On the Maxwell-Lorentz Dynamics of Point Charges

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Abstract

In this work we study the Maxwell-Lorentz system for point charges without self-interaction. The latter is a system of coupled partial and ordinary differential equations that describes the electromagnetic interaction between classical point charges and their electromagnetic fields.

Our first main result states that the initial value problem aiming at continuous solutions is ill-posed for generic initial values.

This is due to the generation of discontinuous or even singular fronts in the electromagnetic fields. These are located along the boundary of the light-cones of the initial charge positions and occur for generic initial values. This phenomenon can be observed with the help of an explicit solution formula of the Maxwell equations which is provided as a second main result. It can be seen as a generalization of the famous formula by Liénard and Wiechert. We argue in mathematical rigorous terms that such fronts are caused by a potential mismatch between the initial positions and momenta and the initial fields.

We show that smooth solutions can only be attained by imposing a system of constraints on initial values in addition to the well-known Maxwell constraints. These extra conditions, however, require knowledge of the history of the solution and, as we discuss, effectively turn the Maxwell-Lorentz system into a system of delay equations.

A large class of solutions fulfilling these constraints is identified and our third main result is an existence and uniqueness result for the Maxwell-Lorentz system for point charges without self-interaction for this class and up to the first time of collision, which may be infinite.

Zusammenfassung

In dieser Arbeit studieren wir das Maxwell-Lorentz-Gleichungssystem für Punktladungen ohne Selbstwechselwirkung. Dabei handelt es sich um ein gekoppeltes System aus partiellen und gewöhnlichen Differentialgleichungen, welches die elektromagnetische Wechselwirkung zwischen klassischen Ladungen und ihren Feldern beschreibt.

Unser erstes Hauptresultat sagt aus, dass das Anfangswertproblem für beliebige Anfangswerte keine stetigen Lösungen besitzt.

Dies ist auf die Existenz von unstetigen sowie singulären Fronten in den elektromagnetischen Feldern zurück zu führen. Diese liegen auf dem Rand der Lichtkegel, welche von den Ursprüngen der Ladungen ausgehen und treten für generische Anfangswerte auf. Dieses Phänomen kann mit Hilfe einer expliziten Lösungsformel der Maxwell-Gleichungen beobachtet werden, welche als zweites Hauptresultat präsentiert wird und als Verallgemeinerung der berühmten Formel von Liénard und Wiechert betrachtet werden kann. Wir legen auf mathematisch rigorose Weise dar, dass diese Fronten durch eine mögliche Inkompatibilität zwischen den Anfangspositionen und -implusen und den Anfangsfeldern verursacht werden.

Wir zeigen, dass stetige Lösungen nur unter Hinzunahme von Bedingungen an die Anfangswerte erreicht werden können, welche über die bekannten Maxwell-Nebenbedingungen hinaus gehen. Diese Zusatzbedingungen erfordern jedoch Wissen über die Historie der Lösung und wandeln somit das Maxwell-Lorentz-Gleichungssystem in ein System aus retardierten Differentialgleichungen um.

Wir präsentieren eine große Klasse von Lösungen, welche die Zusatzbedingungen erfüllt und unser drittes Hauptresultat zeigt die Existenz und Eindeutigkeit von Maxwell-Lorentz-Lösungen für Punktladungen ohne Selbstwechselwirkung für obige Klasse und bis zur ersten Kollisionszeit, welche unendlich sein kann.

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Chapter 1

Introduction

1.1 The Maxwell-Lorentz theory

Classical electrodynamics describes the motion of charged particles in their electromagnetic fields. The theory goes back to 1864, when Maxwell published his famous paper on a dynamical theory of the electromagnetic field. This was, inter alia, preceded by Cavendish's experiments in electrostatics and Coulomb's research published in 1785 (cf. [26]). In 1892 and 1895 Lorentz published his famous works on Maxwell's theory applied to moving objects containing the so-called Lorentz-force law. Until today, classical electrodynamics is the most famous example of a theory about particles and fields. Therein, particles obey the Lorentz equations, and fields obey the Maxwell equations. Its empirical success notwithstanding, the underlying fundamental theory has not been fully understood to this day.

In this work we investigate the dynamics of N charged point particles and their electromagnetic fields, i.e., the existence of a unique solution to the coupled system of Maxwell's and Lorentz's equations. As both equations are of the form of evolution equations, a natural choice of initial value could comprise the positions and momenta of the charges as well as the electromagnetic fields at a predetermined initial time.

Let us introduce these equations together with our notation next. Denoting the index set of the charges by $\mathcal{N} := \{1, \ldots, N\}$ and a charge index by $i \in \mathcal{N}$, the position and momentum of the i-th charge at time $t \in \mathbb{R}$ is denoted by $\boldsymbol{q}_{i,t} \in \mathbb{R}^3$ and $\boldsymbol{p}_{i,t} \in \mathbb{R}^3$. The electromagnetic field due to charge i at time $t \in \mathbb{R}$ will be denoted by $\boldsymbol{f}_{i,t} = (\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})$. To keep it simple, we consider charges of the same mass $m > 0$ and use units such that the speed of light equals one and the vacuum permittivity equals $(4\pi)^{-1}$. As we are interested in the dynamics of point charges, the charge density is rigid and given by the three dimensional Dirac delta distribution centered around the charge position, i.e., at time t the density of charge i is given by $\delta(\cdot - \boldsymbol{q}_{i,t})$ and its current by $\boldsymbol{v}(\boldsymbol{p}_{i,t})\delta(\cdot - \boldsymbol{q}_{i,t})$. Contrary to the text-book presentation, see, e.g. [26, 37, 5], in which one employs only one total electric and magnetic field, it will be convenient for our purpose to associate with each charge i an individual electric and magnetic field $\boldsymbol{f}_{i,t} = (\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})$. Thanks to the linearity of the Maxwell equations this is only a notational convention since summing these individual fields results in a total field which again fulfills the Maxwell equation for the total charge and current, which are defined by summing the individual densities and currents, respectively. Denoting the relativistic velocity by

$$\boldsymbol{v}(\boldsymbol{p}_{i,t}) := \frac{\boldsymbol{p}_{i,t}}{\sqrt{m^2 + \boldsymbol{p}_{i,t}^2}}, \tag{1.1}$$

we introduce the laws of motion of the theory: The dynamics of the electromagnetic fields is governed by the *Maxwell equations*

$$\partial_t \begin{pmatrix} \boldsymbol{E}_{i,t} \\ \boldsymbol{B}_{i,t} \end{pmatrix} = \begin{pmatrix} \nabla \wedge \boldsymbol{B}_{i,t} \\ -\nabla \wedge \boldsymbol{E}_{i,t} \end{pmatrix} + \begin{pmatrix} -4\pi \boldsymbol{v}(\boldsymbol{p}_{i,t})\delta(\cdot - \boldsymbol{q}_{i,t}) \\ 0 \end{pmatrix}, \qquad i \in \mathcal{N}. \tag{1.2}$$

In addition to this law of motion, the electromagnetic fields are required to fulfill the *Maxwell constraints*

$$\nabla \cdot \boldsymbol{E}_{i,0} = 4\pi\, \delta(\cdot - \boldsymbol{q}_{i,0}) \qquad \text{and} \qquad \nabla \cdot \boldsymbol{B}_{i,0} = 0, \qquad i \in \mathcal{N}. \tag{1.3}$$

It turns out that by virtue of (1.2), the constraints (1.3) at $t = 0$ imply that they hold for all times $t \in \mathbb{R}$; see Lemma A.2.1 (Maxwell constraints). Furthermore, the dynamics of the charged particles is governed by the *Lorentz equations*

$$\frac{d}{dt} \begin{pmatrix} \boldsymbol{q}_{i,t} \\ \boldsymbol{p}_{i,t} \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}(\boldsymbol{p}_{i,t}) \\ \sum_j e_{ij} \boldsymbol{L}_{ij,t} \end{pmatrix}, \qquad i \in \mathcal{N}, \tag{1.4}$$

where the *Lorentz force* is given by

$$\boldsymbol{L}_{ij,t} := \boldsymbol{L}(\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{j,t}, \boldsymbol{B}_{j,t}) = \boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}) + \boldsymbol{v}_{i,t} \wedge \boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t}). \tag{1.5}$$

For the choice $e_{ij} = 1$ the Lorentz force (1.4) coincides with the textbook formula; each charge interacts with the total field. Other choices of e_{ij} allow to switch on or off the interaction of the j-th field on the i-th charge.

The Maxwell equations and constraints (1.2)-(1.3) are inhomogeneous, linear partial differential equations. Hence, any of their solution can be represented by a convex combination of special solutions plus a a solution of the homogeneous system which we introduce next. The *homogeneous (or free) Maxwell equations* are given by

$$\partial_t \begin{pmatrix} \boldsymbol{E}_{i,t} \\ \boldsymbol{B}_{i,t} \end{pmatrix} = \begin{pmatrix} \nabla \wedge \boldsymbol{B}_{i,t} \\ -\nabla \wedge \boldsymbol{E}_{i,t} \end{pmatrix}, \qquad i \in \mathcal{N}. \tag{1.6}$$

and the *homogeneous Maxwell constraints* take the form

$$\nabla \cdot \boldsymbol{E}_{i,0} = 0 \qquad \text{and} \qquad \nabla \cdot \boldsymbol{B}_{i,0} = 0, \qquad i \in \mathcal{N}. \tag{1.7}$$

Dropping the index i to keep the notation slim, two special solutions to (1.2)-(1.3) for a fixed charge trajectory $(\boldsymbol{q}, \boldsymbol{p}) : t \mapsto (\boldsymbol{q}_t, \boldsymbol{p}_t)$ are known since the publication [32] by Liénard in 1898 and [48] by Wiechert in 1900. These solutions are the *Liénard-Wiechert fields* $\boldsymbol{f}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}] = (\boldsymbol{e}_t^{\pm}, \boldsymbol{b}_t^{\pm})$, for $\pm$ being a placeholder for $+$ and $-$, in which case one calls the electromagnetic fields *advanced* or *retarded* Liénard-Wiechert fields, respectively. The square bracket notation emphasizes the functional dependence on the charge trajectory $(\boldsymbol{q}, \boldsymbol{p})$. These two special solutions are given by

$$\begin{aligned} \boldsymbol{e}_t^{\pm}(\boldsymbol{x}) &:= \left. \frac{(\boldsymbol{n} \pm \boldsymbol{v})(1 - \boldsymbol{v}^2)}{|\boldsymbol{x} - \boldsymbol{q}|^2 (1 \pm \boldsymbol{n} \cdot \boldsymbol{v})^3} + \frac{\boldsymbol{n} \wedge [(\boldsymbol{n} \pm \boldsymbol{v}) \wedge \boldsymbol{a}]}{|\boldsymbol{x} - \boldsymbol{q}|(1 \pm \boldsymbol{n} \cdot \boldsymbol{v})^3} \right|^{\pm}, \\ \boldsymbol{b}_t^{\pm}(\boldsymbol{x}) &:= \mp \boldsymbol{n}^{\pm} \wedge \boldsymbol{e}_t^{\pm}(\boldsymbol{x}), \end{aligned} \tag{1.8}$$

where we have used the abbreviations

$$\begin{aligned} &\boldsymbol{q}^\pm := \boldsymbol{q}_{t^\pm}, \quad &&\boldsymbol{v}^\pm := \boldsymbol{v}(\boldsymbol{p}_{t^\pm}), \quad &&\boldsymbol{a}^\pm := \tfrac{d}{dt}\boldsymbol{v}(\boldsymbol{p}_t)|_{t=t^\pm}, \\ &\boldsymbol{n}^\pm := \tfrac{\boldsymbol{x}-\boldsymbol{q}^\pm}{|\boldsymbol{x}-\boldsymbol{q}^\pm|}, \quad &&t^\pm := t \pm |\boldsymbol{x}-\boldsymbol{q}^\pm|; \end{aligned} \tag{1.9}$$

cf. [37, 41]. The following notion will be convenient to study the properties of the Maxwell solutions. The future and past light cone at space-time point $(t,\boldsymbol{x}) \in \mathbb{R}^4$, using $\pm$ as a placeholder for either $+$ or $-$, is defined by

$$J^\pm(t,\boldsymbol{x}) := \{(s,\boldsymbol{y}) \in \mathbb{R}^4 \mid |\boldsymbol{y}-\boldsymbol{x}|^2 \le (s-t)^2, \pm s \ge \pm t\}. \tag{1.10}$$

Their union is denoted by

$$J(t,\boldsymbol{x}) := J^+(t,\boldsymbol{x}) \cup J^-(t,\boldsymbol{x}) \tag{1.11}$$

and referred to as light-cone of $(t,\boldsymbol{x})$. The boundaries of the future and past light-cones are denoted by $\partial J^\pm(t,\boldsymbol{x})$ and the boundary of the whole light-cone by $\partial J(t,\boldsymbol{x})$, respectively, cf. Figure 1.1. The inner of the respective light-cones are denoted by $\overset{\circ}{J}{}^\pm(t,\boldsymbol{x})$, $\overset{\circ}{J}(t,\boldsymbol{x})$.

Note that $t^\pm$ is only defined implicitly. Geometrically, t^+ and t^- are the two intersection points of the trajectory $(\boldsymbol{q},\boldsymbol{p})$ with the future and past light-cone boundaries, $\partial J^+(t,\boldsymbol{x})$ and $\partial J^-(t,\boldsymbol{x})$, as illustrated in Figure 1.1.

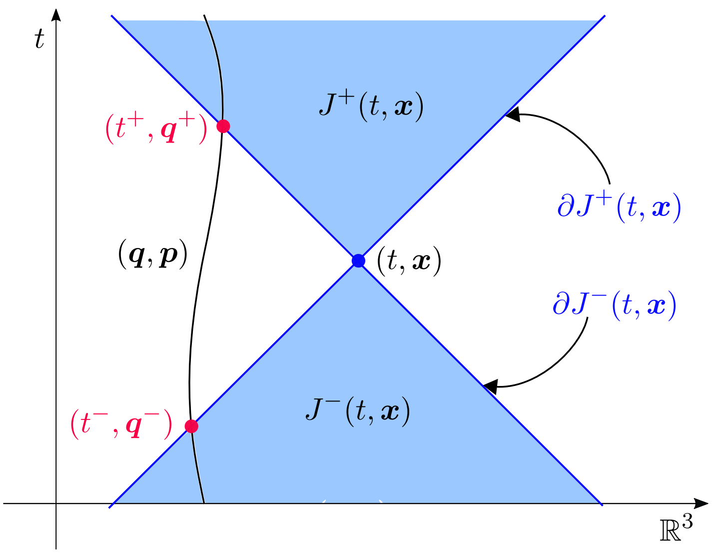

Figure 1.1: Illustration of the future and past light-cones of $(t,\boldsymbol{x}) \in \mathbb{R}^4$. Their boundaries cross the charge trajectory $(\boldsymbol{q},\boldsymbol{p})$ at the space-time points $(t^+,\boldsymbol{q}^+)$ and $(t^-,\boldsymbol{q}^-)$.

All other solutions $\boldsymbol{f}_t$ to (1.2)-(1.3) for the same trajectory $(\boldsymbol{q},\boldsymbol{p})$ can then be represented as convex combinations of the special solutions plus a homogeneous solution, i.e.,

$$\boldsymbol{f}_t = \lambda \boldsymbol{f}_t^-[\boldsymbol{q},\boldsymbol{p}] + (1-\lambda)\boldsymbol{f}_t^+[\boldsymbol{q},\boldsymbol{p}] + \boldsymbol{f}_t^h \tag{1.12}$$

for $\lambda \in [0,1]$, where $\boldsymbol{f}_t^h$ is a solution to the corresponding homogeneous equations (1.6)-(1.7).

1.2 Problems of the theory and state of the art

It is well-known that the Maxwell-Lorentz theory is plagued by several difficulties which we briefly present in this section. The first difficulty we mention is the so-called *self-interaction problem*, which is a conceptual problem of the theory and has been observed very early, e.g. in [1].

The self-interaction problem. In order to calculate the Liénard-Wiechert fields (1.8) due to charge $i \in \mathcal{N}$, one presupposes the trajectory $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ of that particle. But if both, fields and trajectories, are not known, the Maxwell equations have to be coupled with the Lorentz equations and so fields and trajectories have to be computed simultaneously. Assuming $e_{ij} = 1$ for all $i, j \in \mathcal{N}$, to obtain the Lorentz force (1.5) acting on charge i one needs to evaluate the self-field $\boldsymbol{L}_{ii,t}$, i.e., the field due to charge i at its own position $\boldsymbol{q}_{i,t}$. This term is ill-defined because the Maxwell field $\boldsymbol{f}_{i,t}$ is of the form (1.12) and as can be seen from (1.8) has a second order pole at $\boldsymbol{q}_{i,t}$, exactly where it would have to be evaluated. Thus, the coupled system of the Maxwell and Lorentz equations is ill-defined. The problem arises even in the simplest physical system consisting of one moving charge in its own field.

In essence, there are three broad ways to conceive this problem. Either the Lorentz equations as given in (1.4) are wrong or the Maxwell equations as given in (1.2) are wrong or both.

1. In the first conception, the problem is cured by changing the Lorentz force. There is an informal mass renormalization argument by Dirac [15], which effectively replaces the problematic term $\boldsymbol{L}_{ii,t}$ with the finite *Lorentz-Abraham-Dirac* back reaction $\boldsymbol{L}_{ii,t}^{\mathrm{LAD}}$. In the non-relativistic regime, the latter may be approximated by

$$\boldsymbol{L}_{ii,t}^{\mathrm{LAD}} \approx \frac{2}{3} e^2 \dddot{\boldsymbol{q}}_{i,t}, \tag{1.13}$$

with e denoting the electric charge. This procedure cures the original problem, however, introduces a dynamical instability as for almost all but very special initial accelerations, which now must be provided along with initial positions and momenta, the corresponding charge trajectories approach the speed of light exponentially fast; see [41, 30]. However, in certain regimes the center manifold of stable solutions can be studied with the help of singular perturbation theory and the problematic self-interaction term proposed by Dirac can be approximated by the better tampered Landau-Lifshitz radiation reaction term [41].

Another more radical way is to build an action-at-a-distance theory, like the theory of *Wheeler and Feynman*. In the works [46] and [47] it has been shown that this formulation is capable of explaining the irreversible nature of radiation. Historically, electrodynamics by means of direct interaction goes back to works of Gauss [21], Fokker [20], Tetrode [43] and Schwarzschild [39]. Here the Maxwell fields are given by (1.12) for $\lambda = 1/2$ and $\boldsymbol{f}_{i,0}^h = 0$ and represent mere mathematical bookkeepers for the interaction of charged particles. Moreover, each particle interacts solely with the other particles such that the problematic self-interaction summand is not part of the force law.

Nevertheless, global existence has only been established for very special situations. The difficulty of these equations lies in the fact that they contain state-dependent delays.

Such delay equations are considered among the most difficult problems in the area of differential equations [45]. However, this type of equations is currently heavily under investigation in the contemporary mathematics literature (see, e.g. [45] and the references therein) and there is good reason to expect that their solution theory will soon be better understood.

To this day solutions can only be identified uniquely when whole stripes of trajectories are specified. When considering two repelling charges and restricting the motion of the charges to a straight line uniqueness of solutions could be shown; see [16, 17, 6, 3, 9, 13, 25].

2. The second strategy to cure the self-interaction problem is to remedy the field and to adjust the Maxwell fields such that they no longer lead to the self-interaction problem. The *Maxwell-Born-Infeld* theory and the *Bopp-Podolsky* theory are known candidates. In the Bopp-Podolsky, or more precisely, Bopp-Landé-Thomas-Podolsky theory, the field equations are replaced by linear, but higher-order, field equations. The recent work [28] examines this system and establishes local well-posedness.

   In the Maxwell-Born-Infeld theory the Maxwell equations are replaced by non-linear field equations, which are extremely hard to study; see [27] for a uniqueness result for the latter.

3. If both Maxwell's and Lorentz's equations are questioned, an obvious and famous ansatz is the *Abraham model* where particles are modeled as tiny balls with non-zero diameter. This smears out the second order pole in the self-field, however, it breaks Lorentz invariance and introduces another parameter into the theory that changes each particle; see for instance [2, 33, 41].

   The Maxwell-Lorentz system for smeared out charges is widely understood. For one smeared out rigid charged particle, global existence and uniqueness of solutions to the Maxwell-Lorentz system, including the self-interaction summand, has been established in [29] and in [11] by two different methods. In [4] spinning charges have been discussed. When constraining the charge trajectories at times $|t| \geq T$, for arbitrary large but finite T, existence of solutions on $[-T, T]$ was shown for N smoothly extended charges in three dimensions [12, 7]. In [8] global existence and uniqueness of the Maxwell-Lorentz system without self-interaction has been established for smeared out charges with same rigid charge distributions. The semi-relativistic system was considered in [19].

In this work we leave open the question how the term $\boldsymbol{L}_{ii,t}$ has to be changed in order to infer a mathematically well-defined system of equations. Instead, we chose $e_{ij} = 1 - \delta_{ij}$ in (1.4), which simply omits the term. As suggested by the form of the renormalized term proposed by Dirac [15], for a large physically relevant regime (small velocities, jerk and electrical charge), this omission can be justified. Moreover, if well-posedness can not even be shown without the self-interaction summand, there would be no chance to solve the problem including a well-defined self-interaction term $\boldsymbol{L}_{ii,t}$ as, for instance, $\boldsymbol{L}_{ii,t}^{\mathrm{LAD}}$ or the corresponding and less problematic term proposed by Landau and Lifshitz in [30]. In this spirit, this work can also be understood as stepping stone to also include self-interaction.

The conceptual self-interaction problem aside, looking at typical Maxwell solutions (1.12), there are two immediate technical problems.

Collisions. The second problem with Maxwell's fields is the one of collisions, i.e., the charges must not collide, otherwise the factor $\frac{1}{|\boldsymbol{x}-\boldsymbol{q}^{\pm}|^2}$ in (1.8) blows up. This mathematically poses a similarly delicate problem as in the N-particle problem of gravitation, only now with the additional complication that the Coulomb potentials in (1.8) are Lorentz-boosted and to be evaluated at delayed or advanced times $t^{\pm}$ as given in (1.9); see Figure 1.2.

In the N-body problem of Newtonian gravitation, it is known that the collision of particles is not well-defined, see for instance [23]. Since the gravitational force scales like $\frac{1}{|\boldsymbol{x}_i-\boldsymbol{x}_j|^2}$, the dynamics breaks down when particles collide. There is hence no further time evolution after collision. It may be possible to extend the trajectories after collision, but this extension is not unique. Although there may be collisions when N particles move in the gravitational field, the initial conditions resulting in these collisions have measure zero, cf. [38]. We encounter

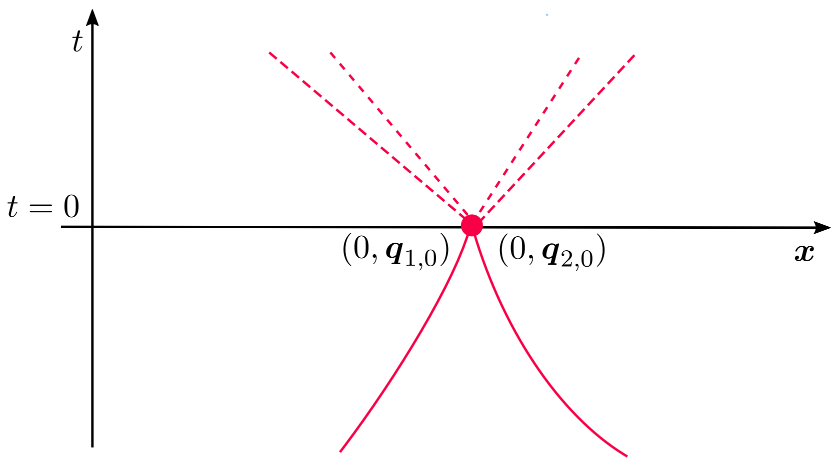

Figure 1.2: Two particles collide at $t=0$. Due to the singular factor $\frac{1}{|\boldsymbol{q}_{1,0}-\boldsymbol{q}_{2,0}|^2}$ in the force law, there is no unique dynamics after collision, which is indicated by the red dashed lines.

the same problem for charged particles. The Liénard-Wiechert field (1.8) contains in the near field the factor $\frac{1}{|\boldsymbol{x}-\boldsymbol{q}^{\pm}|^2}$, which blows up when particles are about to collide. In order to have well-defined dynamics one needs to make sure that particles cannot come arbitrarily close to each other. If they still do, another system of equations would be needed to describe the further motion. Thus, it would be desirable, when investigating initial value problems of the coupled Maxwell-Lorentz system, to show that initial configurations leading to collisions are atypical, that is, have measure 0. Then, one could ignore dynamics with collisions.

Runaway solutions. The third singularity in the equations of motion of classical electrodynamics arises in the term $\frac{1}{(1\pm\boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3}$ of the Liénard–Wiechert fields (1.8), which turns infinite when charges approach the speed of light. Relativistic trajectories fulfill

$$\boldsymbol{v}_t = \frac{\boldsymbol{p}_t}{\sqrt{\boldsymbol{p}_t^2+m^2}}, \tag{1.14}$$

and thus, for masses m being strictly positive, $\boldsymbol{v}_t$ is bounded by the speed of light. Nevertheless, the theory allows for runaway solutions, that is, solutions (of the Lorentz-Dirac equation) that approach the speed of light exponentially fast (see Figure 1.3). There are two

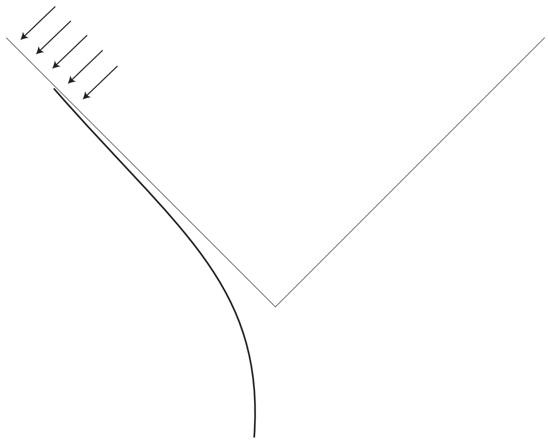

Figure 1.3: A particle approaching the speed of light. In this case the electromagnetic field would accumulate on the light-cone (marked by the arrows).

problems with this kind of solutions. First, we do not observe such accelerating particles. Second, such a particle needs to constantly radiate, and this very radiation will accumulate on the light-cone leading to high-energy radiation (see also Figure 1.3). Such a phenomenon is not observed, either. For technical reasons, however, this phenomenon is not severe, since in the end we propagate solutions to arbitrarily big, but fixed, times T, such that velocities are automatically bounded by a constant smaller than 1. Moreover, when handled with care, it is reasonable to expect that at most only very few initial values $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ lead to catastrophic events.

Finally, setting $e_{ij} = 1 - \delta_{ij}$ one might hope that there are no further obstacles in arriving at a solution theory for the Maxwell-Lorentz system (1.2)-(1.4) up to the first time of collision. However, we ran into a further difficulty which is more subtle and, to the best of our knowledge, has not received attention yet:

Singular fronts. Given a charge trajectory $(\boldsymbol{q}_i, \boldsymbol{p}_i)$, only special initial fields $\boldsymbol{f}_{i,0}$ give rise to solutions $\boldsymbol{f}_{i,t}$ to (1.2)-(1.3) that are sufficiently regular outside a neighborhood of $\boldsymbol{q}_{i,t}$ in order to be evaluated in the terms $\boldsymbol{L}_{ij,t}$ in (1.4) for all times. Generic initial fields will generate singular fronts in the fields traveling at the speed of light, and another charge j moving with velocities below the speed of light is bound to hit such fronts in finite time. This phenomenon has been presented in [14], a joint work with D.-A. Deckert. The mathematical rigorous discussion is content of this work and provides the foundation for the here presented existence and uniqueness result.

1.3 Scope of this work

In this work, we investigate the coupled system of Maxwell's and Lorentz's equations without self-interaction for point charges in terms of an initial value problem. The following points are covered:

1. We demonstrate the existence of singular and discontinuous fronts in the electromagnetic fields, which are located on the light-cone boundaries of the initial charge positions and which occur when charge trajectories $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ do not match the initial electromagnetic fields $\boldsymbol{f}_{i,0}$.

2. From this we infer compatibility conditions between trajectories $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ and initial fields $\boldsymbol{f}_{i,0}$, that prevent the generation of these singular and discontinuous fronts.

3. Since generic initial values of the Maxwell-Lorentz system $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ violate these compatibility conditions, we can infer a no-go result for the Maxwell-Lorentz system of equations.

4. Nevertheless, we are able to identify a large class of potential solutions for Maxwell-Lorentz without self-interaction that do not lead to singular fronts. These solutions, however, are not identified by means of conventional Cauchy data, but by trajectory histories that satisfy the above compatibility condition.

5. Finally, we present an existence and uniqueness result for such solutions. We obtain solutions from an initial time 0 up to the first time of particle collision, which may be infinite.

1.4 Outline

In Chapter 2 we give a heuristic overview of this work including its physical meaning. Starting with a basic example demonstrating the generation of singular fronts located at the light-cone boundary of the initial particle position, we explain this phenomenon by presenting the general formulas for the Maxwell field for given charge trajectories and given initial fields. We argue why these singular fronts occur and how they can be circumvented by additional conditions on the input data. Moreover, this gives rise to two arguments, showing that the initial value problem of the Maxwell-Lorentz system is in general ill-posed. We present a strategy how, nevertheless, Maxwell-Lorentz solutions can be established.

Afterwards, we turn over to the core part of the thesis, which is a mathematical rigorous elaboration motivated by the heuristic discussion in Chapter 2. In Chapter 3 we introduce our notation and the sense in which charge trajectories, fields and solutions have to be understood in this work.

Chapter 4 presents our main results. Our first main result, Theorem 4.1.1 (No-go) states that the initial value problem of Maxwell-Lorentz equations is ill-posed for generic initial values. This is mainly based on our second main result, Theorem 4.2.1 (Explicit Maxwell solutions) which reveals the existence of the singular and discontinuous fronts. Nevertheless, we can show existence and uniqueness of a class of Maxwell-Lorentz solutions in Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions), our third main result. The crucial lemmas for an understanding of these theorems are, as well, included in Chapter 4.

All the proofs including technical lemmas are found in Chapter 5. We start by proving all necessary properties of the Liénard-Wiechert fields in Section 5.1. In Section 5.2 we show existence and uniqueness of Maxwell fields in the distribution sense explained in Chapter 3. In Section 5.3 we derive the explicit representation formula for the Maxwell field generated by a point charge for a given initial field, i.e., Theorem 4.2.1 (Explicit Maxwell solutions); its regularity shall be studied in Section 5.4. In Section 5.5 we can conclude Theorem 4.1.1 (No-go), with the help of the previous sections. Finally, we prove Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions), in Section 5.6.

We end by a conclusion and an outlook of future projects for which this work provides a stepping stone, in Chapter 6.

Several additional and partly well-known results which are used throughout the proofs were collected in the Appendix A for the convenience of the reader.

Chapter 2

Physical motivation

This chapter is in parts adapted from [14], a joint work with D.-A. Deckert. Moreover, this section contains a heuristical discussion only and serves as physical motivation of our work. Therefore, all formulas and calculations mentioned in this section are not rigorous. However, whenever sensible, forward references to the mathematical rigorous part of this work in Chapter 3 –5 are already provided.

2.1 Singular fronts in the Maxwell fields

Example. We start with a basic example that demonstrates the existence of singular fronts in the Maxwell fields. Therefore, consider a system of one point charge with predetermined trajectory for which we solve the initial value problem of Maxwell's equations. To keep it simple we assume the charge to move with the constant velocity $\boldsymbol{v} \in \mathbb{R}^3$ starting at the space-time point $(0, \mathbf{0})$. Its trajectory is then given by $t \mapsto \begin{pmatrix} \boldsymbol{q}_t \\ \boldsymbol{p}_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}t \\ \frac{\boldsymbol{v}}{\sqrt{1-\boldsymbol{v}^2}} m \end{pmatrix}$ with $t \geq 0$. In order to compute the field generated by the latter for any time $t > 0$, we solve Maxwell's equations for the given trajectory and a predetermined initial field $\boldsymbol{f}_0 = (\boldsymbol{E}_0, \boldsymbol{B}_0)$. According to the theory the initial field has to comply with Maxwell's constraints (1.3) and choosing an initial Coulomb field

$$\boldsymbol{f}_0(\boldsymbol{x}) := \begin{pmatrix} \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} \\ 0 \end{pmatrix} \tag{2.1}$$

meets this requirement: For all $\boldsymbol{x} \neq 0$

$$\begin{aligned} \nabla \cdot \boldsymbol{E}_0(\boldsymbol{x}) &= \nabla \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} = (\nabla |\boldsymbol{x}|^{-3}) \cdot \boldsymbol{x} + |\boldsymbol{x}|(\nabla \cdot \boldsymbol{x}) \\ &= (\nabla (x_1^2 + x_2^2 + x_3^2)^{-3/2}) \cdot \boldsymbol{x} + |\boldsymbol{x}|(1+1+1) \\ &= (-\frac{3}{2}(x_1^2 + x_2^2 + x_3^2)^{-5/2} 2\boldsymbol{x}) \cdot \boldsymbol{x} + \frac{3}{|\boldsymbol{x}|^3} = -(\frac{3x_1 + 3x_2 + 3x_3}{|\boldsymbol{x}|^5}) + \frac{3}{|\boldsymbol{x}|^3} \\ &= 0, \end{aligned}$$

whereas for any $R > 0$

$$\begin{aligned}\int_{B_R(\mathbf{0})} d^3x \nabla \cdot \boldsymbol{E}_0(\boldsymbol{x}) &= \int_{\partial B_R(\mathbf{0})} d\sigma(x)\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{E}_0(\boldsymbol{x}) = \int_{\partial B_R(\mathbf{0})} d\sigma(x)\frac{\boldsymbol{x}}{R} \cdot \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} \\ &= \int_{\partial B_R(\mathbf{0})} d\sigma(x)\frac{1}{R^2} = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{1}{R^2} R^2 sin\theta \\ &= 4\pi \neq 0.\end{aligned}$$

The set $B_R(\mathbf{0}) \subset \mathbb{R}^3$ denotes the closed ball of radius R around $\mathbf{0}$ and $\partial B_R(\mathbf{0})$ its boundary. Therefore, $\nabla \cdot \boldsymbol{E}_0 = 4\pi\delta$. For the magnetic field we immediately see $\nabla \cdot \boldsymbol{B}_0 = 0$, and thus, Maxwell's constraints hold. Recasting the Maxwell equations in integral form, the Maxwell field can be written as

$$\boldsymbol{f}_t = \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{f}_0 + 4\pi \int_0^t ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix}, \tag{2.2}$$

where K_t denotes the propagator of the wave equation defined in Definition A.1.1 (Propagator of the d'Alembert operator), ρ_s and $\boldsymbol{j}_s$ denote the charge density and current, i.e. in our example $\rho_s = \delta(\cdot - \boldsymbol{v}s)$ and $\boldsymbol{j}_s = \boldsymbol{v}\delta(\cdot - \boldsymbol{v}s)$. Note that we use the matrix notation, where the first summand, for instance, is equal to

$$\begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{f}_0 = \begin{pmatrix} \partial_t K_t * \boldsymbol{e}_0 + \nabla \wedge K_t * \boldsymbol{b}_0 \\ -\nabla \wedge K_t * \boldsymbol{e}_0 + \partial_t K_t * \boldsymbol{b}_0 \end{pmatrix}. \tag{2.3}$$

Formula (2.2) can be found in [29] and [8]. Moreover, a derivation of the formula is given in Appendix A.1 and the proof of Theorem 4.2.1 (Explicit Maxwell solutions) shows that (2.2) is the unique Maxwell solution for given $\boldsymbol{f}_0$ and given $(\boldsymbol{q}, \boldsymbol{p})$. According to (2.2) the electric field is given by

$$\begin{aligned}\boldsymbol{E}_t(\boldsymbol{x}) &= \partial_t K_t * \boldsymbol{E}_0(\boldsymbol{x}) && \boxed{A} \\ &- 4\pi \int_0^t ds K_{t-s} * \nabla\rho_s(\boldsymbol{x}) - 4\pi \int_0^t ds \partial_t K_{t-s} * \boldsymbol{j}_s(\boldsymbol{x}). && \boxed{B}\end{aligned}$$

Summand $\boxed{A}$ describes how the initial Coulomb field is propagated by the Maxwell evolution and can be simplified as

$$\begin{aligned}\boxed{A} &= (\partial_t K_t * \boldsymbol{E}_0)(\boldsymbol{x}) \overset{FTC}{=} \partial_t K_t * \boldsymbol{E}_0(\boldsymbol{x})|_{t=0} + \int_0^t ds \partial_s^2 \left(K_s * \boldsymbol{E}_0\right)(\boldsymbol{x}) \\ &= \boldsymbol{E}_0(\boldsymbol{x}) + \int_0^t ds \frac{1}{4\pi s} \int_{\partial B_s(\mathbf{0})} d\sigma(y) \Delta \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3} \\ &= \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} + \int_0^t ds \frac{1}{4\pi s} \int_{\partial B_s(\mathbf{0})} d\sigma(y)(\nabla\nabla \cdot - \nabla \wedge \nabla\wedge) \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3} \\ &= \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} + \int_0^t ds \frac{1}{4\pi s} \int_{\partial B_s(\mathbf{0})} d\sigma(y) \nabla 4\pi\delta(\boldsymbol{x}-\boldsymbol{y}) \\ &= \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} + \int_{B_t(\mathbf{0})} d^3y \frac{1}{|\boldsymbol{y}|} \nabla\delta(\boldsymbol{x}-\boldsymbol{y}) \\ &= \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} - \int_{B_t(\mathbf{0})} d^3y \frac{1}{|\boldsymbol{y}|} \nabla_{\boldsymbol{y}}\delta(\boldsymbol{x}-\boldsymbol{y})\end{aligned}$$

$$
\begin{aligned}
&\overset{PI}{=} \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} + \int_{B_t(\boldsymbol{0})} d^3y \nabla_{\boldsymbol{y}} \frac{1}{|\boldsymbol{y}|} \delta(\boldsymbol{x}-\boldsymbol{y}) - \int_{B_t(\boldsymbol{0})} d^3y \nabla_{\boldsymbol{y}} \left[\frac{1}{|\boldsymbol{y}|} \delta(\boldsymbol{x}-\boldsymbol{y}) \right] \\
&\overset{GG}{=} \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} + \int_{B_t(\boldsymbol{0})} d^3y \frac{-\boldsymbol{y}}{|\boldsymbol{y}|^3} \delta(\boldsymbol{x}-\boldsymbol{y}) - \int_{\partial B_t(\boldsymbol{0})} d\sigma(y) \frac{\boldsymbol{y}}{|\boldsymbol{y}|} \frac{1}{|\boldsymbol{y}|} \delta(\boldsymbol{x}-\boldsymbol{y}) \\
&= \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} - \mathbb{1}_{B_t(\boldsymbol{0})}(\boldsymbol{x}) \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} - \int_{\partial B_t(\boldsymbol{0})} d\sigma(y) \frac{\boldsymbol{y}}{|\boldsymbol{y}|^2} \delta(\boldsymbol{x}-\boldsymbol{y}) \\
&= \mathbb{1}_{B_t^c(\boldsymbol{0})}(\boldsymbol{x}) \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} - \int_0^\infty dt \int_{\partial B_t(\boldsymbol{0})} d\sigma(y) \frac{\boldsymbol{y}}{|\boldsymbol{y}|^2} \delta(\boldsymbol{x}-\boldsymbol{y}) \delta(|\boldsymbol{y}|-t) \\
&= \mathbb{1}_{B_t^c(\boldsymbol{0})}(\boldsymbol{x}) \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} - \int d^3y \frac{\boldsymbol{y}}{|\boldsymbol{y}|^2} \delta(|\boldsymbol{y}|-t) \delta(\boldsymbol{x}-\boldsymbol{y}) \\
&= \mathbb{1}_{B_t^c(\boldsymbol{0})}(\boldsymbol{x}) \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} - \delta(|\boldsymbol{x}|-t) \frac{\boldsymbol{x}}{|\boldsymbol{x}|^2},
\end{aligned}
$$

where FTC stands for fundamental theorem of calculus, PI for integration by parts, GG for Gauss-Green Theorem (see Appendix A.4) and $\mathbb{1}_{B_t(\boldsymbol{0})}(\boldsymbol{x})$ denotes the characteristic function being one for $\boldsymbol{x}$ in the closed set $B_t(\boldsymbol{0})$ and zero in the open set $B_t^c(\boldsymbol{0}) = \mathbb{R}^3 \setminus B_t(\boldsymbol{0})$. The electric field component $\boxed{B}$ depending on the charge trajectory can be transformed into

$$
\boxed{B} = \mathbb{1}_{B_t(\boldsymbol{0})}(\boldsymbol{x}) \frac{(\boldsymbol{n}^- - \boldsymbol{v})(1-\boldsymbol{v}^2)}{|\boldsymbol{x}-\boldsymbol{q}^-|^2(1-\boldsymbol{n}^-\cdot\boldsymbol{v})^3} + \delta(\boldsymbol{x}-|t|) \frac{\boldsymbol{n}-\boldsymbol{v}}{(1-\boldsymbol{n}\cdot\boldsymbol{v})|\boldsymbol{x}|}, \tag{2.4}
$$

where we used the abbreviations

$$
\boldsymbol{n}^- := \frac{\boldsymbol{x}-\boldsymbol{q}^-}{|\boldsymbol{x}-\boldsymbol{q}^-|}, \qquad \boldsymbol{q}^- := \boldsymbol{q}_{t^-}, \qquad t^- := t - |\boldsymbol{x}-\boldsymbol{q}^-|, \qquad \boldsymbol{n} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}. \tag{2.5}
$$

Since the computation of $\boxed{B}$ is extremely involving, we refer to the proof of Theorem 4.2.1 (Explicit Maxwell solutions) where it is conducted for general charge trajectories $(\boldsymbol{q}, \boldsymbol{p})$. Thus, the electric field at time $t > 0$ evaluated at space point $\boldsymbol{x}$ is given by

$$
\begin{aligned}
\boldsymbol{E}_t(\boldsymbol{x}) = {} & \mathbb{1}_{B_t(\boldsymbol{0})}(\boldsymbol{x}) \frac{(\boldsymbol{n}^- - \boldsymbol{v})(1-\boldsymbol{v}^2)}{|\boldsymbol{x}-\boldsymbol{q}^-|^2(1-\boldsymbol{n}^-\cdot\boldsymbol{v})^3} && (2.6)\\
& + \mathbb{1}_{B_t^c(\boldsymbol{0})}(\boldsymbol{x}) \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} && (2.7)\\
& + \delta(\boldsymbol{x}-|t|) \left(\frac{\boldsymbol{n}-\boldsymbol{v}}{(1-\boldsymbol{n}\cdot\boldsymbol{v})|\boldsymbol{x}|} - \frac{\boldsymbol{x}}{|\boldsymbol{x}|^2} \right). && (2.8)
\end{aligned}
$$

Summand (2.6) is the electric field component that builds up due to the charge trajectory $(\boldsymbol{q}, \boldsymbol{p})$. As the charge is assumed to move with constant velocity $\boldsymbol{v}$, and thus, without acceleration, this is exactly the retarded Liénard-Wiechert field $\boldsymbol{e}_t^-[\boldsymbol{q}, \boldsymbol{p}]$ introduced in (1.8) due to that charge, supported on the closed ball $B_t(\boldsymbol{0})$. According to (2.7), in this region the initial Coulomb field $\boldsymbol{E}_0$ is displaced. The field $\boldsymbol{E}_0$ then persists only outside of that closed ball. In (2.8) one finds a delta distribution supported on the boundary of $B_t(\boldsymbol{0})$ which depends on the space point $\boldsymbol{x}$ and the initial velocity $\boldsymbol{v}$.

This example demonstrates the existence of a singular front located on the light-cone boundary of the initial charge position $(0, \boldsymbol{0})$ and that the Maxwell field $\boldsymbol{E}_t$ is not necessarily smooth on $\mathbb{R}^3 \setminus \{\boldsymbol{q}_t\}$ although the initial Coulomb field (2.1) is.

For the magnetic field we obtain the same behavior, the initial Coulomb field persists outside of the light-cone $J^+(0,\mathbf{0})$, inside it is displaced by the retarded Liénard-Wiechert field generated by the charge and on the light-cone boundary a singular front can be observed. See Figure 2.1 for illustration. This phenomenon has a rather simple explanation. Morally, the

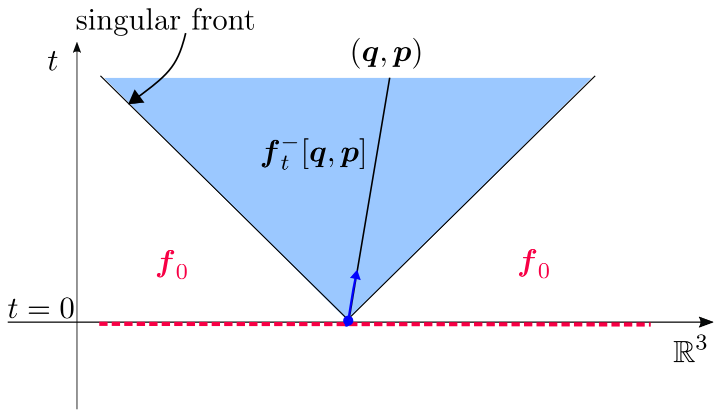

Figure 2.1: An initial Coulomb field for a charge moving with constant velocity $\boldsymbol{v}$ starting at time 0 generates a singular front along the boundary of a light-cone.

initial Coulomb field $\boldsymbol{f}_0$ corresponds to the field generated by a charge at rest at position $\mathbf{0}$; see Figure 2.2. Therefore, there is a kink between the charge trajectory $(\boldsymbol{q}_t,\boldsymbol{p}_t)|_{t\geq 0}$ and the one in the past encoded in the initial field. And this kink is responsible for the singular front observed along the light-cone.

If we look again at equation (2.6)-(2.8), we find that the boundary summand (2.8) cancels if and only if the charge velocity $\boldsymbol{v}$ equals $\mathbf{0}$ – like the velocity of a charge corresponding to the initial Coulomb field.

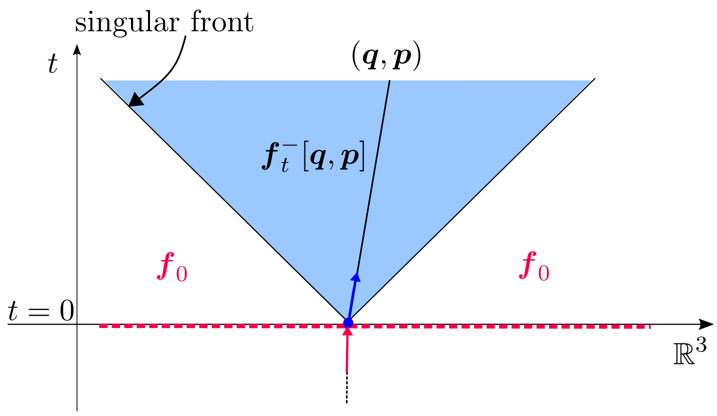

Figure 2.2: An initial Coulomb field corresponds to a past charge that has been at rest. Thus, there is a jump in momentum at time 0.

The general case. We turn over to the general case and present a field formula for any given charge trajectory and any initial field. Moreover, we identify necessary compatibility conditions of initial fields and charge trajectories.

According to the general splitting in (1.12), also any relevant initial field $\boldsymbol{f}_0$, obeying the Maxwell constraint (1.3), can be written as convex combination of the two special solutions and a homogeneous solution, i.e.,

$$\boldsymbol{f}_0 = \lambda \boldsymbol{f}_0^-[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] + (1-\lambda)\boldsymbol{f}_0^+[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] + \boldsymbol{f}_0^h, \tag{2.9}$$

for some $\lambda \in [0,1]$, $\boldsymbol{f}_t^\pm[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}]$ being the Liénard-Wiechert fields (1.8) generated by a auxiliary charge trajectory $(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}})$ fulfilling $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0$. Note that this representation is just a parameterization of the initial field by an initial homogeneous field $\boldsymbol{f}_0^h$ and a trajectory $(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}})$, and the requirement $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0$ is a direct implication of the Maxwell constraints. In this parameterization the initial Coulomb field from the introductory example corresponds to $\lambda = 1$, $(\tilde{\boldsymbol{q}}_t, \tilde{\boldsymbol{p}}_t) = (\boldsymbol{0}, \boldsymbol{0})$ for any time $t \leq 0$ and $\boldsymbol{f}_0^h = 0$.

Moreover, given any general initial field $\boldsymbol{f}_0$ and an auxiliary trajectory $(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}})$, equation (2.9) is merely a definition of $\boldsymbol{f}_0^h$ which must then be a homogeneous field, i.e., one fulfilling the homogeneous Maxwell constraint (1.7). As this free field $\boldsymbol{f}_t^h$ propagates independently of the charges one may, for now, think of it to be smooth (which implies $\boldsymbol{f}_t^h$ to be smooth). In the mathematical part of this work we treat more general cases, cf. Lemma 4.2.2 (Homogeneous Maxwell solutions).

Plugging the actual trajectory $(\boldsymbol{q}, \boldsymbol{p})$ and the initial field $\boldsymbol{f}_0$ in the form of (2.9) into the explicit expressions (2.2) above, for any $t \in \mathbb{R}$, one finds

$$\boldsymbol{f}_t = \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)} \boldsymbol{f}_t^{-\sigma(t)}[\boldsymbol{q}, \boldsymbol{p}] \tag{2.10}$$

$$+ \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)} \lambda \left(\boldsymbol{f}_t^-[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] - \boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] \right) \tag{2.11}$$

$$+ \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)} (1-\lambda) \left(\boldsymbol{f}_t^+[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] - \boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] \right) \tag{2.12}$$

$$+ \mathbb{1}_{B^c_{|t|}(\boldsymbol{q}_0)} \left(\lambda \boldsymbol{f}_t^-[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] + (1-\lambda) \boldsymbol{f}_t^+[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] \right) \tag{2.13}$$

$$+ \boldsymbol{r}^{-\sigma(t)}_{t,(\boldsymbol{q}_0, \boldsymbol{p}_0)} - \boldsymbol{r}^{-\sigma(t)}_{t,(\tilde{\boldsymbol{q}}_0, \tilde{\boldsymbol{p}}_0)} \tag{2.14}$$

$$+ \boldsymbol{f}_t^h, \tag{2.15}$$

using

$$\boldsymbol{r}^\pm_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)}(\boldsymbol{x}) := \frac{\delta(|t| - |\boldsymbol{x} - \boldsymbol{q}_0|)}{(1 \pm \boldsymbol{n}_0 \cdot \boldsymbol{v}_0)|\boldsymbol{x} - \boldsymbol{q}_0|} \begin{pmatrix} \boldsymbol{n}_0 \pm \boldsymbol{v}_0 \\ -\boldsymbol{n}_0 \wedge \boldsymbol{v}_0 \end{pmatrix}, \tag{2.16}$$

together with

$$\boldsymbol{n}_0 := \frac{\boldsymbol{x} - \boldsymbol{q}_0}{|\boldsymbol{x} - \boldsymbol{q}_0|}, \qquad \boldsymbol{v}_0 := \boldsymbol{v}(\boldsymbol{p}_0), \tag{2.17}$$

where $\sigma(t)$ denotes the sign of t, i.e., $\boldsymbol{f}_t^{-\sigma(t)}$ stands for $\boldsymbol{f}_t^-$ if $t \geq 0$ and for $\boldsymbol{f}_t^+$ if $t < 0$. The derivation of this formula is extremely involving. It is the content of Theorem 4.2.1 (Explicit Maxwell solutions) which can be considered a generalization of the Liénard-Wiechert formulas. The first three terms have support inside and on the light-cone of $(0, \boldsymbol{q}_0)$. The term (2.10) describes the field that is generated by the actual charge trajectory $(\boldsymbol{q}, \boldsymbol{p})$ between time 0 and t, and the terms (2.11)-(2.12) describe how the initial advanced and retarded Liénard-Wiechert fields encoded in (2.9) are propagated inside the light-cone. Depending on the sign of t, one of the terms (2.11)-(2.12) will vanish and the respective other will be proportional to the difference $\sigma(t)(\boldsymbol{f}_t^+[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}]-\boldsymbol{f}_t^-[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}])$, which according to Dirac [15] can be interpreted as the radiation emitted or absorbed by the auxiliary charge trajectory $(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}})$ between time 0 and t. Moreover, the term (2.13) is the propagated remainder of the initial retarded and advanced Liénard-Wiechert fields, and therefore, only has support outside the light-cone. The terms in (2.14) are again the distributions given in (2.16) having support on the light-cone, and $\boldsymbol{f}_t^h$ in (2.15) is simply the field $\boldsymbol{f}_0^h$ propagated from 0 to t by the homogeneous Maxwell equations, i.e., (1.6)-(1.7). Note that $\boldsymbol{f}_t^h$ is as regular as $\boldsymbol{f}_0^h$. See Figure 2.3 for an illustration of the trajectories and supports of the terms (2.10)-(2.15). The solution $\boldsymbol{f}_t$ in (2.10)-(2.15)

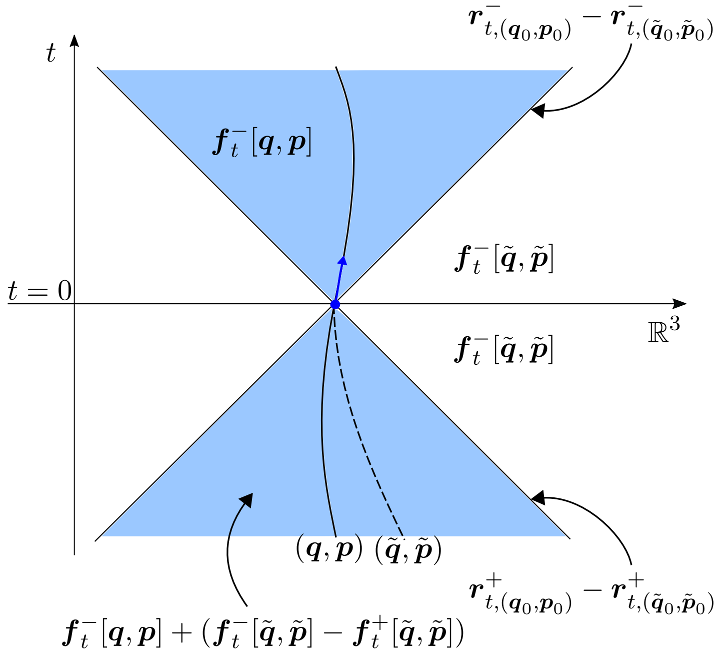

Figure 2.3: An illustration of the charge trajectories $(\boldsymbol{q},\boldsymbol{p})$ and $(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}})$ as well as supports of the corresponding terms in (4.12)-(4.17) for the case $\lambda=1$ and $\boldsymbol{f}_0^0=0$.

can be recast in the more compact form

$$\boldsymbol{f}_t = \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)}\left(\boldsymbol{f}_t^{-\sigma(t)}[\boldsymbol{q},\boldsymbol{p}]-\boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}]\right) \tag{2.18}$$

$$+\lambda \boldsymbol{f}_t^-[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}]+(1-\lambda)\boldsymbol{f}_t^+[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] \tag{2.19}$$

$$+\boldsymbol{r}^{-\sigma(t)}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)} - \boldsymbol{r}^{-\sigma(t)}_{t,(\tilde{\boldsymbol{q}}_0,\tilde{\boldsymbol{p}}_0)} \tag{2.20}$$

$$+\boldsymbol{f}_t^h, \tag{2.21}$$

from which one can read off necessary compatibility conditions between the initial field $\boldsymbol{f}_0$ and the charge trajectory $(\boldsymbol{q},\boldsymbol{p})$ that prevent the development of singular light fronts. These conditions correspond to Lemma 5.4.1 (Compatibility Conditions) and Lemma 4.2.4 (Regularity of $\boldsymbol{f}_t$) in the mathematical part of this work.

(C1) The distributions (2.20) must cancel each other because neither (2.18), (2.19), nor (2.21) contain Dirac delta distributions. This is the case if and only if $(\tilde{\boldsymbol{q}}_0,\tilde{\boldsymbol{p}}_0)=(\boldsymbol{q}_0,\boldsymbol{p}_0)$, where $\tilde{\boldsymbol{q}}_0=\boldsymbol{q}_0$ was already assumed in order to fulfill the Maxwell constraint (1.3).

(C2) Provided (C1) is fulfilled, the field $\boldsymbol{f}_t$ is continuous on $\mathbb{R}^3 \setminus \{\boldsymbol{q}_t\}$ if and only if term (2.18) vanishes on the light-cone of $(0, \boldsymbol{q}_0)$. This can be seen as follows: By virtue of (1.8), for all times t the terms (2.18)-(2.19) are smooth everywhere except maybe on the light-cone of $(0, \boldsymbol{q}_0)$ as well as the points $\boldsymbol{q}_t$ and $\tilde{\boldsymbol{q}}_t$. However, since these terms coincide with (2.10)-(2.13), they must be smooth in $\tilde{\boldsymbol{q}}_t$ as (2.11) and (2.12) are free fields and (2.13) has only support outside of the light-cone of $(0, \boldsymbol{q}_0 = \tilde{\boldsymbol{q}}_0)$. As the free field $\boldsymbol{f}^h_t$ is smooth, and by (C1) terms (2.20), (2.14) vanish, the field $\boldsymbol{f}_t$ is continuous on $\mathbb{R}^3 \setminus \{\boldsymbol{q}_t\}$ if and only if (2.18) vanishes on the light-cone of $(0, \boldsymbol{q}_0)$. This is the case if and only if the accelerations $\ddot{\tilde{\boldsymbol{q}}}_t$ and $\ddot{\boldsymbol{q}}_t$ coincide at $t = 0$. Furthermore, if and only if all l-th derivatives of $\tilde{\boldsymbol{q}}_t$ and $\boldsymbol{q}_t$ for $l = 1, \ldots, k + 2$ coincide at $t = 0$, the field $\boldsymbol{f}_t$ has k spatial derivatives on $\mathbb{R}^3 \setminus \{\boldsymbol{q}_t\}$. Finally, if and only if the trajectories $\tilde{\boldsymbol{q}}_t$ and $\boldsymbol{q}_t$ connect smoothly at time $t = 0$, the field $\boldsymbol{f}_t$ is smooth on $\mathbb{R}^3 \setminus \{\boldsymbol{q}_t\}$.

It was called to our attention that also in [34, 35], where a rigorous electrodynamic point-charge limit was studied in the dipole approximation, a condition relating the initial fields and initial momenta similar to (C1) was needed to ensure convergence.

2.2 Implications for the Maxwell-Lorentz system

In this section we discuss the implications of the observations made in Section 2.1 on the fully coupled system of Maxwell's and Lorentz's equations (1.2)-(1.4) and provide a heuristic explanation for Theorem 4.1.1 (No-go).

First and foremost, we observe that in a system of at least two charges, one charge, say number 2, will inevitably cross the light-cone of the initial space-time point of another charge, say number 1, at a time t^*, which is bounded from below by the minimal distance divided by the speed of light; see Figure 2.4. Thus, at $t = t^*$ the Lorentz force (1.4) felt by charge 2 must

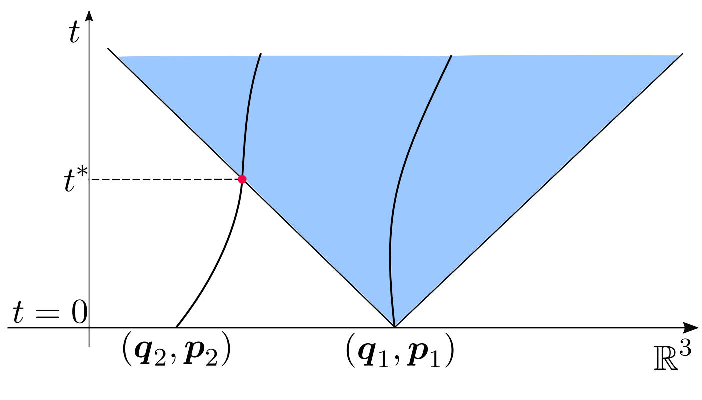

Figure 2.4: Charge 2 on trajectory $(\boldsymbol{q}_2, \boldsymbol{p}_2)$ is bound to cross the light-cone of the initial space-time point $(0, \boldsymbol{q}_{1,0})$ of charge 1 on trajectory $(\boldsymbol{q}_1, \boldsymbol{p}_1)$.

evaluate the field $\boldsymbol{f}_{1,t}$ at some point on the light-cone of $(0, \boldsymbol{q}_{1,0})$. Recall that for an initial field $\boldsymbol{f}_{i,0}$ of the form (2.9) with auxiliary charge trajectory $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$, the propagated field $\boldsymbol{f}_{i,t}$ is given by (2.10)-(2.15). Should condition (C1) of Section 2.1 not be satisfied, this evaluation is ill-defined because of the presence of the distributions (2.14). In this case, the dynamics will cease to exist beyond the time instant t^*. Hence, (C1) is a necessary condition for global existence of solutions to the Maxwell-Lorentz system (1.2)-(1.4). Should condition (C1) hold but not (C2), then the force on charge 2 will undergo a discontinuous jump when traversing the light-cone at time t^*. Therefore, (C2) is a necessary condition for having continuous or smooth solutions to the Maxwell-Lorentz system (1.2)-(1.4).

The following two arguments illustrate that (C1) and (C2) are violated for generic initial data $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ obeying the Maxwell constraints (1.3) only. Precisely, they show that global existence is not stable under arbitrarily small perturbations of the initial data. For this purpose, let us assume that $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i\in\mathcal{N}}$ is a global solution to the Maxwell-Lorentz system (1.2)-(1.4) for some initial value $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ such that the initial fields $\boldsymbol{f}_{i,0}$ are of the form (2.9) for some smooth auxiliary trajectory $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ and some smooth initial free field $\boldsymbol{f}^h_{i,0}$. The corresponding mathematical result is Theorem 4.1.1 (No-go) in Chapter 4, which represents our first main result.

No-go argument (A1): By Maxwell constraints (1.3) and necessary condition (C1) we have $(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0}) = (\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0})$ for $i \in \mathcal{N}$. Then, perturbing the initial momentum of charge 1 by $\boldsymbol{p}_{1,0} \to \boldsymbol{p}'_{1,0} = \boldsymbol{p}_{1,0} + \boldsymbol{\delta}$ for any vector $\boldsymbol{\delta}$ of arbitrarily small norm $|\boldsymbol{\delta}| > 0$ leads to a corresponding local solution $(\boldsymbol{q}'_i, \boldsymbol{p}'_i, \boldsymbol{f}'_i)_{i\in\mathcal{N}}$ with $\boldsymbol{f}'_{1,t}$ taking the form of (2.10)-(2.15), whereas the contribution (2.14) equals the distribution $\boldsymbol{r}^{-\sigma(t)}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0+\boldsymbol{\delta})} - \boldsymbol{r}^{-\sigma(t)}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)}$, which does not vanish. In other words (C1) is violated, and $\boldsymbol{f}'_{1,t}$ manifests a singular light front with support on the light-cone of space-time point $(0, \boldsymbol{q}_{1,0})$, as discussed in Section 2.1. By virtue of (1.2)-(1.4), this perturbation in the initial momentum propagates not faster than the speed of light. In particular, the perturbed field $\boldsymbol{f}'_{1,t}$ of charge 1 and the perturbed trajectory $(\boldsymbol{q}'_{2,t}, \boldsymbol{p}'_{2,t})$ of charge 2 remain identical on $B^c_{|t|}(\boldsymbol{q}_{1,0})$ for $t \in \mathbb{R}$. In consequence, charge 2 is bound to touch the light-cone of $(0, \boldsymbol{q}_{1,0})$ at the very same time t^* as in the unperturbed solution, only now the perturbed field $\boldsymbol{f}'_{1,t}$ contains a singular light front consisting of distributions. In conclusion, the dynamics will cease to exist beyond time t^*, as discussed above. The argument is depicted in Figure 2.5.

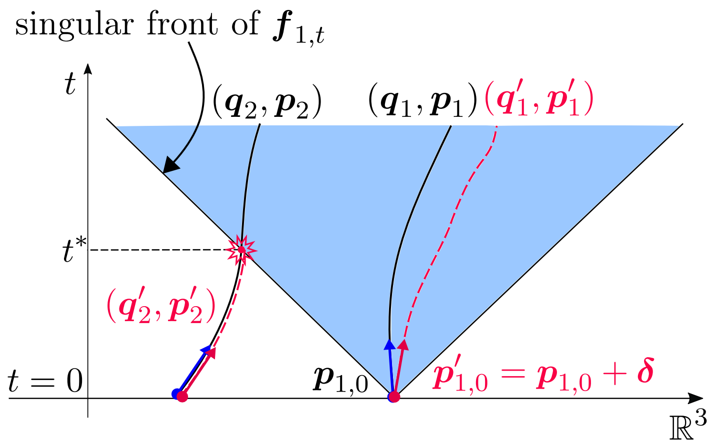

Figure 2.5: Perturbing the initial momentum of charge 1 by $\boldsymbol{p}_{1,0} \to \boldsymbol{p}'_{1,0}$ leads to a singular front supported on the light-cone of $(0, \boldsymbol{q}_{1,0})$ and thus to a sudden stop of the dynamics at time t^* when charge 2 touches the light-cone.

No-go argument (A2): This time, let us assume the global solution is also smooth and $\lambda > 0$ (for $\lambda = 0$ a similar argument can be found). Due to condition (C2), $\dddot{\boldsymbol{q}}_{i,t}$ and $\ddot{\boldsymbol{q}}_{i,t}$ coincide at time $t = 0$. Now, we perturb a little bit the trajectory $(\tilde{\boldsymbol{q}}_2, \tilde{\boldsymbol{p}}_2)$ that defined the initial field $\boldsymbol{f}_{2,0}$ given in (2.9) in an arbitrarily small neighborhood of the retarded time t^- belonging to space-time point $(0, \boldsymbol{q}_{1,0})$. Due to (2.10)-(2.15) this causes a small perturbation $\boldsymbol{f}_{2,0} \to \boldsymbol{f}'_{2,0}$, and we tune this perturbation such that the Lorentz force (1.4) on charge 1 at $t = 0$ changes its value. In consequence, the potential local solution $(\boldsymbol{q}'_i, \boldsymbol{p}'_i, \boldsymbol{f}'_i)_{i\in\mathcal{N}}$ corresponding to this perturbed initial data violates (C2) as the accelerations $\dddot{\boldsymbol{q}}_{1,t}$ and $\ddot{\boldsymbol{q}}'_{1,t}$ do not match anymore at $t = 0$. As discussed in Section 2.1, this leads to a discontinuity on the light-cone of $(0, \boldsymbol{q}_{1,0})$. However, by virtue of (1.2)-(1.4) the perturbed field $\boldsymbol{f}'_{1,t}$ of charge 1 and the perturbed trajectory $(\boldsymbol{q}'_{2,t}, \boldsymbol{p}'_{2,t})$ of charge 2 remain identical on $B^c_{|t|}(\boldsymbol{q}_{1,0})$ for $t \in \mathbb{R}$. Therefore, charge 2 is bound to hit the light-cone of $(0, \boldsymbol{q}_{1,0})$ at the very same time t^* as in the unperturbed solution. At this instant, due to the discontinuity of $\boldsymbol{f}'_{1,t}$, the acceleration of charge 2 will undergo a likewise discontinuous jump. Hence, should the perturbed solution exist globally it can only be piecewise smooth. Furthermore, the discontinuity in the acceleration of charge 2 will give rise to a corresponding discontinuity in the field $\boldsymbol{f}_{2,t}$ on the light-cone of $(t^*, \boldsymbol{q}_{2,t^*})$, which charge 1 is bound to cross eventually. By this mechanism, a whole network of singular light fronts is developed. The argument is depicted in Figure 2.6.

These two arguments indicate that the initial value problem of the Maxwell-Lorentz system (1.2)-(1.4) with renormalized (or without) self-interaction term is ill-posed for general initial values $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ only fulfilling the Maxwell constraints (1.3): Even if a global solution is found, only a small perturbation in the initial values suffices to prevent either global existence, by (A1), or global smoothness, by (A2), of the potential solution corresponding to the perturbed initial values.

One might tend to think that these are all problems connected to the point-like nature of the charges, a concept that could even be considered questionable in the classical regime. Indeed, it is true that for the Maxwell-Lorentz system of smoothly extended charges those mathematical problems do not show up. Nevertheless, the qualitative behavior of generation of singular light fronts for initial conditions that violate (C1) or (C2) remains the same. As the fields of the extended charges with density ρ are of the form $\boldsymbol{\rho} * \boldsymbol{f}_{i,t} + \boldsymbol{f}^h_{i,t}$, the discussed singular fronts are now only smeared out by ρ. For ρ supported on the scale of the classical electron radius, i.e., $r_e \sim 10^{-15}$m, the singular fronts will still result in sharp – though smooth – steps in the fields on the respective light-cones. Other charges are bound to eventually traverse such steps and will suddenly – on time scales of r_e divided by their respective speed – start or stop to radiate, thus, leading to potentially observable though physically questionable phenomena.

2.3 Admissible initial values

If for a moment we also admit piecewise smooth solutions to the Maxwell-Lorentz system (1.2)-(1.4), a sensible restriction on the space of initial values can be taken from condition (C1). If we require the initial value $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ to comprise fields $\boldsymbol{f}_{i,0}$ of the form (2.9) for piecewise smooth auxiliary trajectories $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ fulfilling $(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0}) = (\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0})$, condition (C1) as well as the Maxwell constraints (1.3) are fulfilled by definition and there seems to be no further obstacle concerning mathematical well-posedness of the respective initial value

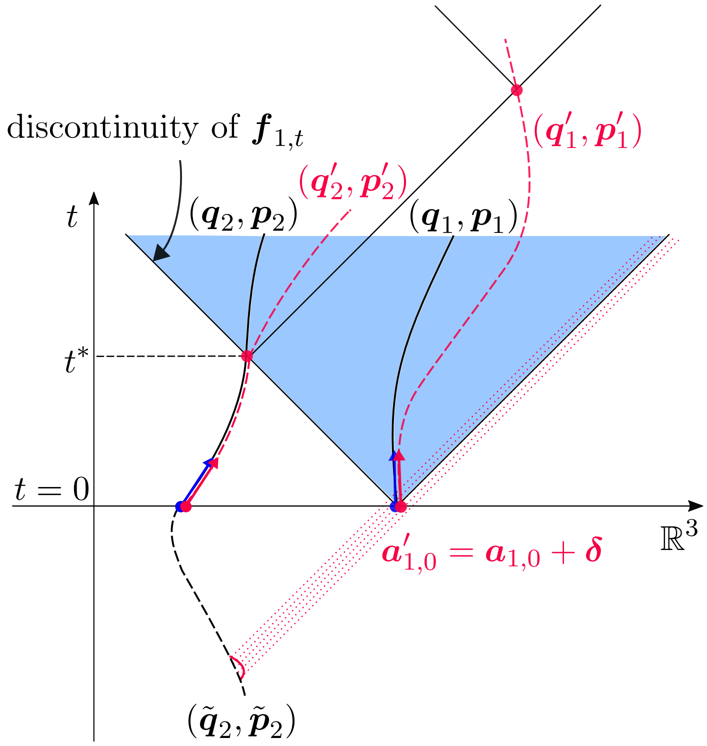

Figure 2.6: Perturbing the initial field of charge 1, $\boldsymbol{f}_{1,0} \to \boldsymbol{f}'_{1,0}$, by a small bump in $(\tilde{\boldsymbol{q}}_2, \tilde{\boldsymbol{p}}_2)$ at the corresponding retarded time, leads to a discontinuity of $\boldsymbol{f}_{1,t}$ supported on the light-cone of $(0, \boldsymbol{q}_{1,0})$ and thus charge 2 experiences a sudden jump in acceleration at time t^*, which causes a discontinuity in $\boldsymbol{f}_{2,t}$.

problem. If, however, we demand smooth global solutions, we would also need to comply with condition (C2). In order to do so we would have to know the derivatives of the charge trajectories $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ at initial time $t = 0$. But those are unknown as they already require knowledge of a local solution in a neighborhood of $t = 0$. Hence, there is no possibility to restrict the space of initial fields a priori in order to ensure well-posedness.

Given initial data $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ fulfilling (1.3) and (C1), it is possible to compute the solution of the Maxwell-Lorentz equations in a sufficiently small time interval $[0, \tau)$. This can be done as the singular fronts live only on the light-cones of the initial space-time points $(0, \boldsymbol{q}_{i,0})$ so that τ only has to be chosen smaller than the smallest time t^* when some charge hits a singular front. For the case $\lambda = 1$ this is the content of Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions), (i), in Chapter 4.

This preliminary local solution allows to compute all derivatives of the charge trajectories $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ at $t = 0$. So, if for some reason $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ in a neighborhood of $t = 0$ should connect smoothly to $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ such that (C2) is fulfilled, we can even establish smooth global solutions, which, for the case $\lambda = 1$ is the content of the second part of Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions).

In order to find initial values that indeed match (C2), one could consider the following approach: Given the local solution on time interval $[0, \tau)$ this would allow to adapt the auxiliary trajectories $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ in a neighborhood of $t = 0$ such that it connects smoothly to $(\boldsymbol{q}_i, \boldsymbol{p}_i)$. This procedure changes the initial fields $\boldsymbol{f}_{i,0} \to \boldsymbol{f}'_{i,0}$ in a spatial neighborhood around the initial positions $\boldsymbol{q}_{i,0}$. Since self-interaction is excluded, the adapted initial values $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}'_{i,0})_{i\in\mathcal{N}}$, however, fulfill the Maxwell constraints (1.3), (C1), and (C2), and therefore, should not bare any further obstacles concerning smooth global solutions.

Though mathematically sound, physically, this is a rather opaque procedure. It is not anymore a formulation of classical electrodynamics in terms of an initial value problem for (1.2)-(1.4) but in terms of an initial guess, that, first, has to be adapted in a quite arbitrary way before a global solution can be inferred at all.

So what is overlooked when naively regarding the Maxwell-Lorentz system (1.2)-(1.4) as an initial value problem? Any inhomogeneous solution $\boldsymbol{f}_{i,t}$ to the Maxwell equations (1.2)-(1.3) is of the form (1.12), which implies that the entire history of the charge trajectory $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ is already encoded in the spatial dependence of the field $\boldsymbol{f}_{i,t}$; recall the $t^{\pm}$ dependence in (1.8). Now, if we set some initial field $\boldsymbol{f}_{i,0}$ by hand, for which the Maxwell constraint (1.3) only requires that we choose it of the form (2.9) with some auxiliary trajectory $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ fulfilling $\tilde{\boldsymbol{q}}_{i,0} = \boldsymbol{q}_{i,0}$, the Maxwell time evolution is fooled to believe that the history of the charge trajectory is given by $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$. But except for $\tilde{\boldsymbol{q}}_{i,0} = \boldsymbol{q}_{i,0}$, the history of the auxiliary trajectory $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ may have nothing in common with the actual one $(\boldsymbol{q}_i, \boldsymbol{p}_i)$, which is to be computed. As a matter of fact, the Maxwell equations propagate such an initial field $\boldsymbol{f}_{i,0}$ as if it was generated by the auxiliary charge trajectory $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ outside the light-cone of $(0, \boldsymbol{q}_{i,0})$ while, inside, a new field is generated according to the actual trajectory $(\boldsymbol{q}_i, \boldsymbol{p}_i)$. It is therefore not surprising that the incompatibilities between the actual charge trajectories $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ and the initial fields $\boldsymbol{f}_{i,0}$ of the solution (1.2)-(1.4) discussed in Section 2.1 occur during the dynamics and that any mismatch between the actual and auxiliary charge trajectories in the sense of (C1) and (C2) expresses itself as a singular light front.

In view of this, it would be desirable to find a formulation of classical electrodynamics that automatically avoids any such incompatibilities. This is possible and in Section 2.4 we discuss a whole class of such formulations having two representatives that are well-known since the beginning of classical electrodynamics.

2.4 A reformulation of the Maxwell-Lorentz system

As demonstrated, the restriction of the solution space of the Maxwell-Lorentz system (1.2)-(1.4) to smooth solutions does not allow a formulation in terms of an initial value problem. Though a potential global solution is uniquely identified by its initial data $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$, only very special initial fields fulfilling the necessary condition (C2) lead to smooth global solutions. Furthermore, the information needed to restrict the initial data according to (C2) would already require knowledge of the unknown solution. These circumstances suggest that we might need to change the way we look at the solution theory for the Maxwell-Lorentz system.

The starting point for such a consideration is the fact that the Maxwell field at one time instant and the entire trajectory of the charge that generated it are intimately intertwined beyond the Maxwell constraint (1.3).

This can be observed best when imagining a single charge i incoming from the remote past $t = -\infty$. Considering, e.g. the case $\lambda = 1$, any auxiliary trajectory $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ in the expression of the field $\boldsymbol{f}_{i,t}$ in (2.10)-(2.15) is forgotten during a time evolution from $t = -\infty$ to any finite time t and so are any potential singular fronts as they escape to spatial infinity with the speed of light. Concerning point-wise evaluation in any finite region of space-time, the Maxwell field in (2.10)-(2.15) reduces to the expression

$$\boldsymbol{f}_{i,t} = \boldsymbol{f}^-_{i,t}[\boldsymbol{q}_i, \boldsymbol{p}_i] + \boldsymbol{f}^h_{i,t}. \tag{2.22}$$

Nothing changes in this argument and in the form of (2.22) when the charge trajectory $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ is not prescribed but also develops simultaneously to the evolution of the Maxwell fields, i.e., according to the fully coupled system (1.2)-(1.4). Hence, stopping the dynamics at time $t = 0$ and starting it again in an initial value problem fashion dictates the natural choice (2.22) for the initial field at $t = 0$. This means that the initial field $\boldsymbol{f}_{i,0}$ should be of the form (2.9) for a auxiliary trajectory $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ that coincides with the actual one $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ and that the free field $\boldsymbol{f}^h_{i,0}$, as it evolves independently of the charges, equals the incoming free field evolved from $t = -\infty$ to $t = 0$.

Hence, in the general case for any $\lambda \in [0, 1]$, where also advanced Liénard-Wiechert fields may occur, one would expect the Maxwell field to take the form

$$\boldsymbol{f}_{i,t} = \lambda \boldsymbol{f}^-_{i,t}[\boldsymbol{q}_i, \boldsymbol{p}_i] + (1 - \lambda)\boldsymbol{f}^+_{i,t}[\boldsymbol{q}_i, \boldsymbol{p}_i] + \boldsymbol{f}^h_{i,t}. \tag{2.23}$$

Any compatibility condition, such as the Maxwell constraint (1.3), (C1), and (C2), is now naturally fulfilled for all times t. But this comes at a high price. By (2.23), the fields $\boldsymbol{f}_{i,0}$ at time $t = 0$ depend on the entire history of the charge trajectories which consequently means letting go of the initial value formulation of classical electrodynamics.

In view of the above, however, such a step seems well grounded. In Section 2.2, it was already indicated when insisting on the merely mathematical property of smoothness of solutions. But there, one might even have been tempted to accept potential kinks in the charge trajectories, say, as long as they decay fast enough. However, the discussion above and in Section 2.3 shows that there is also a physical reason why the initial value formulation is questionable, namely the fact that at each time instant the entire history of a charge trajectory is already encoded in the spatial dependence of its field. Therefore, when entertaining the thought that charges are incoming from the remote past, the form of the Maxwell fields is already presupposed by (2.23) and the space of potential solutions $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i \in \mathcal{N}}$ of the Maxwell-Lorentz system (1.2)-(1.4) should consequently be restricted to solutions having Maxwell fields $\boldsymbol{f}_{i,t}$ that fulfill (2.23).

Such a restriction is easily implemented in the fundamental equations of motion (1.2)-(1.4). It simply means replacing the Maxwell fields on the right-hand side of (1.5) with the explicit form given in (2.23). This makes the Maxwell equations and constraints (1.2)-(1.3) redundant and turns the coupled system of the ordinary differential equations (1.4) and partial differential equations (1.2)-(1.3), only consisting of terms that are all evaluated at the same time instant t, into the following system of ordinary differential equations that involve terms depending on advanced or delayed times $t^{\pm}$ as given in (1.9):

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix}\boldsymbol{q}_{i,t}\\ \boldsymbol{p}_{i,t}\end{pmatrix} &= \begin{pmatrix}\boldsymbol{v}_{i,t} = \boldsymbol{v}(\boldsymbol{p}_{i,t})\\ \sum_{j\neq i}\boldsymbol{L}_{ij,t}\end{pmatrix}, \\
\boldsymbol{L}_{ij,t} &:= \boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}) + \boldsymbol{v}_{i,t}\wedge \boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t}), \\
\boldsymbol{f}_{i,t} &= (\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t}) \\
&= \lambda \boldsymbol{f}^{-}_{i,t}[\boldsymbol{q}_i, \boldsymbol{p}_i] + (1-\lambda)\boldsymbol{f}^{+}_{i,t}[\boldsymbol{q}_i, \boldsymbol{p}_i] + \boldsymbol{f}^{h}_{i,t}.
\end{aligned} \tag{2.24}$$

It is interesting to note that by virtue of (2.2) the free fields $\boldsymbol{f}^{h}_{i,t}$, when prescribed in the remote past, are forgotten should they have some spatial decay at spatial infinity [8, 29].

In this case, for $\lambda = 1/2$ and no initial free fields the system of equations (2.24) is equivalent to the Fokker-Schwarzschild-Tetrode equations [20, 43, 39] as used in Wheeler's and Feynman's investigation of classical radiation reaction [46, 47]. They can be derived from a simple action principle [20, 47], and furthermore, allow a derivation of Dirac's radiation damping term $\boldsymbol{L}^{LAD}_{ii,t}$ without the need of a mass renormalization procedure [46, 9].

Moreover, for $\lambda = 1$ and no initial free fields, the resulting equations are equivalent to the Synge equations [42].

The nature of these equations, involving a priori unbounded state-dependent delays $t^{\pm}$, cf. (4.7), in the definition of the Liénard-Wiechert fields (4.6), renders a general classification of solutions very difficult. In mathematics, this problem is known as the *electrodynamic N-body problem*.

Chapter 3

Notation and Definitions

Before we can state our main results we need to make precise the meaning of solutions to the Maxwell equations (1.2)-(1.3), Lorentz equations (1.4), and the coupled system of Maxwell-Lorentz equations (1.2)-(1.3) and (1.4). We also use this opportunity to introduce the notation, which is mostly the standard one with slight adaptation to our setting.

3.1 Distribution spaces

Throughout this work we denote the natural numbers excluding 0 by $\mathbb{N}$ and write $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$. We consider as test function space the space of smooth functions with compact support $\mathcal{D} := C_c^\infty(\mathbb{R}^3, \mathbb{R})$ and as distribution space the corresponding duals $\overline{\mathcal{D}}^r$, $r \in \mathbb{N}$, where we will generally use only $r = 1$ and $r = 3$. The dual space $\overline{\mathcal{D}}^r$ is the space of linear and continuous maps $l : \mathcal{D} \to \mathbb{R}^r$, where a linear map $l : \mathcal{D} \to \mathbb{R}^r$ is said to be continuous if for all sequences $(\rho^{(n)})_{n\in\mathbb{N}}$ in $\mathcal{D}$ such that there is a compact K with $\operatorname{supp}\rho^{(n)} \subseteq K$ for all $n \in \mathbb{N}$ it holds:

$$\forall \alpha \in \mathbb{N}_0^3 : \lim_{n\to\infty} \sup_{\boldsymbol{y}\in\mathbb{R}^3} |D^\alpha_{\boldsymbol{y}} \rho^{(n)}(\boldsymbol{y})| = 0 \qquad \Rightarrow \qquad \lim_{n\to\infty} l(\rho^{(n)}) = 0. \tag{3.1}$$

Here, for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{N}_0^3$, $D^\alpha_{\boldsymbol{x}}$ denotes the multi-index derivative w.r.t. $\boldsymbol{x} \in \mathbb{R}^3$, i.e., $D^\alpha_{\boldsymbol{x}} := \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}$, where, if unambiguous, we drop the subscript $\boldsymbol{x}$ and mean the derivative w.r.t. the argument of the function. We call a sequence $(\rho^{(n)})_{n\in\mathbb{N}}$ with such properties a *null sequence* in $\mathcal{D}$. Elements $\boldsymbol{l} \in \overline{\mathcal{D}}^3$, i.e. for $r = 3$ or tuples of such elements are expressed in bold letters. We write $\rho_{\boldsymbol{x}} \equiv \rho(\boldsymbol{x} - \cdot)$ to denote the function $\boldsymbol{y} \mapsto \rho(\boldsymbol{x} - \boldsymbol{y})$ on $\mathbb{R}^3$. Moreover, $\rho_{(\cdot)}$ denotes the function $\rho_{(\cdot)} : \boldsymbol{x} \mapsto \rho_{\boldsymbol{x}} \in \mathcal{D}$.

The Dirac delta distribution δ in (1.2)-(1.3) is to be understood as the distribution $\delta \in \overline{\mathcal{D}}^1$ acting as $\delta(\rho) := \rho(0)$ for all test functions $\rho \in \mathcal{D}$. Furthermore, for $t \geq 0$ we will often use another Dirac delta distribution denoted by $\delta(t - |\cdot|) \in \overline{\mathcal{D}}^1$ and acting as $\delta(t - |\cdot|)(\rho) := \int_{\partial B_t(0)} d\sigma(x)\, \rho(\boldsymbol{x})$ for all $\rho \in \mathcal{D}$, where for $\boldsymbol{q} \in \mathbb{R}^3$, $B_t(\boldsymbol{q}) := \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x} - \boldsymbol{q}| \leq t\}$ denotes the closed euclidean ball of radius t around $\boldsymbol{q}$, $\partial B_t(\boldsymbol{q})$ its boundary, and $d\sigma(x)$ the corresponding surface measure. Note that we denote the euclidean norm in $\mathbb{R}^3$ and $\mathbb{R}^6$ by $|\cdot|$, and the norms of the L^p spaces by $\|\cdot\|_p$ for $0 < p \leq \infty$.

Moreover, if for a $\overline{g} \in \overline{\mathcal{D}}^r$ there is a locally integrable function $g \in L^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^r)$ such that $\overline{g}(\rho) = \int d^3y\, g(\boldsymbol{y})\rho(\boldsymbol{y})$ for all $\rho \in \mathcal{D}$ we slightly abuse the notation and identify $\overline{g} \equiv g$, i.e., the distribution in $\overline{\mathcal{D}}$ and the corresponding element in $L^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^r)$. In this case $g(\rho_{\boldsymbol{x}}) \equiv$

$\int d^3y\, g(\boldsymbol{y})\rho(\boldsymbol{x}-\boldsymbol{y}) \equiv g * \rho(\boldsymbol{x})$, where $*$ denotes the common short-hand notation for the convolution. If there is an open set D such that the restriction of $g \in L^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^r)$ to D has a $n \in \mathbb{N}_0$-times continuously differentiable representative we write $g \in C^n(D, \mathbb{R}^r)$, where $C^n(D, \mathbb{R}^r)$ denotes the space of n-times continuously differentiable functions from D to $\mathbb{R}^r$. Finally, we say that $g \in \overline{\mathcal{D}}^r$ can be evaluated at $\boldsymbol{x} \in \mathbb{R}^3$ if there is an open D with $\boldsymbol{x} \in D$ and $g \in C^0(D, \mathbb{R}^r)$. In this case, we abbreviate the evaluation $g(\boldsymbol{x}) = \lim_{n\to\infty} g(\rho^{(n)}_{\boldsymbol{x}})$, where $\rho^{(n)}_{\boldsymbol{x}} := n^3\rho(n(\boldsymbol{x}-\cdot))$ with $\rho \in C^\infty_c(D, \mathbb{R}^r)$ and $\|\rho\|_1 = 1$.

3.2 Electromagnetic fields

The electromagnetic fields $\boldsymbol{f}$ are tuples $\boldsymbol{f} = (\boldsymbol{E}, \boldsymbol{B})$ of electric fields $\boldsymbol{E} \in \overline{\mathcal{D}}^3$ and magnetic fields $\boldsymbol{B} \in \overline{\mathcal{D}}^3$, and hence, take values in $\mathcal{F} := \overline{\mathcal{D}}^3 \times \overline{\mathcal{D}}^3$. We distinguish between electromagnetic fields $\boldsymbol{f} \in \mathcal{F}_{\text{hom}}$, which are defined as those $\boldsymbol{f} \in \mathcal{F}$ fulfilling the homogeneous Maxwell constraints, i.e., for all $\rho \in \mathcal{D}, \boldsymbol{x} \in \mathbb{R}^3$

$$\begin{pmatrix} \nabla_{\boldsymbol{x}}\cdot & 0 \\ 0 & \nabla_{\boldsymbol{x}}\cdot \end{pmatrix} \boldsymbol{f}(\rho_{\boldsymbol{x}}) = \begin{pmatrix} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{E}(\rho_{\boldsymbol{x}}) \\ \nabla_{\boldsymbol{x}} \cdot \boldsymbol{B}(\rho_{\boldsymbol{x}}) \end{pmatrix} = 0, \tag{3.2}$$

cf. [15], and $\boldsymbol{f} \in \mathcal{F}_{\boldsymbol{q}}$, $\boldsymbol{q} \in \mathbb{R}^3$ denoting those $\boldsymbol{f} \in \mathcal{F}$ fulfilling the inhomogeneous Maxwell constraints, i.e., for all $\rho \in \mathcal{D}, \boldsymbol{x} \in \mathbb{R}^3$

$$\begin{pmatrix} \nabla_{\boldsymbol{x}}\cdot & 0 \\ 0 & \nabla_{\boldsymbol{x}}\cdot \end{pmatrix} \boldsymbol{f}(\rho_{\boldsymbol{x}}) = \begin{pmatrix} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{E}(\rho_{\boldsymbol{x}}) \\ \nabla_{\boldsymbol{x}} \cdot \boldsymbol{B}(\rho_{\boldsymbol{x}}) \end{pmatrix} = \begin{pmatrix} 4\pi\rho(\boldsymbol{x}-\boldsymbol{q}) \\ 0 \end{pmatrix}, \tag{3.3}$$

cf. [6]. Note that checking the equality in (3.2) and (3.3) for all $\rho \in \mathcal{D}$ and only one $\boldsymbol{x} \in \mathbb{R}^3$ is already necessary and sufficient. However, leaving open the freedom $\boldsymbol{x} \in \mathbb{R}^3$ is sometimes helpful for a physical interpretation as these equations then describe the situation of a charge density ρ attached to the charge at position $\boldsymbol{q}$; see, e.g. [7, 11, 29].

3.3 Solution sense

The sense in which we consider solutions to the Maxwell and Lorentz equations as well as the coupled Maxwell-Lorentz system is given by the following three definitions. Let Λ always denote an interval on $\mathbb{R}$ containing 0, possibly $\Lambda = \mathbb{R}$. Also for maps on Λ we will sometimes adopt the $\cdot$ notation, e.g. $g_{(\cdot)} : \Lambda \to \overline{\mathcal{D}}, t \mapsto g_t$. Moreover, we will always give priority to the evaluation of the distribution g_t at a specific test function $\rho \in \mathcal{D}$ over operations acting on the argument t, in particular, the partial derivative $\partial_t \boldsymbol{g}_t(\rho) \equiv \partial_t(\boldsymbol{g}_t(\rho))$. Moreover, the derivatives $\frac{d}{dt}$ and ∂_t at a possible boundary of the one-dimensional domain of t are to be interpreted as the corresponding left- or right-hand side derivatives and we often use the short-hand notation $\frac{d}{dt} g_t = \dot{g}_t$.

Definition 3.3.1 (Charge and field trajectories)**.**

(i) Let $n \in \mathbb{N}$. Any tuple $(\boldsymbol{q}, \boldsymbol{p}) : t \mapsto (\boldsymbol{q}_t, \boldsymbol{p}_t)$ for $\boldsymbol{q} \in C^n(\Lambda, \mathbb{R}^3)$ and $\boldsymbol{p} \in C^{n-1}(\Lambda, \mathbb{R}^3)$ fulfilling $\dot{\boldsymbol{q}}_t = \boldsymbol{v}(\boldsymbol{p}_t)$ for all $t \in \Lambda$ is called a <u>charge trajectory on Λ</u>. The space of all such charge trajectories is denoted by $\mathcal{T}^n(\Lambda)$.

(ii) *A charge trajectory* $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^1(\Lambda)$ *such that there is a constant* v_{max} *for which*

$$\sup_{t\in\Lambda} |\boldsymbol{v}(\boldsymbol{p}_t)| \le v_{max} < 1 \tag{3.4}$$

holds is called <u>*strictly time-like*</u> *on* Λ.

(iii) *Let* $n \in \mathbb{N}_0$. *A map* $\boldsymbol{f} : \Lambda \to \mathcal{F}, t \mapsto \boldsymbol{f}_t$ *such that for all* $\rho \in \mathcal{D}$ *the map* $t \mapsto \boldsymbol{f}_t(\rho)$ *is in* $C^n(\Lambda, \mathbb{R}^6)$ *is called a* <u>*field trajectory on* Λ</u>. *The space of all such field trajectories is denoted by* $\mathcal{F}^n(\Lambda)$.

Prescribed initial positions, momenta and fields will be denoted by $\boldsymbol{q}_0$, $\boldsymbol{p}_0$ and $\boldsymbol{f}_0$, to be distinguished from the notation $(\boldsymbol{q}_t,\boldsymbol{p}_t)\big|_{t=0}$ and $\boldsymbol{f}_t\big|_{t=0}$ denoting the evaluation of the charge trajectory $(\boldsymbol{q},\boldsymbol{p})$ and the field trajectory $\boldsymbol{f}$ at time $t = 0$. In addition, the latter may be abbreviated by $\boldsymbol{q}_{t=0}$, $\boldsymbol{p}_{t=0}$ and $\boldsymbol{f}_{t=0}$ to keep the notation slim.

With regard to fields we will encounter the two types of sets

$$S_{\boldsymbol{q}} := \mathbb{R}^3 \backslash \{\boldsymbol{q}\} \quad \forall \boldsymbol{q} \in \mathbb{R}^3 \qquad \text{and} \qquad D^\Lambda_{\boldsymbol{q}} := \{(t,\boldsymbol{x}) \,|\, t \in \Lambda, \boldsymbol{x} \in S_{\boldsymbol{q}_t}\} \quad \forall (\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^0(\mathbb{R}), \tag{3.5}$$

where in case of $\Lambda = \mathbb{R}$ we abbreviate $D^{\mathbb{R}}_{\boldsymbol{q}}$ by $D_{\boldsymbol{q}}$.

For a system of $N \in \mathbb{N}$ charges indexed by $\mathcal{N} := \{1, 2, \ldots, N\}$ the solutions to the Lorentz equations (1.4) we will consider are of the following kind.

Definition 3.3.2 (Lorentz solutions). *Given a tuple* $(\boldsymbol{f}_i)_{i\in\mathcal{N}}$ *of field trajectories* $\boldsymbol{f}_i \in \mathcal{F}^0(\Lambda)$, *we call a tuple* $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{i\in\mathcal{N}}$ *of charge trajectories* $t \mapsto (\boldsymbol{q}_{i,t},\boldsymbol{p}_{i,t})$ *in* $\mathcal{T}^2(\Lambda)$ *a* <u>*solution to the Lorentz equations for* $(\boldsymbol{f}_i)_{i\in\mathcal{N}}$ *on* Λ</u> *if and only if:*

(i) *For all* $t \in \Lambda, i,j \in \mathcal{N}$, $i \neq j$, *the distributions* $\boldsymbol{f}_{j,t}$ *can be evaluated at* $\boldsymbol{q}_{i,t}$ *and* $t \mapsto \boldsymbol{f}_{j,t}(\boldsymbol{q}_{i,t})$ *are in* $C^0(\Lambda, \mathbb{R}^3)$.

(ii) *For all* $t \in \Lambda$, *the tuple* $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{i\in\mathcal{N}}$ *is a solution to* (1.4), *where* $(\boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}), \boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t})) = \boldsymbol{f}_{j,t}(\boldsymbol{q}_{i,t})$.

We say that a Lorentz solution on Λ *has initial value* $(\boldsymbol{q}_0,\boldsymbol{p}_0)$ *if* $(\boldsymbol{q}_t,\boldsymbol{p}_t)|_{t=0} = (\boldsymbol{q}_0,\boldsymbol{p}_0)$.

Similarly, we define solutions to the Maxwell equations (1.2) as follows.

Definition 3.3.3 (Maxwell solutions).

(i) *A field trajectory* $\boldsymbol{f} = (\boldsymbol{E},\boldsymbol{B}) \in \mathcal{F}^1(\Lambda)$ *fulfilling for all* $\rho \in \mathcal{D}, t \in \Lambda, \boldsymbol{x} \in \mathbb{R}^3$

$$\partial_t \boldsymbol{f}_t(\rho_{\boldsymbol{x}}) = \begin{pmatrix} 0 & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & 0 \end{pmatrix} \boldsymbol{f}_t(\rho_{\boldsymbol{x}}) \equiv \begin{pmatrix} \nabla_{\boldsymbol{x}} \wedge \boldsymbol{B}_t(\rho_{\boldsymbol{x}}) \\ -\nabla_{\boldsymbol{x}} \wedge \boldsymbol{E}_t(\rho_{\boldsymbol{x}}) \end{pmatrix} \tag{3.6}$$

is called a <u>*homogeneous Maxwell solution on* Λ</u>.

(ii) *Given a charge trajectory* $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^2(\mathbb{R})$, *a field trajectory* $\boldsymbol{f} \in \mathcal{F}^1(\Lambda)$ *fulfilling for all* $\rho \in \mathcal{D}, t \in \Lambda, \boldsymbol{x} \in \mathbb{R}^3$

$$\partial_t \boldsymbol{f}_t(\rho_{\boldsymbol{x}}) = \begin{pmatrix} 0 & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & 0 \end{pmatrix} \boldsymbol{f}_t(\rho_{\boldsymbol{x}}) + \begin{pmatrix} -4\pi\boldsymbol{v}(\boldsymbol{p}_t)\rho(\boldsymbol{x}-\boldsymbol{q}_t) \\ 0 \end{pmatrix} \tag{3.7}$$

is called a <u>*Maxwell solution for* $(\boldsymbol{q},\boldsymbol{p})$ *on* Λ</u>.

We say that a Maxwell solution has initial value $\boldsymbol{f}_0$ *if* $(\boldsymbol{f}_t)|_{t=0} = \boldsymbol{f}_0$.

By a straight-forward computation it can be seen that the Maxwell constraints (3.2) and (3.3) are preserved by the Maxwell evolutions (3.6) and (3.7), respectively [7]. More precisely, if $\boldsymbol{f}$ is a homogeneous Maxwell solution on Λ, $\boldsymbol{f}_{t=0} \in \mathcal{F}_{\text{hom}} \Leftrightarrow \forall t \in \Lambda : \boldsymbol{f}_t \in \mathcal{F}_{\text{hom}}$, and similarly, if $\boldsymbol{f}$ is a Maxwell solution for $(\boldsymbol{q}, \boldsymbol{p})$ on Λ, $\boldsymbol{f}_{t=0} \in \mathcal{F}_{\boldsymbol{q}_{t=0}} \Leftrightarrow \forall t \in \Lambda : \boldsymbol{f}_t \in \mathcal{F}_{\boldsymbol{q}_t}$; see, e.g. [7, 11, 29], and Lemma A.2.1 (Maxwell constraints).

Finally, the solution sense of the coupled system of Maxwell-Lorentz equations (1.4) and (1.2) considered here is given by the following definition.

Definition 3.3.4 (Maxwell-Lorentz solutions)**.** *We call a tuple* $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i\in\mathcal{N}}$ *of charge trajectories* $(\boldsymbol{q}_i, \boldsymbol{p}_i) \in \mathcal{T}^2(\Lambda)$ *and field trajectories* $\boldsymbol{f}_i \in \mathcal{F}^1(\Lambda)$ *a <u>Maxwell-Lorentz solution on Λ</u> if and only if:*

(i) Given $(\boldsymbol{f}_i)_{i\in\mathcal{N}}$*, the tuple* $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{i\in\mathcal{N}}$ *is a Lorentz solution for* $(\boldsymbol{f}_i)_{i\in\mathcal{N}}$ *on* Λ.

(ii) Given $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{i\in\mathcal{N}}$*, each* $\boldsymbol{f}_i$ *is a Maxwell solution for* $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ *on* Λ.

We say that a Maxwell-Lorentz solution has initial value $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ *if* $(\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{f}_{i,t})|_{t=0} = (\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})$ *for all* $i \in \mathcal{N}$.

It should be noted that though this is a canonical way to make sense out of equations of motion of the Maxwell-Lorentz system (1.2)-(1.4) it is not the most general one. For example, one could regard the Lorentz equation (1.4) in integral form, in which case the continuity assumption $(\boldsymbol{q}_{i,\cdot}, \boldsymbol{p}_{i,\cdot}) \in \mathcal{T}^2(\Lambda)$ as well as the point-wise evaluation $\boldsymbol{f}_{j,t}(\boldsymbol{q}_{i,t})$ of the fields assumption can be weakened. We will briefly come back to this point in Lemma 4.2.1 (Properties of Liénard-Wiechert fields) when discussing the precise regularity of generic Maxwell solutions, however, a weaker sense of solutions than Definitions 3.3.2-3.3.4 is not the focus of this work.

Chapter 4

Main Results

This chapter contains our main theorems. We start by presenting the no-go result for the Maxwell-Lorentz initial value problem in Section 4.1. For a understanding, we present the properties of the Maxwell fields in Section 4.2. With this preparation we may turn to Section 4.3 where our existence and uniqueness result is discussed.

4.1 No-go result for the Maxwell-Lorentz system

It may now seem natural to approach existence and uniqueness of the Maxwell-Lorentz solutions in terms of the initial value problem with initial data specified, e.g. at the equal time hypersurface $\{t\} \times \mathbb{R}^3$: Given an initial value $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ of initial positions and momenta $\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0} \in \mathbb{R}^3$ and initial fields $\boldsymbol{f}_{i,0} \in \mathcal{F}_{\boldsymbol{q}_{i,0}}$, we would say that the Maxwell-Lorentz initial value problem on Λ is well-posed if and only if there exists a unique Maxwell-Lorentz solution $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i\in\mathcal{N}}$ on Λ such that $(\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{f}_{i,t})\big|_{t=0} = (\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})$ for all $i \in \mathcal{N}$. The question to address is then which regularity assumptions on the initial values would be needed in order to guarantee well-posedness in terms of Definition 3.3.4 (Maxwell-Lorentz solutions). As explained in the introduction, an obvious problem that has to be avoided is that of collisions of charges since all Maxwell solutions are singular at the positions of their generating charges, as can be seen from Lemma 4.2.1 (Properties of Liénard-Wiechert fields) below; a problem that is closely related to the N-body problem of Newtonian gravitation. But, even assuming that no collisions occur and that the initial fields are as regular as we would wish for, there is a more fundamental problem which already appears for $N = 2$ charges and that will prevent well-posedness in general. This is the content of our first main result:

Theorem 4.1.1 (No-go). *Let $N = 2$. Assume there is a Maxwell-Lorentz solution*

$$(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i, \tilde{\boldsymbol{f}}_i)_{i=1,2} \tag{4.1}$$

on $[t_1, t_2]$ for times $t_1 < 0 < t_2$ having the initial value

$$(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0}, \tilde{\boldsymbol{f}}_{i,0}) := (\tilde{\boldsymbol{q}}_{i,t}, \tilde{\boldsymbol{p}}_{i,t}, \tilde{\boldsymbol{f}}_{i,t})\big|_{t=0}, \qquad i = 1, 2 \tag{4.2}$$

such that

$$\tilde{\boldsymbol{q}}_{1,t_1} \in \partial J^-(0, \tilde{\boldsymbol{q}}_{2,0}), \qquad \tilde{\boldsymbol{q}}_{2,t_2} \in \partial J^+(0, \tilde{\boldsymbol{q}}_{1,0}), \tag{4.3}$$

and, that $(\tilde{\boldsymbol{q}}_2, \tilde{\boldsymbol{p}}_2)$ is the unique Lorentz solution for $\tilde{\boldsymbol{f}}_1$ with $(\tilde{\boldsymbol{q}}_{2,t}, \tilde{\boldsymbol{p}}_{2,t})\big|_{t=0} = (\tilde{\boldsymbol{q}}_{2,0}, \tilde{\boldsymbol{p}}_{2,0})$ on $[t_1, t_2]$.

Furthermore, let $\boldsymbol{\epsilon} \in \mathbb{R}^3$ *and* $\boldsymbol{\delta} \in \mathcal{F}_{\text{hom}}$*. Then, in the sense of Definition 3.3.4 (Maxwell-Lorentz solutions), there is no Maxwell-Lorentz solution* $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i=1,2}$ *on* $[0, t_2]$ *such that*

$$(\boldsymbol{q}_{1,t}, \boldsymbol{p}_{1,t}, \boldsymbol{f}_{1,t})\big|_{t=0} = (\tilde{\boldsymbol{q}}_{1,0}, \tilde{\boldsymbol{p}}_{1,0} + \boldsymbol{\epsilon}, \tilde{\boldsymbol{f}}_{1,0}) \tag{4.4}$$

$$(\boldsymbol{q}_{2,t}, \boldsymbol{p}_{2,t}, \boldsymbol{f}_{2,t})\big|_{t=0} = (\tilde{\boldsymbol{q}}_{2,0}, \tilde{\boldsymbol{p}}_{2,0}, \tilde{\boldsymbol{f}}_{2,0} + \boldsymbol{\delta}) \tag{4.5}$$

unless $\boldsymbol{\epsilon} = 0$ *and* $\boldsymbol{L}(\tilde{\boldsymbol{q}}_{1,0}, \tilde{\boldsymbol{p}}_{1,0}, \boldsymbol{\delta}) = 0$*, cf.* (1.5).

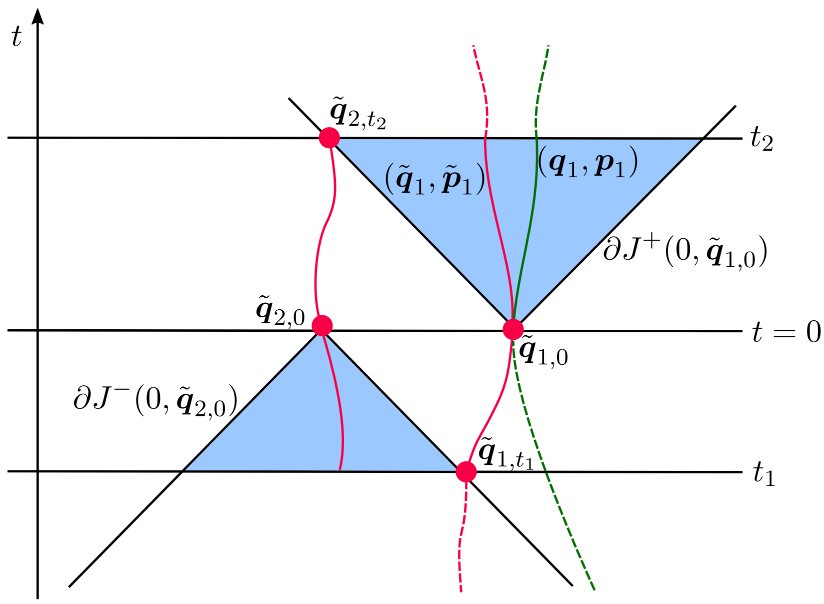

Figure 4.1: Illustration of the setting given in Theorem 4.1.1 (No-go). The left trajectory piece is the unique Lorentz solution of charge 2, the right, red trajectory is the solution of charge 1 extended to whole $\mathbb{R}$ and the right, green trajectory represents the solution of charge 1 to the modified initial value.

See Figure 4.1 for the illustration of the setting in this theorem. In other words, an initial value problem for the Maxwell-Lorentz equations in the sense as discussed above is not well-posed. As we shall see, this behavior is grounded in the fact that for any time t the information about the history of the charges, i.e., $s \mapsto (\boldsymbol{q}_{i,s}, \boldsymbol{p}_{i,s})$ for $s \leq t$, is encoded in the spatial degrees of freedom of the fields $\boldsymbol{f}_{i,t}$ of a Maxwell-Lorentz solution $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i=1,2}$, such as the one in Theorem 4.1.1 (No-go). An arbitrary change of the initial value at $t = 0$ such as (4.4)-(4.5) simply renders the initial momenta or fields incompatible with the history of the charge trajectories stored in the initial fields. As we shall see in the proof of Theorem 4.1.1 (No-go), equation (4.4)-(4.5) causes the fields to become irregular on the boundary of the light-cone $J^+(0, \tilde{\boldsymbol{q}}_{1,0})$ – it turns out that either $\boldsymbol{f}_1$ cannot be evaluated in the point-wise sense or is discontinuous at $\partial J^+(0, \tilde{\boldsymbol{q}}_{1,0})$. In consequence, the dynamics in the sense of Definition 3.3.4 (Maxwell-Lorentz solutions) ceases to exist once the charge trajectory $(\boldsymbol{q}_2, \boldsymbol{p}_2)$ intersects $\partial J^+(0, \tilde{\boldsymbol{q}}_{1,0})$, which is guaranteed thanks to the uniqueness assumption of the Lorentz solution.

Remark 4.1.1. *(i) The uniqueness of the Lorentz equation is a very mild assumption. In the setting of Theorem 4.1.1 (No-go) it holds as soon as the* $\boldsymbol{f}_{i,t}(\boldsymbol{x})$ *are continuous in* t *and Lipschitz continuous in* $\boldsymbol{x}$ *on* $J^+(0, \boldsymbol{q}_{j,0}) \cap \{(t, \boldsymbol{x}) \,|\, t_1 \leq t \leq t_2, x \in \mathbb{R}^3\}$ *for* $j \neq i$*.*

If $(\tilde{\boldsymbol{q}}_2, \tilde{\boldsymbol{p}}_2)$ is not assumed to be the unique Lorentz solution, and thus, the time t_2 is not necessarily the time where $(\boldsymbol{q}_2, \boldsymbol{p}_2)$ hits the light-cone boundary $\partial J^+(0, \tilde{\boldsymbol{q}}_{1,0})$, there are only two possible scenarios: either the dynamics of charge 2, given the modified initial value, stops at some other finite time when it hits the light-cone boundary $\partial J^+(0, \tilde{\boldsymbol{q}}_{1,0})$ so that the dynamics in the sense of Definition 3.3.4 (Maxwell-Lorentz solutions) ceases to exist, or, the trajectory must show "run away" behavior, i.e., asymptotically approach the speed of light. Mathematically, the latter may lead to a solution up to arbitrarily large times without hitting the light-cone boundary. Physically, however, such solutions are also questionable.

(ii) If we allowed the acceleration of charge 2 to be discontinuous at time t_2, the field $\boldsymbol{f}_{2,t}$ of charge 2 would show a corresponding discontinuity located at the light-cone boundary $\partial J^+(t_2, \boldsymbol{q}_{2,t_2})$. If charge 1 does not show runaway behavior it is bound to hit this discontinuous front at a finite time, therefore experiences a jump in acceleration and produces a discontinuous front in its field, as well, and so a whole network of discontinuities would build up; see Figure 4.2 for illustration.

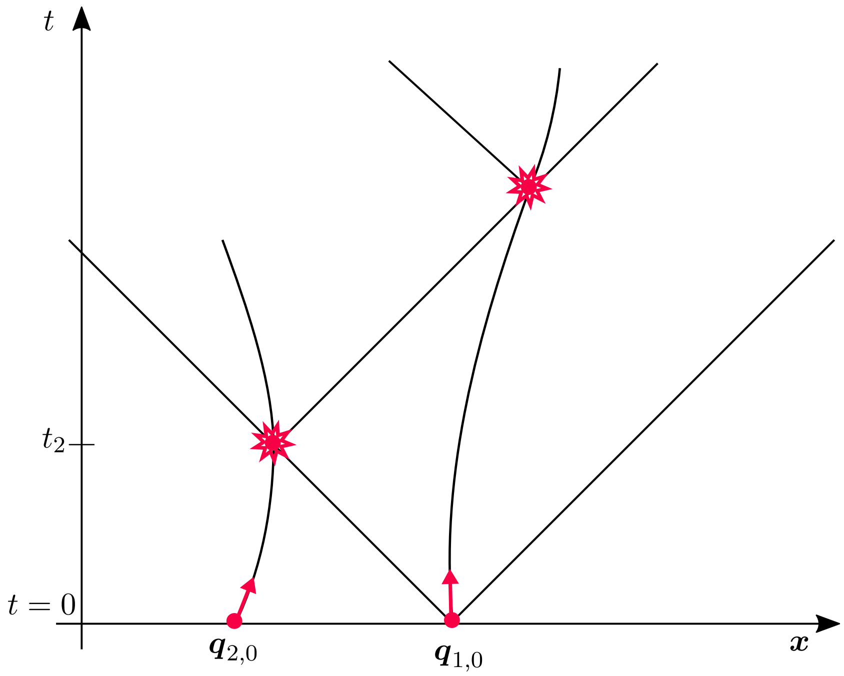

Figure 4.2: Illustration of a network of discontinuous fields and accelerations that builds up whenever the initial field of charge 2 does not fit the initial charge acceleration of charge 1.

4.2 Properties of the Maxwell fields

For the proof of Theorem 4.1.1 (No-go) a detailed analysis of the solution space of the Maxwell equations is necessary. This is the content of the following Definitions and Lemmata.

The explicit formula for general Maxwell solutions are presented in Theorem 4.2.1 (Explicit Maxwell solutions) which is our second main result. It can be seen as a generalization of the

Liénard-Wiechert formulas to the case of prescribed initial fields. For the sake of completeness we start by calling the definition of the Liénard-Wiechert fields (1.8) next.

Definition 4.2.1 (Liénard-Wiechert fields)**.** *Given a strictly time-like charge trajectory* $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{T}^2(\mathbb{R})$*, we define the so-called advanced and retarded Liénard-Wiechert fields* $\boldsymbol{f}^+[\boldsymbol{q}, \boldsymbol{p}] \equiv (\boldsymbol{e}^+, \boldsymbol{b}^-)$ *and* $\boldsymbol{f}^-[\boldsymbol{q}, \boldsymbol{p}] \equiv (\boldsymbol{e}^-, \boldsymbol{b}^-)$ *for* $t \in \mathbb{R}$*, respectively, by point-wise evaluation*

$$\begin{aligned} \boldsymbol{e}_t^\pm(\boldsymbol{x}) &:= \left[\frac{(\boldsymbol{n} \pm \boldsymbol{v})(1 - \boldsymbol{v}^2)}{|\boldsymbol{x} - \boldsymbol{q}|^2 (1 \pm \boldsymbol{n} \cdot \boldsymbol{v})^3} + \frac{\boldsymbol{n} \wedge [(\boldsymbol{n} \pm \boldsymbol{v}) \wedge \boldsymbol{a}]}{|\boldsymbol{x} - \boldsymbol{q}|(1 \pm \boldsymbol{n} \cdot \boldsymbol{v})^3} \right]^\pm, \\ \boldsymbol{b}_t^\pm(\boldsymbol{x}) &:= \mp \boldsymbol{n}^\pm \wedge \boldsymbol{e}_t^\pm(\boldsymbol{x}) \end{aligned} \tag{4.6}$$

for all $(t, \boldsymbol{x}) \in D_{\boldsymbol{q}}^{\mathbb{R}\pm}$*, where we have used the abbreviations*

$$\begin{aligned} \boldsymbol{q}^\pm &:= \boldsymbol{q}_{t^\pm}, & \boldsymbol{v}^\pm &:= \boldsymbol{v}(\boldsymbol{p}_{t^\pm}), & \boldsymbol{a}^\pm &:= \tfrac{d}{dt}\boldsymbol{v}(\boldsymbol{p}_t)|_{t=t^\pm}, \\ \boldsymbol{n}^\pm &:= \tfrac{\boldsymbol{x} - \boldsymbol{q}^\pm}{|\boldsymbol{x} - \boldsymbol{q}^\pm|}, & t^\pm &= t \pm |\boldsymbol{x} - \boldsymbol{q}^\pm|, \end{aligned} \tag{4.7}$$

see, e.g. [41, 37, 26].

As remarked in the introduction, $t^\pm$ in (4.7) is only defined implicitly. The two solutions t^+ and t^- to this quadratic equation can be interpreted geometrically as the intersection times of the charge trajectory $(\boldsymbol{q}, \boldsymbol{p})$ with the forward and backward light-cone boundaries $\partial J^+(t, \boldsymbol{x})$ and $\partial J^-(t, \boldsymbol{x})$, respectively, which, due to the strictly time-like nature of $(\boldsymbol{q}, \boldsymbol{p})$, are guaranteed to exist; see, e.g. [8]. It is sometimes convenient to make the dependence of $t^\pm$ on $t, \boldsymbol{x}$ and the charge trajectory $(\boldsymbol{q}, \boldsymbol{p})$ explicit by writing $t^\pm \equiv t_{\boldsymbol{q}}^\pm(t, \boldsymbol{x})$. Further, note that $|\boldsymbol{x} - \boldsymbol{q}^\pm| = 0$ if and only if $t = t_{\boldsymbol{q}}^\pm(t, \boldsymbol{x})$ and $\boldsymbol{x} = \boldsymbol{q}_t = \boldsymbol{q}^\pm$, see Lemma 5.1.1 ($t^\pm$).

Lemma 4.2.1 (Properties of Liénard-Wiechert fields)**.** *Let* $n, k \in \mathbb{N}_0$*,* $\alpha \in \mathbb{N}_0^3$*. Given a strictly time-like charge trajectory* $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{T}^{2+n}(\mathbb{R})$ *it holds:*

(i) For all $|\alpha| + k \le n$ *the map* $(t, \boldsymbol{x}) \mapsto D_{\boldsymbol{x}}^\alpha \partial_t^k \boldsymbol{f}_t^\pm[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x})$ *is in* $C^{n-|\alpha|-k}(D_{\boldsymbol{q}}, \mathbb{R}^6)$*, and hence, for all* $t \in \mathbb{R}$*,* $\boldsymbol{f}_t^\pm[\boldsymbol{q}, \boldsymbol{p}] \in \mathcal{F}$*.*

(ii) For all $(t, \boldsymbol{x}) \in D_{\boldsymbol{q}}$ *and* $|\alpha| + k \le n$*, the value* $D_{\boldsymbol{x}}^\alpha \partial_t^k \boldsymbol{f}_t^\pm[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x})$ *is a function of* $\boldsymbol{x}$ *and* $\left(\frac{d}{dt}\right)^l \boldsymbol{q}_t|_{t=t^\pm}$ *for* $l = 0, \ldots, |\alpha| + k + 2$ *only.*

(iii) $\boldsymbol{f}^\pm[\boldsymbol{q}, \boldsymbol{p}]$ *are Maxwell solutions for* $(\boldsymbol{q}, \boldsymbol{p})$ *on* $\mathbb{R}$*, in particular,* $\boldsymbol{f}^\pm[\boldsymbol{q}, \boldsymbol{p}] \in \mathcal{F}^1(\mathbb{R})$*.*

Remark 4.2.1. *It turns out that the advanced and retarded Liénard-Wiechert fields are the unique solutions to the Maxwell equations when specifying any initial fields with sufficient spatial decay at times* $t_0 \to +\infty$ *and* $t_0 \to -\infty$*, respectively. The uniqueness is due to the fact that the Maxwell evolution forgets its initial values, provided they have sufficient spatial decay; cf. [41, 8].*

The property of the Liénard-Wiechert fields stated in Lemma 4.2.1 (ii) allows to extend the definition $\boldsymbol{f}_t^\pm[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x})$ to charge trajectories $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{T}^2(I)$ where I is just an interval of $\mathbb{R}$. This generalization will be convenient and it will be used without further notice. However, in this case, great care has to be taken as the resulting expression $\boldsymbol{f}_t^\pm[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x})$ is only defined for $(t, \boldsymbol{x})$ such that $t_{\boldsymbol{q}}^\pm(t, \boldsymbol{x}) \in I$.

For instance, for $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{T}^{2+n}((-\infty, t^{(0)}])$, the retarded field $\boldsymbol{f}_t^-[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x})$ is well-defined for all $(t, \boldsymbol{x}) \in \mathbb{R}^4 \setminus \mathring{J}(t^{(0)}, \boldsymbol{q}_{t^{(0)}})$ by claim (ii) and by (i) it follows $(t, \boldsymbol{x}) \mapsto \boldsymbol{f}_t^-[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x})$ in

$C^n(\mathbb{R}^4 \setminus \overset{\circ}{J}(t^{(0)}, \boldsymbol{q}_{t^{(0)}}), \mathbb{R}^6)$.

Thanks to the linearity of the Maxwell equations, every Maxwell solution can be represented as a sum of any convex combination of the two special inhomogeneous Maxwell solutions $\boldsymbol{f}^{\pm}[\boldsymbol{q}, \boldsymbol{p}]$ in (4.6) and a homogeneous one $\boldsymbol{f}^h_t$. As it turns out in Lemma 4.2.2 (Homogeneous Maxwell solutions), the homogeneous Maxwell solutions are generated from initial fields as follows:

Definition 4.2.2 (Homogeneous Maxwell evolution)**.** *We define a map $\mathcal{W}_t : \mathcal{F} \times \mathcal{D} \to \mathbb{R}^6$ such that for all $\boldsymbol{f}^h_0 \in \mathcal{F}_{\text{hom}}, \rho \in \mathcal{D}, \boldsymbol{x} \in \mathbb{R}^3$:*

$$\mathcal{W}_t(\boldsymbol{f}^h_0, \rho_{\boldsymbol{x}}) := \begin{cases} \boldsymbol{f}^h_0(\rho_{\boldsymbol{x}}) & , t = 0 \\ \begin{pmatrix} \partial_t & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & \partial_t \end{pmatrix} \frac{1}{4\pi t} \int_{\partial B_1(0)} d\sigma(y) \boldsymbol{f}^h_0(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}) & , t \neq 0 \end{cases}. \tag{4.8}$$

Definition (4.8) gives rise to a linear evolution operator W_t on $\mathcal{F}$ as the following Lemma shows.

Lemma 4.2.2 (Homogeneous Maxwell solutions)**.**

(i) For all $t \in \mathbb{R}$

$$W_t : \mathcal{F}_{\text{hom}} \to \mathcal{F}, \boldsymbol{f}^h_0 \mapsto W_t \boldsymbol{f}^h_0 \qquad \textit{with} \qquad (W_t \boldsymbol{f}^h_0)(\rho_{\boldsymbol{x}}) := \mathcal{W}_t(\boldsymbol{f}^h_0, \rho_{\boldsymbol{x}}) \tag{4.9}$$

for all $\rho \in \mathcal{D}, \boldsymbol{x} \in \mathbb{R}^3$ is a well-defined linear operator.

(ii) For all $\boldsymbol{f}^h_0 \in \mathcal{F}_{\text{hom}}$, $t \mapsto W_t \boldsymbol{f}^h_0$ is in $\mathcal{F}^\infty(\mathbb{R})$.

(iii) Let $\boldsymbol{f}^h_0 \in \mathcal{F}_{\text{hom}}$ and $\boldsymbol{f}^h$ be a homogeneous Maxwell solution on Λ such that $\boldsymbol{f}^h_t\big|_{t=0} = \boldsymbol{f}^h_0$. Then,

$$\boldsymbol{f}^h_t = W_t \boldsymbol{f}^h_0, \qquad \forall\, t \in \mathbb{R}. \tag{4.10}$$

Henceforth, we therefore write $\boldsymbol{f}^h_t := W_t \boldsymbol{f}^h_0$ for all $t \in \mathbb{R}$ to denote the unique homogeneous Maxwell solution for initial value $\boldsymbol{f}^h_0 \in \mathcal{F}_{\text{hom}}$.

(iv) If for $n \in \mathbb{N}_0$, $\boldsymbol{f}^h_0 \in \mathcal{F}_{\text{hom}}$ has a representative $\boldsymbol{f}^h_0 \in C^{1+n}(\mathbb{R}^3, \mathbb{R}^6)$ it follows that at least $(t, \boldsymbol{x}) \mapsto \boldsymbol{f}^h_t(\boldsymbol{x}) \in C^n(\mathbb{R}^4, \mathbb{R}^6)$.

It should be noted that in item (iv) the requirement $\boldsymbol{f}^h_0 \in C^{1+n}(\mathbb{R}^3, \mathbb{R}^6)$, i.e., that we require one more derivative, is only of technical nature as we simply apply formula (4.8) in the proof, which already comprises one derivative.

Collecting these results one infers unique solutions to the inhomogeneous Maxwell solutions for $(\boldsymbol{q}, \boldsymbol{p})$.

Lemma 4.2.3 (Inhomogeneous Maxwell solutions)**.** *Let $n \in \mathbb{N}_0$ and $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{T}^{2+n}(\mathbb{R})$ be strictly time-like and $\boldsymbol{f}_0 \in \mathcal{F}_{\boldsymbol{q}_{t=0}}$. Then:*

(i) There is a Maxwell solution $\boldsymbol{f}$ for $(\boldsymbol{q}, \boldsymbol{p})$ on Λ such that $\boldsymbol{f}_{t=0} = \boldsymbol{f}_0$ in the sense of Definition 3.3.3 (Maxwell solutions).

(ii) Let $\boldsymbol{g}$ be a Maxwell solution for $(\boldsymbol{q}, \boldsymbol{p})$ on Λ with $\boldsymbol{g}_t = \boldsymbol{f}_t$ for $t = 0$. Then $\boldsymbol{g}_t = \boldsymbol{f}_t$ for all $t \in \Lambda$.

(iii) Let $\lambda \in [0,1]$. *Then* $\boldsymbol{f}_0^h = \boldsymbol{f}_0 - (\lambda \boldsymbol{f}^-_{t=0}[\boldsymbol{q},\boldsymbol{p}] + (1-\lambda)\boldsymbol{f}^+_{t=0}[\boldsymbol{q},\boldsymbol{p}])$ *is in* $\mathcal{F}_{\text{hom}}$ *and the unique Maxwell solution for* $(\boldsymbol{q},\boldsymbol{p})$ *on* Λ *such that* $\boldsymbol{f}_{t=0} = \boldsymbol{f}_0$ *is given by*

$$\boldsymbol{f}_t = \lambda \boldsymbol{f}_t^-[\boldsymbol{q},\boldsymbol{p}] + (1-\lambda)\boldsymbol{f}_t^+[\boldsymbol{q},\boldsymbol{p}] + W_t \boldsymbol{f}_0^h. \tag{4.11}$$

Thanks to these Lemmata we can provide a generalization of the Liénard-Wiechert formulas that reveals how the discussed irregular fronts are formed in the Maxwell fields on which our first main result Theorem 4.1.1 (No-go) was based. Contrary to the result in Lemma 4.2.3 (Inhomogeneous Maxwell solutions) (iii) we parameterize the initial homogeneous field $\boldsymbol{f}_0^h$ by an auxiliary trajectory $(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}})$, a notation that is motivated by the fact that in the full Maxwell-Lorentz system the real trajectory $(\boldsymbol{q},\boldsymbol{p})$ is unknown at initial time $t=0$.

Theorem 4.2.1 (Explicit Maxwell solutions). *Let* $n \in \mathbb{N}_0$, $0 \le \lambda \le 1$, $\boldsymbol{f}_0 \in \mathcal{F}_{\boldsymbol{q}_{t=0}}$, *and* $(\boldsymbol{q},\boldsymbol{p}),(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}})$ *strictly time-like charge trajectories in* $\mathcal{T}^{2+n}(\mathbb{R})$ *such that* $\boldsymbol{q}_0 := \boldsymbol{q}_{t=0}$ *and* $\tilde{\boldsymbol{q}}_0 := \tilde{\boldsymbol{q}}_{t=0}$ *coincide, i.e.,* $\boldsymbol{q}_0 = \tilde{\boldsymbol{q}}_0$. *Then the unique Maxwell solution* $\boldsymbol{f}$ *on* $\mathbb{R}$ *for* $(\boldsymbol{q},\boldsymbol{p})$ *with* $\boldsymbol{f}_{t=0} = \boldsymbol{f}_0$ *takes the form*

$$\begin{aligned}
\boldsymbol{f}_t =& \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)} \boldsymbol{f}_t^{-\sigma(t)}[\boldsymbol{q},\boldsymbol{p}] && (4.12)\\
&+ \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)} \lambda \left(\boldsymbol{f}_t^-[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] - \boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] \right) && (4.13)\\
&+ \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)} (1-\lambda) \left(\boldsymbol{f}_t^+[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] - \boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] \right) && (4.14)\\
&+ \mathbb{1}_{B^c_{|t|}(\boldsymbol{q}_0)} \left(\lambda \boldsymbol{f}_t^-[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] + (1-\lambda) \boldsymbol{f}_t^+[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] \right) && (4.15)\\
&+ \boldsymbol{r}^{-\sigma(t)}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)} - \boldsymbol{r}^{-\sigma(t)}_{t,(\tilde{\boldsymbol{q}}_0,\tilde{\boldsymbol{p}}_0)} && (4.16)\\
&+ \boldsymbol{f}_t^h && (4.17)
\end{aligned}$$

for

$$\boldsymbol{f}_t^h = W_t \boldsymbol{f}_0^h \qquad \textit{and} \qquad \boldsymbol{f}_0^h = \boldsymbol{f}_0 - (\lambda \boldsymbol{f}_0^-[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] + (1-\lambda)\boldsymbol{f}_0^+[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}]) \in \mathcal{F}_{\text{hom}} \tag{4.18}$$

and all $t \in \mathbb{R}$, *where we have used the abbreviations*

$$\boldsymbol{r}^{\pm}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)}(\boldsymbol{x}) := \frac{\delta(|t| - |\boldsymbol{x}-\boldsymbol{q}_0|)}{(1 \pm \boldsymbol{n}_0 \cdot \boldsymbol{v}_0)|\boldsymbol{x}-\boldsymbol{q}_0|} \begin{pmatrix} \boldsymbol{n}_0 \pm \boldsymbol{v}_0 \\ -\boldsymbol{n}_0 \wedge \boldsymbol{v}_0 \end{pmatrix}, \qquad \boldsymbol{n}_0 := \frac{\boldsymbol{x}-\boldsymbol{q}_0}{|\boldsymbol{x}-\boldsymbol{q}_0|}, \quad \boldsymbol{v}_0 := \boldsymbol{v}(\boldsymbol{p}_0), \tag{4.19}$$

$\sigma(t) := t/|t|$, $\mathbb{1}_M$ *is the characteristic function taking value* 1 *on the set* $M \subseteq \mathbb{R}^3$ *and value* 0 *on the compliment* $M^c = \mathbb{R}^3 \setminus M$, *and* $\boldsymbol{p}_0 := \boldsymbol{p}_t|_{t=0}, \tilde{\boldsymbol{p}}_0 := \tilde{\boldsymbol{p}}_t|_{t=0}$.

Remark 4.2.2. *(i) It should be noted that* (4.18) *is a very large class of initial fields. As* (1.2)-(1.3) *are linear partial differential equations, any solution can be decomposed in a convex combination of special inhomogeneous solutions and the homogeneous solution and that is why* (4.18) *is a general representation of any solution. Note, that this class is parameterized by* $(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}) \in \mathcal{T}^{2+n}(\mathbb{R})$ *and* $\boldsymbol{f}_0^h \in \mathcal{F}_{\text{hom}}$, *which we refer to as history of a charge and initial inhomogeneous Maxwell field, respectively. One may think of* $(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}})$ *as denoting the history of a charge trajectory that generated the initial field* $\boldsymbol{f}_0$ *and* $(\boldsymbol{q},\boldsymbol{p})$ *as the actual charge trajectory. For example, for* $\lambda = 1$, *the history* $(\tilde{\boldsymbol{q}}_t,\tilde{\boldsymbol{p}}_t)_{t\le 0}$, *by Lemma 4.2.1 (Properties of Liénard-Wiechert fields) (iii), defines the initial field* $\boldsymbol{f}_0$ *completely,* *so that for any actual future trajectory* $(\boldsymbol{q}_t, \boldsymbol{p}_t)_{t\geq 0}$ *of a charge the corresponding field is given by* $\boldsymbol{f}_t$ *for* (4.12)-(4.17). *Mathematically,* $(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}})$ *and* $\boldsymbol{f}_0^h$ *is just a convenient parameterization of the here considered class of initial fields.*

(ii) Theorem 4.2.1 (Explicit Maxwell solutions) will be applied for any initial time $t^{(0)} \geq 0$ *and strictly time-like charge trajectories* $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{T}^{2+n}([t^{(0)}, t^{(1)}])$, $(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}) \in \mathcal{T}^{2+n}((-\infty, t^{(0)}])$.

In this case, the expressions $\boldsymbol{f}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}), \boldsymbol{f}_t^{\pm}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}](\boldsymbol{x})$ *are only defined for* $(t, \boldsymbol{x})$ *such that* $t^{\pm}_{\boldsymbol{q}}(t, \boldsymbol{x}) \in [t^{(0)}, t^{(1)}], t^{\pm}_{\tilde{\boldsymbol{q}}}(t, \boldsymbol{x}) \in (-\infty, t^{(0)}]$. *In Section 5.6 this generalization will be used without further notice.*

(iii) Note that the requirement $\boldsymbol{q}_0 = \tilde{\boldsymbol{q}}_0$ *on* $(\boldsymbol{q}, \boldsymbol{p})$ *and* $(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}})$ *together with* $\boldsymbol{f}_0 \in \mathcal{F}_{\boldsymbol{q}_0}$ *implies* $\boldsymbol{f}_0^h \in \mathcal{F}_{\text{hom}}$ *by Lemma 4.2.3 (Inhomogeneous Maxwell Solutions). If we assumed an homogeneous initial free field* $\boldsymbol{f}_0^h \in \mathcal{F}_{\text{hom}}$ *to be given,* $\boldsymbol{f}_0 \in \mathcal{F}_{\boldsymbol{q}_0}$ *is equivalent to* $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0$*, because*

$$\boldsymbol{f}_0 = \lambda \boldsymbol{f}_0^-[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] + (1-\lambda)\boldsymbol{f}_0^+[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] + \boldsymbol{f}_0^h \tag{4.20}$$

and the requirement

$$\begin{aligned} \nabla \cdot \boldsymbol{f}_0(\rho_{\boldsymbol{x}}) &= \lambda \nabla \cdot \boldsymbol{f}_0^-[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}](\rho_{\boldsymbol{x}}) + (1-\lambda)\nabla \cdot \boldsymbol{f}_0^+[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}](\rho_{\boldsymbol{x}}) \\ &= \begin{pmatrix} 4\pi\rho(\boldsymbol{x} - \tilde{\boldsymbol{q}}_0) \\ 0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 4\pi\rho(\boldsymbol{x} - \boldsymbol{q}_0) \\ 0 \end{pmatrix} \qquad \forall \boldsymbol{x} \in \mathbb{R}^3, \rho \in \mathcal{D}, \end{aligned}$$

is equivalent to $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0$.

Moreover, note that the Maxwell solution $\boldsymbol{f}$ *given by* (4.12)-(4.17) *fulfills Maxwell constraints for all times* $t \in \mathbb{R}$*, which is shown in Lemma A.2.1 (Maxwell constraints).*

The next lemma is a statement about the regularity of the Maxwell field as given in equation (4.12)-(4.17). The formula directly suggests that the regularity of the Liénard-Wiechert fields $\boldsymbol{f}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}], \boldsymbol{f}_t^{\pm}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}]$ is transferred to the whole expression, however, the light-cone boundary, where the distributions in (4.16) and the boundaries of the supports of (4.12)-(4.15) are located, have to be investigated in more detail.

As we have presented in in Section 2.1, condition (C1), (4.19) vanishes if and only if $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0$ and $\tilde{\boldsymbol{p}}_0 = \boldsymbol{p}_0$. For continuity or even higher regularity one needs further additional conditions on $(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}})$ and $(\boldsymbol{q}, \boldsymbol{p})$ as we have argued in (C2) in Section 2.1. Hence, proposition (ii) of the next lemma is the mathematical formulation of (C1) and (iii) corresponds to (C2).

Lemma 4.2.4 (Regularity of $\boldsymbol{f}_t$)**.** *Let* $n \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $\boldsymbol{f}_0 \in \mathcal{F}_{\boldsymbol{q}_{t=0}}$*, and* $(\boldsymbol{q}, \boldsymbol{p}), (\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}})$ *strictly time-like charge trajectories in* $\mathcal{T}^{2+n}(\mathbb{R})$ *such that* $\boldsymbol{q}_0 := \boldsymbol{q}_{t=0}$ *and* $\tilde{\boldsymbol{q}}_0 := \tilde{\boldsymbol{q}}_{t=0}$ *coincide. Define* $\boldsymbol{p}_0 := \boldsymbol{p}_{t=0}$ *and* $\tilde{\boldsymbol{p}}_0 := \tilde{\boldsymbol{p}}_{t=0}$ *and let*

$$\boldsymbol{f}_0^h = \boldsymbol{f}_0 - (\lambda \boldsymbol{f}_0^-[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] + (1-\lambda)\boldsymbol{f}_0^+[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}]) \in C^{1+n}(\mathbb{R}^3, \mathbb{R}^6). \tag{4.21}$$

Then, for the Maxwell solution $\boldsymbol{f}$ *given by* (4.12)-(4.17) *it holds*

(i) $\boldsymbol{f} \in C^n((D_{\boldsymbol{q}} \cap D_{\tilde{\boldsymbol{q}}}) \setminus \partial J(0, \boldsymbol{q}_0), \mathbb{R}^6)$.

For the special case $\lambda = 1$ *and the restriction on* $t \geq 0$ *we get* $\boldsymbol{f} \in C^n(D_{\boldsymbol{q}}^{[0,\infty)} \setminus \partial J^+(0, \boldsymbol{q}_0), \mathbb{R}^6)$.

(ii) $\boldsymbol{f}$ *can be evaluated point-wise on* $D_{\boldsymbol{q}} \cap D_{\tilde{\boldsymbol{q}}}$ *if and only if* $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0$ *and* $\tilde{\boldsymbol{p}}_0 = \boldsymbol{p}_0$.

(iii) *And for all* $0 \le m \le n$ *one has also* $\boldsymbol{f} \in C^m(D_{\boldsymbol{q}} \cap D_{\tilde{\boldsymbol{q}}}, \mathbb{R}^6)$ *if and only if*

$$\lim_{t\to 0} \frac{d^l}{dt^l} \boldsymbol{q}_t = \lim_{t\to 0} \frac{d^l}{dt^l} \tilde{\boldsymbol{q}}_t, \qquad \forall\, 0 \le l \le 2+m. \tag{4.22}$$

For the special case $\lambda = 1$ *and the restriction on* $t \ge 0$ *we have* $\boldsymbol{f} \in C^m(D_{\boldsymbol{q}}^{[0,\infty)}, \mathbb{R}^6)$ *if and only if*

$$\lim_{t\searrow 0} \frac{d^l}{dt^l} \boldsymbol{q}_t = \lim_{t\nearrow 0} \frac{d^l}{dt^l} \tilde{\boldsymbol{q}}_t, \qquad \forall\, 0 \le l \le 2+m \tag{4.23}$$

holds true.

Remark 4.2.3. *(i) Actually, with little more effort, one could even show* $\boldsymbol{f} \in C^m(D_{\boldsymbol{q}}, \mathbb{R}^6)$ *as inside of the light-cone* $J(0, \boldsymbol{q}_0)$ *formula* (4.12)-(4.17) *comprises Liénard-Wiechert fields of* $(\boldsymbol{q}, \boldsymbol{p})$ *and* $(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}})$ *which are singular at the respective charge positions the singularities at* $(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}})$ *in* (4.13) *and* (4.14) *cancel each other. Physically, this is to be expected, since the auxiliary trajectory serves only to parameterize the initial fields. However, as we will not need regularity of* (4.12)-(4.17) *inside the light-cone, in this work the above result suffices.*

(ii) Note that Lemma 4.2.4 (Regularity of $\boldsymbol{f}_t$*) (iii) indicates already the mechanism behind Theorem 4.1.1 (No-go)– namely, generic initial fields generate singular or discontinuous fronts located along the light-cone of the initial charge positions. If other charges hit those fronts their dynamics cannot be uniquely determined beyond the hitting.*

4.3 Existence of Maxwell-Lorentz solutions

Even though Theorem 4.1.1 (No-go) reveals that for generic initial values the initial value problem of Maxwell's and Lorentz's equations is ill-posed, we can nevertheless arrive at an existence and uniqueness result of a large class of Maxwell-Lorentz solutions from initial time to the time of the first collision, which may be infinite. This is the content of our third main result Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions).

As we have seen in Theorem 4.1.1 (No-go) the standard initial value formulation fails to provide solutions for generic initial values $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$. This is due to the existence of singular and discontinuous fronts in the Maxwell fields for generic initial fields which was content of Theorem 4.2.1 (Explicit Maxwell solutions). Thus, finding solutions is now a question of eliminating these fronts.

In Lemma 4.2.4 (Regularity of $\boldsymbol{f}_t$), (iii), a condition on how to obtain regular fields has been given, cf. (4.23). In order to encode this condition into initial values $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ it is convenient to parameterize the initial fields $(\boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ by means of $\boldsymbol{f}_{i,0} := \boldsymbol{f}_0^-[\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i] + \boldsymbol{f}_{i,0}^h$ for suitable trajectories $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ with $\tilde{\boldsymbol{q}}_{i,0} = \boldsymbol{q}_{i,0}$ and initial homogeneous fields $\boldsymbol{f}_{i,0}^h \in \mathcal{F}_{\text{hom}}, i \in \mathcal{N}$, cf. (4.18) for $\lambda = 1$. In this case, (4.23) reduces to the special case where for all $i \in \mathcal{N}$, $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ is defined on $(-\infty, 0]$ and $(\boldsymbol{q}_i, \boldsymbol{p}_i)$, should it exist, for non-negative times. In this sense, we speak of $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ as a given history of charges and $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{i\in\mathcal{N}}$ as charge trajectories of a potential solution of the Maxwell-Lorentz system. The distinction between $[0, \infty)$ as future and $(-\infty, 0)$ as past is just a convention and has no technical implication.

However, applying condition (4.23) to the initial configuration is still problematic since this condition is formulated with the help of the unknown charge trajectory $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ of a potential solution for $t > 0$. Nevertheless, in terms of the parameterization $\boldsymbol{f}_{i,0} = \boldsymbol{f}_0^-[\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i] + \boldsymbol{f}_{i,0}^h$ we are able to formulate an equivalent condition for (4.23) by making use of the Lorentz force law (1.4). Thereby, the condition on $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ and the solution $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ at initial time $t^{(0)} = 0$ is translated to a condition on the past trajectories $(\tilde{\boldsymbol{q}}_j, \tilde{\boldsymbol{p}}_j)$ of all other charges $j \neq i$ only. In conclusion, we have a possibility to define admissible initial values through conditions on the history $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i \in \mathcal{N}}$ to, as we shall show, ensure the absence of singular and discontinuous fronts.

In the following, we make these conditions and all other restrictions precise and visualize the difference between (4.23) and the new condition, introduced in Definition 4.3.4 (H3) below, in Figure 4.3. We restrict ourselves to initial values which are specified by special choices of $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i \in \mathcal{N}}$ and $(\boldsymbol{f}_{i,0}^h)_{i \in \mathcal{N}}$. These restrictions are formulated in the following definitions for arbitrary initial times $t^{(0)} \geq 0$, whereas for this section one may think of $t^{(0)} = 0$.

Moreover, let $d > 0$ and $n \in \mathbb{N}$ – not to be confused with $n \in \mathbb{N}_0$ – be fixed numbers in the rest of this section and the corresponding proofs in Section 5.6. The parameter d "detects collisions" by stopping the dynamics when a distance between two charges attains d and n will control the regularity of the corresponding solutions. As d may be chosen arbitrarily small we talk about collisions whenever charges attain this minimum fixed distance.

To define the initial fields, in terms of the parameterization discussed in Section 4.2, we make use of the following type of charge histories:

Definition 4.3.1 (History for $t^{(0)}$). *For the initial time $t^{(0)} \geq 0$, a tuple $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i \in \mathcal{N}}$ of charge trajectories in $\mathcal{T}^1((-\infty, t^{(0)}])$ is called a history for $t^{(0)}$. For single charges i, $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ is called a charge history for $t^{(0)}$. Moreover, we define the following properties:*

(H0) $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i) \in \mathcal{T}^{2+n}((-\infty, t^{(0)}])$ *for all* $i \in \mathcal{N}$.

(H1) $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i) : (-\infty, t^{(0)}] \to \mathbb{R}^6$ *is strictly time-like for all* $i \in \mathcal{N}$.

(H2) $|\tilde{\boldsymbol{q}}_{i,t^{(0)}} - \tilde{\boldsymbol{q}}_{j,t^{(0)}}| > d$ *for all* $i, j \in \mathcal{N}$ *with* $i \neq j$.

The fields we rely on in Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions) below and the corresponding section of proof, Section 5.6, are given in the following definitions.

Definition 4.3.2 (Initial fields). *Let $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i \in \mathcal{N}}$ denote a history for time $t = 0$. We only consider initial fields $(\boldsymbol{f}_{i,0})_{i \in \mathcal{N}}$, parameterized and restricted by*

$$\boldsymbol{f}_{i,0} = \boldsymbol{f}_0^-[\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i] + \boldsymbol{f}_{i,0}^h, \qquad \textit{for} \qquad \boldsymbol{f}_{i,0}^h \in \mathcal{F}_{\text{hom}} \cap C^{1+n}(\mathbb{R}^3, \mathbb{R}^6), \qquad i \in \mathcal{N}. \tag{4.24}$$

Moreover, we set

$$\boldsymbol{f}_{i,t}^h := W_t \boldsymbol{f}_{i,0}^h, \qquad \forall\, t \in \mathbb{R},\, i \in \mathcal{N}. \tag{4.25}$$

Note that $\boldsymbol{f}_{i,t}^h$ has a representative

$$(t, \boldsymbol{x}) \mapsto \boldsymbol{f}_{i,t}^h(\boldsymbol{x}) \in C^n(\mathbb{R}^4, \mathbb{R}^6), \qquad \forall i \in \mathcal{N}, \tag{4.26}$$

cf. Lemma 4.2.2 (Homogeneous Maxwell solutions), (iv).

With the above definition of the considered initial fields, we can turn over to the next definition of considered Maxwell fields. It is important to note that this definition depends on a initial time $t^{(0)}$ and a history for $t^{(0)}$, which will both be changing throughout Section 5.6.

Definition 4.3.3 (Maxwell field for $t^{(0)}$)**.** *Let* $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ *denote a history for* $t^{(0)} \geq 0$*. For all* $i \in \mathcal{N}$*, we define*

$$\boldsymbol{f}_{i,t}(\boldsymbol{x}) := \boldsymbol{f}_t^-[\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i](\boldsymbol{x}) + \boldsymbol{f}_{i,t}^h(\boldsymbol{x}), \qquad \forall\, (t,\boldsymbol{x}) \in \mathbb{R}^4 \setminus \overset{\circ}{J}{}^+(t^{(0)}, \tilde{\boldsymbol{q}}_{i,t^{(0)}}),\ i \in \mathcal{N}, \tag{4.27}$$

where $\boldsymbol{f}_{i,t}^h$ *is given by* (4.26)*.*

Remark 4.3.1. *(i) By Theorem 4.2.1 (Explicit Maxwell solutions), regarding also Remark 4.2.2, (ii), the field* $\boldsymbol{f}_i$ *is the unique Maxwell solution for charge* i *with the initial field* $\boldsymbol{f}_{i,0}$ *in* (4.24) *restricted to* $(t,\boldsymbol{x}) \in \mathbb{R}^4 \setminus \overset{\circ}{J}{}^+(t^{(0)}, \tilde{\boldsymbol{q}}_{i,t^{(0)}})$*.*

(ii) By virtue of (4.26) *its regularity depends on the regularity of the history* $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ *only and is given by Lemma 4.2.1 (Properties of Liénard-Wiechert fields), (i).*

(iii) Given the parameterization (4.24) *of the initial fields* $\boldsymbol{f}_{i,0}$*, the Maxwell-Lorentz system of equations can be considered as the system of Lorentz equations* (2.24) *for* $\lambda = 1$ *in which the formula* (4.27) *is plugged in for the fields in the Lorentz force. This point of view naturally results in a delay system which we exploit in Section 5.6 of the proof in order to construct Lorentz solutions, first, and by means of these we derive Maxwell-Lorentz solutions.*

The key criterion for the histories to guarantee the absence of singular fronts is given by the next definition:

Definition 4.3.4 (H3)**.** *We say that a history* $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ *for* $t^{(0)}$ *fulfilling (H0)-(H1) meets condition (H3) if and only if for all* $i \in \mathcal{N}$ *and all* $k \in \{1, \ldots, 1+n\}$ *the charge histories obey*

$$\lim_{t\nearrow t^{(0)}} \frac{d^k}{dt^k} \begin{pmatrix} \tilde{\boldsymbol{q}}_{i,t} \\ \tilde{\boldsymbol{p}}_{i,t} \end{pmatrix} = \lim_{t\nearrow t^{(0)}} \frac{d^{k-1}}{dt^{k-1}} \begin{pmatrix} \boldsymbol{v}(\tilde{\boldsymbol{p}}_{i,t}) \\ \sum_{j\neq i} \boldsymbol{E}_{j,t}(\tilde{\boldsymbol{q}}_{i,t}) + \boldsymbol{v}(\tilde{\boldsymbol{p}}_{i,t}) \wedge \boldsymbol{B}_{j,t}(\tilde{\boldsymbol{q}}_{i,t}) \end{pmatrix}, \tag{H3}$$

where $(\boldsymbol{f}_{i,t} = (\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t}))_{i\in\mathcal{N}}$ *is given by* (4.27)*.*

We can finally turn to our third main result.

Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions)**.** *Let* $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ *be a history for* $t^{(0)} = 0$ *fulfilling (H0)-(H2) and* $(\boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ *be initial fields of the form* (4.24)*. Then, the following propositions hold true:*

(i) (Local existence) There is a time $T_{\max} > \frac{d}{2}$ *such that there is a Maxwell-Lorentz solution* $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i\in\mathcal{N}}$ *on* $[0, T_{\max}]$ *for initial value* $(\tilde{\boldsymbol{q}}_{i,t=0}, \tilde{\boldsymbol{p}}_{i,t=0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ *in the sense of Definition 3.3.4 (Maxwell-Lorentz solutions). Furthermore, this local solution fulfills* $(\boldsymbol{q}_i, \boldsymbol{p}_i) \in \mathcal{T}^{2+n}([0, T_{\max}])$ *and* $\boldsymbol{f}_i \in C^n(D_{\boldsymbol{q}_i}^{[0,T_{\max}]} \setminus \partial J^+(0, \tilde{\boldsymbol{q}}_{i,0}), \mathbb{R}^6)$*.*

(ii) *(Existence until collision) In addition, if the history $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ fulfills also (H3), then for some $T \in \mathbb{R}^+$ there are charge trajectories $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ in $\mathcal{T}^2([0,T])$, $i \in \mathcal{N}$, such that for*

$$T_{\max} := \sup\{t \in [0,T] \mid \forall\, i,j \in \mathcal{N}, i \neq j, \forall s \in [0,t] : |\boldsymbol{q}_{i,s} - \boldsymbol{q}_{j,s}| > d\} \tag{4.28}$$

there are fields $(\boldsymbol{f}_i)_{i\in\mathbb{N}} \in \mathcal{F}^1([0,T_{\max}])$ with the property that $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i\in\mathcal{N}}$ is a Maxwell-Lorentz solution on $[0,T_{\max}]$ for initial value $(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ in the sense of Definition 3.3.4 (Maxwell-Lorentz solutions). Furthermore, the resulting solution $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i\in\mathcal{N}}$ fulfills $(\boldsymbol{q}_i, \boldsymbol{p}_i) \in \mathcal{T}^{2+n}([0,T_{\max}])$ and $\boldsymbol{f}_i \in C^n(D^{[0,T_{\max}]}_{\boldsymbol{q}_i}, \mathbb{R}^6)$ for $i \in \mathcal{N}$.

(iii) *(Uniqueness) Let Λ be an interval. Should there be another Maxwell-Lorentz solution $(\hat{\boldsymbol{q}}_i, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{f}}_i)_{i\in\mathcal{N}}$ on Λ, then*

$$(\hat{\boldsymbol{q}}_{i,t}, \hat{\boldsymbol{p}}_{i,t}, \hat{\boldsymbol{f}}_{i,t})|_{t=0} = (\tilde{\boldsymbol{q}}_{i,t=0}, \tilde{\boldsymbol{p}}_{i,t=0}, \boldsymbol{f}_{i,0}) \tag{4.29}$$

if and only if

$$(\hat{\boldsymbol{q}}_{i,t}, \hat{\boldsymbol{p}}_{i,t}, \hat{\boldsymbol{f}}_{i,t}) = (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{f}_{i,t}) \qquad \forall\, t \in \Lambda \cap [0, T_{\max}]. \tag{4.30}$$

(iv) *The solution $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i\in\mathcal{N}}$ fulfills the Maxwell constraints* (3.3) *for all $t \in [0,T_{\max}]$, i.e., for all $t \in [0,T_{\max}]$, $i \in \mathcal{N}$ we have $\boldsymbol{f}_{i,t} \in \mathcal{F}_{\boldsymbol{q}_{i,t}}$.*

(v) *((R) ⇔(H3)) The Maxwell-Lorentz solution $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i\in\mathcal{N}}$ on $[0,T_{\max}]$ and the respective history $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ fulfills the condition (R) given by*

$$\lim_{t\searrow 0} \frac{d^k}{dt^k} \begin{pmatrix} \boldsymbol{q}_{i,t} \\ \boldsymbol{p}_{i,t} \end{pmatrix} = \lim_{t\nearrow 0} \frac{d^k}{dt^k} \begin{pmatrix} \tilde{\boldsymbol{q}}_{i,t} \\ \tilde{\boldsymbol{p}}_{i,t} \end{pmatrix}, \qquad \forall\, k = 0, \ldots, 1+n,\ i \in \mathcal{N} \tag{R}$$

if and only if the history $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ fulfills (H3).

Condition (R) corresponds to (4.23) from Lemma 4.2.4 (Regularity of $\boldsymbol{f}_t$). One should note that we use the notation $\begin{pmatrix} \boldsymbol{q}_{i,t} \\ \boldsymbol{p}_{i,t} \end{pmatrix}$ in (R) whereas in (4.23) we write $\boldsymbol{q}_t$, which implies the difference between $1+n$ in (R) and $2+m$ in (4.23). Moreover, in the setting of Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions), for all $i \in \mathcal{N}$ $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i) \in \mathcal{T}^{2+n}((-\infty,0])$ for $n \in \mathbb{N}$ is solely the history of charge i and $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i) \in \mathcal{T}^{2+n}([0,t^{(1)}])$ denotes the local solution of charge i, and thus, the limit on the left hand side of (R) exists only from above whereas the right hand side limit exists only from below.

Remark 4.3.2. *(i) The fact that we need trajectories to be $1+n$ times continuously differentiable for $n \in \mathbb{N}$ throughout the past and at $t = 0$ is only a technical assumption needed in our strategy of proof as we exploit the additional derivative to control the Lipschitz continuity.*

(ii) *Statement (i) shows that there are general local solutions at least up to time $t^{(1)} \geq \min_{i,j\in\mathcal{N}} |\boldsymbol{q}_{i,t=0} - \boldsymbol{q}_{j,t=0}| > d/2$, however, as Theorem 4.1.1 (No-go) states, most cannot be continued beyond $t^{(1)}$. In fact, the time $t^{(1)}$ is exactly the first time, where a solution trajectory impinges the light-cone boundary $\partial J(0, \boldsymbol{q}_{j,0})$ of another charge j. If, however, (H3) is met the solution trajectories connect regular to the charge histories, and then statement (ii) provides existence and uniqueness of solutions for all times such that the charges maintain a minimum distance d.*

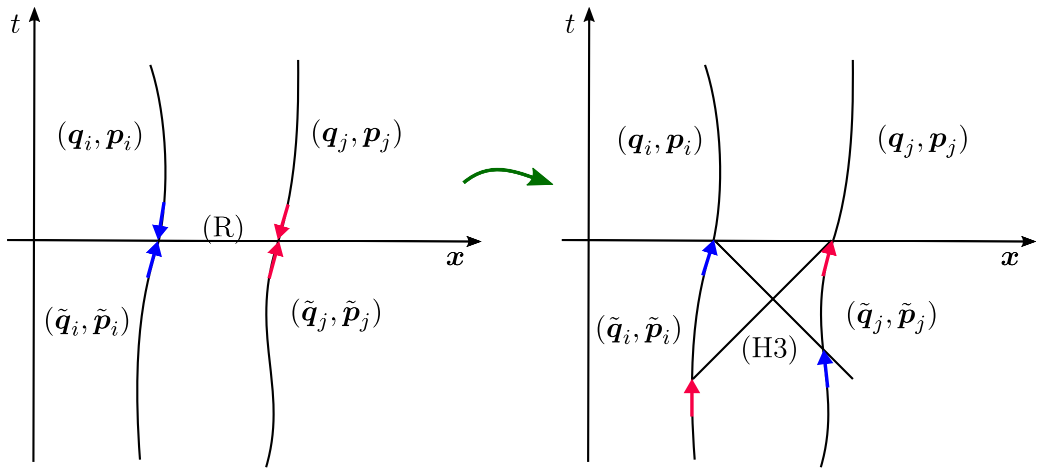

Figure 4.3: The figure illustrates how the condition (R) defined in Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions) on the future and past trajectories can be replaced by the condition (H3) from Definition 4.3.4 (H3) on the past trajectories only by making use of the Lorentz force law. The arrows denote the necessary limits in expressions (R) and (H3).

(iii) The set of charge histories fulfilling (H0)-(H3) is not empty, which we illustrate by the following construction:

For each $i \in \mathcal{N}$ choose $(t^{(0)}, \tilde{\boldsymbol{q}}_{i,t^{(0)}})$, such that (H2) holds. The minimal distance $> d$ guarantees that later, when elaborating on the solution theory of the equations, the minimal time $t^{(1)}$ up to which one can propagate solution trajectories from the given admissible history fulfills $t^{(1)} > d/2$. Secondly, we need to construct histories ending in the points $(t^{(0)}, \tilde{\boldsymbol{q}}_{i,t^{(0)}})$ such that (H3) holds. For each $i \in \mathcal{N}$ set

$$t_i^{(-1)} := \max\{t < t^{(0)} | \exists\, j \neq i : \partial B_{|t^{(0)}-t|}(\tilde{\boldsymbol{q}}_{i,t^{(0)}}) \cap \partial B_{|t^{(0)}-t|}(\tilde{\boldsymbol{q}}_{j,t^{(0)}}) \neq \emptyset\}, \tag{4.31}$$

the first time in history, where the light-cone boundary of the initial position of charge i crosses one of the other light-cones, cf. the dashed lines in the upper image in Figure 4.4.

Afterwards, chose a strictly time-like charge history $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i) \in \mathcal{T}^{2+n}((-\infty, t_i^{(-1)}])$ lying in the inner of $J^-(t^{(0)}, \tilde{\boldsymbol{q}}_{i,t^{(0)}})$, i.e., (H0) and (H1) are met on $(-\infty, t_i^{(-1)}]$. By the choice of the times $t_i^{(-1)}$ we made sure that for each $i \in \mathcal{N}$ the trajectory piece $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ eventually crosses $\partial J^-(t^{(0)}, \tilde{\boldsymbol{q}}_{j,t^{(0)}})$ for each $j \neq i$. The blue dots in the upper image of Figure 4.4 illustrate these intersection points.

Therefore, each parameter needed to compute the right hand side of (H3) *is known, computable and allows us to set the left hand side of* (H3)*, i.e. the derivatives up to degree $1+n$ at the initial positions. The lower image in Figure 4.4 visualizes the dependence of the derivatives at position $(t^{(0)}, \boldsymbol{q}_{i,t^{(0)}})$ on the histories of all charges $j \neq i$ at the positions where the backward light-cone $\partial J^-(t^{(0)}, \boldsymbol{q}_{i,t^{(0)}})$ crosses these histories.*

Finally, we can extend $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i) : (-\infty, t_i^{(-1)}] \to \mathbb{R}^6$ to a trajectory on $(-\infty, t^{(0)}]$ which is $2+n$ times continuously differentiable, strictly time like, and matches the derivatives at time $t^{(0)}$ computed in the previous step. Doing so for each i finishes the construction.

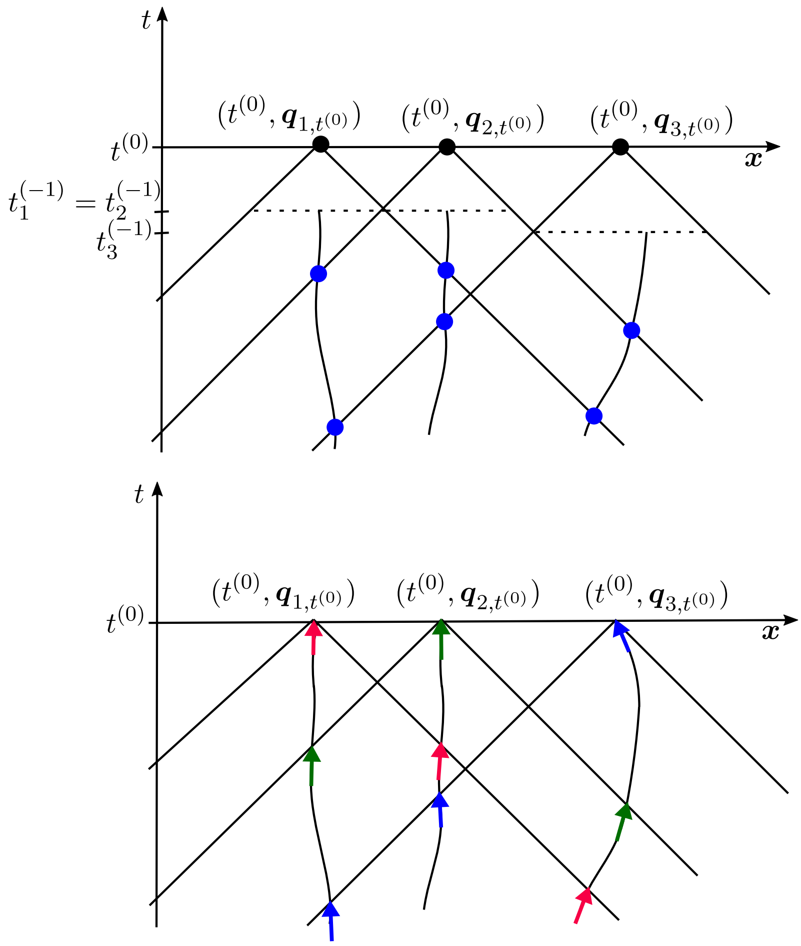

Figure 4.4: Illustration of the suggested construction from Remark 4.3.2, (iii), of an admissible history for 3 charges.

(iv) It should be noted that, by construction in (iii), the set of valid charge histories is a very large set as one may freely choose strictly time-like trajectories in $\mathcal{T}^{2+n}((-\infty, t_i^{(-1)}]$ *for each* i*. Therefore, the global existence and uniqueness result holds for a respectively large set of initial values.*

(v) Moreover, we remark that in [10, 14, 24] it was shown that collisions never occur for two repulsive charges which are initially prepared with initial values that constrain the charge motion on a straight-line (which is preserved under the Maxwell-Lorentz dynamics) for vanishing initial homogeneous fields $\boldsymbol{f}_0^h = 0$*. By definition, the solutions in [6, 14, 24] are contained in the class of solutions considered in Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions), and therefore, the set of global solutions, i.e., fulfilling* $T_{\max} = \infty$*, is not empty.*

Chapter 5

Proofs

5.1 Properties of the Liénard-Wiechert fields

In order to prove Lemma 4.2.1 (Properties of Liénard-Wiechert fields) we need the following auxiliary result:

Lemma 5.1.1 ($t^\pm$)**.** *For all strictly time-like charge trajectories $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^m(\mathbb{R})$, $m \in \mathbb{N}$, solutions $t^\pm$ to equation (4.7) in Definition 4.2.1 (Liénard-Wiechert fields), recall the notation $t^\pm = t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x})$, fulfill the following propositions for all $t \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^3$:*

(i) Both $t^+_{\boldsymbol{q}}(t,\boldsymbol{x})$ and $t^-_{\boldsymbol{q}}(t,\boldsymbol{x})$ exist, and $t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x}) = t$ if and only if $\boldsymbol{x} = \boldsymbol{q}_t = \boldsymbol{q}_{t^\pm}$.

(ii) For all $\alpha \in \mathbb{N}_0^3$, $k \in \mathbb{N}_0$ such that $k + |\alpha| \le m$ the map $(t,\boldsymbol{x}) \mapsto D^\alpha_{\boldsymbol{x}} \partial^k_t t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x})$ is in $C^{m-|\alpha|-k}(D_{\boldsymbol{q}},\mathbb{R})$ and $D^\alpha_{\boldsymbol{x}} \partial^k_t t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x})$ is a function of $\boldsymbol{x}$, $\frac{d^l}{ds^l}\boldsymbol{q}_s\big|_{s=t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x})}$, $l = 0,\ldots,k+|\alpha|$ only.

Proof. (i) Let $m \in \mathbb{N}$ and $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^m(\mathbb{R})$. By assumption, $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^1(\mathbb{R})$ and strictly time-like, see Definition 3.3.1 (Charge and field trajectories). Thus, by geometrical reasoning and the intermediate value theorem, there exists exactly one intersection point of $(\boldsymbol{q},\boldsymbol{p})$ with each of the light-cone boundaries $\partial J^+(t,\boldsymbol{x})$ and $\partial J^-(t,\boldsymbol{x})$. By the definition in (4.7), the times of intersection correspond to $t^+_{\boldsymbol{q}}(t,\boldsymbol{x})$ and $t^-_{\boldsymbol{q}}(t,\boldsymbol{x})$. Hence, $t^\pm_{\boldsymbol{q}}$ is a well-defined function $t^\pm : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, $(t,\boldsymbol{x}) \mapsto t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x})$.

If $t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x}) = t$ both light-cone boundaries $\partial J^+(t,\boldsymbol{x})$ and $\partial J^-(t,\boldsymbol{x})$ intersect the trajectory at the same time t, and thus, at space point $\boldsymbol{x} = \boldsymbol{q}_t = \boldsymbol{q}_{t^\pm}$. If on the other hand $\boldsymbol{x} = \boldsymbol{q}_t = \boldsymbol{q}_{t^\pm}$, it follows $t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x}) = t$.

(ii) By induction we show the following statement:

For all $\alpha \in \mathbb{N}_0^3$, $k \in \mathbb{N}_0$ such that $|\alpha| + k \in \{1,\ldots,m\}$ there exists a function $g^\pm_{\alpha,k} \in C^{m-|\alpha|-k}(D_{\boldsymbol{q}},\mathbb{R})$ such that

$$(t,\boldsymbol{x}) \mapsto D^\alpha_{\boldsymbol{x}} \partial^k_t t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x}) = g^\pm_{\alpha,k}(s,\boldsymbol{x})\big|_{s=t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x})} \in C^0(D_{\boldsymbol{q}},\mathbb{R}). \tag{5.1}$$

Moreover, $g^\pm_{\alpha,k}(s,\boldsymbol{x})$ is a function of $\boldsymbol{x}$, $\frac{d^l}{ds^l}\boldsymbol{q}_s$, $l = 0,\ldots,|\alpha|+k$.

Base case: Since $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^1(\mathbb{R})$, the definition in (4.7) implies

$$\partial_t t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x}) = \frac{1}{1 \pm \boldsymbol{n}\cdot\boldsymbol{v}}\Big|^{\pm} \qquad \text{and} \qquad \partial_{x_i} t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x}) = \frac{\pm n_i}{1 \pm \boldsymbol{n}\cdot\boldsymbol{v}}\Big|^{\pm}, i = 1,\dots,3, \tag{5.2}$$

which are well-defined since, by assumption, $(\boldsymbol{q},\boldsymbol{p})$ is strictly time-like, there is a $v_{\max}$ such that $|\boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm}| \le v_{\max} < 1$. Therefore, $t^{\pm}$ is differentiable in each component of $(t,\boldsymbol{x})$, hence, continuous and in conclusion for $m = 1$, $(t,\boldsymbol{x}) \mapsto D^{\alpha}_{\boldsymbol{x}}\partial^k_t t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x}) \in C^0(D_{\boldsymbol{q}},\mathbb{R})$.

Moreover, we set

$$g^{\pm}_{\alpha,k} : D_{\boldsymbol{q}} \to \mathbb{R}, \qquad (s,\boldsymbol{x}) \mapsto \begin{cases} \frac{1}{1\pm\boldsymbol{n}_s\cdot\boldsymbol{v}_s} & , k = 1 \\ \frac{\pm n_{s,i}}{1\pm\boldsymbol{n}_s\cdot\boldsymbol{v}_s} & , \alpha_i = 1 \end{cases}. \tag{5.3}$$

Then, $g^{\pm}_{\alpha,k}$ is a composition of the maps $(s,\boldsymbol{x}) \mapsto \boldsymbol{n}_s \in C^m(D_{\boldsymbol{q}},\mathbb{R}^3)$ and $(s,\boldsymbol{x}) \mapsto \boldsymbol{v}_s \in C^{m-1}(\mathbb{R}^4,\mathbb{R}^3)$, and thus, $g^{\pm}_{\alpha,k} \in C^{m-1}(D_{\boldsymbol{q}},\mathbb{R})$ and depends on $\boldsymbol{x},\boldsymbol{q}_s,\boldsymbol{v}_s$ only.

Inductive step: Assume the hypothesis holds for $|\alpha| + k < m$. We show, that it holds for α', k' with $|\alpha'| + k' = |\alpha| + k + 1$. Thereby we distinguish two cases: Either an additional time-derivative ∂_t or an additional spatial derivative ∂_{x_i} for $i \in \{1,2,3\}$ is applied. In the first case we obtain

$$\begin{aligned} D^{\alpha'}_{\boldsymbol{x}}\partial^{k'}_t t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x}) &= \partial_t D^{\alpha}_{\boldsymbol{x}}\partial^k_t t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x}) \overset{(IH)}{=} \partial_t g^{\pm}_{\alpha,k}(s,\boldsymbol{x})|_{s=t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x})} \\ &= \Big[\partial_s g^{\pm}_{\alpha,k}(s,\boldsymbol{x})\partial_t t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x})\Big]_{s=t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x})} \\ &\overset{(IH)}{=} \Big[\partial_s g^{\pm}_{\alpha,k}(s,\boldsymbol{x}) g^{\pm}_{0,1}(s,\boldsymbol{x})\Big]_{s=t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x})}, \end{aligned}$$

and in the second case

$$\begin{aligned} D^{\alpha'}_{\boldsymbol{x}}\partial^{k'}_t t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x}) &= \partial_{x_i} D^{\alpha}_{\boldsymbol{x}}\partial^k_t t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x}) \overset{(IH)}{=} \partial_{x_i} g^{\pm}_{\alpha,k}(s,\boldsymbol{x})|_{s=t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x})} \\ &= \Big[\partial_s g^{\pm}_{\alpha,k}(s,\boldsymbol{x})\partial_{x_i} t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x}) + \partial_{x_i} g^{\pm}_{\alpha,k}(s,\boldsymbol{x})\Big]_{s=t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x})} \\ &\overset{(IH)}{=} \Big[\partial_s g^{\pm}_{\alpha,k}(s,\boldsymbol{x}) g^{\pm}_{\alpha'-\alpha,0}(s,\boldsymbol{x}) + \partial_{x_i} g^{\pm}_{\alpha,k}(s,\boldsymbol{x})\Big]_{s=t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x})}. \end{aligned}$$

Thus, we define

$$\begin{aligned} (s,\boldsymbol{x}) \mapsto g^{\pm}_{\alpha',k'}(s,\boldsymbol{x}) :=& \partial_s g^{\pm}_{\alpha,k}(s,\boldsymbol{x}) g^{\pm}_{\alpha'-\alpha,k'-k}(s,\boldsymbol{x}) \\ &+ \partial^{\alpha'_1-\alpha_1}_{x_1}\partial^{\alpha'_2-\alpha_2}_{x_2}\partial^{\alpha'_3-\alpha_3}_{x_3} g^{\pm}_{\alpha,k}(s,\boldsymbol{x}). \end{aligned}$$

Applying the induction hypothesis, we obtain

$$(s,\boldsymbol{x}) \mapsto \partial_s g^{\pm}_{\alpha,k}(s,\boldsymbol{x}), \partial^{\alpha'_1-\alpha_1}_{x_1}\partial^{\alpha'_2-\alpha_2}_{x_2}\partial^{\alpha'_3-\alpha_3}_{x_3} g^{\pm}_{\alpha,k}(s,\boldsymbol{x}) \in C^{m-|\alpha|-k-1}(D_{\boldsymbol{q}},\mathbb{R}), \tag{5.4}$$

and, by the base case,

$$(s,\boldsymbol{x}) \mapsto g^{\pm}_{\alpha'-\alpha,k'-k}(s,\boldsymbol{x}) \in C^{m-1}(D_{\boldsymbol{q}},\mathbb{R}). \tag{5.5}$$

This implies $g^{\pm}_{\alpha,k} \in C^{m-|\alpha|-k-1}(D_{\boldsymbol{q}},\mathbb{R})$. Moreover,

$$(t,\boldsymbol{x}) \mapsto D^{\alpha'}_{\boldsymbol{x}} \partial^{k'}_t t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x}) = g^{\pm}_{\alpha',k'}(s,\boldsymbol{x})\big|_{s=t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x})} \in C^0(D_{\boldsymbol{q}},\mathbb{R}) \tag{5.6}$$

because $(t,\boldsymbol{x}) \mapsto t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x})$ is continuous on $\mathbb{R}\times\mathbb{R}^3$.

Furthermore, by the induction hypothesis, $g^{\pm}_{\alpha,k}$ is a function of $\boldsymbol{x}, \frac{d^l}{ds^l}\boldsymbol{q}_s, l=0,\ldots,|\alpha|+k$ only, i.e. $\partial_s g^{\pm}_{\alpha,k}(s,\boldsymbol{x})$ and $\partial^{\alpha'_1-\alpha_1}_{x_1}\partial^{\alpha'_2-\alpha_2}_{x_2}\partial^{\alpha'_3-\alpha_3}_{x_3} g^{\pm}_{\alpha,k}(s,\boldsymbol{x})$ depends on $\boldsymbol{x}, \frac{d^l}{ds^l}\boldsymbol{q}_s, l = 0,\ldots,|\alpha|+k+1$ only. By the base case, $g^{\pm}_{\alpha'-\alpha,k'-k}(s,\boldsymbol{x})$ is a function of $\boldsymbol{x},\boldsymbol{q}_s,\boldsymbol{v}_s$, and therefore, $g^{\pm}_{\alpha',k'}$ depends on $\boldsymbol{x}, \frac{d^l}{ds^l}\boldsymbol{q}_s, l=0,\ldots,|\alpha|+k+1$ only. Hence, the induction hypothesis holds which concludes the proof.

□

With the help of Lemma 5.1.1 ($t^{\pm}$) we can now move on to the proof of Lemma 4.2.1 (Properties of Liénard-Wiechert fields). Note that in in this proof Lemma 5.1.1 ($t^{\pm}$) is needed in the case $m = 2+n$ with $n \in \mathbb{N}_0$.

Proof of Lemma 4.2.1. (Properties of Liénard-Wiechert fields)

Let $n \in \mathbb{N}_0$ and assume $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^{2+n}(\mathbb{R})$ is strictly time-like. We start by proving item (i)+(ii) together:

1. We write the Liénard-Wiechert fields $\boldsymbol{e}^{\pm}[\boldsymbol{q},\boldsymbol{p}]$ as composition of an auxiliary, explicit field function $\widehat{\boldsymbol{e}}^{\pm}[\boldsymbol{q},\boldsymbol{p}]$ and the advanced/retarded time function $t^{\pm}$, i.e.,

$$(t,\boldsymbol{x}) \mapsto \boldsymbol{e}^{\pm}_t[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) = (\widehat{\boldsymbol{e}}^{\pm}[\boldsymbol{q},\boldsymbol{p}] \circ \tau^{\pm})(t,\boldsymbol{x}), \tag{5.7}$$

for

$$\tau^{\pm} : D_{\boldsymbol{q}} \to \mathbb{R}^4, \qquad (t,\boldsymbol{x}) \mapsto (t^{\pm}_{\boldsymbol{q}}(t,\boldsymbol{x}),\boldsymbol{x}) \tag{5.8}$$

and

$$\begin{aligned} \widehat{\boldsymbol{e}}^{\pm}[\boldsymbol{q},\boldsymbol{p}] : \quad & D_{\boldsymbol{q}} \to \mathbb{R}^3, \\ (s,\boldsymbol{x}) \mapsto{} & \frac{(\boldsymbol{n}_s \pm \boldsymbol{v}_s)(1-\boldsymbol{v}_s^2)}{|\boldsymbol{x}-\boldsymbol{q}_s|^2(1\pm\boldsymbol{n}_s\cdot\boldsymbol{v}_s)^3} + \frac{\boldsymbol{n}_s \wedge [(\boldsymbol{n}_s\pm\boldsymbol{v}_s)\wedge\boldsymbol{a}_s]}{|\boldsymbol{x}-\boldsymbol{q}_s|(1\pm\boldsymbol{n}_s\cdot\boldsymbol{v}_s)^3}. \end{aligned} \tag{5.9}$$

2. For this proof it is convenient to regard the expressions $\boldsymbol{q}_s,\boldsymbol{v}_s,\boldsymbol{a}_s$ and $\boldsymbol{n}_s$ as functions of $(s,\boldsymbol{x})$ although they depend on s only. Since by assumption we have $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^{2+n}(\mathbb{R})$ it follows:

   - The expressions $\boldsymbol{q}_s,\boldsymbol{v}_s,\boldsymbol{a}_s$ are well-defined on $\mathbb{R}^4$ with $(s,\boldsymbol{x}) \mapsto \boldsymbol{q}_s$ in $C^{2+n}(\mathbb{R}^4,\mathbb{R}^3)$, $(s,\boldsymbol{x}) \mapsto \boldsymbol{v}_s$ in $C^{1+n}(\mathbb{R}^4,\mathbb{R}^3)$, $(s,\boldsymbol{x}) \mapsto \boldsymbol{a}_s$ in $C^n(\mathbb{R}^4,\mathbb{R}^3)$.
   - The expression $\boldsymbol{n}_s = \frac{\boldsymbol{x}-\boldsymbol{q}_s}{|\boldsymbol{x}-\boldsymbol{q}_s|}$ is well defined on $D_{\boldsymbol{q}}$ with $(s,\boldsymbol{x}) \mapsto \boldsymbol{n}_s$ in $C^{2+n}(D_{\boldsymbol{q}},\mathbb{R}^3)$.

   Moreover, the denominators are finite:

   - $\sup_{s\in\mathbb{R}} |1 \pm \boldsymbol{n}_s\cdot\boldsymbol{v}_s| > 0$, because $(\boldsymbol{q},\boldsymbol{p})$ is assumed to by strictly time-like, cf. Definition 3.3.1 (Charge and field trajectories), and hence, there is a $v_{\max} < 1$ such that $|\boldsymbol{n}_s\cdot\boldsymbol{v}_s| \le |\boldsymbol{v}_s| \le v_{\max}$ for all $s \in \mathbb{R}$.

- For all $(s, \boldsymbol{x}) \in D_{\boldsymbol{q}}$, $|\boldsymbol{x} - \boldsymbol{q}_s| > 0$.

Therefore, the auxiliary field function $\widehat{\boldsymbol{e}}^{\pm}[\boldsymbol{q}, \boldsymbol{p}](s, \boldsymbol{x})$ is well-defined on $D_{\boldsymbol{q}}$.

3. Collecting these facts about $\widehat{\boldsymbol{e}}^{\pm}[\boldsymbol{q}, \boldsymbol{p}]$ defined in (5.9), we know that it is a composition of functions in $C^n(D_{\boldsymbol{q}}, \mathbb{R}^6)$, which by chain rule ensures that for all $\alpha \in \mathbb{N}_0^3$ and all $k \leq n$ it holds $(s, \boldsymbol{x}) \mapsto D_{\boldsymbol{x}}^{\alpha} \partial_s^k \widehat{\boldsymbol{e}}^{\pm}[\boldsymbol{q}, \boldsymbol{p}](s, \boldsymbol{x}) \in C^{n-k}(D_{\boldsymbol{q}}, \mathbb{R}^3)$ and $D_{\boldsymbol{x}}^{\alpha} \partial_s^k \widehat{\boldsymbol{e}}^{\pm}[\boldsymbol{q}, \boldsymbol{p}](s, \boldsymbol{x})$ depends on $\boldsymbol{x}$ and $\left(\frac{d}{ds}\right)^l \boldsymbol{q}_s$ for $l = 0, \ldots, k+2$ only.

4. By the definition in (5.7), item 3., and again by chain rule, for all $|\alpha| + k \leq n$, it holds $(t, \boldsymbol{x}) \mapsto D_{\boldsymbol{x}}^{\alpha} \partial_t^k \boldsymbol{e}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}) \in C^{n-|\alpha|-k}(D_{\boldsymbol{q}}, \mathbb{R}^3)$. This is due to Lemma 5.1.1 ($t^{\pm}$) according to which

   - $(t, \boldsymbol{x}) \in D_{\boldsymbol{q}}$ implies $t_{\boldsymbol{q}}^{\pm}(t, \boldsymbol{x}) := t \pm |\boldsymbol{x} - \boldsymbol{q}_{t^{\pm}}| \neq t$, and thus, $|\boldsymbol{x} - \boldsymbol{q}_{t^{\pm}}| > 0$.
   - $(t, \boldsymbol{x}) \mapsto D_{\boldsymbol{x}}^{\alpha} \partial_t^k t_{\boldsymbol{q}}^{\pm}(t, \boldsymbol{x}) \in C^{2+n-|\alpha|-k}(D_{\boldsymbol{q}}, \mathbb{R})$ for all $|\alpha| + k \leq 2 + n$.

   Moreover, $D_{\boldsymbol{x}}^{\alpha} \partial_t^k \boldsymbol{e}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x})$ depends on $\boldsymbol{x}$, $\frac{d^l}{dt^l} \boldsymbol{q}_t|_{t=t^{\pm}}, l = 0, \ldots, |\alpha| + k + 2$ only, because:

   - By Lemma 5.1.1 ($t^{\pm}$), $D_{\boldsymbol{x}}^{\alpha} \partial_t^k t^{\pm}(t, \boldsymbol{x})$ depends on $t, \boldsymbol{x}, \frac{d^l}{dt^l} \boldsymbol{q}_t, l = 0, \ldots, |\alpha| + k$ only.
   - By item 3., $D_{\boldsymbol{x}}^{\alpha} \partial_s^k \widehat{\boldsymbol{e}}^{\pm}[\boldsymbol{q}, \boldsymbol{p}](s, \boldsymbol{x})$ depends on $\boldsymbol{x}$ and $\left(\frac{d}{dt}\right)^l \boldsymbol{q}_s$ for $l = 0, \ldots, k+2$ only.

5. The corresponding assertion for the magnetic field can be derived from the Liénard-Wiechert formula (4.6)

$$\boldsymbol{b}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}) = \mp \boldsymbol{n}^{\pm} \wedge \boldsymbol{e}_t^{\pm}(\boldsymbol{x}), \tag{5.10}$$

   which together with the product rule implies that for all $|\alpha| + k \leq n$

$$D_{\boldsymbol{x}}^{\alpha} \partial_t^k \boldsymbol{b}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}) = \sum_{\substack{\alpha_1 + \alpha_2 = \alpha, \\ k_1 + k_2 = k}} \mp D_{\boldsymbol{x}}^{\alpha_1} \partial_t^{k_1} \sigma(t) \boldsymbol{n}^{\pm} \wedge D_{\boldsymbol{x}}^{\alpha_2} \partial_t^{k_2} \boldsymbol{e}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}). \tag{5.11}$$

   The first factor of the cross product can be controlled directly and is in the space $C^{2+n-|\alpha_1|-k_1}(D_{\boldsymbol{q}}, \mathbb{R}^3) \subset C^{2+n-|\alpha|-k}(D_{\boldsymbol{q}}, \mathbb{R}^3)$, depending on the values $\boldsymbol{x}$ and $\frac{d^l}{dt^l} \boldsymbol{q}_t|_{t=t^{\pm}}$ for $l = 0, \ldots, |\alpha_1| + k_1$ only. The second factor is in $C^{n-|\alpha_2|-k_2}(D_{\boldsymbol{q}}, \mathbb{R}^3)$, and thus, in $C^{n-|\alpha|-k}(D_{\boldsymbol{q}}, \mathbb{R}^3)$ and depends on $\boldsymbol{x}$, $\frac{d^l}{dt^l} \boldsymbol{q}_t|_{t=t^{\pm}}, l = 0, \ldots, |\alpha_2| + k_2 + 2$, as shown in item 4.

   Thus, for all $|\alpha| + k \leq n$ the map $(t, \boldsymbol{x}) \mapsto D_{\boldsymbol{x}}^{\alpha} \partial_t^k \boldsymbol{f}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](t, \boldsymbol{x})$ is in $C^{n-|\alpha|-k}(D_{\boldsymbol{q}}, \mathbb{R}^6)$ and for all $(t, \boldsymbol{x}) \in D_{\boldsymbol{q}}$ the value $D_{\boldsymbol{x}}^{\alpha} \partial_t^k \boldsymbol{f}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x})$ is a function of $\boldsymbol{x}$ and $\left(\frac{d}{dt}\right)^l \boldsymbol{q}_t|_{t=t^{\pm}}$ for $l = 0, \ldots, |\alpha| + k + 2$ only.

6. It remains to show, that for all $t \in \mathbb{R}$ it holds $\boldsymbol{f}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}] \in \mathcal{F}$. For this it suffices to show integrability of $\boldsymbol{f}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}]$ on compact domains. By assumption $(\boldsymbol{q}, \boldsymbol{p})$ is strictly time-like, and thus, $\sup_{(t,\boldsymbol{x}) \in \mathbb{R}^4} |1 \pm \boldsymbol{n}^{\pm} \cdot \boldsymbol{v}^{\pm}| \geq 1 - v_{\max} > 0$. Moreover, we can estimate $|\boldsymbol{x} - \boldsymbol{q}^{\pm}|$ for all $(t, \boldsymbol{x}) \in \mathbb{R}^4$ by

$$|\boldsymbol{x} - \boldsymbol{q}^{\pm}| \geq |\boldsymbol{x} - \boldsymbol{q}_t| + |\boldsymbol{q}_t - \boldsymbol{q}^{\pm}| \geq |\boldsymbol{x} - \boldsymbol{q}_t| - v_{\max}|t^{\pm} - t| = |\boldsymbol{x} - \boldsymbol{q}_t| - v_{\max}|\boldsymbol{x} - \boldsymbol{q}^{\pm}|, \tag{5.12}$$

   using Definition (4.7) in the last step. This implies

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{q}^{\pm}|} \leq \frac{1 - v_{\max}}{|\boldsymbol{x} - \boldsymbol{q}_t|}. \tag{5.13}$$

Moreover, for all $\boldsymbol{x} \in B_1(\boldsymbol{q}_t) \setminus \{\boldsymbol{q}_t\}$ there exists a $C < \infty$ such that the electric Liénard-Wiechert field can be estimated by

$$\left|\boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x})\right| \leq \frac{C}{|\boldsymbol{x}-\boldsymbol{q}_t|^2}. \tag{5.14}$$

This follows from the subsequent estimation of (4.6):

$$\begin{aligned}\left|\boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x})\right| &= \left|\frac{(\boldsymbol{n}\pm\boldsymbol{v})(1-\boldsymbol{v}^2)}{|\boldsymbol{x}-\boldsymbol{q}|^2(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^3} + \frac{\boldsymbol{n}\wedge[(\boldsymbol{n}\pm\boldsymbol{v})\wedge\boldsymbol{a}]}{|\boldsymbol{x}-\boldsymbol{q}|(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^3}\right|^{\pm} \\ &\leq \left[\frac{1+|\boldsymbol{v}|+|\boldsymbol{x}-\boldsymbol{q}||\boldsymbol{n}\pm\boldsymbol{v}||\boldsymbol{a}|}{|\boldsymbol{x}-\boldsymbol{q}|^2(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^3}\right]^{\pm} \\ &\leq \frac{1+v_{\max}+2a_{\max}}{|\boldsymbol{x}-\boldsymbol{q}^{\pm}|^2(1-v_{\max})^3},\end{aligned}$$

where in the last step we used that $\boldsymbol{x} \in B_1(\boldsymbol{q}_t) \setminus \{\boldsymbol{q}_t\}$ and since $B_1(\boldsymbol{q}_t)$ is compact, also the range of the continuous map $t_{\boldsymbol{q}}^{\pm}(t,\cdot)$ of $B_1(\boldsymbol{q}_t)$ is. Hence, there is a bound on acceleration $\boldsymbol{a}^{\pm}$ called $a_{\max}$. Eventually, with estimate (5.13) the electric Liénard-Wiechert field estimate (5.14) can be found.

From the continuity shown in item 5., we get that for all $t \in \mathbb{R}$, $\boldsymbol{f}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}] \in C(S_{\boldsymbol{q}_t},\mathbb{R}^6)$. Thus, there is a upper bound for each $\boldsymbol{x}$ arbitrarily close to the singularity $\boldsymbol{q}_t$ that is locally integrable in three dimensions. By dominated convergence, we can therefore conclude that $\boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}]$ is locally integrable on $\mathbb{R}^3$.

By (5.10) the same upper bound can be found for the magnetic field $\left|\boldsymbol{b}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x})\right|$. Therefore, $\boldsymbol{f}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}] \in L^1_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{R}^6)$ and thus $\boldsymbol{f}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}] \in \mathcal{F}$ for all $t \in \mathbb{R}$.

Having proved item (i) and (ii), we turn over to the proof of item (iii):

By assumption we have $n \in \mathbb{N}_0$ and $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^{2+n}(\mathbb{R})$. In order to show that $\boldsymbol{f}^{\pm}[\boldsymbol{q},\boldsymbol{p}]$ is a Maxwell solution, we need to verify that for all $\rho \in \mathcal{D}$, $t \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^3$ the expression $\boldsymbol{f}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\rho_{\boldsymbol{x}}) = \boldsymbol{f}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\rho(\boldsymbol{x}-\cdot))$ solves (3.3) and (3.7). This is guaranteed by Theorem 3.10. and Theorem 3.12. in [8]. Namely, in Theorem 3.10. it has been shown that for all $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^2(\mathbb{R})$

$$\boldsymbol{f}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\rho_{\boldsymbol{x}}) = 4\pi \int_{\pm\infty}^{t} ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho(\cdot-\boldsymbol{q}_s) \\ \boldsymbol{v}_s\rho(\cdot-\boldsymbol{q}_s) \end{pmatrix}(\boldsymbol{x}), \tag{5.15}$$

where we have used the notation K_t from Definition A.1.1 (Propagator of the d'Alembert operator) for the propagator of the wave equation, and according to Theorem 3.12. in [8] the right hand side of equation (5.15) is a solution to the Maxwell equations including Maxwell constraints for all $t \in \mathbb{R}$. Moreover, Theorem 3.10. in [8] states that $\boldsymbol{x} \mapsto \boldsymbol{f}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\rho_{\boldsymbol{x}})$ is in $C^\infty(\mathbb{R}^3,\mathbb{R}^6)$ for all $t \in \mathbb{R}$.

To verify, that (5.15) is indeed in $\mathcal{F}^1(\mathbb{R})$ and thus a Maxwell solution in terms of Definition

3.3.1 (Charge and field trajectories) we compute

$$
\begin{aligned}
t \mapsto\ & \partial_t 4\pi \int_{\pm\infty}^{t} ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho(\cdot - \boldsymbol{q}_s) \\ \boldsymbol{v}_s \rho(\cdot - \boldsymbol{q}_s) \end{pmatrix} (\boldsymbol{x}) \\
&= \partial_t 4\pi \int_{\pm\infty}^{t} ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} (t-s) \fint_{B_{|t-s|}(0)} d\sigma(y) \begin{pmatrix} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_s) \\ \boldsymbol{v}_s \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_s) \end{pmatrix} \\
&= \partial_t 4\pi \int_{\pm\infty}^{t} ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} (t-s) \fint_{B_1(0)} d\sigma(y) \begin{pmatrix} \rho(\boldsymbol{x} - |t-s|\boldsymbol{y} - \boldsymbol{q}_s) \\ \boldsymbol{v}_s \rho(\boldsymbol{x} - |t-s|\boldsymbol{y} - \boldsymbol{q}_s) \end{pmatrix} \\
&= 4\pi \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} (t-s) \fint_{B_1(0)} d\sigma(y) \begin{pmatrix} \rho(\boldsymbol{x} - |t-s|\boldsymbol{y} - \boldsymbol{q}_s) \\ \boldsymbol{v}_s \rho(\boldsymbol{x} - |t-s|\boldsymbol{y} - \boldsymbol{q}_s) \end{pmatrix} \Bigg|_{t=s} \\
&+ 4\pi \int_{\pm\infty}^{t} ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} \fint_{B_1(0)} d\sigma(y) \begin{pmatrix} \rho(\boldsymbol{x} - |t-s|\boldsymbol{y} - \boldsymbol{q}_s) \\ \boldsymbol{v}_s \rho(\boldsymbol{x} - |t-s|\boldsymbol{y} - \boldsymbol{q}_s) \end{pmatrix} \\
&+ 4\pi \int_{\pm\infty}^{t} ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} (t-s) \fint_{B_1(0)} d\sigma(y) \begin{pmatrix} \partial_t \rho(\boldsymbol{x} - |t-s|\boldsymbol{y} - \boldsymbol{q}_s) \\ \boldsymbol{v}_s \partial_t \rho(\boldsymbol{x} - |t-s|\boldsymbol{y} - \boldsymbol{q}_s) \end{pmatrix} .
\end{aligned}
$$

Now, the first summand vanishes, since $t - s = 0$ for $t = s$. The second summand equals $\boldsymbol{f}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\rho_{\boldsymbol{x}})$ and is therefore well-defined. In the third summand the partial time derivative ∂_t acts on test function ρ and since the latter is smooth it is well defined and continuous. □

5.2 Existence of Maxwell solutions

In this section we prove Lemma 4.2.2 (Homogeneous Maxwell solutions) and 4.2.3 (Inhomogeneous Maxwell solutions).

Proof of Lemma 4.2.2. (Homogeneous Maxwell solutions)

(i) First, we show that for all $t \in \mathbb{R}$ the mapping $W_t : \mathcal{F}_{\mathrm{hom}} \to \mathcal{F}$, $\boldsymbol{f}_0^h \mapsto W_t \boldsymbol{f}_0^h$ given by

$$(W_t \boldsymbol{f}_0^h)(\rho_{\boldsymbol{x}}) := \mathcal{W}_t(\boldsymbol{f}_0^h, \rho_{\boldsymbol{x}}) = \begin{cases} \boldsymbol{f}_0^h(\rho_{\boldsymbol{x}}) & , t = 0 \\ \begin{pmatrix} \partial_t & \nabla_{\boldsymbol{x}} \wedge \\ -\nabla_{\boldsymbol{x}} \wedge & \partial_t \end{pmatrix} \frac{1}{4\pi t} \int_{\partial B_1(0)} d\sigma(y) \boldsymbol{f}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}) & , t \neq 0 \end{cases} \tag{5.16}$$

(cf. (4.8) in Definition 4.2.2 (Homogeneous Maxwell evolution)) for all $\boldsymbol{x} \in \mathbb{R}^3, \rho \in \mathcal{D}$ gives rise to a well-defined linear operator.

Therefore, let $\boldsymbol{f}_0^h \in \mathcal{F}_{\mathrm{hom}} \subset \mathcal{F}$ and write $\boldsymbol{f}_0^h = (\boldsymbol{e}_0^h, \boldsymbol{b}_0^h)$ for the electric and magnetic field components. The expression $\mathcal{W}_t(\boldsymbol{f}_0^h, \rho_{\boldsymbol{x}})$ in (5.16) is well-defined for all $\boldsymbol{x} \in \mathbb{R}^3, \rho \in \mathcal{D}$ because:

- For $t = 0$ it follows by the definition of $\mathcal{F}$ and $\boldsymbol{f}_0^h \in \mathcal{F}$.
- For $t \neq 0$ is follows by the facts that $\boldsymbol{f}_0^h(\rho_{(\cdot)}) \in C^\infty(\mathbb{R}^3, \mathbb{R}^6)$ and that the integration domain $\partial B_1(0)$ is compact. Therefore, the integral in (5.16) exists, the time derivative acting on $1/(4\pi t)$ and on $\boldsymbol{f}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}})$ can be applied and the curl acting on $\boldsymbol{e}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}})$ and $\boldsymbol{b}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}})$ can be applied, respectively.

Next, we show $W_t \boldsymbol{f}_0^h \in \mathcal{F}$ for all $t \in \mathbb{R}$, i.e., $W_t \boldsymbol{f}_0^h : \mathcal{D} \to \mathbb{R}^6$, $\rho_{\boldsymbol{x}} \mapsto \mathcal{W}_t(\boldsymbol{f}_0^h, \rho_{\boldsymbol{x}})$ is linear and continuous in $\rho_{\boldsymbol{x}}$ for all $\boldsymbol{x} \in \mathbb{R}^3$. This is true because:

- For $t = 0$ we have $\mathcal{W}_t(\boldsymbol{f}_0^h, \rho_{\boldsymbol{x}}) = \boldsymbol{f}_0^h(\rho_{\boldsymbol{x}})$ for all $\rho \in \mathcal{D}$, $\boldsymbol{x} \in \mathbb{R}^3$ by definition, and hence, $W_t \boldsymbol{f}_0^h = \boldsymbol{f}_0^h \in \mathcal{F}$ by assumption.
- For $t \neq 0$ the mapping is linear and continuous with respect to $\rho_{\boldsymbol{x}}$ for all $\boldsymbol{x} \in \mathbb{R}^3$, which shall be shown now:

  Linearity w.r.t. $\rho_{\boldsymbol{x}}$ follows by linearity of $\boldsymbol{f}_0^h$, linearity of the integral and linearity of the differential operators $\partial_t, \nabla_{\boldsymbol{x}} \wedge$.

  Thus, it remains to verify continuity of $W_t \boldsymbol{f}_0^h$. Let therefore $(\rho^{(n)})_{n \in \mathbb{N}}$ be a null sequence in $\mathcal{D}$, i.e., as introduced in Chapter 3, there is a compact $K \subset \mathbb{R}^3$ such that for all $n \in \mathbb{N}$ supp $\rho^{(n)} \subseteq K$ and for all $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^3$

$$\lim_{n \to \infty} \sup_{\boldsymbol{y} \in \mathbb{R}^3} \left| D_{\boldsymbol{y}}^\alpha \rho^{(n)}(\boldsymbol{y}) \right| = 0. \tag{5.17}$$

  In consequence, for all $\boldsymbol{x} \in \mathbb{R}^3$ also $(\rho_{\boldsymbol{x}}^{(n)})_{n \in \mathbb{N}}$ is a null sequence in $\mathcal{D}$; cf. Lemma A.3.1 (Distributions), (i).

We need to show, $\lim_{n\to\infty}(W_t\boldsymbol{f}_0^h)(\rho_{\boldsymbol{x}}^{(n)}) = 0$ for all $\boldsymbol{x} \in \mathbb{R}^3$. For this purpose, we consider

$$
\begin{aligned}
&|(W_t\boldsymbol{f}_0^h)(\rho_{\boldsymbol{x}}^{(n)})| = \left|\begin{pmatrix} \partial_t & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & \partial_t \end{pmatrix} \frac{1}{4\pi t}\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{f}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| \\
&\leq \left|\partial_t \frac{1}{4\pi t}\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{e}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)}) + \frac{1}{4\pi t}\int_{\partial B_1(0)} d\sigma(y)\nabla_{\boldsymbol{x}}\wedge\boldsymbol{b}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| \\
&+ \left|\partial_t \frac{1}{4\pi t}\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{b}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)}) - \frac{1}{4\pi t}\int_{\partial B_1(0)} d\sigma(y)\nabla_{\boldsymbol{x}}\wedge\boldsymbol{e}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| \\
&\leq \left|\partial_t \frac{1}{4\pi t}\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{e}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| + \left|\partial_t \frac{1}{4\pi t}\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{b}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| \\
&+ \left|\frac{1}{4\pi t}\int_{\partial B_1(0)} d\sigma(y)\nabla_{\boldsymbol{x}}\wedge\boldsymbol{b}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| + \left|\frac{1}{4\pi t}\int_{\partial B_1(0)} d\sigma(y)\nabla_{\boldsymbol{x}}\wedge\boldsymbol{e}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| \\
&= \frac{1}{4\pi t^2}\left|\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{e}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| + \frac{1}{4\pi t^2}\left|\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{b}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| \\
&+ \frac{1}{4\pi|t|}\left|\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{e}_0^h(\partial_t\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| + \frac{1}{4\pi|t|}\left|\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{b}_0^h(\partial_t\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| \\
&+ \frac{1}{4\pi|t|}\left|\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{b}_0^h(\nabla_{\boldsymbol{x}}\wedge\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| + \frac{1}{4\pi|t|}\left|\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{e}_0^h(\nabla_{\boldsymbol{x}}\wedge\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| \\
&\leq 2\frac{1+|t|}{4\pi t^2}\left[\left|\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{f}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right| + \sum_{i=1}^{3}\left|\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{f}_0^h(\partial_{x_i}\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)}y_i)\right|\right. \\
&\qquad\left. + \left|\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{f}_0^h(\nabla_{\boldsymbol{x}}\wedge\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right|\right], \qquad (5.18)
\end{aligned}
$$

where we have used the inequality $|\boldsymbol{f}| \leq |\boldsymbol{e}| + |\boldsymbol{b}| \leq 2|\boldsymbol{f}|$ for $\boldsymbol{f} = (\boldsymbol{e}, \boldsymbol{b}) \in \mathbb{R}^3 \times \mathbb{R}^3$ in the first and in the last step of the estimation and in the second step the triangle inequality. In the third step the product rule was applied for the time derivative. Furthermore, we used

$$
\begin{aligned}
\partial_t\boldsymbol{e}_0^h(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)}) = \boldsymbol{e}_0^h(\partial_t\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)}) &= -\boldsymbol{e}_0^h(\nabla\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)} \cdot \boldsymbol{y}) \\
&= -\sum_{i=1}^{3}\boldsymbol{e}_0^h(\partial_i\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)}y_i),
\end{aligned}
$$

which can be verified by direct computation. A respective equation can be found for the magnetic field component. Now, each of the summands in the last line can be written as $\left|\int_{\partial B_1(0)} d\sigma(y)\boldsymbol{f}_0^h(D_{\boldsymbol{x}}^{\alpha}\varphi_{\boldsymbol{x}-|t|\boldsymbol{y}}^{(n)})\right|$ for $|\alpha| \leq 1$ and a null sequence $(\varphi^{(n)})_{n\in\mathbb{N}}$ since for $\boldsymbol{y} \in \partial B_1(0)$, $(\rho^{(n)}y_i)_{n\in\mathbb{N}}$ is again a null sequence. Then, for all $t \neq 0$,

$\boldsymbol{x} \in \mathbb{R}^3$ we compute

$$\begin{aligned}
\lim_{n\to\infty} \left| \int_{\partial B_1(0)} d\sigma(y) \boldsymbol{f}_0^h (D^\alpha_{\boldsymbol{x}} \varphi^{(n)}_{\boldsymbol{x}-|t|\boldsymbol{y}}) \right| &= \lim_{n\to\infty} \left| \int_{\partial B_{|t|}(\boldsymbol{x})} d\sigma(z) \boldsymbol{f}_0^h (D^\alpha_{\boldsymbol{z}} \varphi^{(n)}_{\boldsymbol{z}}) \right| \\
&= \lim_{n\to\infty} \left| \int_{\mathbb{R}^3} d^3z \mathbb{1}_{\partial B_{|t|}(\boldsymbol{x})}(\boldsymbol{z}) \boldsymbol{f}_0^h (D^\alpha_{\boldsymbol{z}} \varphi^{(n)}_{\boldsymbol{z}}) \right| \\
&= \lim_{n\to\infty} \left| \boldsymbol{f}_0^h \left(\int_{\mathbb{R}^3} d^3z \mathbb{1}_{\partial B_{|t|}(\boldsymbol{x})}(\boldsymbol{z}) D^\alpha_{\boldsymbol{z}} \varphi^{(n)}_{\boldsymbol{z}} \right) \right| \\
&= \lim_{n\to\infty} \left| \boldsymbol{f}_0^h (\mathbb{1}_{\partial B_{|t|}(\boldsymbol{x})} * D^\alpha_{(\cdot)} \varphi^{(n)}_{(\cdot)}(0)) \right| \\
&= 0,
\end{aligned}$$

where the following arguments have been applied:
The assumption of $(\varphi^{(n)})_{n\in\mathbb{N}}$ being a null sequence in $\mathcal{D}$ implies that for all $\boldsymbol{z} \in \mathbb{R}^3$, $(\varphi^{(n)}_{\boldsymbol{z}})_{n\in\mathbb{N}}$ is a null sequence, $(D^\alpha_{\boldsymbol{z}} \varphi^{(n)}_{\boldsymbol{z}})_{n\in\mathbb{N}}$ is a null sequence and for compact sets K also $(\mathbb{1}_K * D^\alpha_{(\cdot)} \varphi^{(n)}_{(\cdot)}(0))_{n\in\mathbb{N}}$ is a null sequence; see Lemma A.3.1 (Distributions), (i). As $\partial B_{|t|}(\boldsymbol{x})$ is compact and by assumption $\boldsymbol{f}_0^h \in \mathcal{F}$ for any $\boldsymbol{x} \in \mathbb{R}^3$ the last equality holds. Furthermore, in the second last equality we made use of the fact that for any $\boldsymbol{f} \in \mathcal{F}$, $\varphi \in \mathcal{D}$, and any $\psi \in L^1(\mathbb{R}^3, \mathbb{R})$ with compact support convolutions and distributions can be interchanged, i.e.,

$$\int d^3z \psi(\boldsymbol{z}) \boldsymbol{f}(\varphi_{\boldsymbol{z}}) = \boldsymbol{f}(\psi * \varphi_0) \tag{5.19}$$

as shown in Lemma A.3.1 (Distributions), (iii).
Finally, each summand in (5.18) converges to 0 as $n \to \infty$, and thus, for all $\rho \in \mathcal{D}$, $\boldsymbol{x} \in \mathbb{R}^3$ $\mathcal{W}_t(\boldsymbol{f}_0^h, \rho_{\boldsymbol{x}})$ is also continuous in $\rho_{\boldsymbol{x}}$.
In conclusion, $\rho \mapsto \mathcal{W}_t(\boldsymbol{f}_0^h, \rho_{\boldsymbol{x}})$ is in $\mathcal{F}$.

Furthermore, by definition, the map $\boldsymbol{f}_0^h \mapsto (W_t \boldsymbol{f}_0^h)(\rho_{\boldsymbol{x}})$ is linear for all $\rho \in \mathcal{D}, \boldsymbol{x} \in \mathbb{R}^3$. This can directly be read off the defining equation (5.16) together with the linearity of convolution, integration, and the differential operators.

We may therefore conclude that for all $t \in \mathbb{R}$, $W_t : \mathcal{F} \to \mathcal{F}$ is a well-defined linear operator.

(ii) We need to show that for all $\rho \in \mathcal{D}, \boldsymbol{x} \in \mathbb{R}^3$:

$$t \mapsto (W_t \boldsymbol{f}_0^h)(\rho_{\boldsymbol{x}}) = \mathcal{W}_t(\boldsymbol{f}_0^h, \rho_{\boldsymbol{x}}) \in C^\infty(\mathbb{R}, \mathbb{R}^6). \tag{5.20}$$

For a slimmer notation, we write the operator W_t in terms of the propagator K_t of the wave equation, introduced in Definition A.1.1 (Propagator of the d'Alembert operator). Therewith, the action of W_t on $\boldsymbol{f}_0^h$ evaluated at test function $\rho_{\boldsymbol{x}}$ reads

$$(W_t \boldsymbol{f}_0^h)(\rho_{\boldsymbol{x}}) = \begin{cases} \boldsymbol{f}_0^h(\rho_{\boldsymbol{x}}) & , t = 0 \\ \begin{pmatrix} \partial_t & \nabla_{\boldsymbol{x}} \wedge \\ -\nabla_{\boldsymbol{x}} \wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{f}_0^h(\rho_\cdot)(\boldsymbol{x}) & , t \neq 0 \end{cases}. \tag{5.21}$$

Let $n \in \mathbb{N}_0$, then for $t \neq 0$

$$\partial_t^n(W_t\boldsymbol{f}_0^h)(\rho_x) = \partial_t \begin{pmatrix} \partial_t & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{f}_0^h(\rho_{(\cdot)})(\boldsymbol{x})$$
$$= \begin{pmatrix} \partial_t^{1+n} K_t * \boldsymbol{e}_0^h(\rho_{(\cdot)})(\boldsymbol{x}) + \partial_t^n K_t * \nabla \wedge \boldsymbol{b}_0^h(\rho_{(\cdot)})(\boldsymbol{x}) \\ \partial_t^n K_t * \nabla \wedge \boldsymbol{e}_0^h(\rho_{(\cdot)})(\boldsymbol{x}) + \partial_t^{1+n} K_t * \boldsymbol{b}_0^h(\rho_{(\cdot)})(\boldsymbol{x}) \end{pmatrix}.$$

Since for any test function $\rho \in \mathcal{D}$, $\boldsymbol{e}_0^h(\rho_{(\cdot)}), \boldsymbol{b}_0^h(\rho_{(\cdot)}) \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$, and thus, $\nabla \wedge \boldsymbol{e}_0^h(\rho_{(\cdot)}), \nabla \wedge \boldsymbol{b}_0^h(\rho_{(\cdot)}) \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$, we can apply Lemma 3.4. (iii) from [8] which states, that for all $f \in C^\infty(\mathbb{R}^3)$ the map $(t, \boldsymbol{x}) \mapsto (K_t * f)(\boldsymbol{x})$ can uniquely be extended at $t = 0$ to give rise to a $C^\infty(\mathbb{R} \times \mathbb{R}^3)$ function fulfilling $K_t * f\big|_{t=0} = 0$ and $\partial_t K_t * f\big|_{t=0} = f$ and $\partial_t^2 K_t * f\big|_{t=0} = \Delta f$; cf. also Lemma A.1.1 (Properties of K_t). Hence, for all $n \in \mathbb{N}_0$ the mapping $t \mapsto \partial_t^n(W_t\boldsymbol{f}_0^h)(\rho_{\boldsymbol{x}})$ is continuous on $\mathbb{R} \setminus \{0\}$ and can uniquely be extended smoothly at $t = 0$ by choice (5.21).

(iii) First we shall show that $W_t\boldsymbol{f}_0^h$ is a homogeneous Maxwell solution on $\mathbb{R}$, and second, that it is uniquely defined by the initial value $\boldsymbol{f}_0^h$.

Solution: We need to show that $W_t\boldsymbol{f}_0^h$ is a homogeneous Maxwell solution for the initial value $\boldsymbol{f}_0^h \in \mathcal{F}_{\text{hom}}$. We thus have to verify equality (3.6) and (3.2), i.e.,

$$\partial_t W_t\boldsymbol{f}_0^h(\rho_{\boldsymbol{x}}) = \begin{pmatrix} 0 & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & 0 \end{pmatrix} W_t\boldsymbol{f}_0^h(\rho_{\boldsymbol{x}}) \tag{5.22}$$

and

$$\begin{pmatrix} \nabla_{\boldsymbol{x}}\cdot & 0 \\ 0 & \nabla_{\boldsymbol{x}}\cdot \end{pmatrix} W_t\boldsymbol{f}_0^h(\rho_{\boldsymbol{x}}) = 0 \tag{5.23}$$

for all $\rho \in \mathcal{D}$, $\boldsymbol{x} \in \mathbb{R}^3$, $t \in \mathbb{R}$, and verify

$$\lim_{t \to 0} W_t\boldsymbol{f}_0^h = \boldsymbol{f}_0^h. \tag{5.24}$$

Let $t \neq 0$, $\boldsymbol{x} \in \mathbb{R}^3$, $\rho \in \mathcal{D}$. Then, we have

$$\left(\partial_t - \begin{pmatrix} 0 & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & 0 \end{pmatrix}\right) W_t\boldsymbol{f}_0^h(\rho_{\boldsymbol{x}})$$
$$= \left(\begin{pmatrix} \partial_t^2 & \partial_t\nabla_{\boldsymbol{x}}\wedge \\ -\partial_t\nabla_{\boldsymbol{x}}\wedge & \partial_t^2 \end{pmatrix} - \begin{pmatrix} -\nabla_{\boldsymbol{x}} \wedge \nabla_{\boldsymbol{x}}\wedge & \nabla_{\boldsymbol{x}} \wedge \partial_t \\ -\nabla_{\boldsymbol{x}} \wedge \partial_t & -\nabla_{\boldsymbol{x}} \wedge \nabla_{\boldsymbol{x}}\wedge \end{pmatrix}\right) K_t * \boldsymbol{f}_0^h(\rho_{(\cdot)})(\boldsymbol{x})$$
$$= \begin{pmatrix} \Delta_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}} \wedge \nabla_{\boldsymbol{x}}\wedge & 0 \\ 0 & \Delta_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}} \wedge \nabla_{\boldsymbol{x}}\wedge \end{pmatrix} K_t * \boldsymbol{f}_0^h(\rho_{(\cdot)})(\boldsymbol{x})$$
$$= K_t * \begin{pmatrix} \nabla\nabla\cdot & 0 \\ 0 & \nabla\nabla\cdot \end{pmatrix} \boldsymbol{f}_0^h(\rho_{(\cdot)})(\boldsymbol{x})$$
$$= K_t * \begin{pmatrix} \nabla\nabla \cdot \boldsymbol{e}_0^h(\rho_{(\cdot)}) \\ \nabla\nabla \cdot \boldsymbol{b}_0^h(\rho_{(\cdot)}) \end{pmatrix} (\boldsymbol{x})$$
$$= 0,$$

where the last equality is due to the assumption $\boldsymbol{f}_0^h \in \mathcal{F}_{\text{hom}}$, i.e., the homogeneous Maxwell constraints (3.2) hold, and the second equality uses the property $\partial_t^2 K_t * g =$

$K_t * \Delta g$ for functions $g \in C^2(\mathbb{R}^3, \mathbb{R})$ (cf. Lemma A.1.1 (Properties of K_t)) and the identity $\Delta + \nabla \wedge \nabla \wedge = \nabla \nabla \cdot$. Therefore, (5.22) holds for all $t \neq 0$, and hence, by the smoothness shown in (ii) for all $t \in \mathbb{R}$.

Equation (5.23) is met because Lemma A.2.1 (Maxwell constraints) states that once (5.22) holds under the assumption $\boldsymbol{f}_0^h \in \mathcal{F}_{\text{hom}}$ the homogeneous Maxwell constraints (3.2) hold for all $t \in \Lambda$, i.e., $\boldsymbol{f}_t^h \in \mathcal{F}_{\text{hom}}$ for all $t \in \Lambda$.

Next, we compute the initial value

$$\begin{aligned}
\lim_{t\to 0} W_t \boldsymbol{f}_0^h(\rho_{\boldsymbol{x}}) &= \lim_{t\to 0} \begin{pmatrix} \partial_t & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{f}_0^h(\rho_{(\cdot)})(\boldsymbol{x}) \\
&= \begin{pmatrix} \lim_{t\to 0} \partial_t K_t * \boldsymbol{e}_0^h(\rho_{(\cdot)})(\boldsymbol{x}) + \lim_{t\to 0} K_t * \nabla \wedge \boldsymbol{b}_0^h(\rho_{(\cdot)})(\boldsymbol{x}) \\ -\lim_{t\to 0} K_t * \nabla \wedge \boldsymbol{e}_0^h(\rho_{(\cdot)})(\boldsymbol{x}) + \lim_{t\to 0} \partial_t K_t * \boldsymbol{b}_0^h(\rho_{(\cdot)})(\boldsymbol{x}) \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{e}_0^h(\rho_{(\cdot)})(\boldsymbol{x}) \\ \boldsymbol{b}_0^h(\rho_{(\cdot)})(\boldsymbol{x}) \end{pmatrix} = \boldsymbol{f}_0^h(\rho_{\boldsymbol{x}}),
\end{aligned}$$

where we have used $\lim_{t\to 0} \partial_t K_t * g = g$ and $\lim_{t\to 0} K_t * g = 0$ for $g \in C^2(\mathbb{R}^3, \mathbb{R})$ (see again Lemma A.1.1 (Properties of K_t)).

Uniqueness: Let $\Lambda \subseteq \mathbb{R}$ be an interval including point $t = 0$. By assumption $\boldsymbol{f}_0^h \in \mathcal{F}_{\text{hom}}$ and $\boldsymbol{f}^h$ is a homogeneous Maxwell solution on Λ such that $\boldsymbol{f}_t^h = (\boldsymbol{e}_t^h, \boldsymbol{b}_t^h) \in \mathcal{F}^1$ and $\boldsymbol{f}_t^h\big|_{t=0} = \boldsymbol{f}_0^h$. According to the paragraph above it follows $\boldsymbol{f}_t^h \in \mathcal{F}_{\text{hom}}$ for all $t \in \Lambda$.

Firstly for all $\rho \in \mathcal{D}$, $\boldsymbol{x} \in \mathbb{R}^3$, $(-t) \in \Lambda$

$$\begin{aligned}
\lim_{t\to 0} W_t \boldsymbol{f}_{-t}^h(\rho_{\boldsymbol{x}}) &= \lim_{t\to 0} \begin{pmatrix} \partial_t & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{f}_{-s}^h(\rho_{(\cdot)})(\boldsymbol{x})\big|_{s=t} \\
&= \lim_{t\to 0} \begin{pmatrix} \lim_{t\to 0} \partial_t K_t * \boldsymbol{e}_{-t}^h(\rho_{(\cdot)})(\boldsymbol{x}) + \lim_{t\to 0} K_t * \nabla \wedge \boldsymbol{b}_{-t}^h(\rho_{(\cdot)})(\boldsymbol{x}) \\ -\lim_{t\to 0} K_t * \nabla \wedge \boldsymbol{e}_{-t}^h(\rho_{(\cdot)})(\boldsymbol{x}) + \lim_{t\to 0} \partial_t K_t * \boldsymbol{b}_{-t}^h(\rho_{(\cdot)})(\boldsymbol{x}) \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{e}_0^h(\rho_{(\cdot)})(\boldsymbol{x}) \\ \boldsymbol{b}_0^h(\rho_{(\cdot)})(\boldsymbol{x}) \end{pmatrix} = \boldsymbol{f}_0^h(\rho_{\boldsymbol{x}}),
\end{aligned}$$

where we used $\boldsymbol{f}_t^h \in \mathcal{F}^1$ and $\lim_{t\to 0} \partial_t K_t * g_t = g_0$, $\lim_{t\to 0} K_t * g_t = 0$ for $g_t \in C^2(\mathbb{R} \times \mathbb{R}^3, \mathbb{R})$ (cf. Lemma A.1.1 (Properties of K_t)), and the computation rules for limits of compositions of continuous maps. And, secondly, exploiting (5.22) and (3.6) we find

$$\begin{aligned}
\partial_t (W_t \boldsymbol{f}_{-t}^h)(\rho_{\boldsymbol{x}}) &= \left[\partial_t W_t \boldsymbol{f}_{-s}^h(\rho_{\boldsymbol{x}}) + W_t \partial_s \boldsymbol{f}_{-s}^h(\rho_{\boldsymbol{x}})\right]\big|_{s=t} \\
&= \left[\partial_t W_t \boldsymbol{f}_{-s}^h(\rho_{\boldsymbol{x}}) - W_t \begin{pmatrix} 0 & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & 0 \end{pmatrix} \boldsymbol{f}_{-s}^h(\rho_{\boldsymbol{x}})\right]\big|_{s=t} \\
&= \left(\begin{pmatrix} \partial_t^2 & \partial_t \nabla_{\boldsymbol{x}}\wedge \\ -\partial_t \nabla_{\boldsymbol{x}}\wedge & \partial_t^2 \end{pmatrix} - \begin{pmatrix} -\nabla_{\boldsymbol{x}} \wedge \nabla_{\boldsymbol{x}}\wedge & \partial_t \nabla_{\boldsymbol{x}}\wedge \\ -\partial_t \nabla_{\boldsymbol{x}}\wedge & -\nabla_{\boldsymbol{x}} \wedge \nabla_{\boldsymbol{x}}\wedge \end{pmatrix}\right) \\
&\quad K_t * \boldsymbol{f}_{-s}^h(\rho_{\cdot})(\boldsymbol{x})\big|_{s=t}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} \Delta_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}} \wedge \nabla_{\boldsymbol{x}} \wedge & 0 \\ 0 & \Delta_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}} \wedge \nabla_{\boldsymbol{x}} \wedge \end{pmatrix} K_t * \boldsymbol{f}^h_{-s}(\rho_{(\cdot)})(\boldsymbol{x})\big|_{s=t} \\
&= K_t * \begin{pmatrix} \nabla\nabla\cdot & 0 \\ 0 & \nabla\nabla\cdot \end{pmatrix} \boldsymbol{f}^h_{-s}(\rho_{(\cdot)})(\boldsymbol{x})\big|_{s=t} \\
&= K_t * \begin{pmatrix} \nabla\nabla \cdot \boldsymbol{e}^h_{-s}(\rho_{(\cdot)}) \\ \nabla\nabla \cdot \boldsymbol{b}^h_{-s}(\rho_{(\cdot)}) \end{pmatrix} (\boldsymbol{x})\big|_{s=t} \\
&= 0,
\end{aligned}
$$

where in the last step we have used $\boldsymbol{f}^h_t \in \mathcal{F}_{\text{hom}}$ for all $t \in \Lambda$. Respectively, we have shown $W_{-t}\boldsymbol{f}^h_t = \boldsymbol{f}^h_0$. This implies

$$
\boldsymbol{f}^h_t = id_{\mathcal{F}_{\text{hom}}} \boldsymbol{f}^h_t = W_t W_{-t} \boldsymbol{f}^h_t = W_t \boldsymbol{f}^h_0 \tag{5.25}
$$

for all $t \in \Lambda$, i.e., $\boldsymbol{f}^h_t = W_t\boldsymbol{f}^h_0$ is the unique solution on $t \in \Lambda$, provided $W_tW_{-t} = id_{\mathcal{F}_{\text{hom}}}$ holds true, which we show next. We use the abbreviation $A := \begin{pmatrix} 0 & \nabla\wedge \\ -\nabla\wedge & 0 \end{pmatrix}$. For all $\rho \in \mathcal{D}$, $\boldsymbol{x} \in \mathbb{R}^3$, $\boldsymbol{h}^h \in \mathcal{F}_{\text{hom}}$ we have for $t = 0$ $W_tW_t\boldsymbol{h}^h = \boldsymbol{h}^h$, and for $t \neq 0$:

$$
\begin{aligned}
\partial_t(W_tW_{-t})\boldsymbol{h}^h &= \partial_s W_s W_{-t}\boldsymbol{h}^h + W_t\partial_s W_{-s}\boldsymbol{h}^h\big|_{s=t} = AW_tW_{-t}\boldsymbol{h}^h - W_t\partial_s W_s\boldsymbol{h}^h\big|_{s=-t} \\
&= AW_tW_{-t}\boldsymbol{h}^h - W_tAW_s\boldsymbol{h}^h\big|_{s=-t} = AW_tW_{-t}\boldsymbol{h}^h - AW_tW_{-t}\boldsymbol{h}^h = 0,
\end{aligned}
$$

where we have used that:

- $\boldsymbol{h}^h \in \mathcal{F}_{\text{hom}}$ implies $W_{-t}\boldsymbol{h}^h \in \mathcal{F}_{\text{hom}}$, by Lemma A.2.1 (Maxwell constraints) and thus $W_t(W_{-t}\boldsymbol{h}^h)$ is a Maxwell solution by the paragraph solution above.
- $\boldsymbol{h}^h \in \mathcal{F}_{\text{hom}}$ implies $W_s\boldsymbol{h}^h$ is a Maxwell solution also by the paragraph above.
- We can interchange the differential operator A with the operator W_t, since the spacial derivatives in A are applied to the respective test function, only.

(iv) Let $\boldsymbol{f}^h_0 \in C^{1+n}(\mathbb{R}^3, \mathbb{R}^6)$. We show that for the solution $t \mapsto \boldsymbol{f}^h_t = W_t\boldsymbol{f}^h_0$ it holds $\boldsymbol{f}^h \in C^n(\mathbb{R}^4, \mathbb{R}^6)$. By definition, for all $\rho \in \mathcal{D}$, $\boldsymbol{x} \in \mathbb{R}^3$, $t \neq 0$:

$$
\begin{aligned}
\boldsymbol{f}^h_t(\rho_{\boldsymbol{x}}) &= \begin{pmatrix} \partial_t & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & \partial_t \end{pmatrix} \frac{1}{4\pi t} \int_{\partial B_1(0)} d\sigma(y) \boldsymbol{f}^h_0(\rho_{\boldsymbol{x}-|t|\boldsymbol{y}}) \\
&= \begin{pmatrix} \partial_t & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & \partial_t \end{pmatrix} \frac{1}{4\pi t} \int_{\partial B_1(0)} d\sigma(y) \int d^3z \boldsymbol{f}^h_0(\boldsymbol{z})\rho(\boldsymbol{x} - \boldsymbol{z} - |t|\boldsymbol{y}) \\
&= \begin{pmatrix} \partial_t & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & \partial_t \end{pmatrix} \frac{1}{4\pi t} \int_{\partial B_1(0)} d\sigma(y) \int d^3z \boldsymbol{f}^h_0(\boldsymbol{z} - |t|\boldsymbol{y})\rho(\boldsymbol{x} - \boldsymbol{z}),
\end{aligned}
$$

where we used $\boldsymbol{f}^h_0 \in C^{1+n}(\mathbb{R}^3, \mathbb{R}^6)$ in the second line. Moreover, this implies that the left hand side is a continuous function as well and we can write

$$
\begin{aligned}
&\int d^3z \boldsymbol{f}^h_t(\boldsymbol{z})\rho(\boldsymbol{x} - \boldsymbol{z}) \\
&= \begin{pmatrix} \partial_t & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & \partial_t \end{pmatrix} \frac{1}{4\pi t} \int_{\partial B_1(0)} d\sigma(y) \int d^3z \boldsymbol{f}^h_0(\boldsymbol{z} - |t|\boldsymbol{y})\rho(\boldsymbol{x} - \boldsymbol{z}) \\
&= \int d^3z \rho(\boldsymbol{x} - \boldsymbol{z}) \begin{pmatrix} \partial_t & \nabla_{\boldsymbol{x}}\wedge \\ -\nabla_{\boldsymbol{x}}\wedge & \partial_t \end{pmatrix} \frac{1}{4\pi t} \int_{\partial B_1(0)} d\sigma(y) \boldsymbol{f}^h_0(\boldsymbol{z} - |t|\boldsymbol{y}),
\end{aligned}
$$

by the use of Fubini in the second line. Hence, for almost every $\boldsymbol{z} \in \mathbb{R}^3$ and $t \neq 0$

$$\boldsymbol{f}_t^h(\boldsymbol{z}) = \begin{pmatrix} \partial_t & \nabla_{\boldsymbol{x}} \wedge \\ -\nabla_{\boldsymbol{x}} \wedge & \partial_t \end{pmatrix} \frac{1}{4\pi t} \int_{\partial B_1(0)} d\sigma(y) \boldsymbol{f}_0^h(\boldsymbol{z} - |t|\boldsymbol{y}). \tag{5.26}$$

This representation shows that for $\boldsymbol{f}_0^h \in C^{1+n}(\mathbb{R}^3, \mathbb{R}^6)$ we have $(t, \boldsymbol{x}) \mapsto \boldsymbol{f}_t^h(\boldsymbol{x}) \in C^n(\mathbb{R} \setminus \{0\} \times \mathbb{R}^3, \mathbb{R}^6)$.

For $t = 0$, we make use of the properties of propagator K_t. According to item (iii) of Lemma A.1.1 (Properties of K_t) $(t, \boldsymbol{x}) \mapsto \boldsymbol{f}_t^h(\boldsymbol{x})$ as given in (5.26) can be n times continuously extended at $t = 0$.

□

We move on to the proof of Lemma 4.2.3(Inhomogeneous Maxwell solutions), which can be deduced from the previous results on the homogeneous Maxwell equations.

Proof of Lemma 4.2.3. (Inhomogeneous Maxwell solutions)

Let $n \in \mathbb{N}_0$, $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{T}^{2+n}$ strictly time-like, and $\boldsymbol{f}_0 \in \mathcal{F}_{\boldsymbol{q}_{t=0}}$.

(ii) We start by proving the uniqueness of inhomogeneous Maxwell solutions, which is a straight forward consequence of the uniqueness result for the homogeneous Maxwell equations from Lemma 4.2.2 (Homogeneous Maxwell solutions), (iii) as we shall see now.

Assume that $t \mapsto \boldsymbol{f}_t$ and $t \mapsto \boldsymbol{g}_t$ are Maxwell solutions on the interval Λ containing $t = 0$ with initial values $\boldsymbol{f}_{t=0} = \boldsymbol{g}_{t=0} = \boldsymbol{f}_0 \in \mathcal{F}_{\boldsymbol{q}_{t=0}}$. By linearity of Maxwell's equations $t \mapsto \boldsymbol{h}_t := \boldsymbol{f}_t - \boldsymbol{g}_t$, $t \in \Lambda$ is a homogeneous Maxwell solution on Λ uniquely characterized by its initial value $\boldsymbol{h}_{t=0} = \boldsymbol{f}_{t=0} - \boldsymbol{g}_{t=0}$, which vanishes by assumption and lies in $\mathcal{F}_{\text{hom}}$. Now, Lemma 4.2.2 (Homogeneous Maxwell solutions), (iii), guarantees that for all $t \in \Lambda$ it holds $\boldsymbol{h}_t = W_t \boldsymbol{h}_{t=0} = 0$, and hence, for all $t \in \Lambda$, $\boldsymbol{f}_t = \boldsymbol{g}_t$.

(i)+(iii) By (ii) it suffices to show that $\boldsymbol{f}$ given by (4.11) is a Maxwell solution on $\mathbb{R}$ for $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{T}^{2+n}(\mathbb{R})$ with $n \in \mathbb{N}_0$.

Let $\lambda \in [0, 1]$. By Lemma 4.2.1 (Properties of Liénard-Wiechert fields), (iii), $\boldsymbol{f}^{\pm}[\boldsymbol{q}, \boldsymbol{p}]$ are Maxwell solutions on $\mathbb{R}$ for $(\boldsymbol{q}, \boldsymbol{p})$, i.e., $\boldsymbol{f}^{\pm}_{t=0}[\boldsymbol{q}, \boldsymbol{p}] \in \mathcal{F}_{\boldsymbol{q}_{t=0}}$ as well as $\lambda \boldsymbol{f}^{-}_{t=0}[\boldsymbol{q}, \boldsymbol{p}] + (1-\lambda) \boldsymbol{f}^{+}_{t=0}[\boldsymbol{q}, \boldsymbol{p}] \in \mathcal{F}_{\boldsymbol{q}_{t=0}}$ for any $\lambda \in [0, 1]$. Together with the condition $\boldsymbol{f}_0 \in \mathcal{F}_{\text{hom}}$ and the linearity of the divergence operator in Maxwell's constraints (3.2) and (3.3), this implies

$$\boldsymbol{f}^h_0 = \boldsymbol{f}_0 - (\lambda \boldsymbol{f}^{-}_{t=0}[\boldsymbol{q}, \boldsymbol{p}] + (1-\lambda) \boldsymbol{f}^{+}_{t=0}[\boldsymbol{q}, \boldsymbol{p}]) \in \mathcal{F}_{\text{hom}}. \tag{5.27}$$

Hence, by Lemma 4.2.2 (Homogeneous Maxwell solutions), (iii), for all $t \in \mathbb{R}$, $\boldsymbol{f}^h_t = W_t \boldsymbol{f}^h_0$ is a homogeneous Maxwell solution on $\mathbb{R}$ with initial value $\boldsymbol{f}^h_0$. Again, due to Lemma 4.2.1 (Properties of Liénard-Wiechert fields), (iii), and the linearity of the Maxwell equations

$$t \mapsto \boldsymbol{f}_t = \lambda \boldsymbol{f}^{-}_t[\boldsymbol{q}, \boldsymbol{p}] + (1-\lambda) \boldsymbol{f}^{+}_t[\boldsymbol{q}, \boldsymbol{p}] + W_t \boldsymbol{f}^h_0 \tag{5.28}$$

(4.11) is a Maxwell solution on $\mathbb{R}$ for $(\boldsymbol{q}, \boldsymbol{p})$. By definition of $\boldsymbol{f}^h_0$ in (5.27) we have $\boldsymbol{f}_{t=0} = \boldsymbol{f}_0$.

□

5.3 Explicit Maxwell solutions

This section consists of the proof of our second main result which is Theorem 4.2.1 (Explicit Maxwell solutions).

Proof of Theorem 4.2.1. (Explicit Maxwell solutions)

Let $n \in \mathbb{N}$, $(\boldsymbol{q},\boldsymbol{p}),(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}) \in \mathcal{T}^{2+n}(\mathbb{R})$ be strictly time-like charge trajectories with $\boldsymbol{q}_{t=0} =: \boldsymbol{q}_0 = \tilde{\boldsymbol{q}}_0 := \tilde{\boldsymbol{q}}_{t=0}$ and $\boldsymbol{f}_0 \in \mathcal{F}_{\boldsymbol{q}_0}$. We show that $\boldsymbol{f} : t \mapsto \boldsymbol{f}_t$ where $\boldsymbol{f}_t$ is given by (4.12)-(4.17) is the unique Maxwell solution on $\mathbb{R}$ for $(\boldsymbol{q},\boldsymbol{p})$ with $\boldsymbol{f}_{t=0} = \boldsymbol{f}_0$.

The strategy is to, at first, provide a solution of the Maxwell equation in which $\delta_{\boldsymbol{x}}$ is replaced by $\rho_{\boldsymbol{x}}$, i.e., a solution of the initial value problem

$$
\begin{aligned}
&\square \begin{pmatrix} \boldsymbol{E}_t \\ \boldsymbol{B}_t \end{pmatrix} = 4\pi \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_t \\ \boldsymbol{j}_t \end{pmatrix} \\
&\begin{pmatrix} \boldsymbol{E}_0 \\ \boldsymbol{B}_0 \end{pmatrix} := \begin{pmatrix} \boldsymbol{E}_t \\ \boldsymbol{B}_t \end{pmatrix} \Big|_{t=0} \\
&\partial_t \begin{pmatrix} \boldsymbol{E}_0 \\ \boldsymbol{B}_0 \end{pmatrix} := \partial_t \begin{pmatrix} \boldsymbol{E}_t \\ \boldsymbol{B}_t \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} \nabla \wedge \boldsymbol{B}_0 - 4\pi \boldsymbol{j}_0 \\ -\nabla \wedge \boldsymbol{E}_0 \end{pmatrix}
\end{aligned}
\quad , \tag{5.29}
$$

cf. (A.11), Appendix A.1. In a second step we show that this solution gives rise to a distribution and in this way a Maxwell solution according to Definition 3.3.3 (Maxwell solutions).

We divide the proof into the following steps:

1. Given $(\boldsymbol{q},\boldsymbol{p})$, $\boldsymbol{f}_0$ and any test function $\rho \in \mathcal{D}, t \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^3$, we define

$$\boldsymbol{g}_\rho(t,\boldsymbol{x}) := \boldsymbol{g}_\rho^{(1)}(t,\boldsymbol{x}) + \boldsymbol{g}_\rho^{(2)}(t,\boldsymbol{x}), \tag{5.30}$$

$$\boldsymbol{g}_\rho^{(1)}(t,\boldsymbol{x}) := \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{F}_0(\boldsymbol{x}), \tag{5.31}$$

$$\boldsymbol{g}_\rho^{(2)}(t,\boldsymbol{x}) := 4\pi \int_0^t ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho(\cdot - \boldsymbol{q}_s) \\ \boldsymbol{v}_s \rho(\cdot - \boldsymbol{q}_s) \end{pmatrix} (\boldsymbol{x}), \tag{5.32}$$

where we use the notation

$$\boldsymbol{x} \mapsto \boldsymbol{F}_0(\boldsymbol{x}) := \boldsymbol{f}_0(\rho_{\boldsymbol{x}}) \equiv \boldsymbol{f}_0(\rho(\boldsymbol{x} - \cdot)). \tag{5.33}$$

Thanks to Lemma A.1.3 (Kirchhoff's formula) the formula (5.30)-(5.32) gives rise to a solution of (5.29). Moreover, with regard to formula (4.18), $\boldsymbol{f}_0$ can be written in the parameterization

$$\boldsymbol{f}_0 = \lambda \boldsymbol{f}_0^-[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] + (1-\lambda)\boldsymbol{f}_0^+[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] + \boldsymbol{f}_0^h, \tag{5.34}$$

which is merely a definition of $\boldsymbol{f}_0^h$. Note that $\boldsymbol{f}_0^h \in \mathcal{F}_{\text{hom}}$ by (4.18), the assumption $\boldsymbol{f}_0 \in \mathcal{F}_{\boldsymbol{q}_0}$, and the linearity of the Maxwell constraints.

2. We verify that $(t,\boldsymbol{x}) \mapsto \boldsymbol{g}_\rho(t,\boldsymbol{x})$ is well-defined and in $C^\infty(\mathbb{R}^4,\mathbb{R}^6)$.

3. We show, that for all $\rho \in \mathcal{D}$, $\boldsymbol{g}_\rho$ solves (3.7) and (3.3) in the strong sense fulfilling the initial condition $\boldsymbol{g}_\rho(t,\boldsymbol{x})|_{t=0} = \boldsymbol{F}_0(\boldsymbol{x}) = \boldsymbol{f}_0(\rho(\boldsymbol{x} - \cdot))$.

4. For all $t \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^3$, $\rho \in \mathcal{D}$ we compute an explicit formula for $\boldsymbol{g}_\rho^{(2)}(t,\boldsymbol{x})$, which gives rise to a distribution $\boldsymbol{g}_t^{(2)}$ for all $t \in \mathbb{R}$ with $\boldsymbol{g}_t^{(2)}(\rho_{\boldsymbol{x}}) = \boldsymbol{g}_\rho^{(2)}(t,\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^3$, $\rho \in \mathcal{D}$.

5. Making use of the parameterization (5.34) for the initial field $\boldsymbol{f}_0$, for all $t \in \mathbb{R}$ we derive an explicit representation for the distribution $\boldsymbol{g}_t^{(1)}$ which is given by $\boldsymbol{g}_t^{(1)}(\rho_{\boldsymbol{x}}) = \boldsymbol{g}_\rho^{(1)}(t,\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^3$, $\rho \in \mathcal{D}$. Thereby, we exploit the formula for $\boldsymbol{g}_t^{(2)}$ derived in item 4 and the fact that the Liénard-Wiechert fields are Maxwell solutions (cf. Lemma 4.2.1 (Properties of Liénard-Wiechert fields)).

6. We conclude that for all $t \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^3$, $\rho \in \mathcal{D}$ $\boldsymbol{g}_\rho(t,\boldsymbol{x})$ is of the form

$$\boldsymbol{g}_\rho(t,\boldsymbol{x}) = \int d^3z \rho(\boldsymbol{x}-\boldsymbol{z})\boldsymbol{f}_t(\boldsymbol{z}) = \boldsymbol{f}_t(\rho(\boldsymbol{x}-\cdot)), \tag{5.35}$$

   where for all $t \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^3$, $\boldsymbol{f}_t(\boldsymbol{x})$ is given by (4.12)-(4.17) and $\boldsymbol{f} : t \mapsto \boldsymbol{f}_t \in \mathcal{F}^1(\mathbb{R})$.

   This implies, that $\boldsymbol{f}$ is a Maxwell solution with initial value $\boldsymbol{f}_{t=0} = \boldsymbol{f}_0$ according to Definition 3.3.3 (Maxwell solutions).

   Moreover, from Lemma 4.2.3 (Inhomogeneous Maxwell solutions), (ii), we then know that $\boldsymbol{f} : t \mapsto \boldsymbol{f}_t$ is the unique Maxwell solution for initial value $\boldsymbol{f}_0$.

ad 2. The mapping $(t,\boldsymbol{x}) \mapsto \boldsymbol{g}_\rho^{(1)}(t,\boldsymbol{x})$, defined in (5.31), is well-defined and in $C^\infty(\mathbb{R}^4,\mathbb{R}^6)$ because:

The map $\boldsymbol{x} \mapsto \boldsymbol{F}_0(\boldsymbol{x}) = \boldsymbol{f}_0(\rho_{\boldsymbol{x}})$ is in $C^\infty(\mathbb{R}^3,\mathbb{R}^6)$ by Lemma A.3.1 (Distributions), (ii). From Lemma A.1.1 (Properties of K_t) we obtain that $(t,\boldsymbol{x}) \mapsto K_t * \boldsymbol{F}_0(\boldsymbol{x})$ is in $C^\infty(\mathbb{R}\setminus\{0\}\times\mathbb{R}^3,\mathbb{R}^6)$ with smooth extension at time $t=0$. The same holds for $(t,\boldsymbol{x}) \mapsto \partial_t K_t * \boldsymbol{F}_0(\boldsymbol{x})$ and $(t,\boldsymbol{x}) \mapsto K_t * \nabla \wedge \boldsymbol{F}_0(\boldsymbol{x})$, which gives that $(t,\boldsymbol{x}) \mapsto \boldsymbol{g}_\rho^{(1)}(t,\boldsymbol{x})$ is in $C^\infty(\mathbb{R}^4,\mathbb{R}^6)$ and in particular well-defined.

The mapping $(t,\boldsymbol{x}) \mapsto \boldsymbol{g}_\rho^{(2)}(t,\boldsymbol{x})$, defined by (5.32) is well-defined because:
The electric component of $\boldsymbol{g}_\rho^{(2)}$ is given by

$$(t,\boldsymbol{x}) \mapsto -4\pi \int_0^t ds K_{t-s} * \nabla\rho(\cdot - \boldsymbol{q}_s)(\boldsymbol{x}) - 4\pi \int_0^t ds \partial_t K_{t-s} * \boldsymbol{v}_s\rho(\cdot - \boldsymbol{q}_s)(\boldsymbol{x}) \tag{5.36}$$

and the magnetic component of $\boldsymbol{g}_\rho^{(2)}$ is

$$(t,\boldsymbol{x}) \mapsto 4\pi \int_0^t ds K_{t-s} * \nabla \wedge \boldsymbol{v}_s\rho(\cdot - \boldsymbol{q}_s)(\boldsymbol{x}). \tag{5.37}$$

We note that for fixed $s \in \mathbb{R}$, $\boldsymbol{x} \mapsto \nabla\rho(\boldsymbol{x}-\boldsymbol{q}_s) \in C_c^\infty(\mathbb{R}^3,\mathbb{R}^3)$ and $\boldsymbol{x} \mapsto \boldsymbol{v}_s\rho(\boldsymbol{x}-\boldsymbol{q}_s) \in C_c^\infty(\mathbb{R}^3,\mathbb{R}^3)$. Hence, for fixed $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^3$ the maps $s \mapsto K_{t-s} * \nabla\rho(\cdot - \boldsymbol{q}_s)(\boldsymbol{x})$, $s \mapsto \partial_t K_{t-s} * \boldsymbol{v}_s\rho(\cdot - \boldsymbol{q}_s)(\boldsymbol{x})$, and $s \mapsto K_{t-s} * \nabla \wedge \boldsymbol{v}_s\rho(\cdot - \boldsymbol{q}_s)(\boldsymbol{x})$ are continuous which implies the existence of the integral.

In order to prove that $(t,\boldsymbol{x}) \mapsto \boldsymbol{g}_\rho^{(2)}(t,\boldsymbol{x})$ is in $C^\infty(\mathbb{R}^4,\mathbb{R}^6)$, we transform (5.36) into

$$\begin{aligned}
(t,\boldsymbol{x}) \mapsto &- \int_0^t ds \frac{1}{t-s} \int_{\partial B_{|t-s|}(0)} d\sigma(y) \nabla_x \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_s) \\
&- \int_0^t ds \partial_t \frac{1}{t-s} \int_{\partial B_{|t-s|}(0)} d\sigma(y) \boldsymbol{v}_s \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_s) \\
\overset{TT}{=} &- \int_0^t ds (t-s) \int_{\partial B_1(0)} d\sigma(y) \nabla_x \rho(\boldsymbol{x}-|t-s|\boldsymbol{y}-\boldsymbol{q}_s) \\
&- \int_0^t ds \partial_t (t-s) \int_{\partial B_1(0)} d\sigma(y) \boldsymbol{v}_s \rho(\boldsymbol{x}-|t-s|\boldsymbol{y}-\boldsymbol{q}_s),
\end{aligned}$$

where TT denotes transformation theorem, cf. Appendix A.4. Since $\boldsymbol{x} \mapsto \nabla_x\rho(\boldsymbol{x}) \in C_c^\infty(\mathbb{R}^3,\mathbb{R}^3)$ and $\boldsymbol{x} \mapsto \boldsymbol{v}_s\rho(\boldsymbol{x}) \in C_c^\infty(\mathbb{R}^3,\mathbb{R}^3)$ the above function is smooth in the $\boldsymbol{x}$-component. For the t-component the smoothness follows as well by the smoothness of ρ and the fundamental theorem of calculus. Here one should note that there is no t-dependence in the trajectory elements $\boldsymbol{q}_s$ and $\boldsymbol{v}_s$, which may destroy smoothness.

The magnetic component of $\boldsymbol{g}_\rho^{(2)}$ can be transformed analogously to the first summand of the electric part and is smooth by the same arguments. We only need to interchange the ∇_x-operator with the $\nabla_x\wedge$-operator.

And therefore, we can conclude that the object $(t,\boldsymbol{x}) \mapsto \boldsymbol{g}_\rho(t,\boldsymbol{x})$ is a well-defined smooth function.

ad 3. In order to show that $t \mapsto \boldsymbol{g}_\rho(t,\cdot)$ is indeed a solution to the Maxwell equations, we need to verify that for all $\rho \in \mathcal{D}$, $t \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^3$ equation (3.7), or respectively,

$$\left(\partial_t - \begin{pmatrix} 0 & \nabla\wedge \\ -\nabla\wedge & 0 \end{pmatrix}\right) \boldsymbol{g}_\rho(t,\boldsymbol{x}) = \begin{pmatrix} -4\pi \boldsymbol{v}(\boldsymbol{p}_t)\rho(\boldsymbol{x}-\boldsymbol{q}_t) \\ 0 \end{pmatrix} \tag{5.38}$$

is fulfilled. Note that by virtue of item 1. of this proof, $\boldsymbol{g}_\rho$ is smooth and hence the expression on the left hand side is well defined and we can compute

$$\begin{aligned}
\boxed{A} := &\left(\partial_t - \begin{pmatrix} 0 & \nabla\wedge \\ -\nabla\wedge & 0 \end{pmatrix}\right) \boldsymbol{g}_\rho(t,\boldsymbol{x}) \\
= &\left(\begin{pmatrix} \partial_t^2 & \partial_t\nabla\wedge \\ -\partial_t\nabla\wedge & \partial_t^2 \end{pmatrix} - \begin{pmatrix} -\nabla\wedge\nabla\wedge & \nabla\wedge\partial_t \\ -\nabla\wedge\partial_t & -\nabla\wedge\nabla\wedge \end{pmatrix}\right) K_t * \boldsymbol{F}_0(\boldsymbol{x}) \\
&+ 4\pi\partial_t \int_0^t ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho(\cdot - \boldsymbol{q}_s) \\ \boldsymbol{v}(\boldsymbol{p}_s)\rho(\cdot - \boldsymbol{q}_s) \end{pmatrix}(\boldsymbol{x}) \\
&- 4\pi \int_0^t ds \begin{pmatrix} -\partial_t\nabla & \nabla\wedge\nabla\wedge \\ 0 & \partial_t\nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho(\cdot - \boldsymbol{q}_s) \\ \boldsymbol{v}(\boldsymbol{p}_s)\rho(\cdot - \boldsymbol{q}_s) \end{pmatrix}(\boldsymbol{x}) \\
= &\begin{pmatrix} \Delta + \nabla\wedge\nabla\wedge & 0 \\ 0 & \Delta+\nabla\wedge\nabla\wedge \end{pmatrix} K_t * \boldsymbol{F}_0(\boldsymbol{x}) \\
&+ 4\pi \lim_{s\to t} \left(\begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho(\cdot - \boldsymbol{q}_s) \\ \boldsymbol{v}(\boldsymbol{p}_s)\rho(\cdot - \boldsymbol{q}_s) \end{pmatrix}(\boldsymbol{x})\right) \\
&+ 4\pi \int_0^t ds \partial_t \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho(\cdot - \boldsymbol{q}_s) \\ \boldsymbol{v}(\boldsymbol{p}_s)\rho(\cdot - \boldsymbol{q}_s) \end{pmatrix}(\boldsymbol{x})
\end{aligned}$$

$$- 4\pi \int_0^t ds \begin{pmatrix} -\partial_t \nabla & \nabla \wedge \nabla \wedge \\ 0 & \partial_t \nabla \wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho(\cdot - \boldsymbol{q}_s) \\ \boldsymbol{v}(\boldsymbol{p}_s)\rho(\cdot - \boldsymbol{q}_s) \end{pmatrix} (\boldsymbol{x})$$

Thereby, we interchanged derivatives which can be done by Schwarz's theorem, we applied that the rotation of a gradient field is zero and the total differentiation theorem, which states that for continuously differentiable functions h

$$\partial_t \int_0^t dsh(t,s) = \lim_{s \to t} h(t,s) + \int_0^t ds \partial_t h(t,s). \tag{5.39}$$

By assumption, $\boldsymbol{f}_0 \in \mathcal{F}_{\boldsymbol{q}_0}$, i.e., $\boldsymbol{f}_0$ fulfills Maxwell constraints (3.3), and respectively, for each $\rho \in \mathcal{D}$, $\boldsymbol{x} \mapsto \boldsymbol{F}_0(\boldsymbol{x}) = \boldsymbol{f}_0(\rho_{\boldsymbol{x}})$ fulfills Maxwell constraints in the strong sense. Thus, writing $\boldsymbol{F}_0 = (\boldsymbol{E}_0, \boldsymbol{B}_0)$, we have $\nabla \cdot \boldsymbol{B}_0 = 0$ and $\nabla \cdot \boldsymbol{E}_0 = 4\pi\rho(\cdot - \boldsymbol{q}_0)$.

Applying $\Delta + \nabla \wedge \nabla \wedge = \nabla \nabla \cdot$ in the first summand, as well as Lemma A.1.1 (Properties of K_t) stating that $\lim_{t \to 0}(K_t * h_t) = 0$ and $\lim_{t \to 0}(\partial_t K_t * h_t) = h_0$ for all h with $h_t \in C^2(\mathbb{R}^3, \mathbb{R})$, $h_{(\cdot)}(\boldsymbol{x}) \in C(\mathbb{R}, \mathbb{R})$ in the second summand, and bringing the third and the fourth summand together, we obtain

$$\begin{aligned}
\boxed{A} &= \nabla K_t * \begin{pmatrix} \nabla \cdot \boldsymbol{E}_0 \\ 0 \end{pmatrix} (\boldsymbol{x}) \\
&+ \begin{pmatrix} -4\pi \boldsymbol{v}(\boldsymbol{p}_t)\rho(\boldsymbol{x} - \boldsymbol{q}_t) \\ 0 \end{pmatrix} \\
&+ 4\pi \int_0^t ds \begin{pmatrix} -\partial_t \nabla & -\partial_t^2 - \nabla \wedge \nabla \wedge \\ 0 & \partial_t \nabla \wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho(\cdot - \boldsymbol{q}_s) \\ \boldsymbol{v}(\boldsymbol{p}_s)\rho(\cdot - \boldsymbol{q}_s) \end{pmatrix} (\boldsymbol{x}).
\end{aligned}$$

Given $-\partial_t^2 - \nabla \wedge \nabla \wedge = -\Box - \nabla \nabla \cdot$, whereas the d'Alembert Operator applied on the propagator K_t equals zero (cf. Lemma A.1.1 (Properties of K_t), (iv)), we get

$$\begin{aligned}
\boxed{A} &= \begin{pmatrix} 4\pi K_t * \nabla \rho(\cdot - \boldsymbol{q}_0) \\ 0 \end{pmatrix} (\boldsymbol{x}) + \begin{pmatrix} -4\pi \boldsymbol{v}(\boldsymbol{p}_t)\rho(\boldsymbol{x} - \boldsymbol{q}_t) \\ 0 \end{pmatrix} \\
&+ 4\pi \int_0^t ds \begin{pmatrix} -\partial_t K_{t-s} * \nabla \rho(\cdot - \boldsymbol{q}_s) + K_{t-s} * \partial_s \nabla \rho(\cdot - \boldsymbol{q}_s) \\ 0 \end{pmatrix} (\boldsymbol{x}) \\
&= \begin{pmatrix} 4\pi K_t * \nabla \rho(\cdot - \boldsymbol{q}_0) \\ 0 \end{pmatrix} (\boldsymbol{x}) + \begin{pmatrix} -4\pi \boldsymbol{v}(\boldsymbol{p}_t)\rho(\boldsymbol{x} - \boldsymbol{q}_t) \\ 0 \end{pmatrix} \\
&- 4\pi \int_0^t ds \begin{pmatrix} \partial_t K_{t-s} * \nabla \rho(\cdot - \boldsymbol{q}_s) \\ 0 \end{pmatrix} (\boldsymbol{x}) + 4\pi \int_0^t ds \begin{pmatrix} K_{t-s} * \partial_s \nabla \rho(\cdot - \boldsymbol{q}_s) \\ 0 \end{pmatrix} (\boldsymbol{x}).
\end{aligned}$$

Making use of integration by parts the last summand can be transformed as

$$\begin{aligned}
&4\pi \int_0^t ds \begin{pmatrix} K_{t-s} * \partial_s \nabla \rho(\cdot - \boldsymbol{q}_s) \\ 0 \end{pmatrix} (\boldsymbol{x}) \\
&= 4\pi \left[K_{t-s} * \begin{pmatrix} \nabla \rho(\cdot - \boldsymbol{q}_s) \\ 0 \end{pmatrix} (\boldsymbol{x}) \right]_{s=0}^t - 4\pi \int_0^t ds \begin{pmatrix} \partial_s K_{t-s} * \nabla \rho(\cdot - \boldsymbol{q}_s) \\ 0 \end{pmatrix} (\boldsymbol{x}) \\
&= \begin{pmatrix} -4\pi K_t * \nabla \rho(\cdot - \boldsymbol{q}_0) \\ 0 \end{pmatrix} (\boldsymbol{x}) + 4\pi \int_0^t ds \begin{pmatrix} \partial_t K_{t-s} * \nabla \rho(\cdot - \boldsymbol{q}_s) \\ 0 \end{pmatrix} (\boldsymbol{x})
\end{aligned}$$

and therefore it cancels with the first and the last summand, which shows that

$$\left((\partial_t - \begin{pmatrix} 0 & \nabla\wedge \\ -\nabla\wedge & 0 \end{pmatrix}\right) \boldsymbol{g}_\rho(t,\boldsymbol{x}) = \begin{pmatrix} -4\pi \boldsymbol{v}(\boldsymbol{p}_t)\rho(\boldsymbol{x}-\boldsymbol{q}_t) \\ 0 \end{pmatrix}$$

for any $\rho \in \mathcal{D}$, $(t,\boldsymbol{x}) \in \mathbb{R}^4$.

Next, we verify $\boldsymbol{g}_\rho(t,\boldsymbol{x})|_{t=0} = \boldsymbol{F}_0(\boldsymbol{x}) = \boldsymbol{f}_0(\rho_{\boldsymbol{x}})$ for all $\boldsymbol{x} \in \mathbb{R}^3$, $\rho \in \mathcal{D}$. For $t \to 0$ the integral term $\boldsymbol{g}_\rho^{(2)}(t,\boldsymbol{x})$ vanishes and it remains to compute the first term, which corresponds to the homogeneous case from Lemma 4.2.2 (Homogeneous Maxwell solutions):

$$\begin{aligned}
\lim_{t\to 0} \boldsymbol{g}_\rho(t,\boldsymbol{x}) &= \lim_{t\to 0} \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{F}_0(\boldsymbol{x}) \\
&= \begin{pmatrix} \lim_{t\to 0} \partial_t K_t * \boldsymbol{E}_0(\boldsymbol{x}) + \lim_{t\to 0} K_t * \nabla\wedge \boldsymbol{B}_0(\boldsymbol{x}) \\ -\lim_{t\to 0} K_t * \nabla \wedge \boldsymbol{E}_0(\boldsymbol{x}) + \lim_{t\to 0} \partial_t K_t * \boldsymbol{B}_0(\boldsymbol{x}) \end{pmatrix} \\
&= \boldsymbol{F}_0(\boldsymbol{x}),
\end{aligned}$$

where we have used Lemma A.1.1 (Properties of K_t), (i) and (ii).

Finally, we show that Maxwells constraints (3.3) are satisfied likewise. Given that $\boldsymbol{g}_\rho$ solves (3.7) in the strong sense, $\boldsymbol{g}_\rho(t,\boldsymbol{x})\big|_{t=0} = \boldsymbol{f}_0(\rho_{\boldsymbol{x}})$, and the assumption $\boldsymbol{f}_0 \in \mathcal{F}_{\boldsymbol{q}_0}$, or respectively, for all $\rho \in \mathcal{D}$ $\boldsymbol{g}_\rho(t=0,\cdot) \in \mathcal{F}_{\boldsymbol{q}_0}$, it follows that for all $t \in \mathbb{R}$ we have $\boldsymbol{g}_\rho(t,\cdot) \in \mathcal{F}_{\boldsymbol{q}_t}$. This is due to Lemma A.2.1 (Maxwell constraints).

ad 4. For any $\rho \in \mathcal{D}, (t,\boldsymbol{x}) \in \mathbb{R}^4$ we compute $\boldsymbol{g}_\rho^{(2)}(t,\boldsymbol{x})$. In the first place, we introduce the objects and notations that will be frequently used throughout the computations.

- Effectively, we compute $\boldsymbol{g}_\rho^{(2)}(t,\boldsymbol{x})$ solely for $t \neq 0$. The formula for $\boldsymbol{g}_\rho(t,\boldsymbol{x})$ that we obtain in the end can be continuously extended at $t = 0$ which has been subject to item 2. Moreover, whenever the case distinction $\pm$ or $\mp$ appears in the computations it is to be read such that the upper case corresponds to times $t > 0$ and the lower case to $t < 0$.

- We use a slightly modified set of abbreviations than in (4.7) and (4.19), namely

$$\boldsymbol{n} := \frac{\boldsymbol{y}}{|\boldsymbol{y}|} \qquad \boldsymbol{v} := \boldsymbol{v}_{t\pm|\boldsymbol{y}|} \qquad \boldsymbol{a} := \boldsymbol{a}_{t\pm|\boldsymbol{y}|} \tag{5.40}$$

$$\begin{aligned}
t^\pm := t \pm \left|\boldsymbol{z} - \boldsymbol{q}^\pm\right| \qquad \boldsymbol{q}^\pm := \boldsymbol{q}_{t^\pm} \\
\boldsymbol{n}^\pm := \frac{\boldsymbol{z}-\boldsymbol{q}^\pm}{|\boldsymbol{z}-\boldsymbol{q}^\pm|} \qquad \boldsymbol{v}^\pm := \boldsymbol{v}_{t^\pm} \qquad \boldsymbol{a}^\pm := \boldsymbol{a}_{t^\pm}
\end{aligned} \tag{5.41}$$

$$\boldsymbol{q}_0 := \boldsymbol{q}_{t=0} \qquad \boldsymbol{n}_0 := \frac{\boldsymbol{z}-\boldsymbol{q}_0}{|\boldsymbol{z}-\boldsymbol{q}_0|} \qquad \boldsymbol{v}_0 := \boldsymbol{v}_{t=0} \qquad \boldsymbol{a}_0 := \boldsymbol{a}_{t=0}. \tag{5.42}$$

Note that (5.41) corresponds to (4.7) and (5.42) to (4.19), however, with the parameter $\boldsymbol{z}$ instead of $\boldsymbol{x}$.

- It will be convenient to transform derivatives $\nabla_x \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|})$ into $\nabla_y \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|})$, where $\boldsymbol{y}$ is the integration variable, and thus, this will allow us to integrate by parts. For this purpose we shall exploit the following computation

$$\begin{aligned}\partial_{y_i}\rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|}) &= \nabla_x\rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|})\cdot\partial_{y_i}(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|})\\ &= \sum_j \partial_{x_j}\rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|})(-\delta_{ij}-\boldsymbol{v}_{t\pm|\boldsymbol{y}|,j}\partial_{y_i}(t\pm|\boldsymbol{y}|))\\ &= \sum_j \partial_{x_j}\rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|})\underbrace{\left(-\delta_{ij}\mp\boldsymbol{v}_{t\pm|\boldsymbol{y}|,j}\frac{y_i}{|\boldsymbol{y}|}\right)}_{=:L^{-1}(\boldsymbol{y})^{\pm}_{ij}}\end{aligned}$$

$$\begin{aligned}\Rightarrow \nabla_y\rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|}) &= L^{-1}(\boldsymbol{y})^{\pm}\cdot\nabla_x\rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|})\\ \nabla_x\rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|}) &= L(\boldsymbol{y})^{\pm}\cdot\nabla_y\rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|}),\end{aligned}$$

where

$$\begin{aligned}L^{-1}(\boldsymbol{y})^{\pm}_{ij} &= -\delta_{ij}\mp v_j n_i\\ L(\boldsymbol{y})^{\pm}_{ij} &= -\delta_{ij}\pm\frac{n_i v_j}{1\pm\boldsymbol{n}\cdot\boldsymbol{v}}.\end{aligned}$$

- Furthermore, we make use of the transformation T given by

$$T(\boldsymbol{y}) := \boldsymbol{z} = \boldsymbol{y}+\boldsymbol{q}_{t\pm|\boldsymbol{y}|}. \tag{5.43}$$

Since by assumption the trajectory $(\boldsymbol{q},\boldsymbol{p})$ is strictly time-like, we find that T is a diffeomorphism, and thus, the equation $\boldsymbol{z}=\boldsymbol{y}+\boldsymbol{q}_{t\pm|\boldsymbol{y}|}$ holds for a unique pair $\boldsymbol{z},\boldsymbol{y}\in\mathbb{R}^3$, cf. Figure 5.1, namely,

$$T^{-1}(\boldsymbol{z}) = \boldsymbol{z}-\boldsymbol{q}^{\pm}. \tag{5.44}$$

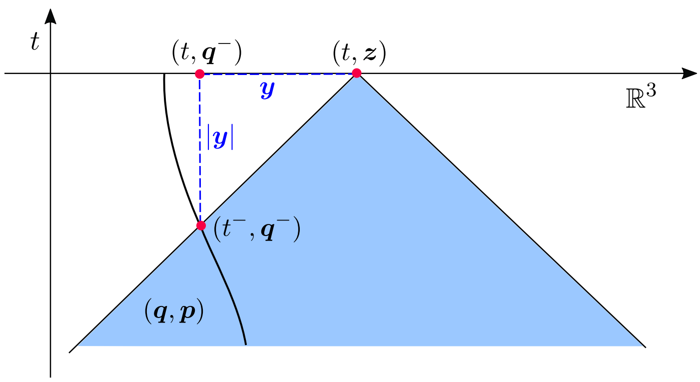

Figure 5.1: This figure illustrates for the retarded case that the transformation T is bijective; i.e., for each $\boldsymbol{z}\in\mathbb{R}^3$ there is exactly one intersection point of the backward light-cone of $(t,\boldsymbol{z})$ and the charge trajectory $(\boldsymbol{q},\boldsymbol{p})$ such that $T^{-1}(\boldsymbol{z})=\boldsymbol{y}$ is uniquely determined.

The Jacobi determinant of the transformation T is given by

$$\begin{aligned}\det DT(\boldsymbol{y}) = \det D(\boldsymbol{y}+\boldsymbol{q}_{t\pm|\boldsymbol{y}|}) &= \det\begin{pmatrix}1\pm v_1\frac{y_1}{|\boldsymbol{y}|} & \pm v_1\frac{y_2}{|\boldsymbol{y}|} & \pm v_1\frac{y_3}{|\boldsymbol{y}|}\\ \pm v_2\frac{y_1}{|\boldsymbol{y}|} & 1\pm v_2\frac{y_2}{|\boldsymbol{y}|} & \pm v_2\frac{y_3}{|\boldsymbol{y}|}\\ \pm v_3\frac{y_1}{|\boldsymbol{y}|} & \pm v_3\frac{y_2}{|\boldsymbol{y}|} & 1\pm v_3\frac{y_3}{|\boldsymbol{y}|}\end{pmatrix}\\ &= 1\pm\boldsymbol{n}\cdot\boldsymbol{v},\end{aligned}$$

which can be computed with Sarrus' rule. And therefore, we find

$$\frac{1}{|\det DT(T^{-1}(\boldsymbol{z}))|} = \frac{1}{1 \pm \boldsymbol{n}^{\pm} \cdot \boldsymbol{v}^{\pm}}. \tag{5.45}$$

Transforming the domain $B_{|t|}(0)$ by T gives

$$\begin{aligned} T(B_{|t|}(0)) &= \{\boldsymbol{y} + \boldsymbol{q}_{t\pm|\boldsymbol{y}|} | \boldsymbol{y} \in B_{|t|}(0)\} \\ &= \cup_{r\in[0,|t|]} \{\boldsymbol{y} + \boldsymbol{q}_{t\pm r} | \boldsymbol{y} \in \partial B_{|t|-r}(0)\} \\ &= B_{|t|}(\boldsymbol{q}_0). \end{aligned}$$

Whenever this transformation is applied, it is denoted by $\overset{T}{=}$.

- Moreover, the following derivatives will be needed:

$$\begin{aligned} \partial_{y_j} \frac{1}{|\boldsymbol{y}|} &= -\frac{y_j}{|\boldsymbol{y}|^3} = -\frac{n_j}{|\boldsymbol{y}|^2} \\ \partial_{y_j} \boldsymbol{y} &= \frac{y_j}{|\boldsymbol{y}|} = n_j \\ \partial_{y_j} \boldsymbol{n} &= \frac{1}{|\boldsymbol{y}|} \left(\begin{pmatrix} \delta_{1j} \\ \delta_{2j} \\ \delta_{3j} \end{pmatrix} - n_j \boldsymbol{n} \right) \\ \partial_{y_j}(\boldsymbol{n} \cdot \boldsymbol{v}) &= \frac{v_j}{|\boldsymbol{y}|} - \frac{\boldsymbol{y} \cdot \boldsymbol{v} y_j}{|\boldsymbol{y}|^3} \pm \frac{\boldsymbol{y} \cdot \boldsymbol{a} y_j}{|\boldsymbol{y}|^2} = \frac{v_j}{|\boldsymbol{y}|} - \frac{\boldsymbol{n} \cdot \boldsymbol{v} n_j}{|\boldsymbol{y}|} \pm (\boldsymbol{n} \cdot \boldsymbol{a}) n_j \\ \partial_{y_j} \boldsymbol{v} &= \pm n_j \boldsymbol{a} \\ \partial_{y_m}(n_j v_m) &= \frac{\delta_{jm} - n_j n_m}{|\boldsymbol{y}|} v_m \pm n_j n_m a_m = \frac{v_j - n_j(\boldsymbol{n} \cdot \boldsymbol{v})}{|\boldsymbol{y}|} \pm n_j (\boldsymbol{n} \cdot \boldsymbol{a}) \\ \partial_{y_m}(v_m v_k n_j) &= \pm(\boldsymbol{n} \cdot \boldsymbol{a}) v_k n_j \pm (\boldsymbol{n} \cdot \boldsymbol{v}) a_k n_j + \frac{v_k v_j}{|\boldsymbol{y}|} - \frac{(\boldsymbol{n} \cdot \boldsymbol{v}) n_j v_k}{|\boldsymbol{y}|} \\ \partial_{y_m}(|\boldsymbol{y}| (1 \pm \boldsymbol{n} \cdot \boldsymbol{v})) &= n_m (1 \pm \boldsymbol{n} \cdot \boldsymbol{v}) \pm v_m \mp (\boldsymbol{n} \cdot \boldsymbol{v}) n_m + (\boldsymbol{n} \cdot \boldsymbol{a}) n_m |\boldsymbol{y}| \end{aligned}$$

Having collected the necessary tools, we pass over to the computation of $\boldsymbol{g}_\rho^{(2)}$ defined in (5.32). We divide the expression into three integrals to be computed separately, namely,

$$\boldsymbol{g}_\rho^{(2)}(t, \boldsymbol{x}) = 4\pi \int_0^t ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho(\cdot - \boldsymbol{q}_s) \\ \boldsymbol{v}_s \rho(\cdot - \boldsymbol{q}_s) \end{pmatrix} (\boldsymbol{x}) := \begin{pmatrix} \boxed{1} + \boxed{2} \\ \boxed{3} \end{pmatrix}. \tag{5.46}$$

Thereby, expression $\boxed{1}+\boxed{2}$ is referred to as electric field component and $\boxed{3}$ as magnetic field component. Now,

$$\begin{aligned}
\boxed{1} &= -4\pi \int_0^t ds (K_{t-s} * \nabla_x \rho(\cdot - \boldsymbol{q}_s))(\boldsymbol{x}) \\
&= -4\pi \int_0^t ds \frac{1}{4\pi(t-s)} \int_{\partial B_{|t-s|}(0)} d\sigma(y) \nabla_x \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_s) \\
&= - \int_0^t ds \frac{1}{(t-s)} \int_{\partial B_{|t-s|}(0)} d\sigma(y) \nabla_x \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_s) \\
&= + \int_t^0 dr \frac{1}{r} \int_{\partial B_{|r|}(0)} d\sigma(y) \nabla_x \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t-r}) \\
&= - \int_0^t dr \frac{1}{r} \int_{\partial B_{|r|}(0)} d\sigma(y) \nabla_x \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t-r}) \\
&= - \underbrace{\int_0^{|t|} dr (\mp 1) \int_{\partial B_{|r|}(0)} d\sigma(y)}_{\int_{B_{|t|}(0)} d^3y} \frac{1}{\mp r} \nabla_x \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t \pm r}) \\
&= - \int_{B_{|t|}(0)} d^3y \frac{1}{|\boldsymbol{y}|} \nabla_x \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t \pm |\boldsymbol{y}|}).
\end{aligned}$$

We denote the ith component of vector $\boxed{1}$ by $\boxed{1}_i$ and compute the latter. Note that we make use of the Einstein summation convention such that $a_j b_j$ is to be read as $\sum_{j=1}^3 a_j b_j$ in the sequel.

$$\begin{aligned}
\boxed{1}_i &= - \int_{B_{|t|}(0)} d^3y \frac{1}{|\boldsymbol{y}|} \partial_{x_i} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t \pm |\boldsymbol{y}|}) \\
&= - \int_{B_{|t|}(0)} d^3y \frac{1}{|\boldsymbol{y}|} L(\boldsymbol{y})^{\pm}_{ij} \partial_{y_j} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t \pm |\boldsymbol{y}|}) \\
&\overset{PI}{=} \int_{B_{|t|}(0)} d^3y \partial_{y_j} \left[\frac{1}{|\boldsymbol{y}|} L(\boldsymbol{y})^{\pm}_{ij} \right] \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t \pm |\boldsymbol{y}|}) && \boxed{1a}_i \\
&\quad - \int_{B_{|t|}(0)} d^3y \partial_{y_j} \left[\frac{1}{|\boldsymbol{y}|} L(\boldsymbol{y})^{\pm}_{ij} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t \pm |\boldsymbol{y}|}) \right], && \boxed{1b}_i
\end{aligned}$$

where the two summands in the last equation are named $\boxed{1a}_i$ and $\boxed{1b}_i$. Moreover, PI denotes integration by parts. For the expression $\boxed{1a}_i$, which is the ith component of the vector $\boxed{1a}$ we compute

$$\begin{aligned}
\boxed{1a}_i &= \int_{B_{|t|}(0)} d^3y \partial_{y_j} \left[\frac{1}{|\boldsymbol{y}|} L(\boldsymbol{y})^{\pm}_{ij} \right] \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t \pm |\boldsymbol{y}|}) \\
&\overset{T}{=} \int_{T(B_{|t|}(0))} d^3z \frac{1}{1 \pm \boldsymbol{n}^{\pm} \cdot \boldsymbol{v}^{\pm}} \partial_{y_j} \left[\frac{1}{|\boldsymbol{y}|} L(\boldsymbol{y})^{\pm}_{ij} \right]_{\boldsymbol{y} := \boldsymbol{z} - \boldsymbol{q}^{\pm}} \rho(\boldsymbol{x} - \boldsymbol{z}),
\end{aligned}$$

where the derivative computes as

$$\begin{aligned}
\partial_{y_j}\left[\frac{1}{|\boldsymbol{y}|}L(\boldsymbol{y})^{\pm}_{ij}\right] &= \partial_{y_j}\left[-\frac{\delta_{ij}}{|\boldsymbol{y}|} \pm \frac{y_i v_j}{|\boldsymbol{y}|^2(1\pm\boldsymbol{n}\cdot\boldsymbol{v})}\right] \\
&= \frac{\delta_{ij}y_j}{|\boldsymbol{y}|^3} \pm \frac{\partial_{y_j}(y_i v_j)}{|\boldsymbol{y}|^2(1\pm\boldsymbol{n}\cdot\boldsymbol{v})} \mp \frac{y_i v_j \partial_{y_j}(|\boldsymbol{y}|^2(1\pm\boldsymbol{n}\cdot\boldsymbol{v}))}{|\boldsymbol{y}|^4(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2} \\
&= \frac{y_i}{|\boldsymbol{y}|^3} \pm \frac{\delta_{ij}v_j - y_i y_j |\boldsymbol{y}|^{-1} a_j}{|\boldsymbol{y}|^2(1\pm\boldsymbol{n}\cdot\boldsymbol{v})} \\
&\mp \frac{y_i v_j \left(2y_j(1\pm\boldsymbol{n}\cdot\boldsymbol{v}) \pm |\boldsymbol{y}|^2 \partial_{y_i}(\boldsymbol{n}\cdot\boldsymbol{v})\right)}{|\boldsymbol{y}|^4(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2}.
\end{aligned}$$

For the sake of convenience we continue in vector notation. The derivative then equals

$$\begin{aligned}
&\frac{\boldsymbol{n}}{|\boldsymbol{y}|^2} \pm \frac{\boldsymbol{v}}{|\boldsymbol{y}|^2(1\pm\boldsymbol{n}\cdot\boldsymbol{v})} + \frac{\boldsymbol{n}(\boldsymbol{a}\cdot\boldsymbol{n})}{|\boldsymbol{y}|(1\pm\boldsymbol{n}\cdot\boldsymbol{v})} \mp \frac{2\boldsymbol{n}(\boldsymbol{v}\cdot\boldsymbol{n})}{|\boldsymbol{y}|^2(1\pm\boldsymbol{n}\cdot\boldsymbol{v})} \\
&- \frac{v^2\boldsymbol{n}}{|\boldsymbol{y}|^2(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2} + \frac{(\boldsymbol{n}\cdot\boldsymbol{v})^2\boldsymbol{n}}{|\boldsymbol{y}|^2(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2} \mp \frac{(\boldsymbol{n}\cdot\boldsymbol{a})(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n}}{|\boldsymbol{y}|(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2} \\
&= \frac{\boldsymbol{n}\pm\boldsymbol{v}+(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{v} - v^2\boldsymbol{n}}{|\boldsymbol{y}|^2(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2} + \frac{(\boldsymbol{n}\cdot\boldsymbol{a})\boldsymbol{n}}{|\boldsymbol{y}|(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2}.
\end{aligned}$$

Plugging this into the last formula for expression $\boxed{1a}$ we obtain

$$\begin{aligned}
\boxed{1a} = \int_{B_{|t|}(\boldsymbol{q}_0)} d^3z\rho(\boldsymbol{x}-\boldsymbol{z}) &\left[\frac{\boldsymbol{n}^{\pm}-\boldsymbol{v}^{\pm}}{|\boldsymbol{z}-\boldsymbol{q}^{\pm}|^2(1\pm\boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^2} + \frac{(\boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})\boldsymbol{n}^{\pm} - (v^{\pm})^2\boldsymbol{n}^{\pm}}{|\boldsymbol{z}-\boldsymbol{q}^{\pm}|^2(1\pm\boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3}\right. \\
&\left. + \frac{(\boldsymbol{n}^{\pm}\cdot\boldsymbol{a}^{\pm})\boldsymbol{n}^{\pm}}{|\boldsymbol{z}-\boldsymbol{q}^{\pm}|(1\pm\boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3}\right].
\end{aligned}$$

We turn over to compute $\boxed{1b}_i$, which is the ith component of expression $\boxed{1b}$.

$$\begin{aligned}
\boxed{1b}_i &= -\int_{B_{|t|}(0)} d^3y\partial_{y_j}\left[\frac{1}{|\boldsymbol{y}|}L(\boldsymbol{y})^{\pm}_{ij}\rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|})\right] \\
&\overset{GG}{=} -\int_{\partial B_{|t|}(0)} d\sigma(y) n_j\left[-\delta_{ij} \pm \frac{n_i v_j}{1\pm\boldsymbol{n}\cdot\boldsymbol{v}}\right]\frac{1}{|\boldsymbol{y}|}\rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|}) \\
&= \int_{\partial B_{|t|}(0)} d\sigma(y)\frac{n_i}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})|\boldsymbol{y}|}\rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_0) \\
&\overset{S}{=} \int_{\partial B_{|t|}(\boldsymbol{q}_0)} d\sigma(z)\frac{n_{0,i}}{(1\pm\boldsymbol{n}_0\cdot\boldsymbol{v}_0)|\boldsymbol{z}-\boldsymbol{q}_0|}\rho(\boldsymbol{x}-\boldsymbol{z}),
\end{aligned}$$

where the Gauss-Green theorem (GG) has been applied and the substitution $\boldsymbol{z} := \boldsymbol{y}+\boldsymbol{q}_0$ (S). Hence, we obtain

$$\boxed{1b} = \int_{\partial B_{|t|}(\boldsymbol{q}_0)} d\sigma(z)\frac{\boldsymbol{n}_0}{(1\pm\boldsymbol{n}_0\cdot\boldsymbol{v}_0)|\boldsymbol{z}-\boldsymbol{q}_0|}\rho(\boldsymbol{x}-\boldsymbol{z}).$$

This concludes the computation of the first summand. Note that it is convenient not to sum $\boxed{1a}$ and $\boxed{1b}$ at that point. The second summand of (5.46) is computed as follows:

$$
\begin{aligned}
\boxed{2} &= 4\pi \int_0^t ds(-\partial_t) K_{t-s} * \boldsymbol{v}_s \rho(\cdot - \boldsymbol{q}_s)(\boldsymbol{x}) \\
&= -4\pi \int_0^t dr (\partial_r K_r) * \boldsymbol{v}_{t-r} \rho(\cdot - \boldsymbol{q}_{t-r})(\boldsymbol{x}) \\
&= -4\pi \int_0^t dr \partial_r r \fint_{\partial B_{|r|}(0)} d\sigma(y) \boldsymbol{v}_{t-r} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t-r}) \\
&= -4\pi \int_0^t dr \left[\fint_{\partial B_{|r|}(0)} d\sigma(y) \boldsymbol{v}_{t-r} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t-r}) \right. \\
&\quad \left. + r\partial_r \fint_{\partial B_1(0)} d\sigma(y) \boldsymbol{v}_{t-r} \rho(\boldsymbol{x} - |r|\, \boldsymbol{y} - \boldsymbol{q}_{t-r}) \right] \\
&= -4\pi \int_0^t dr \left[\fint_{\partial B_{|r|}(0)} d\sigma(y) \boldsymbol{v}_{t-r} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t-r}) \right. \\
&\quad \left. + r \fint_{\partial B_1(0)} d\sigma(y) \boldsymbol{v}_{t-r} (\pm \boldsymbol{y}) \cdot \nabla_x \rho(\boldsymbol{x} - |r|\, \boldsymbol{y} - \boldsymbol{q}_{t-r}) \right] \\
&= -4\pi \int_0^t dr \left[\fint_{\partial B_{|r|}(0)} d\sigma(y) \boldsymbol{v}_{t-r} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t-r}) \right. \\
&\quad \left. + r \fint_{\partial B_{|r|}(0)} d\sigma(y) \boldsymbol{v}_{t-r} (\pm \frac{\boldsymbol{y}}{|r|}) \cdot \nabla_x \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t-r}) \right] \\
&= -4\pi \int_0^t dr \left[\fint_{\partial B_{|r|}(0)} d\sigma(y) \boldsymbol{v}_{t-r} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t-r}) \right. \\
&\quad \left. - \fint_{\partial B_{|r|}(0)} d\sigma(y) \boldsymbol{v}_{t-r} \boldsymbol{y} \cdot \nabla_x \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t-r}) \right] \\
&= -4\pi \int_0^{|t|} dr (\mp 1) \fint_{\partial B_{|r|}(0)} d\sigma(y) \left[\boldsymbol{v}_{t\pm|r|} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t\pm|r|}) \right. \\
&\quad \left. - \boldsymbol{v}_{t\pm|r|} \boldsymbol{y} \cdot \nabla_x \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t\pm|r|}) \right] \\
&= \pm \int_{B_{|t|}(0)} d^3y \frac{1}{|\boldsymbol{y}|^2} \left[\boldsymbol{v}_{t\pm|\boldsymbol{y}|} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t\pm|\boldsymbol{y}|}) \right. \\
&\quad \left. - \boldsymbol{v}_{t\pm|\boldsymbol{y}|} \boldsymbol{y} \cdot \nabla_x \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t\pm|\boldsymbol{y}|}) \right] \\
&= \pm \int_{B_{|t|}(0)} d^3y \frac{1}{|\boldsymbol{y}|^2} \left[\boldsymbol{v}_{t\pm|\boldsymbol{y}|} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t\pm|\boldsymbol{y}|}) \right. \\
&\quad \left. - \boldsymbol{v}_{t\pm|\boldsymbol{y}|} \boldsymbol{y} \cdot L(\boldsymbol{y})^{\pm} \cdot \nabla_{\boldsymbol{y}} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t\pm|\boldsymbol{y}|}) \right] \\
&= \pm \int_{B_{|t|}(0)} d^3y \frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_{t\pm|\boldsymbol{y}|} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t\pm|\boldsymbol{y}|})
\end{aligned}
$$

$$\mp \int_{B_{|t|}(0)} d^3y \frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_{t\pm|\boldsymbol{y}|} y_k L(\boldsymbol{y})^{\pm}_{kj} \partial_{y_j} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t\pm|\boldsymbol{y}|})$$

We apply the integration by parts formula (PI) first, secondly we make use of the Gauss-Green theorem (GG) applied to the second summand, and thus, we obtain

$$\begin{aligned}
\boxed{2} &\overset{PI,GG}{=} \pm \int_{B_{|t|}(0)} d^3y \frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_{t\pm|\boldsymbol{y}|} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t\pm|\boldsymbol{y}|}) && \boxed{2c} \\
&\pm \int_{B_{|t|}(0)} d^3y \partial_{y_m} \left[\frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_{t\pm|\boldsymbol{y}|} y_j L(\boldsymbol{y})^{\pm}_{jm} \right] \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t\pm|\boldsymbol{y}|}) && \boxed{2a} \\
&\mp \int_{\partial B_{|t|}(0)} d\sigma(y) n_m \left[\frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_{t\pm|\boldsymbol{y}|} y_j L(\boldsymbol{y})^{\pm}_{jm} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t\pm|\boldsymbol{y}|}) \right] && \boxed{2b}
\end{aligned}$$

Again, computing the three summands in a row, gives

$$\begin{aligned}
\boxed{2c} &= \pm \int_{B_{|t|}(0)} d^3y \frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_{t\pm|\boldsymbol{y}|} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t\pm|\boldsymbol{y}|}) \\
&\overset{T}{=} \pm \int_{T(B_{|t|}(0))} d^3z \frac{1}{(1 \pm \boldsymbol{n}^{\pm} \cdot \boldsymbol{v}^{\pm}) |\boldsymbol{z} - \boldsymbol{q}^{\pm}|^2} \boldsymbol{v}^{\pm} \rho(\boldsymbol{x} - \boldsymbol{z})
\end{aligned}$$

for the summand $\boxed{2c}$, the summand $\boxed{2a}$ simplifies as follows

$$\begin{aligned}
\boxed{2a} &\overset{T}{=} \pm \int_{T(B_{|t|}(0))} d^3z \frac{1}{(1 \pm \boldsymbol{n}^{\pm} \cdot \boldsymbol{v}^{\pm})} \partial_{y_m} \left[\frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_{t\pm|\boldsymbol{y}|} y_j L(\boldsymbol{y})^{\pm}_{jm} \right]_{\boldsymbol{y}=\boldsymbol{z}-\boldsymbol{q}^{\pm}} \rho(\boldsymbol{x} - \boldsymbol{z}) \\
&= \int_{B_{|t|}(\boldsymbol{q}_0)} d^3z \frac{1}{(1 \pm \boldsymbol{n}^{\pm} \cdot \boldsymbol{v}^{\pm})} \\
&\quad \partial_{y_m} \left[\frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_{t\pm|\boldsymbol{y}|} y_j (\mp\delta_{jm} + \frac{n_j v_m}{1 \pm \boldsymbol{n} \cdot \boldsymbol{v}}) \right]_{\boldsymbol{y}=\boldsymbol{z}-\boldsymbol{q}^{\pm}} \rho(\boldsymbol{x} - \boldsymbol{z}) \\
&\overset{(*)}{=} \int_{B_{|t|}(\boldsymbol{q}_0)} d^3z \left[\frac{\mp \boldsymbol{v}^{\pm}}{(1 \pm \boldsymbol{n}^{\pm} \cdot \boldsymbol{v}^{\pm}) |\boldsymbol{z} - \boldsymbol{q}^{\pm}|^2} + \frac{\mp (\boldsymbol{v}^{\pm})^2 \boldsymbol{v}^{\pm} - (\boldsymbol{n}^{\pm} \cdot \boldsymbol{v}^{\pm}) \boldsymbol{v}^{\pm}}{(1 \pm \boldsymbol{n}^{\pm} \cdot \boldsymbol{v}^{\pm})^3 |\boldsymbol{z} - \boldsymbol{q}^{\pm}|^2} \right. \\
&\quad \left. \pm \frac{(\boldsymbol{n}^{\pm} \cdot \boldsymbol{a}^{\pm}) \boldsymbol{v}^{\pm}}{(1 \pm \boldsymbol{n}^{\pm} \cdot \boldsymbol{v}^{\pm})^3 |\boldsymbol{z} - \boldsymbol{q}^{\pm}|} - \frac{\boldsymbol{a}^{\pm}}{(1 \pm \boldsymbol{n}^{\pm} \cdot \boldsymbol{v}^{\pm})^2 |\boldsymbol{z} - \boldsymbol{q}^{\pm}|} \right] \rho(\boldsymbol{x} - \boldsymbol{z}),
\end{aligned}$$

where in the step marked by $(*)$ the following calculation for the derivative in $\boxed{2a}$ has been taken into account.

$$\begin{aligned}
\partial_{y_m} \left[\frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_{t\pm|\boldsymbol{y}|} y_j (\mp\delta_{jm} + \frac{n_j v_m}{1 \pm \boldsymbol{n} \cdot \boldsymbol{v}}) \right] &= \mp \partial_{y_m} \left[\frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_{t\pm|\boldsymbol{y}|} y_m \delta_{jm} \right] && \boxed{\alpha} \\
&+ \partial_{y_m} \left[\frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_{t\pm|\boldsymbol{y}|} y_j \frac{n_j v_m}{1 \pm \boldsymbol{n} \cdot \boldsymbol{v}}) \right] && \boxed{\beta},
\end{aligned}$$

where $\boxed{\alpha}$ simplifies to

$$\begin{aligned}\boxed{\alpha} &= \mp\left(\partial_{y_m}\frac{1}{|\boldsymbol{y}|^2}\boldsymbol{v}_{t\pm|\boldsymbol{y}|}y_j + \frac{1}{|\boldsymbol{y}|^2}\partial_{y_m}\boldsymbol{v}_{t\pm|\boldsymbol{y}|}y_j + \frac{1}{|\boldsymbol{y}|^2}\boldsymbol{v}_{t\pm|\boldsymbol{y}|}\partial_{y_m}y_j\right)\delta_{jm}\\ &= \mp\left(\frac{-2y_m}{|\boldsymbol{y}|^4}\boldsymbol{v}y_j \pm \frac{y_m\boldsymbol{a}}{|\boldsymbol{y}|^3}y_j + \frac{\boldsymbol{v}\delta_{jm}}{|\boldsymbol{y}|^2}\right)\delta_{jm}\\ &= \frac{\pm 2y^2\boldsymbol{v}}{|\boldsymbol{y}|^4} - \frac{y^2\boldsymbol{a}}{|\boldsymbol{y}|^3} \mp \frac{3\boldsymbol{v}}{|\boldsymbol{y}|^2}\\ &= \frac{\mp\boldsymbol{v}}{|\boldsymbol{y}|^2} - \frac{\boldsymbol{a}}{|\boldsymbol{y}|} = \frac{\mp\boldsymbol{v}}{|\boldsymbol{y}|^2} + \frac{(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{a}-\boldsymbol{a}}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})\,|\boldsymbol{y}|}\end{aligned}$$

and $\boxed{\beta}$ simplifies to

$$\begin{aligned}\boxed{\beta} &= \left[\frac{-2n_m n_j\boldsymbol{v}}{|\boldsymbol{y}|^2} \pm \frac{n_m n_j\boldsymbol{a}}{|\boldsymbol{y}|} + \frac{\boldsymbol{v}\delta_{jm}}{|\boldsymbol{y}|^2}\right]\frac{n_j v_m}{1\pm\boldsymbol{n}\cdot\boldsymbol{v}}\\ &\quad + \frac{\boldsymbol{v}y_j}{|\boldsymbol{y}|^2}\left[\frac{\partial_{y_m}(n_j v_m)}{1\pm\boldsymbol{n}\cdot\boldsymbol{v}} \mp \frac{n_j v_m\partial_{y_m}(\boldsymbol{n}\cdot\boldsymbol{v})}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2}\right]\\ &= \frac{-2(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{v}}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})\,|\boldsymbol{y}|^2} \pm \frac{(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{a}}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})\,|\boldsymbol{y}|} + \frac{(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{v}}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})\,|\boldsymbol{y}|^2}\\ &\quad + \frac{\boldsymbol{v}n_j}{|\boldsymbol{y}|}\left[\frac{v_j - n_j(\boldsymbol{n}\cdot\boldsymbol{v})}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})\,|\boldsymbol{y}|} \pm \frac{n_j(\boldsymbol{n}\cdot\boldsymbol{a})}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})} \mp \frac{n_j v^2 - (\boldsymbol{n}\cdot\boldsymbol{v})^2 n_j}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2\,|\boldsymbol{y}|} - \frac{n_j(\boldsymbol{n}\cdot\boldsymbol{a})(\boldsymbol{n}\cdot\boldsymbol{v})}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2}\right]\\ &= \frac{-(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{v} \mp v^2\boldsymbol{v}}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2\,|\boldsymbol{y}|^2} \pm \frac{(\boldsymbol{n}\cdot\boldsymbol{a})\boldsymbol{v}}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2\,|\boldsymbol{y}|} - \frac{(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{a}}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})\,|\boldsymbol{y}|}.\end{aligned}$$

Their sum is equal to

$$\boxed{\alpha} + \boxed{\beta} = \frac{\mp\boldsymbol{v}}{|\boldsymbol{y}|^2} + \frac{-(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{v} \mp v^2\boldsymbol{v}}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2\,|\boldsymbol{y}|^2} \pm \frac{(\boldsymbol{n}\cdot\boldsymbol{a})\boldsymbol{v}}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^2\,|\boldsymbol{y}|} - \frac{\boldsymbol{a}}{(1\pm\boldsymbol{n}\cdot\boldsymbol{v})\,|\boldsymbol{y}|}.$$

Therefore, the sum of $\boxed{2a}$ and $\boxed{2c}$ yields

$$\begin{aligned}\boxed{2a} + \boxed{2c} = \int_{B_{|t|}(\boldsymbol{q}_0)} d^3z \Bigg[&\frac{(\boldsymbol{v}^\pm)^2\boldsymbol{v}^\pm - (\boldsymbol{n}^\pm\cdot\boldsymbol{v}^\pm)\boldsymbol{v}^\pm}{(1\pm\boldsymbol{n}^\pm\cdot\boldsymbol{v}^\pm)^3\,|\boldsymbol{z}-\boldsymbol{q}^\pm|^2} - \frac{(\boldsymbol{n}^\pm\cdot\boldsymbol{a}^\pm)\boldsymbol{v}^\pm}{(1\pm\boldsymbol{n}^\pm\cdot\boldsymbol{v}^\pm)^3\,|\boldsymbol{z}-\boldsymbol{q}^\pm|}\\ &-\frac{\boldsymbol{a}^\pm}{(1\pm\boldsymbol{n}^\pm\cdot\boldsymbol{v}^\pm)^2\,|\boldsymbol{z}-\boldsymbol{q}^\pm|}\Bigg]\rho(\boldsymbol{x}-\boldsymbol{z}).\end{aligned}$$

It remains to compute summand $\boxed{2b}$:

$$
\begin{aligned}
\boxed{2b} &= \mp \int_{\partial B_{|t|}(0)} d\sigma(y) n_m \left[\frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_{t\pm|\boldsymbol{y}|} y_j L^{\pm}(\boldsymbol{y})_{jm} \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|}) \right] \\
&= - \int_{\partial B_{|t|}(0)} d\sigma(y) n_m \frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_0 y_j (\mp \delta_{jm} + \frac{n_j v_m}{1 \pm \boldsymbol{n}\cdot\boldsymbol{v}}) \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_0) \\
&= - \int_{\partial B_{|t|}(0)} d\sigma(y) \frac{1}{|\boldsymbol{y}|^2} \boldsymbol{v}_0 y_j \frac{n_j}{1 \pm \boldsymbol{n}\cdot\boldsymbol{v}} \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_0) \\
&= \pm \int_{\partial B_{|t|}(0)} d\sigma(y) \frac{1}{|\boldsymbol{y}|} \boldsymbol{v}_0 \frac{1}{1 \pm \boldsymbol{n}\cdot\boldsymbol{v}_0} \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_0) \\
&\overset{S}{=} \pm \int_{\partial B_{|t|}(\boldsymbol{q}_0)} d\sigma(y) \frac{\boldsymbol{v}_0}{(1 \pm \boldsymbol{n}_0\cdot\boldsymbol{v}_0)\,|\boldsymbol{z}-\boldsymbol{q}_0|} \rho(\boldsymbol{x}-\boldsymbol{z}),
\end{aligned}
$$

where in the last step we substituted $\boldsymbol{z} := \boldsymbol{y} + \boldsymbol{q}_0$ (S). Finally, we sum up the summands in the first component of (5.46), which are supported in the inner of the closed ball $B_{|t|}(\boldsymbol{q}_0)$, and obtain:

$$
\begin{aligned}
&\boxed{1a} + \boxed{2a} + \boxed{2c} \\
&= \int_{B_{|t|}(\boldsymbol{q}_0)} d^3z \rho(\boldsymbol{x}-\boldsymbol{z}) \Bigg[\frac{\boldsymbol{n}^{\pm} \pm \boldsymbol{v}^{\pm}}{|\boldsymbol{z}-\boldsymbol{q}^{\pm}|^2 (1 \pm \boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^2} + \frac{\mp(\boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})\boldsymbol{n}^{\pm} - (v^{\pm})^2 \boldsymbol{n}^{\pm}}{|\boldsymbol{z}-\boldsymbol{q}^{\pm}|^2 (1 \pm \boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3} \\
&\quad + \frac{(\boldsymbol{n}^{\pm}\cdot\boldsymbol{a}^{\pm})\boldsymbol{n}^{\pm}}{|\boldsymbol{z}-\boldsymbol{q}^{\pm}| (1 \pm \boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3} + \frac{\mp(\boldsymbol{v}^{\pm})^2\boldsymbol{v}^{\pm} - (\boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})\boldsymbol{v}^{\pm}}{(1 \pm \boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3 |\boldsymbol{z}-\boldsymbol{q}^{\pm}|^2} \\
&\quad \pm \frac{(\boldsymbol{n}^{\pm}\cdot\boldsymbol{a}^{\pm})\boldsymbol{v}^{\pm}}{(1 \pm \boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3 |\boldsymbol{z}-\boldsymbol{q}^{\pm}|} - \frac{\boldsymbol{a}^{\pm}}{(1 \pm \boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^2 |\boldsymbol{z}-\boldsymbol{q}^{\pm}|} \Bigg] \\
&= \int_{B_{|t|}(\boldsymbol{q}_0)} d^3z \rho(\boldsymbol{x}-\boldsymbol{z}) \Bigg[\frac{\boldsymbol{n}^{\pm} \pm \boldsymbol{v}^{\pm} - (\boldsymbol{v}^{\pm})^2 \boldsymbol{n} \mp (\boldsymbol{v}^{\pm})^2 \boldsymbol{v}^{\pm}}{(1 \pm \boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3 |\boldsymbol{z}-\boldsymbol{q}^{\pm}|^2} \\
&\quad + \frac{(\boldsymbol{n}^{\pm} \pm \boldsymbol{v}^{\pm})(\boldsymbol{n}^{\pm}\cdot\boldsymbol{a}^{\pm}) - (\boldsymbol{a}^{\pm}(\boldsymbol{n}^{\pm}\cdot(\boldsymbol{n}^{\pm} \pm \boldsymbol{v}^{\pm})}{(1 \pm \boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3 |\boldsymbol{z}-\boldsymbol{q}^{\pm}|} \Bigg] \\
&= \int_{B_{|t|}(\boldsymbol{q}_0)} d^3z \rho(\boldsymbol{x}-\boldsymbol{z}) \left[\frac{(\boldsymbol{n}^{\pm} \pm \boldsymbol{v}^{\pm})(1-(\boldsymbol{v}^{\pm})^2)}{(1 \pm \boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3 |\boldsymbol{z}-\boldsymbol{q}^{\pm}|^2} + \frac{\boldsymbol{n}^{\pm} \wedge [(\boldsymbol{n}^{\pm} \pm \boldsymbol{v}^{\pm}) \wedge \boldsymbol{a}^{\pm}]}{(1 \pm \boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3 |\boldsymbol{z}-\boldsymbol{q}^{\pm}|} \right] \\
&= \int d^3z \rho(\boldsymbol{x}-\boldsymbol{z}) \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)}(\boldsymbol{z}) \left[\frac{(\boldsymbol{n}^{\pm} \pm \boldsymbol{v}^{\pm})(1-(\boldsymbol{v}^{\pm})^2)}{(1 \pm \boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3 |\boldsymbol{z}-\boldsymbol{q}^{\pm}|^2} + \frac{\boldsymbol{n}^{\pm} \wedge [(\boldsymbol{n}^{\pm} \pm \boldsymbol{v}^{\pm}) \wedge \boldsymbol{a}^{\pm}]}{(1 \pm \boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3 |\boldsymbol{z}-\boldsymbol{q}^{\pm}|} \right]
\end{aligned}
$$

Summands $\boxed{1b}$ and $\boxed{2b}$ are referred to as boundary terms. These sum up to

$$
\begin{aligned}
\boxed{1b} + \boxed{2b} &= \int_{\partial B_{|t|}(\boldsymbol{q}_0)} d\sigma(y) \rho(\boldsymbol{x}-\boldsymbol{z}) \frac{\boldsymbol{n}_0 \pm \boldsymbol{v}_0}{(1 \pm \boldsymbol{n}_0\cdot\boldsymbol{v}_0)\,|\boldsymbol{z}-\boldsymbol{q}_0|} \\
&= \int d^3z \rho(\boldsymbol{x}-\boldsymbol{z}) \delta(|t| - |\boldsymbol{z}-\boldsymbol{q}_0|) \frac{\boldsymbol{n}_0 \pm \boldsymbol{v}_0}{(1 \pm \boldsymbol{n}_0\cdot\boldsymbol{v}_0)\,|\boldsymbol{z}-\boldsymbol{q}_0|},
\end{aligned}
$$

where the δ in the last line denotes the distribution introduced in Chapter 3.

The second component of $\boldsymbol{g}_\rho^{(2)}(t,\boldsymbol{x})$, which corresponds to the magnetic part, computes as

$$\begin{aligned}\boxed{3} &= 4\pi \int_0^t ds K_{t-s} * \nabla \wedge \boldsymbol{v}_s \rho(\cdot - \boldsymbol{q}_s)(\boldsymbol{x}) \\ &= \int_0^t ds \frac{1}{t-s} \int_{\partial B_{|t-s|}(0)} d\sigma(y) \nabla \wedge \boldsymbol{v}_s \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_s) \\ &= \int_0^t dr \frac{1}{r} \int_{\partial B_{|r|}(0)} d\sigma(y) \nabla \wedge \boldsymbol{v}_{t-r} \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t-r}) \\ &= \int_0^{|t|} (\mp 1) dr \frac{1}{\mp r} \int_{\partial B_{|r|}(0)} d\sigma(y) \nabla \wedge \boldsymbol{v}_{t\pm r} \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm r})\end{aligned}$$

The ith component of $\boxed{3}$ is equal to:

$$\begin{aligned}\boxed{3}_i &= \int_{B_{|t|}(0)} d^3y \frac{1}{|\boldsymbol{y}|} \epsilon_{ijk} v_k \partial_{x_j} \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|}) \\ &= \int_{B_{|t|}(0)} d^3y \frac{1}{|\boldsymbol{y}|} \epsilon_{ijk} v_k L^\pm(\boldsymbol{y})_{jm} \partial_{y_m} \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|}) \\ &= \int_{B_{|t|}(0)} d^3y \frac{1}{|\boldsymbol{y}|} \epsilon_{ijk} v_k (-\delta_{jm} \pm \frac{n_j v_m}{1 \pm \boldsymbol{n}\cdot\boldsymbol{v}}) \partial_{y_m} \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|}) \\ &\overset{PI}{=} \mp \int_{B_{|t|}(0)} d^3y \partial_{y_m} \left[\frac{1}{|\boldsymbol{y}|} \epsilon_{ijk} v_k (\mp\delta_{jm} + \frac{n_j v_m}{1 \pm \boldsymbol{n}\cdot\boldsymbol{v}})\right] \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|}) \qquad \boxed{3a}_i \\ &\quad \pm \int_{B_{|t|}(0)} d^3y \partial_{y_m} \left[\frac{1}{|\boldsymbol{y}|} \epsilon_{ijk} v_k (\mp\delta_{jm} + \frac{n_j v_m}{1 \pm \boldsymbol{n}\cdot\boldsymbol{v}}) \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|})\right]. \qquad \boxed{3b}_i\end{aligned}$$

Again we compute the two summands $\boxed{3a}_i$ and $\boxed{3b}_i$ in a row:

$$\begin{aligned}\boxed{3a}_i &= \mp \int_{B_{|t|}(0)} d^3y \partial_{y_m} \left[\frac{1}{|\boldsymbol{y}|} \epsilon_{ijk} v_k (\mp\delta_{jm} + \frac{n_j v_m}{1 \pm \boldsymbol{n}\cdot\boldsymbol{v}})\right] \rho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}_{t\pm|\boldsymbol{y}|}) \\ &\overset{T}{=} \mp \int_{T(B_{|t|}(0))} d^3z \rho(\boldsymbol{x}-\boldsymbol{z}) \frac{1}{1 \pm \boldsymbol{n}^\pm \cdot \boldsymbol{v}^\pm} \epsilon_{ijk} \\ &\qquad \partial_{y_m} \left[\frac{1}{|\boldsymbol{y}|} v_k (\mp\delta_{jm} + \frac{n_j v_m}{1 \pm \boldsymbol{n}\cdot\boldsymbol{v}})\right]_{\boldsymbol{y}=\boldsymbol{z}-\boldsymbol{q}^\pm} \\ &\overset{(**)}{=} \int_{B_{|t|}(\boldsymbol{q}_0)} d^3z \rho(\boldsymbol{x}-\boldsymbol{z}) \Bigg[\mp \boldsymbol{n} \wedge \\ &\qquad \left(\frac{\mp(v^2-1)\boldsymbol{v}}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3 |\boldsymbol{z}-\boldsymbol{q}|^2} \pm \frac{(\boldsymbol{n}\cdot\boldsymbol{a})\boldsymbol{v}}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3 |\boldsymbol{z}-\boldsymbol{q}|} - \frac{\boldsymbol{a}}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^2 |\boldsymbol{z}-\boldsymbol{q}|} \right) \\ &\qquad + \frac{\boldsymbol{v}\wedge\boldsymbol{v}}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v}) |\boldsymbol{z}-\boldsymbol{q}|} \Bigg]^\pm \\ &= \int_{B_{|t|}(\boldsymbol{q}_0)} d^3z \rho(\boldsymbol{x}-\boldsymbol{z}) \\ &\qquad \left[\mp \boldsymbol{n} \wedge \left(\frac{(\boldsymbol{n}\pm\boldsymbol{v})(1-v^2)}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3 |\boldsymbol{z}-\boldsymbol{q}|^2} + \frac{(\boldsymbol{n}\pm\boldsymbol{v})(\boldsymbol{n}\cdot\boldsymbol{a}) - \boldsymbol{a}(\boldsymbol{n}\cdot(\boldsymbol{n}\pm\boldsymbol{v}))}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3 |\boldsymbol{z}-\boldsymbol{q}|} \right)\right]^\pm\end{aligned}$$

$$= \int_{B_{|t|}(\boldsymbol{q}_0)} d^3z \rho(\boldsymbol{x}-\boldsymbol{z}) \left[\mp \boldsymbol{n} \wedge \left(\frac{(\boldsymbol{n} \pm \boldsymbol{v})(1-v^2)}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3 |\boldsymbol{z}-\boldsymbol{q}|^2} + \frac{\boldsymbol{n} \wedge ((\boldsymbol{n} \pm \boldsymbol{v}) \wedge \boldsymbol{a})}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3 |\boldsymbol{z}-\boldsymbol{q}|} \right) \right]^{\pm}$$
$$= \int_{B_{|t|}(\boldsymbol{q}_0)} d^3z \rho(\boldsymbol{x}-\boldsymbol{z}) \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)}(\boldsymbol{z})(\mp \boldsymbol{n} \wedge \boldsymbol{e}^{\pm}(\boldsymbol{z})),$$

where in the step marked with (∗∗) the following auxiliary calculation for the derivative has been exploited:

$$\begin{aligned}
&\partial_{y_m} \left[\frac{1}{|\boldsymbol{y}|} v_k (\mp \delta_{jm} + \frac{n_j v_m}{1 \pm \boldsymbol{n}\cdot\boldsymbol{v}}) \right] \\
&= \partial_{y_j} \frac{\mp v_k}{|\boldsymbol{y}|} + \partial_{y_m} \frac{v_m v_k n_j}{|\boldsymbol{y}|(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})} \\
&= \frac{\pm y_j v_k}{|\boldsymbol{y}|^2} - \frac{a_k n_j}{|\boldsymbol{y}|} + \frac{\partial_{y_m}(v_m v_k n_j)}{|\boldsymbol{y}|(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})} - \frac{v_m v_k n_j \partial_{y_m}(|\boldsymbol{y}|(1 \pm \boldsymbol{n}\cdot\boldsymbol{v}))}{|\boldsymbol{y}|^2 (1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^2} \\
&= \frac{\pm n_j v_k}{|\boldsymbol{y}|^3} - \frac{a_k n_j}{|\boldsymbol{y}|} \pm \frac{(\boldsymbol{n}\cdot\boldsymbol{a}) v_k n_j}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})|\boldsymbol{y}|} \pm \frac{(\boldsymbol{n}\cdot\boldsymbol{v}) a_k n_j}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})|\boldsymbol{y}|} \\
&\quad + \frac{v_k v_j}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})|\boldsymbol{y}|^2} - \frac{(\boldsymbol{n}\cdot\boldsymbol{v}) v_k n_j}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})|\boldsymbol{y}|^2} - \frac{v_m v_k n_j n_m}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})|\boldsymbol{y}|^2} \\
&\quad \mp \frac{v_m v_m v_k n_j}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^2 |\boldsymbol{y}|^2} \pm \frac{v_m n_m v_k n_j (\boldsymbol{n}\cdot\boldsymbol{v})}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^2 |\boldsymbol{y}|^2} - \frac{v_m n_m v_k n_j (\boldsymbol{n}\cdot\boldsymbol{a})}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^2 |\boldsymbol{y}|} \\
&= n_j v_k \left(\frac{\mp v^2 \pm 1}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^2 |\boldsymbol{y}|^2} \pm \frac{\boldsymbol{n}\cdot\boldsymbol{a}}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^2 |\boldsymbol{y}|} \right) \\
&\quad - n_j a_k \frac{1}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})|\boldsymbol{y}|} + v_j v_k \frac{1}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})|\boldsymbol{y}|^2}.
\end{aligned}$$

Note that the last summand cancels in vector notation, since for any $\boldsymbol{v} \in \mathbb{R}^3$ it holds $\boldsymbol{v} \wedge \boldsymbol{v} = 0$. We turn over to the ith component of the second summand of $\boxed{3}$.

$$\begin{aligned}
\boxed{3b}_i &= \pm \int_{\partial B_{|t|}(0)} d\sigma(y) n_m \frac{1}{|\boldsymbol{y}|} \epsilon_{ijk} v_k (\mp \delta_{jm} + \frac{n_j v_m}{1 \pm \boldsymbol{n}\cdot\boldsymbol{v}}) \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t \pm |\boldsymbol{y}|}) \\
&= \pm \int_{\partial B_{|t|}(0)} d\sigma(y) \frac{1}{|\boldsymbol{y}|} \epsilon_{ijk} \frac{\mp n_j v_k}{1 \pm \boldsymbol{n}\cdot\boldsymbol{v}} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t \pm |\boldsymbol{y}|})
\end{aligned}$$

and returning to the full vector we obtain

$$\begin{aligned}
\boxed{3b} &= - \int_{\partial B_{|t|}(0)} d\sigma(y) \frac{\boldsymbol{n} \wedge \boldsymbol{v}}{|\boldsymbol{y}|(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})} \rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t \pm |\boldsymbol{y}|}) \\
&\overset{S}{=} \int_{\partial B_{|t|}(\boldsymbol{q}_0)} d\sigma(z) \frac{-\boldsymbol{n}_0 \wedge \boldsymbol{v}_0}{|\boldsymbol{z} - \boldsymbol{q}_0|(1 \pm \boldsymbol{n}_0 \cdot \boldsymbol{v}_0)} \rho(\boldsymbol{x} - \boldsymbol{z}) \\
&= \int d^3z \rho(\boldsymbol{x} - \boldsymbol{z}) \delta(|t| - |\boldsymbol{z} - \boldsymbol{q}_0|) \frac{-\boldsymbol{n}_0 \wedge \boldsymbol{v}_0}{|\boldsymbol{z} - \boldsymbol{q}_0|(1 \pm \boldsymbol{n}_0 \cdot \boldsymbol{v}_0)},
\end{aligned}$$

(S) denoting the substitution $\boldsymbol{z} := \boldsymbol{y} + \boldsymbol{q}_0$. Summing up each component of the electric part gives

$$\boxed{1}+\boxed{2} = \int_{\mathbb{R}^3} d^3z\rho(\boldsymbol{x}-\boldsymbol{z})\Bigg[\delta(|t|-|\boldsymbol{z}-\boldsymbol{q}_0|)\frac{\boldsymbol{n}_0-\boldsymbol{v}_0}{(1\pm\boldsymbol{n}_0\cdot\boldsymbol{v}_0)\,|\boldsymbol{z}-\boldsymbol{q}_0|}$$
$$+\mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)}(\boldsymbol{z})\left(\frac{(\boldsymbol{n}^\pm\pm\boldsymbol{v}^\pm)(1-(\boldsymbol{v}^\pm)^2)}{(1\pm\boldsymbol{n}^\pm\cdot\boldsymbol{v}^\pm)^3\,|\boldsymbol{z}-\boldsymbol{q}^\pm|^2}+\frac{\boldsymbol{n}^\pm\wedge[(\boldsymbol{n}^\pm\pm\boldsymbol{v}^\pm)\wedge\boldsymbol{a}^\pm]}{(1\pm\boldsymbol{n}^\pm\cdot\boldsymbol{v}^\pm)^3\,|\boldsymbol{z}-\boldsymbol{q}^\pm|}\right)\Bigg]$$
$$=\int_{\mathbb{R}^3} d^3z\rho(\boldsymbol{x}-\boldsymbol{z})\left[\delta(|t|-|\boldsymbol{z}-\boldsymbol{q}_0|)\frac{\boldsymbol{n}_0\pm\boldsymbol{v}_0}{(1\pm\boldsymbol{n}_0\cdot\boldsymbol{v}_0)\,|\boldsymbol{z}-\boldsymbol{q}_0|}+\mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)}(\boldsymbol{z})\boldsymbol{e}_t^\pm(\boldsymbol{z})\right]$$

and for the magnetic part we get

$$\boxed{3} = \int_{\mathbb{R}^3} d^3z\rho(\boldsymbol{x}-\boldsymbol{z})$$
$$\left[\delta(|t|-|\boldsymbol{z}-\boldsymbol{q}_0|)\frac{-\boldsymbol{n}_0\wedge\boldsymbol{v}_0}{(1\pm\boldsymbol{n}_0\cdot\boldsymbol{v}_0)\,|\boldsymbol{z}-\boldsymbol{q}_0|}+\mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)}(\boldsymbol{z})\left(\mp\boldsymbol{n}^\pm\wedge\boldsymbol{e}_t^\pm(\boldsymbol{z})\right)\right]$$
$$=\int_{\mathbb{R}^3} d^3z\rho(\boldsymbol{x}-\boldsymbol{z})\left[\delta(|t|-|\boldsymbol{z}-\boldsymbol{q}_0|)\frac{-\boldsymbol{n}_0\wedge\boldsymbol{v}_0}{(1\pm\boldsymbol{n}_0\cdot\boldsymbol{v}_0)\,|\boldsymbol{z}-\boldsymbol{q}_0|}+\mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)}(\boldsymbol{z})\boldsymbol{b}_t^\pm(\boldsymbol{z})\right],$$

or in a more compact representation

$$\boldsymbol{g}_\rho^{(2)}(t,\boldsymbol{x}) = \begin{pmatrix}\boxed{1}+\boxed{2}\\ \boxed{3}\end{pmatrix} = \int_{\mathbb{R}^3} d^3z\rho(\boldsymbol{x}-\boldsymbol{z})\left[\boldsymbol{r}^\pm_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)}(\boldsymbol{z})+\mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)}(\boldsymbol{z})\boldsymbol{f}_t^\pm(\boldsymbol{z})\right],$$

with

$$\boldsymbol{r}^\pm_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)}(\boldsymbol{z}) := \frac{\delta(|t|-|\boldsymbol{z}-\boldsymbol{q}_0|)}{(1\pm\boldsymbol{n}_0\cdot\boldsymbol{v}_0)|\boldsymbol{z}-\boldsymbol{q}_0|}\begin{pmatrix}\boldsymbol{n}_0\pm\boldsymbol{v}_0\\ -\boldsymbol{n}_0\wedge\boldsymbol{v}_0\end{pmatrix},$$

and $\boldsymbol{f}_t^\pm$ denoting the advanced/retarded Liénard-Wiechert fields, where we write $\boldsymbol{f}_t^\pm = \boldsymbol{f}_t^\pm[\boldsymbol{q},\boldsymbol{p}]$ to emphasize the functional dependence of the charge trajectory $(\boldsymbol{q},\boldsymbol{p})$. Moreover $\boldsymbol{r}^\pm_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)}$ is referred to as boundary term since it is supported only on the light-cone boundary $\partial B_{|t|}(\boldsymbol{q}_0)$, and, $\boldsymbol{r}^\pm_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)}$ depends on the initial position and momentum of the charge, only.

Finally, we write $\boldsymbol{g}_\rho^{(2)}(t,\boldsymbol{x})$ for all $t\in\mathbb{R}$ as distribution in $\mathcal{F}$, which we denote by $\boldsymbol{g}_t^{(2)}$, i.e. for all $\rho\in\mathcal{D}$, $\boldsymbol{x}\in\mathbb{R}$ we have $\boldsymbol{g}_t^{(2)}(\rho_{\boldsymbol{x}}) = \boldsymbol{g}_\rho^{(2)}(t,\boldsymbol{x})$. Then, we obtain

$$\boldsymbol{g}_t^{(2)} = \boldsymbol{g}_t^{(2)}[\boldsymbol{q},\boldsymbol{p}] = \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)}\boldsymbol{f}_t^{-\sigma(t)}[\boldsymbol{q},\boldsymbol{p}]+\boldsymbol{r}^{-\sigma(t)}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)}. \tag{5.47}$$

ad 5. In order to compute $\boldsymbol{g}_\rho^{(1)}$ it is essential to make explicit the functional dependence on the trajectory $(\boldsymbol{q},\boldsymbol{p})$ in each component of $\boldsymbol{g}_\rho^{(2)}$, since $\boldsymbol{g}_\rho^{(1)}$ depends on the initial field which, in our parameterization is a functional of the trajectory $(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}})\in\mathcal{T}^{2+n}(\mathbb{R})$, cf. (5.34). The distribution $\boldsymbol{g}_t^{(1)}\in\mathcal{F}$ with

$$\boldsymbol{g}_t^{(1)}:\rho\mapsto\boldsymbol{g}_\rho^{(1)}(t,\boldsymbol{x}) \tag{5.48}$$

is given by

$$\boldsymbol{g}_t^{(1)} = \begin{pmatrix}\partial_t & \nabla\wedge\\ -\nabla\wedge & \partial_t\end{pmatrix}K_t*\boldsymbol{f}_0. \tag{5.49}$$

Plugging in the explicit representation of $\boldsymbol{f}_0$ from (5.34) depending on $(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}})$, an initial free field $\boldsymbol{f}_0^h$, and a parameter $\lambda \in [0,1]$, we obtain

$$\boldsymbol{g}_t^{(1)} = \lambda \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{f}_0^-[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] \tag{5.50}$$

$$+ (1-\lambda) \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{f}_0^+[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] \tag{5.51}$$

$$+ \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{f}_0^h. \tag{5.52}$$

Making use of Lemma 4.2.1 (Properties of Liénard-Wiechert fields) which guarantees that the Liénard-Wiechert fields $\boldsymbol{f}_t^\pm[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}]$ solve the Maxwell equations, of item 3., which shows that $\boldsymbol{g}_\rho$ from (5.30) solves the Maxwell equations (3.7), of Lemma 4.2.3 (Inhomogeneous Maxwell solutions) which states that the Maxwell solution is unique for predetermined initial values, and of Lemma 4.2.2 (Homogeneous Maxwell solutions), stating that the homogeneous Maxwell equations with initial value $\boldsymbol{f}_0^h$ have a unique solution $\boldsymbol{f}_{(\cdot)}^h : t \mapsto \boldsymbol{f}_t^h$, we can simplify the three summands (5.50)-(5.52) with the aid of the following two equations

$$\begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{f}_0^\pm[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] = \boldsymbol{f}_t^\pm[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] - \boldsymbol{f}_t^{(2)}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] \tag{5.53}$$

$$\begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{f}_0^h = \boldsymbol{f}_t^h. \tag{5.54}$$

Plugging in the solution formula in (5.47) for the field component $\boldsymbol{g}_t^{(2)}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}]$ in (5.53) we get

$$\begin{aligned} \boldsymbol{g}_t^{(1)} = & \lambda \left(\boldsymbol{f}_t^-[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] - \boldsymbol{r}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)}^{-\sigma(t)} - \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)} \boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] \right) \\ & + (1-\lambda) \left(\boldsymbol{f}_t^+[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] - \boldsymbol{r}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)}^{-\sigma(t)} - \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)} \boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] \right) \\ & + \boldsymbol{f}_t^h. \end{aligned}$$

ad 6. Plugging this result together with (5.47) into (5.30), we arrive at the representation

$$\boldsymbol{f}_t = \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)} \left(\boldsymbol{f}_t^{-\sigma(t)}[\boldsymbol{q}, \boldsymbol{p}] - \boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] \right) \tag{5.55}$$

$$+ \lambda \boldsymbol{f}_t^-[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] + (1-\lambda) \boldsymbol{f}_t^+[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}] \tag{5.56}$$

$$+ \boldsymbol{r}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)}^{-\sigma(t)} - \boldsymbol{r}_{t,(\tilde{\boldsymbol{q}}_0,\tilde{\boldsymbol{p}}_0)}^{-\sigma(t)} \tag{5.57}$$

$$+ \boldsymbol{f}_t^h, \tag{5.58}$$

for the field $\boldsymbol{f}_t \in \mathcal{F}$ which is equal to the representation (4.12)-(4.17). □

5.4 Regularity of the Maxwell fields

In this section we aim to prove Lemma 4.2.4 (Regularity of $\boldsymbol{f}_t$). Therefore, Lemma 5.4.1 (Compatibility Conditions) is needed. Before presenting this Lemma we need to introduce some notation that will be used throughout this section, first.

Definition 5.4.1 (Light-cone limit)**.** *Let $(t,\boldsymbol{x}) \mapsto h(t,\boldsymbol{x})$ be a function, $(\boldsymbol{q},\boldsymbol{p})$ a strictly time-like charge trajectory in $\mathcal{T}^{2+n}(\mathbb{R})$, and $(t^*,\boldsymbol{x}^*) \in \partial J(0,\boldsymbol{q}_0) \setminus \{(0,\boldsymbol{q}_0)\}$. We write*

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} h(t,\boldsymbol{x}) = L \tag{5.59}$$

if and only if for all $(t_n,\boldsymbol{x}_n)_{n\in\mathbb{N}}$ with $(t_n,\boldsymbol{x}_n) \in \overset{\circ}{J}{}^{\sigma(t^*)}(0,\boldsymbol{q}_0) \cap D_{\boldsymbol{q}} \cap D_{\tilde{\boldsymbol{q}}}$

$$\lim_{n\to\infty}(t_n,\boldsymbol{x}_n) = (t,\boldsymbol{x}) \qquad \Rightarrow \qquad \lim_{n\to\infty} h(t_n,\boldsymbol{x}_n) = L. \tag{5.60}$$

Analogously, we write

$$\lim_{(t,\boldsymbol{x})\downarrow(t^*,\boldsymbol{x}^*)} h(t,\boldsymbol{x}) = L \tag{5.61}$$

if and only if for all $(t_n,\boldsymbol{x}_n)_{n\in\mathbb{N}}$ with $(t_n,\boldsymbol{x}_n) \in (\mathbb{R}^{\sigma(t^*)} \times \mathbb{R}^3) \setminus J^{\sigma(t^*)}(0,\boldsymbol{q}_0)$

$$\lim_{n\to\infty}(t_n,\boldsymbol{x}_n) = (t,\boldsymbol{x}) \qquad \Rightarrow \qquad \lim_{n\to\infty} h(t_n,\boldsymbol{x}_n) = L, \tag{5.62}$$

where for $t^* < 0$, $\mathbb{R}^{\sigma(t^*)} \equiv (-\infty,0)$ and for $t^* > 0$, $\mathbb{R}^{\sigma(t^*)} \equiv (0,\infty)$.
If both $\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} h(t,\boldsymbol{x}) = L$ and $\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} h(t,\boldsymbol{x}) = L$ coincide, we write

$$\lim_{(t,\boldsymbol{x})\to(t^*,\boldsymbol{x}^*)} h(t,\boldsymbol{x}) = L. \tag{5.63}$$

It should be noted that whenever $(t,\boldsymbol{x}) \uparrow (t^*,\boldsymbol{x}^*)$ occurs in this section, $(t^*,\boldsymbol{x}^*)$ is located on the boundary of the light-cone of the initial charge position of the considered charge trajectory, $\partial J(0,\boldsymbol{q}_0) \setminus \{(0,\boldsymbol{q}_0)\}$, and $(t,\boldsymbol{x})$ approaches $(t^*,\boldsymbol{x}^*)$ from the inner of that light-cone. Moreover, by Definition 5.4.1 (Light-cone limit), $(t,\boldsymbol{x})$ is always in the same half-space as $(t^*,\boldsymbol{x}^*)$, i.e., for $t^* > 0$ only sequences $(t_n,\boldsymbol{x}_n)_{n\in\mathbb{N}}$ in the forward light-cone are considered and for $t^* < 0$ sequences in the backward light-cone are considered. Thus, the notation from Definition 5.4.1 (Light-cone limit) goes along with $\sigma(t) = \sigma(t^*)$. Respectively, $(t,\boldsymbol{x}) \downarrow (t^*,\boldsymbol{x}^*)$ denotes the limit from outside of the light-cone.

Secondly, we shall use the equivalent notations in order to distinguish the two cases of $t^* > 0$ and $t^* < 0$: The expression

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} h^{-\sigma(t)}(t,\boldsymbol{x}) = \lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} h^{-\sigma(t^*)}(t,\boldsymbol{x}) \tag{5.64}$$

shall be written as

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} h^{\pm}(t,\boldsymbol{x}), \tag{5.65}$$

where $\pm$ (or later also $\mp$) serves as placeholder for the two cases $t^*, t \lessgtr 0$. This short-hand notation will be convenient, since we consider only retarded Liénard-Wiechert fields on the forward light-cone and advanced Liénard-Wiechert fields on the backward light-cone of the initial charge position. Note that an analogous notation has already been applied in the proof of Theorem 4.2.1 (Explicit Maxwell solutions) in order to distinguish the two cases of positive and negative times.

Lemma 5.4.1 (Compatibility Conditions). *Let $n, m, k \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^3$, and $(\boldsymbol{q},\boldsymbol{p}), (\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}})$ strictly time-like charge trajectories in $\mathcal{T}^{2+n}(\mathbb{R})$ such that $\boldsymbol{q}_0 := \boldsymbol{q}_{t=0}$ and $\tilde{\boldsymbol{q}}_0 := \tilde{\boldsymbol{q}}_{t=0}$ coincide. Further denote $\boldsymbol{p}_0 := \boldsymbol{p}_{t=0}$ and $\tilde{\boldsymbol{p}}_0 := \tilde{\boldsymbol{p}}_{t=0}$.*

(i) The expression $\boldsymbol{r}^{-\sigma(t)}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)} - \boldsymbol{r}^{-\sigma(t)}_{t,(\tilde{\boldsymbol{q}}_0,\tilde{\boldsymbol{p}}_0)}$ (cf. (4.16)) vanishes if and only if $\boldsymbol{p}_0 = \tilde{\boldsymbol{p}}_0$.

(ii) Let $\boldsymbol{p}_0 = \tilde{\boldsymbol{p}}_0$. For all $(t^*, \boldsymbol{x}^*) \in \partial J(0,\boldsymbol{q}_0) \setminus \{(0,\boldsymbol{q}_0)\}$ and all $|\alpha| + k \le m$ with $m \le n$

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \partial_t^k D_{\boldsymbol{x}}^\alpha \left(\boldsymbol{f}_t^{-\sigma(t)}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) - \boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}](\boldsymbol{x}) \right) = 0 \tag{5.66}$$

if and only if

$$\lim_{t\to 0} \frac{d^l}{dt^l} \boldsymbol{q}_t = \lim_{t\to 0} \frac{d^l}{dt^l} \tilde{\boldsymbol{q}}_t, \qquad \forall\, 0 \le l \le 2+m. \tag{5.67}$$

In order to prove Lemma 5.4.1 (Compatibility Conditions) we make use of the auxiliary Lemma 5.4.2 (Explicit form of derivatives) below in which the algebraic form of derivatives of the Liénard-Wiechert fields is studied.

In its formulation and the corresponding proof we make use of the following notation. Recall the definition $t^\pm = t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x})$ from (4.7) and below. For a compact notation, for any $l \in \mathbb{N}_0$, $t \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^3$ we use the abbreviation, $\boldsymbol{q}_0^{(l)} := \left(\frac{d}{ds}\right)^l \boldsymbol{q}_s|_{s=0}$ and $\boldsymbol{q}^{(l)\pm} := \left(\frac{d}{ds}\right)^l \boldsymbol{q}_s\big|_{s=t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x})}$. For $l = 0$ we omit the upper index and write $\boldsymbol{q}_0$ and $\boldsymbol{q}^\pm$. For the first derivative we shall also stick to the common notation introduced in (4.7) and (4.19) and use $\boldsymbol{v}_0 \equiv \boldsymbol{q}_0^{(1)}$ and $\boldsymbol{v}^\pm \equiv \boldsymbol{q}^{(1)\pm}$. For the second derivatives we use $\boldsymbol{a}_0 \equiv \boldsymbol{q}_0^{(2)}$ and $\boldsymbol{a}^\pm \equiv \boldsymbol{q}^{(2)\pm}$. Moreover, we recycle the notation $\boldsymbol{n}_0 := \frac{\boldsymbol{x}-\boldsymbol{q}_0}{|\boldsymbol{x}-\boldsymbol{q}_0|}$ and $\boldsymbol{n}^\pm := \frac{\boldsymbol{x}-\boldsymbol{q}^\pm}{|\boldsymbol{x}-\boldsymbol{q}^\pm|}$ from (4.7) and (4.19). Whenever $[\ldots]^\pm$ occurs all arguments in the squared bracket that require a time argument are evaluated at the time $t^\pm = t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x})$.

Lemma 5.4.2 (Explicit form of derivatives). *Let $n, m, k \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^3$, $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^{2+n}(\mathbb{R})$ be a charge trajectory, and the map $(t,\boldsymbol{x}) \mapsto \boldsymbol{e}_t^\pm[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x})$ denote the electric advanced and retarded Liénard-Wiechert field of $(\boldsymbol{q},\boldsymbol{p})$; see (4.6).*

(i) Then, for each $|\alpha| + k \le n$, there exists a function

$$(t,\boldsymbol{x}) \mapsto h^\pm_{\alpha,k}(t,\boldsymbol{x}) = h^\pm_{\alpha,k}(\boldsymbol{x},\boldsymbol{q}^\pm,\ldots,\boldsymbol{q}^{(|\alpha|+k+1)\pm}) \in C^{1+n-|\alpha|-k}(D_{\boldsymbol{q}},\mathbb{R}^3) \tag{5.68}$$

such that for all $(t,\boldsymbol{x}) \in D_{\boldsymbol{q}}$

$$\begin{aligned} \partial_t^k D_{\boldsymbol{x}}^\alpha \boldsymbol{e}_t^\pm[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) &= h^\pm_{\alpha,k}(\boldsymbol{x},\boldsymbol{q}^\pm,\ldots,\boldsymbol{q}^{(|\alpha|+k+1)\pm}) \\ &\quad + \left(\partial_t t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x})\right)^k \prod_{i=1}^3 \left(\partial_{x_i} t^\pm_{\boldsymbol{q}}(t,\boldsymbol{x})\right)^{\alpha_i} \\ &\quad \left[\frac{\boldsymbol{n} \wedge \left[(\boldsymbol{n} \pm \boldsymbol{v}) \wedge \boldsymbol{q}^{(|\alpha|+k+2)}\right]}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3 |\boldsymbol{x}-\boldsymbol{q}|} \right]^\pm . \end{aligned} \tag{5.69}$$

(ii) In particular, for any $(t^*, \boldsymbol{x}^*) \in \partial J(0, \boldsymbol{q}_0) \setminus \{(0, \boldsymbol{q}_0)\}$

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \partial_t^k D_{\boldsymbol{x}}^\alpha \boldsymbol{e}_t^\pm[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) = h_{\alpha,k}^\pm(\boldsymbol{x}^*, \boldsymbol{q}_0, \ldots, \boldsymbol{q}_0^{(|\alpha|+k+1)}) + \lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \left(\partial_t t_{\boldsymbol{q}}^\pm(t,\boldsymbol{x})\right)^k \prod_{i=1}^3 \left(\partial_{x_i} t_{\boldsymbol{q}}^\pm(t,\boldsymbol{x})\right)^{\alpha_i} \frac{\boldsymbol{n}_0 \wedge \left[(\boldsymbol{n}_0 \pm \boldsymbol{v}_0) \wedge \boldsymbol{q}_0^{(|\alpha|+k+2)}\right]}{(1 \pm \boldsymbol{n}_0 \cdot \boldsymbol{v}_0)^3 |\boldsymbol{x} - \boldsymbol{q}_0|}, \tag{5.70}$$

where the index $\pm$ *corresponds to the cases* $t^*, t \lessgtr 0$.

One should note that in the function $h_{\alpha,k}^\pm$ there is no further dependence on the trajectory but the given arguments, and therefore for $\boldsymbol{q}^\pm = \tilde{\boldsymbol{q}}^\pm, \ldots, \boldsymbol{q}^{(|\alpha|+k+1)\pm} = \tilde{\boldsymbol{q}}^{(|\alpha|+k+1)\pm}$ it follows

$$h_{\alpha,k}^\pm(\boldsymbol{x}, \boldsymbol{q}^\pm, \ldots, \boldsymbol{q}^{(|\alpha|+k+1)\pm}) = h_{\alpha,k}^\pm(\boldsymbol{x}, \tilde{\boldsymbol{q}}^\pm, \ldots, \tilde{\boldsymbol{q}}^{(|\alpha|+k+1)^\pm}). \tag{5.71}$$

Moreover, Lemma 5.4.2 (Explicit form of derivatives) is the essential tool for showing (5.66) in Lemma 5.4.1 (Compatibility conditions), (ii). The idea is to extract the parameter with the highest derivative of the charge trajectory from the other terms such that in an inductive argument only the extracted term needs to be investigated.

Proof. (i) By assumption $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{T}^{2+n}(\mathbb{R})$ is strictly time-like. In order to study the regularity of the function $(t, \boldsymbol{x}) \mapsto h_{\alpha,k}^\pm(t, \boldsymbol{x})$ we shall need the following ingredients:

(*1) Lemma 5.1.1 ($t^\pm$), (ii), states that for all $\alpha \in \mathbb{N}_0^3$, $k \in \mathbb{N}_0$ such that $k+|\alpha| \le 2+n$ the map $(t, \boldsymbol{x}) \mapsto D_{\boldsymbol{x}}^\alpha \partial_t^k t_{\boldsymbol{q}}^\pm(t, \boldsymbol{x})$ is in $C^{2+n-|\alpha|-k}(D_{\boldsymbol{q}}, \mathbb{R})$ and $D_{\boldsymbol{x}}^\alpha \partial_t^k t_{\boldsymbol{q}}^\pm(t, \boldsymbol{x})$ is a function of $\boldsymbol{x}$, $\frac{d^l}{ds^l}\boldsymbol{q}_s\big|_{s=t_{\boldsymbol{q}}^\pm(t,\boldsymbol{x})}$, $l = 0, \ldots, k + |\alpha|$, only.
In particular, $(t, \boldsymbol{x}) \mapsto t_{\boldsymbol{q}}^\pm(t, \boldsymbol{x}) \in C^{2+n}(D_{\boldsymbol{q}}, \mathbb{R})$ and $t_{\boldsymbol{q}}^\pm(t, \boldsymbol{x})$ depends on $\boldsymbol{x}, \boldsymbol{q}^\pm$.

(*2) The function $(t, \boldsymbol{x}) \mapsto \boldsymbol{n}^\pm = \boldsymbol{n}_s|_{s=t_{\boldsymbol{q}}^\pm(t,\boldsymbol{x})} \in C^{2+n}(D_{\boldsymbol{q}}, \mathbb{R}^3)$ because it is composed of $s \mapsto \boldsymbol{n}_s \in C^{2+n}(D_{\boldsymbol{q}}, \mathbb{R}^3)$ and $(t, \boldsymbol{x}) \mapsto t_{\boldsymbol{q}}^\pm(t, \boldsymbol{x}) \in C^{2+n}(D_{\boldsymbol{q}}, \mathbb{R})$ (cf. (*1)) and $\boldsymbol{n}^\pm$ depends on $\boldsymbol{x}, \boldsymbol{q}^\pm$ by definition.
The first partial derivatives ∂_t, $\partial_{x_i}, i = 1, 2, 3$ applied to $(t, \boldsymbol{x}) \mapsto \boldsymbol{n}^\pm$ are each functions in $C^{1+n}(D_{\boldsymbol{q}}, \mathbb{R}^3)$ depending on $\boldsymbol{x}, \boldsymbol{q}^\pm, \boldsymbol{v}^\pm$.
Correspondingly, $(t, \boldsymbol{x}) \mapsto \boldsymbol{v}^\pm = \boldsymbol{v}_s|_{s=t_{\boldsymbol{q}}^\pm(t,\boldsymbol{x})} \in C^{2+n}(D_{\boldsymbol{q}}, \mathbb{R}^3)$ because it is composed of $s \mapsto \boldsymbol{v}_s \in C^{2+n}(D_{\boldsymbol{q}}, \mathbb{R}^3)$ and $(t, \boldsymbol{x}) \mapsto t_{\boldsymbol{q}}^\pm(t, \boldsymbol{x}) \in C^{2+n}(D_{\boldsymbol{q}}, \mathbb{R})$.
For $0 \le l \le 2 + n$, $(t, \boldsymbol{x}) \mapsto \boldsymbol{q}^{(l)\pm} = \frac{d^l}{ds^l}\boldsymbol{q}_s\big|_{s=t_{\boldsymbol{q}}^\pm(t,\boldsymbol{x})} \in C^{2+n-l}(D_{\boldsymbol{q}}, \mathbb{R}^3)$ because it is composed of $s \mapsto \frac{d^l}{ds^l}\boldsymbol{q}_s \in C^{2+n-l}(D_{\boldsymbol{q}}, \mathbb{R}^3)$ and $(t, \boldsymbol{x}) \mapsto t_{\boldsymbol{q}}^\pm(t, \boldsymbol{x}) \in C^{2+n}(D_{\boldsymbol{q}}, \mathbb{R})$.

(*3) Let $p \in \{1, 2\}$. The function $(t, \boldsymbol{x}) \mapsto 1/((1 \pm \boldsymbol{n}^\pm \cdot \boldsymbol{v}^\pm)^3 |\boldsymbol{x} - \boldsymbol{q}^\pm|^p) = 1/((1 \pm \boldsymbol{n}_s \cdot \boldsymbol{v}_s)^3 |\boldsymbol{x} - \boldsymbol{q}_s|^p)|_{s=t_{\boldsymbol{q}}^\pm(t,\boldsymbol{x})}$ is in $C^{1+n}(D_{\boldsymbol{q}}, \mathbb{R})$ because of (*1) , (*2), and the fact that $\sup_{s\in\mathbb{R}} |1 \pm \boldsymbol{n}^\pm \cdot \boldsymbol{v}^\pm| > 0$ for strictly time like charge trajectories $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{T}^{2+n}(\mathbb{R})$. The expression depends on $\boldsymbol{x}, \boldsymbol{q}^\pm, \boldsymbol{v}^\pm$, only.
The first derivative, i.e., ∂_t or $\partial_{x_i}, i = 1, 2, 3$ applied to $(t, \boldsymbol{x}) \mapsto 1/((1 \pm \boldsymbol{n}^\pm \cdot \boldsymbol{v}^\pm)^3 |\boldsymbol{x} - \boldsymbol{q}^\pm|^p)$, is in $\in C^n(D_{\boldsymbol{q}}, \mathbb{R})$ and depends on the parameters $\boldsymbol{x}, \boldsymbol{q}^\pm, \boldsymbol{v}^\pm \equiv \boldsymbol{q}^{(1)\pm}, \boldsymbol{q}^{(2)\pm}$, only.

(cf. proof of Lemma 4.2.1 (Properties of Liénard-Wiechert fields), where this expression has been discussed in more detail. Moreover, recall from this proof and the proof of Lemma 5.1.1 ($t^{\pm}$) that the charge trajectory needs to be excluded from the domain and is the only singularity to deal with in the above expression.)

With these ingredients we turn to the actual proof, which is done by means of induction over the order $|\alpha| + k$.

Base case: Let $|\alpha| + k = 0$. Then, by definition of the Liénard-Wiechert field, cf. (4.6) and (4.7),

$$\partial_t^k D_{\boldsymbol{x}}^\alpha \boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) = \boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) = \left[\frac{(\boldsymbol{n} \pm \boldsymbol{v})(1-\boldsymbol{v}^2)}{|\boldsymbol{x}-\boldsymbol{q}|^2(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3} + \frac{\boldsymbol{n} \wedge [(\boldsymbol{n} \pm \boldsymbol{v}) \wedge \boldsymbol{a}]}{|\boldsymbol{x}-\boldsymbol{q}|(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3}\right]^{\pm}. \tag{5.72}$$

Since $k = \alpha_1 = \ldots = \alpha_3 = 0$ the second summand corresponds to the second summand in equation (5.69), namely,

$$\left[\frac{\boldsymbol{n} \wedge [(\boldsymbol{n} \pm \boldsymbol{v}) \wedge \boldsymbol{a}]}{|\boldsymbol{x}-\boldsymbol{q}|(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3}\right]^{\pm} = \left(\partial_t t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})\right)^k \prod_{i=1}^{3} \left(\partial_{x_i} t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})\right)^{\alpha_i} \left[\frac{\boldsymbol{n} \wedge \left[(\boldsymbol{n} \pm \boldsymbol{v}) \wedge \boldsymbol{q}^{(|\alpha|+2)}\right]}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3 |\boldsymbol{x}-\boldsymbol{q}|}\right]^{\pm}. \tag{5.73}$$

Therefore, we set

$$(t,\boldsymbol{x}) \mapsto h_{0,0}^{\pm}(t,\boldsymbol{x}) = h_{0,0}^{\pm}(\boldsymbol{x},\boldsymbol{q}^{\pm},\boldsymbol{q}^{(1)\pm}) := \left[\frac{(\boldsymbol{n} \pm \boldsymbol{v})(1-\boldsymbol{v}^2)}{|\boldsymbol{x}-\boldsymbol{q}|^2(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3}\right]^{\pm}. \tag{5.74}$$

According to (*2) and (*3) (5.74) is in $C^{1+n}(D_{\boldsymbol{q}}, \mathbb{R}^3)$. Note that the $(t,\boldsymbol{x})$-dependence is encoded in $t^{\pm} = t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})$.

Inductive step: Assume that for $|\alpha| + k < n$ the claim holds true and consider the derivative $\partial_t^{k'} D_{\boldsymbol{x}}^{\alpha'}$ of order $|\alpha'| + k' = |\alpha| + k + 1$. Then, by inductive hypothesis, there exists a function

$$(t,\boldsymbol{x}) \mapsto h_{\alpha,k}^{\pm}(t,\boldsymbol{x}) = h_{\alpha,k}^{\pm}(\boldsymbol{x},\boldsymbol{q}^{\pm},\ldots,\boldsymbol{q}^{(|\alpha|+k+1)\pm}) \in C^{1+n-k-|\alpha|}(D_{\boldsymbol{q}}, \mathbb{R}^3) \tag{5.75}$$

such that

$$\begin{aligned}
\partial_t^{k'} D_{\boldsymbol{x}}^{\alpha'} \boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) &= \partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha} \partial_t^k D_{\boldsymbol{x}}^\alpha \boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) \\
&= \partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha} \Bigg(h_{\alpha,k}^{\pm}(\boldsymbol{x},\boldsymbol{q}^{\pm},\ldots,\boldsymbol{q}^{(|\alpha|+k+1)\pm}) \\
&\quad + \left(\partial_t t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})\right)^k \prod_{i=1}^{3} \left(\partial_{x_i} t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})\right)^{\alpha_i} \left[\frac{\boldsymbol{n} \wedge \left[(\boldsymbol{n} \pm \boldsymbol{v}) \wedge \boldsymbol{q}^{(|\alpha|+k+2)}\right]}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3 |\boldsymbol{x}-\boldsymbol{q}|}\right]^{\pm} \Bigg).
\end{aligned}$$

Applying the outer derivative on the single components gives

$$
\begin{aligned}
\partial_t^{k'} D_{\boldsymbol{x}}^{\alpha'} \boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) = {} & \partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha} h_{\alpha,k}^{\pm}(\boldsymbol{x},\boldsymbol{q}^{\pm},\dots,\boldsymbol{q}^{(|\alpha|+k+1)\pm}) && (5.76)\\
& + \left(\partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha}\left[\frac{\left(\partial_t t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})\right)^k \prod_{i=1}^3 \left(\partial_{x_i} t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})\right)^{\alpha_i}}{(1\pm \boldsymbol{n}^{\pm}\cdot \boldsymbol{v}^{\pm})^3|\boldsymbol{x}-\boldsymbol{q}^{\pm}|}\right]\right) && (5.77)\\
& \left(\boldsymbol{n}^{\pm}\wedge\left[(\boldsymbol{n}^{\pm}\pm\boldsymbol{v}^{\pm})\wedge \boldsymbol{q}^{(|\alpha|+k+2)\pm}\right]\right) && (5.78)\\
& + \frac{\left(\partial_t t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})\right)^k \prod_{i=1}^3 \left(\partial_{x_i} t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})\right)^{\alpha_i}}{(1\pm \boldsymbol{n}^{\pm}\cdot \boldsymbol{v}^{\pm})^3|\boldsymbol{x}-\boldsymbol{q}^{\pm}|} && (5.79)\\
& \partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha}\left(\boldsymbol{n}^{\pm}\wedge\left[(\boldsymbol{n}^{\pm}\pm\boldsymbol{v}^{\pm})\wedge \boldsymbol{q}^{(|\alpha|+k+2)\pm}\right]\right), && (5.80)
\end{aligned}
$$

where by the product rule (5.80) is given by

$$
\begin{aligned}
(5.80) = {} & \partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha}\boldsymbol{n}^{\pm}\wedge\left[(\boldsymbol{n}^{\pm}\pm\boldsymbol{v}^{\pm})\wedge \boldsymbol{q}^{(|\alpha|+k+2)\pm}\right] && (5.81)\\
& + \boldsymbol{n}^{\pm}\wedge\left[\partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha}(\boldsymbol{n}^{\pm}\pm\boldsymbol{v}^{\pm})\wedge \boldsymbol{q}^{(|\alpha|+k+2)\pm}\right] && (5.82)\\
& + \boldsymbol{n}^{\pm}\wedge\left[(\boldsymbol{n}^{\pm}\pm\boldsymbol{v}^{\pm})\wedge \partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha}\boldsymbol{q}^{(|\alpha|+k+2)\pm}\right]. && (5.83)
\end{aligned}
$$

Making use of (*1)-(*3) we can study the regularity of $(t,\boldsymbol{x})\mapsto \partial_t^{k'} D_{\boldsymbol{x}}^{\alpha'} \boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x})$ given by formula (5.76)-(5.80). Each regularity statement is referred to parameter $(t,\boldsymbol{x})$.

- Term (5.76) is in $C^{1+n-|\alpha|-k-1}(D_{\boldsymbol{q}},\mathbb{R}^3)$ by inductive hypothesis and $|\alpha'|-|\alpha|+k'-k=1$.
- Term (5.79), which is equal to the expression in the squared bracket in (5.77), consists of factor $1/((1\pm\boldsymbol{n}^{\pm}\cdot\boldsymbol{v}^{\pm})^3|\boldsymbol{x}-\boldsymbol{q}^{\pm}|)$ which is in $C^{1+n}(D_{\boldsymbol{q}},\mathbb{R})$ by (*3) and the factors $\partial_t t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x}),\partial_{x_i} t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})$ being in $C^{1+n}(D_{\boldsymbol{q}},\mathbb{R})$ by (*1), and hence, is in $C^{1+n}(D_{\boldsymbol{q}},\mathbb{R})$.

  Moreover, the expression depends on $\boldsymbol{x},\boldsymbol{q}^{\pm},\boldsymbol{v}^{\pm}$, only, by (*1) and (*3).
- Applying the derivative $\partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha}$ in (5.77) on the expression in the squared bracket, where $|\alpha'|-|\alpha|+k'-k=1$ implies that this expression is in $C^n(D_{\boldsymbol{q}},\mathbb{R})$ by (*1) and (*3).

  Because of (*1) and (*3) it depends on the parameters $\boldsymbol{x},\boldsymbol{q}^{\pm},\dots,\boldsymbol{q}^{(2)\pm}$, only.
- Term (5.78) is in $C^{2+n-(|\alpha|+k+2)}(D_{\boldsymbol{q}},\mathbb{R}^3)$, since $\boldsymbol{n}^{\pm}$ and $\boldsymbol{v}^{\pm}$ are in $C^{1+n}(D_{\boldsymbol{q}},\mathbb{R}^3)$ by (*2), and therefore, the expression is as regular as $\boldsymbol{q}^{(|\alpha|+k+2)\pm}$, i.e., (5.78) is in $C^{n-|\alpha|-k}(D_{\boldsymbol{q}},\mathbb{R}^3)$ by (*2).

  It depends on the parameters $\boldsymbol{x},\boldsymbol{q}^{\pm},\dots,\boldsymbol{q}^{(|\alpha|+k+2)\pm}$.
- For expression (5.80) we study (5.81) and (5.82) first and treat (5.83), which is the crucial component, in the end.

  Both, (5.81) and (5.82) are composed of $\boldsymbol{n}^{\pm}$ in $C^{2+n}(D_{\boldsymbol{q}},\mathbb{R}^3)$, $\partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha}\boldsymbol{n}^{\pm}$ in $C^{1+n}(D_{\boldsymbol{q}},\mathbb{R}^3)$, $\boldsymbol{v}^{\pm}$ in $C^{1+n}(D_{\boldsymbol{q}},\mathbb{R}^3)$, and, the map $\boldsymbol{q}^{(|\alpha|+k+2)\pm}$ in $C^{2+n-(|\alpha|+k+2)}(D_{\boldsymbol{q}},\mathbb{R}^3)$.

  Thus, they are both functions in $C^{n-|\alpha|-k}(D_{\boldsymbol{q}},\mathbb{R}^3)$. Moreover, they depend on the parameters $\boldsymbol{x},\boldsymbol{q}^{\pm},\dots,\boldsymbol{q}^{(|\alpha|+k+2)\pm}$, only. This item is due to (*2).

Putting this together, we define the function

$$
\begin{aligned}
(t, \boldsymbol{x}) \mapsto h^{\pm}_{\alpha',k'}(\boldsymbol{x}, t) &= h^{\pm}_{\alpha',k'}(\boldsymbol{x}, \boldsymbol{q}^{\pm}, \ldots, \boldsymbol{q}^{(|\alpha|+k+2)\pm}) \\
&= (5.76) + (5.77) \cdot (5.78) + (5.79) \cdot (5.81) + (5.79) \cdot (5.82)
\end{aligned}
$$

and find $h^{\pm}_{\alpha',k'} \in C^{n-|\alpha|-k}(D_{\boldsymbol{q}}, \mathbb{R}^3)$.

Now, carrying out the derivation in (5.83), we obtain

$$
\begin{aligned}
\partial_t^{k'} D_{\boldsymbol{x}}^{\alpha'} \boldsymbol{e}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}) &= h^{\pm}_{\alpha',k'}(\boldsymbol{x}, \boldsymbol{q}^{\pm}, \ldots, \boldsymbol{q}^{(|\alpha|+k+2)\pm}) \\
&+ \frac{\left(\partial_t t^{\pm}_{\boldsymbol{q}}(t, \boldsymbol{x})\right)^k \cdot \ldots \cdot \left(\partial_{x_3} t^{\pm}_{\boldsymbol{q}}(t, \boldsymbol{x})\right)^{\alpha_3} \cdot \partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha} t^{\pm}_{\boldsymbol{q}}(t, \boldsymbol{x})}{(1 \pm \boldsymbol{n}^{\pm} \cdot \boldsymbol{v}^{\pm})^3 |\boldsymbol{x} - \boldsymbol{q}^{\pm}|} \\
&\quad \left(\boldsymbol{n}^{\pm} \wedge \left[(\boldsymbol{n}^{\pm} \pm \boldsymbol{v}^{\pm}) \wedge \boldsymbol{q}^{(|\alpha|+k+3)\pm}\right]\right),
\end{aligned}
$$

where the factor $\partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha} t^{\pm}_{\boldsymbol{q}}(t, \boldsymbol{x})$ arises by applying the chain-rule

$$
\partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha} \boldsymbol{q}^{(k+|\alpha|+2)\pm} = \boldsymbol{q}^{(|\alpha|+k+3)\pm} \partial_t^{k'-k} D_{\boldsymbol{x}}^{\alpha'-\alpha} t^{\pm}_{\boldsymbol{q}}(t, \boldsymbol{x}), \tag{5.84}
$$

and thus,

$$
\begin{aligned}
\partial_t^{k'} D_{\boldsymbol{x}}^{\alpha'} \boldsymbol{e}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}) &= h^{\pm}_{\alpha',k'}(\boldsymbol{x}, \boldsymbol{q}^{\pm}, \ldots, \boldsymbol{q}^{(|\alpha|+k+2)\pm}) \\
&+ \frac{\left(\partial_t t^{\pm}_{\boldsymbol{q}}(t, \boldsymbol{x})\right)^k \cdot \ldots \cdot \left(\partial_{x_3} t^{\pm}_{\boldsymbol{q}}(t, \boldsymbol{x})\right)^{\alpha'_3}}{(1 \pm \boldsymbol{n}^{\pm} \cdot \boldsymbol{v}^{\pm})^3 |\boldsymbol{x} - \boldsymbol{q}^{\pm}|} \\
&\quad \left(\boldsymbol{n}^{\pm} \wedge \left[(\boldsymbol{n}^{\pm} \pm \boldsymbol{v}^{\pm}) \wedge \boldsymbol{q}^{(|\alpha|+k+3)\pm}\right]\right),
\end{aligned}
$$

respectively. This concludes the induction.

(ii) It remains to prove the last statement of Lemma 5.4.2 (Explicit form of derivatives), namely, (5.70). Let therefore $(t^*, \boldsymbol{x}^*) \in \partial J(0, \boldsymbol{q}_0) \setminus \{(0, \boldsymbol{q}_0)\} \subset D_{\boldsymbol{q}}$. Again, $\pm$ corresponds to the cases $t^*, t \lessgtr 0$.

For the limit $(t, \boldsymbol{x}) \uparrow (t^*, \boldsymbol{x}^*)$ it follows

$$
\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} t^{\pm}_{\boldsymbol{q}}(t, \boldsymbol{x}) = t^{\pm}_{\boldsymbol{q}}(t^*, \boldsymbol{x}^*) = 0, \tag{5.85}
$$

by geometrical reasoning and the intermediate value theorem, cf. proof of Lemma 5.1.1 ($t^{\pm}$), (i).

By the assumption $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{T}^{2+n}(\mathbb{R})$, the continuity of $(t, \boldsymbol{x}) \mapsto h^{\pm}_{\alpha,k}(t, \boldsymbol{x})$, and (5.85), it follows

$$
\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} h^{\pm}_{\alpha,k}(t, \boldsymbol{x}) = h^{\pm}_{\alpha,k}(t^*, \boldsymbol{x}^*), \tag{5.86}
$$

which, writing it as function of the arguments, gives

$$h_{\alpha,k}^{\pm}\left(\boldsymbol{x}^*, \lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \boldsymbol{q}^{\pm}, \ldots, \lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \boldsymbol{q}^{(|\alpha|+k+1)\pm}\right) = h_{\alpha,k}^{\pm}(\boldsymbol{x}^*, \boldsymbol{q}_0, \ldots, \boldsymbol{q}_0^{(|\alpha|+k+1)}). \tag{5.87}$$

Therefore, by $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^{2+n}(\mathbb{R})$ and (5.85)

$$\begin{aligned}
\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \partial_t^k D_{\boldsymbol{x}}^{\alpha} \boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) &= \lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} h_{\alpha,k}^{\pm}(\boldsymbol{x}, \boldsymbol{q}^{\pm}, \ldots, \boldsymbol{q}^{(|\alpha|+k+1)\pm}) \\
&\quad + \lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \left(\partial_t t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})\right)^k \prod_{i=1}^{3} \left(\partial_{x_i} t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})\right)^{\alpha_i} \\
&\quad \lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \left[\frac{\boldsymbol{n} \wedge \left[(\boldsymbol{n} \pm \boldsymbol{v}) \wedge \boldsymbol{q}^{(|\alpha|+k+2)}\right]}{(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3 |\boldsymbol{x} - \boldsymbol{q}|}\right]^{\pm} \\
&= h_{\alpha,k}^{\pm}(\boldsymbol{x}^*, \boldsymbol{q}_0, \ldots, \boldsymbol{q}_0^{(|\alpha|+k+1)}) \\
&\quad + \lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \left(\partial_t t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})\right)^k \prod_{i=1}^{3} \left(\partial_{x_i} t_{\boldsymbol{q}}^{\pm}(t,\boldsymbol{x})\right)^{\alpha_i} \\
&\quad \frac{\boldsymbol{n}_0 \wedge \left[(\boldsymbol{n}_0 \pm \boldsymbol{v}_0) \wedge \boldsymbol{q}_0^{(|\alpha|+k+2)}\right]}{(1 \pm \boldsymbol{n}_0\cdot\boldsymbol{v}_0)^3 |\boldsymbol{x} - \boldsymbol{q}_0|}.
\end{aligned} \tag{5.88}$$

□

Proof of Lemma 5.4.1. (Compatibility Conditions)

(i) We show, that $\boldsymbol{r}^{-\sigma(t)}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)} - \boldsymbol{r}^{-\sigma(t)}_{t,(\tilde{\boldsymbol{q}}_0,\tilde{\boldsymbol{p}}_0)} = 0$ (cf. (4.16)) if and only if $\boldsymbol{p}_0 = \tilde{\boldsymbol{p}}_0$.

We shall use the notation from Theorem 4.2.1 (Explicit Maxwell solutions), namely,

$$\boldsymbol{n}_0 := \frac{\boldsymbol{x} - \boldsymbol{q}_0}{|\boldsymbol{x} - \boldsymbol{q}_0|}, \quad \boldsymbol{v}_0 := \boldsymbol{v}(\boldsymbol{p}_0), \qquad \tilde{\boldsymbol{v}}_0 := \boldsymbol{v}(\tilde{\boldsymbol{p}}_0). \tag{5.89}$$

By the definition in (4.19), $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0$, and (4.16) it is sufficient to show that

$$\frac{1}{1 \pm \boldsymbol{n}_0 \cdot \boldsymbol{v}_0} \begin{pmatrix} \boldsymbol{n}_0 \pm \boldsymbol{v}_0 \\ -\boldsymbol{n}_0 \wedge \boldsymbol{v}_0 \end{pmatrix} = \frac{1}{1 \pm \boldsymbol{n}_0 \cdot \tilde{\boldsymbol{v}}_0} \begin{pmatrix} \boldsymbol{n}_0 \pm \tilde{\boldsymbol{v}}_0 \\ -\boldsymbol{n}_0 \wedge \tilde{\boldsymbol{v}}_0 \end{pmatrix}, \tag{5.90}$$

for all $(t, \boldsymbol{x}) \in D_{\boldsymbol{q}} \cap \partial J(0, \boldsymbol{q}_0)$ is equivalent to $\boldsymbol{p}_0 = \tilde{\boldsymbol{p}}_0$, where $\pm$ serves as placeholder for either $+$ in the case $t < 0$ and $-$ in the case $t > 0$.

First, let us assume (5.90) holds for all $(t, \boldsymbol{x}) \in D_{\boldsymbol{q}} \cap \partial J(0, \boldsymbol{q}_0)$. Since, there is no t-dependence in (5.90) it is equivalent to state that (5.90) holds for all $\boldsymbol{x} \in \mathbb{R}^3 \setminus \{\boldsymbol{q}_0\}$. This implies, that $\boldsymbol{n}_0 \neq 0$. Moreover, $1 \pm \boldsymbol{n}_0 \cdot \boldsymbol{v}_0, 1 \pm \boldsymbol{n}_0 \cdot \tilde{\boldsymbol{v}}_0 \neq 0$ by the assumption $(\boldsymbol{q}, \boldsymbol{p}), (\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}})$ strictly time-like. We distinguish the following two cases:

- Let $\boldsymbol{v}_0 = 0$. Then, $\boldsymbol{n}_0 \pm \boldsymbol{v}_0 = \boldsymbol{n}_0 \neq 0$ and therefore $\boldsymbol{n}_0 \pm \tilde{\boldsymbol{v}}_0 \neq 0$. The first line of (5.90) then gives

$$\boldsymbol{n}_0 \quad \| \quad \boldsymbol{n}_0 \pm \tilde{\boldsymbol{v}}_0, \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^3 \setminus \{\boldsymbol{q}_0\}. \tag{5.91}$$

  Assume $\tilde{\boldsymbol{v}}_0 \neq 0$. Then, $\exists\, c \in \mathbb{R} \setminus \{0\}$ such that $(1 - c)\boldsymbol{n}_0 = \pm c \boldsymbol{v}_0$. Since this needs to hold for all $\boldsymbol{x} \in \mathbb{R}^3 \setminus \{\boldsymbol{q}_0\}$ the assumption leads to a contradiction, and thus, $\tilde{\boldsymbol{v}}_0 = 0$ and $\boldsymbol{p}_0 = \tilde{\boldsymbol{p}}_0$.
- Let $\boldsymbol{v}_0 \neq 0$. Let $M := \{\boldsymbol{x} \in \mathbb{R}^3 \setminus \{\boldsymbol{q}_0\} : \boldsymbol{n}_0, \boldsymbol{v}_0 \text{ linearly independent}\}$. For $\boldsymbol{x} \in M$ it holds $\boldsymbol{n}_0 \wedge \boldsymbol{v}_0 \neq 0$ and therefore by the second line of (5.90) $\boldsymbol{n}_0 \wedge \tilde{\boldsymbol{v}}_0 \neq 0$. This implies

$$\boldsymbol{n}_0 \wedge \boldsymbol{v}_0 \quad \| \quad \boldsymbol{n}_0 \wedge \tilde{\boldsymbol{v}}_0, \qquad \forall\, \boldsymbol{x} \in M. \tag{5.92}$$

  By definition of the outer product $\tilde{\boldsymbol{v}}_0$ has to lie on the surface spanned by the vectors $\boldsymbol{n}_0$ and $\boldsymbol{v}_0$. Consequently, there exist reals $\alpha, \beta \in \mathbb{R}$ with $\tilde{\boldsymbol{v}}_0 = \alpha \boldsymbol{n}_0 + \beta \boldsymbol{v}_0$. As for $\boldsymbol{x} \in M$ it follows $\boldsymbol{n}_0 \pm \boldsymbol{v}_0 \neq 0$ and in return $\boldsymbol{n}_0 \pm \tilde{\boldsymbol{v}}_0 \neq 0$, we find

$$\boldsymbol{n}_0 \pm \boldsymbol{v}_0 \quad \| \quad \boldsymbol{n}_0 \pm (\alpha \boldsymbol{n}_0 + \beta \boldsymbol{v}_0) = (1 \pm \alpha)\boldsymbol{n}_0 \pm \beta \boldsymbol{v}_0 \qquad \forall\, \boldsymbol{x} \in M \tag{5.93}$$

  by the first line of equation (5.90). Since there exists a $\boldsymbol{x} \in M$ such that $\boldsymbol{n}_0$ and $\boldsymbol{v}_0$ are linearly independent, it follows $\beta = 1 \pm \alpha$, and in return, $\tilde{\boldsymbol{v}}_0 = \alpha(\boldsymbol{n}_0 \pm \boldsymbol{v}_0) + \boldsymbol{v}_0$ for all $\boldsymbol{x} \in M$. Since $\boldsymbol{n}_0 \pm \boldsymbol{v}_0 \neq 0$ for $\boldsymbol{x} \in M$ and by the choice of $\boldsymbol{x} \in M$, $\boldsymbol{n}_0 \pm \boldsymbol{v}_0$ may point in almost any direction, it follows $\alpha = 0$, $\beta = 1$, and hence, $\tilde{\boldsymbol{v}}_0 = \boldsymbol{v}_0$ and $\tilde{\boldsymbol{p}}_0 = \boldsymbol{p}_0$.

On the other hand, $\tilde{\boldsymbol{p}}_0 = \boldsymbol{p}_0$ implies immediately that (5.90) is true for all $(t, \boldsymbol{x}) \in D_{\boldsymbol{q}} \cap \partial J(0, \boldsymbol{q}_0)$.

(ii) By assumption of the claim, we have $(\boldsymbol{q},\boldsymbol{p}),(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}) \in \mathcal{T}^{2+n}(\mathbb{R})$ strictly time-like with $\boldsymbol{q}_0 = \tilde{\boldsymbol{q}}_0$ and $\boldsymbol{p}_0 = \tilde{\boldsymbol{p}}_0$. Further fix $m \le n$ throughout this proof.

We show, that for all $(t^*,\boldsymbol{x}^*) \in \partial J(0,\boldsymbol{q}_0) \setminus \{(0,\boldsymbol{q}_0)\}$ and $|\alpha| + k \le m$

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \partial_t^k D_{\boldsymbol{x}}^\alpha \left(\boldsymbol{f}_t^{-\sigma(t)}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) - \boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}](\boldsymbol{x}) \right) = 0 \tag{5.94}$$

is equivalent to

$$\boldsymbol{q}_0^{(l)} = \tilde{\boldsymbol{q}}_0^{(l)}, \qquad \forall\, 0 \le l \le 2+m. \tag{5.95}$$

Note that by virtue of (5.64)-(5.65), we continue in the equivalent notation, where $\pm$ or $\mp$ correspond to the cases $t^*, t \lessgtr 0$, i.e., we write $\pm$ for $-\sigma(t)$ and $\mp$ for $\sigma(t)$.

First, we show, that the equivalence holds for the electric field component, i.e., that the following statements are equivalent:

(*1) For all $|\alpha| + k \le m$ and all $(t^*,\boldsymbol{x}^*) \in \partial J(0,\boldsymbol{q}_0) \setminus \{(0,\boldsymbol{q}_0)\}$ it holds

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \partial_t^k D_{\boldsymbol{x}}^\alpha \left(\boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) - \boldsymbol{e}_t^{\pm}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}](\boldsymbol{x}) \right) = 0. \tag{5.96}$$

(*2) $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0, \ldots, \tilde{\boldsymbol{q}}_0^{(2+m)} = \boldsymbol{q}_0^{(2+m)}$.

Assume statement (*1) holds. We show by induction over m that statement (*2) holds:

Base case: For $m = 0$ item (*1) reads

$$\lim_{(t,\boldsymbol{x})\to(t^*,\boldsymbol{x}^*)} \boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) - \boldsymbol{e}_t^{\pm}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}](\boldsymbol{x}) = 0. \tag{5.97}$$

For all $(t^*,\boldsymbol{q}^*) \in \partial J(0,\boldsymbol{q}_0)\setminus\{(0,\boldsymbol{q}_0)\}$ it holds that $t_{\boldsymbol{q}}^{\pm}(t^*,\boldsymbol{x}^*) = 0 = t_{\tilde{\boldsymbol{q}}}^{\pm}(t^*,\boldsymbol{x}^*)$. Moreover, by assumption we have $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0$ and $\tilde{\boldsymbol{p}}_0 = \boldsymbol{p}_0$. Plugging this in, the first summands of the Liénard-Wiechert fields cancel each other and it remains

$$\frac{\boldsymbol{n}_0^* \wedge [(\boldsymbol{n}_0^* \pm \boldsymbol{v}_0) \wedge (\boldsymbol{a}_0 - \tilde{\boldsymbol{a}}_0)]}{(1 \pm \boldsymbol{n}_0^* \cdot \boldsymbol{v}_0)^3 |\boldsymbol{x}^* - \boldsymbol{q}_0|} = 0 \tag{5.98}$$

where we used the notation $\boldsymbol{n}_0^* := (\boldsymbol{x}^* - \boldsymbol{q}_0)/|\boldsymbol{x}^* - \boldsymbol{q}_0|$.

It should be noted, that though the light-cone $\partial J(0,\boldsymbol{q}_0) \setminus \{(0,\boldsymbol{q}_0)\}$ covers each space coordinate in $\mathbb{R}^3 \setminus \{\boldsymbol{q}_0\}$, the corresponding time coordinate does not appear but in form of advanced and retarded times, which are 0 in this limit. Thus, (5.98) holds true for all $\boldsymbol{x}^* \in \mathbb{R}^3 \setminus \{\boldsymbol{q}_0\}$. And hence, at least one of the following statements is true:

(a) $\boldsymbol{a}_0 - \tilde{\boldsymbol{a}}_0 = 0$

(b) $\forall\, \boldsymbol{x}^* \in \mathbb{R}^3 \setminus \{\boldsymbol{q}_0\} : \boldsymbol{n}_0^* \pm \boldsymbol{v}_0 \parallel \boldsymbol{a}_0 - \tilde{\boldsymbol{a}}_0$

(c) $\forall\, \boldsymbol{x}^* \in \mathbb{R}^3 \setminus \{\boldsymbol{q}_0\} : \boldsymbol{n}_0^* \parallel (\boldsymbol{n}_0^* \pm \boldsymbol{v}_0) \wedge (\boldsymbol{a}_0 - \tilde{\boldsymbol{a}}_0)$

Item b) and c) cannot be fulfilled because in case on finds an $\boldsymbol{x}^*$ for which the statement is true, there are other $\boldsymbol{x}^*$ such that the statement is violated. Therefore, it follows $\boldsymbol{a}_0 - \tilde{\boldsymbol{a}}_0 = 0$, which implies statement (*2) to be true for $m = 0$.

Inductive step: Now, assume that the implication (∗1) ⇒ (∗2) is true for $m < n$. We show, that this implies the claim to be true for $1+m$.

Assume (*1) is true for $1+m$, i.e., for all $k+|\alpha| \le 1+m$ it holds

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \partial_t^k D_{\boldsymbol{x}}^\alpha \left(\boldsymbol{e}_t^\pm[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) - \boldsymbol{e}_t^\pm[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}](\boldsymbol{x})\right) = 0. \tag{5.99}$$

Recall, that the limit is such that $(t,\boldsymbol{x})$ always lies in the same half space as $(t^*,\boldsymbol{x}^*)$.

By induction hypothesis this implies $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0, \dots, \tilde{\boldsymbol{q}}_0^{(2+m)} = \boldsymbol{q}_0^{(2+m)}$. Therefore, making use of Lemma 5.4.2 (Explicit form of derivatives) we can transform (5.99) for $|\alpha|+k = 1+m$ by

$$\begin{aligned}
&\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \partial_t^k D_{\boldsymbol{x}}^\alpha \left(\boldsymbol{e}_t^\pm[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) - \boldsymbol{e}_t^\pm[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}](\boldsymbol{x})\right)\\
&\quad = h_{\alpha,k}^\pm(\boldsymbol{x}^*, \boldsymbol{q}_0, \dots, \boldsymbol{q}_0^{(2+m)}) - h_{\alpha,k}^\pm(\boldsymbol{x}^*, \boldsymbol{q}_0, \dots, \boldsymbol{q}_0^{(2+m)})\\
&\quad + \lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \left(\partial_t t_{\boldsymbol{q}}^\pm(t,\boldsymbol{x})\right)^k \prod_{i=1}^3 \left(\partial_{x_i} t_{\boldsymbol{q}}^\pm(t,\boldsymbol{x})\right)^{\alpha_i}\\
&\qquad \frac{\boldsymbol{n}_0^* \wedge \left[(\boldsymbol{n}_0^* \pm \boldsymbol{v}_0) \wedge (\boldsymbol{q}_0^{(3+m)} - \tilde{\boldsymbol{q}}_0^{(3+m)})\right]}{(1 \pm \boldsymbol{n}_0^* \cdot \boldsymbol{v}_0)^3 |\boldsymbol{x} - \boldsymbol{q}_0|}\\
&\quad = \lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \left(\partial_t t_{\boldsymbol{q}}^\pm(t,\boldsymbol{x})\right)^k \prod_{i=1}^3 \left(\partial_{x_3} t_{\boldsymbol{q}}^\pm(t,\boldsymbol{x})\right)^{\alpha_3}\\
&\qquad \frac{\boldsymbol{n}_0^* \wedge \left[(\boldsymbol{n}_0^* \pm \boldsymbol{v}_0) \wedge (\boldsymbol{q}_0^{(3+m)} - \tilde{\boldsymbol{q}}_0^{(3+m)})\right]}{(1 \pm \boldsymbol{n}_0^* \cdot \boldsymbol{v}_0)^3 |\boldsymbol{x} - \boldsymbol{q}_0|}\\
&\quad = 0,
\end{aligned} \tag{5.100}$$

where, again, the abbreviation $\boldsymbol{n}_0^* := (\boldsymbol{x}^* - \boldsymbol{q}_0)/|\boldsymbol{x}^* - \boldsymbol{q}_0|$ has been used.

We recall formula (5.2) for the derivatives of the advanced and retarded time function, which gives $\partial_t t^\pm(t,\boldsymbol{x}) = \frac{1}{1\pm\boldsymbol{n}\cdot\boldsymbol{v}}\Big|^\pm$ and $\partial_{x_i} t^\pm(t,\boldsymbol{x}) = \frac{\pm n_i}{1\pm\boldsymbol{n}\cdot\boldsymbol{v}}\Big|^\pm$ for $i = 1,2,3$ and in particular

$$\begin{aligned}
&\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \left(\partial_t t_{\boldsymbol{q}}^\pm(t,\boldsymbol{x})\right)^{\alpha_0} \prod_{i=1}^3 \left(\partial_{x_i} t_{\boldsymbol{q}}^\pm(t,\boldsymbol{x})\right)^{\alpha_i}\\
&\quad = \left(\frac{1}{1 \pm \boldsymbol{n}_0^* \cdot \boldsymbol{v}_0}\right)^k \prod_{i=1}^3 \left(\frac{\pm n_{0,i}^*}{1 \pm \boldsymbol{n}_0^* \cdot \boldsymbol{v}_0}\right)^{\alpha_i}\\
&\quad \neq 0
\end{aligned}$$

since $\boldsymbol{x}^* \in \mathbb{R}^3 \setminus \{\boldsymbol{q}_0\}$ for $(t^*,\boldsymbol{x}^*) \in J(0,\boldsymbol{q}_0) \setminus \{(0,\boldsymbol{q}_0)\}$.

Therefore, it follows from (5.100)

$$\boldsymbol{n}_0^* \wedge \left[(\boldsymbol{n}_0^* \pm \boldsymbol{v}_0) \wedge (\boldsymbol{q}_0^{(3+m)} - \tilde{\boldsymbol{q}}_0^{(3+m)})\right] = 0. \tag{5.101}$$

Analogously to the the base case, (5.101) needs to be met for all $\boldsymbol{x}^* \in \mathbb{R}^3 \setminus \{\boldsymbol{q}_0\}$ because the light-cone boundary $\partial J(0,\boldsymbol{q}_0) \setminus \{(0,\boldsymbol{q}_0)\}$ covers exactly these space points and the dependence on t^* disappears since it only appears in the advanced and retarded times and $t^{\pm}_{\boldsymbol{q}}(t^*, \boldsymbol{x}^*) = 0 = t^{\pm}_{\tilde{\boldsymbol{q}}}(t^*, \boldsymbol{x}^*)$.

Accordingly to the base case of the induction, this implies that at least one of the following statements needs to hold true:

(a) $\boldsymbol{q}_0^{(3+m)} - \tilde{\boldsymbol{q}}_0^{(3+m)} = 0$
(b) $\forall \boldsymbol{x}^* \in \mathbb{R}^3 \setminus \{\boldsymbol{q}_0\} : \boldsymbol{n}_0^* \pm \boldsymbol{v}_0 \parallel \boldsymbol{a}_0 - \tilde{\boldsymbol{a}}_0$
(c) $\forall \boldsymbol{x}^* \in \mathbb{R}^3 \setminus \{\boldsymbol{q}_0\} : \boldsymbol{n}_0^* \parallel (\boldsymbol{n}_0^* \pm \boldsymbol{v}_0) \wedge (\boldsymbol{a}_0 - \tilde{\boldsymbol{a}}_0)$

Again, analogously to the base case, item b) and item c) can not be fulfilled for all $\boldsymbol{x}^* \in \mathbb{R}^3 \setminus \{\boldsymbol{q}_0\}$, and therefore, $\tilde{\boldsymbol{q}}_0^{(3+m)} = \boldsymbol{q}_0^{(3+m)}$. Hence, we have shown that $(*1)$ implies $(*2)$.

Second, it is left to show $(*2) \Rightarrow (*1)$. By Lemma the explicit representation of the derivatives in 5.4.2 (Explicit form of derivatives), item $(*2)$, i.e., $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0, \ldots, \tilde{\boldsymbol{q}}_0^{(2+m)} = \boldsymbol{q}_0^{(2+m)}$, implies that for all $k + |\alpha| \leq m$

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \partial_t^k D_{\boldsymbol{x}}^\alpha \boldsymbol{e}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}) \tag{5.102}$$

and

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \partial_t^k D_{\boldsymbol{x}}^\alpha \boldsymbol{e}_t^{\pm}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}](\boldsymbol{x}) \tag{5.103}$$

coincide, and therefore, $(*1)$ holds true.

To conclude the proof, it remains to prove that the assumption $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0, \ldots, \tilde{\boldsymbol{q}}_0^{(2+m)} = \boldsymbol{q}_0^{(2+m)}$ also implies

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \partial_t^k D_{\boldsymbol{x}}^\alpha \left(\boldsymbol{b}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}) - \boldsymbol{b}_t^{\pm}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}](\boldsymbol{x})\right) = 0 \tag{5.104}$$

for all $(t^*, \boldsymbol{x}^*) \in J(0, \boldsymbol{q}_0) \setminus \{(0, \boldsymbol{q}_0)\}$ and $k + |\alpha| \leq m$.

By Definition 4.2.1 (Liénard-Wiechert fields) the magnetic field due to some charge with trajectory $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{T}^{2+n}(\mathbb{R})$ at $(t, \boldsymbol{x}) \in J(0, \boldsymbol{q}_0) \cap D_{\boldsymbol{q}}$ is given by

$$\boldsymbol{b}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}) := \mp \boldsymbol{n}^{\pm} \wedge \boldsymbol{e}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}), \tag{5.105}$$

and therefore, by product rule, for $|\alpha| + k \leq n$ we have

$$\partial_t^k D_{\boldsymbol{x}}^\alpha \boldsymbol{b}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}) = \sum_{\substack{k_1+k_2=k \\ \alpha_1+\alpha_2=\alpha}} \mp \partial_t^{k_1} D_{\boldsymbol{x}}^{\alpha_1} \boldsymbol{n}^{\pm} \wedge \partial_t^{k_2} D_{\boldsymbol{x}}^{\alpha_2} \boldsymbol{e}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}). \tag{5.106}$$

With (5.106) the left hand side of (5.104) can be transformed into

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \sum_{\substack{k_1+k_2=k \\ \alpha_1+\alpha_2=\alpha}} \Big(\mp \partial_t^{k_1} D_{\boldsymbol{x}}^{\alpha_1} \boldsymbol{n}^{\pm} \wedge \partial_t^{k_2} D_{\boldsymbol{x}}^{\alpha_2} \boldsymbol{e}_t^{\pm}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x})$$
$$\pm \partial_t^{k_1} D_{\boldsymbol{x}}^{\alpha_1} \tilde{\boldsymbol{n}}^{\pm} \wedge \partial_t^{k_2} D_{\boldsymbol{x}}^{\alpha_2} \boldsymbol{e}_t^{\pm}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}](\boldsymbol{x}) \Big),$$

which can be written as

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \sum_{\substack{k_1+k_2=k\\ \alpha_1+\alpha_2=\alpha}} \mp \partial_t^{k_1} D_{\boldsymbol{x}}^{\alpha_1} \boldsymbol{n}^{\pm} \wedge \partial_t^{k_2} D_{\boldsymbol{x}}^{\alpha_2} (\boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) - \boldsymbol{e}_t^{\pm}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}](\boldsymbol{x})) \\ \mp \partial_t^{k_1} D_{\boldsymbol{x}}^{\alpha_1} (\boldsymbol{n}^{\pm} - \tilde{\boldsymbol{n}}^{\pm}) \wedge \partial_t^{k_2} D_{\boldsymbol{x}}^{\alpha_2} \boldsymbol{e}_t^{\pm}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}](\boldsymbol{x}). \tag{5.107}$$

Since by assumption $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0, \ldots, \tilde{\boldsymbol{q}}_0^{(2+m)} = \boldsymbol{q}_0^{(2+m)}$ it follows $\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \partial_t^{k_1} D_{\boldsymbol{x}}^{\alpha_1} \left(\boldsymbol{e}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}](\boldsymbol{x}) - \boldsymbol{e}_t^{\pm}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}](\boldsymbol{x})\right) = 0$ for all $|\alpha_1| + k_1 \leq m$ and $\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} \partial_t^{k_2} D_{\boldsymbol{x}}^{\alpha_2} \left(\boldsymbol{n}^{\pm} - \tilde{\boldsymbol{n}}^{\pm}\right) = 0$ for all $|\alpha_2| + k_2 \leq m$ because there is no trajectory dependence beyond the $\alpha_2 + k$th derivative, and thus, (5.107) vanishes.

□

Proof of Lemma 4.2.4. (Regularity of $\boldsymbol{f}_t$)

By condition we have $(\boldsymbol{q},\boldsymbol{p}),(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}) \in \mathcal{T}^{2+n}(\mathbb{R})$, $\boldsymbol{f}_0^h \in C^{1+n}(\mathbb{R}^3,\mathbb{R}^6)$, and thus, by Lemma 4.2.2 (Homogeneous Maxwell solutions) $\boldsymbol{f}^h \in C^n(\mathbb{R}^4,\mathbb{R}^6)$. We write the Maxwell solution $\boldsymbol{f}$ given by (4.12)-(4.17) in the shorter form

$$\boldsymbol{f}_t = \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)} \left(\boldsymbol{f}_t^{-\sigma(t)}[\boldsymbol{q},\boldsymbol{p}] - \boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] \right) \tag{5.108}$$

$$+ \lambda \boldsymbol{f}_t^{-}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] + (1-\lambda)\boldsymbol{f}_t^{+}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] \tag{5.109}$$

$$+ \boldsymbol{r}^{-\sigma(t)}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)} - \boldsymbol{r}^{-\sigma(t)}_{t,(\tilde{\boldsymbol{q}}_0,\tilde{\boldsymbol{p}}_0)} \tag{5.110}$$

$$+ \boldsymbol{f}_t^h. \tag{5.111}$$

(i) The property $\boldsymbol{f} \in C^n(D_{\boldsymbol{q}} \cap D_{\tilde{\boldsymbol{q}}} \setminus \partial J(0,\boldsymbol{q}_0),\mathbb{R}^6)$ is a consequence of our previous results: By Lemma 4.2.1 (Properties of Liénard-Wiechert fields), for $(\boldsymbol{q},\boldsymbol{p}),(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}) \in \mathcal{T}^{2+n}(\mathbb{R})$ the Liénard-Wiechert fields fulfill $\boldsymbol{f}_t^{\pm}[\boldsymbol{q},\boldsymbol{p}] \in C^n(D_{\boldsymbol{q}},\mathbb{R}^6)$ and $\boldsymbol{f}_t^{\pm}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] \in C^n(D_{\tilde{\boldsymbol{q}}},\mathbb{R}^6)$.

By definition (4.19), the boundary terms $(t,\boldsymbol{x}) \mapsto \boldsymbol{r}^{\pm}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)}(\boldsymbol{x})$ and $(t,\boldsymbol{x}) \mapsto \boldsymbol{r}^{\pm}_{t,(\tilde{\boldsymbol{q}}_0,\tilde{\boldsymbol{p}}_0)}(\boldsymbol{x})$ are only supported on the light-cone boundary $\partial J(0,\boldsymbol{q}_0)$.

Furthermore, since $\boldsymbol{q}_0 = \tilde{\boldsymbol{q}}_0$, (5.108) is in $C^n(D_{\boldsymbol{q}} \cap D_{\tilde{\boldsymbol{q}}} \setminus J(0,\boldsymbol{q}_0),\mathbb{R}^6)$, (5.109) is in $C^n(D_{\tilde{\boldsymbol{q}}} \setminus J(0,\boldsymbol{q}_0),\mathbb{R}^6)$, and, (5.110) is in $C^n(D_{\boldsymbol{q}} \cap D_{\tilde{\boldsymbol{q}}} \setminus \partial J(0,\boldsymbol{q}_0),\mathbb{R}^6)$. As stated in the beginning of the proof (5.111) is in $C^n(\mathbb{R}^4,\mathbb{R}^6)$, which concludes the claim.

For the special case $\lambda = 1$ for $t \geq 0$ the field writes

$$\boldsymbol{f}_t = \mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)} \boldsymbol{f}_t^{-\sigma(t)}[\boldsymbol{q},\boldsymbol{p}] \tag{5.112}$$

$$+ \mathbb{1}_{B^c_{|t|}(\boldsymbol{q}_0)} \boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}},\tilde{\boldsymbol{p}}] \tag{5.113}$$

$$+ \boldsymbol{r}^{-\sigma(t)}_{t,(\boldsymbol{q}_0,\boldsymbol{p}_0)} - \boldsymbol{r}^{-\sigma(t)}_{t,(\tilde{\boldsymbol{q}}_0,\tilde{\boldsymbol{p}}_0)} \tag{5.114}$$

$$+ \boldsymbol{f}_t^h \tag{5.115}$$

and fulfills $\boldsymbol{f} \in C^n(D_{\boldsymbol{q}}^{[0,\infty)} \setminus \partial J^+(0,\boldsymbol{q}_{i,0}),\mathbb{R}^6)$, which concludes the proof.

(ii) By (i), to show that $\boldsymbol{f}$ can be evaluated in the point-wise sense on $D_{\boldsymbol{q}} \cap D_{\tilde{\boldsymbol{q}}}$, it suffices to prove that $\boldsymbol{f}$ can be evaluated point-wise on the boundary of the light-cone $\partial J(0,\boldsymbol{q}_0)$ because $\boldsymbol{f} \in C^n(D_{\boldsymbol{q}} \cap D_{\tilde{\boldsymbol{q}}} \setminus \partial J(0,\boldsymbol{q}_0),\mathbb{R}^6)$ by (i). On the light-cone the only singular term arises from (5.110) and by Lemma 5.4.1 (Compatibility Conditions) the latter vanishes if and only if $\tilde{\boldsymbol{p}}_0 = \boldsymbol{p}_0$.

(iii) Let $m \leq n$ be given. Given $\tilde{\boldsymbol{q}}_0 = \boldsymbol{q}_0$, $\tilde{\boldsymbol{p}}_0 = \boldsymbol{p}_0$, which is necessary and sufficient for point-wise evaluation and cancellation of (5.110), we have that (5.111) is in $C^n(\mathbb{R}^4,\mathbb{R}^6)$, (5.109), as function of $(t,\boldsymbol{x})$, is in $C^n(D_{\tilde{\boldsymbol{q}}},\mathbb{R}^6)$ due to Lemma 4.2.1 (Properties of Liénard-Wiechert fields), and therefore, it remains to show that summand (5.108) as function of $(t,\boldsymbol{x})$ is in $C^n(D_{\boldsymbol{q}} \cap D_{\tilde{\boldsymbol{q}}},\mathbb{R}^6)$. Since (5.108) is in $C^n(J(0,\boldsymbol{q}_0) \cap D_{\boldsymbol{q}} \cap D_{\tilde{\boldsymbol{q}}},\mathbb{R}^6)$, and, due to the indicator function, in $C^\infty(J^c(0,\boldsymbol{q}_0),\mathbb{R}^6)$, it suffices to verify that the light-cone limit of (5.108) exists, i.e., we need to prove:

For all $|\alpha| + k \le m$ and all $(t^*, \boldsymbol{x}^*) \in \partial J(0, \boldsymbol{q}_0) \setminus \{(0, \boldsymbol{q}_0)\}$ the limit

$$\lim_{(t,\boldsymbol{x})\to(t^*,\boldsymbol{x}^*)} D_{\boldsymbol{x}}^{\alpha} \partial_t^k \left[\mathbb{1}_{B_{|t|}(\boldsymbol{q}_0)} \left(\boldsymbol{f}_t^{-\sigma(t)}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}) - \boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}](\boldsymbol{x}) \right) \right] \tag{5.116}$$

exists if and only if

$$\lim_{t\to 0} \frac{d^l}{dt^l} \boldsymbol{q}_t = \lim_{t\to 0} \frac{d^l}{dt^l} \tilde{\boldsymbol{q}}_t, \qquad \forall\, 0 \le l \le 2 + m. \tag{5.117}$$

Let $(t^*, \boldsymbol{x}^*) \in \partial J(0, \boldsymbol{q}_0) \setminus \{(0, \boldsymbol{q}_0)\}$, then for the outer limit, $(t, \boldsymbol{x}) \downarrow (t^*, \boldsymbol{x}^*)$ (cf. Definition 5.4.1 (Light-cone limit)) we find

$$\lim_{(t,\boldsymbol{x})\downarrow(t^*,\boldsymbol{x}^*)} D_{\boldsymbol{x}}^{\alpha} \partial_t^k 0 = 0 \tag{5.118}$$

for all $|\alpha| + k \le m$, $(t^*, \boldsymbol{x}^*) \in \partial J(0, \boldsymbol{q}_0) \setminus \{(0, \boldsymbol{q}_0)\}$ and according to Lemma 5.4.1 (Compatibility conditions), (ii),

$$\lim_{(t,\boldsymbol{x})\uparrow(t^*,\boldsymbol{x}^*)} D_{\boldsymbol{x}}^{\alpha} \partial_t^k \left(\boldsymbol{f}_t^{-\sigma(t)}[\boldsymbol{q}, \boldsymbol{p}](\boldsymbol{x}) - \boldsymbol{f}_t^{-\sigma(t)}[\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}](\boldsymbol{x}) \right) = 0 \tag{5.119}$$

if and only if (5.117).

In conclusion, $\boldsymbol{f} \in C^m(D_{\boldsymbol{q}} \cap D_{\tilde{\boldsymbol{q}}}, \mathbb{R}^6)$ if and only if (5.117).

Moreover, for the special case $\lambda = 1$ and the restriction to $t \ge 0$ we obtain $\boldsymbol{f} \in C^n(D_{\boldsymbol{q}}^{[0,\infty)}, \mathbb{R}^6)$ if and only if

$$\lim_{t\searrow 0} \frac{d^l}{dt^l} \boldsymbol{q}_t = \lim_{t\nearrow 0} \frac{d^l}{dt^l} \tilde{\boldsymbol{q}}_t, \qquad \forall\, 0 \le l \le 2 + m. \tag{5.120}$$

□

5.5 No-go theorem

In this section we prove Theorem 4.1.1 (No-go). The proof is essentially based on Theorem 4.2.1 (Explicit Maxwell solutions) and Lemma 5.4.1 (Compatibility Conditions), which was proven in the previous section. We recall the Definitions 3.3.2 (Lorentz solutions), 3.3.3 (Maxwell solutions), and 3.3.4 (Maxwell-Lorentz solutions) in which we make precise the notion of solutions to the Maxwell, Lorentz and Maxwell-Lorentz equations.

Proof of Theorem 4.1.1. (No-go) Let $N = 2$, $t_1 < 0 < t_2$, $\boldsymbol{\epsilon} \in \mathbb{R}^3$ and $\boldsymbol{\delta} \in \mathcal{F}_{\text{hom}}$. By assumption there is a Maxwell-Lorentz solution

$$(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i, \tilde{\boldsymbol{f}}_i)_{i=1,2} \tag{5.121}$$

on $[t_1, t_2]$ having the initial value

$$(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0}, \tilde{\boldsymbol{f}}_{i,0}) := (\tilde{\boldsymbol{q}}_{i,t}, \tilde{\boldsymbol{p}}_{i,t}, \tilde{\boldsymbol{f}}_{i,t})\big|_{t=0}, \qquad i = 1, 2 \tag{5.122}$$

such that

$$\tilde{\boldsymbol{q}}_{1,t_1} \in \partial J^-(0, \tilde{\boldsymbol{q}}_{2,0}), \qquad \tilde{\boldsymbol{q}}_{2,t_2} \in \partial J^+(0, \tilde{\boldsymbol{q}}_{1,0}), \tag{5.123}$$

and, that $(\tilde{\boldsymbol{q}}_2, \tilde{\boldsymbol{p}}_2)$ is the unique Lorentz solution for $\tilde{\boldsymbol{f}}_1$ and $(\tilde{\boldsymbol{q}}_{2,t}, \tilde{\boldsymbol{p}}_{2,t})\big|_{t=0} = (\tilde{\boldsymbol{q}}_{2,0}, \tilde{\boldsymbol{p}}_{2,0})$.

We shall now show that there is no Maxwell-Lorentz solution $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i=1,2}$ on $[0, t_2]$ such that

$$(\boldsymbol{q}_{1,t}, \boldsymbol{p}_{1,t}, \boldsymbol{f}_{1,t})\big|_{t=0} = (\tilde{\boldsymbol{q}}_{1,0}, \tilde{\boldsymbol{p}}_{1,0} + \boldsymbol{\epsilon}, \tilde{\boldsymbol{f}}_{1,0}) \tag{5.124}$$

$$(\boldsymbol{q}_{2,t}, \boldsymbol{p}_{2,t}, \boldsymbol{f}_{2,t})\big|_{t=0} = (\tilde{\boldsymbol{q}}_{2,0}, \tilde{\boldsymbol{p}}_{2,0}, \tilde{\boldsymbol{f}}_{2,0} + \boldsymbol{\delta}) \tag{5.125}$$

unless $\boldsymbol{\epsilon} = 0$ and $\boldsymbol{L}(\tilde{\boldsymbol{q}}_{1,0}, \tilde{\boldsymbol{p}}_{1,0}, \boldsymbol{\delta}) = 0$, cf. (1.5). (Recall Figure 4.1 for an illustration of the setting.)

We aim at a proof by contradiction. Therefore, for either $\boldsymbol{\epsilon} \neq 0$ or $\boldsymbol{L}(\tilde{\boldsymbol{q}}_{1,0}, \tilde{\boldsymbol{p}}_{1,0}, \boldsymbol{\delta}) \neq 0$ or both, let us assume that there is a Maxwell-Lorentz solution $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i=1,2}$ on $[0, t_2]$ fulfilling $(\boldsymbol{q}_{1,0}, \boldsymbol{p}_{1,0}, \boldsymbol{f}_{1,0}) := (\boldsymbol{q}_{1,t}, \boldsymbol{p}_{1,t}, \boldsymbol{f}_{1,t})\big|_{t=0} = (\tilde{\boldsymbol{q}}_{1,0}, \tilde{\boldsymbol{p}}_{1,0} + \boldsymbol{\epsilon}, \tilde{\boldsymbol{f}}_{1,0})$ and $(\boldsymbol{q}_{2,0}, \boldsymbol{p}_{2,0}, \boldsymbol{f}_{2,0}) := (\boldsymbol{q}_{2,t}, \boldsymbol{p}_{2,t}, \boldsymbol{f}_{2,t})\big|_{t=0} = (\tilde{\boldsymbol{q}}_{2,0}, \tilde{\boldsymbol{p}}_{2,0}, \tilde{\boldsymbol{f}}_{2,0} + \boldsymbol{\delta})$. The proof is divided into the following five steps:

1. As preparation, which allows the application on Theorem 4.2.1 (Explicit Maxwell solutions), we extend the two solution charge trajectories of charge 1:

   - Given the Maxwell-Lorentz solution $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i, \tilde{\boldsymbol{f}}_i)_{i=1,2}$ on the interval $[t_1, t_2]$ we extend $(\tilde{\boldsymbol{q}}_1, \tilde{\boldsymbol{p}}_1) \in \mathcal{T}^2([t_1, t_2])$ to a strictly time-like $(\tilde{\boldsymbol{q}}_1, \tilde{\boldsymbol{p}}_1) \in \mathcal{T}^2(\mathbb{R})$ in an arbitrary smooth way; red dashed line in Figure 4.1.
   - Given $(\boldsymbol{q}_1, \boldsymbol{p}_1) \in \mathcal{T}^2([0, t_2])$ we extend it smoothly to a strictly time-like $(\boldsymbol{q}_1, \boldsymbol{p}_1) \in \mathcal{T}^2(\mathbb{R})$ in an arbitrary smooth way; green dashed line in Figure 4.1.
   - Note that by definition, $\boldsymbol{q}_{1,0} = \tilde{\boldsymbol{q}}_{1,0}$ and $\boldsymbol{f}_{1,0} \in \mathcal{F}_{\boldsymbol{q}_{1,0}}$.

2. The strictly time-like charge trajectories $(\boldsymbol{q}_1, \boldsymbol{p}_1), (\tilde{\boldsymbol{q}}_1, \tilde{\boldsymbol{p}}_1) \in \mathcal{T}^2(\mathbb{R})$ with $\boldsymbol{q}_{1,0} = \tilde{\boldsymbol{q}}_{1,0}$ and $\boldsymbol{f}_{1,0} \in \mathcal{F}_{\boldsymbol{q}_{1,0}}$ as ingredients allow to apply Theorem 4.2.1 (Explicit Maxwell solutions) for $\lambda = 1$ and $n = 0$ to express the field $\tilde{\boldsymbol{f}}_{1,t}$ for all $t > 0$ as follows:
$$\begin{aligned} \tilde{\boldsymbol{f}}_{1,t} = &\boldsymbol{f}_t^-[\tilde{\boldsymbol{q}}_1, \tilde{\boldsymbol{p}}_1] \\ &+ \tilde{\boldsymbol{f}}_{1,t}^h \end{aligned} \tag{5.126}$$
with
$$\tilde{\boldsymbol{f}}_{1,t}^h = W_t \left[\tilde{\boldsymbol{f}}_{1,0} - \boldsymbol{f}_0^-[\tilde{\boldsymbol{q}}_1, \tilde{\boldsymbol{p}}_1] \right], \tag{5.127}$$
where the latter is well-defined as $\tilde{\boldsymbol{f}}_{1,0} \in \mathcal{F}_{\boldsymbol{q}_{1,0}}$, $\boldsymbol{f}_0^-[\tilde{\boldsymbol{q}}_1, \tilde{\boldsymbol{p}}_1] \in \mathcal{F}_{\boldsymbol{q}_{1,0}}$, and thus, $\tilde{\boldsymbol{f}}_{1,0} - \boldsymbol{f}_0^-[\tilde{\boldsymbol{q}}_1, \tilde{\boldsymbol{p}}_1] \in \mathcal{F}_{\text{hom}}$ which follows directly by definition of the homogeneous and inhomogeneous Maxwell constraints (3.2) and (3.3).

   Applying Theorem 4.2.1 (Explicit Maxwell solutions) again results in the following representation of the field $\boldsymbol{f}_{1,t}$:
$$\begin{aligned} \boldsymbol{f}_{1,t} = &\mathbb{1}_{B_t(\boldsymbol{q}_0)} \boldsymbol{f}_t^-[\boldsymbol{q}_1, \boldsymbol{p}_1] \\ &+ \mathbb{1}_{B_t^c(\boldsymbol{q}_0)} \boldsymbol{f}_t^-[\tilde{\boldsymbol{q}}_1, \tilde{\boldsymbol{p}}_1] \\ &+ \boldsymbol{r}^-_{t,(\boldsymbol{q}_{1,0}, \boldsymbol{p}_{1,0})} - \boldsymbol{r}^-_{t,(\tilde{\boldsymbol{q}}_{1,0}, \tilde{\boldsymbol{p}}_{1,0})} \\ &+ \tilde{\boldsymbol{f}}_{1,t}^h, \end{aligned} \tag{5.128}$$
where we used $\boldsymbol{f}_{1,t}^h = \tilde{\boldsymbol{f}}_{1,t}^h$, which holds true because:

   - By assumption it holds $\boldsymbol{f}_{1,0} = \tilde{\boldsymbol{f}}_{1,0}$, and thus, $\boldsymbol{f}_{1,0}^h = \tilde{\boldsymbol{f}}_{1,0}^h$ by the definition in (4.18) for $\lambda = 1$.
   - By Lemma 4.2.2 (Homogeneous Maxwell solutions) the homogeneous Maxwell evolution is unique, i.e., $\boldsymbol{f}_{1,t}^h = W_t \boldsymbol{f}_{1,0}^h = W_t \tilde{\boldsymbol{f}}_{1,0}^h = \tilde{\boldsymbol{f}}_{1,t}^h$.

3. For the difference of the two fields we obtain
$$\begin{aligned} \boldsymbol{f}_{1,t} - \tilde{\boldsymbol{f}}_{1,t} = &\mathbb{1}_{B_t(\boldsymbol{q}_0)} (\boldsymbol{f}_t^-[\boldsymbol{q}_1, \boldsymbol{p}_1] - \boldsymbol{f}_t^-[\tilde{\boldsymbol{q}}_1, \tilde{\boldsymbol{p}}_1]) \\ &+ \boldsymbol{r}^-_{t,(\tilde{\boldsymbol{q}}_{1,0}, \tilde{\boldsymbol{p}}_{1,0} + \boldsymbol{\epsilon})} - \boldsymbol{r}^-_{t,(\tilde{\boldsymbol{q}}_{1,0}, \tilde{\boldsymbol{p}}_{1,0})}, \end{aligned} \tag{5.129}$$
where we used $\boldsymbol{q}_{1,0} = \tilde{\boldsymbol{q}}_{1,0}$ and $\boldsymbol{p}_{1,0} = \tilde{\boldsymbol{p}}_{1,0} + \boldsymbol{\epsilon}$. In particular, $\boldsymbol{f}_{1,t} = \tilde{\boldsymbol{f}}_{1,t}$ on the open set $J(0, \boldsymbol{q}_{1,0})^c \cap (\mathbb{R}^+ \times \mathbb{R}^3)$ and by Lemma 5.4.1 (Compatibility Conditions) we have

   (∗1) $\boldsymbol{\epsilon} \neq 0 \Rightarrow \boldsymbol{f}_{1,t} - \tilde{\boldsymbol{f}}_{1,t}$ cannot be evaluated at $\boldsymbol{x} \in \partial J^+(0, \boldsymbol{q}_{1,0})$;

   (∗2) $L(\tilde{\boldsymbol{q}}_{1,0}, \tilde{\boldsymbol{p}}_{1,0}, \boldsymbol{\delta}) \neq 0$, i.e., $\dot{\boldsymbol{p}}_{1,t}|_{t=0} \neq \dot{\tilde{\boldsymbol{p}}}_{1,t}|_{t=0} \Rightarrow \boldsymbol{f}_{1,t} - \tilde{\boldsymbol{f}}_{1,t}$ is not continuous at $\boldsymbol{x} \in \partial J^+(0, \boldsymbol{q}_{1,0})$.

4. By assumption $(\tilde{\boldsymbol{q}}_2, \tilde{\boldsymbol{p}}_2)$ is the unique Lorentz solution for $\tilde{\boldsymbol{f}}_1$ on $[0, t_2)$, but, as for all $t \in [0, t_2)$ we have
$$L(\tilde{\boldsymbol{q}}_{2,t}, \tilde{\boldsymbol{p}}_{2,t}, \tilde{\boldsymbol{f}}_{1,t}) = L(\tilde{\boldsymbol{q}}_{2,t}, \tilde{\boldsymbol{p}}_{2,t}, \boldsymbol{f}_{1,t}), \tag{5.130}$$
it is also the unique Lorentz solution for $\boldsymbol{f}_1$ on $[0, t_2)$. Hence, for all $t \in [0, t_2)$ we have $(\tilde{\boldsymbol{q}}_{2,t}, \tilde{\boldsymbol{p}}_{2,t}) = (\boldsymbol{q}_{2,t}, \boldsymbol{p}_{2,t})$.

Since $(\tilde{\boldsymbol{q}}_2, \tilde{\boldsymbol{p}}_2), (\boldsymbol{q}_2, \boldsymbol{p}_2) \in \mathcal{T}([t_1, t_2])$ we may compute

$$\lim_{t \to t_2} \left[\begin{pmatrix} \boldsymbol{q}_{2,t} \\ \boldsymbol{p}_{2,t} \\ \dot{\boldsymbol{p}}_{2,t} \end{pmatrix} - \begin{pmatrix} \tilde{\boldsymbol{q}}_{2,t_2} \\ \tilde{\boldsymbol{p}}_{2,t_2} \\ \dot{\tilde{\boldsymbol{p}}}_{2,t_2} \end{pmatrix} \right] = 0. \tag{5.131}$$

5. By item 3., 4., the Lorentz equation and the linearity of the latter

$$\begin{aligned} \frac{d}{dt}\boldsymbol{p}_{2,t} - \frac{d}{dt}\tilde{\boldsymbol{p}}_{2,t} &= L(\boldsymbol{q}_{2,t}, \boldsymbol{p}_{2,t}, \boldsymbol{f}_{1,t}) - L(\tilde{\boldsymbol{q}}_{2,t}, \tilde{\boldsymbol{p}}_{2,t}, \tilde{\boldsymbol{f}}_{1,t}) \\ &= L(\boldsymbol{q}_{2,t}, \boldsymbol{p}_{2,t}, \boldsymbol{f}_{1,t} - \tilde{\boldsymbol{f}}_{1,t}) \\ &= 0 \end{aligned}$$

even for $t = t_2$. This leads to a contradiction since for $\boldsymbol{\epsilon} \neq 0$, $(\boldsymbol{f}_{1,t_2} - \tilde{\boldsymbol{f}}_{1,t_2})(\boldsymbol{x})$ does not exist at $\boldsymbol{x} = \boldsymbol{q}_{2,t_2}$ by $(*1)$ and, for $L(\tilde{\boldsymbol{q}}_{1,0}, \tilde{\boldsymbol{p}}_{1,0}, \boldsymbol{\delta}) \neq 0$, it is not continuous at $\boldsymbol{x} = \boldsymbol{q}_{2,t_2}$, by $(*2)$.

Hence, $(\boldsymbol{q}_2, \boldsymbol{p}_2) \notin \mathcal{T}^2([0, t_2])$, and therefore, there is no Maxwell-Lorentz solution in the sense of Definition 3.3.4 (Maxwell-Lorentz solutions) $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i=1,2}$ on $[0, t_2]$ with initial value (4.4)-(4.5), unless,

$$\boldsymbol{\epsilon} = 0 \qquad \wedge \qquad L(\tilde{\boldsymbol{q}}_{1,0}, \tilde{\boldsymbol{p}}_{1,0}, \boldsymbol{\delta}) = 0. \tag{5.132}$$

□

We recall the physical explanation, why the modification in Theorem 4.1.1 (No-go) leads to an ill-posed initial value problem: As can be taken from the field representation formula in Theorem 4.2.1 (Explicit Maxwell solutions), after the modification, the field generated by charge 1 features an irregularity located on the boundary of the light-cone $\partial J(0, \boldsymbol{q}_{1,0})$; cf. Lemma 4.2.4 (Regularity of $\boldsymbol{f}_t$). Unless charge 2 moves with the speed of light, there exists a finite time $t_2 \in \mathbb{R}^+$, where the solution charge trajectory $(\tilde{\boldsymbol{q}}_2, \tilde{\boldsymbol{p}}_2)$ would cross the light-cone boundary $\partial J(0, \boldsymbol{q}_{1,0})$. In the perturbed case the force on charge 2 due to charge 1 does not change until that time t_2 and if existent up to that point the solution trajectory $(\boldsymbol{q}_2, \boldsymbol{p}_2)$ would fulfill

$$\forall\, t \in [0, t_2): \qquad \boldsymbol{q}_{2,t} = \tilde{\boldsymbol{q}}_{2,t}. \tag{5.133}$$

At time t_2, however, the modified dynamics stop as $\boldsymbol{f}_{1,t_2}(\boldsymbol{q}_{2,t_2})$ can not be evaluated or is discontinuous, which prevents us from computing the force on charge 2. Thus, the dynamics ceases to exist beyond that time instant. At most one can have piece-wise well-posedness in the sense of Definition 3.3.4 (Maxwell-Lorentz solutions) when extending this solution sense in an appropriate way.

5.6 Solutions to the Maxwell-Lorentz equations

In this section we prove Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions) and start by recalling the setting of the theorem, which was presented in Chapter 4.

The parameters $n \in \mathbb{N}$ and $d > 0$ are considered to be fixed in this section and control the regularity of the solutions and potential particle collisions by stopping the dynamics when particle distances attain the value d. Moreover, $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ denotes a predetermined history for initial time $t^{(0)} = 0$ fulfilling (H0)-(H2). The tuple $(\boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ of initial fields is parameterized and restricted by

$$\boldsymbol{f}_{i,0} = \boldsymbol{f}^-[\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i] + \boldsymbol{f}^h_{i,0}, \qquad \text{for} \qquad \boldsymbol{f}^h_{i,0} \in \mathcal{F}_{\text{hom}} \cap C^{1+n}(\mathbb{R}^3, \mathbb{R}^6), \qquad i \in \mathcal{N},$$

cf. (4.24), which implies, by means of Theorem 4.2.1 (Explicit Maxwell solutions) and Remark 4.2.2, (ii), that for $i \in \mathcal{N}$ the unique Maxwell solution $\boldsymbol{f}_i$ with initial value $\boldsymbol{f}_{i,0}$ is given by

$$\boldsymbol{f}_{i,t}(\boldsymbol{x}) := \boldsymbol{f}^-_t[\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i](\boldsymbol{x}) + \boldsymbol{f}^h_{i,t}(\boldsymbol{x}), \qquad \forall\, (t, \boldsymbol{x}) \in \mathbb{R}^4 \setminus \overset{\circ}{J}{}^+(t^{(0)}, \tilde{\boldsymbol{q}}_{i,t^{(0)}}),\ i \in \mathcal{N},$$

where $\boldsymbol{f}^h_{i,t} := W_t \boldsymbol{f}^h_{i,0}$ is the unique homogeneous Maxwell solution with initial value $\boldsymbol{f}^h_{i,0}$; cf. (4.27).

Furthermore, for any charge $i \in \mathcal{N}$, we denote the union of two trajectory pieces $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i) \in \mathcal{T}^2((-\infty, t^{(0)}]$ and $(\boldsymbol{q}_i, \boldsymbol{p}_i) \in \mathcal{T}^2([t^{(0)}, t^*])$ by

$$(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i) \cup (\boldsymbol{q}_i, \boldsymbol{p}_i) : (-\infty, t^*] \to \mathbb{R}^6$$
$$t \mapsto \begin{cases} (\tilde{\boldsymbol{q}}_{i,t}, \tilde{\boldsymbol{p}}_{i,t}), & t \in (-\infty, t^{(0)}] \\ (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}), & t \in (t^{(0)}, t^*] \end{cases} .$$

Recall from Chapter 4 that, even though the actual solution trajectory of charge $j \in \mathcal{N}$, $(\boldsymbol{q}_j, \boldsymbol{p}_j)$, is unknown for $t > t^{(0)}$, the field due to charge j can be evaluated for all $(t, \boldsymbol{x}) \in \mathbb{R}^4 \setminus \overset{\circ}{J}{}^+(t^{(0)}, \tilde{\boldsymbol{q}}_{j,t^{(0)}})$ by virtue of $t^-_{\tilde{\boldsymbol{q}}_j}(t, \boldsymbol{x})$ and the fact that the charge history $(\tilde{\boldsymbol{q}}_j, \tilde{\boldsymbol{p}}_j)$ is strictly time-like; cf. Definition 4.2.1 (Liénard-Wiechert fields) and Lemma 4.2.1 (Properties of Liénard-Wiechert fields), (ii). Thus, the force due to charge j acting on charge $i \neq j$ can be computed as long as charge i does not cross the light-cone boundary $\partial J^+(t^{(0)}, \boldsymbol{q}_{j,t^{(0)}})$.

This motivates the strategy of the proof, in which, in a first step, Lorentz solutions are constructed up to the time, say $t^{(1)}$, where the first light-cone boundary $\partial J^+(t^{(0)}, \tilde{\boldsymbol{q}}_{i,t^{(0)}}), i \in \mathcal{N}$ is hit by a solution trajectory. As we shall show in Lemma 5.6.3 (Local existence of Lorentz solutions) this time can be estimated by $t^{(1)} > \frac{d}{2}$.

Using $t^{(1)}$ as the new initial time, one observes that the Maxwell field as given in (4.27) can again be extended to the larger space $\mathbb{R}^4 \setminus \overset{\circ}{J}{}^+(t^{(1)}, \boldsymbol{q}_{j,t^{(1)}})$.

This allows us to propagate the Lorentz solution one step further provided one can control the regularity at the times $t^{(m)}$ between the steps, and hence, to compute the Maxwell fields again on a larger set, and so on. Note that this procedure is similar to the so-called method of steps, known from the theory of delay differential equations. However, in this case, the step length is not fixed, as it depends on the solution trajectories from the corresponding step.

However, as the procedure is stopped at the time of a collision only, it is guaranteed that the step length until a potential collision is always greater than $\frac{d}{2}$.

We start our proof with Lemma 5.6.1 (Lorentz solution of charge i on G_i), which states the existence and uniqueness of Lorentz solutions for a single charge i on an interval $[t^{(0)}, t_i^{(1)}]$ given the a history $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ for $t^{(0)}$. Note that $t^{(0)} \geq 0$ can be any initial time. This lemma is the key ingredient for the next claim in this section, Lemma 5.6.3 (Local existence of Lorentz solutions), which shows the existence of a unique Lorentz solution $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{i\in\mathcal{N}}$ for $(\boldsymbol{f}_i)_{i\in\mathcal{N}}$ given by (4.27) with initial value $(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0})_{i\in\mathcal{N}}$ on a time interval $[t^{(0)}, t^{(1)}]$. Finally, Lemma 5.6.3 (Local existence of Lorentz solutions) allows to conclude the proof of Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions).

In order to formulate Lemma 5.6.1 (Lorentz solution of charge i on G_i), we define the following set: Let $i \in \mathcal{N}$ be a charge index, then,

$$G_i := J^+(t^{(0)}, \tilde{\boldsymbol{q}}_{i,t^{(0)}}) \setminus \cup_{j\neq i} \overset{\circ}{J}{}^+(t^{(0)}, \tilde{\boldsymbol{q}}_{j,t^{(0)}}) \subset \mathbb{R}^4, \tag{5.134}$$

represents the space-time region in which the retarded Liénard-Wiechert fields of all charges $j \neq i$ can be evaluated, as they depend on the predetermined history $(\tilde{\boldsymbol{q}}_j, \tilde{\boldsymbol{p}}_j)$, i.e., G_i is the domain where the Lorentz equation for charge i can be solved in terms of an ordinary differential equation. Note that, in order to keep the notation simple, the dependence on the initial time $t^{(0)}$ is omitted in the symbol G_i.

Lemma 5.6.1. *(Lorentz solution of charge i on G_i) Let $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{1\leq i\leq N}$ be a history for $t^{(0)}$ fulfilling (H0)-(H2). For charge i the following propositions hold true:*

(i) (Existence) For all $T > t^{(0)}$, there exists a map $(\boldsymbol{q}_i, \boldsymbol{p}_i) \in \mathcal{T}^2([t^{(0)}, T])$ such that $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ is a Lorentz solution of charge i for $(\boldsymbol{f}_i)_{i\in\mathcal{N}}$ given by (4.27) on $[t^{(0)}, t_i^{(1)}]$ with

$$t_i^{(1)} := \sup\{t \in [t^{(0)}, T] \mid (t, \boldsymbol{q}_{i,t}) \in G_i\}. \tag{5.135}$$

(ii) (Uniqueness) Moreover, let $\Lambda \subset \mathbb{R}$ be an interval containing $t^{(0)}$. For any Lorentz solution $(\hat{\boldsymbol{q}}_i, \hat{\boldsymbol{p}}_i) \in \mathcal{T}^1(\Lambda, \mathbb{R}^6)$ for $(\boldsymbol{f}_i)_{i\in\mathcal{N}}$ given by (4.27) on Λ with $(\hat{\boldsymbol{q}}_{i,t}, \hat{\boldsymbol{p}}_{i,t})|_{t=t^{(0)}} = (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t})|_{t=t^{(0)}}$ it follows

$$(\hat{\boldsymbol{q}}_{i,t}, \hat{\boldsymbol{p}}_{i,t}) = (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}), \qquad \forall\, t \in \Lambda \cap [t^{(0)}, t_i^{(1)}]. \tag{5.136}$$

(iii) (Regularity) $(\boldsymbol{q}_i, \boldsymbol{p}_i) \in \mathcal{T}^{2+n}([t^{(0)}, t_i^{(1)}])$.

Assume T is chosen arbitrarily large. Then, the Lorentz solution $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ either hits the boundary of the light-cone of $(t^{(0)}, \boldsymbol{q}_{j,t^{(0)}})$ for some $j \neq i$ at a finite time – according to (5.135), the first time of intersection is called $t_i^{(1)}$ – or it approaches the speed of light and never leaves the domain G_i. In the latter case, one has $t_i^{(1)} = T$, and as T is arbitrarily large, one infers a global Lorentz solution; see Figure 5.2. Such a runaway behavior is however not expected.

For two charges, G_i is always unbounded, for more than two charges the domain is, for most charges i, in general bounded by the light-cones of initial positions of charges $j \neq i$, and then, the time $t_i^{(1)}$ lies in between $t^{(0)} + d/2$ and the time instant in which the boundary

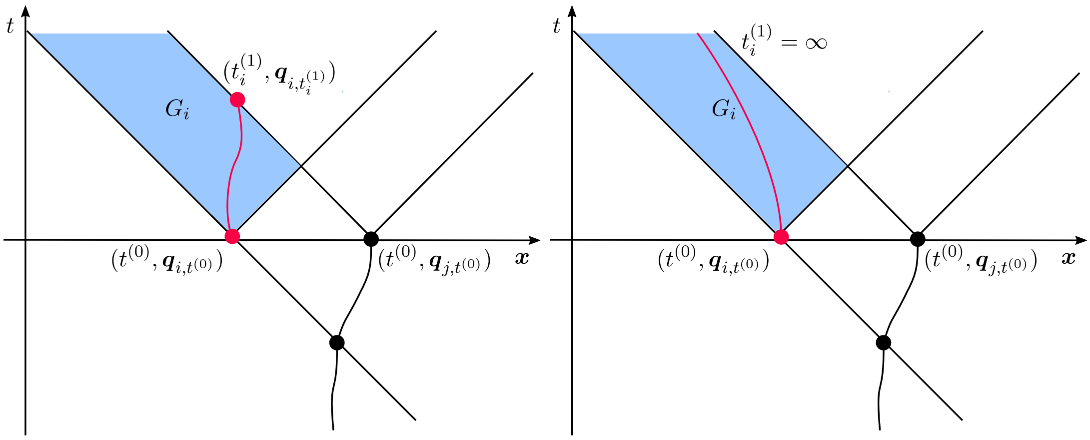

Figure 5.2: In the case of 2 charges i and j, the Lorentz solution of charge i either hits the boundary of the future light-cone $\partial J^+(t^{(0)}, \boldsymbol{q}_{j,t^{(0)}})$ or it's velocity approaches the speed of light and never leaves the domain G_i.

light-cones of G_i cross each other. Only those charges located on the edge of the system may have a corresponding domain G_i that is infinite.

Outside G_i solutions cannot be obtained because retarded Liénard-Wiechert fields due to charges $j \neq i$ are not defined, cf. Definition 4.2.1 (Liénard-Wiechert fields) and Lemma 4.2.1 (Properties of Liénard-Wiechert fields), (ii).

Proof. (i) Let $T > t^{(0)} \geq 0$ be given. For a compact notation we abbreviate the position and momentum of the fixed charge i at time t by $\boldsymbol{\varphi}_t := (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t})$, for the initial configuration we write $\boldsymbol{\varphi}^0 := (\boldsymbol{q}_{i,t^{(0)}}, \boldsymbol{p}_{i,t^{(0)}})$, respectively, and the Lorentz equation for charge i will be abbreviated by the short hand notation

$$\frac{d}{dt}\boldsymbol{\varphi}_t = \boldsymbol{L}(t, \boldsymbol{\varphi}_t) \quad \text{with} \quad \boldsymbol{L}(t, \boldsymbol{\varphi}_t) := \begin{pmatrix} \boldsymbol{v}(\boldsymbol{p}_{i,t}) \\ \sum_{j \neq i} \boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}) + \boldsymbol{v}(\boldsymbol{p}_{i,t}) \wedge \boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t}) \end{pmatrix}. \tag{5.137}$$

We define the sets

$$L_i^T := \{(t, \boldsymbol{x}) \in J^+(t^{(0)}, \boldsymbol{q}_{i,t^{(0)}}) | t \in [t^{(0)}, T]\} \qquad \text{and} \qquad G_i^T := L_i^T \cap G_i, \tag{5.138}$$

In general, neither G_i nor L_i^T is a subset of each other. Just in the case in which G_i is bounded and T is sufficiently large, it holds $G_i \subset L_i^T$.

First, for all $j \neq i$ we extend the charge history $(\tilde{\boldsymbol{q}}_j, \tilde{\boldsymbol{p}}_j)$ to a history defined up to time $T > t^{(0)}$. We will use the same symbol for the extension, i.e., for $t \leq t^{(0)}$, $t \mapsto (\tilde{\boldsymbol{q}}_{j,t}, \tilde{\boldsymbol{p}}_{j,t})$ denotes the original history and for $t \in (t^{(0)}, T]$, $t \mapsto (\tilde{\boldsymbol{q}}_{j,t}, \tilde{\boldsymbol{p}}_{j,t})$ denotes the extension. By means of Definition 4.2.1 (Liénard-Wiechert fields) and Lemma 4.2.1 (Properties of Liénard-Wiechert fields) this allows us to compute the Maxwell field of charge $j \neq i$ on $[t^{(0)}, T] \times \mathbb{R}^3$, whereas on the subset $J(t^{(0)}, \tilde{\boldsymbol{q}}_{j,t^{(0)}}) \cap ([t^{(0)}, T] \times \mathbb{R}^3)$ the field $\boldsymbol{f}_j$ depends on the extension and outside it depends on the history.

We fix the extension such that for each $j \neq i$

(a) the extended history $(\tilde{\boldsymbol{q}}_j, \tilde{\boldsymbol{p}}_j)$ is in $\mathcal{T}^{2+n}((-\infty, T], \mathbb{R}^6)$, i.e., in particular it connects smoothly to the original history at time $t^{(0)}$.

(b) for each time $t \in (t^{(0)}, T]$ the distance between the extended history of charge j and the forward light-cone of charge i is greater than $\frac{d}{2}$.

(c) the extended history $(\tilde{\boldsymbol{q}}_j, \tilde{\boldsymbol{p}}_j)$ is strictly time-like.

For each charge history such an extension exists because at time $t^{(0)}$ the distance $|\tilde{\boldsymbol{q}}_{i,t^{(0)}} - \tilde{\boldsymbol{q}}_{j,t^{(0)}}| > d$, thus, the extension of j may be chosen within a tube of diameter $\frac{d}{2}$ in order to fulfill b) without leaving the light-cone $J(t^{(0)}, \tilde{\boldsymbol{q}}_{j,t^{(0)}})$. Since time T is fixed and finite, the charge can stay in this tube without exceeding the speed of light. Thus, c) is feasible and we can chose a trajectory strip in $\mathcal{T}^{2+n}((-\infty, T], \mathbb{R}^6)$. See Figure 5.3 for an illustration of the extension. Note that by construction the extended history again fulfills (H0)-(H2).

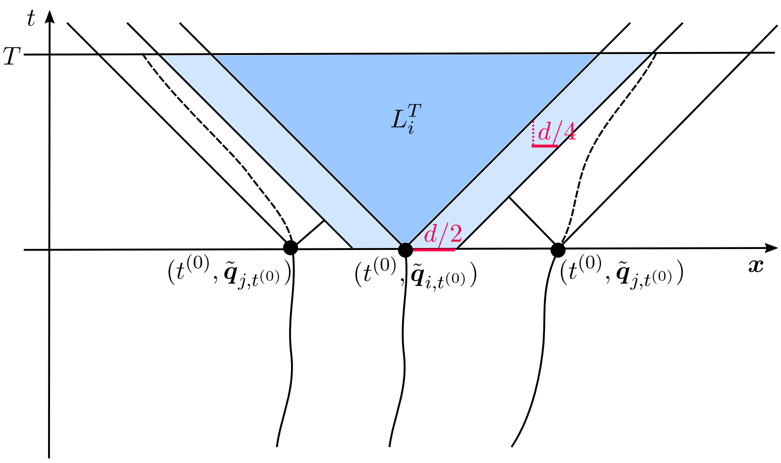

Figure 5.3: Illustration of the extension of the histories of to charges $j \neq i$ indicated by the dashed lines. Thereby, the extension has to connect regular to the history and needs to maintain a distance of $d/2$ to the light-cone of charge i (grey). This requirement implies that $|\boldsymbol{x} - \tilde{\boldsymbol{q}}_j^-| > d/4$ for all $(t, \boldsymbol{x}) \in L_i^T$.

The notation $\boldsymbol{L}(t, \boldsymbol{\varphi}_t)$ for the force term, as well as $\boldsymbol{f}_{j,t}$ for the fields will be maintained, not distinguishing whether the input values come from the original history or the extension of the history. For the fields $\boldsymbol{f}_j$ defined in (4.27) given the extended history $(\tilde{\boldsymbol{q}}_j, \tilde{\boldsymbol{p}}_j)_{j\in\mathcal{N}}$, we obtain

$$\boldsymbol{f}_j \in C^n(D_{\tilde{\boldsymbol{q}}_j} \setminus \overset{\circ}{J}{}^+(T, \tilde{\boldsymbol{q}}_{j,T}), \mathbb{R}^6). \tag{5.139}$$

This holds true because the homogeneous field is governed by (4.26) and the retarded Liénard-Wiechert field is n times continuously differentiable on its domain thanks to Lemma 4.2.1 (Properties of Liénard-Wiechert fields) (i).

Since by construction the extension of the history of charge j maintains a spatial distance greater than $\frac{d}{2}$ from the light-cone $J^+(t^{(0)}, \tilde{\boldsymbol{q}}_{i,t^{(0)}})$ it follows

$$L_i^T \cap (D_{\tilde{\boldsymbol{q}}_j} \setminus \overset{\circ}{J^+}(T, \tilde{\boldsymbol{q}}_{j,T})) = L_i^T, \tag{5.140}$$

and hence,

$$\boldsymbol{f}_j \in C^n(L_i^T, \mathbb{R}^6). \tag{5.141}$$

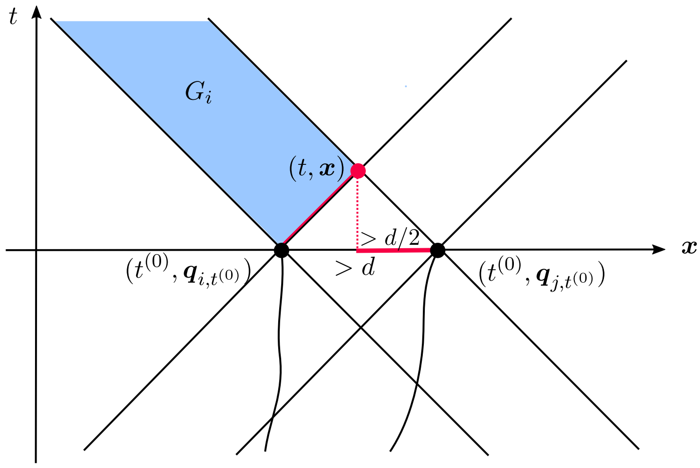

Figure 5.4: Illustration of the argument that (H2) implies that for each $(t, \boldsymbol{x}) \in G_i$ the distance $|\boldsymbol{x} - \boldsymbol{q}_{j,t^-(t,\boldsymbol{x})}| > \frac{d}{2}$, considering the worst case solution trajectory heading towards charge j along the light-cone (red).

The strategy is to apply Banach's fixed point theorem: Let M be a nonempty, closed set in some Banach space X. Consider an operator $S : M \to X$. If S is a contraction and a self mapping on M, i.e., $S(M) \subset M$, there is a unique solution $x^* \in M$ to the equation $x = S(x)$ (see, for instance, [44]).

Hence, the following points need to be addressed:

1. We define a suitable Banach space.
2. We define a self-mapping on a subspace.
3. We show that the self-mapping is a contraction.

ad 1. For $\alpha \in \mathbb{R}^+$, we define the normed, linear space

$$X_T := \left\{ \boldsymbol{\varphi}_{(\cdot)} : [t^{(0)}, T] \to \mathbb{R}^6 \middle| \boldsymbol{\varphi}_{(\cdot)} \in C^0([t^{(0)}, T]), \mathbb{R}^6), \right.$$
$$\left. \left\| \boldsymbol{\varphi}_{(\cdot)} \right\|_{X_T} := \sup_{t \in [t^{(0)}, T]} e^{-\alpha|t|} \left| \boldsymbol{\varphi}_t \right|_{\mathbb{R}^6} < \infty \right\}. \tag{5.142}$$

First of all, X_T is complete thanks to the completeness of $\mathbb{R}^6$:

Therefore, consider a Cauchy sequence $(\boldsymbol{\varphi}_{(\cdot)}^{(n)})_{n \in \mathbb{N}}$ in $(X_T, \|\cdot\|_{X_T})$. By definition, this sequence is uniformly Cauchy w.r.t. the metric space $(\mathbb{R}^6, |\cdot|_{\mathbb{R}^6})$. Since $(\mathbb{R}^6, |\cdot|_{\mathbb{R}^6})$ is complete, the sequence of functions is also uniformly convergent w.r.t. that space, i.e. there exists a limit function $\boldsymbol{\varphi}_{(\cdot)}$. Moreover, the fact that $(\boldsymbol{\varphi}^{(n)}_{(\cdot)})_{n\in\mathbb{N}}$ converges uniformly implies that the limit function $\boldsymbol{\varphi}_{(\cdot)}$ is again continuous. To show, that $\boldsymbol{\varphi}_{(\cdot)}$ is indeed in X_T it remains to verify that $\left\|\boldsymbol{\varphi}_{(\cdot)}\right\|_{X_T}$ is bounded. This follows by the compactness of the interval $[t^{(0)}, T]$ and continuity of the limit function, which renders $|\boldsymbol{\varphi}_t|_{\mathbb{R}^6}$ to be finite for all t and by $\left|e^{-\alpha|t|}\right| < 1$ for $t \in [t^{(0)}, T]$.

Consequently, X_T is a Banach space.

Furthermore, for any $\boldsymbol{\varphi}^0 \in \mathbb{R}^6$ we define the subspace

$$M_{T,\boldsymbol{\varphi}^0} := \left\{\boldsymbol{\varphi}_{(\cdot)} \in X_T \mid \boldsymbol{\varphi}_t|_{t=t^{(0)}} = \boldsymbol{\varphi}^0\right\}, \tag{5.143}$$

which is closed in X_T since for any convergent series $(\boldsymbol{\varphi}^{(n)})_{n\in\mathbb{N}}$ with $\boldsymbol{\varphi}^{(n)}_t|_{t=t^{(0)}} = \boldsymbol{\varphi}^0$ for $n \in \mathbb{N}$ also $\lim_{n\to\infty} \boldsymbol{\varphi}^{(n)}_t|_{t=t^{(0)}} = \boldsymbol{\varphi}^0$.

ad 2. On the space $M_{T,\boldsymbol{\varphi}^0}$ we define the mapping

$$S_{\boldsymbol{\varphi}^0} : M_{T,\boldsymbol{\varphi}^0} \to M_{T,\boldsymbol{\varphi}^0} \qquad \boldsymbol{\varphi}_{(\cdot)} \mapsto S_{\boldsymbol{\varphi}^0}[\boldsymbol{\varphi}_{(\cdot)}] := \boldsymbol{\varphi}^0 + \int_{t^{(0)}}^{(\cdot)} ds \boldsymbol{L}(s, \boldsymbol{\varphi}_s). \tag{5.144}$$

Elements in the range of $S_{\boldsymbol{\varphi}^0}$ fulfill $\left|\dot{\boldsymbol{q}}_{i,t}\right| = \left|\boldsymbol{v}(\boldsymbol{q}_{i,t})\right| < 1$ by the definition of the force $\boldsymbol{L}$, and thus, trajectory pieces in $S_{\boldsymbol{\varphi}^0}[M_{T,\boldsymbol{\varphi}^0}]$ stay in the inner of the light-cone L_i^T bounded by T, or in other words for all $t \in [t^{(0)}, T]$ the corresponding charge position $\boldsymbol{q}_{i,t}$ is in the inner of $B_{t-t^{(0)}}(\boldsymbol{q}_{i,t^{(0)}})$.

$S_{\boldsymbol{\varphi}^0}$ is well-defined: for $t \in [t^{(0)}, T]$ the integral,

$$\int_{t^{(0)}}^{t} ds \boldsymbol{L}(s, \boldsymbol{\varphi}_s), \tag{5.145}$$

is well defined for the following reason: for $\boldsymbol{\varphi}_{(\cdot)} \in X_T$ the mapping $t \mapsto (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t})$ is in $C^0([t^{(0)}, T], \mathbb{R}^3)$ and the integral domain is compact.

By (5.141) it follows $t \mapsto \boldsymbol{f}^-_{j,t}(\boldsymbol{q}_{i,t}) \in C^1([t^{(0)}, T], \mathbb{R}^6)$ and $t \mapsto \boldsymbol{f}^h_{j,t}(\boldsymbol{q}_{i,t}) \in C^1([t^{(0)}, T], \mathbb{R}^6)$. Consequently, $s \mapsto \boldsymbol{L}(s, \boldsymbol{\varphi}_s)$ is continuous on the compact set $[t^{(0)}, T]$ and its anti-derivative exists by the fundamental theorem of calculus.

Furthermore $S_{\boldsymbol{\varphi}^0}$ is a self-mapping, because $S_{\boldsymbol{\varphi}^0}[\boldsymbol{\varphi}_{(\cdot)}]\,|_{t=t^{(0)}} = \boldsymbol{\varphi}^0$ by definition, the right hand side of (5.144) is continuous on $[t^{(0)},T]$, and $S_{\boldsymbol{\varphi}^0}[\boldsymbol{\varphi}_{(\cdot)}]$ is bounded:

$$
\begin{aligned}
&\left\| S_{\boldsymbol{\varphi}^0}[\boldsymbol{\varphi}_{(\cdot)}] \right\|_{X_T} = \sup_{t\in[t^{(0)},T]} e^{-\alpha|t|} \left| \boldsymbol{\varphi}^0 + \int_{t^{(0)}}^t ds \boldsymbol{L}(s,\boldsymbol{\varphi}_s) \right|_{\mathbb{R}^6} \\
&\le |\boldsymbol{\varphi}^0|_{\mathbb{R}^6} + \sup_{t\in[t^{(0)},T]} e^{-\alpha|t|} \left| \int_{t^{(0)}}^t ds \boldsymbol{L}(s,\boldsymbol{\varphi}_s) \right|_{\mathbb{R}^6} \\
&\le |\boldsymbol{\varphi}^0|_{\mathbb{R}^6} + \sup_{t\in[t^{(0)},T]} e^{-\alpha|t|} \int_{t^{(0)}}^t ds \, |\boldsymbol{L}(s,\boldsymbol{\varphi}_s)|_{\mathbb{R}^6} \\
&\le |\boldsymbol{\varphi}^0|_{\mathbb{R}^6} + T \sup_{t\in[t^{(0)},T]} e^{-\alpha|t|} \, |\boldsymbol{L}(t,\boldsymbol{\varphi}_t)|_{\mathbb{R}^6} \\
&\le |\boldsymbol{\varphi}^0|_{\mathbb{R}^6} + T \sup_{t\in[t^{(0)},T]} |\boldsymbol{L}(t,\boldsymbol{\varphi}_t)|_{\mathbb{R}^6} \\
&\le |\boldsymbol{\varphi}^0|_{\mathbb{R}^6} + T \sup_{t\in[t^{(0)},T]} \left| \begin{pmatrix} \boldsymbol{v}(\boldsymbol{p}_{i,t}) \\ \sum_{j\neq i} \boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}) + \boldsymbol{v}(\boldsymbol{p}_{i,t}) \wedge \boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t})) \end{pmatrix} \right|_{\mathbb{R}^6} \\
&\le |\boldsymbol{\varphi}^0|_{\mathbb{R}^6} + T \sup_{t\in[t^{(0)},T]} \left(1 + \sum_{j\neq i} \left|\boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t})\right|_{\mathbb{R}^3} + \left|\boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t})\right|_{\mathbb{R}^3} \right) < \infty,
\end{aligned}
$$

since by (5.141) $\boldsymbol{f}_j$ is continuous in t on the interval $[t^{(0)},T]$, and thus, the supremum over the compact set $[t^{(0)},T]$ exists.

ad 3. Next, it remains to show that $S_{\boldsymbol{\varphi}^0}$ is a contraction. Therefore, let $\boldsymbol{\varphi}_{(\cdot)}$ and $\hat{\boldsymbol{\varphi}}_{(\cdot)}$ be in $M_{T,\boldsymbol{\varphi}^0}$. Then,

$$
\begin{aligned}
&\left\| S_{\boldsymbol{\varphi}^0}[\boldsymbol{\varphi}_{(\cdot)}] - S_{\boldsymbol{\varphi}^0}[\hat{\boldsymbol{\varphi}}_{(\cdot)}] \right\|_{X_T} = \sup_{t\in[t^{(0)},T]} e^{-\alpha|t|} \left| \int_{t^{(0)}}^t ds \boldsymbol{L}(s,\boldsymbol{\varphi}_s) - \boldsymbol{L}(s,\hat{\boldsymbol{\varphi}}_s) \right|_{\mathbb{R}^6} \\
&\le \sup_{t\in[t^{(0)},T]} e^{-\alpha|t|} \int_{t^{(0)}}^t ds e^{\alpha|s|} e^{-\alpha|s|} \, |\boldsymbol{L}(s,\boldsymbol{\varphi}_s) - \boldsymbol{L}(s,\hat{\boldsymbol{\varphi}}_s)|_{\mathbb{R}^6} \\
&\le \sup_{t\in[t^{(0)},T]} e^{-\alpha|t|} \frac{e^{\alpha|t|}}{\alpha} \sup_{s\in[t^{(0)},t]} e^{-\alpha|s|} \, |\boldsymbol{L}(s,\boldsymbol{\varphi}_s) - \boldsymbol{L}(s,\hat{\boldsymbol{\varphi}}_s)|_{\mathbb{R}^6} \\
&= \frac{1}{\alpha} \sup_{t\in[t^{(0)},T]} e^{-\alpha|t|} \, |\boldsymbol{L}(t,\boldsymbol{\varphi}_t) - \boldsymbol{L}(t,\hat{\boldsymbol{\varphi}}_t)|_{\mathbb{R}^6} \\
&= \frac{1}{\alpha} \sup_{t\in[t^{(0)},T]} e^{-\alpha|t|} \left| \begin{pmatrix} \boldsymbol{v}(\boldsymbol{p}_{i,t}) \\ \sum_{j\neq i} \boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}) + \boldsymbol{v}(\boldsymbol{p}_{i,t}) \wedge \boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t})) \end{pmatrix} \right. \\
&\qquad\qquad \left. - \begin{pmatrix} \boldsymbol{v}(\hat{\boldsymbol{p}}_{i,t}) \\ \sum_{j\neq i} \boldsymbol{E}_{j,t}(\hat{\boldsymbol{q}}_{i,t}) + \boldsymbol{v}(\hat{\boldsymbol{p}}_{i,t}) \wedge (\boldsymbol{B}_{j,t}(\hat{\boldsymbol{q}}_{i,t})) \end{pmatrix} \right|_{\mathbb{R}^6} \\
&= \frac{1}{\alpha} \sup_{t\in[t^{(0)},T]} e^{-\alpha|t|} \Bigg(\left| \boldsymbol{v}(\boldsymbol{p}_{i,t}) - \boldsymbol{v}(\hat{\boldsymbol{p}}_{i,t}) \right|^2_{\mathbb{R}^3} \\
&\quad + \left| \sum_{j\neq i} \boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}) + \boldsymbol{v}(\boldsymbol{p}_{i,t}) \wedge \boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t}) - \sum_{j\neq i} (\boldsymbol{E}_{j,t}(\hat{\boldsymbol{q}}_{i,t}) + \boldsymbol{v}(\hat{\boldsymbol{p}}_{i,t}) \wedge \boldsymbol{B}_{j,t}(\hat{\boldsymbol{q}}_{i,t})) \right|^2_{\mathbb{R}^3} \Bigg) \\
&= \boxed{c}
\end{aligned}
$$

and by first binomial formula we obtain

$$\boxed{c} \leq \frac{1}{\alpha} \sup_{t \in [t^{(0)},T]} e^{-\alpha|t|}$$

$$\left|\boldsymbol{v}(\boldsymbol{p}_{i,t}) - \boldsymbol{v}(\hat{\boldsymbol{p}}_{i,t})\right|_{\mathbb{R}^3} \qquad \boxed{1}$$

$$+ \sum_{j \neq i} \left|\boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}) - \boldsymbol{E}_{j,t}(\hat{\boldsymbol{q}}_{i,t})\right|_{\mathbb{R}^3} \qquad \boxed{2}$$

$$+ \sum_{j \neq i} \left|\boldsymbol{v}(\boldsymbol{p}_{i,t}) \wedge \boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t}) - \boldsymbol{v}(\hat{\boldsymbol{p}}_{i,t}) \wedge \boldsymbol{B}_{j,t}(\hat{\boldsymbol{q}}_{i,t}))\right|_{\mathbb{R}^3} \qquad \boxed{3}$$

According to the mean value theorem there exists a $\boldsymbol{k} \in \{\boldsymbol{p}_{i,t} + \lambda(\hat{\boldsymbol{p}}_{i,t} - \boldsymbol{p}_{i,t})|\lambda \in (0,1)\}$ such that summand $\boxed{1}$ equals

$$\boxed{1} = \left|D\boldsymbol{v}(\boldsymbol{k}) \cdot (\boldsymbol{p}_{i,t} - \hat{\boldsymbol{p}}_{i,t})\right|_{\mathbb{R}^3} \leq \|D\boldsymbol{v}(\boldsymbol{k})\|_{\max} \left|\boldsymbol{p}_{i,t} - \hat{\boldsymbol{p}}_{i,t}\right|_{\mathbb{R}^3},$$

where $D\boldsymbol{v}(\boldsymbol{k})$ denotes the Jacobimatrix of $\boldsymbol{v}$ at point $\boldsymbol{k} \in \mathbb{R}^3$ and for matrices $A \in \mathbb{R}^{3\times 3}$ we consider the maximum norm $\|A\|_{\max} := \max_{i,j=1,2,3} |a_{ij}|$.

For estimating $\|D\boldsymbol{v}(\boldsymbol{k})\|_{\max}$ we make use of the auxiliary calculation

$$\begin{aligned}
[D\boldsymbol{v}(\boldsymbol{k})]_{ij} = \partial_{k_i} \boldsymbol{v}_j(\boldsymbol{k}) = \partial_{k_i} \frac{k_j}{\sqrt{\boldsymbol{k}^2 + m^2}} &= \frac{\delta_{ij}}{\sqrt{\boldsymbol{k}^2 + m^2}} - \frac{k_j \partial_{k_i} \sqrt{\boldsymbol{k}^2 + m^2}}{\sqrt{\boldsymbol{k}^2 + m^2}^2} \\
&= \frac{\delta_{ij}}{\sqrt{\boldsymbol{k}^2 + m^2}} - \frac{k_j k_i}{\sqrt{\boldsymbol{k}^2 + m^2}^3} \\
&= \frac{1}{\sqrt{\boldsymbol{k}^2 + m^2}} \left(\delta_{ij} - \frac{k_j k_i}{\boldsymbol{k}^2 + m^2}\right).
\end{aligned}$$

This gives

$$\begin{aligned}
\|D\boldsymbol{v}(\boldsymbol{k})\|_{\max} &= \max_{ij} \left| \frac{1}{\sqrt{\boldsymbol{k}^2 + m^2}} \left(\delta_{ij} - \frac{k_j k_i}{\boldsymbol{k}^2 + m^2}\right) \right| \\
&\leq \max_{ij} \frac{1}{\sqrt{\boldsymbol{k}^2 + m^2}} \left(|\delta_{ij}| + \left| \frac{k_i k_j}{\boldsymbol{k}^2 + m^2} \right| \right) \\
&\leq \frac{2}{\sqrt{\boldsymbol{k}^2 + m^2}} \leq \frac{2}{m},
\end{aligned}$$

where we used $|k_i k_j| \leq \boldsymbol{k}^2$. And thus,

$$\boxed{1} \leq \frac{2}{m} \left|\boldsymbol{p}_{i,t} - \hat{\boldsymbol{p}}_{i,t}\right|_{\mathbb{R}^3} =: C_1 \left|\boldsymbol{p}_{i,t} - \hat{\boldsymbol{p}}_{i,t}\right|_{\mathbb{R}^3}. \tag{5.146}$$

The second summand can as well be estimated by the mean value theorem, i.e., there exists a $\boldsymbol{k} \in \{\boldsymbol{q}_{i,t} + \lambda(\hat{\boldsymbol{q}}_{i,t} - \boldsymbol{q}_{i,t})|\lambda \in (0,1)\}$ such that

$$\boxed{2} = \sum_{j \neq i} \left|\boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}) - \boldsymbol{E}_{j,t}(\hat{\boldsymbol{q}}_{i,t})\right|_{\mathbb{R}^3} = \sum_{j \neq i} \left|D\boldsymbol{E}_{j,t}(\boldsymbol{k}) \cdot (\boldsymbol{q}_{i,t} - \hat{\boldsymbol{q}}_{i,t})\right|_{\mathbb{R}^3}. \tag{5.147}$$

Then, we find

$$\begin{aligned}
\boxed{2} &\leq \sum_{j\neq i} \|D\boldsymbol{E}_{j,t}(\boldsymbol{k})\|_{\mathbb{R}^{3\times 3}} \left|\boldsymbol{q}_{i,t} - \hat{\boldsymbol{q}}_{i,t}\right|_{\mathbb{R}^3} \\
&\leq \sum_{j\neq i} \sup_{(t,\boldsymbol{k})\in L_i^T} \|D\boldsymbol{E}_{j,t}(\boldsymbol{k})\|_{\mathbb{R}^{3\times 3}} \left|\boldsymbol{q}_{i,t} - \hat{\boldsymbol{q}}_{i,t}\right|_{\mathbb{R}^3} \\
&=: C_2 \left|\boldsymbol{q}_{i,t} - \hat{\boldsymbol{q}}_{i,t}\right|_{\mathbb{R}^3},
\end{aligned}$$

where we used $\boldsymbol{q}_{i,t}, \hat{\boldsymbol{q}}_{i,t} \in B_{t-t^{(0)}}(\boldsymbol{q}_{i,t^{(0)}})$ and thus $\boldsymbol{k} \in B_{t-t^{(0)}}(\boldsymbol{q}_{i,t^{(0)}})$ in the second line. Since $t \in [t^{(0)}, T]$ and $\boldsymbol{k} \in \{\boldsymbol{q}_{i,t} + \lambda(\hat{\boldsymbol{q}}_{i,t} - \boldsymbol{q}_{i,t}) | \lambda \in (0,1)\}$ we can estimate $\|D\boldsymbol{E}_{j,t}(\boldsymbol{k})\|_{\mathbb{R}^{3\times 3}}$ by the supremum over $(t,\boldsymbol{k}) \in L_i^T$ for each $j \neq i$. Since L_i^T is compact and $(t,\boldsymbol{k}) \mapsto D\boldsymbol{E}_{j,t}(\boldsymbol{k})$ is continuous (cf. (5.141)) $\sup_{(t,\boldsymbol{k})\in L_i^T} \|D\boldsymbol{E}_{j,t}(\boldsymbol{k})\|_{\mathbb{R}^{3\times 3}}$ exists. Therefore, the sum over all $j \neq i$ exists as well and can be estimated by a finite constant, which we call C_2.

$$\begin{aligned}
\boxed{3} &= \sum_{j\neq i} \left|\boldsymbol{v}(\boldsymbol{p}_{i,t}) \wedge \boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t}) - \boldsymbol{v}(\hat{\boldsymbol{p}}_{i,t}) \wedge \boldsymbol{B}_{j,t}(\hat{\boldsymbol{q}}_{i,t})\right|_{\mathbb{R}^3} \\
&\leq \sum_{j\neq i} \left|\boldsymbol{v}(\boldsymbol{p}_{i,t}) \wedge (\boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t}) - \boldsymbol{B}_{j,t}(\hat{\boldsymbol{q}}_{i,t}))\right|_{\mathbb{R}^3} + \sum_{j\neq i} \left|(\boldsymbol{v}(\boldsymbol{p}_{i,t}) - \boldsymbol{v}(\hat{\boldsymbol{p}}_{i,t})) \wedge \boldsymbol{B}_{j,t}(\hat{\boldsymbol{q}}_{i,t}))\right|_{\mathbb{R}^3} \\
&\leq \sum_{j\neq i} \left|\boldsymbol{v}(\boldsymbol{p}_{i,t})\right|_{\mathbb{R}^3} \left|\boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t}) - \boldsymbol{B}_{j,t}(\hat{\boldsymbol{q}}_{i,t})\right|_{\mathbb{R}^3} + \sum_{j\neq i} \left|\boldsymbol{B}_{j,t}(\hat{\boldsymbol{q}}_{i,t})\right|_{\mathbb{R}^3} \left|\boldsymbol{v}(\boldsymbol{p}_{i,t}) - \boldsymbol{v}(\hat{\boldsymbol{p}}_{i,t})\right|_{\mathbb{R}^3} \\
&\leq \sum_{j\neq i} \|D\boldsymbol{B}_{j,t}(\boldsymbol{k})\|_{\mathbb{R}^{3\times 3}} \left|\boldsymbol{q}_{i,t} - \hat{\boldsymbol{q}}_{i,t}\right|_{\mathbb{R}^3} + \sum_{j\neq i} |\boldsymbol{B}_{j,t}(\boldsymbol{k})|_{\mathbb{R}^3} \|D\boldsymbol{v}(\boldsymbol{k})\|_{\mathbb{R}^{3\times 3}} \left|\boldsymbol{p}_{i,t} - \hat{\boldsymbol{p}}_{i,t}\right|_{\mathbb{R}^3} \\
&\leq \sum_{j\neq i} \sup_{(t,\boldsymbol{k})\in L_i^T} \|D\boldsymbol{B}_{j,t}(\boldsymbol{k})\|_{\mathbb{R}^{3\times 3}} \left|\boldsymbol{q}_{i,t} - \hat{\boldsymbol{q}}_{i,t}\right|_{\mathbb{R}^3} \\
&+ \sum_{j\neq i} \sup_{(t,\boldsymbol{k})\in L_i^T} \|\boldsymbol{B}_{j,t}(\boldsymbol{k})\|_{\mathbb{R}^{3\times 3}} \frac{2}{m} \left|\boldsymbol{p}_{i,t} - \hat{\boldsymbol{p}}_{i,t}\right|_{\mathbb{R}^3} \\
&=: C_{3a} \left|\boldsymbol{q}_{i,t} - \hat{\boldsymbol{q}}_{i,t}\right|_{\mathbb{R}^3} + C_{3b} C_1 \left|\boldsymbol{p}_{i,t} - \hat{\boldsymbol{p}}_{i,t}\right|_{\mathbb{R}^3}.
\end{aligned}$$

The constants C_{3a} and C_{3b} exist and are finite because by (5.141) the expressions $(t,\boldsymbol{k}) \mapsto \boldsymbol{B}_{j,t}(\boldsymbol{k}), D\boldsymbol{B}_{j,t}(\boldsymbol{k})$ are continuous maps on L_i^T and since L_i^T is compact the suprema exist.

Bringing all the partial estimates together we obtain the following Lipschitz estimation:

$$\begin{aligned}
&\left\| S_{\boldsymbol{\varphi}^0}[\boldsymbol{\varphi}_{(\cdot)}] - S_{\boldsymbol{\varphi}^0}[\hat{\boldsymbol{\varphi}}_{(\cdot)}] \right\|_{X_T} \\
&\leq \frac{1}{\alpha} \sup_{t\in[t^{(0)},T]} e^{-\alpha|t|} \\
&\qquad \Big(C_1(1+C_{3b}) \left| \boldsymbol{p}_{i,t} - \hat{\boldsymbol{p}}_{i,t} \right|_{\mathbb{R}^3} + (C_2 + C_{3a}) \left| \boldsymbol{q}_{i,t} - \hat{\boldsymbol{q}}_{i,t} \right|_{\mathbb{R}^3} \Big) \\
&\leq \frac{1}{\alpha} \sup_{t\in[t^{(0)},T]} e^{-\alpha|t|} \\
&\qquad \Big(C_1(1+C_{3b}) \left| \boldsymbol{p}_{i,t} - \hat{\boldsymbol{p}}_{i,t} \right|_{\mathbb{R}^3} + (C_2 + C_{3a}) \left| \boldsymbol{q}_{i,t} - \hat{\boldsymbol{q}}_{i,t} \right|_{\mathbb{R}^3} \Big) \\
&\leq \frac{1}{\alpha} (C_1(1+C_{3b}) + C_2 + C_{3a}) \\
&\qquad \sup_{t\in[t^{(0)},T]} e^{-\alpha|t|} \sqrt{ \left| \boldsymbol{q}_{i,t} - \hat{\boldsymbol{q}}_{i,t} \right|^2_{\mathbb{R}^3} + \left| \boldsymbol{p}_{i,t} - \hat{\boldsymbol{p}}_{i,t} \right|^2_{\mathbb{R}^3} } \\
&\leq \frac{1}{\alpha} (C_1(1+C_{3b}) + C_2 + C_{3a}) \left\| \boldsymbol{\varphi}_{(\cdot)} - \hat{\boldsymbol{\varphi}}_{(\cdot)} \right\|_{X_T} .
\end{aligned}$$

Each of the constants C_1, C_2, C_{3a}, C_{3b} is finite, thus finite sums and products of these are finite. For any $T > 0$ one can define $\alpha > 0$ such that $\frac{1}{\alpha}(C_1(1 + C_{3b}) + C_2 + C_{3a})$ is smaller than one, i.e., for each $T > 0$ one can choose $\alpha > 0$ such that it defines a norm $\|\cdot\|_{X_T}$ with respect to which $S_{\boldsymbol{\varphi}^0}$ is a contraction.

By Banach's fixed point theorem this implies the existence of a unique solution in $M_{\boldsymbol{\varphi}^0}$ to the equation $\boldsymbol{\varphi}_{(\cdot)} = S_{\boldsymbol{\varphi}^0}[\boldsymbol{\varphi}_{(\cdot)}]$ and thus, for a given initial value $\boldsymbol{\varphi}^0 \in \mathbb{R}^6$ and a given time T there exists a unique continuous solution trajectory $t \mapsto \boldsymbol{\varphi}_t, t \in [0, T]$ to the differential equation (5.137). On the domain $G_i^T = G_i \cap L_i^T$ this solution is the unique solution to the original initial value problem with predetermined history $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$.

We denote the solution by $\boldsymbol{\varphi}$ in the remaining proof.

(ii) Uniqueness is implied in Banach's fixed point theorem in the proof of (i).

(iii) We show that $\boldsymbol{\varphi}_{(\cdot)}$ fulfills $\boldsymbol{\varphi}_{(\cdot)} \in \mathcal{T}^{2+n}([t^{(0)}, t_i^{(1)}])$. Continuity follows directly by the definition of the Banach space X_T and the subspace $M_{\boldsymbol{\varphi}^0}$, which assures that $\boldsymbol{\varphi}_{(\cdot)} \in C^0([t^{(0)}, t_i^{(1)}], \mathbb{R}^6)$, and hence, by $\dot{\boldsymbol{q}}_{i,t} = \boldsymbol{v}(\boldsymbol{p}_{i,t})$ it follows directly $\boldsymbol{\varphi}_{(\cdot)} \in \mathcal{T}^1([t^{(0)}, t_i^{(1)}])$. For $t \in [t^{(0)}, t_i^{(1)}]$ the solution reads

$$t \mapsto \boldsymbol{\varphi}_t = \boldsymbol{\varphi}^0 + \int_{t^{(0)}}^t ds \boldsymbol{L}(s, \boldsymbol{\varphi}_s) \tag{5.148}$$

and for $k = 1, \ldots, 1 + n$ the kth derivative reads

$$\frac{d^k}{dt^k} \boldsymbol{\varphi}_t = \frac{d^{k-1}}{dt^{k-1}} \begin{pmatrix} \boldsymbol{v}(\boldsymbol{p}_{i,t}) \\ \sum_{j\neq i} \boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}) + \boldsymbol{v}(\boldsymbol{p}_{i,t}) \wedge (\boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t})) \end{pmatrix} . \tag{5.149}$$

The right hand side of (5.149) consists of the velocity function $\boldsymbol{p} \mapsto \boldsymbol{v}(\boldsymbol{p})$, which is smooth, and the field $(t, \boldsymbol{x}) \mapsto (\boldsymbol{E}_{j,t}, \boldsymbol{B}_{j,t})(\boldsymbol{x})$ which is in $C^n(L_i^T, \mathbb{R}^6)$ by (5.141). Both,

$\boldsymbol{v}$ and the fields $(\boldsymbol{E}_{j,t}, \boldsymbol{B}_{j,t})$ are composed with the solution trajectory $\boldsymbol{\varphi} = (\boldsymbol{q}_i, \boldsymbol{p}_i)$. We prove that the right hand side of (5.149) is well-defined by induction.

For $k = 1$ the right hand side of (5.149) is continuous as composition of continuous functions and thus $\boldsymbol{\varphi} \in \mathcal{T}^2([t^{(0)}, t_i^{(1)}])$.

Now, assume that for any $k \leq 1+n$ it holds $\boldsymbol{\varphi} \in \mathcal{T}^k([t^{(0)}, t_i^{(1)}])$, then the right hand side of (5.149) is in $\mathcal{T}^1([t^{(0)}, t_i^{(1)}])$ as it is composed of continuous functions and thus $\frac{d^k}{dt^k}\boldsymbol{\varphi}_t$ is in $\mathcal{T}^1([t^{(0)}, t_i^{(1)}])$, as well, and respectively, it follows $\boldsymbol{\varphi} \in \mathcal{T}^{k+1}([t^{(0)}, t_i^{(1)}])$. By induction, we therefore obtain $\boldsymbol{\varphi} \in \mathcal{T}^{2+n}([t^{(0)}, t_i^{(1)}])$.

□

The next Lemma corresponds to statement (v) in Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions) with the slight difference that it is formulated for arbitrary initial times $t^{(0)} \geq 0$ and corresponding histories $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$. Since the assertion is needed in the following proofs, it is positioned at this place.

Lemma 5.6.2 ((H3)⇔(R))**.** *Let $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ denote a history for $t^{(0)}$ fulfilling (H0)-(H2) for initial time $t^{(0)} \geq 0$. Further, let $t^{(1)} > t^{(0)}$ and $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{i\in\mathcal{N}}$ denote a local Lorentz solution for the fields $(\boldsymbol{f}_i = (\boldsymbol{E}_i, \boldsymbol{B}_i))_{i\in\mathcal{N}}$ given by (4.27) on $[t^{(0)}, t^{(1)}]$ such that $(\boldsymbol{q}_{i,t=t^{(0)}}, \boldsymbol{p}_{i,t=t^{(0)}}) = (\tilde{\boldsymbol{q}}_{i,t=t^{(0)}}, \tilde{\boldsymbol{p}}_{i,t=t^{(0)}})$ and $(\boldsymbol{q}_i, \boldsymbol{p}_i) \in \mathcal{T}^{2+n}([t^{(0)}, t^{(1)}])$. Then, (H3) is fulfilled if and only if (R) is fulfilled.*

Proof. Assume (H3) is met. Let $i \in \mathcal{N}$ be any charge index. Then, for all $k \in \{1, \ldots, 1+n\}$ it holds

$$\begin{aligned}
\lim_{t\nearrow t^{(0)}} \frac{d^k}{dt^k}\begin{pmatrix}\tilde{\boldsymbol{q}}_{i,t}\\ \tilde{\boldsymbol{p}}_{i,t}\end{pmatrix} &\overset{(H3)}{=} \lim_{t\nearrow t^{(0)}} \frac{d^{k-1}}{dt^{k-1}}\begin{pmatrix}\boldsymbol{v}(\tilde{\boldsymbol{p}}_{i,t})\\ \sum_{j\neq i}\boldsymbol{E}_{j,t}(\tilde{\boldsymbol{q}}_{i,t}) + \boldsymbol{v}(\tilde{\boldsymbol{p}}_{i,t})\wedge \boldsymbol{B}_{j,t}(\tilde{\boldsymbol{q}}_{i,t})\end{pmatrix}\\
&= \lim_{t\searrow t^{(0)}} \frac{d^{k-1}}{dt^{k-1}}\begin{pmatrix}\boldsymbol{v}(\boldsymbol{p}_{i,t})\\ \sum_{j\neq i}\boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}) + \boldsymbol{v}(\boldsymbol{p}_{i,t})\wedge \boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t})\end{pmatrix}\\
&= \lim_{t\searrow t^{(0)}} \frac{d^k}{dt^k}\begin{pmatrix}\boldsymbol{q}_{i,t}\\ \boldsymbol{p}_{i,t}\end{pmatrix},
\end{aligned}$$

and hence, (R) holds. This is true because: The third equality is due to the Lorentz force law and the second equality is valid for the following reasons:

- In the proof of Lemma 5.6.1 it has been shown that $\boldsymbol{f}_j \in C^n(D_{\tilde{\boldsymbol{q}}_j} \setminus \overset{\circ}{J^+}(T, \tilde{\boldsymbol{q}}_{j,T}), \mathbb{R}^6)$ for all $j \in \mathcal{N}$ holds, cf. (5.139).
- For $k = 1$ the equality holds by assumption $(\boldsymbol{q}_{i,t=t^{(0)}}, \boldsymbol{p}_{i,t=t^{(0)}}) = (\tilde{\boldsymbol{q}}_{i,t=t^{(0)}}, \tilde{\boldsymbol{p}}_{i,t=t^{(0)}})$. Assume that the second equality is true for all $k \leq l \in \{1, \ldots, n\}$, i.e., the three equalities are true for $k \leq l$, and in particular

$$\lim_{t\nearrow t^{(0)}} \frac{d^l}{dt^l}\begin{pmatrix}\tilde{\boldsymbol{q}}_{i,t}\\ \tilde{\boldsymbol{p}}_{i,t}\end{pmatrix} = \lim_{t\searrow t^{(0)}} \frac{d^l}{dt^l}\begin{pmatrix}\boldsymbol{q}_{i,t}\\ \boldsymbol{p}_{i,t},\end{pmatrix} \tag{5.150}$$

  and thus, the second equality is true for $k \leq l+1$. Therefore, by induction the second equality is true for all $k \in \{1, \ldots, 1+n\}$.

If on the other hand we assume (R) to be true, it follows for all $i \in \mathcal{N}$ and all $k \in \{1, \ldots, 1+n\}$ that

$$\begin{aligned}
\lim_{t\nearrow t^{(0)}} \frac{d^k}{dt^k}\begin{pmatrix}\tilde{\boldsymbol{q}}_{i,t}\\ \tilde{\boldsymbol{p}}_{i,t}\end{pmatrix} &\overset{(R)}{=} \lim_{t\searrow t^{(0)}} \frac{d^k}{dt^k}\begin{pmatrix}\boldsymbol{q}_{i,t}\\ \boldsymbol{p}_{i,t}\end{pmatrix}\\
&= \lim_{t\searrow t^{(0)}} \frac{d^{k-1}}{dt^{k-1}}\begin{pmatrix}\boldsymbol{v}(\boldsymbol{p}_{i,t})\\ \sum_{j\neq i}\boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}) + \boldsymbol{v}(\boldsymbol{p}_{i,t})\wedge \boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t})\end{pmatrix}\\
&= \lim_{t\nearrow t^{(0)}} \frac{d^{k-1}}{dt^{k-1}}\begin{pmatrix}\boldsymbol{v}(\tilde{\boldsymbol{p}}_{i,t})\\ \sum_{j\neq i}\boldsymbol{E}_{j,t}(\tilde{\boldsymbol{q}}_{i,t}) + \boldsymbol{v}(\tilde{\boldsymbol{p}}_{i,t})\wedge \boldsymbol{b}^-_{j,t}(\tilde{\boldsymbol{q}}_{i,t})\end{pmatrix}.
\end{aligned}$$

Here, the second equality is due to the force law and the third equality is due to assumption (H0) on the histories together with condition (R). □

Lemma 5.6.3 (Local existence of Lorentz solutions). *Let $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ be a history for $t^{(0)}$ fulfilling (H0)-(H2). The following propositions hold true:*

(i) (Existence) There exists a time $t^{(1)}$ with $t^{(1)} - t^{(0)} > \frac{d}{2}$ such that there exists a Lorentz solution $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{i\in\mathcal{N}}$ for $(\boldsymbol{f}_i)_{i\in\mathcal{N}}$ given by (4.27) on $[t^{(0)}, t^{(1)}]$.

(ii) (Uniqueness) Moreover, let $\Lambda \subset \mathbb{R}$ be an interval containing $t^{(0)}$. For any Lorentz solution $(\hat{\boldsymbol{q}}_i, \hat{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ for $(\boldsymbol{f}_i)_{i\in\mathcal{N}}$ given by (4.27) on Λ with $(\hat{\boldsymbol{q}}_{i,t}, \hat{\boldsymbol{p}}_{i,t})|_{t=t^{(0)}} = (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t})|_{t=t^{(0)}}$ for all $i \in \mathcal{N}$ it follows

$$(\hat{\boldsymbol{q}}_{i,t}, \hat{\boldsymbol{p}}_{i,t}) = (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}), \qquad \forall\, t \in \Lambda \cap [t^{(0)}, t^{(1)}],\ i \in \mathcal{N}. \tag{5.151}$$

(iii) (Regularity) The Lorentz solutions fulfill $(\boldsymbol{q}_i, \boldsymbol{p}_i) \in \mathcal{T}^{2+n}([t^{(0)}, t^{(1)}])$ for all $i \in \mathcal{N}$.

(iv) (Connection) If in addition the history fulfills (H3) for initial time $t^{(0)}$, the tuple $((\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i) \cup (\boldsymbol{q}_i, \boldsymbol{p}_i))_{i\in\mathcal{N}}$ is a history fulfilling (H0), (H1), and (H3) for initial time $t^{(1)}$.

Proof. (i) - (iii) By assumption the requirements for Lemma 5.6.1 (Lorentz solution of charge i on G_i) are met to apply it for all $i \in \mathcal{N}$ and any $T > \frac{d}{2}$. Thus, we obtain the existence of a unique Lorentz solution $(\boldsymbol{q}_i, \boldsymbol{p}_i) \in \mathcal{T}^{2+n}([t^{(0)}, t_i^{(1)}])$ on G_i for each charge $i \in \mathcal{N}$, where $t_i^{(1)}$ is defined by (5.135). Setting

$$t^{(1)} := \min\{t_1^{(1)}, \ldots, t_N^{(1)}\}, \tag{5.152}$$

we obtain Lorentz solutions for $(\boldsymbol{f}_i)_{i\in\mathcal{N}}$ on $[t^{(0)}, t^{(1)}]$ according to Definition 3.3.2 (Lorentz solutions).

Moreover, $t^{(1)} > \frac{1}{2}\min_{j\neq i} |\tilde{\boldsymbol{q}}_{i,0} - \tilde{\boldsymbol{q}}_{j,0}| > \frac{d}{2}$ due to the light-cone geometry (see Figure 5.4) and the choice of the time $T > \frac{d}{2}$.

(iv) We need to verify that the solutions $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{i\in\mathcal{N}}$ fulfill (H0), (H1), (H3):

(H0) Since $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)$ and $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ are strictly time-like and $1+n$ times continuously differentiable for all $i \in \mathcal{N}$ it remains to check the regularity at the connection time $t^{(0)}$. Since, for $t^{(0)}$, condition (H3) holds by assumption and this is equivalent to (R) by Lemma 5.6.2 ((H3)⇔(R)), it follows

$$(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i) \cup (\boldsymbol{q}_i, \boldsymbol{p}_i) \in \mathcal{T}^{2+n}((-\infty, t^{(1)}], \mathbb{R}^6). \tag{5.153}$$

(H1) Let $t \in [t^{(0)}, t^{(1)}]$. It holds

$$\begin{aligned}
\left|\boldsymbol{p}_{i,t}\right| &= \left|\boldsymbol{p}_{i,t^{(0)}} + \int_{t^{(0)}}^{t} ds \boldsymbol{E}_{j,s}(\boldsymbol{q}_{i,s}) + \boldsymbol{v}(\boldsymbol{p}_{i,s}) \wedge \boldsymbol{B}_{j,s}(\boldsymbol{q}_{i,s})\right| \\
&\leq \left|\boldsymbol{p}_{i,t^{(0)}}\right| + (t^{(1)} - t^{(0)}) \max_{t\in[t^{(0)},t^{(1)}]} \left|\boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t})\right| + \left|\boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t})\right| \\
&\leq c < \infty,
\end{aligned}$$

since $t \mapsto \boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t})$ and $t \mapsto \boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t})$ are continuous on the compact interval $[t^{(0)}, t^{(1)}]$. Thus, there exists a constant $c_v < 1$ such that

$$\dot{\boldsymbol{q}}_{i,t} = \boldsymbol{v}(\boldsymbol{p}_{i,t}) := \frac{\boldsymbol{p}_{i,t}}{\sqrt{m_i^2 + \boldsymbol{p}_{i,t}^2}} \leq c_v < 1 \tag{5.154}$$

holds for $t \in [t_i^{(0)}, t_i^{(1)}]$, and since the history is strictly time-like by condition, the union $((\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i) \cup (\boldsymbol{q}_i, \boldsymbol{p}_i))$ is strictly time like for each $1 \leq i \leq N$.

(H3) By the proof of Lemma 5.6.1 (Lorentz solution of charge i on G_i), the Lorentz solution can be written in the integral form (5.148). Therefore, for $k = 1, \ldots, 1+n$ it holds

$$
\begin{aligned}
&\lim_{t \nearrow t^{(1)}} \frac{d^k}{dt^k} \begin{pmatrix} \boldsymbol{q}_{i,t} \\ \boldsymbol{p}_{i,t} \end{pmatrix} \\
&= \lim_{t \nearrow t^{(1)}} \frac{d^k}{dt^k} \left[\begin{pmatrix} \boldsymbol{q}_{i,t^{(0)}} \\ \boldsymbol{p}_{i,t^{(0)}} \end{pmatrix} + \int_{t^{(0)}}^{t} ds \begin{pmatrix} \boldsymbol{v}(\boldsymbol{p}_{i,s}) \\ \sum_{j \neq i} \boldsymbol{E}_{j,s}(\boldsymbol{q}_{i,s}) + \boldsymbol{v}(\boldsymbol{p}_{i,s}) \wedge \boldsymbol{B}_{j,s}(\boldsymbol{q}_{i,s}) \end{pmatrix} \right] \\
&= \lim_{t \nearrow t^{(1)}} \frac{d^{k-1}}{dt^{k-1}} \begin{pmatrix} \boldsymbol{v}(\boldsymbol{p}_{i,t}) \\ \sum_{j \neq i} \boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}) + \boldsymbol{v}(\boldsymbol{p}_{i,t}) \wedge \boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t}) \end{pmatrix} .
\end{aligned}
$$

□

We are now able to conclude the proof of Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions). The proof can be deduced from Definition 3.3.4 (Maxwell-Lorentz solutions) and Lemma 5.6.3 (Local existence of Lorentz solutions) applied step by step until particles collide.

Proof of Theorem 4.3.1. (Existence of Maxwell-Lorentz solutions)

Let $T \in \mathbb{R}^+$, $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ be a history for the initial time $t^{(0)} = 0$ fulfilling (H0)-(H2) and $(\boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ be initial fields of the form $\boldsymbol{f}_{i,0} = \boldsymbol{f}_0^-[\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i] + \boldsymbol{f}_{i,0}^h$ as given in (4.24). Recall, that by virtue of (4.25)-(4.26) the homogeneous field $\boldsymbol{f}_{i,t}^h := W_t \boldsymbol{f}_{i,0}^h$ is uniquely determined for all $t \in \mathbb{R}$ with a representative in $C^n(\mathbb{R}^4, \mathbb{R}^6)$.

Let us start by proving item (i).

Given $t^{(0)} = 0$ and setting $(\tilde{\boldsymbol{q}}_i^{(0)}, \tilde{\boldsymbol{p}}_i^{(0)})_{i\in\mathcal{N}} := (\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ which fulfill (H0)-(H2), we can define

$$\boldsymbol{f}_{i,t}^{(1)}(\boldsymbol{x}) := \boldsymbol{f}_t^-[\tilde{\boldsymbol{q}}_i^{(0)}, \tilde{\boldsymbol{p}}_i^{(0)}](\boldsymbol{x}) + \boldsymbol{f}_{i,t}^h(\boldsymbol{x}), \qquad \forall\, (t, \boldsymbol{x}) \in \mathbb{R}^4 \setminus \mathring{J}^+(t^{(0)}, \tilde{\boldsymbol{q}}_{i,t^{(0)}}),\ i \in \mathcal{N}, \tag{5.155}$$

cf. (4.27), which is the unique Maxwell solution for the initial field $\boldsymbol{f}_{i,0}$ restricted to $(t, \boldsymbol{x}) \in \mathbb{R}^4 \setminus \mathring{J}^+(t^{(0)}, \tilde{\boldsymbol{q}}_{i,t^{(0)}})$ according to Theorem 4.2.1 (Explicit Maxwell solutions) with regard to Remark 4.2.2, (ii).

Moreover, we may apply Lemma 5.6.3 (Local existence of Lorentz solutions), (i)-(iii), and obtain a time $t^{(1)} > \frac{d}{2}$ and a unique Lorentz solution $(\boldsymbol{q}_i^{(1)}, \boldsymbol{p}_i^{(1)})_{i\in\mathcal{N}}$ for $(\boldsymbol{f}_i^{(1)})_{i\in\mathcal{N}}$ with initial value $(\boldsymbol{q}_{i,t^{(0)}}^{(1)}, \boldsymbol{p}_{i,t^{(0)}}^{(1)})_{i\in\mathcal{N}} = (\tilde{\boldsymbol{q}}_{i,t^{(0)}}^{(0)}, \tilde{\boldsymbol{p}}_{i,t^{(0)}}^{(0)})_{i\in\mathcal{N}}$ on $[t^{(0)}, t^{(1)}]$ such that

$$(\boldsymbol{q}_i^{(1)}, \boldsymbol{p}_i^{(1)}) \in \mathcal{T}^{2+n}([t^{(0)}, t^{(1)}]), \qquad \forall\, i \in \mathcal{N}. \tag{5.156}$$

With the aid of this Lorentz solution we may extend the field $\boldsymbol{f}_i^{(1)}$ for all $i \in \mathcal{N}$ by virtue of Theorem 4.2.1 (Explicit Maxwell solutions) and we obtain a unique Maxwell solution for $(\boldsymbol{q}_i^{(1)}, \boldsymbol{p}_i^{(1)})$ with initial value $\boldsymbol{f}_{i,0}$ on the interval $[t^{(0)}, t^{(1)}]$ given by

$$\begin{aligned} \boldsymbol{f}_{i,t}^{(1)} &:= \mathbb{1}_{B_{|t|}(\tilde{\boldsymbol{q}}_{i,t^{(0)}}^{(0)})} \boldsymbol{f}_t^-[\boldsymbol{q}_i^{(1)}, \boldsymbol{p}_i^{(1)}] \\ &\quad + \mathbb{1}_{B_{|t|}^c(\tilde{\boldsymbol{q}}_{i,t^{(0)}}^{(0)})} \boldsymbol{f}_t^-[\tilde{\boldsymbol{q}}_i^{(0)}, \tilde{\boldsymbol{p}}_i^{(0)}] \\ &\quad + \boldsymbol{f}_{i,t}^h, \end{aligned} \tag{5.157}$$

and, since charge trajectories can be smoothly extended to all $\mathbb{R}$, cf. Lemma 4.2.1 (Properties of Liénard-Wiechert fields), (ii), we may apply Lemma 4.2.4 (Regularity of $\boldsymbol{f}_t$), (i), for the special case $\lambda = 1$ and obtain

$$\boldsymbol{f}_i^{(1)} \in C^n(D_{\boldsymbol{q}_i^{(1)}}^{[t^{(0)}, t^{(1)}]} \setminus \partial J^+(t^{(0)}, \tilde{\boldsymbol{q}}_{i,t^{(0)}}), \mathbb{R}^6), \qquad \forall\, i \in \mathcal{N}. \tag{5.158}$$

Collecting the results we have

- a unique Lorentz solution $(\boldsymbol{q}_i^{(1)}, \boldsymbol{p}_i^{(1)})_{i\in\mathcal{N}}$ for $(\boldsymbol{f}_i^{(1)})_{i\in\mathcal{N}}$ with initial value $(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0})_{i\in\mathcal{N}}$ on $[t^{(0)}, t^{(1)}]$,

- unique Maxwell solutions $\boldsymbol{f}_i^{(1)}$ for $(\boldsymbol{q}_i^{(1)}, \boldsymbol{p}_i^{(1)})$ with initial value $\boldsymbol{f}_{i,0}$ on $[t^{(0)}, t^{(1)}]$ for all $i \in \mathcal{N}$,
- and thus, a unique Maxwell-Lorentz solution $(\boldsymbol{q}_i^{(1)}, \boldsymbol{p}_i^{(1)}, \boldsymbol{f}_i^{(1)})_{i\in\mathcal{N}}$ on $[t^{(0)}, t^{(1)}]$ with initial value
$$(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}} \tag{5.159}$$
fulfilling the regularity properties (5.156) and (5.158).

Setting $T_{\max} := t^{(1)}$, this concludes the proof of claim (i).

Next, we prove claim (ii). Should no collision occur and assuming in addition that the history $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ fulfills (H3), we shall show next that we can construct the solution up to the first collision, or in the absence of collision also globally, by induction: In the following we will construct, among other objects, the following ones:
$$t^{(m)} \in \mathbb{R},\ (\boldsymbol{q}_i^{(m)}, \boldsymbol{p}_i^{(m)}) \in \mathcal{T}^{2+m}([t^{(m-1)}, t^{(m)}]),\ \forall\, i \in \mathcal{N}, \tag{5.160}$$
inductively for certain $m \in \mathbb{N}$ and with their help define
$$T_{\max}^{(m)} := \sup\{t \in [t^{(m-1)}, t^{(m)}] \cap [0,T] \mid \forall\, i,j \in \mathcal{N}, i \neq j, \forall s \in [0,t] : |\boldsymbol{q}_{i,s}^{(m)} - \boldsymbol{q}_{j,s}^{(m)}| > d\}, \tag{5.161}$$
which detects whether a collision occurred in the m-th step of the induction.

First, we complete the base case $m = 1$.

Under the additional assumption of (H3), by Lemma 5.6.2 ((H3)⇔(R)), the Maxwell-Lorentz solution $(\boldsymbol{q}_i^{(1)}, \boldsymbol{p}_i^{(1)}, \boldsymbol{f}_i^{(1)})_{i\in\mathcal{N}}$ on $[t^{(0)}, t^{(1)}]$ and the respective history $(\tilde{\boldsymbol{q}}_i^{(0)}, \tilde{\boldsymbol{p}}_i^{(0)})_{i\in\mathcal{N}}$ fulfill (R), and hence, also (4.23). As we are in the special case $\lambda = 1$ and the respective charge trajectories can be smoothly extended to all $\mathbb{R}$ without changing the fields on the interval $[t^{(0)}, t^{(1)}]$ we can apply Lemma 4.2.4 (Regularity of $\boldsymbol{f}_t$), (iii), and find
$$\boldsymbol{f}_i^{(1)} \in C^n(D_{\boldsymbol{q}_i^{(1)}}^{[t^{(0)}, t^{(1)}]}, \mathbb{R}^6), \qquad \forall\, i \in \mathcal{N}. \tag{5.162}$$

If $T_{\max}^{(1)} < t^{(1)}$ we cannot continue the induction and conclude the proof for $T_{\max} := T_{\max}^{(1)}$.

Otherwise, for $T_{\max}^{(1)} = t^{(1)}$, we continue the induction and define a new history for $t^{(1)}$ by
$$(\tilde{\boldsymbol{q}}_i^{(1)}, \tilde{\boldsymbol{p}}_i^{(1)}) := (\tilde{\boldsymbol{q}}_i^{(0)}, \tilde{\boldsymbol{p}}_i^{(0)}) \cup (\boldsymbol{q}_i^{(1)}, \boldsymbol{p}_i^{(1)}), \qquad i \in \mathcal{N}. \tag{5.163}$$
Due to Lemma 5.6.3 (Local existence of Lorentz solutions), (iv), and (H3) we know that the new history $(\tilde{\boldsymbol{q}}_i^{(1)}, \tilde{\boldsymbol{p}}_i^{(1)})_{i\in\mathcal{N}}$ fulfills (H0), (H1), (H3), and furthermore, by virtue of $T_{\max}^{(1)} = t^{(1)}$ also (H2).

Finally, note that by construction $((\tilde{\boldsymbol{q}}_i^{(1)}, \tilde{\boldsymbol{p}}_i^{(1)})|_{[t^{(0)}, t^{(1)}]})_{i\in\mathcal{N}}$ is the unique Lorentz solution for $(\boldsymbol{f}_i^{(1)})_{i\in\mathcal{N}}$ with initial value $(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0})_{i\in\mathcal{N}}$ on $[t^{(0)}, t^{(1)}]$ and in terms of the new history $(\tilde{\boldsymbol{q}}_i^{(1)}, \tilde{\boldsymbol{p}}_i^{(1)})_{i\in\mathcal{N}}$, $\boldsymbol{f}_{i,t}^{(1)}$ from (5.157) is equal to
$$\boldsymbol{f}_{i,t}^{(1)} = \boldsymbol{f}_t^{-}[\tilde{\boldsymbol{q}}_i^{(1)}, \tilde{\boldsymbol{p}}_i^{(1)}] + \boldsymbol{f}_{i,t}^{h}, \tag{5.164}$$
for all $t \in [t^{(0)}, t^{(1)}]$ and $i \in \mathcal{N}$. Summing up, for $m = 1$ we have shown:

(*1) $t^{(m)} > m\frac{d}{2}$.

(*2) $(\tilde{\boldsymbol{q}}_i^{(m)}, \tilde{\boldsymbol{p}}_i^{(m)})_{i\in\mathcal{N}}$ is a history for $t^{(m)}$ fulfilling (H0)-(H3).

(*3) $((\tilde{\boldsymbol{q}}_i^{(m)}, \tilde{\boldsymbol{p}}_i^{(m)})|_{[t^{(0)},t^{(m)}]})_{i\in\mathcal{N}}$ is the unique Lorentz solution for $(\boldsymbol{f}_i^{(m)})_{i\in\mathcal{N}}$ with initial value $(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0})_{i\in\mathcal{N}}$ on $[t^{(0)}, t^{(m)}]$, where for all $t \in [t^{(0)}, t^{(m)}]$

$$\boldsymbol{f}_{i,t}^{(m)} := \boldsymbol{f}_t^-[\tilde{\boldsymbol{q}}_i^{(m)}, \tilde{\boldsymbol{p}}_i^{(m)}] + \boldsymbol{f}_{i,t}^h. \tag{5.165}$$

We turn to the inductive step. Therefore, let us assume that for some $m \in \mathbb{N}$ the propositions (*1)-(*3) hold true.

Correspondingly to (4.27), we can define the extension of the field $\boldsymbol{f}_{i,t}^{(m)}$ as

$$\boldsymbol{f}_{i,t}^{(m+1)}(\boldsymbol{x}) := \boldsymbol{f}_t^-[\tilde{\boldsymbol{q}}_i^{(m)}, \tilde{\boldsymbol{p}}_i^{(m)}](\boldsymbol{x}) + \boldsymbol{f}_{i,t}^h(\boldsymbol{x}), \qquad \forall\, (t,\boldsymbol{x}) \in \mathbb{R}^4 \setminus \mathring{J}^+(t^{(m)}, \tilde{\boldsymbol{q}}_{i,t^{(m)}}),\ i \in \mathcal{N}, \tag{5.166}$$

by Theorem 4.2.1 (Explicit Maxwell solutions).

Moreover, we may apply Lemma 5.6.3 (Local existence of Lorentz solutions), (i)-(iii), for the history $(\tilde{\boldsymbol{q}}_i^{(m)}, \tilde{\boldsymbol{p}}_i^{(m)})_{i\in\mathcal{N}}$ and the initial time $t^{(m)}$ and obtain a time

$$t^{(m+1)} > t^{(m)} + \frac{d}{2} > (m+1)\frac{d}{2} \tag{5.167}$$

such that there is a

(*L) unique Lorentz solution $(\boldsymbol{q}_i^{(m+1)}, \boldsymbol{p}_i^{(m+1)})_{i\in\mathcal{N}}$ for $(\boldsymbol{f}_i^{(m+1)})_{i\in\mathcal{N}}$ with initial value

$$(\boldsymbol{q}_{i,t^{(m)}}^{(m+1)}, \boldsymbol{p}_{i,t^{(m)}}^{(m+1)})_{i\in\mathcal{N}} = (\tilde{\boldsymbol{q}}_{i,t^{(m)}}^{(m)}, \tilde{\boldsymbol{p}}_{i,t^{(m)}}^{(m)})_{i\in\mathcal{N}} \tag{5.168}$$

on $[t^{(m)}, t^{(m+1)}]$ such that

$$(\boldsymbol{q}_i^{(m+1)}, \boldsymbol{p}_i^{(m+1)}) \in \mathcal{T}^{2+n}([t^{(m)}, t^{(m+1)}]), \qquad \forall\, i \in \mathcal{N}. \tag{5.169}$$

Given (*L), we define a new history for $t^{(m+1)}$ by

$$(\tilde{\boldsymbol{q}}_i^{(m+1)}, \tilde{\boldsymbol{p}}_i^{(m+1)}) := (\tilde{\boldsymbol{q}}_i^{(m)}, \tilde{\boldsymbol{p}}_i^{(m)}) \cup (\boldsymbol{q}_i^{(m+1)}, \boldsymbol{p}_i^{(m+1)}), \qquad i \in \mathcal{N}. \tag{5.170}$$

Due to Lemma 5.6.3 (Local existence of Lorentz solutions), (iv), and (H3) we know that the new history $(\tilde{\boldsymbol{q}}_i^{(m+1)}, \tilde{\boldsymbol{p}}_i^{(m+1)})_{i\in\mathcal{N}}$ up to time $t^{(m+1)}$ fulfills (H0), (H1), (H3).

Because of assumption (*3), (*L), and the fact that the history $(\tilde{\boldsymbol{q}}_i^{(m+1)}, \tilde{\boldsymbol{p}}_i^{(m+1)})_{i\in\mathcal{N}}$ fulfills (H0), so that it is $2+n$ times continuously differentiable at $t^{(m)}$, we can conclude that

$$((\tilde{\boldsymbol{q}}_i^{(m+1)}, \tilde{\boldsymbol{p}}_i^{(m+1)})|_{[t^{(0)},t^{(m+1)}]})_{i\in\mathcal{N}} \tag{5.171}$$

is the unique Lorentz solution for $(\boldsymbol{f}_i^{(m+1)})_{i\in\mathcal{N}}$ with initial value $(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0})_{i\in\mathcal{N}}$ on $[t^{(0)}, t^{(m+1)}]$.

With the aid of the obtained history $(\tilde{\boldsymbol{q}}_i^{(m+1)}, \tilde{\boldsymbol{p}}_i^{(m+1)})_{i\in\mathcal{N}}$, which was shown to fulfill the regularity condition (H0) up to $t^{(m+1)}$, we may extend the field $\boldsymbol{f}_i^{(m+1)}$ for all $i \in \mathcal{N}$ by virtue of Theorem 4.2.1 (Explicit Maxwell solutions) for all $t \le t^{(m+1)}$ and obtain

$$\boldsymbol{f}_{i,t}^{(m+1)} = \boldsymbol{f}_t^-[\tilde{\boldsymbol{q}}_i^{(m+1)}, \tilde{\boldsymbol{p}}_i^{(m+1)}] + \boldsymbol{f}_{i,t}^h \tag{5.172}$$

as the unique Maxwell solution $\boldsymbol{f}_i^{(m+1)}$ for $(\tilde{\boldsymbol{q}}_i^{(m+1)}, \tilde{\boldsymbol{p}}_i^{(m+1)})$ with initial value $\boldsymbol{f}_{i,0}$ on the interval $[t^{(0)}, t^{(m+1)}]$. Moreover, exploiting property (H0) again, Lemma 4.2.1 (Properties of Liénard-Wiechert fields), (i), and (4.26) guarantee

$$\boldsymbol{f}_i^{(m+1)} \in C^n(D_{\tilde{\boldsymbol{q}}_i^{(m+1)}}^{[t^{(0)},t^{(m+1)}]}, \mathbb{R}^6), \qquad \forall\, i \in \mathcal{N}. \tag{5.173}$$

Summing up the results we have

- a unique Lorentz solution $((\tilde{\boldsymbol{q}}_i^{(m+1)}, \tilde{\boldsymbol{p}}_i^{(m+1)})|_{[t^{(0)},t^{(m+1)}]})_{i\in\mathcal{N}}$ for $(\boldsymbol{f}_i^{(m+1)})_{i\in\mathcal{N}}$ with initial value $(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0})_{i\in\mathcal{N}}$ on the interval $[t^{(0)}, t^{(m+1)}]$,
- unique Maxwell solutions $\boldsymbol{f}_i^{(m+1)}$ for $(\tilde{\boldsymbol{q}}_i^{(m+1)}, \tilde{\boldsymbol{p}}_i^{(m+1)})$ with initial value $\boldsymbol{f}_{i,0}$ on the interval $[t^{(0)}, t^{(m+1)}]$ for all $i \in \mathcal{N}$,
- and thus, a unique Maxwell-Lorentz solution $(\boldsymbol{q}_i^{(m+1)}, \boldsymbol{p}_i^{(m+1)}, \boldsymbol{f}_i^{(m+1)})_{i\in\mathcal{N}}$ on the interval $[t^{(0)}, t^{(m+1)}]$ with initial value

  $$(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}} \tag{5.174}$$

  fulfilling the regularity properties (5.169) and (5.173).

If $T_{\max}^{(m+1)} < t^{(m+1)}$ we cannot continue the induction and conclude the proof for $T_{\max} := T_{\max}^{(m+1)}$.

Otherwise, for $T_{\max}^{(m+1)} = t^{(m+1)}$, the history $(\tilde{\boldsymbol{q}}_i^{(m+1)}, \tilde{\boldsymbol{p}}_i^{(m+1)})_{i\in\mathcal{N}}$ for initial time $t^{(m+1)}$ fulfills (H2), and hence, proposition (*2). Furthermore, (5.167) and the definition of $T_{\max}^{(m+1)}$ implies (*1). Summing up, for given $m \in \mathbb{N}$, we have shown that proposition (*1)-(*3) are also fulfilled for m replaced by $m + 1$. This concludes the induction until the first collision or globally in case no collision occurs.

Thus, we have proven item (ii) and (iii). Furthermore, $\boldsymbol{f}_{i,0} \in \mathcal{F}_{\boldsymbol{q}_{i,0}}$ because: $\boldsymbol{f}_{i,0}^h \in \mathcal{F}_{\text{hom}}$ by assumption and $\boldsymbol{f}_0^-[\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i] \in \mathcal{F}_{\tilde{\boldsymbol{q}}_{i,0}} = \mathcal{F}_{\boldsymbol{q}_{i,0}}$ by Lemma 4.2.1 (Properties of Liénard-Wiechet fields), (iii). Therefore, Lemma A.2.1 (Maxwell constraints) implies that the Maxwell-Lorentz solution $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{f}_i)_{i\in\mathcal{N}}$ fulfills $\boldsymbol{f}_{i,t} \in \mathcal{F}_{\boldsymbol{q}_{i,t}}$ for all $t \in [0, T_{\max}]$, and hence, claim (iv) is met. By virtue of (i) and (ii) we have

$$(\tilde{\boldsymbol{q}}_{i,0}, \tilde{\boldsymbol{p}}_{i,0}) = (\boldsymbol{q}_{i,t=0}, \boldsymbol{p}_{i,t=0}), \qquad \forall\, i \in \mathcal{N} \tag{5.175}$$

so that Lemma 5.6.2 ((H3)⇔(R)) guarantees that claim (v) holds true.

□

Remark 5.6.1. *(i) One should note that condition (H2) solely checks the smallest particle difference at the end time of a propagation step (the initial time of the next step), whereas the implicit definition* (4.28) *of* $T_{\max}$ *checks the distance at any time. Therefore, it might happen, that charges approach each other but separate again so that we can obtain solutions up to a time* t *which is greater than* $T_{\max}$*. However, one may think of* d *as a arbitrary small entity detecting collisions and then, in the time-like regime, a collision in a interval* $[t^{(m)}, t^{(m+1)}]$ *goes hand in hand with the violation of (H2) for the time* $t^{(m+1)}$.

(ii) If T *in Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions), (ii), is chosen arbitrarily big and no collision occurs, the Lorentz solution can be obtained at a finite number of propagation steps. This is due to the required condition (H2), which implies that in each step where Lemma 5.6.1 (Lorentz solution of charge* i *on* G_i*) is applied to each charge* i*, the solution is extended by an interval greater than* $\frac{d}{2}$ *(see Figure 5.4 for illustration of the step length). So after the* m*th step the trajectories of all charges go further than the time* $\frac{d}{2}n$*. Thus, for any arbitrarily big, fixed, time* $T > 0$*, we can find a finite step number* $m(T) \in \mathbb{N}$ *such that*

$$\frac{d}{2}m(T) > T. \tag{5.176}$$

Chapter 6

Conclusion and Outlook

Our first main result has shown that generic initial configurations $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0}, \boldsymbol{f}_{i,0})_{i\in\mathcal{N}}$ lead to an ill-defined initial value problem, if one aims at solutions with continuous momenta, since, as we have shown, the initial particle configuration and the initial electromagnetic field are highly intertwined beyond the Maxwell constraints.

In fact, the introduced system of constraints necessary to infer sufficiently regular solutions suggests that the coupled system of Maxwell's and Lorentz's equations (1.2)-(1.4), a set of ordinary differential equations and partial differential equations, should rather be read as a system of ordinary delay differential equations

$$
\begin{aligned}
\frac{d}{dt}\begin{pmatrix}\boldsymbol{q}_{i,t}\\ \boldsymbol{p}_{i,t}\end{pmatrix} &= \begin{pmatrix}\boldsymbol{v}_{i,t} = \boldsymbol{v}(\boldsymbol{p}_{i,t})\\ \sum_{j\neq i}\boldsymbol{L}_{ij,t}\end{pmatrix}, \qquad i\in\mathcal{N} \qquad (6.1)\\
\boldsymbol{L}_{ij,t} &:= \boldsymbol{E}_{j,t}(\boldsymbol{q}_{i,t}) + \boldsymbol{v}_{i,t}\wedge\boldsymbol{B}_{j,t}(\boldsymbol{q}_{i,t}),\\
\boldsymbol{f}_{i,t} &= (\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})\\
&= \lambda\boldsymbol{f}^{-}_{i,t}[\boldsymbol{q}_i,\boldsymbol{p}_i] + (1-\lambda)\boldsymbol{f}^{+}_{i,t}[\boldsymbol{q}_i,\boldsymbol{p}_i] + \boldsymbol{f}^{h}_{i,t},
\end{aligned}
$$

where solely the initial charge positions and momenta $(\boldsymbol{q}_{i,0}, \boldsymbol{p}_{i,0})$ and initial homogeneous fields $(\boldsymbol{f}^{h}_{i,0})_{i\in\mathcal{N}}$ have to be specified. The arbitrary parameter λ just makes precise how the homogeneous fields $\boldsymbol{f}^{h}_{i,0}$ are to be interpreted. This has already been pointed in the physical discussion in Section 2.4, (2.24).
In fact, our strategy of proof of our existence of solutions result in Theorem 4.3.1 (Existence of Maxwell-Lorentz solutions) is based on this formulation.

This observation seems to be in line with Rohrlich's work [37], where is was also emphasized that the Maxwell and Lorentz equations cannot be treated separately and that initial configurations on a Cauchy surface for the coupled Maxwell-Lorentz system need to be constrained. Moreover, it seems that he is also well-aware of the fact that one needs to tackle a delay problem instead of a Cauchy problem, as can be taken from the following quotation:

> If one wants to specify a Cauchy problem at $t = 0$ together with the current for $t > 0$, the problem will separate into two problems: (a) the Cauchy problem with Cauchy data on $t = 0$; this will determine the fields for $t > 0$ *outside* the light-cone whose vertex is Q_0 (Fig. 4-2); (b) the retarded field problem due to the

> current at $t = 0$; this will determine the fields *inside* and on the future light-cone with vertex at Q_0. The Cauchy data for problem (a), however, are not known and must be found by solving a problem of type (b) for $t < 0$. Thus one simply has a retarded field problem [type (b)] *for all space-time*. It is very essential to realize that the finite propagation velocity of the field forces one into a problem posed for *all* space-time which would be very difficult (and physically awkward) to specify as (partially) a Cauchy problem. ([37], p. 78)

We feel that this work has provided a mathematical explanation of the above quotation by a rigorous analysis of the Maxwell fields. In addition, we have demonstrated that generic initial fields lead to singular fronts in the fields located along the light-cone boundaries of the initial charge positions, and thus, to an ill-posed initial value problem, which let to the original question in which sense one can still solve (6.1).

Our third main result partially answered this question and provided an existence result for the Maxwell-Lorentz system, that can be read as an existence result for the delay Lorentz system (6.1) for $\lambda = 1$ and propagation in future direction, i.e., on the positive half-line.

Therefore, we considered initial data defined by means of initial homogeneous fields $(\boldsymbol{f}^h_{i,0})_{i\in\mathcal{N}}$ and trajectory histories $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$. In our proof we assumed the whole histories to be given. It would be sufficient, though, to consider trajectory histories that for each charge only reach back to the earliest time where a backward light-cone of all other charges crosses its history; cf. red dots in Figure 6.1. In order to avoid the phenomenon of singular or discontinuous light fronts in the fields, these initial trajectory pieces had to be compatible with the Lorentz forces at time $t = 0$.

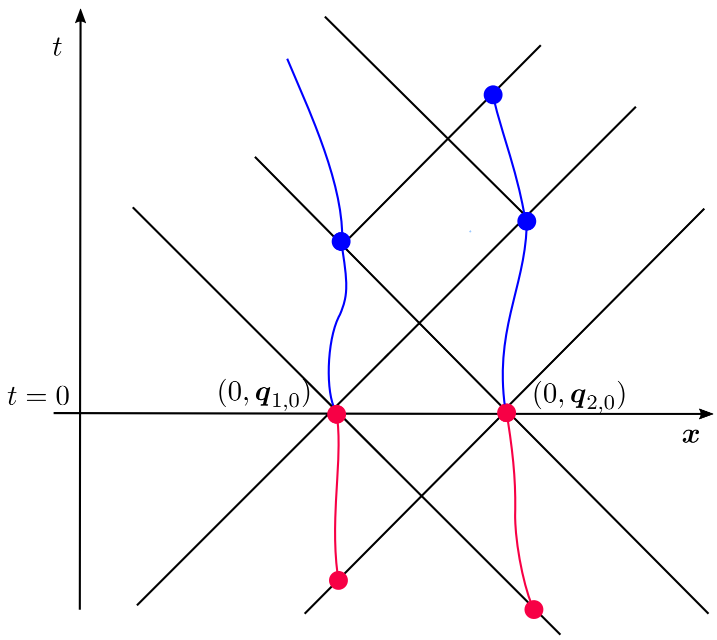

Figure 6.1: Illustration of the method of steps. Initial data depicted colored red; iterated solution trajectories colored blue.

Under these conditions we were able to construct solutions via a method of steps until a time $T_{\max}$, the first time of collision, or in case of no collision until a predetermined, arbitrarily large time T.

Since our proof of Theorem 4.3.1 was based on an iterative construction, it would be obvious to implement an algorithm that delivers Lorentz solutions for the considered class of initial data, which we briefly sketch. In contrast to the proof, such an algorithm would propagate each charge trajectory until the maximum time, until which the Lorentz force can be computed; cf. blue dots in Figure 6.1.

Data: fixed minimal distance d ;
maximal propagation time $T > 0$;
history[i]:= $(\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{p}}_i)_{i\in\mathcal{N}}$ fulfilling (H0)-(H3) ;
homfield[i]:= $\boldsymbol{f}^h_{i,0} \in \mathcal{F}_{\text{hom}} \cap C^{1+n}(\mathbb{R}^3, \mathbb{R}^6), i \in \mathcal{N}$;
Result: Unique solution on interval $[0, T_{\max}]$

```
I_T = ∅;
while |I_T| < |N| do
    for i ∈ N \ I_T do
        [solution[i], t[i]] = solution-of-charge-i-on-G_i^T (history, homfield);
        if t[i] = T then
            I_T = I_T ∪ {i};
        end
        history[i] = history[i] ∪ solution[i];
    end
    T_max = collision-time(history, d) ;
    if T_max < T then
        for i ∈ N \ I_T do
            I_T = I_T ∪ {i};
        end
    end
end
return history
```

Unfortunately, with this method, Maxwell-Lorentz solutions are constructed on the half axis only. Even though the initial trajectory pieces avoid singular and discontinuous fronts in the fields, they will in general not solve the delay Lorentz system (6.1) on the whole time axis. This becomes clear, when propagating the obtained Lorentz solution backwards. The constructed solutions can therefore be only understood as solutions whose past was constrained, e.g., by mechanical forces.

However, this approach may serve as a starting point for obtaining solutions on the entire time axis $\mathbb{R}$. Namely, when the future solutions are propagated into the past, the obtained past solutions can serve as new initial data for future propagation. The new future solutions can be propagated back again, and the past trajectories forth again. If this kind of iteration converges at some point, i.e., the obtained solutions do not differ anymore from the ones in the previous step, one indeed ends up with global solutions to the delay problem by numerical methods. If and for which initial trajectory pieces such an iteration converges is unclear. Moreover, the suggested approach is valid solely for the choice $\lambda = 1$ or $\lambda = 0$.

If both advanced and retarded fields are taken into account the initial fields $\boldsymbol{f}^h_{i,0}$ have to be replaced accordingly in each step. For any choice of $\lambda \in [0,1]$ and N smeared out charges, the iterative construction mentioned above was carried out in [8] on finite but arbitrarily large time intervals. Solving the delay problem in particular for $\lambda = 1/2$ would therefore also be mathematical highly interesting. This dynamics complies with the Wheeler-Feynman dynamics [46, 47]. From [24], where existence of solutions has been established for the special case of two charges on a straight line, it becomes clear how hard it is to build a corresponding solution theory on the entire time axis.

Another step towards a better understanding of classical electrodynamics for point charges is to take into account radiation reaction. Whereas in this work the self-interaction summand $\boldsymbol{L}_{ii,t}$ has been completely ignored, it would be desirable to extend it including a suitable self-interaction; in general one should think of $\boldsymbol{L}_{ii,t}$ as a functional of the charge trajectory $(\boldsymbol{q}_i, \boldsymbol{p}_i)$. As mentioned in Section 1.2, the Lorentz-Abraham-Dirac term $\boldsymbol{L}^{\text{LAD}}_{ii,t}$ may be one of the physically derivable candidates. However, as pointed out it is not too well-tempered as it allows for run-away solutions, and there are concerns on the correctness of this term; see e.g. [36]. A more promising term might therefore be the Landau-Lifshitz term (cf. formula (9.10) in [41]) first mentioned in [30] and later derived in [40, 36]. As this term does only involve a second derivative in time it may be controlled with the same ODE techniques and we have high hopes that we will be able to extend our existence proof in this direction in the near future.

Appendix A

Appendices

A.1 The Kirchhoff formulas

Definition A.1.1 (Propagator of the d'Alembert operator)**.** *The propagator of the d'Alembert operator* $\square := \partial_t^2 - \Delta$ *is denoted by* K_t*. It is given by*

$$K_t := K_t^- - K_t^+ \quad \text{for} \quad K_t^\pm := \frac{\delta(|\cdot| \pm t)}{4\pi|\cdot|} \in \mathcal{D}', \tag{A.1}$$

where $K_t^\pm$ *are the advanced and retarded Green's functions of the d'Alembert operator, i.e., symbolically,* $K_t^\pm$ *fulfills the equation*

$$\square K_t^\pm(\boldsymbol{x}) = \delta(t)\delta(\boldsymbol{x}), \tag{A.2}$$

where δ *denotes either the one dimensional Dirac delta distribution for input values in* $\mathbb{R}$ *or the three dimensional Dirac delta distribution for input values in* $\mathbb{R}^3$*.*

Remark A.1.1. *By Fourier transformation, for instance, the explicit representation can be derived. Note that* K_t *fulfills the homogeneous wave equation* $\square K_t = 0$ *and thus it is also called the propagator of the homogeneous wave equation.*

Lemma A.1.1 (Properties of K_t)**.** *Let* $n \in \mathbb{N}_0$ *and* $g \in C^{2+n}(\mathbb{R}^3, \mathbb{R})$ *and* $h_{(\cdot)} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ *such that for all* $t \in \mathbb{R}$ *and all* $\boldsymbol{x} \in \mathbb{R}^3$ *it holds* $h_t \in C^{2+n}(\mathbb{R}^3, \mathbb{R})$ *and* $h_{(\cdot)}(\boldsymbol{x}) \in C(\mathbb{R}, \mathbb{R})$*. Then, the distribution* K_t *introduced in Definition* (A.1.1) *(Propagator of the d'Alembert operator) has the following properties:*

(i) The mapping $(t, \boldsymbol{x}) \mapsto (K_t * g)(\boldsymbol{x})$ *is in* $C^{2+n}(\mathbb{R} \setminus \{0\} \times \mathbb{R}^3, \mathbb{R})$ *and for* $t \neq 0$ *it is given by*

$$K_t * g = \frac{1}{4\pi t} \int_{\partial B_{|t|}(0)} d\sigma(y) g(\cdot - \boldsymbol{y}). \tag{A.3}$$

Furthermore,

$$\lim_{t \to 0} K_t * g = 0 \qquad \text{and} \qquad \lim_{t \to 0} K_t * h_t = 0. \tag{A.4}$$

(ii) The mapping $(t, \boldsymbol{x}) \mapsto (\partial_t K_t * g)(\boldsymbol{x})$ *is in* $C^n(\mathbb{R} \setminus \{0\} \times \mathbb{R}^3, \mathbb{R})$ *and for* $t \neq 0$ *it is given by*

$$\partial_t K_t * g = \frac{1}{4\pi t^2} \int_{\partial B_{|t|}(0)} d\sigma(y) g(\cdot - \boldsymbol{y}) + \frac{1}{12\pi} \int_{B_{|t|}(0)} d^3y \Delta g(\cdot - \boldsymbol{y}). \tag{A.5}$$

Furthermore,

$$\lim_{t\to 0} \partial_t K_t * g = g \qquad \text{and} \qquad \lim_{t\to 0} \partial_t K_t * h_t = h_0. \tag{A.6}$$

(iii) For any $g \in C^\infty(\mathbb{R}^3, \mathbb{R})$ *the mapping* $\mathbb{R}\backslash\{0\}\times\mathbb{R}^3 \to \mathbb{R}$, $(t, \boldsymbol{x}) \mapsto (K_t * g)(\boldsymbol{x})$ *is continuously extendable to a* $C^\infty(\mathbb{R}\times\mathbb{R}^3, \mathbb{R})$ *function.*

(iv) $\partial_t^2 K_t * g = \Delta K_t * g = K_t * \Delta g$

Note that K_t is also subject to [29] and [8]. The following proof is in parts taken from the proof of Lemma 4.11. in [12].

Proof. (i) Let $\mp t > 0$ and $g \in C^{2+n}(\mathbb{R}^3, \mathbb{R})$. Then,

$$\begin{aligned}
(K_t^\pm * g)(\boldsymbol{x}) &= \int d^3y K_t^\pm(\boldsymbol{y}) g(\boldsymbol{x}-\boldsymbol{y}) = \int d^3y \frac{\delta(|\boldsymbol{y}| \pm t)}{4\pi|\boldsymbol{y}|} g(\boldsymbol{x}-\boldsymbol{y}) \\
&= \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi\, r^2 \sin\theta \frac{\delta(r\pm t)}{4\pi r} g\left(\boldsymbol{x} - r\begin{pmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{pmatrix}\right) \\
&= \int_0^\infty dr \int_{\partial B_1(0)} d\sigma(z) r^2 \frac{\delta(r\pm t)}{4\pi r} g(\boldsymbol{x} - r\boldsymbol{z}) \\
&= \int_{\partial B_1(0)} d\sigma(z) \int_0^\infty dr \frac{r}{4\pi}\delta(r\pm t) g(\boldsymbol{x} - r\boldsymbol{z}) \\
&= \frac{\mp t}{4\pi}\int_{\partial B_1(0)} d\sigma(z) g(\boldsymbol{x} \pm t\boldsymbol{z}) = \mp\frac{1}{4\pi t}\int_{\partial B_{\mp t}(0)} d\sigma(y) g(\boldsymbol{x}-\boldsymbol{y}) \\
&= \mp t \fint_{\partial B_{\mp t}(0)} d\sigma(y) g(\boldsymbol{x}-\boldsymbol{y}).
\end{aligned}$$

For $t\mp < 0$ the expression is 0 by definition. For the sum $K_t = K_t^- - K_t^+$ and all $t \neq 0$ we therefore obtain

$$\begin{aligned}
K_t * g &= K_t^- * g - K_t^+ * g \\
&= t \fint_{\partial B_{|t|}(0)} d\sigma(y) g(\cdot - \boldsymbol{y}) \\
&= t \fint_{\partial B_1(0)} d\sigma(y) g(\cdot - t\boldsymbol{y}),
\end{aligned} \tag{A.7}$$

which is a well-defined expression since it is a integral over a continuous function on a compact domain and $\boldsymbol{x} \mapsto (K_t^\pm * g)(\boldsymbol{x})$ is in $C^{2+n}(\mathbb{R}^3, \mathbb{R})$. Moreover, the function t, $t \mapsto (K_t^\pm * g)(\boldsymbol{x})$ is in $C^{1+n}(\mathbb{R}\setminus\{0\}, \mathbb{R})$.

The equation $\lim_{t\to 0} K_t * g = 0$ can be read of directly the above representation formula (A.7).

Next, we show $\lim_{t\to 0} K_t * h_t = 0$ by the following estimation. Let $\boldsymbol{x} \in \mathbb{R}^3$. Then,

$$\begin{aligned}\lim_{t\to 0} |(K_t * h_t)(\boldsymbol{x})| &\leq \lim_{t\to 0} |(K_t * (h_t - h_0))(\boldsymbol{x})| + \lim_{t\to 0} |(K_t * h_0)(\boldsymbol{x})| \\ &= \lim_{t\to 0} |(K_t * (h_t - h_0))(\boldsymbol{x})| + 0 \\ &\leq C \lim_{t\to 0} \sup_{y\in B_\delta(\boldsymbol{x})} |(h_t - h_0))(\boldsymbol{y})| = 0,\end{aligned}$$

where the last inequality holds for $|t|$ small enough and $C > |t|$ and the equality above is due to the continuity of the mapping $t \mapsto h_t(\boldsymbol{x})$.

(ii) For $t > 0$ we have

$$\begin{aligned}\partial_t ⨍_{\partial B_t(0)} d\sigma(y) g(\boldsymbol{x}-\boldsymbol{y}) &= \partial_t ⨍_{\partial B_1(0)} d\sigma(y) g(\boldsymbol{x}-t\boldsymbol{y}) = ⨍_{\partial B_1(0)} d\sigma(y) \partial_t g(\boldsymbol{x}-t\boldsymbol{y}) \\ &= \frac{1}{4\pi} \int_{\partial B_1(0)} d\sigma(y) \nabla g(\boldsymbol{x}-t\boldsymbol{y}) \cdot \partial_t(\boldsymbol{x}-t\boldsymbol{y}) \\ &= -\frac{1}{4\pi t^2} \int_{\partial B_t(0)} d\sigma(y) \nabla g(\boldsymbol{x}-\boldsymbol{y}) \cdot \frac{\boldsymbol{y}}{t} \\ &= \frac{1}{4\pi t^2} \int_{\partial B_t(0)} d\sigma(y) \nabla_y g(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{y}) \\ &\overset{GG}{=} \frac{1}{4\pi t^2} \int_{B_t(0)} d^3y \nabla_y \cdot \nabla_y g(\boldsymbol{x}-\boldsymbol{y}) \\ &= \frac{1}{4\pi t^2} \int_{B_t(0)} d^3y \Delta_y g(\boldsymbol{x}-\boldsymbol{y}) = \frac{1}{4\pi t^2} \int_{B_t(0)} d^3y \Delta g(\boldsymbol{x}-\boldsymbol{y}) \\ &= \frac{t}{3} ⨍_{B_t(0)} d^3y \Delta g(\boldsymbol{x}-\boldsymbol{y}),\end{aligned}$$

where GG abbreviates Gauss-Green Theorem, cf. Appendix A.4. Using this, for all $\mp t > 0$ we compute

$$\begin{aligned}(\partial_t K_t^\pm * g)(\boldsymbol{x}) &= \partial_t \left(\mp t ⨍_{\partial B_{\mp t}(0)} d\sigma(y) g(\boldsymbol{x}-\boldsymbol{y}) \right) \\ &= \mp ⨍_{\partial B_{\mp t}(0)} d\sigma(y) g(\boldsymbol{x}-\boldsymbol{y}) \mp t \partial_t ⨍_{\partial B_{\mp t}(0)} d\sigma(y) g(\boldsymbol{x}-\boldsymbol{y}) \\ &= \mp ⨍_{\partial B_{\mp t}(0)} d\sigma(y) g(\boldsymbol{x}-\boldsymbol{y}) \mp \frac{t^2}{3} ⨍_{B_{\mp t}(0)} d^3y \Delta g(\boldsymbol{x}-\boldsymbol{y}).\end{aligned}$$

For the sum $K_t = K_t^- - K_t^+$, we therefore obtain

$$\begin{aligned}\partial_t K_t * g &= \partial_t K_t^- * g - \partial_t K_t^+ * g \\ &= ⨍_{\partial B_{|t|}(0)} d\sigma(y) g(\cdot - \boldsymbol{y}) + \frac{t^2}{3} ⨍_{B_{|t|}(0)} d^3y \Delta g(\cdot - \boldsymbol{y}), \qquad t \neq 0.\end{aligned}$$

Next, we show, $\lim_{t\to 0} \partial_t K_t * g = g$.

$$\begin{aligned}(\partial_t K_t * g)(\boldsymbol{x}) &= \fint_{\partial B_{|t|}(0)} d\sigma(y) g(\boldsymbol{x}-\boldsymbol{y}) + \frac{t^2}{3}\fint_{B_{|t|}(0)} d^3y \Delta g(\boldsymbol{x}-\boldsymbol{y}) \\ &= \fint_{\partial B_1(0)} d\sigma(y) g(\boldsymbol{x}-|t|\boldsymbol{y}) \\ &\quad + \frac{3t^2}{4\pi|t|^3\cdot 3}\int_0^{|t|} dr\, r^2 \int_{\partial B_1(0)} d\sigma(y)\Delta_x g(\boldsymbol{x}-r\boldsymbol{y}),\end{aligned}$$

where for the first summand we have

$$\lim_{t\to 0}\fint_{\partial B_1(0)} d\sigma(y) g(\boldsymbol{x}-|t|\boldsymbol{y}) = \fint_{\partial B_1(0)} d\sigma(y) g(\boldsymbol{x}) = g(\boldsymbol{x})$$

and the second summand can be estimated by

$$\begin{aligned}&\left|\frac{1}{4\pi|t|}\int_0^{|t|} dr\, r^2 \int_{\partial B_1(0)} d\sigma(y)\Delta_x g(\boldsymbol{x}-r\boldsymbol{y})\right| \\ &\le \frac{|t|^3}{12\pi|t|}\int_{\partial B_1(0)} d\sigma(y) \max_{\boldsymbol{y}\in B_{|t|}(\boldsymbol{x})}|\Delta g(\boldsymbol{y})| \\ &= \frac{t^2}{3}\max_{\boldsymbol{y}\in B_{|t|}(\boldsymbol{x})}|\Delta g(\boldsymbol{y})| \to 0 \qquad \text{for } t\to 0,\end{aligned}$$

because $\max_{\boldsymbol{y}\in B_{|t|}(\boldsymbol{x})}|\Delta g(\boldsymbol{y})|$ is finite by the condition $g\in C^{2+n}(\mathbb{R}^3,\mathbb{R})$.

It remains to show $\lim_{t\to 0}\partial_t K_t * h_t = h_0$. Let $\boldsymbol{x}\in\mathbb{R}^3$. Similar to the argument in (i) we estimate

$$\begin{aligned}&\lim_{t\to 0}|(\partial_t K_t * h_t)(\boldsymbol{x}) - h_0(\boldsymbol{x})| \\ &\le \lim_{t\to 0}|(\partial_t K_t * (h_t-h_0))(\boldsymbol{x})| + \lim_{t\to 0}|(\partial_t K_t * h_0)(\boldsymbol{x}) - h_0(\boldsymbol{x})| \\ &= \lim_{t\to 0}|(\partial_t K_t * (h_t-h_0))(\boldsymbol{x})| + 0 \\ &\le \lim_{t\to 0}\sup_{y\in B_\delta(\boldsymbol{x})}\left(|h_t(\boldsymbol{y})-h_0(\boldsymbol{y})| + \frac{C^2}{3}|\Delta h_t(\boldsymbol{y}) - \Delta h_0(\boldsymbol{y})|\right) \\ &= 0,\end{aligned}$$

where the last inequality holds for $|t|$ small enough and some constant $C>|t|$ and the last equality is due to the continuity of the mapping $t\mapsto h_t(\boldsymbol{x})$.

(iii) From (i) and (ii), for all $n\in\mathbb{N}$, one can compute the nth partial derivative of $\mathbb{R}\setminus\{0\}\times\mathbb{R}^3\to\mathbb{R}$, $(t,\boldsymbol{x})\mapsto(K_t * g)(\boldsymbol{x})$ inductively making use of Lebesgue's Differentiation Theorem (cf. Appendix A.4) and finds that for any $g\in C^\infty(\mathbb{R}^3,\mathbb{R})$ the mapping is in $C^\infty(\mathbb{R}\setminus\{0\}\times\mathbb{R}^3,\mathbb{R})$.

Moreover, the limits $\lim_{t\to 0} K_t * g$ and $\lim_{t\to 0} \partial_t K_t * g$ exist by (i) and (ii) and for all $n \geq 2$ we obtain

$$\lim_{t\to 0} \partial_t^n K_t * g = \begin{cases} 0 & n \text{ even} \\ \Delta^{\frac{n-1}{2}} g & n \text{ odd} \end{cases}, \tag{A.8}$$

by induction; i.e., the limit from the right and from the left coincides for all $n \in \mathbb{N}_0$ and therefore, the mapping can be continuously extended at $t = 0$.

(iv) Trivial by the fact that K_t fulfills the homogeneous wave equation $\square K_t = 0$.

□

The next lemma gives a solution formula for the homogeneous wave equation for given initial values; cf. Corollary 4.13. in [12] and [18].

Lemma A.1.2. *Consider the homogeneous wave equation $\square A_t = 0$ to the initial values $A_0 := A_t|_{t=0}$ and $\dot{A}_0 := \partial_t A_t|_{t=0}$. Then $(A_t)_{t\in\mathbb{R}}$ given by*

$$A_t = \partial_t K_t * A_0 + K_t * \dot{A}_0 \tag{A.9}$$

represents a solution to the given initial value problem.

Proof. Given Lemma A.1.1 (Properties of K_t) the assertion can easily be verified:

$$\begin{aligned} \square(\partial_t K_t * A_0 + K_t * \dot{A}_0) &= \partial_t^3 K_t * A_0 + \partial_t^2 K_t * \dot{A}_0 - \partial_t K_t * \Delta A_0 - K_t * \Delta \dot{A}_0 \\ &= \partial_t K_t * \Delta A_0 + K_t * \Delta \dot{A}_0 - \partial_t K_t * \Delta A_0 - K_t * \Delta \dot{A}_0 \\ &= 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \lim_{t\to 0}(\partial_t K_t * A_0 + K_t * \dot{A}_0) &= A_0 + 0 = A_0 \\ \lim_{t\to 0} \partial_t(\partial_t K_t * A_0 + K_t * \dot{A}_0) &= \lim_{t\to 0} K_t * \Delta A_0 + \partial_t K_t * \dot{A}_0 = \dot{A}_0. \end{aligned}$$

□

Derivation of the solution formula for the Maxwell fields We consider one charge and its electromagnetic field. The trajectory $(\boldsymbol{q}, \boldsymbol{p}) \in \mathcal{T}^2(\mathbb{R})$ is assumed to be known. Furthermore, let $\rho \in \mathcal{D}$, $\boldsymbol{x} \in \mathbb{R}^3$. Then ρ can be seen as a test function or a smeared out rigid charge distribution, i.e., either $\rho(\boldsymbol{x} - \boldsymbol{q}_t) = \delta(\cdot - \boldsymbol{q}_t)(\rho_{\boldsymbol{x}})$ is the point charge distribution at time t applied to the shifted test function $\rho_{\boldsymbol{x}}$, or it represents the charge distribution at time t evaluated at $\boldsymbol{x}$. For better readability we use the abbreviations $\rho_t := \rho(\cdot - \boldsymbol{q}_t)$ for the charge distribution at time t and $\boldsymbol{j}_t := \boldsymbol{v}_t \rho(\cdot - \boldsymbol{q}_t)$ for the charge current at time t. Then, the Maxwell equations and constraints read

$$\begin{aligned} \partial_t \boldsymbol{E}_t &= \nabla \wedge \boldsymbol{B}_t - 4\pi \boldsymbol{j}_t \\ \partial_t \boldsymbol{B}_t &= -\nabla \wedge \boldsymbol{E}_t \\ \nabla \cdot \boldsymbol{E}_t &= 4\pi \rho_t \\ \nabla \cdot \boldsymbol{B}_t &= 0 \end{aligned} \tag{A.10}$$

Now, we transform Maxwell's equations taking into account the constraints into a inhomogeneous wave equation. Therefore, let $\Box = \partial_t^2 - \Delta$ denote the d'Alembert operator. Then, (A.10) implies

$$\begin{aligned}
\Box \begin{pmatrix} \boldsymbol{E}_t \\ \boldsymbol{B}_t \end{pmatrix} &= \begin{pmatrix} \partial_t(\nabla \wedge \boldsymbol{B}_t) - 4\pi\partial_t \boldsymbol{j}_t - \Delta \boldsymbol{E}_t \\ -\partial_t(\nabla \wedge \boldsymbol{E}_t) - \Delta \boldsymbol{B}_t \end{pmatrix} \\
&= \begin{pmatrix} -\nabla \wedge \nabla \wedge \boldsymbol{E}_t - 4\pi\partial_t \boldsymbol{j}_t - \nabla\nabla \cdot \boldsymbol{E}_t + \nabla \wedge \nabla \wedge \boldsymbol{E}_t \\ \nabla \wedge (-\nabla \wedge \boldsymbol{B}_t + 4\pi \boldsymbol{j}_t) - \nabla\nabla \cdot \boldsymbol{B}_t + \nabla \wedge \nabla \wedge \boldsymbol{B}_t \end{pmatrix} \\
&= 4\pi \begin{pmatrix} -\nabla \rho_t - \partial_t \boldsymbol{j}_t \\ \nabla \wedge \boldsymbol{j}_t \end{pmatrix}.
\end{aligned}$$

Thus, any solution to the initial value problem

$$\begin{aligned}
&\Box \begin{pmatrix} \boldsymbol{E}_t \\ \boldsymbol{B}_t \end{pmatrix} = 4\pi \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_t \\ \boldsymbol{j}_t \end{pmatrix} \\
&\begin{pmatrix} \boldsymbol{E}_0 \\ \boldsymbol{B}_0 \end{pmatrix} := \begin{pmatrix} \boldsymbol{E}_t \\ \boldsymbol{B}_t \end{pmatrix} |_{t=0} \\
&\partial_t \begin{pmatrix} \boldsymbol{E}_0 \\ \boldsymbol{B}_0 \end{pmatrix} := \partial_t \begin{pmatrix} \boldsymbol{E}_t \\ \boldsymbol{B}_t \end{pmatrix} |_{t=0} = \begin{pmatrix} \nabla \wedge \boldsymbol{B}_0 - 4\pi \boldsymbol{j}_0 \\ -\nabla \wedge \boldsymbol{E}_0 \end{pmatrix}
\end{aligned} \tag{A.11}$$

is a solution to (A.10) for initial value $\begin{pmatrix} \boldsymbol{E}_0 \\ \boldsymbol{B}_0 \end{pmatrix}$.

Since wave equations are well understood by methods of partial differential equations, we find a solution representation formula to (A.11). In the proof of Theorem 4.2.1 (Explicit Maxwell solutions), we show that this solution is indeed the unique solution to (1.2)-(1.3) in the weak sense.

Lemma A.1.3 (Kirchhoff's formula)**.** *A solution to the initial value problem* (A.11) *is given by*

$$\boldsymbol{F}_t = \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{F}^0 - 4\pi K_t * \begin{pmatrix} \boldsymbol{j}_0 \\ 0 \end{pmatrix} + 4\pi \int_0^t ds K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix}. \tag{A.12}$$

After a integration by parts in the time variable s formula (A.12) *can be transformed into*

$$\boldsymbol{F}_t = \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{F}_0 + 4\pi \int_0^t ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix}. \tag{A.13}$$

Proof. We abbreviate the inhomogeneity in the wave equation (A.11) by

$$I_t := 4\pi \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_t \\ \boldsymbol{j}_t \end{pmatrix} \tag{A.14}$$

and a solution to the corresponding homogeneous wave equation with initial value $\boldsymbol{F}_0$ by $\boldsymbol{F}_t^h$.

For $t > 0$ a solution to (A.11) is given by

$$\boldsymbol{F}_t = \boldsymbol{F}_t^h + \int_0^\infty ds K_{t-s}^- * I_s. \tag{A.15}$$

This is true because

$$\begin{aligned}\Box \boldsymbol{F}_t &= 0 + \int_0^\infty ds \Box_t K^-_{t-s} * I_s \\ &= \int_0^\infty ds \int d^3y \Box_{t-s} K^-_{t-s}(\cdot - \boldsymbol{y}) I_s(\boldsymbol{y}) \\ &= \int_0^\infty ds \int d^3y \delta(t-s)\delta(\cdot - \boldsymbol{y}) I_s(\boldsymbol{y}) \\ &= I_t,\end{aligned}$$

where the third equality is due to Definition A.1.1 (Propagator of the d'Alembert operator), and the initial value is taken since

$$\begin{aligned}\lim_{t\to 0} \boldsymbol{F}_t &= \lim_{t\to 0} \boldsymbol{F}^h_t + \int_0^\infty ds K^-_{t-s} * I_s \\ &= \boldsymbol{F}_0 + \int_0^\infty ds K^-_{-s} * I_s \\ &= \boldsymbol{F}_0,\end{aligned}$$

where the the second summand vanishes since K^-_{-s} is 0 for $s > 0$. According to the solution formula for homogeneous wave equations in Lemma A.1.2, $\boldsymbol{F}^h_t = \partial_t K_t * \boldsymbol{F}_0 + K_t * \dot{\boldsymbol{F}}_0$. Plugging this into (A.15) we obtain

$$\begin{aligned}\boldsymbol{F}_t &= \partial_t K_t * \boldsymbol{F}_0 + K_t * \begin{pmatrix} \nabla \wedge \boldsymbol{B}_0 - 4\pi \boldsymbol{j}_0 \\ -\nabla \wedge \boldsymbol{E}_0 \end{pmatrix} + \int_0^\infty ds K^-_{t-s} * 4\pi \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix} \\ &= \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{F}_0 - 4\pi K_t * \begin{pmatrix} \boldsymbol{j}_0 \\ 0 \end{pmatrix} + 4\pi \int_0^t ds K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix} \\ &= \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \boldsymbol{F}_0 + 4\pi \int_0^t ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix}.\end{aligned}$$

For $t < 0$ the proof works analogously, only replacing (A.15) by

$$\boldsymbol{F}_t = \boldsymbol{F}^h_t + \int_{-\infty}^0 ds K^+_{t-s} * I_s. \tag{A.16}$$

□

A.2 The Maxwell constraints

Lemma A.2.1. *(Maxwell constraints) Let $\Lambda \subset \mathbb{R}$.*

(i) Let $\boldsymbol{f}_0^h \in \mathcal{F}_{\text{hom}}$ and $\boldsymbol{f}^h : \Lambda \to \mathcal{F}$ be a homogeneous Maxwell solution on Λ with $\boldsymbol{f}_t^h|_{t=0} = \boldsymbol{f}_0^h$. Then, for all $t \in \Lambda$ it holds $\boldsymbol{f}_t^h \in \mathcal{F}_{\text{hom}}$.

(ii) Let $\boldsymbol{f}_0 \in \mathcal{F}_{\boldsymbol{q}_{t=0}}$ and $\boldsymbol{f} : \Lambda \to \mathcal{F}$ be a Maxwell solution for $(\boldsymbol{q}, \boldsymbol{p})$ on Λ with $\boldsymbol{f}_t|_{t=0} = \boldsymbol{f}_0$. Then, for all $t \in \Lambda$ it holds $\boldsymbol{f}_t \in \mathcal{F}_{\boldsymbol{q}_t}$.

Proof. (i) It suffices to check the Maxwell constraints (3.2) for a $\rho \in \mathcal{D}$ and $\boldsymbol{x} \in \mathbb{R}^3$. Since $\boldsymbol{f}^h$ is a homogeneous Maxwell solution it holds

$$\begin{aligned}
\partial_t \begin{pmatrix} \nabla_{\boldsymbol{x}} \cdot & 0 \\ 0 & \nabla_{\boldsymbol{x}} \cdot \end{pmatrix} \boldsymbol{f}_t^h(\rho_{\boldsymbol{x}}) &= \begin{pmatrix} \nabla_{\boldsymbol{x}} \cdot & 0 \\ 0 & \nabla_{\boldsymbol{x}} \cdot \end{pmatrix} \partial_t \boldsymbol{f}_t^h(\rho_{\boldsymbol{x}}) \\
&= \begin{pmatrix} \nabla \cdot & 0 \\ 0 & \nabla \cdot \end{pmatrix} \begin{pmatrix} 0 & \nabla_{\boldsymbol{x}} \wedge \\ -\nabla_{\boldsymbol{x}} \wedge & 0 \end{pmatrix} \boldsymbol{f}_t^h(\rho_{\boldsymbol{x}}) \\
&= \begin{pmatrix} 0 & \nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{x}} \wedge \\ -\nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{x}} \wedge & 0 \end{pmatrix} \boldsymbol{f}_t^h(\rho_{\boldsymbol{x}}) = 0,
\end{aligned}$$

and, by the condition it holds $\begin{pmatrix} \nabla_{\boldsymbol{x}} \cdot & 0 \\ 0 & \nabla_{\boldsymbol{x}} \cdot \end{pmatrix} \boldsymbol{f}_0^h(\rho_{\boldsymbol{x}}) = 0$. Therefore,

$$\begin{pmatrix} \nabla_{\boldsymbol{x}} \cdot & 0 \\ 0 & \nabla_{\boldsymbol{x}} \cdot \end{pmatrix} \boldsymbol{f}_t^h(\rho_{\boldsymbol{x}}) = 0 \tag{A.17}$$

for any time $t \in \Lambda$.

(ii) Analogously, we can compute

$$\begin{aligned}
&\partial_t \begin{pmatrix} \nabla_{\boldsymbol{x}} \cdot & 0 \\ 0 & \nabla_{\boldsymbol{x}} \cdot \end{pmatrix} \boldsymbol{f}_t(\rho_{\boldsymbol{x}}) = \begin{pmatrix} \nabla_{\boldsymbol{x}} \cdot & 0 \\ 0 & \nabla_{\boldsymbol{x}} \cdot \end{pmatrix} \partial_t \boldsymbol{f}_t(\rho_{\boldsymbol{x}}) \\
&= \begin{pmatrix} \nabla_{\boldsymbol{x}} \cdot & 0 \\ 0 & \nabla_{\boldsymbol{x}} \cdot \end{pmatrix} \left[\begin{pmatrix} 0 & \nabla_{\boldsymbol{x}} \wedge \\ -\nabla_{\boldsymbol{x}} \wedge & 0 \end{pmatrix} \boldsymbol{f}_t(\rho_{\boldsymbol{x}}) - \begin{pmatrix} 4\pi \boldsymbol{v}(\boldsymbol{p}_t) \rho(\boldsymbol{x} - \boldsymbol{q}_t) \\ 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} 0 & \nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{x}} \wedge \\ -\nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{x}} \wedge & 0 \end{pmatrix} \boldsymbol{f}_t(\rho_{\boldsymbol{x}}) - \begin{pmatrix} \nabla_{\boldsymbol{x}} \cdot & 0 \\ 0 & \nabla_{\boldsymbol{x}} \cdot \end{pmatrix} \begin{pmatrix} 4\pi \boldsymbol{v}(\boldsymbol{p}_t) \rho(\boldsymbol{x} - \boldsymbol{q}_t) \\ 0 \end{pmatrix} \\
&= 0 - \begin{pmatrix} 4\pi \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}(\boldsymbol{p}_t) \rho(\boldsymbol{x} - \boldsymbol{q}_t) \\ 0 \end{pmatrix} \\
&= - \begin{pmatrix} 4\pi \boldsymbol{v}(\boldsymbol{p}_t) \cdot \nabla_{\boldsymbol{x}} \rho(\boldsymbol{x} - \boldsymbol{q}_t) \\ 0 \end{pmatrix},
\end{aligned}$$

as $\boldsymbol{f}$ is a Maxwell solution for $(\boldsymbol{q}, \boldsymbol{p})$. On the other hand, if we compute the partial time derivative of the right hand side of (3.3) we obtain

$$\begin{aligned}
\partial_t \begin{pmatrix} 4\pi \rho(\boldsymbol{x} - \boldsymbol{q}_t) \\ 0 \end{pmatrix} &= \begin{pmatrix} 4\pi \nabla_{\boldsymbol{x}} \rho(\boldsymbol{x} - \boldsymbol{q}_t) \cdot (-\boldsymbol{v}(\boldsymbol{p}_t)) \\ 0 \end{pmatrix} \\
&= - \begin{pmatrix} 4\pi \boldsymbol{v}(\boldsymbol{p}_t) \cdot \nabla_{\boldsymbol{x}} \rho(\boldsymbol{x} - \boldsymbol{q}_t) \\ 0 \end{pmatrix}.
\end{aligned}$$

By condition, for $t = 0$ we have

$$\begin{pmatrix} \nabla_{\boldsymbol{x}} \cdot & 0 \\ 0 & \nabla_{\boldsymbol{x}} \cdot \end{pmatrix} \boldsymbol{f}_0(\rho_{\boldsymbol{x}}) = \begin{pmatrix} 4\pi\rho(\boldsymbol{x} - \boldsymbol{q}_0) \\ 0 \end{pmatrix}. \tag{A.18}$$

Therefore the time derivative for all $t \in \Lambda$ of the right and left hand side of (3.3) coincide, as well as their values at time $t = 0$, and thus, Maxwell's constraints hold for any time $t \in \Lambda$.

□

A.3 Distributions

Lemma A.3.1 (Distributions)**.** *Let $\rho \in \mathcal{D}$, $(\rho^{(n)})_{n\in\mathbb{N}}$ be a null sequence in $\mathcal{D}$, and $\boldsymbol{f} \in \mathcal{F}$. Moreover, $\boldsymbol{x} \in \mathbb{R}^3$ and $\alpha \in \mathbb{N}_0^3$. Then:*

(i) $(D^\alpha_{\boldsymbol{x}} \rho^{(n)}_{\boldsymbol{x}})_{n\in\mathbb{N}}$ is again a null sequence.

(ii) The function $\boldsymbol{x} \mapsto \boldsymbol{f}(\rho_{\boldsymbol{x}})$ is in $C^\infty(\mathbb{R}^3, \mathbb{R}^6)$ and $(D^\alpha_{\boldsymbol{x}} \boldsymbol{f})(\rho_{\boldsymbol{x}}) := D^\alpha_{\boldsymbol{x}} \boldsymbol{f}(\rho_{\boldsymbol{x}}) = \boldsymbol{f}(D^\alpha_{\boldsymbol{x}} \rho_{\boldsymbol{x}})$.

(iii) For compactly supported $\psi \in L^1(\mathbb{R}^3)$ it holds

$$\int d^3x \psi(\boldsymbol{x}) \boldsymbol{f}(\rho_{\boldsymbol{x}}) = \boldsymbol{f}(\psi * \rho_0), \tag{A.19}$$

where $\rho_0 = \rho(0 - \cdot) = \rho(- \cdot)$.

In the proof of item (ii) and (iii) we follow the proof of Lemma 6.8. (Interchanging convolutions with distributions) in [31].

Proof. (i) By condition $(\rho^{(n)})_{n\in\mathbb{N}}$ is a null sequence in $\mathcal{D}$, i.e. for all $\alpha \in \mathbb{N}_0^3$

$$\lim_{n\to\infty} \sup_{\boldsymbol{y}\in\mathbb{R}^3} |D^\alpha_{\boldsymbol{y}} \rho^{(n)}(\boldsymbol{y})| = 0. \tag{A.20}$$

For the sequence $(D^\alpha_{\boldsymbol{x}} \rho^{(n)}_{\boldsymbol{x}})_{n\in\mathbb{N}}$ and all $\gamma \in \mathbb{N}_0^3$ we get

$$\begin{aligned} \lim_{n\to\infty} \sup_{\boldsymbol{y}\in\mathbb{R}^3} |D^\gamma_{\boldsymbol{y}} D^\alpha_{\boldsymbol{x}} \rho^{(n)}_{\boldsymbol{x}}(\boldsymbol{y})| &= \lim_{n\to\infty} \sup_{\boldsymbol{y}\in\mathbb{R}^3} |D^{\gamma+\alpha}_{\boldsymbol{y}} \rho^{(n)}(\boldsymbol{x} - \boldsymbol{y})| \\ &= \lim_{n\to\infty} \sup_{\boldsymbol{y}\in\mathbb{R}^3} |D^{\gamma+\alpha}_{\boldsymbol{y}} \rho^{(n)}(\boldsymbol{y})| \\ &= 0. \end{aligned}$$

Therefore, $(D^\alpha_{\boldsymbol{x}} \rho^{(n)}_{\boldsymbol{x}})_{n\in\mathbb{N}}$ is again a null sequence.

(ii) Let $\boldsymbol{x} \in \mathbb{R}^3$, $\rho \in \mathcal{D}$, and $\epsilon > 0$. For all $|\boldsymbol{z}| > \epsilon$ there exists a $C < \infty$ such that

$$|\rho_{\boldsymbol{x}}(\boldsymbol{y}) - \rho_{\boldsymbol{x}+\boldsymbol{z}}(\boldsymbol{y})| = |\rho(\boldsymbol{x} - \boldsymbol{y}) - \rho(\boldsymbol{x} + \boldsymbol{z} - \boldsymbol{y})| < C\epsilon, \tag{A.21}$$

because ρ is compactly supported, has continuous derivatives, and therefore, the derivatives are uniformly continuous. Analogously, for each $\alpha \in \mathbb{N}_0^3$ there exists a $C_\alpha < \infty$ such that

$$\left|D^\alpha_{\boldsymbol{y}} \rho_{\boldsymbol{x}}(\boldsymbol{y}) - D^\alpha_{\boldsymbol{y}} \rho_{\boldsymbol{x}+\boldsymbol{z}}(\boldsymbol{y})\right| = \left|D^\alpha_{\boldsymbol{y}} \rho(\boldsymbol{x} - \boldsymbol{y}) - D^\alpha_{\boldsymbol{y}} \rho(\boldsymbol{x} + \boldsymbol{z} - \boldsymbol{y})\right| < C_\alpha \epsilon. \tag{A.22}$$

Therefore,

$$\lim_{\boldsymbol{z}\to 0}\sup_{\boldsymbol{y}\in\mathbb{R}^3}\left|D^\alpha_{\boldsymbol{y}}\rho_{\boldsymbol{x}}(\boldsymbol{y})-D^\alpha_{\boldsymbol{y}}\rho_{\boldsymbol{x}+\boldsymbol{z}}(\boldsymbol{y})\right|=0 \tag{A.23}$$

and by definition $\rho_{\boldsymbol{x}+\boldsymbol{z}}\to\rho_{\boldsymbol{x}}$ in $\mathcal{D}$ for $\boldsymbol{z}\to 0$. This implies $\boldsymbol{f}(\rho_{\boldsymbol{x}+\boldsymbol{z}})\to\boldsymbol{f}(\rho_{\boldsymbol{x}})$ for $\boldsymbol{z}\to 0$ because $\boldsymbol{f}\in\mathcal{F}$ and $\boldsymbol{f}$ is continuous. Therefore, $\boldsymbol{x}\mapsto\boldsymbol{f}(\rho_{\boldsymbol{x}})$ is continuous on $\mathbb{R}^3$.

Moreover, for all $\boldsymbol{x}\in\mathbb{R}^3$, $\rho\in\mathcal{D}$, $\epsilon>0$, $|\boldsymbol{z}|<\epsilon$ there exists a $C'<\infty$ such that for the directional derivative along $\boldsymbol{z}$ we obtain

$$\left|\frac{\rho(\boldsymbol{x}+t\boldsymbol{z}-\boldsymbol{y})-\rho(\boldsymbol{x}-\boldsymbol{y})}{t}-\nabla\rho(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{z}\right|<C't\epsilon. \tag{A.24}$$

As above, this implies that for all $|\alpha|=1$ it holds $D^\alpha_{\boldsymbol{x}}\rho_{\boldsymbol{x}+\boldsymbol{z}}\to D^\alpha_{\boldsymbol{x}}\rho_{\boldsymbol{x}}$ in $\mathcal{D}$. And thus, by continuity of $\boldsymbol{f}$ it holds, $\boldsymbol{f}(D^\alpha_{\boldsymbol{x}}\rho_{\boldsymbol{x}+\boldsymbol{z}})\to\boldsymbol{f}(D^\alpha_{\boldsymbol{x}}\rho_{\boldsymbol{x}})$, which means that $\boldsymbol{x}\mapsto\boldsymbol{f}(D^\alpha_{\boldsymbol{x}}\rho_{\boldsymbol{x}})$ is continuous. For $|\alpha|>1$ the argument can be shown by induction using similar arguments which we omit here.

(iii) As ψ can be approximated by a C^∞_c-function, cf. Theorem 2.16 in [31]), we prove the assertion for $\psi\in\mathcal{D}$.

Then, $\boldsymbol{x}\mapsto\psi(\boldsymbol{x})\boldsymbol{f}(\rho_{\boldsymbol{x}})$ is a product of two functions in $\mathcal{D}$. Thus, $\int d^3x\psi(\boldsymbol{x})\boldsymbol{f}(\rho_{\boldsymbol{x}})$ can be approximated by a Riemann sum

$$\sum_{i=1}^{n}(\boldsymbol{x}_{i+1}-\boldsymbol{x}_i)\psi(\boldsymbol{x}_i)\boldsymbol{f}(\rho_{\boldsymbol{x}_i}). \tag{A.25}$$

Likewise, for any $\alpha\in\mathbb{N}_0^3$,

$$D^\alpha(\psi*\rho_0)=\int d^3x\psi(\boldsymbol{x})D^\alpha\rho(-(\cdot-\boldsymbol{x}))=\int d^3x\psi(\boldsymbol{x})D^\alpha\rho_{\boldsymbol{x}} \tag{A.26}$$

can be approximated by

$$\sum_{i=1}^{n}(\boldsymbol{x}_{i+1}-\boldsymbol{x}_i)\psi(\boldsymbol{x}_i)D^\alpha\rho_{\boldsymbol{x}_i}, \tag{A.27}$$

which implies that $\sum_{i=1}^n(\boldsymbol{x}_{i+1}-\boldsymbol{x}_i)\psi(\boldsymbol{x}_i)\rho_{\boldsymbol{x}_i}\to\psi*\rho_0$ as $n\to\infty$ in $\mathcal{D}$. Setting $\Delta^{(n)}:=(\boldsymbol{x}_{i+1}-\boldsymbol{x}_i)$ it holds

$$\begin{aligned}\int d^3x\psi(\boldsymbol{x})\boldsymbol{f}(\rho_{\boldsymbol{x}})&=\lim_{n\to\infty}\Delta^{(n)}\sum_{i=1}^{n}\psi(\boldsymbol{x}_i)\boldsymbol{f}(\rho_{\boldsymbol{x}_i})\\&=\lim_{n\to\infty}\boldsymbol{f}\left(\Delta^{(n)}\sum_{i=1}^{n}\psi(\boldsymbol{x}_i)\rho_{\boldsymbol{x}_i}\right)\\&=\boldsymbol{f}\left(\lim_{n\to\infty}\Delta^{(n)}\sum_{i=1}^{n}\psi(\boldsymbol{x}_i)\rho_{\boldsymbol{x}_i}\right)\\&=\boldsymbol{f}(\psi*\rho_0),\end{aligned}$$

where the second equality holds by linearity of $\boldsymbol{f}$ and the third equality holds by continuity of $\boldsymbol{f}$ w.r.t. $\mathcal{D}$.

□

A.4 Calculus

In this section we collect the calculation rules and theorems from (vector) analysis that are frequently used throughout the computations in this work (cf. [22, 18, 31]).

Rotation and cross product For $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$ it holds

$$\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$$

and therefore, for each $\boldsymbol{v} \in \mathbb{R}^3$ it follows

$$\nabla \wedge (\nabla \wedge \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \Delta \boldsymbol{v}.$$

With ϵ_{ijk} denoting the Levi-Cevita-symbol and δ_{ij} the Cronecker-delta for all $k = 1, 2, 3$ it holds

$$(\boldsymbol{a} \wedge \boldsymbol{b})_k = \sum_{i,j} \epsilon_{ijk} a_i b_j$$

since

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}.$$

Moreover, the rotation of a gradient field is zero

$$\nabla \wedge (\nabla f) = 0$$

and the divergence of a rotation field is zero

$$\nabla \cdot (\nabla \wedge \boldsymbol{v}) = 0.$$

Gauss-Green theorem Let $U \subset \mathbb{R}^3$ be bounded. Suppose $\boldsymbol{F} \in C^1(\bar{U}, \mathbb{R}^3)$. Then

$$\int_U d^3y \nabla \cdot \boldsymbol{F}(\boldsymbol{y}) = \int_{\partial U} d\sigma(y) \boldsymbol{n}(\boldsymbol{y}) \cdot \boldsymbol{F}(\boldsymbol{y}), \tag{A.28}$$

where $\boldsymbol{n}(\boldsymbol{y})$ denotes the outward pointing unit normal field of the boundary ∂U.

In particular this holds for any $\boldsymbol{F} = f\boldsymbol{c}$, where f is a scalar function and $\boldsymbol{c}$ a constant vector, and hence,

$$\int_U d^3y \nabla f(\boldsymbol{y}) = \int_{\partial U} d\sigma(y) \boldsymbol{n}(\boldsymbol{y}) f(\boldsymbol{y}) \tag{A.29}$$

or respectively for single components $i = 1, 2, 3$

$$\int_U d^3y \partial_{y_i} f(\boldsymbol{y}) = \int_{\partial U} d\sigma(y) n_i(\boldsymbol{y}) f(\boldsymbol{y}). \tag{A.30}$$

Note that the theorem is also known as Divergence Theorem.

Integration by parts formula Let $U \subset \mathbb{R}^3$ be bounded. Further, let $f, g \in C^1(U, \mathbb{R})$. Then

$$\int_U d^3x \partial_{x_i} f(\boldsymbol{x}) g(\boldsymbol{x}) = -\int_U d^3x f(\boldsymbol{x}) \partial_{x_i} g(\boldsymbol{x}) + \int_U d^3x \partial_{x_i} [f(\boldsymbol{x}) g(\boldsymbol{x})] \tag{A.31}$$

Transformation theorem Let $\Phi : \Omega \to \mathbb{R}^3$ be a diffeomorohism. Then f is integrable on the domain $\Phi(\Omega))$ if the function $|\det(D\Phi)(\boldsymbol{y})|f(\Phi(\boldsymbol{y}))$ is integrable on Ω and it holds

$$\int_{\Phi(\Omega)} d^3y f(\boldsymbol{y}) = \int_{\Omega} d^3y |\det(D\Phi)(\boldsymbol{y})| f(\Phi(\boldsymbol{y}))$$

Equivalently it holds

$$\int_{\Phi(\Omega)} d^3y \frac{1}{|\det D\Phi(\Phi^{-1}(\boldsymbol{y}))|} f(\boldsymbol{y}) = \int_{\Omega} d^3y f(\Phi(\boldsymbol{y})).$$

Special cases which are frequently used in our calculations are

$$\int_{B_t(0)} d^3y f(\boldsymbol{y}) = \int_0^t dr \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\varphi f\left(r \begin{pmatrix} cos\theta sin\varphi \\ cos\theta cos\varphi \\ sin\theta \end{pmatrix} \right) r^2 cos\theta$$

and

$$\frac{1}{4\pi t^2} \int_{\partial B_t(0)} d\sigma(y) f(\boldsymbol{y}) = \frac{1}{4\pi} \int_{\partial B_1(0)} d\sigma(y) f(t\boldsymbol{y})$$

or in normalized notation

$$⨍_{\partial B_t(0)} d\sigma(y) f(\boldsymbol{y}) = ⨍_{\partial B_1(0)} d\sigma(y) f(t\boldsymbol{y}).$$

Lebesgue's differentiation theorem Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally summable w.r.t. some $1 \le p < \infty$, i.e. $f \in L^p_{loc}(\mathbb{R}^n)$. Then, for almost every $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$⨍_{B_\epsilon(\boldsymbol{x})} d^n y \, |f(\boldsymbol{y}) - f(\boldsymbol{x})| \to 0 \text{ as } \epsilon \to 0. \tag{A.32}$$

Fundamental theorem of calculus For all $(t, \boldsymbol{x}) \in \mathbb{R}^4$ and let $s \mapsto f_s(\boldsymbol{x})$ be in $C^1([0,t], \mathbb{R})$ such that $s \mapsto \partial_s f_s(\boldsymbol{x})$ is integrable on $[0,t]$. Then,

$$f_t(\boldsymbol{x}) = f_0(\boldsymbol{x}) + \int_0^t ds \partial_s f_s(\boldsymbol{x}). \tag{A.33}$$

Note that this theorem is used in order to transform surface integrals over a set $\partial B_t(\mathbf{0})$ into volume integrals over $B_t(\mathbf{0})$.

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