with a generic t_*. In other words, we can sum the definition up in a short relation that has to be fulfilled. Let again t_0 be the time where the (regular) initial conditions are set and consider

[7]In Hilbert spaces m-accretiveness is sufficient in order to act as generator for a contraction semi-group.

a Gaussian wave-functional of a free scalar field theory $\Psi^{(0)}[\phi](t)$. Quantum completeness requires the following relation to hold:

$$\|\Psi^{(0)}\|^2(t) \leq \|\Psi^{(0)}\|^2(t_0), \quad \forall 0 \leq t \leq t_0. \tag{4.78}$$

Here, the initial time is arbitrary, however, it should be part of the space-time. This reflects the property that completeness should hold under arbitrary initial conditions. To make it concrete, $\|\Psi^{(0)}\|^2(t)$ should be a monotonically decreasing function. Note, for excited states $\Psi^{(\text{exc.})}[\phi](t)$ might have nodes, hence the envelope should go to zero monotonically.

Precisely the occurrence of a contraction semi-group in the time-evolution forces the norm to decrease, this culminates into a mathematically definition

Definition 15 (Quantum completeness)**.** *Let $\mathcal{M}$ be a globally hyperbolic manifold and g a metric, we call a space-time $(\mathcal{M}, g)$ with spacelike singularity at 0 **quantum complete** with respect to a free field theory if the functional Hamilton operator $H[\Phi](t)$ is (maximal) accretive and serves as the generator for a contraction semi-group decreasing to zero towards the singularity.*

In this definition we claim the singular hypersurface Σ_0 will not be populated by any field configuration which agrees to the idea of completeness that the end-point cannot be reached

$$\lim_{t \to 0} \|\Psi^{(0)}\|^2(t) = 0. \tag{4.79}$$

With the contraction groups we can formulate our criterion very smoothly. The above definition of quantum completeness is exactly that the time-evolution is ruled by a contraction semi-group generated by the m-accretive operator.

Let us elaborate on the criterion made in Definition 14. Quantum field theory, as well as quantum mechanics, support a probabilistic interpretation while classical physics are purely deterministic. The normalisability of a quantum theory should be preserved because otherwise the quantum theory becomes meaningless. If the norm diverged because of a probability gain, we would interpret as the significant influence of the geometrical singularity. In other words, the pathology of the geometric singularity would become measurable/visible in quantum field theory by a divergent norm.

A gain of probability is interpreted as a massive particle production from the background resulting in a backreaction that deforms the geometry; if backreaction becomes important, the question whether a previously given space-time is quantum complete will become obsolete. From the point of view of a Cauchy problem, regular initial conditions running into inconsistencies can only be caused by pathologies in the theory.

Wrapping it up, our motivation was to find a criterion for quantum field theory in order to probe spacelike singularities. The basic idea is the singular configuration can not be reached under arbitrary initial conditions through a consistent/unique time-evolution. Our definition is in accordance with this. Consistent evolution of arbitrary field configuration under arbitrary initial conditions agrees with the basic idea. Moreover, we specify explicitly how (i.e. with respect to what state) we measure our observables. In Minkowski space-time this might be no big issue but in curved space-times it is essential. The probability[8] to be zero at the singular hypersurface coincides with the demand that the end-point can not be reached. This shows that Definition 14 is a well proposed notion in the spirit of completeness.

In Minkowski space-time is no spacelike singularity but nevertheless a unitary time-evolution still guarantees normalisability and the probabilistic interpretation is preserved; (4.78) holds in Minkowski. It is time to apply our criterion to a space-time with spacelike singularity. We will begin with Schwarzschild and then proceed with generalised Kasner space-time.

4.3. Quantum probing of Schwarzschild

The first thing an arbitrary person on the street connects with a black hole is its incompleteness, though they might not name it like that. It is usually seen as a hole where nothing can escape once it has crossed the horizon. In the semi-classical picture Hawking found that a quite strange effect occur, the black hole emits radiation with a thermal spectrum [Hawking, 1975]. However, the thermal spectrum might be a relic of the considered approximations, but it is up to now not explained to full satisfaction. Potentially all horizons should emit particles, for example the Rindler space-time, which is nothing but a piece of Minkowski due to acceleration of the observer, shows this effect of particle production [Einstein and Rosen, 1935]. Note, the particles in the Rindler space-time are described by the so-called Unruh effect [Unruh, 1976], which might be interpreted as a fictitious force analogue of quantum field theory.

The point of this is that the horizon causes some pathologies although the curvature scale does not blow up. The horizon itself is a null surface, since it is spanned only by two

[8]In German there is a brilliant word to describe the probability to be at a specific point, it is called Aufenthaltswahrscheinlichkeit. Unfortunately, there is no such expression in English.

tangent vectors for Schwarzschild, namely ∂_ϑ and ∂_φ. As we mentioned in 3.1 the light-cone swaps, i.e. the timelike Killing vector outside (3.1) becomes spacelike inside, the same counts for the Killing vector of the interior metric.

To avoid this peculiarity and the arising confusion we could change our point of view. Consider of being inside the black hole, the interior space-time (3.4) is globally hyperbolic and admits a timelike Killing vector ∂_t; this is only true without imposing matching conditions with respect to the exterior solution. Nevertheless, we could safely assume we are close to the singularity, such that the outside region will not affect the interior geometry severely. For the sake of completeness, and for the lazy reader we state again our test object, the interior of the Schwarzschild black hole in the classical picture of general relativity

$$g = -\frac{1}{\left|1-\frac{2M}{t}\right|}dt \otimes dt + \left|1-\frac{2M}{t}\right| dr \otimes dr + t^2\left(d\vartheta \otimes d\vartheta + \sin^2(\vartheta) d\varphi \otimes d\varphi\right) \qquad (4.80)$$

with black hole mass M. We assume for the sake of simplicity an eternal black hole. The Schwarzschild time t, which admit a Killing vector ∂_t due to global hyperbolicity, is only defined within the range $t \in (2M, 0)$. Spatial coordinates have the topology of $\mathbb{R} \times S^2$. Note that $r \in \mathbb{R}$ is not really a radius but it is a spatial coordinate. The interior Schwarzschild metric loses its spherical symmetry, it has the shape of a cigar which is getting streched and thinner during time evolution. The angles are defined as usual: azimuthal $\varphi \in [0, 2\pi)$ and polar $\vartheta \in [0, \pi)$. The singularity is located at $t = 0$ where the prefactor of the radial part becomes infinite and all others zero. The gravitational singularity in the interior Schwarzschild space-time is a naked one, but not similar to the singularity in a Friedmann universe because of the symmetries: a Friedmann is isotropic and the Schwarzschild interior anisotropic. The cosmic censorship hypothesis is not violated since the interior is seen as a detached part from our universe and for the outside observer the singularity is still censored by the horizon.

4.3.1. Ground state analysis

In this subsection we probe the Schwarzschild interior with quantum fields and apply our criterion directly. Our probing device is a free scalar field $\Phi(x)$ with mass m given by the action

$$S = -\tfrac{1}{2}\int d^4x\sqrt{-\det(g)}\left\{\partial_\mu\Phi(x)\partial^\mu\Phi(x) + m^2\Phi^2(x)\right\}. \tag{4.81}$$

One could now argue that a scalar field might not be the most general case and fermionic fields could behave differently. This is true for most parts of physics, however it has been shown, that the spin of a particle can not prevent from falling into the black-hole singularity [Stewart and Hájíček, 1973]. However, we do not know how a confining phase like quantum chromodynamics behave in this set-up. This question is postponed to future research.

Following the steps mentioned in 4.1.3 we build the Hamilton operator (4.57) and the canonical commutation relation (4.58) for the field $\Phi(x)$ and its conjugate momentum $\pi(x)$. Note, the spatial hypersurfaces Σ_t are conformally flat, while the whole space-time admits a non-vanishing Weyl tensor. Immediate consequences can be seen by taking the limit $t \to 0$. The Schwarzschild space-time has one diverging coordinate and collapses in two other spatial directions. This feature is also indicative for a Kasner space-time. With a few modifications, (3.1) can be transformed into a Kasner type-D metric. We emphasise this again, because in section 3.1 we showed that it can be transformed into (3.14). This purely time-depending metric will show the same behaviour as the Schwarzschild metric when approaching the singularity (cf. appendix F).
We construct the Hamilton operator with aid of (4.57)

$$H[\Phi](t) = \frac{1}{2}\int_{\Sigma_t} d\mu(x)\left[\frac{1}{\left(\frac{2M}{t}-1\right)^{\frac{3}{2}} t^4 \sin^2(\vartheta)}\frac{\delta^2}{\delta\Phi(x)^2} + \phi(x)\left(\Delta - m^2\right)\Phi(x)\right] \tag{4.82}$$

where we used the relations (4.59) and the canonical commutation relation (4.58). For a vacuum solution ($Ric = 0$) the Ricci term in the Hamilton operator vanishes. The Laplace operator is with respect to the hypersurface Σ_t but we should keep in mind Δ is also time-dependent. The ground state defined as a Gaussian wave-functional $\Psi^{(0)}[\phi](t)$ is given by the equations (4.62), (4.64), and (4.158).

Now we follow the steps we have introduced in the general discussion about the Schrödinger representation and it is no surprise that the Schrödinger equation (4.65) yields again here an integro-differential equation as in (4.68). What makes it tough to solve this equation is its additional dependence on the spatial coordinate ϑ. The polar angle should actually bear a subscript depending on where it comes from, either x or y in (4.68). This turns our calculation more cumbersome and we get some immediate consequence concerning the spatial Fourier transform. We have to think about, how to define it properly. We split the Laplace operator into a radial part $R(r)$ and the Laplace-Beltrami operator $B(\vartheta, \varphi)$ on the spherical shell S_r^2 at a specific r. Eigenfunctions of $B(\vartheta, \varphi)$ are given by the spherical harmonics $Y_m^l(\vartheta, \varphi)$. A more elegant way is to define the Fourier transform by an exponential function containing the coordinate vector (r, ϑ, φ). Since the hypersurfaces are conformally flat, we use the Fourier transform (4.72) to derive the following Riccati equation from the Schrödinger equation

$$i\partial_t \left[\sqrt{\left(\frac{2M}{t} - 1\right)} t^2 \widehat{\sin(\vartheta)} \widehat{K(t,k)} \right] = \left(\frac{2M}{t} - 1\right) \left(t^2 \widehat{\sin(\vartheta)}\right)^{3/2} \widehat{K(t,k)}^2 - t^2 \widehat{\sin(\vartheta)} \widehat{\Omega^2(t,k)} \tag{4.83}$$

with $\widehat{\Omega^2(t,k)}$ the Fourier transformed Laplace operator

$$\widehat{\Omega^2(t,k)} = \widehat{g^{-1}(\partial,\partial)} = -\frac{k_r^2}{\left|1 - \frac{2M}{t}\right|} - \frac{k_\vartheta^2}{t^2} + \frac{i}{t^2}\cot(\vartheta) k_\vartheta - \frac{k_\varphi^2}{t^2 \sin^2(\vartheta)} \tag{4.84}$$

that contains an imaginary part which might cause problems. The δ-distribution is defined by (4.70) such that

$$\delta^{(3)}(x, y) = \frac{\delta^{(3)}(x - y)}{\sqrt{(2M - t) t^3 \sin(\vartheta)}}. \tag{4.85}$$

Putting this into (4.83) and substituting $\widehat{K'(t,k)} = \widehat{\det(g_\Sigma)} \widehat{K(t,k)}$, where g_Σ is the induced metric on the hypersurfaces Σ, we end up with the final equation for our kernel

$$i\partial_t \widehat{K'(t,k)} = A(t) \widehat{K'(t,k)}^2 - B(t,k). \tag{4.86}$$

This equation is again a Riccati equation (cf. B). The coefficients, here, are given by

$$A(t) = -i\frac{\sqrt{-\det(g)}}{\det(g_\Sigma)}, \tag{4.87}$$

$$B(t,k) = i\sqrt{-\det(g)}\Omega(t,k). \tag{4.88}$$

Solving this equation is highly non-trivial. With the following transformation, we have already mentioned in (4.75)

$$\widehat{K(t,k)} = -\frac{1}{2\det(g_\Sigma)} \partial_t \ln(A(t) F^2(t,k)) \tag{4.89}$$

we are able to rewrite this equation in a form of an ordinary linear second order differential equation

$$\left(\partial_t^2 + \omega^2(t,k)\right) F(t,k) = 0, \tag{4.90}$$

with the complicated dispersion relation

$$\omega^2(t,k) \equiv \frac{1}{16 g_{\vartheta\vartheta}}(1 - 2g_{tt} + g_{tt}^2) - g_{tt}\Omega^2(t,k). \tag{4.91}$$

The dispersion relation (4.91) is singular at the horizon at the Schwarzschild radius r_s as well as at the geometrical singularity at $t = 0$ but our main interest is focussed on the behaviour near the singularity at $t = 0$. Therefore, we can express $\omega^2(t,k)$ asymptotically in leading orders of the singular behaviour ($t \ll 1$), that is, the most divergent part goes like $\omega_0^2(t) \propto \frac{1}{t}$. Corrections of order $\mathcal{O}(\sqrt{t}^{-1})$ are omitted. It has to be remarked that the dominant contribution does not depend on the momenta k, it is purely time-dependent. As a consequence, the full solution must show a behaviour which coincides with the above mentioned property of the differential equation, namely

$$K(t,x,y) = \kappa(t) K_\Sigma(x,y) \propto \kappa(t)\delta(x-y). \tag{4.92}$$

We see we get for the asymptotic limit that the differential equation contains just the time coordinate while the momentum parts have disappeared. The solution to (4.90) is given by

$$F(t) = C_2\sqrt{t}\left(C_1 + \ln\left(\frac{t}{2M}\right)\right) \tag{4.93}$$

with C_1, C_2 constants of integration. Note, when considering more terms of lesser order of divergence the solution gets more and more complicated, like Bessel functions (all divergent terms), or Whittaker functions (all non vanishing terms). This will turn the inverse Fourier transform to be impossible. Nevertheless, staying in the limit of small t it all reduces to (4.93).

Inversion of the transformation (4.89) leads to the kernel

$$K(t, x-y) = \frac{-i\delta^{(3)}(x-y)}{\sin(\vartheta)t^3\left|\ln\left(\frac{t}{2M}\right)\right|}\left(1 + \frac{i|\mathrm{Im}(C_1)|}{\left|\ln\left(\frac{t}{2M}\right)\right|}\right). \tag{4.94}$$

The kernel (4.94) is highly divergent in the limit $t \to 0$. Additionally, we see it has a real and an imaginary part. As we have already expected, dynamical space-times do not in general admit a unitary time-translation operator, this can we see here explicitly, because

the real part destroys unitarity of time-evolution. The immediate consequence will be a dissipation[9] effect causing a continuous probability transfer to the background. Note, the real part $\mathrm{Re}(K) \ll \mathrm{Im}(K)$ near the singularity.
We can now begin with the construction of the wave-functional $\Psi^{(0)}[\phi](t)$, first the normalisation, with (4.158) we get

$$N^{(0)}(t) = \frac{\mathcal{N}_0}{\sqrt{\left|\ln\left(\frac{t}{2M}\right)\right|}^{\,\mathrm{vol}(\Sigma_t)\Lambda}}, \tag{4.95}$$

where $\mathcal{N}_0$ is a collection of all constants. Before we go to the wave part, we briefly discuss (4.95). Since we are working in the semi-classical picture, we introduced an ultra-violet cut-off Λ and an infrared cut-off by the volume of the hypersurface $\mathrm{vol}(\Sigma_t) = \int d^3x$. Those quantities can of course be regularised, however, we can see that this will not change the picture because they are both positive and huge numbers therefore the normalisation goes to zero - this is what we expect from a sensible normalisation (a schematic plot of $N^{(0)}(t)$ is provided in figure 4.3).

The normalisation is a monotonically decreasing function, which goes to zero at $t \to 0$. However, the question is, what does $\mathcal{G}^{(0)}[\phi](t)$ and in the end $\Psi^{(0)}[\phi](t)$. The wave part is oscillating from the contribution of the imaginary part, the real part in the vicinity of the singularity will steadily decrease due to the measure functions in the exponential

$$\Psi^{(0)}[\phi](t) = \frac{\mathcal{N}_0}{\sqrt{\left|\ln\left(\frac{t}{2M}\right)\right|}^{\,\mathrm{vol}(\Sigma_t)\Lambda}} \exp\left(-\frac{-i|2M-t|}{t^{3/2}\left|\ln\left(\frac{t}{2M}\right)\right|}\left(1+\frac{i|\mathrm{Im}(C_1)|}{\left|\ln\left(\frac{t}{2M}\right)\right|}\right)\int_{\Sigma_t} d^3x\,\phi^2(x)\right). \tag{4.96}$$

The normalisation itself gives zero in the limit of small times as well as the non-oscillating part of $\mathcal{G}$ and so does the ground state $\Psi^{(0)}[\phi](t)$ as well. Interpreting this, we come to the result that the wave functional vanishes continuously when approaching the singular hypersurface.

No matter how promising the result looks, the Schrödinger formulation of quantum field theory is not formulated in observable quantities, which is similar to quantum mechanics. Hence we will later analyse observables like the expectation value of the energy density with respect to the Schrödinger states. For the reason how we formulated the quantum

[9]The dissipation does not occur when we consider the Klein-Gordon product. However, we see a similar effect for the measurement vertices when we consider the theory of classical/quantum measurements.

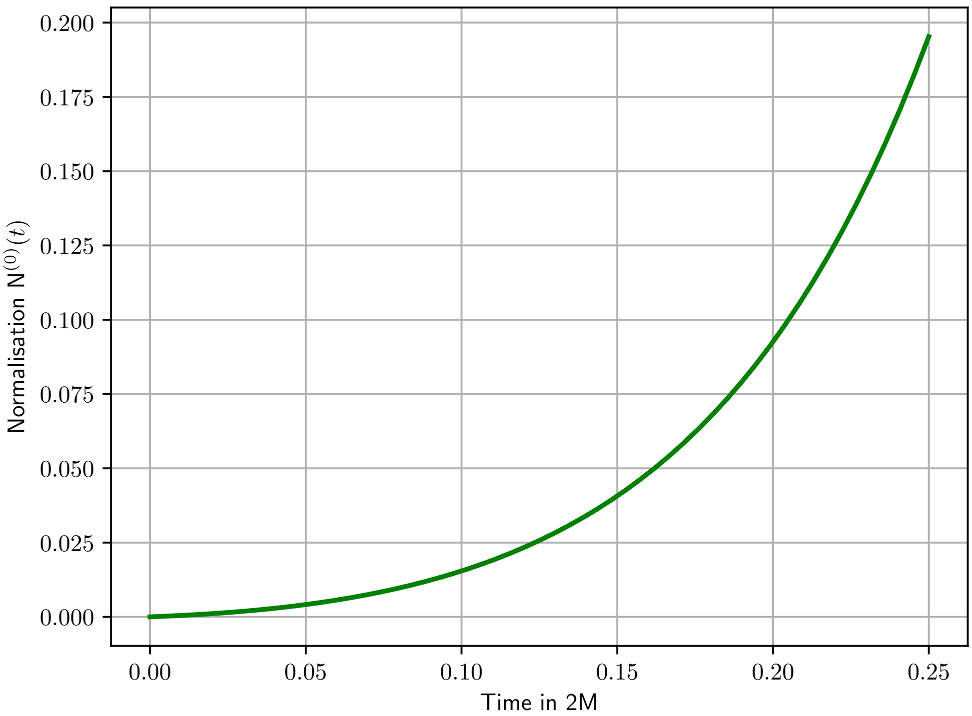

Figure 4.3.: Plot of the normalisation $N^{(0)}(t)$ for $t \in (0, 0.25M)$

completeness criterion 14, we have a look at the norm of the state (4.63) with (4.8)

$$\|\Psi^{(0)}\|^2(t) = \frac{|\mathcal{N}_0|^2}{\left|\ln\left(\frac{t}{2M}\right)\right|^{\mathrm{vol}(\Sigma_t)\Lambda}} \left(t^{\frac{3}{4}} \left|\ln\left(\frac{t}{2M}\right)\right|\right)^{N(\Lambda)} \tag{4.97}$$

with number of momentum modes $N(\Lambda)$ with $|k| \in [0, \Lambda^{\frac{1}{3}}]$. This is a huge number corresponding to the ultraviolet cut-off Λ which could be regularised as well. Again, regularisation would not change the result, whatsoever. A close look at (4.97) offers the two relevant things we need in order to have quantum completeness: $\|\Psi^{(0)}\|^2(t)$ is a monotonically decreasing continuous function which is important for the consistency of the time-evolution. In other words, the probabilistic interpretation is guaranteed and protected under time-evolution.

The second criterion is the question of the population of the singular hypersurface Σ_0.

Therefore we consider the limit of (4.97)

$$\lim_{t\to 0} \|\Psi^{(0)}\|^2(t) = 0 \tag{4.98}$$

which is zero and the second requirement is fulfilled. What does this mean for the physics of the system? Concerning the interpretation we consider a set of observables localised on a hypersurface Σ_t. It could be expected that an observable exists with an expectation value with respect to the ground state that is diverging, for example the trace anomaly of the stress-energy tensor for conformally coupled fields. However, this is not the case because the ground state does not populate Σ_0. In other words, no field configuration is living on Σ_0 because the probability measure vanishes there. The consequence is, that quantum fields detect no significance of the singularity. It seems that the geometrical singularity bordering on the physical space-time (the space-time which could be measured by quantum fields) is detached. Although it is beyond our reckoning, we want to mention that this is a powerful argument against the singularity theorems as a physical statement and it gives also good reason to even doubt the dynamical formation of a black hole singularity.

The publication related to this calculation can be found in appendix C.

4.3.2. Gaussian deviations: Excited states

The ground state analysis showed no inconsistencies, however, this might not be the whole story yet. Dynamical space-times support emission and absorption processes which lead to excitations with respect to the ground state. Excitations are not an integral component in the definition of quantum completeness [Hofmann and Schneider, 2017]; they can deform the system such that a new lowest energy configuration will be realised.

Higher excitations $\Psi^{(n)}[\phi](t)$ may admit a diverging norm; this could be the case if they were populated close to the singularity while the ground state depletes.

Hence, it is a natural question to investigate also the norm of the excited states. They are analogously defined to quantum mechanics by creation and destruction operators; the destruction operator can be defined by

$$a[f](t)\Psi^{(0)}[\phi](t) = 0. \tag{4.99}$$

Quantum mechanics provides us with the same kind of equation where the destruction operator $a(k)$ has a local description which erases a field in the Hilbert space. When applied on a state $\psi(k_1, \dots, k_n)$ we get [Reed and Simon, 1975]

$$(a(p)\psi)^{(n)}(k_1, \dots, k_n) = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \delta(p - k_l)\psi^{(n-1)}(k_1, \dots k_{l-1}, k_{l+1}, \dots, k_n). \tag{4.100}$$

Note, the destruction (or annihilation) operator can be defined via an eigenvalue equation with eigenvalue 0. Here, we showed how to exclude the state with momentum $p = k_l$. The creation operator is defined in a similar manner

$$\left(a^\dagger(p)\psi\right)^{(n)}(k_1, \dots, k_n) = \sqrt{n+1}\psi^{(n+1)}(p, k_1, \dots, k_n). \tag{4.101}$$

We see, a state with momentum p has been included in the n-particle state $\psi^{(n)}$.

The Schrödinger representation provides a functional analogy of the creation and annihilation operators. Similar to quantum mechanics, we use (4.99) as the defining equation for $a[f](t)$ which destroys an on-shell field $f(t, x)$ in the Fock space; the vacuum state is free of on-shell fields $f(t, x)$ because the fields in the wave-functional are restricted to Σ_t hence we get zero. Formally, the operator is defined such that the ground state lies in the kernel Ker$(a[f])$

$$a[f](t) = \int_{\Sigma_t} d\mu(x) f^*(t, x) a(t, x). \tag{4.102}$$

The annihilation operator consists of two ingredients: the adjoint on-shell field $f^*(t, x)$, or the harmonic function on the background, and the ultra-local description of the annihilation operator $a(t, x)$ which is in a free scalar field theory represented by

$$a(t, x) = \frac{i}{\sqrt{\det(g_\Sigma)(x)}}\pi(x) + \int_{\Sigma_t} d\mu(y) K(t, x, y)\Phi(y) \tag{4.103}$$

with $K(t, x, y)$ given in (4.94). From the canonical quantisation prescription we find the first term to be the functional derivative with respect to the field $\phi(x)$, like for Minkowski shown in (4.32) and (4.33), while the second is a scalar product with the multiplication operator Φ mediated through the kernel $K(t, x, y)$. Conjugating the destruction operator gives the creation operator which we apply in order to get the first excited state

$$\Psi^{(1)}[\phi, f](t) = (a[f](t))^* \, \Psi^{(0)}[\phi](t). \tag{4.104}$$

As usual, on Σ_t the following algebraic relation holds:

$$\left[a[f](t), a^\dagger[f'](t)\right] = 2\mathrm{Re}\left(\mathcal{K}_2[f, f']\right)(t)\ . \tag{4.105}$$

The first excited state can be interpreted as a functional re-normalisation[10] of the ground state, because we could rewrite this equation when absorbing the creation operator in the normalisation function

$$\Psi^{(1)}[f, \phi](t) = N[f, \phi](t)\Psi^{(0)}(t). \tag{4.106}$$

Excitations should be sensitive to the overlap between $\phi(x)$ and the on-shell fields $f(t, x)$. To make it precise, let $\mathfrak{F}$ be the set of all fields and $\mathfrak{F}_{os}$ the set of all fields obeying the on-shell condition with $\mathfrak{F}_{os} \subset \mathfrak{F}$. All on-shell fields $f(t, x) \in \mathfrak{F}_{os}$ whereas $\phi(x) \in \mathfrak{F} \restriction \Sigma_t$. Because the on-shell fields are not covered by the time-evolution of the Schrödinger formalism, i.e. they are not restricted to a hypersurface like $\phi(x)$, they obey the time-evolution given by the harmonic equation $\Box f(t, x) = 0$. The d'Alembert operator is due to the non-trivial geometry of the interior Schwarzschild space-time

$$\Box f(t, x) = g^{-1}(df, df) = -\frac{1}{\sqrt{-\det(g)}}\frac{\partial}{\partial t}\left[\sqrt{-\det(g)}\frac{\partial f}{\partial t}(t, x)\right] + g^{ii}\frac{\partial^2}{\partial x_i^2}f(t, x). \tag{4.107}$$

A purely time-dependent and diagonal metric simplifies the calculation a lot, such that we can use a separation ansatz. The function $f(t, x)$ can hence be split into spatial $X(x)$ and time part $T(t)$. Our ansatz could be to employ a Fourier transform on the spatial hypersurfaces Σ in order to solve the harmonic equation [Simon, 2015b] but there is a smarter way in this set-up.

First of all, we use the definition (4.99) in order to state the annihilation operator which has the same form as (4.102) and (4.103). Then, (4.104) is the defining equation for the first excitation.

The fist excited state is given by (4.106), that is by using (4.105), it acts as a functional derivative with respect to the fields $\phi(x)$; application of the creation operator on $\Psi^{(0)}[\phi](t)$ yields

$$\Psi^{(1)}[f, \phi](t) = \langle\phi|f\rangle_K(t)\Psi^{(0)}[\phi](t). \tag{4.108}$$

The formal equation is of course identical for all space-times. At least, however, the system is specified by the metric as well as the on-shell fields $f(t, x)$ because they fulfil the harmonic's condition $\Box f = 0$ on Schwarzschild for massless case, or for massive $f(t, x)$s the

[10]See footnote 4 on page 55 in case this expression puzzles you.

Klein-Gordon equation

$$(\Box - m^2)f(t, x) = 0 \tag{4.109}$$

The d'Alembert operator for (3.1) is much more complicated - compared to Minkowski - as we see in (4.107)

$$\Box = -\partial_t^2 - \frac{1}{t}\partial_t + \frac{1}{\frac{2M}{t} - 1}\partial_r^2 + \frac{1}{t^2}\partial_\vartheta^2 + \frac{1}{t^2}\cot(\vartheta)\partial_\vartheta + \frac{1}{t^2\sin^2(\vartheta)}\partial_\varphi^2. \tag{4.110}$$

This d'Alembert operator has the property of admitting two single derivatives, one for time and the other for the polar angle ϑ. These terms will produce friction terms in (4.109) which could lead to an imaginary part in the spectrum of the Hamilton operator. Via a special reformulation of the field $f(t, x)$ we find the solution to equation (4.109) after Boulware [Boulware, 1975]

$$f(t, x) = \sum\!\!\!\!\!\!\int dk f_{ml}(t, k) e^{i2\pi kr} Y_m^l(\varphi, \vartheta). \tag{4.111}$$

The above expression is a Fourier transform in the radial component combined with an expansion in the eigenfunction of the angular part of the d'Alembert operator, the spherical harmonics $Y_m^l(\vartheta, \varphi)$. This smart representation of $f(t, x)$ allows for the aforementioned smart solution of the Klein-Gordon equation. Inserting (4.111) yields as solution after redefinition $t \equiv M\tau$

$$f(\tau, x) = \sum\!\!\!\!\!\!\int dk e^{i2\pi kr}\left[C_1 I_0\left(\sqrt{2l(l+1)\tau}\right) + C_2 K_0\left(\sqrt{2l(l+1)\tau}\right)\right] Y_m^l(\vartheta, \varphi) \tag{4.112}$$

The Bessel functions I_0 of first and K_0 second kind solve (4.109) up to order $\mathcal{O}(\tau^{-1})$, i.e. we have asymptotically expanded the Klein-Gordon equation for small time. The constants of integration are named C_1 and C_2. In general the constants of integration are complex and could depend on the spatial coordinates. When expanding around the singularity, i.e. for small times, we can simplify the Bessel functions [Gradshteyn and Ryzhik, 2014]

$$I_0(x) \sim 1 + \frac{x^2}{4} + \frac{x^4}{64} + \mathcal{O}(x^6), \tag{4.113}$$

$$K_0(x) \sim -\ln(x) - \gamma. \tag{4.114}$$

Applying the asymptotic forms to (4.112) yields

$$f(t, x) \sim \sum\!\!\!\!\!\!\int dk e^{i2\pi kr} C(r)\ln(\tau) Y_m^l(\vartheta, \varphi). \tag{4.115}$$

Note, it is tricky to see why this should be consistent with the differential equation, in other words the logarithm will not solve the asymptotic expansion for the time-dependent part

$$(t\partial_t^2 + \partial_t)f(t,k) = \kappa f(t,k) \tag{4.116}$$

with κ coming from the spatial part. For vanishing Schwarzschild time $f(t,k)$ not a solution to (4.116). Therefore, we introduce $\xi \equiv \zeta\tau$ and the double-scaling limit $\tau \to 0$ and $\zeta \to \infty$ while ξ stays constant. The rescaled equation of motion

$$(\xi\partial_\xi^2 + \partial_\xi)f(\xi,k) = 0 \tag{4.117}$$

will then be solved by the logarithm. We saw that a careful asymptotic expansion has to be imposed on the level of the equation of motion. Only in this case, an expansion of the solution is consistent with the expansion of the Klein-Gordon equation.

Note, that this solution is a classical one, therefore, we have no duty towards a specific way to normalise the fields; in quantum mechanics we have to respect the probabilistic interpretation. The classical solution is divergent in the limit $t \to 0$. For quantum fields the picture changes, proper normalisation of quantum fields on a Schwarzschild background will result in a regular field amplitude, even in the vicinity of the singularity. Elizalde [Elizalde, 1987, Elizalde, 1988] has calculated massive scalar quantum fields in the Heisenberg picture and received a regular amplitude.

Heisenberg and Schrödinger representation should per construction agree but to see this is not as easy as one might think because here, the Stone-von Neumann theorem is inapplicable. In the Heisenberg representation we need to consider the theory of measurements (classical or quantum) [DeWitt, 2003], then we formulate the detection mechanism with a vertex density describing the absorption of the test field. We evaluate then the vertex density in the limit of sending the detector towards the singular hypersurface. Consistency of Heisenberg and Schrödinger representation requires the vertex density to vanish in case of quantum completeness; we show such a Heisenberg analysis[11] in the charged scalar case in G.

Let us close this intermezzo about the Heisenberg formulation and turn our attention to our calculation of the excitations and especially the norm $\|\Psi^{(1)}\|^2(t)$ from (4.108) under

[11] This is a very important consistency requirement, because people are more familiar with the Heisenberg picture.

consideration of (4.105) we can deduce the n^{th} excitation

$$\|\Psi^{(n)}\|^2(t) = \langle\Psi^{(0)}|(a[f](t))^n((a[f](t))^*)^n|\Psi^{(0)}\rangle = \kappa_n\langle f, f\rangle_K(\tau)\|\Psi^{(0)}\|^2(t) \tag{4.118}$$

where κ_n is just a combinatorial factor. The norm is given by the K-mediated scalar product of the on-shell fields with $M\tau = t$

$$\langle f, f\rangle_K(\tau) \sim \iint_{\Sigma_\tau} d\mu(x,y) f^*(\tau, x)\mathrm{Re}(K)(\tau, x, y)f(\tau, y) \propto \frac{|\mathrm{Im}(C)|M(2-\tau)}{4|\ln^2(\tau)|}\ln^2(2l(l+1)\tau)\,; \tag{4.119}$$

the scalar product mediated by the integral kernel becomes a constant when we send $t \to 0$. We see, $\langle f, f\rangle_K$ preserves the validity of the ground state analysis, and consequently all excitations vanish on the singular hypersurface Σ_0 because the ground state stays the dominant part even neighbouring the singular hypersurface.

We found again that the interior of a Schwarzschild black hole is quantum complete. In this geometry, quantum field theory has a consistent evolution and preserves the probabilistic interpretation. Moreover, the singular hypersurface is not populated with any field configurations. Therefore, the properties for quantum completeness were fulfilled.

Excitations do not influence the ground-state result such that completeness is violated because all excitations are teared to zero by the probability measure. The ground state analysis is hence a robust physical statement. In other words, the singularity does not affect free quantum theory.

4.3.3. Influence of polynomial self-interactions

In the last two sections we showed that the black hole is quantum complete, but only with respect to free field theory; the interesting physics, however, occurs when we impose an interaction term. Here, we will investigate a self-interaction. It might be that free field theory is too special to make a sensible statement about completeness, at least in quantum mechanics the shape of the potential is significant for quantum mechanical completeness [Simon, 1971].

In quantum field theory, one could intuitively guess that including an interaction term in the action changes the picture, possibly, this terms could cause trouble if the field reaches a strong coupling regime which lead to a deformation of the Schwarzschild space-time rendering the free field analysis void.

The question we have to answer is: can an initially weakly coupled configuration run into a regime of strong coupling? Let us answer this is two ways, first the brute force method where we calculate the contribution from interaction terms in the wave functional, second where we will give a heuristic argument about self-interaction terms. Afterwards we connect both arguments, the brute-force derivation and the elegant thoughts.

We start with the straightforward part, the calculation: Our theory shall be a Φ^4-interaction which is stable in four dimensions, nevertheless, all other polynomial interations of $\Phi(x)$ would be possible as well

$$S_{int} = \frac{\lambda}{4!} \int d^4x \sqrt{\det(g)} \Phi^4(x). \tag{4.120}$$

The dimensionless coupling constant λ is small, and we choose the initial data such that perturbation theory is applicable. Our ground states must change and include somehow the non-Gaussianity induced from the interaction [Hatfield, 1992]

$$\Psi_{int}[\phi](t) = N_{int}(t)\mathcal{G}^{(0)}[\phi](t) \exp\left(\lambda D[\phi](t)\right). \tag{4.121}$$

All information about the interaction is encoded in the interaction (or deformation) functional $D[\phi](t)$ which is a polynomial in the field consistent with the power-counting in S_{int}. Note, also the normalisation acquires additional terms from the interaction part. We will restrict ourself to first order in perturbation theory. The deformation term in (4.121) is given by

$$\begin{aligned} D[\phi](t) =& \iint d\mu(x,y)\phi(x)D_2(t,x,y)\phi(y) \\ &+ \iiiint d\mu(w,x,y,z)\phi(w)\phi(x)D_4(t,w,x,y,z)\phi(y)\phi(z). \end{aligned} \tag{4.122}$$

D_2 and D_4 represent the modifications of $\mathcal{O}(\lambda)$ and obey different kernel equations motivated by power counting in the field variable. The first term describes the mass renormalisation and the second the contribution from the four-point vertex. After plugging (4.121) into the Schrödinger equation and asymptotical expansion for small times we write down the equations proportional to λ; afterwards we perform the rescalings $\tilde{D}_4 = \det^2(g_\Sigma)D_4$

and $\tilde{D}_2 = \det(g_\Sigma)D_2$, then the two equations read

$$i\frac{\partial \tilde{D}_4}{\partial \tau}(\tau) = \sqrt{\det(g)}\left[\tilde{D}_4(\tau)\widehat{K}(\tau)\right], \tag{4.123}$$

$$i\frac{\partial \tilde{D}_2}{\partial \tau}(\tau) = \sqrt{\det(g)}\left[2\widehat{K}(\tau)\tilde{D}_2(\tau) + \det(g_\Sigma)\tilde{D}_4(\tau)\right]. \tag{4.124}$$

Like before we have defined the dimensionless time variable $\tau = t/M$ and $K(\tau)$ is the time dependent part of the ground-state kernel. For an asymptotic expansion in small times, we can solve for the four-point vertex contribution D_4

$$D_4(\tau) \sim \tau^6 \left|\ln(\tau)\right|^{-1} \xrightarrow{\tau\to 0} 0. \tag{4.125}$$

We see, this equation goes to zero in the limit of small times. Applying this to (4.124) the equation for D_2 simplifies a lot:

$$i\frac{\partial \tilde{D}_2}{\partial \tau}(\tau) = \sqrt{\det(g)}\left[\widehat{K}(\tau)\tilde{D}_2(\tau)\right]. \tag{4.126}$$

Note, in the limit of small times, the ground-state kernel $K(t,x,y)$ factorises its time-dependence $K(t,x,y) \to K(t)\delta^{(3)}(x,y)$; this fact is consistent with the BKL conjecture since the spatial correlation becomes trivial while the time-dependent function dominates (here it diverges). We solve (4.126) and get a very similar result

$$\widehat{D_2}(\tau) \sim \tau^3 \left|\ln(\tau)\right|^{-1} \xrightarrow{\tau\to 0} 0. \tag{4.127}$$

Taking limit for small times leads to a vanishing deformation kernel. The four-point modification D_4 of the wave functional becomes negligible near the origin, the same behaviour is shown by the two-point modification D_2. As a rule of thumb, for each field in the polynomial interaction we get a factor of $\sqrt{\det(g_\Sigma)} \propto \tau^{\frac{3}{2}}$.

If we perform our asymptotic expansion less radical, the equations for the rescaled deformation kernels are given by two coupled nonlinear integro-differential equations which are solved via integral exponential functions which reduce to $\ln(\tau)^{-1}$ for small times.

The behaviour of the deformation kernels affect the interaction wave-functional such that the normalisation $N_{\text{int}}(\tau) \to N^{(0)}(\tau)$ for $\tau \to 0$, the same happens for $\mathcal{G}$ because the integral kernels $D_2(t,x,y)$ and $D_4(t,w,x,y,z)$ vanish the functional $D[\phi](t)$ goes to zero as well and we receive the ground state wave functional of the free theory $\Psi_{\text{int}} \to \Psi_0$.

The brute force calculation shows that the contributions from the self-interaction term vanish close to the singularity. However, we can make a general argument about the interaction terms. Whatever the polynomial interaction looks like, the exponent has to be at least out of $\mathbb{N}$.

Now we come to our heuristic argument which starts with considering self-interactions which are polynomials in the fields $S_{int} = \frac{\lambda_n}{n!}\sqrt{g_{tt}}\int d\mu(x)\Phi^n(x)$. Their time-dependent prefactor is the given by $\sqrt{\det(g)} \propto t^2$ which vanishes in the limit of time going to zero. Consequently the whole term becomes less and less important the closer the hypersurfaces are to the singular one. We will take this as a hint that all contributions which only contain multiplication operators will not affect regularity. For interactions involving derivatives of $\Phi(x)$ we get terms proportional to the conjugate momentum which is identified with the functional derivative. These terms will come with two problems, first of all they admit a singular prefactor and additionally those terms correspond usually to non-local interactions.

From Schrödinger's equation (4.159) we deduce the formal solution for the wave-functional for a general Hamilton operator with interaction term and time ordering $\mathcal{T}$

$$\Psi[\phi](t) = \mathcal{T}\exp\left(-i\int_{t_0}^{t} d\tau H[\Phi](\tau)\right)\psi[\phi](t_0). \tag{4.128}$$

The structure shows we separated out a time-ordered time-evolution operator and a part evaluated at initial time t_0. The Hamilton operator can be split into two parts, one containing the derivative operators and one the multiplication operators.

$$H[\Phi](t) = H_D\left[\frac{\delta}{\delta\phi}\right] + H_M[\Phi] \tag{4.129}$$

All self-interaction is contained in the latter part which just gives real contributions which vanish for $t \to 0$ due to the prefactor. More significant is the asymptotic form of the Hamilton operator $H[\Phi](t)$ which becomes $H_D[\Phi](t)$ in the limit of $t \to 0$ because of the diverging prefactor: $\csc(\vartheta)/(2Mt - t^2) \sim t^{-2}$. In other words, such multiplication operator contributions do not affect self-adjointness of the Hamilton operator, or at least not dominantly; we have identified the functional derivative operator part as dominant. This creates an imaginary part in the time evolution of (4.128) hindering it to be unitary.

Let us do a short excursus: For the functional version we could find an (heuristic) argument very analogue to the example of a free Schrödinger particle where one can show that for the operator $-\frac{d^2}{dx^2}$ on $L^2(0,\infty)$ we can find a non-zero solution $\chi(x)$ on the domain

$\mathcal{C}^\infty(0,\infty)$ such that $-\chi'' = \pm i\chi$ (cf. [Reed and Simon, 1975]). The solutions can be found as:

$$\chi_{1+}(x) = \exp\left(\frac{1}{\sqrt{2}}(-1+i)x\right), \tag{4.130}$$

$$\chi_{2+}(x) = \exp\left(\frac{1}{\sqrt{2}}(1-i)x\right), \tag{4.131}$$

$$\chi_{1-}(x) = \exp\left(\frac{1}{\sqrt{2}}(-1-i)x\right), \tag{4.132}$$

$$\chi_{2-}(x) = \exp\left(\frac{1}{\sqrt{2}}(1+i)x\right). \tag{4.133}$$

These solutions are strong solutions, i.e. they are infinitely many times continuously differentiable. Although they do not lie in the Hilbert space $L^2(\mathbb{R}, d\mu)$ they lie in the domain of the derivative. For unbounded operators the domain and the Hilbert space will in general not coincide, it is sufficient that the domain is dense in the Hilbert space. For closed unbounded operators it can be shown that the domain is never the Hilbert space itself, but the closure of the domain is [Simon, 2015c]. Of course we can find an interval on $\mathbb{R}$ such that the operator will be self-adjoint but that is not the idea of a physical system.

Nevertheless, a self-adjoint extension can be derived but the squared derivative admit infinitely many self-adjoint extensions on $\mathcal{C}^\infty(0,\infty)$, or any Hilbert space $L^2(a,b)$. Note, that $-\frac{d^2}{dx^2}$ admits a unique self-adjoint extension on the whole field $\mathbb{R}$ [Bonneau et al., 2001]. Let us try to follow a similar logic the functional derivative. At least, however, it is to say that for the functional derivative we can take the argument only partially since the spaces we are working in are much more general and need mathematical justification but we could make some statements about their structure. In this sense the following argument should be taken as heuristic idea. The interior Schwarzschild space-time is defined for the finite time interval $(0, 2M)$ due to the time-dependence of the metric, which is the same for all time dependent functions. Quantum fields $\phi(x)$ are living at least in the space of two times continuously differentiable functions $C^2(\Sigma_t, d\mu)$ and are defined only on the spatial hypersurface Σ_t which means they are purely depending on spatial variables, as usual in the Schrödinger picture. The wave-functional $\Psi[\phi](t)$ collects these fields and therefore should lie in a space similar to the L^2 in quantum-mechanics, which we call $\mathcal{L}^2(C^2(\sigma_t), \mathcal{D}\phi)$, the space of „square integrable“ functionals. The term square integrable means with respect to the functional measure $\mathcal{D}\phi$.

Let us do some bookkeeping and describe the structure of the aforementioned space, although it needs mathematical legitimation (existence of such spaces has been proven by [Amour et al., 2015]): We call the space of all functionals $\Psi[\phi]$ obeying:

$$||\Psi||^2 = \int \mathcal{D}\phi \Psi^*[\phi]\Psi[\phi] < \infty \tag{4.134}$$

in relation to the L^2 Hilbert space, $\mathcal{L}^2(C^2(\Sigma_t, d\mu), \mathcal{D}\phi)$. It contains all normalisable functionals, or square integrable functionals with respect to the functional measure. Although there is a lack of mathematics, we will still work in this space. Let the domain of the functional derivative $D(\frac{\delta^2}{\delta\phi^2})$ be dense in $\mathcal{L}^2(C^2(\Sigma_t, d\mu), \mathcal{D}\phi)$. We could construct functions fulfilling the condition given by the von Neumann criterion

$$\upsilon_{1+}[\phi] = \exp\left(\frac{1}{\sqrt{2}}(-1+i)\int d\mu(x)\phi(x)\right), \tag{4.135}$$

$$\upsilon_{2+}[\phi] = \exp\left(\frac{1}{\sqrt{2}}(1-i)\int d\mu(x)\phi(x)\right), \tag{4.136}$$

$$\upsilon_{1-}[\phi] = \exp\left(\frac{1}{\sqrt{2}}(-1-i)\int d\mu(x)\phi(x)\right), \tag{4.137}$$

$$\upsilon_{2-}[\phi] = \exp\left(\frac{1}{\sqrt{2}}(1+i)\int d\mu(x)\phi(x)\right). \tag{4.138}$$

However, the solutions do not lie in the domain of the derivative Hamiltonian $\mathcal{D}(H_\pi)$. The problem is that the measure can not be adapted as in the quantum mechanical case because the functional derivation will produce $\delta(0)$ terms. We may therefore expect that the functional derivatives do not admit unique self-adjoint extensions.

What has instead to be checked is the sign of the ground-state kernel's imaginary part sgn(Im(K)).

$$\text{sgn}(\text{Im}(K)) = \begin{cases} -1 \text{ (implies increasing norm)}, \\ 1 \text{ (implies decreasing norm)}. \end{cases} \tag{4.139}$$

We want to empasise that this condition is mandatory for a consistent and complete evolution, it guarantees that neither probability can be gained from the background, nor the singular hypersurface will be populated if K diverges in the limit t approaches zero.

We want to mention here, that the Schwarzschild kernel admits the desired positive sign in the imaginary part:

$$\text{Im}(K)(t,x,y) \sim \frac{-i\delta^{(3)}(x-y)}{\sin(\vartheta)t^3 \ln\left(\frac{t}{2M}\right)} \tag{4.140}$$

The above expression seems to have a negative sign, however, the logarithm for arguments smaller than zero admits a negative sign, therefore the whole expression is positive. This leads then to the result of quantum completeness as we have just seen.

If the sign of the imaginary part of the kernel is negative, it will be possible that normalisablility fails and probability can be gained from the background. Then the system would be quantum incomplete.

It has to be emphasised that the occurrence of an imaginary part originates only from the functional derivative part H_π of $H[\Phi](t)$ while the multiplication operators do not contribute in the limit $t \to 0$ because of the vanishing prefactor

$$H[\Phi](t) \to H_\pi \,, \text{ for } t \ll 1. \tag{4.141}$$

Perhaps we could adjust the domain such that there is a unique extension but this would certainly not describe our system.

Altogether, Schwarzschild quantum probing shows a consistent evolution for the ground-state probability amplitude; the imaginary eigenvalue fulfils the completeness criterion for the free theory. Polynomial self-interactions respect quantum completeness with the above argument which is consistent in the sense that a regular quantum theory will not evolve into a strong coupling regime, if not initially imposed. Quantum chromodynamics could presumably modify our analysis, but this is postponed to future research.

Quantum completeness might be rephrased in the following sense: quantum completeness means that an initially well-defined Cauchy problem, stays well-defined during time-evolution. Regular initial conditions will not evolve into singular values. Quantum regularity stays robust against self-interaction of the probes.

The calculation and the result in (4.127) and (4.125) show immediately coincidence with our heuristic argumentation. The statement that the polynomial interaction term vanishes for small times fits in the picture of the asymptotic expansion of the Hamilton operator and the vanishing interaction term. All contributions from polynomial interactions will conclude with this. Nevertheless, all interactions involving conjugate momenta are excluded from this reasoning.

Note, the dominating part comes from the kinetic part of the Hamilton operator H_π, and $\pi \propto \partial_t \phi$, hence, it is the time gradient which dominates the behaviour near spacelike singularities [Belinskii et al., 1970]. Quantum completeness of H_π cannot be destroyed by any interaction which strongly supports the validity of the BKL conjecture.

4.3.4. Stress-energy tensor of quantum probes

The squared amplitude $\|\Psi\|^2(t)$ of the wave-functional is a good criterion to check the consistency but unfortunately, it is not measurable, therefore the desired quantity for physicists is usually energy because we can get some intuition when we compare the energy of the system with the energy scale of the theory which in our case would could be the Planck energy.

In curved space-time there are some proposals how to measure the energy of a manifold because it might not be a well-defined quantity. In principle the Kodama vector provides one notion of energy for space-times with spherical symmetry. Since we are interested in the energy of the probing field and not of the whole system, i.e. the field content on the space-time, the stress-energy tensor will do the job. It is a very good candidate, nevertheless, in curved space-time the renormalisation might cause some difficulties. Some components might diverging due to the coincidence limit which even occurs in flat space. Especially the energy density $\varepsilon = T_{00}(x)$ is a quantity for which we have some intuition. For a scalar field the stress-energy tensor is given by

$$T = d\Phi \otimes d\Phi - \frac{g}{2} g^{-1}(d\Phi, d\Phi) + m^2 \Phi \otimes \Phi \tag{4.142}$$

Note, this is a local quantity and it has to be renormalised because it is defined at one space-time point, i.e. all fields $\Phi(x)$ are evaluated at the same space-time point x; for example we could perform the axiomatic renormalisation procedure which has been developed by Wald [Wald, 1978].

Let us sketch an instructive example: we take the expectation value $\langle \phi(x)\phi(y) \rangle$ in order to see how we can fulfil those axioms. In general the expectation value $\langle T \rangle$ consists of parts like $\langle \phi^2(x) \rangle$ which if naïvely evaluated imply a multiplication of two δ-distributions at the same point; this would not yield any finite result. Nevertheless, when calculating the expectation value under consideration of a local bi-distribution $\beta(x, y)$ with suitable singular structure, the expression $\langle \phi(x)\phi(y) \rangle$ makes totally sense. The construction can be seen by Hadamard [Hadamard, 1925] or Wald [Wald, 1994]. Basically, this is the idea of a point-split prescription: a single point is replaced by two points but we will impose the limit that the points coincide. For further reading consider the book of Wald [Wald, 1994] or Birell and Davies [Birrell and Davies, 1984].

In the Schrödinger representation this coincidence limit causes problems but a subtraction of the Minkowski divergence (this is the divergence which is independent from the space-time curvature) will regularise T after a point-split. Furthermore a normal ordering procedure renormalises the energy-momentum tensor and the resulting singular contributions arise purely from the metric.

Another advantage of the Schrödinger representation is that the fields are defined on the spatial hypersurface Σ_t but the time dependence of the stress-energy tensor is not on the same footing as the spatial dependence; it is more like a label for the hypersurfaces Σ_t. The renormalisation will therefore affect the spatial part, the divergence occurring from the time coordinate is identified as purely geometrical.

In quantum field theory all observables are evaluated with respect to the quantum states, hence, we take the expectation value $\langle T_{\mu\nu}\rangle(x)$. There is a relation between the different notions of energy, the energy density $\langle T_{00}\rangle$ can be identified with the expectation value of the Hamilton operator $\langle H\rangle$ [Traschen and Brandenberger, 1990]. Note, that these two quantities are also related to the number of produced particles $\langle N\rangle$, and consequently to the Bogolubov transformation and its coefficient, usually named as β

$$\langle N_j\rangle = \sum_i |\beta_{ij}|^2, \tag{4.143}$$

where we referred to the energy produced from one single mode here indicated with the subscript j. The leading behaviour (in time) of all three quantities is for that reason similar. In our calculation we choose the expectation value of the Hamilton operator. In the Schrödinger representation

$$\langle\Psi|H[\phi]|\Psi\rangle(t) = \int \mathcal{D}\phi\Psi^*[\phi](t)\,(H[\Phi](t)\Psi[\phi](t)) \tag{4.144}$$

and it should be said that in the Schrödinger representation the stress-energy tensor acts as an operator consisting of functional derivatives and field operators. To apply the Hamilton operator and perform the path integral might get cumbersome, alternatively we may use the Schrödinger equation in order to identify $H[\Phi](t)\Psi[\phi](t)$ with $i\partial_t\Psi[\phi](t)$. It might or might not be that the time derivative of the ground-state wave-functional in (4.62)

$$\langle\Psi^{(0)}|i\partial_t|\Psi^{(0)}\rangle \tag{4.145}$$

is less complicated. No matter how we calculate the result will always include the expectation value $\langle\Phi^2\rangle(t)$ and the time-dependence of the kernel function $K(t,x,y)$. Let us have a look at the purely time-depending function in (4.145): in the limit of approaching the singularity at $t/M = \tau \equiv 0$ we get a diverging contribution $\propto \tau^{-1}$. Nevertheless, we get also a contribution which is proportional to $\|\Psi\|^2(\tau)$, the norm of the ground state which is decreasing for the Schwarzschild space-time. Together with (4.62) the time dependence of (4.145) can be asymptotically written as

$$\langle i\partial_t \rangle \sim \tau^{\frac{3}{4}N(\Lambda)-1} |\ln(\tau)|^{N(\Lambda)-2} \tag{4.146}$$

which is not decreasing in general. This can be seen in figure 4.4. We would have to distinguish between two cases, either the case where $N(\Lambda) \leq \frac{4}{3}$ then the singularity at zero will cause a blow up (green line) of $\langle H \rangle$, or $\langle T_{00} \rangle$ respectively. In diagram 4.4 we see

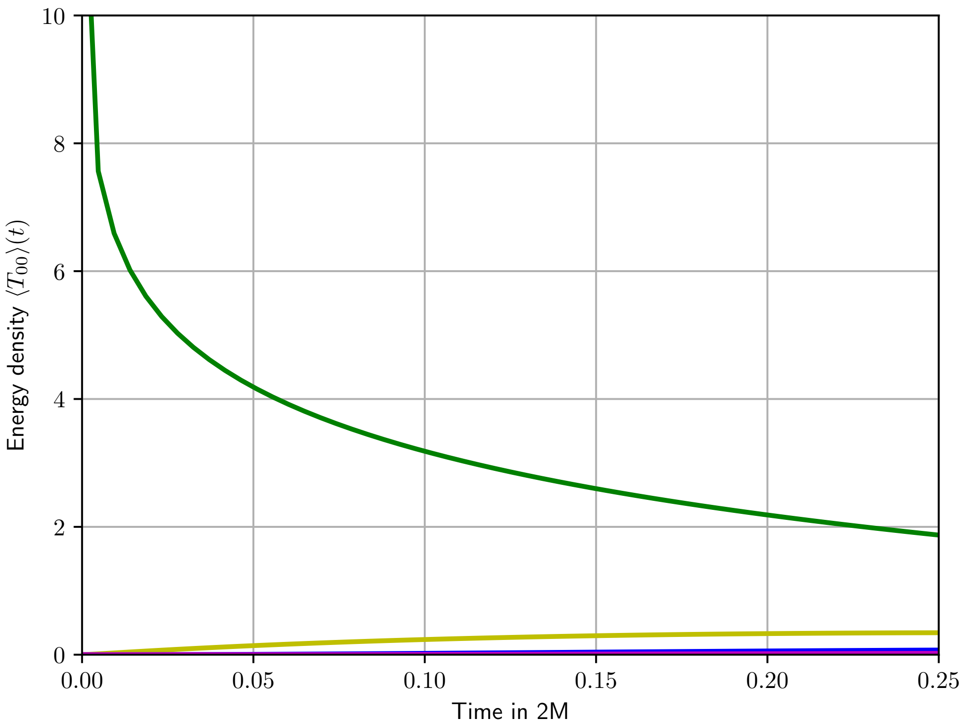

Figure 4.4.: Plot of the stress-energy tensor $\langle T_{00} \rangle$ for $N(\Lambda) \in \{1, 5\}$ and $t \in [0, 0.25]$

for small values of $N(\Lambda)$, that is $N(\Lambda) = 0$ (not plotted) and $N(\Lambda) = 1$ (green line) the expectation value of the 00 component of the stress-energy tensor diverges. All number of momentum modes $\mathbb{N} \ni N(\Lambda) > 1$ are regular at $t = 0$. Figure 4.4 is not suitable to observe the higher-$N(\Lambda)$ modes's behaviour; they more or less look like straight lines. Nevertheless, we see the most important property: they go to zero for $t \to 0$. In order to investigate their behaviour we take a deeper look at higher $N(\Lambda)$ modes.

In this case, when $N(\Lambda) \geq \frac{4}{3}$, then no divergence will occur, whatsoever. From Figure 4.5 it is visible that for $N(\Lambda) > 1$ the qualitative behaviour stays the same, i.e. the expectation value of $\langle T_{00} \rangle$ increases up to some point, then it starts to decrease, and vanishes in the limit $t \to 0$. This means the energy density vanishes at the singular hypersurface which supports our quantum completeness calculation and the physical relevance of the quantum completeness criterion. The red curve shows $N(\Lambda) = 5$ and the green $N(\Lambda) = 13$, the bigger $N(\Lambda)$ the lower is the maximum of the curve. We should add that the maximum of $\langle T_{00} \rangle$ neither exceeds the Planck scale nor are the values neighbouring. It is to say that if the Planck scale had been reached, then we would have mistrusted the predictability beyond this value.

Returning to our discussion about $N(\Lambda)$, the question is, how strong is this restriction. The answer is: it does not matter. The number of momentum states in between the range of the cut-off and zero is huge; the spectrum of the momentum operator is continuous. Therefore, infinitely many states with momentum are in the range $|k| \in [0, \sqrt[3]{\Lambda}]$. To assume $N(\Lambda)$ be small, is very poor. It is a huge number which has to be regularised but the result stays valid regardless of the regularisation, because regularising $N(\Lambda)$ to a finite value which is still big, will not change the picture. The stress-energy tensor is in agreement with our results of the persistence amplitude $\|\Psi\|^2(t)$ and the physical interpretation is clear, a zero probability at Σ_0 should not result in an infinite amount of energy at this point. The energy density has proved to be a brilliant diagnostic tool for the validity of our argumentation.

Nevertheless, one might wonder about the trace anomaly of the stress-energy tensor for conformally coupled fields which is divergent for $t = 0$ because it scales with t^{-6}. Here, we refer to the same argumentation, we have already employed when we explained the connection between Schrödinger and Heisenberg representation: in the latter, we have to check, whether or not the divergent energy can be measured in a quantum-mechanical or classical measurement process [DeWitt, 2003]. Moreover, DeWitt explains that the trace anomaly is a non-critical anomaly which can not influence the consistency of quantum field theory even if the background is dynamically resolved [DeWitt, 2003].

We expect two effects which act contrariwise: on the one hand side the squeezing of the hyperplane and the fields on it causing the energy density to grow, and on the other the probability measure which goes to zero at the singularity. In other words, quantum theory

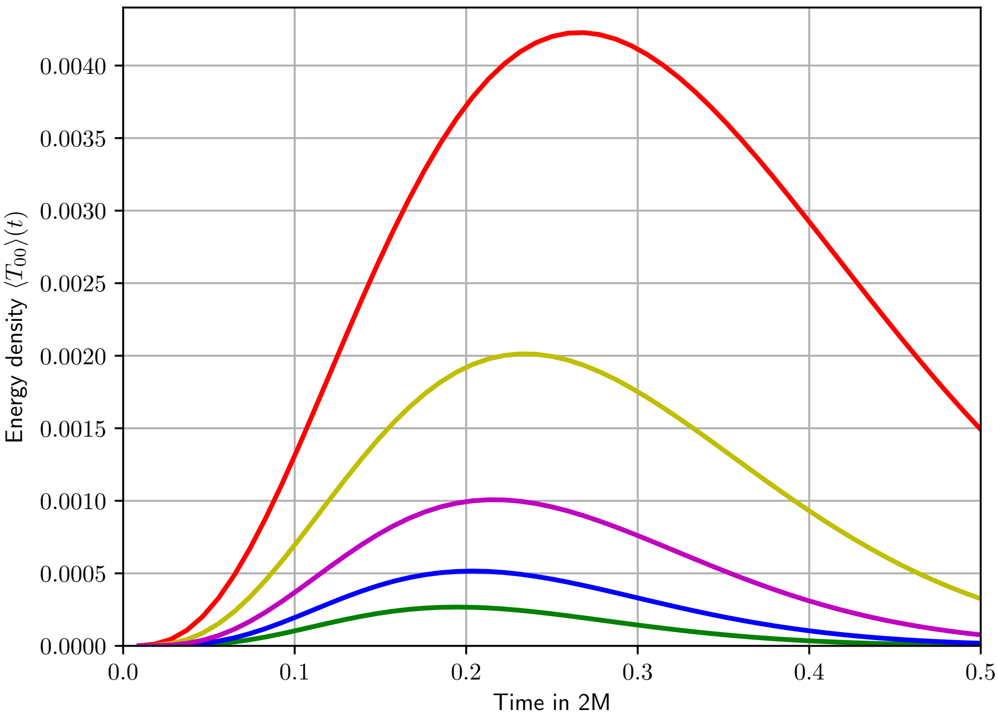

Figure 4.5.: Plot of the stress-energy tensor $\langle T_{00} \rangle$ for $t \in [0, 0.5]$.

protects itself from running into a coincident limit by its probability measure. These two effects fight each other in the energy density, but in the end the probabilistic measure dominates.

From (4.142) follows, the stress-energy tensor $\langle T \rangle$ is proportional to two quantities: the expectation value of the fields $\langle \Phi^2 \rangle$ and of its conjugate variable $\langle \Pi^2 \rangle$ because observables are built from the variable $\Phi(f)$ and its conjugate momentum $\Pi(f)$, where f is a continuous function with compact support. Each of those expectation values can be shown to be proportional to the ground state probability amplitude $\|\Psi^{(0)}\|^2(t)$.

First we introduce an auxiliary source functional $\mathcal{J}$ describing the absorption and emission of fields ϕ minimally coupled to the associated local current density J, and define $\Psi_0^J[\phi](t) := \langle \phi | \exp(\mathcal{J})[\Phi](t) | \Psi_0 \rangle_{\Sigma_t}$, which allows to replace compositions of the configuration operator Φ by the corresponding succession of functional derivatives with respect to the current. In the presence of the auxiliary source,

$$\langle \Psi_0^J | \Phi^2(f) | \Psi_0^J \rangle_{\Sigma_t} = [f]\delta_J^2 \exp\left\{ \frac{1}{4} \frac{1}{\sqrt{\det(g_\Sigma)}} [J] \left[\mathrm{Re}(\mathcal{K})\right]^{-1} [J] \right\} \mathcal{P}_0(t) , \tag{4.147}$$

where $[f]\delta_J^2$ denotes the second functional derivative with respect to J, smeared with an appropriate field configuration f. In the absence of the auxiliary source, the ground-state expectation (4.147) is real and semi-positive definite. Towards the singularity Σ_0, (4.147) approaches zero due to the temporal support granted by the probability density $\mathcal{P}_0(t) := \|\Psi_0[\phi]\|^2(t)$. Similarly,

$$\langle \Psi_0^J | \Pi^2(f) | \Psi_0^J \rangle = (\mathrm{Re}(K))(t) \|\Psi_0\|^2(t) + \sqrt{\det(g_\Sigma)} |k|^2(t) \langle \Psi_0^J | \Phi^2(f) | \Psi_0^J \rangle_{\Sigma_t} . \tag{4.148}$$

The ground-state expectation value (4.148) is real, semi-positive definite, and approaches zero towards Σ_0. Therefore, $\langle \Psi_0^J | T_{00}(f) | \Psi_0^J \rangle$ is always semi-positive definite and vanishes towards the black-hole singularity. We present the scaling in short

$$\langle \Psi_0^J | \Phi^2(f) | \Psi_0^J \rangle_{\Sigma_t} \sim t^{\frac{9}{2}} \|\Psi_0\|^2(t) \tag{4.149}$$

$$\langle \Psi_0^J | \Pi^2(f) | \Psi_0^J \rangle_{\Sigma_t} \sim \left(1 + \frac{1}{t^3}\right) \|\Psi_0\|^2(t) \tag{4.150}$$

which makes clear that the expectation value of Φ vanishes for $t \to 0$ while the conjugate momentum part $(\mathrm{Re}(K))(t)\|\Psi_0\|^2(t)$ poses a condition on $N(\Lambda)$ which we have just seen before in the analysis of $\langle H \rangle$. Nevertheless for a suitable, and reasonable $N(\Lambda)$ the expectation value vanishes. It can be shown that these qualifications remain true for arbitrarily excited states.

4.4. Charge conservation inside the black hole

This section covers charged fields inside the black-hole geometry [Eglseer et al., 2017]; although we might repeat a lot of the steps from previous analyses we will explicitly derive that the charge is conserved by the time-evolution inside. First of all, we come up with the formalism derived in the previous sections but now our degree of freedom will be a $U(1)$-charge of mass m.

The charged scalar $\Phi(x)$ act as test fields on the background and will not disturb the geometry in such an amount that the Schwarzschild metric is transformed into a different metric. Note that the $U(1)$-charged fields could act as a toy model for information carriers. Ohanian and Ruffini [Ohanian and Ruffini, 2013] explain, how an infalling charge in the Schwarzschild background looks for an outside observer. During the infall the spherical symmetry is broken and hence the set-up of a test charge inside a Schwarzschild black-hole is properly described by the Schwarzschild background[12]. Conclusively, our system is a charged test field placed inside a black-hole. The corresponding action is

$$S = -\int d^4x\sqrt{-g}\ \left[\partial_\mu\Phi\partial^\mu\Phi^* + m^2\Phi\Phi^*\right]. \tag{4.151}$$

with g the metric determinant. We proceed as usual and construct the Hamilton operator and solving the Schrödinger equation; we restrict to only the essential steps. To emphasise it again, the Schrödinger representation generalises the quantum-mechanical picture to infinite numbers of degrees of freedom that include fields.

The metric (3.4) is globally hyperbolic, we perform a 1+3-split and foliate $\mathcal{M}$ along the timelike vector field ∂_t into spacelike hypersurfaces Σ_t, and impose the canonical commutation relations (for the charged fields)

$$[\Pi(x), \Phi(y)] = [\Pi^*(x), \Phi^*(y)] = -i\delta^{(3)}(x-y). \tag{4.152}$$

in order to formulate the Hamilton operator. Using the quantisation prescription we identify the conjugate momenta by functional derivatives

$$\Pi(x) = -i\frac{\delta}{\delta\phi(x)}. \tag{4.153}$$

Note, we can find an analogue formula for $\Pi^*(x)$ by application of the Hermitian conjugation. The Hamilton operator is then derived by a Legendre transformation similarly to the real scalar field

$$H\left[\Phi^*, \Phi\right](t) = \int_{\Sigma_t} d^3x\sqrt{-\det(g)}\left[-\frac{1}{\det(g_\Sigma)}\left|\frac{\delta}{\delta\phi}\right|^2 + |\partial\Phi|^2 + m^2|\Phi|^2\right]. \tag{4.154}$$

[12]Suppose an electric charge reaches the singular hypersurface the background should transform into Reißner-Nordström. Of course we could have also tested this metric, but we want to give a heuristic argument why the Schwarzschild background threatens completeness as a worst case scenario. Reißner-Nordström has a singularity which in leading order is dominated by the term Q^2/r^2. This purely timelike singularity can be probed via quantum mechanics but will not lead to any incompleteness due to Theorem X.11 in Reed and Simon 2 [Reed and Simon, 1975].

The metric of the hypersurface is, as usual, denoted by g_Σ and the determinant by $\det(g_\Sigma)$. We define the ground-state functional by a Gaussian wave package which contains the fields as variables through

$$\Psi_0[\phi^*, \phi](t) = \mathcal{N}(t) \exp\left(-[V^*]\mathcal{K}[V](t)\right), \text{with} \quad V = \begin{pmatrix} \Phi \\ \Phi^* \end{pmatrix} \tag{4.155}$$

normalisation function $\mathcal{N}(t)$, and the kernel $\mathcal{K}$ represents here a 2×2 matrix. We expect we would have to investigate the full theory which includes excitations and self-interaction of the theory, nevertheless, we saw for the Schwarzschild metric that close to the singularity (4.154) shows a very remarkable behaviour; diagonal Hamilton operators can be decomposed into a part consisting of derivatvives H_Π and one containing only polynomials of multiplication operators $\mathcal{P}[\Phi](t)$. After we applied the Schwarzschild coordinate neighbourhood to (4.154) the relevant part of the system is given by H_Π which is obviously not self-adjoint since it consists of functional derivatives. It holds for all terms polynomial in Φ that $\mathcal{P}[\Phi](t) \to 0$ when $t \to 0$. Therefore, the ground state analysis, and moreover, the analysis of H_Π is sufficient.

A complex Gaussian distribution is most easily expressed by a quadratic form of a vector v containing the fields contracted with a matrix K describing the spatial correlations and the time translation

$$[V^*]\mathcal{K}[V](t) := \frac{1}{2} \int d\mu(x, y) v^\dagger(x) K(t, x, y) v(y) , \tag{4.156}$$

where $v^\dagger(x) = (\phi^*(x), \phi(x))$, the kernel matrix is

$$K(t, x, y) = \begin{pmatrix} \mathcal{K}(t, x, y) & L(t, x, y) \\ L(t, x, y) & \mathcal{J}(t, x, y) \end{pmatrix} , \tag{4.157}$$

and the functional state $\Psi_0[\phi^*, \phi]$ contains the field configurations. The norm of the state is given by a functional integration over all field configurations localised on the hypersurface Σ_t. The normalisation $\mathcal{N}(t)$ is purely time dependent (and therefore field-independent) and can be expressed by the trace of $K(t, x, x)$, which we denote $\mathrm{tr}(K)(x, y, t) \cong k(t)\delta^{(3)}(x, y)$[13]

$$\mathcal{N}(t) = \mathcal{N}_0 \exp\left(-i \int_{t_0}^{t} dt' \int d^3x \sqrt{-g} \;\; k(t)\delta^{(3)}(x, x)\right) . \tag{4.158}$$

[13]This relation holds for asymptotic expanded kernel functions in the limit of small times.

In principle the $\delta^{(3)}(x,x)$ will need some regularisation procedure, but for our argumentation this is not needed. Again, the time evolution of the quantum states is governed by the Schrödinger equation

$$i\partial_t \Psi_0[\phi^*,\phi](t) = H[\Phi^*,\Phi](t)\Psi_0[\phi^*,\phi](t). \tag{4.159}$$

The real scalar field analysis should be consistent with the results of the complex fields. We could split the complex field in its real and imaginary part and perform the calculations in [Hofmann and Schneider, 2015] for each sector. We might expect a factorisation of the wave functional $\Psi_0[\phi] \to \Psi_0[\mathrm{Re}(\phi)] \times \Psi_0[\mathrm{Im}(\phi)]$ into real and imaginary part of the field which are both real and could be evaluated separately, and for real scalar field we know that the system is complete.

Taking this as a motivation we perform the calculation and solve for the kernel matrix explicitly. In the absence of interactions, $K(t,x,y)$ becomes diagonal and again only the trace enters. The result, after asymptotical expansion for small times $t \to 0$, is

$$k(t) = \frac{i4M}{(2M-t)t^3 \sin(\vartheta)(iC_0 - \ln(t))}, \tag{4.160}$$

where C_0 is an integration constant. Because the Hamilton operator is not essentially self-adjoint, which we know because H_Π is not self-adjoint, we will get complex eigenvalues which leads to a non-unitary time-evolution. Apart from this, when the Hamilton operator admits a contraction semi-group a consistent evolution can be assured because the wave functional decreases to zero towards the singularity (quantum completeness). This guarantees that the amplitude $||\Psi_0||^2(t)$ for $t \to 0$ goes to zero as well and the validity of the probabilistic interpretation is protected by a bounded time-evolution.

We want to assume throughout the analysis that backreaction can be ignored because its inclusion could deform the background which makes investigations of the original spacetime obsolete. In other words, we take the same assumptions as for the real scalar field and no further assumptions than Hawking and Penrose in the derivation of the singularity theorems. Recalling the quantum completeness criterion, the presence of a geometrical singularity does not need to pose a problem, the important fact is whether or not the singular hypersurface can be populated with any field configuration. For the matrix $K(t,x,y)$ we get the following amplitude

$$\|\Psi_0\|^2(t) \propto \frac{(t^{3/2}\ln^2(t))^{N(\Lambda)}}{|\ln(t)|^{2vol(\Sigma_t)}} \xrightarrow{t\to 0} 0 \tag{4.161}$$

with volume regularisation $vol(\Sigma_t)$ and a regulator concerning the momenta $N(\Lambda)$ with momentum cut-off Λ. We see, the Hamilton operator admits a contraction semi-group which takes the probability amplitude to zero when we approach the singular hypersurface.

The regularisation does not sensitively influence the result. Nevertheless, it might be seen as a sign for the existence of a fundamental parent theory. Quantum gravity should be able to explain how field theory is protected from classical singularities. We interpret our result to the favour of quantum gravity because if quantum field theory cannot reach a classical singularity, this state should not even be formed. A valid quantum gravity should provide a dynamical resolution of the black hole formation not ending in a singularity.
We conclude that the Schwarzschild interior is quantum complete for charged fields. When we compare the integral kernels corresponding to the real and the complex field we see that up to constant factors there are identical.

By now we only know that the probability of populating spatial hypersurfaces decreases when we approach the singularity. Investigations of the charged scalar field help to understand what exactly happens to the fields inside the black-hole.

This result is rather intuitive: The ultimate reason behind the consistency of local quantum physics inside a black hole, even in a semi-classical set-up, is quantum completeness, which also renders charge conservation sacrosanct. The geodesic information sink at the singularity is closed because the probabilistic measure keeps Σ_0 void of any charges and information carriers. As a consequence Σ_0 cannot be probed by local quantum physics, not even indirectly in the sense of allowing the black hole interior $\mathcal{B}$ to leak. Quantum fields are totally ignorant about the presence of Σ_0 and the corresponding complete event space can be interpreted as a physical space-time which is regular.

The absence of an information sink in the quantum theory can be reconsidered as follows. Conservation laws are connected to balance equations. Since K is diagonal, we only consider ϕ-configurations for the sake of brevity. The probability-current density is given by

$$\mathcal{S}_0(x) := \frac{\sqrt{-g_{tt}}}{g_\Sigma}\left(\Psi_0\,\Pi(x)\,\Psi_0^* - \text{h.c.}\right)\ , \tag{4.162}$$

and satisfies the functional generalisation of differential probability conservation, $\partial_t\mathcal{P}_0 +$

$\text{div}\mathcal{S}_0 = 0$, where

$$\text{div}\mathcal{S}_0 := \int_{\Sigma_t} d\mu_x \; i\Pi\mathcal{S}_0 \; , \tag{4.163}$$

on any spatial hypersurface Σ_t , $t \in (0, t_0)$. Integrating this divergence over the field configuration space, conservation of probability amounts to

$$\partial_t \mathcal{W}(t) = i \int_{\Sigma_t} d\mu_x \frac{\sqrt{-g_{tt}}}{g_\Sigma} \left(\langle \Psi_0 | \Pi^2(x) | \Psi_0 \rangle - \text{h.c.} \right) \; , \tag{4.164}$$

where $\mathcal{W}(t) = ||\Psi||^2(t)$ denotes the total probability for populating Σ_t with any field configuration, on- and off-shell. Towards the black-hole singularity $\langle \Psi_0 | \Pi^2(x) | \Psi_0 \rangle \in \mathbb{R}_0^+$, see (4.148), and so $\mathcal{W}(t)$ is conserved. The probability current cannot reach the geodesic information sink because for $t \to 0$ the expectation value $\langle \Pi^2 \rangle \to 0$. Therefore no probability leakage occurs which is in accordance with our former statement that Σ_0 cannot be populated with scalar fields. This result suggests that the geodesic information sink is closed for quantum fields. We conclude that charges are conserved and then no information can be destroyed by the black hole interior. This is illustrated by figure 4.6, we see that towards the singular hypersurface Σ_0 the probabilistic current depletes and there is no passing through this hypersurface.

Instead of populating the interior $\mathcal{B}$ with arbitrary information carriers, consider a population originating from an Unruh state $|\mathcal{U}\rangle \equiv \sum |\Psi_{\text{in}}\rangle \otimes |\Psi_{\text{out}}\rangle$, where $|\Psi_{\text{out}}\rangle$ denotes a state associated with Hawking radiation (i.e. a state describing the outgoing fields/particles) and $|\Psi_{\text{in}}\rangle$ is the corresponding ingoing state. Let us choose an initial Cauchy hypersurface Σ_{t_0} inside the black hole $\mathcal{B}$ at time t_0 such that the matter content of the interior is $|\Psi_\mathcal{B}\rangle = |\Psi_\text{S}\rangle \otimes |\Psi_{\text{in}}\rangle$ on this hypersurface [Horowitz and Maldacena, 2004]. The configuration space contains dual states such as

$$\langle \text{BH}| = \langle \text{M}| \otimes \langle \text{R}|, \tag{4.165}$$

where M represents the matter fields that have participated in the gravitational collapse, and R denotes the ingoing Hawking quanta. Evolving the states from Σ_{t_0} towards the singular hypersurface to $\Sigma_{\varepsilon\tau}$, the wave functional in configuration representation is given by

$$\Psi_\mathcal{B}[\text{BH}](\varepsilon\tau) = {}_{\Sigma_{\varepsilon\tau}}\langle \text{BH} | \mathcal{E}(\varepsilon\tau, t_0) | \Psi_\mathcal{B} \rangle_{\Sigma_{t_0}}. \tag{4.166}$$

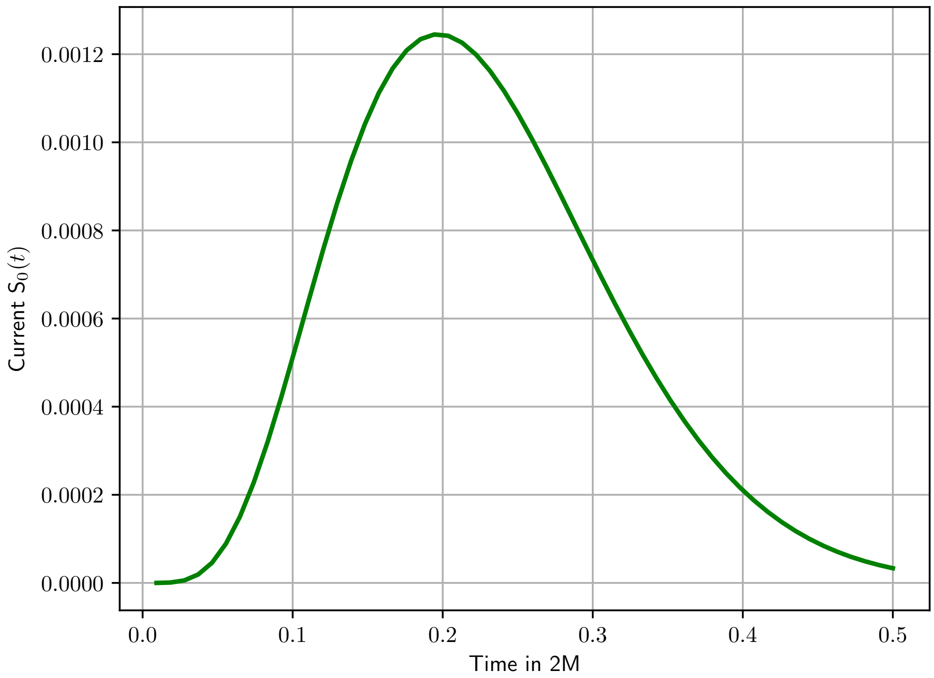

Figure 4.6.: Plot of the probabilistic current.

Note that ε is a smallness parameter. In order to allow for a probabilistic interpretation, the evolution operator is required to satisfy the contraction property $\|\mathcal{E}(\varepsilon\tau, t_0)\| \leq 1$ towards Σ_0. Equivalently, its generator, the Hamilton operator

$$\mathcal{H} = \mathcal{H}_{\text{coll}}(M) + \mathcal{H}_{\text{rad}}(R) + H_{\text{coup}}(M, R) \tag{4.167}$$

needs to be accretive. Here, H_{coup} describes the coupling between the quanta that participated in the gravitation collapse $\langle M|$ (corresponding to $\mathcal{H}_{\text{coll}}(M)$) and the ingoing Hawking radiation $\langle R|$ (corresponding to $\mathcal{H}_{\text{rad}}(R)$) . If a weak coupling regime is assumed, then $\Psi_{\mathcal{B}}[BH](\varepsilon\tau) \approx \Psi_S[M](\varepsilon\tau) \times \Psi_{\text{in}}[R](\varepsilon\tau)$ to leading order in the coupling. The results of our analysis show that $\Psi_{\text{in}}[R](\varepsilon\tau)$ vanishes towards Σ_0. Provided $\Psi_S[M](\varepsilon\tau)$ is sufficiently well behaved, it then follows that $\Psi_{\mathcal{B}}[BH]$ vanishes at the border Σ_0 of physical space-time. Hence, our calculation provides some foundation to the black hole final state proposal by [Horowitz and Maldacena, 2004] which has been suggested as a candidate to solve the black-hole information paradox. We want to mention that this proposal is not a very fruitful approach, since there is no physical motivation except the closing of the wanna-be information sink.

However, we showed that this arises naturally without imposing boundary conditions, the wave functional of the black-hole interior satisfies a trivial Dirichlet boundary condition. This boundary condition restricts information configurations to $\mathcal{B}$ in which information processing is described by a contraction semi-group $\mathcal{E}(t, t_0)$ with an accretive generator $\mathcal{H}$. This generator shares properties with Dirichlet operators: close to the boundary the dynamics trivialises to free evolution, corresponding to a geometrically induced asymptotic freedom, and information processing is only supported away from the boundary.

The whole derivation of charge conservation could also be performed in the Heisenberg picture which is shown in appendix G.

4.5. Quantum probing of the Kasner space-time

In chapter 3.2 we motivated that the Kasner space-time is of paramount importance because of the BKL conjecture, moreover, Kasner space-times serve also as model for an early stage of the universe which is followed by an inflationary stage [Kofman et al., 2011]. One of the defining properties are that it is anisotropic and homogeneous on a given scale. For our universe, satellite measurements hint towards a flat universe [de Bernardis et al., 2000, Gomero et al., 2016]. This unusual flatness - known as the flatness problem - is one of the naturalness problems of cosmology [Guth, 1981]. This flatness is very well explained by the inflationary paradigm; the fast expansion in the inflationary epoch has flatten our universe such that it is consistent with observational data.

However, the space-time describing inflationary scenarios (for example de Sitter space) are bothered by the presence of an initial singularity. This type of singularity is different from the black hole singularity because the degrees of freedom will inevitably come from a coincidence limit, i.e. all fields are collected at one specific point (cosmological singularity) where observables like the energy density diverge. A remedy can be found in an anisotropic pre-inflationary phase which does not support a coincidence limit, because vacuum solutions do not support such a limit due to the vanishing Ricci tensor. In order to get a de Sitter phase (which describes inflationary space-times)

$$g = -dt \otimes dt + e^{2H_0 t} \left(dx \otimes dx + dy \otimes dy + dz \otimes dz \right) \tag{4.168}$$

with Hubble parameter H_0, out of Kasner we need an isotropisation which has been proposed by [Gümrükçüoğlu et al., 2008]. In the presence of a cosmological constant we find a relation for the scale factors $a_i(t)$ to be

$$a_i(t) = a_i^{\text{in}} \left(\sinh(3H_0 t)\right)^{\frac{1}{3}} \left(\tanh(\tfrac{3}{2}H_0 t)\right)^{p_i - \frac{1}{3}} . \tag{4.169}$$

Here, each coordinate has a different scale factor, e.g. $a_x(t)$, $a_y(t)$, and $a_z(t)$. The characteristic time scale of isotropisation is given by H_0 [Gümrükçüoğlu et al., 2008]. We see from (4.169), that for small times, $t \ll H_0^{-1}$, the scale factors exhibit Kasner like behaviour $a_i(t) \sim a_0^{\text{in}}\, t^{2p_i}$. After crossing the scale, for $t \gg H_0^{-1}$ the scale factors approach a de Sitter behaviour $a_i(t) \sim a_0^{\text{in}} \exp(H_0 t)$ and the spacetime becomes isotopic. A preinflationary Kasner epoch is consistent with the number of e-folds [Kofman et al., 2011].

Nevertheless, the question arises whether or not the Kasner background provides a consistent evolution for quantum fields, i.e. it is quantum complete.

This section discusses only the ground state analysis, that is, the evolution of a Gaussian wave-functional. In appendix F, we show that a Schwarzschild analysis can be connected to a Kasner type-D solution. Hence, non-Gaussian fluctuations and excitations will follow the same schemes as in the previous section and do not necessarily be repeated.

4.5.1. Ground state analysis

Our aim is to check consistency for a scalar field on a general Kasner background. Usual argument against inflation is the singularity at $\mathcal{I}^-$ (past infinity). The past infinite incompleteness poses a severe threat to the inflationary scenario. Here, we caught the idea of [Kofman et al., 2011] and proposed an anisotropic preinflationary phase which is given by a Kasner space-time. This family of space-times has an initial singularity but quantum completeness of Schwarzschild is indicative that these space-times might be quantum complete as well.

Our space-time is given by the Kasner metric

$$g = -dt \otimes dt + t^{2p_1} dx \otimes dx + t^{2p_2} dy \otimes dy + t^{2p_3} dz \otimes dz \tag{4.170}$$

with Kasner exponents fulfilling the Kasner sphere ($\sum_i p_i^2 = 1$) and plane ($\sum_i p_i = 1$) condition. Due to the Bianchi I form of the space-time (4.168) the metric components are purely time-dependent and the hypersurface is conformally flat (vanishing Cotton-Bach tensor); one huge advantage is its easy form, because the determinant of the full metric is up to a sign identical to the determinant of the hypersurface Σ: $\det(g)(t) = -\det(g_\Sigma)(t) = t^2$.

In the proceeding we will see how this simplifies calculations.
Our probing device is a free scalar field $\Phi(x)$ with mass m given by the action

$$S = -\frac{1}{2}\int d^4x\sqrt{-\det(g)}\left\{\partial_\mu\Phi(x)\partial^\mu\Phi(x) + m^2\Phi^2(x)\right\}. \tag{4.171}$$

In order to describe the inflationary paradigm correctly, we would have to analyse a spatial potential which fulfils some conditions, for example a slow-roll condition. For now we will restrict ourself to test whether the Kasner singularity could harm scalar quantum fields. Note the scalar field case is totally sufficient because in the most promising attempts the inflaton field is a scalar field.

Following the steps mentioned in 4.1.3 we construct the Hamilton operator (4.57) and the canonical commutation relation (4.58) for the field $\Phi(x)$ and its conjugate momentum $\pi(x)$. Again the conjugate momentum is identified with the functional derivative (4.59). When we explicitly plug the Kasner metric into the formula of $H[\Phi](t)$ we get

$$H[\Phi](t) = \frac{1}{2}\int d^3x\left[\frac{1}{t}\frac{\delta^2}{\delta\phi(x)^2} + t\left(\sum_i\frac{(\partial_i\Phi(x))^2}{t^{2p_i}} + m^2\Phi^2(x)\right)\right]. \tag{4.172}$$

Following the Schwarzschild treatment of the last subsection we are interested in the most divergent contributions to the Hamilton operator which is the kinetic contribution due to the BKL conjecture.

The p_i are not linearly independent, they can be arranged such that $p_1 < p_2 < p_3$

$$-\frac{1}{3} \leq p_1 \leq 0 \tag{4.173}$$

$$0 \leq p_2 \leq \frac{2}{3} \tag{4.174}$$

$$\frac{2}{3} \leq p_3 \leq 1. \tag{4.175}$$

We can find special choices where only one exponent is different from the others; the most special is the choice $(0, 0, 1)$ which has only an artificial singularity. This choice is nothing but a part of Minkowski space, the so-called Milne universe which is an extremal and outstanding case of Kasner.

In principle we could do the analysis in the parameter λ because all exponents are depending on each other but we stay in the description of the p_i in order to make comparison with Schwarzschild easy. From our Schwarzschild analysis we know H_Π does not affect consistency of the evolution.

What about the new polynomial part $\mathcal{P}[\Phi]$? The divergence goes the worst (worst means here the lowest possible power or the most divergent) with $1/t$ which is equal of setting one exponent to one in which case we recover Minkowski.

More interesting cases are the configurations with a real singularity. The polynomial part contributes with t^{1-2p_i} which is less dominant than the kinetic part. Respecting the ordering of the Kasner exponents we are left with

$$\mathcal{P}[\Phi](t) \xrightarrow{t\to 0} \frac{1}{t^{2p_3-1}}(\partial_z\Phi)^2 + \left(\frac{1}{t^{2p_2-1}}(\partial_y\Phi)^2\right) \tag{4.176}$$

where the contribution of the last term strongly depends on the choice of p_2. Our ground states are defined as in (4.64) and (4.158) with the specific measures in the integration. Schrödinger's equation (4.159) yield an explicit equation for the kernel $K(t,x,y)$ in the fashion of (4.68)

$$i\partial_t K(t,x,y) = \frac{1}{t}\int d^3z K(t,x,z)K(t,z,y) - \frac{1}{t^{2p_3-1}}\partial_z^2\delta^{(3)}(x-y), \tag{4.177}$$

where the determinant of the metric is just given by t^2 and consequently the kernel is transformed like $K \to \det(g_\Sigma)K$. As expected, we get an integro-differential equation which can be transformed into a Riccati equation via a spatial Fourier transform (4.71) to get (4.57) for the Kasner case.
A closed form for the solution cannot be given unless the value of the p_i is specified. Calculations show that modulo constants, the time-dependence of the kernel and therefore the norm always scales like in the Schwarzschild case. For all but $(0,0,1)$ we get the same result and the Kasner space-time family is quantum complete.

4.5.2. Mode functions

For Kasner space-times we restrict ourself to show that the mode functions scale similar with t to the harmonics of Schwarzschild from which we determine a verdict about excited states. The kernel of both Kasner and Schwarzschild are very similar; the reason is they are connected by Huygens's principle to the solutions of the wave equation $\hat{\varphi}(k,t)$ (which are similar close to the singular hypersurface). For the kernel we find

$$\hat{K}(k,t) = -\frac{i}{\sqrt{-g}}\partial_t \ln(\hat{\varphi}(k,t)). \tag{4.178}$$

The (Fourier transformed) harmonics on a Kasner background, to which we wish to apply (4.178), is for small times $t \to 0$ of the form:

$$\hat{\varphi}(\vec{k}, t) = c_1(\vec{k}) \ln\left(\frac{t}{t_0}\right) + c_2(\vec{k}), \tag{4.179}$$

with momentum dependent constants $c_{1,2}$. We find the modes to be logarithmically divergent, which is the same degree of divergence we get for the Schwarzschild harmonics (4.115). Due to the BKL conjecture and the fact that the Schwarzschild metric approaches a Kasner type-D close to the singularity, we would have expected to get similar results. With (4.179) and (4.178) we may once more derive an asymptotic equation for the Kernel that will be given by:

$$\hat{K}(\vec{k}, t) = \frac{-i}{t^2|\ln(t)|}\left(1 - \frac{|d(k)|}{|\ln(t)|}\right), \tag{4.180}$$

where $d(k)$ is a complex constant in time. Let us emphasise that the kernel function is similar to the Schwarzschild kernel and it meets the criterion we have stated for a complete evolution (4.139). Therefore, close to the singularity the wave-functional shows the same behaviour compared to the evolution of a scalar field on Schwarzschild which again reflects the BKL conjecture.

We could perform the analysis of excitations similar to Subsection 4.3.2 which would conclude with the same result. It is important to say that the transformation from Schwarzschild to Kasner time is $t \to t^{2/3}$ which we see immediately is reflected in the time dependence of the kernels (4.180) and (4.94). Here, we saw explicitly that both space-times behave similar. Kasner spaces are hence quantum complete with respect to excitations and additionally self-interactions vanish because of the BKL conjecture; the conjugate momentum part dominates over the gradient and polynomial part because $\partial_t \phi \to \Pi$. BKL states that the time gradient dominates so must the conjugate momentum created from the time gradient.

5 Conclusion

In this chapter we briefly wrap up all relevant findings of this thesis and give an overview about future directions in this field of research. The main purpose was to develop a notion of completeness for quantum field theory on curved space-times which can be used to investigate singular dynamical and also static space-times. Inspired by the work of Horowitz and Marolf, we adapted their completeness criterion to situations where the only adequate description is in terms of quantum field theory on curved space-times. The central point in their notion was the self-adjointness of the Hamilton operator or the unitarity of time evolution which would not be appropriate in a set-up where emission and absorption of particles occur; these processes happen in dynamical space-times where friction-like terms perturb self-adjointness of the Hamilton operator and hence unitarity of time-evolution.

In this regard, the pair unitarity and self-adjointness is replaced by the more generic pair contractivity and accretiveness which leads to a decreasing norm towards the singularity; the contraction group hence guarantees that the physical space-time is detached from the singular hypersurface and that observables stay finite. We require that a consistent evolution of quantum field theory on a specific background should respect the probabilistic interpretation of quantum theory, i.e. that states are normalisable.

To achieve our goal we chose the Schrödinger representation of quantum field theory which is most effective as framework in order to formulate the criterion: first, this the wave-functions are explicitly time-dependent, and second the formulation is close to quantum mechanics and uses a functional generalisation of the Hamilton operator. We propose as a criterion for quantum completeness that the probability amplitude shall respect the probabilistic interpretation of quantum theory, i.e. if once normalised it should not exceed its initial value. Moreover, the singular hypersurface should not get populated by any field configuration. In other words, quantum field probes are detached from the geometric singularity and the resulting physical space-time is regular.

The quantum completeness analysis is a Cauchy problem: we start with initial conditions given by a regular and renormalised quantum field theory on an arbitrary hypersurface of the space-time; we observe the system under time-evolution and study the behaviour of this field configuration as well as of observables when they approach the singularity.

Taking Schwinger's argument that, in the presence of an external source, quantum field theory will experience a depletion of the vacuum persistency, we could interpret our results similar: in some sense, the classical background plays the role of the external source because it is not resolved into dynamical degrees of freedom. The loss of probability which might occur is explained such that it is transferred to the background but not through the singularity.

For the Schwarzschild metric as well as Kasner space-times we get consistent evolutions since both norms of the ground state wave-functionals (as well as the functionals on its own) go monotonically towards zero. This leaves two ways of interpretation: either Schwinger's suggestion, or it could have been the decaying ground state which is not stable under the particle production of the source.

Therefore we tested excitations with on-shell particles which led to the result that also excitations respect completeness. The ground-state amplitude stays stable under deformation of the Gaussian shape which emphasises the validity and the robustness of our criterion.

We investigated the black-hole singularity, assuming that the geometry is given by the interior Schwarzschild metric and analysed the probability amplitude of the wave-functional and the stress-energy tensor close to the singular hypersurface. Finally, we found quantum completeness inside the black hole and stability of the ground state under on-shell perturbations. In other words, all excited states with respect to the ground state admit a consistent evolution, i.e. the quantum fields do not experience the presence of a singularity and quantum fields cannot reach the end-point, because the singular hypersurface is not populated with any field configuration. This conclusion holds so far for free field theory.

The analysis of the Schwarzschild space-time unveils further that only the free part will matter. All included (polynomial) self-interactions do not have the power of destroying the conclusion. Intuitively one could have thought that an initially weakly coupled theory evolves into a strong coupling regime. This is, however, a wrong conclusion because the kinetic and the potential part of the Hamilton operator are sourced differently; close to the singular hypersurface, self-interaction terms vanish while the kinetic part dominates. We gave a strong argument supporting this result coming from the Kasner space-times.

Although it might look rather counterintuitive, however, the analysis of Kasner space-times shows instead that the vanishing of self-interations can be motivated by the conjecture of Belinskii, Khalatnikov, and Lifshitz. Generalised Kasner space-times show exactly the same behaviour, that is, the wave-functional for the ground-state goes to zero close to the singularity. The Kasner class of metrics include one specific Kasner type-D space-time which is connected to the Schwarzschild space-time. Close to the singularity Schwarzschild and Kasner type-D are more or less similar because generalised Kasner obeys the BKL conjecture, which says, in the vicinity of spacelike singularities time-derivatives are favoured compared to spatial gradients. With this conjecture at hand the fading of the self-interaction is not more than an immediate consequence of BKL's conjecture because the kinetic part of the Hamilton operator dominates, i.e. the conjugate momentum. We found an additional confirmation of this conjecture by our results on quantum completeness for general Kasner space-times.

We think that quantum completeness has physical relevance and presents a physical characterisation of space-time singularities. In contrast to the singularity theorems of Hawking and Penrose this criterion is related to a physical measurement process with respect to the field configurations.

The behaviour of observables, like the expectation value of the stress-energy tensor, towards the geometric singularity support our argument and give more insight into ongoing physics. The anisotropy of both space-times pulls the fields apart. While the fields approach the singularity the spacing between them becomes larger due to the divergent prefactor in one spatial coordinate. The stress-energy tensor applied on the functional states decreases towards the singularity and renders quantum completeness sacrosanct. The normalisability condition turns out to be the major criterion prevailing over the geometrical singularity.

Decreasing wave-functionals show that no field configuration can live on the singular hypersurface. A vanishing probability is consistent with a vanishing stress-energy tensor. The singular hypersurface is void of quantum fields, there can be no contribution from the field content. Therefore, the stress-energy tensor shows a behaviour in total agreement with our interpretation. In the Heisenberg picture, we derive a similar result if the measurement process for the stress-energy tensor is considered. The vertex density for the measurement depletes and the singular hypersurface is free from quantum fields.

We think that this view on spacelike singularities is valid and consistent and should be evaluated further. We sketched a procedure how to generalise the whole argument by looking at the kernel in the exponent but we expect that some geometrical properties can be used in order to formulate mathematically rigorous quantum regularity theorems. This should be in the fashion of the theorems of Hawking and Penrose; it is plausible that they can complement or weaken the physical significance of the singularity theorems, though they will still be a valid diagnostic tool for manifolds in general relativity; just their physical relevance might be doubted.

The singularity in both scenarios, Kasner and Schwarzschild, does not affect the consistency of the probing quantum fields. It might have been guessed that the concept of a singular structure is not favoured by nature. Quantum completeness of black holes strongly supports this because if we cannot reach the singular configuration by quantum fields the formation of such a configuration seems to be against the principles of nature (or at least quantum theory).

Our results are on the verge to open a new field of research. Not only could it be used as classification criterion for manifolds, it serves as a technique to tackle time-dependent problems in quantum field theory. Moreover, the formulation could be extended to field theories which have the property to be in a confining phase, for example quantum chromodynamics. Our future research is devoted also to inflationary and other physically relevant space-times but with the focus on learning about the very structure of gravitation and its interplay with quantum theory. A possible extension is to replace the geometric sector by a minisuperspace and quantise it in order to see the influences of the background's dynamical resolution. It would also be possible to couple existent quantum gravity models to the field theoretic sector in order to bring clarity in this vivid field of research.

Quantum completeness of black holes change the way we think about black holes. Since there is no problem with its geometrical singularity, we see that quantum field theory is protected from the influence of the geometrical singularity. Moreover, in this regard quantum field theory seems to protect general relativity. This has severe consequences with respect to the information paradox. We know that the black hole interior conserves charges, or information, and there is no leakage through the singularity. Moreover, the singular hypersurface bordering on the regular physical manifold is dynamically detached and could be interpreted as a Dirichlet boundary. Consequently, the black hole itself preserves the information which has fallen inside. The only way to destroy information can only be due to the Hawking process or to some effect outside the black hole; the interior is safe.

Quantum completeness is a viable criterion to probe consistency of quantum theories on curved manifolds. Moreover, it has physical relevance and opens a wide window of opportunity for studies concerning the interdependence of general relativity and quantum field theory.

A
Fourier transformation

Here, we present some basics on Fourier transformations on curved space-times taken from [Hörmander, 1990, Hörmander, 2009], [Reed and Simon, 1975] and [Simon, 2015a]. In general it is a very cumbersome topic and it is not even guaranteed that a Fourier transform can be defined.

Usually, Fourier transforms are defined on Schwartz spaces $\mathcal{S}(\mathbb{R}^\nu)$ (for a definition cf. [Reed and Simon, 1975]) in the following [Simon, 2015a]

Definition 16. *A Fourier transform of a function* $f \in \mathcal{S}(\mathbb{R}^\nu)$ *is given by*

$$\hat{f}(k) = \left(\frac{1}{2\pi}\right)^{\frac{\nu}{2}} \int \exp(-ikx) f(x) d^\nu x \tag{A.1}$$

and the inverse Fourier transform by

$$\check{f}(k) = \left(\frac{1}{2\pi}\right)^{\frac{\nu}{2}} \int \exp(ikx) f(x) d^\nu x. \tag{A.2}$$

This is the general formula for Fourier transformations. When we have a curved manifold the measure is non-trivial because the determinant is a function of the coordinates as well as the harmonic functions do not need to form a complete set. The trick for the measure is to see $\mu(x)$ as a distribution which brings us in the comfortable situation of defining the prescription [Simon, 2015a]

$$\hat{\mu}(k) = \left(\frac{1}{2\pi}\right)^{\frac{\nu}{2}} \int \exp(-ikx) d\mu(x). \tag{A.3}$$

We can use the convolution formula of the Fourier transform [Reed and Simon, 1975]

$$\widehat{f * g} = (2\pi)^{\frac{\nu}{2}} \hat{f} \cdot \hat{g} \tag{A.4}$$

or conversely,

$$\widehat{f \cdot g} = \left(\frac{1}{2\pi}\right)^{\frac{\nu}{2}} \hat{f} * \hat{g}. \tag{A.5}$$

The star $*$ denotes the usual convolution. These formulae turn out to be important when solving the Riccati equation. In our case, the transformation of the measure will play a crucial role.

A Fourier expansion consists of a complete set of functions. One can see it in the sense that one decomposes arbitrary functions into sums of eigenfunctions for translations [Hörmander, 1990]. In fact harmonic functions $u(x)$ are defined

Definition 17. *A real-valued function, u, on a region Ω, is called harmonics if u is $\mathcal{C}^2$ and*

$$\Delta u(x) = 0 \tag{A.6}$$

In generically curved space-times a natural expansion will not be given through the exponential function. What we have to use instead are the harmonic functions of the curved space-time's Laplace operator. For spherically symmetric space-times it can be performed by the spherical harmonics $Y^l_m(\vartheta, \varphi)$ at least for the angular part which are the harmonics of the angular Laplace-Beltrami operator.

In order to give an example we take the case of the Schwarzschild metric, under consideration of the conformal flatness of the hypersurfaces, we could use the plane wave expansion in equation (4.72)

$$f(t, z) = \int dk_r dk_\vartheta dk_\varphi e^{i2\pi(rk_r + \vartheta k_\vartheta + \varphi k_\varphi)} \widehat{f(t, k)}. \tag{A.7}$$

We work in the coordinate neighbourhood of the Schwarzschild space-time (3.1) given by the coordinate vector $(t, r, \vartheta, \varphi)$. The angular coordinates could here be interpreted as arc length in the coordinates ϑ and φ.

B Riccati differential equation

Equation (4.3.1) is called a Riccati differential equation. Finding a solution is very hard in general case. Some methods in order to solve this type of differential equations have been developed, for example the homotopy analysis method [Odibat and Momani, 2008] or Adomian's decomposition method [Abbasbandy, 2006]. If one special solution is found, it is possible to construct the full space of solutions [Reid, 1972].

Although here, we will only restrict to the differential equation, we want to mention that there is an algebraic version of this equation (see for example the book of Leicester and Rodman [Lancaster and Rodman, 1995]). Let $f(x)$ be a function, the generic form of this type of non-linear differential equation is

$$\frac{df}{dx} = a(x)f^2(x) + b(x)f(x) + c(x) \tag{B.1}$$

with $a(x)$, $b(x)$, and $c(x)$ arbitrary functions. Those functions are of course known. For our specific equation (4.3.1) we could identify these functions as:

$$a(t,k) = -i\sqrt{\det(g)}\det(q), \tag{B.2}$$

$$b(t,k) = -\partial_t \ln(\det(q)), \tag{B.3}$$

$$c(t,k) = i\frac{\sqrt{g_{tt}}}{\sqrt{\det(q)}}\Omega^2(t,k). \tag{B.4}$$

Under special circumstances, one can perform a transformation into a ordinary differential equation [Friedman, 1981]. This is achieved by guaranteeing that no singular transformation has been made. To make it clear, none of the three given functions $a(x)$, $b(x)$, and

$c(x)$ should become zero in the domain of the solution f. Otherwise it should be clear that the inverse transformation is ill-defined [Gradshteyn and Ryzhik, 2014].

The first step is to redefine $f(x)$ by the solution of the linear second order differential equation $F(x)$

$$f(x) = -\frac{1}{a(x)} \frac{d}{dx} \ln(F(x)). \tag{B.5}$$

Having done this, it is easy to verify that by inverting (B.5) and plugging into

$$\frac{d^2F}{dx^2}(x) - \left[b(x) + \frac{d}{dx} \ln(a(x))\right] \frac{dF}{dx}(x) + (a(x)c(x))\, F(x) = 0 \tag{B.6}$$

yields (B.1). For this type of differential equation we have more and more powerful methods [Polyanin and Zaitsev, 1995] to solve and to perform an asymptotic expansion. This makes it easier to go for the solution. It is possible to reduce the above equation further to an equation resembling a harmonic oscillator with complicated frequency term. This should, for our purposes, be enough in order to find a solution.

C
Publication: Classical versus quantum completeness

Here, we present our first publication in Physical Review D which is the basis of this thesis. It includes the idea of the formalism and shows quantum completeness of the black hole space-time with respect to the ground state formulation.

PHYSICAL REVIEW D **91,** 125028 (2015)

Classical versus quantum completeness

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(Received 22 April 2015; published 22 June 2015)

The notion of quantum-mechanical completeness is adapted to situations where the only adequate description is in terms of quantum field theory in curved space-times. It is then shown that Schwarzschild black holes, although geodesically incomplete, are quantum complete.

DOI: 10.1103/PhysRevD.91.125028 PACS numbers: 03.65.Db, 04.20.Dw, 04.70.-s, 11.10.-z

I. INTRODUCTION

Completeness is a very important concept in classical and quantum physics. The classical motion on a half-line is called complete at the end point if there are no initial conditions such that the trajectory runs off to the end point in a finite time. If the potential satisfies certain regularity conditions, then the classical motion is complete at the end point if and only if the potential grows unbounded from above near the end point [1]. In general relativity, a space-time is called geodesically complete if every maximal geodesic is defined on the entire real line. If the space-time is timelike or null geodesic incomplete, it is said to be singular [2]. The physical relevance of this geometrical notion is provided upon identifying geodesics with trajectories of free test particles. In quantum mechanics on a half-line, a time-independent potential is called quantum-mechanically complete [1] if the associated Hamiltonian is essentially self-adjoint on the space of C^∞ functions of compact support on the half-line with the origin excluded.

Horowitz and Marolf [3] showed that there are geodesically incomplete static space-times, with timelike curvature singularities, which are quantum-mechanically complete. Their work stimulated a lot of research concerning geodesically incomplete but quantum-mechanically complete spacetimes, e.g. [4–6]. As a working analogue, they suggested the nonrelativistic hydrogen atom. The classical motion of the electron in the Coulomb potential is incomplete at the origin, because the potential is bounded from above near the origin and thus the origin can be reached by the electron in a finite time. The Coulomb potential is, however, quantum-mechanically complete when probed by the non-relativistic bound-state electron. In other words, the classical singularity of the Coulomb potential is not reflected in any observable related to the bound-state electron.

Quantum field theory in a static, globally hyperbolic space-time allows us to define a consistent quantum theory for a single relativistic particle, where the energy of each one-particle state is equal to that of the corresponding classical field [7]. Horowitz and Marolf [3] showed that this is still the case for certain static space-times with timelike singularities. Their result is based on a work by Wald [8,9], who proved that the problem of defining the evolution of a Klein-Gordon scalar field in an arbitrary static space-time (with arbitrary singularities consistent with statics) can be reformulated as the problem of constructing self-adjoint extensions of the spatial part of the wave operator.

For a general time-dependent space-time, there is no consistent quantum theory of a single free particle, and the only adequate description is in terms of quantum field theory. This requires to study the evolution of classical test fields in a singular space-time. In static space-times, the evolution of quantum fields is unitary and represents an endomorphism of the physical Hilbert space. In particular, unitarity preserves state normalization. If dynamical space-times are treated as external backgrounds, the quantum theory does not require a unitary evolution [10]. Therefore, the notion of quantum-mechanical completeness needs to be adapted to include this case.

In discussing geodesic completeness, it usually suffices to consider geodesics defined on $(0, t_0]$, right end points can be treated similarly. A convenient topological criterion for the inextendibility of a geodesic $\gamma(t), t \in (0, t_0]$ is the following: There is a parameter sequence $\{t_n\} \to 0$ such that $\{\gamma(t_n)\}$ does not converge. As is well known, geodesic parametrizations have geometric significance. If a curve has a reparametrization as a geodesic, it is called a pregeodesic. In particular, any spacelike or timelike curve is pregeodesic if and only if its reparametrization by its arc-length yields a geodesic. A spacelike or timelike pregeodesic $\alpha(t), t \in (0, t_0]$ is complete (to the left) if and only if it has infinite length [11].

We call a globally hyperbolic space-time quantum complete (to the left) with respect to a free field theory if the Schrödinger wave functional of the free test fields can be normalized at the initial time t_0 and if the normalization is bounded from above by its initial value for any $t \in (0, t_0)$. Note that neglecting backreaction is no severe restriction here since if backreaction becomes important, the question of whether a previously given space-time is quantum complete becomes obsolete.

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Intuitively, the notion of quantum complete space-times refers to the following: A space-time background can be considered as an external source coupled to quantum fields. This coupling is consistent provided that the norm of the vacuum-to-vacuum transition amplitude does not exceed the corresponding norm in Minkowski space-time, i.e. unity. There is no conceptual problem if the transition is less probable than in Minkowski space-time, although the evolution is then nonunitary. In this case, the ground state is not persistent, but its norm is reduced by transferring probability to the space-time background, which is not resolved into dynamical degrees of freedom. If, in contrast, the transition is more probable, then unitarity is violated in such a way that the quantum theory becomes meaningless. This intuition is stated more precisely in our definition of quantum completeness, which is based on the functional Schrödinger approach to quantum field theory. As opposed to an asymptotic framework pertinent to a scattering description, the functional Schrödinger approach allows us to analyze unitarity violations occurring during a finite amount of time, and, in particular, during the time interval $(0, t_0]$.

For a Schwarzschild black hole, a Cauchy hypersurface is given by $\{t_0\} \times \mathbb{R} \times S^2$, where $t_0 \in (0, 2M)$ and M denotes the black hole mass. It follows that the black hole interior is globally hyperbolic and foliated by smooth spacelike Cauchy hypersurfaces [12]. The purpose of this article is to show that the interior of a Schwarzschild black hole is quantum complete, although it is geodesically incomplete.

II. SETUP

We briefly review the functional Schrödinger formulation for quantum field theory in generic space-times, when backreaction can be neglected (for a detailed discussion in Minkowski space-time, see [13]). This formulation will prove to be efficient for investigating qualitative features such as the stability of ground states and the quantum (in)completeness of generic space-times.

Due to a theorem by Geroch [14], a globally hyperbolic space-time is diffeomorphic to $\mathbb{R} \times \Sigma$, and foliates into hypersurfaces $\Sigma_t, t \in \mathbb{R}$. In the $(1+3)$-split formulation, the classical theory for a free scalar field of mass m is given by the Hamiltonian

$$H = \int_{\Sigma_t} \mathrm{d}\mu(x)(N_\perp \mathcal{H}^\perp + N_\parallel{}^a \mathcal{H}^\parallel{}_a). \quad (1)$$

Here, $\mathrm{d}\mu(x) \equiv \mathrm{d}^3x\sqrt{\det(q)}$, with q denoting the spatial part of the metric, $\mathcal{H}^\parallel{}_a = \pi\partial_a\Phi/\sqrt{\det(q)}$, where $\pi = \partial\mathcal{L}/\partial\dot{\Phi} = \sqrt{\det(q)}(\partial_t\Phi - N_\parallel^a\partial_a\Phi)/N_\perp$ denotes the canonical momentum field, and

$$\mathcal{H}^\perp = \frac{1}{2}\left[\frac{1}{\det(q)}\pi^2 + q^{ab}\partial_a\Phi\partial_b\Phi + (m^2 + \zeta R)\Phi^2\right]. \quad (2)$$

Here, all tensors are pulled-back to the hypersurface Σ_t, and ζ is a numerical factor representing the nonminimal coupling to gravity. Adapting the space-time coordinates to the foliation, $N_\parallel = 0$ and $N_\perp = \sqrt{-g_{tt}}$.

Each hypersurface Σ_t is equipped with a Fock space. In the Schrödinger representation, the basis of this Fock space is constructed from the time-independent operator $\Phi(x)$. Its spectrum contains the classical fields $\phi(x)$ as eigenvalues [13]. The ϕ representation of an arbitrary state $|\Psi\rangle$ in the Fock space is a (nonlinear) wave functional $\Psi[\phi](t)$. For the momentum field π canonically conjugated to Φ, the functional version of the quantization prescription is given by $\pi(x) \to -\mathrm{i}\delta/\delta\Phi(x)$.

$\Psi[\phi](t)$ satisfies a functional generalization of the Schrödinger equation,

$$\mathrm{i}\partial_t\Psi[\phi](t) = H[\Phi](t)\Psi[\phi](t), \quad (3)$$

$$H[\Phi](t) = \int_{\Sigma_t} \mathrm{d}\mu(x)\mathcal{H}(\Phi(x); t, x), \quad (4)$$

where $H[\Phi](t)$ denotes an operator valued functional constructed from the Hamilton density

$$\mathcal{H} = \frac{1}{2}\left[\frac{\sqrt{-g_{tt}}}{\det(q)}\frac{\delta^2}{\delta\Phi^2} + q^{ab}\partial_a\Phi\partial_b\Phi + (m^2 + \zeta R)\Phi^2\right]. \quad (5)$$

Note that any explicit time dependence is due to the space-time geometry, which can be thought of as an external source nonminimally coupled to the quantum field.

Wave functionals are normalized in the usual sense,

$$\|\Psi\|^2(t) = \int \mathrm{D}\phi\Psi^*[\phi](t)\Psi[\phi](t), \quad (6)$$

where $\mathrm{D}\phi$ denotes the measure over all field configurations in Σ_t. Stability of the state populated with $\phi(x)$ requires that the norm of the wave functional is time-independent. This corresponds to a unitary evolution.

On a dynamical space-time, considered as an external background, however, the evolution is not required to be unitary; i.e., $H[\Phi](t)$ needs not be a self-adjoint operator on the space of wave functionals. Intuitively, probability can be lost to the background (like for dissipative systems when the interaction causing the friction is not fully resolved in the participating degrees of freedom). Consistency of the dynamics is more subtle in this case. Let $\|\Psi[\phi]\|^2(t_0)$ denote the probability density (with respect to the space of field configurations) at the initial hypersurface, and consider the interval $(0, t_0]$ with zero marking the left end point. We call the evolution consistent, even if it violates unitarity, provided that $\|\Psi[\phi]\|^2(t) \leq \|\Psi[\phi]\|^2(t_0), \forall t \in (0, t_0)$. Intuitively, probability must not be gained from a background which is not resolved in dynamical degrees of freedom. If the above consistency relation is violated, then backreaction effects are relevant, and the original space-time geometry is obsolete.

For the time-dependent ground state, a generalized Gaussian ansatz is motivated following the example of the harmonic oscillator in quantum mechanics:

$$\Psi^{(0)}[\phi](t) = N^{(0)}(t)\mathcal{G}^{(0)}[\phi](t),$$
$$\mathcal{G}^{(0)}[\phi](t) = \exp\left[-\frac{1}{2}\int_{\Sigma_t} \mathrm{d}\mu(x)\mathrm{d}\mu(y)\phi(x)K(x,y,t)\phi(y)\right]. \tag{7}$$

Substituting the ansatz (7) in the functional Schrödinger equation (3) gives for the ϕ-independent factor $N^{(0)}(t)$ an evolution equation that can be directly integrated,

$$N^{(0)}(t) = N_0 \exp\left[-\frac{\mathrm{i}}{2}\int_{t_0}^{t}\mathrm{d}t'\int_{\Sigma_{t'}}\sqrt{-g_{tt}}\mathrm{d}\mu(z)K(z,z,t')\right] \tag{8}$$

while the evolution for the kernel $K(x,y,t)$ is described by a ϕ-dependent nonlinear integro-differential equation,

$$\frac{\mathrm{i}\partial_t[\sqrt{\det(q)}(x)\sqrt{\det(q)}(y)K(x,y,t)]}{\sqrt{\det(q)}(x)\sqrt{\det(q)}(y)} = \int_{\Sigma_t}\sqrt{-g_{tt}}(z)\mathrm{d}\mu(z)K(x,z,t)K(z,y,t) + \sqrt{-g_{tt}}(x)(\Delta - m^2 - \zeta R)\delta^{(3)}(x,y). \tag{9}$$

The spatial part of the Laplace-Beltrami operator is defined as $\Delta \equiv \partial_a[\sqrt{\det(q)}q^{ab}\partial_b]/\sqrt{\det(q)}$, and we use the following convention for the Dirac distribution: $\sqrt{\det(q)}(x)\delta^{(3)}(x,y) \equiv \delta^{(3)}(x-y)$.

III. CALCULATION

In this section, we specialize to the interior of Schwarzschild black holes. In the usual Schwarzschild coordinate neighborhood, the Schwarzschild function is given by $h(\tau) = (2-\tau)/\tau$, where $\tau \equiv 2t/r_{\mathrm{g}}$ is dimensionless, and $r_{\mathrm{g}} \equiv 2M$ denotes the Schwarzschild radius ($G_{\mathrm{N}} \equiv 1$). The warped product line element for the Schwarzschild black hole becomes

$$g = -h^{-1}(\tau)\mathrm{d}t^2 + h(\tau)\mathrm{d}r^2 + (\tau r_{\mathrm{g}})^2\mathrm{d}\mathfrak{s}^2/4, \tag{10}$$

where by this normalization, in each rest space $t = \text{constant}$, the surface $r = \text{constant}$ has the induced line element $(\tau r_{\mathrm{g}})^2\mathrm{d}\mathfrak{s}^2/4$, and is thus the two-sphere of radius $\tau r_{\mathrm{g}}/2$ with Gaussian curvature $4/(\tau r_{\mathrm{g}})^2$ and area $\pi(\tau r_{\mathrm{g}})^2$. In this parametrization, the geometry is incomplete to the left, since tidal forces approach infinity along inextendible timelike geodesics as $\tau \to 0$.

Since the Schwarzschild space-time is spherically symmetric, the kernel K introduced in (7) is a function $K(x-y,\tau)$. Our convention for Fourier transforms is

$$K(z,\tau) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\exp(\mathrm{iq(k,z)})\hat{K}(k,\tau), \tag{11}$$

with $q(k,z) \equiv q^a{}_b k_a z^b$. The Fourier amplitudes $\hat{K} \equiv \tilde{K}/\det(q)$ satisfy a Riccati equation,

$$\mathrm{i}\partial_\tau\tilde{K}(k,\tau) = \sqrt{\det(g)}\frac{r_{\mathrm{g}}}{2}[(\det(q))^{-1}\tilde{K}^2(k,\tau) - \Omega^2(k,\tau)]. \tag{12}$$

The inhomogeneous contribution $\Omega^2(k,\tau) \equiv q^{ab}k_a k_b + m^2$ is just the dispersion relation of the free fields.

The kernel can alternatively be described as follows. Suppose $\varphi(x',t')$ is a solution of the equation of motion for the free fields. It is related to a solution at a later time $t > t'$ by Huygens's principle [15,16],

$$\varphi(x,t) = \int_{t_0}^{t}\mathrm{d}t' h^{-1/2}\int_{\Sigma_{t'}}\sqrt{\det(q)}\mathrm{i}K(x-x',t')\varphi(x',t')\mathrm{d}^3x'. \tag{13}$$

Indeed, a kernel fulfilling Huygens's principle for the time-dependent fields φ is a solution of the kernel equation (9). Moreover,

$$(\Box - \Omega^2(k,\tau))\hat{\varphi}(k,\tau) = 0. \tag{14}$$

Of course, from the solutions of (14) the kernel can be calculated directly,

$$\hat{K}(k,\tau) = \frac{-\mathrm{i}}{\sqrt{|\det(g)|}}\partial_t \ln\hat{\varphi}(k,\tau), \tag{15}$$

but it should be clear that this is a less efficient approach than solving the kernel equation (12). With the kernel representation (15), however, it is straightforward to show that the time dependence of $\|\Psi^{(0)}\|$ is not fictitious, even without solving (14). Using (15) in (8), we find

$$|N^{(0)}(\tau)|^2 = |N_0|^2\exp\left(-\frac{v(\Sigma)}{2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\ln\left|\frac{\hat{\varphi}(k,\tau)}{\hat{\varphi}(k,\tau_0)}\right|^2\right), \tag{16}$$

where $v(\Sigma)$ denotes the time-independent coordinate volume of the hypersurfaces. Furthermore,

$$\|\mathcal{G}^{(0)}\|^2(\tau) = \left(\mathrm{Det}\left(\frac{\det(q)}{\sqrt{\det(g)}}\frac{\mathrm{i}}{2}\frac{W(\hat{\varphi},\hat{\varphi}^*)}{|\hat{\varphi}|^2}\right)\right)^{-1/2}, \tag{17}$$

with $W(\hat{\varphi}, \hat{\varphi}^*) \equiv \hat{\varphi}\overset{\leftrightarrow}{\partial}_t \hat{\varphi}^*$ denoting the Wronskian of the solution and its complex conjugate, and Det is the functional determinant. From this result, we can draw two important immediate conclusions. First, for Friedman space-times, Abel's differential equation identity [17] gives that $\sqrt{|\det(g)|}W(\hat{\varphi}, \hat{\varphi}^*)$ is time independent. As a consequence, $\|\Psi^{(0)}\|$ is time-independent (the time-dependent contributions to (16) and (17) cancel), and the ground state is stable in Friedman space-times. By our definition, Friedman space-times are quantum complete, although they are geodesically incomplete. Second, for a Schwarzschild black hole, the situation is different, because $g^{tt}\sqrt{|\det(g)|}W(\hat{\varphi}, \hat{\varphi}^*)$ is time-dependent in this case. Hence, the ground state cannot be stable, but the Schwarzschild black hole can still be quantum complete (with respect to free fields).

In order to show that Schwarzschild black holes are indeed quantum complete, we transform the Riccati equation (12) for the Fourier amplitudes $\hat{K}$ to a homogeneous, second-order ordinary differential equation in normal form,

$$\partial_\tau{}^2 f(k,\tau) + \omega^2(k,\tau) f(k,\tau) = 0, \tag{18}$$

$$\omega^2(k,\tau) \equiv \frac{r_g{}^2}{16 g_{\theta\theta}(\tau)}(1 - 2g_{tt}(\tau) + g_{tt}{}^2(\tau)) - g_{tt}(\tau) M^2 \Omega^2(k,\tau). \tag{19}$$

The Fourier amplitudes $\hat{K}$ are related to f as follows:

$$\hat{K}(k,\tau) = -\frac{1}{2\det(q)}\partial_\tau \ln(\sigma(\tau) f^2(k,\tau)), \tag{20}$$

with $\sigma(\tau) \equiv -\mathrm{i}M\sqrt{|\det(g)|}/\det(q)$.

The dispersion relation for f is singular at the horizon, $\tau = 2$, and at the classical black hole singularity, $\tau = 0$. For our purposes, it suffices to expand f near $\tau = 0$. Let us first give a quick argument and justify it *a posteriori*. The leading singularity in the dispersion relation around $\tau = 0$ is given by $\omega_0 = 1/(2\tau)$, with corrections $\mathcal{O}(1/\sqrt{\tau})$. Near $\tau = 0$, the dynamics is governed by the background; i.e., the dominant contribution in the dispersion relation is momentum-independent. In this regime, $f(\tau) \to C'\sqrt{\tau}(C + \ln\tau)$, which translates to

$$\mathrm{Im}(\hat{K}(\tau)) \to \frac{-1}{M^3 \sin(\theta)} \frac{1}{\tau^3 |\ln\tau|},$$
$$\mathrm{Re}(\hat{K}(\tau)) \to |\mathrm{Im}(C)| \frac{|\mathrm{Im}(\hat{K}(\tau))|}{|\ln\tau|}, \tag{21}$$

near the black hole singularity. Here, $C, C' \in \mathbb{C}$ are constants of integration. Note that $\mathrm{Re}(\hat{K}(\tau)) \ll \mathrm{Im}(\hat{K})$ near the singularity. The real part is taken into account since the dominant contribution gives a phase factor for $\mathcal{G}^{(0)}$. Using (21) in (8), the normalization $N^{(0)}$ goes to zero like

$$N^{(0)}(\tau) \to |\ln\tau|^{-\frac{1}{2}v(\Sigma)\Lambda}. \tag{22}$$

Of course, this evaluation requires a volume as well as an ultraviolet regularization. We simply introduced a coordinate volume and an ultraviolet cut-off, $v(\Sigma)$ and Λ, respectively, since the regularization details have no impact on the limit $N^{(0)}(\tau) \to 0$ as $\tau \to 0$. For $\mathcal{G}^{(0)}$, we find

$$\mathcal{G}^{(0)}(\tau) = \exp\left(-\frac{1}{2}\mathrm{Re}(\hat{K})(\tau)\int_{\Sigma_\tau} \mathrm{d}\mu(x)\phi^2(x)\right) \tag{23}$$

times an irrelevant phase factor.

It is more rigorous to take all contributions in the dispersion relation into account that are singular at $\tau = 0$,

$$\omega_s{}^2(k,\tau) = \omega_0{}^2(\tau) + \left(k_\measuredangle^2(\theta) + \frac{1}{2}\right)\omega_0(\tau) + \mathcal{O}(\tau^0), \tag{24}$$

where $k_\measuredangle^2 \equiv (\tau r_g)^2 \mathrm{d}\mathfrak{s}^2(k,k)/4$. Introducing the variable $z \equiv \sqrt{1 + 2k_\measuredangle^2}\sqrt{\tau}$, we find

$$f(k,\tau) \to -\frac{z}{2}(J_0(z) - 2\mathrm{i}K_0(\mathrm{i}z)), \tag{25}$$

near the black hole singularity, with J_0 denoting the Bessel function of the first kind, and K_0 denoting the modified Bessel function of the second kind. This combination shows the same behavior near $\tau = 0$ as f subject to the dispersion relation ω_0. The momenta $k_\measuredangle$ in angular directions appear only in an overall factor ≥ 1 and do not modify the dominant behavior near $\tau = 0$.

Therefore, it is safe to conclude that

$$\|\Psi^{(0)}\|^2(\tau) \to |\ln(\tau)|^{-v(\Sigma)\Lambda}(\tau^{3/4}|\ln(\tau)|)^{N(\Lambda)} \to 0 \tag{26}$$

as the black hole singularity is approached. Here, $N(\Lambda)$ denotes the number of momentum modes with $|k| \in [0, \Lambda^{1/3}]$. The limit (26) is our main result. In fact, already $\Psi^{(0)}[\phi](\tau) \to 0$ as $\tau \to 0$; i.e., the wave functional has vanishing support towards the singularity.

Let us stress again that we were interested in examining the quantum completeness of Schwarzschild black holes with respect to free quantum fields. The answer to this question is insensitive to the details of volume and short-distance regularization, both of which are required, in principle.

IV. CONCLUSION AND DISCUSSION

In this article we adapted the notion of quantum-mechanical completeness to situations where the only adequate description is in terms of a quantum theory of fields in generic space-times. We showed that according to the advanced consistency criterion, a Schwarzschild black hole is quantum complete with respect to free scalar fields (in the ground state). Moreover, the wave functional has vanishing support towards the black hole singularity.

There are two types of non-Gaussianities that can be introduced to describe processes associated with deviations from free fields in the ground state. First of all, excitations of the ground state can be considered. It should be clear that the term excitation is strictly appropriate for static backgrounds. In general, excitations will depend on eigenfunctions ϵ of $(\Delta - m^2)$ in the background geometry. Excited states are of the form $\Psi^{(n)}[\phi](\tau) = N_n[\phi, \epsilon](\tau)\Psi^{(0)}[\phi](\tau)$. So excitations are reflected in a (functional) renormalization of $N_0(\tau)$. The difference between the ground state and the excited states is the following: $\Psi^{(0)}[\phi](\tau)$ populates the ground state with field configurations that need not satisfy any on-shell criteria. What matters is the spatial support of the scalar fields and the correlation between two fields as communicated by the kernel function. This is why the completeness concept used here poses a rather strong consistency requirement on the kernel function. In contrast, excited states are sensitive, in addition, to the moderated overlap between an arbitrary field configurations and fields obeying on-shell conditions. Moderation indicates that the overlap is evaluated using $\phi(x)K(x, y, \tau)\epsilon(y)$. Intuitively, excitations show an increasing sensitivity on the on-shell conditions.

Secondly, interactions of the Klein-Gordon field with itself and with other fields can be introduced in the Hamiltonian. In this case, we choose the initial data such that the interactions can be treated in the usual perturbative framework. If the Schwarzschild black hole fails to be quantum complete with respect to interacting fields, then the participating fields necessarily entered a strong coupling regime, because the space-time is quantum complete with respect to free fields.

Perhaps not surprisingly, Schwarzschild black holes are enjoying a clash of completeness concepts. The obvious question is how to qualify the importance of quantum completeness relative to classical completeness. We think that this question is related to the measurement process. Let $\gamma(t), t \in (0, t_0]$ be a geodesic, and $\{t_n\} \to 0$ denote a parameter sequence such that $\{\gamma(t_n)\}$ does not converge. The inextendibility of the geodesic can be observed by measuring any classical observable $\mathcal{O}$ along γ: $\{\mathcal{O}(\gamma(t_n))\} \subset \mathbb{R}$ does not converge. Hence, geodesic incompleteness is observable, provided the measurement process associated with $\mathcal{O}$ is known. Certainly the measurement process will involve quantum theory at a more or less obvious but essential level. We can ask whether the geodesic incompleteness has an impact on the quantum theory underlying the measurement process. For instance, if black holes are quantum incomplete with respect to the degrees of freedom employed in the measurement device, then $\mathcal{O}$ cannot be measured, and the geodesic incompleteness is not observable. If this holds for any observable, then the geodesic incompleteness is unobservable in principle. This may sound impractical as a criterion. Measurement processes, however, rely on a few principles and are realized via universal principles such as minimal coupling. This makes it relatively easy to pass from unobservable to observable in principle.

We found that Schwarzschild black holes are quantum complete, and, moreover, the ground state does not support field configurations near the singularity. The logical conflict with the measurement process as described above has a well-known resolution: Near the black hole singularity, observables necessarily are part and parcel of the quantum theory. So consistency of the quantum theory is not only essential for the measurement device, but also for the very construction of observables.

In our opinion, and in conclusion, the concept of quantum completeness as suggested in this work has physical relevance, and presents a physical characterization of space-time singularities and their impact.

ACKNOWLEDGMENTS

It is a great pleasure to thank Cesar Gomez, Andre Franca, Sophia Müller, Florian Niedermann, Tehseen Rug and Robert Schneider for delightful discussions, and Kristina Giesel and Thomas Thiemann for their thoughts on the topic during a cold night. We appreciate financial support of our work by the DFG cluster of excellence "Origin and Structure of the Universe" and by TRR 33 "The Dark Universe."

[1] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II—Fourier Analysis and Self-Adjointness* (Elsevier, New York, 1974).
[2] S. Hawking and R. Penrose, Proc. R. Soc. Lond, A **314**, 529 (1970).
[3] G. T. Horowitz and D. Marolf, Phys. Rev. D **52**, 5670 (1995).
[4] A. Ishibashi and A. Hosoya, Phys. Rev. D **60**, 104028 (1999).
[5] A. Ishibashi and R. M. Wald, Classical Quantum Gravity **20**, 3815 (2003).
[6] D. A. Konkowski, T. M. Helliwell, and C. Wieland, Classical Quantum Gravity **21**, 265 (2004).
[7] A. Ashtekar and A. Magnon, Proc. R. Soc. A **346**, 375 (1975).
[8] R. M. Wald, J. Math. Phys. (N.Y.) **20**, 1056 (1979).
[9] R. M. Wald, J. Math. Phys. (N.Y.) **21**, 2802 (1980).

[10] A. D. Helfer, Classical Quantum Gravity **13**, L129 (1996).
[11] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry I* (Wiley, New York, 1963).
[12] R. M. Wald and V. Iyer, Phys. Rev. D **44**, R3719 (1991).
[13] B. Hatfield, *Quantum Field Theory of Point Particles and Strings* (Westview Press, Boulder, 1991).
[14] R. Geroch, J. Math. Phys. (N.Y.) **11**, 437 (1970).
[15] B. B. Baker and E. T. Copson, *The Mathematical Theory of Huygens' Principle* (AMS Chelsea Publishing, New York, 1987).
[16] M. Reed and B. Simon, *Methods of Modern Mathematical Physics III—Scattering Theory* (Elsevier, New York, 1979).
[17] M. Bohner and A. Peterson, J. Diff. Equ. Appl. **7**, 767 (2001).

D

Publication: Non-Gaussian ground-state deformations near a black-hole singularity

Here, we present our second publication in Physical Review D which is the basis of the analysis of excitations and self-interaction.

PHYSICAL REVIEW D **95,** 065033 (2017)

Non-Gaussian ground-state deformations near a black-hole singularity

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(Received 23 November 2016; published 31 March 2017)

The singularity theorem by Hawking and Penrose qualifies Schwarzschild black holes as geodesic incomplete space-times. Albeit this is a mathematically rigorous statement, it requires an operational framework that allows us to probe the spacelike singularity via a measurement process. Any such framework necessarily has to be based on quantum theory. As a consequence, the notion of classical completeness needs to be adapted to situations where the only adequate description is in terms of quantum fields in dynamical space-times. It is shown that Schwarzschild black holes turn out to be complete when probed by self-interacting quantum fields in the ground state and in excited states. The measure for populating quantum fields on hypersurfaces in the vicinity of the black-hole singularity goes to zero towards the singularity. This statement is robust under non-Gaussian deformations of and excitations relative to the ground state. The physical relevance of different completeness concepts for black holes is discussed.

DOI: 10.1103/PhysRevD.95.065033

I. INTRODUCTION

The singularity theorem [1] by Hawking and Penrose identifies Schwarzschild black holes as incomplete in a precise sense: Black holes incorporate a spacelike singularity, where null and timelike geodesics end prematurely, referring to classical point particles that reach these end points in a finite time, because their potential is bounded from above [2]. This relates the geometric completeness concept to the usual notion of potential completeness. The latter can be lifted to quantum mechanical completeness, which implies the existence of a unique evolution in compliance with unitarity. Unitarity remains the relevant completeness criterion in static space-times and extends to encompass the relativistic domain of a single particle.

In this context, Horowitz and Marolf [3] were the first to point out that geodesic (and hence, potential) incompleteness does not necessarily imply unitarity violation. They gave examples of static space-times with timelike singularities that nevertheless qualified as complete from a quantum mechanical perspective. This was conceptually promising since quantum field theory in a static, globally hyperbolic space-time admits a consistent description of a single relativistic particle, as was shown by Ashtekar and Magnon in [4]. And it was practical, since Wald [5] showed that the dynamics of a Klein-Gordon scalar field in arbitrary static space-times could be examined by asking whether the spatial part of the wave operator admits a self-adjoint extension.

Dynamical space-times in general, however, require a quantum theory with fields as local bookkeeping devices. The interior of Schwarzschild black holes is a dynamical space-time, even though the exterior is thought of as being static. A strictly unitary evolution is no longer necessary, even in the absence of interactions. As a consequence, a new criterion for quantum completeness is required that reduces to the classical one (and its quantum mechanical descendant) in appropriate limits, but extends to quantum field theory. While unitarity reflects the symmetry underlying quantum mechanical evolution, the logically more potent concept is state normalization. Unitarity is replaced by stability, which demands a valid probabilistic interpretation instead of a conserved norm. So stability requires only a semigroup of contractions [6]. At the intuitive level, stability ensures a probabilistic interpretation of the quantum system in a background dynamical space-time. Such a stability notion clearly reflects on the completeness of the background space-time as scrutinized by quantum fields.

Stability investigations are usually pursued in the asymptotic framework pertinent to scattering theory, which is neither an option in generic space-times, nor is it practical given that instabilities are anchored in regions near classical singularities. In this situation, the Schrödinger representation of quantum field theory turns out to be extremely useful, since it conveniently allows us to investigate stability at finite times. Based on this framework, the following completeness criterion [7] has recently been suggested: A globally hyperbolic space-time is called quantum complete to the left with respect to a free field theory, if its Schrödinger wave functional can be normalized at an initial time t_0, and if the normalization is bounded from above by its initial value for all $t \in (0, t_0)$.

In a previous article [7], it has been shown that Schwarzschild black holes are quantum complete. Here, the notion of quantum completeness is extended to include non-Gaussian deformations of the ground state

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induced by self-interactions and excited states. As will be shown, all generalizations respect the concept of quantum completeness as suggested above. Concrete calculations are presented for a Schwarzschild black hole populated by real scalar fields with quartic self-interactions. The wave functionals for the ground state as well as for arbitrary excited states are investigated near the black-hole singularity. The main result is that Schwarzschild black holes are quantum complete even if self-interactions and excitations are permitted. The different completeness concepts employed in physics are logically consistent in their respective domains of validity. Their physical relevance for black-hole interiors is discussed in detail.

II. GEOMETRIC PRELIMINARIES

Let us first clarify our conventions. Consider the space-time $(\mathcal{M}, g)$ with $\mathcal{M} \coloneqq \mathbb{R} \times \mathbb{R}^+ \times S^2$, where S^2 is the unit two sphere. The projections $t: \mathcal{M} \to \mathbb{R}$ and $r: \mathcal{M} \to \mathbb{R}^+$ are called Schwarzschild time and Schwarzschild radius, respectively. The Schwarzschild function $h(r) \coloneqq 1 - r_g/r$ is increasing from minus infinity at $r = 0$ to one as r approaches plus infinity, passing through zero at $r = r_g$. Here, $r_g = 2M$ denotes the gravitational radius of a source of mass M. The physical conditions implied by a static and spherical symmetric source in vacuum, supplemented with asymptotic falloff conditions, give rise to two warped product space-times, the Schwarzschild exterior space-time $\mathcal{E} \coloneqq P_> \times_r S^2$, with $P_>$ denoting the region $r > r_g$ in the (t, r)-half plane $\mathbb{R} \times \mathbb{R}^+$, and the Schwarzschild black hole $\mathcal{B} \coloneqq P_< \times_r S^2$, with $P_<$ denoting the region $r < r_g$. In $\mathcal{B}$, the coordinate vector field ∂_t becomes spacelike and ∂_r becomes timelike. Owing to this, we write for the metric in $\mathcal{B}$

$$g = -s^{-1}(t)\mathrm{d}t \otimes \mathrm{d}t + s(t)\mathrm{d}r \otimes \mathrm{d}r + t^2 w. \tag{1}$$

Here, $s(t) \coloneqq |1 - r_g/t|$ and w denotes the metric on S^2, equipped with the usual spherical coordinates (θ, ϕ).

Spatial hypersurfaces Σ in $\mathcal{B}$ are conformally flat as implied by a vanishing Cotton tensor. In order to appreciate conformal flatness, it suffices here to consider the region $t \ll r_g$ close to the spacelike singularity of $\mathcal{B}$, where the line element of $P_<$ takes the approximate form $-(t/r_g)\mathrm{d}t^2 + (r_g/t)\mathrm{d}r^2$. Following Ehlers and Kundt [8], and demanding in addition $\theta \ll 1$, the metric can be restated as a type-D Kasner solution characterized by the exponents $(p_1, p_2, p_3) = (2/3, 2/3, -1/3)$. The corresponding coordinate transformation is $r \eqqcolon (3/2r_g)^{1/3} z$, $t \eqqcolon (9r_g/4)^{1/3} \tau^{2/3}$, and $\theta \exp(\mathrm{i}\phi) \eqqcolon (4/9r_g)^{1/3}(x + \mathrm{i}y)$. In this coordinate neighborhood, the line element of $\mathcal{B}$ becomes

$$\mathrm{d}s^2 = -(\mathrm{d}\tau)^2 + \tau^{4/3}((\mathrm{d}x)^2 + (\mathrm{d}y)^2) + \tau^{-2/3}(\mathrm{d}z)^2. \tag{2}$$

Harmonic analysis in Σ is similar to Euclidean space. In particular, the Laplace operator in Σ factorizes $\Delta_\Sigma = g_{ab}(\tau)\partial^a\partial^b$, with τ indexing Σ_τ. Generalized eigenfunctions of Δ_Σ are plane waves $\exp(\mathrm{i}g_\Sigma(k, x))$, where g_Σ denotes the induced metric tensor in Σ. This coordinate neighborhood is useful for a quick examination of our results. Moreover, it allows us to relate to the framework suggested by Belinskii *et al.* [9]. Let us stress, however, that all the results in this paper have been derived in the usual Schwarzschild neighborhood.

III. QUANTUM COMPLETENESS OF SCHWARZSCHILD BLACK HOLES

In this section, we briefly review the argument showing that $\mathcal{B}$ is quantum complete. The main result is Eq. (7), which has recently been published in [7], where a considerably more detailed derivation can be found. Subsequently, it is shown that the free Hamilton density in the ground state vanishes towards the classically singular hypersurface Σ_0.

Since $\mathcal{B}$ is a globally hyperbolic space-time, it is diffeomorphic to $\mathbb{R} \times \Sigma$ and foliates into spatial hypersurfaces Σ_t indexed by Schwarzschild time. In $\mathcal{B}$, consider the dynamical system $(\mathcal{H}, \Phi)$, where Φ denotes a real scalar field with Hamilton density $\mathcal{H} = \mathcal{H}_\pi + \mathcal{P}[\Phi]$. Here, $\mathcal{H}_\pi = \sqrt{-g_{tt}}/2\pi^2/\det(g_\Sigma)$ with $\pi = -\mathrm{i}\delta/\delta\Phi$, and $\mathcal{P}[\Phi]$ denotes the effective potential. Quantum completeness refers to free evolution, corresponding to $\mathcal{P}[\Phi] = \sqrt{-g_{tt}}/2g_\Sigma(\mathrm{d}\Phi, \mathrm{d}\Phi)$, possibly supplemented by a mass term. The wave functional of the ground state evolves from the initial Cauchy surface Σ_{t_0} backwards in time to $\Sigma_t (t \in (0, t_0))$ as $\Psi[\phi](t) = \mathcal{E}(t, t_0)\Psi[\phi](t_0)$, with

$$\mathcal{E}(t, t_0) = \exp\left(-\mathrm{i}\int_{t_0}^{t} \mathrm{d}t' \int_{\Sigma_{t'}} \mathrm{d}\mu_z \mathcal{H}[\Phi]\right), \tag{3}$$

where $\mathrm{d}\mu_z$ denotes the covariant volume form with respect to g_Σ, and z refers to the coordinate neighborhood. In $\mathcal{B}$, the evolution operator $\mathcal{E}$ is not unitary. Quantum completeness of $\mathcal{B}$ with respect to $(\mathcal{H}, \Phi)$ requires $\|\Psi[\phi]\|(t) \leq \|\Psi[\phi]\|(t_0)\ \forall\ t \in (0, t_0)$, implying that $\mathcal{B}$ can be a sink for probability but not a source, given it is not resolved in dynamical degrees of freedom.

We expect that a quadratic functional $\mathcal{K}_2[\phi, \phi](t)$ exists such that the wave functional $\Psi^{(0)}[\phi](t)$, corresponding to the ground state of $(\mathcal{H}, \Phi)$, is given by

$$\Psi^{(0)}[\phi](t) = N^{(0)}(t)\exp(-\mathcal{K}_2[\phi, \phi](t)), \tag{4}$$

with $N^{(0)}$ denoting the time-dependent normalization, and $\mathcal{K}_2$ can be expressed in terms of the bilocal kernel function K_2 as

$$\mathcal{K}_2[\phi,\phi](t) = \frac{1}{2}\int_{\Sigma_t} \mathrm{d}\mu_{z_1}\mathrm{d}\mu_{z_2}\phi(z_1)K_2(z_1,z_2,t)\phi(z_2). \quad (5)$$

In the vicinity of the Schwarzschild singularity, the evolution simplifies considerably,

$$\mathcal{E}(t,t_0) \to \exp\left(-\frac{\mathrm{i}c(t_0)}{4M}\ln t\int_{\Sigma_{t_0}} \mathrm{d}\mu_z \frac{1}{\sin^2\theta}\frac{\delta^2}{\delta\phi^2(z)}\right), \quad (6)$$

where $c(t_0)$ is a constant of integration. As a consequence, the kernel function becomes a contact term in this limit, $K_2(z_1,z_2,t) \to k_2(t)\delta^{(3)}(z_1,z_2)$, which is consistent with the conjecture by Belinskii *et al.* [9]: Close to a spacelike singularity, the variation of observables on Σ_t from one location to another becomes irrelevant compared to changes in time. Subleading corrections to the asymptotic form of K_2 deviate from a contact contribution without changing the qualitative result.

In leading order, the evolution of the wave functional is given by

$$\lim_{t\to 0}\Psi^{(0)}[\phi](t) = \lim_{t\to 0}|\ln(t/t_0)|^{-\Lambda v(\Sigma_t)/2} \quad (7)$$

up to constant and phase factors, which are irrelevant for the analysis presented here. In (7), an ultraviolet cutoff Λ and a volume regularization $v(\Sigma_t)$ have been introduced. Clearly, the limit $t\to 0$ is not affected by this simple choice. Hence, the wave functional has vanishing support towards the Schwarzschild singularity, and $\|\Psi^{(0)}[\phi]\|(t) \to 0 \le \|\Psi^{(0)}[\phi]\|(t_0)$ for $t\to 0$, as required for $\mathcal{B}$ to be quantum complete with respect to the dynamical system $(\mathcal{H},\Phi)$.

Concerning an interpretation: Consider the set of observables $\mathcal{A}_{\Sigma_t}$ of $(\mathcal{H},\Phi)$ localized on Σ_t. Following the logic of geodesic incompleteness, it could be expected that an observable $\mathcal{O}_{\Sigma_t}$ exists with an expectation $\langle\mathcal{O}_{\Sigma_t}\rangle_{\Psi^{(0)}}$ in the ground state that is ill-defined. However, this is not the case since the asymptotic surface Σ_0 does not support any population of fields ϕ, because the associated probability measure vanishes there.

As an example, consider a free field theory $(\mathcal{H},\Phi)$ in the ground state described by the Schrödinger wave functional $\Psi^{(0)}[\phi](t)$, where ϕ denotes a classical field configuration over the hypersurface Σ_t, $t\in(0,2M)$. Introducing an auxiliary source $\mathcal{J}$ coupling by $\Psi^{(0)}_{\mathcal{J}}[\phi](t) \coloneqq \Psi^{(0)}[\phi](t)\times\exp(\mathcal{J}[\phi](t))$ facilitates the description of measurement processes. Observables are evaluated in the ground state $|\Omega\rangle^{\mathcal{J}}$ in the presence of the auxiliary source, which is subsequently set to zero. Compositions of the configuration operator are then replaced by the corresponding succession of derivations δ_ϕ, where

$$\delta_\phi \coloneqq \int_{\Sigma_t} \mathrm{d}\mu_x \frac{\phi(x)}{\sqrt{\det(g_\Sigma)}}\frac{\delta}{\delta J(x)} \quad (8)$$

is a directional derivative in field space, with J denoting the ultralocal representation of $\mathcal{J}$. For instance, in the presence of an auxiliary source

$$^{\mathcal{J}}\langle\Omega|\Phi^2(\phi)|\Omega\rangle^{\mathcal{J}} = \delta_\phi^2\exp\left\{\frac{1}{4}\frac{1}{\sqrt{\det(g_\Sigma)}}[\mathrm{Re}(\mathcal{K}_2)]^{-1}[J,J]\right\}\mathcal{W}^{(0)}(t), \quad (9)$$

where $\mathcal{W}^{(0)}(t) \coloneqq \|\Psi^{(0)}[\phi]\|^2(t)$. In the absence of the auxiliary source, the ground-state expectation becomes

$$\langle\Omega|\Phi^2(\phi)|\Omega\rangle = \frac{1}{2}\sqrt{\det(g_\Sigma)}[\mathrm{Re}(\mathcal{K}_2)]^{-1}[\phi,\phi]\mathcal{W}^{(0)}(t), \quad (10)$$

which is real and semipositive definite. In particular, in the vicinity of the limiting hypersurface Σ_0, the ground-state expectation approaches zero, due to the temporal support properties associated with the probability density. Similarly, it can be shown that $\langle\Omega|\pi^2(\phi)|\Omega\rangle$ is real and semipositive definite, and approaches zero towards Σ_0. Therefore, the ground-state expectation of $\mathcal{H}$ is in $\mathbb{R}^+$ and vanishes towards the would-be singular hypersurface Σ_0.

IV. SELF-INTERACTIONS

In this section, polynomial self-interactions are included, and their impact on the stability of the ground state is analyzed. For definiteness, we consider the effective potential $\mathcal{P}_{\mathrm{int}}[\Phi] \coloneqq \mathcal{P}[\Phi] + \sqrt{-g_{tt}}\lambda\Phi^4/4!$. The dimensionless coupling λ is chosen such that perturbation theory is applicable in a neighborhood of Σ_{t_0}. Self-interactions deform the ground-state wave functional away from its Gaussian shape

$$\Psi^{(0)}_{\mathrm{int}}[\phi](t) = \Psi^{(0)}[\phi](t)\times\exp(\lambda\mathcal{D}[\phi](t)). \quad (11)$$

The deformation functional $\mathcal{D} = \mathcal{D}_2 + \mathcal{D}_4$ is a sum of the time-dependent nonlinear functionals $\mathcal{D}_2 : \mathcal{S}^{\otimes 2} \to \mathcal{C}(\mathbb{R}^+)$ and $\mathcal{D}_4 : \mathcal{S}^{\otimes 4} \to \mathcal{C}(\mathbb{R}^+)$, where $\mathcal{S}$ denotes the field space and $\mathcal{C}(\mathbb{R}^+)$ is the space of functions depending smoothly on time. As before, local versions can be introduced via kernel functions D_2 and D_4, respectively. Close to the singularity, $D_j = d_j(t)\Pi_{a=1}^j\delta^{(3)}(z_a)$, $j\in\{2,4\}$, i.e., any spatial information close to Σ_0 is concentrated in a single event. Again, only the temporal gradients matter. In this limit, the kernel functions obey the coupled kernel equations $\mathrm{i}\partial_t\underline{d} = \sqrt{\det(g)}\mathfrak{a}\underline{d}$, where $\underline{d} \coloneqq (d_2,d_4)^{\mathrm{T}}$ and $\mathfrak{a}$ is a two-by-two matrix with coefficients $\mathfrak{a}_{11} = k(t)$, $\mathfrak{a}_{12} = 1$, $\mathfrak{a}_{21} = 0$ and $\mathfrak{a}_{22} = k(t)$. The asymptotic solution is $\underline{d}(t) = (1,1)^{\mathrm{T}}/|\ln(t)| \to 0$ for $t\to 0$.

As a consequence, deformations of the Gaussian ground state, induced by self-interactions, become less and less important towards the black-hole singularity, $\mathcal{D}[\phi](t) \to 0$ for $t \to 0$. In greater detail, asymptotically $\mathcal{D}_j \propto t^{3j/2}/|\ln(t)|$ for $j \in \{2, 4\}$ and, hence,

$$\lim_{t\to 0}\Psi^{(0)}_{\text{int}}[\phi](t) = \lim_{t\to 0}\Psi^{(0)}[\phi](t) = 0. \tag{12}$$

Thus, close to Σ_0 (i.e., for Schwarzschild times $t \ll t_0$), the dynamical systems $(\mathcal{H}, \Phi)$ and $(\mathcal{H}_\pi, \Phi)$ may be identified.

This proves that self-interactions cannot cure the classical black-hole singularity via backreaction effects on the external geometry. Close to the singularity, self-interactions loose their impact on the evolution of the system. The system becomes asymptotically free, and the stability requirement on the quantum theory is too stringent to allow the free theory to destabilize even towards Σ_0. Hence, the quantum fields are totally ignorant about the singularity. From this point of view, the classical singularity needs no resolution since it appears as a mathematical artifact with no observational consequences whatsoever, assuming the measurements are anchored in the framework provided by quantum theory. It seems that quantum completeness of the Schwarzschild black hole protects general relativity against its classical incompleteness. In fact, the potential harmful implications associated with Σ_0 decouple from quantum measurements.

A more abstract reasoning is the following: Consider classical fields ϕ as configurations in $\mathcal{C}^2(\Sigma)$. In order to ensure a probabilistic interpretation, the Schrödinger wave functionals have to be normalizable with respect to some functional measure $\mathrm{D}\phi$. Wave functionals enjoying this property can be collected in a state space $\mathcal{L}^2(\mathcal{C}^2(\Sigma), \mathrm{D}\phi)$, which obviously requires a mathematical justification beyond the scope of this article. Even for these wave functionals, $\mathcal{H}_\pi$ is not self-adjoint, but the spectrum contains only functions with a positive-semidefinite imaginary part. As a consequence, towards the singularity, $\mathcal{E}(t, t_0)$ becomes exponentially damped. Self-interactions cannot harm this regularization of the classical singularity, simply because they are given as compositions of multiplication operators. Furthermore, $\mathcal{H} \to \mathcal{H}_\pi$ towards the singularity, where the limit is taken in a generalization of the strong operator topology appropriate for the functional calculus involved here. From this point of view, a self-interacting quantum probe is totally ignorant about the classical singularity.

V. EXCITATIONS

Excitations of the ground state are not an integral component in the definition of quantum completeness. If the dynamical system $(\mathcal{H}, \Phi)$ is unstable, then excitations might trigger a transition towards a stable ground state.

The ground state is an eigenstate of the conjugated momentum field, $(\pi(t, x) - \mathrm{i}\delta\mathcal{K}_2/\delta\phi(t, x))\Psi^{(0)}[\phi](t) = 0$, and a kernel element of the operator valued functional $a[f](t)$, describing the absorption of a field $f \in \mathcal{S}_{\text{os}}$, where the index "os" implies the restriction to on shell fields. The above eigenstate equation is a ultralocal version of absorption. Emission can be considered accordingly using the adjoint $a^\dagger[f](t)$. As usual, on Σ_t the following algebraic relation holds:

$$[a[f](t), a^\dagger[f'](t)] = 2\mathrm{Re}(\mathcal{K}_2[f, f'])(t). \tag{13}$$

An excitation relative to the ground state is given by $\Psi^{(1)}[f, \phi](t) \coloneqq a^\dagger[f](t)\Psi^{(0)}[\phi](t)$. Note that $\phi \in \mathcal{S}$, while $f \in \mathcal{S}_{\text{os}} \subset \mathcal{S}$, i.e., the emission operator creates on shell information and stores it in the excited state $\Psi^{(1)}[f, \phi](t) = 2\mathrm{Re}(\mathcal{K}_2[f, \phi])(t)\Psi^{(0)}[\phi](t)$. Therefore, exciting the ground state by emitting an on shell quantum simply results in a functional renormalization of the ground state. Owing to the algebraic relation (13), we find

$$\|\Psi^{(1)}\|^2(t) = 2\mathrm{Re}(\mathcal{K}_2[f, f])(t)\|\Psi^{(0)}\|^2(t). \tag{14}$$

So quantum completeness of the ground state is a necessary but not sufficient criterion for the stability of the first excited state. In addition, $\mathrm{Re}(\mathcal{K}_2[f, f])(t) < \infty$ is required for all $f \in \mathcal{S}_{\text{os}}$.

For vanishing Schwarzschild time, the renormalization becomes constant and is therefore inconsequential. This can be seen as follows: Up to subleading contributions, the time dependence of $f(t, x) = T(t)R(x)$ is given by $(t\partial_t^2 + \partial_t)T = \kappa T$ with κ a constant determined by the equation for R. For vanishing Schwarzschild time, T should be singular. Introducing $\tau \coloneqq \zeta t$, and taking the limit $t \to 0, \zeta \to \infty$ such that the rescaled Schwarzschild time τ remains constant, the equation of motion for T becomes $(\tau\partial_\tau^2 + \partial_\tau)T = 0$. Thus, up to an additive constant, $T = \ln(t)$. Therefore, $\mathrm{Re}(\mathcal{K}_2[f, f']) = \text{const}$, because the time dependence of the corresponding kernel function cancels exactly against the time dependence of the mode functions and the volume form. Note that the additive constant poses no problem due to the prescription for taking the asymptotic limit. As a consequence, $\|\Psi^{(1)}\|^2 \to 0$ towards the black-hole singularity. In fact, as can be seen by induction, all excitations $\Psi^{(n)}(n \in \mathbb{N})$ give rise to a vanishing probability measure on Σ_0. Neither the ground state nor any excited states are populated with fields on Σ_0. The natural probability measure protects the stability of any state, and this stability protection can be traced back to a persistent ground state.

VI. DISCUSSION

In this article, the notion of quantum mechanical completeness is adapted to situations where the only adequate description is in terms of (interacting) quantum fields in dynamical space-times. The adaption necessarily generalizes from requiring a unitary evolution by demanding a normalization condition that ensures a probabilistic interpretation. Of course, this condition reduces to unitarity in the absence of dynamical sources. While originally stated for free fields in a Gaussian ground state, it is shown to extend to interacting quantum fields in arbitrary states. It is tempting to expect that this extension is rather trivial if the ground state admits a Gaussian wave functional. This expectation has to be confronted with the dynamics of the external space-time that sources different terms in the Hamiltonian differently. It is important to stress that both, geodesic and quantum completeness, assume a background space-time, which is either diagnosed by test particles or by test fields, respectively. This assumption, however, can only be investigated in a quantum theory of fields.

Whether a given dynamics is consistent with a probabilistic interpretation is usually examined in an asymptotic framework pertinent to scattering theory. There the stability of the ground state is studied in the presence of an external source after an infinite amount of time has passed. This is clearly not an option in arbitrary space-times. Furthermore, it seems intuitive that stability challenges are anchored in the vicinity of space-time singularities, which suggests a more local stability analysis. For these reasons, the Schrödinger representation of quantum field theory is quite convenient, which allows us, in particular, to investigate the stability of a given quantum system in a dynamical space-time after a finite amount of time elapsed.

The Schrödinger representation requires a functional generalization of many quantum-mechanical concepts. In particular, choosing the configuration field as the multiplication operator, the associated momentum field becomes a functional derivative. And the norm of a wave functional requires a functional integral over the configuration fields. Many of these functional techniques can be disputed on mathematical grounds. However, the stability analysis is entirely at the qualitative level and not based on any specific regularization.

The main result of this article is that Schwarzschild black holes are quantum complete, which has a very precise meaning. However, equally precise they are qualified as geodesically incomplete space-times by the singularity theorem of Hawking and Penrose. Of course, both completeness notions are logically consistent within their respective domains of validity. If we are to derive further consequences from these notions, in particular, concerning the consistency of black holes and of general relativity, it is important to understand which domain and therefore which completeness notion is applicable given the physical conditions. Our point of view advocated here is the following: Geodesic completeness is a concept in the category of smooth manifolds as models for space-times. To the extent that we can be certain that these models can be probed by physical events, it is falsifiable. In the vicinity of spacelike singularities, spatial correlations become trivial, i.e., events can only be spatially correlated if they are stacked on top of each other. As might be expected, what matters in the vicinity of a spacelike singularity are temporal correlations. In fact, temporal gradients correspond to a characteristic length scale that is smaller than the length scale characterizing the spatial extent of any conceivable classical measurement device. Therefore, any completeness diagnosis based on classical measurements is inappropriate given the physical conditions. Any measurement process in the vicinity of a black-hole singularity has to rely on quantum field theory. In the context of classical singularity theorems, the only falsifiable completeness notion applicable to black-hole interiors is quantum completeness.

This argument is not in conflict with the logic underlying the usual quantization prescription, precisely because the probability measure is always well-defined. In particular, the Gaussian ground state is respected by self-interactions, provided the system was in a weak-coupling regime initially. This is in accordance with the intuitive expectation that the free dynamics (temporal correlations) dominates in the vicinity of the singularity. Consequently, excitations relative to the ground state cannot change the conclusion. Let us stress that these results are in full accordance with the dynamical stability of classical field configuration in Schwarzschild space-time, as has been established in [10–12]. Temporal support for field configurations is strictly restricted to the interval $(0, t_0]$ with the initial time $t_0 < r_g$, and the field configurations are smooth on this interval.

Black-hole interiors are quantum complete, and this notion is sensible from a physics point of view even in the vicinity of the classically singular hypersurface. In contrast, geodesic incompleteness of black holes, albeit a mathematical rigorous qualification, is not a physical statement since any operative measurement has to employ physics beyond point particle dynamics. As a consequence, the classically singular hypersurface bears no impact on observables based on bookkeeping devices (fields) with sensible dynamics. Less sensible is the argument that geometrical observables such as the Kretschmann scalar would diverge at the origin. This line of argument is already invalidated for simple bound-state problems in quantum mechanics, for instance, the hydrogen atom. Clearly, the Coulomb potential enjoys geodesic and potential incompleteness, which is inconsequential for hydrogen as a quantum bound state. Albeit the singular structure in this case is just a point, quantum completeness is established by arguments related to the support properties of the probability measure, as well. In the case of black holes, the singular structure is spacelike, but corresponds to a limiting instant in time.

ACKNOWLEDGMENTS

It is a great pleasure to thank Ingemar Bengtsson, Kristina Giesel, Maximilian Kögler, Florian Niedermann, and Maximilian Urban for delightful discussions. We thank Robert C. Myers for sharing his ideas about this topic. We appreciate financial support of our work by the DFG cluster of excellence "Origin and Structure of the Universe", the Humboldt Foundation, and by TRR 33 "The Dark Universe".

[1] S. W. Hawking and R. Penrose, Proc. R. Soc. A **314**, 529 (1970).
[2] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness* (Elsevier, New York, 1975), Vol. 2.
[3] G. Horowitz and D. Marolf, Phys. Rev. D **52**, 5670 (1995).
[4] A. Ashtekar and A. Magnon, in Proc. R. Soc. A **346**, 375 (1975).
[5] R. M. Wald, J. Math. Phys. (N.Y.) **20**, 1056 (1978).
[6] This corresponds to relaxing the requirements posed on evolution generators to linear unbounded operators that are closed, have a dense domain, whose resolvent set contains all positive reals and whose resolvent operator, which can be thought of as the Laplace transform of the semigroup, is bounded from above for each element of the resolvent set by its inverse.
[7] S. Hofmann and M. Schneider, Phys. Rev. D **91**, 125028 (2015).
[8] J. Ehlers and W. Kundt, in *The Theory of Gravitation* (John Wiley & Sons, Inc., New York, 1962), p. 49.
[9] V. Belinskii, I. Khalatnikov, and E. Lifshitz, Adv. Phys. **19**, 525 (1970).
[10] B. P. Jensen and P. Candelas, Phys. Rev. D **33**, 1590 (1986).
[11] E. Elizalde, Phys. Rev. D **36**, 1269 (1987).
[12] E. Elizalde, Phys. Rev. D **37**, 2127 (1988).

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Publication: Information carriers closing in on the black-hole singularity

The third publication describes the direct application of the formalism in order to explore the black hole information paradox.

Information carriers closing in on the black-hole singularity

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(Dated: April 19, 2018)

The interiors of Schwarzschild black-holes border on space-like singularities which are considered to serve as geodesic information sinks. It is shown that these sinks are completely decoupled from local quantum physics by a dynamically generated Dirichlet boundary. Towards this boundary, local information carriers become free and receive vanishing probabilistic support. Hence information cannot leak and information processing in black-hole interiors is free from paradoxes.

PACS numbers: 03.70.+k, 04.20.Dw, 04.62.+v, 11.10.-z

Introduction. Schwarzschild black-hole interiors are considered to border on information sinks which are accessible to geodesic information carriers. Any information carriers falling into these space-like singularities are irreversibly lost, their recovery is forbidden by causality. In the purely geometrical description of black-hole interiors, geodesic incompleteness is realized in its most radical version: Any geodesic carriers of information in the interior are destined to reach the singularity after a finite time has elapsed. From this perspective, black holes seem to leak information through their borders. In other words, black holes destroy information [1].

In this letter we show that black-hole interiors cannot loose information carried by local bookkeeping devices. Local quantum physics is decoupled from the space-like singularity by a Dirichlet boundary condition [2]. This boundary condition is not imposed on the information carriers, but emerges dynamically due to a quantum-complete evolution given by a contracting semi-group. The contraction property is reflected in the vanishing probabilistic support that is assigned to local information carriers towards the boundary. As a result, the classical information sink is closed in the quantum regime, and local information carriers are protected from inconsistencies rooted in the purely geometrical description of black-hole interiors.

After some geometrical preliminaries we present an intuitive argument based on scaling relations in the micropædia before giving the exact argument in the macropædia, followed by an interpretation based on Dirichlet decoupling. The latter can be related to the black-hole final state proposal by Horowitz and Maldacena [3]. The emerging Dirichlet boundary supports their argument for a black-hole final state from a semi-classical perspective.

Geometrical Preliminaries. Schwarzschild black-holes are the warped geometries $\mathcal{B} := P_< \times_r S^2$, with $P_<$ denoting the region $r < r_{\rm g} := 2M$ in the (t, r)-half plane $\mathbb{R} \times \mathbb{R}^+$, where the projections $t : \mathcal{B} \to \mathbb{R}$ and $r : \mathcal{B} \to \mathbb{R}^+$ are the Schwarzschild time and Schwarzschild radius, respectively. And S^2 denotes the unit two-sphere. In $\mathcal{B}$ the coordinate vector field ∂_t is space-like, and ∂_r is time-like. Taking this into account, the quadratic form associated with this geometry can be written as $\mathrm{d}s^2 = -s^{-1}(t)\mathrm{d}t^2 + s(t)\mathrm{d}r^2 + t^2 w$, where $s(t) := |1 - r_{\rm g}/t|$ is the Schwarzschild function, and w denotes the line element associated with the Euclidean metric on S^2, equipped with the usual spherical coordinates (ϕ, θ). $\mathcal{B}$ is a globally hyperbolic space-time diffeomorphic to $\mathbb{R} \times \Sigma$, where Σ is the folio of spatial hypersurfaces Σ_t labeled by Schwarzschild time. The metric field associated with the above quadratic form will be denoted by g, and its pull-back to Σ by g_Σ.

In this geometry, the geodesic motion of a point particle that is initially equatorial relative to Schwarzschild spherical coordinates is bound to remain equatorial, $\theta = \pi/2$. The so-called energy equation $E^2 = (\mathrm{d}t/\mathrm{d}s)^2 + V_{\rm eff}$ holds, where $E := s(t)\mathrm{d}r/\mathrm{d}s$ and $L := t^2\mathrm{d}\phi/\mathrm{d}s$ are constants, which have an intuitive interpretation in the exterior as asymptotic energy per unit mass and angular momentum per unit mass, respectively. In fact, in the exterior, the definition of L formally coincides with Kepler's second law. The effective potential is given by $V_{\rm eff} := -(1 + L^2/t^2)s(t)$. Close to the endpoint at $t = 0$, the effective potential is bounded from above, $V_{\rm eff} = -L^2 r_{\rm g}/t^3$ plus less singular contributions. Thus the classical motion generated by $V_{\rm eff}$ is not complete at $t = 0$, and so $\mathcal{B}$ is geodesic incomplete [4]. Geodesic completeness, however, does not imply quantum completeness (and vice versa).

Micropædia. Let us first give an intuitive argument based on scaling relations for information conservation in $\mathcal{B}$, before providing exact statements. In $\mathcal{B}$ consider a dynamical system $(\mathcal{L}, \Phi^*, \Phi)$, where Φ^*, Φ denote scalar fields charged under $U(1)$, and $\mathcal{L}$ is the corresponding Lagrange density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int}$, with the first term denoting the free theory $\mathcal{L}_0 = \Phi^* \Box \Phi$. The intuitive argument will be given in the absence of interactions, $\mathcal{L}_{\rm int} \equiv 0$. Close to the space-like singularity Σ_0 bordering on $\mathcal{B}$, $\mathrm{d}s^2 \cong -(t/r_{\rm g})\mathrm{d}t^2 + (r_{\rm g}/t)\mathrm{d}r^2 + t^2 w$, where $\cong$ means equality up to sub-leading contributions in each term as Σ_0 is approached [5]. In this asymptotic regime, $\Box \cong (-r_{\rm g}/t)(\partial_t^2 + (1/t)\partial_t) + (t/r_{\rm g})\partial_r^2 + (1/t^2)\partial_\sphericalangle^2$. Here $\partial_\sphericalangle^2$ denotes the usual angular part of the Laplace opera-

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tor in $\mathbb{R}^3$ in Schwarzschild spherical coordinates.

The corresponding Green function G is sourced by $\delta(t-t')\delta(\sigma-\sigma')/\sqrt{-\det(g)}$, with σ and σ' denoting Schwarzschild spherical coordinates of events localized on Σ_t and $\Sigma_{t'}$, respectively, and satisfies

$$\begin{aligned} D(t)G(t,\sigma;t',\sigma') &= \delta(t-t')\delta(\sigma-\sigma')\ , \\ D(t) &:= -r_{\rm g}\partial_t(t\partial_t) + (t/r_{\rm g})t^2\partial_r^2 + \partial_\sphericalangle^2\ . \end{aligned} \tag{1}$$

In order to estimate the asymptotic relevance of each term in the differential operator, consider $D(\varepsilon\tau)$ in the limit $\varepsilon \to 0^+$. The effective potential for free fields scales asymptotically as $1/t^2$ and develops a repulsive barrier. It is well-known from quantum mechanics [6] that potentials of this type cannot be penetrated via tunnel processes.

Following the geometrical description of space-like singularities by Belinskii, Khalatnikov and Lifshitz [7], temporal variations dominate over spatial variations in the region bordering on Σ_0. Therefore, $D(\varepsilon\tau) \cong (1/\varepsilon)\partial_\tau(\tau\partial_\tau)$. The time-dependent part of the source distribution scales like $(1/\varepsilon)\delta(\tau - t'/\varepsilon)$. This effectively allows to split the Green function $G = T(t,t')P(\sigma,\sigma')$ in the vicinity of Σ_0, with the asymptotic dynamics given by $\partial_\tau(\tau\partial_\tau T) \cong 0$. Here all identifiers labeling the eigenvalue problem of the Laplace operator have been suppressed for ease of notation. We find the asymptotic solution $T(t,t') \cong C_0(t') + C_1(t')\ln(t/r_{\rm g})$, with $C_{0,1}$ sufficiently well behaved to guarantee a well-posed initial value problem on $\Sigma_{t'}$.

In order to appreciate the rather mild divergence of the asymptotic solution T, we introduce an emitter $Q_{\rm em}$ localized on $\Sigma_{t'}$, $t' \in (0,t_*)$ in the asymptotic domain, and an absorber $Q_{\rm ab}$ on $\Sigma_{\varepsilon\tau}$. For instance, consider $Q_{\rm ab} = \delta(t-\varepsilon\tau)\ q_{\rm ab}(\sigma)$, with $q_{\rm ab}$ encoding the spatial extension of the detector on $\Sigma_{\varepsilon\tau}$. This blueprint effectively replaces part of Σ_0 with a detector volume that can resolve arbitrary frequencies. Note that the asymptotic regime is controlled by the parameter ε while τ represents a constant instant of time.

The classical measurement process is described by the on-shell vertex density $\nu_{\rm obs} = \sqrt{\det(g)}Q^*_{\rm ab}\Phi_{\rm os} + {\rm c.c.}$ with $\Phi_{\rm os}$ denoting a linear functional of $Q_{\rm em}$ with a bi-local kernel given by G [8]. In the region bordering on Σ_0, as specified by the support properties of emitter and absorber, $\nu_{\rm obs} \cong t^2\ln(t)\delta(t-\varepsilon\tau)\sin(\theta)q_{\rm ab}\ F_{\rm em}$, where $F_{\rm em}$ contains the exclusive information on the emission process and depends only on source parameters. In particular, $F_{\rm em}$ is finite in accordance with a globally hyperbolic interior $\mathcal{B}$. For $\varepsilon \to 0$, the measurement of the emitter's influence on the detector gives a vanishing response, $\nu_{\rm obs} \cong 0$, in the distributional sense. This implies that no information carried by local bookkeeping devices can reach Σ_0. It is possible to be more specific about the emitter. As an example, the energy momentum tensor for the complex scalar field scales like $\mathcal{T} \propto 1/(\varepsilon\tau)^2$ on $\Sigma_{\varepsilon\tau}$ and develops a singularity towards Σ_0. It is easy to accommodate this observable in the above naive measurement prescription: $Q_{\rm em} = {\rm tr}\mathcal{T} \propto 1/t'^3$ for t' in the asymptotic domain. Let us consider two detector models in this case. First, again $Q_{\rm ab} = \delta(t-\varepsilon\tau)q_{\rm ab}(\sigma)$, resulting in a measurement of $\nu_{\rm obs} = 0$, as before. Second, $\widetilde{Q}_{\rm ab} = M(\varepsilon\tau)U \otimes U$, where $U \cong \sqrt{\varepsilon\tau/r_{\rm g}}{\rm d}t$, and $M(t)$ denotes the spatial volume integral over an energy density. The principle of minimal coupling underlying a measurement description based on $\widetilde{Q}_{\rm ab}$ is of course the coupling to a gravitational wave, hence $\widetilde{Q}_{\rm em} = \mathcal{T}$. In this case, measuring the influence of the emitter on the absorber located at $\Sigma_{\varepsilon\tau}$ we find the scaling $\widetilde{\nu}_{\rm obs} \propto M(\varepsilon\tau)(\varepsilon\tau)^3\ln(\varepsilon\tau)$. From a phenomenological point of view, $\widetilde{Q}_{\rm ab}$ is required to have nontrivial support towards Σ_0 and $\varepsilon M(\varepsilon\tau)$ needs to be bounded as $\varepsilon \in \mathbb{R}^+$ approaches zero. Then, $\widetilde{\nu} \cong 0$, as well, which only confirms that the asymptotic description of the tree-level measurement process is independent of the tensor providing the principal communication channel.

Before closing the micropædia, let us briefly discuss the asymptotic diagnostics of Noether charges. The $U(1)$-current density is $j = \Phi^* P\Phi - c.c.$, where P denotes the four-momentum. Projecting the current density onto U, we find the following scaling relation for the charge density ρ localized on $\Sigma_{\varepsilon\tau}$: $\rho(\varepsilon\tau) \cong \rho(t_*)(t_*/\varepsilon\tau)^{3/2}$, which formally diverges as Σ_0 is approached. Physical measurements of the charge $Q(\varepsilon\tau)$, however, are fine. In fact $Q(\varepsilon\tau) = Q(t_*)$, where t_* denotes a fiducial time in the asymptotic regime. Thus black-holes cannot be discharged through the geodesic singularity Σ_0 bordering on their interiors. Any active information sink would necessarily lead to charge depletion. Note that this discussion of asymptotic charge conservation is fully based on local physics inside black holes, and no reference to the usual global characterization in the exterior is made.

Macropædia. A more rigorous argument is based on the Schrödinger representation of local quantum physics. In $\mathcal{B}$ consider a dynamical system $(\mathcal{H},\Phi^*,\Phi)$, where Φ^*,Φ denote scalar fields charged under $U(1)$, and $\mathcal{H}$ is the corresponding Hamilton density $\mathcal{H} = :\mathcal{G}(\Pi^*,\Pi) + \mathcal{V}(\Phi^*,\Phi):$ with $\mathcal{G}(\Pi^*,\Pi) := \sqrt{-g_{tt}}\,\Pi^*\Pi/\det(g_{\Sigma_t})$, and $\Pi := -{\rm i}\delta/\delta\Phi$ denoting the momentum field conjugated to Φ. The effective potential density $\mathcal{V}$ is a pure multiplication operator.

Let $|\Psi\rangle_{\Sigma_{t_0}}$ denote a Schrödinger state localized on an initial hypersurface Σ_{t_0}. At a later time t, the initial state has evolved to $|\Psi\rangle_{\Sigma_t} = \mathcal{E}(t,t_0)|\Psi\rangle_{\Sigma_{t_0}}$, and is localized on the hypersurface Σ_t in the folio Σ. Let $\mathcal{C}$ denote the set of all possible field configuration, both on-shell and off-shell. The $\mathcal{C}$-representation of $\mathcal{E}$ is given by

$$\mathcal{E}(t,t_0) = \exp\left(-{\rm i}\int_{t_0}^{t}{\rm d}t'\int_{\Sigma_{t'}}{\rm d}\mu_x\ \mathcal{H}\left(\Phi^*,\Phi\right)\right)\ , \tag{2}$$

where ${\rm d}\mu_x$ denotes the covariant measure with respect to the metric g_Σ, and x refers to the coordinate neighbourhood.

In Minkowski space-time and in the absence of external fields, the evolution operator $\mathcal{E}$ is required to be a unitary representation of time translations. Locally these are generated by a self-adjoint Hamiltonian which is therefore associated with the total energy of the dynamical system. In dynamical space-times such as $\mathcal{B}$, and therefore in general, $\mathcal{E}$ is only required to be a member of a contraction semi-group, which implies, in particular, $\|\mathcal{E}(t,t_0)\| \leq 1$ for all times t later than the initial time t_0, which translates in $\mathcal{B}$ to $\forall t \in (0,t_0)$. Fortunately, $\mathcal{E}$ can still be approximated locally and $\mathcal{H}$ can still be interpreted as a generator density. This is more than nomenclature: $\mathcal{H}$ is called accretive if $\mathrm{Re}(\langle\Psi|\mathcal{H}|\Psi\rangle) \geq 0$ for any state in the domain of $\mathcal{H}$, and it generates a contraction semi-group if and only if $\mathcal{H}$ is accretive. In other words, the pair (unitary, self-adjoint) is superseded by the pair (contractive, accretive) as the complete characterization of consistent quantum dynamics in generic space-times.

The main criterion for the non-existence of sinks for local information carriers is the following: Information associated with configurations of local bookkeeping devices is conserved in a globally hyperbolic space-time bordering on a space-like singularity Σ_0 if and only if the corresponding evolution is given by a contraction semi-group with vanishing norm on Σ_0.

Consider first a free system with effective potential $\mathcal{V} := \sqrt{-g_{tt}}(|\mathrm{grad}\Phi|^2 + m^2|\Phi|^2)$. The ground state of the corresponding dynamical system represented in $\mathcal{C}$ should be given by a Gaussian wave functional

$$\Psi_0[\phi^*,\phi](t) = \mathcal{N}(t)\exp\left(-[V^*]\mathcal{K}[V](t)\right) \ , \qquad (3)$$

where the degrees of freedom are organised in the two-tuple $V := (\phi,\phi^*)^{\mathrm{T}}$, and

$$[V^*]\mathcal{K}[V](t) := \tfrac{1}{2}\int_{\Sigma_t} \mathrm{d}\mu_{x,y}\ V^*(x)K(x,y,t)V(y) \quad (4)$$

is a quadratic functional with a bi-local representation given by the matrix K. In the absence of interactions, K is diagonal and only its trace enters (4). Furthermore, in the vicinity of the Schwarzschild singularity Σ_0, the evolution generator simplifies to $\mathcal{H} \cong \mathcal{G}(\Pi^*,\Pi)$. As a consequence, $\mathrm{tr}(K)$ becomes a spatial contact term towards the singularity, $\mathrm{tr}(K)(x,y,\varepsilon\tau) \cong k(\varepsilon\tau)\delta^{(3)}(x,y)$ with [9]

$$\begin{aligned}\mathrm{Im}\left(k(\varepsilon\tau)\right) &\cong -\frac{2}{(\varepsilon\tau)^3}\frac{1}{|\ln(\varepsilon\tau)|} \ , \\ \mathrm{Re}\left(k(\varepsilon\tau)\right) &\cong |\mathrm{Im}(C)|\,\frac{|\mathrm{Im}\left(k(\varepsilon\tau)\right)|}{|\ln(\varepsilon\tau)|} \ , \end{aligned} \qquad (5)$$

in the limit $\varepsilon \to 0$. This is consistent with the analysis of generalized Kasner space-times by Belinskii, Khalatnikov and Lifshitz: In the vicinity of space-like singularities, but still in the domain of general relativity, the spatial variation of local quantities is insignificant compared to temporal gradients. In the asymptotic region, the asymptotic kernel (5) implies for the wave functional (3) of the Gaussian ground state that

$$\lim_{\varepsilon\to 0}\Psi_0[\phi^*,\phi](\varepsilon\tau) = \lim_{\varepsilon\to 0}|\ln(\varepsilon\tau)|^{-\Lambda v(\Sigma_{\varepsilon\tau})} \ , \qquad (6)$$

where Λ denotes a short-distance cut-off and $v(\Sigma_{\varepsilon\tau})$ a volume regularization. This choice is satisfactory for the purpose at hand and does not affect the limit $\varepsilon \to 0$.

In order to show that $\mathcal{H}$ is accretive, it is sufficient to introduce an auxiliary source functional $\mathcal{J}$ describing the absorption and emission of fields ϕ minimally coupled to the associated local current density J, and define ${\Psi_0}^J[\phi](t) := \langle\phi|\exp(\mathcal{J})[\Phi](t)|\Psi_0\rangle_{\Sigma_t}$, which allows to replace compositions of the configuration operator Φ by the corresponding succession of functional derivatives with respect to the current. In the presence of the auxiliary source,

$$\begin{aligned}&\langle{\Psi_0}^J|\Phi^2(f)|{\Psi_0}^J\rangle_{\Sigma_t} = \\ &[f]\delta_J^{\,2}\exp\left\{\frac{1}{4}\frac{1}{\sqrt{\det(g_\Sigma)}}[J]\left[\mathrm{Re}(\mathcal{K})\right]^{-1}[J]\right\}\mathcal{P}_0(t) \ , \end{aligned} \quad (7)$$

where $[f]\delta_J^{\,2}$ denotes the second functional derivative with respect to J, smeared with an appropriate field configuration f. In the absence of the auxiliary source, the ground-state expectation (7) is real and semi-positive definite. Towards the singularity Σ_0, it approaches zero due to the temporal support granted by the probability density $\mathcal{P}_0(t) := \|\Psi_0[\phi]\|^2(t)$. Similarly,

$$\begin{aligned}&\langle{\Psi_0}^J|\Pi^2(f)|{\Psi_0}^J\rangle = \\ &\sqrt{\det(g_\Sigma)}|k|^2(t)\langle{\Psi_0}^J|\Phi^2(f)|{\Psi_0}^J\rangle_{\Sigma_t} \ . \end{aligned} \qquad (8)$$

The ground-state expectation value (8) is real, semi-positive definite, and approaches zero towards Σ_0. Therefore, $\langle{\Psi_0}^J|\mathcal{H}(f)|{\Psi_0}^J\rangle$ is always semi-positive definite and vanishes towards the black-hole singularity. It can be shown that these qualifications remain true for arbitrarily excited states. Hence, $\mathcal{H}$ is accretive and the quantum evolution is indeed given by a contraction semi-group [10]. Furthermore, information is conserved in $\mathcal{B}$ and cannot leave through the geodesic information sink Σ_0 bordering on $\mathcal{B}$. All these results remain true if (self-) interactions are included. We checked this explicitly [10], but it is also understandable at the qualitative level since still $\mathcal{H} \cong \mathcal{G}(\Pi^*,\Pi)$.

This result is rather intuitive: The ultimate reason behind the consistency of local quantum physics inside a black hole, even in a semi-classical set-up, is quantum completeness, which also renders information conservation sacrosanct. The geodesic information sink is closed because the probabilistic measure keeps Σ_0 void of any information carriers. As a consequence Σ_0 cannot be probed by local quantum physics, not even indirectly in the sense of allowing $\mathcal{B}$ to leak. Quantum fields are totally ignorant about the presence of Σ_0 and the corresponding complete event space can be interpreted as a physical space-time which is regular.

The absence of an information sink in the quantum theory can be reconsidered as follows. Since K is diagonal, we only consider ϕ-configurations for the sake of brevity. The probability-current density is given by

$$\mathcal{S}_0(x) := \frac{\sqrt{-g_{tt}}}{g_\Sigma}\left(\Psi_0\ \Pi(x)\ \Psi_0^* - \mathrm{h.c.}\right) \ , \qquad (9)$$

and satisfies the functional generalization of differential probability conservation, $\partial_t \mathcal{P}_0 + \mathrm{div}\mathcal{S}_0 = 0$, where

$$\mathrm{div}\mathcal{S}_0 := \int_{\Sigma_t} \mathrm{d}\mu_x \; \mathrm{i}\Pi\mathcal{S}_0 \;, \tag{10}$$

on any spatial hypersurface Σ_t , $t \in (0, t_0)$. Integrating this divergence over the field configuration space, continuity of probability amounts to

$$\partial_t \mathcal{W}(t) = \mathrm{i} \int_{\Sigma_t} \mathrm{d}\mu_x \frac{\sqrt{-g_{tt}}}{g_\Sigma} \left(\langle \Psi_0 | \Pi^2(x) | \Psi_0 \rangle - \text{h.c.} \right) , \tag{11}$$

where $\mathcal{W}(t)$ denotes the total probability for populating Σ_t with any field configuration, on- and off-shell. Towards the black-hole singularity $\langle \Psi_0 | \Pi^2(x) | \Psi_0 \rangle \in \mathbb{R}_0^+$, see (8), and so $\mathcal{W}(t)$ is conserved. The probability current cannot reach the geodesic information sink. Therefore there is no probability leakage in accordance with our former statement that Σ_0 cannot be populated with information carriers.

Discussion. In the region bordering on Σ_0 we may expect nontrivial support for the originally pure state $|\Psi_{\mathrm{S}}\rangle$ that formed the black hole. Instead of populating the interior $\mathcal{B}$ with arbitrary information carriers, consider a population originating from an Unruh state $|\mathcal{U}\rangle \equiv \sum |\Psi_{\mathrm{in}}\rangle \otimes |\Psi_{\mathrm{out}}\rangle$, where $|\Psi_{\mathrm{out}}\rangle$ denotes a state associated with Hawking radiation and $|\Psi_{\mathrm{in}}\rangle$ is the corresponding ingoing state. Let us choose a Cauchy initial hypersurface Σ_{t_0} in $\mathcal{B}$ such that the quantum content of the interior is $|\Psi_{\mathcal{B}}\rangle = |\Psi_{\mathrm{S}}\rangle \otimes |\Psi_{\mathrm{in}}\rangle$ on this hypersurface [3]. The configuration space contains dual states such as $\langle C| = \langle X| \otimes \langle R|$, where X represents the configuration fields that participated in the gravitational collapse, and R denotes the ingoing Hawking quanta. Evolving the states from Σ_{t_0} to $\Sigma_{\varepsilon\tau}$, the wave functional in configuration representation is given by $\Psi_{\mathcal{B}}[C](\varepsilon\tau) = {}_{\Sigma_{\varepsilon\tau}}\langle C|\mathcal{E}(\varepsilon\tau, t_0)|\Psi_{\mathcal{B}}\rangle_{\Sigma_{t_0}}$. In order to allow for a probabilistic interpretation, the evolution operator is required to satisfy the contraction property $\|\mathcal{E}(\varepsilon\tau, t_0)\| \leq 1$ towards Σ_0. Equivalently, its generator $\mathcal{H} = \mathcal{H}_{\mathrm{coll}}(X) + \mathcal{H}_{\mathrm{rad}}(R) + H_{\mathrm{coup}}(X, R)$ needs to be accretive. Here, H_{coup} describes the coupling between the quanta that participated in the gravitation collapse and the ingoing Hawking radiation. If a weak coupling regime is assumed, then $\Psi_{\mathcal{B}}[C](\varepsilon\tau) \approx \Psi_{\mathrm{S}}[X](\varepsilon\tau) \times \Psi_{\mathrm{in}}[R](\varepsilon\tau)$ to leading order in the coupling. The results of this letter show that $\Psi_{\mathrm{in}}[R](\varepsilon\tau)$ vanishes towards Σ_0. Provided $\Psi_{\mathrm{S}}[X](\varepsilon\tau)$ is sufficiently well behaved, it then follows that $\Psi_{\mathcal{B}}[C]$ vanishes at the border Σ_0 of physical space-time.

Hence, the wave functional of the black-hole interior satisfies a trivial Dirichlet boundary condition. This boundary condition restricts information configurations to $\mathcal{B}$ in which information processing is described by a contraction semi-group $\mathcal{E}(t, t_0)$ with an accretive generator $\mathcal{H}$. This generator shares properties with Dirichlet operators: Close to the boundary the dynamics trivialises to free evolution, corresponding to a geometrically induced asymptotic freedom, and information processing is only supported away from the boundary. This boundary condition is not imposed on the wave functional, it is rather a direct consequence of a quantum complete evolution that protects the probabilistic interpretation of the theory against the singular structure Σ_0 bordering on the physical space-time $\mathcal{B}$. The only conceivable way the evolution can achieve this within the usual approximation scheme is by depriving Σ_0 from rendering probabilistic support to local bookkeeping devices. As a consequence, the geometrical information sink is closed for local quantum physics, and there can be no leakage of information. The analysis was restricted to $\mathcal{B}$ which is sufficient to address the information paradox.

As a result, black-hole interiors respect information in the sense that information processing is free from paradoxes.

We thank Robert C. Myers for inspiring discussions and for suggesting the idea for this letter to us. We thank Achim Kempf, Maximilian Kögler and Florian Niedermann for fruitful discussions. We appreciate financial support of our work by the DFG cluster of excellence 'Origin and Structure of the Universe' and by TRR 33 'The Dark Universe'.

[1] J. Preskill, Proceedings, Black Holes, membranes, wormholes and superstrings (1992).
[2] P. Deift and B. Simon, Journal of Functional Analysis **23**, 218 (1976).
[3] G. T. Horowitz and J. Maldacena, Journal of High Energy Physics **2004**, 008 (2004).
[4] S. Hawking and R. Penrose, Proceedings of the Royal Society London A **314**, 529 (1970).
[5] For θ small, the quadratic form can be transformed into type-D Kasner line-element with exponents $(p_1, p_2, p_3) = (2/3, 2/3, -1/3)$, corresponding to a spatially anisotropic cosmology.
[6] M. Andrews, American Journal of Physics **44**, 1064 (1976).
[7] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, Advances in Physics **19**, 525 (1970).
[8] B. S. DeWitt, The global approach to quantum field theory, Vol. 114 (Oxford University Press, 2003).
[9] S. Hofmann and M. Schneider, Phys. Rev. D **91**, 125028 (2015).
[10] S. Hofmann and M. Schneider, Physical Review D **95**, 065033 (2017).

F Kasner analysis and connexion to the Schwarzschild case

Analogous to the Schwarzschild case, we determine the integral kernel by solving (4.3.1) in case of the Kasner space-time. We saw in Section 3.1 that the Schwarzschild metric in the interior can be approximated by a Kasner type-D form (3.14) which is given by

$$g = -d\tau \otimes d\tau + \tau^{4/3}(dx \otimes dx + dy \otimes dy) + \tau^{-2/3} dz \otimes dz. \tag{F.1}$$

For this metric the dispersion (Fourier transformed Laplace operator) is $\Omega(k, \tau) = \tau^{-4/3}k_x^2 + \tau^{-4/3}k_y^2 + \tau^{2/3}k_z^2 + m^2$. All Kasner space-times are as well Bianchi I space-times. For Kasner space-times the determinant of the spatial part and the full metric are equal $\det(g) = \det(g_\Sigma) = \tau^2$ which comes from the Kasner plane and Kasner sphere conditions. Another such property is given by two shrinking (or expanding) coordinates while the other is expanding (or shrinking respectively). It can not collapse to one point like a Friedmann universe or de Sitter. Therefore, the black hole, although often mentioned to be like a collapsing Friedmann universe inside, is completely unlike the Friedmann case which is in contrast to Kasner not a vacuum solution of Einstein's equation.

Taking space-time (3.14) and putting it into the calculation scheme of chapter 4 we calculate the ground state wave-functional. This analysis brings a few more insights also into the Schwarzschild case. Kasner space-times admits a kernel in the wave functional which in the limit of small times goes to

$$\widehat{K(\tau, k)} \sim -\frac{i}{\tau^2 \ln(\tau)}. \tag{F.2}$$

It is similar to the result of the full Schwarzschild interior metric (4.94). We have absorbed all real numbers into a constant which is for simplicity not shown here. The different exponent in the Schwarzschild case can be explained by the coordinate transformation where $t \to \tau^{2/3}$. Therefore, the result for the wave functional is consistent with the Schwarzschild case. Another remarkable fact is that the dependence on the polar angle through $\sin(\vartheta)$ has disappeared. The asymptotics are purely time-dependent and the contribution containing the momenta vanishes in the limit of small τ. In the vicinity of the singular hyperplane Σ_0 we find that the dominant contribution is given by the time coordinate. This is what we expected for the Schwarzschild case for geometrical reasons. Now, Kasner space-times allow to draw the same conclusion.

As a side remark this could as well have been guessed by considering the analysis of Belinskii, Khalatnikov and Lifshitz which says that near a singularity oscillations in time become huge compared to oscillations in space. Our analysis totally supports this, no contributions of the momenta arise in Fourier space which is equivalent to the fact that the kernel factorises into a time-dependent function and a δ-distribution.

However, cosistency has to be checked within the framework of quantum completeness. Deriving the wave functional for the Kasner type gives

$$\|\Psi^{(0)}\|^2 = \frac{|\mathcal{N}_0|^2}{\ln(\tau)^{\nu(\Sigma_\tau)\Lambda}} (\tau^{\frac{1}{2}} \ln(\tau))^{N(\Lambda)} \xrightarrow{\tau \to 0} 0. \tag{F.3}$$

All constants have been stored into $\mathcal{N}_0$ which is different from the one in (4.97). The result of the ground state amplitude yields the same behaviour in the vicinity of the singularity and the Kasner analysis is in full agreement with the aforementioned Schwarzschild case. The leading contribution is the same which shows consistency with the Schwarzschild case as well as the BKL conjecture.

G Heisenberg analysis of charge conservation inside a black hole

In this appendix, we show the connection between the Heisenberg and the Schrödinger picture. The derivation has been taken from the article [Eglseer et al., 2017]. The main idea of the correspondence is mediated by the measurement theory of the Heisenberg picture Let us first give an intuitive argument based on scaling relations for charge/information conservation in the black hole interior $\mathcal{B}$, before providing exact statements. In $\mathcal{B}$ consider a dynamical system $(\mathcal{L}, \Phi^*, \Phi)$, where Φ^*, Φ denote scalar fields charged under $U(1)$, and $\mathcal{L}$ is the corresponding Lagrange density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$, with the first term denoting the free theory $\mathcal{L}_0 = \Phi^* \Box \Phi$. The intuitive argument will be given in the absence of interactions, $\mathcal{L}_{\text{int}} \equiv 0$. Close to the space-like singularity Σ_0 bordering on $\mathcal{B}$,

$$ds^2 \cong -(t/r_g)dt^2 + (r_g/t)dr^2 + t^2 w, \tag{G.1}$$

where $\cong$ means equality up to sub-leading contributions in each term as Σ_0 is approached[1]. In this asymptotic regime,

$$\Box \cong (-r_g/t)(\partial_t^2 + (1/t)\partial_t) + (t/r_g)\partial_r^2 + (1/t^2)\partial_\sphericalangle^2. \tag{G.2}$$

Here $\partial_\sphericalangle^2$ denotes the usual angular part of the Laplace operator in $\mathbb{R}^3$ in Schwarzschild spherical coordinates.

The corresponding Green function G is sourced by $\delta(t - t')\delta(\sigma - \sigma')/\sqrt{-\det(g)}$, with σ and σ' denoting Schwarzschild spherical coordinates of events localised on Σ_t and $\Sigma_{t'}$,

[1] For θ small, the quadratic form can be transformed into type-D Kasner line-element with exponents $(p_1, p_2, p_3) = (2/3, 2/3, -1/3)$, corresponding to a spatially anisotropic cosmology.

respectively, and satisfies

$$D(t)G(t,\sigma;t',\sigma') = \delta(t-t')\delta(\sigma-\sigma') ,$$
$$D(t) := -r_g\partial_t(t\partial_t) + (t/r_g)t^2\partial_r^{\,2} + \partial_\triangleleft^{\,2} . \tag{G.3}$$

In order to estimate the asymptotic relevance of each term in the differential operator, consider $D(\varepsilon\tau)$ in the limit $\varepsilon \to 0$. The effective potential for free fields scales asymptotically as $1/t^2$ and develops a repulsive barrier. It is well-known from quantum mechanics [Andrews, 1976] that potentials of this type cannot be penetrated via tunnel processes.

Following the geometrical description of space-like singularities by Belinskii, Khalatnikov and Lifshitz [Belinskii et al., 1970], temporal variations dominate over spatial variations in the region bordering on Σ_0. Therefore, $D(\varepsilon\tau) \cong (1/\varepsilon)\partial_\tau(\tau\partial_\tau)$. The time-dependent part of the source distribution scales like $(1/\varepsilon)\delta(\tau - t'/\varepsilon)$. This effectively allows to split the Green function

$$G = T(t,t')P(\sigma,\sigma') \tag{G.4}$$

in the vicinity of Σ_0, with the asymptotic dynamics given by $\partial_\tau(\tau\partial_\tau T) \cong 0$. Here all identifiers labelling the eigenvalue problem of the Laplace operator have been suppressed for ease of notation. We find the asymptotic solution

$$T(t,t') \cong C_0(t') + C_1(t')\ln(t/r_g), \tag{G.5}$$

with $C_{0,1}$ sufficiently well behaved to guarantee a well-posed initial value problem on $\Sigma_{t'}$.

In order to appreciate the rather mild divergence of the asymptotic solution T, we introduce an emitter Q_{em} localised on $\Sigma_{t'}$, $t' \in (0,t_*)$ in the asymptotic domain, and an absorber Q_{ab} on $\Sigma_{\varepsilon\tau}$. For instance, consider $Q_{ab} = \delta(t-\varepsilon\tau)\ q_{ab}(\sigma)$, with q_{ab} encoding the spatial extension of the detector on $\Sigma_{\varepsilon\tau}$. This blueprint effectively replaces part of Σ_0 with a detector volume that can resolve arbitrary frequencies. Note that the asymptotic regime is controlled by the parameter ε while τ represents a constant instant of time.

The classical measurement process is described by the on-shell vertex density

$$\nu_{obs} = \sqrt{\det(g)}Q_{ab}^{*}\Phi_{os} + \text{c.c.} \tag{G.6}$$

with Φ_{os} denoting a linear functional of Q_{em} with a bi-local kernel given by G [DeWitt, 2003]. In the region bordering on Σ_0, as specified by the support properties of emitter and absorber,

$$\nu_{\rm obs} \cong t^2 \ln(t)\delta(t-\varepsilon\tau)\sin(\theta) q_{ab}\, F_{\rm em}, \tag{G.7}$$

where $F_{\rm em}$ contains the exclusive information on the emission process and depends only on source parameters. In particular, $F_{\rm em}$ is finite in accordance with a globally hyperbolic interior $\mathcal{B}$. For $\varepsilon \to 0$, the measurement of the emitter's influence on the detector gives a vanishing response, $\nu_{\rm obs} \cong 0$, in the distributional sense. This implies that no information carried by local bookkeeping devices can reach Σ_0. It is possible to be more specific about the emitter. As an example, the energy momentum tensor for the complex scalar field scales like $\mathcal{T} \propto 1/(\varepsilon\tau)^2$ on $\Sigma_{\varepsilon\tau}$ and develops a singularity towards Σ_0. It is easy to accommodate this observable in the above naive measurement prescription:

$$Q_{\rm em} = {\rm tr}\mathcal{T} \propto 1/t'^3 \tag{G.8}$$

for t' in the asymptotic domain. Let us consider two detector models in this case. First, again $Q_{ab} = \delta(t-\varepsilon\tau)q_{ab}(\sigma)$, resulting in a measurement of $\nu_{\rm obs} = 0$, as before. Second,

$$\widetilde{Q}_{ab} = M(\varepsilon\tau) U \otimes U, \tag{G.9}$$

where $U \cong \sqrt{\varepsilon\tau/r_g}\,{\rm d}t$, and $M(t)$ denotes the spatial volume integral over an energy density. The principle of minimal coupling underlying a measurement description based on $\widetilde{Q}_{ab}$ is of course the coupling to a gravitational wave, hence $\widetilde{Q}_{\rm em} = \mathcal{T}$. In this case, measuring the influence of the emitter on the absorber located at $\Sigma_{\varepsilon\tau}$ we find the scaling

$$\widetilde{\nu}_{\rm obs} \propto M(\varepsilon\tau)(\varepsilon\tau)^3 \ln(\varepsilon\tau). \tag{G.10}$$

From a phenomenological point of view, $\widetilde{Q}_{ab}$ is required to have nontrivial support towards Σ_0 and $\varepsilon M(\varepsilon\tau)$ needs to be bounded as $\varepsilon \in \mathbb{R}^+$ approaches zero. Then, $\widetilde{\nu} \cong 0$, as well, which only confirms that the asymptotic description of the tree-level measurement process is independent of the tensor providing the principal communication channel.

Before closing this section, let us briefly discuss the asymptotic diagnostics of Noether charges. The $U(1)$-current density is

$$j = \Phi^* P \Phi - {\rm c.c.}, \tag{G.11}$$

where P denotes the four-momentum. Projecting the current density onto U, we find the following scaling relation for the charge density ρ localized on $\Sigma_{\varepsilon\tau}$:

$$\rho(\varepsilon\tau) \cong \rho(t_*)(t_*/\varepsilon\tau)^{3/2}, \tag{G.12}$$

which formally diverges as Σ_0 is approached. Physical measurements of the charge $Q(\varepsilon\tau)$, however, are fine. In fact $Q(\varepsilon\tau) = Q(t_*)$, where t_* denotes a fiducial time in the asymptotic regime. Thus black-holes cannot be discharged through the geodesic singularity Σ_0 bordering on their interiors. Any active information sink would necessarily lead to charge depletion. Note that this discussion of asymptotic charge conservation is fully based on local physics inside black holes, and no reference to the usual global characterisation in the exterior is made.

Literaturverzeichnis

[Abbasbandy, 2006] Abbasbandy, S. (2006). Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method. *Applied Mathematics and Computation*, 172(1):485–490.

[Amour et al., 2015] Amour, L., Jager, L., and Nourrigat, J. (2015). On bounded pseudo-differential operators in Wiener spaces. *Journal of Functional Analysis*, 269(9):2747 – 2812.

[Andrews, 1976] Andrews, M. (1976). Singular potentials in one dimension. *American Journal of Physics*, 44(11):1064–1066.

[Arnowitt et al., 1960] Arnowitt, R., Deser, S., and Misner, C. W. (1960). Canonical variables for general relativity. *Physical Review*, 117(6):1595.

[Ashtekar et al., 2011] Ashtekar, A., Henderson, A., and Sloan, D. (2011). Hamiltonian formulation of the Belinskii-Khalatnikov-Lifshitz conjecture. *Physical Review D*, 83(8):084024.

[Ashtekar and Magnon, 1975] Ashtekar, A. and Magnon, A. (1975). Quantum fields in curved space-times. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 346, pages 375–394. The Royal Society.

[Balcerzak and Dabrowski, 2006] Balcerzak, A. and Dabrowski, M. P. (2006). Strings at future singularities. *Physical Review D*, 73(10):101301.

[Belinskii et al., 1970] Belinskii, V. A., Khalatnikov, I. M., and Lifshitz, E. M. (1970). Oscillatory approach to a singular point in the relativistic cosmology. *Advances in Physics*, 19(80):525–573.

[Bergmann et al., 1965] Bergmann, P., Cahen, M., and Komar, A. (1965). Spherically symmetric gravitational fields. *Journal of Mathematical Physics*, 6(1):1–5.

[Bianchi, 1898] Bianchi, L. (1898). Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti. *Mem. Soc. Ital. delle Scienze (3)*, 11:267–352.

[Birkhoff and Langer, 1923] Birkhoff, G. D. and Langer, R. E. (1923). *Relativity and modern physics*, volume 1. Harvard University Press Cambridge.

[Birrell and Davies, 1984] Birrell, N. D. and Davies, P. C. W. (1984). *Quantum fields in curved space*. Number 7. Cambridge university press.

[Blau et al., 2006] Blau, M., Frank, D., and Weiss, S. (2006). Scalar field probes of power-law space-time singularities. *Journal of High Energy Physics*, 2006(08):011.

[Bonneau et al., 2001] Bonneau, G., Faraut, J., and Valent, G. (2001). Self-adjoint extensions of operators and the teaching of quantum mechanics. *American Journal of physics*, 69(3):322–331.

[Boulware, 1975] Boulware, D. G. (1975). Quantum field theory in Schwarzschild and Rindler spaces. *Phys. Rev. D*, 11:1404–1423.

[Buchdahl, 1971] Buchdahl, H. (1971). Conformal flatness of the Schwarzschild interior solution. *American Journal of Physics*, 39(2):158–162.

[Collins, 1977] Collins, C. (1977). Global structure of the Kantowski–Sachs cosmological models. *Journal of Mathematical Physics*, 18(11):2116–2124.

[Collins, 1984] Collins, J. C. (1984). *Renormalization*. Cambridge university press.

[Damour et al., 2001] Damour, T., Henneaux, M., Julia, B., and Nicolai, H. (2001). Hyperbolic Kac–Moody algebras and chaos in Kaluza–Klein models. *Physics Letters B*, 509(3):323–330.

[de Bernardis et al., 2000] de Bernardis, P., Ade, P. A., Bock, J., Bond, J., Borrill, J., Boscaleri, A., Coble, K., Crill, B., De Gasperis, G., Farese, P., et al. (2000). A flat universe from high-resolution maps of the cosmic microwave background radiation. *Nature*, 404(6781):955–959.

[DeWitt, 1967] DeWitt, B. S. (1967). Quantum theory of gravity. I. The canonical theory. *Physical Review*, 160(5):1113.

[DeWitt, 1975] DeWitt, B. S. (1975). Quantum field theory in curved spacetime. *Physics Reports*, 19(6):295–357.

[DeWitt, 2003] DeWitt, B. S. (2003). *The global approach to quantum field theory*, volume 114. Oxford University Press.

[d'Inverno and Smallwood, 1980] d'Inverno, R. and Smallwood, J. (1980). Covariant 2+2 formulation of the initial-value problem in general relativity. *Physical Review D*, 22(6):1233.

[Dixon, 1970a] Dixon, W. G. (1970a). Dynamics of extended bodies in general relativity. I. momentum and angular momentum. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 314, pages 499–527. The Royal Society.

[Dixon, 1970b] Dixon, W. G. (1970b). Dynamics of extended bodies in general relativity. II. moments of the charge-current vector. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 319, pages 509–547. The Royal Society.

[Dixon, 1974] Dixon, W. G. (1974). Dynamics of extended bodies in general relativity. III. equations of motion. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 277(1264):59–119.

[Dobarro and Ünal, 2005] Dobarro, F. and Ünal, B. (2005). Curvature of multiply warped products. *Journal of Geometry and Physics*, 55(1):75–106.

[Eglseer et al., 2017] Eglseer, L., Hofmann, S., and Schneider, M. (2017). Information carriers closing in on the black-hole singularity. *arXiv preprint arXiv:1710.07645*.

[Ehlers and Kundt, 1962] Ehlers, J. and Kundt, W. (1962). Exact solutions of the gravitational field equations. In *The Theory of Gravitation*, pages 49–101. John Wiley & Sons, Inc.

[Ehlers and Rudolph, 1977] Ehlers, J. and Rudolph, E. (1977). Dynamics of extended bodies in general relativity center-of-mass description and quasirigidity. *General Relativity and Gravitation*, 8(3):197–217.

[Einstein and Rosen, 1935] Einstein, A. and Rosen, N. (1935). The particle problem in the general theory of relativity. *Phys. Rev.*, 48:73–77.

[Elizalde, 1987] Elizalde, E. (1987). Series solutions for the Klein-Gordon equation in Schwarzschild space-time. *Phys. Rev. D*, 36:1269–1272.

[Elizalde, 1988] Elizalde, E. (1988). Exact solutions of the massive Klein-Gordon-Schwarzschild equation. *Phys. Rev. D*, 37:2127–2131.

[Epstein and Glaser, 1973] Epstein, H. and Glaser, V. (1973). The role of locality in perturbation theory. In *Annales de l'IHP Physique théorique*, volume 19, pages 211–295.

[Feynman et al., 2005] Feynman, R. P., Hibbs, A. R., and Styer, D. F. (2005). *Quantum mechanics and path integrals.* Courier Corporation.

[Friedman, 1981] Friedman, C. N. (1981). Second order linear ODE and Riccati equations. *Journal of Mathematical Analysis and Applications*, 81(2):291–296.

[Geroch, 1970] Geroch, R. (1970). Domain of dependence. *Journal of Mathematical Physics*, 11(2):437–449.

[Gibbons et al., 1995] Gibbons, G., Horowitz, G. T., and Townsend, P. (1995). Higher-dimensional resolution of dilatonic black-hole singularities. *Classical and Quantum Gravity*, 12(2):297.

[Giddings and Strominger, 1989] Giddings, S. B. and Strominger, A. (1989). Baby universe, third quantization and the cosmological constant. *Nuclear Physics B*, 321(2):481–508.

[Gielen and Turok, 2016] Gielen, S. and Turok, N. (2016). Perfect quantum cosmological bounce. *Physical review letters*, 117(2):021301.

[Gomero et al., 2016] Gomero, G. I., Mota, B., and Rebouças, M. J. (2016). Limits of the circles-in-the-sky searches in the determination of cosmic topology of nearly flat universes. *Phys. Rev. D*, 94:043501.

[Gradshteyn and Ryzhik, 2014] Gradshteyn, I. S. and Ryzhik, I. M. (2014). *Table of integrals, series, and products.* Academic press.

[Gullstrand, 1922] Gullstrand, A. (1922). Allgemeine Lösung des statischen Einkörperproblems in der Einsteinschen Gravitationstheorie. *Ark. Mat. Astron. Fys*, 16.

[Gümrükçüoğlu et al., 2008] Gümrükçüoğlu, A., Kofman, L., and Peloso, M. (2008). Gravity waves signatures from anisotropic preinflation. *Physical Review D*, 78(10):103525.

[Guth, 1981] Guth, A. H. (1981). Inflationary universe: A possible solution to the horizon and flatness problems. *Phys. Rev. D*, 23:347–356.

[Hadamard, 1925] Hadamard, J. (1925). Lectures on Cauchy's problem in linear partial differential equations.

[Hatfield, 1992] Hatfield, B. (1992). Quantum field theory of point particles and strings. *Frontiers in Physics, Redwood City, CA: Addison-Wesley,— c1992*, 1.

[Hawking, 1965] Hawking, S. (1965). Occurrence of singularities in open universes. *Physical Review Letters*, 15(17):689.

[Hawking, 1966] Hawking, S. W. (1966). The occurrence of singularities in cosmology. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 294, pages 511–521. The Royal Society.

[Hawking, 1975] Hawking, S. W. (1975). Particle creation by black holes. *Commun. Math. Phys.*, 43:199–220.

[Hawking, 1976a] Hawking, S. W. (1976a). Black holes and thermodynamics. *Physical Review D*, 13(2):191.

[Hawking, 1976b] Hawking, S. W. (1976b). Breakdown of predictability in gravitational collapse. *Physical Review D*, 14(10):2460.

[Hawking and Ellis, 1973] Hawking, S. W. and Ellis, G. F. R. (1973). *The large scale structure of space-time*, volume 1. Cambridge university press.

[Hawking and Penrose, 1970] Hawking, S. W. and Penrose, R. (1970). The singularities of gravitational collapse and cosmology. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 314, pages 529–548. The Royal Society.

[Helliwell et al., 2003] Helliwell, T., Konkowski, D., and Arndt, V. (2003). Quantum singularity in quasiregular spacetimes, as indicated by Klein-Gordon, Maxwell and Dirac fields. *General Relativity and Gravitation*, 35(1):79–96.

[Higuchi, 1987] Higuchi, A. (1987). Forbidden mass range for spin-2 field theory in de Sitter spacetime. *Nuclear Physics B*, 282:397–436.

[Hofmann and Schneider, 2015] Hofmann, S. and Schneider, M. (2015). Classical versus quantum completeness. *Physical Review D*, 91(12):125028.

[Hofmann and Schneider, 2017] Hofmann, S. and Schneider, M. (2017). Non-gaussian ground-state deformations near a black-hole singularity. *Physical Review D*, 95(6):065033.

[Holzhey and Wilczek, 1992] Holzhey, C. F. and Wilczek, F. (1992). Black holes as elementary particles. *Nuclear Physics B*, 380(3):447–477.

[Hörmander, 1983] Hörmander, L. (1983). The analysis of linear partial differential operators II. differential operators with constant coefficients. *Die Grundlehren der mathematischen Wissenschaften* .

[Hörmander, 1990] Hörmander, L. (1990). The analysis of linear partial differential operators I, distribution theory and Fourier analysis, Grundlehren der mathematischen Wissenschaften 256.

[Hörmander, 2007] Hörmander, L. (2007). *The analysis of linear partial differential operators III: Pseudo-differential operators*, volume 274. Springer Science & Business Media.

[Hörmander, 2009] Hörmander, L. (2009). *The analysis of linear partial differential operators IV: Fourier integral operators*, volume 274. Springer Science & Business Media.

[Horowitz and Maldacena, 2004] Horowitz, G. T. and Maldacena, J. (2004). The black hole final state. *Journal of High Energy Physics*, 2004(02):008.

[Horowitz and Marolf, 1995] Horowitz, G. T. and Marolf, D. (1995). Quantum probes of spacetime singularities. *Physical Review D*, 52(10):5670.

[Horowitz and Myers, 1995] Horowitz, G. T. and Myers, R. (1995). The value of singularities. *General Relativity and Gravitation*, 27(9):915–919.

[Horowitz and Polchinski, 2002] Horowitz, G. T. and Polchinski, J. (2002). Instability of spacelike and null orbifold singularities. *Physical Review D*, 66(10):103512.

[Horowitz and Steif, 1990] Horowitz, G. T. and Steif, A. R. (1990). Spacetime singularities in string theory. *Physical Review Letters*, 64(3):260.

[Horowitz and Strominger, 1991] Horowitz, G. T. and Strominger, A. (1991). Black strings and p-branes. *Nuclear Physics B*, 360(1):197–209.

[Ishibashi and Hosoya, 1999] Ishibashi, A. and Hosoya, A. (1999). Who's afraid of naked singularities? probing timelike singularities with finite energy waves. *Physical Review D*, 60(10):104028.

[Ishibashi and Wald, 2003] Ishibashi, A. and Wald, R. M. (2003). Dynamics in non-globally-hyperbolic static spacetimes: II. general analysis of prescriptions for dynamics. *Classical and Quantum Gravity*, 20(16):3815.

[Israel, 1967] Israel, W. (1967). Event horizons in static vacuum space-times. *Phys. Rev.*, 164:1776–1779.

[Jackiw, 1990] Jackiw, R. (1990). Analysis on infinite dimensional manifolds: Schrödinger representation for quantized fields. *Field theory and particle physics*, page 731.

[Kasner, 1921] Kasner, E. (1921). Geometrical theorems on Einstein's cosmological equations. *American Journal of Mathematics*, 43(4):217–221.

[Kerr, 1963] Kerr, R. P. (1963). Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.*, 11:237–238.

[Kleinert, 2009] Kleinert, H. (2009). *Path integrals in quantum mechanics, statistics, polymer physics, and financial markets*. World Scientific.

[Kobayashi and Nomizu, 1963] Kobayashi, S. and Nomizu, K. (1963). *Foundations of differential geometry*, volume 1. New York.

[Kobayashi and Nomizu, 1969] Kobayashi, S. and Nomizu, K. (1969). *Foundations of differential geometry, vol. II*, volume 1.

[Kofman et al., 2011] Kofman, L., Uzan, J.-P., and Pitrou, C. (2011). Perturbations of generic Kasner spacetimes and their stability. *Journal of Cosmology and Astroparticle Physics*, 2011(05):011.

[Konkowski and Helliwell, 1985] Konkowski, D. and Helliwell, T. (1985). Cosmologies with quasiregular singularities. II. stability considerations. *Physical Review D*, 31(6):1195.

[Konkowski et al., 1985] Konkowski, D., Helliwell, T., and Shepley, L. (1985). Cosmologies with quasiregular singularities. I. spacetimes and test waves. *Physical Review D*, 31(6):1178.

[Konkowski and Helliwell, 2001] Konkowski, D. A. and Helliwell, T. M. (2001). Letter: quantum singularity of quasiregular spacetimes. *General Relativity and Gravitation*, 33(6):1131–1136.

[Konkowski et al., 2003] Konkowski, D. A., Helliwell, T. M., and Wieland, C. (2003). Quantum singularity of Levi-Civita spacetimes. *Classical and Quantum Gravity*, 21(1):265.

[Kruskal, 1960] Kruskal, M. D. (1960). Maximal extension of Schwarzschild metric. *Physical review*, 119(5):1743.

[Lancaster and Rodman, 1995] Lancaster, P. and Rodman, L. (1995). *Algebraic Riccati equations*. Clarendon press.

[Lasiecka and Triggiani, 1991] Lasiecka, I. and Triggiani, R. (1991). Differential and algebraic Riccati equations with application to boundary/point control problems: continuous theory and approximation theory. *Lecture notes in control and Information Sciences*, 164:1–160.

[Maldacena, 2003] Maldacena, J. (2003). Eternal black holes in anti-de Sitter. *Journal of High Energy Physics*, 2003(04):021.

[Maldacena and Susskind, 2013] Maldacena, J. and Susskind, L. (2013). Cool horizons for entangled black holes. *Fortschritte der Physik*, 61(9):781–811.

[Modesto, 2004] Modesto, L. (2004). Disappearance of the black hole singularity in loop quantum gravity. *Physical Review D*, 70(12):124009.

[Neumann, 1930] Neumann, J. v. (1930). Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren. *Mathematische Annalen*, 102(1):49–131.

[Newman et al., 1963] Newman, E., Tamburino, L., and Unti, T. (1963). Empty-space generalization of the Schwarzschild metric. *Journal of Mathematical Physics*, 4(7):915–923.

[Newman and Janis, 1965] Newman, E. T. and Janis, A. (1965). Note on the Kerr spinning-particle metric. *Journal of Mathematical Physics*, 6(6):915–917.

[Nordström, 1918] Nordström, G. (1918). On the energy of the gravitation field in Einstein's theory. *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences*, 20:1238–1245.

[Odibat and Momani, 2008] Odibat, Z. and Momani, S. (2008). Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order. *Chaos, Solitons & Fractals*, 36(1):167–174.

[Ohanian and Ruffini, 2013] Ohanian, H. C. and Ruffini, R. (2013). *Gravitation and spacetime*. Cambridge University Press.

[Painlevé, 1921] Painlevé, P. (1921). La mécanique classique et la théorie de la relativité. *Comptes Rendus Academie des Sciences (serie non specifiee)*, 173:677–680.

[Penrose, 1965] Penrose, R. (1965). Gravitational collapse and space-time singularities. *Physical Review Letters*, 14(3):57.

[Petrov, 1954] Petrov, A. Z. (1954). Classification of spaces defining gravitational fields. *Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki*, 114(8):55–69.

[Polyanin and Zaitsev, 1995] Polyanin, A. D. and Zaitsev, V. F. (1995). Exact solutions for ordinary differential equations. *Campman and Hall/CRC*, 2.

[Rauch and Reed, 1973] Rauch, J. and Reed, M. (1973). Two examples illustrating the differences between classical and quantum mechanics. *Communications in Mathematical Physics*, 29(2):105–111.

[Raychaudhuri, 1957] Raychaudhuri, A. (1957). Singular state in relativistic cosmology. *Physical Review*, 106(1):172.

[Raychaudhuri and Maiti, 1979] Raychaudhuri, A. and Maiti, S. (1979). Conformal flatness and the Schwarzschild interior solution. *Journal of Mathematical Physics*, 20(2):245–246.

[Reed and Simon, 1975] Reed, M. and Simon, B. (1975). *Methods of modern mathematical physics, Vol. II.* Academic Press, New York.

[Reed and Simon, 1978] Reed, M. and Simon, B. (1978). *Analysis of operators. Methods of modern mathematical physics IV.* Academic Press, New York.

[Reed and Simon, 1979] Reed, M. and Simon, B. (1979). *Methods of mathematical physics III: Scattering Theory.* Academic Press, New York.

[Reed and Simon, 1980] Reed, M. and Simon, B. (1980). *Methods of modern mathematical physics. vol. 1. Functional analysis.* Academic.

[Reid, 1972] Reid, W. T. (1972). *Riccati differential equations.* Elsevier.

[Reissner, 1916] Reissner, H. (1916). Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen theorie. *Annalen der Physik*, 355(9):106–120.

[Schwarzschild, 1916] Schwarzschild, K. (1916). Über das Gravitationsfeld eines Massenpunktes nach der Einstein'schen Theorie. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften*, 1:189.

[Schwinger, 1951] Schwinger, J. (1951). On the Greens functions of quantized fields. I. *Proceedings of the National Academy of Sciences*, 37(7):452–455.

[Shubin, 1998] Shubin, M. (1998). Classical and quantum completeness for the Schrödinger operators on non-compact manifolds. Technical report.

[Simon, 1971] Simon, B. (1971). *Quantum mechanics for Hamiltonians defined as quadratic forms.* Princeton University Press.

[Simon, 1979] Simon, B. (1979). *Functional integration and quantum physics*, volume 86. Academic press.

[Simon, 1982] Simon, B. (1982). Schrödinger semigroups. *Bulletin of the American Mathematical Society*, 7(3):447–526.

[Simon, 1991] Simon, B. (1991). Fifty years of eigenvalue perturbation theory. *Bull. Amer. Math. Soc. (N.S.)*, 24(2):303–319.

[Simon, 2015a] Simon, B. (2015a). *A Comprehensive Course in Analysis.* AMS American Mathematical Society.

[Simon, 2015b] Simon, B. (2015b). *Harmonic Analysis.* AMS American Mathematical Society.

[Simon, 2015c] Simon, B. (2015c). *Operator theory.* AMS American Mathematical Society.

[Simon, 2015d] Simon, B. (2015d). $\mathsf{P}(\Phi)_2$ *Euclidean (Quantum) Field Theory.* Princeton University Press.

[Stewart and Hájíček, 1973] Stewart, J. and Hájíček, P. (1973). Can spin avert singularities? *Nature*, 244(136):96–96.

[Stone, 1929a] Stone, M. (1929a). Linear transformations in Hilbert space: II. analytical aspects. *Proceedings of the National Academy of Sciences*, 15(5):423–425.

[Stone, 1929b] Stone, M. H. (1929b). Linear transformations in Hilbert space. I. geometrical aspects. *Proceedings of the National Academy of Sciences*, 15(3):198–200.

[Stone, 1930] Stone, M. H. (1930). Linear transformations in Hilbert space III. operational methods and group theory. *Proceedings of the National Academy of Sciences*, 16(2):172–175.

[Stone, 1932] Stone, M. H. (1932). On one-parameter unitary groups in Hilbert space. *Annals of Mathematics*, pages 643–648.

[Szekeres, 1960] Szekeres, G. (1960). On the singularities of a Riemannian manifold. *Publicationes Mathematicae Debrecen 7, 285 (1960)*, 7:285.

[Taub, 1951] Taub, A. H. (1951). Empty space-times admitting a three parameter group of motions. *Annals of Mathematics*, pages 472–490.

[Traschen and Brandenberger, 1990] Traschen, J. H. and Brandenberger, R. H. (1990). Particle production during out-of-equilibrium phase transitions. *Physical Review D*, 42(8):2491.

[Unruh, 1976] Unruh, W. G. (1976). Notes on black-hole evaporation. *Physical Review D*, 14(4):870.

[van Baal and Bais, 1983] van Baal, P. and Bais, F. (1983). Lightlike singularities in compactified supergravity. *Physics Letters B*, 133(5):295–299.

[Wald, 1978] Wald, R. M. (1978). Axiomatic renormalization of the stress tensor of a conformally invariant field in conformally flat spacetimes. *Annals of Physics*, 110(2):472–486.

[Wald, 1979] Wald, R. M. (1979). Note on the stability of the Schwarzschild metric. *Journal of Mathematical Physics*, 20(6):1056–1058.

[Wald, 1980] Wald, R. M. (1980). Dynamics in nonglobally hyperbolic, static space-times. *Journal of Mathematical Physics*, 21(12):2802–2805.

[Wald, 1994] Wald, R. M. (1994). *Quantum field theory in curved spacetime and black hole thermodynamics*. University of Chicago Press.

[Weyl, 1910] Weyl, H. (1910). Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen. *Mathematische Annalen*, 68(2):220–269.

Acknowledgements

Zuallererst möchte ich Dir Stefan für die gesamte Zeit danken. Du warst ein großartiger Doktorvater, der diese Bezeichnung mehr als nur verdiente. Danke für Dein Vertrauen, die gewaltige Unterstützung, in persönlicher und insbesondere wissenschaftlicher Hinsicht. Ohne all Deine Mühen hätte ich es sicherlich nicht geschafft. Ich habe die unzähligen Diskussionen über Physik sehr genossen. Du schafftest es, eine Umgebung inmitten einer Umwelt aus Beton und Asbest zu kreieren, in dem die eigene Kreativität wachsen und florieren kann.

Auch bedanke ich mich herzlich bei Herta für die Assistenz bei allen administrativen Angelegenheiten, welche mich an die schneller Grenzen meiner Befähigung brachten als wissenschaftliche Fragen und bei Deiner Hilfe auf der Suche nach Geld.

Many thanks to my colleagues and office mates who went part of the way with me (the order has no meaning I just wrote who came to my mind, I was too lazy to arrange them alphabetically): Sofia Müller, David Licht, Cora Uhlemann, Thomas Haugg, Tehseen Rug, Lukas Gründing, Leila Mirzagholi, Davide Gualdi, Paul Hunt, Tobias Hofbaur, Lasma Alberte, Cecilia Giavoni, Christoph Chiaffrino, Felix Berkhahn, Florian Niedermann, Robert Schneider, Dennis Schimmel, Sebastian Konopka, Ludwig Eglseer, Maximilian Kögler, Maximilian Urban, Alexis Kassiteridis, Ottavia Balducci, Daniel Weiss, Katrin Hammer, Thomas Steingasser, Raoul Letschka, Max Warkentin. I enjoyed our discussions about physics, politics, religion, and bullshit. You all made it a great time in Munich. I also thank those people I forgot to mention explicitly.

Vielen Dank an alle Mitglieder der Prüfungskomitees: Stefan Hofmann, Peter Mayr insbesondere für die Bereitschaft, Zweitkorrektor zu sein, Otmar Biebel und Ilka Brunner, besonderer Dank gilt auch den Ersatzprüfern Ivo Sachs and Dorothee Schaile. I would like to thank all those people I met aside from the LMU who have taught me essential and important lessons in science: Abhay Ashtekar, Jack Borthwick, Bernard Kay, Ingemar Bengtsson, Lisette Jager, Jan Dereziński, Robert Myers, and Yasha Neiman.

Many thanks to my reference letter writers Stefan Hofmann, Ivo Sachs, Ingemar Bengtsson, and Jochen Weller.

Ein ganz besonderer Dank geht an meine Freunde der ersten Stunde des Studiums: Michael Opitsch und Peter Sterflinger. Es war eine grandiose Zeit und ich bedanke mich bei Euch für die Stunden der geistigen Zerstreuung, in denen wir Seite an Seite die Welt vor dem Bösen retteten. Natürlich auch an meine Freunde abseits der Universität: Michael Opitsch, Peter Sterflinger, Veronika Böltl, Robert Heigermoser, Ariane Steinegger, Lena Lämmle,... und allen die ich vergaß.

Maximilian Kögler and Leila Mirzagholi deserve special gratitude for proofreading my thesis and for all the brilliant advice you gave me which improved the manuscript infinitely.

Un ringraziamento speciale va a Cecilia Giavoni, che è stata una luce nella mia ora più buia. Mi hai detto le parole giuste al momento giusto. Pertanto I miei eterni ringraziamenti sono dovuti a te. Sei una delle persone più importanti che ho incontrato nel mio viaggio.

Ich danke dem Exzellenzcluster und der Humboldt-Stiftung für die Finanzierung dieser Arbeit.

Besonderer Dank gilt meinen Eltern Petra und Christoph Schneider für all die Unterstützung während der letzten Jahre. Insbesondere möchte ich meiner Mutter danken, leider kenne ich keine Sprache, die ein Wort besitzt, welches die Größe meiner Dankbarkeit auszudrücken vermag.