Insurance Modeling in Continuous Time

Yinglin Zhang

Dissertation an der Fakultät für Mathematik, Informatik und Statistik der Ludwig-Maximilians-Universität München

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Erstgutachterin:	Prof. Dr. Francesca Biagini
Zweitgutachter:	Prof. Dr. Patrick Cheridito
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Eidesstattliche Versicherung
(Siehe Promotionsordnung vom 12.07.11, §8, Abs. 2 Pkt. 5)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne enerlaubte Beihilfe angefertigt ist.

München, 26.07.2018 Yinglin Zhang

Zusammenfassung

In dieser Dissertation behandeln wir das Pricing und Hedging von Versicherungs-Verbindlichkeiten, indem wir Konzepte und Methoden, die kürzlich in der mathematische Literatur zu Finanzmärkten entwickelt wurden, auf die Modellierung von Lebens- und Schadenversicherungsmärkten erweitern.

Wir stellen zum ersten Mal einen einheitlichen Rahmen für die traditionell getrennt voneinander betrachteten Lebens- und Schadenversicherungen vor, indem wir das klassische Reduced-form Framework verallgemeinern und so eine nicht triviale Abhängigkeitsstruktur zwischen dem Finanz- und dem Versicherungsmarkt einführen. Das Problem des Pricings und Hedgings von Versicherungsprodukten wird durch eine Kombination der Risk-Minimization Methode und des Benchmark Ansatzes gelöst. Der Lebensversicherungsfall wird dann im Detail in einem Polynomial Diffusion Modell untersucht, welches zum einen flexibel ist und zum anderen die Möglichkeit bietet, explizite Formeln für das Pricing und Hedging zu erhalten. Neben einem modellabhängigen Rahmen entwickeln wir zusätzlich auch einen modellfreien Rahmen für Versicherungsmärkte, in welchem wir eine Familie von Wahrscheinlichkeitsmaßen betrachten, die paarweise singulär zueinander sein können. Zum einen betrachten wir zum ersten Mal das Problem des Superhedgings von Zahlungsströmen unter Modellunsicherheit in stetiger Zeit. Zum anderen konstruieren wir explizit einen konsistenten sublinearen bedingten Erwartungswert auf einer progressiv vergrößerten Filtration, der existierende Resultate verallgemeinert, welche nur auf dem kanonischen Raum und seiner natürlichen Filtration gelten. In Anbetracht der Superhedging-Resultate, wird dieser sublineare bedingte Erwartungswert, als Preis-Operator für Versicherungsansprüche genutzt.

Abstract

In this dissertation we consider the problem of pricing and hedging insurance liabilities, by extending concepts and methodologies recently introduced in the mathematical literature for financial markets to the modeling of life and non-life insurance markets.

We propose for the first time a unified framework for both life and non-life insurance, which are traditionally studied separately, by generalizing the classic reduced-form framework, in order to introduce a nontrivial dependence structure between the financial market and the insurance market. The pricing and hedging problem of insurance products is solved by using risk-minimization method combined with the benchmark approach. The case of life insurance is then studied in detail in a polynomial diffusion model, which offers at the same time flexibility and the possibility of obtaining explicit pricing and hedging formulas. Beside model-dependent setting, we develop also a model-free framework for insurance markets, where we consider a family of probability measures, possibly mutually singular to each other. On one hand, we introduce and analyze for the first time the problem of superhedging payment streams under model uncertainty in continuous time. On the other hand, we construct explicitly a consistent sublinear conditional expectation on a progressively enlarged filtration, which generalizes existing results valid only on the canonical space endowed with the natural filtration. This sublinear conditional expectation is then used as pricing operator for insurance claims in view of the superhedging results.

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Contents

Chapter 1

Introduction

This dissertation is based on three papers on insurance market modeling in continuous time, namely [18], [19] and [20]. We consider the problem of pricing and hedging insurance liabilities in continuous time, by extending concepts and methodologies recently introduced in the mathematical literature for financial markets. We propose for the first time a unified framework for both life and non-life insurance, which are traditionally studied separately. In order to introduce a nontrivial dependence structure between the financial market and the insurance market, we introduce a generalization of the classic reduced-form framework and give an explicit bottom-up construction. The pricing and hedging problem is solved by using risk-minimization method combined with the benchmark approach. The case of life insurance is then studied in detail in a polynomial diffusion model, which offers at the same time flexibility and the possibility of obtaining explicit pricing and hedging formulas. Beside model-dependent setting, we develop also a model-free framework for insurance markets, where we consider a family of probability measures, possibly mutually singular to each other. On one hand, we introduce and analyze for the first time the problem of superhedging payment streams under model uncertainty in continuous time. On the other hand, we construct explicitly a consistent sublinear conditional expectation on a progressively enlarged filtration, which generalizes existing results valid only on the canonical space endowed with the natural filtration. This sublinear conditional expectation is then used as pricing operator for insurance claims in view of the superhedging results.

The two broad types of insurance are life and non-life insurance. The first is linked to the decease of persons. The second covers all other forms of insurance, such as theft insurance, motor insurance, flood insurance, etc. Non-life insurance can be further classified in catastrophe insurance[1], which covers low-probability high-cost events such as natural catastrophes, terrorist attacks, etc.; and non-catastrophe insurance, which covers high-probability low-cost events such as car accident, house damage, etc. Unlike life insurance, in the case of non-life insurance there are typically reporting delay, which can be even several years, and further updating and development after the accident itself. So far, the two types of insurance have been studied separately and there is no

[1] See e.g. [27] for the distinction between catastrophe and non-catastrophe insurance.

unified framework for both life and non-life insurance in continuous time. While there is a large amount of literature concerning life insurance market modelling in continuous time, see e.g. [78], [79], [23], [28], [33], [7], [16], [17] and [15], non-life insurance is mostly studied in discrete time and/or state space, see e.g. [66], [69], [54]. Ideas of continuous time modeling for non-life insurance can be found in e.g. [5], [8], [77], [32], [84], [83] and [106]. However, these papers do not consider a nontrivial dependence structure between the insurance market and the financial market. Here we introduce for the first time a unified framework for both life and non-life insurance, where the following common characteristics are considered in modeling generic insurance market throughout this dissertation. Firstly, insurance claims are typically payment streams, not only with a single payoff at the maturity as in the case of contingent claims. Indeed, most of insurance products are combination of the following three building blocks:

- pure endowment: the insurer pays if a particular random event occurs after the maturity of the contract;
- term insurance: the insurer pays if a particular random event occurs before the maturity of the contract;
- annuity: the insurer pays a continuous cash flow as long as a particular random event does not occur, or the contract is valid.

Hence, the pricing and hedging problem must to be understood and solved for generic payment stream. The second feature of insurance market modeling throughout this thesis is the enlargement of filtration. Let filtration $\mathbb{F}$ describe the reference information flow which includes financial market information and other social-economic indicators, $\mathbb{H}$ the internal information flow only available to the insurance company and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ the global information flow. On one hand, we emphasize that filtration $\mathbb{H}$ is not included in $\mathbb{F}$. Indeed, insurance policies are related to individual random events such as the decease of a person, car accidents, house damages, etc., which are information available only to the insurance company and not deducible from reference information. In other words, the occurrence times of these individual events are not $\mathbb{F}$-stopping times and the filtration $\mathbb{H}$ is strictly different from $\mathbb{F}$. On the other hand, we stress that, the recent introduction of insurance linked derivatives on the financial market, such as mortality derivative, weather derivatives, etc., creates a bridge between the capital market and the insurance market, as discussed in [11]. These derivatives are written on some macro-factor linked indexes, which describe the occurrence intensity of a certain type of random events, such as mortality intensity of a given population, or rain intensity of a given region, etc. This allows insurance companies to hedge insurance liabilities by investing on the financial market, which offers much more potential in terms of liquidity and hedging capacity. Hence, it is important to model a nontrivial dependency between the reference filtration $\mathbb{F}$ and the internal insurance filtration $\mathbb{H}$.

Under the above described structure, the combined market is intrinsically incomplete even when the reference market is complete. Similar discussion can be found in e.g. [15]. Perfect hedging of a $\mathbb{G}$-adapted insurance claim by means of $\mathbb{F}$-adapted financial assets is not possible and there is no unique no-arbitrage claim price. Hence, a pricing-hedging method for payment streams in incomplete markets must to be chosen. There are several dynamic pricing-hedging dualities, based on different mathematical decompositions. In this thesis, we concentrate on the following two methods.

- *Risk-minimization method*, based on Galtchouk–Kunita–Watanabe decomposition of a square integrable martingale on orthogonal subspaces. This method provides a mean self-financing hedging strategy which minimizes the expected quadratic risk at any time, and a risk-minimization price process consistent with the strategy. See e.g. [52] for a single payoff and e.g. [79], [78], [33], [7], [17] and [15] for payment stream and application to insurance contracts.
- *Superhedging approach*, based on optional decomposition of supermartingale. This method provides a strategy which superhedges the claim at any time and a superhedging price process consistent with the superhedging strategy. See e.g. [68] and [50] for a single payoff and e.g. [51], [94] and [95] for payment stream in discrete time.

In the first part of the dissertation we discuss our setting under the statistical or real-world probability measure P. In this context, we will mainly use risk-minimization method combined with benchmark approach, developed in e.g. [96], [98] and [99]. That is, we do not assume the existence of martingale measures, but work directly under the real-world measure P assuming only the existence of a benchmark or numéraire portfolio. As shown in [60], this assumption is equivalent to no unbounded profit with bounded risk condition, weaker than the classic risk-neutral condition of no free-lunch with vanishing risk (see e.g. [35]). Combining benchmark approach and risk-minimization method appears to be natural and suitable in modeling a hybrid market, as discussed in [11], rather than selecting a particular equivalent martingale measure. The so called benchmarked risk-minimization method is analyzed in e.g. [97], [99], [44] and [12] for a single payoff. Here we extend these results to the case of payment stream.

Beside model-dependent setting for the insurance market, we develop here also a model-free framework, where no prior is chosen a priori and a generic family of probability measures possibly mutually singular to each other is taken into account. The topic of model uncertainty has become particularly relevant after the financial crisis, and intensive study has been done in different directions. Existing literature for insurance modeling under model uncertainty considers only dominated probability family, e.g. [72], [65] and [29]. When we take into account a non-dominated probability family, it is necessary to go beyond the classic probability theory. Indeed, as discussed in [109], the core of the underlying stochastic analysis is the aggregation problem of stochastic notions defined traditionally under one probability measure (such as conditional expectation, stochastic integral, semimartingale decomposition, etc.) into one independent of the underlying prior. Many independent results have been achieved by using different research approaches and applied to financial market modeling, see e.g. [40], [91], [39], [108], [53], [85], [58], [87], [93], [21], [1] and [82]. However, the above results are valid only on the canonical space endowed with the natural filtration and do not allow structure with general filtrations, as noted in [2]. In [2] the case of initial enlargement of filtration is solved, but other forms of filtration enlargement remain open problems. Here we propose a solution for progressive enlargement of filtration by introducing an external totally inaccessible jump, based on the canonical filtration construction of the reduced-form framework in Section 6.5 of [24], and consistent with the existing construction in [87] on the canonical space endowed with the natural filtration. The pricing and hedging problem in this context is solved by using the superhedging approach, which appears to be a natural choice in a setting under model uncertainty. Superhedging with respect to a non-dominated probability family has been widely studied in recent years, see e.g. [88], [80], [100], [43], [9], [57], [86] and [56]. However, these results do not cover the case of generic payment streams, which are studied only in discrete time in e.g. [51], [94] and [95]. The superhedging problem for a generic payment stream under model uncertainty and in continuous time will be analyzed here for the first time.

More precisely, the structure of the thesis is as follows.

In Chapter 2, based on [20] and partially on [18], we present a unified framework in continuous time for life and non-life insurance by using a direct modeling approach and analyze the benchmarked risk-minimization method for insurance products. While it is common to model life insurance within a setting with nontrivial filtration enlargement, it is however not the case of non-life insurance. In the present literature of non-life insurance in continuous time, the insurance filtration is either not distinguished from the reference filtration, see e.g. [32], [84] and [83], or assumed to be independent from the reference filtration, see e.g. [8]. The existing approach used in these papers is to assume the insurance internal filtration $\mathbb{H}$ as generated by a marked point process, which describes the insurance portfolio movement, and to model its $\mathbb{H}$-compensator. This compensator is then involved in the pricing and hedging formulas. This approach, however, seems to be not convenient in the case of general filtrations. Indeed, with respect to a generic filtration, it does not always exist a marked point process with given compensator, and the compensator does not always determine uniquely the law of the process on a generic σ-algebra. Hence, we introduce here a new framework in order to overcome these difficulties. Our framework uses a direct modeling approach as in Section 5.1 and 9.1.2 of [24] and allows an explicit bottom-up construction to treat more general filtrations. The classic reduced-form framework for life insurance is then included as a special case. We stress that, while for life insurance modeling within the classic reduced-form framework, the compensator approach and the direct modeling approach coincide, see e.g. [24], it is however not the case for non-life insurance. Under our new setting, we consider a homogeneous insurance portfolio and model the accident times of all claims in the portfolio, in a way such that they are not $\mathbb{F}$-stopping times but admit a common $\mathbb{F}$-adapted intensity process μ. Reporting delay is taken into account and further updating of each claim is modelled by independent marked point process, which describes updating times and related losses. The insurance internal information starts only from the moment of the first reporting. This structure includes life-insurance as special case but presents at the same time a significant difference from the classic reduced-form framework. We note that non-life insurance policies are linked to properties and hence have costs sensitive to inflation fluctuation. Consequently, we take in to account to role of inflation as already proposed in [8], [115] and [89]. We assume the presence of derivatives linked to the intensity process μ and to the inflation index on the financial market, which creates a hybrid nature of the combined market. Under the statistical probability measure, the so called real-world pricing formula is used for pricing purpose. The real-world pricing formula is consistent with the benchmarked risk-minimization price for payment streams, as shown in Section 2.4. Detailed study of non-life and life insurance products is postponed respectively to Chapter 3 and Chapter 4. Our framework contributes to give an insight into the attractiveness of a more developed hybrid market and the potential of non-life insurance linked financial products, especially the non-catastrophe non-life insurance linked derivatives which are still not common and do not cover all insurance linked risks.

In Chapter 3, based on [20], we analyze the case of non-life insurance within the general framework presented in Chapter 2. We study in detail the consequences of our new framework and derive useful analytical valuation formulas for non-life insurance claims in term of the basic elements, such as accident intensity μ, the delay distribution and the updating distribution. These results can be considered as an extension of the ones for the classic reduced-form framework, e.g. in Section 5.1 of [24]. In particular, this shows that the new framework for non-life insurance is at the same time conceptually general and computationally tractable. The benchmarked risk-minimization method for non-life insurance is then analyzed in detail in view of the explicit valuation formulas, especially for the reserve problem. We derive explicit pricing formula as well as benchmarked risk-minimizing hedging strategies.

In Chapter 4, based on [18], we study life insurance under polynomial diffusion model, within the generic framework of Chapter 2. Polynomial diffusions are first introduced in [47] as a nontrivial generalization of affine processes. As the biggest advantage of these processes, it is possible to derive explicit formula for conditional expectation of polynomial functions of the state variable. We concentrate in particular on the case of state variables within a compact state space, studied in detail in [70]. Under the compactness assumption, we can ensure the positivity of both risk-free short rate and mortality intensity, as well as use polynomial approximation for pricing and hedging purpose. The compactness assumption implies, nevertheless, also the boundedness of risk-free short rate and mortality intensity. These are however common assumptions in the literature, see e.g. [105], [49], [36], [4], [74] and [72]. Boundedness of mortality intensity is also supported by recent statistical studies in e.g. [38] and [64] and can be understood in terms of confidence region, as shown in [72]. The new approach of combining benchmark methodology and polynomial diffusion model is proposed here for the first time, so that it is possible to have at the same time a general and flexible model together with explicit and tractable pricing-hedging formulas. In a numerical example with a 2-dimensional state variable, we calibrate our model to MSCI and LLMA index under linear specification of the inverse of benchmark and the longevity index. This shows that even under parsimonious specification, our model can already produce a good fit to market data.

In Chapter 5, based on [19], we provide a consistent insurance framework under model uncertainty, when we consider a generic family of priors possibly mutually singular to each other. We mainly follow the pathwise approach of e.g. [85], [87] and [82], since it can be extended naturally in our setting. Stochastic analysis for general filtrations is the main problem when we want to extend the model uncertainty study for financial market to the one for insurance market, which is still missing. Motivated by life insurance modeling, here we propose a solution for the case of progressive enlargement of filtration by introducing an external jump, by following the canonical approach in Section 6.5 of [24]. We note that, since the current construction of sublinear conditional expectation in e.g. [87] relies on the properties of the natural filtration $\mathbb{F}$ of the canonical space, modifications are needed for the enlarged filtration, denoted by $\mathbb{G}$. By exploiting the properties of the canonical filtration construction in Section 6.5 of [24], it is possible to construct explicitly a sublinear conditional expectation on the enlarged filtration $\mathbb{G}$, which consistently extends the one introduced in [87]. Such extension presents however several technical difficulties and additional requirements, which we discuss in detail. In particular, integrability condition is needed in order to have the sublinear conditional expectation well-defined on the enlarged filtration. Only a weak version of dynamic programming principle or tower property holds, similar to [90], and the classic tower property is not satisfied in full generality as we show in a counterexample. However, we have the classic tower property in all cases of often used insurance contracts. Other sufficient conditions are presented as well. Beside this construction, we analyze also the superhedging problem for a generic payment stream in this setting. Superhedging dualities with respect to a nondominated probability family have been studied in several papers, but only limited to the initial time and applicable to European or American type of contingent claims, e.g. [88], [80], [100], [43], [9], [57], [86] and [56]. Here we formulate for the first time the problem of dynamic superhedging for a generic payment stream and determine superhedging dualities useful to solve this problem. We emphasize that the definitions and the results hold also in the case without model uncertainty. The results are illustrated first for the canonical setting and then extended to the reduced-form setting. As a co-product of these results, the constructed sublinear conditional expectation can be considered as a robust pricing operator for insurance cash flows.

Finally, in Appendix A, we show a brief overview of the uncertainty framework in the current literature, covering capacity theory, the G-setting in e.g. [91], [39] and the pathwise setting in e.g. [85], [87] and [82]. A few secondary results not used in the papers [18] [19] and [20] are presented here as well.

1.1 Contribution and declaration

The three articles [18], [19] and [20], on which this thesis is based, are results of joint works of the thesis' author Y. Zhang and Prof. F. Biagini.

The paper "Polynomial Diffusion Models for Life Insurance Liabilities" [18] (F. Biagini and Y. Zhang) is a published journal article. It arises from an idea of Prof. F. Biagini to apply the recently developed polynomial diffusion processes to the insurance modeling and to analyze the risk-minimization method in this case. The detailed structure is result of a close cooperation of F. Biagini and Y. Zhang. Computations and numerical results are carried out by Y. Zhang independently and reviewed together with F. Biagini in regular meetings. The benchmarked approach is incorporated in the setting during an early stage of the paper as an idea of Y. Zhang, in view of other papers of F. Biagini such as [11] and [12], which deal with benchmarked risk-minimization for contingent claims. The extension of the benchmarked risk-minimization method to payment streams is derived by Y. Zhang independently.

The paper "Reduced-form framework under model uncertainty" [19] (F. Biagini and Y. Zhang) is an unpublished preprint. Under suggestion of F. Biagini, the paper aims to extend the recent model uncertainty results with respect to a nondominated probability family to the insurance setting. Current stochastic analysis under uncertainty focus mainly on the canonical space endowed with the natural filtration. Its application to the pricing problem in continuous time is often limited to contingent claims. In joint discussions the two coauthors identify two main technical difficulties, i.e. defining rigorously the pricing problem for payment streams in this context and extending the stochastic analysis under uncertainty to the case of a progressively enlarged filtration. The two coauthors agree on using the superhedging approach for the pricing problem, which appears to be natural in the model uncertainty setting. However, the superhedging problem for payment streams in continuous time is still not addressed in the literature. A rigorous formulation of the problem is missing even in the case of a single prior. With the help of F. Biagini, Y. Zhang introduces in this paper consistent formulations and definitions regarding the superhedging problem for payment streams for the first time, and derives dynamic dualities results which support such definitions. Regarding the problem of progressively enlarged filtration, the two coauthors decide in joint discussions to extend the existing construction of sublinear conditional expectation on the canonical space to the reduced-form framework, which is particularly relevant for insurance modeling. The construction of a new sublinear conditional expectation leans on the classic construction of a progressive enlargement of filtration by introducing an external jump. This can be then used as pricing operator in view of the superhedging results. A weak form of tower property is satisfied by the constructed sublinear conditional expectation. In close cooperation with F. Biagini, Y. Zhang provides a counterexample showing that the classic tower property is not satisfied in full generality and gives sufficient conditions for its validity. Structure details are carried out in joint works and the proofs are mainly derived by Y. Zhang and then reviewed by F. Biagini.

The paper "A Unified Modeling Framework for Life and Non-Life Insurance" [20] (F. Biagini and Y. Zhang) is an unpublished preprint. It was born from the idea of Y. Zhang to create a continuous time framework for non-life insurance within a hybrid market, which is still missing in the literature. Indeed, existing non-life insurance setting in continuous time considers only the case of insurance filtration not distinguished from the reference filtration or the case of independence of the two filtrations, and are based mainly on modeling the compensator of a marked point process which describes the non-life insurance portfolio movement. Y. Zhang creates in this paper for the first time a non-trivial filtration dependence for non-life insurance modeling, which includes both life and non-life insurance setting in the current literature and overcomes the difficulties derived from the compensator approach in the case of progressive enlargement of filtration. The paper structure is developed in joint meetings and detailed computations are carried out by Y. Zhang under suggestions and reviews of F. Biagini.

Chapter 2

Unified framework for life and non-life insurance

2.1 Introduction

In this chapter, based on [20] and partially on [18], we give a unified framework in continuous time for both life and non-life insurance, traditionally studied in a separated way. A direct modeling approach, which generalizes the classic reduced-form framework in e.g. [24], is used to introduce a dependence structure between insurance filtration and reference filtration. The presence of intensity index linked derivatives and inflation index linked derivatives on the capital market determines a hybrid nature of the combined market. The pricing and hedging problem of insurance liabilities is solved by means of benchmarked risk-minimization methodology. This new framework shows the potential of a more developed hybrid market with insurance-linked financial products, especially those linked to non-life non-catastrophe insurance, which are currently still not common.

This chapter is organized as follows. In Section 2.2 we give a bottom-up construction of a generalization of the classic reduced-form framework. In Section 2.3 we specify the hybrid nature of the combined market under the benchmark approach. In Section 2.4, we analyze the benchmarked risk-minimization method for payment stream and show its relation with the real-world pricing formula. Finally, we show in Section 2.5 how to apply our general framework to the cases of both life and non-life insurance and discuss the relation of the present setting with the compensator approach.

2.2 Enlargement of filtration

Let $(\Omega, \mathcal{G}, \mathbb{G}, P)$ be a filtered probability space, where $\mathbb{G} := (\mathcal{G}_t)_{t\geqslant 0}$, $\mathcal{G} = \mathcal{G}_\infty$, and $\mathcal{G}_0$ is trivial. The filtration $\mathbb{G}$ represents the global information flow available to the insurance company and P is interpreted as the statistical or real-world probability measure. Furthermore, assume that the global filtration $\mathbb{G}$ is composed by two subfiltrations, i.e. $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} := (\mathcal{F}_t)_{t \geqslant 0}$ and $\mathbb{H} := (\mathcal{H}_t)_{t \geqslant 0}$ represent respectively the reference information flow and the internal information flow only available to the insurance company. The reference filtration $\mathbb{F}$ is assumed to include information related to the financial market, as well as to environmental, political, social and economic indicators. Without loss of generality, all filtrations are assumed to satisfy the canonical conditions of completeness and right-continuity. If not otherwise specified, all relations in this chapter hold in the P-a.s. sense. While the structure of the reference filtration $\mathbb{F}$ is not specified for now, we assume that the insurance internal filtration $\mathbb{H}$ is generated by a family of marked point processes which represent the insurance portfolio movements, similarly to what is proposed in [5]. For background of marked point processes, we refer to e.g. [71], [34] and [62]. In the following, the classic terminology of non-life insurance is used, see e.g. [115] and [89].

We consider a portfolio with n insurance policies. For i-th policy with $i = 1, ..., n$, an accident occurred at a random time τ_0^i is reported to the insurance company only after a nonnegative random delay θ^i. Let τ_1^i be the first reporting time with

$$\tau_1^i := \tau_0^i + \theta^i. \tag{2.2.1}$$

We stress that information about the accident time τ_0^i, the reporting delay θ^i and the damage size or severity of the accident, described by a nonnegative random variable X_1^i, is available only after the first reporting. Let $\mathbb{N}^+$ be the set of natural numbers without zero. After the first reporting of the accident, there may be some further developments of the case. We use a marked point process $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j \in \mathbb{N}_+}$ with nonnegative marks to describe this further development after τ_1^i. The sequence $(\tilde{\tau}_j^i)_{j \in \mathbb{N}^+}$ is a point process, where

$$\tilde{\tau}_j^i : (\Omega, \mathcal{G}, P) \to (\overline{\mathbb{R}}_+, \mathcal{B}(\overline{\mathbb{R}}_+)), \quad j \in \mathbb{N}^+,$$

and $(\tilde{X}_j^i)_{j \in \mathbb{N}^+}$ is a sequence of nonnegative random variables,

$$\tilde{X}_j^i : (\Omega, \mathcal{G}, P) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)), \quad j \in \mathbb{N}^+.$$

We set that the marked point process $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j \in \mathbb{N}_+}$ is simple, that is

$$\lim_{j \to \infty} \tilde{\tau}_j^i = \infty$$

and $\tilde{\tau}_j^i < \tilde{\tau}_{j+1}^i$, if $\tilde{\tau}_j^i < \infty$, and the following integrability condition holds

$$E\left[\sum_{j=1}^{\infty} \mathbf{1}_{\{\tilde{\tau}_j^i \leqslant t\}} \tilde{X}_j^i\right] < \infty \quad \text{for all } t \geqslant 0. \tag{2.2.2}$$

The global movement of the i-th insurance policy is described by a new marked point process $(\tau_j^i, \Theta_j^i)_{j \in \mathbb{N}^+}$ with 2-dimensional nonnegative marks. That is, the sequence of random times $(\tau_j^i)_{j \in \mathbb{N}^+}$ is a point process

$$\tau_j^i : (\Omega, \mathcal{G}, P) \to (\overline{\mathbb{R}}_+, \mathcal{B}(\overline{\mathbb{R}}_+)), \quad j \in \mathbb{N}^+,$$

and $(\Theta^i_j)_{j\in\mathbb{N}^+}$ is a sequence of 2-dimensional nonnegative random variables

$$\Theta^i_j : (\Omega, \mathcal{G}, P) \to (\mathbb{R}^2_+, \mathcal{B}(\mathbb{R}^2_+)), \qquad j \in \mathbb{N}^+.$$

Every random time τ^i_j describes the reporting time of j-th event related to i-th policy. The mark components Θ^i_j represent the reporting delay and the loss or damage size of the corresponding event, respectively, which are known only if the event is reported. More precisely, we assume

$$\tau^i_1 \quad \text{with mark} \quad \Theta^i_1 = (\theta^i, X^i_1), \tag{2.2.3}$$

and

$$\tau^i_{j+1} = \tau^i_1 + \tilde{\tau}^i_j \quad \text{with mark} \quad \Theta^i_{j+1} = (0, X^i_{j+1}) := (0, \tilde{X}^i_j), \tag{2.2.4}$$

for $j \geqslant 1$. We note that here we assume that only the first reporting delay is different from zero, since we focus mainly on modeling the relation between the first accident times τ^i_0 and the reference filtration $\mathbb{F}$. This setting can be easily generalized if non-zero random delays are considered in (2.2.4). We note that the simplicity of $(\tilde{\tau}^i_j, \tilde{X}^i_j)_{j\in\mathbb{N}_+}$ implies in particular that the random times $(\tau^i_j)_{j\in\mathbb{N}_+}$ are strictly ordered:

$$\begin{array}{l} \tau^1_1 < \tau^1_2 < \cdots < \tau^1_j < \tau^1_{j+1} < \cdots , \\ \tau^2_1 < \tau^2_2 < \cdots < \tau^2_j < \tau^2_{j+1} < \cdots , \\ \quad \vdots \\ \tau^n_1 < \tau^n_2 < \cdots < \tau^n_j < \tau^n_{j+1} < \cdots . \end{array} \tag{2.2.5}$$

We stress that every τ^i_j may eventually assume infinite value, in case it describes an event which never happens. The following conditions are assumed for the sake of simplicity.

Assumption 2.2.1.

1. Homogeneous delay*: the distribution of the random delays θ^i, $i = 1, ..., n$, is the same.*

2. Homogeneous development*: the distribution of the marked point processes $(\tilde{\tau}^i_j, \tilde{X}^i_j)_{j\in\mathbb{N}_+}$, $i = 1, ..., n$ is the same.*

3. Independent first mark*: the first marks X^i_1, $i = 1, ..., n$, are all mutually independent and independent from the σ-algebra $\mathcal{F}_\infty \vee \sigma(\tau^1_0) \vee ... \vee \sigma(\tau^n_0)$.*

4. Independent delay*: the random delays θ^i, $i = 1, ..., n$, are all mutually independent and independent from the σ-algebra $\mathcal{F}_\infty \vee \sigma((\tau^1_0, X^1_1)) \vee ... \vee \sigma((\tau^n_0, X^n_1))$.*

5. Independent development*: the marked point processes $(\tilde{\tau}^i_j, \tilde{X}^i_j)_{j\in\mathbb{N}_+}$, $i = 1, .., n$ are all mutually independent and independent from the σ-algebra $\mathcal{F}_\infty \vee \sigma((\tau^1_1, \theta^1, X^1_1)) \vee ... \vee \sigma((\tau^n_1, \theta^n, X^n_1))$.*

The above assumptions do not compromise the generality of the framework structure. Indeed, the homogeneity assumptions can be always satisfied if the insurance portfolio is opportunely subdivided, and reporting delays θ^i, occurrences and size of the losses after the first reporting time, described by $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j\in\mathbb{N}_+}$, are typically idiosyncratic factors independent to each other and independent from the reference information. Furthermore, we assume the following structure for the distribution of delay variables θ^i, $i = 1, ..., n$.

Assumption 2.2.2. *The common cumulative distribution function G of θ^i, $i = 1, ..., n$, defined by*

$$G(x) := P(\theta^i \leqslant x), \qquad x \in \mathbb{R}, \tag{2.2.6}$$

satisfies

$$G(x) = \alpha_0 + \int_0^{x\vee 0} g(x)\mathrm{d}x, \qquad x \in \mathbb{R}, \tag{2.2.7}$$

where $\alpha_0 = P(\theta^i \leqslant 0) = P(\theta^i = 0)$[1], and g is a nonnegative Lebesgue–integrable function.

The delays θ^i, $i = 1, ..., n$, may have a mixed distribution according to the above assumption. This covers both the case of life insurance without reporting delays by setting $g = 0$, and the case of non-life insurance with non-null delays by setting $g \neq 0$.

Before we describe the structure of the insurance internal filtration $\mathbb{H}$, we need to introduce some more processes. For every $i = 1, ..., n$, the process $(\mathbf{N}_t^i)_{t\geqslant 0}$, with

$$\mathbf{N}_t^i := N^i(t, \mathbb{R}_+^2) = \sum_{j=1}^{\infty} \mathbf{1}_{\{\tau_j^i \leqslant t\}}, \qquad t \geqslant 0,$$

is called *ground process* associated to the marked point process, which counts the number of occurrence of τ_j^i at any time $t \geqslant 0$. The *marked cumulative process N^i* is defined by

$$N^i(t, B)(\omega) := \sum_{j=1}^{\infty} \mathbf{1}_{\{\tau_j^i(\omega)\leqslant t\}} \mathbf{1}_{\{\Theta_j^i(\omega)\in B\}} = \sum_{j=1}^{\mathbf{N}_t^i} \mathbf{1}_{\{\Theta_j^i(\omega)\in B\}},$$

for every $t \geqslant 0$, $B \in \mathcal{B}(\mathbb{R}_+^2)$, $\omega \in \Omega$. In the literature, the process N^i is sometimes also called marked point process. Indeed, by Lemma 2.2.2 of [71], there is a unique correspondence between the marked point process $(\tau_j^i, \Theta_j^i)_{j\in\mathbb{N}^+}$ and its marked cumulative process N^i, i.e.

$$\{\tau_j^i \leqslant t\} = \{\mathbf{N}_t^i \geqslant j\}, \tag{2.2.8}$$

for all $t \geqslant 0$ and

$$\{\Theta_j^i \in B\} = \{\tau_j^i < \infty\} \cap \{N^i(\tau_j^i, B) > 0\} \tag{2.2.9}$$

for all $B \in \mathcal{B}(\mathbb{R}_+^2)$. Therefore, in the sequel with the name marked point process we will refer to N^i and $(\tau_j^i, \Theta_j^i)_{j\in\mathbb{N}^+}$ indifferently . We set the insurance internal information $\mathbb{H} = (\mathcal{H}_t)_{t\geqslant 0}$ to be

$$\mathcal{H}_t := \mathcal{H}_t^1 \vee ... \vee \mathcal{H}_t^n, \qquad t \geqslant 0, \tag{2.2.10}$$

[1] The random delays θ^i are nonnegative.

where $\mathbb{H}^i := (\mathcal{H}_t^i)_{t\geqslant 0}$, $i = 1, ..., n$, is the natural filtration of the marked point process N^i, i.e.

$$\mathcal{H}_t^i = \sigma(N^i(s,B), 0 \leqslant s \leqslant t, \text{ for all } B \in \mathcal{B}(\mathbb{R}_+^2)), \quad t \geqslant 0.$$

Besides, for every $i = 1, ..., n$ and $j \in \mathbb{N}_+$, we consider $\mathbb{H}^{i,1} := (\mathcal{H}_t^{i,1})_{t\geqslant 0}$ with

$$\mathcal{H}_t^{i,1} := \sigma\left(\mathbf{1}_{\{\tau_1^i \leqslant s\}}\mathbf{1}_{\{(\theta^i, X_1^i)\in B\}}, 0 \leqslant s \leqslant t, \text{ for all } B \in \mathcal{B}(\mathbb{R}_+^2)\right), \quad t \geqslant 0,$$

and $\mathbb{H}^{i,j} := (\mathcal{H}_t^{i,j})_{t\geqslant 0}$, $j > 1$, with

$$\mathcal{H}_t^{i,j} := \sigma\left(\mathbf{1}_{\{\tau_j^i \leqslant s\}}\mathbf{1}_{\{X_j^i\in B\}}, 0 \leqslant s \leqslant t, \text{ for all } B \in \mathcal{B}(\mathbb{R}_+)\right), \quad t \geqslant 0.$$

We note that it holds clearly

$$\mathcal{H}_\infty^{i,j} = \sigma(\tau_j^i) \vee \sigma(X_j^i) \quad \text{for } j > 1.$$

In particular, according to (2.2.1) we have

$$\mathcal{H}_\infty^{i,1} = \sigma(\tau_1^i) \vee \sigma((\theta^i, X_1^i)) = \sigma(\tau_0^i) \vee \sigma((\theta^i, X_1^i)).$$

Similarly, all notations related to the marked point processes $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j\in\mathbb{N}_+}$, $i = 1, .., n$, will be denoted with the symbol "$\sim$", e.g. $\tilde{\mathbb{H}}^i$ denotes the corresponding natural filtration, $\tilde{N}^i$ denotes the corresponding marked cumulative processes, etc.

Lemma 2.2.3. *For all $i = 1, ..., n$, it holds that $\mathbb{H}^i = \bigvee_{j\in\mathbb{N}_+} \mathbb{H}^{i,j}$.*

Proof. One inclusion is trivial,

$$\mathcal{H}_t^i \subseteq \bigvee_{j\in\mathbb{N}_+} \mathcal{H}_t^{i,j}.$$

For the other inclusion, we only need to show that for all $0 \leqslant s \leqslant t$ and $B \in \mathcal{B}(\mathbb{R}_+^2)$,

$$\{\tau_j^i \leqslant s\} \cap \{\Theta_j^i \in B\} \in \mathcal{H}_t^i.$$

Indeed, we note that the marked point process N^i is simple, hence by (2.2.8) and (2.2.9) we have

$$\{\tau_j^i \leqslant s\} = \{\mathbf{N}_s^i \geqslant j\} = \{N^i(s, \mathbb{R}_+^2) \geqslant j\} \in \mathcal{H}_t^i,$$

and

$$\{\tau_j^i \leqslant s\} \cap \{\Theta_j^i \in B\} = \{\tau_j^i \leqslant s\} \cap \{N^i(\tau_j^i, B) > 0\} \in \mathcal{H}_t^i.$$

□

The following notations will be used in further discussion. For $i = 1, ..., n$, $j \in \mathbb{N}_+$, let

$$\mathbb{H}^{i,\leqslant j} := \bigvee_{k\leqslant j} \mathcal{H}^{i,k}, \quad \mathbb{H}^{i,\geqslant j} := \bigvee_{k\geqslant j} \mathcal{H}^{i,k},$$

similarly for $\mathbb{H}^{i,>j}$ and $\mathbb{H}^{i,<j}$. If $j = 1$, we set $\mathcal{H}_t^{i,<1} := \{\emptyset, \Omega\}$ for every $t \geqslant 0$. As a direct consequence of Lemma 2.2.3, we have the following corollary.

Corollary 2.2.4. *For every $i = 1, ..., n$, $j \in \mathbb{N}_+$, it holds that*

$$\mathbb{H}^i = \mathbb{H}^{i,\leqslant j} \vee \mathbb{H}^{i,>j} = \mathbb{H}^{i,<j} \vee \mathbb{H}^{i,\geqslant j}.$$

Now we introduce the dependence structure between the filtrations $\mathbb{H}$ and $\mathbb{F}$ by following the reduced-form setting for credit risk and life insurance. We focus on the accident times τ_0^i, $i = 1, ..., n$ and model their relation with the reference filtration $\mathbb{F}$ in a similar way as in Section 9.1.2 of [24]. We assume that random times $(\tau_j^i)_{j\in\mathbb{N}}$, $i = 1, ..., n$, are not $\mathbb{F}$-stopping times. Accident times τ_0^i, $i = 1, ..., n$, are such that for all $i = 1, ..., n$, $\tau_0^i > 0$ P-a.s. and that for $t \in [0, \infty]$ and $s \in [0, t] \cap [0, \infty)$,

$$P\left(\tau_0^i > s \middle| \mathcal{F}_t\right) = P\left(\tau_0^i > s \middle| \mathcal{F}_s\right). \tag{2.2.11}$$

Moreover, for $l, k = 1, .., n$ with $l \neq k$, τ_0^l and τ_0^k are $\mathbb{F}$-conditionally independent, i.e. if $t \in [0, \infty]$ and $r, s \in [0, t] \cap [0, \infty)$, it holds that

$$P\left(\tau_0^l > r, \tau_0^k > s \middle| \mathcal{F}_t\right) = P\left(\tau_0^l > r \middle| \mathcal{F}_t\right) P\left(\tau_0^k > s \middle| \mathcal{F}_t\right). \tag{2.2.12}$$

Remark 2.2.5. *We define $\mathcal{H}_t^{i,0} := \sigma\left(\mathbf{1}_{\{\tau_0^i \leqslant s\}} : 0 \leqslant s \leqslant t\right)$, $i = 1, ..., n$. Then condition (2.2.11) equals the following*

$$E[X|\mathcal{F}_t] = E[X|\mathcal{F}_s],$$

for each $\mathcal{H}_s^{i,0}$-measurable integrable random variable X. Condition (2.2.12) is equals the $\mathcal{F}_t$-conditional independence between the σ-algebras $\mathcal{H}_t^{l,0}$ and $\mathcal{H}_t^{k,0}$.

Furthermore, let $F^i := (F_t^i)_{t\geqslant 0}$ be the $\mathbb{F}$-conditional cumulative process of τ_0^i,

$$F_t^i := P\left(\tau_0^i \leqslant t \middle| \mathcal{F}_t\right), \quad t \geqslant 0,$$

which is a bounded non-negative $\mathbb{F}$-submartingale. We assume that there exists a continuous $\mathbb{F}$-adapted process $\Gamma^i := (\Gamma_t^i)_{t\geqslant 0}$ and a locally integrable and $\mathbb{F}$-progressively measurable process $\mu^i := (\mu_t^i)_{t\geqslant 0}$, such that

$$e^{-\Gamma_t^i} = 1 - F_t^i \quad \text{for all } t \geqslant 0, \tag{2.2.13}$$

$$\Gamma_t^i := \int_0^t \mu_u^i \mathrm{d}u, \quad t \geqslant 0. \tag{2.2.14}$$

The processes Γ^i and μ^i are called respectively *hazard process* and *intensity process* of τ_0^i. Given a family of locally integrable $\mathbb{F}$-progressively measurable process μ^i, $i = 1, ..., n$, by following the explicit construction in Example 9.1.5 of [24], it is always possible to construct random times τ_0^i, $i = 1, ..., n$, such that Γ^i is the hazard process of τ_0^i for every $i = 0, ..., n$, and all the assumptions above are satisfied. For the sake of simplicity, we work under the following homogeneity condition.

Assumption 2.2.6. *The accident times τ_0^i, $i = 1, ..., n$, have the same intensity process.*

Under this homogeneity condition, we denote the common $\mathbb{F}$-conditional cumulative process, hazard process and intensity process respectively by F, Γ and μ. The above assumption can be interpreted in the following way. While a single policy movement described by N^i, $i = 1, ..., n$, may have purely individual and idiosyncratic factors, and not have direct link to the reference information flow $\mathbb{F}$, the accident occurrences τ_0^i, $i = 1, ..., n$, are however influenced by some common systematic risk-factors related to environmental, social and economic conditions, hence the common conditional intensity μ is deducible from the reference information flow $\mathbb{F}$. We emphasize that, unlike the classic assumptions in the reduced-form setting in e.g. [24], here accident times τ_0^i, $i = 1, ..., n$ are not even $\mathbb{H}$-stopping times, unless there is no reporting delay.

Remark 2.2.7. *In an early stage of the paper [20], the thesis' author tried to define hazard process and intensity process for every random event time τ_j^i, with $i = 1, ..., n$ and $j \in \mathbb{N}_+$. However, this structure seems to be unrealistic, since it implies that all updating information of insurance policies are influenced by common factors as well and are related to the reference information flow $\mathbb{F}$. On the contrary, the framework described in this section has more reasonable interpretation and at the same time allows analytical computations as we will see in Section 3.2.*

2.3 Combined market

In this section we describe the hybrid nature of the combined market and introduce the benchmark approach, developed in e.g. [96], [98] and [99]. We consider a finite time horizon T with $0 < T < \infty$. The role of inflation is taken into account and the inflation index process is denoted by $I := (I_t)_{t\in[0,T]}$, which represents the percentage increments of the Consumer Price Index (CPI) and follows a nonnegative $(P, \mathbb{F})$-semimartingale. We distinguish real price value, i.e. inflation-adjusted price value, from nominal price value, i.e. not inflation-adjusted prices value. Nominal value can be converted in real value at any time $t \in [0, T]$, if divided by the inflation index I_t. If not otherwise specified, we express all prices in nominal value.

We assume that the financial market is frictionless and there are l liquidly traded primary assets with price processes $S^i := (S_t^i)_{t\in[0,T]}$, $i = 1, ..., l$, which follow real-valued $(P, \mathbb{F})$-semimartingales. We denote the asset vector by $S := (S^i)_{i=1,...,l}$. For now we do not fix any dynamics of the vector process S, but only specify some macro-categories of primary assets. We assume that there exists a publicly accessible index, based on the intensity process μ and described by the process $L := (L_t)_{t\in[0,T]}$. We follow the approach of [28] and model L in the following way

$$L_t := e^{-\Gamma_t}, \quad t \in [0, T],$$

This index should reflect the underlying systematic risk-factor related to the insurance portfolio, such as mortality risk, weather risk, car accident risk, etc. We distinguish three macro-categories of primary assets as elements of the vector S:

1. classic financial assets, such as the zero-coupon bond, call and put options, futures, etc.;

2. inflation linked derivatives, based on inflation index I, such as inflation linked zero-coupon bond (called also zero-coupon Treasury Inflation Protected Security, TIPS), which pays off I_T (equivalent to 1 real unit) at time T, inflation linked call and put options, etc.;
3. systematic risk-factor linked derivatives based on the risk index L, such as longevity bond which pays off L_T at time T, weather index-based derivatives, etc.

Definition 2.3.1. *A* trading strategy *is a $\mathbb{R}^l$-valued $\mathbb{G}$-predictable S-integrable process $\delta := (\delta_t)_{t\in[0,T]}$.*

The space of all $\mathbb{R}^l$-valued $\mathbb{G}$-predictable S-integrable processes is denoted by $L(S, P, \mathbb{G})$. The following definition is given in [14].

Definition 2.3.2. *We call* portfolio *or* value process $S^\delta := (S^\delta_t)_{t\in[0,T]}$ *associated to a trading strategy δ the following càdlàg adapted process*

$$S^\delta_{t-} = \delta_t^\top S_t = \sum_{i=1}^{l} \delta^i_t S^i_t, \qquad t \in [0, T].$$

It is called self-financing *if*

$$S^\delta_t = S^\delta_0 + \int_0^t \delta_u^\top \mathrm{d}S_u = S^\delta_0 + \sum_{i=1}^{l} \int_0^t \delta^i_u \mathrm{d}S^i_u, \qquad t \in [0.T].$$

According to this definition, for $t \in [0, T]$ and $i = 1, ..., l$, the variable δ^i_t represents the amount of i-th primary asset held at time t. We introduce the following set

$$\mathcal{V}^+_x = \{S^\delta \text{ self-financing} \mid \delta \in L(S, P, \mathbb{G}),\ S^\delta_0 = x > 0,\ S^\delta > 0\}.$$

Definition 2.3.3. *A* benchmark *or* numéraire portfolio $S^* := (S^*_t)_{t\in[0,T]}$ *is an element in the set $\mathcal{V}^+_1$, such that for every portfolio $S^\delta \in \mathcal{V}^+_1$,*

$$\frac{S^\delta_s}{S^*_s} \geqslant E\left[\left.\frac{S^\delta_t}{S^*_t}\right| \mathcal{G}_s\right], \qquad s, t \in [0, T], \quad t \geqslant s.$$

In our framework, we follow the approach of [99] and work under the following assumption, which is weaker than assuming the existence of an equivalent martingale measure, as shown in [60].

Assumption 2.3.4. *There exists a benchmark portfolio S^*.*

As discussed in [11], this weak no-arbitrage assumption is more suitable for modeling a combined market.

Definition 2.3.5. *We call* benchmarked value *the value of any security or portfolio S^δ when discounted by the benchmark portfolio and we denote it by $\hat{S}^\delta$, i.e.*

$$\hat{S}^\delta := \frac{S^\delta}{S^*}.$$

The following lemma is proven in [12].

Lemma 2.3.6. *If the vector process of primary assets* S *is continuous, then the benchmarked vector process* $\hat{S} := S/S^*$ *is a* $(P, \mathbb{G})$*-local martingale.*

For the sake of simplicity, we assume that the following conditions hold in view of the above lemma.

Assumption 2.3.7. *The inflation index* $I = (I_t)_{t\in[0,T]}$ *and the vector of primary assets* S *are continuous processes. The benchmark portfolio* $S^* = (S^*_t)_{t\in[0,T]}$ *is continuous,* $\mathbb{F}$*-adapted. The benchmarked value process* $\hat{S} := S/S^*$ *is an* $(P, \mathbb{F})$*-true martingale.*

We note that according to the above assumption, we exclude that the benchmark may contain assets related to single accident event and/or insurance claim.

The cash flow in real unit, i.e. in inflation adjusted value, received by the policyholder from the insurance company over time can be seen as a dividend payment which is modelled by a process $D := (D_t)_{t\in[0,T]}$ of finite variation or more in general a $(P, \mathbb{G})$-semimartingale. We denote by $A := (A_t)_{t\in[0,T]}$ the nominal benchmarked value of the cumulative liabilities of the insurer towards a policyholder, namely

$$A_t := \int_0^t \frac{I_u}{S^*_u} \, \mathrm{d}D_u, \qquad t \in [0, T], \tag{2.3.1}$$

where we assume that D is defined such that A is square integrable, i.e.

$$\sup_{t\in[0,T]} \mathbb{E}\left[A_t^2\right] < \infty. \tag{2.3.2}$$

Definition 2.3.8. *We call* real-world pricing formula *associated to a dividend process, which settles at time* T*, the following formula*

$$V_t := \frac{S^*_t}{I_t} E\left[A_T - A_t \middle| \mathcal{G}_t\right] = \frac{S^*_t}{I_t} E\left[\int_{]t,T]} \frac{I_u}{S^*_u} \, \mathrm{d}D_u \middle| \mathcal{G}_t\right], \tag{2.3.3}$$

for $t \in [0, T]$.

This definition generalizes the so called ex-dividend price process defined in e.g. [6] and [63], which gives the current value of the future remaining payment in a risk-neutral context, i.e. when a martingale measure is assumed to exist. In our case, the quantity V_t in (2.3.3) is expressed in inflation adjusted value and corresponds to the *benchmarked risk-minimizing price* as we explain in Section 2.4. We note that, for Definition 2.3.8 it is be sufficient to have A is integrable. However, the square integrability (2.3.2) is a technical condition necessary for the risk-minimization approach.

2.4 Benchmarked risk-minimization for payment streams

In this section we give an easy extension of the benchmarked risk-minimizing method for contingent claims, described in [12], to the case for payment streams and discuss its relation with the real-world pricing formula (2.3.3). We illustrate the general definitions and results by following mainly [7] for the presentation.

We introduce the following Hilbert spaces, where $[X] = ([X^i, X^j])_{i,j=1,\ldots,l}$ denotes the quadratic variation matrix process of a vector process X,

$$M_0^2(P,\mathbb{G}) := \{M := (M_t)_{t\in[0,T]} \ \mathbb{G}\text{-martingale} \mid M_0 = 0, \ \sup_{t\in[0,T]} \mathbb{E}\left[M_t^2\right] < \infty\},$$

$$L^2(\hat{S},P,\mathbb{G}) = \left\{ \delta \ \ \mathbb{R}^l\text{-valued } \mathbb{G}\text{-predictable processes} \ \middle| \ \mathbb{E}\left[\int_0^T \delta_u^\top \mathrm{d}[\hat{S}]_u \delta_u\right] < \infty \right\},$$

$$\mathcal{I}(\hat{S},P,\mathbb{G}) = \left\{ \int_0^T \delta_u^\top \mathrm{d}\hat{S}_u \middle| \delta \in L^2(\hat{S},P,\mathbb{G}) \right\},$$

their norms are given respectively by

$$\|M\|_{M_0^2(P,\mathbb{G})} := \sup_{t\in[0,T]} \mathbb{E}\left[M_t^2\right]^{\frac{1}{2}} = \mathbb{E}\left[M_T^2\right]^{\frac{1}{2}} = \sup_{t\in[0,T]} \mathbb{E}\left[[M]_t\right]^{\frac{1}{2}} = \mathbb{E}\left[[M]_T\right]^{\frac{1}{2}},$$

$$\|\delta\|_{L^2(\hat{S},P,\mathbb{G})} := \left(\mathbb{E}\left[\int_0^T \delta_u^\top \mathrm{d}[\hat{S}]_u \delta_u\right]\right)^{\frac{1}{2}},$$

$$\left\|\int_0^T \delta_u^\top \mathrm{d}\hat{S}_u\right\|_{\mathcal{I}(\hat{S},P,\mathbb{G})} := \mathbb{E}\left[\left(\int_0^T \delta_u^\top \mathrm{d}\hat{S}_u\right)^2\right]^{\frac{1}{2}},$$

for $M \in M_0^2(P,\mathbb{G})$ and $\delta \in L^2(\hat{S},P,\mathbb{G})$. It is shown in Lemma 3.4 of [7] (or Lemma 2.1 of [107]) that $\mathcal{I}(\hat{S},P,\mathbb{G})$ is a stable subspace of $M_0^2(P,\mathbb{G})$. In particular, for every $\delta \in L^2(\hat{S},P,\mathbb{G})$ it holds

$$\|\delta\|_{L^2(\hat{S},P,\mathbb{G})} = \left\|\int_0^T \delta_u^\top \mathrm{d}\hat{S}_u\right\|_{\mathcal{I}(\hat{S},P,\mathbb{G})} = \left\|\int_0^T \delta_u^\top \mathrm{d}\hat{S}_u\right\|_{M_0^2(P,\mathbb{G})}.$$

Definition 2.4.1. *We call* L^2-admissible strategy *a process* $\delta := (\delta_t)_{t\in[0,T]}$ *such that* $\delta \in L^2(\hat{S},P,\mathbb{G})$ *and that the associated benchmarked value process* $\hat{S}^\delta$ *with*

$$\hat{S}_{t-}^\delta := \delta_t^\top \hat{S}_t, \qquad t \in [0,T]$$

belongs to $M_0^2(P,\mathbb{G})$*.*

Now we fix a process A as defined in (2.3.1), which models the (nominal) benchmarked cumulative payments towards a policyholder.

Definition 2.4.2. *We call* benchmarked cumulative cost process *of a* L^2*-admissible strategy* δ *associated to* A *a process* $C^\delta := (C_t^\delta)_{t\in[0,T]}$ *defined by*

$$C_t^\delta = \hat{S}_t^\delta - \int_0^t \delta_u \mathrm{d}\hat{S}_u + A_t, \qquad t \in [0,T].$$

Definition 2.4.3. *We call* risk process *of an L^2-admissible strategy δ a process $R^\delta := (R_t^\delta)_{t\in[0,T]}$ defined by*

$$R_t^\delta = E\left[(C_T^\delta - C_t^\delta)^2 \middle| \mathcal{G}_t\right], \qquad t \in [0,T].$$

Definition 2.4.4. *An L^2-admissible strategy $\bar{\delta}$ such that*

(1) $\hat{S}_T^{\bar{\delta}} = 0$ *P-a.s.,*

(2) $R_t^{\bar{\delta}} \leqslant R_t^\delta$ *P-a.s. for every $t \in [0,T]$ and for any L^2-admissible strategy δ such that $\hat{S}_T^{\bar{\delta}} = \hat{S}_T^\delta$ P-a.s., $\bar{\delta}_u = \delta_u$ P-a.s. for all $u \leqslant t$,*

is called benchmarked risk-minimizing *for A.*

Lemma 2.4.5. *The benchmarked cumulative cost process of a benchmarked risk-minimizing strategy is a $(P, \mathbb{G})$-martingale.*

Proof. This lemma is a straightforward consequence of Lemma 3.5 of [12] combined with Lemma A.4 of [78]. □

Lemma 2.4.6. *The benchmarked value process $\hat{S}^{\bar{\delta}}$ associated to a benchmarked risk-minimizing strategy $\bar{\delta}$ for A is given by*

$$\hat{S}_t^{\bar{\delta}} = E\left[A_T - A_t \middle| \mathcal{G}_t\right], \qquad t \in [0,T].$$

Proof. The proof follows from Lemma 2.4.5 and Lemma 3.12 of [78]. □

The following theorem is the core of the benchmarked risk-minimizing method.

Theorem 2.4.7. *Let the following be the Galtchouk–Kunita–Watanabe decomposition*[2] *of A_T*

$$A_T = \mathbb{E}[A_T] + \int_0^T \left(\delta_u^A\right)^\top \mathrm{d}\hat{S}_u + L_T^A, \qquad P-a.s., \tag{2.4.1}$$

where $\int_0^T \delta_u^A \mathrm{d}\hat{S}_u$ is the projection of $(A_T - \mathbb{E}[A_T])$ on the space $\mathcal{I}(\hat{S}, P, \mathbb{G})$ with $\delta^A \in L^2(\hat{S}, P, \mathbb{G})$, and $L^A \in M_0^2(P, \mathbb{G})$ is P-strongly orthogonal to $\mathcal{I}(\hat{S}, P, \mathbb{G})$. There is a unique benchmarked risk-minimizing strategy $\bar{\delta}$ for A, given by $\bar{\delta} = \delta^A$. The associated benchmarked cumulative cost process is given by

$$C_t^{\bar{\delta}} = \mathbb{E}\left[A_T\right] + L_t^A = C_0^{\bar{\delta}} + L_t^A, \qquad t \in [0,T],$$

and the benchmarked value process is given by

$$\hat{S}_t^{\bar{\delta}} = E\left[A_T - A_t \middle| \mathcal{G}_t\right], \qquad t \in [0,T].$$

Proof. See Lemma 2.4.5 and Theorem 2.1 of [78]. □

[2] See [3] for an overview of Galtchouk–Kunita–Watanabe decomposition.

As for the classic risk-minimization method, the crucial point of the solution of the benchmarked risk-minimizing problem is finding the Galtchouk–Kunita–Watanabe decomposition (2.4.1). We emphasize that, the orthogonal projection given in the decomposition (2.4.1) shows that, every benchmarked cumulative payment A_T has a perfectly hedgeable part $\int_0^T \left(\delta_u^A\right)^\top \mathrm{d}\hat{S}_u$ and a totally unhedgeable part $\left(\mathbb{E}\left[A_T\right] + L_T^A\right)$ which is covered by the benchmarked cumulative cost process C. Moreover, by Lemma 2.4.6, the benchmarked value process associated to the unique benchmarked risk-minimizing strategy $\bar{\delta}$ for A coincides with the discounted value of the real-world pricing formula given in (2.3.3), i.e.

$$\hat{S}_t^{\bar{\delta}} = \frac{I_t}{S_t^*} V_t, \qquad t \in [0, T].$$

The benchmarked hedging problem and its relation with the real-world pricing formula have already been discussed in [11] and [12] in the case of a T-contingent claim $\bar{D}$, i.e. when the dividend process D is given by

$$D_t = \mathbf{1}_{\{t=T\}} \bar{D}, \qquad t \in [0, T],$$

with $\bar{D}$ a square integrable $\mathcal{G}$-measurable random variable. The real-world pricing formula in this case is reduced to

$$V_t = \frac{S_t^*}{I_t} E\left[\left.\frac{I_T}{S_T^*}\bar{D}\right| \mathcal{G}_t\right], \qquad t \in [0, T[,$$

which is the original definition of fair price given in e.g. [99] for a T-contingent claim $\bar{D}$. In this case, if the T-contingent claim admits a self-financing strategy, then the supermartingale property of the benchmark portfolio in Definition (2.3.3) yields that, V corresponds to the least expensive self-financing portfolio which replicates $\bar{D}$.

2.5 Application and comments

In this section we show that the general framework described above includes both the cases of life insurance and non-life insurance. We compare in particular our setting and the compensator approach.

2.5.1 Life insurance

Regarding life insurance, reporting delays in this case are often negligible and the policies depend only on accident times τ_0^i, $i = 1, ..., n$, which represent the decease times of persons in this context. This can be easily included in our general framework by setting $\theta^i \equiv 0$, $\tau_j^i \equiv \infty$ for all $j > 1$ and $X_j^i \equiv 1$ for all $j \in \mathbb{N}_+$. The random times $\tau_0^i = \tau_1^i$ are interpreted as decease time of person i, for $i = 1, ..., n$. The filtration $\mathbb{G}$ is consequently reduced to

$$\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee ... \vee \mathbb{H}^n,$$

where $\mathcal{H}^i$, $i = 1, ..., n$, is generated by the jump process of τ_0^i, i.e.

$$\mathcal{H}_t^i = \sigma\left(\mathbf{1}_{\left\{\tau_0^i \leqslant s\right\}}, 0 \leqslant s \leqslant t\right), \qquad t \geqslant 0,$$

and represents the information flow relative to i-th policyholder's life status. In particular, in such case the common $\mathbb{F}$-progressively measurable process μ is interpreted as mortality intensity, for policyholders e.g. belonging to the same age cohort in the same country. The index L corresponds to survival index or longevity index of the given age/country group. The financial market is typically assumed to include longevity index linked derivatives, such as longevity bond, which pays off the longevity index value $e^{-\Gamma_T}$ at maturity T.

All classic results of the reduced-form framework hold in this context. In particular, for $i = 1, ..., n$, we set the process $L^i := (L^i_t)_{t\in[0,T]}$ associated to the i-th policyholder

$$L^i_t := \mathbf{1}_{\{\tau^i_0 > t\}} e^{\Gamma}_t, \qquad t \in [0, T]. \tag{2.5.1}$$

Proposition 5.7 and Proposition 5.8 of [7] show that the process $\left(\mathbf{1}_{\{\tau^i_0>t\}} e^{\Gamma^i_t}\right)_{t\in[0,T]}$ is a $\mathbb{G}$-martingale and satisfies

$$\mathbf{1}_{\{\tau^i_0>t\}} e^{\Gamma^i_t} = 1 - \int_{]0,t]} \mathbf{1}_{\{\tau^i_0 \geqslant u\}} e^{\Gamma^i_u} \mathrm{d}M^i_u, \qquad t \in [0, T],$$

where $M^i := (M^i_t)_{t\in[0,T]}$ is a $\mathbb{G}$-martingale defined by

$$M^i_t = \mathbf{1}_{\{\tau^i_0 \leqslant t\}} - \Gamma^i_{t\wedge\tau^i}, \qquad t \in [0, T].$$

Consequently, the hazard process $(\Gamma_t)_{t\in[0,T]}$ coincides up to τ^i_0 to the $\mathbb{G}$-compensator of the single jump process $\left(\mathbf{1}_{\{\tau^i_0\leqslant t\}}\right)_{t\in[0,T]}$. It is hence indifferent to model the hazard process or the compensator of the jump process. As we will see in the next section, this is however not the case for non-life insurance.

Life insurance within this setting is widely studied in the literature, see e.g. [7], [17] and [15]. In Chapter 4 we will examine the detailed case of life insurance under polynomial diffusion model.

2.5.2 Non-life insurance

The framework in Section 2.2 in its full generality describes the case of non-life insurance. In particular, it includes the setting of e.g. [32], [84], [83] and [8] as special cases. We note that non-life insurance policies typically have reporting delay, i.e. $\theta^i \neq 0$, which can also count to several years. For every $i = 1, ..., n$, we interpret the sequence $(X^i_j)_{j\in\mathbb{N}_+}$ as payment amount at random times $(\tau^i_j)_{j\in\mathbb{N}_+}$ related to the i-th policy. The exact accident time τ^i_0 and first payment amount X^i_1 is known only after the first reporting time τ^i_1. Further updatings and developments may occur after the first reporting and before the settlement of claim. The total number of eventual developments $(\tau^i_j)_{j\in\mathbb{N}_+}$ is unknown as well as the corresponding payment amount $(X^i_j)_{j\in\mathbb{N}_+}$. The cumulative payment up to time t related to i-th policy expressed in real value is given by

$$\sum_{j=1}^{\infty} \mathbf{1}_{\{\tau^i_j\leqslant t\}} X^i_j = \sum_{j=1}^{\mathbf{N}^i_t} X^i_j.$$

Hence, the nominal benchmarked cumulative payment process $A := (A_t)_{t\in[0,T]}$ is given by

$$A_t := \int_0^t \frac{I_s}{S_s^*} \mathrm{d}D_s = \sum_{i=1}^{n} \sum_{j=1}^{\mathbf{N}_t^i} \frac{I_{\tau_j^i}}{S^*_{\tau_j^i}} X_j^i, \qquad t \in [0,T]. \tag{2.5.2}$$

The estimation of A is called *reserve problem* in the non-life insurance sector, see e.g. [5]. We emphasize that, the risk related to non-life insurance policies is not only related to the accident itself, but also to the reporting delay, as well as to the times and impact sizes of developments after the first reporting.

Unlike our direct modeling approach illustrated in Section 2.2, in most of the current literature, e.g. [8], [32], [84], [83] and [106], the study of non-life insurance contracts is based on modeling the compensator of a marked point process N, which is then used in the pricing formula and in the calculation of risk-minimizing strategy. However, unlike the life insurance case where the direct approach and the compensator approach coincide, the second one appears to be not convenient for modelling non-life insurance in a framework with general filtrations, as we explain below. In this discussion we treat only the case of one marked point process and omit the index i for the sake of simplicity. We recall some basic definitions. Here the filtration $\mathbb{H}$ denotes the natural filtration of a marked point process N, $\mathbb{G}$ is a generic enlargement of $\mathbb{H}$ and $\mathcal{H} := \mathcal{H}_\infty$, $\mathcal{G} := \mathcal{G}_\infty$.

Definition 2.5.1. *The* $\mathbb{G}$-mark-predictable σ-algebra *on the space* $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \times \Omega$ *is the σ-algebra generated by sets of the form $(s,t] \times B \times A$ where $0 < s < t$, $B \in \mathcal{B}(\mathbb{R}_+)$ and $A \in \mathcal{G}_s$.*

Definition 2.5.2. *The $\mathbb{G}$-compensator of a marked point process N is any $\mathbb{G}$-mark-predictable, cumulative process $\Lambda(t,B,\omega)$ such that, $(\Lambda(t,B))_{t\geqslant 0}$ with $\Lambda(t,B)(\cdot) := \Lambda(t,B,\cdot)$ is the $\mathbb{G}$-compensator of the point process $(N(t,B))_{t\geqslant 0}$. The $\mathbb{G}$-compensator of the ground process $(N_t)_{t\geqslant 0}$ is denoted by $(\Lambda_t)_{t\geqslant 0}$, $\Lambda_t := \Lambda(t,\mathbb{R}_+)$.*

Theorem 14.2.IV(a) of [34] shows that, $\mathbb{G}$-compensator Λ of a marked point process N with finite first moment measure always exists and is $(l \otimes P)$-a.e. unique, where l denotes the Lebesgue measure on $\mathbb{R}_+$. In particular, for all $(t,B,\omega) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \times \Omega$, it holds

$$\Lambda(t,B,\omega) = \int_0^t \kappa(B|s,\omega)\Lambda(\mathrm{d}s,\omega), \tag{2.5.3}$$

where $\kappa(B|s,\omega)$, $B \in \mathcal{B}(\mathbb{R}_+)$, $s \geqslant 0$, $\omega \in \Omega$, is the unique predictable kernel such that for every $A \in \mathcal{G}_s, 0 < s < t, B \in \mathcal{B}(\mathbb{R}_+)$,

$$\int_A \int_s^t N(u,B)(\omega)\mathrm{d}uP(\mathrm{d}\omega) = \int_A \int_s^t \kappa(B|u,\omega)N_u(\omega)\mathrm{d}uP(\mathrm{d}\omega).$$

However, under general conditons, given a $\mathbb{G}$-mark-predictable and cumulative process Λ, we cannot ensure the existence of a marked point process N with $\mathbb{G}$-compensator Λ. The problem is first raised in [61], where the case with respect to the natural filtration $\mathbb{H}$ of the marked point process is solved. An extention of the existence theorem to the case of $\mathbb{G} = \mathbb{F} \otimes \mathbb{H}$, i.e. when the filtrations $\mathbb{F}$ and $\mathbb{H}$ are independent, can be found in [41]. another problem is that, while the law of N is uniquely determined by the $\mathbb{H}$-compensator, this is not true for the $\mathbb{G}$-compensator where $\mathbb{G}$ is a generic filtration, as discussed in [61] and Section 4.8 of [62]. Hence, the current literature with the compensator approach is only limited to the cases of $\mathbb{G} \equiv \mathbb{H}$, see e.g. [32], [84], [83], or $\mathbb{G} = \mathbb{F} \otimes \mathbb{H}$, see e.g. [8].

Here we provide a sufficient but unnatural condition in the case of $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, such that the law of N is uniquely determined by Λ. Similarly to the setting of e.g. [32], [84] and [83], we assume that the $\mathbb{G}$-compensator of N has the form

$$\Lambda(t, B) = \int_0^t \int_B \lambda_s \eta_s(\mathrm{d}x)\mathrm{d}s \quad \text{for all } t \geqslant 0,\ B \in \mathcal{B}(\mathbb{R}_+), \tag{2.5.4}$$

where $\lambda := (\lambda_t)_{t \geqslant 0}$ is a $\mathbb{G}$-progressively measurable process and the mapping η

$$\begin{aligned} \eta : \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \times \Omega &\longrightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \\ (t, B, \omega) &\mapsto \eta_t(B)(\omega), \end{aligned}$$

is such that for each $t \geqslant 0$, $\omega \in \Omega$, $\eta(t, \cdot, \omega)$ is a probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, and for each $B \in \mathcal{B}(\mathbb{R}_+)$, $(\eta_t(B))_{t \geqslant 0}$ is a $\mathbb{G}$-progressively measurable process. Trivially, we have

$$\Lambda_t = \int_0^t \lambda_s \mathrm{d}s \quad \text{for all } t \geqslant 0.$$

In particular, it is possible to choose a predictable version of both λ and η, see Section 14.3 of [34] for details. The processes λ and η can be interpreted as jump intensity and jump size intensity, respectively. We recall that a marked point process N is said to have *independent marks* if the marks $(X_n)_{n \in \mathbb{N}}$ are mutually independent given N.

Proposition 2.5.3. *On $(\Omega, \mathcal{H})$, the law of a simple marked point process N with finite first moment measure, independent marks and of the form (2.5.4) is uniquely determined by λ and η. If in addition λ is $\mathbb{H}$-measurable, then also the law of N on $(\Omega, \mathcal{G})$ is uniquely defined.*

Proof. The law of marked point process with independent marks is uniquely determined by the kernel κ and the distribution of N according Proposition 6.4.IV(a) of [34]. By relations (2.5.3) and (2.5.4), the kernel κ is given by

$$\kappa(B|t, \omega) = \eta_t(B)(\omega), \qquad (t, B, \omega) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \times \Omega.$$

It follows from Corollary 4.8.5 of [62] and Theorem 14.2.IV(c) of [34] that, if N is simple and of the form (2.5.4), the process $(E[\lambda_t|\mathcal{H}_t])_{t \geqslant 0}$ determines uniquely the distribution of N on $(\Omega, \mathcal{H})$. If furthremore λ is $\mathbb{H}$-adapted, then by Theorem 4.8.1 of [62], also the distribution of N on $(\Omega, \mathcal{G})$ is uniquely determined. □

Proposition 2.5.3 is result of a first attempt of thesis' author to formulate a unified framework for insurance by following the compensator approach. According to Proposition 2.5.3, the jump intensity process λ need to be $\mathbb{H}$-adapted in order to have N uniquely determined on $(\Omega, \mathcal{G})$. However, in our setting it is more natural to have an $\mathbb{F}$-adapted intensity process. Hence, the direct approach proposed in Section 2.2 seems to be more convenient in this context. Indeed, using this approach, it is possible to model the $\mathbb{F}$-adapted intensity process μ directly and then use the explicit bottom-up construction to establish a dependence structure between the filtrations $\mathbb{F}$ and $\mathbb{G}$, hence between the insurance market and the financial market.

Chapter 3

Non-life insurance in continuous time

3.1 Introduction

In this chapter, based on [20], we study in detail the case of non-life insurance within the general setting of Chapter 2. Non-life insurance portfolio movement is characterized by accident times, reporting delays, damage sizes and further updating information. We emphasize that a systematic framework for non-life insurance in continuous time within a hybrid market is new to the literature. Computationally, our framework structure allows to obtain explicit analytical formulas in term of the accident intensity μ, the delay distribution and the updating distribution. These valuation formulas can be used for pricing and hedging purpose. In particular, we mainly focus on the reserve problem of non-life insurance portfolio.

This chapter is organized as follows. In Section 3.2 we compute useful preliminary results under the setting of Chapter 2. In Section 3.3 we solve the pricing-hedging problem for non-life insurance liabilities by means of benchmarked risk-minimization method.

3.2 Valuation formulas for non-life insurance

In this Section, we give some useful preliminary results under the structure assumptions of Section 2.2. The presentation is similar to the one in Section 5.1 of [24].

We first extend relation (2.2.11) and the $\mathbb{F}$-independence (2.2.12) of τ_0^i, $i = 1, ..., n$, which hold for the filtrations $\mathbb{H}^{i,0}$, $i = 1, ..., n$, to the larger filtrations $\mathbb{H}^i$, $i = 1, ..., n$.

Lemma 3.2.1. *For every $t \in [0, \infty]$ and $l, k = 1, ..., n$ with $l \neq k$, the σ-algebras $\mathcal{H}_t^l$ and $\mathcal{H}_t^k$ are $\mathcal{F}_t$-independent.*

Proof. In view of Lemma 2.2.3, it suffices to prove that $\mathcal{H}_t^{k,p}$ and $\mathcal{H}_t^{k,q}$ are $\mathcal{F}_t$-independent for all $p, q \in \mathbb{N}_+$. For the sake of simplicity, we only consider $p \neq 1$ and $q \neq 1$, since the other cases are similar. We want to show

$$E\left[\mathbf{1}_{\{\tau_p^l\leqslant s\}}\mathbf{1}_{\{X_p^l\in B^l\}}\mathbf{1}_{\{\tau_q^k\leqslant r\}}\mathbf{1}_{\{X_q^k\in B^k\}}\,\middle|\,\mathcal{F}_t\right] = E\left[\mathbf{1}_{\{\tau_p^l\leqslant s\}}\mathbf{1}_{\{X_p^l\in B^l\}}\,\middle|\,\mathcal{F}_t\right]\,E\left[\mathbf{1}_{\{\tau_q^k\leqslant r\}}\mathbf{1}_{\{X_q^k\in B^k\}}\,\middle|\,\mathcal{F}_t\right],$$

where $s,r\in[0,t]\cap[0,\infty)$[1] and $B^l,B^k\in\mathcal{B}(\mathbb{R}_+)$. By using (2.2.3) and (2.2.4), the above equality equals

$$E\left[\mathbf{1}_{\{\tau_0^l+\theta^l+\tilde\tau_p^l\leqslant s\}}\mathbf{1}_{\{\tilde X_p^l\in B^l\}}\mathbf{1}_{\{\tau_0^k+\theta^k+\tilde\tau_q^k\leqslant r\}}\mathbf{1}_{\{\tilde X_p^k\in B^k\}}\,\middle|\,\mathcal{F}_t\right] = E\left[\mathbf{1}_{\{\tau_0^l+\theta^l+\tilde\tau_p^l\leqslant s\}}\mathbf{1}_{\{\tilde X_p^l\in B^l\}}\,\middle|\,\mathcal{F}_t\right]\,E\left[\mathbf{1}_{\{\tau_0^k+\theta^k+\tilde\tau_q^k\leqslant r\}}\mathbf{1}_{\{\tilde X_p^k\in B^k\}}\,\middle|\,\mathcal{F}_t\right].$$

By setting the following deterministic functions

$$f^l(x):=E\left[\mathbf{1}_{\{\theta^l+\tilde\tau_p^l\leqslant s-x\}}\mathbf{1}_{\{\tilde X_p^l\in B^l\}}\right],$$

$$f^k(x):=E\left[\mathbf{1}_{\{\theta^k+\tilde\tau_p^l\leqslant r-x\}}\mathbf{1}_{\{\tilde X_q^l\in B^k\}}\right],$$

we have

$$f^l(x)=f^l(x)\mathbf{1}_{\{x\leqslant s\}},$$
$$f^k(x)=f^l(x)\mathbf{1}_{\{x\leqslant r\}}.$$

In particular, $f^l(\tau_0^l)$ and $f^l(\tau_0^k)$ are respectively $\mathcal{H}_t^{l,0}$- and $\mathcal{H}_t^{k,0}$-measurable. This together with Remark 2.2.5 and the independence conditions in Assumption 2.2.1 implies

$$\begin{aligned}
&E\left[\mathbf{1}_{\{\tau_0^l+\theta^l+\tilde\tau_p^l\leqslant s\}}\mathbf{1}_{\{\tilde X_p^l\in B^l\}}\mathbf{1}_{\{\tau_0^k+\theta^k+\tilde\tau_q^k\leqslant r\}}\mathbf{1}_{\{\tilde X_p^k\in B^k\}}\,\middle|\,\mathcal{F}_t\right]\\
&=E\left[E\left[\mathbf{1}_{\{\tau_0^l+\theta^l+\tilde\tau_p^l\leqslant s\}}\mathbf{1}_{\{\tilde X_p^l\in B^l\}}\mathbf{1}_{\{\tau_0^k+\theta^k+\tilde\tau_q^k\leqslant r\}}\mathbf{1}_{\{\tilde X_p^k\in B^k\}}\,\middle|\,\mathcal{F}_t\vee\sigma(\tau_0^l)\vee\sigma(\tau_0^k)\right]\,\middle|\,\mathcal{F}_t\right]\\
&=E\left[E\left[\mathbf{1}_{\{x+\theta^l+\tilde\tau_p^l\leqslant s\}}\mathbf{1}_{\{\tilde X_p^l\in B^l\}}\mathbf{1}_{\{y+\theta^k+\tilde\tau_q^k\leqslant r\}}\mathbf{1}_{\{\tilde X_p^k\in B^k\}}\right]\Big|_{\substack{x=\tau_0^l\\ y=\tau_0^k}}\,\middle|\,\mathcal{F}_t\right]\\
&=E\left[E\left[\mathbf{1}_{\{x+\theta^l+\tilde\tau_p^l\leqslant s\}}\mathbf{1}_{\{\tilde X_p^l\in B^l\}}\right]\Big|_{x=\tau_0^l}\,E\left[\mathbf{1}_{\{y+\theta^k+\tilde\tau_q^k\leqslant r\}}\mathbf{1}_{\{\tilde X_p^k\in B^k\}}\right]\Big|_{y=\tau_0^k}\,\middle|\,\mathcal{F}_t\right]\\
&=E\left[f^l(\tau_0^l)f^l(\tau_0^k)\,\middle|\,\mathcal{F}_t\right]\\
&=E\left[f^l(\tau_0^l)\,\middle|\,\mathcal{F}_t\right]\,E\left[f^l(\tau_0^k)\,\middle|\,\mathcal{F}_t\right]\\
&=E\left[E\left[\mathbf{1}_{\{x+\theta^l+\tilde\tau_p^l\leqslant s\}}\mathbf{1}_{\{\tilde X_p^l\in B^l\}}\right]\Big|_{x=\tau_0^l}\,\middle|\,\mathcal{F}_t\right]\,E\left[E\left[\mathbf{1}_{\{y+\theta^k+\tilde\tau_q^k\leqslant r\}}\mathbf{1}_{\{\tilde X_p^k\in B^k\}}\right]\Big|_{y=\tau_0^k}\,\middle|\,\mathcal{F}_t\right]\\
&=E\left[E\left[\mathbf{1}_{\{\tau_0^l+\theta^l+\tilde\tau_p^l\leqslant s\}}\mathbf{1}_{\{\tilde X_p^l\in B^l\}}\,\middle|\,\mathcal{F}_t\vee\sigma(\tau_0^l)\vee\sigma(\tau_0^k)\right]\,\middle|\,\mathcal{F}_t\right]\\
&\quad\cdot E\left[E\left[\mathbf{1}_{\{\tau_0^k+\theta^k+\tilde\tau_q^k\leqslant r\}}\mathbf{1}_{\{\tilde X_p^k\in B^k\}}\,\middle|\,\mathcal{F}_t\vee\sigma(\tau_0^l)\vee\sigma(\tau_0^k)\right]\,\middle|\,\mathcal{F}_t\right]\\
&=E\left[\mathbf{1}_{\{\tau_0^l+\theta^l+\tilde\tau_p^l\leqslant s\}}\mathbf{1}_{\{\tilde X_p^l\in B_l\}}\,\middle|\,\mathcal{F}_t\right]\,E\left[\mathbf{1}_{\{\tau_0^k+\theta^k+\tilde\tau_q^k\leqslant r\}}\mathbf{1}_{\{\tilde X_p^k\in B_k\}}\,\middle|\,\mathcal{F}_t\right].
\end{aligned}$$

[1] We note that t may assume ∞.

This concludes the proof. □

Lemma 3.2.2. *For every $t \in [0,\infty]$, $s \in [0,t]$ and $i = 1,...,n$, if X is $\mathcal{H}_s^i$-measurable and integrable, then*

$$E\left[X\middle|\,\mathcal{F}_t\right] = E\left[X\middle|\,\mathcal{F}_s\right].$$

Proof. The proof of the Lemma is analogue to the one of Lemma 3.2.1. Indeed, without loss of generality, it is sufficient to restrict to the case of $j \neq 1$ and to show

$$E\left[\mathbf{1}_{\{\tau_j^i \leqslant s\}}\mathbf{1}_{\{X_j^i \in B\}}\middle|\,\mathcal{F}_t\right] = E\left[\mathbf{1}_{\{\tau_j^i \leqslant s\}}\mathbf{1}_{\{X_j^i \in B\}}\middle|\,\mathcal{F}_s\right],$$

where $s \in [0,t] \cap [0,\infty)$[2] and $B \in \mathcal{B}(\mathbb{R}_+)$. By using (2.2.3), (2.2.4) and Remark 2.2.5, we have

$$\begin{aligned}
&E\left[\mathbf{1}_{\{\tau_j^i \leqslant s\}}\mathbf{1}_{\{X_j^i \in B\}}\middle|\,\mathcal{F}_t\right] = E\left[\mathbf{1}_{\{\tau_0^i+\theta^i+\tilde{\tau}_j^i \leqslant s\}}\mathbf{1}_{\{\tilde{X}_j^i \in B\}}\middle|\,\mathcal{F}_t\right]\\
&= E\left[E\left[\mathbf{1}_{\{\tau_0^i+\theta^i+\tilde{\tau}_j^i \leqslant s\}}\mathbf{1}_{\{\tilde{X}_j^i \in B\}}\middle|\,\mathcal{F}_t \vee \sigma(\tau_0^l) \vee \sigma(\tau_0^k)\right]\middle|\,\mathcal{F}_t\right]\\
&= E\left[E\left[\mathbf{1}_{\{x+\theta^i+\tilde{\tau}_j^i \leqslant s\}}\mathbf{1}_{\{\tilde{X}_j^i \in B\}}\right]\Big|_{x=\tau_0^i}\middle|\,\mathcal{F}_t\right]\\
&= E\left[f^i(\tau_0^i)\middle|\,\mathcal{F}_t\right]\\
&= E\left[f^i(\tau_0^i)\middle|\,\mathcal{F}_s\right]\\
&= E\left[E\left[\mathbf{1}_{\{x+\theta^i+\tilde{\tau}_j^i \leqslant s\}}\mathbf{1}_{\{\tilde{X}_j^i \in B\}}\right]\Big|_{x=\tau_0^i}\middle|\,\mathcal{F}_s\right]\\
&= E\left[E\left[\mathbf{1}_{\{\tau_0^i+\theta^i+\tilde{\tau}_j^i \leqslant s\}}\mathbf{1}_{\{\tilde{X}_j^i \in B\}}\middle|\,\mathcal{F}_s \vee \sigma(\tau_0^l) \vee \sigma(\tau_0^k)\right]\middle|\,\mathcal{F}_s\right]\\
&= E\left[\mathbf{1}_{\{\tau_0^i+\theta^i+\tilde{\tau}_j^i \leqslant s\}}\mathbf{1}_{\{\tilde{X}_j^i \in B\}}\middle|\,\mathcal{F}_s\right]\\
&= E\left[\mathbf{1}_{\{\tau_j^i \leqslant s\}}\mathbf{1}_{\{X_j^i \in B\}}\middle|\,\mathcal{F}_s\right],
\end{aligned}$$

where

$$f^i(x) := E\left[\mathbf{1}_{\{\theta^i+\tilde{\tau}_j^i \leqslant s-x\}}\mathbf{1}_{\{\tilde{X}_j^i \in B\}}\right],$$

and $f^i(\tau_0^i)$ is $\mathcal{H}_s^{i,0}$-measurable. □

As a consequence of the above two lemmas, the $\mathbb{G}$-conditional expectation can be reduced to $\mathbb{F} \vee \mathbb{H}^i$-conditional expectation in most cases as we show in the following corollary.

Corollary 3.2.3. *Let $0 \leqslant t \leqslant T < \infty$, and Y be an integrable $(\mathcal{F}_T \vee \mathcal{H}_T^i)$-measurable random variable, then*

$$E\left[Y\middle|\,\mathcal{G}_t\right] = E\left[Y\middle|\,\mathcal{F}_t \vee \mathcal{H}_t^i\right].$$

Proof. It suffices to prove the statement for the indicator functions of the form $Y = \mathbf{1}_A\mathbf{1}_B$, where $A \in \mathcal{F}_T$ and $B \in \mathcal{H}_T^i$. We observe that

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^1 \vee ... \vee \mathcal{H}_t^n.$$

[2] We note that t may assume ∞.

Let $C \in \mathcal{F}_t$, $D^j \in \mathcal{H}_t^j$, $j = 1, ..., n$. It is enough to show that

$$\int_{C \cap D^1 \cap \ldots \cap D^n} \mathbf{1}_A \mathbf{1}_B \mathrm{d}P = \int_{C \cap D^1 \cap \ldots \cap D^n} E\left[\mathbf{1}_A \mathbf{1}_B \middle| \mathcal{F}_t \vee \mathcal{H}_t^i\right] \mathrm{d}P.$$

By Lemma 3.2.1 and Lemma 3.2.2, it holds

$$\begin{aligned}
\int_{C \cap D^1 \cap \ldots \cap D^n} \mathbf{1}_A \mathbf{1}_B \mathrm{d}P &= \int_{C \cap D^i \cap A \cap B} \prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \mathrm{d}P \\
&= \int_{C \cap D^i \cap A \cap B} E\left[\prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \middle| \mathcal{F}_T \vee \mathcal{H}_T^i \right] \mathrm{d}P \\
&= \int_{C \cap D^i \cap A \cap B} E\left[\prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \middle| \mathcal{F}_T \right] \mathrm{d}P \\
&= \int_{C \cap D^i \cap A \cap B} E\left[\prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \middle| \mathcal{F}_t \right] \mathrm{d}P \\
&= \int_{C \cap D^i \cap A \cap B} E\left[\prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \middle| \mathcal{F}_t \vee \mathcal{H}_t^i \right] \mathrm{d}P \\
&= \int_{C \cap D^i} \mathbf{1}_A \mathbf{1}_B \, E\left[\prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \middle| \mathcal{F}_t \vee \mathcal{H}_t^i \right] \mathrm{d}P \\
&= \int_{C \cap D^i} E\left[\mathbf{1}_A \mathbf{1}_B \middle| \mathcal{F}_t \vee \mathcal{H}_t^i\right] \, E\left[\prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \middle| \mathcal{F}_t \vee \mathcal{H}_t^i \right] \mathrm{d}P \\
&= \int_{C \cap D^i} \prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \, E\left[\mathbf{1}_A \mathbf{1}_B \middle| \mathcal{F}_t \vee \mathcal{H}_t^i\right] \mathrm{d}P \\
&= \int_{C \cap D^1 \cap \ldots \cap D^n} E\left[\mathbf{1}_A \mathbf{1}_B \middle| \mathcal{F}_t \vee \mathcal{H}_t^i\right] \mathrm{d}P.
\end{aligned}$$

□

We present another important corollary of Lemma 3.2.1 and Lemma 3.2.2: the *H-hypothesis* between filtrations $\mathbb{F}$ and $\mathbb{G}$, i.e. the property that every $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale.

Corollary 3.2.4. *The H-hypothesis holds between filtrations $\mathbb{F}$ and $\mathbb{G}$.*

Proof. By Lemma 6.1.1 of [24], an property equivalent to the H-hypothesis between two filtrations $\mathbb{F} \subseteq \mathbb{G}$, is that for any $t \geqslant 0$ and any bounded, $\mathcal{G}_t$-measurable random variable η, it holds

$$E\left[\eta \middle| \mathcal{F}_\infty\right] = E\left[\eta \middle| \mathcal{F}_t\right].$$

It is enough to prove the above relation for indicator functions of the form $\mathbf{1}_A \mathbf{1}_{B^1} \dots \mathbf{1}_{B^n}$, where $A \in \mathcal{F}_t$, $B^i \in \mathcal{H}_t^i$, $i = 1, ..., n$. By applying Lemma 3.2.1 and Lemma 3.2.2 more times, we get

$$\begin{aligned}
E\left[\mathbf{1}_A \mathbf{1}_{B^1} \dots \mathbf{1}_{B^n} \middle| \mathcal{F}_\infty\right] &= \mathbf{1}_A E\left[\mathbf{1}_{B^1} \dots \mathbf{1}_{B^n} \middle| \mathcal{F}_\infty\right] \\
&= \mathbf{1}_A \prod_{i=1}^n E\left[\mathbf{1}_{B^i} \middle| \mathcal{F}_\infty\right] \\
&= \mathbf{1}_A \prod_{i=1}^n E\left[\mathbf{1}_{B^i} \middle| \mathcal{F}_t\right] \\
&= \mathbf{1}_A E\left[\mathbf{1}_{B^1} \dots \mathbf{1}_{B^n} \middle| \mathcal{F}_t\right] \\
&= E\left[\mathbf{1}_A \mathbf{1}_{B^1} \dots \mathbf{1}_{B^n} \middle| \mathcal{F}_t\right].
\end{aligned}$$

□

Corollary 3.2.3 shows that, in most cases $\mathbb{G}$-conditional expectation equals the $\mathbb{F} \vee \mathbb{H}^i$-conditional expectation. Now we want to derive some more explicit representations. We note that the following decomposition holds for every integrable random variable Y, $t \geqslant 0$, $i = 1, ..., n$ and $j \in \mathbb{N}_+$

$$E\left[Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] + E\left[\mathbf{1}_{\{\tau_j^i \leqslant t\}} Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right]. \tag{3.2.1}$$

In the following we give separate valuation to the two components on the right-hand side of (3.2.1). The following lemma is important for a representation of the first component.

Lemma 3.2.5. *For every $t \geqslant 0$, $i = 1, ..., n$ and $j \in \mathbb{N}_+$, we have*

$$\mathcal{H}_t^i \vee \mathcal{F}_t \subseteq \mathcal{G}_t^{i,j},$$

where

$$\mathcal{G}_t^{i,j} := \left\{ A \in \mathcal{G} : \exists C \in \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t, A \cap \{\tau_j^i > t\} = C \cap \{\tau_j^i > t\} \right\}. \tag{3.2.2}$$

Proof. By Corollary 2.2.4, we have

$$\mathcal{H}_t^i = \mathcal{H}_t^{i,<j} \vee \mathcal{H}_t^{i,\geqslant j}.$$

Hence, it is enough to check that both $\mathcal{H}_t^{i,\geqslant j}$ and $\mathcal{H}_t^{i,<j} \vee \mathcal{F}_t$ belong to $\mathcal{G}_t^{i,j}$. In the first case, if $i > 1$ and $A = \{\tau_k^i \leqslant s\} \cap \{X_k^i \in B\}$ for some $k \geqslant j$, $0 \leqslant s \leqslant t$ and $B \in \mathcal{B}(\mathbb{R})$, it suffices to take $C = \emptyset$. Similarly for $i = 1$ and $A = \{\tau_k^1 \leqslant s\} \cap \{(\theta_k, X_k^1) \in B\}$ for $k \geqslant j$, $0 \leqslant s \leqslant t$ and $B \in \mathcal{B}(\mathbb{R}_+^2)$. In the second case, if $A \in \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t$ it suffices to take $C = A$. □

Proposition 3.2.6 gives two representations of the first component on the right-hand side of (3.2.1). Representation (3.2.3) is similar to the one in Lemma 5.1.2. in [24], representation (3.2.4) is new and is useful for further discussion.

Proposition 3.2.6. *Let $t \geqslant 0$, $i = 1, ..., n$, $j \in \mathbb{N}_+$ and Y be an integrable $\mathcal{G}$-measurable random variable, then*

$$E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau_j^i > t\}} \frac{E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right]}{P\left(\tau_j^i > t \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right)} \tag{3.2.3}$$

$$= \mathbf{1}_{\{\tau_j^i > t\}} E\left[Y \middle| \mathcal{H}_t^{i,\leqslant j} \vee \mathcal{F}_t\right]. \tag{3.2.4}$$

Proof. Equality (3.2.3) can be also written as

$$E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \, P\left(\tau^i > t \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right) \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau_j^i > t\}} \, E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right].$$

We observe that the right-hand side is $(\mathcal{H}_t^i \vee \mathcal{F}_t)$-measurable. Hence, it is sufficient to show that for any $A \in \mathcal{H}_t^i \vee \mathcal{F}_t$,

$$\int_A \mathbf{1}_{\{\tau_j^i > t\}} Y \, P\left(\tau_j^i > t \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right) \mathrm{d}P = \int_A \mathbf{1}_{\{\tau_j^i > t\}} \, E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P. \tag{3.2.5}$$

According to Lemma 3.2.5, there is an event $C \in \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t$ such that

$$A \cap \{\tau_j^i > t\} = C \cap \{\tau_j^i > t\},$$

thus,

$$\begin{aligned}
&\int_A \mathbf{1}_{\{\tau_j^i>t\}} Y\ P\left(\tau_j^i > t\middle|\, \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right) \mathrm{d}P \\
=&\int_{A\cap\{\tau_j^i>t\}} Y\ P\left(\tau_j^i > t\middle|\, \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right) \mathrm{d}P \\
=&\int_{C\cap\{\tau_j^i>t\}} Y\ P\left(\tau_j^i > t\middle|\, \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right) \mathrm{d}P \\
=&\int_C \mathbf{1}_{\{\tau_j^i>t\}} Y\ P\left(\tau_j^i > t\middle|\, \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right) \mathrm{d}P \\
=&\int_C E\left[\mathbf{1}_{\{\tau_j^i>t\}} Y\middle|\, \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] E\left[\mathbf{1}_{\{\tau_j^i>t\}}\middle|\, \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P \\
=&\int_C E\left[\mathbf{1}_{\{\tau_j^i>t\}}\ E\left[\mathbf{1}_{\{\tau_j^i>t\}} Y\middle|\, \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right]\middle|\, \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P \\
=&\int_C \mathbf{1}_{\{\tau_j^i>t\}}\ E\left[\mathbf{1}_{\{\tau_j^i>t\}} Y\middle|\, \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P \\
=&\int_{C\cap\{\tau_j^i>t\}} E\left[\mathbf{1}_{\{\tau_j^i>t\}} Y\middle|\, \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P \\
=&\int_{A\cap\{\tau_j^i>t\}} E\left[\mathbf{1}_{\{\tau_j^i>t\}} Y\middle|\, \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P \\
=&\int_A \mathbf{1}_{\{\tau_j^i>t\}}\ E\left[\mathbf{1}_{\{\tau_j^i>t\}} Y\middle|\, \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P.
\end{aligned}$$

Equality (3.2.4) can be shown in the same way. We only need to note that

$$\mathcal{G}_t^{i,j} \subseteq \left\{A \in \mathcal{G} : \exists C \in \mathcal{H}_t^{i,\leqslant j} \vee \mathcal{F}_t, A \cap \{\tau_j^i > t\} = C \cap \{\tau_j^i > t\}\right\}.$$

Consequently, the σ-algebra $\mathcal{H}_t^{i,<j}$ in (3.2.5) can be replaced by $\mathcal{H}_t^{i,\leqslant j}$. That is,

$$\int_A \mathbf{1}_{\{\tau_j^i>t\}} Y \mathrm{d}P = \int_A \mathbf{1}_{\{\tau_j^i>t\}}\ E\left[\mathbf{1}_{\{\tau_j^i>t\}} Y\middle|\, \mathcal{H}_t^{i,\leqslant j} \vee \mathcal{F}_t\right] \mathrm{d}P,$$

for all $A \in \mathcal{H}_t^i \vee \mathcal{F}_t$. Hence,

$$E\left[\mathbf{1}_{\{\tau_j^i>t\}} Y\middle|\, \mathcal{H}_t^i \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau_j^i>t\}} E\left[Y\middle|\, \mathcal{H}_t^{i,\leqslant j} \vee \mathcal{F}_t\right].$$

□

Now we concentrate on the second component on the right-hand side of (3.2.1). A slightly more general result is given in the following lemma.

Lemma 3.2.7. *Let $t \geqslant 0$, $i = 1, ..., n$, $j \in \mathbb{N}_+$, $\mathcal{A} \subseteq \mathcal{G}$ be a σ-algebra and Y an integrable $\mathcal{G}$-measurable random variable, then*

$$E\left[\mathbf{1}_{\{\tau_j^i\leqslant t\}} Y\middle|\, \mathcal{H}_t^{i,\leqslant j} \vee \mathcal{A}\right] = E\left[\mathbf{1}_{\{\tau_j^i\leqslant t\}} Y\middle|\, \mathcal{H}_\infty^{i,\leqslant j} \vee \mathcal{A}\right].$$

Proof. The left-hand side is clearly $(\mathcal{H}_\infty^{i,\leqslant j} \vee \mathcal{A})$-measurable. Note that the marked point process N^i is simple, i.e. the strict monotonicy (2.2.5) holds. If $A \in \mathcal{H}_\infty^{i,\leqslant j} \vee \mathcal{A}$, then $A \cap \{\tau_j^i \leqslant t\} \in \mathcal{H}_t^{i,\leqslant j} \vee \mathcal{A}$, and

$$\begin{aligned}\int_A \mathbf{1}_{\{\tau_j^i \leqslant t\}} Y \mathrm{d}P &= \int_{A\cap\{\tau_j^i \leqslant t\}} Y \mathrm{d}P = \int_{A\cap\{\tau_j^i \leqslant t\}} E\left[Y \middle| \mathcal{H}_t^{i,\leqslant j} \vee \mathcal{A}\right] \mathrm{d}P \\ &= \int_A E\left[\mathbf{1}_{\{\tau_j^i \leqslant t\}} Y \middle| \mathcal{H}_t^{i,\leqslant j} \vee \mathcal{A}\right] \mathrm{d}P,\end{aligned}$$

which concludes the proof. □

Remark 3.2.8. *We recall that*

$$\mathcal{H}_\infty^{i,\leqslant j} = \sigma\left(\tau_h^i, h = 1, ..., j\right),$$

and the random times $\tau_h^i, h = 1, ..., j$ are strictly ordered,

$$\tau_1^i < ... < \tau_j^i.$$

Lemma 3.2.7 can be interpreted in the following way: if τ_j^i has already occurred before time t, then having partial information about τ_j^i up to t is equals having full information about all the random times τ_h^i, $h = 1, ..., j$. This implies in particular that, if Y is a function of $\tau_1^i, ..., \tau_j^i$, i.e. $Y = f(\tau_1^i, ..., \tau_j^i)$, then the conditional expectation is reduced to

$$E\left[\mathbf{1}_{\{\tau_j^i \leqslant t\}} Y \middle| \mathcal{H}_t^{i,\leqslant j} \vee \mathcal{A}\right] = \mathbf{1}_{\{\tau_j^i \leqslant t\}} Y.$$

The above results are summarized in the following representation theorem.

Theorem 3.2.9. *Let $t \geqslant 0$, $i = 1, ..., n$, $j \in \mathbb{N}_+$ and Y be an integrable $\mathcal{G}$-measurable random variable, then*

$$E\left[Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau_j^i \leqslant t\}} E\left[Y \middle| \mathcal{H}_\infty^{i,\leqslant j} \vee \mathcal{H}_t^{i,>j} \vee \mathcal{F}_t\right] + \mathbf{1}_{\{\tau_j^i > t\}} E\left[Y \middle| \mathcal{H}_t^{i,\leqslant j} \vee \mathcal{F}_t\right].$$

If furthermore Y is $(\mathcal{H}_T^i \vee \mathcal{F}_T)$-measurable, then it holds

$$E\left[Y \middle| \mathcal{G}_t\right] = \mathbf{1}_{\{\tau_j^i \leqslant t\}} E\left[Y \middle| \mathcal{H}_\infty^{i,\leqslant j} \vee \mathcal{H}_t^{i,>j} \vee \mathcal{F}_t\right] + \mathbf{1}_{\{\tau_j^i > t\}} E\left[Y \middle| \mathcal{H}_t^{i,\leqslant j} \vee \mathcal{F}_t\right].$$

Proof. Since

$$E\left[Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = E\left[\mathbf{1}_{\{\tau_j^i \leqslant t\}} Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] + E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right],$$

the first part is a direct consequence of Proposition 3.2.6 and Lemma 3.2.7 applied to $\mathcal{A} = \mathcal{H}_t^{i,>j} \vee \mathcal{F}_t$. For the second part, it is sufficient to use Corollary 3.2.3. □

We now present some results which play an important role for the reserve estimation problem introduced in Section 2.5.2. Let $0 \leqslant t \leqslant T < \infty$ and $Z := (Z_t)_{t\in[0,T]}$ be a continuous, bounded and $\mathbb{F}$-adapted process. For $i = 1, ..., n$, we consider the random variable

$$Y = \sum_{j=\mathbf{N}_t^i}^{\mathbf{N}_T^i} X_j^i Z_{\tau_j^i} = \sum_{j=1}^{\infty} \mathbf{1}_{\left\{t<\tau_j^i \leqslant T\right\}} X_j^i Z_{\tau_j^i}, \tag{3.2.6}$$

and want to compute

$$E\left[Y \middle| \mathcal{G}_t\right] = E\left[\sum_{j=\mathbf{N}_t^i}^{\mathbf{N}_T^i} X_j^i Z_{\tau_j^i} \middle| \mathcal{G}_t\right]. \tag{3.2.7}$$

Similarly to before, we decompose (3.2.7) with respect to the first reporting time τ_1^i and study separately the two components,

$$E\left[\sum_{j=\mathbf{N}_t^i}^{\mathbf{N}_T^i} X_j^i Z_{\tau_j^i} \middle| \mathcal{G}_t\right] = E\left[\mathbf{1}_{\left\{\tau_1^i>t\right\}} \sum_{j=\mathbf{N}_t^i}^{\mathbf{N}_T^i} X_j^i Z_{\tau_j^i} \middle| \mathcal{G}_t\right] + E\left[\mathbf{1}_{\left\{\tau_1^i \leqslant t\right\}} \sum_{j=\mathbf{N}_t^i}^{\mathbf{N}_T^i} X_j^i Z_{\tau_j^i} \middle| \mathcal{G}_t\right]. \tag{3.2.8}$$

We aim to derive more explicit formulas in terms of the intensity process μ, the distribution of delay θ^i, and the distribution of development N^i after the first reporting. As a preliminary step, we calculate the $\mathbb{F}$-conditional expectation of τ_1^i.

Lemma 3.2.10. *For any $i = 1, ..., n$ and $t \geqslant 0$, it holds*

$$P\left(\tau_1^i > t \middle| \mathcal{F}_t\right) = e^{-\int_0^t \mu_u \mathrm{d}u} + \int_0^t \bar{G}(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u, \tag{3.2.9}$$

and

$$P\left(\tau_1^i \leqslant t \middle| \mathcal{F}_t\right) = \int_0^t G(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u, \tag{3.2.10}$$

where G is the common cumulative distribution function of θ^i defined in (2.2.6), i.e.

$$G(x) := P(\theta^i \leqslant x), \qquad x \in \mathbb{R},$$

and

$$\bar{G}(x) := 1 - G(x) = P(\theta^i > x), \qquad x \in \mathbb{R}. \tag{3.2.11}$$

Proof. We prove first equality (3.2.9). By Assumption 2.2.1, θ^i is independent from $\mathcal{F}_t \vee \sigma(\tau_0^i)$. Moreover, both θ^i and τ_0^i are P-a.s. nonnegative. Thus, it holds

$$\begin{aligned}
P\left(\tau_1^i > t \middle| \mathcal{F}_t\right) &= E\left[\mathbf{1}_{\left\{\tau_0^i+\theta^i>t\right\}} \middle| \mathcal{F}_t\right] \\
&= E\left[\mathbf{1}_{\left\{\tau_0^i>t\right\}} + \mathbf{1}_{\left\{\tau_0^i \leqslant t\right\}} \mathbf{1}_{\left\{\tau_0^i+\theta^i>t\right\}} \middle| \mathcal{F}_t\right] \\
&= E\left[\mathbf{1}_{\left\{\tau_0^i>t\right\}} \middle| \mathcal{F}_t\right] + E\left[\mathbf{1}_{\left\{\tau_0^i \leqslant t\right\}} \mathbf{1}_{\left\{\tau_0^i+\theta^i>t\right\}} \middle| \mathcal{F}_t\right] \\
&= e^{-\int_0^t \mu_u \mathrm{d}u} + E\left[E\left[\mathbf{1}_{\left\{\tau_0^i \leqslant t\right\}} \mathbf{1}_{\left\{\tau_0^i+\theta^i>t\right\}} \middle| \mathcal{F}_t \vee \sigma(\tau_0^i)\right] \middle| \mathcal{F}_t\right] \\
&= e^{-\int_0^t \mu_u \mathrm{d}u} + E\left[\mathbf{1}_{\left\{\tau_0^i \leqslant t\right\}} E\left[\mathbf{1}_{\{\theta^i>t-x\}}\right]\Big|_{x=\tau_0^i} \middle| \mathcal{F}_t\right] \\
&= e^{-\int_0^t \mu_u \mathrm{d}u} + E\left[\mathbf{1}_{\left\{\tau_0^i \leqslant t\right\}} \bar{G}(t-\tau_0^i) \middle| \mathcal{F}_t\right].
\end{aligned}$$

To conclude we only need to prove

$$E\left[\mathbf{1}_{\{\tau_0^i\leqslant t\}}\bar{G}(t-\tau_0^i)\Big|\,\mathcal{F}_t\right]=\int_0^t\bar{G}(t-u)e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u\mathrm{d}u. \tag{3.2.12}$$

This can be done in the analogue way as for Proposition 5.1.1 of [24], in view of relation (2.2.11) and the fact that G is continuous by Assumption 2.2.2. Equality (3.2.10) is a straightforward consequence. Indeed,

$$\begin{aligned}P\left(\tau_1^i\leqslant t\big|\,\mathcal{F}_t\right)=&1-P\left(\tau_1^i>t\big|\,\mathcal{F}_t\right)\\=&1-e^{-\int_0^t\mu_u\mathrm{d}u}-\int_0^t\bar{G}(t-u)e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u\mathrm{d}u\\=&-\left.e^{-\int_0^u\mu_v\mathrm{d}v}\right|_0^t-\int_0^t\bar{G}(t-u)e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u\mathrm{d}u\\=&\int_0^t e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u\mathrm{d}u-\int_0^t\bar{G}(t-u)e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u\mathrm{d}u\\=&\int_0^t\left(1-\bar{G}(t-u)\right)e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u\mathrm{d}u\\=&\int_0^t G(t-u)e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u\mathrm{d}u\end{aligned}$$

□

Remark 3.2.11. *We note that (3.2.12) is the conditional probability that the accident has incurred, but not yet reported. In this last case, the events are called IBNR in the terminology used in the insurance sector.*

In expression (3.2.10) of Lemma 3.2.10, the parameter t is present also in the integrand. The following corollary give an improvement of relation (3.2.10) and shows that the process of conditional expectation $(P\left(\tau_1^i\leqslant t\big|\,\mathcal{F}_t\right))_{t\geqslant 0}$ is absolutely continuous with respect to the Lebesgue measure.

Corollary 3.2.12. *Let $i=1,...,n$, it holds*

$$P\left(\tau_1^i\leqslant t\big|\,\mathcal{F}_t\right)=\int_0^t\left(\alpha_0 e^{-\int_0^s\mu_v\mathrm{d}v}\mu_s+\int_0^s g(s-u)e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u\mathrm{d}u\right)\mathrm{d}s, \tag{3.2.13}$$

where α_0 and g are defined in (2.2.7).

Proof. By Assumption 2.2.2, relation (3.2.10) and Leibniz integral rule, we have immediately

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}P\left(\tau_1^i\leqslant t\big|\,\mathcal{F}_t\right)&=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\bar{G}(t-u)e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u\mathrm{d}u\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha_0 e^{-\int_0^s\mu_v\mathrm{d}v}\mu_s+\int_0^s g(s-u)e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u\mathrm{d}u\right).\end{aligned}$$

□

Lemma 3.2.13. *For any $i = 1, ..., n$ and $t \in [0,T]$, if the sequence of random variables $Z := (Z_u)_{u\in[t,T]}$ is left-continuous and bounded and Z_t is $\mathcal{F}_T$-measurable for all $t \geqslant 0$, and $\tilde{Z} := (\tilde{Z}_u)_{u\in[t,T]}$ is independent from $\mathcal{F}_T \vee \sigma(\tau_1^i)$ and such that $(E[\tilde{Z}_u])_{u\in[t,T]}$ is left-continuous and bounded, then we have*

$$E\left[\mathbf{1}_{\left\{t<\tau_1^i\leqslant T\right\}}\tilde{Z}_{\tau_1^i}Z_{\tau_1^i}\middle|\,\mathcal{F}_t\right] = E\left[\int_t^T E[\tilde{Z}_u]Z_u \mathrm{d}P\left(\tau_1^i\leqslant u\middle|\,\mathcal{F}_u\right)\middle|\,\mathcal{F}_t\right].$$

Proof. The argument is similar to Proposition 5.1.1 of [24]. As a first step, we assume that both Z and $\tilde{Z}$ are stepwise constant, i.e. we assume without loss of generality that

$$Z_u = \sum_{j=0}^n Z_{t_j}\mathbf{1}_{\{t_j<u\leqslant t_{j+1}\}}, \qquad \tilde{Z}_u = \sum_{j=0}^n \tilde{Z}_{t_j}\mathbf{1}_{\{t_j<u\leqslant t_{j+1}\}},$$

for $t < u \leqslant T$, where $t_0 = t < ... < t_{j+1} = T$, Z_{t_j} is $\mathcal{F}_T$-measurable and $\tilde{Z}_{t_j}$ is independent from $\mathcal{F}_T \vee \sigma(\tau_1^i)$ for all $j = 0, ..., n$. By Lemma 3.2.2, it holds that

$$\begin{aligned}
&E\left[\mathbf{1}_{\left\{t<\tau_1^i\leqslant T\right\}}\tilde{Z}_{\tau_1^i}Z_{\tau_1^i}\middle|\,\mathcal{F}_t\right]\\
=&E\left[\sum_{j=0}^n \tilde{Z}_{t_j}Z_{t_j}\mathbf{1}_{\left\{t_j<\tau_1^i\leqslant t_{j+1}\right\}}\middle|\,\mathcal{F}_t\right]\\
=&E\left[\sum_{j=0}^n E\left[\tilde{Z}_{t_j}Z_{t_j}\mathbf{1}_{\left\{t_j<\tau_1^i\leqslant t_{j+1}\right\}}\middle|\,\mathcal{F}_T\right]\middle|\,\mathcal{F}_t\right]\\
=&E\left[\sum_{j=0}^n E[\tilde{Z}_{t_j}]Z_{t_j}E\left[\mathbf{1}_{\left\{t_j<\tau_1^i\leqslant t_{j+1}\right\}}\middle|\,\mathcal{F}_T\right]\middle|\,\mathcal{F}_t\right]\\
=&E\left[\sum_{j=0}^n E[\tilde{Z}_{t_j}]Z_{t_j}\left(E\left[\mathbf{1}_{\left\{\tau_1^i\leqslant t_{j+1}\right\}}\middle|\,\mathcal{F}_T\right]-E\left[\mathbf{1}_{\left\{\tau_1^i\leqslant t_j\right\}}\middle|\,\mathcal{F}_T\right]\right)\middle|\,\mathcal{F}_t\right]\\
=&E\left[\sum_{j=0}^n E[\tilde{Z}_{t_j}]Z_{t_j}\left(E\left[\mathbf{1}_{\left\{\tau_1^i\leqslant t_{j+1}\right\}}\middle|\,\mathcal{F}_{t_{j+1}}\right]-E\left[\mathbf{1}_{\left\{\tau_1^i\leqslant t_j\right\}}\middle|\,\mathcal{F}_{t_j}\right]\right)\middle|\,\mathcal{F}_t\right] \qquad (3.2.14)\\
=&E\left[\int_t^T E[\tilde{Z}_u]Z_u \mathrm{d}P\left(\tau_1^i\leqslant u\middle|\,\mathcal{F}_u\right)\middle|\,\mathcal{F}_t\right]. \qquad (3.2.15)
\end{aligned}$$

In the general case, it is possible to find stepwise constant approximations for Z and $\tilde{Z}$. Since Z is bounded and $E[\tilde{Z}]$ is continuous and bounded on $[t,T]$, the Riemann sum under the sign of conditional expectation in (3.2.14) converges to the Lebesgue-Stieltjes integral in expression (3.2.15). The convergence of the conditional expectations follows as well. □

Remark 3.2.14. *We note that the above Lemma does note involves the notion of Itô integral, but only Lebesgue-Stieltjes integral in (3.2.15), which coincides with Lebesgue* *integral in view of Corollary 3.2.12. Hence, it is not necessary that Z is $\mathbb{F}$-adapted for the proof to hold. Furthermore, we emphasize that the boundedness condition is imposed only for the sake of simplicity, the results hold without changes if the processes are sufficiently integrable.*

Now we are able to compute the first component on the right-hand side of (3.2.8). For $i = 1, ..., n$, let $\tilde{\mathbf{N}}$ be the ground process of $(\tilde{\tau}^i_j, \tilde{X}^i_j)_{j\in\mathbb{N}_+}$, i.e.

$$\tilde{\mathbf{N}}_t := \sum_{j=1}^{\infty} \mathbf{1}_{\{\tilde{\tau}^i_j \leqslant t\}}, \qquad t \geqslant 0.$$

We define

$$\begin{aligned} \tilde{m}(t) &:= E\left[\sum_{j=1}^{\tilde{\mathbf{N}}_t} \tilde{X}^i_j\right], && \text{if } t \geqslant 0, \\ \tilde{m}(t) &:= 0, && \text{if } t < 0. \end{aligned} \tag{3.2.16}$$

We note that the function $\tilde{m}$ is independent of the index i because of the homogeneous condition in Assumption 2.2.1(2).

Proposition 3.2.15. *Let $Z := (Z_t)_{t\in[0,T]}$ be a continuous, bounded and $\mathbb{F}$-adapted*[3] *process and Y be as in (3.2.6) Then we have for any $t \in [0, T]$,*

$$\begin{aligned} &E\left[\mathbf{1}_{\{\tau^i_1>t\}} Y \middle| \mathcal{H}^i_t \vee \mathcal{F}_t\right] \\ &= \mathbf{1}_{\{\tau^i_1>t\}} \frac{E\left[\int_t^T \left(E[X^i_1]Z_u + \int_u^T Z_v \mathrm{d}\tilde{m}(v-u)\right) \mathrm{d}P\left(\tau^i_1 \leqslant u \middle| \mathcal{F}_u\right) \middle| \mathcal{F}_t\right]}{P\left(\tau^i_1 > t \middle| \mathcal{F}_t\right)}. \end{aligned}$$

Proof. The representation (3.2.4) in Proposition 3.2.6 applied to Y defined in (3.2.6) yields

$$\begin{aligned} &E\left[\mathbf{1}_{\{\tau^i_1>t\}} Y \middle| \mathcal{H}^i_t \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau^i_1>t\}} E\left[Y \middle| \mathcal{H}^{i,1}_t \vee \mathcal{F}_t\right] \\ =&\mathbf{1}_{\{\tau^i_1>t\}} E\left[\sum_{j=1}^{\infty} \mathbf{1}_{\{\tau^i_j \leqslant T\}} X^i_j Z_{\tau^i_j} \middle| \mathcal{H}^{i,1}_t \vee \mathcal{F}_t\right] \\ =&\mathbf{1}_{\{\tau^i_1>t\}} E\left[\mathbf{1}_{\{\tau^i_1 \leqslant T\}} X^i_1 Z_{\tau^i_1} \middle| \mathcal{H}^{i,1}_t \vee \mathcal{F}_t\right] + \mathbf{1}_{\{\tau^i_1>t\}} E\left[\sum_{j=2}^{\infty} \mathbf{1}_{\{\tau^i_j \leqslant T\}} X^i_j Z_{\tau^i_j} \middle| \mathcal{H}^{i,1}_t \vee \mathcal{F}_t\right]. \end{aligned} \tag{3.2.17}$$

For the first component of (3.2.17), it suffices to use (3.2.3) in Proposition 3.2.6 and Lemma 3.2.13, considering the independence condition in Assumption 2.2.1 (3). We

[3]Note that the result holds without changes if the process is sufficiently integrable.

have hence

$$\begin{aligned}
&\mathbf{1}_{\{\tau_1^i>t\}}E\left[\mathbf{1}_{\{\tau_1^i\leqslant T\}}X_1^iZ_{\tau_1^i}\middle|\,\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right]\\
=&E\left[\mathbf{1}_{\{t<\tau_1^i\leqslant T\}}X_1^iZ_{\tau_1^i}\middle|\,\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right]\\
=&\mathbf{1}_{\{\tau_1^i>t\}}\frac{E\left[\mathbf{1}_{\{t<\tau_1^i\leqslant T\}}X_1^iZ_{\tau_1^i}\middle|\,\mathcal{F}_t\right]}{P\left(\tau_1^i>t\middle|\,\mathcal{F}_t\right)}\\
=&\mathbf{1}_{\{\tau_1^i>t\}}\frac{E\left[\int_t^T E[X_1^i]Z_u\mathrm{d}P\left(\tau_1^i\leqslant u\middle|\,\mathcal{F}_u\right)\middle|\,\mathcal{F}_t\right]}{P\left(\tau_1^i>t\middle|\,\mathcal{F}_t\right)}.
\end{aligned}$$

Now we concentrate on the second component of (3.2.17). Firstly, we assume first that on the interval $[t,T]$, Z is a bounded, stepwise, $\mathbb{F}$-predictable process, i.e.

$$Z_u=\sum_{i=0}^{n}Z_{t_i}\mathbf{1}_{\{t_i<u\leqslant t_{i+1}\}},\tag{3.2.18}$$

for $t<u\leqslant T$, where $t_0=t<...<t_{n+1}=T$ and Z_{t_i} is $\mathcal{F}_{t_i}$-measurable for all $i=0,...,n$. In this case, we obtain

$$\begin{aligned}
&\mathbf{1}_{\{\tau_1^i>t\}}E\left[\sum_{j=2}^{\infty}\mathbf{1}_{\{\tau_j^i\leqslant T\}}X_j^iZ_{\tau_j^i}\middle|\,\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right]\\
=&\mathbf{1}_{\{\tau_1^i>t\}}E\left[\sum_{j=1}^{\infty}\mathbf{1}_{\{t<\tau_1^i+\tilde{\tau}_j^i\leqslant T\}}\tilde{X}_j^iZ_{\tau_j^i}\middle|\,\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right]\\
=&\mathbf{1}_{\{\tau_1^i>t\}}E\left[\sum_{i=0}^{n}\sum_{j=1}^{\infty}\mathbf{1}_{\{t_i<\tau_1^i+\tilde{\tau}_j^i\leqslant t_{i+1}\}}\tilde{X}_j^iZ_{t_i}\middle|\,\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right]\\
=&\mathbf{1}_{\{\tau_1^i>t\}}E\left[\sum_{i=0}^{n}Z_{t_i}\;E\left[\sum_{j=1}^{\infty}\mathbf{1}_{\{t_i<\tau_1^i+\tilde{\tau}_j^i\leqslant t_{i+1}\}}\tilde{X}_j^i\middle|\,\mathcal{H}_t^{i,1}\vee\mathcal{F}_{t_i}\vee\sigma(\tau_1^i)\right]\middle|\,\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right]\\
=&\mathbf{1}_{\{\tau_1^i>t\}}E\left[\sum_{i=0}^{n}Z_{t_i}\;E\left[\sum_{j=1}^{\infty}\mathbf{1}_{\{t_i<x+\tilde{\tau}_j^i\leqslant t_{i+1}\}}\tilde{X}_j^i\middle|\,\mathcal{H}_t^{i,1}\vee\mathcal{F}_{t_i}\vee\sigma(\tau_1^i)\right]\Bigg|_{x=\tau_1^i}\middle|\,\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right]\\
=&\mathbf{1}_{\{\tau_1^i>t\}}E\left[\sum_{i=0}^{n}Z_{t_i}\;E\left[\sum_{j=1}^{\infty}\mathbf{1}_{\{t_i<x+\tilde{\tau}_j^i\leqslant t_{i+1}\}}\tilde{X}_j^i\right]\Bigg|_{x=\tau_1^i}\middle|\,\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right]\\
=&\mathbf{1}_{\{\tau_1^i>t\}}E\left[\sum_{i=0}^{n}Z_{t_i}\left(\tilde{m}(t_{i+1}-\tau_1^i)-\tilde{m}(t_i-\tau_1^i)\right)\middle|\,\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right],
\end{aligned}\tag{3.2.19}$$

where in the second last equality we make use of the independence between the marked point process $(\tilde{\tau}_j^i,\tilde{X}_j^i)_{j\in\mathbb{N}_+}$ and the σ-algebra $\mathcal{H}_\infty^{i,1}\vee\mathcal{F}_\infty$ in Assumption 2.2.1. This shows that for any bounded, stepwise, $\mathbb{F}$-predictable process Z, it holds

$$\mathbf{1}_{\{\tau_1^i>t\}} E\left[\sum_{j=2}^{\infty} \mathbf{1}_{\{\tau_j^i \leqslant T\}} X_j^i Z_{\tau_j^i} \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau_1^i>t\}} E\left[\int_t^T Z_u \mathrm{d}\tilde{m}(u-\tau_1^i) \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right].$$

If Z is a continuous bounded process, Z can be approximated by a sequence of bounded, stepwise and $\mathbb{F}$-predictable processes, i.e. there exists a sequence Z^n of the form (3.2.18) such that

$$Z^n \longrightarrow Z \quad \text{and} \quad |Z^n| \leqslant M,$$

with $M > 0$. We note that $\tilde{m}$ is right-continuous and monotone, hence the Lebesgue–Stieltjes integral

$$\int_t^T Z_u \mathrm{d}\tilde{m}(u-\tau_1^i) \tag{3.2.20}$$

is well defined. By Lebesgue Theorem, we have the convergence

$$\int_t^T Z_u^n \mathrm{d}\tilde{m}(u-\tau_1^i) \longrightarrow \int_t^T Z_u \mathrm{d}\tilde{m}(u-\tau_1^i).$$

Moreover,

$$\left|\int_t^T Z_u^n \mathrm{d}\tilde{m}(u-\tau_1^i)\right| \leqslant M \left|\int_t^T \mathrm{d}\tilde{m}(u-\tau_1^i)\right| = M|\tilde{m}(T-\tau_1^i) - \tilde{m}(t-\tau_1^i)|. \tag{3.2.21}$$

The right-hand side of (3.2.21) is uniformly bounded in view of (2.2.2). By applying again Lebesgue Theorem, the convergence of the conditional expectations also holds

$$E\left[\int_t^T Z_u^n \mathrm{d}\tilde{m}(u-\tau_1^i) \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] \longrightarrow E\left[\int_t^T Z_u \mathrm{d}\tilde{m}(u-\tau_1^i) \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right].$$

We note that $\tilde{m}(u) = 0$ for $u < 0$, thus we get

$$\begin{aligned} &\mathbf{1}_{\{\tau_1^i>t\}} E\left[\sum_{j=2}^{\infty} \mathbf{1}_{\{\tau_j^i \leqslant T\}} X_j^i Z_{\tau_j^i} \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] \\ &= \mathbf{1}_{\{\tau_1^i>t\}} E\left[\int_t^T Z_u \mathrm{d}\tilde{m}(u-\tau_1^i) \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] \\ &= E\left[\mathbf{1}_{\{t<\tau_1^i \leqslant T\}} \int_t^T Z_u \mathrm{d}\tilde{m}(u-\tau_1^i) \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right]. \end{aligned}$$

The representation (3.2.3) in Proposition 3.2.6 applied to the above expression yields

$$\begin{aligned} &\mathbf{1}_{\{\tau_1^i>t\}} E\left[\sum_{j=2}^{\infty} \mathbf{1}_{\{\tau_j^i \leqslant T\}} X_j^i Z_{\tau_j^i} \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] \\ &= \mathbf{1}_{\{\tau_1^i>t\}} \frac{E\left[\mathbf{1}_{\{t<\tau_1^i \leqslant T\}} \int_t^T Z_u \mathrm{d}\tilde{m}(u-\tau_1^i) \middle| \mathcal{F}_t\right]}{P\left(\tau_1^i > t \middle| \mathcal{F}_t\right)}. \end{aligned}$$

We set $\tilde{Z}_s := \int_t^T Z_u \mathrm{d}\tilde{m}(u-s)$, $s \in [0,T]$. Since $\tilde{m}$ is right-continuous and monotone, n one hand, for fixed $s \in [0,T]$, the function $d_s(u) := \tilde{m}(u-s)$, $u \in [0,T]$, is also right-continuous and monotone and is the cumulative distribution function of a finite positive measure, by (2.2.2). One the other hand, for fixed $u \in [0,T]$, the function $\tilde{m}(u-s)$, $s \in [0,T]$, is left-continuous in s, that is, for every series $s_n \nearrow s$, we have the following pointwise convergence

$$\lim_{s_n \nearrow s} d_{s_n}(u) = d_s(u) \quad \text{for all } u \in [0,T]$$

of the cumulative distribution functions, which is equivalent to the convergence in distribution or weak convergence in measure. Indeed, we recall that a series of positive finite measures $(\nu_n)_{n\in\mathbb{N}}$ converges weakly to a positive finite measure ν, if for all bounded continuous functions f, it holds

$$\int f \mathrm{d}\nu_n \longrightarrow \int f \mathrm{d}\nu.$$

This yields the following convergence

$$\tilde{Z}_{s_n} \longrightarrow \tilde{Z}_s, \quad P-\text{a.s.}.$$

In other words, $\tilde{Z}_s := \int_t^T Z_u \mathrm{d}\tilde{m}(u-s)$, $s \in [0,T]$, is left-continuous. We note that it is also bounded. Now we apply Lemma 3.2.13 and obtain

$$\begin{aligned}
&\mathbf{1}_{\{\tau_1^i > t\}} \frac{E\left[\mathbf{1}_{\{t<\tau_1^i\leqslant T\}} \int_t^T Z_u \mathrm{d}\tilde{m}(u-\tau_1^i) \middle| \mathcal{F}_t\right]}{P\left(\tau_1^i > t \middle| \mathcal{F}_t\right)} \\
&= \mathbf{1}_{\{\tau_1^i > t\}} \frac{E\left[\mathbf{1}_{\{t<\tau_1^i\leqslant T\}} \tilde{Z}_{\tau_1^i} \middle| \mathcal{F}_t\right]}{P\left(\tau_1^i > t \middle| \mathcal{F}_t\right)} \\
&= \mathbf{1}_{\{\tau_1^i > t\}} \frac{E\left[\int_t^T \tilde{Z}_u \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u\right) \middle| \mathcal{F}_t\right]}{P\left(\tau_1^i > t \middle| \mathcal{F}_t\right)} \\
&= \mathbf{1}_{\{\tau_1^i > t\}} \frac{E\left[\int_t^T \left(\int_t^T Z_v \mathrm{d}\tilde{m}(v-u)\right) \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u\right) \middle| \mathcal{F}_t\right]}{P\left(\tau_1^i > t \middle| \mathcal{F}_t\right)}.
\end{aligned}$$

We emphasize that by Corollary 3.2.12, the integrals under the sign of conditional expectation in the last two steps are well defined as Lebesgue–Stieltjes integrals, hence we do not need that the integrands are $\mathbb{F}$-adapted. Finally, we note that for $u < s$, $\int_t^T Z_u \mathrm{d}\tilde{m}(u-s) = \int_s^T Z_u \mathrm{d}\tilde{m}(u-s)$ since $\tilde{m}(u-s) = 0$. This concludes the proof. □

Remark 3.2.16. *An anther sufficient condition for the above Proposition, alternative to (3.2.16), would be that $\tilde{m}$ is a continuous function, e.g. in the case of a compound Poisson process or a Cox process with continuous intensity process and in-*

tegrable marks. In such case, since $\tilde{m}(u) = 0$ for $u < 0$, we have

$$\mathbf{1}_{\{\tau_1^i > t\}} \left| \int_t^T Z_u^n \mathrm{d}\tilde{m}(u - \tau_1^i) \right| \leqslant \mathbf{1}_{\{\tau_1^i > t\}} M \left| \int_t^T \mathrm{d}\tilde{m}(u - \tau_1^i) \right|$$
$$= \mathbf{1}_{\{t < \tau_1^i \leqslant T\}} M |\tilde{m}(T - \tau_1^i) - \tilde{m}(t - \tau_1^i)|,$$

and the right-hand side is bounded if $\tilde{m}$ is continuous.

A representation of the second component on the right-hand side of (3.2.8) is given in the following proposition.

Proposition 3.2.17. *Under the same assumptions of Proposition 3.2.15, if furthermore the process $\left(\sum_{j=1}^{\tilde{\mathbf{N}}_t} \tilde{X}_j^i\right)_{t\in[0,T]}$ has independent increments with respect to its natural[4] filtration $\tilde{H}^i$, then for $t \in [0,T]$ and Y as in (3.2.6), we have*

$$E\left[\mathbf{1}_{\{\tau_1^i \leqslant t\}} Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau_1^i \leqslant t\}} \left. E\left[\int_t^T Z_u \mathrm{d}\tilde{m}(u - x) \middle| \mathcal{H}_\infty^{i,1} \vee \tilde{\mathcal{H}}_{t-x}^i \vee \mathcal{F}_t\right]\right|_{x=\tau_1^i}.$$

Proof. Lemma 3.2.7 yields that

$$E\left[\mathbf{1}_{\{\tau^i \leqslant t\}} Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = E\left[\mathbf{1}_{\{\tau_1^i \leqslant t\}} Y \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{F}_t\right].$$

Using the same argument of the proof of Proposition 3.2.15, we assume first Z of the form (3.2.18). In such case, we get

$$\begin{aligned}
&E\left[\mathbf{1}_{\{\tau_1^i \leqslant t\}} Y \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{H}_t^{i,>1} \vee \mathcal{F}_t\right] \\
=&\mathbf{1}_{\{\tau_1^i \leqslant t\}} E\left[\sum_{j=1}^\infty \mathbf{1}_{\{t < \tau_1^i + \tilde{\tau}_j^i \leqslant T\}} \tilde{X}_j^i Z_{\tau_j^i} \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{H}_t^{i,>1} \vee \mathcal{F}_t\right] \\
=&\mathbf{1}_{\{\tau_1^i \leqslant t\}} E\left[\sum_{i=0}^n \sum_{j=1}^\infty \mathbf{1}_{\{t_i < \tau_1^i + \tilde{\tau}_j^i \leqslant t_{i+1}\}} \tilde{X}_j^i Z_{t_i} \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{H}_t^{i,>1} \vee \mathcal{F}_t\right] \\
=&\mathbf{1}_{\{\tau_1^i \leqslant t\}} \left. E\left[\sum_{i=0}^n \sum_{j=1}^\infty \mathbf{1}_{\{t_i < x + \tilde{\tau}_j^i \leqslant t_{i+1}\}} \tilde{X}_j^i Z_{t_i} \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{H}_t^{i,>1} \vee \mathcal{F}_t\right]\right|_{x=\tau_1^i} \\
=&\mathbf{1}_{\{\tau_1^i \leqslant t\}} \left. E\left[\sum_{i=0}^n \sum_{j=1}^\infty \mathbf{1}_{\{t_i < x + \tilde{\tau}_j^i \leqslant t_{i+1}\}} \tilde{X}_j^i Z_{t_i} \middle| \mathcal{H}_\infty^{i,1} \vee \tilde{\mathcal{H}}_{t-x}^i \vee \mathcal{F}_t\right]\right|_{x=\tau_1^i},
\end{aligned}$$

[4] We note that for $t \geqslant 0$,

$$\begin{aligned}
\tilde{H}_t^i &= \sigma\left(\sum_{j=1}^\infty \mathbf{1}_{\{\tilde{\tau}_j^i \leqslant s\}} \mathbf{1}_{\{\tilde{X}_j^i \in B\}}, 0 \leqslant s \leqslant t, \text{ for all } B \in \mathcal{B}(\mathbb{R}_+)\right) \\
&= \sigma\left(\sum_{j=1}^\infty \mathbf{1}_{\{\tilde{\tau}_j^i \leqslant s\}} \tilde{X}_j^i,\ 0 \leqslant s \leqslant t\right) = \sigma\left(\sum_{j=1}^{\tilde{\mathbf{N}}_s} \tilde{X}_j^i,\ 0 \leqslant s \leqslant t\right).
\end{aligned}$$

where in the last step we use the definitions of the filtrations. By using tower property, the independence between the marked point process $(\tilde{\tau}^i_j, \tilde{X}^i_j)_{j\in\mathbb{N}_+}$ and $\mathcal{F}_\infty \vee \mathcal{H}^{i,1}_\infty$ (see Assumption 2.2.1), and the independence of increments of the process $\left(\sum_{j=1}^{\tilde{\mathbf{N}}_t} \tilde{X}^i_j\right)_{t\in[0,T]}$ with respect to its natural filtration, we obtain furthermore

$$\begin{aligned}
& E\left[\mathbf{1}_{\{\tau^i_1\leqslant t\}} Y \middle| \mathcal{H}^{i,1}_\infty \vee \mathcal{H}^{i,>1}_t \vee \mathcal{F}_t\right] \\
=& \mathbf{1}_{\{\tau^i_1\leqslant t\}}\ E\left[\sum_{i=0}^n Z_{t_i}\left(E\left[\sum_{j=1}^\infty \mathbf{1}_{\{\tilde{\tau}^i_j\leqslant t_{i+1}-x\}}\tilde{X}^i_j \middle| \mathcal{H}^{i,1}_\infty \vee \tilde{\mathcal{H}}^i_{t-x} \vee \mathcal{F}_{t_i}\right]\right.\right. \\
& \left.\left.\left. - \ E\left[\sum_{j=1}^\infty \mathbf{1}_{\{\tilde{\tau}^i_j\leqslant t_i-x\}}\tilde{X}^i_j \middle| \mathcal{H}^{i,1}_\infty \vee \tilde{\mathcal{H}}^i_{t-x} \vee \mathcal{F}_{t_i}\right]\right) \middle| \mathcal{H}^{i,1}_\infty \vee \tilde{\mathcal{H}}^i_{t-x} \vee \mathcal{F}_t\right]\right|_{x=\tau^i_1} \\
=& \mathbf{1}_{\{\tau^i_1\leqslant t\}}\ E\left[\sum_{i=0}^n Z_{t_i}\left(E\left[\sum_{j=1}^\infty \mathbf{1}_{\{\tilde{\tau}^i_j\leqslant t_{i+1}-x\}}\tilde{X}^i_j \middle| \tilde{\mathcal{H}}^i_{t-x}\right]\right.\right. \\
& \left.\left.\left. - \ E\left[\sum_{j=1}^\infty \mathbf{1}_{\{\tilde{\tau}^i_j\leqslant t_i-x\}}\tilde{X}^i_j \middle| \tilde{\mathcal{H}}^i_{t-x}\right]\right) \middle| \mathcal{H}^{i,1}_\infty \vee \tilde{\mathcal{H}}^i_{t-x} \vee \mathcal{F}_t\right]\right|_{x=\tau^i_1} \\
=& \mathbf{1}_{\{\tau^i_1\leqslant t\}}\ E\left[\sum_{i=0}^n Z_{t_i}\left(E\left[\sum_{j=1}^\infty \mathbf{1}_{\{\tilde{\tau}^i_j\leqslant t_{i+1}-x\}}\tilde{X}^i_j\right]\right.\right. \\
& \left.\left.\left. - \ E\left[\sum_{j=1}^\infty \mathbf{1}_{\{\tilde{\tau}^i_j\leqslant t_i-x\}}\tilde{X}^i_j\right]\right) \middle| \mathcal{H}^{i,1}_\infty \vee \tilde{\mathcal{H}}^i_{t-x} \vee \mathcal{F}_t\right]\right|_{x=\tau^i_1} \\
=& \mathbf{1}_{\{\tau^i_1\leqslant t\}}\ E\left[\sum_{i=0}^n Z_{t_i}\left(\tilde{m}(t_{i+1}-x) - \sum_{j=1}^{\tilde{\mathbf{N}}_{t-x}} \tilde{X}^i_j - \tilde{m}(t_i-x) + \sum_{j=1}^{\tilde{\mathbf{N}}_{t-x}} \tilde{X}^i_j\right) \middle| \mathcal{H}^{i,1}_\infty \vee \tilde{\mathcal{H}}^i_{t-x} \vee \mathcal{F}_t\right]\Bigg|_{x=\tau^i_1} \\
=& \mathbf{1}_{\{\tau^i_1\leqslant t\}}\ E\left[\sum_{i=0}^n Z_{t_i}\left(\tilde{m}(t_{i+1}-x) - \tilde{m}(t_i-x)\right) \middle| \mathcal{H}^{i,1}_\infty \vee \mathcal{F}_t\right]\Bigg|_{x=\tau^i_1}
\end{aligned} \tag{3.2.22}$$

This yields that for any bounded, stepwise, $\mathbb{F}$-predictable process Z, it holds

$$E\left[\mathbf{1}_{\{\tau^i_1\leqslant t\}} Y \middle| \mathcal{H}^i_t \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau^i_1\leqslant t\}}\ E\left[\int_t^T Z_u \mathrm{d}\tilde{m}(u-x) \middle| \mathcal{H}^{i,1}_\infty \vee \mathcal{F}_t\right]\Bigg|_{x=\tau^i_1}.$$

The same arguments of Proposition 3.2.15 shows that, if Z is continuous, bounded and $\mathbb{F}$-adapted, then we can approximate Z by a sequence of bounded, stepwise and $\mathbb{F}$-predictable processes. This together with the fact that $\tilde{m}$ is right-continuous and monotone guarantees that the Riemann sum in (3.2.22) under the sign of conditional expectation converges to Lebesgue-Stieltjes integral. The convergence of the conditional expectation follows from the boundedness of Z and the integrability condition (2.2.2) of $\tilde{m}$. □

The following theorem summarizes the results above and gives an explicit representation of $\mathbb{G}$-conditional expectation with respect to the first reporting time τ_1^i.

Theorem 3.2.18. *Let $Z := (Z_t)_{t\in[0,T]}$ be a continuous, bounded and $\mathbb{F}$-adapted process*[5]*, $i = 1,\dots,n$ and Y of the form (3.2.6). If the process $\left(\sum_{j=1}^{\tilde{\mathbf{N}}_t} \tilde{X}_j^i\right)_{t\in[0,T]}$ has independent increments with respect to its natural filtration $\tilde{H}^i$ and $\tilde{m}$ is defined as in (3.2.16), then it holds*

$$E\left[Y\middle|\mathcal{G}_t\right] = \mathbf{1}_{\{\tau_1^i\leqslant t\}}\ E\left[\int_t^T Z_u \mathrm{d}\tilde{m}(u-x)\middle|\mathcal{H}_\infty^{i,1}\vee\tilde{\mathcal{H}}_{t-x}^i\vee\mathcal{F}_t\right]\Bigg|_{x=\tau_1^i}$$
$$+\mathbf{1}_{\{\tau_1^i>t\}}\frac{E\left[\int_t^T\left(E[X_1^i]Z_u+\int_u^T Z_v\mathrm{d}\tilde{m}(v-u)\right)\mathrm{d}P\left(\tau_1^i\leqslant u\middle|\mathcal{F}_u\right)\middle|\mathcal{F}_t\right]}{P\left(\tau_1^i>t\middle|\mathcal{F}_t\right)},$$

where

$$P\left(\tau_1^i\leqslant t\middle|\mathcal{F}_t\right)=\int_0^t\left(\alpha_0 e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u+\int_0^u g(u-v)e^{-\int_0^v\mu_s\mathrm{d}s}\mu_v\mathrm{d}v\right)\mathrm{d}u,$$

and

$$P\left(\tau_1^i>t\middle|\mathcal{F}_t\right)=e^{-\int_0^t\mu_u\mathrm{d}u}+\int_0^t\bar{G}(t-u)e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u\mathrm{d}u,$$

where α_0 and g are defined in (2.2.7) and $\bar{G}$ is defined in (3.2.11).

Proof. It is sufficient to combine Corollary 3.2.3, Lemma 3.2.10, Corollary 3.2.12, Proposition 3.2.15 and Proposition 3.2.17. □

Compared to Theorem 3.2.18, Theorem 3.2.9 gives a more explicit representation which is expressed as function of μ, the distribution of θ^i and the distribution of $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j\in\mathbb{N}_+}$. This is helpful for the pricing and hedging problem as we show in the next section.

3.3 Pricing and hedging non-life insurance

In this section, we focus on the problem of total reserve for a non-life insurance portfolio introduced in Section 2.5.2, by using the results in Section 3.2. For any time $t\in[0,T]$, we aim to price and hedge the nominal remaining payment $A_T - A_t$, where

$$A_t=\sum_{i=1}^n\sum_{j=1}^{\mathbf{N}_t^i}\frac{I_{\tau_j^i}}{S^*_{\tau_j^i}}X_j^i,\qquad t\in[0,T],$$

where we recall that I is the inflation index, S^* the benchmark portfolio, and X_j^i the payment amount in real value related to the random time τ_j^i.

By Assumption 2.3.7, the price process $S=(S_t)_{t\in[0,T]}$, the inflation index $I=(I_t)_{t\in[0,T]}$ and the benchmark portfolio $S^*=(S_t^*)_{t\in[0,T]}$ are continuous and $\mathbb{F}$-adapted,

[5]Note that the result of Theorem 3.2.18 holds without changes if Z is sufficiently integrable.

and that benchmarked value process $\hat{S} := S/S^*$ is an $(P, \mathbb{F})$-true martingale. Moreover, we assume that cumulative payment related to marked point processes $(\tilde{\tau}^i, \tilde{X}^i_j)_{j\in\mathbb{N}_+}$, $i = 1, ..., n$,

$$\sum_{j=1}^{\tilde{\mathbf{N}}^i_t} \tilde{X}^i_j, \qquad t \in [0,T], \qquad i = 1, ..., n,$$

which describe development after the first reports τ^i_1, are i.i.d. square integrable compound Poisson processes. That is, $\tilde{\mathbf{N}}^i$ are Poisson processes with parameter λ mutually independent, and $\tilde{X}^i_j$, are i.i.d. square integrable nonnegative random variables independent from $\tilde{\mathbf{N}}^i$, with expectation $E[\tilde{X}^i_j] = m$. In particular, we have

$$\tilde{m}(t) = E\left[\sum_{j=1}^{\tilde{\mathbf{N}}^i_t} \tilde{X}^i_j\right] = \lambda m t, \qquad t \in [0,T].$$

Under the above assumptions, all conditions in Theorem 3.2.18 are satisfied for $Y = A_T - A_t$, $t \in [0,T]$. Let l_t be the number of reported claims at time t, i.e.

$$l_t := \sum_1^n \mathbf{1}_{\left\{\tau^i_1 \leqslant t\right\}}, \qquad t \in [0,T].$$

The real-world pricing formula (2.3.3) combined with Corollary 3.2.3 and Theorem 3.2.18 yields

$$\begin{aligned} V_t \frac{I_t}{S^*_t} =& E\left[A_T - A_t \middle| \mathcal{G}_t\right] = E\left[\sum_{i=1}^n \sum_{j=\mathbf{N}^i_t}^{\mathbf{N}^i_T} \frac{I_{\tau^i_j}}{S^*_{\tau^i_j}} X^i_j \middle| \mathcal{G}_t\right] \\ =& \sum_{i=1}^n E\left[\sum_{j=\mathbf{N}^i_t}^{\mathbf{N}^i_T} \frac{I_{\tau^i_j}}{S^*_{\tau^i_j}} X^i_j \middle| \mathcal{F}_t \vee \mathcal{H}^i_t\right] \\ =& \lambda m l_t \, E\left[\int_t^T \frac{I_u}{S^*_u} \mathrm{d}u \middle| \mathcal{H}^{i,1}_\infty \vee \mathcal{F}_t\right] \\ &+ (n - l_t) \frac{E\left[\int_t^T \left(E[X^i_1] \frac{I_u}{S^*_u} + \lambda m \int_u^T \frac{I_v}{S^*_v} \mathrm{d}v\right) \mathrm{d}P\left(\tau^i_1 \leqslant u \middle| \mathcal{F}_u\right) \middle| \mathcal{F}_t\right]}{e^{-\int_0^t \mu_u \mathrm{d}u} + \int_0^t \bar{G}(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u}, \end{aligned} \quad (3.3.1)$$

where the conditional probability function $P\left(\tau^i_1 \leqslant u \middle| \mathcal{F}_u\right)$ is given in (3.2.13), i.e.

$$P\left(\tau^i_1 \leqslant t \middle| \mathcal{F}_t\right) = \int_0^t \left(\alpha_0 e^{-\int_0^s \mu_v \mathrm{d}v} \mu_s + \int_0^s g(s-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u\right) \mathrm{d}s.$$

We note that the first component on the right-hand side of (3.3.1)

$$\lambda m l_t \, E\left[\int_t^T \frac{I_u}{S^*_u} \mathrm{d}u \middle| \mathcal{H}^{i,1}_\infty \vee \mathcal{F}_t\right]$$

represents reported claims and does not involve the updating information after the first reporting. In particular, if we assume furthermore that the inflation linked zero-coupon bond (or TIPS) is a primary asset, i.e. an element of the vector S, and that the process I/S^* is $\mathbb{F}$-conditionally independent from τ_1^i, for every $i = 1, ..., n$, then we get

$$\lambda m l_t \, E\left[\int_t^T \frac{I_u}{S_u^*} \mathrm{d}u \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{F}_t \right] = \lambda m l_t \int_t^T E\left[\frac{I_u}{S_u^*} \middle| \mathcal{F}_t \right] \mathrm{d}u$$
$$= \lambda m l_t (T-t) \frac{I_t}{S_t^*}, \tag{3.3.2}$$

where the last step follows from the martingale property of the process I/S^*. The second component on the right-hand side of (3.3.1)

$$(n-l_t) \frac{E\left[\int_t^T \left(E[X_1^i] \frac{I_u}{S_u^*} + \lambda m \int_u^T \frac{I_v}{S_v^*} \mathrm{d}v \right) \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u \right) \middle| \mathcal{F}_t \right]}{e^{-\int_0^t \mu_u \mathrm{d}u} + \int_0^t \bar{G}(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u}, \tag{3.3.3}$$

corresponds to not reported claims and can be further explicitly calculated. These includes both cases of incurred but not reported (IBNR claims) and those not yet incurred. The standard non-life insurance literature is mainly focused on IBNR claims. However, we stress that since the occurrence of accident is unknown unless the claim is reported and even after the reporting there still might be updating events, the entire expression (3.3.1) should be taken into account for pricing purpose.

Now we use the notations in Section 2.4 and solve explicitly the hedging problem related to the total reserve by means of benchmarked risk-minimization method. We note that hedging strategy is additive with respect to claims, and the first component (3.3.2) related to reported claims is totally hedgeable by trading inflation linked zero-coupon bonds. Hence, here we mainly focus on the second component (3.3.3) related to not reported claims

$$\sum_{i=1}^n \mathbf{1}_{\{\tau_1^i > t\}} \sum_{j=\mathbf{N}_t^i}^{\mathbf{N}_T^i} \frac{I_{\tau_j^i}}{S_{\tau_j^i}^*} X_j^i, \qquad t \in [0, T],$$

For $t \in [t, T]$, its associated real-world pricing value V_t is given by

$$V_t = (n-l_t) \frac{\frac{S_t^*}{I_t} E\left[\int_t^T \left(E[X_1^i] \frac{I_u}{S_u^*} + \lambda m \int_u^T \frac{I_v}{S_v^*} \mathrm{d}v \right) \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u \right) \middle| \mathcal{F}_t \right]}{e^{-\int_0^t \mu_u \mathrm{d}u} + \int_0^t \bar{G}(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u}.$$

As already mentioned in Section 2.3, this price coincides with the benchmarked risk-minimizing price. That is, the benchmarked value process $\hat{S}^{\bar{\delta}} := (\hat{S}_t^{\bar{\delta}})_{t \in [0,T]}$ associated to the benchmarked risk-minimizing strategy $\bar{\delta} := (\bar{\delta}_t)_{t \in [0,T]}$ is

$$\hat{S}_t^{\bar{\delta}} = \frac{I_t}{S_t^*} V_t = (n-l_t) \frac{E\left[\int_t^T \left(E[X_1^i] \frac{I_u}{S_u^*} + \lambda m \int_u^T \frac{I_v}{S_v^*} \mathrm{d}v \right) \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u \right) \middle| \mathcal{F}_t \right]}{e^{-\int_0^t \mu_u \mathrm{d}u} + \int_0^t \bar{G}(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u}, \quad t \in [0, T].$$

Using the same arguments of Proposition 4.11 in [7], the associated benchmarked risk minimizing strategy $\bar{\delta}$ is given by

$$\bar{\delta}_t = (n - l_{t_-}) \left(e^{-\int_0^t \mu_u \mathrm{d}u} + \int_0^t \bar{G}(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u \right)^{-1} \phi_t, \qquad t \in [0,T],$$

where ϕ_t is the amount at t of risky assets of the benchmarked risk-minimizing strategy related to the purely financial contingent claim

$$U_t := E\left[\int_0^T \left(E[X_1^i] \frac{I_u}{S_u^*} + \lambda m \int_u^T \frac{I_v}{S_v^*} \mathrm{d}v \right) \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u \right) \middle| \mathcal{F}_t \right], \qquad t \in [0,T]. \tag{3.3.4}$$

In particular, the vector process $\phi := (\phi_t)_{t\in[0,T]}$ is obtained by the Galtchouk–Kunita–Watanabe decomposition of $(U_t)_{t\in[0,T]}$,

$$U_t = E\left[\int_0^T \left(E[X_1^i] \frac{I_u}{S_u^*} + \lambda m \int_u^T \frac{I_v}{S_v^*} \mathrm{d}v \right) \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u \right) \right] + \int_0^t \phi_u^\top \mathrm{d}\hat{S}_u + L_t^U,$$

where $\phi \in L^2(\hat{S}, P, \mathbb{G})$ and $L^U \in M_0^2(P, \mathbb{G})$ is strongly orthogonal to $\mathcal{I}^2(\hat{S}, P, \mathbb{G})$. The form of V suggests how to design derivatives which can be used to hedge risks in this market model. In particular, the purely financial contingent claim U_t involves only the random values of S^*, I and μ, this justifies the setting in Section 2.3 which proposes to introduce three kinds of primary assets as hedging assets, including pure financial assets, inflation linked derivatives and macro risk-factor linked derivatives. Heuristically, these three kinds of assets should be used to hedge risks derived from S^*, I and μ respectively.

Chapter 4

Life insurance in continuous time

4.1 Introduction

In this chapter, based on [18], we analyze the case of life insurance under polynomial diffusion model within the setting of Chapter 2. Since the general life insurance framework in continuous time have been already intensively studied in the literature, here we focus only on the particular polynomial diffusion model specification, which generalizes the classic affine model and allows at the same time explicit computations. We neglect in this chapter the inflation effect and assume that the reference market is driven by a possibly multi-dimensional state variable which follows a polynomial diffusion on a compact state space, and represents underlying risk factors, such as macro-economic variables, environmental and social indicators. The primary assets are composed by risk-free zero-coupon bonds and longevity bonds, both modelled as functions of the state variable. A parsimonious numerical example with calibration to real data is provided in this model setting and explicit results for real-world pricing formulas and benchmarked risk-minimizing strategies for relevant life insurance products are derived.

The chapter is structured as follows. Section 4.2 gives a brief introduction of polynomial diffusions. Section 4.3 describes the polynomial diffusion model assumptions for a portfolio of life insurance policies. In Section 4.4, a 2-dimensional state variable is used to calibrate our model to MSCI and LLMA index. In Section 4.5, we provide pricing-hedging formulas for the three building blocks of life insurance products (pure endowment, term insurance, annuity).

4.2 Polynomial diffusion process

In this section, we give a synthetic summary of the most important results for polynomial diffusions presented in [47], which will be used in our discussion.

We consider a compact set with nonempty internal part $E \subset \mathbb{R}^d$ as state space. The space of real symmetric $d \times d$ matrices is denoted by $\mathbb{S}^d$, and the convex cone of positive semidefinite symmetric matrices by $\mathbb{S}^d_+$. For every $n \in \mathbb{N}$, let $\mathrm{Pol}_n(E)$ be the following finite-dimensional vector space

$$\mathrm{Pol}_n(E) := \{\text{polynomials on } E \text{ of degree} \leqslant n\},$$

where N_n is the dimension of $\mathrm{Pol}_n(E)$. Let $[0,T]$ be a fixed time horizon and $(\Omega, \mathcal{F}, \mathbb{F}, P)$ a generic filtered probability space. Let $Z := (Z_t)_{t\in[0,T]}$ be an E-valued $\mathbb{F}$-adapted process with initial value Z_0 constant and belonging to E, which follows

$$\mathrm{d}Z_t = b(Z_t)\mathrm{d}t + \sigma(Z_t)\mathrm{d}W_t, \tag{4.2.1}$$

where $W := (W_t)_{t\in[0,T]}$ is a d-dimensional $\mathbb{F}$-Brownian motion, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ a continuous function, $a := \sigma\sigma^\top$ and

$$a : \mathbb{R}^d \to \mathbb{S}^d, \qquad b : \mathbb{R}^d \to \mathbb{R}^d,$$

with

$$a_{ij} \in \mathrm{Pol}_2(\mathbb{R}^d), \qquad b_i \in \mathrm{Pol}_1(\mathbb{R}^d),$$

for every $i, j = 1, ..., d$.

Definition 4.2.1. *An E-valued process Z following (4.2.1) is called a* polynomial diffusion on E *if*

$$\mathbf{G}Pol_n(E) \subseteq Pol_n(E), \qquad \textit{for all} \quad n \in \mathbb{N},$$

where the operator

$$\mathbf{G} : \mathcal{C}^2(\mathbb{R}^d) \longrightarrow \mathbb{R},$$

is defined by

$$\mathbf{G}f(z) := \frac{1}{2}Tr(a(z)\nabla^2 f(z)) + b(z)^\top \nabla f(z), \qquad z \in \mathbb{R}^d. \tag{4.2.2}$$

Proposition 4.2.2. *A necessary and sufficient condition for Z defined in (4.2.1) to be a polynomial diffusion on E is that the components of the maps a and b restricted to E lie in $Pol_2(E)$ and $Pol_1(E)$, respectively, i.e.*

$$a_{ij}\big|_E \in Pol_2(E), \qquad b_i\big|_E \in Pol_1(E).$$

Proof. See Lemma 2.2 of [47]. □

In particular, for every fixed $n \in \mathbb{N}$, the operator $\mathbf{G}$ associated to a polynomial diffusion Z has a unique matrix representation $G_n \in \mathbb{R}^{N_n\times N_n}$ restricted to $\mathrm{Pol}_n(E)$. In other words, if $p \in \mathrm{Pol}_n(E)$ has coordinate representation

$$p(z) = H_n(z)^\top \vec{p}, \qquad z \in \mathbb{R}^d, \tag{4.2.3}$$

where $H_n(z)$ is a basis vector of $\mathrm{Pol}_n(E)$ and $\vec{p} \in \mathbb{R}^{N_n}$, then

$$\mathbf{G}p(z) = H_n(z)^\top G_n\vec{p}, \qquad z \in \mathbb{R}^d.$$

In the following proposition, we show one of the most relevant results for polynomial diffusions: the conditional expectation of a polynomial function of Z_T is again given by a polynomial function of the state variable, where the polynomial coefficients are deterministic and time-dependent.

Proposition 4.2.3. *Let $n \in \mathbb{N}$ and (4.2.3) the coordinate representation of $p \in Pol_n(E)$ with $0 \leqslant t \leqslant T$. If Z is an E-valued solution to (4.2.1), then we have*

$$E\left[p(Z_T)|\,\mathcal{F}_t\right] = H_n(Z_t)^\top e^{(T-t)G_n}\vec{p}. \tag{4.2.4}$$

Proof. See Theorem 3.1 of [47]. □

Theorem 4.2.4 and Theorem 4.2.6 give sufficient conditions on the state space E, such that an E-valued state variable Z admits weak uniqueness and existence.

Theorem 4.2.4. *If Z is an E-valued solution to (4.2.1) and E is compact, then Z is weakly unique*[1]*.*

Proof. See Theorem 5.1 of [47]. □

Remark 4.2.5. *Strong uniqueness (or pathwise uniqueness) for an E-valued solution to (4.2.1) holds for $d = 1$, see [103]. A more detailed discussion about the strong uniqueness in a generic dimension is provided in [70].*

Theorem 4.2.6. *If the boundary of the state space E is defined by a family $\mathbf{P}$ of polynomials on $\mathbb{R}^d$, i.e.*

$$E = \{x \in \mathbb{R}^d : p(x) \leqslant 0 \text{ for all } p \in \mathbf{P}\},$$

then the following conditions on the parameters a and b are sufficient for the existence of an E-valued solution to (4.2.1):

1. $a \in \mathbb{S}^d_+$;
2. $a\nabla p = 0$ *on* $\{p = 0\}$ *for all* $p \in \mathbf{P}$;
3. $\mathbf{G}p > 0$ *on* $E \cap \{p = 0\}$ *for all* $p \in \mathbf{P}$.

Proof. This theorem is a special case of Theorem 5.3 of [47]. □

Remark 4.2.7. *We note that, due to the compactness of the state space and the weak uniqueness given by Theorem 4.2.4, every E-solution to (4.2.1) is a strong Markov process, see e.g. Theorem 4.6 of [46]. Consequently, the operator defined in (4.2.2) is the extended Markov generator of Z.*

[1] That is, any other E-valued solution to (4.2.1) with initial value Z_0 has the same law as Z.

4.3 Polynomial diffusion model assumptions

Now we specify a polynomial diffusion model under intensity-based reduced-form approach in e.g. [24] and Section 2.5.1, in order to describe life insurance derivatives linked to a portfolio of life insurance contracts. We fix a finite time horizon $[0,T]$ and work under the setting and use the notations of Chapter 2. For the sake of simplicity, throughout this chapter we neglect the inflation I and denote the decease time τ_0^i of policyholders by τ^i, since they are the only random times involved in the discussion. Let $H^i := (H_t^i)_{t\in[0,T]}$ with $H_t^i = \mathbf{1}_{\{\tau^i \leqslant t\}}$, $t \in [0,T]$, be the jump process[2] of τ^i, $i = 1, ..., n$, and $l := (l_t)_{t\in[0,T]}$ be the death counting process

$$l_t = \sum_{i=1}^{n} \mathbf{1}_{\{\tau^i \leqslant t\}}, \qquad t \in [0,T]. \tag{4.3.1}$$

We recall the homogeneity assumption and set that

$$F_t = P\left(\tau^i \leqslant t \middle| \mathcal{F}_t\right), \qquad t \in [0,T]$$

is the common conditional cumulative distribution function of τ^i, $i = 1, ..., n$;

$$\Gamma_t = -\ln\left(1 - F_t\right), \qquad t \in [0,T]$$

is the common hazard process; the common mortality intensity μ is a nonnegative $\mathbb{F}$-progressively measurable process with integrable sample paths such that

$$\Gamma_t = \int_0^t \mu_u \mathrm{d}u, \qquad t \in [0,T].$$

We may take a $\mathbb{F}$-predictable version of μ^i, see Lemma 1.36 of [62]. We recall furthermore that the longevity index L, is modelled by

$$L_t = 1 - F_t = e^{-\Gamma_t} = e^{-\int_0^t \mu_u \mathrm{d}u}, \qquad t \in [0,T].$$

Let Z be a state variable process which represents the underlying risk factors, and $m_1, m_2 \in \mathbb{N}$ be such that $m_1 + m_2 = d$. We assume Z to be a polynomial diffusion of the form

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix},$$

on the compact state space $E \subset \mathbb{R}^d$ with nonempty internal part given by

$$E = E^X \times E^Y \ , \ E^X \subseteq \mathbb{R}^{m_1} \text{ and } \ E^Y \subseteq \mathbb{R}^{m_2} \ ,$$

where E^X and E^Y are respectively the state space of process X and Y. The dynamics of the components X and Y is described by

$$\begin{cases} \mathrm{d}X_t &= b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \\ \mathrm{d}Y_t &= \bar{b}(X_t, Y_t)\mathrm{d}t, \end{cases} \tag{4.3.2}$$

[2] We note that, since here the marked point process of Chapter 2 is reduced to a single jump, the process $(H_t^i)_{t\in[0,T]}$ coincides with the ground process $(\mathbf{N}_t^i)_{t\in[0,T]}$ of Section 2.2 in this case.

for $t \in [0,T]$, where $W := (W_t)_{t\in[0,T]}$ is a d-dimensional $\mathbb{F}$-Brownian motion, $\sigma : \mathbb{R}^{m_1} \to \mathbb{R}^{m_1\times d}$ a continuous function, $a := \sigma\sigma^\top$ and

$$\begin{aligned} a &: \mathbb{R}^{m_1} \longrightarrow \mathcal{S}^{m_1}, \quad b : \mathbb{R}^{m_1} \longrightarrow \mathbb{R}^{m_1}, \quad \bar{b} : \mathbb{R}^{d} \longrightarrow \mathbb{R}^{m_2}, \\ a_{ij} &\in \mathrm{Pol}_2(\mathbb{R}^{m_1}), \quad b_i \in \mathrm{Pol}_1(\mathbb{R}^{m_1}), \quad \bar{b}_k \in \mathrm{Pol}_1(\mathbb{R}^{d}), \end{aligned}$$

for all $i, j = 1, ..., m_1$ and $k = 1, ..., m_2$. As shown in the following proposition, if E^X is a compact set of $\mathbb{R}^{m_1}$, then E^Y is also compact.

Proposition 4.3.1. *If there exists a constant C such that for all $t \in [0,T]$,*

$$\|X_t\| \leqslant C,$$

then Y is uniformly bounded.

Proof. We note that $\bar{b}$ is a linear function of Z, hence the Y-dynamics can be written as

$$\mathrm{d}Y_t = (AX_t + BY_t + c)\mathrm{d}t,$$

where $A \in \mathbb{R}^{m_2\times m_1}$, $B \in \mathbb{R}^{m_2\times m_2}$, $c \in \mathbb{R}^{m_2}$. In particular, this yields

$$\|Y_t\| \leqslant b\int_0^t \|Y_u\|\,\mathrm{d}u + aCt + \|c\|\,t + \|Y_0\|\,,$$

with a and b some matrix norms of A and B. By the Grönwall's inequality, we have

$$\|Y_t\| \leqslant \bar{C} + b\int_o^t e^{b(t-s)}ds \leqslant \bar{C} + b\int_o^T e^{b(T-s)}ds < \infty,$$

for all $t \in [0,T]$, with $\bar{C}$ a suitable constant. This concludes the proof. □

For the sake of simplicity, we denote the degree of a generic polynomial function p by $\bar{p}$. If $p \in \mathrm{Pol}_{\bar{p}}(E)$ and $t \in [0,T]$, we denote the coordinate representation (4.2.4) of $E[p(Z_T)|\mathcal{F}_t]$ by

$$\mathbf{p}_{(t,T)}(Z_t) := H_{\bar{p}}(Z_t)^\top e^{(T-t)G_{\bar{p}}}\vec{p}, \qquad t \in [0,T].$$

We model the benchmark portfolio in the following way

$$\frac{1}{S^*_t} := e^{-\alpha t}p(Z_t), \quad \alpha \in \mathbb{R} \text{ and } p \text{ strictly positive polynomial on } E, \tag{4.3.3}$$

for every $t \in [0,T]$. In [47], a similar dynamics is specified for a state price density, while here we choose to model the benchmark portfolio. We assume that primary assets include risk-free OIS bond and longevity bond maturing at T and indicate their value processes respectively by $(P(t,T))_{t\in[0,T]}$ and $(P^l(t,T))_{t\in[0,T]}$. Since the state space E is assumed to be compact, the restricted polynomial $p|_E$ admits a strictly positive minimum value. That is, there exists a strictly positive number ε such that

$$E \subseteq \{z \in \mathbb{R}^d : p(z) \geqslant \varepsilon\}. \tag{4.3.4}$$

As we will see below, by adjusting the parameter α, the condition (4.3.4) ensures the continuity of both risk-free OIS bond and longevity bond, as well as the non-negativity of the risk-free short rate.

A risk-free OIS bond maturing in T is defined as a zero-coupon bond with unit payment at time T, whose value at $t \in [0, T]$ is given by

$$P(t,T) = S_t^* E\left[\frac{1}{S_T^*} \middle| \mathcal{G}_t \right].$$

This can be calculated using (4.3.3) and Proposition 4.2.3,

$$\begin{aligned} S_t^* E\left[\frac{1}{S_T^*} \middle| \mathcal{G}_t \right] &= e^{-\alpha(T-t)} \frac{E\left[p(Z_T) \middle| \mathcal{G}_t \right]}{p(Z_t)} = e^{-\alpha(T-t)} \frac{E\left[p(Z_T) \middle| \mathcal{F}_t \right]}{p(Z_t)} \\ &= e^{-\alpha(T-t)} \frac{H_{\bar{p}}(Z_t)^\top e^{(T-t)G_{\bar{p}}} \vec{p}}{p(Z_t)} = e^{-\alpha(T-t)} \frac{\mathbf{p}_{(t,T)}(Z_t)}{p(Z_t)}. \end{aligned}$$

In particular, the second equality follows by Lemma 6.1.1 of [24]. Thanks to (4.3.4), the process above is well-defined and continuous. Therefore, we have

$$P(t,T) = e^{-\alpha(T-t)} \frac{\mathbf{p}_{(t,T)}(Z_t)}{p(Z_t)}, \qquad t \in [0, T]. \tag{4.3.5}$$

The dynamics of risk-free short rate process $r := (r_t)_{t \in [0,T]}$ follows immediately from the risk-free OIS bond dynamics (4.3.5). That is, for $t \in [t, T]$

$$\begin{aligned} r_t :&= -\left. \partial_T \log P(t,T) \right|_{T=t} \\ &= -\left. \partial_T \log \left(e^{-\alpha(T-t)} \frac{\mathbf{p}_{(t,T)}(Z_t)}{p(Z_t)} \right) \right|_{T=t} \\ &= -\left. \partial_T \log \left(e^{-\alpha(T-t)} \right) \right|_{T=t} - \left. \partial_T \log \left(\frac{H_{\bar{p}}(Z_t)^\top e^{(T-t)G_{\bar{p}}} \vec{p}}{p(Z_t)} \right) \right|_{T=t} \\ &= \alpha - \left. \left[\frac{p(Z_t)}{H_{\bar{p}}(Z_t)^\top e^{(T-t)G_{\bar{p}}} \vec{p}} \, \partial_T \left(\frac{H_{\bar{p}}(Z_t)^\top e^{(T-t)G_{\bar{p}}} \vec{p}}{p(Z_t)} \right) \right] \right|_{T=t} \\ &= \alpha - \left. \left[\frac{p(Z_t)}{H_{\bar{p}}(Z_t)^\top e^{(T-t)G_{\bar{p}}} \vec{p}} \frac{H_{\bar{p}}(Z_t)^\top G_{\bar{p}} e^{(T-t)G_{\bar{p}}} \vec{p}}{p(Z_t)} \right] \right|_{T=t} \\ &= \alpha - \frac{p(Z_t)}{H_{\bar{p}}(Z_t)^\top \vec{p}} \frac{H_{\bar{p}}(Z_t)^\top G_{\bar{p}} \vec{p}}{p(Z_t)} \\ &= \alpha - \frac{H_{\bar{p}}(Z_t)^\top G_{\bar{p}} \vec{p}}{p(Z_t)}, \end{aligned}$$

since $e^{(T-t)G_{\bar{p}}} = 1$ when $T = t$. We note that, the compactness of the state space E and (4.3.4) insure that

$$\frac{H_{\bar{p}}(Z_t)^\top G_{\bar{p}} \vec{p}}{p(Z_t)}$$

has an upper bound $\overline{\alpha}$ and a lower bound $\underline{\alpha}$ uniformly in $t \in [0, T]$. If we choose $\alpha = \overline{\alpha}$, then short rate takes positive value in $[0, \overline{\alpha} - \underline{\alpha}]$.

According to the definition in [28] and [26], a longevity bond maturing at T is a longevity index-linked zero-coupon bond with payment at T equal to the value of the longevity or survival index L at T. Here we model first the longevity index L and then derive the dynamics of the mortality intensity μ, unlike the usual intensity-based approach. We make use of the Y-component of the state variable Z to model the longevity index

$$L_t := e^{-\gamma t} q(Y_t), \quad \gamma \in \mathbb{R} \text{ and } q \text{ strictly positive polynomial on } E^Y, \tag{4.3.6}$$

for $t \in [0,T]$. The parameter γ is used to adjust the value level of mortality intensity. In the same way as before, there exists a strictly positive number δ such that

$$E^Y \subseteq \{y \in \mathbb{R}^{m_2} : q(y) \geqslant \delta\}. \tag{4.3.7}$$

We derive the formula for the mortality intensity $(\mu) := (\mu_t)_{t\in[0,T]}$

$$\mu_t = -\left.\partial_T \log\left(L_T\right)\right|_{T=t} = \gamma - \frac{\nabla q(Y_t)^\top \overline{b}(X_t, Y_t)}{q(Y_t)}, \tag{4.3.8}$$

for all $t \in [0.T]$. Similarly to the case of risk-free short rate r, the compactness of E^Y and the condition (4.3.7) insure that, uniformly in t the quantity

$$\frac{\nabla q(Y_t)^\top \overline{b}(X_t, Y_t)}{q(Y_t)}$$

has an upper bound $\overline{\gamma}$ and a lower bound $\underline{\gamma}$. By setting $\gamma = \overline{\gamma}$, the mortality intensity has a positive value range $[0, \overline{\gamma} - \underline{\gamma}]$. By using definition (4.3.6) of longevity index and Proposition 4.2.3, it is possible to calculate explicitly the value of a T-longevity bond $P^l(t,T)$ at time $t \in [0,T]$,

$$\begin{aligned} P^l(t,T) &= S_t^* E\left[\left.\frac{L_T}{S_T^*}\right| \mathcal{G}_t\right] = S_t^* E\left[\left.(S_T^*)^{-1} e^{-\gamma_T} q(Y_T)\right| \mathcal{G}_t\right] \\ &= L_t e^{-(\alpha+\gamma)(T-t)} \frac{E\left[\left. p(Z_T) q(Y_T)\right| \mathcal{F}_t\right]}{p(Z_t) q(Y_t)} \\ &= L_t e^{-(\alpha+\gamma)(T-t)} \frac{\mathbf{pq}_{(t,T)}(Z_t)}{p(Z_t) q(Y_t)} \\ &= e^{-\gamma T - \alpha(T-t)} \frac{\mathbf{pq}_{(t,T)}(Z_t)}{p(Z_t)}, \end{aligned} \tag{4.3.9}$$

where in the second equality we use Lemma 6.1.1 of [24]. Due to condition (4.3.4), the process above is clearly continuous.

4.4 Calibration example

We now give a parsimonious numerical example with calibration to real data. We set $m_1 = m_2 = 1$ and assume without loss of generality that $E^X = [-1,1]$ and

$Z_0 = (X_0, Y_0)^\top = 0$. In particular, by E^Y is also bounded in view of Proposition 4.3.1. Further discussion of polynomial diffusions on unit ball in higher dimension can be found in [70]. In view of Theorem 2.1 of [70], we consider the following dynamics of Z,

$$\mathrm{d}Z_t = \mathrm{d}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \Psi & 0 \\ d & \kappa \end{pmatrix}\left(\begin{pmatrix} b \\ \eta \end{pmatrix} - \begin{pmatrix} X_t \\ Y_t \end{pmatrix}\right)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t,$$

where W is a 1-dimensional Brownian motion,

$$a(x) = \sigma(x)^2 = \sigma(1 - x^2), \qquad \text{for all } x \in [-1, 1],$$

and

$$b\Psi x - \Psi x^2 \leqslant 0, \qquad \text{for } x \in \{1, -1\}.$$

We note that the above condition equals $|b\Psi| \leqslant \Psi$ or

$$\begin{cases} |b| \leqslant 1, \\ \Psi \geqslant 0. \end{cases} \tag{4.4.1}$$

In particular, the dynamics of component X satisfies

$$\mathrm{d}X_t = \Psi(b - X_t)\mathrm{d}t + \sigma\sqrt{1 - X_t^2}\mathrm{d}W_t.$$

Moreover, we assume that the polynomials p and q are both linear and positive on E,

$$p(x) = \rho + cx, \qquad p > 0 \text{ on } E^X, \tag{4.4.2}$$

$$q(y) = \delta + \nu y, \qquad q > 0 \text{ on } E^Y, \tag{4.4.3}$$

where $\rho, \delta, \nu, c \in \mathbb{R}$, analogue to the specification in [48]. The above assumptions imply

$$\frac{1}{S_t^*} = e^{-\alpha t}\left(\rho + cX_t\right),$$

$$P(t,T) = \frac{(\rho + cb)\, e^{-\alpha(T-t)} + ce^{-(\alpha+\Psi)(T-t)}(X_t - b)}{\rho + cpX_t},$$

$$r_t = \alpha - \frac{c\Psi(b - X_t)}{\rho + cX_t},$$

$$L_t = e^{-\gamma t}\left(\delta + \nu Y_t\right),$$

$$\mu_t = \gamma - \frac{\nu\left(db + \kappa\eta - dX_t - \kappa Y_t\right)}{\delta + \nu Y_t},$$

$$P^l(t,T) = e^{-\gamma T - \alpha(T-t)}\frac{\mathbf{pq}_{(t,T)}(Z_t)}{\rho + cX_t},$$

with dynamics described by

$$\frac{\mathrm{d}(S_t^*)^{-1}}{(S_t^*)^{-1}} = -r_t\mathrm{d}t - \lambda_t\mathrm{d}W_t,$$

where $\lambda_t = -\sqrt{a(X_t)}c/(\rho + cX_t)$,

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = (r_t + \nu(t,T)\lambda_t)\mathrm{d}t + \nu(t,T)\mathrm{d}W_t,$$

where $\nu(t,T) = \sqrt{a(X_t)}\nabla P(t,T)/P(t,T)$,

$$\frac{\mathrm{d}L_t}{L_t} = -\mu_t \mathrm{d}t.$$

We calibrate the above described model to the inverse of benchmark portfolio and the longevity index. As discussed in [98] and [99], the benchmark portfolio can be identified with a sufficiently diversified portfolio such as Morgan Stanley capital weighted world stock accumulation index, called MSCI world index. For longevity index, we take data from LLMA index related to German population. We consider a sample period ranging from January 1970 to January 2013, with 517 monthly observations of MSCI world index and 44 annual observations of LLMA Germany male graduated initial rate of mortality, relating to the cohort of male population aged 20 in 1970. The summary statistics of the two data sets is reported in the following table. The inverse of benchmark portfolio data is shown in basis points and longevity index data in percentages.

	Mean	Median	Std.	min	MAX
1/MSCI index ($1/S^*$)	38.612	19.468	35.397	5.9441	134.31
Longevity index (L)	98.628	98.984	1.1337	95.606	99.803

We use the symbol Φ to denote the model parameter vector and the series $t_1, t_2, ..., t_N$, with $N = 517$, to denote the times of observation. For every t_k with $1 \leqslant k \leqslant N$, we may have an 1-dimensional v_{t_k} if only MSCI index is observable, or a 2-dimensional observation vector v_{t_k} if both MSCI and LLMA indexes are observable. In this last case, the measurement equation is given below; when only the MSCI index is observable, v_{t_k} is reduced to the first component:

$$\begin{aligned} v_{t_k} &= f(Z_{t_k}, \Phi) + \varepsilon_{t_k} = \left[\frac{1}{S^*_{t_k}},\ L_{t_k}\right]^\top + \varepsilon_{t_k} \\ &= \left[e^{-\alpha t_k}\left(\rho + cX_{t_k}\right), e^{-\gamma t_k}\left(\delta + \nu Y_{t_k}\right)\right]^\top + \varepsilon_{t_k} \\ &= \Theta_{k0} + \Theta_{k1} Z_{t_k} + \varepsilon_{t_k}, \end{aligned}$$

where

$$\Theta_{k0} = \begin{pmatrix} e^{-\alpha t_k}\rho \\ e^{-\gamma t_k}\delta \end{pmatrix}, \qquad \Theta_{k1} = \begin{pmatrix} e^{-\alpha t_k}c & 0 \\ 0 & e^{-\gamma t_k}\nu \end{pmatrix}.$$

As in [75] and in [48], the measurement error vector is assumed[3] to be

$$\varepsilon_{t_k} \sim N\left(0, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\right),$$

[3]In general cases, ε_{t_k} is a random error vector with $\mathbb{E}[\varepsilon_{t_k}] = 0$.

where σ_1^2 indicates the measurement error variance related to the inverse of benchmark portfolio, and σ_2^2 the one associated to the longevity index. If we assume that the longevity index does not have relevant influence on the benchmark portfolio, according to [70] and [45] the discrete time transition equation of the component X of the unobserved state variable at t_k, $1 < k \leqslant N$, can be approximated by,

$$X_{t_k} = \mathbb{E}\left[X_{t_k} \middle| X_{t_{k-1}}\right] + \mathrm{Var}\left[X_{t_k} \middle| X_{t_{k-1}}\right]^{\frac{1}{2}} \eta_{t_k},$$

with η_{t_k} an 1-dimensional error term of zero mean and unit variance, independent from $X_{t_{k-1}}$. We stress that $\mathbb{E}\left[X_{t_k} \middle| X_{t_{k-1}}\right]$ and $\mathrm{Var}\left[X_{t_k} \middle| X_{t_{k-1}}\right]^{\frac{1}{2}} \eta_{t_k}$ are both conditionally and unconditionally independent. Moreover, by Proposition 4.2.3 the conditional expectation $\mathbb{E}\left[X_{t_k} \middle| X_{t_{k-1}}\right]$ is an affine function of $X_{t_{k-1}}$, and the conditional variance $\mathrm{Var}\left[X_{t_k} \middle| X_{t_{k-1}}\right]$ is a second degree polynomial function of $X_{t_{k-1}}$. More precisely,

$$\mathbb{E}\left[X_{t_k} \middle| X_{t_{k-1}}\right] = \Phi_{k0} + \Phi_{k1} X_{t_{k-1}},$$

where

$$\Phi_{k0} = b\left(1 - e^{-\Psi(t_k - t_{k-1})}\right),$$
$$\Phi_{k1} = e^{-\Psi(t_k - t_{k-1})}.$$

By following [75], we approximate $\mathrm{Var}\left[X_{t_k} \middle| X_{t_{k-1}}\right]^{\frac{1}{2}} \eta_{t_k}$ with a normal distribution error term u_{k-1} independent from $\mathbb{E}\left[X_{t_k} \middle| X_{t_{k-1}}\right]$,

$$X_{t_k} = \Phi_{k0} + \Phi_{k1} X_{t_{k-1}} + u_{k-1}. \tag{4.4.4}$$

$$u_{k-1} \sim N(0, Q_{k-1}), \qquad Q_{k-1} = \mathbb{E}\left[\mathrm{Var}\left[X_{t_k} \middle| X_{t_{k-1}}\right]\right].$$

We note that the component Y of state variable at time t can be expressed explicitly as function depending on X,

$$Y_t = e^{-\kappa t} \int_0^t \left(-dX_s + db + \kappa\eta\right) e^{\kappa s} \mathrm{d}s.$$

For t equal to an observation time of the LLMA index, we use the following approximation

$$Y_t = e^{-\kappa t} \sum_{0 \leqslant t_{k_i} < t} \left(-dX_{t_{k_i}} + db + \kappa\eta\right) e^{\kappa t_{k_i}} (t_{k_{i+1}} - t_{k_i}),$$

where $X_{t_{k_i}}$ are the monthly values computed by using the transition equation (4.4.4) of X. We observe that both the inverse of the benchmark portfolio $1/S^*$ and the longevity index L are affine functions of the state variable. If we neglect for now the state space restrictions (see [75] for details regarding this assumption), we are in the case of a linear Gaussian state space model. As described in [55], we apply linear Kalman filter and maximum likelihood estimation under these approximations. For the sake of simplicity, we apply these methods only to estimate parameters of the X component. For the remaining parameters of the Y component, we use least squares estimation since the longevity index can be considered as a linear regression of X

under our approximation. Let V_{t_k} denote the information at time t_k regarding the benchmark portfolio, namely

$$V_{t_k} = (v^1_{t_1}, v^1_{t_2}, ..., v^1_{t_k}).$$

For $1 < k \leqslant N$ we set

$$\bar{X}_{t_k|t_{k-1}} := \mathbb{E}\left[X_{t_k} \middle| V_{t_{k-1}}\right], \qquad \Sigma_{t_k|t_{k-1}} := \mathrm{Var}\left[X_{t_k} \middle| V_{t_{k-1}}\right],$$

where $\bar{X}_{t_k|t_{k-1}}$ is the optimal predictor of X_{t_k} and $\Sigma_{t_k|t_{k-1}}$ is the mean square error. Similary, for $1 \leqslant k \leqslant N$ we denote

$$\bar{X}_{t_k} := \mathbb{E}\left[X_{t_k} \middle| V_{t_k}\right], \qquad \Sigma_{t_k} := \mathrm{Var}\left[X_{t_k} \middle| V_{t_k}\right].$$

For every $1 < k \leqslant N$, the *prediction step* of linear Kalman filter is hence

$$\bar{X}_{t_k|t_{k-1}} = \Phi_{k0} + \Phi_{k1}\bar{X}_{t_{k-1}},$$

with mean square error

$$\Sigma_{t_k|t_{k-1}} = \Phi^2_{k1}\Sigma_{t_{k-1}} + Q_{k-1},$$

and the *update step* is described by

$$\bar{X}_{t_k} = \bar{X}_{t_k|t_{k-1}} + \Sigma_{t_k|t_{k-1}}\Theta^1_{k1}\left(F_{t_k}\right)^{-1} w_{t_k},$$

$$\Sigma_{t_k} = \left(\left(\Sigma_{t_k|t_{k-1}}\right)^{-1} + \left(\Theta^1_{k1}\right)^2 \sigma_1^{-2}\right)^{-1},$$

where

$$w_{t_k} = v^1_{t_k} - \mathbb{E}\left[v^1_{t_k} \middle| V_{t_{k-1}}\right] = v^1_{t_k} - \left(\Theta^1_{k0} + \Theta^1_{k1}\bar{X}_{t_k|t_{k-1}}\right),$$

$$F_{t_k} = \mathrm{Var}(w_{t_k}) = \left(\Theta^1_{k1}\right)^2 \Sigma_{t_k|t_{k-1}} + \sigma_1^2.$$

The (approximated) log-likelihood function is of the form

$$\log L(v_{t_1}, v_{t_2}, ..., v_{t_N}; \Phi) = \sum_{k=1}^{N} -\log(2\pi) - \frac{1}{2}\log|F_{t_k}| - \frac{1}{2}w^\top_{t_k}F^{-1}_{t_k}w_{t_k}.$$

We note that due to the annual observation of longevity index, the component Y can be updated only annually. For $k = 1 + 12 * h$ with $h = 0, ..., 43$, the approximated value of Y_{t_k} is given by

$$\bar{Y}_{t_k} = e^{-\kappa t_k}\sum_{s=0}^{k-1}\left(-d\bar{X}_{t_s} + db + \kappa\eta\right)e^{\kappa t_s}(t_{s+1} - t_s). \tag{4.4.5}$$

We fix $\rho = 0.01, c = 0.006, \delta = 0.998, \nu = -0.00044$, so that conditions (4.4.2) and (4.4.3) are satisfied. In particular, since $Z_0 = (X_0, Y_0)^\top = 0$, the value of ρ and δ is forced to be (almost) equal to the first value of the inverse of benchmark portfolio and longevity index respectively, while the values of c and ν can be arbitrarily chosen within the condition that (4.4.2) and (4.4.3) are fulfilled. Theorem 5 of [48] shows that a different choice of c and ν will result in a scaling of the state variable Z. The following table shows the calibrated parameters.

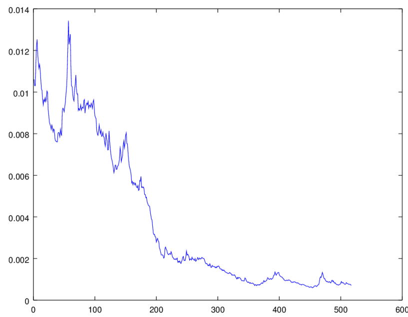
Ψ	b	σ	d	κ	η
14.98581	-0.79506	1.25299	5.18417	-5.87517	-5.05117

The correspondent values of α, γ and the log-likelihood value are reported below.

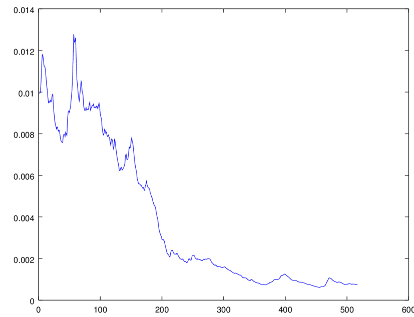
α	γ	L
4.6068	0.0045607	2347.5

Figure 4.1 shows the graphics related the the calibrated benchmark portfolio S^*. Figures 4.1 (a) and (b) plot respectively the observed inverse of benchmark portfolio data and the fit produced by Kalman filter. Figures 4.1 (c) and (d) display in basis point respectively the pricing error generated by Kalman filter and the root mean square pricing error (RMSE) computed over 100 Monte Carlo replications. Figure 4.1 (e) plots time series of estimated state variable component X which drives the benchmark portfolio dynamics and takes value in the compact interval $[-1, 1]$. Figure 4.1 (f) shows time series of estimated short rate r adjusted by the level parameter α. We observe that the one dimensional component X, with a mean RMSE equal to 15.24 bps, has already sufficient explanation power for the inverse of benchmark portfolio dynamics structure, and is able to produce a reasonable fit to the observed data. A better fit is shown in the tail, which is a desirable situation since we are fitting the inverse of LLMA world index value.

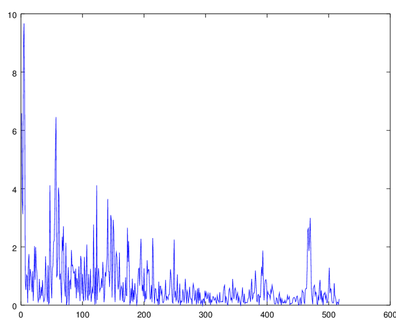
Similarly, in Figure 4.2 we see the graphics related to the calibrated longevity index L. Figure 4.2 (a) and (b) plot respectively the observed longevity index data and the fit produced by (4.4.5) with estimated parameter sets. Figures 4.2(c) and (d), with unit in basis point, display respectively the pricing error associated to (4.4.5) and the root mean square pricing error (RMSE) computed over 100 Monte Carlo replications. Figure 4.2(e) plots time series of estimated state variable component Y. Figure 4.2(f) shows time series of estimated mortality intensity μ. Smooth paths of Y and of the longevity index fit are due to the absence of the diffusion term in the Y dynamics. This is reasonable since oscillations along the trend of longevity index data is very slight, with a mean RMSE value of 15.39 bps. Nevertheless, the poor data set of the longevity index, which has always less than 50 annual observations for one age cohort, and the long time frame between two consecutive data may be a drawback for calibration.



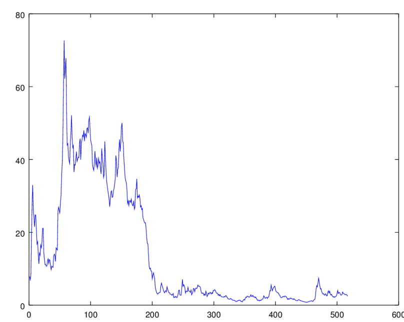
(a) Inverse of benchmark portfolio data



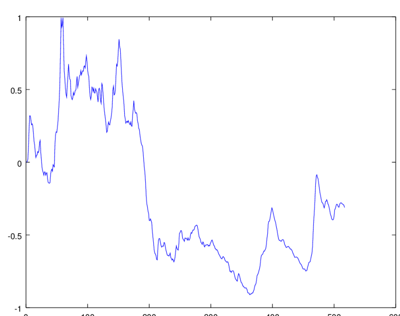
(b) Inverse of benchmark portfolio fit



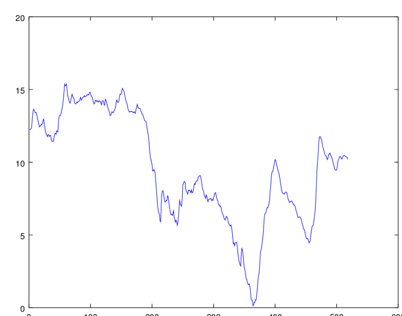
(c) Pricing error generated by Kalman filter



(d) Inverse of benchmark portfolio RMSE

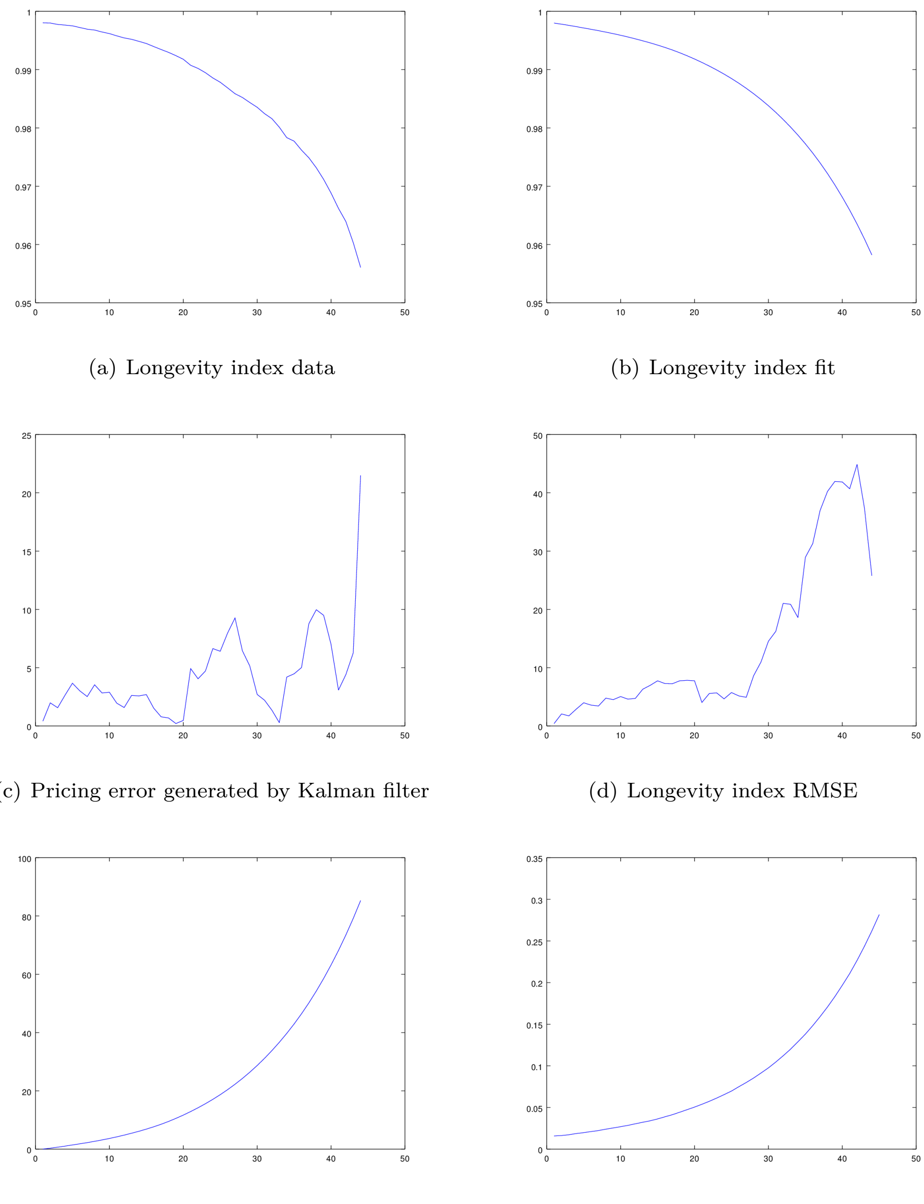


(e) State variable component X



(f) Short rate r

Figure 4.1: Benchmark portfolio S^* data and fit.



(a) Longevity index data

(b) Longevity index fit

(c) Pricing error generated by Kalman filter

(d) Longevity index RMSE

(e) State variable component Y

(f) Mortality intensity μ

Figure 4.2: Longevity index L data and fit.

4.5 Pricing and hedging life insurance under polynomial diffusion model

In this section, we calculate under our polynomial diffusion model specification the real-world pricing formula and the benchmarked risk-minimizing strategy for the three building blocks of life insurance liabilities, namely pure endowment contract, term insurance contract and annuity contract. Theorems and notations related to the benchmarked risk-minimization method for payment streams are provided in Section 2.4. We show that the property in Proposition 4.2.3 gives explicit formulas in the case of polynomial payments, and good approximations in the case of continuous payments. For the sake of simplicity, we assume that the financial market is composed only by the OIS T-bond and the longevity T-bond, i.e.

$$\hat{S}_t = \left(\frac{P(t,T)}{S_t^*}, \frac{P^l(t,T)}{S_t^*} \right)^\top, \quad t \in [0,T].$$

Explicit benchmarked risk-minimizing strategy is calculated only for pure endowment contract with $\dim W = n = 2$. The computations of general dimension and of the other two building blocks are similar. The following lemma is frequently used.

Lemma 4.5.1. *Let p be a polynomial in $Pol_{\bar{p}}(E)$ with coordinate representation*

$$p(z) = H_{\bar{p}}(z)^\top \vec{p},$$

for $z \in E$. It holds that for $0 \leqslant t \leqslant T$,

$$\mathbf{p}_{(t,T)}(Z_t) = E\left[p(Z_T)|\, \mathcal{G}_t\right] = H_{\bar{p}}(Z_0)^\top e^{TG_{\bar{p}}}\vec{p} + \int_0^t \nabla \mathbf{p}_{(u,T)}(Z_u)^\top \sigma(Z_u) \mathrm{d}W_u.$$

Proof. We have

$$\begin{aligned}
&\mathbf{p}_{(t,T)}(Z_t)\\
=&\mathbf{p}_{(0,T)}(Z_0) + \int_0^t \left(\frac{\partial}{\partial u} \mathbf{p}_{(u,T)}(Z_u) \right) \mathrm{d}u + \int_0^t \mathbf{G}\mathbf{p}_{(u,T)}(Z_u) \mathrm{d}u\\
&+ \int_0^t \nabla \mathbf{p}_{(u,T)}(Z_u)^\top \sigma(Z_u) \mathrm{d}W_u\\
=&H_{\bar{p}}(Z_0)^\top e^{TG_{\bar{p}}}\vec{p} - \int_0^t \left(H_{\bar{p}}(Z_u)^\top G_{\bar{p}} e^{(T-u)G_{\bar{p}}}\vec{p} \right) \mathrm{d}u + \int_0^t \left(H_{\bar{p}}(Z_u)^\top G_{\bar{p}} e^{(T-u)G_{\bar{p}}}\vec{p} \right) \mathrm{d}u\\
&+ \int_0^t \nabla \mathbf{p}_{(u,T)}(Z_u)^\top \sigma(Z_u) \mathrm{d}W_u\\
=&H_{\bar{p}}(Z_0)^\top e^{TG_{\bar{p}}}\vec{p} + \int_0^t \nabla \mathbf{p}_{(u,T)}(Z_u)^\top \sigma(Z_u) \mathrm{d}W_u,
\end{aligned}$$

where the first equality is given by the Itô's formula and the second one follows from Proposition 4.2.3. □

4.5.1 Pure endowment

A pure endowment contract provides a payment at the term T of contract in case the insured person is still alive. For $i = 1, ..., n$, the payoff at T associated to i-th policyholder is given by

$$\mathbf{1}_{\{\tau^i > T\}} g_T,$$

where the random variable g_T is assumed to be a $\mathcal{F}_T$-measurable and square integrable. At the portfolio level we have

$$\sum_{i=1}^{n} \mathbf{1}_{\{\tau^i > T\}} g_T = (n - l_T) g_T.$$

The benchmarked cumulative payment A is hence

$$A_t = \sum_{i=1}^{n} (S_T^*)^{-1} \mathbf{1}_{\{\tau^i > T\}} g_T \mathbf{1}_{\{t=T\}} = (S_T^*)^{-1} (n - l_T) g_T \mathbf{1}_{\{t=T\}},$$

for $t \in [0, T]$.

Let $V^T := (V_t^T)_{t\in[0,T]}$ denote the price process given by the real-world pricing formula (2.3.3) in this case. Under our model assumptions of Section 4.3, at time $t \in [0, T]$ it holds

$$\begin{aligned}
V_t^T =& S_t^* E\left[(S_T^*)^{-1} \sum_{i=1}^{n} \mathbf{1}_{\{\tau^i > T\}} g_T \middle| \mathcal{G}_t \right] \\
=& \sum_{i=1}^{n} S_t^* E\left[(S_T^*)^{-1} \mathbf{1}_{\{\tau^i > T\}} g_T \middle| \mathcal{G}_t \right] \\
=& \sum_{i=1}^{n} \mathbf{1}_{\{\tau^i > t\}} S_t^* E\left[(S_T^*)^{-1} e^{-\int_t^T \mu_u \mathrm{d}u} g_T \middle| \mathcal{F}_t \right] \\
=& (n - l_t) e^{-(\gamma+\alpha)(T-t)} \frac{E\left[p(Z_T) q(Y_T) g_T \middle| \mathcal{F}_t \right]}{p(Z_t) q(Y_t)},
\end{aligned}$$

where the third equality follows from Proposition 5.5 of [7] combined with Corollary 5.1.1 of [24]. The benchmarked value process $\hat{S}^{\bar{\delta}^T} := (\hat{S}_t^{\bar{\delta}^T})_{t\in[0,T]}$ associated to the benchmarked risk-minimizing strategy $\bar{\delta}^T = (\bar{\delta}_t^T)_{t\in[0,T]}$ of the given portfolio is hence

$$\hat{S}_t^{\bar{\delta}^T} = (S_t^*)^{-1} V_t^T = (n - l_t) e^{-\alpha T - \gamma(T-t)} \frac{E\left[p(Z_T) q(Y_T) g_T \middle| \mathcal{F}_t \right]}{q(Y_t)},$$

for $t \in [0, T]$. Proposition 5.11 of [7] can be easily adapted to our case. This together with (4.3.3) shows that the benchmarked risk-minimizing strategy is given by $\bar{\delta}^T$ with

$$\bar{\delta}_t^T = (n - N_{t-}) e^{-\alpha T - \gamma(T-t)} q^{-1}(Y_t) \phi_t, \tag{4.5.1}$$

for $t \in [0, T]$, where the vector process $\phi := (\phi_t)_{t\in[0,T]}$ is given by the Galtchouk–Kunita–Watanabe decomposition of $U_t := E\left[p(Z_T) q(Y_T) g_T \middle| \mathcal{F}_t \right]$

$$U_t = \mathbb{E}\left[p(Z_T) q(Y_T) g_T \right] + \int_0^t \phi_u^\top \mathrm{d}\hat{S}_u + L_t^U, \quad t \in [0, T], \tag{4.5.2}$$

where $\phi \in L^2(\hat{S}, P, \mathbb{G})$ and $L^U \in M_0^2(P, \mathbb{G})$ is strongly orthogonal to $\mathcal{I}^2(\hat{S}, P, \mathbb{G})$. Furthermore, the benchmarked cumulative cost process is

$$\begin{aligned} C_t^{\bar{\delta}^T} =& ne^{-(\alpha+\gamma)T}\mathbb{E}\left[p(Z_T)q(Y_T)g_T\right] + \int_0^t (n - N_{u-})e^{-\alpha T - \gamma(T-u)}q^{-1}(Y_u)\mathrm{d}L_u^U \\ &+ \int_0^t U_{u-}e^{-\alpha T-\gamma(T-u)}q^{-1}(Y_u)\mathrm{d}M_u, \end{aligned}$$

for $t \in [0, T]$, where the $\mathbb{G}$-martingale M is given by

$$M_t = l_t - (n - l_{t-})\Gamma_t, \quad t \in [0, T].$$

Polynomial payoff

We start from the simple case when the payoff is given by a polynomial function of the state variable, i.e.

$$g_T = g(Z_T), \qquad \text{with } g \text{ polynomial function.}$$

In this case, the pricing formula is clearly reduced to

$$V_t^T = (n - l_t)e^{-(\gamma+\alpha)(T-t)}\frac{\mathbf{pqg}_{(t,T)}(Z_t)}{p(Z_t)q(Y_t)}, \quad t \in [0, T]. \tag{4.5.3}$$

We note that this covers many realistic cases for an insurance contract, e.g. with constant payoff $g_T = k$, $k \in \mathbb{R}^+$, or with an index-linked payoff, which can be proportional to the longevity index at time T, i.e. $g_T = kL_T = ke^{-\gamma T}q(Y_T)$, $k \in \mathbb{R}^+$. In this case, it holds

$$U_t = \mathbf{pqg}_{(t,T)}(Z_t), \quad t \in [0, T]. \tag{4.5.4}$$

Lemma 4.5.1 applied to (4.5.4), (4.3.5) and (4.3.9) yields

$$\mathbf{pqg}_{(t,T)}(Z_t) = \mathbf{pqg}_{(0,T)}(Z_0) + \int_0^t \nabla_x \mathbf{pqg}_{(u,T)}(Z_u)^\top \sigma(X_u)\mathrm{d}W_u,$$

$$(S_t^*)^{-1}P(t,T) = e^{-\alpha T}\mathbf{p}_{(0,T)}(X_0) + \int_0^t e^{-\alpha T}\nabla_x \mathbf{p}_{(u,T)}(Z_u)^\top \sigma(X_u)\mathrm{d}W_u,$$

$$(S_t^*)^{-1}P^l(t,T) = e^{-(\alpha+\gamma)T}\mathbf{pq}_{(0,T)}(Z_0) + \int_0^t e^{-(\alpha+\gamma)T}\nabla_x \mathbf{pq}_{(u,T)}(Z_u)^\top \sigma(X_u)\mathrm{d}W_u,$$

where $t \in [0, T]$. We set the 2-dimensional square matrix process $\theta := (\theta_t)_{t\in[0,T]}$

$$\theta_t := \left[\begin{array}{cc} e^{-\alpha T}\sigma(X_t)^\top \nabla_x \mathbf{p}_{(t,T)}(Z_t), & e^{-(\alpha+\gamma)T}\sigma(X_t)^\top \nabla_x \mathbf{pq}_{(t,T)}(Z_t) \end{array}\right], \quad t \in [0, T], \tag{4.5.5}$$

and the 2-dimensional vector process $\phi := (\phi_t)_{t\in[0,T]}$

$$\phi_t = \begin{pmatrix} \phi_t^1 \\ \phi_t^2 \end{pmatrix}, \quad t \in [0, T],$$

with

$$\theta_t \phi_t = \sigma(X_t)^\top \nabla \mathbf{pqg}_{(t,T)}(Z_t), \qquad t \in [0,T].$$

In the case that the matrix θ_t is a.s. invertible for all $t \in [0,T]$, we have

$$\phi_t = \theta_t^{-1} \sigma(X_t)^\top \nabla \mathbf{pqg}_{(t,T)}(Z_t), \qquad t \in [0,T].$$

Then

$$\mathbf{pqg}_{(t,T)}(Z_t) = \mathbf{pqg}_{(0,T)}(Z_0) + \int_0^t \phi_u^1 \mathrm{d}\left((S_u^*)^{-1} P(u,T)\right) + \int_0^t \phi_u^2 \mathrm{d}\left((S_u^*)^{-1} P^l(u,T)\right),$$

thus, for $t \in [0,T]$, the benchmarked risk-minimizing strategy is given by

$$\bar{\delta}_t^T = \left((n - N_{t-}) e^{-\alpha T - \gamma(T-t)} q^{-1}(Y_t) \phi_t^1, (n - N_{t-}) e^{-\alpha T - \gamma(T-t)} q^{-1}(Y_t) \phi_t^2\right),$$

and the benchmarked cumulative cost process satisfies

$$C_t^{\bar{\delta}^T} = n e^{-(\alpha+\gamma)T} \mathbf{pqg}_{(0,T)}(Z_0) + \int_0^t \mathbf{pqg}_{(u,T)}(Z_u) e^{-\alpha T - \gamma(T-u)} q^{-1}(Y_u) \mathrm{d}M_u.$$

Continuous payoff

If the payoff is a generic continuous function of the state variable, i.e.

$$g_T = g(Z_T), \qquad \text{with } g \text{ continuous function on } E,$$

then it is not always possible to find an explicit form of the conditional expectation as in the previous polynomial case. This class includes a large family of longevity linked contracts, e.g. options on survival index or longevity bond. However, providing that the state space E is compact, it is always possible to find a uniform polynomial approximation $\{g_m\}_{m\in\mathbb{N}}$ of g on E, i.e.

$$\|g_m - g\|_\infty \xrightarrow{m\to\infty} 0 \qquad \text{on} \quad E, \tag{4.5.6}$$

where the norm $\|\cdot\|_\infty$ is defined by

$$\|f\|_\infty := \sup_{\substack{x\in E \\ \|x\|=1}} |f(x)|,$$

for any $f \in \mathcal{C}(E)$. In the following, we show that this approximation of the payoff function induces a good approximation of the real-world pricing formula, the benchmarked risk-minimizing strategy and the benchmarked cumulative cost process. We start from a preliminary lemma.

Lemma 4.5.2. *If $\{g_m\}_{m\in\mathbb{N}}$ is a uniform polynomial approximation of the continuous function g on E as in (4.5.6), it holds*

$$\sup_{t\in[0,T]} |E\left[g_m(Z_T) - g(Z_T)|\,\mathcal{F}_t\right]| \xrightarrow{m\to\infty} 0, \qquad a.s.,$$

$$\sup_{t\in[0,T]} \|E\left[g_m(Z_T) - g(Z_T)|\,\mathcal{F}_t\right]\|_{L^p(\Omega,P)} \xrightarrow{m\to\infty} 0,$$

for all $p \geqslant 1$.

Proof. We first show the a.s. approximation,

$$\begin{aligned}
&\sup_{t\in[0,T]} \left| E\left[g_m(Z_T) - g(Z_T) \middle| \mathcal{F}_t \right] \right| \leqslant \sup_{t\in[0,T]} E\left[\left| g_m(Z_T) - g(Z_T) \right| \middle| \mathcal{F}_t \right] \\
&\leqslant \sup_{t\in[0,T]} E\left[\left\| g_m - g \right\|_\infty \middle| \mathcal{F}_t \right] = \left\| g_m - g \right\|_\infty \xrightarrow{m\to\infty} 0, \qquad \text{a.s..}
\end{aligned}$$

Similarly we get the $L^p(\Omega, P)$ approximation uniformly in $t \in [0,T]$ for any $p \geqslant 1$,

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\left[\left| E\left[g_m(Z_T) - g(Z_T) \middle| \mathcal{F}_t \right] \right|^p \right] \leqslant \mathbb{E}\left[\left(\sup_{t\in[0,T]} \left| E\left[g_m(Z_T) - g(Z_T) \middle| \mathcal{F}_t \right] \right| \right)^p \right] \\
&\leqslant \left\| g_m - g \right\|_\infty^p \xrightarrow{m\to\infty} 0.
\end{aligned}$$

□

Now we show that the sequence of pricing formulas related to $\{g_m\}_{m\in\mathbb{N}}$ is a good approximation of the one related to the original payoff function g. We define $V^T := (V_t^T)_{t\in[0,T]}$, where

$$V_t^T = (n - l_t) e^{-(\gamma+\alpha)(T-t)} \frac{E\left[p(Z_T) q(Y_T) g(Z_T) \middle| \mathcal{F}_t \right]}{p(Z_t) q(Y_t)}, \tag{4.5.7}$$

for $t \in [0,T]$.

Proposition 4.5.3. *If $\{g_m\}_{m\in\mathbb{N}}$ is a uniform polynomial approximation of the continuous function g on E as in (4.5.6), and for every $m \in \mathbb{N}$*

$$V^{T,m} := \left(V_t^{T,m} \right)_{t\in[0,T]} = \left((n - l_t) e^{-(\gamma+\alpha)(T-t)} \frac{\mathbf{pqg}_{\mathbf{m}(t,T)}(Z_t)}{p(Z_t) q(Y_t)} \right)_{t\in[0,T]},$$

then $\left\{ V^{T,m} \right\}_{m\in\mathbb{N}}$ provides both a pathwise and $L^p(\Omega, P)$ approximation of V^T in (4.5.7) for any $p \geqslant 1$ uniformly in $t \in [0,T]$, i.e.

$$\sup_{t\in[0,T]} \left| V_t^{T,m} - V_t^T \right| \xrightarrow{m\to\infty} 0, \qquad a.s.,$$

$$\sup_{t\in[0,T]} \left\| V_t^{T,m} - V_t^T \right\|_{L^p(\Omega,P)} \xrightarrow{m\to\infty} 0.$$

Proof. It is an immediate consequence of Lemma 4.5.2. □

As the second step, we show that both the benchmarked risk-minimizing strategies and the benchmarked cumulative cost processes associated to $\{g_m\}_{m\in\mathbb{N}}$ provide a good approximation of the ones associated to g as well.

Lemma 4.5.4. *Let $\{g_m\}_{m\in\mathbb{N}}$ be a uniform polynomial approximation of the continuous function g on E as in (4.5.6) and for every $m \in \mathbb{N}$*

$$U_m := ((U_m)_t)_{t\in[0,T]} = \left(\mathbf{pqg}_{\mathbf{m}(t,T)}(Z_t) \right)_{t\in[0,T]}, \tag{4.5.8}$$

with the following Galtchouk–Kunita–Watanabe decomposition

$$(U_m)_t = (U_m)_0 + \int_0^t (\phi_m)_u^\top \mathrm{d}\hat{S}_u + L_t^{U_m}, \qquad t \in [0,T].$$

If ϕ and L^U are the two processes given by the Galtchouk–Kunita–Watanabe decomposition of U in (4.5.2) with respect to $\hat{S}$, then it holds

$$\|\phi - \phi_m\|_{L^2(\hat{S},P,\mathbb{G})} \xrightarrow{m\to\infty} 0, \tag{4.5.9}$$

and

$$\left\|L^U - L^{U_m}\right\|_{M_0^2(P,\mathbb{G})} \xrightarrow{m\to\infty} 0. \tag{4.5.10}$$

If furthermore the matrix process θ defined in (4.5.5) is such that, for all $t \in [0,T]$, θ_t is a.s. invertible with $\theta^{-1} \in L^2(\Omega \times [0,T], P \otimes \mathrm{d}t)$, then we have

$$\|\phi - \phi_m\|_{L^1(\Omega\times[0,T],P\otimes\mathrm{d}t)} \xrightarrow{m\to\infty} 0, \tag{4.5.11}$$

and $L^U = L^{U_m} = 0$ for all $m \in \mathbb{N}$.

Proof. Proposition 4.5.3 implies in particular the following convergence in $M_0^2(P,\mathbb{G})$,

$$\|U - U_m\|^2_{M_0^2(P,\mathbb{G})} = \sup_{t\in[0,T]} \mathbb{E}\left[\left(U_t - (U_m)_t\right)^2\right] \xrightarrow{m\to\infty} 0.$$

We note that L^U and $\left(L^{U_m}\right)_{m\in\mathbb{N}}$ are strongly orthogonal to the space $\mathcal{I}(\hat{S},P,\mathbb{G})$, hence

$$\begin{aligned}
&\left\|\int_0^\cdot \left(\phi_u^\top - (\phi_m)_u^\top\right)\mathrm{d}\hat{S}_u + \left(L^U - L^{U_m}\right)\right\|^2_{M_0^2(P,\mathbb{G})}\\
=&\left\|\int_0^\cdot \left(\phi_u^\top - (\phi_m)_u^\top\right)\mathrm{d}\hat{S}_u\right\|^2_{M_0^2(P,\mathbb{G})} + \left\|L^U - L^{U_m}\right\|^2_{M_0^2(P,\mathbb{G})}\\
=&\|\phi - \phi_m\|^2_{L^2(\hat{S},P,\mathbb{G})} + \left\|L^U - L^{U_m}\right\|^2_{M_0^2(P,\mathbb{G})},
\end{aligned}$$

which yields

$$\left\|L^U - L^{U_m}\right\|_{M_0^2(P,\mathbb{G})} \xrightarrow{m\to\infty} 0, \tag{4.5.12}$$

and

$$\|\phi - \phi_m\|_{L^2(\hat{S},P,\mathbb{G})} \xrightarrow{m\to\infty} 0.$$

Furthermore, for each $m \in \mathbb{N}$, it follows from the Itô isometry

$$\begin{aligned}
\|\phi - \phi_m\|_{L^2(\hat{S},P,\mathbb{G})} =&\mathbb{E}\left[\int_0^T \left(\phi_u - (\phi_m)_u\right)^\top \mathrm{d}[\hat{S}]_u \left(\phi_u - (\phi_m)_u\right)\right]\\
=&\mathbb{E}\left[\int_0^T \left(\phi_u - (\phi_m)_u\right)^\top \theta_u^\top \mathrm{d}[W]_u \theta_u \left(\phi_u - (\phi_m)_u\right)\right]\\
=&\mathbb{E}\left[\int_0^T \left(\phi_u - (\phi_m)_u\right)^\top \theta_u^\top \theta_u \left(\phi_u - (\phi_m)_u\right)\mathrm{d}u\right]\\
=&\|\theta(\phi - \phi_m)\|_{L^2(\Omega\times[0,T],P\otimes\mathrm{d}t)}.
\end{aligned}$$

If furthermore the matrix θ_u is invertible for all $u \in [0,T]$ a.s. with

$$\theta^{-1} \in L^2(\Omega \times [0,T], P \otimes \mathrm{d}t),$$

then Cauchy-Schwarz inequality yields

$$\mathbb{E}\left[\int_0^T |(\phi)_u - (\phi_m)_u| \, du\right] \leqslant \|\theta(\phi - \phi_m)\|_{L^2(\Omega\times[0,T],P\otimes \mathrm{d}t)} \cdot \left\|\theta^{-1}\right\|_{L^2(\Omega\times[0,T],P\otimes \mathrm{d}t)} \xrightarrow{m\to\infty} 0.$$

In particular, it holds that $L^{U_m} = 0$ for all $m \in \mathbb{N}$ by Lemma 4.5.1. Then (4.5.12) leads to $L = 0$. □

Remark 4.5.5. *We stress that, if g is given by a continuous function, via a convergence argument similar to the one in Lemma 4.5.4, the Galtchouk–Kunita–Watanabe decomposition of U with projection on the subspace $\mathcal{I}(W, P, \mathbb{G})$ is given by*

$$U_t = U_0 + \int_0^t \psi_u^\top \mathrm{d}W_u, \qquad t \in [0,T],$$

where $\psi := (\psi_t)_{t\in[0,T]}$ is a predictable W-integrable vector process, according to Lemma 4.5.1. In other words, U contains no orthogonal component even without the assumption $\mathbb{F} = \mathbb{F}^W$.

Proposition 4.5.6. *Let the series $\{g_m\}_{m\in\mathbb{N}}$ be a uniform polynomial approximation of the continuous function g on E, as in (4.5.6). Let $\bar{\delta}^T$ and $C^{\bar{\delta}^T}$ be respectively the benchmarked risk-minimizing strategy and benchmarked cumulative cost process associated to g, and $\bar{\delta}^T_m$ and $C^{\bar{\delta}^T_m}$ be the ones associated to g_m, then*

$$\left\|\bar{\delta}^T - \bar{\delta}^T_m\right\|_{L^2(\hat{S},P,\mathbb{G})} \xrightarrow{m\to\infty} 0, \tag{4.5.13}$$

$$\left\|C^{\bar{\delta}^T} - C^{\bar{\delta}^T_m}\right\|_{M_0^2(P,\mathbb{G})} \xrightarrow{m\to\infty} 0. \tag{4.5.14}$$

If in addition the matrix process θ given by (4.5.5) is a.s. invertible, then it holds

$$\left\|\bar{\delta}^T - \bar{\delta}^T_m\right\|_{L^1(\Omega\times[0,T],P\otimes \mathrm{d}t)} \xrightarrow{m\to\infty} 0. \tag{4.5.15}$$

Proof. Convergences (4.5.13) and (4.5.15) are straightforward consequence of (4.5.9) and (4.5.11) in Lemma 4.5.4. Now we prove the convergence 4.5.14 in $M_0^2(P,\mathbb{G})$ of the benchmarked cumulative cost process. It holds

$$\mathbb{E}\left[\left(C_t^{\bar{\delta}^T} - C_t^{\bar{\delta}^T_m}\right)^2\right] \leqslant c_m + 2\mathbb{E}\left[\int_0^t \left((n - N_{u-})e^{-\alpha T - \gamma(T-u)} q^{-1}(Y_u)\right)^2 \mathrm{d}[L^U - L^{U_m}]_u\right] + 2\mathbb{E}\left[\int_0^t \left((U_u - (U_m)_u)e^{-\alpha T - \gamma(T-u)} q^{-1}(Y_u)\right)^2 \mathrm{d}[M]_u\right],$$

for every $t \in [0,T]$, where

$$c_m = 2\left(C_0^{\bar{\delta}^T} - C_0^{\bar{\delta}_m^T}\right) = 2\left(ne^{-(\alpha+\gamma)T}\mathbb{E}\left[p(Z_T)q(Y_T)g(Z_T) - p(Z_T)q(Y_T)g_m(Z_T)\right]\right).$$

We note that

$$c_m \xrightarrow{m\to\infty} 0.$$

For the first addend, the compactness of the state space yields

$$2\mathbb{E}\left[\int_0^t \left((n-N_{u-})e^{-\alpha T-\gamma(T-u)}q^{-1}(Y_u)\right)^2 \mathrm{d}[L^U - L^{U_m}]_u\right] \leqslant \bar{c}\mathbb{E}\left[(L_t^U - L_t^{U_m})^2\right],$$

for every $t \in [0,T]$, where $\bar{c}$ is a suitable constant. This quantity turns to zero uniformly in t according to (4.5.10) in Lemma 4.5.4. For the second addend, since $(U^m)_{m\in\mathbb{N}}$ is a pathwise approximation of U uniformly in $t \in [0,T]$, the dominated convergence theorem combined with the boundedness of the integrand process yields

$$2\mathbb{E}\left[\int_0^t \left((U_u - (U_m)_u)e^{-\alpha T-\gamma(T-u)}q^{-1}(Y_u)\right)^2 \mathrm{d}[M]_u\right] \xrightarrow{m\to\infty} 0, \qquad t \in [0,T],$$

that concludes the proof. □

4.5.2 Term insurance

A term insurance contract provides payoff in the case of a policyholder's decease before the term T of contract. We assume the payment process $R := (R_t)_{t\in[0,T]}$ to be $\mathbb{F}$-predictable and square integrable. The amount paid at T to the i-th policyholder is given by

$$\mathbf{1}_{\{0<\tau^i\leqslant T\}}R_{\tau^i},$$

where $i = 1,...,n$. For a homogeneous portfolio of policies, we have

$$\sum_{i=1}^n \mathbf{1}_{\{0<\tau^i\leqslant T\}}R_{\tau^i}.$$

The associated benchmarked payment process A is hence

$$A_t = \sum_{i=1}^n \int_0^t (S_u^*)^{-1}\mathrm{d}D_u = \sum_{i=1}^n (S_{\tau^i}^*)^{-1}\mathbf{1}_{\{0<\tau^i\leqslant t\}}R_{\tau^i},$$

for $t \in [0,T]$.

The price process associated to a homogeneous portfolio of term insurance contracts is denoted by $V^\tau := (V_t^\tau)_{t\in[0,T]}$. The real-world pricing formula (2.3.3) combined with

(4.3.3) and (4.3.8) yields

$$
\begin{aligned}
V_t^\tau =& S_t^* E\left[\sum_{i=1}^n (S_{\tau^i}^*)^{-1}\mathbf{1}_{\{t<\tau^i\leqslant T\}} R_{\tau^i}\middle|\, \mathcal{G}_t\right]\\
=& \sum_{i=1}^n S_t^* E\left[(S_{\tau^i}^*)^{-1}\mathbf{1}_{\{t<\tau^i\leqslant T\}} R_{\tau^i}\middle|\, \mathcal{G}_t\right]\\
=& \sum_{i=1}^n \mathbf{1}_{\{\tau^i>t\}} S_t^* E\left[\int_t^T (S_u^*)^{-1} R_u e^{-\int_t^T \mu_u \mathrm{d}u}\mu_u \mathrm{d}u\middle|\, \mathcal{F}_t\right]\\
=& (n-l_t)e^{(\gamma+\alpha)t}\frac{E\left[\int_t^T e^{-(\gamma+\alpha)u} R_u p(Z_u) q(Y_u)\mu_u \mathrm{d}u\middle|\, \mathcal{F}_t\right]}{p(Z_t)q(Y_t)}\\
=& (n-l_t)e^{(\gamma+\alpha)t}\frac{E\left[\int_t^T e^{-(\gamma+\alpha)u} R_u p(Z_u)\left(\gamma q(Y_u) - \nabla q(Y_u)^\top \bar{b}(Z_u)\right)\mathrm{d}u\middle|\, \mathcal{F}_t\right]}{p(Z_t)q(Y_t)},
\end{aligned}
$$

where in the third equality, Proposition 5.5 of [7] is used combined with Corollary 5.1.3 of [24]. We note that the requirement of bounded R in Corollary 5.1.3 of [24] can be easily relaxed by using a localization argument together with the dominated convergence theorem for conditional expectation, if R is sufficiently integrable.

Continuous payoff

We start directly with the case of continuous payoff by assuming

$$
R_t = R(Z_t),
$$

for $t \in [0,T]$, where R is a continuous function on the compact state space E. The stochastic Fubini–Tonelli Theorem leads to

$$
\begin{aligned}
V_t^\tau =& (n-l_t)e^{(\gamma+\alpha)t}\frac{E\left[\int_t^T e^{-(\gamma+\alpha)u} R(Z_u) p(Z_u)(\gamma q(Y_u) - \nabla q(Y_u)^\top \bar{b}(Z_u))\mathrm{d}u\middle|\, \mathcal{F}_t\right]}{p(Z_t)q(Y_t)}\\
=& (n-l_t)e^{(\gamma+\alpha)t}\frac{\int_t^T e^{-(\gamma+\alpha)u} E\left[R(Z_u) p(Z_u)(\gamma q(Y_u) - \nabla q(Y_u)^\top \bar{b}(Z_u))\middle|\, \mathcal{F}_t\right]\mathrm{d}u}{p(Z_t)q(Y_t)},
\end{aligned}
$$

for $t \in [0,T]$. As before, this expression can be approximated by explicated pricing formulas related to polynomial payoff as we show in the following.

Proposition 4.5.7. *Let $\{R_m\}_{m\in\mathbb{N}}$ be a sequence of polynomials functions which approximates uniformly the continuous function R on the state space E. For each $m \in \mathbb{N}$, we set $V^{\tau,m} := (V_t^{\tau,m})_{t\in[0,T]}$ with*

$$
\begin{aligned}
V_t^{\tau,m} :=& (n-l_t)e^{(\gamma+\alpha)t}\frac{\int_t^T e^{-(\gamma+\alpha)u} E\left[\gamma r_m(Z_u) - s_m(Z_u)\middle|\, \mathcal{F}_t\right]\mathrm{d}u}{p(Z_t)q(Y_t)}\\
=& (n-l_t)e^{(\gamma+\alpha)t}\frac{\int_t^T e^{-(\gamma+\alpha)u}\left(\gamma \mathbf{r}_{\mathbf{m}(}t,u)(Z_t) - \mathbf{s}_{\mathbf{m}(t,u)}(Z_t)\right)\mathrm{d}u}{p(Z_t)q(Y_t)},
\end{aligned}
$$

for every $t \in [0,T]$, where the polynomial functions r_m and s_m are respectively $r_m := R_m pq$ and $s_m := R_m p\left(\nabla q^\top \bar{b}\right)$. Then the series $\{V^{\tau,m}\}_{m\in\mathbb{N}}$ provides both a pathwise and $L^p(\Omega, P)$ approximation of V^τ uniformly in $t \in [0,T]$.

Proof. Analogous to Proposition 4.5.3. □

Analogue approximation results hold for the benchmarked risk-minimizing strategy and the benchmarked cumulative cost process.

4.5.3 Annuity

An annuity is a continuous cash flow paid by the insurer as long as the policyholder is alive. Let C_t denote its cumulated payoff value up to time t. We assume that the process $C := (C_t)_{t\in[0,T]}$ is a right continuous increasing $\mathbb{F}$-adapted and square integrable process, with $C_0 = 0$ and $C_{T-} = C_T$. The total payoff at T associated to the i-th policyholder is hence

$$\int_{]0,T]} (1 - H^i_u) dC_u = \int_{]0,T]} \mathbf{1}_{\{\tau^i > u\}} dC_u = C_T \mathbf{1}_{\{\tau^i > T\}} + C_{\tau^i -} \mathbf{1}_{\{0 < \tau^i \leqslant T\}}. \qquad (4.5.16)$$

Similarly, the total payoff at T of a homogeneous portfolio of annuity contracts is given by

$$\sum_{i=1}^n \int_{]0,T]} (1 - H^i_u) dC_u.$$

The benchmarked cumulated payment process at time t with $t \in [0,T]$ equals

$$A_t = \sum_{i=1}^n \int_0^t (S^*_u)^{-1} (1 - H^i_u) dC_u.$$

We set $V^C := (V^C_t)_{t\in[0,T]}$ to be the price process given by the real-world pricing formula (2.3.3) for a homogeneous portfolio of annuity contracts. Using (4.3.3) and (4.3.8) we have at any $t \in [0,T]$

$$\begin{aligned} V^C_t :=& S^*_t E\left[\sum_{i=1}^n \int_t^T (S^*_u)^{-1} (1 - H_u) dC_u \middle| \mathcal{G}_t\right] \\ =& \sum_{i=1}^n S^*_t E\left[\int_t^T (S^*_u)^{-1} (1 - H_u) dC_u \middle| \mathcal{G}_t\right] \\ =& \sum_{i=1}^n \mathbf{1}_{\{\tau^i > t\}} S^*_t E\left[\int_{]t,T]} (S^*_u)^{-1} e^{-\int_t^u \mu_u \mathrm{d}u} dC_u \middle| \mathcal{F}_t\right] \\ =& (n - l_t) e^{(\gamma+\alpha)t} \frac{E\left[\int_t^T e^{-(\gamma+\alpha)u} p(Z_u) q(Y_u) dC_u \middle| \mathcal{F}_t\right]}{p(Z_t) q(Y_t)}, \end{aligned}$$

where in the third equality we apply Proposition 5.5 of [7] and Proposition 5.1.2 of [24]. Though Proposition 5.1.2 of [24] requires that the process C is bounded, this condition can be relaxed as in Section 4.5.2, by using a localization argument combined with the theorem of dominated convergence for conditional expectation.

Continuous payoff

Under our assumptions, if C is in addition a continuous process, then it is also an $\mathbb{F}$-predictable process. Hence, according to (4.5.16) we get

$$\sum_{i=1}^{n} \int_{]0,T]} (1 - H_u^i) dC_u = C_T(n - l_t) + \sum_{i=1}^{n} C_{\tau^i} \mathbf{1}_{\{0<\tau^i \leqslant T\}}.$$

In other words, a homogeneous annuity portfolio can be considered as the sum of a homogeneous pure endowment portfolio and a homogeneous term insurance portfolio as defined in Section 4.5.1 and 4.5.2 respectively, where $g_T = C_T$ and $R = C$. In particular, the linearity of the pricing formula leads to

$$V^C = V^T + V^\tau.$$

By assuming

$$C_t = \bar{C}(Z_t),$$

for $t \in [0,T]$, with $\bar{C}$ a continuous function on the compact state space E, we have the following proposition.

Proposition 4.5.8. *Let $\{C_m\}_{m\in\mathbb{N}}$ be a sequence of polynomials approximating uniformly the continuous function $\bar{C}$ on E. For each $m \in \mathbb{N}$, we consider $V^{C,m} := (V_t^{C,m})_{t\in[0,T]}$ with*

$$V^{C,m} := V^{T,m} + V^{\tau,m}, \tag{4.5.17}$$

where $V^{T,m}$ and $V^{\tau,m}$ are defined in Proposition 4.5.3 and 4.5.7 respectively, with

$$g_m = R_m = C_m,$$

for all $m \in \mathbb{N}$. Then $\{V^{C,m}\}_{m\in\mathbb{N}}$ is both a pathwise and $L^p(\Omega, P)$ approximation of V^C uniformly in $t \in [0,T]$.

Proof. This is an immediate consequence of (4.5.17), Proposition 4.5.3 and Proposition 4.5.7. □

Similar approximation results hold for the benchmarked risk-minimizing strategy and the benchmarked cumulative cost process.

Chapter 5

Insurance framework under model uncertainty

5.1 Introduction

In this chapter, based on [19], we develop a model-free framework for insurance market, when a generic family of priors possibly mutually singular to each other is considered. While financial market under model uncertainty has been recently introduced and intensively studied, a corresponding study for insurance market was still missing. The main issues are the pricing problem of payment streams in continuous time with respect to a non-dominated probability family, and the stochastic analysis on a progressively enlarged filtration under model uncertainty. We solve the first problem by defining superhedging of payment streams and providing several equivalent dynamic robust superhedging dualities in continuous time. For the second problem, instead of the generic filtration structure in Chapter 2, we consider for the sake of simplicity the classic reduced-form case, i.e. when the progressively enlarged filtration $\mathbb{G}$ is generated by an external jump. On this enlarged filtration, we construct a consistent sublinear conditional expectation, which can be used as a pricing operator in view of the superhedging results. We stress that, even though our analysis is motivated by insurance setting, it can be applied to credit risk setting as well.

The chapter is organized as follows. In Section 5.2, we introduce the notations and recall briefly some useful theorems in the existing literature. As the first main result, we formulate the superhedging problem for payment streams, provide equivalent dynamic robust superhedging dualities in continuous time for payment streams in the financial market, determine the robust superhedging price and show the existence of optimal robust superhedging strategies. In Section 5.3, we construct a consistent robust reduced-form framework. As the second main result, we define explicitly sublinear conditional expectation on the filtration enlarged according to our construction and study its properties. We discuss in detail difficulties arisen from our construction and the fact that in the general case, the constructed sublinear conditional satisfies only a weak form of tower property and do not always preserve the integrability condition. In Section 5.4, we show that the above results can be applied to insurance cash flows, and in these cases the integrability condition is satisfied and the classic tower property holds. As a consequence, the superhedging problem for payment streams in this robust reduced-form framework can be solved.

5.2 Canonical setting under model uncertainty

In this section, we recall some existing results within the canonical setting under model uncertainty with applications to financial market. Afterwards, we introduce and analyze for the first time the superhedging problem for payment streams. We set $\Omega = D_0(\mathbb{R}_+, \mathbb{R}^d)$, i.e. the space of càdlàg functions $\omega = (\omega_t)_{t\geqslant 0}$ in $\mathbb{R}^d$ which start from zero. The space Ω is Polish, i.e. a complete separable metrizable space, if equipped with metric induced by the Skorokhod topology. Let $\mathcal{F} := \mathcal{B}(\Omega)$ be its Borel σ-algebra and $\mathcal{P}(\Omega)$ the set of all probability measures on $(\Omega, \mathcal{F})$. We consider the topology of weak convergence on $\mathcal{P}(\Omega)$. We note that by Prokhorov's theorem (see e.g. [101], [37] and [25]), $\mathcal{P}(\Omega)$ inherits from Ω the property of being a Polish space with the Lévy-Prokhorov metric. We emphasize that all results in this chapter also hold if the space $D_0(\mathbb{R}_+, \mathbb{R}^d)$ is replaced by $C_0(\mathbb{R}_+, \mathbb{R}^d)$, i.e. the space of continuous functions $\omega = (\omega_t)_{t\geqslant 0}$ in $\mathbb{R}^d$ which start from zero, equipped with the topology of locally uniform convergence. In case of no ambiguity, we keep the notations B and $\mathbb{F}$ respectively for the canonical process on $C_0(\mathbb{R}_+, \mathbb{R}^d)$ and its natural filtration.

Let $B := (B_t)_{t\geqslant 0}$ be the canonical process which $B_t(\omega) := \omega_t$, $t \geqslant 0$. We denote its raw filtration by $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$. In particular, we have $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_\infty := \bigvee_{t\geqslant 0} \mathcal{F}_t = \mathcal{F}$. For every $P \in \mathcal{P}(\Omega)$ and $t \in \overline{\mathbb{R}}_+$, $\mathcal{N}_t^P$ denotes the collection of sets which are $(P, \mathcal{F}_t)$-null. We define

$$\mathcal{F}_t^* := \mathcal{F}_t \vee \mathcal{N}_t^*, \quad \mathcal{N}_t^* := \bigcap_{P\in\mathcal{P}(\Omega)} \mathcal{N}_t^P,$$

and the corresponding universally completed filtration by $\mathbb{F}^* := (\mathcal{F}_t^*)_{t\geqslant 0}$. Moreover, for every $P \in \mathcal{P}(\Omega)$ we denote the usual P-augmentation by $\mathbb{F}_+^P$, i.e. $\mathbb{F}_+^P$ is the right continuous version of $\mathbb{F}^P := (\mathcal{F}_t^P)_{t\geqslant 0}$, with

$$\mathcal{F}_t^P := \mathcal{F}_t \vee \mathcal{N}_\infty^P, \quad t \geqslant 0.$$

Clearly, the above enlargements of the raw filtration are ordered as follows

$$\mathcal{F}_t \subseteq \mathcal{F}_t^* \subseteq \mathcal{F}_t^P \subseteq \mathcal{F}_{t+}^P, \quad t \geqslant 0, \quad P \in \mathcal{P}. \tag{5.2.1}$$

If $\mathcal{P} \subseteq \mathcal{P}(\Omega)$ is a generic nonempty set, we define the following σ-algebras

$$\mathcal{F}^\mathcal{P} := \mathcal{F} \vee \mathcal{N}_\infty^\mathcal{P}, \quad \mathcal{N}_\infty^\mathcal{P} := \bigcap_{P\in\mathcal{P}} \mathcal{N}_\infty^P.$$

The space of all real-valued $\mathcal{F}^\mathcal{P}$-measurable functions is denoted by $L^0(\Omega)$ and the upper expectation $\mathcal{E} : L^0(\Omega) \to \overline{\mathbb{R}}$ associated to $\mathcal{P}$ is defined by

$$\mathcal{E}(X) := \sup_{P\in\mathcal{P}} E^P[X], \quad X \in L^0(\Omega), \tag{5.2.2}$$

where we use the convention in [109]: for every $P \in \mathcal{P}$, we set $E^P[X] := E^P[X^+] - E^P[X^-]$ if $E^P[X^+]$ or $E^P[X^-]$ is finite, and $E^P[X] := -\infty$ if $E^P[X^+] = E^P[X^-] = +\infty$.

5.2.1 $(\mathcal{P}, \mathbb{F})$-conditional expectation

In this section, we summarize the pathwise construction in [87] of conditional expectation with respect to the filtration $\mathbb{F}$ and a probability measure family $\mathcal{P}$. The results hold both on the space $D_0(\mathbb{R}_+, \mathbb{R}^d)$ and on the space $C_0(\mathbb{R}_+, \mathbb{R}^d)$, as noted in e.g. [21], [82] and [86]. For the sake of simplicity, we consider only the case when the parametrized families in Assumption 2.1 of [87] have no dependence on the parameters. The following notations are the same as in [87]. If τ is a finite-valued $\mathbb{F}$-stopping time and $\omega \in \Omega$, for every $\omega' \in \Omega$, the concatenation $\omega \otimes_\tau \omega' := ((\omega \otimes_\tau \omega')_t)_{t \geqslant 0}$ of (ω, ω') at τ is defined by

$$(\omega \otimes_\tau \omega')_t := \omega_t \mathbf{1}_{[0,\tau(\omega))}(t) + \Big(\omega_{\tau(\omega)} + \omega'_{t-\tau(\omega)}\Big) \mathbf{1}_{[\tau(\omega),+\infty)}(t), \qquad t \geqslant 0.$$

For every function X on Ω, let

$$X^{\tau,\omega}(\omega') := X(\omega \otimes_\tau \omega'), \qquad \omega' \in \Omega. \tag{5.2.3}$$

Analogously, for every probability measure P we define

$$P^{\tau,\omega}(A) := P^\omega_\tau(\omega \otimes_\tau A), \qquad A \in \mathcal{B}(\Omega),$$

where $\omega \otimes_\tau A := \{\omega \otimes_\tau \omega' : \omega' \in A\}$ and P^ω_τ is the $\mathcal{F}_\tau$-conditional probability measure chosen to be

$$P^\omega_\tau\left(\omega' \in \Omega : \omega' = \omega \text{ on } [0, \tau(\omega)]\right) = 1.$$

We note that $P^{\tau,\omega}$ is still a probability measure.

Definition 5.2.1. *A set of a Polish space is called* analytic*, if it is the image of a Borel set of another Polish space under a Borel-measurable mapping.*

Definition 5.2.2. *A $\overline{\mathbb{R}}$-valued function f on a Polish space is called* upper semianalytic*, if $\{f > c\}$ is analytic for all $c \in \mathbb{R}$.*

Remark 5.2.3. *We stress that all Borel sets are analytic and all Borel-measurable functions are upper semianalytic.*

We assume the following conditions.

Assumption 5.2.4. *For every finite-valued $\mathbb{F}$-stopping time τ, we assume that the family $\mathcal{P}$ satisfies the following conditions:*

1. *measurability: the set $\mathcal{P} \in \mathcal{P}(\Omega)$ is analytic;*
2. *invariance: $P^{\tau,\omega} \in \mathcal{P}$ for P-a.e. $\omega \in \Omega$;*

3. *stability under pasting: for all $\mathcal{F}_\tau$-measurable kernel $\kappa : \Omega \to \mathcal{P}(\Omega)$ such that $\kappa(\omega) \in \mathcal{P}$ for P-a.e. $\omega \in \Omega$, the following measure*

$$\overline{P}(A) := \iint (\mathbf{1}_A)^{\tau,\omega}(\omega')\kappa(\mathrm{d}\omega';\omega)P(\mathrm{d}\omega), \qquad A \in \mathcal{B}(\Omega),$$

still belongs to $\mathcal{P}$.

Remark 5.2.5. *According to [82], Assumption 5.2.4 is satisfied if the family $\mathcal{P}$ is generated by all semimartingale laws with differential characteristics taking values in a Borel-measurable set $\theta \subseteq \mathbb{R}^d \times \mathbb{S}^d_+ \times \mathcal{L}$, where $\mathbb{S}^d_+$ is the set of symmetric nonnegative definite $(d \times d)$-matrices and $\mathcal{L}$ is the set of all Lévy measures. In particular, this includes the G-expectations introduced in [91] as a special case. A brief introduction to the G-expectations is provided in Appendix A.*

The following proposition is a simplified version of Theorem 2.3 of [87], when an unparametrized family $\mathcal{P}$ satisfying Assumption 5.2.4 is considered.

Proposition 5.2.6. *For all finite-valued $\mathbb{F}$-stopping times σ, τ such that $\sigma \leqslant \tau$ and for every upper semianalytic function X, the function $\mathcal{E}_\tau(X)$ with*

$$\mathcal{E}_\tau(X)(\omega) := \mathcal{E}(X^{\tau,\omega}) = \sup_{P \in \mathcal{P}} E^P[X^{\tau,\omega}], \qquad \omega \in \Omega \tag{5.2.4}$$

is $\mathcal{F}^*_\tau$-measurable, upper semianalytic and satisfies the following consistency condition

$$\mathcal{E}_\tau(X) = \operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(\tau;P)} E^{P'}[X|\mathcal{F}_\tau] \quad P\text{-a.s. for all } P \in \mathcal{P}, \tag{5.2.5}$$

where $\mathcal{P}(\tau;P) := \{P' \in \mathcal{P} : P' = P \text{ on } \mathcal{F}_\tau\}$. Moreover, the tower property holds, i.e.

$$\mathcal{E}_\sigma(X)(\omega) = \mathcal{E}_\sigma(\mathcal{E}_\tau(X))(\omega) \quad \text{for all } \omega \in \Omega. \tag{5.2.6}$$

Definition 5.2.7. *The family of sublinear conditional expectations $(\mathcal{E}_t)_{t \geqslant 0}$ is called* $(\mathcal{P}, \mathbb{F})$-conditional expectation.

In the case of G-setting of [91], G-martingales are càdlàg, see e.g. [113]. However, under generic assumptions, we cannot always guarantee that the process $(\mathcal{E}_t(X))_{t\geqslant 0}$ with X upper semianalytic is càdlàg. In the following proposition, we give an independent result which gives sufficient conditions such that $(\mathcal{E}_t(X))_{t\geqslant 0}$ becomes càdlàg. We recall that, by Prokhorov's theorem, the tightness of a family of probability measures is equivalent to the compactness of its weak closure. In particular, the probability measure family generating the G-expectation is tight, see Proposition 49 in [39] and Appendix A.

Proposition 5.2.8. *If the family $\mathcal{P}$ is a tight and X is an upper semianalytic function which is bounded and continuous on a set $A \in \mathcal{B}(\Omega)$ such that $P(A^c) = 0$ for every $P \in \mathcal{P}$, then the process $(\mathcal{E}_t(X))_{t\geqslant 0}$ is càdlàg.*

Proof. We show first the right continuity. Let $t \geqslant 0$ and $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ with $t_n \downarrow t$. We want to prove that for all $\omega \in \Omega$,

$$\mathcal{E}_t(X)(\omega) = \lim_{n\to\infty} \mathcal{E}_{t_n}(X)(\omega).$$

Let $\omega \in \Omega$. Definitions (5.2.4) and (5.2.3) yield

$$\mathcal{E}_t(X)(\omega) = \mathcal{E}(X^{t,\omega}) = \sup_{P \in \mathcal{P}} E^P[X^{t,\omega}] = \sup_{P \in \mathcal{P}} \int X(\omega \otimes_t \omega') P(\mathrm{d}\omega').$$

For every t and ω, the concatenation function $c^{t,\omega} : \Omega \to \Omega$ defined by

$$c^{t,\omega}(\omega') := \omega \otimes_t \omega', \qquad \omega' \in \Omega,$$

is uniformly continuous in ω' with respect to Skorokhod topology on $\Omega = D_0(\mathbb{R}_+, R^d)$, or the topology induced by the locally uniform convergence on $\Omega = C_0(\mathbb{R}_+, R^d)$. That is, if d is the distance function associated to the metric on Ω, then for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $\omega', \omega'' \in \Omega$ with $d(\omega', \omega'') < \delta$, it holds

$$d(\omega \otimes_t \omega', \omega \otimes_t \omega'') < \varepsilon.$$

Indeed, we note that it is sufficient to take $\delta = \varepsilon$. In particular, $\delta = \varepsilon$ does not depend on the choice of t, thus the sequence of functions $(c^{t_n,\omega})_{n \in \mathbb{N}}$ is equicontinuous. Besides, the sequence $(c^{t_n,\omega})_{n \in \mathbb{N}}$ converges to $c^{t,\omega}$ pointwisely,

$$d\left(\omega \otimes_{t_n} \omega', \omega \otimes_t \omega'\right) \xrightarrow{n \to \infty} 0 \quad \text{for all } \omega' \in \Omega,$$

since $D_0(\mathbb{R}_+, R^d)$ is the space of càdlàg paths. Thus, Ascoli-Arzelà Theorem yields that the sequence $(c^{t_n,\omega})_{n \in \mathbb{N}}$ converges to $c^{t,\omega}$ uniformly on every compact set $K \subseteq \Omega$, i.e. we have

$$\sup_{\omega' \in K} d\left(c^{t_n,\omega}(\omega'), c^{t,\omega}(\omega')\right) = \sup_{\omega' \in K} d\left(\omega \otimes_{t_n} \omega', \omega \otimes_t \omega'\right) \xrightarrow{n \to \infty} 0.$$

In particular, for every compact set $K \in \mathcal{B}(\Omega)$, the composition $X^{t,\omega} = X \circ c^{t,\omega}$ is bounded and continuous on $A \cap K$, and $X^{t,\omega}$ is the uniform limit of $(X^{t_n,\omega})_{n \in \mathbb{N}}$, i.e. for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$|X(\omega \otimes_{t_n} \omega') - X(\omega \otimes_t \omega')| < \varepsilon \quad \text{for every } \omega' \in A \cap K.$$

Consequently, on one hand, for every $n \in \mathbb{N}$, the function f^n with

$$f^n(P) := E^P[X^{t_n,\omega}], \qquad P \in \mathcal{P}(\Omega),$$

is continuous in P with respect to Lévy-Prokhorov metric on $\mathcal{P}(\Omega)$, since this coincides with the metric induced by weak convergence of measures. Thus, the restriction $f^n|_{\mathcal{P}}$ is still continuous. On the other hand, it follows from the tightness of $\mathcal{P}$ that there is a compact set $K \in \mathcal{B}(\Omega)$ such that

$$P(K^c) < \frac{\varepsilon}{4C} \qquad \text{for all } P \in \mathcal{P},$$

where C is such that $|X(\omega)| \leqslant C$ for every $\omega \in A$. If n is big enough, since $X^{t,\omega}$ is the P-a.s. uniform limit of $(X^{t_n,\omega})_{n \in \mathbb{N}}$ on $A \cap K$, we get

$$\begin{aligned} |E^P[X^{t_n,\omega}] - E^P[X^{t,\omega}]| &\leqslant E^P[|X^{t_n,\omega} - X^{t,\omega}|] \\ &= E^P[\mathbf{1}_{A \cap K}|X^{t_n,\omega} - X^{t,\omega}|] + E^P[\mathbf{1}_{A \setminus K}|X^{t_n,\omega} - X^{t,\omega}|] \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4C} \cdot 2C = \varepsilon \quad \text{for all } P \in \mathcal{P}. \end{aligned}$$

As a consequence, for all $\omega \in \Omega$,

$$\begin{aligned}\mathcal{E}(X^{t,\omega}) &= \sup_{P\in\mathcal{P}} E^P[\lim_{n\to\infty} X^{t_n,\omega}] = \sup_{P\in\mathcal{P}} \lim_{n\to\infty} E^P[X^{t_n,\omega}] \\ &= \lim_{n\to\infty} \sup_{P\in\mathcal{P}} E^P[X^{t_n,\omega}] = \lim_{n\to\infty} \mathcal{E}(X^{t_n,\omega}).\end{aligned}$$

Similarly, with the same argument we can show the existence and finiteness of the left limit, which concludes the proof. □

Remark 5.2.9. *It is shown Proposition 4.5 in [88] and in the proof of Theorem 3.2 of [86] that a modified process of $(\mathcal{E}_t(X))_{t\geqslant 0}$ is càdlàg pathwisely. However, such process is adapted to a filtration different from the filtration $\mathbb{F}^*$, i.e. adapted to*

$$\left(\mathbb{F}_{t+} \cup \mathcal{N}_T^{\mathcal{P}}\right)_{t\in[0,T]},$$

where $\mathcal{N}_T^{\mathcal{P}}$ are the sets which are $(P, \mathcal{F}_T)$-null for all $P \in \mathcal{P}$. This is not consistent with our framework, where the filtration $\mathbb{F}^*$ is interpreted as information available to the agents.

5.2.2 Robust optional decomposition

Let $[0,T]$ with $T > 0$ be a finite time horizon. We recall in this section the results of Section 2 in [86], which hold for an arbitrary measurable space Ω equipped with an arbitrary filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$. We stress that throughout this section "sigma martingale" can be replaced by "local martingale".

Let m be a positive integer and $S := (S_t)_{t\in[0,T]}$ an m-dimensional $\mathbb{F}$-adapted process with càdlàg paths. If under a probability measure P the process S is a $(P,\mathbb{F})$-semimartingale, then we denote its characteristics by (B^P, C^P, ν^P).

Remark 5.2.10. *According to Proposition 2.2 of [81], the process S is also a $(P, \mathbb{F}_+^P)$-semimartingale with the same characteristics.*

If δ is an m-dimensional $\mathbb{F}$-predictable process which is S-integrable under P, we denote its usual Itô integral under P by

$$^{(P)}\!\!\int \delta \mathrm{d}S := \left(^{(P)}\!\!\int_0^t \delta \mathrm{d}S \right)_{t\in[0,T]}.$$

Furthermore, if S is a $(P,\mathbb{F})$-semimartingale for all $P \in \mathcal{P}$, we set

$$\begin{aligned}&L(S,\mathbb{F},\mathcal{P}) \\ &:= \left\{ \delta \ m\text{-dimensional } \mathbb{F}\text{-predictable process} : \ ^{(P)}\!\!\int_0^t |\delta| \mathrm{d}S < \infty \ \text{ for all } P \in \mathcal{P} \right\}.\end{aligned}$$

We assume the following conditions.

Assumption 5.2.11. *1. $\mathcal{P}$ is a set of sigma martingale measures for S: the process S is a $(P, \mathbb{F}_+^P)$-sigma martingale for all $P \in \mathcal{P}$;*

2. *$\mathcal{P}$ is saturated: all equivalent sigma martingale measures of its element still belong to $\mathcal{P}$;*

3. *S has dominating diffusion under every $P \in \mathcal{P}$: it holds that $\nu^P \ll (C^P)^{ii}$ P-a.s. for all $i = 1, ..., m$ and for all $P \in \mathcal{P}$.*

Remark 5.2.12. *We note that if S has continuous paths, it always has dominating diffusion under a sigma martingale measure P. Indeed, its characteristics are reduced to $(0, C^P, 0)$. In particular, it is a continuous local martingale under P.*

Remark 5.2.13. *In the case of $m = d$ and $S = B$, Lemma 4.2 and Proposition 4.3 of [86] show a sufficient condition such that Assumption 5.2.4 and Assumption 5.2.11 are both satisfied.*

The following fundamental result, called Optional Decomposition Theorem, is proved in Theorem 2.4 of [86].

Theorem 5.2.14. *If $Y := (Y_t)_{t\in[0,T]}$ is a real-valued, $\mathbb{F}$-adapted process with càdlàg paths, which is a $(P, \mathbb{F}^P_+)$-local supermartingale for all $P \in \mathcal{P}$, then there exists an $\mathbb{F}$-predictable process $\delta := (\delta_t)_{t\in[0,T]}$ in $L(S, \mathbb{F}, \mathcal{P})$ such that*

$$Y - Y_0 - {}^{(P)}\!\!\int \delta \mathrm{d}S \text{ is nonincreasing } P\text{-a.s. for all } P \in \mathcal{P}.$$

5.2.3 Robust superhedging for payment streams

We now introduce and analyze the problem of dynamic superhedging for payment streams in continuous time. Let the filtration $\mathbb{F}^{\mathcal{P}} := (\mathcal{F}^{\mathcal{P}}_t)_{t\in[0,T]}$ be defined by

$$\mathcal{F}^{\mathcal{P}}_t := \mathcal{F}^*_t \vee \mathcal{N}^{\mathcal{P}}_T, \quad t \in [0, T],$$

where $\mathcal{N}^{\mathcal{P}}_T$ is the collection of sets which are $(P, \mathcal{F}_T)$-null for all $P \in \mathcal{P}$. Let $A := (A_t)_{t\in[0,T]}$ be an $\mathbb{F}^{\mathcal{P}}$-adapted process with nonnegative nondecreasing paths such that $A_t(\omega)$, $\omega \in \Omega$, is upper semianalytic for all $t \geqslant 0$. We assume $A_0 = 0$ without loss of generality. Let $S := (S_t)_{t\in[0,T]}$ be an m-dimensional $\mathbb{F}^{\mathcal{P}}$-adapted process with càdlàg paths which is a $(P, \mathbb{F}^P)$-semimartingale for all $P \in \mathcal{P}$. The two processes A and S represent respectively an (eventually discounted) cumulative payment stream and (eventually discounted) liquidly tradable assets on the market. We denote by $L(S, \mathbb{F}^{\mathcal{P}}, \mathcal{P})$ the set[1]

$$\begin{aligned} &L(S, \mathbb{F}^{\mathcal{P}}, \mathcal{P}) \\ &:= \left\{ \delta \ m\text{-dimensional } \mathbb{F}^{\mathcal{P}}\text{-predictable process} : \ {}^{(P)}\!\!\int_0^t |\delta| \mathrm{d}S < \infty \ \text{ for all } P \in \mathcal{P} \right\}. \end{aligned}$$

Definition 5.2.15. *The elements of the set*

$$\Delta := \left\{ \delta \in L(S, \mathbb{F}^{\mathcal{P}}, \mathcal{P}) : {}^{(P)}\!\!\int \delta \mathrm{d}S \text{ is a } (P, \mathbb{F}^P_+)\text{-supermartingale for all } P \in \mathcal{P} \right\}$$

[1] Later we will apply the Optional Decomposition Theorem 5.2.14 to the filtration $\mathbb{F}^{\mathcal{P}}$.

are called admissible strategies*.*

Definition 5.2.16. *We call* robust global superhedging strategy for a cumulative payment stream A *a process* $\delta \in \Delta$ *such that there exists* $v \in \mathbb{R}$ *satisfying*

$$v + {}^{(P)}\!\!\int_0^\tau \delta_u \mathrm{d}S_u \geqslant A_\tau \qquad P\text{-a.s. for all } P \in \mathcal{P},$$

for every $[0,T]$*-valued* $\mathbb{F}$*-stopping time* τ*.*

Definition 5.2.17. *Let* σ, τ *be two* $[0,T]$*-valued* $\mathbb{F}$*-stopping times such that* $\sigma \leqslant \tau$*. We call* robust local superhedging strategy for a cumulative payment stream A on the random interval $[\sigma, \tau]$ *a process* $\delta \in \Delta$ *such that there exists a real-valued* $\mathcal{F}_\sigma^{\mathcal{P}}$*-measurable function* v *satisfying*

$$v + {}^{(P)}\!\!\int_\sigma^{\sigma'} \delta_u \mathrm{d}S_u \geqslant A_{\sigma'} - A_\sigma \qquad P\text{-a.s. for all } P \in \mathcal{P},$$

for all $[0,T]$*-valued* $\mathbb{F}$*-stopping time* σ' *with* $\sigma \leqslant \sigma' \leqslant \tau$*.*

Our Definition 5.2.17 agrees with the definition of superhedging strategies given in e.g. [51], [94] and [95] in discrete time and without model uncertainty. Moreover, an admissible strategy δ is a robust global superhedging strategy if and only if it is a robust local superhedging strategy on all random intervals in $[0,T]$. Analogously, we define global and local superhedging prices as follows.

Definition 5.2.18. *A value* $\pi_0^T \in \mathbb{R}$ *is called* robust global superhedging price for A *if*

$$\begin{aligned}\pi_0^T = \inf \Big\{ & v \in \mathbb{R} : \exists \delta \in \Delta \text{ such that for every } [0,T]\text{-valued } \mathbb{F}\text{-stopping time } \tau, \\ & v + {}^{(P)}\!\!\int_0^\tau \delta_u \mathrm{d}S_u \geqslant A_\tau \ P\text{-a.s. for all } P \in \mathcal{P} \Big\}.\end{aligned} \tag{5.2.7}$$

Definition 5.2.19. *Let* σ, τ *be two* $[0,T]$*-valued* $\mathbb{F}$*-stopping times such that* $\sigma \leqslant \tau$*. A real-valued* $\mathcal{F}_\sigma^{\mathcal{P}}$*-measurable function* π_σ^τ *is called* robust local superhedging price for A over the random interval $[\sigma, \tau]$ *if*

$$\begin{aligned}\pi_\sigma^\tau = \text{ess inf}^P \Big\{ & v \text{ is } \mathcal{F}_\sigma^{\mathcal{P}}\text{-measurable} : \exists \delta \in \Delta \text{ such that for every } \mathbb{F}\text{-stopping time } \sigma' \\ & \text{with } \sigma \leqslant \sigma' \leqslant \tau,\ v + {}^{(P)}\!\!\int_\sigma^{\sigma'} \delta_u \mathrm{d}S_u \geqslant A_{\sigma'} - A_\sigma \ P\text{-a.s. for all } P \in \mathcal{P} \Big\} \\ & P\text{-a.s. for all } P \in \mathcal{P}.\end{aligned} \tag{5.2.8}$$

Our Definition 5.2.19 agrees with the definition of superhedging price (or superhedging premium) given in e.g. [51], [94] and [95] in discrete time and without model uncertainty. We stress that the robust local superhedging price is unique only up to a set $N \in \mathcal{N}^{\mathcal{P}}$.

The dynamic superhedging for payment streams can be formulated in the following two problems.

1. Show that robust global and local superhedging prices as defined in Definition 5.2.18 and Definition 5.2.19 exist and determine their value.

2. Show that global and local superhedging strategies for a payment stream associated to robust global and local superhedging prices exist. We call *optimal superhedging strategies for A*, if it exists, a robust global superhedging strategy δ for A such that, for all $[0,T]$-valued $\mathbb{F}$-stopping times σ,σ',τ with $\sigma \leqslant \sigma' \leqslant \tau$, we have

$$\pi_\sigma^\tau + {}^{(P)}\!\!\int_\sigma^{\sigma'} \delta_u \mathrm{d}S_u \geqslant A_{\sigma'} - A_\sigma \qquad P\text{-a.s. for all } P \in \mathcal{P}.$$

The first is a *pricing problem*. The robust global (or resp. local) superhedging price of A can be interpreted both as the minimal amount of money the company should keep in order to be able to pay out in the future, and as the minimal price the product should be sold. The second issue is a *hedging problem*. We stress the importance of distinguishing robust global and local superhedging problems. Obviously, for products with single payoff such as European contingent claims, only the global problem is relevant. However, if we consider a generic payment stream, investors may be interested in the superhedging problem over a particular time interval.

Remark 5.2.20. *We note that all the above definitions are independent of the initial choice of Ω, $\mathbb{F}$ and $\mathcal{P}$.*

The following theorem gives several equivalent dynamic dualities and is a crucial intermediate step for our further discussion.

Theorem 5.2.21. *Let σ,τ be two $[0,T]$-valued $\mathbb{F}$-stopping times such that $\sigma \leqslant \tau$, and $A := (A_t)_{t\in[0,T]}$ a cumulative payment stream with $\mathcal{E}(A_T) < \infty$. If there exists an $\mathbb{F}^{\mathcal{P}}$-adapted process $Y = (Y_t)_{t\in[0,T]}$ with càdlàg path, such that for all $t \in [0,T]$*

$$Y_t = \mathcal{E}_t(A_\tau) \qquad P\text{-a.s. for all } P \in \mathcal{P},$$

then the following equivalent dualities hold for every $P \in \mathcal{P}$:

$$\begin{aligned}
&\mathcal{E}_\sigma(A_\tau)\\
=&\operatorname{ess\,inf}^P \Big\{ v \text{ is } \mathcal{F}_\sigma^{\mathcal{P}}\text{-measurable} : \exists \delta \in \Delta \text{ such that } v + {}^{(P')}\!\!\int_\sigma^\tau \delta_u \mathrm{d}S_u \geqslant A_\tau \\
&P'\text{-a.s. for all } P' \in \mathcal{P}\Big\} \qquad P\text{-a.s.} \qquad (5.2.9)\\
=&\operatorname{ess\,inf}^P \Big\{ v \text{ is } \mathcal{F}_\sigma^{\mathcal{P}}\text{-measurable} : \exists \delta \in \Delta \text{ such that } v + {}^{(P')}\!\!\int_\sigma^\tau \delta_u \mathrm{d}S_u \geqslant A_\tau \\
&P'\text{-a.s. for all } P' \in \mathcal{P}(\sigma;P)\Big\} \qquad P\text{-a.s.,} \qquad (5.2.10)
\end{aligned}$$

and

$$\begin{aligned}&\mathcal{E}_\sigma(A_\tau - A_\sigma)\\ =&\operatorname{ess\,inf}{}^P \left\{ v \text{ is } \mathcal{F}^{\mathcal{P}}_\sigma\text{-measurable} : \exists \delta \in \Delta \text{ such that } v + {}^{(P')}\!\!\int_\sigma^\tau \delta_u \mathrm{d}S_u \geqslant A_\tau - A_\sigma \right.\\ &\left. P'\text{-a.s. for all } P' \in \mathcal{P} \right\} \quad P\text{-a.s.} \qquad (5.2.11)\\ =&\operatorname{ess\,inf}{}^P \left\{ v \text{ is } \mathcal{F}^{\mathcal{P}}_\sigma\text{-measurable} : \exists \delta \in \Delta \text{ such that } v + {}^{(P')}\!\!\int_\sigma^\tau \delta_u \mathrm{d}S_u \geqslant A_\tau - A_\sigma \right.\\ &\left. P'\text{-a.s. for all } P \in \mathcal{P}(\sigma; P) \right\} \quad P\text{-a.s.} \qquad (5.2.12)\end{aligned}$$

Proof. The proof follows similar argument as in Theorem 3.2 of [86] and Theorem 3.4 of [13].

Firstly, we note that dualities (5.2.9) and (5.2.10) are equivalent to (5.2.11) and (5.2.12). Indeed,

$$\begin{aligned}\mathcal{E}_\sigma(A_\tau) - A_\sigma := \operatorname*{ess\,sup}_{P' \in \mathcal{P}(\sigma;P)}{}^P E^{P'}[A_\tau|\mathcal{F}_\sigma] - A_\sigma &= \operatorname*{ess\,sup}_{P' \in \mathcal{P}(\sigma;P)}{}^P E^{P'}[A_\tau - A_\sigma|\mathcal{F}_\sigma]\\ &= \mathcal{E}_\sigma(A_\tau - A_\sigma) \quad P\text{-a.s. for all } P \in \mathcal{P}. \qquad (5.2.13)\end{aligned}$$

Thus, we prove here only dualities (5.2.9) and (5.2.10).

For every $P \in \mathcal{P}$ and σ, τ two $[0,T]$-valued $\mathbb{F}$-stopping times such that $\sigma \leqslant \tau$, we define the following sets:

$$\begin{aligned}{}^{(P)}D^\tau_\sigma := \Bigg\{ & v \text{ is } \mathcal{F}^{\mathcal{P}}_\sigma\text{-measurable} : \exists \delta \in \Delta \text{ such that } v + {}^{(P')}\!\!\int_\sigma^\tau \delta_u \mathrm{d}S_u \geqslant A_\tau \ P'\text{-a.s.}\\ & \text{for all } P' \in \mathcal{P}(\sigma; P) \Bigg\},\end{aligned}$$

and

$$\begin{aligned}D^\tau_\sigma := \Bigg\{ & v \text{ is } \mathcal{F}^{\mathcal{P}}_\sigma\text{-measurable} : \exists \delta \in \Delta \text{ such that } v + {}^{(P')}\!\!\int_\sigma^\tau \delta_u \mathrm{d}S_u \geqslant A_\tau \ P'\text{-a.s.}\\ & \text{for all } P' \in \mathcal{P} \Bigg\}.\end{aligned}$$

One inclusion is obvious,

$${}^{(P)}D^\tau_\sigma \supseteq D^\tau_\sigma.$$

This yields

$$\operatorname{ess\,inf}{}^P \{{}^{(P)}D^\tau_\sigma\} \leqslant \operatorname{ess\,inf}{}^P \{D^\tau_\sigma\} \quad P\text{-a.s.}.$$

We first show the following inequality

$$\mathcal{E}_\sigma(A_\tau) \leqslant \operatorname{ess\,inf}{}^P \{{}^{(P)}D^\tau_\sigma\} \quad P\text{-a.s.}, \qquad (5.2.14)$$

where the convention $\inf \emptyset = \infty$ is used. If $v \in {}^{(P)}D^\tau_\sigma$, then for each $P' \in \mathcal{P}(\sigma; P)$, we have

$$E^{P'}\left[v + {}^{(P')}\!\!\int_\sigma^\tau \delta_u \mathrm{d}S_u \,\middle|\, \mathcal{F}_\sigma \right] \geqslant E^{P'}\left[A_\tau | \, \mathcal{F}_\sigma \right] \quad P'\text{-a.s.}$$

By the supermartingale property of the Itô integral ${}^{(P')}\!\int \delta \mathrm{d}S$, which is in particular $\mathcal{F}^{P'}_{\sigma+}$-measurable, we have

$$E^{P'}[v|\mathcal{F}_\sigma] \geqslant E^{P'}\left[v + E^{P'}\left[{}^{(P')}\!\!\int_\sigma^\tau \delta_u \mathrm{d}S_u \middle| \mathcal{F}^{P'}_{\sigma+} \right] \middle| \mathcal{F}_\sigma \right] \geqslant E^{P'}\left[A_\tau | \, \mathcal{F}_\sigma \right] \quad P'\text{-a.s.}$$

By relation (5.2.5), this implies

$$\operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(\sigma;P)} E^{P'}[v|\mathcal{F}_\sigma] \geqslant \operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(\sigma;P)} E^{P'}\left[A_\tau | \, \mathcal{F}_\sigma \right] = \mathcal{E}_\sigma(A_\tau) \quad P\text{-a.s.}$$

We note that v is a version of $E^{P'}[v|\mathcal{F}_\sigma]$ under each $P' \in \mathcal{P}(\sigma;P)$, i.e.

$$v = E^{P'}[v|\mathcal{F}_\sigma] \qquad P' - \text{a.s. for all } P' \in \mathcal{P}(\sigma;P).$$

Consequently,

$$v \geqslant \mathcal{E}_\sigma(A_\tau) \quad P\text{-a.s.}$$

It follows that

$$\text{ess inf }^P\{{}^{(P)}D^\tau_\sigma\} \geqslant \mathcal{E}_\sigma(A_\tau) \quad P\text{-a.s.},$$

which shows relation (5.2.14).

We now prove that for all $[0,T]$-valued $\mathbb{F}$-stopping times σ and τ such that $\sigma \leqslant \tau$, we have $\mathcal{E}_\sigma(A_\tau) \in D^\tau_\sigma$. In this way we get equalities

$$\mathcal{E}_\sigma(A_\tau) = \text{ess inf }^P\{{}^{(P)}D^\tau_\sigma\} = \text{ess inf }^P\{D^\tau_\sigma\} \quad P\text{-a.s.}$$

For the sake of simplicity, we show only the case of $\tau = T$, since the proof is the same for a generic $[0,T]$-valued $\mathbb{F}$-stopping time τ. Since $\mathcal{E}(A_T) < \infty$ holds by assumption, we obtain

$$\sup_{P \in \mathcal{P}} E^P[|\mathcal{E}_t(A_T)|] < \infty \qquad \text{for all } t \in [0,T],$$

as shown in Step 1 of the proof of Theorem 2.3 of [80]. Hence, the process Y with $Y_t = \mathcal{E}_t(A_T)$, P-a.s. for all $P \in \mathcal{P}$ and for all $t \in [0,T]$, satisfies

$$\sup_{P \in \mathcal{P}} E^P[|Y_t|] < \infty \quad \text{for all } t \in [0,T].$$

We can thus apply directly Theorem 5.2.14 to the filtered space $(\Omega, \mathbb{F}^{\mathcal{P}})$ and to the càdlàg process Y, which is a $(P, \mathbb{F}^P_+)$-supermartingale for every $P \in \mathcal{P}$, by (5.2.6) and Remark 2.1 of [86]. This yields that there exists an $\mathbb{F}^{\mathcal{P}}$-predictable process $\delta \in L(S, \mathbb{F}^{\mathcal{P}}, \mathcal{P})$ such that

$$Y - {}^{(P)}\!\!\int_0^\cdot \delta_u \mathrm{d}S_u \text{ is nonincreasing } P\text{-a.s. for all } P \in \mathcal{P}.$$

As a consequence, for all $[0,T]$-valued $\mathbb{F}$-stopping time σ,

$$\begin{aligned}\mathcal{E}(A_T) = Y_0 \geqslant Y_\sigma - {}^{(P)}\!\!\int_0^\sigma \delta_u \mathrm{d}S_u \\ =& \mathcal{E}_\sigma(A_T) - {}^{(P)}\!\!\int_0^\sigma \delta_u \mathrm{d}S_u \\ \geqslant& A_T - {}^{(P)}\!\!\int_0^T \delta_u \mathrm{d}S_u \quad P\text{-a.s. for all } P \in \mathcal{P}.\end{aligned}$$

The first inequality yields that $\delta \in \Delta$, i.e. the process ${}^{(P)}\!\int \delta \mathrm{d}S$ is a $(P, \mathbb{F}^P_+)$-supermartingale for all $P \in \mathcal{P}$, since the $(P, \mathbb{F}^P_+)$-sigma martingale ${}^{(P)}\!\int \delta \mathrm{d}S$ is P-a.s. bounded from below by $(\mathcal{E}_t(A_T) - \mathcal{E}(A_T))_{t\in[0,T]}$. It follows from the second inequality that

$$\mathcal{E}_\sigma(A_T) + {}^{(P)}\!\!\int_\sigma^T \delta_u \mathrm{d}S_u \geqslant A_T \quad P\text{-a.s. for all } P \in \mathcal{P}.$$

Hence, we have

$$\mathcal{E}_\sigma(A_T) \in D^T_\sigma.$$

□

Theorem 5.2.21 can be considered as an extension of Theorem 3.4 of [13] to the case of payment streams and a dynamic version of Theorem 3.2 of [86]. It includes as special cases the static robust superhedging dualities in e.g. [100], [43] and [9]. However, Theorem 5.2.21 alone do not directly solve our superhedging problem for payment streams. Indeed, the robust global superhedging price of A as defined in Definition 5.2.18 may be higher than $\mathcal{E}(A_T)$ and the robust local superhedging price of A on the interval $[\sigma,\tau]$ as defined in Definition 5.2.19 may be higher than $\mathcal{E}_\sigma(A_\tau - A_\sigma)$. Nevertheless, in the following we will see that equality holds. For every $[0,T]$-valued $\mathbb{F}$-stopping times σ,τ such that $\sigma \leqslant \tau$, we define the following set:

$$\mathcal{C}^\tau_\sigma := \Big\{ \delta \in \Delta : \mathcal{E}_{\sigma_1}(A_\tau) + {}^{(P)}\!\!\int_{\sigma_1}^{\sigma_2} \delta_u \mathrm{d}S_u \geqslant A_{\sigma_2}\ P\text{-a.s. for all } [0,T]\text{-valued} \\ \mathbb{F}\text{-stopping times } \sigma_1,\sigma_2 \text{ such that } \sigma \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \tau, \quad \text{for all } P \in \mathcal{P} \Big\}.$$

If $\sigma, \sigma', \tau, \tau'$ are $[0,T]$-valued $\mathbb{F}$-stopping times such that $\sigma \leqslant \sigma' \leqslant \tau \leqslant \tau'$, it clearly holds by definition

$$\mathcal{C}^T_0 \subseteq \mathcal{C}^{\tau'}_\sigma \subseteq \mathcal{C}^\tau_\sigma \subseteq \mathcal{C}^\tau_{\sigma'}. \tag{5.2.15}$$

The solution of both pricing and hedging problems for a payment stream is given in the following theorem.

Theorem 5.2.22. *Under the same assumptions as in Theorem 5.2.21, the following statements hold:*

1. *the set $\mathcal{C}^T_0$ is not empty;*

2. *the robust global superhedging price of A coincides with $\mathcal{E}(A_T)$ and the robust local superhedging price of A on the interval $[\sigma,\tau]$ coincides with $\mathcal{E}_\sigma(A_\tau - A_\sigma)$;*
3. *the infimum value in (5.2.7) and (5.2.8) is attained, that is, optimal superhedging strategies exist.*

Proof. By (5.2.13), every set $\mathcal{C}_\sigma^\tau$ has the following equivalent representation

$$\mathcal{C}_\sigma^\tau = \Bigg\{ \delta \in \Delta : \mathcal{E}_{\sigma_1}(A_\tau - A_{\sigma_1}) + {}^{(P)}\!\!\int_{\sigma_1}^{\sigma_2} \delta_u \mathrm{d}S_u \geqslant A_{\sigma_2} - A_{\sigma_1} \;\; P\text{-a.s. for all } [0,T]\text{-valued}$$
$$\mathbb{F}\text{-stopping times } \sigma_1, \sigma_2 \text{ such that } \sigma \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \tau, \text{ for all } P \in \mathcal{P} \Bigg\}.$$

This yields that the second and the third point follow from the first point together with dualities (5.2.9), (5.2.11) and inclusion (5.2.15).

Now we concentrate on the first point. According to the proof of Theorem 5.2.21, there exists an $\mathbb{F}^{\mathcal{P}}$-predictable process $\delta \in L(S, \mathbb{F}^{\mathcal{P}}, \mathcal{P})$ such that for every $[0,T]$-valued $\mathbb{F}$-stopping time σ we have

$$\mathcal{E}_\sigma(A_T) + {}^{(P)}\!\!\int_{\sigma}^{T} \delta_u \mathrm{d}S_u \geqslant A_T \qquad P\text{-a.s. for all } P \in \mathcal{P}.$$

In particular, if σ' is another $[0,T]$-valued $\mathbb{F}$-stopping time such that $\sigma \leqslant \sigma'$, it holds that

$$\mathcal{E}_\sigma(A_T) + {}^{(P)}\!\!\int_{\sigma}^{\sigma'} \delta_u \mathrm{d}S_u + {}^{(P)}\!\!\int_{\sigma'}^{T} \delta_u \mathrm{d}S_u \geqslant A_T \qquad P\text{-a.s. for all } P \in \mathcal{P}.$$

Since ${}^{(P)}\!\!\int \delta \mathrm{d}S$ is a $(P, \mathbb{F}_+^P)$-supermartingale, we can apply conditional expectation on both hand sides and obtain

$$\begin{aligned}
&E^{\mathcal{P}}\left[\mathcal{E}_\sigma(A_T) + {}^{(P)}\!\!\int_{\sigma}^{\sigma'} \delta_u \mathrm{d}S_u + {}^{(P)}\!\!\int_{\sigma'}^{T} \delta_u \mathrm{d}S_u \,\middle|\, \mathcal{F}_{\sigma'+}^P \right] \\
=&\mathcal{E}_\sigma(A_T) + {}^{(P)}\!\!\int_{\sigma}^{\sigma'} \delta_u \mathrm{d}S_u \\
\geqslant& E^P[A_T | \mathcal{F}_{\sigma'+}^P] \qquad P\text{-a.s. for all } P \in \mathcal{P}.
\end{aligned}$$

We note that since A is nondecreasing, we get

$$E^P[A_T | \mathcal{F}_{\sigma'+}^P] - A_{\sigma'} = E^P[A_T - A_{\sigma'} | \mathcal{F}_{\sigma'+}^P] \geqslant 0 \qquad P\text{-a.s. for all } P \in \mathcal{P}.$$

This yields

$$\mathcal{E}_\sigma(A_T) + {}^{(P)}\!\!\int_{\sigma}^{\sigma'} \delta_u \mathrm{d}S_u \geqslant A_{\sigma'} \qquad P\text{-a.s. for all } P \in \mathcal{P}.$$

In this way, we show that the set $\mathcal{C}_0^T$ is not empty. □

We emphasis that Theorem 5.2.21 and Theorem 5.2.22 can be carried out without changes also in the case without model uncertainty, i.e. when we consider a single prior P which is a sigma (or local) martingale measure for S.

5.3 Reduce-form setting under model uncertainty

In the current section we introduce the reduced-form setting under model uncertainty. The previous framework with $\Omega = D_0(\mathbb{R}_+, \mathbb{R}^d)$ or $\Omega = C_0(\mathbb{R}_+, \mathbb{R}^d)$ endowed with the natural filtration $\mathbb{F}$ of the canonical process B does not allow to treat more general filtrations, as emphasized in [2]. In [2], a solution for the case of initial enlargement, but the case of progressive enlargement of filtration remains open. This last problem is particularly relevant in life-insurance modeling[2], when we want to model a decease event which occurs as a surprise and is itself not observable under the reference filtration $\mathbb{F}$, but admits an $\mathbb{F}$-adapted intensity process.

To this end, we follow the canonical construction in Section 6.5 of [24] in the classic context of a single prior, and introduce a random time $\tilde{\tau}$, which is not an $\mathbb{F}$-stopping time but has an $\mathbb{F}$-progressively measurable intensity process μ, to represent a totally unexpected decease time under model uncertainty. The structure constructed in this way is a special case of the framework in Chapter 2, when $n = 1$ and $\tilde{\tau} = \tau_0^1$. For the sake of clarity, the notations are slightly different from the ones in Chapter 2 in order to emphasize the product space structure which we describe in the following section.

5.3.1 Space construction

We keep the same setting and notations as in Section 5.2. Let $\hat{\Omega}$ denote an additional Polish space equipped with its Borel σ-algebra $\mathcal{B}(\hat{\Omega})$. We consider the product measurable space

$$(\tilde{\Omega}, \mathcal{G}) := (\Omega \times \hat{\Omega}, \mathcal{B}(\Omega) \otimes \mathcal{B}(\hat{\Omega}))^3,$$

and denote $\tilde{\omega} = (\omega, \hat{\omega})$ for $\omega \in \Omega$ and $\hat{\omega} \in \hat{\Omega}$. The following standard conventions are used on the product space $(\tilde{\Omega}, \mathcal{G})$. For a function or process X on $(\Omega, \mathcal{B}(\Omega))$, we consider its natural immersion into the product space, i.e. $X(\tilde{\omega}) := X(\omega)$ for all $\omega \in \Omega$, similarly for $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$. For a sub-$\sigma$-algebra $\mathcal{A}$ of $\mathcal{B}(\Omega)$, we consider its natural extension $\mathcal{A} \otimes \{\emptyset, \hat{\Omega}\}$ as a sub-σ-algebra of $\mathcal{G}$ on the product space, similarly for sub-σ-algebras of $\mathcal{B}(\hat{\Omega})$. When there is no ambiguity, $\mathcal{A} \otimes \{\emptyset, \hat{\Omega}\}$ is still denoted by $\mathcal{A}$ in order to avoid cumbersome notations. The following lemma is trivial.

Lemma 5.3.1. *Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{B}(\Omega)$, a random variable X on the product space $(\tilde{\Omega}, \mathcal{B}(\tilde{\Omega}))$ is $\mathcal{A}$-measurable if and only if $X(\tilde{\omega}) = X(\omega)$ for every $\tilde{\omega} = (\omega, \omega^*) \in \tilde{\Omega}$. Similarly for the space $(\Omega^*, \mathcal{B}(\Omega^*))$.*

On $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$ we consider a probability measure $\hat{P}$ such that $(\hat{\Omega}, \mathcal{B}(\hat{\Omega})), \hat{P})$ is an atomless probability space, i.e. there exists a random variable with an absolutely continuous distribution. Let ξ be a Borel-measurable surjective random variable

$$\xi : (\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{P}) \to ([0,1], \mathcal{B}([0,1])),$$

with uniform distribution, i.e.

$$\xi \sim U([0,1]).$$

Without loss of generality we set $\mathcal{B}(\hat{\Omega}) = \sigma(\xi)$.

[2]The same framework can be applied to credit risk modeling as well.

[3]We note that $\mathcal{B}(\Omega) \otimes \mathcal{B}(\Omega^*) = \mathcal{B}(\Omega \times \Omega^*)$ since we have a countable topology base.

Remark 5.3.2. *We note that the space* $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{P})$ *has a canonical form*

$$([0,1], \mathcal{B}([0,1]), U([0,1]))\,,$$

with ξ *the identity function on* $[0,1]$.

Let $\mathcal{P}(\tilde{\Omega})$ denotes the set of all probability measures on $(\tilde{\Omega}, \mathcal{G})$. We consider the following family of probability measures

$$\tilde{\mathcal{P}} := \left\{ \tilde{P} \in \mathcal{P}(\tilde{\Omega}) : \tilde{P} = P \otimes \hat{P},\ P \in \mathcal{P} \right\}. \tag{5.3.1}$$

We use the corresponding notations of Section 5.2 and denote the associated upper expectation by $\tilde{\mathcal{E}}$, i.e.

$$\tilde{\mathcal{E}}(\tilde{X}) := \sup_{\tilde{P} \in \tilde{\mathcal{P}}} E^{\tilde{P}}[\tilde{X}], \qquad \tilde{X} \in L^0(\tilde{\Omega}), \tag{5.3.2}$$

where for every $\tilde{P} \in \tilde{\mathcal{P}}$, we set $E^{\tilde{P}}[\tilde{X}] := E^{\tilde{P}}[\tilde{X}^+] - E^{\tilde{P}}[\tilde{X}^-]$ if $E^{\tilde{P}}[\tilde{X}^+]$ or $E^{\tilde{P}}[\tilde{X}^-]$ is finite, and $E^{\tilde{P}}[\tilde{X}] := -\infty$ if $E^{\tilde{P}}[\tilde{X}^+] = E^{\tilde{P}}[\tilde{X}^-] = +\infty$.

Let $\Gamma := (\Gamma_t)_{t \geqslant 0}$ be a real-valued, $\mathbb{F}$-adapted, continuous and increasing process on $(\Omega, \mathcal{B}(\Omega))$, such that $\Gamma_0 = 0$ and $\Gamma_\infty = +\infty$. In particular, Γ has the representation

$$\Gamma_t := \int_0^t \mu_s \mathrm{d}s, \qquad t \geqslant 0,$$

where $\mu := (\mu_t)_{t \geqslant 0}$ is a nonnegative $\mathbb{F}$-progressively measurable process such that for all $t \geqslant 0$ and for all $\omega \in \Omega$,

$$\int_0^t |\mu_s|(\omega) \mathrm{d}s < \infty.$$

We set

$$\tilde{\tau} := \inf\{t \geqslant 0 : e^{-\Gamma_t} \leqslant \xi\} = \inf\{t \geqslant 0 : \Gamma_t \leqslant -\ln \xi\}$$

on $\tilde{\Omega} = \Omega \times \hat{\Omega}$, with the convention $\inf \emptyset = \infty$.

Example 5.3.3. *In the case of* $\Omega = C_0([0,T], \mathbb{R})$, *the following is a simple example of dynamics of* μ

$$\mu_t = \mu_0 \exp(kB_t), \qquad t \geqslant 0, \tag{5.3.3}$$

where $\mu_0 \geqslant 0$, $k \in \mathbb{R}$ *and* $B := (B_t)_{t \in [0,T]}$ *denotes the G-Brownian motion. This is a reasonable setting when* μ *is interpreted as the mortality intensity in the context of life insurance, since it is well known that mortality intensity has exponential behavior, see e.g. [72], [104] and [18]. According to the results in [76], the process (5.3.3) can be expressed as the solution of the following SDE driven by G-Brownian motion*

$$\mu_t = \mu_0 + \int_0^t k \mu_s \mathrm{d}B_s, \qquad t \geqslant 0.$$

Remark 5.3.4. *As an immediate consequence of the above assumptions, we have that* $\tilde{\tau}(\omega, \cdot)$ *is a surjective function on* $\mathbb{R}_+$ *for every fixed* $\omega \in \Omega$.

Lemma 5.3.5. *For all* $t \geqslant 0$*, we have* $\{\tilde{\tau} \leqslant t\} = \{e^{-\Gamma_t} \leqslant \xi\}$.

Proof. The inclusion $\{e^{-\Gamma_t} \leqslant \xi\} \subseteq \{\tilde{\tau} \leqslant t\}$ always holds. The other inclusion follows immediately from

$$\tilde{\tau} = \min\{s \geqslant 0 : e^{-\Gamma_s} \leqslant \xi\},$$

since Γ is continuous. □

Corollary 5.3.6. $\tilde{\tau}$ *is* $\mathcal{B}(\tilde{\Omega})$*-measurable, that is* $\tilde{\tau}$ *is a random time on* $(\tilde{\Omega}, \mathcal{B}(\tilde{\Omega}))$.

Lemma 5.3.7. *For every* $t \in [0, T]$ *and for every* $\tilde{P}$, $\tilde{P}' \in \tilde{\mathcal{P}}$*, with* $\tilde{P} = P \otimes \hat{P}$, $\tilde{P}' = P' \otimes \hat{P}$*, we have* $\tilde{P} = \tilde{P}'$ *on* $\mathcal{G}_t$ *if and only if* $P = P'$ *on* $\mathcal{F}_t$.

Proof. One implication is immediate. It is sufficient to show that $P = P'$ on $\mathcal{F}_t$ implies $\tilde{P} = \tilde{P}'$ on $\mathcal{G}_t$. We note that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ by definition, then $\tilde{P} = \tilde{P}'$ on $\mathcal{G}_t$ is equivalent to $P \otimes \hat{P} = P' \otimes \hat{P}$ on $\mathcal{H}_t$, which follows from the definition of the σ-algebra $\mathcal{H}_t$ and Lemma 5.3.5. □

Under every $\tilde{P} \in \tilde{\mathcal{P}}$, we denote the $\tilde{P}$-hazard process by $\Gamma^{\tilde{P}} := (\Gamma_t^{\tilde{P}})_{t \geqslant 0}$, i.e.

$$\Gamma_t^{\tilde{P}} := -\ln \tilde{P}\left(\tilde{\tau} > t \middle| \mathcal{F}_t\right), \qquad t \geqslant 0.$$

We state the following proposition which is a natural but important consequence of the above construction.

Proposition 5.3.8. *The process* Γ *is a* $\tilde{P}$*-a.s. version of* $\tilde{P}$*-hazard process* $\Gamma^{\tilde{P}}$ *for each* $\tilde{P} \in \tilde{\mathcal{P}}$.

Proof. By Lemma 5.3.5, we have

$$\{\tilde{\tau} > t\} = \{e^{-\Gamma_t} > \xi\} \quad \text{ for all } t \geqslant 0.$$

Thus, for every $t \geqslant 0$ and for every $\tilde{P} \in \tilde{\mathcal{P}}$ with $\tilde{P} = P \otimes \hat{P}$, it holds

$$\begin{aligned} e^{-\Gamma_t^{\tilde{P}}(\omega)} &= \tilde{P}\left(\tilde{\tau} > t \middle| \mathcal{F}_t\right)(\omega) = \tilde{P}\left(e^{-\Gamma_t} > \xi \middle| \mathcal{F}_t\right)(\omega) \\ &\overset{(i)}{=} \tilde{P}\left(e^{-x} > \xi\right)\Big|_{x=\Gamma_t(\omega)} = \hat{P}\left(e^{-x} > \xi\right)\Big|_{x=\Gamma_t(\omega)} \\ &\overset{(ii)}{=} e^{-x}\Big|_{x=\Gamma_t(\omega)} \\ &= e^{-\Gamma_t(\omega)} \quad \text{ for } \tilde{P}\text{-a.e. } \omega, \end{aligned}$$

where equality (i) is a consequence of the independence between ξ and $\mathcal{F}_t$ under each $\tilde{P} \in \tilde{\mathcal{P}}$, and equality (ii) is due to the fact that ξ has uniform distribution on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. Furthermore, the continuity of Γ yields

$$\Gamma^{\tilde{P}} = \Gamma \quad \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}},$$

which concludes the proof. □

On the product space $\tilde{\Omega}$, we consider the filtration $\mathbb{H} := (\mathcal{H}_t)_{t \geqslant 0}$ generated by the single jump process $H := (H_t)_{t \geqslant 0}$ with

$$H_t := \mathbf{1}_{\{\tilde{\tau} \leqslant t\}}, \qquad t \geqslant 0,$$

and the enlarged filtration $\mathbb{G} := (\mathcal{G}_t)_{t \geqslant 0}$ with $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$, $t \geqslant 0$. In particular, we have

$$\mathcal{G} = \mathcal{F}_\infty \otimes \sigma(\xi) = \mathcal{H}_\infty \vee \mathcal{F}_\infty = \sigma(\tilde{\tau}) \vee \mathcal{F}_\infty.$$

We recall that by construction $\tilde{\tau}$ is an $\mathbb{H}$-stopping time as well as a $\mathbb{G}$-stopping time, but not an $\mathbb{F}$-stopping time. As before, the filtration $\mathbb{F}$ represents the reference information flow including financial market information, and the filtration $\mathbb{G}$ represents the minimal information flow of the extended market including accident information. Similarly to Section 5.2, for every $\tilde{P} \in P(\tilde{\Omega})$ we denote by $\mathbb{G}^*$, $\mathbb{G}^{\tilde{P}}$ and $\mathbb{G}_+^{\tilde{P}}$ the corresponding enlargements of the raw filtration $\mathbb{G}$. As in (5.2.1), we have

$$\mathcal{G}_t \subseteq \mathcal{G}_t^* \subseteq \mathcal{G}_t^{\tilde{P}} \subseteq \mathcal{G}_{t+}^{\tilde{P}}, \qquad t \geqslant 0, \qquad \tilde{P} \in \tilde{\mathcal{P}}.$$

We introduce also the following σ-algebras which will be used in the sequel.

$$\mathcal{G}^P := \mathcal{G} \vee \mathcal{N}_\infty^P, \qquad P \in \mathcal{P},$$

and

$$\mathcal{G}^{\mathcal{P}} := \mathcal{G} \vee \mathcal{N}_\infty^{\mathcal{P}}.$$

Remark 5.3.9. *We emphasize that the filtration $\mathbb{H}$ is automatically right continuous since it is generated by a right continuous jump process. See e.g. Theorem 25 Chap. I Sect. 3 of [102].*

Lemma 5.3.10. *For every $t \in [0,T]$ and for every $\tilde{P}$, $\tilde{P}' \in \tilde{\mathcal{P}}$, with $\tilde{P} = P \otimes P^*$, $\tilde{P}' = P' \otimes P^*$, we have $\tilde{P} = \tilde{P}'$ on $\mathcal{G}_t$ if and only if $P = P'$ on $\mathcal{F}_t$.*

Proof. One implication is immediate. It is sufficient to show that $P = P'$ on $\mathcal{F}_t$ implies $\tilde{P} = \tilde{P}'$ on $\mathcal{G}_t$. We note that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ by definition, then $\tilde{P} = \tilde{P}'$ on $\mathcal{G}_t$ is equivalent to $P \otimes P^* = P' \otimes P^*$ on $\mathcal{H}_t$, which follows from the definition of the σ-algebra $\mathcal{H}_t$ and Lemma 5.3.5. □

5.3.2 $(\tilde{\mathcal{P}}, \mathbb{G})$-conditional expectation

Motivated by the results in Section 5.2.3, we give in this section a construction of sublinear conditional expectations with respect to the enlarged filtration $\mathbb{G}$ and the family of probability $\tilde{\mathcal{P}}$ introduced in (5.3.1). We denote these by $(\tilde{\mathcal{E}}_t)_{t \geqslant 0}$ and call them $(\tilde{\mathcal{P}}, \mathbb{G})$-conditional expectation. As shown in e.g. [109], [31], [110], [111] and [87], the family $(\tilde{\mathcal{E}}_t)_{t \geqslant 0}$ should satisfy the following necessary consistency condition: for every $t \geqslant 0$ and $\mathcal{G}$-measurable function $\tilde{X}$ on $\tilde{\Omega}$,

$$\tilde{\mathcal{E}}_t(\tilde{X}) = \operatorname*{ess\,sup}^{\tilde{P}}_{\tilde{P}' \in \tilde{\mathcal{P}}(t;\tilde{P})} E^{\tilde{P}'}[\tilde{X}|\mathcal{G}_t] \qquad \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}}, \tag{5.3.4}$$

where $\tilde{\mathcal{P}}(t;\tilde{P}) := \left\{\tilde{P}' \in \tilde{\mathcal{P}} : \tilde{P}' = \tilde{P} \text{ on } \mathcal{G}_t\right\}$. We stress that this is not possible to achieve by using exactly the same method proposed in [87] and summarized in Section 5.2.1, even if we choose $\hat{\Omega} = D_0(\mathbb{R}+, \mathbb{R}^d)$ or $\hat{\Omega} = C_0(\mathbb{R}+, \mathbb{R}^d)$, which was the direction tried by the thesis' author in an early stage of the paper [19]. The method in [87] is indeed based on some special properties of the natural filtration of the canonical process, e.g. Galmarino's test for stopping times, which holds for the product filtration $\mathbb{F} \otimes \mathbb{F}$ on the product space $\tilde{\Omega}$, but not for the nontrivial filtration $\mathbb{G}$. However, we are still able to extend the results of [87] to the setting of Section 5.3.1, and construct a consistent $(\tilde{\mathcal{P}}, \mathbb{G})$-conditional expectation. Besides, we are able to prove, as in [90], that the family $(\tilde{\mathcal{E}}_t(\tilde{X}))_{t\geqslant 0}$ satisfies a weak form of time-consistency, called also dynamic programming principle or tower property, i.e.

$$\tilde{\mathcal{E}}_s(\tilde{\mathcal{E}}_t(\tilde{X})) \geqslant \tilde{\mathcal{E}}_s(\tilde{X}) \quad \text{for all } 0 \leqslant s \leqslant t \ \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}}. \tag{5.3.5}$$

From an interpretation point of view, if we use $(\tilde{\mathcal{E}}_t)_{t\geqslant 0}$ as pricing functional, the weak tower property (5.3.5) can be means: valuation of an evaluated future price is more conservative than direct valuation of the price. Counterexample 5.3.26 shows that the classic tower property does not hold in the general case. Nevertheless, in all cases of practical interest, we are able to prove the classic tower property as we will show in Section 5.4. For the sake of simplicity, from this section onwards we focus on deterministic times.

Let $\mathcal{G}^P := \mathcal{G} \vee \mathcal{N}^P_\infty$, $P \in \mathcal{P}$, and $\mathcal{G}^{\mathcal{P}} := \mathcal{G} \vee \mathcal{N}^{\mathcal{P}}_\infty$. We introduce the following sets

$$\begin{aligned} L^1_{\tilde{P}}(\tilde{\Omega}) :=\{ &\tilde{X} \mid \tilde{X} : (\tilde{\Omega}, \mathcal{G}^P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ measurable function such that} \\ &E^{\tilde{P}}[|\tilde{X}|] < \infty\}, \end{aligned}$$

for every $\tilde{P} \in \tilde{\mathcal{P}}$, and

$$\begin{aligned} L^1(\tilde{\Omega}) := \{ &\tilde{X} \mid \tilde{X} : (\tilde{\Omega}, \mathcal{G}^{\mathcal{P}}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ measurable function such that} \\ &\tilde{\mathcal{E}}(|\tilde{X}|) < \infty\Big\}, \end{aligned}$$

where $\tilde{\mathcal{E}}$ is the upper expectation defined in (5.3.2). We stress that in the above definitions we only take into account $(\Omega, \mathcal{G}^P)$-measurable (or $(\Omega, \mathcal{G}^{\mathcal{P}})$-measurable resp.) functions, and not $(\Omega, \mathcal{G}^{\tilde{P}})$-measurable (or $(\Omega, \mathcal{G}^{\tilde{\mathcal{P}}})$-measurable resp.) functions. The reason is explained in Remark 5.3.13. Given $t \geqslant 0$, we have the following decomposition for every real-valued function $\tilde{X}$ on $\tilde{\Omega}$

$$\tilde{X} = \mathbf{1}_{\{\tilde{\tau}\leqslant t\}}\tilde{X} + \mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}.$$

Corollary 5.1.2 of [24], which holds without the usual conditions on the filtrations, together with Proposition 5.3.8 yields that if $\tilde{X} \in L^1(\tilde{\Omega})$, then for every $\tilde{P} \in \tilde{\mathcal{P}}$,

$$E^{\tilde{P}}[\tilde{X}|\mathcal{G}_t] = \mathbf{1}_{\{\tilde{\tau}\leqslant t\}} E^{\tilde{P}}[\tilde{X}|\sigma(\tilde{\tau}) \vee \mathcal{F}_t] + \mathbf{1}_{\{\tilde{\tau}>t\}} e^{\Gamma_t} E^{\tilde{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}|\mathcal{F}_t] \quad \tilde{P}\text{-a.s.} \tag{5.3.6}$$

We aim to find a representation of (5.3.6) where the right-hand side is reduced to conditional expectations restricted to Ω. This is particularly important for the definition of conditional expectation on $\tilde{\Omega}$. The following Lemma gives a solution of the problem for the second term on the right-hand side of (5.3.6). For the sake of simplicity, we denote with a slight abuse of notation

$$E^{\hat{P}}[\tilde{X}](\omega) := \int_{\hat{\Omega}} \tilde{X}(\omega, \hat{\omega}) \hat{P}(\mathrm{d}\hat{\omega}), \qquad \omega \in \Omega. \tag{5.3.7}$$

Lemma 5.3.11. *If $t \geqslant 0$, $\tilde{P} = P \otimes \hat{P}$ and $\tilde{X} \in L^1_{\tilde{P}}(\tilde{\Omega})$, then*

$$E^{\tilde{P}}[\tilde{X}|\mathcal{F}_t] = E^P[E^{\hat{P}}[\tilde{X}]|\mathcal{F}_t] \qquad \tilde{P}\text{-a.s.}$$

Proof. Let $\tilde{X} \in L^1_{\tilde{P}}(\tilde{\Omega})$. It is enough to see that for any $A \in \mathcal{F}_t$, the Fubini–Tonelli theorem yields

$$\begin{aligned}\int_{A \times \hat{\Omega}} \tilde{X}(\omega, \hat{\omega}) \tilde{P}(\mathrm{d}(\omega, \hat{\omega})) &= \int_A \int_{\hat{\Omega}} \tilde{X}(\omega, \hat{\omega}) \hat{P}(\mathrm{d}\hat{\omega}) P(\mathrm{d}\omega) \\ &= \int_A E^{\hat{P}}[\tilde{X}](\omega) P(\mathrm{d}\omega) \\ &= \int_{A \times \hat{\Omega}} E^P[E^{\hat{P}}[\tilde{X}]|\mathcal{F}_t](\omega) \tilde{P}(\mathrm{d}(\omega, \hat{\omega})),\end{aligned}$$

where the notation is introduced in (5.3.7). □

Now we concentrate on the first term on the right-hand side of (5.3.6).

Lemma 5.3.12. *Let $t \in \overline{\mathbb{R}}_+$ and $\tilde{X}$ be a real-valued $\sigma(\tilde{\tau}) \vee \mathcal{F}_t$-measurable function. There exists a unique measurable function*

$$\varphi : (\mathbb{R}_+ \times \Omega\ ,\ \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_t) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})),$$

such that

$$\tilde{X}(\omega, \hat{\omega}) = \varphi(\tilde{\tau}(\omega, \hat{\omega}), \omega), \qquad (\omega, \hat{\omega}) \in \tilde{\Omega}. \tag{5.3.8}$$

Proof. If φ satisfies (5.3.8), its uniqueness follows directly from the surjectivity of $\tilde{\tau}$ for every fixed $\omega \in \Omega$, see Remark 5.3.4. Indeed, if φ and ψ are two functions with

$$\varphi(\tilde{\tau}(\omega, \hat{\omega}), \omega) = \psi(\tilde{\tau}(\omega, \hat{\omega}), \omega) \qquad \text{for all } (\omega, \hat{\omega}) \in \tilde{\Omega},$$

then for every $(x, \omega) \in \mathbb{R}_+ \times \Omega$, it follows from the surjectivity of $\tilde{\tau}$ for every fixed $\omega \in \Omega$ that there is an $\hat{\omega} \in \hat{\Omega}$ such that $\tau(\omega, \hat{\omega}) = x$. Hence,

$$\varphi(x, \omega) = \psi(x, \omega) \qquad \text{for all } (x, \omega) \in \mathbb{R}_+ \times \Omega.$$

Now we consider the set

$$E = \{\tilde{X} \mid (\tilde{\Omega}, \sigma(\tilde{\tau}) \vee \mathcal{F}_t) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})),\ \tilde{X} \text{ of the form (5.3.8)}\},$$

and show that it includes a monotone class. Clearly, the set E contains all constants and is closed under linear operations. Moreover, all indicator functions of a π-system which generates $\sigma(\tilde{\tau}) \vee \mathcal{F}_t$ belong to E. Let $(\tilde{X}_n)_{n \in \mathbb{N}}$ be a sequence in E such that

$$\tilde{X}_n(\tilde{\omega}) \uparrow \tilde{X}(\tilde{\omega}) \qquad \text{for all } \tilde{\omega} \in \tilde{\Omega},$$

where $\tilde{X}$ is a bounded function. For each $n \in \mathbb{N}$, we have

$$\tilde{X}_n(\omega,\hat{\omega}) = \varphi_n(\tilde{\tau}(\omega,\hat{\omega}),\omega) \qquad \text{for all } (\omega,\hat{\omega}) \in \tilde{\Omega},$$

where φ_n is a measurable function

$$\varphi_n : (\mathbb{R}_+ \times \Omega\ ,\ \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_t) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})).$$

It follows from Remark 5.3.4 and the boundedness of $\tilde{X}$ that the function

$$\varphi(z,\omega) := \lim_{n\to\infty} \varphi_n(z,\omega), \qquad z \in \mathbb{R}_+,\ \omega \in \Omega, \tag{5.3.9}$$

is well defined and finite. In particular, φ is $(\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_t)$-measurable as well. If we apply again Remark 5.3.4, we can represent $\tilde{X}$ by

$$\tilde{X}(\omega,\hat{\omega}) = \varphi(\tilde{\tau}(\omega,\hat{\omega}),\omega), \qquad (\omega,\hat{\omega}) \in \tilde{\Omega}.$$

Thus, X belongs to E as well. The Monotone Class theorem yields that the set E contains all bounded $\sigma(\tilde{\tau}) \vee \mathcal{F}_t$-measurable functions.

Furthermore, we note that every nonnegative $\sigma(\tilde{\tau}) \vee \mathcal{F}_t$-measurable function $\tilde{X}$ is the pointwise limit of a nondecreasing sequence of simple functions, that is, there exists a sequence of simple functions $(\tilde{X}_n)_{n\in\mathbb{N}}$ such that

$$\tilde{X}_n(\tilde{\omega}) \uparrow \tilde{X}(\tilde{\omega}) \qquad \text{for all } \tilde{\omega} \in \tilde{\Omega}.$$

In particular, by the argument above, if

$$\tilde{X}_n(\omega,\hat{\omega}) = \varphi_n(\tilde{\tau}(\omega,\hat{\omega}),\omega), \qquad (\omega,\hat{\omega}) \in \tilde{\Omega},$$

and we define φ as the pointwise limit of $(\varphi_n)_{n\in\mathbb{N}}$ as in (5.3.9), it follows that all nonnegative $\sigma(\tilde{\tau}) \vee \mathcal{F}_t$-measurable functions have representation (5.3.8). The results can be easily extended to all $\sigma(\tilde{\tau}) \vee \mathcal{F}_t$-measurable functions, since $\tilde{X} = \tilde{X}^+ + \tilde{X}^-$. □

Remark 5.3.13. *We stress that Lemma 5.3.12 can be carried out without changes if $\tilde{X}$ is $\mathcal{G}^P$-measurable or $\mathcal{G}^{\mathcal{P}}$-measurable, respectively. In such case, φ is $(\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}^P_\infty)$-measurable or $(\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}^{\mathcal{P}}_\infty)$-measurable, respectively. However, the result does not hold if $\tilde{X}$ is $\mathcal{G}^{\tilde{P}}$-measurable with $\mathcal{G}^{\tilde{P}} := \mathcal{G} \vee \mathcal{N}^{\tilde{P}}_\infty$ or $\mathcal{G}^{\tilde{\mathcal{P}}}$-measurable with $\mathcal{G}^{\tilde{\mathcal{P}}} = \mathcal{G} \vee \mathcal{N}^{\tilde{\mathcal{P}}}_\infty$, respectively. The reason is similar to the case of the classic Doob-Dynkin lemma, which states that if X, Y are two real-valued measurable functions with Y $\sigma(X)$-measurable, then there is a Borel-measurable function f such that $Y = f(X)$. This representation does not hold pathwisely in general if $\sigma(X)$ is completed with null sets of some measure Q, i.e. if $\sigma(X)$ is replaced by $\sigma(X) \vee \mathcal{N}^Q$. It is sufficient to take $Y = \mathbf{1}_A$ with $A \in \mathcal{N}^Q$ as a counterexample.*

Lemma 5.3.14. *For $t \geqslant 0$ and $\tilde{P} = P \otimes \hat{P}$, if $\tilde{X} \in L^1_{\tilde{P}}(\tilde{\Omega})$, then*

$$\mathbf{1}_{\{\tilde{\tau} \leqslant t\}} E^{\tilde{P}}[\tilde{X} | \sigma(\tilde{\tau}) \vee \mathcal{F}_t] = \mathbf{1}_{\{\tilde{\tau} \leqslant t\}}\ E^P[\varphi(x,\cdot)|\mathcal{F}_t]\big|_{x=\tilde{\tau}} \qquad \tilde{P}\text{-a.s.}, \tag{5.3.10}$$

where φ is the measurable function

$$\varphi : (\mathbb{R}_+ \times \Omega\ ,\ \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}^P_\infty) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})),$$

such that

$$\tilde{X}(\omega,\hat{\omega}) = \varphi(\tilde{\tau}(\omega,\hat{\omega}),\omega), \qquad (\omega,\hat{\omega}) \in \tilde{\Omega}. \tag{5.3.11}$$

Proof. It follows from Lemma 5.3.12 and Remark 5.3.13 that there exists a unique representation (5.3.11) and the right-hand side of (5.3.10) is $\sigma(\tilde{\tau}) \vee \mathcal{F}_t$-measurable. Firstly, we show that relation (5.3.10) holds for indicator functions of a π-system, which generates $\mathcal{G} = \sigma(\tilde{\tau}) \vee \mathcal{F}_\infty$. Let $s \geqslant 0$ and $A \in \mathcal{F}_\infty$, we show

$$\mathbf{1}_{\{\tilde{\tau}\leqslant t\}} E^{\tilde{P}}[\mathbf{1}_{\{\tilde{\tau}\leqslant s\}\cap\{A\times\hat{\Omega}\}}|\sigma(\tilde{\tau})\vee\mathcal{F}_t] = \mathbf{1}_{\{\tilde{\tau}\leqslant t\}}\mathbf{1}_{\{\tilde{\tau}\leqslant s\}} E^P[\mathbf{1}_A|\mathcal{F}_t] \quad \tilde{P}\text{-a.s.}$$

Indeed, if $u \geqslant 0$ and $B \in \mathcal{F}_t$,

$$\begin{aligned}
\int_{\{\tilde{\tau}\leqslant u\}\cap\{B\times\hat{\Omega}\}} \mathbf{1}_{\{\tilde{\tau}\leqslant t\}}\mathbf{1}_{\{\tilde{\tau}\leqslant s\}}\mathbf{1}_{A\times\hat{\Omega}}\mathrm{d}\tilde{P} &= \int_{B\times\hat{\Omega}} \mathbf{1}_{\{\tilde{\tau}\leqslant t\wedge s\wedge u\}}\mathbf{1}_{A\times\hat{\Omega}}\mathrm{d}\tilde{P} \\
&= \int_{B\times\hat{\Omega}} E^{\tilde{P}}[\mathbf{1}_{\{\tilde{\tau}\leqslant t\wedge s\wedge u\}}\mathbf{1}_{A\times\hat{\Omega}}|\mathcal{F}_t]\mathrm{d}\tilde{P} \\
&= \int_{B\times\hat{\Omega}} E^{\tilde{P}}[\mathbf{1}_{\{\tilde{\tau}\leqslant t\wedge s\wedge u\}}|\mathcal{F}_t]\ E^{\tilde{P}}[\mathbf{1}_{A\times\hat{\Omega}}|\mathcal{F}_t]\mathrm{d}\tilde{P} \\
&= \int_{B\times\hat{\Omega}} E^{\tilde{P}}[\mathbf{1}_{\{\tilde{\tau}\leqslant t\wedge s\wedge u\}}|\mathcal{F}_t]\ E^{P}[\mathbf{1}_{A}|\mathcal{F}_t]\mathrm{d}\tilde{P} \\
&= \int_{B\times\hat{\Omega}} E^{\tilde{P}}[\mathbf{1}_{\{\tilde{\tau}\leqslant t\wedge s\wedge u\}}E^{P}[\mathbf{1}_{A}|\mathcal{F}_t]|\mathcal{F}_t]\mathrm{d}\tilde{P} \\
&= \int_{B\times\hat{\Omega}} \mathbf{1}_{\{\tilde{\tau}\leqslant t\wedge s\wedge u\}}E^{P}[\mathbf{1}_{A}|\mathcal{F}_t]\mathrm{d}\tilde{P} \\
&= \int_{\{\tilde{\tau}\leqslant u\}\cap\{B\times\hat{\Omega}\}} \mathbf{1}_{\{\tilde{\tau}\leqslant t\}}\mathbf{1}_{\{\tilde{\tau}\leqslant s\}}E^{P}[\mathbf{1}_{A}|\mathcal{F}_t]\mathrm{d}\tilde{P},
\end{aligned}$$

where in the third equality we make use of the $\mathcal{F}_t$-conditional independence between $\mathcal{H}_t$ and $\mathcal{F}_\infty$, see pp.166 of [24]. Lemma 5.3.12 and the conditional monotone convergence yield that the set of bounded measurable functions $\tilde{X} \in L^1_{\tilde{P}}(\tilde{\Omega})$, which satisfy relation (5.3.10), contains a monotone class. Hence, relation (5.3.10) holds for all bounded measurable functions $\tilde{X} \in L^1_{\tilde{P}}(\tilde{\Omega})$ by Monotone Class theorem. By applying conditional monotone convergence theorem to $\tilde{X}^+$ and $\tilde{X}^-$ respectively, the result can be extended to every $\tilde{X} \in L^1_{\tilde{P}}(\tilde{\Omega})$, since every nonnegative measurable function is the pointwise limit of a sequence of nonnegative and nondecreasing simple functions. □

Remark 5.3.15. *We note that Lemma 5.3.11 and Lemma 5.3.14 hold clearly also for $\tilde{X}$ which is $\mathcal{G}^{\mathcal{P}}$-measurable and nonnegative.*

The above results are summarized in the following proposition.

Proposition 5.3.16. *For $t \geqslant 0$ and $\tilde{P} = P\otimes\hat{P}$, if $\tilde{X} \in L^1_{\tilde{P}}(\tilde{\Omega})$ or $\tilde{X}$ is $\mathcal{G}^{\mathcal{P}}$-measurable and nonnegative, then*

$$E^{\tilde{P}}[\tilde{X}|\mathcal{G}_t] = \mathbf{1}_{\{\tilde{\tau}\leqslant t\}}\ E^P[\varphi(x,\cdot)|\mathcal{F}_t]\big|_{x=\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>t\}}e^{\Gamma_t}E^P[E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]|\mathcal{F}_t] \quad \tilde{P}\text{-a.s.},$$

where φ is the measurable function

$$\varphi : (\mathbb{R}_+\times\Omega\ ,\ \mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}^P_\infty) \to (\mathbb{R},\mathcal{B}(\mathbb{R})),$$

such that

$$\tilde{X}(\omega,\hat{\omega}) = \varphi(\tilde{\tau}(\omega,\hat{\omega}),\omega), \qquad (\omega,\hat{\omega})\in\tilde{\Omega}. \tag{5.3.12}$$

Proof. It suffices to apply Lemma 5.3.11, Lemma 5.3.14 and Remark 5.3.15 to decomposition (5.3.6). □

The following properties of upper semianalytic functions are useful for our main results.

Lemma 5.3.17. *Let X, Y be two Polish spaces.*

1. *If $f : X \to Y$ is Borel-measurable and $A \subseteq X$ is an analytic set, then $f(A)$ is analytic. If $B \subseteq Y$ is an analytic set, then $f^{-1}(B)$ is analytic.*

2. *If $f_n : X \to \bar{\mathbb{R}}$, $n \in \mathbb{N}$, is a sequence of upper semianalytic functions and $f_n \to f$, then f is also upper semianalytic.*

3. *If $f : X \to Y$ is Borel-measurable and $g : Y \to \bar{\mathbb{R}}$ is upper semianalytic, then the composition $g \circ f$ is also upper semianalytic. If $f : X \to Y$ is surjective and Borel-measurable and there exists a function $g : Y \to \bar{\mathbb{R}}$ such that $g \circ f$ is upper semianalytic, then g is upper semianalytic.*

4. *If f, $g : X \to \bar{\mathbb{R}}$ are two upper semianalytic functions, then the sum $f + g$ is upper semianalytic.*

5. *If $f : X \to \bar{\mathbb{R}}$ is upper semianalytic, $g : X \to \bar{\mathbb{R}}$ is Borel-measurable and $g \geqslant 0$, then the product $f \cdot g$ is upper semianalytic.*

6. *If $f : X \times Y \to \bar{\mathbb{R}}$ is upper semianalytic and $\kappa(\mathrm{d}y; x)$ is a Borel-measurable stochastic kernel on Y given X, then the function $g : X \to \bar{\mathbb{R}}$ with*

$$g(x) = \int f(x, y)\kappa(\mathrm{d}y; x), \qquad x \in X,$$

is upper semianalytic.

Proof. For points 1, 2, 4, 5 and 6, see Proposition 7.40, Lemma 7.30 and Proposition 7.48 of [10][4]. For the third point, the first implication is proved in Lemma 7.30 (3) of [10]. Hence we only have to prove the second implication. If $g \circ f$ is upper semianalytic, then for every $c \in \mathbb{R}$, the set

$$A := \{x \in X : g \circ f(x) > c\}$$

is analytic. Moreover, if we set

$$B := \{y \in Y : g(y) > c\},$$

we have $f(A) \subseteq B$. Since f is a surjective function, it holds that for all $y \in B$, there exists $x \in X$ such that $y = f(x)$ and $g(f(x)) > c$. Thus $f(A) \supseteq B$. By the first point, the set B is analytic. This yields that g is upper semianalytic. □

[4]The discussion in [10] only considers lower semianalytic functions. Nevertheless, the results hold without changes also for upper semianalytic functions.

Theorem 5.3.18. *We consider an upper semianalytic function $\tilde{X}$ on $\tilde{\Omega}$ such that $\tilde{X} \in L^1(\tilde{\Omega})$ or $\tilde{X}$ is $\mathcal{G}^{\mathcal{P}}$-measurable and nonnegative. For $t \geqslant 0$, the following function*

$$\tilde{\mathcal{E}}_t(\tilde{X}) := \mathbf{1}_{\{\tilde{\tau} \leqslant t\}}\ \mathcal{E}_t(\varphi(x,\cdot))|_{x=\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>t\}} \mathcal{E}_t(e^{\Gamma_t} E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]) \tag{5.3.13}$$

is well defined, where φ is the measurable function

$$\varphi : (\mathbb{R}_+ \times \Omega\ ,\ \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}^{\mathcal{P}}_{\infty}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})),$$

such that

$$\tilde{X}(\omega,\hat{\omega}) = \varphi(\tilde{\tau}(\omega,\hat{\omega}),\omega), \qquad (\omega,\hat{\omega}) \in \tilde{\Omega}.$$

Moreover, $\tilde{\mathcal{E}}_t(\tilde{X})$ satisfies the consistency condition (5.3.4).

Proof. According to points 5 and 6 of Lemma 5.3.17, $e^{\Gamma_t} E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]$ is an upper semianalytic function on Ω. Consequently, the second component on the right-hand side of (5.3.13) is well defined. For the first component, it suffices to prove that for every fixed $x \in \mathbb{R}_+$, the function

$$\varphi_x(\omega) := \varphi(x,\omega), \qquad \omega \in \Omega,$$

is upper semianalytic. We note first that

$$\tilde{X}(\omega,\hat{\omega}) = \varphi \circ (\tau, id|_{\Omega})(\omega,\hat{\omega}), \qquad (\omega,\hat{\omega}) \in \Omega \times \hat{\Omega},$$

is upper semianalytic, hence φ as function of $(x,\omega) \in \mathbb{R}_+ \times \Omega$ is upper semianalytic by Remark 5.3.4 and the second implication of point 3 of Lemma 5.3.17. Furthermore, for every fixed $x \in \mathbb{R}_+$, we have $\varphi_x = \varphi \circ \psi_x$ where

$$\psi_x(\omega) := (x,\omega), \qquad \omega \in \Omega,$$

and the function ψ_x is Borel-measurable. Hence, we have that by the first implication of point 3 of Lemma 5.3.17, φ_x as function of $\omega \in \Omega$ is also upper semianalytic.
Now we prove that consistency condition (5.3.4) holds. By Proposition 5.2.6, under every $\tilde{P} \in \tilde{\mathcal{P}}$ we obtain

$$\mathbf{1}_{\{\tilde{\tau} \leqslant t\}}\ \mathcal{E}_t(\varphi(x,\cdot))|_{x=\tilde{\tau}} = \mathbf{1}_{\{\tilde{\tau} \leqslant t\}} \operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(t;P)} E^{P'}[\varphi(x,\cdot)|\mathcal{F}_t]\Big|_{x=\tilde{\tau}} \qquad \tilde{P}\text{-a.s.},$$

$$\mathbf{1}_{\{\tilde{\tau}>t\}} \mathcal{E}_t(e^{\Gamma_t} E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]) = \mathbf{1}_{\{\tilde{\tau}>t\}} \operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(t;P)} E^{P'}[e^{\Gamma_t} E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]|\mathcal{F}_t] \qquad \tilde{P}\text{-a.s.}$$

Furthermore, by Lemma 5.3.10, it holds $\tilde{P}$-a.s.

$$\operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(t;P)} E^{P'}[\varphi(x,\cdot)|\mathcal{F}_t]\Big|_{x=\tilde{\tau}} = \operatorname*{ess\,sup}^{\tilde{P}}_{\tilde{P}' \in \tilde{\mathcal{P}}(t;\tilde{P})} E^{P'}[\varphi(x,\cdot)|\mathcal{F}_t]\Big|_{x=\tilde{\tau}},$$

$$\operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(t;P)} E^{P'}[e^{\Gamma_t} E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]|\mathcal{F}_t] = \operatorname*{ess\,sup}^{\tilde{P}}_{\tilde{P}' \in \tilde{\mathcal{P}}(t;\tilde{P})} E^{P'}[e^{\Gamma_t} E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]|\mathcal{F}_t].$$

We stress that $\{\tilde{\tau} \leqslant t\}$ and $\{\tilde{\tau} > t\}$ are disjoint events, thus $\tilde{P}$-a.s.

$$\mathbf{1}_{\{\tilde{\tau}\leqslant t\}} \operatorname*{ess\,sup}^{\tilde{P}}_{\tilde{P}'\in\tilde{\mathcal{P}}(t;\tilde{P})} E^{P'}[\varphi(x,\cdot)|\mathcal{F}_t]\Big|_{x=\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>t\}} \operatorname*{ess\,sup}^{\tilde{P}}_{\tilde{P}'\in\tilde{\mathcal{P}}(t;\tilde{P})} e^{\Gamma_t} E^{P'}[e^{\Gamma_t} E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]|\mathcal{F}_t]$$
$$= \operatorname*{ess\,sup}^{\tilde{P}}_{\tilde{P}'\in\tilde{\mathcal{P}}(t;\tilde{P})} \left(\mathbf{1}_{\{\tilde{\tau}\leqslant t\}} E^{P'}[\varphi(x,\cdot)|\mathcal{F}_t]\Big|_{x=\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>t\}} E^{P'}[e^{\Gamma_t} E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]|\mathcal{F}_t]\right).$$

As the final step, the integrability conditions on $\tilde{X}$ guarantee the use of Fubini-Tonelli Theorem, Proposition 5.3.16 thus yields

$$\tilde{\mathcal{E}}_t(\tilde{X}) = \operatorname*{ess\,sup}^{\tilde{P}}_{\tilde{P}'\in\tilde{\mathcal{P}}(t;\tilde{P})} E^{\tilde{P}'}[\tilde{X}|\mathcal{G}_t] \quad \tilde{P}\text{-a.s. for all } \tilde{P}\in\tilde{\mathcal{P}}.$$

□

Remark 5.3.19. *As a fundamental difference with respect to the construction in [87], where only measurability conditions are required for the definition of $(\mathcal{P},\mathbb{F})$-conditional expectation (5.2.4), here in Theorem 5.3.18 integrability conditions are required to define the sublinear operator $\tilde{\mathcal{E}}_t$ as well. Indeed, this is necessary for the validity of Fubini-Tonelli Theorem, which is crucial in the proof.*

Remark 5.3.20. *Let $t \geqslant 0$ and let $\tilde{X}$ satisfy the conditions in Theorem 5.3.18. The following statements holds:*

1. *If $\tilde{X}(\omega,\hat{\omega}) = X(\omega)$ for every $\hat{\omega}\in\hat{\Omega}$, then $\tilde{\mathcal{E}}_t(X)$ defined in (5.3.13) is reduced to $\mathcal{E}_t(X)$ defined in (5.2.4).*
2. *The function $\tilde{\mathcal{E}}_t(\tilde{X})$ as defined in (5.3.13) is sublinear in $\tilde{X}$.*
3. *If $\tilde{Y}$ is an upper semianalytic function on $\tilde{\Omega}$, such that $\tilde{Y}\in L^1(\tilde{\Omega})$ and*

$$\operatorname*{ess\,sup}^{\tilde{P}}_{\tilde{P}'\in\tilde{\mathcal{P}}(t;\tilde{P})} E^{\tilde{P}'}[\tilde{X}|\mathcal{G}_t] = \operatorname*{ess\,sup}^{\tilde{P}}_{\tilde{P}'\in\tilde{\mathcal{P}}(t;\tilde{P})} E^{\tilde{P}'}[\tilde{Y}|\mathcal{G}_t] \quad \tilde{P}\text{-a.s. for all } \tilde{P}\in\tilde{\mathcal{P}},$$

   then we have $\tilde{\mathcal{E}}_t(\tilde{X}) = \tilde{\mathcal{E}}_t(\tilde{Y})$ $\tilde{P}$-a.s. for all $\tilde{P}\in\tilde{\mathcal{P}}$.
4. *Let $A\in\mathcal{G}_t$, then $\tilde{\mathcal{E}}_t(\mathbf{1}_A\tilde{X}) = \mathbf{1}_A\tilde{\mathcal{E}}_t(\tilde{X})$. This is a direct consequence of Lemma 5.1.1 of [24] and the above point.*
5. *The following pathwise relations hold:*

$$\tilde{\mathcal{E}}_t(\mathbf{1}_{\{\tilde{\tau}\leqslant t\}}\tilde{X}) = \mathbf{1}_{\{\tilde{\tau}\leqslant t\}}\tilde{\mathcal{E}}_t(\tilde{X}),$$

$$\tilde{\mathcal{E}}_t(\mathbf{1}_{\{\tilde{\tau}> t\}}\tilde{X}) = \mathbf{1}_{\{\tilde{\tau}> t\}}\tilde{\mathcal{E}}_t(\tilde{X}),$$

$$\tilde{\mathcal{E}}_t(\tilde{X}) = \tilde{\mathcal{E}}_t(\mathbf{1}_{\{\tilde{\tau}\leqslant t\}}\tilde{X}) + \tilde{\mathcal{E}}_t(\mathbf{1}_{\{\tilde{\tau}> t\}}\tilde{X}).$$

We adopt the following notations for the sake of simplicity in the sequel.

$$E^P[\tilde{X}|\mathcal{F}_t](\omega,\hat{\omega}) := E^P[\tilde{X}(\cdot,\hat{\omega})|\mathcal{F}_t](\omega), \quad (\omega,\hat{\omega}) \in \tilde{\Omega},\ t \geqslant 0, \tag{5.3.14}$$

$$\mathcal{E}_t(\tilde{X})(\omega,\hat{\omega}) := \mathcal{E}_t(\tilde{X}(\cdot,\hat{\omega}))(\omega), \quad (\omega,\hat{\omega}) \in \tilde{\Omega},\ t \geqslant 0. \tag{5.3.15}$$

We stress that the right-hand side of (5.3.15) is well defined by (5.2.4) and points 3 and 6 of Lemma 5.3.17, since the concatenation function is Borel-measurable.

Proposition 5.3.21. *Let $\tilde{X}$ be an upper semianalytic function on $\tilde{\Omega}$, such that $\tilde{X} \in L^1(\tilde{\Omega})$ or $\tilde{X}$ is $\mathcal{G}^{\mathcal{P}}$-measurable and nonnegative. Then for each $t \geqslant 0$, the function $\tilde{\mathcal{E}}_t(\tilde{X})$ defined in (5.3.13) is $\mathcal{G}_t^*$- and $\mathcal{G}^{\mathcal{P}}$-measurable and upper semianalytic.*

Proof. Let $t \geqslant 0$. We have that $\tilde{\mathcal{E}}_t(\tilde{X})$ is $(\mathcal{F}_t^* \vee \sigma(\tau))$-measurable by definition (5.3.13) and Proposition 5.2.6, hence it is also $\mathcal{G}_t^*$- and $\mathcal{G}^{\mathcal{P}}$-measurable. It is upper semianalytic according to points 3, 4, 5 of Lemma 5.3.17 and Proposition 5.2.6. □

Remark 5.3.20 together with Proposition 5.3.21 shows that $(\tilde{\mathcal{E}}_t)_{t\geqslant 0}$ is a family of sublinear conditional expectations which extends $(\mathcal{E}_t)_{t\geqslant 0}$ defined for functions on Ω. We now present that the family $(\tilde{\mathcal{E}}_t)_{t\geqslant 0}$ satisfies a weak form of dynamic programming principle or tower property, similarly to the one of [90].

Theorem 5.3.22. *If $\tilde{X}$ is an upper semianalytic function on $\tilde{\Omega}$ such that $\tilde{X}$ is $\mathcal{G}^{\mathcal{P}}$-measurable and nonnegative, and $0 \leqslant s \leqslant t$, then*

$$\tilde{\mathcal{E}}_s(\tilde{\mathcal{E}}_t(\tilde{X})) \geqslant \tilde{\mathcal{E}}_s(\tilde{X}) \quad \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}}. \tag{5.3.16}$$

Proof. We recall that notations (5.3.7), (5.3.14) and (5.3.15) are used. The left-hand side of (5.3.16) is well defined by Proposition 5.3.21 and the nonnegativity of $\tilde{X}$. According to definition (5.3.13), relation (5.3.16) is equivalent to the following

$$\begin{aligned}&\mathbf{1}_{\{\tilde{\tau}\leqslant s\}}\ \mathcal{E}_s(\bar{\varphi}(x,\cdot))\big|_{x=\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>s\}}\mathcal{E}_s(e^{\Gamma_s}E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{\mathcal{E}}_t(\tilde{X})])\\ \geqslant&\mathbf{1}_{\{\tilde{\tau}\leqslant s\}}\ \mathcal{E}_s(\varphi(x,\cdot))\big|_{x=\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>s\}}\mathcal{E}_s(e^{\Gamma_s}E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{X}]),\end{aligned} \tag{5.3.17}$$

where φ is the measurable function

$$\varphi : (\mathbb{R}_+ \times \Omega\ ,\ \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_\infty^{\mathcal{P}}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})),$$

such that

$$\tilde{X}(\omega,\hat{\omega}) = \varphi(\tilde{\tau}(\omega,\hat{\omega}),\omega), \quad (\omega,\hat{\omega}) \in \tilde{\Omega},$$

and

$$\bar{\varphi}(x,\omega) = \mathbf{1}_{\{x\leqslant t\}}\mathcal{E}_t(\varphi(x,\cdot))(\omega) + \mathbf{1}_{\{x>t\}}\mathcal{E}_t(e^{\Gamma_t}E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}])(\omega),$$

for every $(x,\omega) \in \mathbb{R}_+ \times \Omega$. We prove first that the first terms on both hand sides of (5.3.17) are equal, by using (5.3.13) and the tower property (5.2.6) of $(\mathcal{P},\mathbb{F})$-conditional expectation:

$$
\begin{aligned}
&\mathbf{1}_{\{\tilde\tau\leqslant s\}}\ \mathcal{E}_s(\bar\varphi(x,\cdot))\big|_{x=\tilde\tau}\\
=&\mathbf{1}_{\{\tilde\tau\leqslant s\}}\ \mathcal{E}_s\left(\mathbf{1}_{\{x\leqslant t\}}\mathcal{E}_t(\varphi(x,\cdot))+\mathbf{1}_{\{x>t\}}\mathcal{E}_t(e^{\Gamma_t}E^{\hat P}[\mathbf{1}_{\{\tilde\tau>t\}}\tilde X])\right)\Big|_{x=\tilde\tau}\\
=&\mathbf{1}_{\{\tilde\tau\leqslant s\}}\ \left(\mathbf{1}_{\{x\leqslant t\}}\mathcal{E}_s\left(\mathcal{E}_t(\varphi(x,\cdot))\right)+\mathbf{1}_{\{x>t\}}\mathcal{E}_s\left(\mathcal{E}_t(e^{\Gamma_t}E^{\hat P}[\mathbf{1}_{\{\tilde\tau>t\}}\tilde X])\right)\right)\Big|_{x=\tilde\tau}\\
=&\mathbf{1}_{\{\tilde\tau\leqslant s\}}\ \mathcal{E}_s\left(\mathcal{E}_t(\varphi(x,\cdot))\right)\big|_{x=\tilde\tau}\\
=&\mathbf{1}_{\{\tilde\tau\leqslant s\}}\ \mathcal{E}_s(\varphi(x,\cdot))\big|_{x=\tilde\tau}\,.
\end{aligned}
$$

For the second terms, we need some preliminary considerations. For every fixed $\hat\omega\in\hat\Omega$, $\tilde\tau(\cdot,\hat\omega)$ is an $\mathbb{F}$-stopping time. Galmarino's test yields that, on the event $\{\tilde\tau\leqslant t\}$ it holds

$$\tilde\tau(\omega\otimes_t\omega',\hat\omega)=\tilde\tau(\omega,\hat\omega)\quad\text{ for all }\omega'\in\Omega.$$

Hence on the event $\{\tilde\tau\leqslant t\}$, for every fixed $\hat\omega\in\hat\Omega$, by definitions (5.2.3), (5.2.4) and representation (5.3.8), we obtain

$$
\begin{aligned}
\mathcal{E}_t(\tilde X)(\omega,\hat\omega)&=\sup_{P\in\mathcal{P}}\int_\Omega\tilde X(\omega\otimes_t\omega',\hat\omega)P(\mathrm{d}\omega')\\
&=\sup_{P\in\mathcal{P}}\int_\Omega\varphi(\tilde\tau(\omega\otimes_t\omega',\hat\omega),\omega\otimes_t\omega')P(\mathrm{d}\omega')\\
&=\sup_{P\in\mathcal{P}}\int_\Omega\varphi(\tilde\tau(\omega,\hat\omega),\omega\otimes_t\omega')P(\mathrm{d}\omega')\\
&=\sup_{P\in\mathcal{P}}\int_\Omega\varphi(x,\omega\otimes_t\omega')P(\mathrm{d}\omega')\Big|_{x=\tilde\tau(\omega,\hat\omega)}\\
&=\mathcal{E}_t(\varphi(x,\cdot))(\omega)|_{x=\tilde\tau(\omega,\hat\omega)}\quad\text{ for all }\omega\in\Omega,
\end{aligned}
$$

in other words,

$$\mathbf{1}_{\{\tilde\tau\leqslant t\}}\ \mathcal{E}_t(\varphi(x,\cdot))\big|_{x=\tilde\tau}=\mathbf{1}_{\{\tilde\tau\leqslant t\}}\mathcal{E}_t(\tilde X)\quad\text{ for every fixed }\hat\omega\in\hat\Omega.\tag{5.3.18}$$

Moreover, we note that by (5.2.5), for each $P\in\mathcal{P}$

$$\mathcal{E}_t(e^{\Gamma_t}E^{\hat P}[\mathbf{1}_{\{\tilde\tau>t\}}\tilde X])=e^{\Gamma_t}\mathcal{E}_t(E^{\hat P}[\mathbf{1}_{\{\tilde\tau>t\}}\tilde X])\quad P\text{-a.s.}\tag{5.3.19}$$

By using (5.3.13), (5.3.18), (5.3.19) and Remark 2.4 (iii) of [87], it holds

$$
\begin{aligned}
&\mathcal{E}_s(e^{\Gamma_s}E^{\hat P}[\mathbf{1}_{\{\tilde\tau>s\}}\tilde{\mathcal{E}}_t(\tilde X)])\\
=&e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat P}\left[\mathbf{1}_{\{\tilde\tau>s\}}(\mathbf{1}_{\{\tilde\tau\leqslant t\}}\ \mathcal{E}_t(\varphi(x,\cdot))|_{x=\tilde\tau}+\mathbf{1}_{\{\tilde\tau>t\}}\mathcal{E}_t(e^{\Gamma_t}E^{\hat P}[\mathbf{1}_{\{\tilde\tau>t\}}\tilde X]))\right]\right)\\
=&e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat P}\left[\mathbf{1}_{\{s<\tilde\tau\leqslant t\}}\ \mathcal{E}_t(\varphi(x,\cdot)|_{x=\tilde\tau}+\mathbf{1}_{\{\tilde\tau>t\}}\mathcal{E}_t(e^{\Gamma_t}E^{\hat P}[\mathbf{1}_{\{\tilde\tau>t\}}\tilde X]))\right]\right)\\
=&e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat P}\left[\mathbf{1}_{\{s<\tilde\tau\leqslant t\}}\mathcal{E}_t(\tilde X)+\mathbf{1}_{\{\tilde\tau>t\}}e^{\Gamma_t}\mathcal{E}_t(E^{\hat P}[\mathbf{1}_{\{\tilde\tau>t\}}\tilde X])\right]\right)\\
=&e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat P}[\mathbf{1}_{\{s<\tilde\tau\leqslant t\}}\mathcal{E}_t(\tilde X)]+E^{\hat P}[\mathbf{1}_{\{\tilde\tau>t\}}e^{\Gamma_t}\mathcal{E}_t(E^{\hat P}[\mathbf{1}_{\{\tilde\tau>t\}}\tilde X])]\right)\quad P\text{-a.s. for all }P\in\mathcal{P}.
\end{aligned}
$$

We observe that $e^{\Gamma_t}\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}])$ depends only on the first component ω. Then by using the definition of Γ, (5.3.7) and Lemma 5.3.5 we have

$$\begin{aligned} E^{\hat{P}}\left[\mathbf{1}_{\{\tilde{\tau}>t\}}e^{\Gamma_t}\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}])\right] =& E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}]e^{\Gamma_t}\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]) \\ =& e^{-\Gamma_t}e^{\Gamma_t}\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]) \\ =& \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]). \end{aligned} \tag{5.3.20}$$

This yields

$$E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}e^{\Gamma_t}\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}])] = \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]),$$

which implies

$$\begin{aligned} \mathcal{E}_s(e^{\Gamma_s}E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{\mathcal{E}}_t(\tilde{X})]) =& e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat{P}}[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\mathcal{E}_t(\tilde{X})] + \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}])\right) \\ =& e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat{P}}[\mathcal{E}_t(\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X})] + \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}])\right) \\ \geqslant& e^{\Gamma_s}\mathcal{E}_s\left(\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X}]) + \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}])\right) \quad (5.3.21) \\ \geqslant& e^{\Gamma_s}\mathcal{E}_s\left(\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X}] + E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}])\right) \quad (5.3.22) \\ =& e^{\Gamma_s}\mathcal{E}_s(\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{X}])) \\ =& e^{\Gamma_s}\mathcal{E}_s(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{X}]) \\ =& \mathcal{E}_s(e^{\Gamma_s}E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{X}]) \quad P\text{-a.s. for all } P\in\mathcal{P}. \end{aligned}$$

In the second equality we observe that for every fixed $\hat{\omega}\in\hat{\Omega}$, $\{s<\tilde{\tau}(\cdot,\hat{\omega})\leqslant t\}\in\mathcal{F}_t$ and $\mathcal{E}_t(\mathbf{1}_A X) = \mathbf{1}_A\mathcal{E}_t(X)$, if $A\in\mathcal{F}_t$ and X is upper semianalytic, see Remark 2.4 (iv) of [87]. The inequality (5.3.21) is a consequence of (5.2.5) and the conditional Fubini–Tonelli Theorem. Indeed, by using notation (5.3.7) we have

$$\begin{aligned} E^{\hat{P}}[\mathcal{E}_t(\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X})] =& E^{\hat{P}}\left[\operatorname*{ess\,sup}^P_{P'\in\mathcal{P}(t;P)} E^{P'}[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X}|\mathcal{F}_t]\right] \\ \geqslant& E^{\hat{P}}[E^P[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X}|\mathcal{F}_t]] \\ =& E^P[E^{\hat{P}}[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X}]|\mathcal{F}_t] \quad P\text{-a.s. for all } P\in\mathcal{P}. \end{aligned}$$

Thus,

$$\begin{aligned} E^{\hat{P}}[\mathcal{E}_t(\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X})] \geqslant& \operatorname*{ess\,sup}^P_{P'\in\mathcal{P}(t;P)} E^{P'}[E^{\hat{P}}[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X}]|\mathcal{F}_t] \\ =& \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X}]) \quad P\text{-a.s. for all } P\in\mathcal{P}. \end{aligned}$$

The inequality (5.3.22) is a consequence of the sublinearity of $(\mathcal{P},\mathbb{F})$-conditional expectation. In the second last equality, the tower property (5.2.6) is used. This concludes the proof. □

Corollary 5.3.23. *If $\tilde{X}$ is an upper semianalytic function on $\tilde{\Omega}$ such that $\tilde{X} \in L^1(\tilde{\Omega})$, and $\tilde{\mathcal{E}}_t(\tilde{X}) \in L^1(\tilde{\Omega})$ for $t \geqslant 0$, then it holds that*

$$\tilde{\mathcal{E}}_s(\tilde{\mathcal{E}}_t(\tilde{X})) \geqslant \tilde{\mathcal{E}}_s(\tilde{X}) \quad \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}}, \tag{5.3.23}$$

with $0 \leqslant s \leqslant t$.

Proof. Under the condition that $\tilde{\mathcal{E}}_t(\tilde{X}) \in L^1(\tilde{\Omega})$ for $t \geqslant 0$, the left-hand side of (5.3.23) is well defined. The rest of the proof is the same as for Theorem 5.3.22. □

The above results are summarized in the following theorem which extends Proposition 5.2.6 to the reduced-form setting under model uncertainty.

Theorem 5.3.24. *If $\tilde{X}$ is an upper semianalytic function on $\tilde{\Omega}$ such that $\tilde{X} \in L^1(\tilde{\Omega})$ or $\mathcal{G}^{\mathcal{P}}$-measurable and nonnegative, then for each $t \geqslant 0$, the function $\tilde{\mathcal{E}}_t(\tilde{X})$ defined in (5.3.13) is $\mathcal{G}_t^*$- and $\mathcal{G}^{\mathcal{P}}$-measurable, upper semianalytic and satisfies the consistency condition (5.3.4). Moreover, the family of functions $(\tilde{\mathcal{E}}_t(\tilde{X}))_{t\geqslant 0}$ satisfies the weak tower property (5.3.16).*

Definition 5.3.25. *The family of sublinear conditional expectations $(\tilde{\mathcal{E}}_t)_{t\geqslant 0}$ is called* $(\tilde{\mathcal{P}}, \mathbb{G})$-conditional expectation.

5.3.3 Further considerations on the $(\tilde{\mathcal{P}}, \mathbb{G})$-conditional expectation

In this section we discuss in detail the problem of dynamic programming principle and integrability in the case of $(\tilde{\mathcal{P}}, \mathbb{G})$-conditional expectation, and the difficulties arisen from the construction in Section 5.3.2.

We note that inequalities (5.3.21) and (5.3.22) in the proof of Theorem 5.3.22 imply only a weak form of tower property. This cannot be improved in full generality as we see in the following Counterexample 5.3.26.

Counterexample 5.3.26. *We restrict our attention to the case of $\Omega = C_0(\mathbb{R}_+, \mathbb{R}^d)$ with G-conditional expectation, see e.g. [91], [114] for reference. Since the $(\tilde{\mathcal{P}}, \mathbb{F})$-conditional expectation is only sublinear, it is possible to find $t \geqslant 0$ and X, Y functions on Ω, which are sufficiently regular for the definition of G-conditional expectation and satisfy the following strict inequality on a measurable set A with $P(A) > 0$ for all $P \in \mathcal{P}$,*

$$\mathcal{E}_t(X)(\omega) + \mathcal{E}_t(Y)(\omega) > \mathcal{E}_t(X+Y)(\omega) \quad \text{for all } \omega \in A. \tag{5.3.24}$$

If there exists a measurable subset $B \subseteq A$ with $P(B) > 0$ for all $P \in \mathcal{P}$, such that for all $s < t$ it holds

$$\mathcal{E}_s(\mathcal{E}_t(X) + \mathcal{E}_t(Y)) = \mathcal{E}_s(\mathcal{E}_t(X+Y)) \quad P\text{-a.s. for all } P \in \mathcal{P} \text{ on } B,$$

As shown in e.g. [108] and [114], the operator $\mathcal{E}_t$ is continuous in t in the case of the G-conditional expectation. By taking the limit for $s \uparrow t$, this yields

$$\mathcal{E}_t(\mathcal{E}_t(X) + \mathcal{E}_t(Y)) = \mathcal{E}_t(\mathcal{E}_t(X+Y)) \quad P\text{-a.s. for all } P \in \mathcal{P} \text{ on } B,$$

which is equivalent to

$$\mathcal{E}_t(X)+\mathcal{E}_t(Y)=\mathcal{E}_t(X+Y) \quad P\text{-a.s. for all } P\in\mathcal{P} \text{ on } B,$$

by (5.2.5). This contradicts clearly (5.3.24). Hence, there exists s *with* $s<t$ *such that*

$$\mathcal{E}_s(\mathcal{E}_t(X)+\mathcal{E}_t(Y))>\mathcal{E}_s(\mathcal{E}_t(X+Y)) \quad P\text{-a.s. for all } P\in\mathcal{P} \text{ on } A. \tag{5.3.25}$$

For r,l *with* $s<r\leqslant t\leqslant l$*, we now define*

$$\bar{X}:=\frac{X}{e^{-\Gamma_s}-e^{-\Gamma_r}}, \qquad \bar{Y}:=\frac{Y}{e^{-\Gamma_l}}.$$

Inequality (5.3.25) can be rewritten as follows

$$\begin{aligned}&\mathcal{E}_s\left((e^{-\Gamma_s}-e^{-\Gamma_r})\ \mathcal{E}_t(\bar{X})+\mathcal{E}_t(e^{-\Gamma_l}\bar{Y})\right)\\ >\ &\mathcal{E}_s\left(\mathcal{E}_t((e^{-\Gamma_s}-e^{-\Gamma_r})\bar{X}+e^{-\Gamma_l}\bar{Y})\right) \quad P\text{-a.s. for all } P\in\mathcal{P} \text{ on } A.\end{aligned} \tag{5.3.26}$$

By setting

$$\tilde{X}:=\mathbf{1}_{\{\tilde{\tau}\leqslant r\}}\bar{X}+\mathbf{1}_{\{\tilde{\tau}>l\}}\bar{Y},$$

we check now that the classic tower property does not hold for $\tilde{X}$*. According to the proof of Theorem 5.3.22, this equals to showing that one of (5.3.21) and (5.3.22) is strict inequality on* A*. By (5.3.25), we have indeed*

$$\begin{aligned}&e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat{P}}[\mathcal{E}_t(\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X})]+\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}])\right)\\ =&e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat{P}}[\mathcal{E}_t(\mathbf{1}_{\{s<\tilde{\tau}\leqslant r\}}\bar{X})]+\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>l\}}\bar{Y}])\right)\\ =&e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat{P}}[\mathbf{1}_{\{s<\tilde{\tau}\leqslant r\}}]\mathcal{E}_t(\bar{X})+\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>l\}}]\bar{Y})\right)\\ =&e^{\Gamma_s}\mathcal{E}_s\left((e^{-\Gamma_s}-e^{-\Gamma_r})\mathcal{E}_t(\bar{X})+\mathcal{E}_t(e^{-\Gamma_l}\bar{Y})\right)\\ >&e^{\Gamma_s}\mathcal{E}_s\left(\mathcal{E}_t((e^{-\Gamma_s}-e^{-\Gamma_r})\bar{X}+e^{-\Gamma_l}\bar{Y})\right)\\ =&e^{\Gamma_s}\mathcal{E}_s(\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{s<\tilde{\tau}\leqslant r\}}\bar{X}+\mathbf{1}_{\{\tilde{\tau}>l\}}\bar{Y}]))\\ =&e^{\Gamma_s}\mathcal{E}_s(\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{X}])) \quad P\text{-a.s. for all } P\in\mathcal{P} \text{ on } A.\end{aligned}$$

Hence, in this case we have only a strictly weak tower property.

The following Yan's commutability Theorem can be found in [116] and Theorem a3 of [90], and is useful for our discussion in the sequel.

Theorem 5.3.27 (Yan's commutability Theorem)**.** *Let* $(\Omega,\mathcal{F},P)$ *be an arbitrary probability space and* H *be a subset of* $L^1(\Omega,\mathcal{F},P)$ *such that* $\inf_{\xi\in H}E^P[\xi]>-\infty$*. The following statements are equivalent.*

1. *For all* $\varepsilon>0$ *and* ξ_1, $\xi_2\in H$*, there exists a* $\xi_3\in H$ *such that*

$$E^P[(\xi_3-\xi_1\wedge\xi_2)^+]\leqslant\varepsilon.$$

2. $E^P\left[\underset{\xi\in H}{ess\ inf}^{\,P}\xi\right] = \underset{\xi\in H}{inf} E^P[\xi]$.

3. *For any sub-σ-algebra $\mathcal{J}$ of $\mathcal{F}$, we have*

$$E^P\left[\left.\underset{\xi\in H}{ess\ inf}^{\,P}\xi\right|\mathcal{J}\right] = \underset{\xi\in H}{ess\ inf}^{\,P} E^P\left[\xi|\mathcal{J}\right].$$

This can be equivalently formulated in terms of supremum in the following way.

Theorem 5.3.28 (Yan's commutability Theorem, sup version)**.** *Let $(\Omega,\mathcal{F},P)$ be an arbitrary probability space and H be a subset of $L^1(\Omega,\mathcal{F},P)$ such that $\sup_{\xi\in H} E^P[\xi] < +\infty$. The following statements are equivalent.*

1. *For all $\varepsilon > 0$ and ξ_1, $\xi_2 \in H$, there exists a $\xi_3 \in H$ such that*

$$E^P[(\xi_1 \vee \xi_2 - \xi_3)^+] \leqslant \varepsilon.$$

2. $E^P\left[\underset{\xi\in H}{ess\ sup}^{P}\xi\right] = \underset{\xi\in H}{sup} E^P[\xi]$.

3. *For any sub-σ-algebra $\mathcal{J}$ of $\mathcal{F}$, we have*

$$E^P\left[\left.\underset{\xi\in H}{ess\ sup}^{P}\xi\right|\mathcal{J}\right] = \underset{\xi\in H}{ess\ sup}^{P} E^P\left[\xi|\mathcal{J}\right].$$

Yan's commutability Theorem gives a sufficient condition such that the conditional expectation and the essential supremum are exchangeable. It is hence a sufficient condition such that the classic tower property holds for a family of sublinear conditional expectations. In the general case, it is however not a necessary condition. We show in the following proposition that Yan's commutability condition 1 is satisfied by the $(\mathcal{P},\mathbb{F})$-conditional expectation constructed in [87] and summarized in Section 5.2.1.

Proposition 5.3.29. *Let $t \geqslant 0$ and $P \in \mathcal{P}$. For every $\varepsilon > 0$ and P_1, P_2 in $\mathcal{P}(t;P)$, there exists a $P_3 \in \mathcal{P}(t;P)$ such that*

$$E^P\left[\left(E^{P_1}[X|\mathcal{F}_t] \vee E^{P_2}[X|\mathcal{F}_t] - E^{P_3}[X|\mathcal{F}_t]\right)^+\right] \leqslant \varepsilon.$$

Proof. According to step 3 in the proof of Theorem 2.3 in [87], there is a probability $P_\varepsilon \in \mathcal{P}(t;P)$ such that

$$E^{P_\varepsilon}[X|\mathcal{F}_t] \geqslant (V_t - \varepsilon) \wedge \frac{1}{\varepsilon} \quad P\text{-a.s.},$$

where

$$V_t = \underset{P'\in\mathcal{P}(t;P)}{\text{ess sup}}^{P} E^{P'}[X|\mathcal{F}_t].$$

Hence, in particular

$$E^{P_\varepsilon}[X|\mathcal{F}_t] \geqslant \big(E^{P_1}[X|\mathcal{F}_t] \vee E^{P_2}[X|\mathcal{F}_t] - \varepsilon\big) \wedge \frac{1}{\varepsilon} \quad P\text{-a.s.}$$

This implies

$$\big(E^{P_1}[X|\mathcal{F}_t] \vee E^{P_2}[X|\mathcal{F}_t] - E^{P_\varepsilon}[X|\mathcal{F}_t]\big)^+ \leqslant \varepsilon \quad P\text{-a.s..}$$

It follows

$$E^P\left[\big(E^{P_1}[X|\mathcal{F}_t] \vee E^{P_2}[X|\mathcal{F}_t] - E^{P_\varepsilon}[X|\mathcal{F}_t]\big)^+\right] \leqslant \varepsilon.$$

That is, it is sufficient to choose $P_3 = P_\varepsilon$. □

We emphasize that, the above argument is however not sufficient for proving Yan's property for the $(\tilde{\mathcal{P}}, \mathbb{G})$-conditional expectation. For fixed ε and $\tilde{X}$ on the product space $\tilde{\Omega}$, one would be tempted to conclude that there exists $P_\varepsilon \in \mathcal{P}(t; P)$ such that

$$E^{\hat{P}}[E^{P_\varepsilon}[\tilde{X}|\mathcal{F}_t]] \geqslant (\tilde{V}_t - \varepsilon) \wedge \frac{1}{\varepsilon} \quad P\text{-a.s.,} \tag{5.3.27}$$

where

$$\tilde{V}_t = \operatorname*{ess\,sup}^P_{P' \in \mathcal{P}(t;P)} E^{\hat{P}}[E^{P'}[\tilde{X}|\mathcal{F}_t]],$$

by using the same argument of step 3 in the proof of Theorem 2.3 in [87]. This would eventually lead to condition 2 of Proposition 5.3.30 which we state below, and hence the validity of classic tower property for $(\tilde{\mathcal{P}}, \mathbb{G})$-conditional expectation in full generality. This is however not possible, since P_ε depends on the choice of $\hat{\omega} \in \hat{\Omega}$, while in (5.3.27) we need a P_ε homogeneous in $\hat{\omega}$ or at least a family $\{P_\varepsilon^{\hat{\omega}} : \hat{\omega} \in \hat{\Omega}\}$ such that $E^{P_\varepsilon^{\hat{\omega}}}[\tilde{X}|\mathcal{F}_t]$ is $\mathcal{B}(\hat{\Omega})$-measurable in $\hat{\omega} \in \hat{\Omega}$. Both cases are not obtainable in full generality. We furthermore observe that in general there is no guarantee that the function $\mathcal{E}_t(\tilde{X})$ itself is $\mathcal{B}(\hat{\Omega})$-measurable, unless $\mathcal{E}_t(\tilde{X}) = \mathbf{1}_{\{\tau \leqslant t\}}\mathcal{E}_t(\tilde{X})$, as shown in (5.3.18) and Theorem 5.3.18.

Some general sufficient conditions for the classic tower property in the case of $(\tilde{\mathcal{P}}, \mathbb{G})$-conditional expectation are stated in the following proposition.

Proposition 5.3.30. *Under the same assumptions of Theorem 5.3.22 or Corollary 5.3.23, if one of the following conditions holds*

1. $\tilde{X}(\omega, \hat{\omega}) = X(\omega)$ *for all* $\omega \in \Omega$*, i.e.* $\tilde{X}$ *does not depend on* $\hat{\omega} \in \hat{\Omega}$*;*
2. *the function* $\mathcal{E}_t(\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{X})$ *is* $\mathcal{B}(\hat{\Omega})$*-measurable and it holds that*

$$\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{X}]) = E^{\hat{P}}[\mathcal{E}_t(\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{X})] \quad P\text{-a.s.}$$

   for all $P \in \mathcal{P}$ *and for all* $0 \leqslant s \leqslant t$*;*
3. *for all* $P \in \mathcal{P}$*,* P*-a.e.* ω*,* $0 \leqslant s \leqslant t$*,* $\varepsilon > 0$ *and probability measures* $P_1, P_2 \in \mathcal{P}$*, there exists a third probability measure* $P_3 \in \mathcal{P}$ *such that*

$$E^{\hat{P}}[(\xi_1 \vee \xi_2 - \xi_3)^+] \leqslant \varepsilon \quad P\text{-a.s.,}$$

where

$$\xi_i(\hat{\omega}) = \int_\Omega \tilde{Y}(\omega \otimes_t \omega', \hat{\omega}) \mathrm{d}P_i(\omega'), \qquad \hat{\omega} \in \hat{\Omega},\ i = 1, 2, 3,$$

and

$$\tilde{Y} := \mathbf{1}_{\{\tilde{\tau} > s\}} \tilde{X},$$

then we have

$$\tilde{\mathcal{E}}_s(\tilde{\mathcal{E}}_t(\tilde{X})) = \tilde{\mathcal{E}}_s(\tilde{X}) \quad \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}}, \tag{5.3.28}$$

for $0 \leqslant s \leqslant t$.

Proof. Condition 1 follows directly from point 1 of Remark 5.3.20 and the tower property for $(\mathcal{P}, \mathbb{F})$-conditional expectation (5.2.6) in Proposition 5.2.6.

If condition 2 holds, by the proof of Theorem 5.3.22 we need only to check that (5.3.21) and (5.3.22) are equalities,

$$\begin{aligned}
& e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat{P}}[\mathcal{E}_t(\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X})] + \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}])\right) \\
=& e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat{P}}[\mathcal{E}_t(\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X})] + E^{\hat{P}}[\mathcal{E}_t(\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X})]\right) \\
=& e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat{P}}[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\mathcal{E}_t(\tilde{X})] + E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\mathcal{E}_t(\tilde{X})]\right) \\
=& e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat{P}}[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\mathcal{E}_t(\tilde{X}) + \mathbf{1}_{\{\tilde{\tau}>t\}}\mathcal{E}_t(\tilde{X})]\right) \\
=& e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>s\}}\mathcal{E}_t(\tilde{X})]\right) \\
=& e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat{P}}[\mathcal{E}_t(\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{X})]\right) \\
=& e^{\Gamma_s}\mathcal{E}_s\left(\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{X}])\right) \\
=& e^{\Gamma_s}\mathcal{E}_s(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{X}]) \quad P\text{-a.s. for all } P \in \mathcal{P}.
\end{aligned}$$

Condition 3 is equivalent to condition 2 by Yan's Commutability Theorem 5.3.28. For fixed $P \in \mathcal{P}$, $t \geqslant 0$ and $\omega \in \Omega$, it is sufficient to consider the the probability space $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{P})$ and the family

$$H := \left\{\xi(\hat{\omega}) := \int_\Omega \tilde{X}(\omega \otimes_t \omega', \hat{\omega})\mathrm{d}P'(\omega') : t \geqslant 0,\ P' \in \mathcal{P}\right\}.$$

In view of the equivalence between statements 1 and 2 of Theorem 5.3.28, condition 3 is equivalent to the following

$$\int_{\hat{\Omega}}\left(\sup_{P\in\mathcal{P}}\int_\Omega \tilde{X}(\omega \otimes_t \omega', \hat{\omega})\mathrm{d}P'(\omega')\right)\mathrm{d}P(\hat{\omega}) = \sup_{P\in\mathcal{P}}\int_{\hat{\Omega}}\left(\int_\Omega \tilde{X}(\omega \otimes_t \omega', \hat{\omega})\mathrm{d}P'(\omega')\right)\mathrm{d}P(\hat{\omega}),$$

which is exactly condition 2 by the definition (5.2.4) of $(\mathcal{P}, \mathbb{F})$-conditional expectation. □

We note that the classic tower property in full generality fails to hold for $(\tilde{\mathcal{P}}, \mathbb{G})$-conditional expectation due to the nature of the progressively enlarged filtration $\mathbb{G}$ itself. While the "path-pasting" construction of $(\mathcal{P}, \mathbb{F})$-conditional expectation in [87] and Section 5.2.1 is consistent with the tower property on the canonical filtration $\mathbb{F}$ by Galmarino's test, it is however not the case for the progressively enlarged filtration $\mathbb{G}$.

Similarly, we observe that the property of belonging to $L^1(\tilde{\Omega})$ is not always valid under the operator $\tilde{\mathcal{E}}_t$, for the same reason which causes the dynamic programming property to fail. Step 1 in the proof of Theorem 2.3 of [80] shows that $(\mathcal{P}, \mathbb{F})$-conditional expectation maps integrable function into integrable function. Indeed, if $t \geqslant 0$ and X on Ω is integrable under every $P \in \mathcal{P}$, then there exists a sequence $(P_n)_{n\in\mathbb{N}}$ such that $P_n \in \mathcal{P}(t;P)$ and

$$E^{P_n}[|X||\mathcal{F}_t] \nearrow \operatorname*{ess\,sup}^P_{P'\in\mathcal{P}(t;P)} E^{P'}[|X||\mathcal{F}_t] \qquad P\text{-a.s.}$$

Monotone convergence theory yields consequently

$$\begin{aligned}
E^P[|\mathcal{E}_t(X)|] \leqslant & E^P\left[\operatorname*{ess\,sup}^P_{P'\in\mathcal{P}(t;P)} E^{P'}[|X||\mathcal{F}_t]\right] \\
= & \lim_{n\to\infty} E^P\left[E^{P_n}[|X||\mathcal{F}_t]\right] \\
= & \lim_{n\to\infty} E^{P_n}\left[E^{P_n}[|X||\mathcal{F}_t]\right] \\
= & \lim_{n\to\infty} E^{P_n}[|X|] \\
\leqslant & \mathcal{E}(|X|),
\end{aligned}$$

where the second equality is valid since by the definition of $\mathcal{P}(t;P)$, it holds that $P_n = P$ on $\mathcal{F}_t$ for all $n \in \mathbb{N}$. A misleading argument similar to the one for the tower property would be using the above conclusion to obtain

$$\begin{aligned}
E^{\tilde{P}}\left[\operatorname*{ess\,sup}^P_{P'\in\mathcal{P}(t;P)} E^{P'}[|\tilde{X}||\mathcal{F}_t]\right] &= \lim_{n\to\infty} E^{\tilde{P}}\left[E^{P_n}[|\tilde{X}||\mathcal{F}_t]\right] \\
&= \lim_{n\to\infty} E^{\hat{P}}\left[E^{P}\left[E^{P_n}[|\tilde{X}||\mathcal{F}_t]\right]\right] \\
&= \lim_{n\to\infty} E^{\hat{P}}\left[E^{P_n}\left[E^{P_n}[|\tilde{X}||\mathcal{F}_t]\right]\right] \\
&= \lim_{n\to\infty} E^{\hat{P}}\left[E^{P_n}[|\tilde{X}|]\right] \\
&= \lim_{n\to\infty} E^{\tilde{P}_n}[|\tilde{X}|],
\end{aligned}$$

by Fubini–Tonelli Theorem. However, this cannot be done since P_n in such case depends on $\hat{\omega}$, and there is no guarantee in the general case that the family $\{P_n^{\hat{\omega}} : \hat{\omega} \in \hat{\Omega}\}$ is such that $E^{P_n^{\hat{\omega}}}[|\tilde{X}||\mathcal{F}_t]$ is measurable in $\hat{\omega} \in \hat{\Omega}$.

Nevertheless, in Section 5.4, we show that the classic tower property holds for all cases of often used insurance contracts and the $(\tilde{\mathcal{P}}, \mathbb{G})$-conditional expectation maps $L^1(\tilde{\Omega})$ into $L^1(\tilde{\Omega})$ in these cases.

5.4 Pricing and hedging insurance products under model uncertainty

Let $[0,T]$ with $0 < T < \infty$ be a fixed time horizon. Now we apply the results of the previous sections to the building blocks of life insurance contracts. In Proposition 5.4.14 we show that the classic tower property is valid for $(\tilde{\mathcal{P}},\mathbb{G})$-conditional expectation and the property of belonging to $L^1(\tilde{\Omega})$ is invariant under $(\tilde{\mathcal{P}},\mathbb{G})$-conditional expectation in all these cases. The constructed family of sublinear conditional expectations can be hence used as pricing operators in view of the superhedging results in Section 5.2.3.

Let the filtration $\mathbb{G}^{\tilde{\mathcal{P}}} := (\mathcal{G}_t^{\tilde{\mathcal{P}}})_{t\in[0,T]}$ be defined by

$$\mathcal{G}_t^{\tilde{\mathcal{P}}} := \mathcal{G}_t^* \vee \mathcal{N}_T^{\tilde{\mathcal{P}}}, \qquad t \in [0,T],$$

where $\mathcal{N}_T^{\tilde{\mathcal{P}}}$ is the collection of sets which are $(\tilde{P},\mathcal{G}_T)$-null for all $\tilde{P} \in \tilde{\mathcal{P}}$, and $\tilde{A} := (\tilde{A}_t)_{t\in[0,T]}$ be a nonnegative $\mathbb{G}^{\tilde{\mathcal{P}}}$-adapted process with nondecreasing paths, such that $\tilde{A}_t$ is upper semianalytic for all $t \in [0,T]$ and $\tilde{A}_0 = 0$. The process $\tilde{A}$ is interpreted as an (eventually discounted) cumulative payment stream on the extended market. Let S be an m-dimensional $\mathbb{G}^{\tilde{\mathcal{P}}}$-adapted process with càdlàg paths, which is a $(\tilde{P},\mathbb{G}^{\tilde{P}})$-semimartingale for all $\tilde{P} \in \tilde{\mathcal{P}}$, representing (eventually discounted) tradable assets on the enlarged market and $L(S,\mathbb{G}^{\tilde{\mathcal{P}}},\tilde{\mathcal{P}})$ be the set of all m-dimensional $\mathbb{G}^{\tilde{\mathcal{P}}}$-predictable processes which are S-integrable for all $\tilde{P} \in \tilde{\mathcal{P}}$. We define the following set of admissible strategies on the extended market,

$$\tilde{\Delta} := \left\{ L(S,\mathbb{G}^{\tilde{\mathcal{P}}},\tilde{\mathcal{P}}) : {}^{(\tilde{P})}\!\!\int \tilde{\delta}\mathrm{d}S \text{ is a } (\tilde{P},\mathbb{G}_+^{\tilde{P}})\text{-supermartingale for all } \tilde{P} \in \tilde{\mathcal{P}} \right\},$$

where

$$^{(\tilde{P})}\!\!\int \tilde{\delta}\mathrm{d}S := \left({}^{(\tilde{P})}\!\!\int_0^t \tilde{\delta}\mathrm{d}S \right)_{t\in[0,T]}$$

denotes the usual Itô integral under $\tilde{P}$. Robust global and local superhedging strategies, robust global and local superhedging prices and the sets $\tilde{\mathcal{C}}_s^t$ with $0 \leqslant s \leqslant t \leqslant T$ are defined analogously as in Section 5.2.3.
Theorem 5.4.1 and Theorem 5.4.2 can be proved similarly to Theorem 5.2.21 and Theorem 5.2.22 for the $\mathbb{F}$-filtration.

Theorem 5.4.1. *Let Assumption 5.2.11 hold for $\tilde{\mathcal{P}}$ and $\tilde{A} := (\tilde{A}_t)_{t\in[0,T]}$ be a cumulative payment stream with $\tilde{\mathcal{E}}_t(\tilde{A}_T) < \infty$ for all $t \in [0,T]$. If the tower property holds for A_t with $t \in [0,T]$, i.e.*

$$\tilde{\mathcal{E}}_r(\tilde{A}_t) = \tilde{\mathcal{E}}_r(\tilde{\mathcal{E}}_s(\tilde{A}_t)) \quad \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}}, \quad 0 \leqslant r \leqslant s \leqslant t,$$

and if there is a $\mathbb{G}^{\mathcal{P}}$-adapted process $\tilde{Y} = (\tilde{Y}_s)_{s\in[0,T]}$ with càdlàg paths, such that for all $s \in [0,T]$ it holds

$$\tilde{Y}_s = \tilde{\mathcal{E}}_s(\tilde{A}_t) \quad \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}},$$

then the following equivalent dualities hold for all $\tilde{P} \in \tilde{\mathcal{P}}$ *and* $0 \leqslant s \leqslant t \leqslant T$*:*

$$\begin{aligned}
&\tilde{\mathcal{E}}_s(\tilde{A}_t)\\
=&\text{ess inf}^{\tilde{P}}\{\tilde{v} \text{ is } \mathcal{G}_s^{\tilde{\mathcal{P}}}\text{-measurable} : \exists \tilde{\delta} \in \tilde{\Delta} \text{ such that } \tilde{v} + {}^{(\tilde{P}')}\!\!\int_s^t \tilde{\delta}_u \mathrm{d}S_u \geqslant \tilde{A}_t \ \tilde{P}'\text{-a.s.}\\
&\text{for all } \tilde{P}' \in \tilde{\mathcal{P}}\} \quad \tilde{P}\text{-a.s.}\\
=&\text{ess inf}^{\tilde{P}}\{\tilde{v} \text{ is } \mathcal{G}_s^{\tilde{\mathcal{P}}}\text{-measurable} : \exists \tilde{\delta} \in \tilde{\Delta} \text{ such that } \tilde{v} + {}^{(\tilde{P}')}\!\!\int_s^t \tilde{\delta}_u \mathrm{d}S_u \geqslant \tilde{A}_t \ \tilde{P}'\text{-a.s.}\\
&\text{for all } \tilde{P}' \in \tilde{\mathcal{P}}(s;\tilde{P})\} \quad \tilde{P}\text{-a.s.},
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\mathcal{E}}_s(\tilde{A}_t - \tilde{A}_s)\\
=&\text{ess inf}^{\tilde{P}}\{\tilde{v} \text{ is } \mathcal{G}_s^{\tilde{\mathcal{P}}}\text{-measurable} : \exists \tilde{\delta} \in \tilde{\Delta} \text{ such that } \tilde{v} + {}^{(\tilde{P}')}\!\!\int_s^t \tilde{\delta}_u \mathrm{d}S_u \geqslant \tilde{A}_t - \tilde{A}_s\\
&\tilde{P}'\text{-a.s. for all } \tilde{P}' \in \tilde{\mathcal{P}}\} \quad \tilde{P}\text{-a.s.}\\
=&\text{ess inf}^{\tilde{P}}\{\tilde{v} \text{ is } \mathcal{G}_s^{\tilde{\mathcal{P}}}\text{-measurable} : \exists \tilde{\delta} \in \tilde{\Delta} \text{ such that } \tilde{v} + {}^{(\tilde{P}')}\!\!\int_s^t \tilde{\delta}_u \mathrm{d}S_u \geqslant \tilde{A}_t - \tilde{A}_s\\
&\tilde{P}'\text{-a.s. for all } \tilde{P}' \in \tilde{\mathcal{P}}(s;\tilde{P})\} \quad \tilde{P}\text{-a.s.}
\end{aligned}$$

Proof. It is sufficient to apply Theorem 5.2.14 to the measurable space $\tilde{\Omega}$ with filtration $\mathbb{G}^{\tilde{\mathcal{P}}}$ and to the processes $\tilde{Y}$ and $\tilde{Z}$, as in the proof of Theorem 5.2.21. □

Theorem 5.4.2. *Under the same assumptions in Theorem 5.4.1, for* $0 \leqslant s \leqslant t \leqslant T$*, the following statements hold.*

1. *The set* $\tilde{\mathcal{C}}_0^T$ *is not empty.*
2. *The robust global superhedging price of* $\tilde{A}$ *is given by* $\tilde{\mathcal{E}}(\tilde{A}_T)$ *and the robust local superhedging price of* $\tilde{A}$ *on the interval* $[s,t]$ *is given by* $\tilde{\mathcal{E}}_s(\tilde{A}_t - \tilde{A}_s)$*.*
3. *Optimal superhedging strategies exist.*

Proof. Analogue to Theorem 5.2.22. □

5.4.1 Pure endowment

We recall that the payoff of a pure endowment contract is given by $\mathbf{1}_{\{\tilde{\tau}>T\}}Y$, where Y is an $\mathcal{F}_T^{\mathcal{P}}$-measurable nonnegative upper semianalytic function such that $\mathcal{E}(|Y|) < \infty$. The associated cumulative payment stream $\tilde{A} := (\tilde{A}_t)_{t\in[0,T]}$ is hence

$$\tilde{A}_t = \mathbf{1}_{\{\tilde{\tau}>T\}} Y \mathbf{1}_{\{t=T\}}, \quad t \in [0,T]. \tag{5.4.1}$$

Lemma 5.4.3. *For every $t \in [0,T]$, the functions*

$$\mathbf{1}_{\{\tilde{\tau}>T\}}Y \quad \text{and} \quad Ye^{-\int_t^T \mu_u \mathrm{d}u}$$

are upper semianalytic and $\mathcal{G}^{\mathcal{P}}$-measurable. Moreover, it holds pathwisely for every $t \in [0,T]$ that

$$\tilde{\mathcal{E}}_t\left(\mathbf{1}_{\{\tilde{\tau}>T\}}Y\right) = \mathbf{1}_{\{\tilde{\tau}>t\}}\mathcal{E}_t\left(Ye^{-\int_t^T \mu_u \mathrm{d}u}\right). \tag{5.4.2}$$

Proof. Firstly, $\mathbf{1}_{\{\tilde{\tau}>T\}}$ and $e^{-\int_t^T \mu_u \mathrm{d}u}$ are nonnegative Borel-measurable functions. Point 5 of Lemma 5.3.17 implies that

$$\mathbf{1}_{\{\tilde{\tau}>T\}}Y \quad \text{and} \quad Ye^{-\int_t^T \mu_u \mathrm{d}u}$$

are upper semianalytic and $\mathcal{G}^{\mathcal{P}}$-measurable. Equality (5.4.2) is a direct consequence of (5.3.13) and the fact that Y does not depend on $\hat{\omega} \in \hat{\Omega}$:

$$\begin{aligned}\tilde{\mathcal{E}}_t(\mathbf{1}_{\{\tilde{\tau}>T\}}Y) =&\mathbf{1}_{\{\tilde{\tau}>t\}}\mathcal{E}_t(e^{\Gamma_t}E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>T\}}Y])\\ =&\mathbf{1}_{\{\tilde{\tau}>t\}}\mathcal{E}_t(Ye^{\Gamma_t-\Gamma_T})\\ =&\mathbf{1}_{\{\tilde{\tau}>t\}}\mathcal{E}_t(Ye^{-\int_t^T \mu_u \mathrm{d}u}).\end{aligned}$$

□

Proposition 5.4.4. *If the family of probability measures $\mathcal{P}$ is tight and the processes μ and Y are bounded and continuous in ω on a set $A \in \mathcal{B}(\Omega)$ such that $P(A^c) = 0$ for each $P \in \mathcal{P}$, then the process*

$$\left(\tilde{\mathcal{E}}_t\left(\mathbf{1}_{\{\tilde{\tau}>T\}}Y\right)\right)_{t\in[0,T]}$$

is $\mathbb{G}^*$-adapted and equivalent to a càdlàg process $Y := (Y_t)_{t\in[0,T]}$ $\tilde{P}$-a.s. for all $\tilde{P} \in \tilde{\mathcal{P}}$.

Proof. Clearly, the process is $\mathbb{G}^*$-adapted by definition. For every $t \in [0,T]$, Lemma 5.4.3 yields

$$\begin{aligned}\tilde{\mathcal{E}}_t\left(\mathbf{1}_{\{\tilde{\tau}>T\}}Y\right) &= \mathbf{1}_{\{\tilde{\tau}>t\}}\mathcal{E}_t\left(Ye^{-\int_t^T \mu_u \mathrm{d}u}\right)\\ &= \mathbf{1}_{\{\tilde{\tau}>t\}}e^{\int_0^t \mu_u \mathrm{d}u}\mathcal{E}_t\left(Ye^{-\int_0^T \mu_u \mathrm{d}u}\right) \qquad \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}},\end{aligned}$$

Under the above assumptions, Proposition 5.2.8 shows that

$$\left(\mathcal{E}_t\left(Ye^{-\int_0^T \mu_u \mathrm{d}u}\right)\right)_{t\in[0,T]}$$

is càdlàg. The thesis follows immediately. □

Lemma 5.4.5. *It holds that*

$$\tilde{\mathcal{E}}_t(\mathbf{1}_{\{\tilde{\tau}>T\}}Y) \in L^1(\tilde{\Omega}),$$

and

$$\tilde{\mathcal{E}}_s(\tilde{\mathcal{E}}_t(\mathbf{1}_{\{\tilde{\tau}>T\}}Y)) = \tilde{\mathcal{E}}_s(\mathbf{1}_{\{\tilde{\tau}>T\}}Y) \qquad \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}},$$

for all $s, t \in [0,T]$ with $s \leqslant t$.

Proof. Firstly, (5.4.2) in Lemma 5.4.3 yields

$$\tilde{\mathcal{E}}_t(\mathbf{1}_{\{\tilde{\tau}>T\}}Y) = \mathbf{1}_{\{\tilde{\tau}>t\}}\mathcal{E}_t(e^{\Gamma_t-\Gamma_T}Y),$$

where the right-hand side belongs to $L^1(\tilde{\Omega})$ by using step 1 of the proof of Theorem 2.3 in [80]. Regarding the tower property, on one hand we have

$$\tilde{\mathcal{E}}_s(\mathbf{1}_{\{\tilde{\tau}>T\}}Y) = \mathbf{1}_{\{\tilde{\tau}>s\}}\mathcal{E}_s(e^{\Gamma_s-\Gamma_T}Y),$$

On the other hand,[5] we have

$$\begin{aligned}\tilde{\mathcal{E}}_s(\tilde{\mathcal{E}}_t(\mathbf{1}_{\{\tilde{\tau}>T\}}Y)) &= \tilde{\mathcal{E}}_s(\mathbf{1}_{\{\tilde{\tau}>t\}}\mathcal{E}_t(e^{\Gamma_t-\Gamma_T}Y)) \\ &= \mathbf{1}_{\{\tilde{\tau}>s\}}\mathcal{E}_s(e^{\Gamma_s-\Gamma_t}\mathcal{E}_t(e^{\Gamma_t-\Gamma_T}Y)) \\ &= \mathbf{1}_{\{\tilde{\tau}>s\}}\mathcal{E}_s(\mathcal{E}_t(e^{\Gamma_s-\Gamma_t+\Gamma_t-\Gamma_T}Y)) \\ &= \mathbf{1}_{\{\tilde{\tau}>s\}}\mathcal{E}_s(e^{\Gamma_s-\Gamma_T}Y) \quad \tilde{P}\text{-a.s. for all } \tilde{P}\in\tilde{\mathcal{P}}.\end{aligned}$$

□

In view of the above results, we can apply Theorem 5.4.1 and Theorem 5.4.2 to solve the pricing and hedging problem for pure endowment contracts.

Corollary 5.4.6. *For the cumulative payment stream $\tilde{A}$ as in (5.4.1), the operator $\tilde{\mathcal{E}}_t(\mathbf{1}_{\{\tilde{\tau}>T\}}Y)$ defines a robust local superhedging price for every $t\in[0,T]$ and optimal superhedging strategy for $\tilde{A}$ exists.*

5.4.2 Term insurance

A term insurance contract has payoff represented by $\mathbf{1}_{\{0<\tilde{\tau}\leqslant T\}}Z_{\tilde{\tau}}$, where $Z := (Z_t)_{t\in[0,T]}$ is an $\mathbb{F}^{\mathcal{P}}$-predictable nonnegative process, such that the function $Z(t,\omega) := Z_t(\omega), (t,\omega) \in [0,T]\times\Omega$, is upper semianalytic and $\sup_{t\in[0,T]}\mathcal{E}(|Z_t|) < \infty$. The associated cumulative payment stream $\tilde{A}$ is given by

$$\tilde{A}_0 = 0, \qquad \tilde{A}_t = \mathbf{1}_{\{0<\tilde{\tau}\leqslant t\}}Z_{\tilde{\tau}}, \qquad t\in[0,T]. \tag{5.4.3}$$

Lemma 5.4.7. *It holds under every $\tilde{P}\in\tilde{\mathcal{P}}$ with $\tilde{P} = P\otimes\hat{P}$ that*

$$E^{\tilde{P}}\left[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}Z_{\tilde{\tau}}\middle|\,\mathcal{G}_s\right] = \mathbf{1}_{\{\tilde{\tau}>s\}}E^P\left[\int_s^t Z_u e^{-\int_s^u \mu_v \mathrm{d}v}\mu_u \mathrm{d}u\middle|\,\mathcal{F}_s\right] \qquad \tilde{P}\text{-}a.s., \tag{5.4.4}$$

for $s,t\in[0,T]$ with $s\leqslant t$.

Proof. Let $\tilde{P}\in\tilde{\mathcal{P}}$ and $0\leqslant s\leqslant t\leqslant T$. By using Proposition 5.3.8, Proposition 5.1.1 and Corollary 5.1.3 of [24], which hold without the usual conditions on the filtrations, we get

$$E^{\tilde{P}}\left[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}Z_{\tilde{\tau}}\middle|\,\mathcal{G}_s\right] = \mathbf{1}_{\{\tilde{\tau}>s\}}E^{\tilde{P}}\left[\int_s^t Z_u e^{-\int_s^u \mu_v \mathrm{d}v}\mu_u \mathrm{d}u\middle|\,\mathcal{F}_s\right] \qquad \tilde{P}\text{-a.s.}$$

Hence, the $\tilde{P}$-a.s. equality (5.4.4) follows from $P\otimes\hat{P}|_{(\Omega,\mathcal{F})} = P$. □

[5] We note that the first two equalities below hold also pathwisely, but we can plug $(e^{\Gamma_s-\Gamma_t})$ into $\mathcal{E}_t$ only in the $\tilde{P}$-a.s. sense for each $\tilde{P}\in\tilde{\mathcal{P}}$.

Corollary 5.4.8. *The functions*

$$\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}} Z_{\tilde{\tau}} \quad \text{and} \quad \int_s^t Z_u e^{-\int_s^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u$$

are upper semianalytic and $\mathcal{G}^{\mathcal{P}}$-measurable, for all $s, t \in [0,T]$ with $s \leqslant t$. Furthermore,

$$\tilde{\mathcal{E}}_s\left(\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}} Z_{\tilde{\tau}}\right) = \mathbf{1}_{\{\tilde{\tau}>s\}} \mathcal{E}_s\left(\int_s^t Z_u e^{-\int_s^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u\right) \quad \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}}, \tag{5.4.5}$$

for all $s, t \in [0,T]$ with $s \leqslant t$.
If in addition Z is stepwise and $\mathbb{F}$-predictable, i.e.

$$Z_t = \sum_{i=0}^n Z_{t_i} \mathbf{1}_{\{t_i<t\leqslant t_{i+1}\}}, \qquad t \in [0,T],$$

where $t_0 = s < \cdots < t_{n+1} = t$, Z_{t_i} is $\mathcal{F}_{t_i}$-measurable for all $i = 0, ..., n$, then equality (5.4.5) holds pathwisely, i.e.

$$\tilde{\mathcal{E}}_s\left(\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}} Z_{\tilde{\tau}}\right) = \mathbf{1}_{\{\tilde{\tau}>s\}} \mathcal{E}_s\left(\int_s^t Z_u e^{-\int_s^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u\right). \tag{5.4.6}$$

Proof. We stress that point 6 of Lemma 5.3.17 holds also for $Y = [0,T]$, $\kappa(\mathrm{d}y; x) \equiv \mathrm{d}y$. This together with points 3 and 5 of Lemma 5.3.17 yields that

$$\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}} Z_{\tilde{\tau}} \quad \text{and} \quad \int_s^t Z_u e^{-\int_s^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u$$

are upper semianalytic and $\mathcal{G}^{\mathcal{P}}$-measurable. Equality (5.4.5) is a consequence of Lemma 5.4.7 and point 3 of Remark 5.3.20.

If Z is a stepwise $\mathbb{F}$-predictable process, it holds by (5.3.13)

$$\begin{aligned}
\tilde{\mathcal{E}}_s\left(\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}} Z_{\tilde{\tau}}\right) &= \mathbf{1}_{\{\tilde{\tau}>s\}} \mathcal{E}_s\left(e^{\Gamma_s} E^{\hat{P}}\left[\sum_{i=0}^n Z_{t_i} \mathbf{1}_{\{t_i<\tilde{\tau}\leqslant t_{i+1}\}}\right]\right)\\
&= \mathbf{1}_{\{\tilde{\tau}>s\}} \mathcal{E}_s\left(e^{\Gamma_s} \sum_{i=0}^n Z_{t_i} E^{\hat{P}}\left[\mathbf{1}_{\{t_i<\tilde{\tau}\leqslant t_{i+1}\}}\right]\right)\\
&= \mathbf{1}_{\{\tilde{\tau}>s\}} \mathcal{E}_s\left(e^{\Gamma_s} \sum_{i=0}^n Z_{t_i} \left(E^{\hat{P}}\left[\mathbf{1}_{\{\tilde{\tau}>t_i\}}\right] - E^{\hat{P}}\left[\mathbf{1}_{\{\tilde{\tau}>t_{i+1}\}}\right]\right)\right)\\
&= \mathbf{1}_{\{\tilde{\tau}>s\}} \mathcal{E}_s\left(e^{\Gamma_s} \sum_{i=0}^n Z_{t_i} (e^{-\Gamma_{t_i}} - e^{-\Gamma_{t_{i+1}}})\right)\\
&= \mathbf{1}_{\{\tilde{\tau}>s\}} \mathcal{E}_s\left(\int_s^t Z_u e^{\Gamma_s - \Gamma_u} \mathrm{d}\Gamma_u\right)\\
&= \mathbf{1}_{\{\tilde{\tau}>s\}} \mathcal{E}_s\left(\int_s^t Z_u e^{-\int_s^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u\right),
\end{aligned}$$

where the integrals above are pathwise Lebesgue–Stieltjes integrals. □

Proposition 5.4.9. *If the family $\mathcal{P}$ is tight and the processes μ and Z are bounded and continuous in ω on a set $A \in \mathcal{B}(\Omega)$ such that $P(A^c) = 0$ for each $P \in \mathcal{P}$, then the process*

$$\left(\tilde{\mathcal{E}}_t\left(\mathbf{1}_{\{0<\tilde{\tau}\leqslant T\}} Z_{\tilde{\tau}}\right)\right)_{t\in[0,T]},$$

is $\mathbb{G}^*$-adapted and equivalent to a càdlàg process $Y := (Y_t)_{t\in[0,T]}$ $\tilde{P}$-a.s. for all $\tilde{P} \in \tilde{\mathcal{P}}$.

Proof. Similar to the proof of Proposition 5.4.4 in view of Corollary 5.4.8. □

Lemma 5.4.10. *We have*

$$\tilde{\mathcal{E}}_t(\mathbf{1}_{\{0<\tilde{\tau}\leqslant T\}} Z_{\tilde{\tau}}) \in L^1(\tilde{\Omega}),$$

and

$$\tilde{\mathcal{E}}_s(\tilde{\mathcal{E}}_t(\mathbf{1}_{\{0<\tilde{\tau}\leqslant T\}} Z_{\tilde{\tau}})) = \tilde{\mathcal{E}}_s(\mathbf{1}_{\{0<\tilde{\tau}\leqslant T\}} Z_{\tilde{\tau}}) \qquad \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}},$$

for all $s, t \in [0, T]$ with $s \leqslant t$,

Proof. By (5.3.13), Remark 5.3.20 (5) and (5.4.6) in Corollary 5.4.8, we obtain

$$\begin{aligned}\tilde{\mathcal{E}}_s(\mathbf{1}_{\{0<\tilde{\tau}\leqslant T\}} Z_{\tilde{\tau}}) &= \mathbf{1}_{\{\tilde{\tau}\leqslant s\}} Z_{\tilde{\tau}} + \tilde{\mathcal{E}}_s\left(\mathbf{1}_{\{s<\tilde{\tau}\leqslant T\}} Z_{\tilde{\tau}}\right) \\ &= \mathbf{1}_{\{\tilde{\tau}\leqslant s\}} Z_{\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>s\}} \mathcal{E}_s\left(\int_s^T Z_u e^{-\int_s^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u\right),\end{aligned}$$

where the right-hand side belongs to $L^1(\tilde{\Omega})$ by the assumption on Z and step 1 of the proof of Theorem 2.3 in [80]. Moreover, computations similar to the ones in the proof of Corollary 5.4.8 shows that for $s \leqslant t$,

$$
\begin{aligned}
&\tilde{\mathcal{E}}_s(\tilde{\mathcal{E}}_t(\mathbf{1}_{\{0<\tilde{\tau}\leqslant T\}}Z_{\tilde{\tau}}))\\
=&\tilde{\mathcal{E}}_s\left(\mathbf{1}_{\{\tilde{\tau}\leqslant t\}}Z_{\tilde{\tau}}+\mathbf{1}_{\{\tilde{\tau}>t\}}\mathcal{E}_t\left(\int_t^T Z_u e^{-\int_t^u \mu_v \mathrm{d}v}\mu_u \mathrm{d}u\right)\right)\\
=&\mathbf{1}_{\{\tilde{\tau}\leqslant s\}}Z_{\tilde{\tau}}\\
&+\mathbf{1}_{\{\tilde{\tau}>s\}}\mathcal{E}_s\left(e^{\Gamma_s}E^{\hat{P}}\left[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}Z_{\tilde{\tau}}+\mathbf{1}_{\{\tilde{\tau}>t\}}\mathcal{E}_t\left(\int_t^T Z_u e^{-\int_t^u \mu_v \mathrm{d}v}\mu_u \mathrm{d}u\right)\right]\right)\\
=&\mathbf{1}_{\{\tilde{\tau}\leqslant s\}}Z_{\tilde{\tau}}\\
&+\mathbf{1}_{\{\tilde{\tau}>s\}}\mathcal{E}_s\left(e^{\Gamma_s}E^{\hat{P}}\left[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}Z_{\tilde{\tau}}\right]+e^{\Gamma_s}E^{\hat{P}}\left[\mathbf{1}_{\{\tilde{\tau}>t\}}\right]\mathcal{E}_t\left(\int_t^T Z_u e^{-\int_t^u \mu_v \mathrm{d}v}\mu_u \mathrm{d}u\right)\right)\\
=&\mathbf{1}_{\{\tilde{\tau}\leqslant s\}}Z_{\tilde{\tau}}\\
&+\mathbf{1}_{\{\tilde{\tau}>s\}}\mathcal{E}_s\left(\int_s^t Z_u e^{\Gamma_s-\Gamma_u}\mathrm{d}\Gamma_u+e^{\Gamma_s-\Gamma_t}\mathcal{E}_t\left(\int_t^T Z_u e^{\Gamma_t-\Gamma_u}\mathrm{d}\Gamma_u\right)\right)\\
=&\mathbf{1}_{\{\tilde{\tau}\leqslant s\}}Z_{\tilde{\tau}}+\mathbf{1}_{\{\tilde{\tau}>s\}}\mathcal{E}_s\left(\mathcal{E}_t\left(\int_s^T Z_u e^{\Gamma_s-\Gamma_u}\mathrm{d}\Gamma_u\right)\right)\\
=&\mathbf{1}_{\{\tilde{\tau}\leqslant s\}}Z_{\tilde{\tau}}+\mathbf{1}_{\{\tilde{\tau}>s\}}\mathcal{E}_s\left(\mathcal{E}_t\left(\int_s^T Z_u e^{-\int_s^u \mu_v \mathrm{d}v}\mu_u \mathrm{d}u\right)\right)\\
=&\mathbf{1}_{\{\tilde{\tau}\leqslant s\}}Z_{\tilde{\tau}}+\mathbf{1}_{\{\tilde{\tau}>s\}}\mathcal{E}_s\left(\int_s^T Z_u e^{-\int_s^u \mu_v \mathrm{d}v}\mu_u \mathrm{d}u\right)\qquad \tilde{P}\text{-a.s. for all } \tilde{P}\in\tilde{\mathcal{P}},
\end{aligned}
$$

as wanted. □

Similar to the case of pure endowment, the following corollary gives a solution to the pricing and hedging problem for term insurance contracts in view of the above results combined with Theorem 5.4.1 and Theorem 5.4.2.

Corollary 5.4.11. *For the cumulative payment stream $\tilde{A}$ as in (5.4.3), the operator $\tilde{\mathcal{E}}_s\left(\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}Z_{\tilde{\tau}}\right)$ defines a robust local superhedging price for every $s,t\in[0,T]$ with $s<t$ and optimal superhedging strategy for $\tilde{A}$ exists.*

5.4.3 Annuity

The payoff structure of an annuity contract is given by

$$\int_0^T (1-H_u)\mathrm{d}C_u{}^6=\mathbf{1}_{\{\tilde{\tau}>T\}}C_T+\mathbf{1}_{\{0<\tilde{\tau}\leqslant T\}}C_{\tilde{\tau}},$$

where $C:=(C_t)_{t\in[0,T]}$ is a nonnegative $\mathbb{F}^{\mathcal{P}}$-adapted nondecreasing and continuous process, with $C(t,\omega):=C_t(\omega),(t,\omega)\in[0,T]\times\Omega$, upper semianalytic and $\sup_{t\in[0,T]}\mathcal{E}(|C_t|)<$

[6] This integral is a pathwisely defined Lebesgue–Stieltjes integral.

∞, which represents the cumulative payment. The associated cumulative payment stream $\tilde{A}$ is hence

$$\tilde{A}_0 = 0, \qquad \tilde{A}_t = \mathbf{1}_{\{0<\tilde{\tau}\leqslant t\}} C_{\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>t\}} C_t, \qquad t \in [0,T]. \tag{5.4.7}$$

Lemma 5.4.12. *For every $\tilde{P} \in \tilde{\mathcal{P}}$ with $\tilde{P} = P \otimes \hat{P}$, it holds*

$$\begin{aligned} &E^{\tilde{P}}\left[\int_s^t (1-H_u)\mathrm{d}C_u \middle| \mathcal{G}_s\right] \\ =&\mathbf{1}_{\{\tilde{\tau}>s\}} E^P\left[\int_s^t C_u e^{-\int_s^u \mu_v \mathrm{d}v}\mu_u \mathrm{d}u + C_t e^{-\int_s^t \mu_u \mathrm{d}u} \middle| \mathcal{F}_s\right] \qquad \tilde{P}\text{-a.s.}, \end{aligned} \tag{5.4.8}$$

for all $s, t \in [0,T]$ with $s \leqslant t$.

Proof. Let $\tilde{P} \in \tilde{\mathcal{P}}$ and $0 \leqslant s \leqslant t \leqslant T$. The same proof of the first part of Proposition 5.1.2 of [24], which hold without the usual conditions on the filtrations, together with Proposition 5.3.8 yields

$$\begin{aligned} &E^{\tilde{P}}\left[\int_s^t (1-H_u)\mathrm{d}C_u \middle| \mathcal{G}_s\right] \\ =&\mathbf{1}_{\{\tilde{\tau}>s\}} E^{\tilde{P}}\left[\int_s^t C_u e^{-\int_s^u \mu_v \mathrm{d}v}\mu_u \mathrm{d}u + C_t e^{-\int_s^t \mu_u \mathrm{d}u} \middle| \mathcal{F}_s\right] \qquad \tilde{P}\text{-a.s.} \end{aligned}$$

Hence, $\tilde{P}$-a.s. equality (5.4.8) follows from $P \otimes \hat{P}|_{(\Omega,\mathcal{F})} = P$. ☐

Corollary 5.4.13. *The functions*

$$\int_s^t (1-H_u)\mathrm{d}C_u \quad \text{and} \quad \int_s^t C_u e^{-\int_s^u \mu_v \mathrm{d}v}\mu_u \mathrm{d}u + C_t e^{-\int_s^t \mu_u \mathrm{d}u}$$

are upper semianalytic and $\mathcal{G}^{\mathcal{P}}$-measurable for all $s, t \in [0,T]$ with $s \leqslant t$. Moreover, we have

$$\begin{aligned} &\tilde{\mathcal{E}}_s\left(\int_s^t (1-H_u)\mathrm{d}C_u\right) \\ =&\mathbf{1}_{\{\tilde{\tau}>s\}} \mathcal{E}_s\left(\int_s^t C_u e^{-\int_s^u \mu_v \mathrm{d}v}\mu_u \mathrm{d}u + C_t e^{-\int_s^t \mu_u \mathrm{d}u}\right) \qquad \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}}, \end{aligned} \tag{5.4.9}$$

for all $s, t \in [0,T]$ with $s \leqslant t$.

Proof. Since

$$\int_s^t (1-H_u)\mathrm{d}C_u = \mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}} C_{\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>t\}} C_t,$$

points 2, 4, 5 and 6 of Lemma 5.3.17 yield that

$$\int_s^t (1-H_u)\mathrm{d}C_u \quad \text{and} \quad \int_s^t C_u e^{-\int_s^u \mu_v \mathrm{d}v}\mu_u \mathrm{d}u + C_t e^{-\int_s^t \mu_u \mathrm{d}u}$$

are upper semianalytic and $\mathcal{G}^{\mathcal{P}}$-measurable. Furthermore, equality (5.4.9) follows from Lemma 5.4.12 and point 3 of Remark 5.3.20. ☐

Proposition 5.4.14. *If the family $\mathcal{P}$ is tight and the processes μ and C are bounded and continuous in ω on a set $A \in \mathcal{B}(\Omega)$ such that $P(A^c) = 0$ for each $P \in \mathcal{P}$, then the process*

$$\left(\tilde{\mathcal{E}}_t \left(\int_0^T (1 - H_u) \mathrm{d}C_u \right) \right)_{t \in [0,T]}$$

is $\mathbb{G}^*$-adapted and equivalent to a càdlàg process $Y := (Y_t)_{t\in[0,T]}$ $\tilde{P}$-a.s. for all $\tilde{P} \in \tilde{\mathcal{P}}$.

Proof. Similar to Proposition 5.4.4 in view of Corollary 5.4.13. □

The following proposition generalizes the results in Lemma 5.4.5 and Lemma 5.4.10. In particular, it shows that the $(\tilde{P}, \mathbb{G})$-conditional expectation maps $L^1(\tilde{\Omega})$ into $L^1(\tilde{\Omega})$ and the classic tower property holds for annuity contracts.

Proposition 5.4.15. *Let $Z := (Z_t)_{t\in[0,T]}$ be an $\mathbb{F}^{\mathcal{P}}$-predictable process and Y an $\mathcal{G}^{\mathcal{P}}$-measurable upper semianalytic function. If*

$$\tilde{X} = \mathbf{1}_{\{0<\tau\leqslant T\}} Z_{\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>T\}} Y,$$

then

$$\tilde{\mathcal{E}}_t(\tilde{X}) \in L^1(\tilde{\Omega}),$$

and the tower property holds, i.e.

$$\tilde{\mathcal{E}}_s(\tilde{\mathcal{E}}_t(\tilde{X})) = \tilde{\mathcal{E}}_s(\tilde{X}) \qquad \tilde{P}\text{-a.s. for all } \tilde{P} \in \tilde{\mathcal{P}},$$

for all $s, t \in [0,T]$ with $s \leqslant t$.

Proof. Let $t \in [0,T]$. Similar to Lemma 5.4.3 and Corollary 5.4.8, it is immediate to prove that the above $(\tilde{\mathcal{P}}, \mathbb{G})$-conditional expectations are well defined and $\tilde{\mathcal{E}}(|\tilde{X}|) < \infty$. Now we show that

$$\tilde{\mathcal{E}}(|\tilde{\mathcal{E}}_t(\tilde{X})|) < \infty.$$

By computations analogue to the ones in Theorem 5.3.22 and Corollary 5.4.8, we have

$$\begin{aligned}
&\sup_{\tilde{P}\in\tilde{\mathcal{P}}} E^{\tilde{P}} \left[\left| \tilde{\mathcal{E}}_t(\tilde{X}) \right| \right] \\
&\leqslant \sup_{\tilde{P}\in\tilde{\mathcal{P}}} E^{\tilde{P}} \left[\left| \mathbf{1}_{\{\tilde{\tau}\leqslant t\}} \; \mathcal{E}_t(\varphi(x,\cdot))|_{x=\tilde{\tau}} \right| \right] + \sup_{\tilde{P}\in\tilde{\mathcal{P}}} E^{\tilde{P}} \left[\left| \mathbf{1}_{\{\tilde{\tau}>t\}} \mathcal{E}_t \left(e^{\Gamma_t} E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}} \tilde{X}] \right) \right| \right] \\
&= \sup_{\tilde{P}\in\tilde{\mathcal{P}}} E^{\tilde{P}} \left[\left| \mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}} Z_{\tilde{\tau}} \right| \right] + \sup_{P\in\mathcal{P}} E^P \left[E^{\hat{P}} \left[\left| \mathbf{1}_{\{\tilde{\tau}>t\}} \mathcal{E}_t \left(e^{\Gamma_t} E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>T\}} Y] \right) \right| \right] \right] \\
&= \sup_{P\in\mathcal{P}} E^P \left[E^{\hat{P}} \left[\left| \mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}} Z_{\tilde{\tau}} \right| \right] \right] + \sup_{P\in\mathcal{P}} E^P \left[\left| \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>T\}} Y]) \right| \right] \\
&\leqslant \sup_{P\in\mathcal{P}} E^P \left[\int_s^t |Z_u| e^{-\Gamma_u} \mathrm{d}\Gamma_u \right] + \sup_{P\in\mathcal{P}} E^P \left[\left| E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>T\}} Y] \right| \right] \\
&\leqslant \int_s^t \sup_{P\in\mathcal{P}} E^P[|Z_u|] e^{-\Gamma_u} \mathrm{d}\Gamma_u + \sup_{P\in\mathcal{P}} E^P \left[|Y| \right] \\
&< \infty,
\end{aligned}$$

where in the second inequality we use Step 1 of the proof of Theorem 2.3 in [80] applied to the second component. Hence, for every $t > 0$, $\tilde{\mathcal{E}}_t(\tilde{X})$ still belongs to $L^1(\tilde{\Omega})$.

Now we show the tower property. Let $\tilde{P} \in \tilde{\mathcal{P}}$, according to the proof of Theorem 5.3.22, the classic tower property equals equalities in (5.3.21) and (5.3.22). That is

$$e^{\Gamma_s}\mathcal{E}_s\left(E^{\hat{P}}[\mathcal{E}_t(\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X})] + \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}])\right) = e^{\Gamma_s}\mathcal{E}_s(\mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{X}])) \quad \tilde{P}\text{-a.s.}$$

Calculations similar to the ones in Corollary 5.4.8 yield

$$\begin{aligned}
& E^{\hat{P}}[\mathcal{E}_t(\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}\tilde{X})] + \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>t\}}\tilde{X}]) \\
=& E^{\hat{P}}[\mathcal{E}_t(\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}Z_{\tilde{\tau}})] + \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{t<\tilde{\tau}\leqslant T\}}Z_{\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>T\}}Y]) \\
=& E^{\hat{P}}[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}Z_{\tilde{\tau}}] + \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{t<\tilde{\tau}\leqslant T\}}Z_{\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>T\}}Y]) \\
=& \int_s^t Z_u e^{-\Gamma_u}\mathrm{d}\Gamma_u + \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{t<\tilde{\tau}\leqslant T\}}Z_{\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>T\}}Y]) \\
=& \mathcal{E}_t\left(\int_s^t Z_u e^{-\Gamma_u}\mathrm{d}\Gamma_u + E^{\hat{P}}[\mathbf{1}_{\{t<\tilde{\tau}\leqslant T\}}Z_{\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>T\}}Y]\right) \\
=& \mathcal{E}_t\left(E^{\hat{P}}[\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}Z_{\tilde{\tau}}] + E^{\hat{P}}[\mathbf{1}_{\{t<\tilde{\tau}\leqslant T\}}Z_{\tilde{\tau}} + \mathbf{1}_{\{\tilde{\tau}>T\}}Y]\right) \\
=& \mathcal{E}_t(E^{\hat{P}}[\mathbf{1}_{\{\tilde{\tau}>s\}}\tilde{X}]) \quad \tilde{P}\text{-a.s..}
\end{aligned}$$

We emphasize that for fixed $\hat{\omega}$, $\mathbf{1}_{\{s<\tilde{\tau}\leqslant t\}}Z_{\tilde{\tau}}$ is $\mathcal{F}_t^{\mathcal{P}}$-measurable, and $\int_s^t Z_u e^{-\Gamma_u}\mathrm{d}\Gamma_u$ is $\mathcal{F}_t^{\mathcal{P}}$-measurable as well. □

The superhedging problem can be hence solved also for annuity contracts.

Corollary 5.4.16. *For the cumulative payment stream $\tilde{A}$ as in (5.4.7), the operator $\tilde{\mathcal{E}}_s\left(\int_s^t (1-H_u)\mathrm{d}C_u\right)$ defines a robust local superhedging price for every $s,t \in [0,T]$ with $s < t$ and optimal superhedging strategy for $\tilde{A}$ exists.*

Appendix A

Background uncertainty framework

Here we present briefly a background of the robust stochastic theory with respect to a nondominated probability family, ranging from capacity theory to the G-setting in e.g. [91], [39] and the pathwise setting in e.g. [85], [87] and [82]. Some secondary results not included in the paper [20] are present as well. If not otherwise specified, we adopt the same notations in Chapter 5.

Capacity theory is a generalization of the usual measure theory and is the starting point of the stochastic calculus under uncertainty. Let $\mathcal{P} \subseteq \mathcal{P}(\Omega)$ be a generic nonempty set of probability measures. The upper probability associated to $\mathcal{P}$ is defined by,

$$\nu(A) := \sup \{P(A) : P \in \mathcal{P}\}, \quad \text{for every } A \in \mathcal{B}(\Omega). \tag{A.0.1}$$

The probability family $\mathcal{P}$ is called a set that represents the upper probability ν. We note that there can be more probability families which represent the same upper probability ν and the maximum set is given by

$$\mathcal{P}_\nu := \{P \in \mathcal{P}(\Omega) : P(A) \leqslant \nu(A) \text{ for all } A \in \mathcal{B}(\Omega)\}. \tag{A.0.2}$$

We denote the convex hull of $\mathcal{P}$ by $\hat{\mathcal{P}}$, then the following inclusions hold

$$\mathcal{P} \subseteq \hat{\mathcal{P}} \subseteq \mathcal{P}_\nu.$$

Some examples in [59] shows that these inclusions can be strict. The following definition can be found in [30] and [37].[1]

Definition A.0.1. *A* Choquet capacity *or* capacity *c is a function $c : \mathcal{B}(\Omega) \to \mathbb{R}^+$ such that*

1. *$c(\emptyset) = 0$;*
2. *if $A \subseteq B$, then $c(A) \leqslant c(B)$;*

[1] We stress that the general capacity theory does not require any structure on the set Ω.

3. *if $A_n \uparrow A$, then $c(A_n) \uparrow c(A)$;*

4. *If $F_n \downarrow F$, F_n closed, then $c(F_n) \downarrow c(F)$.*

It is easy to check that $\nu(\Omega) = 1$ and ν satisfies properties 1, 2 and 3 of Definition A.0.1. If $\mathcal{P}$ is weakly compact, i.e. compact under the topology of weakly convergence, then property 4 of Definition A.0.1 also holds, hence ν is a proper Choquet capacity. See e.g. Lemma 2.3 in [59]. Following [109], we use a slight abuse of terminology[2], and call a set A a $\mathcal{P}$-polar set or ν-polar set if $P(A) = 0$ for all $P \in \mathcal{P}$; we say that a property holds $\mathcal{P}$-quasi-surely or ν-quasi surely if it holds outside a $\mathcal{P}$-polar set, i.e. it holds almost surely for all $P \in \mathcal{P}$.
We define the upper expectation $E^{\mathcal{P}} : L^0(\Omega) \to \overline{\mathbb{R}}$ associated to $\mathcal{P}$,

$$E^{\mathcal{P}}[X] := \sup_{P \in \mathcal{P}} E^P[X], \qquad \text{for every } X \in L^0(\Omega), \tag{A.0.3}$$

where $L^0(\Omega)$ is the space of all $\mathcal{B}(\Omega)$-measurable functions. The following definitions are given in [39] and [92][3].

Definition A.0.2. *Let $\mathcal{H}$ be a vector lattice of real-valued functions defined on Ω and containing constants. A* sublinear expectation E on $\mathcal{H}$ *is a functional $E : \mathcal{H} \to \overline{\mathbb{R}}$ with the following properties*

1. *monotonicity: $E[X] \geqslant E[Y]$ if $X \geqslant Y$;*

2. *sub-additivity: $E[X + Y] \leqslant E[X] + E[Y]$;*

3. *positive homogeneity: $E[\lambda X] = \lambda E[X]$ for $\lambda \geqslant 0$;*

4. *constant translatability: $E[X + c] = E[X] + c$ for all $c \in \mathbb{R}$.*

E is called regular *if for each sequence $\{X_n\}_{n\in\mathbb{N}}$ in $C_b(\Omega)$ with $X_n \downarrow 0$ on Ω, we have $E[X_n] \downarrow 0$.*

Clearly, every upper expectation $E^{\mathcal{P}}$ is a sublinear expectation on $L^0(\Omega)$. Theorem 2.1 and Remark 2.2 of Chapter I in [92] show that every regular sublinear expectation E can be represented as an upper expectation as in (A.0.3), where $\mathcal{P}$ is a family of probability measures. We give the following theorem for completeness.

Theorem A.0.3. *Given a weakly closed set $\mathcal{P} \subseteq \mathcal{P}(\Omega)$, its associated upper probability ν and upper expectation $E^{\mathcal{P}}$, the following statements are equivalent:*

1. *$\mathcal{P}$ is tight, i.e. for every $\varepsilon > 0$, there exists a compact set $K \in \mathcal{B}(\Omega)$ such that $P(K^c) < \varepsilon$ for all $P \in \mathcal{P}$;*

2. *for every $\varepsilon > 0$, there is a compact set $K \in \mathcal{B}(\Omega)$ such that $\nu(K^c) < \varepsilon$;*

3. *$\mathcal{P}$ is sequentially compact (called also relatively compact), i.e. any sequence of elements in $\mathcal{P}$ has a weakly convergent subsequence;*

[2] Traditionally the notion of *quasi-surely* is associated to a capacity.
[3] In the original definition of sublinear expectation Ω is a generic set.

4. $\mathcal{P}$ *is weakly compact;*

5. $\mathcal{P}_\nu$ *is weakly compact;*

6. ν *is a Choquet capacity;*

7. $E^{\mathcal{P}}$ *is a regular sublinear expectation.*

Proof. The equivalence among statements 1, 3 and 4 is the Prokhorov's Theorem. Equivalence between statement 1 and 2 is a direct consequence of definition (A.0.1) of ν. By Lemma 2.2 in [59], 6 implies 5. Statement 5 implies 4 because $\mathcal{P}$ is a weakly closed subset of $\mathcal{P}_\nu$. The implication from 4 to 6 is given by Lemma 2.3 in [59]. Theorem 12 in [39] shows the equivalence between 3 and 7. □

We state here an observation related to our construction in Section 5.3.1. If $\mathcal{P}$ is a generic probability family, $\hat{P}$ is a probability measure on another Polish space $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$ and the family $\tilde{\mathcal{P}}$ is defined by

$$\tilde{\mathcal{P}} := \left\{ \tilde{P} \in \mathcal{P}(\tilde{\Omega}) : \tilde{P} = P \otimes \hat{P},\ P \in \mathcal{P} \right\},$$

then the following holds.

Proposition A.0.4. *$\tilde{\mathcal{P}}$ is weakly compact if and only if $\mathcal{P}$ is weakly compact.*

Proof. we note that according to Prokhorov's Theorem, weak compactness is equivalent to weak closeness and tightness. Theorem 3.8 in [25] shows that $\tilde{\mathcal{P}}$ is weakly closed if and only if $\mathcal{P}$ is weakly closed.
Now we suppose that $\mathcal{P}$ is weakly compact and we want to prove that $\tilde{\mathcal{P}}$ is tight. By Theorem A.0.3, this is equivalent to find a compact set $\tilde{K}$ on the product space $\tilde{\Omega}$ such that $\tilde{\nu}(\tilde{K}^c) < \varepsilon$ for every $\varepsilon > 0$. We note that it is enough to assume ε small enough, i.e. $0 < \varepsilon < 1$. Since $\mathcal{P}$ is compact, there is a compact set K on Ω such that

$$\nu\left(K^c\right) = \sup_{P \in \mathcal{P}} P\left(K^c\right) < \frac{\sqrt{\varepsilon}}{3}.$$

We observe that the singleton $\hat{P}$ is weakly compact. Indeed, since $\hat{\Omega}$ is complete separable, by Theorem 1.3 of [25] $\hat{P}$ is tight. It is also closed as a point of the Fréchet space $\hat{P}$, hence weakly compact by Theorem A.0.3. In view of the compactness of $\hat{P}$, we can find a compact set K^* on Ω^* such that

$$P^*\left((K^*)^c\right) < \frac{\sqrt{\varepsilon}}{3}.$$

Then $\tilde{K} = K \times K^*$ is the compact set on the product space we are looking for. Indeed

$$\begin{aligned}
\tilde{\nu}\left((K \times K^*)^c\right) &= \sup_{P \otimes P^* \in \tilde{\mathcal{P}}} P \otimes P^*\left((K \times K^*)^c\right) = \sup_{P \in \mathcal{P}} P \otimes P^*\left((K \times K^*)^c\right) \\
&= \sup_{P \in \mathcal{P}} P \otimes P^*\left((K^c \times (K^*)^c) \cup (K \times (K^*)^c) \cup (K^c \times K^*)\right) \\
&\leqslant \sup_{P \in \mathcal{P}} P\left(K^c\right) P^*\left((K^*)^c\right) + \sup_{P \in \mathcal{P}} P\left(K\right) P^*\left((K^*)^c\right) \\
&\quad + \sup_{P \in \mathcal{P}} P\left(K^c\right) P^*\left(K^*\right) \\
&\leqslant \sup_{P \in \mathcal{P}} P\left(K^c\right) P^*\left((K^*)^c\right) + P^*\left((K^*)^c\right) + \sup_{P \in \mathcal{P}} P\left(K^c\right) \\
&< \frac{\sqrt{\varepsilon}}{3} \cdot \frac{\sqrt{\varepsilon}}{3} + \frac{\sqrt{\varepsilon}}{3} + \frac{\sqrt{\varepsilon}}{3} \leqslant \varepsilon.
\end{aligned}$$

The third last inequality follows from the fact that every probability measure assigns mass less or equal than one.
Now we suppose that $\tilde{\mathcal{P}}$ is weakly compact. Still by Theorem A.0.3, for every $\varepsilon > 0$ there is a compact set $\tilde{K}$ on $\tilde{\Omega}$ such that

$$\tilde{\nu}\left(\tilde{K}^c\right) = \sup_{P \otimes P^* \in \tilde{\mathcal{P}}} P \otimes P^*\left(\tilde{K}^c\right) < \varepsilon.$$

Let K be the continuous projection of $\tilde{K}$ on Ω, in particular we have $\tilde{K} \subseteq K \times \Omega^*$. Then

$$\begin{aligned}
\nu\left(K^c\right) = \sup_{P \in \mathcal{P}} P\left(K^c\right) &= \sup_{P \in \mathcal{P}} P\left(K^c\right) P^*(\Omega^*) \\
&= \sup_{P \otimes P^* \in \tilde{\mathcal{P}}} P \otimes P^*\left((K \times \Omega^*)^c\right) \\
&\leqslant \sup_{P \otimes P^* \in \tilde{\mathcal{P}}} P \otimes P^*\left(\tilde{K}^c\right) < \varepsilon.
\end{aligned}$$

This concludes the proof.

□

Capacity theory is however not sufficient for a deeper stochastic analysis under uncertainty, which requires extension of notions such as conditional expectation, martingale, stochastic integral, etc. S. Peng introduced first in [91] the notions of G-expectation and G-Brownian motion. The G-setting is then developed into a systematic robust stochastic calculus system, with G-martingale decomposition, Itô-type calculus, etc. See e.g. [91], [73], [112], [113], [53], [58], [92] and [93]. We show here a basic construction of the G-setting and its relation with the capacity theory.
Let P^B denote the Wiener measure on $(\Omega, \mathcal{B}(\Omega))$ where $\Omega = C_0([0,T], \mathbb{R})$. In particular, the canonical process B is a Brownian motion on the probability space $(\Omega, \mathcal{B}(\Omega), P^B)$. We denote by $\tilde{\mathbb{F}}$ the filtration generated by B and completed with all P^B-null sets. We fix two numbers $\underline{\sigma}$ and $\overline{\sigma}$ such that $\underline{\sigma} \leqslant \overline{\sigma}$. Let $\mathcal{A}$ be the following set

$$\begin{aligned}
\mathcal{A} = \{&\tilde{\mathbb{F}} - \text{progressively measurable processes on } (\Omega, \mathcal{B}(\Omega), P^B) \text{ which} \\
&\text{take values in } [\underline{\sigma}^2, \overline{\sigma}^2] \text{ and are } P^B - \text{a.s. square integrable}\}.
\end{aligned}$$

For every $\alpha := (\alpha_t)_{t\in[0,T]}$ such that $\alpha \in \mathcal{A}$, the following process $X^\alpha := (X_t^\alpha)_{t\in[0,T]}$ with

$$X_t^\alpha := \int_0^t \alpha_s \mathrm{d}B_s, \quad \text{for every } t \in [0,T],$$

is well defined and continuous outside a P^B-null set A. By setting $X^\alpha(\omega) \equiv 0$ for every $\omega \in A$, X^α can be considered as a random variable taking values in $(\Omega, \mathcal{B}(\Omega))$

$$X^\alpha : (\Omega, \mathcal{B}(\Omega), P^B) \to (\Omega, \mathcal{B}(\Omega)),$$

with

$$X^\alpha(\omega)(t) := X_t^\alpha(\omega).$$

We consider the probability measure family $\mathcal{P} \subseteq \mathcal{P}(\Omega)$ defined by

$$\mathcal{P} := \overline{\mathcal{P}_0},$$

$$\mathcal{P}_0 = \left\{ P^B \circ (X^\alpha)^{-1} : \alpha \in \mathcal{A} \right\},$$

that is, $\mathcal{P}$ is the weak closure of the family of probability measures induced by X^α for all $\alpha \in \mathcal{A}$. Proposition 49 in [39] shows that $\mathcal{P}_0$ is tight, then by Theorem A.0.3 $\mathcal{P}$ is weakly compact, the associated upper expectation c is a Choquet capacity and $E^{\mathcal{P}}$ is a regular sublinear expectations. The same probability family can be also represented as in the sequel. By Lemma 3.2 in [87], the following subset of (true) martingale measures

$$\mathcal{P}' = \{P \in \mathcal{P}(\Omega^1) : B \text{ is a ture } P\text{-martingale}, \langle B \rangle^P \text{ is absolutely continuous } P\text{-a.s.}, \\ \mathrm{d}\langle B \rangle^P/\mathrm{d}t \in [\underline{\sigma}^2, \overline{\sigma}^2] \ P \times \mathrm{d}t\text{-a.e.}\},$$

is weakly closed. By Proposition 3.5 of [42], we have furthermore

$$\overline{\hat{\mathcal{P}}_0} = \mathcal{P}'.$$

Since $\overline{\mathcal{P}_0} \subseteq \overline{\hat{\mathcal{P}}_0} \subseteq \mathcal{P}_c$, $\mathcal{P}$ and $\mathcal{P}'$ determine the same capacity and sublinear expectation, in particular

$$E^{\mathcal{P}} = E^{\mathcal{P}'}.$$

According to [39] and [92], the upper expectation $E^{\mathcal{P}}$ constructed above is a G-expectation and the canonical process B is a G-Brownian motion. Here G denotes the following function which characterizes the G-normal distribution $N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$

$$G : \mathbb{R} \to \mathbb{R},$$

$$G(x) = \frac{1}{2} \sup_{\sigma^2 \in [\underline{\sigma}^2, \overline{\sigma}^2]} x\sigma^2 = \frac{1}{2}\overline{\sigma}^2 x^+ - \frac{1}{2}\underline{\sigma}^2 x^-,$$

where x^+ and x^- are respectively the positive and negative parts of x.

Within the G-setting, however, the robust stochastic calculus is always limited to a weakly compact probability family and cannot easily treat processes with jumps. A different method with pathwise approach is recently proposed in e.g. [85], [87] and [82], which is the one we adopted in Chapter 5. Beside the results presented in Chapter 5, we present here an independent result for the aggregation problem of the stochastic integration, which is traditionally defined as limit in probability. Some partial results about its pathwise construction are given e.g. in [22] and [67]. An extension of these results can be found in [85], where the filtration is assumed to be universally augmented and the integration is constructed for all generic predictable processes with respect to a large class of processes which are semimartingale under each $P \in \mathcal{P}$. In the following proposition we state a slightly different version of Theorem 2.2 in [85]. The construction is based on [67]. Comparing to Theorem 2.2 in [85], it requires less regularity of the filtration and more regularity of the processes.

Proposition A.0.5 (Stochastic integral). *Let $Z := (Z_t)_{t\in[0,T]}$ be an $\mathbb{F}$-adapted pathwise continuous process and $X := (X_t)_{t\in[0,T]}$ a pathwise continuous process which is $\mathbb{F}$-local martingale under every $P \in \mathcal{P}$, where $\mathcal{P}$ is a generic probability family. Then there exists an $\mathbb{F}$-adapted $\mathcal{P}$-q.s. continuous process, denoted by $\int Z\mathrm{d}X = (\int_0^t Z\mathrm{d}X)_{t\in[0,T]}$, such that*

$$\int Z\mathrm{d}X = {}^{(P)}\!\!\int Z\mathrm{d}X, \qquad P-a.s.\ \textit{for all}\ \ P \in \mathcal{P},$$

where ${}^{(P)}\!\!\int Z\mathrm{d}X$ denotes the Itô integral under P.

Proof. The construction mainly follows [67]. A similar proof can be found in [88, Proposition 4.11] with slightly different measurability result.

For every $n \geqslant 1$, we define pathwisely a sequence of random times $(\tau_i^n)_{i\geqslant 0}$ where $\tau_0^n = 0$ and for $i \geqslant 0$

$$\tau_{i+1}^n := \inf\left\{t \geqslant \tau_i^n : \left|Z_t - Z_{\tau_i^n}\right| \geqslant 2^{-n}\right\}.$$

Since Z is pathwise continuous, $(\tau_i^n)_{i\geqslant 0}$ is a sequence of $\mathbb{F}$-stopping time even when $\mathbb{F}$ is a raw filtration[4]. Consequently, the simple process $I^n := (I_t^n)_{t\in[0,T]}$ defined as follows,

$$I_t^n := \sum_{i=0}^{k-1} Z_{\tau_i^n}(X_{\tau_{i+1}^n} - X_{\tau_i^n}) + Z_{\tau_k^n}(X_t - X_{\tau_k^n}),$$

where $\tau_k^n \leqslant t < \tau_{k+1}^n$, $k \geqslant 0$, is $\mathbb{F}$-adapted. It is also continuous thanks to the continuity of X. We define pathwisely

$$\int_0^t Z\mathrm{d}X := \lim_{n\to\infty} I_t^n, \qquad \text{for all } t \in [0,T],$$

which is still $\mathbb{F}$-adapted. By the same arguments in the proof of [67, Theorem 1], we have for every $P \in \mathcal{P}$

$$\sup_{0\leqslant t\leqslant T}\left|I_t^n - {}^{(P)}\!\!\int_0^t Z\mathrm{d}X\right| \to 0, \qquad P-\text{a.s.}.$$

In particular, $\int Z\mathrm{d}X$ is a limit of $(I^n)_{n\geqslant 0}$ uniformly in t outside a $\mathcal{P}$-polar set. This yields at the same time the $\mathcal{P}$-q.s. continuity of $\int Z\mathrm{d}X$ and the fact that $\int Z\mathrm{d}X$ coincides P-a.s. with the Itô integral ${}^{(P)}\!\!\int Z\mathrm{d}X$ for all $P \in \mathcal{P}$. □

[4]When Z is only càdlàg, the filtration needs to be universally completed.

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