Topological Aspects of Classical and Quantum Black Hole Hair

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München, 2018

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Dissertation
an der Fakultät für Physik
der Ludwig–Maximilians–Universität
München

vorgelegt von
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aus Filderstadt

München, den 14.05.2018

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Zweitgutachter: Prof. Dr. Stefan Hofmann
Tag der mündlichen Prüfung: 02.07.2018

Contents

III Summary of Project 2 and Related Aspects:

IV Summary of Project 3:

V Summary of Project 4:

VI Summary of Project 5:

List of Figures

Zusammenfassung

Das Studium Schwarzer Löcher - das sind Objekte mit einer Fluchtgeschwindigkeit größer als die Lichtgeschwindigkeit, die in Folge eines Gravitationskollapses oder in hochenergetischen Teilchenkollisionen entstehen können - ist ein faszinierendes Forschungsfeld in der Astrophysik, in der Teilchenphysik und in der Kosmologie. Aus Sicht der beobachtenden Astrophysik wurden in den letzten Jahrzehnten große Fortschritte erzielt. Insbesondere wurden viele (Kandidaten für) astrophysikalische Schwarze Löcher in unserem Universum entdeckt und kürzlich wurde die Detektion von Gravitationswellen verschmelzender Schwarzer Löcher gefeiert.

In der theoretischen Physik Schwarzer Löcher, auf die wir uns in dieser Arbeit beschränken, wurden Hypothesen formuliert und Theoreme bewiesen, die viel darüber aussagen wie Schwarze Löcher entstehen und was für Eigenschaften sie haben. Die berühmte Cosmic Censorship Hypothese besagt, dass Schwarze Löcher in *jedem* Gravitationskollaps, der zu einer Raumzeitsingularität führt, entstehen. Die No-Hair Hypothese besagt, dass man stationäre Schwarze Löcher eindeutig durch Parameter, die einem klassischen Gaußschen Gesetz folgen, charakterisieren kann. Semiklassisch ist bekannt, dass Schwarze Löcher Objekte sind, denen eine Entropie zugeordnet werden kann und die thermische Hawkingstrahlung emittieren.

Haare Schwarzer Löcher, das sind Parameter, die Schwarze Löcher charakterisieren, aber keinem klassischen Gaußschen Gesetz folgen, existieren in Übereinstimmung mit der No-Hair Hypothese in vielen konkreten Modellen nicht - ein Faktum, das in sogenannten Eindeutigkeits- und No-Hair Theoremen (unter bestimmten Voraussetzungen) bewiesen wurde. Heutzutage wissen wir jedoch, dass im Allgemeinen Gegenbeispiele zur No-Hair Hypothese existieren - es gibt Modelle, in denen haarige Schwarze Löcher als Lösungen existieren. Das sind auf der einen Seite Schwarze Löcher mit klassischen Haaren und auf der anderen Seite Schwarze Löcher mit quantenmechanischen Haaren (letzteres sind Parameter, die nicht klassisch, sondern nur quantenmechanisch gemessen werden können). Einiger dieser

bekannten Schwarzen Löcher mit Haaren sind nicht relevant in dem Sinne dass sie dynamisch instabil sind und man deshalb nicht erwartet, dass sie in einem realistischen physikalischen Prozess auftreten. Es gibt allerdings auch Schwarze Löcher mit Haaren, von denen wir wissen, dass sie dynamisch stabil sind oder die zumindest Stabilitätssymptome zeigen, die stabil gegen lineare Störungen sind.

Viele dieser Schwarzen Löcher mit Haaren, die dynamisch stabil zu sein scheinen, stehen eng in Verbindung mit dem Konzept der Topologie: Auf der einen Seite haben diejenigen asymptotisch flachen und sphärisch symmetrischen Schwarzen Löcher mit klassischen Haaren, von denen wir wissen, dass sie dynamisch stabil gegen lineare Störungen sind, es alle gemeinsam, dass sie Lösungen der Einsteinschen Feldgleichungen in einer Theorie mit an die Gravitation gekoppelter Lagrangefunktion, welche topologische Solitonen als Lösungen hat, sind. Auf der anderen Seite können viele Schwarze Löcher mit quantenmechanischen Haaren in Beziehung zu einer nicht verschwindenden Aharonov-Bohm-Phase gebracht werden.

In dieser Arbeit untersuchen wir verschiedene topologische Aspekte von Schwarzen Löchern mit Haaren und darüber hinaus. Erstens argumentieren wir für vorgegebene asymptotische Parameter, dass man ein Schwarzes Loch ohne klassische Haare mit diesen Parametern von einem Schwarzen Loch mit klassischen Haaren und mit denselben Parametern durch die Streuung von Wellen unterscheiden kann. Wir demonstrieren dies, indem wir die differenziellen Wirkungsquerschnitte für viele konkrete Beispiele von Schwarzen Löchern mit und ohne Haare numerisch bestimmen. Zweitens studieren wir das konkrete Beispiel von skyrmionischen Haaren Schwarzer Löcher. Wir zeigen, dass es neben den bekannten Schwarzen Löchern mit klassischen skyrmionischen Haaren auch Schwarze Löcher mit quantenmechanischen skyrmionischen Haaren vom Aharonov-Bohmschen Typ gibt. Die Kombination der Schwarzen Löcher mit klassischen skyrmionischen Haaren und derjenigen mit quantenmechanischen skyrmionischen Haaren hat viele interessante physikalische Konsequenzen, die wir im Detail untersuchen; vor allem weisen wir auf eine Lücke in dem Beweis der sogenannten Folk Theoreme hin, die es erlaubt, ein selbstkonsistentes Szenario zu formulieren, in dem Schwarze Löcher Baryonenzahl nicht verletzen. Drittens diskutieren wir verschiedene Aspekte von kosmischen Strings in berandeten Raumzeiten, insbesondere argumentieren wir aus Sicht des Randes unter welchen Bedingungen diese kosmischen Strings fraktionale Statistik haben können und stellen heraus wie solche kosmische Strings in Situationen mit Schwarzen Löchern mit Aharonov-Bohm Z_N Haaren auftreten.

Abstract

The study of black holes, that are objects with escape velocity larger than the speed of light which can be formed in gravitational collapse and in high energy particle collisions, is a fascinating research topic in particle physics, astrophysics and cosmology. From the observational point of view a lot of progress has been achieved within the last decades. In particular, many (candidates for) astrophysical black holes have been discovered in our universe and recently the detection of gravitational waves from a black hole merger has been celebrated.

In theoretical black hole physics, which we will focus on in this thesis, conjectures have been stated and theorems have been proven which tell us a lot about black hole formation and black hole properties. According to the famous Cosmic Censorship conjecture, black holes are formed in *every* gravitational collapse which leads to a spacetime with a singularity. According to the no-hair conjecture stationary black holes can be uniquely characterized by parameters associated to a classical Gauss law. Semi-classically, black holes are known to be thermal objects which carry entropy and emit thermal "Hawking" radiation.

Black hole hairs, that are parameters which characterize a black hole but are not associated to a classical Gauss law, do, in agreement with the no-hair conjecture, not exist in many concrete models, a fact which has been proven (under certain assumptions) in so-called uniqueness and no-hair theorems. Today, we know however that in general there are counterexamples to the no-hair conjecture, there are models in which black hole solutions with hair do exist! These hairy black holes are black holes with classical hair on the one hand and black holes with quantum hair (that are black hole parameters not measurable classically but only quantum mechanically) on the other hand. Some of these known hairy black holes are not very relevant in the sense that they are dynamically unstable and are therefore not expected to be formed in any realistic physical process. There are however also hairy black holes which are known to be dynamically stable or which at least show symptoms of dynamical stability, that is they are known to be stable against linear perturbations.

Many of these hairy black holes which seem to be dynamically stable are known to be tightly related to the concept of topology: On the one hand, the asymptotically flat and spherically symmetric black holes with classical hair which are known to be stable against perturbations all have in common that they are obtained as solutions of the Einstein field equations in a theory with a matter Lagrangian coupled to gravity which allows for topological solitons as solutions. On the other hand, many black hole quantum hairs can be related to a non-vanishing Aharonov-Bohm phase shift.

In this thesis we study several topological aspects of hairy black holes and beyond.

First, we argue that, for given asymptotic characteristics, one can distinguish a black hole with classical hair with these asymptotic characteristics from a black hole without classical hair with the same asymptotic characteristics via classical scattering of waves. We demonstrate the viability of this proposal by calculating differential scattering cross sections for many concrete examples of hairy and non-hairy black holes numerically.

Second, we study the particular case of skyrmion black hole hair. We show that on top of the known black holes with classical skyrmion hair also black holes with quantum Aharonov-Bohm-type skyrmion hair do exist. The connection of these black holes with classical skyrmion hair on the one hand and with quantum skyrmion hair on the other hand has many interesting physical consequences which we discuss in detail. Most importantly, we point out a loophole in the black hole folk theorems argument which allows for a self-consistent possibility of baryon number conservation by semi-classical black holes.

Third, we discuss several aspects of cosmic strings in spacetimes with boundary, in particular we argue from a boundary point of view under which conditions these cosmic strings can obey fractional anyon-type statistics and point out how such anyonic cosmic strings appear in situations of black holes with discrete quantum Z_N Aharonov-Bohm-type hair.

Conference Talks

The work for this thesis was carried out at the Arnold Sommerfeld Center for theoretical physics at LMU Munich under the supervision of Prof. Georgi Dvali. Some parts of this work have been presented by the author in many internal seminar talks as well as in the talks presented at the following conferences/workshops:

- Conference "Recent Developments in Strings and Gravity" 2016 at the Corfu Summer Institute (plenary talk)
- "LMU Particle Physics Meeting" 2016 in Garmisch-Patenkirchen (plenary talk)
- DESY Theory Workshop "Rethinking Quantum Field Theory" 2016 in Hamburg (parallel talk)
- "LMU Particle Physics Retreat" 2017 in Bayrischzell (plenary talk)
- Conference "3rd Karl-Schwarzschild Meeting" 2017 in Frankfurt (plenary talk, honorably mentioned as one of the best talks of the meeting given by a PhD student)

Thesis Publications

This thesis is in part based on the following publications. The thesis also contains many (so far) unpublished results.

- Skyrmion Black Hole Hair: Conservation of Baryon Number by Black Holes and Observable Manifestations,
  by G. Dvali and A. Gußmann,
  published in: **Nucl. Phys. B913** (2016) 1001-1036

- Scattering of Massless Scalar Waves by Magnetically Charged Black Holes in Einstein-Yang-Mills-Higgs Theory,
  by A. Gußmann,
  published in: **Class. Quant. Grav. 34** (2017) no.6, 065007

- Aharonov-Bohm Protection of Black Hole's Baryon/Skyrmion Hair,
  by G. Dvali and A. Gußmann,
  published in: **Phys. Lett. B768** (2017) 274-279

- Aspects of Skyrmion Black Hole Hair,
  by A. Gußmann,
  Conference Proceedings,
  published in: **PoS CORFU2016** (2017) 089

Part I

Introduction

In this introduction, topics, on which our research papers and our unpublished results which we will summarize in the next parts of this thesis are based on, are reviewed in a self-consistent way.

In chapter 1, we start with discussing the forces, both gravitational and non-gravitational, and the kinds of matter we are dealing with when describing the physics of our universe. We briefly review some elements of the fundamental theories which are nowadays used to describe these forces and different kinds of matter in our universe. We put most emphasis on the paradigm of effective descriptions which implies that on different length scales (and different energy scales) matter and forces in our universe can be properly described by different "effective theories". We discuss topological solitons and review how one particular class of topological solitons, so-called "skyrmions", can be viewed as effective descriptions of baryons at low energies.

In chapter 2, we focus on particular objects in our universe, on black holes. We review several aspects of stationary and asymptotically-flat black holes, in particular black hole uniqueness theorems, the black hole no-hair conjecture and black hole no-hair theorems. We review certain types of classical hair of asymptotically-flat black holes and discuss the role of topological solitons in the context of classical black hole hair. We review the formalism of classical wave scattering by black holes and point out that scattering of classical waves can be useful to detect classical black hole hair. We then discuss semi-classical and quantum black holes, black hole thermodynamics and several types of quantum black hole hair. We briefly review the idea of holography in the context of black hole physics and mention further developments of this idea which became known as "gauge gravity duality".

In chapter 3, we give a short outline about the following parts of this thesis.

1. Forces and Matter

1.1 Matter and Forces in our Universe

Since today, we have learned a lot about the universe we are living in [Muk05]: In particular we know that our universe is approximately 15 billion years old, it is isotropic and homogeneous on large scales, has structure on smaller scales and is expanding. The expansion of our universe is driven by its matter content. We know that our universe is composed out of less than 10 percent baryonic matter. The rest we call dark matter (the non-baryonic part of the matter content which has zero pressure) and dark energy (the part which has negative pressure). Currently we are living in a period of time in which dark energy starts to be the dominating kind of matter for driving the expansion of the universe.

How a given matter component drives the expansion of the universe is described by the laws of gravity. Nowadays we use the laws of Einstein's (field) theory of general relativity as laws of gravity on scales which are significantly larger than the Planck size.

On top of the gravitational interactions, there are also other (non-gravitational) interactions between the matter components of our universe which we are describing today on a fundamental level in terms of gauge field theories.

The history of our universe can be parametrized by energy-scale, 15 billion years ago the typical energies were very high, today they are much smaller. When describing the interactions between the matter components of our universe, depending on the energy scale we are working, we use different low energy "effective" field theories.

Let us now, in the next section, briefly comment on the concept of a field theory and the concept of a low energy "effective" field theory and then, in the following sections, review the particular field theories and the particular low energy effective field theories which are nowadays used in order to describe the physics of our universe.

1.2 Field Theories and Effective Field Theories

The concept of a classical field theory was historically first introduced by Faraday (see e.g. [Far52]) as a theory which has, in contrast to classical theories for point particles, uncountably infinitely many degrees of freedom. Field theories therefore allow us to describe infinitely near actions (and thus allow us to avoid "actions at a distance") by using field quantities which can take different values at different points of the classical spacetime continuum. If we quantize these field quantities in classical field theories, we formally end up with quantum field theories. We shall review the most important quantum field theories which are nowadays used in order to describe the physics of our universe in the next sections. As we will review in these concrete examples, the field quantities can themselves have a fixed definite number of degrees of freedom per spacetime point.

An effective low energy (or large distance) field theory is a field theory which is valid only in a certain domain of large distance (and low energy) scales. If such effective field theories exist for describing properly the physics of our universe, they can therefore be used to properly describe the relevant physical processes happening in a certain domain of low energy (and large distance) scales but typically cannot be used for describing physical processes happening at smaller distance (and higher energy) scales. This implies that, even if we do not know anything about a (fundamental) field theory describing the physics at very high energies, we nevertheless still have a chance to properly describe the physics happening at much lower energy scales by using (low energy) effective theories which are not applicable at the higher energy scales. In particular when our universe was still much younger than today and energies were typically very high, field theories can be used for describing the universe at that time which are different than the (effective) low energy field theories which we use in order to describe the universe today in which typically processes with much lower energies involved are happening.

That such low energy effective theories for properly describing the physics of our universe in fact have a chance to exist needs to be justified in the sense that it has to be shown that the physics of our universe at large distance (and low energy) scales decouples. In nature, as an experimental fact, it indeed seems to be the case that the physics describing phenomena at large distance (and low energy) scales decouples. Thus, adequate physical theories should implement this decoupling property of nature and we have a chance to construct effective (field) theories.[1]

In the context of quantum field theory there are two standard ways for constructing a low energy effective field theory, the so-called "top down approach" and the so-called "bottom up approach". The basic ideas and recipes are very simple and only (especially for the top down approach) require some knowledge about the renormalization group (see [Wil75] and references therein) which we will not review here. On the one hand, in the top down approach to construct a low energy effective field theory, one starts with a known theory valid up to a certain high energy scale and ends up with a theory valid only for lower energy scales by "integrating out" heavy fields with masses above some definite mass scale and by "flowing down" to some smaller energy scale using the renormalization group flow. On the other hand, in the bottom up approach, one constructs an effective theory at lower energy scales without knowing what the appropriate theory valid at higher energy scales is (or if such a theory exists at all) or without knowing how to obtain the low energy effective field theory from a known high energy theory via the top down approach. In this approach the low energy effective field theory can for example be constructed by demanding that the theory contains certain definite fields which are known to exist from experiments and by demanding that the theory obeys certain symmetries.

For discussions on the decoupling properties of quantum field theories as well as for detailed explanations on how to construct effective quantum field theories we refer to original works such as the works of Appelquist and Carrazzone [AC75], of Witten [Wit77] and of Weinberg [Wei80] as well as to reviews such as [Geo93], [Kap95], [Bur07] and [Wei09] and references therein. Here we will not discuss these aspects of effective field theories further. Rather, in the next sections, we will briefly discuss the field theory which we nowadays use in order to describe the non-gravitational interactions (the so-called "standard model of particle physics") and then discuss what kind of effective theories can be used in order to describe some of the interactions of the standard model at lower energy scales. Here we will put particular emphasis on so-called "chiral theories" of strong interactions.[2]

[1] In the simplest classical case, in order to obtain an effective theory for a given domain of energy scales, one can set the parameters of a theory which are much smaller than these energy scales to zero and the parameters of the theory which are much bigger than these energy scales to infinity. In this way an approriate effective theory can be obtained.

[2] Historically, effective chiral theories for the strong interactions were known and used long before the standard model of particle physics was fully invented and understood.

1.3 Standard Model of Particle Physics and Beyond

On a fundamental level we can today describe all the known non-gravitational interactions in our universe successfully with the so-called standard model of particle physics. This standard model needs several input parameters and can describe (almost) all experimental data of particle physics given these input parameters. The standard model is a (quantum) field theory. In the following we will briefly review some basic aspects of this field theory. We will separately discuss the different sectors of the standard model: the electromagnetic sector which describes the electromagnetic interactions via the quantum field theory of quantum electrodynamics, a $U(1)$ gauge theory, the electroweak sector which contains the electromagnetic interactions in a form unified with the weak interactions, a $SU(2) \otimes U(1)$ gauge theory, and the strong sector which describes the strong interactions via the quantum field theory of quantum chromodynamics, a $SU(3)$ gauge theory. We also comment on classical aspects of electromagnetism, on the Aharonov-Bohm effect, on the concept of spontaneous symmetry breaking, on the Georgi-Glashow model, on effective chiral theories of the strong interactions as well as on physics beyond the standard model. For more details we refer to standard textbooks such as [BD65], [PS95], [Wei05], [Wei13b], [Zee03] and to the references therein.

1.3.1 Elements of Classical Electrodynamics, Quantum Electrodynamics and the Aharonov-Bohm Effect

The field theory of classical electromagnetism which describes the dynamics of the classical electromagnetic field degrees of freedom in our universe can be obtained from the Lagrangian of a 4-component field A_μ ($\mu = 0, 1, 2, 3$),

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \tag{1.1}$$

where $F_{\mu\nu} \equiv \partial_{[\mu} A_{\nu]}$.[3] The electromagnetic "vector potential" A_μ is related to the electric field E_i and the magnetic field B_i ($i = 1, 2, 3$) as

$$\begin{aligned} E_i &= -F_{0i} \,, \\ B_i &= \epsilon_{ijk} \partial_j A_k \,. \end{aligned} \tag{1.2}$$

[3]An external source J_μ can couple to A_μ via $\delta\mathcal{L} = A_\mu J^\mu$. (1.1) is therefore often referred to as the Lagrangian of "source-free electrodynamics".

The Lagrangian (1.1) is invariant under

$$A_\mu \longrightarrow A_\mu + \partial_\mu \alpha(x)\,, \tag{1.3}$$

where $\alpha(x)$ is an arbitrary (continuous and single-valued) function. This transformation does not affect the physical quantities E_i and B_i and is often referred to as a "redundancy" or "gauge redundancy". The transformation (1.3) is called a "gauge transformation".

A_μ carries two physical propagating degrees of freedom per spacetime point: The four components A_0, A_1, A_2 and A_3 are in general four degrees of freedom. If A_μ has to be invariant under (1.3), only three out of these four degrees of freedom are left as physical degrees of freedom. On top of that, the Maxwell equations which can be obtained from (1.1) via the principle of least action include the constraint $\text{div}E = 0$ which, when expanded out in terms of A_μ, directly tells us that A_0 is constrained by the spatial components of A_μ for any time, or in other words, that one degree of freedom of A_μ is not propagating. This leaves us with two propagating physical degrees of freedom per spacetime point for A_μ.

In classical electrodynamics it is equivalent to use either the fields E_i and B_i or the vector potential A_μ for describing a given electromagnetic field. In quantum theory this however is not the case as was first pointed out in [AB59]. In fact, there are physical effects which can be captured by using the vector potential but not by using only the electric and magnetic fields E_i and B_i. These are so-called Aharonov-Bohm effects (or generalisations of it) which have been verified experimentally (see e.g. [Cha60]). One of the simplest cases in which this effect can be illustrated is the case in which there is a solonoid with magnetic flux going through it such that outside of the solonoid the magnetic field B_i is completely vanishing (which implies that the vector potential A_μ outside of the solonoid is pure gauge). If we take a charged particle once (or several times) around this solonoid, this process induces a shift in the action $\Delta\mathcal{S}$ of the form

$$\Delta\mathcal{S} = \oint dx^\mu A_\mu\,, \tag{1.4}$$

where dx^μ parameterizes the world line of the particle and A_μ is the vector potential (which outside of the solonoid is pure gauge). In quantum theory such a shift induces a phase shift in the wave function. If we therefore for example perform an interference experiment, the interference pattern gets shifted once an Aharonov-Bohm phase shift is induced. For a more detailed explanation of the Aharonov-Bohm effect we refer to [AB59].

In a quantum description of electromagnetism, the field A_μ has to be quantized. In addition to the Maxwell Lagrangian (1.1) which describes the (quantized) electromagnetic field degrees of freedom, in quantum electrodynamics also the particles which interact via the electromagnetic interactions, for example electrons and positrons, are described by fields which have to be taken into account in the Lagrangian and which have to be quantized as well. Electrons and positrons with mass m are spin-$\frac{1}{2}$ particles which can be described by a 4-component Dirac spinor Ψ [Dir28]. The corresponding Lagrangian for the free Dirac spinor takes the form

$$\mathcal{L} = \bar{\Psi}\left(i\gamma^\mu \partial_\mu - m\right)\Psi \,. \tag{1.5}$$

Here $\bar{\Psi} \equiv \Psi^+\gamma^0$ and γ^μ are the 4×4 Dirac gamma matrices which satisfy the Clifford algebra [Dir28]

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \,. \tag{1.6}$$

The particles with spin $\frac{1}{2}$ obey Fermi-Dirac statistics [Pau40].

There are different procedures which can be used in order to quantize the Lagrangians (1.1) and (1.5) (see e.g. the textbooks [BD65], [PS95], [Wei05], [Wei13b], [Zee03] for some detailed discussions). Let us briefly sketch the so-called "canonical quantization procedure" to quantize Ψ: One promotes the fields Ψ (and their canonical momenta) to operators and imposes the equal time canonical anti-commutation relations on these operators

$$\{\hat{\Psi}_i(x), \hat{\Psi}_j^+(y)\} = \delta^{(3)}(x-y)\delta_{ij} \,, \tag{1.7}$$

with $\hat{\Psi}_i$ the i-th component of the spinor $\hat{\Psi}$. These anti-commutation relations imply that Ψ obeys Fermi-Dirac statistics [PS95]. (For particles obeying Bose-Einstein statistics, the equal time anti-commutation relations have to be replaced by equal time commutation relations [PS95].) One can expand Ψ and $\bar{\Psi}$ in terms of eigenfunctions of the Hamiltonean which corresponds to the Lagrangian (1.5),

$$u_s(p)e^{ipx} \,, \tag{1.8}$$

with eigenvalues E_p and

$$v_s(p)e^{-ipx} \,, \tag{1.9}$$

with eigenvalues $-E_p$, as

$$\hat{\Psi}(x) = \int d^3p \sum_s \frac{1}{\sqrt{2E_p}} \left(\hat{a}_p^s u_s(p) e^{-ipx} + \hat{b}_p^s v_s(p) e^{ipx}\right) \,,$$

$$\hat{\bar{\Psi}}(x) = \int d^3p \sum_s \frac{1}{\sqrt{2E_p}} \left(\hat{a}_p^{s+} \bar{u}_s(p) e^{ipx} + \hat{b}_p^{s+} \bar{v}_s(p) e^{-ipx} \right) . \tag{1.10}$$

The canonical anticommutation relations (1.7) are fulfilled if the operators $\hat{a}_p$ and $\hat{b}_p$ satisfy

$$\{\hat{a}_p^i, \hat{a}_k^{j+}\} = \{\hat{b}_p^i, \hat{b}_k^{j+}\} = \delta^{(3)}(p-k)\delta_{ij} . \tag{1.11}$$

The vacuum state $|0\rangle$ of the theory, occupied by no particles, can then be defined as the state which satisfies

$$\hat{a}_p^s |0\rangle = \hat{b}_p^s |0\rangle = 0 \tag{1.12}$$

and states which are occupied by more particles can be obtained from the vacuum state by acting with the "creation operators" $\hat{a}_p^+$ (creating electrons with energy E_p and momentum p) and $\hat{b}_p^+$ (creating positrons). In this way the complete "Fock space" [Foc32] which spans the whole Hilbert space can be constructed for the theory (1.15).

The quantization of the Maxwell fields of (1.1) follows similar lines as the quantization of the Ψ fields which we just have briefly recapulated. The quantization of A_μ is however a bit more involved because of the gauge redundancy of (1.1) which has to be properly taken into account. We refer to the textbooks [BD65], [PS95], [Wei05], [Wei13b], [Zee03] for discussions on how to quantize the Maxwell field A_μ.

The Lagrangians (1.1) and (1.5) describe non-interacting photons and non-interacting electrons (and positrons). We can describe interactions between these particles if we add a coupling between Ψ and A_μ of the form

$$eA_\mu J^\mu , \tag{1.13}$$

with

$$J_\mu \equiv \bar{\Psi}\gamma_\mu\Psi \tag{1.14}$$

and e a coupling constant. The interacting Lagrangian of quantum electrodynamics becomes then

$$\mathcal{L}_{QED} = \bar{\Psi}\left(i\gamma^\mu D_\mu - m\right)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} , \tag{1.15}$$

where $D_\mu \equiv \partial_\mu - ieA_\mu$ is the covariant derivative.

This Lagrangian is invariant under the gauge transformation

$$A_\mu \longrightarrow A_\mu + \frac{1}{e}\partial_\mu \alpha(x) ,$$

$$\Psi \longrightarrow e^{i\alpha(x)}\Psi\,, \tag{1.16}$$

where $\alpha(x)$ is a gauge transformation function. These transformations can be understood as transformations generated by a $U(1)$ group, (1.15) is therefore often referred to as a $U(1)$ gauge theory.

If the coupling constant e is small enough, the interaction term (1.13) can be considered as a small correction to the Lagrangian without any interaction between Ψ and A_μ and one can do calculations by performing perturbation theory. These perturbative calculations are often visualized using Feynman diagrams [Fey49]. Such a perturbative treatment is often done in quantum field theory and especially in quantum electrodynamics. Many interesting results have been obtained using such perturbative methods [BD65], [PS95], [Wei05], [Wei13b], [Zee03].

The value of the coupling constant e depends on the energy scale physical processes are happening. In nature, e becomes larger when the energy scale increases. Therefore, for small enough energies the perturbative treatment is justified.

In this thesis we however do not focus on these "perturbative aspects" of quantum electrodynamics. Therefore we neither review any of these perturbative techniques and perturbative results nor the quantization procedures in the interacting case here, but refer to the standard textbooks such as [BD65], [PS95], [Wei05], [Wei13b], [Zee03] and the references therein. We will now briefly review some aspects of the other sectors of the standard model of particle physics and then, in the next section, focus on non-perturbative aspects of quantum field theory.

1.3.2 Elements of Electroweak Interactions and Spontaneous Symmetry Breaking

The electromagnetic interactions $U(1)_{EM}$ can be understood as part of a larger unified class of interactions, the class of "electroweak interactions" which is characterized by the gauge group $SU(2)_L \otimes U(1)_Y$. At high enough energies, which were typical in nature for example when our universe was very young, $SU(2)_L \otimes U(1)_Y$ was in fact manifest, or, in other words, it was respected by the ground state of the theory. At low energies $SU(2)_L \otimes U(1)_Y$ gets however "spontaneously broken" down to $U(1)_{EM}$, that means that the ground state developed by the theory is not invariant under the whole group $SU(2)_L \otimes U(1)_Y$ anymore. At low energies, the electroweak interactions can be separated into "weak interactions" which are mediated by massive gauge fields and the electromagnetic interactions which we reviewed above in such a way that weak and electromagnetic interactions have different coupling strengths. The transition from $SU(2)_L \otimes U(1)_Y$ to $U(1)_{EM}$ which happend for example in our early universe when the typical energies decreased, is known as the "electroweak phase transition".

The electroweak Lagrangian can, according to Glashow [Gla61] who found the structure of this Lagrangian and to Weinberg [Wei67] and Salam [Sal68] who wrote it down in its complete form, be written as

$$\mathcal{L} = -\frac{1}{4}W^{a\mu\nu}W^a_{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \mathcal{L}_f + \mathcal{L}_Y + (D^\mu\phi)^+ D_\mu\phi - V(\phi)\,, \quad (1.17)$$

where

$$W^a_{\mu\nu} \equiv \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g\epsilon^{abc}W^b_\mu W^c_\nu\,, \quad (1.18)$$

$$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu\,, \quad (1.19)$$

$$\mathcal{L}_f \equiv \sum_k \left(\bar{\Psi}_{kL} i\gamma^\mu D_\mu \Psi_{kL} + \bar{\Psi}_{kR} i\gamma^\mu D_\mu \Psi_{kR}\right)\,, \quad (1.20)$$

$$\mathcal{L}_Y = \Gamma^{(1)}_{ij}\bar{q}_{jL}\tilde{\phi}q_{iR} + \Gamma^{(2)}_{ij}\bar{q}_{iL}\phi q_{jR} + \Gamma^{(3)}_{ij}\bar{l}_{iL}\phi l_{jR} + \Gamma^{(4)}_{ij}\bar{l}_{jL}\tilde{\phi}\nu_{iR} + \text{h.c.}\,, \quad (1.21)$$

$$V(\phi) \equiv -\frac{1}{2}\mu^2\phi^+\phi + \lambda\left(\phi^+\phi\right)^2\,. \quad (1.22)$$

Here the index a runs from 1 to 3. The scalar field ϕ stands for the two components

$$\phi \equiv \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix}\,, \quad (1.23)$$

with the two complex scalar fields ϕ_+ and ϕ_0. $\tilde{\phi}$ is defined as $\tilde{\phi}_i \equiv \epsilon_{ij}\phi^*_j$ where ϕ^* is the complex conjugate of ϕ. λ and μ are parameters with $\lambda > 0$. The spinors Ψ_R and Ψ_L stand for so-called left- and right-handed Weyl spinors defined as

$$\Psi_L \equiv \frac{1}{2}\left(1-\gamma^5\right)\Psi\,, \quad (1.24)$$

$$\Psi_R \equiv \frac{1}{2}\left(1+\gamma^5\right)\Psi\,, \quad (1.25)$$

with $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. The sums of the left-handed spinors in (1.20) and (1.21) run over the three families of left-handed quark doublets q_L with left-handed spinors for the up-quark u, the down quark d, the charm quark c, the strange quark s, the top quark t and the bottom quark b and the three families of left handed lepton doublets l_L with left-handed spinors for the electron e, the electron neutrino ν_e, the muon μ, the muon neutrino ν_μ, the tauon τ and tau neutrino ν_τ:

$$\Psi_{kL} \equiv \begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L\,. \quad (1.26)$$

The sums of the right-handed spinors in (1.20) and (1.21) run over the right handed quarks q_R and the right handed leptons l_R

$$\Psi_{kR} \equiv u_R, d_R, c_R, s_R, t_R, b_R, e_R, \mu_R, \tau_R \,. \tag{1.27}$$

$\Gamma^{(i)}$ ($i = 1, 2, 3, 4$) are matrices which describe the so-called Yukawa couplings between ϕ and the quarks and leptons. In the quark sector these matrices are not diagonal in the space of families but lead to mixing between the different generations of quarks [KM73].[4]

The covariant derivatives D_μ are defined as

$$D_\mu \phi \equiv \left(\partial_\mu + i\frac{g}{2}\sigma^a W_\mu^a + i\frac{1}{2}\tilde{g} B_\mu \right) \phi \,, \tag{1.28}$$

$$D_\mu \Psi_L \equiv \left(\partial_\mu + i\frac{g}{2}\sigma^a W_\mu^a + i\frac{1}{2}\tilde{g} Y B_\mu \right) \Psi_L \,, \tag{1.29}$$

$$D_\mu \Psi_R \equiv \left(\partial_\mu + i\frac{1}{2}\tilde{g} Y B_\mu \right) \Psi_R \,, \tag{1.30}$$

where Y is the so-called weak hypercharge which is different for different particle species: For left-handed quarks $Y = \frac{1}{6}$, for right-handed up, charm and top quarks $Y = \frac{2}{3}$ and for right-handed down, strange and bottom quarks $Y = -\frac{1}{3}$. For left-handed leptons $Y = -\frac{1}{2}$ and for right-handed leptons $Y = -1$.

The electroweak Lagrangian is invariant under the gauge transformation

$$B_\mu \longrightarrow B_\mu - \frac{1}{\tilde{g}} \partial_\mu \beta(x) \,, \tag{1.31}$$

$$\sigma^a W_\mu^a \longrightarrow e^{i\frac{\sigma^a}{2}\alpha^a(x)} \sigma^a W_\mu^a \left(e^{i\frac{\sigma^a}{2}\alpha^a(x)} \right)^+ + \frac{1}{g} \partial_\mu e^{i\frac{\sigma^a}{2}\alpha^a(x)} \left(e^{i\frac{\sigma^a}{2}\alpha^a(x)} \right)^+ \,, \tag{1.32}$$

$$\Psi_L \longrightarrow e^{iY\beta(x)} e^{i\frac{\sigma^a}{2}\alpha^a(x)} \Psi_L \,, \tag{1.33}$$

$$\Psi_R \longrightarrow e^{iY\beta(x)} \Psi_R \,, \tag{1.34}$$

$$\phi \longrightarrow e^{i\beta(x)} e^{i\frac{\sigma^a}{2}\alpha^a(x)} \phi \,, \tag{1.35}$$

where $\beta(x)$ and $\alpha^i(x)$ are gauge transformation functions. These transformations can be understood as transformations of the non-Abelian group $SU(2)_L \otimes U(1)_Y$.

[4] Note that here we did not take into account the existence of neutrino masses.

The kinetic term of the gauge field in the Lagrangian (1.17) is known as a "Yang-Mills Lagrangian" [YM54].

The fields in this Lagrangian can also, similarly as in the case of quantum electrodynamics, be quantized. We refer to the textbooks such as [BD65], [PS95], [Wei05], [Wei13b], [Zee03] for the techniques needed.

The scalar field ϕ develops a vacuum expectation value because the potential $V(\phi)$ has a minimum at the point

$$\phi^+\phi = \frac{\mu^2}{2\lambda}\,. \tag{1.36}$$

Therefore, classically ϕ is expected to take this non-trivial minimal value instead of the value 0. In such a case, usually fluctuations around this value are considered and quantized. Because of the $SU(2)_L$ invariance, this vacuum expectation value can be chosen on the second component of ϕ and fluctuations around the vacuum expectation value can be parameterized as

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\gamma e^{i\delta} & \sin\gamma e^{i\kappa} \\ -\sin\gamma e^{-i\kappa} & \cos\gamma e^{-i\delta} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{\mu}{\sqrt{\lambda}} + h \end{pmatrix}, \tag{1.37}$$

where h is the so-called Higgs boson and γ, δ, κ are so-called Goldstone angles.

The vacuum state is invariant under a $U(1)$ transformation whereas the Lagrangian is, as we pointed out above, invariant under $SU(2)_L \otimes U(1)_Y$. This effect is called "spontaneous symmetry breaking". The symmetry breaking pattern in this case is often denoted as

$$SU(2)_L \otimes U(1)_Y \longrightarrow U(1)_{QED}\,. \tag{1.38}$$

Spontaneous symmetry breaking can occur also for different Lagrangians in different setups. (We will recall several other examples in the next section when reviewing aspects of topological defects.)

According to Goldstones theorem [Nam60, Gol61] whenever a continuous symmetry is spontaneously broken, massless particles, so-called "Nambu Goldstone bosons", emerge. In general the number of massless Goldstone bosons depend on the number of generators of the broken symmetry group and the number of generators of the subgroup to which the original symmetry group is broken down. In the case of the above-mentioned electroweak symmetry breaking (1.38), three massless Nambu Goldstone bosons emerge. Due to the Higgs effect [EB64, Hig64], the combinations

$$W^\pm_\mu \equiv \frac{1}{\sqrt{2}}\left(W^1_\mu \mp iW^2_\mu\right), Z_\mu \equiv \frac{1}{\sqrt{g^2+\tilde{g}^2}}\left(gW^3_\mu - \tilde{g}B_\mu\right) \tag{1.39}$$

aquire masses and the Goldstone bosons γ, δ and κ become the longitudinal components of these massive gauge fieldes. These massive gauge fields, Z_μ and $W_\mu^\pm$, mediate the so-called weak interactions. The combination

$$A_\mu \equiv \frac{1}{\sqrt{g^2+\tilde{g}^2}}\left(\tilde{g}W_\mu^3 + gB_\mu\right) \tag{1.40}$$

remains massless and can be identified with the gauge field of QED if the electric coupling e is identified with

$$e = \frac{g\tilde{g}}{\sqrt{g^2+\tilde{g}^2}}\,. \tag{1.41}$$

. Also the fermions aquire masses because of the couplings of the Higgs to the fermions in $\mathcal{L}_Y$. One can read off the masses of the gauge fields and fermions by plugging in the expansion of the Higgs field (1.37) into the electroweak Lagrangian.

At energy scales which are much lower than the masses of the Z_μ and $W_\mu^\pm$, these particles can be "integrated out" and we can obtain a low energy effective theory which properly describes weak interactions on these low energy scales. This is the theory of Fermi [Fer34, Wil68] which was known long before the full electroweak Lagrangian has been discovered.

The model proposed by Glashow, Weinberg and Salam was not the only model which went beyond Fermi's theory of the weak interactions and which has been proposed in order to describe the weak and electromagnetic interactions in a unified way. In fact, Georgi and Glashow proposed a different model for describing the electroweak interactions [GG72]. Although this model has been ruled out by experiments as a realistic model for electroweak interactions, it is important both from the point of view of the theory of topological defects which we will discuss in the next section as well as from the point of view of the physics beyond the standard model. The Georgi-Glashow Lagrangian is given by:

$$\mathcal{L} = \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - \frac{\lambda}{4}\left(\phi^a\phi^a - v^2\right)^2 - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}\,. \tag{1.42}$$

Here ϕ^a is a scalar triplett $(a = 1, 2, 3)$, $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\epsilon^{abc}A^b_\mu A^c_\nu$ and $D_\mu\phi^a \equiv \partial_\mu\phi^a - e\epsilon^{abc}A^b_\mu\phi^c$. This Lagrangian is invariant under $SU(2)$, the fields transform in the adjoint representation of $SU(2)$. The symmetry is spontaneously broken down to $U(1)$ by configurations with the scalar field condensed.

The main difference when compared to the model of Glashow, Weinberg and Salam is that in this model the scalar field ϕ transforms as a vector under the adjoint representation of $SU(2)$ whereas ϕ transforms as a spinor under the fundamental representation of $SU(2)$ in the Glashow-Weinberg-Salam model. From this point it follows that, in contrast to the model of Glashow, Weinberg and Salam, no additional $U(1)$ field is needed in (1.42) in order for ending up with one massless gauge field (which could in principle play the role of the massless photon) after spontaneous symmetry breaking [GG72].

1.3.3 Elements of Strong Interactions

The strong interactions are described by the Lagrangian of quantum chromodynamics (QCD),

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_k \bar{\Psi}^i_k \left(i\gamma^\mu \left(D_\mu\right)_{ij} - m_k \delta_{ij} \right) \Psi^j_k + \theta G^a_{\mu\nu} G^a_{\alpha\beta} \epsilon^{\mu\nu\alpha\beta} \,. \tag{1.43}$$

Here a runs from 1 to 8 and i, j run from 1 to 3. Ψ_k are tripletts of Dirac spinors,

$$\Psi_k = \begin{pmatrix} \Psi_{kr} \\ \Psi_{kg} \\ \Psi_{kb} \end{pmatrix} , \tag{1.44}$$

where k runs over all the different quark species. In fact, each quark species k which appeared in the electroweak Lagrangian now comes as a set of three quarks which differ by the new quantum number "color" (N_C): each quark can carry the quantum number $N_C =$ "red", $N_C =$ "green" or $N_C =$ "blue" ("r", "g" or "b"). The quark masses m_k are induced by the Higgs mechanism as pointed out above. The covariant derivative $\left(D_\mu\right)_{ij}$ is defined as

$$\left(D_\mu\right)_{ij} \equiv \delta_{ij} \partial_\mu - 2ig_s \lambda^a_{ij} G^a_\mu \,. \tag{1.45}$$

Here g_s is the strong gauge coupling constant and λ^a_{ij} are the components of the Gell-Mann matrices [GM62] which are generators of $SU(3)$. G^a_μ are the so-called gluon fields and the field strength $G^a_{\mu\nu}$ is defined as

$$G^a_{\mu\nu} \equiv \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu \,. \tag{1.46}$$

The Lagrangian (1.43) is invariant under the $SU(3)$ gauge transformation

$$\Psi \longrightarrow e^{-i\alpha_a(x)\frac{\lambda_a}{2}} \Psi \,, \tag{1.47}$$

$$G^a_\mu \longrightarrow G^a_\mu - \frac{1}{g_s} \partial_\mu \alpha^a(x) + f^{abc} \alpha^b(x) G^c_\mu \,, \tag{1.48}$$

where $\alpha_a(x)$ are gauge transformation functions and f^{abc} are the structure constants of $SU(3)$.

In contrast to the electroweak couplings, the strong coupling g_s gets bigger once the distance scales become larger (once the energy scales become smaller). Therefore, perturbation theory in g_s in quantum chromodynamics is only applicable on small enough distance scales.

Perturbation theory in QCD has also been applied using a different expansion parameter: In that case the number of colors N_C is taken as a free parameter and perturbation theory has been worked out using N_C^{-1} as expansion parameter keeping the product $N_C g_S^2$ (the so-called "t Hooft coupling") fixed. This expansion was introduced by 't Hooft [tH74a] (see also the lecture notes [tH02] and [Col88]). It turned out to be very useful because in the large-N_C limit,

$$N_C \longrightarrow \infty, N_C g_S^2 \,\text{fix}\,, \tag{1.49}$$

many simplifications occur and qualitative results for meson and baryon dynamics can be found. We will later comment on some useful results which Witten obtained for baryons using this limit [Wit79a].

The Lagrangian (1.43) is invariant under some global symmetry transformations: It is invariant under the $U(1)$ transformation

$$\Psi_k \longrightarrow e^{i\theta}\Psi_k\,. \tag{1.50}$$

The corresponding Noether current J_B^μ is

$$J_B^\mu = \sum_k \bar{\Psi}_k \gamma^\mu \Psi_k\,. \tag{1.51}$$

This symmetry is known as baryon number symmetry and the corresponding conserved charge, $\int J_B^0$, is the baryon number.

If $m_u = m_d$, (1.43) is invariant under the $SU(2)$ transformation

$$\begin{pmatrix} u \\ d \end{pmatrix} \longrightarrow e^{i\sigma^a\theta^a} \begin{pmatrix} u \\ d \end{pmatrix}\,. \tag{1.52}$$

This symmetry is known as isospin symmetry. In the case when more than the two quark masses m_u and m_d are considered to be equal, the symmetry can be extended.

In the approximation where the three lightest quarks u, d and s are considered to be massless, the Lagrangian is invariant under $SU(3)_R \otimes$

$SU(3)_L$. In case only the two lightest quarks are considered to be massless, it is invariant under $SU(2)_R \otimes SU(2)_L$:

$$\begin{aligned} \Psi_R &\longrightarrow e^{i\omega^a \theta^a_R} \Psi_R \,, \\ \Psi_L &\longrightarrow e^{i\omega^a \theta^a_L} \Psi_L \,. \end{aligned} \tag{1.53}$$

Here ω^a are either the Pauli matrices (for the case of $SU(2)_R \otimes SU(2)_L$) or the Gell-Mann matrices (for the case of $SU(3)_R \otimes SU(3)_L$). This symmetry is known as chiral symmetry. It is explicitly broken by non-vanishing quark masses and is therefore (only) an approximate symmetry if we take into account the non-vanishing quark masses.

Since in nature at low energies we have discovered strongly interacting particles which are "almost" massless, the so-called pions (π^0, π^+ and π^-), it is often expected that due to non-perturbative effects $SU(2)_R \otimes SU(2)_L$ chiral symmetry is broken spontaneously and that these pions are the pseudo Nambu Goldstone bosons which appear due to this symmetry breaking according to Goldstones theorem [Nam60, Gol61]. The order paramter for this spontaneous symmetry breaking is considered to be a non-trivial ground state for the quarks (the so-called "quark condensate" or "QCD condensate"),

$$\langle \bar{\Psi} | \Psi \rangle \neq 0 \,. \tag{1.54}$$

Low energy effective theories for these pions, "chiral theories" for strong interactions, have been studied for example in [Wei79b, GL84] and have been reviewed for example in [Leu94, SS12].

At very high energies, which were typical in nature for example in the very early universe, chiral symmetry is restored and the relevant degrees of freedom are not pions (and other hadrons) but quarks and gluons. When the typical energies decreased and chiral symmetry gets spontaneously broken, the relevant degrees of freedom are pions (and other hadrons). The transition between these two regimes which is expected to have taken place in the very early universe is known as "QCD phase transition".

At low energies many techniques have been developed in order to do particle physics using meson degrees of freedom instead of the fundamental quark degrees of freedom. If, for example, a pion π^- decays into leptons as

$$\pi^- \longrightarrow e^- \bar{\nu}_e \,, \tag{1.55}$$

the amplitude of this weak process can be parameterized by only demanding certain symmetry relations without taking into account the high energetic electroweak and strong interactions. In fact, the matrix element for this process mediated by a current J_μ^5, $\langle 0| J_\mu^5 |\pi^-(p)\rangle$, can, demanding Lorentz invariance, be parametrized as

$$\langle 0| J_\mu^5 \left|\pi^-(p)\right\rangle = i p_\mu f_\pi \,, \tag{1.56}$$

with a constant f_π, a so-called "form factor", which is known as the pion decay constant and which is a free parameter of such a theory.

For a detailed explanation of these techniques we refer to [Leu94, SS12]. Let us now describe in some more details the effective chiral theories for the strong interactions and in particular the effective sigma models for the meson degrees of freedom (in our analysis we shall restrict only to pions and do not take into account other meson degress of freedom).

1.3.4 Sigma Models

The so-called linear sigma model for the pions can be obtained from the Lagrangian with four real scalar fields ϕ_1, ϕ_2, ϕ_3 and ϕ_4 of the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^k\partial^\mu\phi^k + \frac{\mu^2}{2}|\phi|^2 - \frac{\lambda}{4}|\phi|^4 \,, \tag{1.57}$$

with $\lambda > 0$ and $k = 1, 2, 3, 4$. This Lagrangian leads to spontaneous symmetry breaking since the scalar fields develop the vacuum expectation value $|\phi|^2 = \frac{\mu^2}{\lambda} \equiv v^2$. The Lagrangian of the linear sigma model which was introduced in [GML60] can be obtained from (1.57) if we identify ϕ_1, ϕ_2 and ϕ_3 with the Goldstone bosons π_a $(a = 1, 2, 3)$ and expand the last component of ϕ around the vacuum expectation value v as $\phi_4 = v + \sigma$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \mu^2\sigma^2 - \frac{\lambda}{4}\left(\pi^2+\sigma^2\right)^2 - \mu\sqrt{\lambda}\sigma\left(\pi^2+\sigma^2\right) \,, \tag{1.58}$$

where $\pi^2 \equiv \pi^a\pi^a$.

This pion Lagrangian is part of the Lagrangian which was introduced in order to describe the interactions of pions with nucleons. The corresponding nucleon Lagrangian and the corresponding pion-nucleon interaction Lagrangian were introduced in [GML60] as

$$\mathcal{L} = \bar{\Psi}\left(i\gamma^\mu\partial_\mu + g\left(v + \sigma + i\sigma^a\pi^a\gamma_5\right)\right)\Psi \,, \tag{1.59}$$

with the Pauli matrices σ^a and the coupling constant g.

According to Noethers theorem a conserved axial vector current in this model is given by

$$J_\mu^{a5} = \bar{\Psi}\gamma_\mu\gamma_5\frac{\sigma^a}{2}\Psi + \pi^a\partial_\mu\sigma - v\partial_\mu\pi^a - \sigma\partial_\mu\pi^a \,. \tag{1.60}$$

From this current, together with (1.56), it follows that v can be identified with the pion decay constant f_π.

In the double scaling limit $\lambda \to \infty$ but $v^2 \equiv \frac{\mu^2}{\lambda}$ fixed in which σ becomes infinitely heavy, or equivalently when $|\phi|^2 = f_\pi^2$, the fields σ and π_a satisfy the constraint

$$\pi^2 + \sigma^2 = -2 f_\pi \sigma \,. \tag{1.61}$$

Plugging this constraint into the pion Lagrangian of the linear sigma model we obtain the so-called non-linear sigma model Lagrangian [Wei68, CWZ69, CCWZ69]

$$\mathcal{L} = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{2}\frac{\pi^a \partial_\mu \pi^a \pi^b \partial^\mu \pi^b}{f_\pi^2 - \pi^2}\,, \tag{1.62}$$

which can be expanded as

$$\mathcal{L} = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + \frac{1}{6 f_\pi^2}\left((\pi^a \partial_\mu \pi^a)^2 - \pi^a \pi^a \partial_\mu \pi^a \partial^\mu \pi^a\right) + \mathcal{O}(\pi^6)\,. \tag{1.63}$$

The Lagrangian of the non-linear sigma model can also be written in other representations, one commonly used representation is the so-called exponential representation where an $SU(2)$ field U is introduced as

$$U = e^{i\frac{\pi^a \sigma^a}{f_\pi}}\,. \tag{1.64}$$

By Taylor-expanding U, one can see that

$$U \equiv \cos\left(\frac{|\pi|}{f_\pi}\right)\mathbb{1} + i \sin\left(\frac{|\pi|}{f_\pi}\right)\frac{\pi_a \sigma_a}{|\pi|}\,. \tag{1.65}$$

The Lagrangian of the non-linear sigma model in this representation reads

$$\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{Tr}\left(\partial_\mu U^+ \partial^\mu U\right)\,. \tag{1.66}$$

It is invariant under the $SU(2)\otimes SU(2)$ chiral transformations under which U transforms as

$$U \longrightarrow A U B^+\,, \tag{1.67}$$

where A and B are unitary matrices. To this Lagrangian a mass term for the pions can be added as [AN84]

$$\mathcal{L} = \frac{1}{2} m_\pi^2 f_\pi^2 \left(\mathrm{Tr} U - 2\right)\,, \tag{1.68}$$

where m_π is the pion mass.

This chiral Lagrangian can be generalized if we take into account not only pions but also other meson degrees of freedom which we know to exist in nature. Since at low energies in nature we do not only observe mesons but also baryons, one important question is how baryons appear in this low energy effective field theory. One option which one could follow would be to try to add the baryons as additional fields in the chiral Lagrangians. Another option, suggested for example by Witten's analysis in [Wit79a], is that baryons could appear as solitons in a meson Lagrangian. We will discuss this latter option in detail after we have reviewed some aspects of topological solitons in the next section.

1.3.5 Beyond the Standard Model

Before reviewing topological solitons, let us briefly note that there are good reasons to believe that the standard model of particle physics does not completely describe all the matter and non-gravitational interactions in our universe. In particular, as we pointed out in the first section, most of the matter content in our universe is so-called dark matter or dark energy which is not included in the standard model. On top of that, the baryon-antibaryon asymmetry in our universe cannot be explained within the standard model. There are also aestetical reasons to go beyond the standard model, in particular the idea of unification which was so successful in the electroweak case could in principle also work in order to unify the strong and electroweak interactions $SU(2) \otimes U(1)_Y \otimes SU(3)$ in a simpler gauge group. Finally, questions of naturalness like the hierarchy and the strong CP problem are sometimes used as motivations to go beyond the standard model.

There are many concrete ideas of how we can go beyond the standard model. In particular there are so-called grand unified theories [Gra84, Zee82], there are supersymmetric extensions of the standard model [WB92, Wei13a] as well as higher dimensional models [ACF87, AHDD98].

1.4 Topological Solitons and Skyrmions as Effective Descriptions of Baryons

In the previous section we reviewed several aspects of quantum field theories. Historically, aspects of quantum field theories which can be studied by doing a perturbative analysis in small coupling constants, so-called "perturbative aspects", dominated the discussion of quantum field theories for a long time. Later, it however has been realized that in certain quantum field theories also “non-perturbative” classical finite energy solutions which cannot be continuously deformed to the vacuum state of the theory, so-called “topological solitons”, can exist as classical ground states of the theory.[5] These classical solutions can have important physical consequences which cannot be captured by perturbative methods. Topological solitons might have been formed in phase transitions in the very early universe [VS00] and also arise in systems of condensed matter physics [AS10]. There are two classes of topological solitons, so-called “topological defects” and so-called “textures”. Let us now first discuss the structure and some properties of topological defects and textures in general (assuming the existence of such configurations) and then review several concrete examples of defects and of textures. Finally, we will recall how one particular class of textures, so-called “skyrmions”, can be understood as a low energy effective description of baryons.

Topological Defects and Textures

Topological defects can arise as classical solutions in certain quantum field theories if a symmetry of the quantum field theory is spontaneously broken in a certain way. Let us discuss in detail how topological defects arise in (gauged) scalar field theories with Lagrangians of the form

$$\mathcal{L} = \frac{1}{2}\left(D_\mu\phi^a\right)^+\left(D^\mu\phi^a\right) - V(\phi) - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}\,, \tag{1.69}$$

where ϕ^a is a (set of) scalar field(s) and $F^a_{\mu\nu}$ is a field strength tensor of some Abelian or non-Abelian gauge field A^a_μ. D_μ is the corresponding covariant derivative and $V(\phi)$ a potential which has certain degenerate vacua and which therefore allows for configurations which break some symmetry of the Lagrangian spontaneously. The symmetry breaking pattern which is necessary and sufficient for the existence of topological defects in a given theory of the form (1.69) can be characterized by the so-called “homotopy groups”. This characterisation can be understood as follows.

If a Lagrangian of the form (1.69) is invariant under the action of a symmetry group G and if there exists a solution to the equations of motion of (1.69) which minimizes $V(\phi)$ and breaks this symmetry group spontaneously down to a subgroup H, $G \longrightarrow H$, then there is a one-to-one correspondence between the degenerate minima of $V(\phi)$ and the coset space $\frac{G}{H}$: If a field configuration minimizes $V(\phi)$, then the action of any group element of G on that particular field configuration produces a field configuration which also minimizes $V(\phi)$. If that group element is not part

[5]There are also so-called non-topological solitons. Here we however focus on topological solitons.

of H, the new minimizing field configuration is different than the original one. If that group element however is part of H, the new minimizing field configuration is the same as the original configuration.

In order for having finite energy, the field configuration of a classical solution of a quantum field theory (1.69) has to be such that at each point at spatial infinity the scalar field takes one of the vacuum values (one minimum of the potential $V(\phi)$ of the theory). Therefore one characteristic of such a configuration is the (continuous) map which maps the field value at each point at spatial infinity onto one vacuum state of $V(\phi)$, or equivalently, onto one point of $\frac{G}{H}$. All these mappings from the d-sphere S_d which sets spatial infinity in $(d+2)$ spacetime dimensions onto the group $\frac{G}{H}$ (mapping different field values onto different points of $\frac{G}{H}$) can be supplemented with a group structure: roughly speaking, two maps that can be continuously deformed into each other are said to be in the same "homotopy class" and the product of two classes is defined to be connecting the (multidimensional) pathes of the representatives of the two classes. These groups of homotopy classes are so-called homotopy groups $\pi_d(\frac{G}{H})$. For a more rigorous definition of the group structure and the homotopy groups one might consult the textbooks [Raj82], [MS07], [Shi12], [VS00] and [Wei15].

A given classical field configuration with finite energy can be characterized by one element of the homotopy group. If the homotopy group is isomorphic to the integer numbers, which is the case for example for $\pi_n(S^n)$ (for any n), the field configurations can be characterized by one integer number which is sometimes called the "winding number" or "topological charge".

It follows from Derricks theorem [Der64] that in a gauged scalar field theory with Lagrangian (1.69), topological defects with finite energy have as codimension-1 objects a chance to exist even if the gauge field is zero everywhere (these codimension-1 configurations are characterized by homotopy group π_0), as codimension-2 and codimension-3 objects they only have a chance to exist if the gauge field is non-vanishing (codimension-2 objects are characterized by π_1, codimension-3 objects by π_2). For codimension-4 only for pure Yang-Mills (without any scalar) such topological nontrivial configurations have a chance to exist. For higher codimensions such configurations are not possible. These conclusions can be obtained using a scaling argument [Der64]:

Let a solitonic solution with finite energy in D space dimensions be given by $\phi_{(sol)}(x)$ and $A^j_{(sol)}(x)$ such that $V(\phi_{(sol)})$ is not vanishing everywhere. Let us define the scaled field configurations $\psi^{(\lambda)}(x) \equiv \phi_{(sol)}(\lambda x)$ and $\mathcal{A}^{i(\lambda)}(x) \equiv \lambda A_{i(sol)}(\lambda x)$ with some real parameter λ. In $A_0 = 0$ gauge the energy E of the (static) solitonic configuration $\phi_{(sol)}(x)$, $A^j_{(sol)}(x)$ is given by

$$E = \frac{1}{4}\int d^D x F^a_{ij(sol)} F^{aij(sol)} + \frac{1}{2}\int d^D x \left(D_i \phi_{(sol)}\right)^+ \left(D^i \phi_{(sol)}\right) + \int d^D x V(\phi_{(sol)})\,. \tag{1.70}$$

The energy functional evaluated on the rescaled configuration can be expressed in terms of the unscaled field configurations $\phi_{(sol)}$ and $A_{i(sol)}$ as

$$E(\lambda) = \frac{1}{4}\lambda^{4-D}\int d^D x F^a_{(sol)ij} F^{aij}_{(sol)} + \frac{1}{2}\lambda^{2-D}\int d^D x \left(D_i \phi_{(sol)}\right)^+ \left(D^i \phi_{(sol)}\right) + \lambda^{-D}\int d^D x V(\phi_{(sol)})\,. \tag{1.71}$$

Since $\phi_{(sol)}(x)$, $A^j_{(sol)}(x)$ is a solitonic solution, it is a stationary point of the energy functional among all the rescaled configurations. In fact $\partial_\lambda E(\lambda)|_{\lambda=1} = 0$. This requirement gives

$$0 = (D-4)\frac{1}{4}\int d^D x F^a_{(sol)ij} F^{aij}_{(sol)} + (D-2)\frac{1}{2}\int d^D x \left(D_i \phi_{(sol)}\right)^+ \left(D^i \phi_{(sol)}\right) + D\int d^D x V(\phi_{(sol)})\,. \tag{1.72}$$

For $D = 2, 3$ this equation implies that for vanishing gauge field the only possible configuration is the trivial configuration where ϕ is in a minimum of $V(\phi)$ everywhere. Thus, for $D = 2, 3$ topologically non-trivial topological defects without a gauge field cannot exist. For $D = 1$ this equation with vanishing gauge field gives the so-called virial identity which can be fulfilled by a non trivial topological defect, the so-called kink, as we will point out below. Therefore for $D = 1$ topological defects do exist even with vanishing gauge field. For $D = 4$ it is impossible to fulfill the equation for non-vanishing scalar field leaving us only with the option of pure Yang-Mills topological soliton. For $D > 4$ no nontrivial solutions of this equation exist. Therefore for $D > 4$ no topological defects can exist.

So far, we have restricted our discussions to topological solitons in theories with Lagrangians of the form (1.69). There are however also topological solitons as finite energy configurations in scalar field theories with Lagrangians which have no scalar field potential and no gauge field but which have non-linear terms of the scalar field. Such configurations with the scalar field taking values in some closed manifold X are sometimes called "textures". In order for such configurations to have a chance to have finite energy, the scalar field has to approach a constant at spatial infinity (only in that case the gradient energy does not diverge). This is the main difference when compared to topological defects where the scalar field does not have to take a single constant value at spatial infinity but could vary over the whole set of minima of the potential V at spatial infinity. Because in the case of textures the scalar field has a constant value at spatial infinity, the physical space of dimension d can be compactified to the d-sphere S^d by adding one point at spatial infinity (for example via stereographic projection). If a scalar field ϕ is mapped into a closed manifold X, these maps can therefore be characterized by the homotopy groups $\pi_n(X)$. In contrast to the cases of topological defects, in these cases of textures the homotopy groups therefore describe mappings of the whole physical space on a manifold (in the case of topological defects the mapping is from the boundary of the physical space). Therefore, textures with codimension-n are characterized by the homotopy group π_n whereas, as pointed out above, topological defects with codimension-n are characterized by homotopy group π_{n-1}.

So far, we have not shown if such topological defects and textures in fact exist. In order to prove the existence, below, we will review certain particular examples of textures and defects. Many of the defects have the property that they are singular at one point whereas textures are typically non-singular.

To (almost) all topological solitons which we will review below a typical classical length scale, L, can be associated around which most of the energy of the soliton is localized. In addition one can associate a mass to all topological solitions and therefore a Compton wavelength L_C. In case the Compton wavelength of a given topological soliton is much smaller than its typical classical size,

$$L_C \ll L \,, \tag{1.73}$$

it is indeed justified, as a good approximation, to treat the topological soliton as classical. As we will point out below, this criterion (1.73) is typically justified if and only if the topological soliton is weakly coupled. In this thesis we restrict to such weakly-coupled topological solitons.[6] In case (1.73) is satisfied, in the literature often the approach is followed where perturbations around this classical background are studied and quantized (see e.g. [GJ75, CG75, RW75] and the discussions in the textbooks [Raj82], [MS07], [Shi12], [VS00] and [Wei15]). The perturbations are often expanded in normal modes which are then quantized canonically. Whenever the soliton

[6] For discussions on solitons for which (1.73) is not satisfied, we refer for example to [DGGR15].

breaks symmetries of the Lagrangian, zero modes (normal modes with zero frequency) appear which are treated seperately: Each zero mode is usually taken into account by introducing a "collective coordinate" or "modulus" $\beta(t)$ and by essentially replacing the static soliton coordinate x by $(x - \beta(t))$. These collective coordinates then appear separately in the Lagrangian and can be quantized canonically. Whenever these zero modes are excited in a time dependent manner, the conjugate momenta of the collective coordinates are non-vanishing. We will illustrate this procedure below on some particular examples.

In what follows we briefly recall some aspects of the topological defects of codimension-1 (known as "kinks" in $(1 + 1)$ spacetime dimensions and "domain walls" in higher spacetime dimensions), of codimension-2 (known as "vortices" in $(2+1)$ spacetime dimensions and "cosmic strings" in higher spacetime dimensions) and of codimension-3 (known as "magnetic monopoles"). We then review some aspects of textures with codimension-1 (known as "(nonlinear) kinks"), textures with codimension-2 (known as "lumps" and "baby skyrmions") and textures with codimension-3 (known as "skyrmions"). On these examples we will illustrate some of the previous-mentionned points. For more details on topological solitons, as well as instantons which we will not review, we refer to the standard textbooks [Raj82], [MS07], [Shi12], [VS00] and [Wei15] and the references therein.

Kinks and Domain Walls

Codimension-1 topological defects arise as topologically non-trivial lowest energy configurations in theories with Lagrangians of the form (1.69) in the case of a scenario with spontaneous symmetry breaking pattern $G \longrightarrow H$ which is such that the homotopy group $\pi_0\left(\frac{G}{H}\right)$ is non-trivial. In the case of $(1 + 1)$ spacetime dimensions these topological defects are known as "kinks" (and "anti-kinks"). They can arise for example in the theory with the following prototype Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2 . \tag{1.74}$$

This Lagrangian has a Z_2 symmetry, $\phi \to -\phi$, which is spontaneously broken by the potential $V(\phi) = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2$ which has two degenerate minima ($\phi = \pm v$). The corresponding homotopy group is $\pi_0(S^0)$. Static solutions of the equations of motion are given by

$$\phi_{kink}(x) = \pm v \tanh\left(\frac{m}{\sqrt{2}}\left(x - x_0\right)\right) , \tag{1.75}$$

where $m^2 \equiv v^2\lambda$. The solutions with plus sign are called "kinks" whereas the solutions with minus sign are called "anti-kinks". With the "topological current"

$$J^\mu \equiv \frac{1}{2v}\epsilon^{\mu\nu}\partial_\nu\phi \tag{1.76}$$

the kink can be characterized by topological charge,

$$Q \equiv \int_{-\infty}^{\infty} J^0 dx \,, \tag{1.77}$$

plus one ($Q_{kink} = 1$) whereas the anti-kink carries topological charge minus one ($Q_{antikink} = -1$). Due to the antisymmetric property of the epsilon tensor the current (1.76) is trivially conserved (without having to impose the equations of motion).

By considering the Hamiltonean which corresponds to the Lagrangian (1.74) and evaluating it on the kink solution, it is easy to show that the total energy (mass) M of a the kink (1.75) is given by

$$M = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} \,. \tag{1.78}$$

Most energy is concentrated in a localized region with size of order $L \equiv m^{-1}$. Therefore, the criterion (1.73) is satisfied if $\frac{\lambda}{m^2}$ is very small, or in other words if the kink is weakly coupled. Thus, it is justified to treat such weakly coupled kinks classically as a good approximation.

In this regime one can study perturbations $\beta(x,t)$ around the classical kink solution:

$$\phi(x,t) = \phi_{kink}(x) + \beta(x,t) \,, \tag{1.79}$$

with $|\beta(x,t)| \ll |\phi_{kink}(x)|$. The function $\beta(x,t)$ can be expanded in normal modes $f_k(x)$ which diagonalize the quadratic part of the perturbed Lagrangian,

$$\left(-\frac{d^2}{dx^2} + V''(\phi_{kink})\right) f_k(x) = w_k^2 f_k(x) \,, \tag{1.80}$$

and which correspond to solutions

$$\beta_k(x,t) = f_k(x) e^{-iw_k t} \,. \tag{1.81}$$

The expansion of the perturbations can be written as

$$\beta(x,t) = \sum_k \left(c_k(t) f_k(x) + \text{h.c.}\right) \,, \tag{1.82}$$

with operators c_k. There is the zero mode $f_0 = \partial_x\phi_{kink}$ which originated because the kink breaks translational invariance of the Lagrangian (see e.g.

the discussion in [Wei15]). In order to take this into account one usually introduces a "collective coordinate" $z(t)$ as

$$\phi(x,t) = \phi_{kink}(x - z(t)) + \beta(x - z(t))\,, \tag{1.83}$$

which leads to the quadratically perturbed kink Lagrangian

$$\mathcal{L} = \frac{1}{2}M\dot{z}^2 + \sum_k \left(\frac{1}{2}\dot{c}_k^2 - \frac{1}{2}w_k^2 c_k^2\right)\,. \tag{1.84}$$

In this way, the collective coordinate can be interpreted as the position of the kink which can be quantized canonically by introducing the canonical momentum $P \equiv M\dot{z}$ and imposing standard commutation relations between z and P. Whenever the zero mode is excited in a time dependent manner, P is non-zero.

One can trivially embedd the kinks in higher dimensional spacetimes with the kink solution being independent of all coordinates but one (therefore "codimension-1 objects"). These higher dimensional kinks are known as "domain walls".

Vortices and Cosmic Strings

Codimension-2 topological defects arise as topologically non-trivial lowest energy configurations in theories of the form (1.69) in scenarios with a spontaneous symmetry breaking pattern $G \longrightarrow H$ which is such that the homotopy group $\pi_1\left(\frac{G}{H}\right)$ is non-trivial. In the case of $(2+1)$ spacetime dimensions these are "vortices" which can arise for example in the gauged scalar field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(D_\mu\phi\right)^+\left(D^\mu\phi\right) - \frac{\lambda}{4}\left(\phi^+\phi - v^2\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{1.85}$$

where $D_\mu\phi \equiv \partial_\mu\phi + ieA_\mu\phi$ is the covariant derivative and $F_{\mu\nu} \equiv \partial_{[\mu}A_{\nu]}$. ϕ is a complex scalar field which can be parameterized as

$$\phi(x) = \rho(x)e^{i\alpha(x)}\,, \tag{1.86}$$

with the two real valued functions $\rho(x)$ and $\alpha(x)$. This Lagrangian has a $U(1)$ symmetry under which the fields transform as $\phi \to e^{iw(x)}\phi$ and $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu w(x)$ with $w(x)$ a gauge transformation function. This symmetry is spontaneously broken by the potential $V(\phi) = \frac{\lambda}{4}\left(\phi^+\phi - v^2\right)^2$. The corresponding homotopy group is $\pi_1(U(1)) = \pi_1(S^1) = Z$. A topological current can be defined as

$$J^\mu \equiv \epsilon^{\mu\nu\gamma}\partial_\nu\partial_\gamma\alpha(x)\,, \tag{1.87}$$

where (1.86) was used. Using Stokes theorem the charge which corresponds to this current, $\int J_0 d^2x$, can be rewritten as

$$\int J_0 d^2x = \oint dx^\mu \partial_\mu \alpha(x) \, . \tag{1.88}$$

Therefore the topological charge, $\int d^2x J_0$, measures the number of times α winds around S^1.

One can show that the topological charge $\oint dx^\mu \partial_\mu \alpha(x)$ is nothing but the magnetic flux $\int d^2xB$ [NO73].

Topological non-trivial lowest energy configurations ("vortices") with topological charge n can be obtained numerically by minimizing the energy functional which corresponds to (1.85) with the ansatzes

$$\phi(r,\theta,\phi) = ve^{in\theta} f(evr), A_0 = 0, A_i = \epsilon_{ik} e^k \frac{a(evr)}{er} \, , \tag{1.89}$$

where e^k is a unit vektor in $k-th$ direction and $f(evr)$ and $a(evr)$ are two real-valued ansatz-functions which take the boundary conditions

$$f(0) = 0, a(0) = 0, f(\infty) = 1, a(\infty) = n \, . \tag{1.90}$$

These boundary conditions have to be chosen for the configurations to be regular at the origin and to have finite energy. (Finite energy is obtained in particular because for these boundary conditions the covariant derivative $D_\mu \phi$ which appears in the energy density vanishes at spatial infinity.) Solution profile functions for certain choices of parameters can be obtained numerically and can be found for example in [NO73].

The mass of a vortex can be obtained by evaluating the Hamiltonean which corresponds to the Lagrangian (1.85) on the vortex solution. It is of order

$$M = v^2 \, . \tag{1.91}$$

A typical length scale L in which most of the energy is concentrated is given by

$$L = \frac{1}{ev} \, . \tag{1.92}$$

Vortices can safely be considered as classical to a good approximation whenever (1.73) holds. This is the case if and only if $\frac{e}{v} \ll 1$.

In this regime one can study perturbations around the vortex solution $A^\mu_{(vortex)}$, $\phi_{(vortex)}$,

$$A^\mu = A^\mu_{(vortex)} + \delta A^\mu \, ,$$

$$\phi = \phi_{(vortex)} + \delta\phi\,, \tag{1.93}$$

with $|\delta A^{\mu}| \ll |A^{\mu}_{(vortex)}|$ and $|\delta\phi| \ll |\phi_{(vortex)}|$ and quantize them canonically. In contrast to the case of the a kink, since the theory with Lagrangian (1.85) is a gauge theory, one has to take into account that not all perturbations which fullfull the perturbed equations of motion are physical. Instead, some of the perturbations are gauge redundancies. One should therefore only quantize the physical perturbations and only study the physical zero modes. One can show that in the case of the vortex there are two physical zero modes (see e.g. the discussion in [Wei15]) which, in complete analogy to the kink case, originate from the broken translational invariance. As in the case of the kink, they can be taken into account by introducing two collective coordinates.

Julia and Zee showed in [JZ75] that all $(2+1)$ dimensional static finite energy vortices which can be obtained from (1.85) are electrically-neutral. Electrically charged vortices can however exist as static finite energy configurations in the theory with Lagrangian (1.85) if a Chern-Simons term is added to this Lagrangian [PK86, dVS86]. The electric charge of such Chern-Simons vortices is then proportional to the winding number and the coefficient introduced in front of the Chern-Simons term. These Chern-Simons vortices are so-called “anyons” [Wil82], indeed they carry fractional statistics [FM89].

As in the case of kinks, the vortices can also be trivially embedded in higher dimensions by letting the vortex solution be independent of the other coordinates (therefore “codimension-2 objects”). These higher dimensional vortices are known as cosmic strings.

Vortices (and cosmic strings) can also arise for example for Lagrangians which are slightly modified when compared to (1.85) such that the $U(1)$ symmetry is not completely broken but broken down to a discrete subgroup Z_N. Such a symmetry breaking pattern can be realized for a Lagrangian

$$\mathcal{L} = \frac{1}{2}|\left(\partial_\mu + iNeA_\mu\right)\phi|^2 - \frac{\lambda}{4}\left(\phi^+\phi - v^2\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{1.94}$$

which is invariant under $\phi \to e^{iNw(x)}\phi$ and $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu w(x)$ where $w(x)$ is a gauge transformation function. When the scalar field ϕ which has charge N condenses and develops a vacuum expectation value, $\langle\phi(x)\rangle = v$, the ground state is invariant under the above shift (only) when $w(x)$ is an integer multiple of $\frac{2\pi}{N}$. Any other scalar field which is charged under A_μ such that it has carries charge 1 and does not condense transforms under this “residual” Z_N symmetry.

Vortices can be obtained as lowest energy configurations of the energy functional which corresponds to the Lagrangian (1.94) exactly in the same way as in the case of (1.85) except that now the boundary condition for the gauge field has to be chosen such that it goes as $A_\theta \to \frac{n}{Ne}$ for $r \to \infty$ instead of as $\frac{n}{e}$ as in the case of vortices in (1.85). For such an asymptotic behaviour, the discrete Z_N charges can be detected via Aharonov-Bohm experiment if we for example take a probe particle at infinity once around the cosmic string: Such a process induces a change in the action

$$\Delta\mathcal{S} = \oint_{S_\infty} A_\mu dx^\mu = \frac{2\pi}{N}\,, \tag{1.95}$$

where dx^μ parametrizes the world line of the particle. This change in the action leads to a measurable Aharonov-Bohm phase shift whenever $N > 1$.

For some applications it is useful to note that a Z_N gauge theory is dual to a "topological BF-theory" which might be more convenient to use in certain applications. We refer for example to [BS11] for a detailed explanation of what this means.

Magnetic Monopoles

Magnetic monopoles are codimension-3 topological defects which arise as topologically non-trivial lowest energy configurations in theories of the form (1.69) in scenarios with spontaneous symmetry breaking pattern $G \longrightarrow H$ which is such that the homotopy group $\pi_2\left(\frac{G}{H}\right)$ is non-trivial. In $(3+1)$ spacetime dimensions a famous example where such a scenario is realized is the model of Georgi and Glashow [GG72] which has a symmetry $SU(2)$ spontaneously broken down to $U(1)$ as reviewed in (1.42). In this model 't Hooft [tH74b] and Polyakov [Pol74] found topological defects as classical solutions. As reviewed in (1.42), the Georgi-Glashow Lagrangian is given by:

$$\mathcal{L} = \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - \frac{\lambda}{4}\left(\phi^a\phi^a - v^2\right)^2 - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}\,. \tag{1.96}$$

Here ϕ^a is a scalar triplett $(a = 1,2,3)$, $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\epsilon^{abc}A^b_\mu A^c_\nu$ and $D_\mu\phi^a \equiv \partial_\mu\phi^a - e\epsilon^{abc}A^b_\mu\phi^c$. This Lagrangian is invariant under $SU(2)$ transformations. The $SU(2)$ symmetry is broken down to $U(1)$ by configurations with the scalar field condensed. The homotopy group which corresponds to this symmetry breaking pattern in $(3+1)$ spacetime dimensions is $\pi_2\left(\frac{SU(2)}{U(1)}\right) = \pi_2(S^2) = Z$. A topological current can be defined as

$$J^\mu = \frac{\epsilon^{\mu\nu\alpha\beta}\epsilon^{abc}\partial_\nu\phi^a\partial_\alpha\phi^b\partial_\beta\phi^c}{8\pi|\phi|^3}\,. \tag{1.97}$$

Topological non-trivial lowest energy configurations with topological charge one ("magnetic monopoles") can be obtained numerically by minimizing the energy functional which corresponds to (1.42) with the ansatzes

$$\phi^a = vh(r)e_r^a, A_0^a = 0, A_i^a = \epsilon_{iak}\left(\frac{1-u(r)}{er}\right)e_r^k\,, \tag{1.98}$$

where e_r^a is a unit vector in radial direction (in fact ϕ^a assumes a hedgehog ansatz). $h(r)$ and $u(r)$ are two ansatz-functions which take the boundary conditions

$$h(0) = 0, u(0) = 1, h(\infty) = v, u(\infty) = 0\,. \tag{1.99}$$

One can find the solution functions for several parameter choices for example in [tH74b].

Since there is an unbroken $U(1)$ subgroup in the above model when $SU(2)$ gets spontaneously broken, one can define $U(1)$ magnetic and electric fields for the solutions: Defining $B_i^a \equiv -\frac{1}{2}\epsilon_{ijk}F^{ajk}$ and $E_i^a \equiv F_{i0}^a$, the components of B_i^a and E_i^a that lie in the unbroken $U(1)$ subgroup can be projected out by multiplying them with the normalized field of ϕ^a. This leads to the electric and magnetic charges

$$Q_M = \int dS^i \frac{\phi^a}{|\phi|}B_i^a\,, \tag{1.100}$$

$$Q_E = \int dS^i \frac{\phi^a}{|\phi|}E_i^a\,. \tag{1.101}$$

The magnetic monopole solutions have vanishing electric field and at large distances, $r \to \infty$, a magentic field of the form [tH74b]

$$B_i^a \equiv -\frac{1}{2}\epsilon^{ijk}F_{jk}^a \to e_r^a e_r^i \frac{1}{er^2}\,, \tag{1.102}$$

which is exactly the magnetic field of a magnetic monopole (therefore these solitons are called "magnetic monopoles").

In contrast to the case of vortices, spherically-symmetric magnetic monopoles with topological charge higher than one do not exist (except in the so-called "BPS limit" $\frac{\lambda}{e^2} = 0$ [Bog76, PS75, Wei79a]) [Wei15].

One can define several characteristic mass and length scales of such a magnetic monopole solution with topological charge one as

$$M_m = \frac{v}{e}, M_H = \sqrt{\lambda}v, L = \frac{1}{ev}, L_C = \frac{1}{\sqrt{\lambda}v}\,. \tag{1.103}$$

Up to numerical constants of order one these scales can be identified as follows: M_m can be identified with the mass or total energy of a magnetic monopole, M_H can be identified with the mass of the Higgs field and therefore L_C with the Compton wavelength of the Higgs field. L sets the Compton wavelength of the gauge field. The criterion (1.73) is satisfied if and only if $e^2 \ll 1$. Thus, in this case of $e^2 \ll 1$ magnetic monopoles can be considered as classical objects as a good approximation.

As in the cases of the kink and the vortex, one can study perturbations around the classical monopole solution

$$A^a_\mu = A^a_{\mu(mon)} + \delta A^a_\mu\,,$$
$$\phi^a = \phi^a_{(mon)} + \delta\phi^a\,, \tag{1.104}$$

with $|\delta A^a_\mu| \ll |A^a_{\mu(mon)}|$ and $|\delta\phi^a| \ll |\phi^a_{(mon)}|$ and can quantize the perturbations. The monopole solution has four zero modes. Three zero modes correspond to spatial translations in complete analogy to the cases of the kink and vortex. The fourth zero mode can be understood as a result from the unbroken $U(1)$ subgroup.

We discuss the fourth zero mode in some more details: This zero mode (which in particular is not pure gauge [Wei15]) is for a small parameter α given by [Wei15]

$$\delta\phi = 0, \delta A^a_\mu = \frac{1}{v} D_\mu \phi^a_{(mon)}\,. \tag{1.105}$$

In the case of small α this transformation can be understood as the infinitesimal form of the global $U(1)$ rotation of $A^a_{\mu(mon)}$ and $\phi^a_{(mon)}$ via the matrix

$$U = e^{i\alpha\frac{\phi^a\sigma^a}{v}}\,. \tag{1.106}$$

Thus, α and $\alpha + 2\pi$ can be identified and α is in this sense an angular variable. If we introduce a collective coordinate, α becomes the angular collective coordinate (see e.g. the discussion in [Wei15]).

Thus, the generator of these transformations is $e^{i\alpha k}$ where k is the integer Noether charge. The Noether charge can be written as integral over the zeroth component of the Noether current corresponding to these transformations [Wit79b]:

$$k = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_0 A_\mu\right)} \delta A_\mu + \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi\right)} \delta\phi \right) = \int d^3x \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A_i\right)} D_i \phi\,. \tag{1.107}$$

Using (1.101), one obtains

$$k = \frac{Q_E}{e}\,. \tag{1.108}$$

If a θ term is added to the Lagrangian (1.42):

$$\Delta\mathcal{L} = \frac{e^2\theta}{32\pi^2}\epsilon^{\alpha\beta\gamma\delta}F^a_{\alpha\beta}F^a_{\gamma\delta}\,, \tag{1.109}$$

the Noether charge gets modified and becomes

$$k = \frac{Q_E}{e} + \frac{e\theta}{8\pi^2}g\,. \tag{1.110}$$

This modification leads, using $eg = -4\pi$, to an electric charge

$$Q_E = ke + \frac{e\theta}{2\pi}\,. \tag{1.111}$$

Therefore, even if there is a configuration which has no electric charge in the case no theta term is taken into account, $k = 0$, (for example the case of the ‘t Hooft Polyakov monopole solution) after a theta term is added to the Lagrangian (1.42), an electric charge is automatically induced and the magnetic monopole becomes a dyon. This is the celebrated Witten effect [Wit79b].

A nontrivial homotopy group π_2 which leads to magnetic monopoles can also be obtained in Lagrangians with symmetry breaking patterns which are different than the one in the ‘t Hooft Polyakov case. In fact, in grand unified theories, magnetic monopoles are therefore quite generically predicted (see e.g. the discussion in [Wei15]). The existence of such GUT magnetic monopoles (formed for example in GUT phase transitions in the very early universe) would have important physical implications, for example the existence of magnetic monopoles predicts the quantization of electric charge (see for example [Wei15] for an explanation).

Nonlinear Kinks

So-called “Sine-Gordon kinks” can appear as static topologically non-trivial finite energy configurations in the (1+1) dimensional scalar field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - (1 - \cos\phi)\ . \tag{1.112}$$

Sine-Gordon kinks are the lowest energy configurations of this theory which take the form

$$\phi(x) = 4\tan^{-1}\left(e^{x-a}\right)\ , \tag{1.113}$$

where a is some constant. In the same way as the ϕ^4 kinks which we discussed above, these Sine-Gordon kinks can as topological defects be characterized by the homotopy group $\pi_0(Z)$.

Alternatively, Sine-Gordon kinks can also be understood as textures (see e.g. the discussion in [MS07]). This can be seen introducing a two-component scalar field $\tilde{\phi}^a$ $(a = 1, 2)$ which is related to the field ϕ via

$$\tilde{\phi} = (\sin\phi, \cos\phi) . \tag{1.114}$$

The Lagrangian (1.112) written in terms of $\tilde{\phi}$ takes the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\tilde{\phi}^a\partial^\mu\tilde{\phi}^2 - \left(1 - \tilde{\phi}_2\right) + \lambda\left(1 - \tilde{\phi}^a\tilde{\phi}^a\right) , \tag{1.115}$$

with a Lagrange multiplier λ which enforces the constraint $\tilde{\phi}^a\tilde{\phi}^a = 1$ on the scalar field which therefore takes values in S^1. The homotopy group of textures in this model is thus given by $\pi_1(S^1) = Z$.

A static lowest energy configuration with topological charge one and boundary conditions $\tilde{\phi} \to (0, 1)$ at spatial infinity is given by

$$\tilde{\phi}_1 = \sin\left(\phi(x)\right), \tilde{\phi}_2 = \cos\left(\phi(x)\right) , \tag{1.116}$$

where $\phi(x)$ is the solution (1.113).

Lumps and Baby Skyrmions

The simplest codimension-2 textures are so-called "lumps". They arise as topologically non-trivial static classical finite energy configurations for example in the (2+1) dimensional scalar field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a + \lambda\left(1 - \phi^a\phi^a\right) , \tag{1.117}$$

where $a = 1, 2, 3$ and λ is a Lagrange multiplier which enforces the constraint $\phi^a\phi^a = 1$ on the scalar field and therefore implies that the scalar field takes values in S^2. This model is invariant under $O(3)$ rotations, $\phi \longrightarrow O\phi$ (where O is a $O(3)$ matrix), and is known as "$O(3)$ sigma model". Topological non-trivial configurations in this model are characterized by the homotopy group $\pi_2(S^2) = Z$ and the topological current can be written as

$$J_\mu = \frac{1}{4\pi}\epsilon_{\mu\nu\alpha}\epsilon^{abc}\phi^a\partial^\nu\phi^b\partial^\alpha\phi^c . \tag{1.118}$$

Lowest energy configurations with given topological charge can be found as solutions of the equations of motions when boundary conditions are chosen such that ϕ takes a constant value at spatial infinity. The energy of a solution with given topological charge N satisfies the so-called Bogomolny bound [Bog76, PS75, Wei79a]

$$E \geq 2\pi|N| . \tag{1.119}$$

In the case of $E = 2\pi|N|$ the solutions of the equations of motion are the same as the solutions of the so-called Bogomolny equation [Bog76, PS75, Wei79a] which is a first order differential equation. These solutions of the Bogomolny equation can be obtained analytically and are given in terms of so-called "rational maps" in the case of topological charge one as

$$R(z) = \frac{\alpha e^{i\beta}}{z-\gamma}\,, \tag{1.120}$$

where $\alpha > 0$ is a real consant and β a complex constant. The function $R(z)$ is related to the scalar field triplett ϕ^a as

$$R(z) \equiv \frac{\phi_1 + i\phi_2}{1+\phi_3} \tag{1.121}$$

and z is the complex coordinate $z \equiv x + iy$ with x and y the standard eucledian coordinates on the two dimensional plane. The solution is peaked around $z = \gamma$.

The energy of this lump is given by $E = 2\pi$. This is independent of the typical radius of the configuration and in fact leads to an instability which lets the lump to collapse in a finite amount of time (see e.g. [MS07]).

Stable textures in $(2 + 1)$ spacetime dimensions can be obtained for example if the Lagrangian (1.117) is modified and takes the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{8}\left(\partial_\mu\phi^a \times \partial_\nu\phi^a\right)\left(\partial^\mu\phi^a \times \partial^\nu\phi^a\right) - \frac{m^2}{2}\left(1-\phi_3\right)\,. \tag{1.122}$$

Topologically non-trivial static finite energy configurations are characterized by the same topological current as the lumps discussed above. These configurations can, in the case of topological charge one, be obtained from (1.122) by making the ansatz

$$\phi_1 = \sin f(r)\cos\theta, \phi_2 = \sin f(r)\sin\theta, \phi_3 = \cos f(r)\,, \tag{1.123}$$

where r, θ are the polar coordinates on the two dimensional plane and $f(r)$ is a "profile function" taking the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$. By minimizing the energy functional which corresponds to (1.122), one can numerically obtain solutions for $f(r)$ [PSZ95]. These solutions are sometimes called "baby skyrmions" because the terms in (1.122) are lower dimensional forms of the so-called "Skyrme term" which appears in the case of codimension-3 textures ("skyrmions") which we will recall in what follows. The instability which is present in the case of lumps is absent in the case of baby skyrmions (see e.g. [MS07]). As we will point out below, in contrast to the case of codimension-2 textures, for the codimension-3 case lumps do not exist and thus for codimension-3 the addition of the Skyrme term is essential in order to have topologically non-trivial static finite energy configurations.

Skyrmions

Skyrmions in $(3+1)$ spacetime dimensions were originally introduced by Skyrme [Sky61, Sky62] as topological solitons which can appear as non-trivial static lowest energy configurations in the model given by the effective chiral Lagrangian of Gell-Mann and Levy [GML60] which we reviewed in (1.57) and (1.58) with a term of fourth order in derivatives (sometimes referred to as the "Skyrme term") added. This Skyrme term is the unique term with four derivatives which is invariant under chiral transformations (1.53) and has two time derivatives.[7]

This Lagrangian is given by the linear sigma model

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2\,, \tag{1.124}$$

with

$$\mathcal{L}_1 \equiv \partial_\mu \phi^k \partial^\mu \phi^k + \lambda\left(\phi^k\phi^k - f_\pi^2\right) \tag{1.125}$$

and the Skyrme term

$$\mathcal{L}_2 \equiv -\frac{1}{2}\left(\partial_\mu \phi^k \partial^\mu \phi^k\right)^2 + \frac{1}{2}\left(\partial_\mu \phi^k \partial_\nu \phi^k\right)\left(\partial^\mu \phi^k \partial^\nu \phi^k\right)\,. \tag{1.126}$$

Here $k = 1,2,3,4$ and λ is a Lagrange multiplier which enforces the constraint $\phi^k\phi^k = f_\pi^2$ on the scalar field and therefore implies that the scalar field takes values in S^3. This is the linear sigma model (1.57) with the Skyrme term added. The associated non-linear sigma model Lagrangian is obtained when the field ϕ_4 is expanded around the expectation value f_π, $\phi_4 = f_\pi + \sigma$, and the fields σ and π_a satisfy the constraint (1.61).

In the exponential representation which we introduced in (1.66) this Lagrangian can be written as:

$$\mathcal{L}_S = \frac{f_\pi^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^+\right) + \frac{1}{32e^2}\mathrm{Tr}[\partial_\mu U U^+, \partial_\nu U U^+]^2\,. \tag{1.127}$$

As pointed out in the context of (1.66), a mass term for the pions with mass m_π can be added:

$$\mathcal{L}_m = \frac{1}{2}m_\pi^2 f_\pi^2\left(\mathrm{Tr}U - 2\right)\,. \tag{1.128}$$

[7] We only know how to quantize Lagrangians with not more than two time derivatives. Therefore, in this sense, the Skyrme term is the only term with four derivatives which is meaningful.

Here, as in (1.64), U is in the case of $SU(2) \otimes SU(2)$ chiral theory defined as

$$U \equiv e^{\frac{i}{f_\pi}\pi^a\sigma^a}\,, \tag{1.129}$$

with π_a the pion fields and σ_a the Pauli matrices. f_π is the pion decay constant with dimensionality

$$[f_\pi] = \sqrt{\frac{[\text{mass}]}{[\text{length}]}} \tag{1.130}$$

and e is the "Skyrme coupling constant" with dimensionality

$$[e] = \frac{1}{\sqrt{[\text{mass}][\text{length}]}}\,. \tag{1.131}$$

Using these two parameters one can define a length scale L and a mass scale M_S as

$$L \equiv \frac{1}{ef_\pi}\,, \tag{1.132}$$

$$M_S \equiv \frac{f_\pi}{e}\,. \tag{1.133}$$

Length scales and masses of solitonic configurations of (1.127) can therefore be measured in units of $(ef_\pi)^{-1}$, $\frac{f_\pi}{e}$ respectively.

For the case of chiral $SU(3) \otimes SU(3)$ the Lagrangian (1.127) has to be modified (see e.g. [Shi12]). In fact, one has to add the so-called Wess Zumino Witten term [WZ71] to the Lagrangian. We however restrict to the case of $SU(2) \otimes SU(2)$ chiral in this thesis and refer to [Wit83b] and [Wit83a] for the discussion on cases with three instead of two quark flavors.

The Skyrme Lagrangian (1.127) can also be understood as an expansion of pion interactions in the inverse number of colors N_C^{-1} if we take into account that for any N_C the Skyrme model is the low energy effective model of QCD. This is easy to see: In the large-N_C expansion of QCD it was shown that the quartic meson interactions scale like N_C^{-1} [Wit79a]. Since quartic meson interactions scale like N_C^{-1} in large-N_C QCD and the chiral model with Lagrangian (1.63) is supposed to be the low energy effective theory of large-N_C QCD, it follows by matching that

$$f_\pi^2 \sim N_C\,, \tag{1.134}$$

$$e^2 \sim N_C^{-1}\,. \tag{1.135}$$

Therefore the Lagrangian (1.63) is an expansion of the pion interactions in powers of N_C^{-1}.

Topologically non-trivial static finite energy configurations which minimize the energy E corresponding to (1.127),

$$E = \int d^3x \mathcal{H}_S \,, \tag{1.136}$$

where $\mathcal{H}_S$ is the Skyrme Hamiltonean density, so-called "skyrmions", can be obtained numerically in the spherically-symmetric case when making a hedgheog ansatz for the pions

$$\frac{\pi_a}{f_\pi} = F(r) n_a \,, \tag{1.137}$$

where n_a is a unit vector in radial direction and $F(r)$ an ansatz "profile" function and minimizing E with the boundary conditions

$$F(0) = B\pi \,, \tag{1.138}$$

$$F(\infty) = 0 \,, \tag{1.139}$$

with B a natural number. For such configurations to exist the existence of the Skyrme term in (1.127) is essential. Without the presence of this term there are no topologically non-trivial static lowest energy configurations due to Derricks theorem [Der64]: Assuming that such a configuration $U_{lowest}(x)$ does exist for the energy E corresponding to (1.127) with the Skyrme term left out, leads to a contradiction. This is because the function $U_{lowest}(\lambda x)$ for some $\lambda > 1$ would be a configuration which has lower energy than $U_{lowest}(x)$:

$$E(U_{lowest}(\lambda x)) = \frac{1}{\lambda} E(U_{lowest})(x) \,. \tag{1.140}$$

This is in contrast to the $(2+1)$ dimensional case where lumps do in fact exist as reviewed above.

For the whole Skyrme Lagrangian (1.127) lowest energy configurations however do exist. The asymptotic behaviour of the solution profile function of these configurations is in the case of $m_\pi = 0$ (for large r) given as [ANW83]

$$F(r) \to \frac{1}{r^2} \,, \tag{1.141}$$

whereas for a finite pion mass m_π it is [AN84]

$$F(r) \to \frac{e^{-m_\pi r}}{r} \,. \tag{1.142}$$

For illustration, plotted solution profile functions for some parameter choices can be found for example in [ANW83, AN84]. The configurations can be classified by the homotopy group $\pi_3(S^3)$.

These skyrmions can be treated classically if the criterion (1.73) is satisfied. This is the case if, using (1.133) and (1.132), $e^2 \ll 1$. According to (1.135) this is the case if and only if $N_C \gg 1$. This implies that in the large-N_C expansion it is justified to treat skyrmions classically.

One can define a topological current for such configurations in the exponential representation as

$$J_\mu \equiv -\frac{\epsilon_{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{Tr}\left(U^+\partial^\nu U U^+\partial^\alpha U U^+\partial^\beta U\right) . \tag{1.143}$$

In terms of the linear sigma model fields ϕ^a, this current (1.143) can be written as [HY]

$$J^\mu = \frac{1}{12\pi^2|\phi|^4}\epsilon^{ijkl}\epsilon^{\mu\nu\alpha\beta}\phi^i\partial_\nu\phi^j\partial_\alpha\phi^k\partial_\beta\phi^l . \tag{1.144}$$

The above discussed skyrmions with ansatz (1.137) and boundary conditions (1.138) and (1.139) have topological charge B. However, only in the case of $B = 1$ these hedgehog skyrmions are minimal energy configurations. For some $B > 1$ also lowest energy topologically non-trivial static configurations in the Skyrme model have been found [KS87, Man87, Ver87, BS97, BTC90, BS02]. These are however not of hedgehog form but have a more complicated symmetry structure. We refer to [MS07] and references therein for a discussion on lowest energy configurations in the Skyrme model with topological charge higher than one.

Since skyrmions appear as non-trivial topological configurations in a chiral theory which is a low energy effective theory of QCD, it is an important questions if and how we can understand skyrmions from the QCD point of view. Skyrme introduced the skyrmions originally as models for baryons [Sky61, Sky62] and later it was shown by Witten [Wit83b, Wit83a, Wit84] in the framework of QCD with a large number of colors N_C that skyrmions are really nothing than baryons. Let us now briefly review the main steps which lead to this conclusion.

One important step which shows that the idea that skyrmions are nothing than baryons in the framework of QCD with a large number of colors is consistent, is to show that the masses and length scales of baryons and skyrmions scale in the same way with N_C and that the cross sections which can be calculated in large-N_C QCD also scale in the same way with N_C as the anologous cross sections in the chiral theory. On the one hand it was established by Witten in [Wit79a] that in large-N_C QCD the masses of baryons scale linearly with N_C whereas the size and the baryon-meson scattering cross sections are independent of N_C. That, on the other hand, the analogous quantities in the Skyrme model have the same scaling behavior with N_C can be seen easily from (1.132), (1.133), (1.135), (1.134).

That parameters in the Skyrme models have the same scaling behavior in N_C as the analogous parameters for baryons in large-N_C QCD does not show that skyrmions are in fact nothing than baryons. In order to show this, it has to be shown that baryons and skyrmions have the same quantum numbers (spin, isospin and flavor) and in particular that skyrmions are fermions for N_C odd (as baryons with odd number of colors are) and bosons for N_C even (as baryons with even number of colors are). If two objects (in this case baryons and skyrmions) have the same quantum numbers, they can be considered to be the same objects. In fact, it has been shown that in the case of two quark flavors, skyrmions can be treated as fermions for N_C odd [Wit83a, FR68].[8] Let us now briefly go through the arguments which establish these claims in the case of two quark flavors.

That skyrmions can be quantized as fermions in the case of two quark flavors is based on the fact that $\pi_4(SU(2)) = Z_2$ [Wit83a, FR68]. Therefore, the maps from the 4 dimensional spacetime to $SU(2)$ fall into two classes. If we consider a process where a skyrmion which is created by skyrmion-antiskyrmion pair creation is rotated around the antiskyrmion with an angle 2π and then annihilated again, this process is topologically different to the same process with the skyrmion not rotated. One process can be weightend with a factor (-1) meaning that the skyrmion is a fermion. Therefore, in the case of two quark flavors, in this sense, skyrmions can be treated as fermions.

That, for large number of colors, skyrmions have the same spin and isospin quantum numbers as baryons in large-N_C QCD was shown by Witten in [Wit83a] (see als [BNRS83]). This can be seen once we quantize the collective coordinates which have to be introduced due to the appearance of zero modes when quantizing fluctuations around the classical skyrmion solution. There are three zero modes since skyrmions break the translational invariance (in all three directions of space), in complete analogy to the translational zero modes of the kinks, vortices and monopoles which we

[8] For cases with more quark flavors which we are not considering here, the stronger statement that skyrmions have to be treated as fermions if and only if N_C is odd was established as a consequence of the Wess Zumino Witten term added to the Lagrangian [Wit83a, Wit83b].

mentionned above, and there are additional three zero modes due to the rotation in real space and the rotation in $SU(2)$ space which are coupled in the case at hand (see e.g. [Shi12]) in analogy to the $U(1)$ related zero mode in the monopole case. Introducing the collective coordinates x_i and w_i $(i = 1, 2, 3)$ leads to a Hamiltonean which can be written as (see e.g. [Shi12])

$$\mathcal{H} = M + \frac{1}{2}M\dot{x}^2 + \frac{1}{2I}\nabla_3^2\,, \tag{1.145}$$

where I is the momentum of inertia which can be calculated (see e.g. [Shi12]) and ∇_3^2 is the Laplace operator on the 3-sphere with respect to the canonical momenta of the collective coordinates w_i. The eigenvalues of the last term of $\mathcal{H}$ are thus

$$E_{mm'j} = \frac{j(j+1)}{2I}\,, \tag{1.146}$$

where j is an integer which stands for the angular momentum and $m, m' \in (-j, j)$ are magnetic quantum numbers. Using Noethers theorem it can be shown that spin (the Noether charge with respect to rotations in real space) and isospin (the Noether charge with respect to rotations in isospin space) are related by $I^2 = J^2$ and that these are nothing but ∇_3^2. Eigenstates of the Hamiltonean are characterized by the quantum numbers spin $i =$ isospin j, m and m'. For bosons (even number of colors) j is even and for fermions j is odd. These are exactly the same quantum numbers baryons have.

Let us finally emphasize what the restrictions are which are made when skyrmions are identified with baryons. First, skyrmions and baryons were identified (only) in the framework of large N_C QCD. We did not argue if and how this correspondence between skyrmions and baryons is modified once we work with a finite numter of colors. It has however been argued [Wit84] that even for finite N_C baryons still correspond to skyrmions, that however, as pointed out below (1.142), for small N_C the skyrmions cannot be treated classically anymore. Second, the Skyrme Lagrangian (only) describes pion degrees of freedom. In nature, we however have many more meson degrees of freedom which have to be present in a proper low energy description of QCD. In this sense, the Skyrme model with only pions taken into account is an approximation to a (unknown) large N_C effective meson Lagrangian which is equivalent to QCD and which takes into account all meson degrees of freedom. It is known that some calculations done in the Skyrme model are around 30 percent off from experimental data (see e.g. [Wit84] and references therein). It is expected that this error comes from the fact that in the calculation we only take into account the existence of pions and neglect the other meson degrees of freedom [Wit84]. It is expected that calculations would coincide with experimental data once we use the (unknown) effective meson Lagrangian (for large N_C) which takes into account all the meson degrees of freedom and which is really equivalent to (large-N_C) QCD [Wit84].

1.5 Classical Gravity

On top of the non-gravitational interactions described by the quantum field theories which we reviewed in the previous sections, all objects which carry energy-momentum do also interact gravitationally. These gravitational interactions were originally introduced in a geometric way but can also be understood in terms of the language of quantum field theory of a massless spin-2 field. We briefly review a few aspects of these two viewpoints.

Geometric Point of View

The main idea which underlies general relativity as theory of gravity is to use Riemann's insights about geometry (see for example [Rie54]) and to apply them to Minkowski spacetime [Min08]. In particular, every spacetime point of the Minkowski spacetime is assigned a quadratic differential form whose coefficients $g_{\mu\nu}$ are postulated to be a dynamical field such that the proper time ds along a given world line is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,. \tag{1.147}$$

This implies that $g_{\mu\nu} = g_{\nu\mu}$. The evolution of this dynamical field is set by the Einstein field equations [Ein15]

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \,, \tag{1.148}$$

which can be obtained from the Einstein-Hilbert action,

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left(R + g^{\mu\nu} T_{\mu\nu} \right) \,, \tag{1.149}$$

with $g \equiv \det(g)$, via the principle of least action. Here $T_{\mu\nu}$ is the energy-momentum tensor of some kind of matter and $G_{\mu\nu}$ is the Einstein tensor which is defined via $g_{\mu\nu}$ as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,, \tag{1.150}$$

where the Ricci tensor $R_{\mu\nu}$ is defined as $R_{\mu\nu} \equiv R^\alpha_{\mu\alpha\nu}$ and the Ricci scalar R as $R \equiv g^{\mu\nu} R_{\mu\nu}$ with the Riemann tensor

$$R^\alpha_{\mu\beta\nu} \equiv \partial_\beta \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\beta\gamma} \Gamma^\gamma_{\nu\mu} - \Gamma^\alpha_{\nu\gamma} \Gamma^\gamma_{\beta\mu} \tag{1.151}$$

and the Christoffel symbols

$$\Gamma^{\alpha}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta} \left(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu} \right) . \tag{1.152}$$

Therefore, in this sense, the evolution of the field $g_{\mu\nu}$ is determined by the matter, described by $T_{\mu\nu}$, which is present in the spacetime.

The metric tensor $g_{\mu\nu}$ in four spacetime dimensions has 16 components among which are only 10 independent ones because of the property $g_{\mu\nu} = g_{\nu\mu}$. Under the change of coordinate system, $x^a \to \tilde{x}^b = f^b(x^a)$, these components transform as

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = g_{\alpha\beta}(x) \frac{dx^\alpha}{d\tilde{x}^\mu} \frac{dx^\beta}{d\tilde{x}^\nu} . \tag{1.153}$$

Since we are free to choose four functions f^a to do a coordinate transformation, only 6 out of the 10 independent components of $g_{\mu\nu}$ can be physical degrees of freedom in the sense that they are independent of the choice of coordinate system.

The Riemann tensor in four spacetime dimensions has 256 components. Because by definition the Riemann tensor fulfills the relations $R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha} = -R_{\nu\mu\alpha\beta}$, $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$ and $R_{\mu\nu\alpha\beta} + R_{\mu\alpha\beta\nu} + R_{\mu\beta\nu\alpha} = 0$, out of these 256 components 20 independent components are left. Since the metric has 6 physical degrees of freedom, 8 out of these 20 components are left to describe curvature.[9]

In order to determine how many of the 6 physical degrees of freedom of the metric $g_{\mu\nu}$ are propagating, we can use the coordinate freedom to fix for example the four components g_{00} and g_{0i}, as it is often done in the so-called "ADM formalism" [ADM59]. In the case of vanishing energy-momentum tensor, $T_{\mu\nu} = 0$, we are then left with 6 second order differential equations $G_{ij} = 0$ to deterime the physical components g_{ij}. This requires 12 initial conditions. 4 of these 12 initial conditions are fixed by the first order constraint equations $G_{0i} = 0$ which fix the 4 components for all times (this can be seen from the Bianchi identies in complete analogy to the constraint $\mathrm{div}E = 0$ in the source-free electromagnetic case). This leaves us with 8 initial conditions, or 4 degrees of freedom. Since we can fix the 4 functions g_{00} and g_{i0} arbitarily, in total we have 2 physical degrees of freedom for $g_{\mu\nu}$.

[9] This statement heavily depends on the number of spacetime dimensions. For example, in three spacetime dimensions the Riemann tensor has only 6 independent components and the metric has 3 physical degrees of freedom leaving us with no free component to describe curvature.

Gravity Viewed as a Quantum Field Theory

Instead of thinking in terms of geometric concepts such as curvature of spacetime, one can also understand general relativity by using the concepts of quantum field theory. This viewpoint is nicely reviewed for example in [Duf73] and [Fey96].

Perturbatively this is done by expanding the metric $g_{\mu\nu}$ in small perturbations $h_{\mu\nu}$ around a background metric $\bar{g}_{\mu\nu}$,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,, \tag{1.154}$$

by inserting this expansion into the Einstein-Hilbert action (1.149) and by applying the rules of quantum field theory to the linearized action. Using (1.153) it is easy to see that under coordinate transformations, $x^\mu \longrightarrow x^\mu + \epsilon^\mu$, $h_{\mu\nu}$ transforms as

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \,. \tag{1.155}$$

The linearized Einstein Hilbert action is the unique action of a massless spin-2 field $h_{\mu\nu}$ which both transforms as (1.155) and is consistent with Newton's theory of gravity [Wei65]. For the Minkoswki case, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, this linearized Einstein Hilbert action takes the form

$$\int d^4x \Big[\frac{1}{32\pi G_N} \left(\frac{1}{2} \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} - \frac{1}{2} \partial_\lambda h \partial^\lambda h - \partial_\lambda h^{\lambda\nu} \partial^\mu h_{\mu\nu} + \partial^\nu h \partial^\mu h_{\mu\nu} \right) - \frac{1}{2} h_{\mu\nu} T^{\mu\nu} \Big] \,, \tag{1.156}$$

with $h \equiv h^\mu_\mu$. For convenience the spin-2 field $h_{\mu\nu}$ is usually "canonically normalized" via $h_{\mu\nu} \to \sqrt{G_N} h_{\mu\nu}$.

The corresponding linearized Einstein field equations are given by

$$\Pi h_{\mu\nu} = 8\pi G_N T_{\mu\nu} \,, \tag{1.157}$$

where $\Pi h_{\mu\nu}$ is the linearized Einstein tensor

$$\Pi h_{\mu\nu} \equiv \Box h_{\mu\nu} - \eta_{\mu\nu} h - \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta} + \partial_\mu \partial_\nu h \,. \tag{1.158}$$

The full non-linear theory can be obtained iteratively by considering the first order perturbations $h_{\mu\nu}$ as a source for higher order perturbations (see e.g. the discussion in [DG14b]).

Using the linearized Einstein Hilbert action one can quantize the field $h_{\mu\nu}$ and one can do perturbative calculations in the "coupling constant" G_N analogously as in other quantum field theories. For a detailed discussion with many examples of this we refer to [Fey96] and [Duf73].

Since gravity viewed as quantum field theory is not UV complete, the perturbative results can only be trusted on low enough energies or on scales much larger than the Planck size.

1.6 Classical Singularities and Quantum Gravity

Some exact analytical solutions to the classical Einstein field equations have been discovered (see for example [Wal84]). Historically, the first solution which was found is the spherically symmetric metric with the following line element which solves the Einstein field equations $G_{\mu\nu} = 0$:

$$ds^2 = \left(1 - \frac{2MG_N}{r}\right) dt^2 - \left(1 - \frac{2MG_N}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 . \qquad (1.159)$$

This metric which is characterized by one free parameter denoted as M is well known as the "Schwarzschild metric" due to Karl Schwarzschild who discovered it in 1916 [Sch16].

The metric (1.159) has a curvature singularity at the point $r = 0$. In fact, at this point curvature invariants diverge.

Spacetime singularities occur in many classical solutions of the Einstein field equations. In general the unboundedness of curvature invariants however turned out not to be a complete satisfactory criterium to characterize singular spacetimes in general (see e.g. [Wal84] for a detailed discussion). Therefore, different criteria to characterize the existence of classical spacetime singularities have been proposed. For example, a spacetime is sometimes said to be singular if it possesses (at least) one geodesic which is incomplete [HP70]. This criterium is used in the context of the well known "singularity theorems" [HE11].

At spacetime singularites the classical description of general relativity is expected to break down and quantum effects of gravity are expected to become dominant. In the case of curvature singularities this breakdown of the classical description happens at scales at which the curvature invariants become of Planck size.[10]

1.7 Classical and Quantum Matter on Classical Backgrounds

In the cases the spacetime can be treated classically, matter coupled to this classical background, both in the case matter is treated classically and

[10] Since the classical description breaks down at (or close to) a spacetime singularity, it is in fact questionable if classical criteria such as the above mentionned criterium of geodesic incompleteness are adequate to characterize spacetime singularities or if quantum criteria should better be used (see e.g. [HS15] for a recent discussion).

in the case matter is treated as quantum, has been studied a lot. Let us briefly review some aspects of both cases.

Classical Matter

The equations of motion which describe classical matter on curved backgrounds are given by the Euler Lagrange equations which can be derived from the Einstein Hilbert action (1.149) by varying this action with respect to the matter fields. For the example of a minimally-coupled scalar field ϕ with potential $V(\phi)$ on a spacetime described by a metric $g_{\mu\nu}$ the Einstein Hilbert action takes the form

$$S = \int d^4x\sqrt{-g}\left(R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) \tag{1.160}$$

and the Euler Lagrange equations for the scalar field ϕ become

$$\Box_g\phi = \frac{\partial V(\phi)}{\partial\phi}\,, \tag{1.161}$$

with

$$\Box_g\phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right)\,. \tag{1.162}$$

Similar equations can be obtained for higher spin fields coupled to gravity.

Quantum Matter

A regime which is often studied in the literature is the regime in which the gravitational degrees of freedom are treated as classical whereas the matter fields which are coupled to the classical spacetime are treated as quantum. This “semiclassical” regime is supposed to arise from the full theory of quantum gravity where both gravitational and matter degrees of freedom are quantized in some semi-classical limit. This can be seen in analogy to the regime of “strong electromagnetic fields”, considered for example in the book [GMR85] and in the references therein.

One of the most important physical effects which can appear in this regime is the effect of particle antiparticle creation. We briefly review the canonical formalism which is usually used in semiclassical gravity for the case of a minimally-coupled scalar field.[11] We focus on how the effect of particle antiparticle creation is implemented in this formalism. For more details we refer to reviews in the literature, for example [MW07], [BD84] and [DeW75].

[11] Alternatively, one can use path integral techniques.

Given a background solution $g_{\mu\nu}$ of the (vacuum) Einstein equations, one usually starts with the classical action (1.160) with the classical solution $g_{\mu\nu}$ inserted and, in complete analogy to the canonical quantization of scalar fields in flat spacetimes, performs a mode expansion

$$\phi(t,x,y,z) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \left(\hat{a}_k v_k^*(t) + \hat{a}_{-k}^+ v_k(t)\right) e^{ikx} , \tag{1.163}$$

with

$$\begin{aligned} [\hat{a}_{k_1}, \hat{a}_{k_2}^+] &= \delta(k_1 - k_2) , \\ [\hat{a}_{k_1}, \hat{a}_{k_2}] &= [\hat{a}_{k_1}^+, \hat{a}_{k_2}^+] = 0 . \end{aligned} \tag{1.164}$$

One then derives the equations of motions for the "mode functions" $v_k(t)$ (the "mode equations") via (1.161) and normalizes these mode functions such that

$$\frac{d}{dt} v_k v_k^* - v_k \frac{d}{dt} v_k^* = 2i . \tag{1.165}$$

The left hand side is known as the (time independent) Wronskian. This normalisation condition is needed for ϕ and its canonical momentum to satisfy the standard canonical commutation relations. The vacuum state $|0\rangle$ is defined by

$$\hat{a}_k |0\rangle = 0 . \tag{1.166}$$

The mode functions $v_k(t)$ are by the mode equations and the normalisation condition (1.165) however only defined up to a Bogoliubov transformation, indeed one can also choose alternative mode functions $\tilde{v}_k'(t)$ which can be expressed in terms of the "old" mode functions v_k as

$$\tilde{v}_{k'} = \int (\alpha_{kk'} v_k + \beta_{kk'} v_k^*) \, dk , \tag{1.167}$$

whenever the time independent complex coefficients α_k and β_k satisfy

$$\int dk \, (\alpha_{kk'} \alpha_{kk''}^* - \beta_{kk'} \beta_{kk''}^*) = \delta(k' - k'') . \tag{1.168}$$

Therefore also the vacuum (and the corresponding Fock space) is not uniquely defined by (1.166). In fact, using the "new" mode functions implies the use of "new" annihilation (and creation) operators $\hat{b}_k$ (and $\hat{b}_k^+$) which can be expressed in terms of the "old" ones $\hat{a}_k$ and $\hat{a}_k^+$ as

$$\hat{b}_k = \int dk' \left(\alpha_{kk'} \hat{a}_{k'} + \beta_{kk'} \hat{a}_{k'}^+\right) . \tag{1.169}$$

The equality $\hat{a}_k |0\rangle = 0$ does not imply the equality $\hat{b}_k |0\rangle = 0$. This is formally the source of particle antiparticle creation in spacetimes which do not have a global timelike killing vector field. In such spacetimes different observers use different (local) timelike killing vector fields and therefore identify different mode functions (all related by a Bogolubov transformation) as positive and negative frequency functions which they use to define what they mean with "particles" and "antiparticles". In other words, there is no invariant meaning of positive and negative frequency functions in spacetimes without a global timelike killing vector field and different observers in such spacetimes disagree on what they call vacuum. Thus, if one observer observs a state occupied with no particles, some other observers observing the same state interpret it as a state containing particles.

Hawking applied this reasoning to the case of the Schwarzschild spacetime (1.159) and in this way discovered the famous Hawking radiation which we will recall after having introduced classical black holes in the next chapter.

2. Black Holes

Due to J. A. Wheeler objects with escape velocity larger than the speed of light are called “black holes”. After some early ideas about such objects with escape velocity larger than the speed of light, based on Newtonian gravity (for example by Michell [Mic84] and by Laplace [Lap96]), the study of black holes became a popular research topic only later in the context of general relativity. The Schwarzschild metric (1.159) characterizes the simplest known classical black hole solution in general relativity (it can be completely described by one parameter, its mass M). This metric has an event horizon at $r = 2MG_N$, indeed a surface such that light emitted from the inside can never reach an outside observer. (Due to the existence of this surface, the Schwarzschild metric describes therefore a black hole.[12]) More complicated black hole solutions (completely describable only by more than one parameter) are also known in general relativity. In Einstein-Maxwell theory (without sources) the number of these different black hole solutions is however limited; under the assumption of analyticity, all classical stationary non-extremal asymptotically-flat black holes which can be found as black hole solutions with regular event horizon in source-free Einstein-Maxwell theory are completely describable by the three parameters mass, electric (and magnetic) charge and angular momentum (“uniqueness theorems”) [Isr67, Isr68, Car71, Rob74, Rob75, Maz82] (see also [Heu96], [CLCH12], [Maz00] and [Car79] for some reviews and for further references therein). These black holes are known as “Kerr-Newman black holes”.

It has been conjectured that black holes (and no “naked singularities”) are formed in every gravitational collapse which leads to a spacetime singularity [Pen69].

Wheeler and Ruffini conjectured in 1971 that every asymptotically-flat black hole (which is formed by gravitational collapse) is completely describable by its mass, electric (and magnetic) charge and angular momentum

[12] Note that the point $r = 2MG_N$ in the metric (1.159) is a “coordinate singularity”, indeed the solution is perfectly regular at $r = 2MG_N$. This can be seen for example by using Eddington-Finkelstein coordinates [Edi24, Fin58]. From the historical point of view it took a long time to realize this.

- not only black holes in (source-free) Einstein-Maxwell theory [RW71]. They emphasized that electric charge for example is distinguished when compared to many other physical parameters in the sense that it is associated to a Gauss law. Therefore, the association of a Gauss law to a physical parameter is in the spirit of Wheeler and Ruffini sometimes conjectured to be a nessecary and sufficient condition for the parameter to describe a black hole ("no-hair conjecture"). This conjecture has been proven classically for several special types of non-Maxwellian matter ("no-hair theorems"[13]), most prominently by Bekenstein [Bek72c, Bek72b, Bek72d], Teitelboim [Tei72a, Tei72b] and Hartle [Har71]. However, from the late 80s on many classical black hole solutions of the Einstein field equations (with different matter sources) with primary hair (with parameters not associable to a Gauss law) have been discovered (see e.g. [VG99, HR15b, Vol17, Biz94, Bek96] for some reviews). Although many of these solutions turned out to be dynamically unstable (in Lyapunov sense), some of them are known to be stable against perturbations (on the linear level). All of the asymptotically-flat and spherically-symmetric black hole solutions of this kind which are known to be stable against spherically-symmetric linear perturbations are black hole solutions of the Einstein field equations where the matter source is played by a topological soliton. In this sense, it seems therefore that topology is a necessary condition for spherically-symmetric and asymptotically-flat classical hairy black holes to be dynamically stable against linear perturbations.

Classical black hole hair can be detected via classical experiments, for example via classical scattering of waves.

On top of classical hair, black holes can also carry certain types of quantum hair, indeed parameters which desribe a black hole but become invisible in the classical limit and which therefore can only be detected via quantum experiments. Different types of quantum hairs have been discovered, for example black hole parameters which characterize a black hole but can only be detected via quantum Aharonov-Bohm-type experiments ("Aharonov-Bohm-type black hole hair"). Recently, in the literature, also "soft black hole hair" [HPS16, ADGL16] and black hole hair which are due to a proposed quantum microscopic substructure of the black hole ("quantum $1/N$ hair", [DG13a]) have been discussed.

In the following subsections we briefly recapulate the Kerr-Newman black hole solutions and the uniqueness theorems in Einstein-Maxwell theory without sources, the no-hair conjecture and no-hair theorems, classical black hole hair and the role of topology in this context, scattering of clas-

[13]Sometimes in the literature also some of the black hole uniqueness theorems in Einstein-Maxwell theory are referred to as "no-hair theorems".

sical waves by black holes as one way to detect classical black hole hair, quantum Aharonov-Bohm-type black hole hair and $1/N$ black hole hair.

2.1 Kerr-Newman Black Holes and Uniqueness Theorems

In addition to the Schwarzschild black hole solution (1.159), electrically-charged "Reissner-Nordstroem black holes" [Rei16, Nor18] and rotating (axisymmetric) "Kerr black holes" [Ker63] have been discovered as asymptotically-flat black hole solutions of the source-free Einstein-Maxwell equations. Newman discovered the "Kerr-Newman black holes" which are black holes which are both electrically-charged and rotating [NJ65, NCC⁺65]. These Kerr-Newman black holes are described by the metric with line element (in Boyer-Lindquist coordinates [BL67])

$$ds^2 = -\left(\frac{dr^2}{A} + d\theta^2\right)\rho^2 + \left(dt - \frac{J}{M}\sin^2\theta d\phi\right)^2 \frac{A}{\rho^2}$$

$$- \left(\left(r^2 + \left(\frac{J}{M}\right)^2\right) d\phi - \frac{J}{M}dt\right)^2 \frac{\sin^2\theta}{\rho^2}\,, \tag{2.170}$$

with

$$\rho^2 \equiv r^2 + \left(\frac{J}{M}\right)^2 \cos^2\theta\,,$$

$$A \equiv r^2 - 2G_N M r + \left(\frac{J}{M}\right)^2 + Q^4 G_N^2\,. \tag{2.171}$$

Here Q can be identified as the electric charge of the black hole, M as its mass and J can be interpreted as its angular momentum. For $J = 0$ this is the line element of the Reissner-Nordstroem metric, for $Q = 0$ this is the line element of the Kerr metric and for $Q = J = 0$ it reduces to the line element of the Schwarzschild metric. For $M^2 = \frac{J^2}{M^2} + Q^2$ the corresponding black holes are known as "extremal black holes".

All these spacetimes are stationary, that is they admit an asymptotically timelike killing vector field. The Scharzschild and Reissner-Noerdstroem spacetimes are static, that is the killing vector field k wich generates the stationary symmetry is hypersurface-orthorgonal (that is $k \wedge dk = 0$). The Kerr-Newman spacetime is axisymmetric, that is it has a rotation axis and is invariant under the $SO(2)$ transformations with the rotation axis as fix point.

Under the assumption of an analytic spacetime, it has been proven that all non-extremal asymptotically-flat stationary black hole solutions with regular event horizon of the source-free Einstein-Maxwell equations can classically be completely characterized by the parameters mass, electric (and magnetic) charge and angular momentum J; in other words, assuming analyticity of the spacetime, that the non-extremal Kerr-Newman metric completely characterizes any stationary asymptotically-flat classical non-extremal black hole solution of Einstein-Maxwell theory without sources [Isr67, Isr68, Car71, Rob74, Rob75, Maz82]. The according theorems are the "black hole uniqueness theorems" (see also [Heu96], [CLCH12], [Maz00] and [Car79] for some reviews). In fact, Israel gave a first proof of the statment that all static asymptotically-flat non-extremal black hole solutions of Einstein-Maxwell theory without sources which have a horizon with spherical topology can classically be completely described by the Schwarzschild or Reissner-Nordstroem metric [Isr67, Isr68] (see also [MzHRJa, MzHRJb, Sim85, Bun, BuAAKM87] for different proofs and technical assumptions involved). Carter [Car71], Robinson [Rob74, Rob75] and Mazur [Maz82] showed that all non-extremal axisymmetric asymptotically-flat stationary black hole solutions of the source-free Einstein-Maxwell equations which have a horizon with spherical topology can classically be completely described by the Kerr-Newman metric. Since, assuming that the spacetime is analytic, all stationary asymptotically-flat black hole solutions of Einstein-Maxwell theory (without sources) are either static or axisymmetric and have a horizon with spherical topology ("rigidity and staticity theorems" [Haw72a, HE11, Chr97]) this proves the uniqueness of the non-extremal Kerr-Newman metric for stationary black hole solutions in Einstein-Maxwell theory (without sources) in the asymptotically-flat case in the case of analytic spacetimes.

2.2 No-Hair Conjecture and No-Hair Theorems

In the spirit of Wheeler and Ruffini [RW71] it was sometimes conjectured that even beyond Einstein-Maxwell theory, for any kind of matter, all black hole solutions of the Einstein field equations (with any conserved energy momentum tensor) can classically be completely described by parameters which are associable to a Gauss law. Parameters which are not associable to a classical Gauss law and which nevertheless can characterize a black hole are sometimes referred to as "primary black hole hair". Thus, one formulation of the conjecture of Wheeler and Ruffini is that black holes do not carry classical primary hair ("no-hair conjecture").[14]

For certain energy momentum tensors coupled to Einstein gravity, the no-hair conjecture in fact was proven to be correct, for example in [Bek72c, Bek72b, Bek72d], Teitelboim [Tei72a, Tei72b] and Hartle [Har71]. Let us, for illustration, briefly review the simple and elegant proof of one such "no-hair theorem". Namely, we will recall one proof which shows that, under the assumption of analyticity, for a stationary asymptotically-flat (rotating) black hole, a minimally-coupled real scalar field Φ with mass m which obeys $\partial_t \Phi = \partial_\phi \Phi = 0$ cannot appear as classical black hole hair [Bek72c, HR15b]. This setup is described by the action

$$\mathcal{S} = \int d^4x \sqrt{-g} \left(R - \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{m^2}{2} \Phi^2 \right) , \tag{2.172}$$

where R is the Ricci scalar. The scalar field obeys the Klein Gordon equation on curved spacetime (1.161). Let us now assume that a stationary asymptotically-flat black hole exists as solution of the field equations corresponding to (2.172) and then show that in such a case the scalar field Φ either has to vanish everywhere or becomes singular at the event horizon at the black hole. This conclusion implies that no regular massive minimally-coupled real scalar field can be a classical black hole hair in the case of stationary asymptotically-flat black holes.

As a first step in the proof, let us multiply the equation of motion for Φ by Φ and integrate over the exterior region of the black hole,

$$\int d^4x \sqrt{-g} \left(\Phi \nabla_\mu \nabla^\mu \Phi - \Phi m^2 \Phi \right) = 0 \, . \tag{2.173}$$

Integrating the first term by parts, assuming Φ to be regular at the horizon, gives

$$\int d^4x \sqrt{-g} \left(-\nabla_\mu \Phi \nabla^\mu \Phi - m^2 \Phi^2 \right) + \int d^3x n^\mu \Phi \nabla_\mu \Phi = 0 \, , \tag{2.174}$$

where the second integral is taken over the black hole horizon with normal vector n^μ (the boundary term at spatial infinity vanishes because the spacetime was assumed to be asymptotically-flat). Since, under the assumption

[14] On top of classical primary black hole hair, also so-called "secondary" black hole hair are discussed in the literature. These "secondary hairs" are parameters that characterize a classical black hole and are not associated to a Gauss law which can however completely be re-parametrized (or "re-expressed") in terms of parameters associated to a Gauss law. Famous examples for such classical secondary black hole hairs are conformally coupled scalar fields [BBM70, Bek74a, Bek75]. Another well-known example is discussed in [Gib82].

of analyticity, the event horizon of any asymptotically-flat and stationary spacetime is killing [Haw72a, HE11, Chr97], the boundary term of (2.174) vanishes provided analyticity. This implies

$$\int d^4x\sqrt{-g}\left(\nabla_\mu\Phi\nabla^\mu\Phi + m^2\Phi^2\right) = 0\,. \tag{2.175}$$

This equality can only hold if $\Phi = 0$.

For many other kinds of matter Lagrangians similar theorems and proofs do exist. We however want to emphasize that such theorems do not exist for all possible matter Lagrangians. In the following section we will point out several examples of matter Lagrangians for which classical black hole hair can in fact exist.

2.3 Classical Black Hole Hair and Topology

In the late 1980s it turned out that for certain types of (non-Abelian) matter there exist black hole solutions of the classical Einstein field equations which are classically not completely describable by parameters which are associated to a Gauss law and which therefore violate the no-hair conjecture. Most prominently, Volkov and Gal'tsov discovered numerical hairy black hole solutions in Einstein-Yang-Mills theory with gauge group $SU(2)$ [VG89], indeed black hole solutions of the Lagrangian

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{4}R - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\right)\,, \tag{2.176}$$

where R is the Ricci scalar and $F^a_{\mu\nu} \equiv \partial_{[\mu}A^a_{\nu]} + \epsilon^{abc}A^b_\mu A^c_\nu$, which cannot be completely characterized by parameters associated to a Gauss law. They found these solutions by making the following spherically-symmetric ansatzes for the metric g and the gauge field A^a_μ

$$ds^2 = N^2(r)h(r)dt^2 - \frac{dr^2}{h(r)} - r^2 d\Omega^2\,, \tag{2.177}$$

$$A^a_i = \epsilon_{aik}\frac{x^k}{r^2}\left(1 - w(r)\right)\,, \tag{2.178}$$

with the ansatz-functions $N(r)$, $h(r)$ and $w(r)$, and by solving the Einstein field equations with appropriate boundary conditions for the ansatz-functions. For certain boundary conditions there exist black hole solutions, indeed solutions with event horizon r_h (a point r_h such that $h(r_h) = 0$), where in particular for $r \to \infty$ the Yang-Mills field strength behaves as $F^a_{ij} \to \frac{1}{r^3}$. Therefore, this field strength is not associated to a Gauss law and cannot be seen from far away. From infinity these black holes which were found for arbitrary event horizon sizes in fact are indistinguishable from Schwarzschild black holes. However the solution-metric describing these black holes is in the near-horizon regime different from the Schwarzschild metric. This shows that for given ADM mass a black hole therefore is not uniquely characterized (for given ADM mass there exist both the Schwarzschild black hole metric and the metric found by Volkov and Gal'tsov as black hole solution). Numerically, it is relatively easy to find the black hole solutions outside of the event horizon [VG89, VG90, Biz90] but much more involved to obtain solutions inside of the event horizon [BLM98, DGZ97]. For plots where the solution functions $N(r)$, $h(r)$ and $w(r)$ are visualized both inside and outside of the event horizon we refer for example to the review [VG99].

On top of the hairy black holes in Einstein-Yang-Mills theory with gauge group $SU(2)$ many other hairy black holes have been discovered (for different matter Lagrangians), see e.g. [VG99, HR15b, Vol17, Biz94, Bek96] for some reviews. There are both hairy black holes which are asymptotically-flat and hairy black holes which are asymptotically de Sitter or anti de Sitter. Many of these hairy black holes turned out to be dynamically unstable. To my knowledge the only spherically-symmetric and asymptotically-flat black hole solutions with classical hair which are known to be stable against perturbations have in common that they are obtained as solutions of the Einstein field equations with a matter Lagrangian coupled to gravity which (in flat spacetime) allows for topological solitons as non-trivial lowest-energy configurations. Thus, from what we know today, in this sense topology seems to be a necessary condition for asymptotically-flat and spherically-symmetric hairy black holes to be dynamically stable (at least against perturbations). This can be the Lagrangian which allows for magnetic monopoles as non-trivial lowest energy configurations (the Lagrangian of Einstein-Yang-Mills-Higgs theory) or the Lagrangian which allows for skyrmions as non-trivial lowest energy configurations.

These asymptotically-flat hairy black holes which are known to be dynamically stable against perturbations, all have in common that (in contrast to the $SU(2)$ case by Volkov and Gal'tsov) they are known for parameters and boundary conditions (for the ansatz-functions) chosen such that the black hole event horizon of the hairy black hole is smaller than a maximal possible size which is set by a characteristic length scale associated to the topological soliton. Therefore, for "small" solitons these black holes are typically very "small" and the event horizon is located inside of the core of the soliton; they are thus also sometimes called "horizons inside classical lumps" [KT92].

2.4 Scattering of Probe Waves by Black Holes and Detecting Classical Hair

Scattering of classical probe waves by black holes has been a research topic since the 1960s when Hildreth [Hil64] and Matzner [Mat68] first studied cross sections of probe (scalar) waves scattered by black holes. Since today much work has been done on that topic and detailed scattering and absorption cross sections for many different kinds of probe waves scattered by many different types of black holes have been obtained (see e.g. [FHM12] and references therein and [GA01, DLDH05, DDL06, COHM07, CDO09b, CDO09a, CHO09, OCH11, MLO$^+$13, BdODC14, Spo17] for some more recent works). Typically the cross sections of a probe wave scattered by a black hole differ for the same wave scattered by different black holes with same asymptotic characteristics but different near-horizon geometries. Therefore, since black holes with and without classical hair are characterized by the same asymptotic characteristics but different near-horizon geometries, using classical scattering of (probe) waves is a natural way to discriminate black holes with and without classical hair.

In the simplest case of a probe wave - a monochromatic massless scalar wave Φ - the quantitative analysis to obtain cross sections is usually done as follows. The motion of the scalar field in the background spacetime of a classical black hole which is described by a metric g is given by the Klein-Gordon equation (1.161) with zero mass,

$$\Box_g \Phi = 0 \,, \tag{2.179}$$

where $\Box_g$ is the d'Alambert operator in the spacetime of the black hole with metric g. With the expansion

$$\Phi(t, r, \theta, \phi) = \sum_{lm} \frac{A_{Wl}(r)}{r} Y_{lm}(\theta, \phi) e^{-iWt} \,, \tag{2.180}$$

where $Y_{lm}(\theta, \phi)$ are the standard spherical harmonics, one can separate this equation into a radial part and an angular part. For a black hole metric g which is parametrized by the line element (2.177) with $h(x) \equiv$

$\left(1 - \frac{2M(x)G_N}{x}\right)$,

$$ds^2 = N^2(r)\left(1 - \frac{2M(r)G_N}{r}\right)dt^2 - \left(1 - \frac{2M(r)G_N}{r}\right)^{-1}dr^2 - r^2 d\Omega^2\,, \tag{2.181}$$

the radial part can be written as

$$\partial^2_{x^*}A_{wl}(x) + \left(w^2 - V_{eff}(x)\right)A_{wl}(x) = 0\,, \tag{2.182}$$

where x^* is defined as

$$\partial_{x^*} = N(r)\left(1 - \frac{2M(r)G_N}{r}\right)\partial_r \tag{2.183}$$

and the effective potential $V_{eff}(x)$ is given by

$$V_{eff}(x) = N^2(x)h(x)\frac{l(l+1)}{x^2} + \frac{N(x)}{x}h(x)\partial_x\left(N(x)h(x)\right)\,. \tag{2.184}$$

In this form the radial part has the form of a Schroedinger equation. Cross sections can therefore be analyzed by using the same methods which are used in the study of quantum (one-dimensional) scattering theory.

One can, for example, perform a partial wave analysis in order to obtain scattering (or absorption) cross sections. The effective potentials $V_{eff}(x)$ for black holes usually obey $V_{eff}(x^*) \to 0$ for $x^* \to \infty$. Therefore, for $x^* \to \infty$, one can write

$$A_{wl}(x^*) \to A^{(1)}_{wl}e^{-iwx^*} + A^{(2)}_{wl}e^{iwx^*}\,, \tag{2.185}$$

where $A^{(1)}_{wl}$ and $A^{(2)}_{wl}$ are two complex coefficients. For $x^* \to -\infty$ one can choose the boundary condition

$$A_{wl} \to A^{(3)}_{wl}e^{-iwx^*}\,, \tag{2.186}$$

with a complex coefficient $A^{(3)}$ which satisfies

$$|A^{(1)}_{wl}|^2 + |A^{(2)}_{wl}|^2 = |A^{(3)}_{wl}|^2\,. \tag{2.187}$$

Choosing this boundary condition means that we consider a monochromatic wave which is purely ingoing (at the horizon).

The differential scattering cross section can then be obtained as (see e.g. [New82])

$$\frac{d\sigma}{d\Omega} = |h(\theta)|^2\,, \tag{2.188}$$

where

$$h(\theta) = \frac{1}{2iw} \sum_{l=0}^{\infty} (2l+1) \left(e^{2i\delta_l(w)} - 1 \right) P_l(\cos\theta) \,, \tag{2.189}$$

with P_l the Legendre Polynomials. $\delta_l(w)$ are the phase shifts defined as

$$e^{2i\delta_l(w)} = (-1)^{l+1} \frac{A_{wl}^{(2)}}{A_{wl}^{(1)}} \,. \tag{2.190}$$

Depending on the form of the effective potential, it might be difficult to calculate the sum. There is however a method which has been developed long time ago in the context of classical Coulomb scattering [YRW54] (the "method of reduced series") and which can also be applied in the case of black holes.

Typically scattering cross sections obtained in this way by a partial wave analysis are characterized by certain "glory peaks" and the location of these peaks can characterize a black hole very well.

In certain regimes, in particular for high frequencies and scattering angles $\theta \approx \pi$, there exist approximation methods which can be used in order to obtain scattering cross sections without doing a detailed partial wave analysis (see e.g. [HM80, MDMNZ85]).

In our work we used black holes which can be parametrized by the above-mentioned metric and used both a detailed partial wave analysis and certain approximation methods in order to find diffential scattering cross sections for several monochromatic scalar waves scattered by these black holes. We showed by considering several examples that (for these examples) the scattering cross sections for black holes with and without hair differ. The characteristic glory peaks are located at different scattering angles. Therefore, scattering cross sections can be used in order to discriminate these black holes.

One can generalize this analysis in several directions, for example one can easily study also cross sections caused by waves of higher spin, for example electromagnetic waves or gravitational waves.

2.5 Quantum Aharonov-Bohm-Type Black Hole Hair

On top of the classical black hole hair discussed so far, several so-called Aharonov-Bohm-type (quantum) black hole hair have been discussed in the literature. These hairs are parameters which can characterize a black hole but which do not gravitate in the sense that they do not appear in the classical black hole metric. These black hole parameters are only measurable via (quantum) Aharonov-Bohm-type experiments (and not classically). Let us briefly review some aspects of the different kinds of Aharonov-Bohm-type black hole hair which have been discussed in the literature [KW89, PK90, CPW92, BGH+88, ABL90, KR91, Dva06]: Discrete Z_N Aharonov-Bohm hair, axion Aharonov-Bohm hair (both massless and massive) and massive spin 2 Aharonov-Bohm hair. For more details we refer to [KW89, PK90, CPW92, BGH+88, ABL90, KR91, Dva06].

Z_N Hair:

Discrete (Z_N) gauge theories can be realized in a model with two complex scalar fields Φ and Ψ which are charged under a $U(1)$ gauge field A_μ such that Φ has charge N and condenses at some scale v whereas Ψ has charge 1 and does not condense. The Lagrangian for such a model is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |\left(\partial_\mu - ieNA_\mu\right)\Phi|^2 + |\left(\partial_\mu - ieA_\mu\right)\Psi|^2 - V(\Phi,\Psi)\,, \quad (2.191)$$

where $F_{\mu\nu} \equiv \partial_{[\mu}A_{\nu]}$, e is the gauge coupling and $V(\Phi,\Psi)$ is a potential which is taken such that Φ condenses at the scale v and Ψ does not condense, e.g.

$$V(\Phi,\Psi) = \lambda(\Phi^2 - v^2)^2\,, \quad (2.192)$$

with a constant parameter λ. This Lagrangian (2.301) is invariant under

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha(x)\,,$$
$$\Phi \to e^{iN\alpha(x)}\Phi\,,$$
$$\Psi \to e^{i\alpha(x)}\Psi\,, \quad (2.193)$$

where $\alpha(x)$ is a gauge transformation function.

When Φ condenses and develops a vacuum expectation value (vev), the Lagrangian (2.301) allows for cosmic strings. The vev v of Φ is invariant under the above transformation only when $\alpha(x)$ is an integer-multiple of

$\frac{2\pi}{N}$. Since Ψ transforms nontrivially under these "residual" gauge transformations, the original $U(1)$ theory is broken down to a discrete Z_N theory when Φ condenses. If a quantum of Ψ moves once around the cosmic string, an Aharonov-Bohm phase is induced via (1.4)

$$\oint A_\mu dx^\mu \,. \tag{2.194}$$

Here dx^μ parametrizes the world line of the particle. That is also the case if this quantum is inserted into a black hole because (2.194) does not depend on whether the particle crossed a black hole horizon or not. In this sense the black hole develops a Z_N hair.

Axion Hair:

A different kind of Aharonov-Bohm-type hair has been discussed in the context of axions [BGH$^+$88]. In the case of massless axions, the Lagrangian for the two-form axion field $B_{\mu\nu}$ is given by

$$\mathcal{L}_{axion} = H_{\mu\nu\alpha}H^{\mu\nu\alpha} \,, \tag{2.195}$$

where the field strength H is defined as $H = dB$. The axion charge q can be defined as

$$q = \oint_{S^2} B \,. \tag{2.196}$$

Note that due to the existence of the field B, q can be non-zero also if H is vanishing. This is the case for example if $B_{\mu\nu} = \frac{q\epsilon_{\mu\nu}}{4\pi r^2}$ with $\epsilon_{\mu\nu}$ the volume-form on S^2.

A solution to the coupled Einstein-axion equations,

$$G_{\mu\nu} = \frac{16\pi G_N}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_{axion}\right)}{\delta g^{\mu\nu}} \,, \tag{2.197}$$

is given by

$$ds^2 = -\left(1 - \frac{2MG_N}{r}\right)dt^2 + \left(1 - \frac{2MG_N}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \,,$$

$$B_{\mu\nu} = \frac{q\epsilon_{\mu\nu}}{4\pi r^2} \,. \tag{2.198}$$

The corresponding field strength vanishes everywhere except at the origin,

$$\epsilon^{\mu\nu\alpha}H_{\mu\nu\alpha} = q\delta^{(3)}(x) \,. \tag{2.199}$$

That this is in fact the unique solution was shown in [BGH+88]. Since the axion charge q is defined at infinity, it is irrelevant if the charge-carrying axion crossed the event horizon of the black hole or not. In particular if the axion with charge q is inserted into a black hole, the above solution describes the system correctly.

The axion charge of the black hole can be measured via a stringy-generalisation of the Aharonov-Bohm effect. In fact, if a string couples to the axion field B an Aharonov-Bohm phase shift is induced via

$$\oint B_{\mu\nu} dX^\mu \wedge dX^\nu = q\,, \tag{2.200}$$

where X^μ are the string embedding coordinates. This phase shift leads to a physical effect whenever q is not an integer-multiple of 2π.

A generalisation to the case of massive axions is studied in [ABL90].

Massive Spin-2 Hair:

That black holes can also be charged under massive spin-2 was argued in [Dva06]. Let us here briefly review the essence of that argument.

The unique ghost-free (linearized) theory of massive spin-2 is given by the Pauli-Fierz action [FP39] which leads to the following equation of motion for the massive spin-2 field $h_{\mu\nu}$ with mass m

$$\Pi^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} = m^2 \left(h_{\mu\nu} - \eta_{\mu\nu} h\right)\,. \tag{2.201}$$

Here $h \equiv h^\alpha_\alpha$ and $\Pi^{\alpha\beta}_{\mu\nu}$ is the linearized Einstein tensor (1.158).

The massive spin-2 field $h_{\mu\nu}$ has five degrees of freedom which can be decomposed into two degrees of freedom of a field $\hat{h}_{\mu\nu}$ and three degrees of freedom of a massive "Stueckelberg" gauge field A_μ as

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu\,. \tag{2.202}$$

It is then argued in [Dva06] that among all possible configurations of $h_{\mu\nu}$ (allowed by the equations of motion) there are configurations where $h_{\mu\nu}$ is classically zero (pure-gauge) but nevertheless can produce a non-trivial phase shift at infinity in appropriate Aharonov-Bohm-type experiments. These are configurations where A_μ takes the form of a Dirac magnetic monopole,

$$A_\mu = \frac{(\cos(\theta) - 1)}{r\sin(\theta)} \partial_\mu \phi \tag{2.203}$$

on the upper hemisphere and

$$A_\mu = \frac{(\cos(\theta) + 1)}{r\sin(\theta)} \partial_\mu \phi \tag{2.204}$$

on the lower hemisphere and $\hat{h}_{\mu\nu} \equiv -(\partial_\mu A_\nu + \partial_\nu A_\mu)$.

Although $h_{\mu\nu}$ is pure gauge everywhere, one can detect such a configuration via stringy generalisation of the Aharonov-Bohm effect provided there exists a probe string which couples to $F_{\mu\nu} \equiv \partial_{[\mu} A_{\nu]}$ as

$$gF_{\mu\nu} \int d^2\sigma \partial_a X^\mu \partial_b X^\nu \epsilon^{ab} \delta^{(4)}(x - X)\,, \tag{2.205}$$

with g a coupling constant which is not an integer, σ the world-sheet coordinates of the string and $X^\mu(\sigma)$ the embedding coordinates. In fact, if g is non-integer the change in the action,

$$\Delta\mathcal{S} = 4\pi g\,, \tag{2.206}$$

in a process in which the probe string encloses a black hole which swallowed a particle charged under massive spin-2 induces, can be detected via Aharonov-Bohm effect.

2.6 Semiclassical Black Holes, Hawking Radiation, Quantum Black Holes and $1/N$ Hair

On top of the pure classical studies of black holes, there have been done a lot of studies on quantum fields in classical black hole backgrounds and on quantum black holes. We briefly review some of the aspects studied in these contexts and point out implications which these studies have on the topic of black hole hair.

Semiclassical Black Holes and Hawking Radiation

The Schwarzschild spacetime (1.159) has an event horizon and therefore no global timelike killing vector field. As reviewed in the last section we therefore expect particle anti-particle creation to happen in this spacetime if we consider quantum fields on the classical black hole backround. In general these created particle anti-particle pairs then backreact on the classical spacetime (1.159), there is however a particular double scaling limit in which this backreaction is absent. This is the limit [Dva16]

$$G_N \to 0, M \to \infty, G_N M \,\text{kept fix}\,, \tag{2.207}$$

in which in particular gravity is decoupled. In this "semi-classical limit" the surface area of the black hole event horizon, $A = 16\pi G_N^2 M^2$, is kept fix. Thus, although in this limits a black hole becomes infinitely heavy, it has a finite size. What is known as "black hole thermodynamics", which we will briefly review in what follows, can be understood self-consistently in this limit.

The research topic of black hole thermodynamics was born when Bekenstein conjectured with the help of Gedankenexeriments that a black hole has an intrinsic entropy S_{BH} which is proportional to the surface area of its event horizon [Bek72a, Bek73],

$$S_{BH} \propto \frac{A}{L_P^2}\,, \tag{2.208}$$

where L_P is the Planck length which can be related to Newtons constant G_N via

$$L_P^2 \equiv \hbar G_N\,. \tag{2.209}$$

Therefore, there exists a second law of thermodynamics for the black hole event horizon which can be stated as follows: The black hole event horizon never decreases. This is exactly the statement of the Hawking area theorem (for this theorem and assumptions involved one may consult [Haw71, Haw72b, HE11]).[15] On this level of the discussion the concept of black hole entropy should be understood as a phenomenological concept analogous to the Clausius entropy in thermodynamics and not as a statistical concept analogous to the Boltzmann entropy in thermodynamics.

In the semi-classical limit (2.207) the black hole entropy becomes infinite if $\hbar$ is kept fix. Hawking showed in [Haw74] and [Haw75], by using methods from semi-classical gravity, that in this limit the black hole emits thermal radiation with temperature T_H given by

$$T_H = \frac{\hbar}{8\pi M G_N} \tag{2.210}$$

and that a first law of thermodynamics holds in semi-classical black hole physics:

$$dM = T_H dS_{BH}\,. \tag{2.211}$$

Using this result one can also determine the proportionality constant between the area and the entropy of the black hole event horizon as

$$S_{BH} = \frac{A}{4L_P^2}\,. \tag{2.212}$$

[15] If a physical system consists of a black hole and any other subsystem which carries entropy, the second law states that the total entropy (the sum of the black hole entropy and the entropy of the other subsystem) never decreases.

On the level of this discussion there are many open questions about black hole thermodynamics. For example, from the discussion above, it is not clear if (and how) the entropy of a black hole can be understood as a statistical concept. Such an understanding would require certain black hole microstates. It is often argued that some of the open questions can only be answered if we treat black holes as quantum mechanical objects instead of describing them by a classical metric.

Quantum Black Holes

We will in what follows point out two examples of ideas about quantum black holes.

One line of research follows some early ideas which have been pointed out for example by Bekenstein [Bek74b]. He argued that the black hole event horizon is an adiabatic invariant and should therefore, in the spirit of "old quantum mechanics" [Bor25], only take discrete values in a quantum theory of gravity. We will not elaborate more on this line of research but refer to the review [Bek00] and to the references therein.

In another line of research a microscopic picture for a black hole, inspired by methods which are commonly used in condensed matter physics, has recently been suggested by Dvali and Gomez [DG13b, DG14a, DG14b]. They argued that a quantum black hole can be understood as a Bose-Einstein condensate of certain microscopic spacetime constituents stuck at the critical point of a quantum phase transition. In fact, it was assumed that there exist certain microscopic (quantum) black hole constituents which are the origin of black hole entropy. If we take N of such quantum constituents with de Broglie wavelength λ which interact pairwise with an interaction strength

$$\alpha = \frac{L_P^2}{\lambda^2}\,, \tag{2.213}$$

these constituents can form a self-sustained bound state

$$E_{kin} + V = 0\,. \tag{2.214}$$

This is because each of the N constituents "feels" a potential

$$V = -(N-1)\alpha\frac{\hbar}{\lambda} \approx -N\alpha\frac{\hbar}{\lambda} \tag{2.215}$$

by interaction with all the other constituents. Further, each constituent has a kinetic energy

$$E_{kin} = \frac{\hbar}{\lambda}\,. \tag{2.216}$$

Condition (2.214) is fulfilled whenever

$$\alpha = \frac{1}{N}\,. \tag{2.217}$$

Therefore a black hole can be thought of a self sustained bound state made out of N spacetime constituents of wavelength $\lambda = \frac{N}{L_P}$.

The constituents in this bound state may scatter with each other and constituents may leave the bound state due to this re-scattering. This leads in leading order of $\frac{1}{N}$ to a depletion rate [DG13b]

$$\Gamma = \frac{1}{\sqrt{N}L_P} \tag{2.218}$$

and to

$$\dot{N} = -\frac{1}{\sqrt{N}L_P}\,. \tag{2.219}$$

In the double scaling limit

$$N \to \infty, G_N \to 0, \sqrt{N}G_N \,\text{and}\, \hbar \,\text{kept fix} \tag{2.220}$$

this can be interpreted as a thermal Stefan Boltzmann depletion when we identify $N = S_{BH}$ and $T_H = \frac{\hbar}{\sqrt{N}L_P}$. Therefore, in this limit, Hawkings result of thermal black hole evaporation can be understood consistently. Beyond that limit (for finite N) we however get $\frac{1}{N}$ corrections to this "thermal depletion",

$$\Gamma = \frac{1}{\sqrt{N}L_P} + \frac{1}{L_P}\mathcal{O}(N^{-\frac{3}{2}})\,, \tag{2.221}$$

$$\dot{N} = -\frac{1}{\sqrt{N}L_P} + \frac{1}{L_P}\mathcal{O}(N^{-\frac{3}{2}})\,. \tag{2.222}$$

Therefore, thermality of the radiation in this picture is an emergent property which only arises in the semi-classical limit. These corrections give rise to new kind of "microscopic" quantum hair considered in [DG13a].

Black Hole $\frac{1}{N}$ Hair

In the above described quantum bound state picture of a black hole also other matter ingrediants can be implemented. In that case the bound state consists both of microscopic spacetime constituents and of matter quanta. If the matter quanta rescatter with the microscopic constituents (which is the case if there are appropriate interaction vertices), they can leave the condensate and in this sense appear as black hole hair. If there is a vertex between the matter constituents and the spacetime constituents which contributes to the probability of re-scattering with a factor α_{matter}, the rate for one single matter constituent goes as

$$\Gamma_{matter} = \alpha_{matter}\frac{1}{N^{\frac{3}{2}}L_P} + \frac{1}{L_P}\mathcal{O}(N^{-\frac{5}{2}})\,. \tag{2.223}$$

When compared to the depletion rate (2.218) this is a $\frac{1}{N}$-effect.

These matter constituents can for example be baryons. If there are B baryons in the bound state with N spacetime constituents, the rate will be

$$\Gamma_{baryon} = \alpha_{baryon}\frac{B}{N^{\frac{3}{2}}L_P}\left(1+\mathcal{O}\left(\frac{B}{N}\right)\right)\,. \tag{2.224}$$

This leads to a depletion of the baryon number of the form

$$\dot{B} = -\frac{1}{\sqrt{N}L_P}\frac{B}{N}\left(1+\mathcal{O}\left(\frac{B}{N}\right)\right)\,. \tag{2.225}$$

Therefore, compared to the depletion of spacetime constituents (2.219) this is a $\frac{B}{N}$ effect. This effect vanishes in the semi-classical limit if we keep B fixed. It however remains in the semi-classical limit if we both take N and B to infinity with the ratio $\frac{B}{N}$ kept fixed.

There are also many other ideas related with quantum black holes discussed in the literature which we will however not recall here.

2.7 Holography and Gauge Gravity Correspondence

We have pointed out in (2.208) that the entropy of a black hole scales as the area of its event horizon and not for example as the volume bounded by the black hole event horizon. This behaviour of black hole entropy can motivate the idea that the whole information about the black hole is "holographically" encoded in its event horizon. This idea as well as the generalisation of this idea to other physical systems without black holes (the question whether it is possible to encode the entire information of a theory defined in a given bounded volume in another (effective) theory which is living on the boundary of that volume) have been discussed a lot in the literature starting with the work of ‘t Hooft in 1993 [tH93] (see e.g. [Bou02] for a review).

One approach which implements this "holographic principle" beyond black hole physics in a concrete framework is the AdS/CFT conjecture

[Mal99, Wit98] (see also [AE15] for a pedagogical introduction). According to the AdS/CFT conjecture, it is possible to relate the whole physical content of a certain gravitational "bulk" theory in a dynamical boundary theory which is in this sense dual to the bulk theory. In fact, it is conjectured that $\mathcal{N} = 4$ Super Yang-Mills theory [DP99] with gauge group $SU(N_C)$ and Yang-Mills coupling constant g in $(3+1)$ spacetime dimensions (which is a conformal field theory) is dual to type IIB superstring theory on $AdS_5 \times S^5$ with string length l and coupling constant $g_s \equiv \frac{g^2}{2\pi}$. Several consistency checks have been performed in order to "test" this conjecture in certain limits [AE15]. In many cases it is however not possible to perform exact calculations on both sides of the correspondence. If the AdS/CFT conjecture is correct, this implies that in such cases by doing a pure gravitational calculation one can get results in the strongly coupled field theory which one cannot obtain by doing a pure field theoretical calculation.

3. Outline

In the following parts of this thesis we summarize our research papers as well as many unpublished results [DG16, DG17, Gus17, Guß17, GSW18]. For more details we refer to the original publications.

In part II of this thesis we shall summarize our results of the paper [DG16]. This paper makes use of the following topics which we have reviewed in part I of this thesis: Effective field theories (reviewed in 1.2), sigma models and skyrmions (reviewed in 1.3.3, 1.3.4 and 1.4), the Aharonov-Bohm effect (reviewed in 1.3.1), classical gravity (reviewed in 1.5), classical matter on classical backgrounds (reviewed in 1.7) and all the topics of black hole physics which we reviewed in 2.1, 2.2, 2.3, 2.4, 2.5 and 2.6.

We shall review the known black hole solutions with classical skyrmion hair in the asymptotically flat and spherically symmetric case in great detail. We shall then point out that the so-called black hole bound on the maximal allowed number of particle species for which strong quantum gravity effects can be ignored [Dva10, DR08, DG09] plays an important role in understanding the parameter space of solutions in which black holes with classical skyrmion hair do exist. We shall show that, on top of the known black holes with classical skyrmion hair, there exist also black holes with quantum skyrmion hair of Aharonov-Bohm-type. We shall show that these Aharonov-Bohm-type skyrmion black hole hair can exist for arbitrarily large black holes whereas the classical skyrmion black hole hair has been found only for black holes which are smaller than a critical size. We shall review the so-called black hole folk theorems and argue that the existence of black holes with skyrmion black hole hair provides a loophole in the proof of the black hole folk theorems which allows for a self-consistent possibility of baryon number conservation by black holes. We propose a dynamical process which realizes baryon number conservation by black holes. We shall also point out that the existence of black holes with skyrmion hair implies a loophole in a standard argument which is often used in order to justify the so-called weak-gravity conjecture. Finally, we show that classical skyrmion hair can be detected via classical scattering of waves by calculating scattering cross sections of massless minimally-coupled scalar waves scattered by black holes with and without classical skyrmion hair (which both have the same asymptotic characteristics) numerically. We find that the locations of the characteristic peaks in the differential scattering cross sections depend on whether the black hole carries a classical hair or not and determine the shift in the location of these peaks “induced” by the black hole hair in some concrete cases quantitatively.

In part III of this thesis we shall summarize our results of the paper [DG17] and point out some related unpublished results. This paper is a follow-up paper of [DG16]. In addition to the topics on which [DG16] is based on, [DG17] makes use of the following topics which we have reviewed in part I of this thesis: Cosmic strings (reviewed in 1.4), the Witten effect (reviewed in 1.4) as well as textures and point defects in different spacetime dimensions (reviewed in 1.4).

We shall show that under certain assumptions which we will specify the Aharonov-Bohm-type skyrmion hair can be detected via stringy generalisations of the Aharonov-Bohm effect in experiments in which a probe string encloses the skyrmion. Although it is not important for our purpose what kind of string exactly plays the role of the probe string, we shall show that the role of the probe string can for example be played by a cosmic string. We shall argue that one can understand the Skyrme topological charge of a black hole as coming from a Witten-type effect. Finally, we present a general procedure which allows to construct boundary differential forms for textures in a definite number of spacetime dimensions from topological currents of point defects in one dimension higher.

In part IV of this thesis we shall summarize our results of the paper [Guß17]. This paper makes use of the following topics which we have reviewed in part I of this thesis: The Georgi-Glashow model (reviewed in 1.3.2), magnetic monopoles (reviewed in 1.4), classical gravity (reviewed in 1.5), classical matter on classical backgrounds (reviewed in 1.7) and all topics of black hole physics reviewed in 2.1, 2.2, 2.3 and 2.4.

We shall review aspects of magnetically charged black holes solutions, both with and without classical hair, in the Georgi-Glashow model coupled to gravity in detail. We shall show that one can distinguish black holes with and without classical hair in that model (which both have the same asymptotic characteristics) by scattering of classical waves. We shall determine differential scattering cross sections of massless minimally-coupled scalar waves scattered by these black holes explicitly for some concrete cases. We show that the location of the characteristic peaks in the differential scattering cross sections depend on whether the black hole carries a classical hair or not and find that (and explain why) the shift in the location of the peaks "induced" by the hair is qualitatively different as the analogous shifts which we found in the case of classical skyrmion black hole hair. We also point out that the proposed method of classical wave scattering can have interesting astrophysical implications for finding out if a given black hole in nature carries some classical hair or not.

In part V of this thesis we mention our conference proceedings paper [Gus17]. In this paper we discussed some of the results of the papers which we reviewed in part II, part III and part IV. The presentation of the topics in [Gus17] is, in part, given from a different perspective when compared to the presentation of the topics in [DG16], [DG17] and [Guß17].

In part VI of this thesis we summarize some preliminary results of the ongoing project [GSW18]. This project makes use of the following topics which we have reviewed in part I of this thesis: Vortices and cosmic strings (reviewed in 1.4) and holography (reviewed in 2.7).

We shall review in detail the anyon-type statistics of Chern-Simons vortices in $(2+1)$ spacetime dimensions. We shall then consider cosmic strings in $(3+1)$ dimensional spacetimes with boundary. We argue that cosmic strings in $(3+1)$ spacetime dimensions obey fractional statistics whenever the boundary endpoint vortices of the strings are electrically charged and when the upper endpoint vortex carries a different electric charge than the lower endpoint vortex. We demonstrate this claim in several concrete models for cosmic strings in $(3+1)$ spacetime dimensions. We show that in our setups of cosmic strings with fractional statistics, the statistics of the string can be fully understood by studying only the statistics of its boundary endpoints. We shall point out that our results can have interesting applications to the AdS/CFT correspondence as well as to the topic of black holes with Z_N Aharonov-Bohm-type quantum hair.

Part II

Summary of Project 1:

“Skyrmion Black Hole Hair - Conservation of Baryon Number by Black Holes and Observable Manifestations”
authors: Gia Dvali and Alexander Gußmann
published in: Nucl. Phys. B913 (2016) 1001-1036

In the paper "Skyrmion Black Hole Hair - Conservation of Baryon Number by Black Holes and Observable Manifestation" [DG16] we investigated several aspects of skyrmion black hole hair. In the first part of the paper, after carefully reviewing the known classical black hole solutions with skyrmion hair of the Einstein-Skyrme equations and after pointing out how these solutions can be related to the so-called black hole species bound, we showed that black holes can not only carry the known classical skyrmion hair but also quantum Aharonov-Bohm-type skyrmion hair. We argued that the existence of skyrmion black hole hair sheds new light on the question if baryon number can be respected by black holes. In the second part of the paper we showed that classical skyrmion black hole hair can be detected via scattering of classical waves.

We summarize these discussions in what follows. For more details we refer to [DG16].

1. Review of Classical Skyrmion Black Hole Hair

In the late 1980s and in the beginning 1990s it has been realized that skyrmions can play the role of classical black hole hair. After some early ideas by H. Luckock and I. Moss in 1986 who argued in the "probe limit" (by studying probe skyrmions on a fixed Schwarzschild background) that black holes with classical skyrmion hair exist [LM86], the full backreacting problem in the asymptotically-flat and spherically-symmetric case was solved by S. Droz, M. Heusler and N. Straumann in 1991 [DHS91] and was further studied by P. Bizon and T. Chmaj in 1992 [BC92]. It was shown there that black holes with classical skyrmion hair in a certain domain of parameters indeed exist as solutions of the Einstein-Skyrme equations. Both works focused on numerical black hole solutions outside of the event horizon. Dynamical stability of some of the asymptotically-flat and spherically-symmetric black hole solutions with classical skyrmion hair was established on the linear level by M. Heusler, S. Droz and N. Straumann in 1992 [HDS92] and was also discussed by P. Bizon and T. Chmaj [BC92]. Classical skyrmion black hole solutions inside of the event horizon were discussed by T. Tamaki, K. Maeda and T. Torii in 2001 [TMT01]. Axisymmetric black holes with classical skyrmion hair were studied by N. Sawado, N. Shiiki, K. Maeda and T. Torii in 2004 [SSMT04]. Black holes with classical skyrmion hair in anti de Sitter spacetime were studied in 2005 by N. Shiiki and N. Sawado [SS05a, SS05c] and in de Sitter spacetime by Y. Brihaye and T. Delsate in 2006 [BD06]. Classical skyrmion black holes in a generalized Skyrme model (with higher order terms in the effective meson Lagrangian (1.127) taken into account) were studied by S. B. Gudnason, M. Nitta and N. Sawado [GNS16] as well as by C. Adam, O. Kichakova, Y. Shnir and A. Wereszczynski [AKSW16] in 2016. Higher-order derivative terms (as replacement for the Skyrme term) and their relevance for black hole hair have been discussed in 2018 by S. B. Gudnason and M. Nitta [GN18]. A review was written by N. Shiiki and N. Sawado in 2005 [SS05b]. Other related works are for example [Nie06] and [BHRT17]. In this thesis we focus on the spherically-symmetric and asymptotically-flat case in the "minimal" Skyrme model (1.127) in four spacetime dimensions.

1.1 Black Holes with Classical Skyrmion Hair as Solutions of the Einstein-Skyrme Equations

Black holes with classical skyrmion hair can be obtained as classical solutions of the Einstein-Skyrme equations,

$$G_{\mu\nu} = 8\pi G_N T^s_{\mu\nu}\,, \tag{1.226}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T^s_{\mu\nu}$ the energy momentum tensor which corresponds to the Skyrme Lagrangian (1.127),

$$T^s_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_s\right)}{\delta g^{\mu\nu}}\,. \tag{1.227}$$

Black hole solutions with classical skyrmion hair in the asymptotically-flat and spherically-symmetric case have been studied with a hedgehog ansatz for the pions and a spherically-symmetric ansatz for the metric of the following kind

$$\frac{\pi_a}{f_\pi} = F(r) n_a\,,$$
$$ds^2 = -N^2(r)\left(1 - \frac{2M(r)G_N}{r}\right)dt^2 + \left(1 - \frac{2M(r)G_N}{r}\right)^{-1} dr^2 + r^2 d\Omega^2\,, \tag{1.228}$$

where $F(r)$, $N(r)$ and $M(r)$ are three ansatz-functions and n_a is a unit-vector in radial direction. With these ansatz functions the independent components of the Einstein-Skyrme equations can be written as

$$\partial_x\left(\left(x^2 + 2\mathrm{sin}^2 F\right) N(x) h(x) \partial_x F\right)$$
$$= N(x)\left(\mathrm{sin}2F\left(1 + h(x)\left(\partial_x F\right)^2 + \frac{\mathrm{sin}^2 F}{x^2}\right) + \beta^2 x^2 \mathrm{sin}F\right)\,, \tag{1.229}$$

$$\partial_x m = \alpha\Big[\frac{x^2}{2} h(x)\left(\partial_x F\right)^2 + \mathrm{sin}^2 F$$
$$+ \mathrm{sin}^2 F\left(h(x)\left(\partial_x F\right)^2 + \frac{\mathrm{sin}^2 F}{2x^2}\right) - \frac{1}{2}\beta^2 x^2\left(2\mathrm{cos}F - 2\right)\Big]\,, \tag{1.230}$$

$$\partial_x N = \alpha\left(x + \frac{2}{x}\mathrm{sin}^2 F\right) N(x)\left(\partial_x F\right)^2\,, \tag{1.231}$$

where we introduced the dimensionless quantities $x \equiv ef_\pi r$, $\alpha \equiv 4\pi G_N f_\pi^2$, $\beta \equiv \frac{m_\pi}{\hbar e f_\pi}$ and $m(x) \equiv ef_\pi G_N M(r)$ as well as the function $h(r)$ which is defined as

$$h(x) \equiv 1 - \frac{2m(x)}{x} . \tag{1.232}$$

For certain choices of parameters α and β and certain boundary conditions for the ansatz-functions solutions have been found numerically both without event horizon ("gravitating skyrmions") and with event horizon ("black holes with skyrmion hair"). Here we focus on the black hole solutions (with event horizon of size r_h). In order to obtain these solutions, the following boundary conditions can be taken for a given event horizon size x_h:

$$m(x_h) = \frac{x_h}{2}, F(\infty) = 0, N(\infty) = 1 . \tag{1.233}$$

One boundary condition still has to be chosen to fix the boundary value problem completely. This is usually done by choosing $F(x_h)$ as a shooting parameter and to solve the equations (1.229), (1.230) and (1.231) numerically with the shooting method. For certain choices of event horizon sizes x_h numerical solutions of the equations in fact do exist. The values of x_h for which solutions do exist depend on the values of α and β. In order to understand this parameter domain of black hole solutions from a physical point of view, we now elaborate first on the physical meanings of α and β: α and β set the ratios of the relevent length scales in the system, in fact α is proportional to the ratio of the characteristic length scale of the skyrmion L (1.132) and the gravitational radius of the skyrmion $L_g \equiv 2M_S G_N$ with the mass M_S defined in (1.133):

$$\alpha \sim \frac{L_g}{L} \tag{1.234}$$

and β is equal to the ratio of L and of the Compton wavelength of the pion L_C:

$$\beta = \frac{L}{L_C} . \tag{1.235}$$

On top of that, since L_g scales linearly with the number of colors N_C and L is independent of the number of colors, α scales linearly with the number of colors [Wit79a]

$$\alpha \sim N_C , \tag{1.236}$$

whereas β is independent of N_C because both L and L_C do not depend on N_C. For fixed β the parameter space of solutions with boundary conditions $F(\infty) = 0$ and $N(\infty) = 1$ is given as follows: There exists a maximal value α_{max} and hairy black hole solutions of (1.229), (1.230) and (1.231) have only been found for $0 \leq \alpha \leq \alpha_{max}$. According to (1.234) and (1.236), this maximal value α_{max} for α represents both a maximal value for $\frac{L_g}{L}$ and a maximal value of N_C for which hairy black hole solutions of (1.229), (1.230) and (1.231) have been found. Therefore, this means that hairy black hole solutions have only been found as long as the size of the skyrmion L is bigger than its gravitational radius L_g (or, in other words, as long as the skyrmion not itself becomes a black hole). On top of that, for a given value of α (which is smaller than α_{max}) there exists a maximal value for the event horizon size $r_h^{max,\alpha}$ and black holes with skyrmion hair have only been found for event horizon sizes $r_h \leq r_h^{max,\alpha}$. For $\alpha \to \alpha_{max}$, $r_h^{max,\alpha}$ goes to zero, in fact $r_h^{max,\alpha_{max}} = 0$. This implies that the event horizon size r_h of black holes with skyrmion hair is always located inside of a typical length scale associated to the skyrmion (which itself is a function of L and L_g). Therefore, one can think of the black holes with classical skyrmion hair as black holes with event horizon size which is located inside the skyrmion. The skyrmion is therefore a classical hair of the black hole in the sense that it "goes through" the black hole and is not fully swallowed by it.

We plot for illustration some solution functions of black holes with classical skyrmion hair for some allowed values of parameters in Figure 1.1, Figure 1.2, Figure 1.3, Figure 1.4 and Figure 1.5. The solution function $F(r)$ for $\beta = 0$ scales as $1/r^2$ for $r \to \infty$ and as $\frac{1}{r}\exp^{-m_\pi r}$ for $\beta \neq 0$ as $r \to \infty$ (as in the flat spacetime case as mentioned in (1.141) and (1.142)). The solution function $m(x)$ approaches a constant when $r \to \infty$ which sets the ADM mass of the black hole. In the near horizon regime $m(x)$ however, depending on the parameter choices, can significantly differ from this constant. Thus, for given ADM mass, the black holes with skyrmion hair provide black hole solutions which have the same asymptotic characteristics than the Schwarzschild black holes with this ADM mass but can have significantly different near-horizon geometries (therefore, these solutions are called "hairy" black holes).

The graphs in the Figures are in all cases plotted only in regimes outside of the event horizon of the black hole. Finding the solutions inside of the black hole event horizon is numerically much more difficult than finding the plotted solutions in the regime outside of the event horizon. The internal structure of classical skyrmion black holes was studied in [TMT01]. There it was found that in most cases the mass function $m(x)$ has an oscillating behaviour inside of the event horizon but that there are also exceptional cases where no oscillations appear. Plots of the functions inside of the event horizon can be found in [TMT01].

Finally, we want to note that there exist for given parameter choices

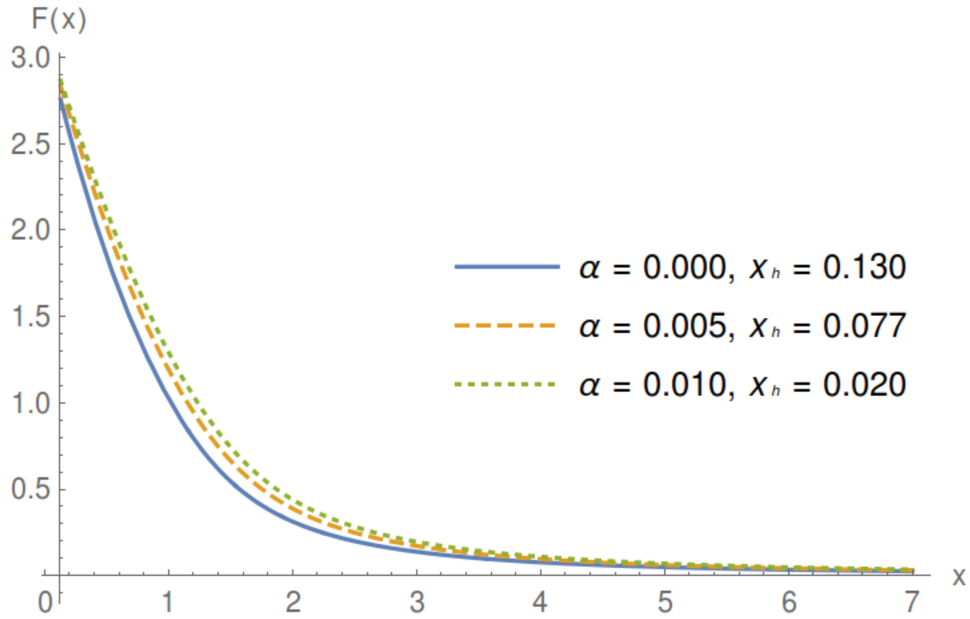

Figure 1.1: Profile function $F(x)$ for black holes with classical skyrmion hair in the case $\beta = 0$ and $m_{ADM} = 0.065$

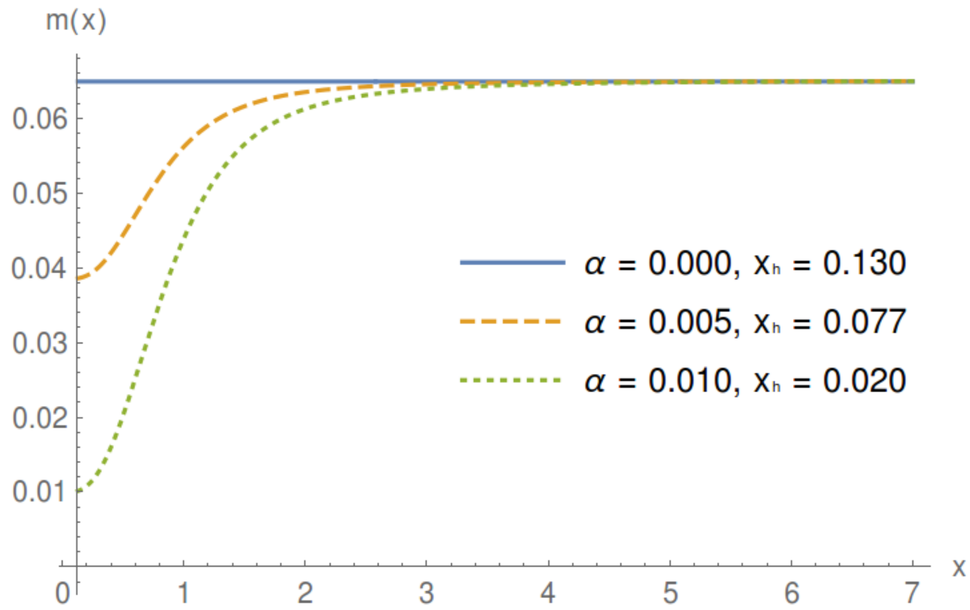

Figure 1.2: $m(x)$ for black holes with classical skyrmion hair in the case $\beta = 0$ and $m_{ADM} = 0.065$

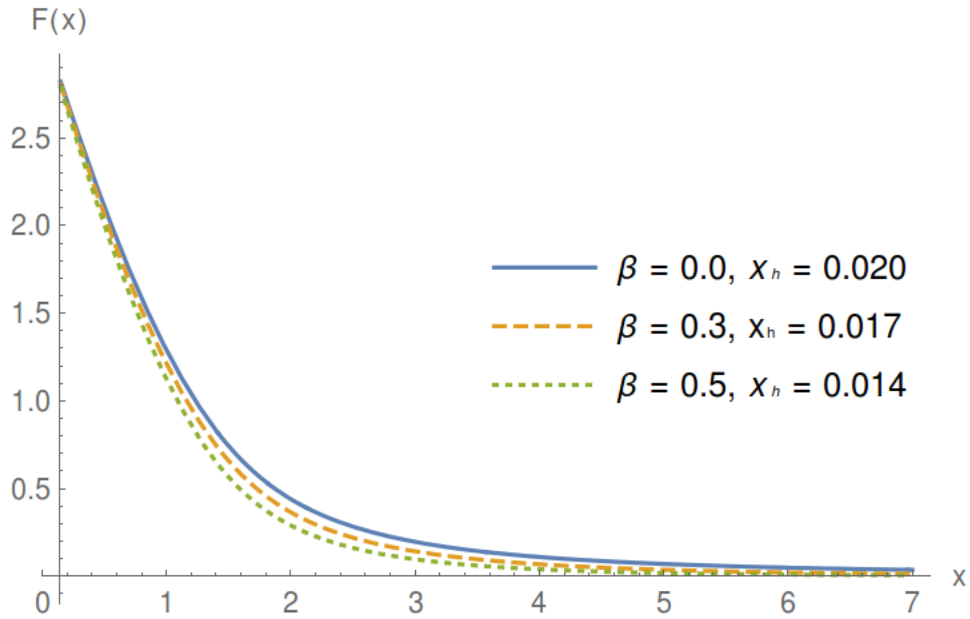

Figure 1.3: Profile function $F(x)$ for black holes with classical skyrmion hair in the case $\alpha = 0.01$ and $m_{ADM} = 0.065$

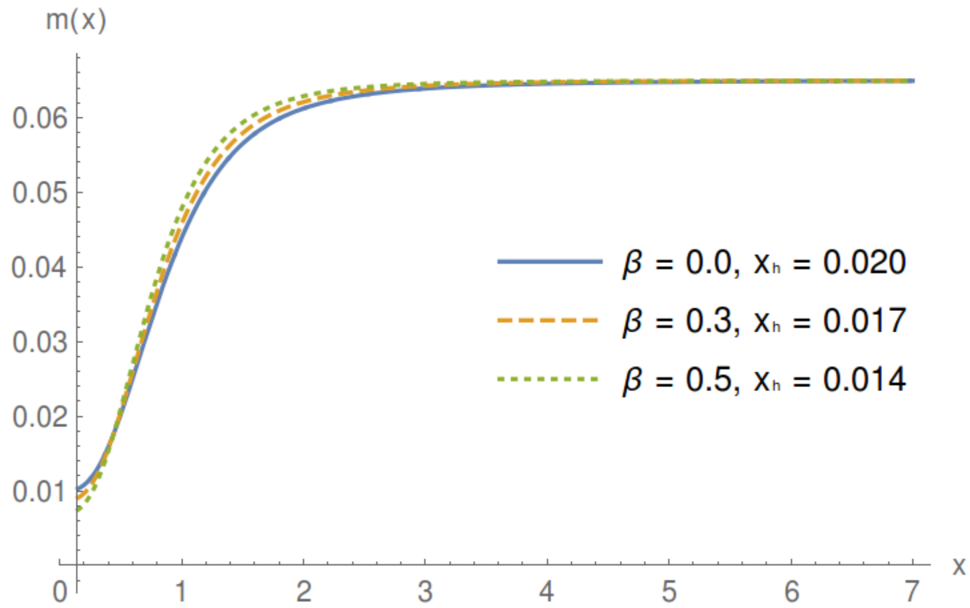

Figure 1.4: $m(x)$ for black holes with classical skyrmion hair in the case $\alpha = 0.01$ and $m_{ADM} = 0.065$

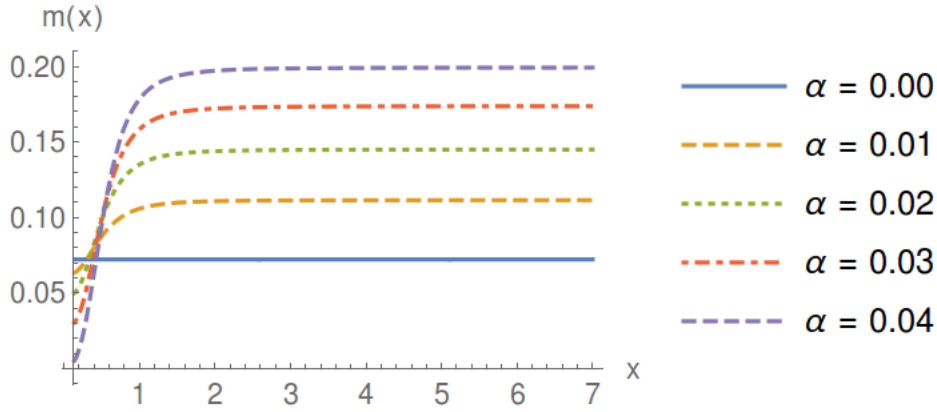

Figure 1.5: $m(x)$ for black holes with classical skyrmion hair in the case $\beta = 0$ and $x_h = x_h^{max,\alpha,\beta=0}$

two different branches of solutions. We will discuss in the next section that only the solution of one branch are dynamically stable (against linear perturabations) whereas the solutions of the other branch are dynamically unstable.

1.2 Stability of Black Holes with Classical Skyrmion Hair

If black holes with classical skyrmion hair are stable against (radial) perturbations or not is usually investigated by studying the linearized Einstein-Skyrme equations. One can derive a closed form of the radial matter perturbations δF which has the form of a one-dimensional radial Schroedinger equation [HDS92]

$$\left(-\frac{d^2}{dy^2} + V_{eff}(y)\right)\zeta = w^2\zeta\,, \tag{1.237}$$

with a "radial coordinate" y which is defined as

$$\frac{dy}{dx} = \frac{1}{N(x)\left(1 - \frac{2M(x)G_N}{x}\right)}\,, \tag{1.238}$$

an effective potential $V_{eff}(y)$ and ζ the Fourier component of the radial perturbations of the profile function

$$\delta F(x,t) = \frac{1}{\sqrt{x^2 + 2\sin^2 F}}\zeta(x)e^{iwt}\,. \tag{1.239}$$

If there exists a negative eigenvalue w^2, the corresponding mode is exponentially growing and the system is unstable. Since the equation for the radial perturbations (1.237) has the form of a (radial) Schroedinger equation, one can apply the theorem that states that the number of negative eigenvalues is equal to the nodes of the zero mode ($w^2 = 0$) solution of the perturbations ζ [RS78]. Using this method it was shown in [HDS92] that for one branch (the "upper branch") of black hole solutions with classical skyrmion hair there is one unstable mode whereas for the other branch (the "lower branch") of black hole solutions with classical skyrmion hair there is no unstable mode. Therefore, the black holes with classical skyrmion hair of the lower branch are stable against linear perturbations whereas the black holes of the upper branch are dynamically unstable.

To our knowledge, a stability analysis beyond the linear level has not yet been performed for black holes with classical skyrmion hair.

2. Classical Skyrmion Black Holes and Species Bound

We argued in [DG16] that the existence of the maximal value of α, α_{max}, in the parameter domain in which black holes with classical skyrmion hair have been discovered can be understood as a particular manifestation of the so-called black hole bound on the maximal allowed number of particle species for which strong quantum gravity effects can be ignored [Dva10, DR08, DG09].

This bound states that for $N \gg 1$ independent particle species, the scale L_{QG} at which quantum gravity effects become strong is given by

$$L_{QG} = \sqrt{N} L_P \,, \tag{2.240}$$

where L_P is the Planck length.

In order to see how this bound is realized in the case of skyrmion black holes, consider a large black hole which evaporates due to Hawking radiation. We argued in [DG16] that one can think about the evaporation process on scales much larger than L_{QCD} as process in which either a quark or an anti-quark is emitted and then hadronizes by combining with a partner from the "quark sea". This implies that the number of emission channels scales as the number of colors N_C. Thus

$$\frac{\dot{T}}{T^2} = N_C \left(\frac{T}{M_P} \right)^2 \,. \tag{2.241}$$

Once the parameter on the left hand side becomes order one, semi-classical gravity can no longer be trusted because if this is the case deviations from thermality become large [Dva16]. This happens at a temperature

$$T = \frac{\hbar}{L_{QG}} \,, \tag{2.242}$$

with $L_{QG} = \sqrt{N} L_P$ from the species bound. The requirement that L_{QG} must be shorter than L which is of order the QCD length L_{QCD} then gives the bound

$$N_C < \left(\frac{L}{L_P} \right)^2 , \tag{2.243}$$

which is nothing then the bound on α.

3. Existence of Aharonov-Bohm-Type Skyrmion Black Hole Hair

We realized in [DG16] that on top of the classical black hole solutions with skyrmion hair (which - as discussed below - can be detected classically) also Aharonov-Bohm-type black hole skyrmion hair which cannot be detected classically can exist. One of the most important differences when compared to the black holes with classical skyrmion hair is, that the black holes with Aharonov-Bohm-type skyrmion black hole hair can exist for arbitrary large event horizon sizes r_h and not only for event horizon sizes below a maximal value r_h^{max}.

In order to see that Aharonov-Bohm-type skyrmion black hole hair can exist, we proceed in two steps. First we study skyrmions in flat spacetime, then we go to the black hole case.

We realized that there exists a differential two-form $S_{\mu\nu}$ such that the Skyrme topological current (in flat spacetime) - understood as a differential one-form - can be written as Hodge dual of the exterior derivative of that two-form. We figured out that this two-form is given as

$$S_{\mu\nu} = -\frac{1}{4\pi^2}\cos^2\gamma\partial_{[\mu}\alpha\partial_{\nu]}\beta\,, \tag{3.244}$$

where α, β and γ are three angles parameterizing the $SU(2)$ matrix U which enters the Skyrme topological current (1.143):

$$U = \begin{pmatrix} \cos\gamma e^{i\alpha} & \sin\gamma e^{i\beta} \\ -\sin\gamma e^{-i\beta} & \cos\gamma e^{-i\alpha} \end{pmatrix}\,. \tag{3.245}$$

Note that every $SU(2)$ matrix can be parameterized via three angles like this, therefore this is not a particular ansatz for U but a generic parametrization. The angles α, β and γ can be identified with the hedgehog ansatz for the pions via

$$\gamma = \arcsin(\sin F \sin\theta)\,,$$
$$\beta = \phi\,,$$
$$\alpha = \arctan(\tan F \cos\theta)\,. \tag{3.246}$$

Evaluated on the hedgehog ansatz this gives for the topological charge B the two-form (up to the exterior derivative of a one-form)

$$S_{\mu\nu} = -\frac{1}{4\pi^2}\left(F(r) - \frac{1}{2}\sin(2F(r)) - B\pi\right)\partial_{[\mu}\cos(\theta)\partial_{\nu]}\phi\,. \tag{3.247}$$

Note that everything is written in the language of differential forms, if one wants to work in the language of functions in the equation for $S_{\mu\nu}$ an overall multiplicative factor of $(r^2\sin\theta)^{-1}$ has to be added. The constant $B\pi$ in the equation is crucial for the form to be well-defined everywhere. Without adding this constant, the two-form would be singular at the origin. For topological charge B this constant of $B\pi$ is the unique constant which makes the two-form well-defined everywhere.

Now, since this two-form exists - via Stokes theorem - one can write the Skyrme topological charge as surface integral over this two-form evaluated on a boundary at spatial infinity, e.g. a two-sphere S^2:

$$\int_{S^2} S_{\mu\nu} dX^\mu \wedge dX^\nu = B\,. \tag{3.248}$$

Let us now consider the black hole case. Since this two-form exists and the charge which corresponds to a conserved current can be defined at infinity, by the same argument as for example in the case of black holes with Aharonov-Bohm-type axion hair (which we have reviewed in section 2.5 of part I in this thesis), even if we locally insert the skyrmion into a black hole, the charge can still be defined at infinity. Or, in other words, there exists a solution of the Einstein-Skyrme equation for which the metric is pure Schwarzschild but still has a non-vanishing Skyrme topological charge due to a non-vanishing two-form $S_{\mu\nu}$. This solution which is there for black holes with arbitrary large event horizon sizes is given by

$$ds^2 = \left(1 - \frac{2MG_N}{r}\right)dt^2 - \left(1 - \frac{2MG_N}{r}\right)^{-1} dr^2 - r^2 d\Omega^2\,,$$
$$S_{\mu\nu} = \frac{B\pi}{4\pi^2}\partial_{[\mu}\cos(\theta)\partial_{\nu]}\phi\,. \tag{3.249}$$

As in the flat spactime case, the Skyrme charge of this configuration can be defined as surface integral over this two-form evaluated on spatial infinity

$$\int_{S^2} S_{\mu\nu} dX^\mu \wedge dX^\nu = B\,. \tag{3.250}$$

One remaining question is if this charge can "only" be defined at spatial infinity or if it can also be detected in some way (under certain assumptions). We will in detail argue in the next part of this thesis that, how (and under what assumptions) this charge can indeed be measured via stringy generalisations of the Aharonov-Bohm effect similarly as in the case of black holes with Aharonov-Bohm-type axion hair.[16] In what follows in this chapter we take it as granted, that an asymptotic observer indeed has the ability to measure the Skyrme topological charge of a black hole of arbitrary event horizon size via Aharonov-Bohm-type experiments.

[16] In [DG16] we also discussed skyrmion black holes with weakly-gauged baryon number and a way to measure such weakly-gauged baryon number of a black hole. In this thesis we will not review this case of a baryon number which is weakly-gauged in detail but only comment on one implication in section 4.3.

4. Baryon Number Conservation by Black Holes

The existence of skyrmion black hole hair (both classical and Aharonov-Bohm-type) sheds new light on the question if baryon number can be conserved by black holes or not. Let us, in this chapter, first briefly recall the black hole folk theorems which claim that global charges such as baryon number are not respected by black holes. Let us then elaborate on how the existence of skyrmion black hole hair changes these “standard” arguments. Finally, we will point out how the existence of skyrmion black hole hair sheds new light on the weak gravity conjecture [AHMNV07].

4.1 Black Hole Folk Theorems

The “standard” folk theorems argument is often presented as follows: Consider some particles which carry a global charge (in our case baryon number) and insert these particles into a black hole which can be arbitrarily large. The resulting black hole metric is, according to the folk theorems argument, classically completely described by the Kerr-Newman metric, irrespective of the baryon charge which has been inserted into the black hole. Then the black hole evaporates (via emitting Hawking radiation) and shrinks until it reaches the Planck size. Since the evaporation is, according to the folk theorems argument, said to be thermal, no (or negligible) baryon number is emitted during the evaporation process. Therefore, the “end product” is a Planck size Kerr-Newman black hole with no sign of baryon number. Usually it is then concluded in the folk theorems argument that the baryon number which has been inserted into the black hole is either lost or “hidden” in a Planck size black hole remnant which can only be described by a theory of quantum gravity operating at the Planck scale.

This argument is based on two main assumptions: First, that black holes cannot carry any (semi)classical hair which carries information about the baryon charge. Second, that the Hawking evaporation is exactly thermal. In what follows we will argue that there is a loophole in the folk theorems argument because, as we will discuss, the first assumption is not correct. (Note that, as discussed in detail in [DG16], the two assumptions are not unrelated and that also the second assumption is not correct since evaporating black holes shrink and can thus not evaporate exactly thermally. Here we shall however focus only on the first assumption which, as we will discuss, is enough to point out a loophole in the folk theorems argument.)

4.2 Baryon Number Conservation due to Black Hole Skyrmion Hair

In a first step we will argue that the existence of classical skyrmion black hole hair makes baryon number conservation in systems with black holes logically possible. We will describe one possible dynamical scenario by which baryon number conservation can possibly be realized. We then argue in a second step that, taking into account also the existence of Aharonov-Bohm-type skyrmion black hole hair, baryon number is respected by black holes (under certain assumptions which we will point out) and that thus (under these assumptions) baryon number is in fact conserved in systems with black holes. We do not claim that the baryon number conservation is necessarily realized in nature by the dynamical scenario which we discuss in the first step (there might in fact be other, more exotic, dynamical scenarios).

For simplicity we restrict in the following to the case of a black hole which swallowed only one baryon. The arguments can be generalized to the case of more baryons. In the arguments we consider two regimes: The regime when the event horizon size r_h of the black hole which swallowed the baryon is larger than the characteristic size L of the skyrmion, $r_h > L$, on the one hand and the regime when the event horizon of the black hole is smaller than L, $r_h < L$, on the other hand.

Let us first only take into account the existence of black holes with classical skyrmion hair. This existence already implies that the first assumption in the folk theorems argument is not correct since (from the point of view of a low energy observer) these black holes with classical skyrmion hair are nothing but black holes with classical baryon hair [Wit84]. Thus, the following dynamical scenario cannot be ruled out by the arguments used in favor of the folk theorems and is therefore a logical possibility which

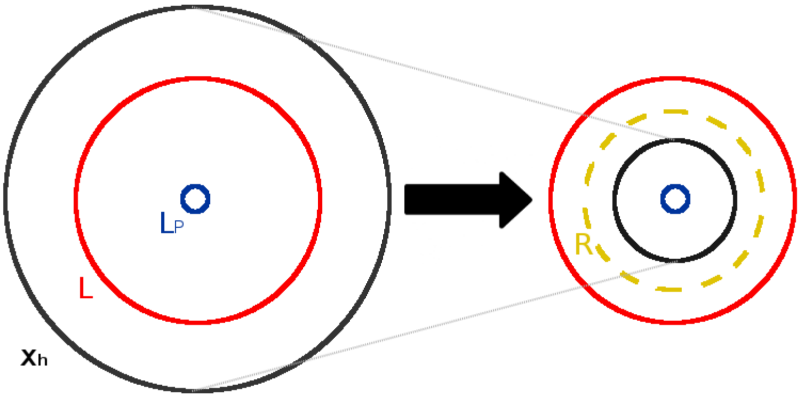

Figure 4.6: Black hole with event horizon size x_h shrinks via Hawking evaporation up to a size L; after reaching size L, it shrinks further inside of the skyrmion/baryon with size L

may or may not be realized in nature: Consider a large black hole in the regime $r_h > L$ which swallows a skyrmion/baryon. Then, as in the folk theorems argument, one concludes (if one is not aware of the existence of Aharonov-Bohm-type skyrmion hair) that the metric of the resulting black hole is of Kerr-Newman type and that thus the resulting black hole of size $r_h > L$ carries from the point of view of an outside observer no measurable baryon number. As in the folk theorems argument the black hole then starts to shrink via emitting Hawking radiation. Since in the regime of a size smaller than L, black holes with classical skyrmion/baryon hair are known (a fact which is not taken into account in the folk theorems argument), it is now in contrast to the folk theorems argument however logically possible that the baryon/skyrmion which was swallowed by the black hole might classically re-emerge in the form of a classical skyrmion hair once the shrinking black hole reaches a size L or, in other words, that when reaching the size L the shrinking Kerr-Newman black hole undergoes a phase transition and becomes a black hole with classical skyrmion hair. Since the size L is much larger than the Planck size (for realistic baryons), the baryon number in such a scenario is revealed long before the black hole reaches the Planck size. After the baryon number re-emerged in such a scenario, the black hole might shrink further down to the Planck size inside of the baryon/skyrmion. This option is visualized in Figure 4.6.

Let us now in a second step take into account the existence of the skyrmion Aharonov-Bohm-type black hole hair. The difference when compared to the discussion in the first step is that an outside observer at spatial infinity can now use the Aharonov-Bohm effect to measure the Skyrme topological charge/the baryon number of a skyrmion/baryon which allows him to monitor the Skyrme topological charge/the baryon number at infinity, also if the baryon/skyrmion which carries this charge gets locally swallowed by a black hole of size $r_h > L$ (this is doable under certain assumptions which we will discuss in detail in the next part of this thesis).[17] Therefore from the point of view of such an outside asymptotic observer black holes of size $r_h > L$ which swallowed a baryon/skyrmion carry a measurable baryon number/Skyrme topological charge and thus baryon number is conserved in the presence of black holes (under the assumptions which have to be fulfilled that an asymptotic observer can do such an Aharonov-Bohm-type experiment which we will discuss in the next part of this thesis). The conservation might dynamically be realized in nature via a process such as the one visualized in Figure 4.6. For such a process at each definite moment of time a static black hole solution with fixed baryon number is known (for black holes with size $r_h > L$ the black holes with Aharonov-Bohm-type skyrmion hair and for black holes with size $r_h < L$ both the black holes with Aharonov-Bohm-type skyrmion hair and the black holes with classical skyrmion hair). It can however also be the case that in nature a (more exotic) dynamical scenario is realized which conserves baryon number in systems with black holes.

4.3 Consequences for the Weak-Gravity Conjecture

One formulation of the so-called "weak gravity conjecture" [AHMNV07] states that due to black hole physics a gauge coupling cannot be arbitrarily weak. Instead, it is conjectured that there is a lower bound on any gauge coupling. One argument leading to this conjecture goes in complete analogy to the folk theorems argument for global charges which we have reviewed in section 4.1: Consider a weakly-gauged symmetry with a gauge coupling which is so tiny that there is only negligible correction to the Schwarzschild metric due to the gauge "electric" field and thus only negligible correction to the assumed thermality of the Hawking radiation. Then - as in the standard folk theorems argument - it is argued in [AHMNV07] that when a particle which is charged under this gauge symmetry is inserted into a black hole, the charge cannot be detected by an outside observer anymore and can also not reappear when the black hole shrinks down to the Planck

[17] We will discuss such possible Aharonov-Bohm-type experiments in detail in the next part of this thesis.

size (at least not in a sufficiently large amount because Hawking radiation is considered to be thermal).

As in the standard folk theorems argument, the existence of black holes with baryon/skyrmion hair implies a loophole in that argument in the case the weakly-gauged symmetry is weakly-gauged baryon number symmetry. In fact, as discussed in the previous section, a logical possible scenario for arbitrarily-weak gauge coupling is that (weakly-gauged) baryon number reappears in form of a classical baryon/skyrmion hair when the black hole shrinks down to a size L.

5. Detecting Classical Skyrmion Black Hole Hair via Classical Scattering of Waves

In the second part of the paper [DG16] we showed, by studying massless minimally-coupled probe scalar waves scattered on the one hand by a Schwarzschild black hole and on the other hand by a black hole with classical skyrmion hair which has the same asymptotic characteristics as the Schwarzschild black hole, that the differential scattering cross sections of the scalar field differ for these two cases. Therefore, given the asymptotic characteristics (the ADM mass) of a black hole, one can find out if this black hole is Schwarzschild or if it carries a classical skyrmion hair by studying scattering of classical waves. In this analysis we used the same techniques which we reviewed in section 2.4 in the introduction (part I) of this thesis. We focused on the effects which are due to gravitational interactions of the probe scalar field with the black hole. We therefore neglected non-gravitational interactions between the scalar and the skyrmion.

We considered two particular examples of black holes with classical skyrmion hair and studied scattering cross sections for these two examples for the case of several massless minimally-coupled monochromatic probe scalar waves with different frequencies. As a first example we used a black hole with classical skyrmion hair which is described by the parameters $\alpha = 0.01$, $\beta = 0$ and $x_h = 0.1263$ (note that this $x_h = 0.1263$ is the maximal value for the given α and β; $x_h = 0.1263 \equiv x_h^{max,\alpha=0.01,\beta=0}$). As a second example we studied a black hole with classical skyrmion hair which is given by the parameters $\alpha = 0.01$, $\beta = 0.5$ and $x_h = 0.116$. We determined differential scattering cross sections of probe monochromatic massless minimally-coupled scalar waves both of frequency $w = 8$ and $w = 25$ scattered by these two hairy black holes and by the Schwarzschild black holes which have the same asymptotic characteristics as the hairy

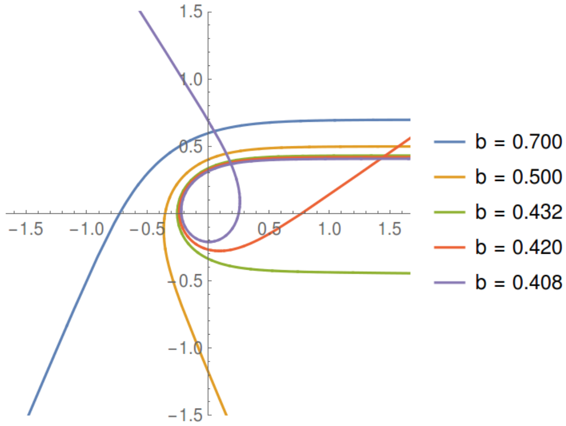

Figure 5.7: Orbits for massless particles on null geodesics in the background of the skyrmion black hole of our first example for different impact parameters b

black holes.[18] We both did a complete partial wave analysis and a glory approximation in order to detemine these cross sections. For this purpose we also studied geodesic motion of massless particles in the background of our two skyrmion black holes. We visualize the results in the following plots.

One can see from the plots that the characteristic peaks in the differential scattering cross sections of the waves scattered by the hairy black holes are located at smaller scattering angles when compared to the analogous peaks in the cross sections of the waves scattered by the Schwarzschild black holes with same asymptotic characteristics as the hairy black holes. This effect comes from the different near-horizon geometries of the hairy black holes and the non-hairy black holes with same asymptotic characteristics.

[18] Here we use the dimensionless frequency $w \equiv W\,(ef_\pi)^{-1}$, where W is the dimensionful frequency.

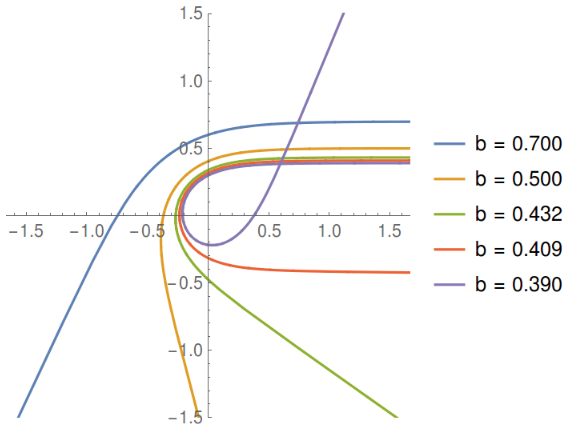

Figure 5.8: Orbits for massless particles on null geodesics in the background of the skyrmion black hole of our second example for different impact parameters b

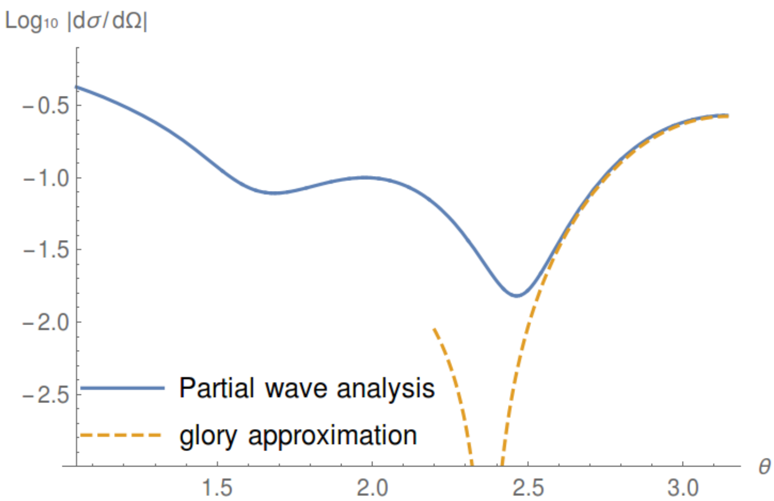

Figure 5.9: Differential scattering cross section of a massless scalar wave with frequency $w = 8$ scattered by the skyrmion black hole of our first example

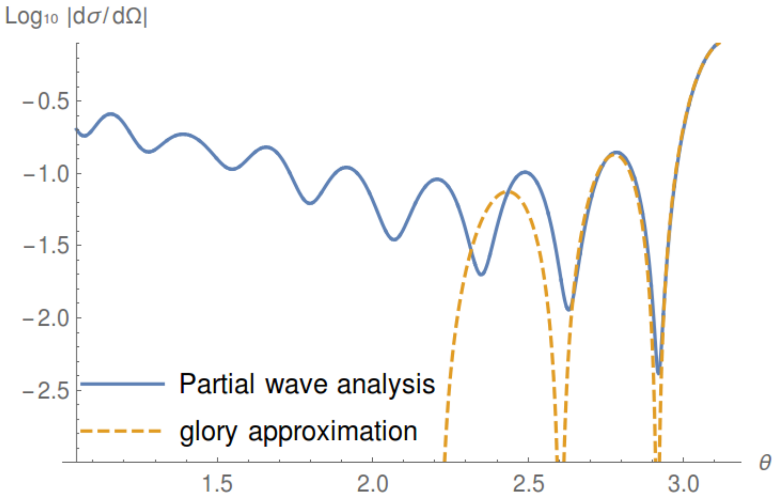

Figure 5.10: Differential scattering cross section of a massless scalar wave with frequency $w = 25$ scattered by the skyrmion black hole of our first example

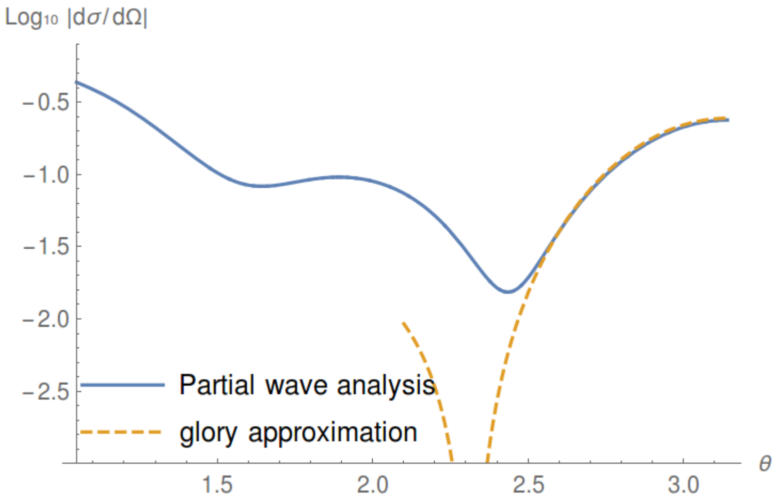

Figure 5.11: Differential scattering cross section of a massless scalar wave with frequency $w = 8$ scattered by the skyrmion black hole of our second example

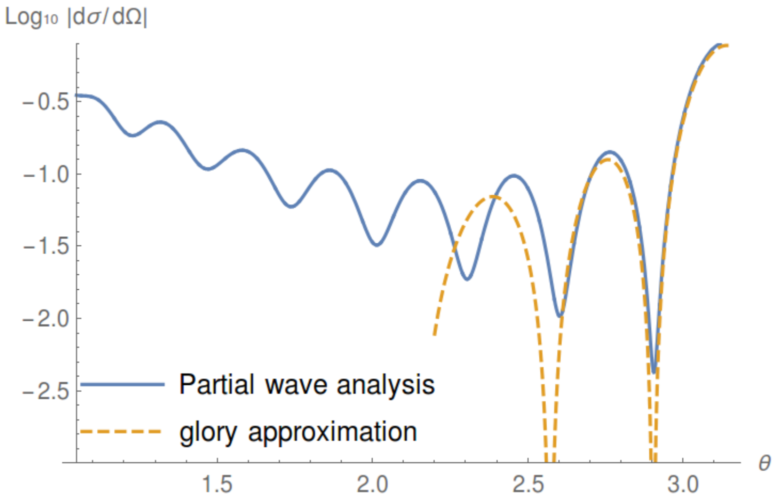

Figure 5.12: Differential scattering cross section of a massless scalar wave with frequency $w = 25$ scattered by the skyrmion black hole of our second example

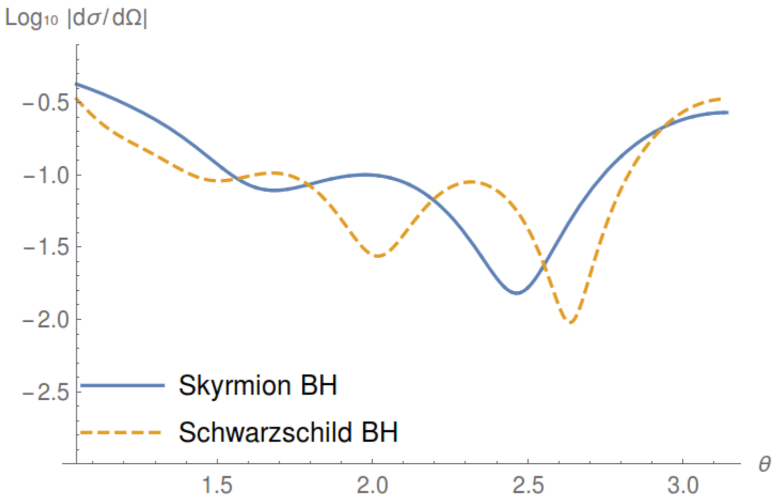

Figure 5.13: Differential scattering cross section of a massless scalar wave with frequency $w = 8$ scattered by the skyrmion black hole of our first example and differential scattering cross section of the same scalar wave scattered by a Schwarzschild black hole with same ADM mass

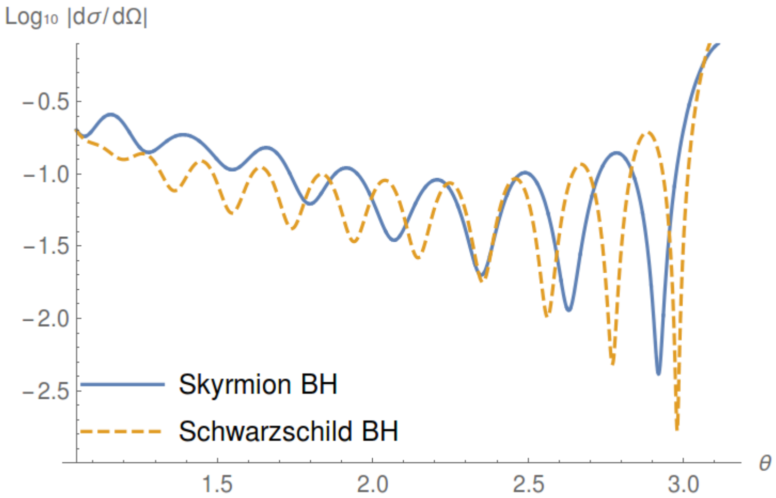

Figure 5.14: Differential scattering cross section of a massless scalar wave with frequency $w = 25$ scattered by the skyrmion black hole of our first example and differential scattering cross section of the same scalar wave scattered by a Schwarzschild black hole with same ADM mass

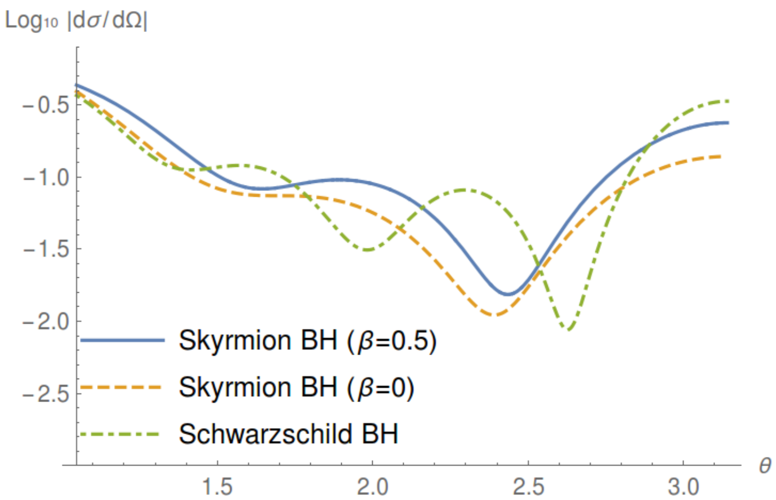

Figure 5.15: Differential scattering cross section of a massless scalar wave with frequency $w = 8$ scattered by the skyrmion black hole of our second example and differential scattering cross section of the same scalar wave scattered by a Schwarzschild black hole with same ADM mass and differential scattering cross section of the same scalar wave scattered by a skyrmion black hole with same ADM mass but with $\beta = 0$

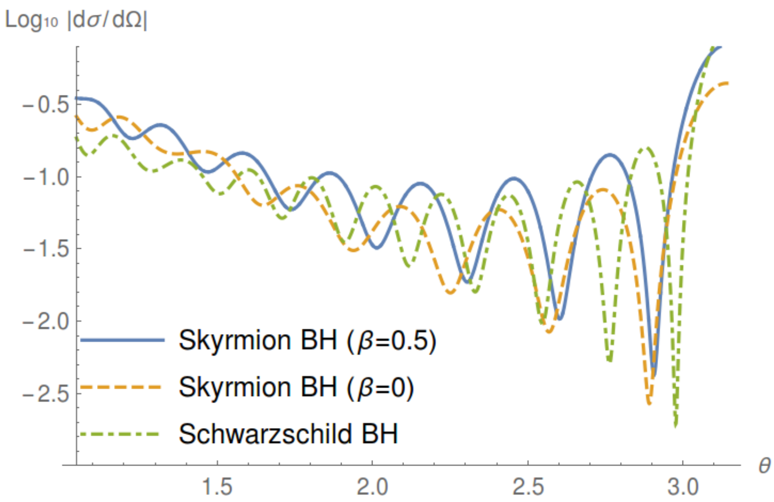

Figure 5.16: Differential scattering cross section of a massless scalar wave with frequency $w = 25$ scattered by the skyrmion black hole of our second example and differential scattering cross section of the same scalar wave scattered by a Schwarzschild black hole with same ADM mass and differential scattering cross section of the same scalar wave scattered by a skyrmion black hole with same ADM mass but with $\beta = 0$

Part III

Summary of Project 2 and Related Aspects:

"Aharonov-Bohm Protection of Black Hole's Baryon/Skyrmion Hair"
authors: Gia Dvali and Alexander Gußmann
published in: Phys. Lett. B768 (2017) 274-279

In the paper “Aharonov-Bohm Protection of Black Hole’s Baryon/Skyrmion Hair” [DG17] we studied several aspects of the Aharonov-Bohm-type skyrmion/ baryon black hole hair which we introduced in [DG16]. We discussed in detail how and under what assumptions the Aharonov-Bohm-type skyrmion/ baryon hair of a black hole can be detected with the help of a probe string. We considered the scenario in which the role of the probe string is played by a cosmic string and pointed out how the Skyrme topological charge/baryon number detected in this way can be understood as a result of a Witten-type effect.

We summarize these discussions in what follows and discuss several unpublished related aspects. For more details we refer to [DG17].

1. Detecting Skyrmion Black Hole Hair of Aharonov-Bohm-Type

Although the black hole solutions (3.249) of the Einstein-Skyrme equations do not have any locally-observable classical asymptotic field, the Skyrme topological charge/baryon number of such configurations is nevertheless measurable via a stringy generalisation of the Aharonov-Bohm effect provided there is a probe string which couples to the two-form $S_{\mu\nu}$ via

$$g \int d^2\sigma \partial_a X^\mu \partial_b X^\nu \epsilon^{ab} \delta^{(4)}(x - X) S_{\mu\nu} \,, \tag{1.251}$$

with a coupling constant g which is not an integer-multiple of 2π, worldsheet coordinates σ_a and embedding coordinates $X^\mu(\sigma)$: If such a coupling exists each process in which the probe string encloses the skyrmion induces a change in the action of

$$\Delta\mathcal{S} = gB \,, \tag{1.252}$$

which is due to the Aharonov-Bohm effect measurable whenever g is not an integer-multiple of 2π. Two of such processes are visualized in Figure 1.17 and Figure 1.18.

On the level of an effective theory consideration the coupling (1.251) has to be included in the Lagrangian, it is however from the effective point of view not clear what value g takes in nature. It can for example be the case that in the high-energy theory there is a superselection rule which forbids any value of g which is not an integer-multiple of 2π. In that case the Skyrme topological charge/baryon number is not measurable via such Aharonov-Bohm-type effects. Therefore, on the level of our effective desription, in this sense it is an assumption that a coupling of the above-mentioned form with g not an integer-multiple of 2π exists. If such a coupling (1.251) with g not an integer-multiple of 2π however does exist, an asymptotic observer who has appropriate probe strings at his disposal

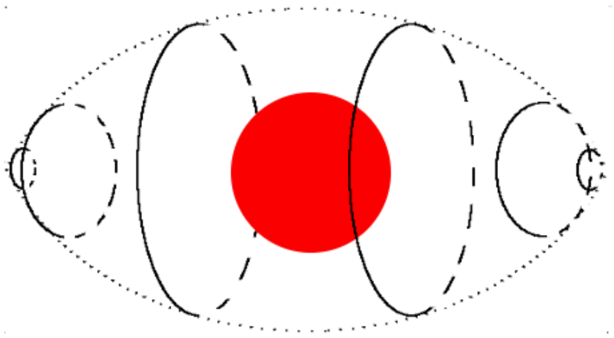

Figure 1.17: A process in which a probe string encloses the (red) skyrmion.

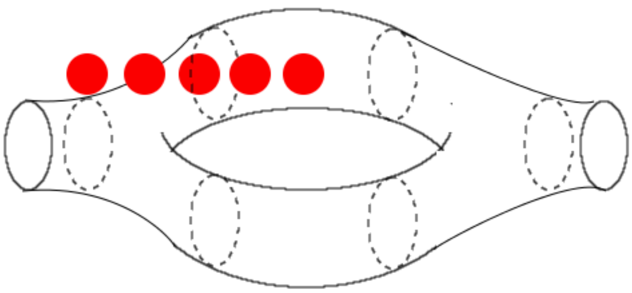

Figure 1.18: Another process in which a probe string encloses the (red) skyrmion.

can measure each baryon number/Skyrme topological charge B of a black hole which is such that the product gB is not an integer-multiple of 2π via the stringy Aharonov-Bohm effect.

It is not important what kind of probe string couples to the two-form. This can in principle for example be a fundamental string or also a composite object such as a cosmic string. In the next chapter we explicitly consider the case of a cosmic string.

2. Manufacturing a Probe String - Cosmic String

We showed in [DG17] that the role of the probe string which couples to $S_{\mu\nu}$ can be played by a cosmic string of Nielsen-Olesen type [NO73]. The Lagrangian which allows for cosmic strings as non-trivial lowest-energy configurations is given by (see equation (1.85))

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu H|^2 - \lambda^2\left(|H|^2 - v^2\right)^2 , \tag{2.253}$$

where D_μ is the covariant derivative, $D_\mu \equiv \partial_\mu - iqA_\mu$, with q the gauge coupling, $F_{\mu\nu} \equiv \partial_{[\mu}A_{\nu]}$ is the field strength and H a complex scalar field.

This Lagrangian admits cosmic "Nielsen-Olesen" strings [NO73] as solutions when the scalar field condenses, $\langle |H| \rangle = v$. This Nielsen-Olesen cosmic string acts as a source for the skyrmion two-form $S_{\mu\nu}$ when a coupling of the following form is introduced

$$cS_{\mu\nu}F_{\alpha\beta}\epsilon^{\mu\nu\alpha\beta} . \tag{2.254}$$

Here c is a parameter. This coupling is a legitimate coupling on the level of an effective field theory discussion. In the case in which the thickness of the string is much smaller than the a string loop, this coupling reduces to

$$c\int d^4x S_{\mu\nu}F_{\alpha\beta}\epsilon^{\mu\nu\alpha\beta} \to \frac{c}{q}\int dX^\mu \wedge dX^\nu S_{\mu\nu} . \tag{2.255}$$

This is nothing but (1.251) which shows that a cosmic string can play the role of the probe string considered before.

3. Skyrme Topological Charge of a Black Hole from Witten-Type Effect

One way to understand how the skyrmion aquires a charge under the Higgsed $U(1)$ symmetry of the cosmic string Lagrangian is in terms of a Witten kind effect [Wit79b]. In order to see this, one has to realize that the asymptotic configuration of the skyrmion two-form takes the form of a magnetic monopole: Asymptotically, $S_{\mu\nu}$ is (for $B = 1$) given as

$$S_{\mu\nu} = \frac{1}{4\pi} \partial_{[\mu} \cos(\theta) \partial_{\nu]} \phi \,. \tag{3.256}$$

This form is nothing but the field strength of a Dirac monopole,

$$S_{\mu\nu} = \partial_{[\mu} \zeta_{\nu]} \,, \tag{3.257}$$

with the one-form ζ_ν given by

$$\zeta_\nu = \frac{1}{4\pi} \left(\cos(\theta) - 1\right) \partial_\nu \phi \tag{3.258}$$

on the upper hemisphere and

$$\zeta_\nu = \frac{1}{4\pi} \left(\cos(\theta) + 1\right) \partial_\nu \phi \tag{3.259}$$

on the lower hemisphere.

Therefore, the coupling $c S_{\mu\nu} F_{\alpha\beta} \epsilon^{\nu\mu\alpha\beta}$ is nothing than the dual coupling between the field strengths of A_μ and ζ_μ. The Witten effect states that the ζ_ν-monopole (and thus the skyrmion via the asymptotic two-form (3.256)) aquires via this coupling an electric charge $\frac{c}{q}$ under the $U(1)$ gauge symmetry of the cosmic string Lagrangian.

4. Related Aspects: Boundary-Form for Textures from Point Defects

In this chapter, we shall outline a procedure which allows in various spacetime dimensions to construct a boundary differential form for a texture in that spacetime dimensions from the topological current of a point defect in one dimension higher. (3.244) will turn out to be the special case in which the texture is $(3+1)$ dimensional. Such a procedure can exist since both point defects in $(d+1)$ spatial dimensions and textures in d spatial dimensions are characterized by the same homotopy group π_d. We shall illustrate our points in the case of the homotopy groups π_2 and π_3. The result obtained in the case of π_3 matches with (3.244).

4.1 Case Homotopy Group π_2

The topological current of a ‘t Hooft Polyakov magnetic monopole in 3 spatial dimensions is (up to constants of normalisation) given by the three-form

$$\epsilon^{abc}\partial_\nu\hat{\phi}^a\partial_\alpha\hat{\phi}^b\partial_\beta\hat{\phi}^c dx^\nu \wedge dx^\alpha \wedge dx^\beta \,, \tag{4.260}$$

where $\hat{\phi}^a$ are normalized fields,

$$\hat{\phi}^a \equiv \frac{\phi^a}{|\phi|} \,. \tag{4.261}$$

These normalized fields can be parameterized in spherical coordinates as

$$\hat{\phi}^a = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) \,. \tag{4.262}$$

Up to constants the three-form (4.260) can be written as the exterior derivative of the two-form

$$\epsilon^{abc}\left(\hat{\phi}^a\partial_\alpha\hat{\phi}^b\partial_\beta\hat{\phi}^c\right) dx^\alpha \wedge dx^\beta \,. \tag{4.263}$$

This two-form can be considered on a boundary two-sphere at spatial infinity. In order to cover the whole sphere, (at least) two coordinate patches are needed. Using (4.262), this two-form on the boundary two-sphere can be written as

$$\mathrm{sin}\theta \mathrm{d}\theta \wedge \mathrm{d}\phi \,. \tag{4.264}$$

This two-form can be written as exterior derivative of a one-form which takes the form

$$(\mathrm{cos}\theta - 1)\, d\phi \tag{4.265}$$

on the patch covering the north pole and

$$(\mathrm{cos}\theta + 1)\, d\phi \tag{4.266}$$

on the patch covering the south pole. In case the two-sphere is decompactified (by stereographic projection), the decompactified sphere can be covered with a single coordinate patch and the one-form can be written as

$$(\mathrm{cos}\theta + 1)\, d\phi \tag{4.267}$$

This is nothing but a texture on a plane, a baby skyrmion, when making the identification

$$\phi = \phi \,, \tag{4.268}$$

$$\theta = f(r) \,, \tag{4.269}$$

where $f(r)$ is the profile function of the baby skyrmion. Using this identification, the two-form (4.264) can be written as

$$\mathrm{sinf(r)f'(r)dr} \wedge \mathrm{d}\phi \tag{4.270}$$

and the one-form (4.267) becomes

$$(\mathrm{cosf(r)} + 1)\, d\phi \,. \tag{4.271}$$

4.2 Case Homotopy Group π_3

In one dimension higher, the topological current is given (up to constants of normalisation) by the four-form

$$\epsilon^{abcd}\partial_\nu \hat{\phi}^a \partial_\alpha \hat{\phi}^b \partial_\beta \hat{\phi}^c \partial_\gamma \hat{\phi}^d dx^\nu \wedge dx^\alpha \wedge dx^\beta \wedge dx^\gamma \tag{4.272}$$

and $\hat{\phi}^a$ can be parametrized in spherical coordinates as

$$\hat{\phi}^a = (\mathrm{cos}\phi \mathrm{sin}\theta_1 \mathrm{sin}\theta_2, \mathrm{sin}\phi \mathrm{sin}\theta_1 \mathrm{sin}\theta_2, \mathrm{sin}\theta_1 \mathrm{cos}\theta_2, \mathrm{cos}\theta_1) \,. \tag{4.273}$$

This four-form can be written as exterior derivative of the three-form

$$\epsilon^{abcd}\left(\hat{\phi}^a\partial_\alpha\hat{\phi}^b\partial_\beta\hat{\phi}^c\partial_\gamma\hat{\phi}^d\right)dx^\alpha\wedge dx^\beta\wedge dx^\gamma\,, \tag{4.274}$$

which on the boundary-sphere can be written as

$$\mathrm{sin}\theta_2\mathrm{sin}^2\theta_1\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\phi\,. \tag{4.275}$$

After decompatifying this sphere (by stereographic projection), this can on a single patch be written as the exterior derivative of the two-form

$$\mathrm{sin}\theta_2\left(\theta_1-\frac{1}{2}\mathrm{sin}2\theta_1-\pi\right)\mathrm{d}\phi\wedge\mathrm{d}\theta_2\,. \tag{4.276}$$

This form matches (up to normalisation which we did not take into account here) with (3.244) after making the identification

$$\phi=\phi\,, \tag{4.277}$$

$$\theta_2=\theta\,, \tag{4.278}$$

$$\theta_1=F(r)\,, \tag{4.279}$$

where $F(r)$ is the profile function of the skyrmion.

Part IV

Summary of Project 3:

"Scattering of Massless Scalar Waves by Magnetically-Charged Black Holes in Einstein-Yang-Mills-Higgs Theory"
author: Alexander Gußmann
published in: Class. Quant. Grav. 34 (2017) no.6, 065007

In the paper “Scattering of Massless Scalar Waves by Magnetically-Charged Black Holes in Einstein-Yang-Mills-Higgs Theory” [Guß17], after carefully reviewing magnetically-charged classical black hole solutions in Einstein-Yang-Mills-Higgs theory with gauge group $SU(2)$ and a Higgs triplett (both with and without classical hair), we studied scattering cross sections of massless minimally-coupled probe scalar waves scattered by certain black holes which can be obtained as solutions of this Einstein-Yang-Mills-Higgs theory. For some “working examples” we compared the scattering cross sections of the same scalar waves scattered by magnetically-charged black holes without hair on the one hand and magnetically-charged black holes with classical hair on the other hand (both having the same asymptotic characteristics). We then discussed, since we studied only certain examples, how general the result that one can distinguish these magnetically-charged black with same asymptotic characteristics via classical scattering of waves, is. In view of our results, we pointed out that classical scattering of waves can provide a useful tool to find out if a given (astrophysical) black hole carries some classical hair or not.

We summarize these discussions in what follows and refer to [Guß17] for more details.

1. Review of Classical Yang-Mills-Higgs Black Hole Hair

About the possibility of classical hairy black holes in Einstein-Yang-Mills-Higgs theory it was first speculated in [Ort92]. That such hairy black hole solutions in Einstein-Yang-Mills-Higgs theory do really exist was then shown and discussed later, for example in [LNW92]. We will now review these black hole solutions in the spherically-symmetric and asymptotically-flat case in Einstein-Yang-Mills-Higgs theory with a Higgs triplett and gauge group $SU(2)$. The corresponding matter Lagrangian is the Georgi-Glashow Lagrangian (1.42) given by

$$\mathcal{L}_{YM} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - \frac{\lambda}{2}\left(\phi^a\phi^a - v^2\right)^2 , \qquad (1.280)$$

where

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\epsilon^{abc}A^b_\mu A^c_\nu , \qquad (1.281)$$

$$D_\mu\phi^a \equiv \partial_\mu\phi^a - e\epsilon^{abc}A^b_\mu\phi^c , \qquad (1.282)$$

with Greek indices referring to spacetime indices and Latin indices refering to $SU(2)$ indices. Here e is the Yang-Mills coupling constant, λ the Higgs coupling constant and v is the vacuum expectation value of the Higgs field ϕ^a. In flat spacetime this Lagrangian allows for topological solitons ('t Hooft Polyakov magnetic monopoles [tH74b, Pol74]) as lowest energy configurations.

This Lagrangian can be coupled to gravity and the Einstein field equations

$$G_{\mu\nu} = 8\pi G_N T^{YM}_{\mu\nu} , \qquad (1.283)$$

with $T^{YM}_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{YM})}{\delta g^{\mu\nu}}$ can be solved numerically. Usually this is done by making the ansatzes

$$\phi^a = vh(r)e^a_r, A_0 = 0, A^a_i = \epsilon_{iak}\left(\frac{1-u(r)}{er}\right)e^k_r \qquad (1.284)$$

for the matter fields and

$$ds^2 = N^2(r)t(r)dt^2 - t(r)^{-1}dr^2 - r^2 d\Omega^2 \tag{1.285}$$

for the metric (here $h(r)$, $u(r)$, $N(r)$ and $t(r)$ are ansatz-functions) and by solving the Einstein field equations numerically with appropriate boundary conditions for these ansatz functions. Besides solutions which do not have an event horizon ("gravitating magnetic monopoles" [BR75, CF75, VNWP76]), there are two different classes of black hole solutions (indeed solutions with event horizon). First, the following functions solve the field equations:

$$u(r) = 0, h(r) = 1, M(r) = M - \frac{4\pi}{2e^2 r}, N(r) = 1\,, \tag{1.286}$$

where $M(r)$ is defined via

$$t(r) \equiv \left(1 - \frac{2M(r)G_N}{r}\right)\,. \tag{1.287}$$

These solutions are nothing but the Reissner-Nordstroem metric which for

$$M^2 \geq \frac{2\pi}{G_N e^2} \tag{1.288}$$

describes black holes.

The other class of black hole solutions are black hole solutions which have the same asymptotic characteristics as the Reissner-Nordstroem solutions but different near-horizon geometries (called "black holes with classical Yang-Mills-Higgs hair"). They can be obtained numerically when using the boundary conditions

$$u(\infty) = 0, h(\infty) = 1, N(\infty) = 1\,. \tag{1.289}$$

In complete analogy to the classical skyrmion black holes, these solutions have been found only in a certain domain of parameters. Namley, there exists a maximal value α_{max} and solutions have for given β only been found for

$$0 \leq \alpha \leq \alpha_{max}(\beta)\,. \tag{1.290}$$

In complete analogy to the case of classical skyrmion black holes, α and β set the ratios of the relevant length scales of the system: α and β are defined as

$$\alpha^2 \equiv 2\pi \frac{L_g}{L} = 4\pi v^2 G_N\,, \tag{1.291}$$

$$\beta \equiv \frac{L}{L_C} = \frac{\sqrt{\lambda}}{e}\,, \tag{1.292}$$

where L_g is the gravitational radius of the ‘t Hooft Polyakov magnetic monopole, L its typical size and L_C the Compton wavelength of the Higgs field (see equation (1.103) for a definition of these scales in terms of parameters of the theory). On top of that, for given α and β these black holes have only been found as solutions of the Einstein-Yang-Mills-Higgs equations for event horizon sizes r_h which are smaller than a maximal a maximal value

$$0 \leq r_h \leq r_h^{max,\alpha,\beta}\,. \tag{1.293}$$

In complete analogy to the classical skyrmion black holes, this means from a physical point of view that hairy black hole solutions of the Einstein-Yang-Mills-Higgs equations are known only when first the magnetic monopole not itself becomes a black hole and second, when the event horizon is located inside of a critical length scale which is set by the Compton wavelength of the gauge field.

There is one limit in which these hairy black holes in Einstein-Yang-Mills-Higgs theory have been studied in much detail, namley when the Higgs field is taken to be infinitely heavy, $M_H \equiv \sqrt{\lambda} v\hbar \to \infty$, by taking the Higgs coupling constant to infinity, but keeping all the other parameters constant (see for example [AB93]). In this limit the Higgs field is frozen in its vacuum expectation value und $h(r) \equiv 1$. In this limit the independent components of the Einstein field equations can be written as

$$\partial_x N(x) = 2\alpha^2 \frac{N(x)}{x} \left(\partial_x u(x)\right)^2\,, \tag{1.294}$$

$$\partial_x m(x) = \alpha^2 \left(\left(\partial_x u(x)\right)^2 t(x) + \frac{\left(1-u(x)^2\right)^2}{2x^2} + u(x)^2 \right)\,, \tag{1.295}$$

$$\partial_x \left(\partial_x u(x) N(x) t(x)\right) = \frac{N(x)u(x)}{x^2} \left(x^2 - \left(1-u(x)^2\right)\right)\,, \tag{1.296}$$

where the dimensionless quantity $x \equiv evr$ and the dimensionless mass function $m(x) \equiv evG_N M(r)$ were used. Hairy black hole solutions in this limit are known to be stable on the linearized level [AB93]. In our work we restricted to this limit.

2. Classical Scattering Cross Sections of Massless Minimally-Coupled Scalar Waves

We studied scattering cross sections of massless minimally-coupled probe scalar waves scattered by hairy and non-hairy classical black holes obtained as solutions of the Einstein-Yang-Mills-Higgs equations in the abovementioned limit of an infinitely heavy Higgs field. We considered the "working example" $\alpha = 0.01$, $\beta = \infty$, $x_h = 0.02$, $m_{ADM} = 0.01018$ and did a detailed partial wave analysis (as well as a glory approximation) for several monochromatic scalar waves to obtain the differential scattering cross sections of these scalar waves scattered by the black hole of our working example and the Reissner-Nordstroem black hole with same asymptotic characteristics. We plot the mass functions of these black holes in Figure 2.19.

We focused on pure gravitational interactions and therefore neglected non-gravitational interactions between the scalar field and the magnetic monopole. We then argued that the qualitative features of the scattering cross sections obtained in our working example when compared to the analogous differential scattering cross sections of the same scalar field scattered by a non-hairy (Reissner-Nordstroem) black hole with same asypmtotic characteristics apply also for all other hairy black holes in the limit of an infinitely-heavy Higgs field.

From the plots of our results one can see that - as in the case of black holes with classical skyrmion hair when compared to Schwarzschild black holes with same asymptotic characteristics - the characteristic glory peaks in the differential scattering cross sections of the probe scalar wave scattered by a hairy black hole are located at different scattering angles as the analogous peaks in the cross sections of the same wave scattered by Reissner-Nordstroem black holes which have the same asymptotic characteristics as the hairy black holes. In contrast to the skyrmion case the

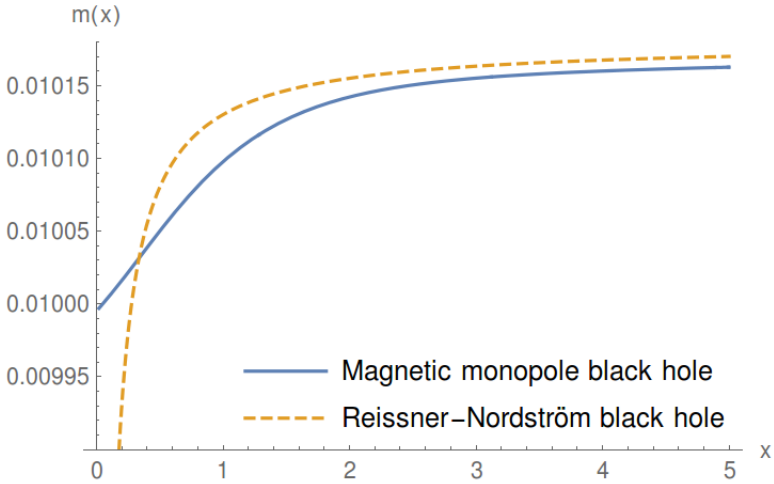

Figure 2.19: solution mass function $m(x)$ in the regime $x > x_h$ for the metric of our working example and the metric of the Reissner-Nordstroem black hole with same asymptotic characteristics

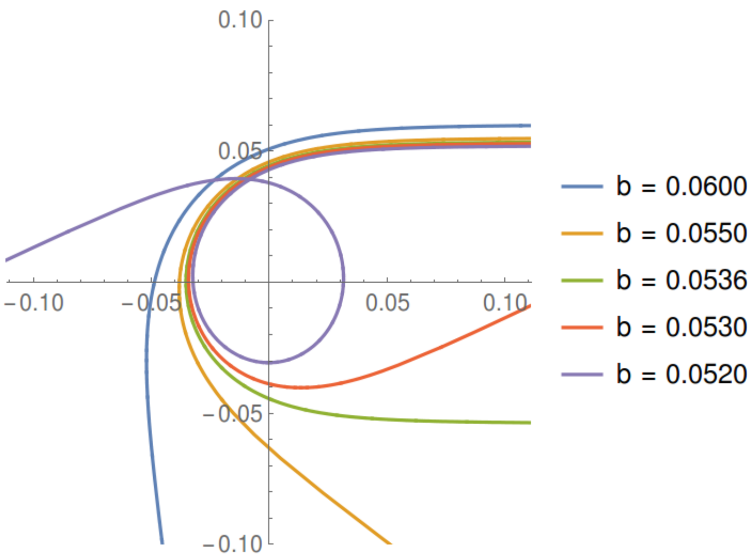

Figure 2.20: Orbits for massless particles moving on null geodesics in the background of the hairy black hole of Einstein-Yang-Mills-Higgs theory of our working example

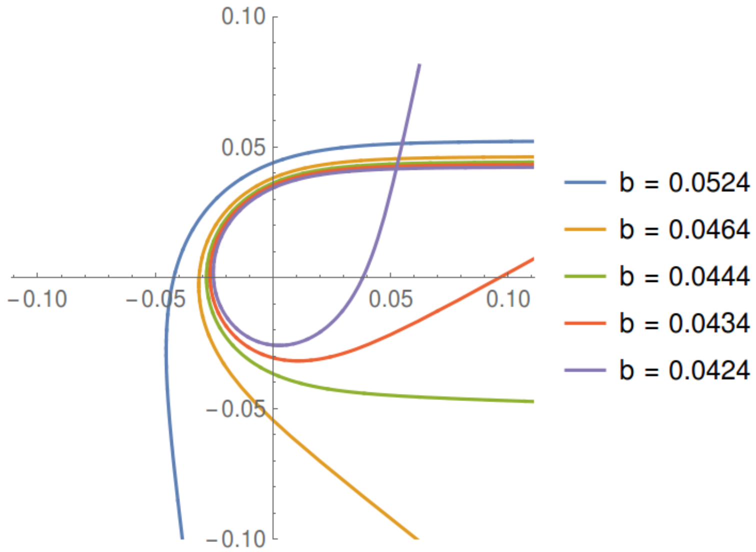

Figure 2.21: Orbits for massless particles moving on null geodesics in the background of the Reissner-Nordstroem black hole which has the same asymptotic characteristics than the hairy black hole of Einstein-Yang-Mills-Higgs theory of our working example

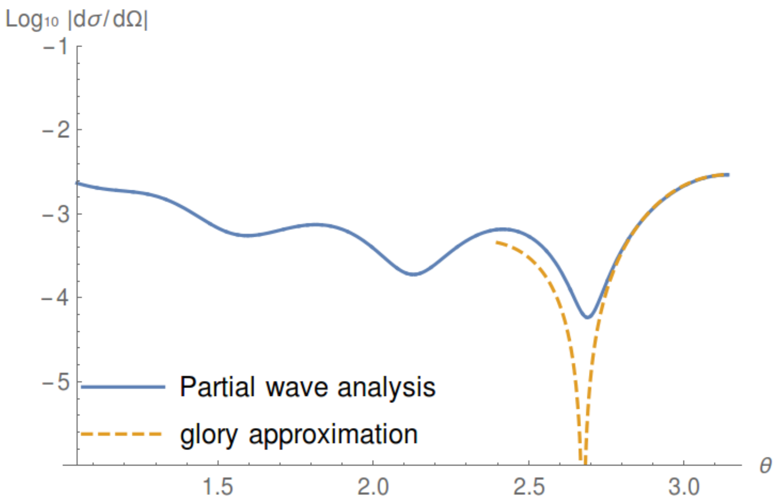

Figure 2.22: Differential scattering cross section of a massless minimally-coupled probe scalar wave with frequency $w = 100$ scattered by the hairy black hole of our working example

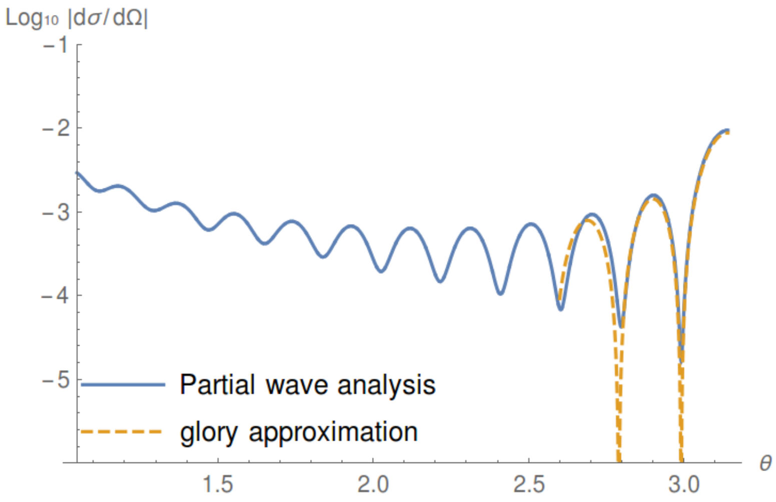

Figure 2.23: Differential scattering cross section of a massless minimally-coupled probe scalar wave with frequency $w = 300$ scattered by the hairy black hole of our working example

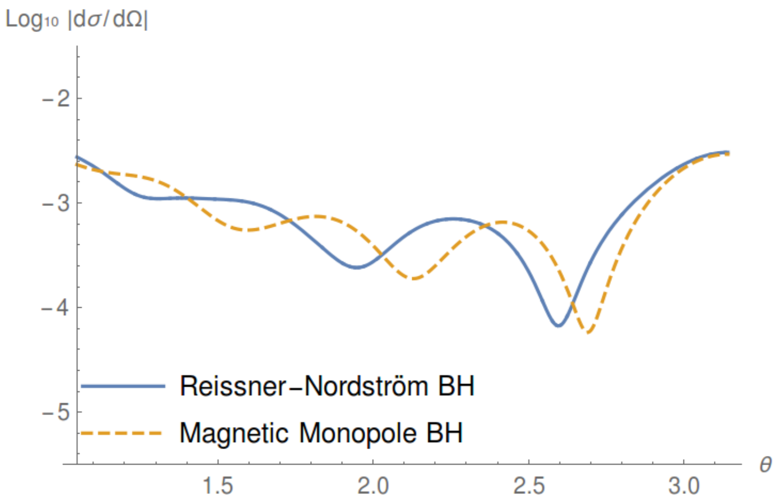

Figure 2.24: Differential scattering cross section of a massless minimally-coupled probe scalar wave with frequency $w = 100$ scattered by the hairy black hole of our working example and differential scattering cross section of the same scalar wave scattered by a Reissner-Nordstroem black hole which has the same asymptotic characteristics than the hairy black hole of our working example

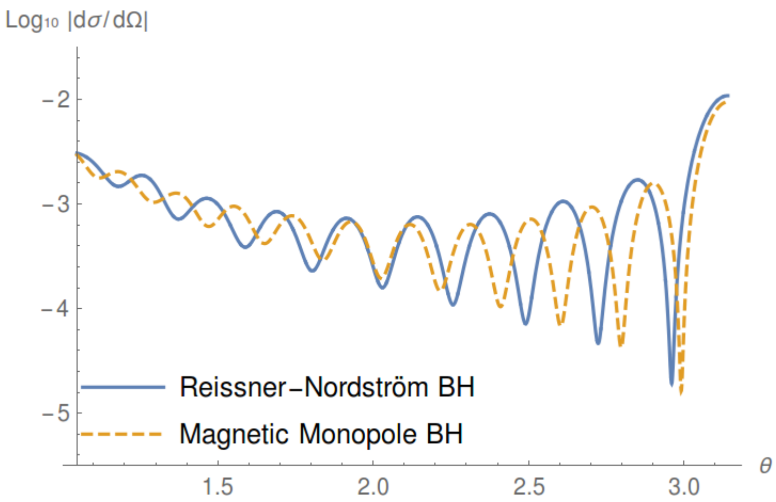

Figure 2.25: Differential scattering cross section of a massless minimally-coupled probe scalar wave with frequency $w = 300$ scattered by the hairy black hole of our working example and differential scattering cross section of the same scalar wave scattered by a Reissner-Nordstroem black hole which has the same asymptotic characteristics than the hairy black hole of our working example

"shift" of the peaks in the differential scattering cross sections of the hairy black holes (when compared to the analogous peaks in the cross sections of the black holes without hair) is in the different direction as in the skyrmion black hole case. That is because the event horizon size of hairy black holes of the Einstein-Yang-Mills-Higgs equations is larger than the event horizon size of a Reissner-Nordstroem black hole with same asymptotic characteristics whereas in the skyrmion black hole case it is the other way around.

Since, to our knowledge, all black holes with Einstein-Yang-Mills-Higgs hair in the considered limit share this property of the event horizon, we expect that the numerical results obtained for our two working examples qualitatively also apply to all these other black holes with Einstein-Yang-Mills-Higgs hair.

3. Outlook: Astrophysical Implications

The studies of how the differential scattering cross sections of waves scattered by hairy black holes differ from the differential scattering cross sections of the same waves scattered by black holes without classical hair which have the same asymptotic characteristics as the hairy black holes can have many interesting applications for example in astrophysics.

For example, for astrophysical black holes these studies can provide a method to check if a given black hole in nature with known asymptotic characteristics carries a classical hair or not:[19] Given the asymptotic characteristics of an astrophyiscal black hole in nature (determined by appropriate experiments), one can see from the cross sections of waves scattered by such a black hole if the black hole carries a classical hair or not. Instead of scalar waves, for such an experiment of course different kinds of waves (for example electromagnetic waves or gravitational waves) should be considered. (A similar analysis as was done here for probe scalar waves can easily be done also for such waves of higher spin.)

[19] See e.g. [HR14, HR15a] for a recently discussed astrophysical black hole candidate with classical hair.

Part V

Summary of Project 4:

“Aspects of Skyrmion Black Hole Hair”
author: Alexander Gußmann
published in: PoS CORFU2016 (2017) 089

In the conference paper "Aspects of Skyrmion Black Hole Hair" [Gus17], prepared for the Proceedings of the Corfu Meeting 2016, we reviewed several aspects of skyrmion/baryon black hole hair which were studied in [DG16, DG17] and which were presented at the Corfu Meeting by the author. Since we have already reviewed these aspects in the previous parts of this thesis, we refer to [Gus17] for the details of this conference talk.

Part VI

Summary of Project 5:

“Bulk-Boundary Correspondence between Charged, Anyonic Strings and Vortices”
authors: Alexander Gußmann, Debajyoti Sarkar, Nico Wintergerst
work in progress

In the paper [GSW18] “Bulk-Boundary Correspondence between Charged, Anyonic Strings and Vortices”, the question under what conditions cosmic strings in $(3+1)$ dimensional theories in spacetimes with boundary can obey fractional statistics was investigated from the point of view of the boundary theory. As one particular example, statistics of cosmic strings in $(3+1)$ dimensional global AdS spacetime was studied from the point of view of the $(2+1)$ dimensional boundary point of view. The appearance of fractional statistics for cosmic strings in spacetimes with boundary can have interesting consequences for the black holes with discrete Aharonov-Bohm-type quantum hair as well as for holography and in particular the AdS/CFT conjecture.

This paper is still work in progress and not all aspects have been worked out in detail so far. In what follows we shall summarize some of the preliminary results.

1. Statistics of Particles and Extended Objects

If the positions of two identical point particles in $(3+1)$ (or in higher) spacetime dimensions are interchanged, the corresponding wave function aquires a multiplicative factor of $(+1)$ or (-1). In the case of $(+1)$ the point particles are known as bosons, in the case of (-1) they are known as fermions. Fermions are half-integer spin particles which distribute themselves according to the Fermi-Dirac statistics and bosons are integer spin particles which distribute themselves according to the Bose-Einstein statistics [Pau40]. It is well known [Wil82] that in $(2+1)$ spacetime dimensions point particles can carry any spin and can obey fractional statistics. These point particles with fractional spin and statistics are known as "anyons" [Wil82]. If the positions of two identical anyons are exchanged, the wave function aquires a factor of $e^{i\alpha}$ where all real values of α can be realized. In (2+1) dimensional quantum field theories such anyons can be realized in certain cases if a Chern-Simons term is present in the field theory Lagrangian [WZ83, ASWZ85]. One particular well known example where such anyons can be realized within quantum field theory is the case of electrically charged Chern-Simons vortices: According to the theorem of Julia and Zee [JZ75] finite energy $(2+1)$ dimensional vortices of Nielsen-Olesen type [NO73] cannot be electrically charged. If however a Chern-Simons term is added to the Nielsen-Olesen Lagrangian, electrically charged vortices do exist as $(2+1)$ dimensional topologically non-trivial static lowest energy configurations in this theory [PK86, dVS86] and these electrically charged Chern-Simons vortices can obey fractional statistics [FM89].

Although in higher spacetime dimensions point-like objects cannot obey fractional statistics because the braid group is trivial in such cases, it is an interesting question if higher dimensional objects can obey fractional statistics in spacetime dimensions higher than $(2+1)$. Since the (generalisations of the) braid group for n dimensional objects in $n+1$ space dimensions are nontrivial [MS89], one can in fact not a priori exclude the option that in certain quantum field theories n dimensional objects obeying fractional statistics in $n + 1$ space dimensions are realized, for example that string-like objects of certain quantum field theories can obey fractional statistics in $(3 + 1)$ spacetime dimensions. Discussions on this topic have already appeared in the literature (see e.g. [ABKS91] and [Har07] for a discussion in a string theory context).

Here, we shall investigate this question for the case of Abelian cosmic strings in $(3 + 1)$ dimensional quantum field theories in spacetimes with boundary from the point of view of the induced boundary theory. For this purpose we consider Abelian cosmic strings in the $(3+1)$ dimensional spacetime which end on the $(2 + 1)$ dimensional boundary of the spacetime and investigate the question if and under what conditions these cosmic strings in $(3 + 1)$ spacetime dimensions obey fractional statistics. We argue that one can answer this question by considering only the endpoint vortices of the cosmic string which live on the $(2 + 1)$ dimensional boundary. Our investigations, when applied to AdS spacetime, can have interesting applications in the context of the AdS_4/CFT_3 duality [Mal99, Wit98].

In order to illustrate our points in the $(3 + 1)$ dimensional case, we investigate two particular setups. First, we study the case of Abelian cosmic strings in the $(3 + 1)$ dimensional spacetime which are obtained as finite energy configurations in the theory with Nielsen-Olesen Lagrangian with a term $\Delta\mathcal{L} = \theta\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ with a θ which is such that it changes discontinuously somewhere (e.g. because of the presence of an axion domain wall) added and which have both endpoints ending on the $(2 + 1)$ dimensional boundary of the spacetime.[20] Second, we study the case of a cosmic string, ending with both endpoints on the $(2 + 1)$ dimensional boundary of the spacetime, which is charged through an additional gauge field which we add to the theory. Such a string is similar to the well-known superconducting Abelian cosmic string which has been introduced in [Wit85]. Throughout, we shall work in the limit in which backreaction of the strings on the spacetime is absent.

The next chapters are organized as follows. In chapter 2 we study certain electrically charged $(2 + 1)$ dimensional Abelian vortices in flat spacetime. In section 2.1 we point out, by considering such vortices from the dual point of view, that $(2 + 1)$ dimensional electrically charged Chern-Simons vortices obey fractional statistics as a consequence of being electri-

[20]Such a discontinuous jump of θ can in certain condensed matter systems also be realized by junctions of matter (see e.g. [NZvdB16]).

cally charged. This is a well-known result [FM89]. In section 2.2 we argue that also $(2+1)$ dimensional vortices with no pure Chern-Simons term can obey fractional statistics in the case a certain additional internal current which we will specify is present. In chapter 3 we study some particular $(3+1)$ dimensional Abelian cosmic strings in spacetimes with boundary. In section 3.1 we consider, in complete analogy to the $(2+1)$ dimensional case which we considered in section 2.1, (3+1) dimensional cosmic strings from the dual point of view and show that cosmic strings obey fractional statistics if they are electrically charged. In section 3.2 we consider two particular cosmic strings which are electrically charged due to the presence of an additional current which we will introduce. In chapter 4 we combine our discussions of chapter 2 and chapter 3 and point out that the boundary endpoints of the particular cosmic strings discussed in section 3.2 are nothing but the vortices which we considered in chapter 2. We conclude with an outlook in chapter 5 where we shall mention possible implications of our findings in different contexts, for example in the context of the AdS/CFT duality.

2. Fractional Statistics of Vortices in $(2+1)$ Dimensions

We consider two cases of electrically charged vortices in (2+1) flat spacetime dimensions and argue that these vortices obey fractional statistics as a consequence of being electrically charged. These are in part well known results (see e.g. [FM89]).

2.1 Case 1: Abelian Chern-Simons Vortices

Electrically charged static Abelian vortices in (2+1) dimensional Minkowski spacetime can be obtained numerically as static lowest energy topologically-nontrivial configurations in the theory with Nielsen-Olesen Lagrangian (1.85) with Chern-Simons term added [PK86]:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(D_\mu\phi\right)^+\left(D^\mu\phi\right) - \frac{\lambda}{4}\left(\phi^+\phi - v^2\right)^2 + \mu\epsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha\,. \tag{2.297}$$

Here $F_{\mu\nu} \equiv \partial_{[\mu}A_{\nu]}$ and $D_\mu\phi \equiv \partial_\mu\phi - ieA_\mu\phi$. ϕ is a complex scalar field which can be parametrized as

$$\phi(x) = \rho(x)e^{i\theta_{(v)}(x)}\,, \tag{2.298}$$

with the two real valued functions $\rho(x)$ and $\theta_{(v)}(x)$.

This Lagrangian is invariant under the $U(1)$ transformation $\phi \to e^{iw(x)}\phi$, $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu w(x)$ with $w(x)$ a gauge transformation function.

The electric current J_μ ("electric" referring to electric charge under the gauge field A_μ) is given by

$$J_\mu = \frac{ie}{2}\left(\phi\left(D_\mu\phi\right)^+ - \phi^+D_\mu\phi\right)\,. \tag{2.299}$$

Electrically charged vortices have been found in [PK86] numerically by minimizing the energy functional which corresponds to this Lagrangian after using appropriate ansatz-functions and boundary conditions for the scalar and gauge field. These electrically charged vortices carry electric charge $\mu\Phi_B$, where $\Phi_B \equiv \int d^2x\epsilon^{ij}\partial_i A_j$ is the magnetic flux [PK86]:

$$\int J_0 d^2x = 2\mu\Phi_B\,. \tag{2.300}$$

At spatial infinity for these vortices the solution-functions behave as $\rho(x) \to v$ and $\left(\partial_\mu\theta_{(v)} - eA_\mu\right) \to 0$ [PK86]. This implies that the vortices have finite energy.

Let us now for simplicity use the approximation that the vortex is point-like, indeed that the vortex solution-function $\rho(x) \equiv v$ everywhere except at $x = 0$. More rigorous treatments without using this approximation can be found for example in [KL94] and in references therein. With this approximation the Lagrangian (2.297) can then be written as

$$\mathcal{L} = \frac{v^2}{2}\left(\partial_\mu\theta_{(v)} - eA_\mu\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mu\epsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha\,. \tag{2.301}$$

The electric current (2.299) is then

$$J_\mu = ev^2\left(\partial_\mu\theta_{(v)} - eA_\mu\right)\,. \tag{2.302}$$

It is well known that, at low energies, one can dualize the Lagrangian (2.301) and that in the dual theory the vortices appear as point charges of a gauge field C_μ. As we will now demonstrate, going to the dual picture is very useful to visualize that (if the constant μ in (2.301) is appropriately chosen) electrically charged vortices obey fractional statistics [WZ89b], a result which is well known [FM89] and which could also be inferred directly from (2.301) [WZ89b].

The Lagrangian (2.301) can be dualized as follows. First we introduce an auxiliary field $J_\mu^{(aux)}$ as

$$J_\mu^{(aux)} = \epsilon_{\mu\nu\alpha}\partial^\nu B^\alpha\,, \tag{2.303}$$

where B_μ is a $U(1)$ gauge field which can later be identified with C_μ.

Using the auxiliary field $J_\mu^{(aux)}$, the Lagrangian (2.301) can be written as

$$\mathcal{L} = -\frac{1}{2v^2}J_\mu^{(aux)}J^{(aux)\mu} + J_\mu^{(aux)}\left(\partial^\mu\theta_{(v)} - eA^\mu\right) + \mu\epsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha\,, \tag{2.304}$$

which in terms of the field B_μ can be written as

$$\mathcal{L} = -\frac{1}{4v^2}\partial_{[\mu}B_{\nu]}\partial^{[\mu}B^{\nu]}+\epsilon_{\mu\nu\alpha}\partial^\nu B^\alpha\left(\partial^\mu\theta_{(v)}-eA^\mu\right)+\mu\epsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha\,. \quad (2.305)$$

Here we neglected the kinetic term for A_μ which is justified at low enough energies.[21] Integrating out A_μ we get

$$\mathcal{L} = -\frac{1}{4v^2}\partial_{[\mu}B_{\nu]}\partial^{[\mu}B^{\nu]}+\epsilon_{\mu\nu\alpha}\partial^\nu B^\alpha\partial^\mu\theta_{(v)}+\tilde{\mu}\epsilon^{\mu\nu\alpha}B_\mu\partial_\nu B_\alpha\,, \quad (2.308)$$

with $\tilde{\mu}\equiv-\frac{e^2}{4\mu}$. We see in particular that the Chern-Simons term manifests itself also in the dual theory at low energies, a well known result sometimes referred to as "Chern-Simons self-duality". The term $\epsilon_{\mu\nu\alpha}\partial^\mu\partial^\nu\theta_{(v)}$ (which for a vortex is non-zero since $\theta_{(v)}$ is 2π-periodic) can be interpreted as a vortex current $j_{(vortex)\alpha}$ (see e.g. [KL94]) and the Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4v^2}\partial_{[\mu}B_{\nu]}\partial^{[\mu}B^{\nu]}+B^\alpha j_{(vortex)\alpha}+\tilde{\mu}\epsilon^{\mu\nu\alpha}B_\mu\partial_\nu B_\alpha\,. \quad (2.309)$$

In fact, the vortex current appears as "electric" current for the gauge potential B_μ.

If we take two of such currents and consider a process in the theory (2.309) where one vortex is moving adiabatically around the other identical one in such a way that finally the initial positions of the vortices are exchanged, both the interaction $j^\mu_{(vortex)}B_\mu$ and the Chern-Simons term induce Aharonov-Bohm phase shifts [KL94, WZ89b, WZ89a]:

The equations of motion for the dual gauge field B_μ are

$$j^\mu_{(vortex)} = -2\tilde{\mu}\epsilon^{\mu\nu\alpha}\partial_\nu B_\alpha\,. \quad (2.310)$$

In order to see the statistical Aharonov-Bohm phase which is induced in the process in which the positions of two identical vortices are adiabatically exchanged it is convenient to introduce a "total current" j^{tot}_μ as [WZ89a]

$$j^{tot}_\mu \equiv j_{(vortex)\mu}+\tilde{\mu}\epsilon_{\mu\nu\alpha}\partial^\nu B^\alpha\,. \quad (2.311)$$

[21] The equations of motion for the auxiliary field give

$$J^{(aux)}_\mu = \frac{1}{e}J_\mu\,. \quad (2.306)$$

Using Stokes theorem, the electric charge can thus be written as

$$2\mu\Phi_B \equiv \int J_0 d^2x = e\int\partial_i B_j\epsilon^{ij}d^2x = e\oint B_\mu dx^\mu\,. \quad (2.307)$$

This total current is the coefficient of $\partial_\mu \Lambda(x)$ in the variation of (2.309) under $B_\mu \to B_\mu + \partial_\mu \Lambda$. The total charge, $Q \equiv \int d^2x j_0^{tot}$, is then, using the equations of motion (2.310), given by

$$Q = \frac{1}{2} \int d^2x j_{(vortex)0} \,. \tag{2.312}$$

If one vortex (1) with charge $Q = \frac{1}{2}$ is moving adiabatically once around another identical vortex (2) at rest which produces the potential $B_\mu^{(2)}$, an Aharonov-Bohm phase

$$e^{i\frac{1}{2}\oint dx^i B_i^{(2)}} = e^{i\frac{1}{2e}\int d^2x J_0^{(2)}} = e^{i\frac{\mu}{e}\Phi_B} \tag{2.313}$$

is induced. Here dx^i parametrizes the contour (worldline) of vortex (1). In (2.313), equation (2.307) was used.

If the positions of the vortices are exchanged, in fact if one vortex is only moved half-way around the contour and the vortices are then parallel-shifted, a change in the action $\Delta\mathcal{S}$ of

$$\Delta\mathcal{S} = \frac{\mu}{2e}\Phi_B \tag{2.314}$$

is induced which can be measured as Aharonov-Bohm phase provided μ is chosen such that $\Delta\mathcal{S}$ is not an integer multiple of 2π. Thus, in that case, the vortices obey fractional statistics.

We want to emphasize that using the total current (2.311) makes sure that both the phase shifts generated by the interaction $j_{(vortex)\mu}B^\mu$ and by the Chern-Simons term are taken into account [GW89]. In our setup the use of the total current instead of the vortex current “only" produces an additional factor of $\frac{1}{2}$, in other similar setups not using the total current can however lead to qualitatively completely wrong conclusions [GW89].

Alternatively, in order to obtain the correct statistical phase which a process in which one vortex is adiabatically moved around another identical one induces, one could have integrated out B_μ in (2.305) or in (2.309). This would have given rise to an effective non-local Lagrangian, known as “Hopf term",

$$\mathcal{L}_{Hopf} = \frac{1}{4\tilde{\mu}} j_{(vortex)\mu} \frac{\epsilon^{\mu\nu\lambda}\partial_\nu}{\partial^2} j_{(vortex)\lambda} \,, \tag{2.315}$$

which leads to the correct quantum statistics [WZ89a].

2.2 Case 2: Electrically Charged Abelian Vortices with Additional Current

Vortices in $(2+1)$ spacetime dimensions which obey fractional statistics can also be obtained in some cases where the Nielsen-Olesen Lagrangian is modified in a different way than by adding a pure Chern-Simons term.

One obvious example is the case of a mixed Chern-Simons term in the dual theory with an additional gauge field E_μ introduced and an internal current $\tilde{J}_\mu$ which is localized on the vortex and is coupled to E_μ. The corresponding dual low energy Lagrangian without kinetic terms is given by

$$\mathcal{L} = B^\mu j_{(vortex)\mu} + E_\mu \tilde{J}^\mu + \kappa\epsilon_{\mu\nu\alpha} E^\mu \partial^\nu B^\alpha \,. \tag{2.316}$$

In a process in which one vortex (1) is taken adiabatically around another identical one (2) in such a way that finally the initial positions of the vortices are interchanged, the interaction term $B^\mu j_{(vortex)\mu}$ induces a change in the action of

$$\Delta\mathcal{S} = \frac{1}{2}\oint B_\mu^{(2)} dx^\mu \,, \tag{2.317}$$

where dx^μ is the worldline of the vortex (1) which is adiabatically taken around the vortex (2) which is at rest and which sources $B_\mu^{(2)}$. The factor of $\frac{1}{2}$ is included in (2.317) because in order to exchange the positions of the vortices, vortex (1) has to be taken *half-way* around vortex (2). Using the equations of motion for $E_\mu^{(2)}$,

$$\tilde{J}_\mu = -\kappa\epsilon_{\mu\nu\alpha}\partial^\nu B^\alpha \,, \tag{2.318}$$

and Stokes theorem this change can be written as

$$\Delta\mathcal{S} \equiv \frac{1}{2}\oint B_\mu^{(2)} dx^\mu = -\frac{1}{2\kappa}\int \tilde{J}_0^{(2)} d^2x \,. \tag{2.319}$$

Therefore, if the charge $\int d^2x \tilde{J}_0^{(2)}$ of the vortex at rest is non-vanishing and if κ is chosen appropriately such that a non-trivial Aharonov-Bohm phase shift is induced by $\Delta\mathcal{S}$, these vortices obey fractional statistics.[22]

[22]Note, that in contrast to the case which we have discussed in section 2.1, here the contributions of the field terms were not taken into account. Taking these into account gives, in complete analogy to the previous case, an additional multiplicative factor of $\frac{1}{2}$ entering $\Delta\mathcal{S}$.

Alternatively, the correct statistical phase can be inferred from integrating out B_μ in (2.316) by using (2.318). This leads to an effective non-local Hopf term which gives rise to the correct statistics.

This setup can also be generalized by adding pure Chern-Simons terms for B_μ and/or for E_μ to (2.316). The statistical phase then changes accordingly.

3. Fractional Statistics of Cosmic Strings

In $(3+1)$ dimensional flat spacetime cosmic strings exist as static topologically non-trivial lowest energy configurations in the theory given by the Nielsen-Olesen Lagrangian [NO73]. Such Nielsen-Olesen cosmic strings have also been studied as solutions in different spacetimes, for example in global AdS spacetime (both with and without the backreaction of the cosmic string on the spacetime taken into account) [DGM02].

An analogous dualization argument as we gave for $(2+1)$ dimensional vortices in section 2.1 also works for cosmic strings in $(3+1)$ spacetime dimensions [Fra07]: Let us consider a cosmic string with electric charge

$$\int J_0 d^3x = ev^2 \int \left(\partial_0 \theta_{(v)} - eA_0\right) d^3x \,. \tag{3.320}$$

As before, an auxiliary field $J_\mu^{(aux)}$ can be introduced as

$$J_\mu^{(aux)} = \epsilon_{\mu\nu\alpha\beta}\partial^\nu B^{\alpha\beta} \tag{3.321}$$

for some two-form $B_{\mu\nu}$. Since $\epsilon_{\mu\nu\alpha\beta}\partial^\mu B^{\alpha\beta} \longrightarrow 0$ at spatial infinity, the two-form $B_{\mu\nu}$ can at spatial infinity locally be written as exterior derivative of a one-form B_μ:

$$B_{\mu\nu} \longrightarrow \partial_{[\mu} B_{\nu]} \,. \tag{3.322}$$

Following an analogous argument as the one before, one can easily see that the electric charge of the cosmic string can be written in terms of $B_{\mu\nu}$ as

$$\int J_0 d^3x = e \int \partial_i B_{jk} \epsilon^{ijk} d^3x = e \oint B_{ij} dx^i \wedge dx^j \,. \tag{3.323}$$

For Nielsen-Olesen cosmic strings this electric charge is zero [JZ75]. Let us now consider two kinds of cosmic strings (not of Nielsen-Olesen type) in spacetimes with boundary (for example AdS_4), which are electrically charged and show, in particular by dualizing the corresponding

Lagrangians, that in these cases the strings obey fractional statistics. As we will discuss in chapter 4, these electrically charged cosmic strings are, in contrast to the pure Nielsen-Olesen cosmic strings, cosmic strings which have boundary vortices/antivortices as endpoints which carry different electric charges.

In our discussion we shall work in the limit in which backreaction of the cosmic string on the spacetime is absent.

3.1 Abelian Cosmic String with θ terms

Let us first consider the theory with Nielsen-Olesen Lagrangian with the term $\delta\mathcal{L} = \theta\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ added to the Nielsen-Olesen Lagrangian,

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(D_\mu\phi\right)^+\left(D^\mu\phi\right) - \frac{\lambda}{4}\left(\phi^+\phi - v^2\right)^2\right) + \theta\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\,. \tag{3.324}$$

Here g is the metric of the spacetime and $\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita symbol (note, that throughout we shall use $\epsilon^{\mu\nu\alpha\beta}$ for the Levi-Civita symbol and not for the Levi-Civita tensor).

In what follows we will study the statistics of cosmic strings which are topologically non-trivial lowest energy configurations of this Lagrangian first in the case of a constant θ parameter and second in the case of θ which is constant everywhere except at one axion domain wall along which θ changes discontinuously.

3.1.1 Constant Parameter θ

If θ is a constant parameter, the term $\theta\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ is a boundary term,

$$\theta\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = \partial_\mu\left(4\theta\epsilon^{\mu\nu\alpha\beta}A_\nu\partial_\alpha A_\beta\right)\,, \tag{3.325}$$

and thus has no effect on the bulk equations of motion. Therefore, in the bulk, the standard Nielsen-Olesen cosmic strings are topologically non-trivial lowest energy configurations of this Lagrangian.

These pure Nielsen-Olesen cosmic strings do not obey fractional statistics as a consequence of not being electrically charged [JZ75]. We shall show in chapter 4 by considering the endpoint vortices of the cosmic string and the boundary term (3.325) that not obeying fractional statistics in this case is consistent with the statistics of the induced boundary theory. In other words, we shall show in chapter 4 that in the case of the Nielsen-Olesen cosmic string solution of (3.324) with constant θ (which does not obey fractional statistics) also the boundary endpoint vortices of that string do not obey fractional statistics.

Let us now consider the case with θ not a constant parameter, but a function.

3.1.2 Non-Constant Parameter θ

In the case of a non-constant θ the term $\delta\mathcal{L} = \theta\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ is not a boundary term, but differs from a boundary term by $-2\partial_\mu\theta\epsilon^{\mu\nu\alpha\beta}A_\nu F_{\alpha\beta}$:

$$\theta\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = \partial_\mu\left(2\theta\epsilon^{\mu\nu\alpha\beta}A_\nu F_{\alpha\beta}\right) - 2\partial_\mu\theta\epsilon^{\mu\nu\alpha\beta}A_\nu F_{\alpha\beta}\,. \qquad (3.326)$$

Let us consider the setup of a cosmic string, oriented in the z-direction, which ends on both sides on some $(2+1)$ dimensional boundary and a θ parameter which is constant everywhere except at one hypersurface along which θ changes discontinuously.[23]

In this setup the Nielsen-Olesen cosmic string gets electrically-charged due to an effect which can be seen in analogy to the Witten effect now applied to $U(1)$ [Wit79b, NZvdB16]. The electric charge density ρ which is induced by the term $\Delta\mathcal{L} = \theta\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ with a non-constant θ is given by

$$\rho = \mathrm{div}(\theta \mathrm{B_{mag}}) = \mathrm{B_{mag}}\partial_\mathrm{z}\theta\,, \qquad (3.327)$$

where B_{mag} is the magnetic field. The electric charge Q_E of the cosmic string is then

$$Q_E \equiv \int d^3x\rho = \int d^2xB_{mag}\int dz\partial_z\theta = \Phi_B\left(\theta_u - \theta_d\right)\,, \qquad (3.328)$$

with Φ_B the magnetic flux and θ_u and θ_d the values of θ at the upper and lower endpoints of the cosmic string.

Let us now argue, from the dual point of view which is analogous to the one used in section 2.1, that such electrically charged cosmic strings obey fractional statistics.

Dualizing the Lagrangian (3.324) at low energies (neglecting kinetic terms for A_μ) in the approximation analogous to the one used in section 2.1 for the $(2+1)$ dimensional vortices gives, using the auxiliary field $J_\mu^{(aux)} \equiv \frac{1}{\sqrt{-g}}\epsilon_{\mu\nu\alpha\beta}\partial^\nu B^{\alpha\beta}$, [Fra07]

$$\mathcal{L} = -\frac{\sqrt{-g}}{4v^2}H_{\mu\nu\alpha}H^{\mu\nu\alpha} + \epsilon^{\mu\nu\alpha\beta}\partial_\nu B_{\alpha\beta}\partial_\mu\theta_{(v)} - e\epsilon^{\mu\nu\alpha\beta}\partial_\nu B_{\alpha\beta}A_\mu + \theta\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\,, \qquad (3.329)$$

[23] Such a hypersurface can for example be realized by an axionic domain wall [DP14].

where $H_{\mu\nu\alpha} \equiv B_{\mu\nu,\alpha} + B_{\mu\alpha,\nu} + B_{\nu\alpha,\mu}$.

Integrating by parts and neglecting the kinetic term for $B_{\mu\nu}$, this Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_{bulk} + \mathcal{L}_{boundary} \,, \tag{3.330}$$

with

$$\mathcal{L}_{bulk} \equiv B_{\mu\nu} j^{\mu\nu}_{(vortex)} + e\epsilon^{\mu\nu\alpha\beta} B_{\alpha\beta}\partial_\nu A_\mu - 4\partial_\mu\theta\epsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta \,, \tag{3.331}$$

where $j^{\mu\nu}_{(vortex)} \equiv \epsilon^{\mu\nu\alpha\beta}\partial_\alpha\partial_\beta\theta_{(v)}$ can be interpreted as a vortex loop current [Fra07], and with the boundary part[24]

$$\mathcal{L}_{boundary} \equiv \partial_\nu \left(\epsilon^{\mu\nu\alpha\beta} B_{\alpha\beta}\partial_\mu\theta_{(v)} - e\epsilon^{\mu\nu\alpha\beta} B_{\alpha\beta} A_\mu - 4\theta\epsilon^{\mu\nu\alpha\beta} A_\mu \partial_\alpha A_\beta \right) . \tag{3.332}$$

We shall consider $\mathcal{L}_{boundary}$ in chapter 4 and now, in this chapter, focus on $\mathcal{L}_{bulk}$. Integrating out $B_{\mu\nu}$ in $\mathcal{L}_{bulk}$ gives the effective Lagrangian

$$\mathcal{L}_{bulk} = -\frac{4}{e^2} j^{\mu\nu}_{(vortex)} \partial_\mu\theta\partial_\nu\theta_{(v)} \,. \tag{3.333}$$

In a process in which one cosmic string (1) is adiabatically taken around another identical one (2) at rest in such a way that the positions of the strings are exchanged, this term induces a change in the action of

$$\Delta\mathcal{S} = -\frac{4}{2e^2} \oint \partial_\mu\theta\partial_\nu\theta^{(2)}_{(v)} dx^\mu \wedge dx^\nu \,, \tag{3.334}$$

where $dx^\mu \wedge dx^\nu$ parametrizes the trajectory of string (1) and $\theta^{(2)}_{(v)}$ is the vortex angle of the string (2) which is at rest. The factor of $\frac{1}{2}$ was included because in a process in which the positions of the two strings are exchanged, one string has to be taken *half-way* around the other one.

Using cylindrical coordinates and a taking into account that the string (1) is oriented in the z-direction at rest, this change in the action can according to Stokes theorem be written as

$$\Delta\mathcal{S} = \frac{4}{2e^2} \int d^3x \partial_z \theta j^{(1)0z}_{(vortex)} \,. \tag{3.335}$$

Using the equations of motion for B_μ from (3.331),

$$j^{(vortex)\mu\nu(1)} = e\epsilon^{\mu\nu\alpha\beta}\partial_\alpha A^{(1)}_\beta \,, \tag{3.336}$$

[24]Note that, as pointed out in (3.322), on the boundary locally $B_{\mu\nu} \equiv \partial_{[\mu} B_{\nu]}$ for some vector field B_μ. Therefore in (3.332) $B_{\mu\nu}$ can locally be replaced by $\partial_{[\mu} B_{\nu]}$. We shall make use of this in chapter 4.

gives

$$\Delta\mathcal{S} = \frac{4}{2e}\int d^3x \partial_z \theta \epsilon^{0z\alpha\beta}\partial_\alpha A^{(1)}_\beta = \frac{4}{2e}\Phi_B\left(\theta_u - \theta_d\right) . \tag{3.337}$$

Whenever $(\theta_u - \theta_d)$ is chosen such that $\Delta\mathcal{S}$ is not an integer multiple of 2π, this change in the action induces a measurable Aharonov-Bohm phase shift in a process in which the positions of two identical strings are exchanged. These strings, in that case, therefore obey fractional statistics.

3.2 Charging $U(1)$ Cosmic Strings Using Additional Current

In complete analogy to the case of $(2+1)$ dimensional vortices which we have considered in section 2.2, cosmic strings in $(3+1)$ spacetime dimensions can also be obtained in the case an additional gauge field E_μ and an internal current $\tilde{J}_\mu$ (localized on the string and coupled to E_μ) are present. In this case, in a setup with constant θ and without any embedded axionic domain wall, the cosmic string can obey fractional statistics. This setup is well-known and has been studied in the past for flat spacetime [ABKS91].

The corresponding dual low energy bulk Lagrangian without kinetic terms is given by[25]

$$\mathcal{L} = B_{\mu\nu} j^{\mu\nu}_{(vortex)} + \tilde{J}_\mu E^\mu + \theta \epsilon^{\mu\nu\alpha\beta}\partial_\mu E_\nu B_{\alpha\beta} . \tag{3.338}$$

This Lagrangian can be seen in analogy to the Lagrangian in $(2+1)$ dimensions which we have considered in section 2.2. As shown in [ABKS91], in complete analogy to the case of the $(2+1)$ dimensional vortices which we have considered in section 2.2, such cosmic strings of (3.338) obey fractional statistics if the constant θ and the current $\tilde{J}_\mu$ in (3.338) are such that the charge $\int \tilde{J}_0$ is not an integer-multiple of $2\pi\theta$. To see this, we note that the equations of motion for E_μ are given by

$$\tilde{J}^\beta = -\theta \epsilon^{\beta\mu\nu\alpha}\partial_\mu B_{\nu\alpha} . \tag{3.339}$$

Therefore, in a process in which one cosmic string is adiabatically taken around another identical one at rest such that the initial positions of the

[25] Just as in section 2.2, the present setup can also be generalized. For example, one can add a term $\epsilon^{\mu\nu\alpha\beta}B_{\mu\nu}B_{\alpha\beta}$ and/or a term $\epsilon^{\mu\nu\alpha\beta}\partial_\mu E_\nu \partial_\alpha E_\beta$ to (3.338). The statistical phase then changes accordingly.

strings get exchanged, the interaction term $B_{\mu\nu} j^{\mu\nu}_{(vortex)}$ induces a change in the action of the form

$$\Delta\mathcal{S} = \frac{1}{2} \oint B_{\mu\nu} dx^\mu \wedge dx^\nu = -\frac{1}{2\theta} \int d^3x \tilde{J}_0 \,, \tag{3.340}$$

where Stokes theorem was used. $B_{\mu\nu}$ is the field corresponding to the string at rest. $dx^\mu \wedge dx^\nu$ parametrizes the worldsheet of the moving string. As in the case of (2.317), the factor of $\frac{1}{2}$ is included because in order to exchange the positions of the strings, one string has to be taken *half-way* around the other one at rest.[26] Note that, once we identify both θ above, with the Chern-Simons coefficient κ which we have introduced in (2.316), then the bulk phase shift (3.340) matches exactly with the boundary phase shift obtained in section 2.2. We will elaborate on this relation between the bulk and boundary phase shifts in chapter 4.

[26]Note that once again, here we did not take into account that there can be an additional change in the action induced by the last two field terms in (3.338). Taking into account these terms when determining $\Delta\mathcal{S}$ gives rise to an additional multiplicative factor of $\frac{1}{2}$ in (3.340) as shown in [ABKS91]. Thus, the correct statistical phase can be obtained from (3.340) when taking this additional factor of $\frac{1}{2}$ into account. Alternatively, the correct statistical phase can also be obtained by integrating out $B_{\mu\nu}$ in (3.338) and by following an analysis analogous to the one performed in section 3.1.

4. Electrically Charged Vortices as Endpoints of Cosmic Strings

In the previous chapters we have independently studied the statistics of certain $(2+1)$ dimensional vortices in flat spacetime and of certain $(3+1)$ dimensional cosmic strings in the corresponding bulk. At the end of chapters 2 and 3, we have already demonstrated how to obtain the boundary statistics of a vortex starting from the bulk statistics in one higher dimension. In this chapter we will argue that the statistics of the cosmic strings which we found in chapter 3, is exactly the same as the combined statistics of the upper and lower endpoint boundary vortices of the corresponding strings (which we discussed in chapter 2). *Thus, the statistics of the cosmic strings in* $(3+1)$ *dimensional bulk spacetime can be fully understood by considering only the statistics of the boundary vortices of the string on the* $(2+1)$ *dimensional boundary.*

This section is organized as follows. First, we shall consider the case of the cosmic string which can be obtained as classical solution of the Nielsen-Olesen Lagrangian with $\Delta\mathcal{L} = \theta\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ added to the Lagrangian (discussed in section 3.1) and the corresponding Chern-Simons vortices (discussed in section 2.1). Second, we consider the correspondences between the cosmic strings of section 3.2 and the boundary vortices of section 2.2.

4.1 Bulk Cosmic Strings with θ-Term and Boundary Chern-Simons Vortices

Let us consider the cosmic strings which are obtained as finite energy configurations of the Nielsen-Olesen Lagrangian with an additional term $\Delta\mathcal{L}$. We will first consider the cosmic strings in the case of a *constant* parameter θ. As mentioned in section 3.1, only for the case of non-constant θ, do we get a non-trivial fractional statistics. Since the boundary term induced by $\delta\mathcal{L}$ is an Abelian Chern-Simons term, the endpoints of the cosmic string are nothing but the Abelian Chern-Simons vortices (or antivortices) which we have discussed in section 2.1. There we noted that generically, separate Abelian Chern-Simons vortices/antivortices can obey fractional statistics. However, as discussed in chapter 2, the upper and lower endpoints of the cosmic string taken together do not obey fractional statistics for a *constant* θ. In what follows, we will reconcile the two by noting that the Aharonov-Bohm phases of the upper and lower endpoint boundary vortices/antivortices always cancel in a process in which one cosmic string is taken around another identical one.

To see this clearly and for concreteness, let us consider the two dimensional spatial boundary sphere of conformally compactified AdS_4 ($\mathbf{S}^2$) and two cosmic strings in AdS_4 which end on this sphere. Let the upper endpoints of the strings end on the northern hemisphere of this 2-sphere and the lower endpoints of the strings end on the southern hemisphere. From the point of view of an observer who is located on this $\mathbf{S}^2$, the upper endpoints are vortices whereas the lower endpoints are antivortices. Since $\theta^{\mu}_{(vortex)} = -\theta^{\mu}_{(antivortex)}$, we obtain

$$j^{\mu}_{(vortex)} = -j^{\mu}_{(antivortex)} \,. \tag{4.341}$$

Here the j^{μ} is once again the vortex current, $j^{\mu}_{(vortex)} \equiv \epsilon^{\mu\nu\alpha}\partial_{\nu}\partial_{\alpha}\theta_{(vortex)}$, which was introduced in (2.309). Using Stokes theorem and the equations of motion $\epsilon^{\mu\nu\alpha}\partial_{\nu}B_{\alpha} \propto j^{\mu}_{vortex}$ (2.310), the change in the action induced by one upper vortex moving around the other identical one goes as (2.313)

$$\oint B_{\mu} dx^{\mu}_{upper} = \int d^2x j^0_{(vortex)} \,, \tag{4.342}$$

whereas the change in the action induced by one lower antivortex moving around the other identical one goes as (2.313)

$$\oint B_{\mu} dx^{\mu}_{lower} = \int d^2x j^0_{(antivortex)} \,. \tag{4.343}$$

Here dx^{μ}_{upper} is the worldline of a vortex current whereas dx^{μ}_{lower} is the worldline of an antivortex current, implying $dx^{\mu}_{lower} = -dx^{\mu}_{upper}$.[27] Since $j^0_{(vortex)} = -j^0_{(antivortex)}$, in total

$$\oint B_{\mu} dx^{\mu}_{upper} + \oint B_{\mu} dx^{\mu}_{lower} = \int d^2x \left(j^0_{(vortex)} + j^0_{(antivortex)}\right) = 0 \,. \tag{4.344}$$

[27]In (4.343) two minus signs cancel: one coming from the change in directions in the curve integration (when compared to the upper case) and other one due to the difference between an antivortex and a vortex: $dx^{\mu}_{upper} = -dx^{\mu}_{lower}$.

Thus, the combined Aharonov-Bohm phase shift of the upper and lower endpoint boundary vortices/antivortices cancel and in this sense the upper and lower boundary endpoint vortices/antivortices of the cosmic string taken together do not obey fractional statistics. Since also the Nielsen-Olesen bulk cosmic strings of the form discussed in section 3.1 do not obey fractional statistics for constant θ, the statistics of the boundary endpoint vortices/antivortices matches with the statistics of the bulk cosmic strings in this case.

Let us now consider the case of a string which ends on both sides on the global boundary of AdS_4 and is piercing an axionic domain wall embedded in AdS_4 along which θ changes discontinuously. In this case, in contrast to the case of a constant θ parameter, the induced Aharonov-Bohm phases of the upper and lower boundary endpoint vortices/antivortices of a cosmic string do not cancel in a process in which one cosmic string is moved around another identical one. This is because in this case the Chern-Simons term on the upper hemisphere of the AdS boundary is induced with a different prefactor than the Chern-Simons term on the lower hemisphere and thus $|j^0_{(vortex)}| \neq |j^0_{(antivortex)}|$. The induced boundary Lagrangian can be written at the boundary i $(i = 1, 2)$ as (3.332)

$$\mathcal{L}_i = -\epsilon^{\nu\alpha\beta}\partial_\nu\theta_{(v)}\partial_\alpha B_\beta + e\epsilon^{\nu\alpha\beta}A_\nu\partial_\alpha B_\beta + 4\theta_i\epsilon^{\nu\alpha\beta}A_\nu\partial_\alpha A_\beta \,, \tag{4.345}$$

where θ_i are the values of the θ parameter at the ith boundary. In our convention, boundary number 1 is the boundary of global AdS_4 on the upper hemisphere and boundary number 2 is the lower hemisphere. Both hemispheres are separated by the domain wall. When we identify $4\theta_i$ in (4.345) with $-\mu$ in (2.305), this Lagrangian (4.345) is nothing but minus the dual Lagrangian (2.305) which describes the Chern-Simons vortices. In the case of the constant θ parameter discussed above, $\theta_1 = \theta_2$ which induces boundary Lagrangians of the type (2.305) with both having the same constant in front of the Chern-Simons term. This leads to the above mentioned cancelation of the induced Aharonov-Bohm phases of the upper and lower boundary vortices/antivortices of the cosmic string since in this case $j^0_{(vortex)} = j^0_{(antivortex)}$. In the case of a non-constant θ parameter (e.g. which jumps discontinuously along the domain wall with $\theta_1 \neq \theta_2$), the induced Chern-Simons terms on the boundaries arise with different prefactors. Therefore, the charges of the upper endpoint vortices are different than the charges of the lower endpoint antivortices (for the same strings) and the Aharonov-Bohm phases do not cancel. In such a case, the upper and lower boundary endpoint (Chern-Simons) vortices/antivortices of the cosmic string taken together obey fractional statistics (as discussed in section 2.2) and the statistics in the bulk and on the boundary matches. In other words, we can say that the statistics of the bulk cosmic string can be obtained by considering only the statistics of the boundary endpoint vortices/antivortices of the string.

4.2 Bulk Cosmic String and Boundary Vortices with Additional Current

Let us now argue that the boundary vortices/antivortices of the (superconducting) cosmic string which we discussed in sections 3.2 and 3.3 are vortices/antivortices of the kind we have discussed in section 2.2. We will again study the Aharonov-Bohm phase of the upper and lower boundary vortices/antivortices from both bulk and boundary perspectives in a process where one (superconducting) string is adiabatically taken around another identical one in such a way that the initial positions of the strings get exchanged.

The low energy dual Lagrangian with constant parameter θ,

$$\mathcal{L} = \epsilon^{\mu\nu\alpha\beta}\partial_\nu B_{\alpha\beta}\partial_\mu\theta_{(v)} + \theta\epsilon^{\mu\nu\alpha\beta}E_\mu\partial_\nu B_{\alpha\beta}\,, \tag{4.346}$$

can be written as

$$\mathcal{L} = \mathcal{L}_{bulk} + \mathcal{L}_{boundary}\,, \tag{4.347}$$

with

$$\mathcal{L}_{bulk} \equiv j^{(vortex)\alpha\beta}B_{\alpha\beta} + \theta\epsilon^{\mu\nu\alpha\beta}\partial_\mu E_\nu B_{\alpha\beta}\,, \tag{4.348}$$

$$\mathcal{L}_{boundary} \equiv \partial_\mu\left(-\epsilon^{\mu\nu\alpha\beta}B_\beta\partial_\nu\partial_\alpha\theta_{(v)} + \theta\epsilon^{\mu\nu\alpha\beta}E_\nu B_{\alpha\beta}\right)\,. \tag{4.349}$$

If we couple the bulk current,

$$\tilde{J}_\mu \equiv \int d^2\sigma\delta^{(4)}(x - x(\sigma))\epsilon^{ab}\partial_a x^\mu\partial_b\gamma\,, \tag{4.350}$$

and its boundary current [BSS79],

$$\tilde{J}_\mu^{boundary} \equiv \int d\sigma^0\delta^{(4)}(x - x(\sigma^0))\partial_0 x^\mu(\sigma^0)\gamma(\sigma^0, \sigma^1 = \sigma^1|_{boundary})\,, \tag{4.351}$$

to (4.348) and (4.349) respectively, then the resulting bulk Lagrangian is the Lagrangian which we have considered in section 3.2 and the resulting boundary Lagrangian is equal to minus the Lagrangian which we have considered in section 2.2 (when we identify θ in (4.349) with the parameter $-\kappa$ used in section 2.2). Thus, in this sense, the vortices which we have considered in section 2.2 can be viewed as the boundary vortices of the cosmic string which we have considered in section 3.2.

If $\gamma(\sigma^0, \sigma^1 = \sigma^1|_{boundary1}) \neq \gamma(\sigma^0, \sigma^1 = \sigma^1|_{boundary2})$, the electric charges of the upper and lower boundary endpoint vortices/antivortices are different and induce different Aharonov-Bohm phase shifts which do not cancel. Thus, as follows from the discussion in section 3.2, the phase shift induced on the boundary is the same as the one in the bulk.

Therefore, the conclusion that the Aharonov-Bohm phase shifts (which are induced on the boundary in a process in which the boundary vortices/antivortices are taken adiabatically around each other, such that their initial positions get exchanged) is equal to the analogous phase shifts of the cosmic strings in the bulk, applies equally for both setups which we have discussed.

5. Summary and Outlook

We have first demonstrated separately in two different setups that electrically charged vortices in $(2+1)$ spacetime dimensions and electrically charged cosmic strings in $(3+1)$ dimensions obey fractional statistics. In both setups, we have explicitly calculated the induced Aharonov-Bohm phase shifts in processes in which two identical vortices or strings are rotated around each other. As we have mentioned throughout the text, some of these results are well-known: It is well known that, as we discussed in section 2.1, electrically charged Chern-Simons vortices in $(2+1)$ dimensions obey fractional statistics [FM89] and it is also well known that, as we discussed in section 3.2, cosmic strings in $(3+1)$ dimensions can obey fractional statistics if a certain additional current is localized on the string [ABKS91]. To our knowledge, the presentations which we gave in section 2.2 and in particular in section 3.1 however have not been appeared in the literature so far, although there are related works such as [Har07]. In chapter 4 we combined the discussions of the previous two chapters and presented a unified way of understanding the statistics of the cosmic strings in a $(3+1)$ dimensional spacetime with boundary and the statistics of corresponding boundary endpoint vortices/antivortices of the string which are located on the boundary of the spacetime. In both setups that we have considered, the cosmic strings obey fractional statistics if and only if their boundary endpoint vortices and antivortices carry different electric charges. This might be a very general criterium, not only applicable to the two ways of charging cosmic strings which we have considered explicitly. Thus, we are naturally led to the following general conjecture: *Cosmic strings in spacetimes with boundary obey fractional statistics if and only if their boundary endpoint vortices and antivortices carry different electric charges.* Since the statistical phase shifts are purely due to the topological terms both at the bulk and on the boundary, it is clear that our result goes through for any suitable manifold $\mathcal{M}$, which can support these topological solutions.

This result might have generalisations to higher dimensional extended objects in higher spacetime dimensions with boundary. In fact, one can wonder under what conditions membranes in concrete theories can obey fractional statistics. Given our results, one can expect that e.g. two-dimensional membranes in five dimensional spacetimes with boundary obey fractional statistics if and only if their boundary endpoint strings carry different electric charges.

Throughout we have worked in the probe limit in which the backreaction of the topological objects on the spacetime is absent. Although, for a given spacetime it is not easy to determine the backreaction effects completely, because this would require to solve the whole coupled Einstein-Higgs equations, in certain approximations backreaction effects have already been studied, for example in [DGM02] for the case of cosmic strings in AdS. It might be interesting to study such backreaction effects in the context of fractional statistics which we have considered.

Our results can have several interesting applications in different contexts. We want to conclude by commenting on some of them.

First, as we have already mentioned several times , our configurations can be naturally extended to global AdS spacetime, since AdS cosmic strings exist as solutions of (3.324) [DGM02]. For us, it means that the fractionally charged cosmic strings are embedded in AdS spacetime with anyonic boundary endpoint vortices/antivortices located on the boundary of AdS. In chapter 4, we have already focused on such setups.

In the literature, e.g. in [DGM02, DHIS14], in the context of the AdS/CFT correspondence, setups with (Nielsen-Olesen type) vortices located on the AdS boundary which are endpoints of cosmic strings in the AdS bulk have already been studied. In [DHIS14] it has been emphasized that in the context of AdS/CFT (which relates a gravitational bulk theory to a conformal field theory on the AdS boundary), these lower dimensional vortices/antivortices can be understood as conformal defects (of the low energy field theory on the boundary). These defects break the full conformal group $SO(3,2)$ of the boundary field theory down to $SO(2,1) \times SO(2)$. So in this case, the boundary field theory is only invariant under the subgroup $SO(2,1) \times SO(2)$. To our knowledge, the possible impact of fractionally charged anyonic vortices on such conformal defects has not yet been studied in the literature. In this setting, it will thus be interesting to investigate this question both from the perspectives of a boundary vortex and also for the bulk string-vortex.

Our results can also have interesting applications at finite temperature and, in the context of AdS/CFT, will closely relate to the studies of holographic superconductors [HHH08a, HHH08b] and to the studies of the fractional quantum hall effect [KVK08, NR16]. Because vortices located at the AdS boundary have already been studied in such contexts in [MPS09, DHIS14], one might hope to learn the effects of fractional statistics on such condensed matter applications.

Finally, our results may have implications in the physics of Aharonov-Bohm-type black hole hair. In fact, it is well-known that black holes can be charged under discrete Z_N symmetry [CPW92], and in those cases, cosmic strings do appear as solutions. It is therefore an interesting question as to whether our studies on the fractional statistics of cosmic strings might have some implications on the physics of black holes.

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Acknowledgements

First of all I would like to thank my supervisor Gia Dvali for taking me as a PhD student as well as for his guidance and support over the years. I am grateful for the opportunity to collaborate with him, for the discussions and for all I have learned from him.

It is a pleasure to thank Stefan Hofmann for agreeing to be the second referee for this thesis as well as for his encouragement and support.

I want to thank my collaborators and all the group members. In particular Alex, Alexis, Andre, Andrea, Andrea, Artem, Claudius, Daniel, Deb, Dimitris, Georgios, Jaba, Lasha, Lena, Lukas, Lukas, Marco, Max, Mischa, Nico, Sebastian, Tehseen.

I want to thank our secretaries Gabriele Bodenmüller and Herta Wiesbeck-Yonis for constant support.

Finally, I want to thank DFG Excellence Cluster Universe and Humboldt Foundation for financial support during my PhD time.