Effective Evolution Equations from Quantum Mechanics

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München 2017

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Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
der Ludwig–Maximilians–Universität
München

vorgelegt von
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aus München

München, den 10.10.2017

Erstgutachter: Prof. Dr. Peter Pickl
Zweitgutachter: Prof. Dr. Volker Bach
Drittgutachter: Prof. Dr. Stefan Teufel
Tag der mündlichen Prüfung: 28.02.2018

ABSTRACT

The goal of this thesis is to provide a mathematical rigorous derivation of the Schrödinger-Klein-Gordon equations, the Maxwell-Schrödinger equations and the defocusing cubic nonlinear Schrödinger equation in two dimensions.

We study the time evolution of the Nelson model (with ultraviolet cutoff) in a limit where the number N of charged particles gets large while the coupling of each particle to the radiation field is of order $N^{-1/2}$. At time zero it is assumed that almost all charges are in the same one-body state (a Bose-Einstein condensate) and that the radiation field is close to a coherent state. We show the persistence of condensation over time and prove that the time evolution is approximately described by the Schrödinger-Klein-Gordon system of equations in the large N limit.
Subsequently, we consider the spinless Pauli-Fierz Hamiltonian which models the interaction between charged bosons and the quantized electromagnetic field. We discuss the limit previously described and prove that the time evolution is approximated by the Maxwell-Schrödinger equations. To our knowledge, this is the first rigorous result concerning a mean-field limit of the Pauli-Fierz Hamiltonian.
We then turn to the evolution of Bose-Einstein condensates in two dimensions and consider N bosons which interact by a repulsive two-body potential. The interaction is given either by $N^{-1+2\beta}V(N^{\beta}x)$ with $\beta \in \mathbb{R}_0^+$ or by $e^{2N}V(e^N x)$, for some spherical symmetric, positive and compactly supported $V \in L^{\infty}(\mathbb{R}^2, \mathbb{R})$. We prove that the dynamics is approximated by the defocusing two-dimensional cubic nonlinear Schrödinger equation in the large N limit. In case of the exponential scaling, we show that a short-scale correlation structure affects the dynamics of the condensate. This is the first rigorous derivation that considers an exponential scaling of the interaction.

All derivations rely on a method developed by Pickl in [Lett. Math. Phys. 97(2), 151–164 (2011)]. The first two results are obtained by an extension of the method to systems which interact with quantized radiation fields. The latter is derived by an appropriate adaption of the proof in three space dimensions [Rev. Math. Phys., 27, 1550005 (2015)].
The crucial insight to derive the Maxwell-Schrödinger equations is to restrict the class of many-body wave functions to a subspace of states whose energy per particle only fluctuates little around the energy functional of the Maxwell-Schrödinger system.
To derive the two-dimensional Gross-Pitaevskii equation it is essential to define a measure of condensation which properly incorporates the correlations that arise from the exponential scaling of the interaction.

This thesis is based on the preprints [54, 47].

ZUSAMMENFASSUNG

Ziel dieser Arbeit ist eine mathematisch präzise Herleitung der Schrödinger-Klein-Gordon Gleichungen, der Maxwell-Schrödinger Gleichungen und der repulsiven kubischen nichtlinearen Schrödingergleichung in zwei Dimensionen.

Wir untersuchen die zeitliche Entwicklung des Nelson Modelles mit Ultraviolett Beschränkung und studieren einen Limes, in dem die Teilchenzahl N der Ladungen wächst, während die Kopplungskonstante zum Strahlungsfeld wie $N^{-1/2}$ abfällt. Zum Anfangszeitpunkt Null setzen wir voraus, dass sich fast alle Ladungen in demselben Zustand (einem Bose-Einstein-Kondensat) befinden und das Strahlungsfeld näherungsweise durch einen kohärenten Zustand beschrieben wird. Dann zeigen wir die Stabilität des Kondensates während der zeitlichen Entwicklung und beweisen, dass diese für große Teilchenzahlen N näherungsweise durch die Schrödinger-Klein-Gordon Gleichungen beschrieben wird.
Anschließend betrachten wir den Pauli-Fierz Hamiltonian ohne Spin, der die Wechselwirkung zwischen bosonischen Ladungen und dem quantisierten elektromagnetischen Feld modelliert. Wir beweisen, dass dessen Zeitentwicklung in dem bereits beschriebenen Limes durch die Maxwell-Schrödinger Gleichungen genähert wird. Nach unserem Kenntnisstand liefert dies das erste mathematisch präzise Resultat über eine Mittlere-Feld Näherung des Pauli-Fierz Hamiltonians.
Als nächstes widmen wir uns der Dynamik von Bose-Einstein-Kondensaten in zwei Dimensionen und studieren N bosonische Teilchen, die durch ein abstoßendes Zwei-Teilchen-Potential miteinander wechselwirken. Das Wechselwirkungspotential wird entweder durch $N^{-1+2\beta}V(N^{\beta}x)$ wobei $\beta \in \mathbb{R}_0^+$ oder durch $e^{2N}V(e^N x)$ mit der Anzahl der Teilchen skaliert. Bei $V \in L^{\infty}(\mathbb{R}^2, \mathbb{R})$ handelt es sich um eine radialsymmetrische und positive Funktion mit kompaktem Träger. Wir beweisen, dass die Dynamik im Grenzwert vieler Teilchen durch die repulsive kubische nichtlineare Schrödingergleichung in zwei Dimensionen genähert wird. Hierbei treten im Falle der exponentiellen Skalierung Korrelationen mit kurzer Reichweite auf, die die Dynamik des Kondensates beeinflussen. Dies ist die erste mathematisch präzise Arbeit, die eine exponentielle Skalierung der Wechselwirkung untersucht.

Unsere Resultate basieren auf einer von Herrn Prof. Dr. P. Pickl entwickelten Methode [Lett. Math. Phys. 97(2), 151–164 (2011)]. Für die Herleitung der ersten beiden Gleichungen musste deren Anwendbarkeit auf Systeme mit quantisierten Strahlungsfeldern erweitert werden. Im Falle der nichtlinearen Schrödingergleichung war eine adäquate Änderung des Beweises in drei Dimensionen [Rev. Math. Phys., 27, 1550005 (2015)] von Nöten. Für die Herleitung der Maxwell-Schrödinger Gleichungen war es unabdingbar die Vielteilchen-Wellenfunktionen auf einen Unterraum von Zuständen zu beschränken, deren Einteilchen-Energien gering um die Energie des Maxwell-Schrödinger Systems fluktuieren. Bei der Herleitung der Gross-Pitaesvkii Gleichung war es notwendig eine Messgröße für Kondensation zu definieren, die auf passende Weise die durch die exponentielle Skalierung des Potentiales hervorgerufenen Korrelationen berücksichtigt. Diese Dissertation basiert auf den Arbeiten [54, 47].

ACKNOWLEDGEMENTS

First of all I want to thank my advisor Prof. Dr. Peter Pickl for his guidance and support over the last years. I am grateful for the fruitful discussions we had and his patience in answering all my questions. Observing his creative approach to solve problems has been a real joy and true inspiration. Moreover, I would like to thank him for proposing an intriguing Ph.D. topic which perfectly matched my background from prior studies.

Furthermore, I want to thank Prof. Dr. Detlef Dürr, whose teaching and enthusiasm about physics inspired me a lot over the years, for his constant support since my Bachelor studies, many good advices and our collaboration concerning the description of arrival time measurements.

Besides, I am grateful and feel honored that Prof. Dr. Volker Bach and Prof. Dr. Stefan Teufel agreed to co-supervise this thesis.

Special thanks belongs to Vytautas Matulevičius, who introduced me in detail to his master thesis on which part of this work is based on and provided valuable insights into the derivation of effective equations.

In addition, I am thankful to Maximilian Jeblick for a ten year long friendship that started at our first day at university. I was able to learn a lot through our various discussions and the joint project concerning the Gross-Pitaevskii equation. Moreover, I am grateful to Dr. David Mitrouskas and Phillip Grass for their great company, precious debates about physics and various explanations in stochastics. Furthermore, I would like to thank Prof. Dr. Sören Petrat for many insights, his great hospitality at Princeton university and our joint work on the Nelson model with electrons. Special thanks also go to Dr. Dirk André Deckert, Dr. Robert Grummt and Dr. Nicola Vona for many interesting debates and good advices.

It is a pleasure to thank all the great people of the group Deckert/Dürr/Pickl that made the last years unforgettable for me: Sergey Aristarkov, Christian Beck, Siddhant Das, Anne Froemel, Felix Hänle, Vera Hartenstein, Florian Hoffmann, Lukas Nickel, Andrej Nikonov, Johannes Nissen-Meyer, Markus Noeth, Martin Oelker, Paula Reichert, Aaron Schaal, Robin Schlenga, Dr. Ward Struyve and Sahand Tokasi.

Also, during the last years, I was given the opportunity to discuss my field of research with excellent researchers from other institutes. In particular, I would like to thank Dr. Marco Falconi, Prof. Dr. Marcel Griesemer, Prof. Dr. Alessandro Michelangeli, Prof. Dr. Benjamin Schlein and Prof. Dr. Robert Seiringer for their hospitality and many rewarding debates.

I am especially thankful to my lovely girlfriend Juliane Reichmann for being so wonderful, her endless support and motivating words in times when I feel lost. Without you my life would for sure not be this cheerful. Special thanks belongs to my parents Erika and Peter Leopold for their love, support, a marvelous childhood and for being there when I need you. Moreover, I would like to thank my brother Gregor Leopold for valuable debates about life and the insight that theoretical concepts must also be measured by their practical realization.

Ultimately, I am thankful to the Cusanuswerk for the financial support and the interesting seminars over the last three years.

PREFACE

This thesis consists of four chapters and one brief outlook. The first chapter provides a short introduction into mean-field equations and can be skipped by readers familiar with the subject. All chapters are self-contained and can be read in an arbitrary order. Nonetheless, chapter two may be seen as a preparation for chapter three. Some of the results presented here has been achieved with coworkers. Therefore, we begin every chapter with a short abstract and a preface which clarifies the contributions of the author. Moreover, the chapters are complemented by an appendix which provides further information or includes parts of the proof. The chapters slightly differ in notation. However, all variations made are stated explicitly. References are classified either by chronological order or by their relevance for the presented content.

CONTENTS

CHAPTER

ONE

INTRODUCTION

Abstract: This chapter introduces the subject of effective evolution equations. We briefly motivate the necessity of effective equations and explain the mean-field idea. Thereafter, the Hartree equation serves as prime example to illustrate the kind of Theorem that is proven in this work. Subsequently, we explain the method of counting and derive the Hartree equation in the simple case of bounded interaction potentials.

Contributions of the author The content of this chapter is common knowledge in the community of mathematical physics. Parts of the discussion closely follow the presentations in [10, 11, 16] and [53, 75].

Introduction

Quantum systems in the non-relativistic regime are considered to be well described by the Schrödinger equation and a suitable Hamiltonian. The Schrödinger equation however is difficult to analyze for systems with many particles. This favors the use of simpler effective theories to predict the outcomes of experiments. These involve fewer degrees of freedom, are less exact but easier to investigate. In physics, effective theories are usually derived by heuristic arguments. Beyond that, a mathematical rigorous derivation is necessary to justify the validity of the applied approximation. The present work is concerned with the rigorous derivation of effective evolution equations from many-body quantum dynamics. This started in the 1970s with the work of Hepp [44] and has since then been an active field of research in the community of mathematical physics. Hereby, great emphasis was put on the derivation of the Hartree equation [84, 31, 38, 80, 75, 67], the fermionic Hartree(-Fock) equations [8, 36, 11, 13, 14, 15, 6, 71, 70, 78] and the Gross-Pitaevskii equation[27, 28, 29, 30, 76, 77, 12, 18]. Moreover, there has been interest in the dynamics of particles which interact with quantized radiation fields [39, 33, 1, 86, 34, 35, 43, 24]. The goal of the present thesis is to derive three effective equations from quantum mechanics/non-relativistic quantum field theory:

(a) the Maxwell-Schrödinger equations from the spinless Pauli-Fierz Hamiltonian,

(b) the Schrödinger-Klein-Gordon equations from the Nelson model and

(c) the time dependent NLS- and Gross-Pitaevskii equation in two dimensions from the many-body Schrödinger equation.

(a) The central result is the derivation of the Maxwell-Schrödinger equations. The motivation of its consideration originates from fundamental physics and is explained as follows:
Most phenomena of light, for instance the diffraction of light on a prisma, are perfectly predicted by Classical Electromagnetism. Here, light is described by an electromagnetic wave which evolves according to Maxwell's equations. However, there exist effects which are not solely explained by the wave character of the electromagnetic field. The most famous is the photoelectric effect, which caused Albert Einstein [25] to postulate the existence of light quanta and led to the invention of Quantum Electrodynamics. Nowadays, Quantum Electrodynamics is considered to be the fundamental theory about the interaction of light and matter. It endows the nature of light with an interpretation in terms of particles and pictures the electromagnetic field as a collection of photons. Nonetheless, the particle character is negligible in many situations and the quantized electromagnetic field appears as a classical wave. This raises the question:

Is it possible to derive Maxwell's equations from Quantum Electrodynamics?

A short yet insufficient answer is the following: The Heisenberg equations of the field operators satisfy Maxwell's equations. Consequently, in situations where quantum effects are negligible it seems plausible to replace the second quantized fields by their classical counterparts. This answer is unsatisfactory because the Heisenberg equations are always valid (even when Classical Electromagnetism leads to wrong predictions) and do not provide a regime in which the applied approximation is justified.
A rigorous approach is to investigate the emergence of Maxwell's equations in a specific scaling regime. From physics literature it is commonly known that coherent states with a high occupation number of photons can be approximated by a classical electromagnetic field [23, Chapter III.C.4]. However, it is less clear which initial configurations of charges lead to the creation of a coherent state. In this work, we show that a condensate of charged bosons leads to the creation of a coherent state whose back reaction on the charges behaves like a classical electromagnetic field. We consider a system, described by a wave function $\Psi_{N,t} \in L^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p$, of N identical charged bosons which interact with a photon field. The time evolution of $\Psi_{N,t}$ is governed by the Schrödinger equation

$$i\partial_t \Psi_{N,t} = H_N \Psi_{N,t},$$

where

$$H_N = \sum_{j=1}^{N} \Big(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}} \Big)^2 + \frac{1}{N} \sum_{1 \leq j < k \leq N} v(x_j - x_k) + H_f$$

is the Pauli-Fierz Hamiltonian. H_f denotes the free Hamiltonian of the photon field, $\hat{\boldsymbol{A}}_\kappa$ the quantized transverse vector potential and v causes a direct interaction between the charged particles. The mean-field scaling $1/N$ in front of the interaction potential and the scaling $1/\sqrt{N}$ in front of the vector potential ensure that the kinetic and potential energy of H_N are of the same order. We are interested in initial conditions of the product form $\varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega$, where $W(\sqrt{N}\alpha_0)\Omega$ denotes a coherent state with a mean particle number $N \left|\left|\alpha_0\right|\right|^2$. Due to the interaction correlations take place and the time evolved state will no longer have an exact product structure. In general, the photon state does not need to be coherent and to behave like a classical field at later times. However, for large N we are able to show that the time evolved state can be approximated in trace norm distance of reduced density matrices by a state of product form $\varphi_t^{\otimes N} \otimes W(\sqrt{N}\alpha_t)\Omega$, where $|k|^{1/2}\alpha_t(k,\lambda) = \frac{1}{\sqrt{2}}\boldsymbol{\epsilon}_\lambda(k) \cdot (|k|\mathcal{FT}[\boldsymbol{A}](k,t) - i\mathcal{FT}[\boldsymbol{E}](k,t))^1$ and $(\varphi_t, \boldsymbol{A}(t), \boldsymbol{E}(t))$ solve the Maxwell-Schrödinger system

$$\begin{cases} i\partial_t\varphi_t(x) &= \big((-i\nabla - (\kappa * \boldsymbol{A})(x,t))^2 + (v * |\varphi_t|^2)(x)\big)\varphi_t(x), \\ \nabla \cdot \boldsymbol{A}(x,t) &= 0, \\ \partial_t \boldsymbol{A}(x,t) &= -\boldsymbol{E}(x,t), \\ \partial_t \boldsymbol{E}(x,t) &= (-\Delta \boldsymbol{A})(x,t) - \big(1 - \nabla \mathrm{div}\Delta^{-1}\big)(\kappa * \boldsymbol{j}_t)(x), \\ \boldsymbol{j}_t(x) &= 2\left(\mathrm{Im}(\varphi_t^* \nabla\varphi_t)(x) - |\varphi_t|^2(x)(\kappa * \boldsymbol{A})(x,t)\right), \end{cases}$$

with initial datum

$$\begin{cases} \varphi_0, \\ \boldsymbol{A}(x,0) = (2\pi)^{-3/2}\sum_{\lambda=1,2}\int d^3k\, \frac{1}{\sqrt{2|k|}}\boldsymbol{\epsilon}_\lambda(k)\left(e^{ikx}\alpha_0(k,\lambda) + e^{-ikx}\alpha_0^*(k,\lambda)\right), \\ \boldsymbol{E}(x,0) = (2\pi)^{-3/2}\sum_{\lambda=1,2}\int d^3k\, \sqrt{\frac{|k|}{2}}\boldsymbol{\epsilon}_\lambda(k) i\left(e^{ikx}\alpha_0(k,\lambda) - e^{-ikx}\alpha_0^*(k,\lambda)\right). \end{cases}$$

This system of equations models the coupling of a non-relativistic particle to the classical electromagnetic field. The precise result is given in Theorem 3.2.1. To our knowledge this is the first rigorous result concerning a mean-field limit of the Pauli-Fierz Hamiltonian.

(b) In order to derive the Maxwell-Schrödinger equations we extended the "method of counting" which was introduced by Pickl in [75]. Formerly, the method was used to derive mean-field dynamics for systems with a fixed particle number. Its extension to systems which interact with quantized radiation fields was achieved by the introduction of an additional functional which measures the closeness of the radiation field to a coherent state. The result can be seen as a combination of the method of counting and the coherent state approach.[2] The strategy turns out to be rather general and we hope it will be useful for the derivation of further mean-field equations. Since quantized radiation fields are not only used to describe photons but also appear in condensed matter physics for the description of quasiparticles and collective excitations this seems to be of physical interest. This observation motivated us to concisely explain the method by means of the Nelson model. The mathematical structure of the Nelson model is similar to the Pauli-Fierz Hamiltonian. However, the mean-field limit of the Pauli-Fierz Hamiltonian is much more complicated because one encounters technical problems that arise from the minimal coupling term. The Nelson model was introduced to describe the interaction of non-relativistic nucleons with a meson field. The state of the system is given by a wave function $\Psi_{N,t} \in L^2(\mathbb{R}^3) \otimes \mathcal{F}$ which evolves according to the Schrödinger equation

$$i\partial_t\Psi_{N,t} = \Big[\sum_{j=1}^N \Big(- \Delta_j + \frac{\widehat{\Phi}_\kappa(x_j)}{\sqrt{N}}\Big) + H_f\Big]\Psi_{N,t}.$$

[1] $\mathcal{FT}[\boldsymbol{A}]$ and $\mathcal{FT}[\boldsymbol{E}]$ denote the Fourier transforms of the vector potential and the electric field.

[2] The coherent state approach is a method for the derivation of mean-field dynamics which is based on a representation of the many-body system on the Fock space. A detailed introduction can be found in [16, Chapter 3].

The non-relativistic particles couple linearly to the second quantized scalar field $\widehat{\Phi}_\kappa$. We again choose initial states of the form $\varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega$ and show that the product structure is preserved during the time evolution $\Psi_{N,t} \approx \varphi_t^{\otimes N} \otimes W(\sqrt{N}\alpha_t)\Omega$. However, this time (φ_t, α_t) solve the Schrödinger-Klein-Gordon system of equations

$$\begin{cases} i\partial_t\varphi_t(x) &= H^{eff}\varphi_t(x) = \left[-\Delta + (\kappa * \Phi)(x,t)\right]\varphi_t(x), \\ i\partial_t\alpha_t(k) &= \omega(k)\alpha_t(k) + (2\pi)^{3/2}\frac{\tilde{\kappa}(k)}{\sqrt{2\omega(k)}}\mathcal{FT}\left[|\varphi_t|^2\right](k), \\ \Phi(x,t) &= \int d^3k\,(2\pi)^{-3/2}\frac{1}{\sqrt{2\omega(k)}}\left(e^{ikx}\alpha_t(k) + e^{-ikx}\alpha_t^*(k)\right), \end{cases}$$

which describes the interaction between a quantum particle and a classical scalar field. The exact statement can be found in Theorem 2.3.1. A comparison with similar results is given in Remark 2.3.2.

(c) Subsequently, we consider the evolution of Bose-Einstein condensates which interact by a repulsive two-body potential. We are interested in experimental setups in which the condensate is strongly confined in one spatial direction. Then, one approximately obtains a two dimensional system without confining potential. The dynamics of N bosons is described by a wave function $\Psi_{N,t} \in L^2(\mathbb{R}^{2N})$, which evolves according to the Schrödinger equation

$$i\partial_t\Psi_{N,t} = \Big[\sum_{j=1}^N(-\Delta_j + V_{ext}(x_j,t)) + \sum_{1\le j<k\le N} U(x_i - x_j)\Big]\Psi_{N,t}.$$

The potential U renders the interaction between the particles and V_{ext} is a time dependent external trapping potential in the unconfined direction. We are interested in a dilute gas where rare but hard collisions take place. This is achieved by strong interactions with small range. More explicitly, we consider either $U(x) = N^{-1+2\beta}W(N^\beta x)$ for any fixed $\beta > 0$ or $U = e^{2N}V(e^N x)$, for some spherical symmetric, positive and compactly supported $V \in L^\infty(\mathbb{R}^2,\mathbb{R})$. At time zero, the Bose gas is assumed to condensate in the ground state of the Hamiltonian, i.e. $\gamma^{(1)}_{\Psi_{N,0}} \to |\varphi_0\rangle\langle\varphi_0|$ for some $\varphi_0 \in L^2(\mathbb{R}^2)$ as $N \to \infty$.[3] Then, we show that the condensate is stable during the time evolution after the trapping potential has been switched off. We prove $\gamma^{(1)}_{\Psi_{N,t}} \to |\varphi_t\rangle\langle\varphi_t|$ as $N \to \infty$, where the condensate wave function evolves according to the nonlinear Schrödinger equation

$$i\partial_t\varphi_t = (-\Delta + V_{ext})\,\varphi_t + b_U|\varphi_t|^2\varphi_t.$$

The coupling constant b_U is $||V||_1$ for $U = N^{-1+2\beta}V(N^\beta x)$ for all $\beta > 0$. In case of the exponential scaling, the ground state has a short-scale correlation structure which affects the dynamics of the condensate. Then, b_U is given by 4π. The fact that the coupling parameter of the effective equation does not depend on the scattering length of the potential V is special in two dimensions and follows from the structure of the zero energy scattering state. The precise result is stated in Theorem 4.2.1. To our knowledge, this is the first rigorous derivation considering an exponential scaling of the interaction.

[3] $\gamma^{(1)}_{\Psi_{N,t}}$ denotes the one particle reduced density matrix with kernel $\gamma^{(1)}_{\Psi_{N,t}}(x,x') = \int \Psi^*_{N,t}(x,x_2,\dots,x_N)\Psi^*_N(x',x_2,\dots,x_N)d^2x_2\dots d^2x_N$.

Structure of the thesis

This thesis is organized as follows.

Chapter 1 The remaining part of this chapter is used to introduce the mean-field regime and heuristically motivate the appearance of the Hartree equation. Subsequently, we introduce the "method of counting" and derive the Hartree equation for bounded interaction potentials.

Chapter 2 We explain how the "method of counting" must be extended in order to derive mean-field limits of systems which interact with quantized radiation. As an example we look at the Nelson model and derive its mean-field limit, the Schrödinger-Klein-Gordon system of equations.

Chapter 3 In Chapter 3 we derive the Maxwell-Schrödinger equations from the Pauli-Fierz Hamiltonian. We explain how the strategy of Chapter 2 must be adapted to solve the difficulties that arise from the minimal coupling term in the Pauli-Fierz Hamiltonian. Subsequently, we provide essential preliminary estimates and prove Theorem 3.2.1.

Chapter 4 Chapter 4 is devoted to study the Gross-Pitaevskii regime in two dimensions. We introduce the short scale correlation structure described by the zero energy scattering state and motivate the exponential scaling of the interaction. Then, we derive the defocusing two-dimensional cubic nonlinear Schrödinger equation.

Chapter 5 Finally, we provide a short outlook for possible future research.

1.1 Mean-field regime

We are interested in bosonic quantum system of N particles in three dimensions that interact with each other by a two-particle interaction potential. At a given time t, the state of the system is described by a wave function $\Psi_{N,t} \in L^2_s(\mathbb{R}^{3N})$. The Hilbert space of the system

$$L^2_s(\mathbb{R}^{3N}) = \{\Psi_N \in L^2(\mathbb{R}^{3N}) : \Psi_N(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = \Psi_N(x_1, \ldots, x_N) \text{ for all } \sigma \in \mathcal{S}_N\}$$

consists of square-integrable functions that are symmetric under the interchange of their arguments. The Hamiltonian of the system is given by

$$H_N = \sum_{j=1}^{N} \big(-\Delta_{x_j} + V_{ext}(x_j, t)\big) + \sum_{i<j}^{N} v_N(x_j - x_i), \tag{1.1}$$

where V_{ext} models a time-dependent trapping potential and v_N is an interaction potential depending on the number of particles. The time evolution of the system is governed by the Schrödinger equation

$$i\partial_t \Psi_{N,t} = H_N \Psi_{N,t} \tag{1.2}$$

with initial data $\Psi_{N,0}$. The specific choice of $v_N = N^{-1}v$, where v is a function independent of N, is referred to as mean-field limit in the literature. It implies that the mean kinetic and potential energy are of the same order and ensures interesting behaviors of systems with a large number of particles. The physical meaning of the mean-field scaling is best motivated for the Coulomb potential $v = |\cdot|^{-1}$. In this case, it is possible to rescale space- and time-coordinates in a way that the interaction potential is no longer dependent on N. Choosing $y_j = N^{-1}x_j$ for all $j \in \{1, \ldots, N\}$ and $\tau = N^{-2}t$ leads to

$$\begin{aligned} &i\partial_\tau \Psi_{N,N^2\tau}(Ny_1, \ldots, Ny_N) \\ &\quad= \Big[\sum_{j=1}^{N} \big(-\Delta_{y_j} + N^2 V_{ext}(Ny_j, N^2\tau)\big) + \sum_{i<j}^{N} |y_1 - y_j|^{-1}\Big] \Psi_{N,N^2\tau}(Ny_1, \ldots, Ny_N). \end{aligned} \tag{1.3}$$

Considering a mean-field system with size of order one thus corresponds to an unscaled system with small support and high density (of order N^3). Moreover, studying the mean-field regime for times of order one allows us to investigate the time evolution of the systems without scaling up to times of order N^{-2}.
Generally, it is difficult to analyze the physical properties and determine the time evolution of systems with many particles. Nevertheless, the problem becomes feasible if one studies mean-field systems near equilibrium. For instance, if one cools bosons in an external trapping potential below a certain critical temperature it has been proven that almost all particles occupy the same quantum state; a phenomenon known as Bose-Einstein condensation. The ground state of the system is approximately described by a factorized state $\varphi_t^{\otimes N}$ and its energy can in good approximation be computed by a simple energy functional, called the Hartree functional (see for instance [42] and [11, p.6]). Moreover, it is assumed that the quantum state remains unchanged if one switches off the external potential. At that particular moment the system is no longer in a static state and evolves according to the Schrödinger equation

$$i\partial_t \Psi_{N,t} = \Big[\sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} N^{-1} v(x_j - x_i)\Big] \Psi_{N,t} \tag{1.4}$$

with initial data $\Psi_{N,0} = \varphi_0^{\otimes N}$. It is obvious that a complete factorization of the initial state will be destroyed during the the time evolution because the particles are getting correlated by the interaction potential. Nevertheless, we will see that the factorization nearly remains

$$\Psi_{N,t}(x_1, \ldots, x_N) \approx \prod_{i=1}^{N} \varphi_t(x_i) \tag{1.5}$$

in the large N limit. Hereby, the one-particle wave function φ_t evolves according to the Hartree equation

$$i\partial_t \varphi_t = \left(-\Delta + v * |\varphi_t|^2\right) \varphi_t. \tag{1.6}$$

The many-body wave function can consequently be approximated by a product of the same one particle wave function whose evolution is given by an effective nonlinear one-particle equation. This substitution simplifies numerical calculations tremendously.
The previously mentioned can be elaborated more explicitly in probabilistic terms. Therefore, we would like to stress that quantum mechanics is a statistical theory that predicts distributions of repeatedly or simultaneously performed experiments. At a given time t there exists a probability space $(\mathbb{R}^{3N}, \mathcal{B}(\mathbb{R}^{3N}), |\Psi_{N,t}|^2)$ and observables which describe the outcomes of experiments. These can be seen as random variables. An effective theory can be called a good approximation to quantum mechanics if it predicts the same distributions of observables.
In the following we will see that this can be shown in form of a law of large number statement. Given a system in the state $\Psi_{N,t}$ and an observable $O^{(k)}$ that describes a measurement involving (the first) k particles one computes its mean value by

$$\left\langle \Psi_{N,t}, O^{(k)} \Psi_{N,t} \right\rangle = \mathrm{Tr}\, O^{(k)} |\Psi_{N,t}\rangle\langle\Psi_{N,t}|. \tag{1.7}$$

Since the operator only acts on the first k particles, we are allowed to trace out the remaining degrees of freedom. Explicitly, we define the k-particle reduced density matrix by

$$\gamma_{N,t}^{(k)} = \mathrm{Tr}_{k+1,\ldots,N} |\Psi_{N,t}\rangle\langle\Psi_{N,t}|, \tag{1.8}$$

where $\mathrm{Tr}_{k+1,\ldots,N}$ denotes the partial trace over the last $(N-k)$ particles. It is a non-negative trace class operator on $L^2(\mathbb{R}^{3k})$ with an integral kernel given by

$$\begin{aligned} &\gamma_{N,t}^{(k)}(x_1, \ldots, x_k; y_1, \ldots, y_k) \\ &\quad = \int dx_{k+1}^3 \ldots dx_N^3\ \Psi_{N,t}(x_1, \ldots, x_k, x_{k+1}, \ldots x_N) \Psi_{N,t}^*(y_1, \ldots, y_k, x_{k+1}, \ldots x_N). \end{aligned} \tag{1.9}$$

The mean value of a k-particle observable is then computed by

$$\left\langle \Psi_{N,t}, O^{(k)} \Psi_{N,t} \right\rangle = \mathrm{Tr}_{1,\ldots,k}\, O^{(k)} \gamma_{N,t}^{(k)}. \tag{1.10}$$

This shows that the statistics of experiments that involve at most k particles are determined by the k-particle reduced density matrix. To motivate the appearance of the mean-field potential we consider for $i \in \{1, \ldots, N\}$ the random variables $x_i : \mathbb{R}^{3N} \to \mathbb{R}^3, (x_1, \ldots, x_N) \mapsto x_i$ describing the positions of the particles. For $\Psi_{N,0} = \varphi_0^{\otimes N}$ the probability measure is a product measure and the positions of the particles are independently, identically distributed with a probability density $|\varphi_0|^2$. Independence is lost at later times as a result of the correlation of the particles. Nonetheless, "[...] they are still identically distributed because of the permutation symmetry of $\Psi_{N,t}$ [...]" [10, p.4] and one could hope that the correlations between the particles are sufficiently weak to prove a law of large number statement for the positions of the particles. For large N this guarantees that the total potential of the i-th particle is typically well approximated by the mean-field potential

$$N^{-1}\sum_{j\neq i}^{N} v(x_i - x_j) \approx (v * |\varphi_t|^2)(x_i). \tag{1.11}$$

To show the emergence of the Hartree equation we consider the evolution of the integral kernel of the one-particle reduced density matrix. By using the many-body Schrödinger equation (1.4) and integration by parts, we compute

$$\begin{aligned}\partial_t \gamma^{(1)}_{N,t}(x,y) &= \partial_t \int dx_2^3 \dots dx_N^3\; \Psi_{N,t}(x,x_2,\dots,x_N)\Psi^*_{N,t}(y,x_2,\dots,x_N)\\ &= -i\int dx_2^3\dots dx_N^3\,\big(-\Delta_x + N^{-1}\sum_{j=2}^{N} v(x-x_j) + \Delta_y - N^{-1}\sum_{j=2}^{N} v(y-x_j)\big)\times\\ &\qquad \times \Psi_{N,t}(x,x_2,\dots,x_N)\Psi^*_{N,t}(y,x_2,\dots,x_N). \end{aligned}\tag{1.12}$$

Substituting (1.11) into (1.12) yields

$$\partial_t \gamma^{(1)}_{N,t}(x,y) \approx -i\big[-\Delta_x + (v*|\varphi_t|^2)(x) + \Delta_y - (v*|\varphi_t|^2)(y)\big]\gamma^{(1)}_{N,t}(x,y) \tag{1.13}$$

and we obtain

$$i\partial_t \gamma^{(1)}_{N,t} \approx \Big[(-\Delta + v*|\varphi_t|^2), \gamma^{(1)}_{N,t}\Big]. \tag{1.14}$$

In case of weak correlations this suggests to approximate the time evolution of the one-particle reduced density matrix by the Hartree equation.
In this thesis, we prove results of the following type.

Theorem 1.1.1. *Let $v \in L^\infty(\mathbb{R}^3,\mathbb{R})$. Let $\Psi_{N,0} \in L^2_s(\mathbb{R}^{3N}) \cap H^2(\mathbb{R}^{3N})$ with $||\Psi_{N,0}|| = 1$ and $\varphi_0 \in H^2(\mathbb{R}^3)$ with $||\varphi_0|| = 1$ such that*

$$a_N := Tr_{L^2(\mathbb{R}^3)}|\gamma^{(1)}_{N,0} - |\varphi_0\rangle\langle\varphi_0|| \to 0 \quad \text{as } N \to 0. \tag{1.15}$$

Let $\Psi_{N,t}$ be the unique solution of the Schrödinger equation (1.4) with initial data $\Psi_{N,0}$ and φ_t the unique solution of (1.6) with initial data φ_0. Then, for any $t \geq 0$ there exists a generic constant C independent of N and t such that

$$Tr_{L^2(\mathbb{R}^3)}|\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|| \leq Ce^{Ct}\sqrt{a_N + N^{-1}}. \tag{1.16}$$

Remark 1.1.2. *It is possible to consider more general interaction potentials and initial conditions. The choices of Theorem 1.1.1 were made to ease the presentation of the proof. Moreover it is possible to add time-dependent external potentials to the Schrödinger and Hartree equation. Further information can be found in [53].*

Remark 1.1.3. *The rate of convergence presented here is in the best case of order $N^{-1/2}$. This is known to be unideal. Regarding the fluctuations around the Hartree dynamics it is possible to derive similar results with a rate of order N^{-1} (see for instance [67, 80]).*

Analogously to Theorem 1.1.1 one derives (see [53, Lemma 2.1 and Lemma 2.3]) the estimate

$$\mathrm{Tr}_{1,\ldots,k}|\gamma^{(k)}_{N,t} - \big(|\varphi_t\rangle\langle\varphi_t|\big)^{\otimes k}| \leq Ce^{Ct}\sqrt{k\,(a_N + N^{-1})}. \tag{1.17}$$

The relation

$$|\mathrm{Tr}_{1,\ldots,k}O^{(k)}\gamma^{(k)}_{N,t} - \mathrm{Tr}_{1,\ldots,k}O^{(k)}\,(|\varphi_t\rangle\langle\varphi_t|)^{\otimes k}\,| \leq \left|\left|O^{(k)}\right|\right|_{op}\mathrm{Tr}_{1,\ldots,k}|\gamma^{(k)}_{N,t} - \big(|\varphi_t\rangle\langle\varphi_t|\big)^{\otimes k}| \tag{1.18}$$

for bounded observable $O^{(k)}$ (acting on the first k particles) shows that mean-values of (bounded) observables can (in the limit of large N) be computed by means of the one-particle wave function φ_t. Additionally, we could think of measuring a physical property (for example the kinetic energy) of all particles in the system. Such a measurement is described by an observable $\sum_{i=1}^N O_j$, where $O_i = 1\otimes\ldots\otimes O\otimes\ldots\otimes 1$ only acts on the i-th particle and O is an operator on $L^2(\mathbb{R}^3)$. For a bounded observable O one derives the weak law of large numbers

$$\limsup_{N\to\infty}\mathbb{P}_{\Psi_{N,t}}\left(|N^{-1}\sum_{i=1}^N O_j - \big\langle\varphi_t, O\varphi_t\big\rangle \geq \epsilon|\right) = 0 \quad \text{for all } \epsilon > 0 \tag{1.19}$$

by Theorem 1.1.1 and Markov's inequality [10, p.4]. This tells us that the empirical mean of a measurement is (for typical configurations of the many-body system) given by the mean value of the corresponding observable with respect to the one-particle wave function φ_t. In [10] it was shown that it is possible to derive a central limit theorem for the observable $N^{-1/2}\sum_{i=1}^N\big(O_i - \big\langle\varphi_t, O\varphi_t\big\rangle\big)$.

Literature: The first rigorous derivation of the Hartree equation with bounded interaction potential was initiated by Hepp [44] and was later extended to singular potentials by Ginibre and Velo [40]. The convergence of the one-particle reduced density matrix to the projector onto the condensate wave function were proven without an explicit rate of convergence by the BBGKY technique [84, 31, 8, 26]. Moreover, precise estimates on the rate has been obtained in [80, 53, 75, 67, 2, 21, 37, 38] by the coherent state approach, the method of counting and other techniques. The derivation of the Hartree equation in the mean-field regime has also been examined by the use of Wigner measures [3, 4]. More information referring to the literature can be found at [16, p.8].

1.2 Method of counting

In this section, we introduce the "method of counting" and prove Theorem 1.1.1. Our sole goal is to emphasize the main ideas of the proof. For a more detailed introduction we refer to the articles [53, 75]. In order to prove Theorem 1.1.1 we have to keep track of the correlations that are developed during the time evolution. The key idea of the method is to pinpoint that correlations emerge because particles leave the condensate. This suggests to quantify the correlations of the quantum state with the help of a functional that counts the relative number of particles which are not in the condensate state φ_t. The advantage of this functional is that its change can easily be controlled in time. Moreover, it provides a notion of condensation which is related to the trace norm convergence of reduced density matrices. The functional is defined as follows:

Definition 1.2.1. *For any $N \in \mathbb{N}$, $\varphi_t \in L^2(\mathbb{R}^3)$ and $1 \leq j \leq N$ we define the time-dependent projectors $p_j^{\varphi_t} : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ and $q_j^{\varphi_t} : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ by*

$$p_j^{\varphi_t} \Psi_N(x_1, \ldots, x_N) := \varphi_t(x_j) \int dx_j^3 \; \varphi_t^*(x_j) \Psi_N(x_1, \ldots, x_N) \quad \text{for all } \Psi_N \in L^2(\mathbb{R}^{3N}) \tag{1.20}$$

and $q_j^{\varphi_t} := 1 - p_j^{\varphi_t}$.[4]
Furthermore, we define the functional $\alpha_N(\Psi_{N,t}, \varphi_t) : L^2(\mathbb{R}^{3N}) \times L^2(\mathbb{R}^3) \to \mathbb{R}_0^+$ as

$$\alpha_N(\Psi_{N,t}, \varphi_t) := \big\langle \Psi_{N,t}, N^{-1} \sum_{j=1}^{N} q_j^{\varphi_t} \Psi_{N,t} \big\rangle_{L^2(\mathbb{R}^{3N})}. \tag{1.21}$$

One should note, that the projector $q_j^{\varphi_t}$ gives the eigenvalue 1 if and only if the j-th coordinate of the many-body wave function is orthogonal to the condensate wave function φ_t. The functional α_N tests the orthogonality of each coordinate of the wave function to the condensate state and counts in this manner the relative number of particles outside the condensate. Using the symmetry of the wave function, the functional can be written as

$$\alpha_N(\Psi_{N,t}, \varphi_t) = \big\langle \Psi_{N,t}, q_1^{\varphi_t} \Psi_{N,t} \big\rangle_{L^2(\mathbb{R}^{3N})}. \tag{1.22}$$

The relation of the functional to the trace-norm distance of the one-particle reduced density matrix is given by the inequalities[5]

$$\alpha_N(\Psi_{N,t}, \varphi_t) \leq \mathrm{Tr}_{L^2(\mathbb{R}^3)} |\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|| \leq \sqrt{8 \alpha_N(\Psi_{N,t}, \varphi_t)}. \tag{1.23}$$

Both indicators are equivalent in the way that the limit $\alpha_N \to 0$ as $N \to \infty$ implies the convergence of the one-particle reduced density matrix to the projector onto the condensate wave function in trace norm and vice versa. Nevertheless, the second inequality shows that both indicators may converge with a different rate [53][p.2]. Theorem 1.1.1 is often (for instance in [11, 64]) depicted as the diagram:

$$\begin{tikzcd} \Psi_{N,0} \arrow[r, "\text{partial trace}"] \arrow[d, "\text{Many-body dynamics}"'] & \gamma_{N,0}^{(1)} \arrow[r, "N \to \infty"] \arrow[d] & |\varphi_0\rangle\langle\varphi_0| \arrow[d, "\text{Hartree equation}"] \\ \Psi_{N,t} \arrow[r, "\text{partial trace}"] & \gamma_{N,t}^{(1)} \arrow[r, "N \to \infty"'] & |\varphi_t\rangle\langle\varphi_t|. \end{tikzcd} \tag{1.24}$$

[4] For ease of notation we mostly omit the superscript φ_t in the following. Additionally, we use the bra-ket notation $p_j^{\varphi_t} = |\varphi_t(x_j)\rangle\langle\varphi_t(x_j)|$.

[5] A proof can be found in [53] and Section 2.7

The convergence of the one-particle reduced density matrix to the projector onto the condensate wave function will be proven in the following way:

(a) We choose appropriate initial states φ_0 and $\Psi_{N,0}$ such that $\alpha_N(\Psi_{N,0}, \varphi_0) \leq \mathrm{Tr}|\gamma^{(1)}_{N,0} - |\varphi_0\rangle\langle\varphi_0|| \to 0$ as $N \to 0$.

(b) For each $t \in \mathbb{R}_0^+$ we control the time-derivative of the functional by $|d_t\alpha_N(\Psi_{N,t}, \varphi_t)| \leq C\left(\alpha_N(\Psi_{N,0}, \varphi_0) + N^{-1}\right)$. Then, $\alpha_N(\Psi_{N,t}, \varphi_t) \leq e^{Ct}\left(\alpha_N(\Psi_{N,0}, \varphi_0) + N^{-1}\right)$ follows by Gronwall's Lemma.

(c) For a given time t we conclude condensation in terms of the one-particle reduced density matrix by means of (1.23).

Proof of Theorem 1.1.1:

Lemma 1.2.2. *Let $v \in L^\infty(\mathbb{R}^3, \mathbb{R})$, $\Psi_{N,0} \in L^2_s(\mathbb{R}^{3N}) \cap H^2(\mathbb{R}^{3N})$ with $||\Psi_{N,0}|| = 1$ and $\varphi_0 \in H^2(\mathbb{R}^3)$ with $||\varphi_0|| = 1$. Let $\Psi_{N,t}$ be the unique solution of the Schrödinger equation* (1.4) *with initial data $\Psi_{N,0}$, φ_t the unique solution of* (1.6) *with initial data φ_0 and α_N defined as in Definition 1.2.1. Then, for any $t \geq 0$ there exists a generic constant C independent of N and t such that*

$$\begin{aligned} |d_t\alpha_N(\Psi_{N,t}, \varphi_t)| &\leq C\left(\alpha_N(\Psi_{N,t}, \varphi_t) + N^{-1}\right) \quad \textit{and} \\ \alpha_N(\Psi_{N,t}, \varphi_t) &\leq e^{Ct}\left(\alpha_N(\Psi_{N,0}, \varphi_0) + N^{-1}\right). \end{aligned} \tag{1.25}$$

From inequality (1.23) we conclude

$$\begin{aligned} \mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|| &\leq Ce^{Ct}\sqrt{\alpha_N(\Psi_{N,0}, \varphi_0) + N^{-1}} \\ &\leq Ce^{Ct}\sqrt{\mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma^{(1)}_{N,0} - |\varphi_0\rangle\langle\varphi_0|| + N^{-1}} \end{aligned} \tag{1.26}$$

and Theorem 1.1.1 follows.

Proof of Lemma 1.2.2. In the following, we use the shorthand notation $\alpha(t) = \alpha_N(\Psi_{N,t}, \varphi_t)$. The symbol C is used as a generic constant independent of N and t. The functional $\alpha(t)$ is time-dependent, because $\Psi_{N,t}$ and φ_t evolve according to (1.4) and (1.6) respectively. Its derivative is given by

$$\dot\alpha(t) = d_t\langle\Psi_{N,t}, q_1^{\varphi_t}\Psi_{N,t}\rangle = i\langle\Psi_{N,t}, \left[\left(H_N - H_1^{eff.}\right), q_1^{\varphi_t}\right]\Psi_{N,t}\rangle, \tag{1.27}$$

where $H_N = \sum_{j=1}^N -\Delta_{x_j} + N^{-1}\sum_{i<j}^N v(x_j - x_i)$ is the many body Hamiltonian without external potential and $H_1^{eff.} := -\Delta_{x_1} + \left(v * |\varphi_t|^2\right)(x_1)$ denotes the mean-field Hamiltonian acting on the first particle. The free evolution of the first particle cancels and all terms in the commutator that do not act on the first coordinate of the many-body wave function vanish. The time-derivative of the functional then simplifies to

$$\begin{aligned} \dot\alpha(t) &= i\langle\Psi_{N,t}, \left[\left(N^{-1}\sum_{j=2}^N v(x_1 - x_j) - \left(v * |\varphi_t|^2\right)(x_1)\right), q_1^{\varphi_t}\right]\Psi_{N,t}\rangle \\ &= i\langle\Psi_{N,t}, \left[\left(N^{-1}(N-1)v(x_1 - x_2) - \left(v * |\varphi_t|^2\right)(x_1)\right), q_1^{\varphi_t}\right]\Psi_{N,t}\rangle, \end{aligned} \tag{1.28}$$

where we used the symmetry of the wave function. This shows that the change of $\alpha(t)$ is due to the difference of the actual potential experienced by the first particle and the mean-field potential. So if we insert the identity $1 = p_1^{\varphi_t} + q_1^{\varphi_t}$ and use the shorthand notation

$$Z(x_1, x_2) := N^{-1}(N-1)v(x_1 - x_2) - \left(v * |\varphi_t|^2\right)(x_1), \tag{1.29}$$

we get

$$\begin{aligned}
\dot\alpha(t) = \quad & i\langle \Psi_{N,t}, Z(x_1,x_2) q_1^{\varphi_t} \Psi_{N,t}\rangle \quad - i\langle \Psi_{N,t}, q_1^{\varphi_t} Z(x_1,x_2)\Psi_{N,t}\rangle \\
= \quad & i\langle \Psi_{N,t}, p_1^{\varphi_t} Z(x_1,x_2) q_1^{\varphi_t} \Psi_{N,t}\rangle + i\langle \Psi_{N,t}, q_1^{\varphi_t} Z(x_1,x_2) q_1^{\varphi_t}\Psi_{N,t}\rangle \\
& - i\langle \Psi_{N,t}, q_1^{\varphi_t} Z(x_1,x_2) q_1^{\varphi_t} \Psi_{N,t}\rangle + i\langle \Psi_{N,t}, q_1^{\varphi_t} Z(x_1,x_2) p_1^{\varphi_t}\Psi_{N,t}\rangle \\
= & -2\mathrm{Im}\langle \Psi_{N,t}, p_1^{\varphi_t} Z(x_1,x_2) q_1^{\varphi_t} \Psi_{N,t}\rangle.
\end{aligned} \tag{1.30}$$

Inserting $1 = p_2^{\varphi_t} + q_2^{\varphi_t}$ then gives

$$\begin{aligned}
\dot\alpha(t) = & - 2\mathrm{Im}\langle \Psi_{N,t}, p_1^{\varphi_t} p_2^{\varphi_t} Z(x_1,x_2) q_1^{\varphi_t} p_2^{\varphi_t} \Psi_{N,t}\rangle \\
& - 2\mathrm{Im}\langle \Psi_{N,t}, p_1^{\varphi_t} p_2^{\varphi_t} Z(x_1,x_2) q_1^{\varphi_t} q_2^{\varphi_t} \Psi_{N,t}\rangle \\
& - 2\mathrm{Im}\langle \Psi_{N,t}, p_1^{\varphi_t} q_2^{\varphi_t} Z(x_1,x_2) q_1^{\varphi_t} p_2^{\varphi_t} \Psi_{N,t}\rangle \\
& - 2\mathrm{Im}\langle \Psi_{N,t}, p_1^{\varphi_t} q_2^{\varphi_t} Z(x_1,x_2) q_1^{\varphi_t} q_2^{\varphi_t} \Psi_{N,t}\rangle.
\end{aligned} \tag{1.31}$$

The third line vanishes due to the symmetry of the wave function under the interchange of x_1 and x_2. The absolute value of $\dot\alpha$ is then bounded by[6]

$$|\dot\alpha(t)| \le 2|\langle \Psi_{N,t}, p_1 p_2 Z(x_1,x_2) q_1 p_2 \Psi_{N,t}\rangle| \tag{1.32}$$

$$+2|\langle \Psi_{N,t}, p_1 p_2 Z(x_1,x_2) q_1 q_2 \Psi_{N,t}\rangle| \tag{1.33}$$

$$+2|\langle \Psi_{N,t}, p_1 q_2 Z(x_1,x_2) q_1 q_2 \Psi_{N,t}\rangle|. \tag{1.34}$$

The first term can be interpreted as a process where two particles in the condensate interact with each other such that one particle leaves the condensate. The other two lines might likewise be seen as interactions (between two particles within the condensate or one particle inside the condensate with another particle outside the coordinate) which cause both particles to leave the condensate. The first line contains the dominant part of the interaction since most of the particles are in the condensate state and collisions are weak. It is small because the many-body potential experienced by the first particle is well approximated by the mean-field potential. This is seen by

$$\begin{aligned}
p_2 Z(x_1,x_2) p_2 &= p_2\left[N^{-1}(N-1)v(x_1-x_2) - \left(v * |\varphi_t|^2\right)(x_1)\right]p_2 \\
&= \quad \left[N^{-1}(N-1) - 1\right]\left(v * |\varphi_t|^2\right)(x_1) p_2 = -N^{-1}\left(v * |\varphi_t|^2\right)(x_1) p_2
\end{aligned} \tag{1.35}$$

and

$$\begin{aligned}
|(1.32)| &\le 2N^{-1}\left|\langle \Psi_{N,t}, p_1 \left(v * |\varphi_t|^2\right)(x_1) q_1 p_2 \Psi_{N,t}\rangle\right| \\
&\le 2N^{-1}\left|\left|\left(v * |\varphi_t|^2\right)(x_1) p_1 \Psi_{N,t}\right|\right| \, \left|\left|q_1 p_2 \Psi_{N,t}\right|\right| \le 2N^{-1}\left|\left|v * |\varphi_t|^2\right|\right|_\infty \\
&\le 2N^{-1}\left|\left|v\right|\right|_\infty \left|\left|\varphi_t\right|\right|_2 \le 2N^{-1}\left|\left|v\right|\right|_\infty .
\end{aligned} \tag{1.36}$$

In order to estimate the remaining terms we note that

$$\left|\left|Z(x_1,x_2)\right|\right|_{op} \le (N-1)N^{-1}\left|\left|v\right|\right|_\infty + \left|\left|v * |\varphi_t|^2\right|\right|_\infty \le 2\left|\left|v\right|\right|_\infty \tag{1.37}$$

[6]To ease notation we omit the superscript φ_t of p^{φ_t} and q^{φ_t} in the following.

holds for bounded interaction potentials. By use of symmetry and Schwarz's inequality we estimate

$$\begin{aligned}|(1.33)| &= 2|\langle \Psi_{N,t}, p_1 p_2 Z(x_1,x_2) q_1 q_2 \Psi_{N,t}\rangle| \\ &= 2(N-1)^{-1}|\langle \sum_{j=2}^{N} q_1 q_j Z(x_1,x_j) p_1 p_j \Psi_{N,t}, q_1 \Psi_{N,t}\rangle| \\ &\leq 2(N-1)^{-1}\, ||q_1\Psi_{N,t}||\, \Big|\Big| \sum_{j=2}^{N} q_1 q_j Z(x_1,x_j) p_1 p_j \Psi_{N,t}\Big|\Big|. \end{aligned} \tag{1.38}$$

Due to $(N-1)^{-1} \leq 2N^{-1}$ and $2ab \leq a^2 + b^2$ this becomes

$$\begin{aligned}|(1.33)| &\leq 4\,||q_1\Psi_{N,t}||^2 + N^{-2}\langle \sum_{j=2}^{N} q_1 q_j Z(x_1,x_j) p_1 p_j \Psi_{N,t}, \sum_{k=2}^{N} q_1 q_k Z(x_1,x_j) p_1 p_k \Psi_{N,t}\rangle \\ &\leq 4\alpha(t) + N^{-1}\,||q_1 q_2 Z(x_1,x_2) p_1 p_2 \Psi_{N,t}||^2 \\ &+ \langle q_1 q_2 Z(x_1,x_2) p_1 p_2 \Psi_{N,t}, q_1 q_3 Z(x_1,x_3) p_1 p_3 \Psi_{N,t}\rangle. \end{aligned} \tag{1.39}$$

Schwarz's inequality and the fact that p_i, q_i (for $i \in \{1,2,\ldots,N\}$) are projectors leads to

$$\begin{aligned}|(1.33)| &\leq 4\alpha(t) + N^{-1}\,||Z(x_1,x_2)||_{op}^2 + \langle q_1 Z(x_1,x_2) p_1 p_2 q_3 \Psi_{N,t}, q_1 Z(x_1,x_3) p_1 p_3 q_2 \Psi_{N,t}\rangle \\ &\leq 4\alpha(t) + ||Z(x_1,x_2)||_{op}^2 \left(||q_1\Psi_{N,t}||^2 + N^{-1}\right) \\ &\leq C\,||v||_\infty^2 \left(\alpha(t) + N^{-1}\right). \end{aligned} \tag{1.40}$$

The last term is simply estimated by

$$\begin{aligned}|(1.34)| &= 2|\langle p_1 q_2 \Psi_{N,t}, Z(x_1,x_2) q_1 q_2 \Psi_{N,t}\rangle| \leq 2\,||Z(x_1,x_2) q_1 q_2 \Psi_{N,t}||\,||p_1 q_2 \Psi_{N,t}|| \\ &\leq ||Z(x_1,x_2)||_{op}\,||q_2\Psi_{N,t}||^2 \leq 2\,||v||_\infty\,\alpha(t). \end{aligned} \tag{1.41}$$

Since $v \in L^\infty(\mathbb{R}^3, \mathbb{R})$ there exists a constant independent of N and t such that

$$\dot\alpha(t) \leq |\dot\alpha(t)| \leq C\left(\alpha(t) + N^{-1}\right). \tag{1.42}$$

By means of Gronwall's Lemma 1.3.1 we obtain

$$\alpha(t) \leq e^{C\int_0^t ds}\alpha(0) + N^{-1}\Big(e^{C\int_0^t ds} - 1\Big) \leq e^{Ct}\left(\alpha(0) + N^{-1}\right) \tag{1.43}$$

for any $t \geq 0$. □

1.3 Appendix: Gronwall's inequality

In this section, we state and prove Gronwall's inequality as it is found in [70].

Lemma 1.3.1 (Gronwall's inequality)**.** *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies the estimate*

$$(d_t\alpha)(t) \leq C(t)(\alpha(t) + \epsilon) \quad \textit{for all}\, t \geq 0 \tag{1.44}$$

with $C : \mathbb{R} \to \mathbb{R}$ continuous and $\epsilon \in \mathbb{R}$. Then, for all $t \geq 0$ one has

$$\alpha(t) \leq e^{\int_0^t C(s)ds}\alpha(0) + \left(e^{\int_0^t C(s)ds} - 1\right)\epsilon. \tag{1.45}$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies

$$(d_t f)(t) \leq C(t)f(t) \quad \text{for all } t \geq 0 \tag{1.46}$$

and let $g : \mathbb{R} \to \mathbb{R}_0^+$ be defined by

$$g(t) = e^{\int_0^t C(s)ds}. \tag{1.47}$$

From the estimate

$$d_t\left(\frac{f}{g}\right)(t) = \frac{(d_t f)(t)g(t) - f(t)(d_t g)(t)}{g^2(t)} \leq \frac{C(t)f(t)g(t) - f(t)C(t)g(t)}{g^2(t)} = 0 \tag{1.48}$$

for all $t \geq 0$ and $\left(\frac{f}{g}\right)(0) = f(0)$ one obtains $\left(\frac{f}{g}\right)(t) \leq f(0)$. For all $t \geq 0$ this leads to the inequality

$$f(t) \leq g(t)f(0) = f(0)e^{\int_0^t C(s)ds}. \tag{1.49}$$

Next, we define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(t) = e^{\int_0^t C(s)ds}\alpha(0) + \left(e^{\int_0^t C(s)ds} - 1\right)\epsilon \tag{1.50}$$

The function h is differentiable (due to the continuity of C) with derivative

$$(d_t h)(t) = C(t)(h(t) + \epsilon) \tag{1.51}$$

and $h(0) = \alpha(0)$. This gives

$$(d_t(\alpha - h))(t) \leq C(t)(\alpha(t) + \epsilon) - C(t)(h(t) + \epsilon) = C(t)(\alpha(t) - h(t)). \tag{1.52}$$

By means of (1.49) we obtain

$$\alpha(t) - h(t) \leq (\alpha(0) + h(0))\, e^{\int_0^t C(s)ds} = 0, \tag{1.53}$$

hence $\alpha(t) \leq h(t)$. □

CHAPTER

TWO

DERIVATION OF THE SCHRÖDINGER-KLEIN-GORDON EQUATIONS FROM THE NELSON MODEL

Abstract We report on a simple strategy to treat mean-field limits of quantum mechanical systems in which a large number of particles weakly couple to a second-quantized radiation field. Extending the method of counting, introduced in [75], with ideas inspired by [63] and [33] leads to a technique that can be seen as a combination of the method of counting and the coherent state approach. The strategy is similar to the coherent state approach but might be slightly better suited to systems in which a fixed number of particles couple to radiation. It is effective and provides explicit error bounds. As an instructional example we derive the Schrödinger-Klein-Gordon system of equations from the Nelson model with ultraviolet cutoff. Furthermore, we derive explicit bounds on the rate of convergence of the one-particle reduced density matrix of the non-relativistic particles in Sobolev norm. More complicated models like the Pauli-Fierz Hamiltonian can be treated by similar manner (see chapter 3).

Contributions of the author and Acknowledgements The content of this chapter is presented here for the first time and has not been published elsewhere. I am grateful to Prof. Dr. Peter Pickl for many valuable discussions. Moreover, I would like to thank Dr. Ioannis Anapolitanos, Dr. Dirk André Deckert, Prof. Dr. Detlef Dürr, Dr. Marco Falconi, Dr. David Mitrouskas, Prof. Dr. Benjamin Schlein and Prof. Dr. Robert Seiringer for helpful remarks. N.L. gratefully acknowledges support from the Cusanuswerk.

2.1 Introduction

Quantum systems with many degrees of freedom can be very complicated and difficult to analyze. This becomes especially severe in the presence of quantized radiation fields which are described by Fock spaces with infinitely many degrees of freedom. It is thus not surprising that there has been interest in the derivation of effective dynamics for particles that interact with quantized radiation fields [39, 33, 1, 86, 34, 35, 43, 24]. The general setting in these works is given by the tensor product of two Hilbert spaces

$$\mathcal{H}^{(N)} = \mathcal{H}_p^{(N)} \otimes \mathcal{F}. \tag{2.1}$$

The space $\mathcal{H}_p^{(N)}$ describes N non-relativistic particles and $\mathcal{F}$ (usually a bosonic Fock space) models the quantized radiation field in terms of gauge bosons. The dynamics of the system is governed by the Schrödinger equation with a Hamiltonian of the form

$$H^N \coloneqq H_0^N + H_f + \sum_{j=1}^{N} H_{int,j}. \tag{2.2}$$

Here, H_0^N and H_f (solely acting on $\mathcal{H}_p^{(N)}$ and $\mathcal{F}$) denote the free Hamiltonians of the particles and the radiation field. The term $H_{int,j}$ establishes an interaction between the j-th particle and the radiation field. This couples the dynamics of the particles with the gauge bosons. A typical question of interest is, whether the quantized radiation field can be approximated by a classical field and the evolution of the whole system described by a system of simple effective equations. Usually one considers initial data $\Psi_{N,0} = \Phi_{N,0} \otimes W(\gamma^{1/2}\alpha_0)\Omega$[1] with no correlations between the particles and the gauge bosons, sometimes referred to as Pekar product state [34]. The state $W(\gamma^{1/2}\alpha_0)\Omega \in \mathcal{F}$ denotes gauge bosons in the coherent state α_0 with a mean particle number $\gamma \left|\left|\alpha_0\right|\right|^2$, see (2.16). Hereby, γ is a model dependent scaling parameter, for instance the number of particles [33, 1, 54] or the strong coupling parameter in the Polaron model [34, 35, 43]. From physics literature it is commonly known that coherent states with a high occupation number of gauge bosons can approximately be described by a classical radiation field [23, Chapter III.C.4]. This allows us to describe the system in the limit $\gamma \to \infty$ (in a suitable sense, see Section 2.3) effectively by the state of the particles $\Phi_{N,0}$ and a classical radiation field with mode function α_0. The arising question is, if at later time t one can still approximate the system by the pair $(\Phi_{N,t}, \alpha_t)$ which evolves according to a set of simple effective equations with initial datum $(\Phi_{N,0}, \alpha_0)$. The diagram (1.24) then generalizes to

$$\begin{tikzcd}
\Psi_{N,0} \arrow[r, "\gamma\to\infty"] \arrow[d, "\text{Many-body dynamics}"'] & (\Phi_{N,0}, \alpha_0) \arrow[d, "\text{Effective dynamics}"] \\
\Psi_{N,t} \arrow[r, "\gamma\to\infty"] & (\Phi_{N,t}, \alpha_t).
\end{tikzcd} \tag{2.3}$$

This only holds, if the radiation sector of $\Psi_{N,t}$ is approximately given by a coherent state, i.e. if the gauge bosons, that are created during the time evolution, are either in a coherent state or subleading in the number of particles with respect to γ. The effect of the particles on the radiation field is typically negligible, if one considers a fixed number of particles, a coupling constant that tends to zero in a suitable sense and a coherent state, whose mean particle number scales with the parameter γ [39]. Otherwise, the state of the particles must have a special structure to ensure that the contributing gauge bosons are coherent [23, Complement $\mathrm{B_{III}}$]. This is expected, if one considers slow and heavy particles [86] or a condensate of particles that weakly couple to the radiation field. In this work, we are interested in the latter situation. More explicitly, we study the dynamics of initial states $\Psi_{N,0} = \varphi_0^{\otimes N} \otimes W(N^{1/2}\alpha_0)\Omega$ with one particle wave function φ_0 in the limit $N = \gamma \to \infty$ where the fields in the interaction Hamiltonian $H_{int,j}$ are multiplied by $N^{-1/2}$ (see Section 2.2). We refer to this limit as mean-field limit, because its key feature is that the source term of the radiation field is replaced by its mean value in the effective description. So far, such kind of limits have been studied either by the coherent state approach [39, 32, 33] or by means of Wigner measures [1].[2] While the method of Wigner measures allows us to derive limiting equations for an extensive class

[1]The reader might note that the letter α refers to the mode function of the radiation field while the counting functional is denoted by β in the present chapter.

[2]These approaches usually embed the N particle states of $\mathcal{H}_p^{(N)}$ in a bosonic Fock space for the particles $\mathcal{F}_p$ and consider the Hilbert space $\mathcal{F}_p \otimes \mathcal{F}$.

of initial states it does in contrast to the coherent state approach not provide quantitative bounds on the rate of convergence. In the following, we present a strategy, similar to the coherent state approach, which is designed for systems with fixed particle number. Such systems usually arise in the non-relativistic limit when the creation and annihilation of the (charged) particles is suppressed.[3] The method provides explicit bounds on the rate of convergence and can be seen as a combination of the method of counting and the coherent state approach. As an instructional example we derive the Schrödinger-Klein-Gordon system of equations from the Nelson model with ultraviolet cutoff. Our strategy is effective and we hope it will be useful when treating more complicated models. As shown in Chapter 3 it can also be applied to derive the Maxwell-Schrödinger system of equations from the Pauli-Fierz Hamiltonian.

2.2 Setting of the problem

We consider a system of N identical charged bosons interacting with a scalar field, described by a wave function $\Psi_{N,t} \in \mathcal{H}^{(N)}$. The Hilbert space is given by

$$\mathcal{H}^{(N)} := L^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}, \tag{2.4}$$

where the scalar field is represented by elements of the Fock space $\mathcal{F} := \bigoplus_{n\geq 0} L^2(\mathbb{R}^3)^{\otimes_s^n}$. The subscript s indicates symmetry under interchange of variables. An element $\Psi_N \in \mathcal{H}^{(N)}$ is a sequence $\{\Psi_N^{(n)}\}_{n\in\mathbb{N}_0}$ in $L^2(\mathbb{R}^{3N+3n})$ with[4]

$$||\Psi_N||^2 = \sum_{n=0}^{\infty} \int d^{3N}x \, d^{3n}k \, |\Psi_N^{(n)}(x_1, \ldots, x_N, k_1, \ldots, k_n)|^2 < \infty. \tag{2.5}$$

The time evolution of $\Psi_{N,t}$ is governed by the Schrödinger equation

$$i\partial_t \Psi_{N,t} = H_N \Psi_{N,t}. \tag{2.6}$$

Here,

$$H_N = \sum_{j=1}^{N} \left(-\Delta_j + \frac{\widehat{\Phi}_\kappa(x_j)}{\sqrt{N}} \right) + H_f \tag{2.7}$$

denotes the Nelson Hamiltonian and

$$\widehat{\Phi}_\kappa(x) = \int d^3k \, \frac{\tilde{\kappa}(k)}{\sqrt{2\omega(k)}} \left(e^{ikx} a(k) + e^{-ikx} a^*(k) \right). \tag{2.8}$$

The scalar bosons evolve according to the dispersion relation $\omega(k) = (|k|^2 + m_b^2)^{1/2}$ with mass $m_b \geq 0$ and

$$\tilde{\kappa}(k) = (2\pi)^{-3/2} \, \mathbb{1}_{|k|\leq\Lambda}(k), \quad \text{with } \mathbb{1}_{|k|\leq\Lambda}(k) = \begin{cases} 1 & \text{if } |k| \leq \Lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{2.9}$$

[3] For the sake of clarity, we want to stress that only the number of the non-relativistic particles is fixed while gauge bosons are created and destroyed during the time evolution.

[4] Note that $\Psi_N^{(n)}$ is symmetric in the variables $k_1, \ldots k_n$. For notational convenience we will use the shorthand notation $\Psi_N^{(n)}(X_N, K_n) = \Psi_N^{(n)}(x_1, \ldots, x_N, k_1, \ldots k_n)$.

cuts off the high frequency modes of the radiation field. The (pointwise) annihilation operator $a(k)$ and creation operator $a^*(k)$ are given by[5]

$$\begin{aligned}
\left(a(k)\Psi_N\right)^{(n)}(X_N, k_1, \dots, k_n) &= (n+1)^{1/2}\Psi_N^{(n+1)}(X_N, k, k_1, \dots, k_n), \\
\left(a^*(k)\Psi_N\right)^{(n)}(X_N, k_1, \dots, k_n) &= n^{-1/2}\sum_{j=1}^{n}\delta(k-k_j)\Psi_N^{(n)}(X_N, k_1, \dots, \hat{k}_j, \dots, k_n).
\end{aligned} \tag{2.10}$$

They are operator valued distributions and satisfy the commutation relations

$$[a(k), a^*(l)] = \delta(k-l), \quad [a(k), a(l)] = [a^*(k), a^*(l)] = 0. \tag{2.11}$$

On the domain

$$\mathcal{D}(H_f) = \left\{\Psi_N \in \mathcal{H}^{(N)} : \sum_{n=1}^{\infty}\int d^{3N}x\, d^{3n}k \,|\sum_{j=1}^{n} w(k_j)|^2 |\Psi_N^{(n)}(X_N, K_n)|^2 < \infty\right\} \tag{2.12}$$

the free Hamiltonian of the scalar field is defined by

$$(H_f\Psi_N)^{(n)} = \sum_{j=1}^{n} w(k_j)\Psi_N^{(n)}. \tag{2.13}$$

By means of the creation and annihilation operators it can be written as

$$H_f = \int d^3k\, \omega(k) a^*(k) a(k). \tag{2.14}$$

The Nelson model was originally introduced to describe the interaction of non-relativistic nucleons with a meson field. By standard estimates of the field operator and Kato's theorem it is easily shown that H_N is a self-adjoint operator with $\mathcal{D}\left(H_N\right) = \mathcal{D}\big(\sum_{j=1}^N -\Delta_j + H_f\big)$ [69]. The mean-field scaling in front of the interaction ensures that the kinetic and potential energy of H_N are of the same order. For simplicity, we are first interested in the evolution of initial states of the product form

$$\varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega. \tag{2.15}$$

Here, Ω denotes the vacuum in $\mathcal{F}$ and $W(f)$ is the Weyl operator

$$W(f) := \exp\left(\int d^3k\, f(k)a^*(k) - f^*(k)a(k)\right), \tag{2.16}$$

where $f \in L^2(\mathbb{R}^3)$. This choice of initial data corresponds to situations in which no correlations among the particles and the gauge bosons are present. Nevertheless, it should be noted that Theorem 2.3.1 holds for larger class of initial data. Due to the interaction between the particles and the gauge bosons correlations take place and the time evolved state will no longer have an exact product structure. However, for large N and times of order one it can be approximated, in a sense more specified below, by a state of the form $\varphi_t^{\otimes N} \otimes W(\sqrt{N}\alpha_t)\Omega$, where (φ_t, α_t) solves the Schrödinger-Klein-Gordon system of equations[6]

$$\begin{cases}
i\partial_t\varphi_t(x) &= H^{eff}\varphi_t(x) = \left[-\Delta + (\kappa * \Phi)(x,t)\right]\varphi_t(x), \\
i\partial_t\alpha_t(k) &= \omega(k)\alpha_t(k) + (2\pi)^{3/2}\frac{\tilde{\kappa}(k)}{\sqrt{2\omega(k)}}\mathcal{FT}\left[|\varphi_t|^2\right](k), \\
\Phi(x,t) &= \int d^3k\, (2\pi)^{-3/2}\frac{1}{\sqrt{2\omega(k)}}\left(e^{ikx}\alpha_t(k) + e^{-ikx}\alpha_t^*(k)\right),
\end{cases} \tag{2.17}$$

[5]Here, $\hat{k}_j$ means that k_j is left out in the argument of the function.

[6]We use the shorthand notation $(\kappa * \Phi)(x,t) = \int d^3k\, e^{ikx}\tilde{\kappa}(k)\mathcal{FT}[\Phi](k,t)$.

with initial data $(\varphi_0, \alpha_0) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. In this work we will assume global existence and smoothness of the following type.[7]

Conjecture 2.2.1. *Let* $(\varphi_0, \alpha_0) \in \left(H^{2n}(\mathbb{R}^3) \oplus L^2_n(\mathbb{R}^3)\right)$ *for* $1 \leq n \leq 2$.[8] *Then there is a strongly differentiable* $\left(H^{2n}(\mathbb{R}^3) \oplus L^2_n(\mathbb{R}^3)\right)$*-valued function* $(\varphi(t), \alpha(t))$ *on* $[0, \infty)$ *that satisfies* (2.17).

This system of equations determines the evolution of a single quantum particle in interaction with a classical scalar field. In the literature it is better known in its formally equivalent form

$$\begin{cases} i\partial_t \varphi_t(x) & = \left[-\Delta + (\kappa * \Phi)(x,t)\right] \varphi_t(x), \\ \left[\partial_t^2 - \Delta + m_b^2\right] \Phi(x,t) & = -\left(\kappa * |\varphi_t|^2\right)(x). \end{cases} \tag{2.18}$$

2.3 Main result

The physical situation we are interested in is the dynamical description of a Bose-Einstein condensate of charges. We start initially with a product state (2.15) and show that the condensate persists during the time evolution, i.e. correlations are small also at later times. Let $\Psi_{N,t} \in \left(L^2_s\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}\right) \cap \mathcal{H}^{(N)} \cap \mathcal{D}(\mathcal{N})$ with $||\Psi_{N,t}|| = 1$. On the Hilbert space $L^2(\mathbb{R}^3)$ we define the "one-particle reduced density matrix of the charges" by

$$\gamma^{(1,0)}_{N,t} := \mathrm{Tr}_{2,\dots,N} \otimes \mathrm{Tr}_{\mathcal{F}} |\Psi_{N,t}\rangle\langle\Psi_{N,t}|, \tag{2.19}$$

where $\mathrm{Tr}_{2,\dots,N}$ denotes the partial trace over the coordinates $x_2, \dots, x_N$ and $\mathrm{Tr}_{\mathcal{F}}$ the trace over Fock space. Then, the charged particles of the many-body state $\Psi_{N,t}$ are said to exhibit complete asymptotic Bose-Einstein condensation at time t, if there exists $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$, such that

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)} |\gamma^{(1,0)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|| \to 0, \tag{2.20}$$

as $N \to \infty$. Such φ_t is called the condensate wave function. For other indicators of condensation and their relation we refer to [65]. Moreover, we introduce the "one-particle reduced density matrix of the gauge bosons" with kernel

$$\gamma^{(0,1)}_{N,t}(k,k') := N^{-1} \left\langle \Psi_{N,t}, a^*(k')a(k)\Psi_{N,t} \right\rangle_{\mathcal{H}^{(N)}}. \tag{2.21}$$

Let

$$\mathcal{N} := \int d^3k \, a^*(k)a(k) \tag{2.22}$$

be the number (of gauge bosons) operator with domain

$$\mathcal{D}(\mathcal{N}) = \left\{ \Psi_N \in \mathcal{H}^{(N)} : \sum_{n=1}^{\infty} n^2 \int d^{3N}x \, d^{3n}k \, |\Psi_N^{(n)}(X_N, K_n)|^2 < \infty \right\}. \tag{2.23}$$

[7] We expect that Conjecture 2.2.1 can be proven by a standard fixed-point argument. Especially due to the cutoff in the radiation field it seems possible to make use of Theorem X.74 in [79].

[8] Here, $H^m(\mathbb{R}^3)$ denotes the Sobolev space of order m and $L^2_m(\mathbb{R}^3)$ a weighted L^2-space with norm $||\alpha||_{L^2_m(\mathbb{R}^3)} = \left|\left|(1 + |\cdot|^2)^{m/2}\alpha\right|\right|_{L^2(\mathbb{R}^3)}$.

Then, $\gamma^{(0,1)}_{N,t}$ is a positive trace class operator with $\mathrm{Tr}_{L^2(\mathbb{R}^3)}(\gamma^{(0,1)}_{N,t}) = N^{-1}\left\langle \Psi_{N,t}, \mathcal{N}\Psi_{N,t}\right\rangle_{\mathcal{H}^{(N)}}$. It should be noted, that (2.21) differs from the usual definition (e.g. [80, p.8]) by the weight factor $\left\langle \Psi_N, \mathcal{N}\Psi_N\right\rangle_{\mathcal{H}^{(N)}}/N$. Our choice ensures that we only measure deviations from the classical mode function that are at least of order N. This is reasonable because Fock space vectors with a mean particle number smaller than of order N only have a subleading effect on the dynamics of the charged particles. We say the gauge bosons exhibit "asymptotic Bose-Einstein condensation", if there exists a state $\alpha_t \in L^2(\mathbb{R}^3)$, such that

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma^{(0,1)}_{N,t} - |\alpha_t\rangle\langle\alpha_t|| \to 0, \tag{2.24}$$

as $N \to \infty$.

Theorem 2.3.1. *Let $(\varphi_0, \alpha_0) \in (H^2(\mathbb{R}^3) \oplus L^2_1(\mathbb{R}^3))$ with $||\varphi_0|| = 1$ and assume that Conjecture 2.2.1 holds. Let $\Psi_{N,0} \in \left(L^2_s(\mathbb{R}^{3N}) \otimes \mathcal{F}\right) \cap \mathcal{D}(\mathcal{N}) \cap \mathcal{D}(\mathcal{N}H_N)$ with $||\Psi_{N,0}|| = 1$ such that* [9]

$$\begin{aligned} a_N =& Tr_{L^2(\mathbb{R}^3)}|\gamma^{(1,0)}_{N,0} - |\varphi_0\rangle\langle\varphi_0|| \to 0 \text{ and} & (2.25)\\ b_N =& N^{-1}\left\langle W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}, \mathcal{N}W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}\right\rangle_{\mathcal{H}^{(N)}} \to 0 & (2.26)\end{aligned}$$

as $N \to \infty$. Let $\Psi_{N,t}$ be the unique solution of (2.6) with initial data $\Psi_{N,0}$ and let (φ_t, α_t) be the unique solution of (2.17) with initial data (φ_0, α_0). Then, there exists a generic constant C independent of N, Λ and t such that

$$\begin{aligned} Tr_{L^2(\mathbb{R}^3)}|\gamma^{(1,0)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|| &\leq \sqrt{a_N + b_N + N^{-1}}e^{\Lambda^2 Ct}, & (2.27)\\ Tr_{L^2(\mathbb{R}^3)}|\gamma^{(0,1)}_{N,t} - |\alpha_t\rangle\langle\alpha_t|| &\leq \sqrt{a_N + b_N + N^{-1}}e^{\Lambda^2 Ct}C\left(1 + ||\alpha_t||\right) & (2.28)\end{aligned}$$

for any $t \in \mathbb{R}^+_0$. [10] *In particular, for $\Psi_{N,0} = \varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega$ one obtains*

$$\begin{aligned} Tr_{L^2(\mathbb{R}^3)}|\gamma^{(1,0)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|| &\leq N^{-1/2}e^{C\Lambda^2 t}, & (2.29)\\ Tr_{L^2(\mathbb{R}^3)}|\gamma^{(0,1)}_{N,t} - |\alpha_t\rangle\langle\alpha_t|| &\leq N^{-1/2}e^{\Lambda^2 Ct}C\left(1 + ||\alpha_t||\right). & (2.30)\end{aligned}$$

Moreover, let $(\varphi_0, \alpha_0) \in (H^4(\mathbb{R}^3) \oplus L^2_2(\mathbb{R}^3))$ and $\Psi_{N,0} \in \left(L^2_s(\mathbb{R}^{3N}) \otimes \mathcal{F}\right) \cap \mathcal{D}(\mathcal{N}) \cap \mathcal{D}(\mathcal{N}H_N) \cap \mathcal{D}\left(H_N^2\right)$ such that

$$c_N = \left|\left|\nabla_1\left(1 - |\varphi_0\rangle\langle\varphi_0| \otimes \mathbb{1}_{L^2(\mathbb{R}^{3(N-1)})} \otimes \mathbb{1}_{\mathcal{F}}\right)\Psi_{N,0}\right|\right|^2_{\mathcal{H}^{(N)}} \to 0 \tag{2.31}$$

as $N \to \infty$. Then, there exists a positive monotone increasing function $C(s)$ of the norms $||\alpha_s||_{L^2(\mathbb{R}^3)}$ and $||\varphi_s||_{H^1(\mathbb{R}^3)}$ such that

$$Tr_{L^2(\mathbb{R}^3)}|\sqrt{1-\Delta}\left(\gamma^{(1,0)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\right)\sqrt{1-\Delta}| \leq \sqrt{a_N + b_N + c_N + N^{-1}}C(t)e^{\Lambda^4\int_0^t C(s)ds}. \tag{2.32}$$

For $\Psi_{N,0} = \varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega$ one obtains

$$Tr_{L^2(\mathbb{R}^3)}|\sqrt{1-\Delta}\left(\gamma^{(1,0)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\right)\sqrt{1-\Delta}| \leq N^{-1/2}C(t)e^{\Lambda^4\int_0^t C(s)ds}. \tag{2.33}$$

[9] Here, $W^{-1}(\sqrt{N}\alpha_0) = W(-\sqrt{N}\alpha_0)$ is the inverse of the unitary Weyl operator $W(\sqrt{N}\alpha_0)$, see Section 2.9.
[10] To ease the presentation we have chosen for given t the scaling parameter N large enough such that $0 \leq \beta(t) \leq 1$ and $0 \leq \beta_2(t) \leq 1$ (see Subsections 2.8.2 and 2.8.3).

Remark 2.3.2. *The trace norm convergence of the reduced density matrices was already obtained in [33] for special classes of initial states.*[11] *This was established by quantitative bounds with a rate of order N^{-1}. Theorem 2.3.1 generalizes this result to a larger class of initial states. However, it provides a slower rate of convergence. Additionally, we present the first explicit bounds on the rate of convergence of the one-particle reduced density matrix of the charges in Sobolev norm. It seems possible to improve the rate of convergence if one combines the strategy of the present work with techniques from [67].*

2.4 Comparison with the literature

In [39], Ginibre, Nironi and Velo derived the Schrödinger-Klein-Gordon system of equations from the Nelson model with cutoff. They considered a finite number of charged bosons, a coupling constant that tends to zero and a coherent state of gauge bosons whose particle number goes to infinity. The number of gauge bosons that are created during the time evolution is negligible in this case and it is possible to approximate the quantized scalar field by an external potential which evolves according to the Klein-Gordon equation without source term. Falconi [33] derived the Schrödinger-Klein-Gordon system of equations in the setting of the present paper by means of the coherent state approach. A comparison between his result and Theorem 2.3.1 is given in Remark 2.3.2. Making use of a Wigner measure approach Ammari and Falconi [1] were able to establish the classical limit (without quantitative bounds on the rate of convergence) of the renormalized Nelson model without cutoff. Teufel [86] considered the adiabatic limit of the Nelson model and showed that the interaction mediated by the quantized radiation field is well approximated by a direct Coulomb interaction.

2.5 Notations

The Fourier transform of a function f is denoted by $\tilde{f}$ and $\mathcal{FT}[f]$. $H^s(\mathbb{R}^3)$ stands for the Sobolev space with norm $||f||_{H^s(\mathbb{R}^3)} = \big|\big|(1+|\cdot|^2)^{s/2}\mathcal{FT}[f]\big|\big|_{L^2(\mathbb{R}^3)}$ and $L^2_m(\mathbb{R}^3)$ is the weighted L^2 space with $||f||_{L^2_m(\mathbb{R}^3)} = \big|\big|(1+|\cdot|^2)^{m/2} f\big|\big|_{L^2(\mathbb{R}^3)}$. Moreover, we use $||A||_{HS} = \sqrt{TrA^*A}$ to denote the Hilbert-Schmidt norm. With a slight abuse of notation we write Φ and F to indicate the scalar and auxiliary field but also their respective Fourier transforms. If we use $\Phi(t)$ or $F(t)$, we always refer to the coordinate representation of the fields. Furthermore, we apply the shorthand notation $\Phi_\kappa(x,t) := (\kappa * \Phi)(x,t)$.

2.6 The strategy

We are interested in the evolution of product states of the form (2.15) under the dynamics (2.6). The scalar field in the Nelson Hamiltonian establishes an interaction between the charges and the field modes with wave vectors smaller than Λ.[12] This changes the state of the charges, leads to the creation and annihilation of gauge bosons and causes initially factorized states to build correlations between the charges, the gauge bosons as well as

[11] The considered initial states are of the form $\varphi_0^{\otimes N} \otimes C(\sqrt{N}\alpha_0)\Omega \in \mathcal{H}^{(N)}$, $C(\sqrt{N}\varphi_0, \sqrt{n}\alpha_0)\Omega \in \mathcal{F}_p \otimes \mathcal{F}$ and $\varphi_0^{\otimes N} \otimes \alpha_0^{\otimes n} \in L^2(\mathbb{R}^{3N+3n})$. For a precise definition we refer to [33][Theorem 3].

[12] One should note that the high frequency modes of the radiation field do not interact with the non-relativistic particles and evolve according to the free dynamics.

among charges and gauge bosons. To study the emergence of these correlations we combine the "method of counting", introduced in [75], with ideas from [63] and [33]. The result can be seen as a fusion of the "method of counting" and the coherent state approach, as used for instance in [33, 80]. The key idea is to prove condensation not in terms of reduced density matrices but to consider a different indicator of condensation. To study the correlations between the charges we introduce a functional β^a, which counts the relative number of particles that are not in the state of the condensate wave function φ_t.

Definition 2.6.1. *For any $N \in \mathbb{N}$, $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$ and $1 \leq j \leq N$ we define the time-dependent projectors $p_j^{\varphi_t} : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ and $q_j^{\varphi_t} : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ by*

$$p_j^{\varphi_t} f(x_1, \ldots, x_N) \coloneqq \varphi_t(x_j) \int d^3x_j \, \varphi_t^*(x_j) f(x_1, \ldots, x_N) \quad \textit{for all } f \in L^2(\mathbb{R}^{3N}) \tag{2.34}$$

and $q_j^{\varphi_t} \coloneqq 1 - p_j^{\varphi_t}$.[13] *Let $\Psi_{N,t} \in \mathcal{H}^{(N)}$. Then $\beta^a : \mathcal{H}^{(N)} \times L^2(\mathbb{R}^3) \to \mathbb{R}_0^+$ is given by*

$$\beta^a \left(\Psi_{N,t}, \varphi_t\right) \coloneqq \left\langle \Psi_{N,t}, q_1^{\varphi_t} \otimes \mathbb{1}_{\mathcal{F}} \Psi_{N,t} \right\rangle. \tag{2.35}$$

Remark 2.6.2. *The functional β^a was denoted by α in Chapter 1. It was already used in [75, 76, 77, 47, 53, 67, 66, 5] and others to derive the Hartree and Gross-Pitaevskii equation.*

The situation is slightly different in the radiation sector because the number of gauge bosons is not preserved during the time evolution. Moreover, it is known from physics literature [23, Chapter III.C.4] that the radiation field must be in a coherent state with a high occupation number of gauge bosons to behave classically. This is a state not only with little correlations but also a Poisson distributed number of gauge bosons. In order to investigate if the state of the radiation field is coherent we define a functional, referred to as β^b, which measures the fluctuations of the field modes around the classical mode function α_t for each time.

Definition 2.6.3. *Let $\alpha_t \in L^2(\mathbb{R}^3)$ and $\Psi_{N,t} \in \mathcal{H}^{(N)} \cap \mathcal{D}(\mathcal{N})$. Then $\beta^b : \mathcal{H}^{(N)} \cap \mathcal{D}(\mathcal{N}) \times L^2(\mathbb{R}^3) \to \mathbb{R}_0^+$ is given by*

$$\beta^b \left(\Psi_{N,t}, \alpha_t\right) \coloneqq \int d^3k \, \Big\langle \left(\frac{a(k)}{\sqrt{N}} - \alpha_t(k)\right) \Psi_{N,t}, \left(\frac{a(k)}{\sqrt{N}} - \alpha_t(k)\right) \Psi_{N,t} \Big\rangle. \tag{2.36}$$

Remark 2.6.4. *Let $\alpha_0 \in L^2(\mathbb{R}^3)$ and $\Psi_{N,0} = W(\sqrt{N}\alpha_0)\Psi$ for some $\Psi \in \mathcal{H}^{(N)} \cap \mathcal{D}(\mathcal{N})$. Then, the functional β^b can be written as*

$$\beta^b \left(\Psi_{N,t}, \alpha_t\right) = N^{-1} \langle \mathcal{U}_N(t;0)\Psi, \mathcal{N}\mathcal{U}_N(t;0)\Psi \rangle, \tag{2.37}$$

where $\mathcal{U}_N(t;0) = W^*(\sqrt{N}\alpha_t) e^{-iH_N t} W(\sqrt{N}\alpha_0)$ denotes the fluctuation dynamics of the coherent state approach (as used for example in [16, p.18]).[14] *Thus, β^b measures the number of gauge boson fluctuations around the effective evolution.*

Remark 2.6.5. *It seems that β^a is the natural quantity to consider for condensates with fixed particle number. The functional β^b, which usually arises in the coherent state approach as used in [80, 33, 16] and others, is perfectly suited to keep track if the state of the radiation field remains coherent.*

[13] For ease of notation we occasionally omit the superscript φ_t. Additionally, we use the bra-ket notation $p_j^{\varphi_t} = |\varphi_t(x_j)\rangle\langle\varphi_t(x_j)| = |\varphi_t\rangle\langle\varphi_t|_j$.

[14] This is a simple consequence of $W(\sqrt{N}\alpha_t)$ being unitary and $W^*(\sqrt{N}\alpha_t)a(k) = a(k)W^*(\sqrt{N}\alpha_t) + \sqrt{N}W^*(\sqrt{N}\alpha_t)\alpha_t(k)$, see (2.128).

Finally, the counting functional is defined by

Definition 2.6.6. *Let* $N \in \mathbb{N}$, $\varphi_t \in L^2(\mathbb{R}^3)$ *with* $||\varphi_t|| = 1$, $\alpha_t \in L^2(\mathbb{R}^3)$ *and* $\Psi_{N,t} \in \mathcal{H}^{(N)} \cap \mathcal{D}(\mathcal{N})$. *Then* $\beta : \mathcal{H}^{(N)} \cap \mathcal{D}(\mathcal{N}) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \to \mathbb{R}_0^+$ *is defined by*[15]

$$\beta(\Psi_{N,t}, \varphi_t, \alpha_t) := \beta^a(\Psi_{N,t}, \varphi_t) + \beta^b(\Psi_{N,t}, \alpha_t). \tag{2.38}$$

In summary, the functional has the following properties:

(i) β^a measures if the non-relativistic particles exhibit condensation.

(ii) β^b examines whether the radiation field is in a coherent state.

(iii) $\beta(\Psi_{N,t}, \varphi_t, \alpha_t) \to 0$ as $N \to \infty$ implies condensation in terms of reduced density matrices (Lemma 2.7.1).

(iv) $\beta(\Psi_{N,t}, \varphi_t, \alpha_t) = 0$ if $\Psi_{N,t} = \varphi_t^{\otimes N} \otimes W(\sqrt{N}\alpha_t)\Omega$ (see Lemma 2.9.2).

In order to show that the product structure (2.15) is preserved during the time evolution we apply the same strategy as in Chapter 1:

(a) We choose initial states φ_0, α_0 and $\Psi_{N,0}$ such that $\beta(\Psi_{N,0}, \varphi_0, \alpha_0) \leq a_N + b_N \to 0$ as $N \to \infty$.

(b) For each $t \in \mathbb{R}_0^+$ we estimate $|d_t \beta(\Psi_{N,t}, \varphi_t, \alpha_t)| \leq C\Lambda^2 \left(\beta(\Psi_{N,t}, \varphi_t, \alpha_t) + N^{-1}\right)$ for some $C \in \mathbb{R}_0^+$. Then, Grönwall's Lemma establishes the bound $\beta(\Psi_{N,t}, \varphi_t, \alpha_t) \leq e^{C\Lambda^2 t}\left(\beta(\Psi_{N,0}, \varphi_0, \alpha_0) + N^{-1}\right)$.

(c) By means of property (iii) we conclude condensation in terms of reduced density matrices.

To show the convergence of $\gamma_{N,t}^{(1,0)}$ to the projector onto the condensate wave function in Sobolev norm we include $\beta^c(\Psi_{N,t}, \varphi_t) := ||\nabla_1 q_1^{\varphi_t} \Psi_{N,t}||^2$ in the definition of the functional. This allows us to control the kinetic energy of the non-relativistic particles which are not in the condensate.

Definition 2.6.7. *Let* $N \in \mathbb{N}$, $\varphi_t \in H^2(\mathbb{R}^3)$ *with* $||\varphi_t|| = 1$, $\alpha_t \in L^2(\mathbb{R}^3)$ *and* $\Psi_{N,t} \in \mathcal{D}(H_N) \cap \mathcal{D}(\mathcal{N})$. *Then* $\beta_2 : \mathcal{D}(H_N) \cap \mathcal{D}(\mathcal{N}) \times H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \to \mathbb{R}_0^+$ *is defined by*

$$\begin{aligned}\beta_2(\Psi_{N,t}, \varphi_t, \alpha_t) &:= \beta(\Psi_{N,t}, \varphi_t, \alpha_t) + \beta^c(\Psi_{N,t}, \varphi_t) \\ &= \beta(\Psi_{N,t}, \varphi_t, \alpha_t) + ||\nabla_1 q_1^{\varphi_t} \Psi_{N,t}||^2.\end{aligned} \tag{2.39}$$

We would like to remark, that the ultraviolet cutoff (2.9) is essential for the proof because:

(a) The finiteness of $||\eta||_2$ (see (2.65)) is needed to establish a connection between the difference of the radiation fields and the functional β^b by means of the auxiliary fields (2.62).

(b) The cutoff Λ imposes regularity on the radiation fields which will be used to estimate the time derivative of $||\nabla_1 q_1 \Psi_{N,t}||^2$. In spirit, this is opposite to the usual treatment of the polaron [61], where regularity of the electron state is used to obtain a sufficient decay in the field modes with large wave vectors.

[15] We sometimes apply the shorthand notation $\beta(t) = \beta(\Psi_{N,t}, \varphi_t, \alpha_t)$.

2.7 Relation to reduced density matrices

In this section, we relate the functional β to the trace norm distance of the one-particle reduced density matrices.

Lemma 2.7.1. *Let $N \in \mathbb{N}$, $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$, $\alpha_t \in L^2(\mathbb{R}^3)$ and $\Psi_{N,t} \in \mathcal{H}^{(N)} \cap \mathcal{D}(\mathcal{N})$. Then,*

$$\beta^a(\Psi_{N,t}, \varphi_t) \leq Tr_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|| \leq \sqrt{8\beta^a(\Psi_{N,t}, \varphi_t)}, \tag{2.40}$$

$$Tr_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle\alpha_t|| \leq 3\beta^b(\Psi_{N,t}, \alpha_t) + 6\,||\alpha_t||_{L^2(\mathbb{R}^3)} \sqrt{\beta^b(\Psi_{N,t}, \alpha_t)}. \tag{2.41}$$

For $\varphi_t \in H^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$ and $\Psi_{N,t} \in \mathcal{H}^{(N)} \cap \mathcal{D}(H_N)$, we have

$$\begin{aligned} &Tr_{L^2(\mathbb{R}^3)}|\sqrt{1-\Delta}\left(\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|\right)\sqrt{1-\Delta}| \leq \left(1 + ||\varphi_t||^2_{H^1(\mathbb{R}^3)}\right)\times \\ &\times \left(\beta^a(\Psi_{N,t}, \varphi_t) + \beta^c(\Psi_{N,t}, \varphi_t)\right) + 2\,||\varphi_t||_{H^1(\mathbb{R}^3)} \sqrt{\beta^a(\Psi_{N,t}, \varphi_t) + \beta^c(\Psi_{N,t}, \varphi_t)}. \end{aligned} \tag{2.42}$$

Proof. The lower bound of (2.40) is proven by

$$\begin{aligned} \beta^a(t) &= 1 - \left\langle \Psi_{N,t}, p_1^{\varphi_t}\Psi_{N,t}\right\rangle = 1 - \left\langle \varphi_t, \gamma_{N,t}^{(1,0)}\varphi_t\right\rangle = \mathrm{Tr}_{L^2(\mathbb{R}^3)}(|\varphi_t\rangle\langle\varphi_t| - |\varphi_t\rangle\langle\varphi_t|\gamma_{N,t}^{(1,0)}) \\ &\leq ||p_1||_{\mathrm{op}}\,\mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|| = \mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t||. \end{aligned} \tag{2.43}$$

To obtain the upper bound we use that

$$\mathrm{Tr}|\gamma - p| \leq 2\,||\gamma - p||_{HS} + \mathrm{Tr}(\gamma - p) \tag{2.44}$$

is valid for any one-dimensional projector p and non-negative density matrix γ. The original argument of the proof was first observed by Robert Seiringer, see [80]. We present a version that is found in [5]: Let $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of eigenvalues of the trace class operator $A := \gamma - p$. Since p is a rank one projection, A has at most one negative eigenvalue. If there is no negative eigenvalue, $\mathrm{Tr}|A| = \mathrm{Tr}(A)$ and (2.44) holds. If there is one negative eigenvalue λ_1, we have $\mathrm{Tr}|A| = |\lambda_1| + \sum_{n\geq 2}\lambda_n = 2|\lambda_1| + \mathrm{Tr}(A)$. Inequality (2.44) then follows from $|\lambda_1| \leq ||A||_{\mathrm{op}} \leq ||A||_{HS}$.
This shows

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|| \leq 2\left|\left|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|\right|\right|_{HS} \tag{2.45}$$

because $\mathrm{Tr}_{L^2(\mathbb{R}^3)}(\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|) = 0$. The upper bound of (2.40) is obtained by

$$\begin{aligned} \mathrm{Tr}_{L^2(\mathbb{R}^3)}(\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|)^2 =&1 - 2\mathrm{Tr}_{L^2(\mathbb{R}^3)}(|\varphi_t\rangle\langle\varphi_t|\gamma_{N,t}^{(1,0)}) + \mathrm{Tr}_{L^2(\mathbb{R}^3)}((\gamma_{N,t}^{(1,0)})^2) \\ \leq&2(1 - \mathrm{Tr}_{L^2(\mathbb{R}^3)}(|\varphi_t\rangle\langle\varphi_t|\gamma_{N,t}^{(1,0)})) = 2\beta^a(t). \end{aligned} \tag{2.46}$$

To prove (2.41) it is useful to write the kernel of $\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle\alpha_t|$ as

$$\begin{aligned} (\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle\alpha_t|)(k,l) &= N^{-1}\left\langle\Psi_N, a^*(l)a(k)\Psi_N\right\rangle - \alpha_t^*(l)\alpha_t(k) \\ &= \left\langle \left(N^{-1/2}a(l) - \alpha_t(l)\right)\Psi_N, \left(N^{-1/2}a(k) - \alpha_t(k)\right)\Psi_N\right\rangle \\ &+ \alpha_t(k)\left\langle \left(N^{-1/2}a(l) - \alpha_t(l)\right)\Psi_N, \Psi_N\right\rangle \\ &+ \alpha_t^*(l)\left\langle\Psi_N, \left(N^{-1/2}a(k) - \alpha_t(k)\right)\Psi_N\right\rangle. \end{aligned} \tag{2.47}$$

By means of Schwarz's inequality we have

$$|(\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle\alpha_t|)(k,l)|^2 \leq \left|\left|\left(N^{-1/2}a(k) - \alpha_t(k)\right)\Psi_N\right|\right|^2 \left|\left|\left(N^{-1/2}a(l) - \alpha_t(l)\right)\Psi_N\right|\right|^2$$
$$+|\alpha_t(l)|^2 \left|\left|\left(N^{-1/2}a(k) - \alpha_t(k)\right)\Psi_N\right|\right|^2 + |\alpha_t(k)|^2 \left|\left|\left(N^{-1/2}a(l) - \alpha_t(l)\right)\Psi_N\right|\right|^2 \tag{2.48}$$

and

$$\begin{aligned}\left|\left|\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle\alpha_t|\right|\right|_{HS}^2 &= \int d^3k \int d^3l \, |(\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle\alpha_t|)(k,l)|^2 \\ &\leq (\beta^b(t))^2 + 2\,||\alpha_t||_{L^2(\mathbb{R}^3)}^2 \,\beta^b(t).\end{aligned} \tag{2.49}$$

Similarly, one obtains

$$\begin{aligned}\mathrm{Tr}_{L^2(\mathbb{R}^3)}(\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle\alpha_t|) \leq& \int d^3k \, |(\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle\alpha_t|)(k,k)| \\ \leq& \int d^3k \left|\left|\left(N^{-1/2}a(k) - \alpha_t(k)\right)\Psi_N\right|\right|_{\mathcal{H}^{(N)}}^2 \\ &+ 2\int d^3k \, |\alpha_t(k)| \left|\left|\left(N^{-1/2}a(k) - \alpha_t(k)\right)\Psi_N\right|\right|_{\mathcal{H}^{(N)}}.\end{aligned} \tag{2.50}$$

Applying Schwarz's inequality in the second line leads to

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)}(\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle\alpha_t|) \leq \beta^b(t) + 2\,||\alpha_t||_{L^2(\mathbb{R}^3)}\sqrt{\beta^b(t)}. \tag{2.51}$$

Inequality (2.41) follows from the monotonicity of the square root and (2.44). The estimate (2.42) originates from [67]. One starts with the relation

$$\begin{aligned}&Tr_{L^2(\mathbb{R}^3)}|\sqrt{1-\Delta}(\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|)\sqrt{1-\Delta}| \\ &\quad= \sup_{||A_1||\leq 1} |Tr_{L^2(\mathbb{R}^3)}(A_1\sqrt{1-\Delta}(\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|)\sqrt{1-\Delta})|,\end{aligned} \tag{2.52}$$

where the supremum is applied to all compact operators A_1 on $L^2(\mathbb{R}^3)$ with norm smaller or equal to one. Then, one continues with

$$Tr_{L^2(\mathbb{R}^3)}(A_1\sqrt{1-\Delta_1}(\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|)\sqrt{1-\Delta_1}) \tag{2.53}$$
$$= \left\langle\Psi_N, p_1^{\varphi_t}\sqrt{1-\Delta_1}A_1\sqrt{1-\Delta_1}p_1^{\varphi_t}\Psi_N\right\rangle - \left\langle\varphi_t, \sqrt{1-\Delta_1}A_1\sqrt{1-\Delta_1}\varphi_t\right\rangle \tag{2.54}$$
$$+ \left\langle\Psi_N, q_1^{\varphi_t}\sqrt{1-\Delta_1}A_1\sqrt{1-\Delta_1}p_1^{\varphi_t}\Psi_N\right\rangle + \left\langle\Psi_N, p_1^{\varphi_t}\sqrt{1-\Delta_1}A_1\sqrt{1-\Delta_1}q_1^{\varphi_t}\Psi_N\right\rangle \tag{2.55}$$
$$+ \left\langle\Psi_N, q_1^{\varphi_t}\sqrt{1-\Delta_1}A_1\sqrt{1-\Delta_1}q_1^{\varphi_t}\Psi_N\right\rangle. \tag{2.56}$$

By means of

$$\left|\left|\sqrt{1-\Delta_1}q_1^{\varphi_t}\Psi_N\right|\right|^2 = ||q_1^{\varphi_t}\Psi_N||^2 + ||\nabla_1 q_1^{\varphi_t}\Psi_N||^2 \leq \beta^a(t) + \beta^c(t) \tag{2.57}$$

and

$$\left|\left|\sqrt{1-\Delta_1}p_1^{\varphi_t}\right|\right|_{op}^2 \leq \left\langle\varphi_t, (1-\Delta_1)\,\varphi_t\right\rangle = ||\varphi_t||_{H^1(\mathbb{R}^3)}^2 \tag{2.58}$$

we estimate

$$\begin{aligned}
|(2.54)| \leq & |\langle \varphi_t, \sqrt{1-\Delta} A_1 \sqrt{1-\Delta} \varphi_t \rangle| |\langle \Psi_N, p_1^{\varphi_t} \Psi_N \rangle - 1| \leq ||A_1||_{op} ||\varphi_t||^2_{H^1(\mathbb{R}^3)} \beta^a(t), \\
|(2.55)| \leq & 2 ||A_1||_{op} ||\varphi_t||_{H^1(\mathbb{R}^3)} \sqrt{\beta^a(t) + \beta^c(t)}, \\
|(2.56)| \leq & ||A_1||_{op} (\beta^a(t) + \beta^c(t)) \, . \qquad (2.59)
\end{aligned}$$

This leads to

$$\begin{aligned}
\mathrm{Tr}_{L^2(\mathbb{R}^3)} |\sqrt{1-\Delta} \left(\gamma^{(1,0)}_{N,t} - |\varphi_t\rangle\langle\varphi_t| \right) \sqrt{1-\Delta}| \leq & \left(1 + ||\varphi_t||^2_{H^1(\mathbb{R}^3)} \right) (\beta^a(t) + \beta^c(t)) \\
& + 2 ||\varphi_t||_{H^1(\mathbb{R}^3)} \sqrt{\beta^a(t) + \beta^c(t)}. \qquad (2.60)
\end{aligned}$$

□

2.8 Estimates on the time derivative

2.8.1 Preliminary estimates

In the following, we control the change of β in time by separately estimating the time derivative of β^a and β^b. On the one hand a change in β^a is caused by the fraction of particles which are not in the condensate state φ_t. This behavior is analogous to the growth of diseases, where the infection rate of cells (or particles that will leave the condensate) at a given time is proportional to the number of already infected cells. On the other hand there will be a change due to the fact that the particles of the many-body system couple to the quantized radiation field, whereas the condensate wave function is in interaction with the classical field. To control the difference between the quantized and classical field by the functional β^b we will have to split the radiation fields in their positive and negative frequency parts.

$$\begin{aligned}
\widehat{\Phi}^+_\kappa(x) &:= \int d^3k \, \frac{\tilde{\kappa}(k)}{\sqrt{2\omega(k)}} e^{ikx} a(k), \qquad \widehat{\Phi}^-_\kappa(x) := \int d^3k \, \frac{\tilde{\kappa}(k)}{\sqrt{2\omega(k)}} e^{-ikx} a^*(k), \\
\Phi^+_\kappa(x,t) &:= \int d^3k \, \frac{\tilde{\kappa}(k)}{\sqrt{2\omega(k)}} e^{ikx} \alpha_t(k), \; \Phi^-_\kappa(x,t) := \int d^3k \, \frac{\tilde{\kappa}(k)}{\sqrt{2\omega(k)}} e^{-ikx} \alpha^*_t(k). \qquad (2.61)
\end{aligned}$$

For technical reason it is then helpful to introduce the following (less singular) auxiliary fields

$$\begin{aligned}
\hat{F}^+_\kappa(x) &:= \int d^3k \, \tilde{\kappa}(k) e^{ikx} a(k), \qquad \hat{F}^-_\kappa(x) := \int d^3k \, \tilde{\kappa}(k) e^{-ikx} a^*(k), \\
F^+_\kappa(x,t) &:= \int d^3k \, \tilde{\kappa}(k) e^{ikx} \alpha_t(k), \; F^-_\kappa(x,t) := \int d^3k \, \tilde{\kappa}(k) e^{-ikx} \alpha^*_t(k). \qquad (2.62)
\end{aligned}$$

By means of the cutoff function

$$\tilde{\eta}(k) := \frac{\tilde{\kappa}(k)}{\sqrt{2\omega(k)}} = \frac{(2\pi)^{-3/2}}{\sqrt{2\omega(k)}} \mathbb{1}_{|k|\leq\Lambda}(k) \qquad (2.63)$$

we are able to express the scalar fields in terms of the auxiliary fields.

Lemma 2.8.1. *Let η be the Fourier transform of* (2.63), *then*

$$\begin{aligned}
\widehat{\Phi}^+_\kappa(x) &= \left(\eta * \hat{F}^+_\kappa \right)(x), \qquad \widehat{\Phi}^-_\kappa(x) = \left(\eta * \hat{F}^-_\kappa \right)(x), \\
\Phi^+_\kappa(x,t) &= \left(\eta * F^+_\kappa \right)(x,t), \; \Phi^-_\kappa(x,t) = \left(\eta * F^-_\kappa \right)(x,t). \qquad (2.64)
\end{aligned}$$

Proof. The proof is a simple application of convolutions theorem. □

In the following, we will integrate the form-factor η of the radiation field and estimate the difference in the auxiliary fields. This requires that the L^2-norms of the cutoff functions

$$||\kappa||_2^2 = \Lambda^3/(6\pi^2) \quad \text{and} \quad ||\eta||_2^2 \leq \Lambda^2/(4\pi^2) \tag{2.65}$$

are finite. Subsequently, we use Plancherel's theorem and estimate the difference in the positive frequency parts of the auxiliary fields by

$$\begin{aligned}&\int d^3y\, \left|\left|\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_{N,t}\right|\right|^2 = \int d^3k\, \left|\left|\left(N^{-1/2}\hat{F}_\kappa^+(k) - F_\kappa^+(k,t)\right)\Psi_{N,t}\right|\right|^2 \\ =&\int_{|k|\leq\Lambda} d^3k\, \langle \left(N^{-1/2}a(k) - \alpha_t(k)\right)\Psi_{N,t}, \left(N^{-1/2}a(k) - \alpha_t(k)\right)\Psi_{N,t}\rangle \leq \beta^b\left(\Psi_{N,t},\alpha_t\right).\end{aligned} \tag{2.66}$$

Pulling the pieces together we get

Lemma 2.8.2. *Let $\alpha_t \in L^2(\mathbb{R}^3)$ and $\Psi_{N,t} \in \mathcal{H}^{(N)} \cap \mathcal{D}(\mathcal{N})$. Then, there exists a generic constant C independent of N, Λ and t such that*

$$\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\Psi_{N,t}\right|\right|^2 \leq C\Lambda^2\left(\beta^b\left(\Psi_{N,t},\alpha_t\right) + N^{-1}\right), \tag{2.67}$$

$$\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa^-(x_1) - \Phi_\kappa^-(x_1,t)\right)\Psi_{N,t}\right|\right|^2 \leq C\Lambda^2\left(\beta^b\left(\Psi_{N,t},\alpha_t\right) + N^{-1}\right), \tag{2.68}$$

$$\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa^+(x_1) - \Phi_\kappa^+(x_1,t)\right)p_1\Psi_{N,t}\right|\right|^2 \leq C\Lambda^2\beta^b\left(\Psi_{N,t},\alpha_t\right). \tag{2.69}$$

Proof. From the canonical commutation relations (2.11), we obtain

$$\left[\left(N^{-1/2}\widehat{\Phi}_\kappa^+(x) - \widehat{\Phi}_\kappa^+(x,t)\right), \left(N^{-1/2}\widehat{\Phi}_\kappa^-(x) - \widehat{\Phi}_\kappa^-(x,t)\right)\right] = N^{-1}\, ||\eta||_2^2 \tag{2.70}$$

and estimate

$$\begin{aligned}&\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\Psi_N\right|\right|^2 \\ \leq& 2\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa^+(x_1) - \Phi_\kappa^+(x_1,t)\right)\Psi_N\right|\right|^2 + 2\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa^-(x_1) - \Phi_\kappa^-(x_1,t)\right)\Psi_N\right|\right|^2 \\ \leq& 4\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa^+(x_1) - \Phi_\kappa^+(x_1,t)\right)\Psi_N\right|\right|^2 + 2N^{-1}\, ||\eta||_2^2.\end{aligned} \tag{2.71}$$

By means of Lemma 2.8.1 we have

$$\begin{aligned}&\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa^+(x_1) - \Phi_\kappa^+(x_1,t)\right)\Psi_N\right|\right|^2 \\ =&\langle \int d^3y\, \eta(x_1-y)\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_N, \int d^3z\, \eta(x_1-z)\left(N^{-1/2}\hat{F}_\kappa^+(z) - F_\kappa^+(z,t)\right)\Psi_N\rangle \\ \leq& \int d^3y \int d^3z\, |\langle \eta^*(x_1-z)\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_N, \eta^*(x_1-y)\left(N^{-1/2}\hat{F}_\kappa^+(z) - F_\kappa^+(z,t)\right)\Psi_N\rangle|.\end{aligned} \tag{2.72}$$

Cauchy-Schwarz inequality and the estimate $ab \leq 1/2\left(a^2+b^2\right)$ give rise to

$$
\begin{aligned}
&\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa^+(x_1) - \Phi_\kappa^+(x_1,t)\right)\Psi_N\right|\right|^2 \\
\leq &\int d^3y \int d^3z \left|\left|\eta^*(x_1-z)\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_N\right|\right| \left|\left|\eta^*(x_1-y)\left(N^{-1/2}\hat{F}_\kappa^+(z) - F_\kappa^+(z,t)\right)\Psi_N\right|\right| \\
\leq &\int d^3y \int d^3z \left|\left|\eta^*(x_1-z)\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_N\right|\right|^2 \\
= &\int d^3y \left\langle \left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_N, \int d^3z |\eta(x_1-z)|^2 \left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_N\right\rangle \\
= &\,||\eta||_2^2 \int d^3y \left|\left|\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_N\right|\right|^2 \leq ||\eta||_2^2 \,\beta^b\left(\Psi_{N,t},\alpha_t\right). \qquad (2.73)
\end{aligned}
$$

In total, we get

$$
\begin{aligned}
\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\Psi_{N,t}\right|\right|^2 &\leq ||\eta||_2^2\left(4\beta^b\left(\Psi_{N,t},\alpha_t\right) + 2N^{-1}\right) \\
&\leq C\Lambda^2\left(\beta^b\left(\Psi_{N,t},\alpha_t\right) + N^{-1}\right). \qquad (2.74)
\end{aligned}
$$

The second and third inequality are shown analogously. Hereby, it is helpful to recall that $\left[p_1, \hat{F}_\kappa^+(y)\right] = [p_1, F_\kappa^+(y)] = 0$. ☐

2.8.2 Estimate on the time derivative of β

Subsequently, we control the change of $\beta\left(\Psi_{N,t},\varphi_t,\alpha_t\right)$ in time.

Lemma 2.8.3. *Let* $\Psi_{N,0} \in \left(L_s^2(\mathbb{R}^{3N})\otimes\mathcal{F}\right)\cap\mathcal{D}\left(\mathcal{N}\right)\cap\mathcal{D}\left(\mathcal{N}H_N\right)$ *with* $||\Psi_{N,0}||=1$, $(\varphi_0,\alpha_0) \in (H^2(\mathbb{R}^3)\oplus L_1^2(\mathbb{R}^3))$ *with* $||\varphi_0||=1$ *and assume that Conjecture 2.2.1 holds. Let* $\Psi_{N,t}$ *be the unique solution of* (2.6) *with initial data* $\Psi_{N,0}$ *and let* (φ_t,α_t) *be the unique solution of* (2.17) *with initial data* (φ_0,α_0)*. Then*

$$
\begin{aligned}
d_t\beta^a(t) &= -2\mathrm{Im}\left\langle\Psi_{N,t}, \left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right) q_1^{\varphi_t}\Psi_{N,t}\right\rangle, \\
d_t\beta^b(t) &= \ \ 2\mathrm{Im}\left\langle\Psi_{N,t}, \left(\int d^3k\, \tilde{\eta}(k)(2\pi)^{3/2}\mathcal{FT}^*[|\varphi_t|^2](k)\left(N^{-1/2}a(k) - \alpha_t(k)\right)\right)\Psi_{N,t}\right\rangle \\
&\quad - 2\mathrm{Im}\left\langle\Psi_{N,t}, \left(\int d^3k\, \tilde{\eta}(k)e^{ikx_1}\left(N^{-1/2}a(k) - \alpha_t(k)\right)\right)\Psi_{N,t}\right\rangle. \qquad (2.75)
\end{aligned}
$$

Proof. The structure of the proof is best understood as presented in the following. Since some manipulations are only formal, we provide a more detailed derivation in Appendix 2.11. There, we also show the invariance of the domain $\mathcal{D}\left(\mathcal{N}\right)\cap\mathcal{D}\left(\mathcal{N}H_N\right)$.
The functional $\beta^a(t)$ is time-dependent, because $\Psi_{N,t}$ and φ_t evolve according to (2.6) and (2.17) respectively. The derivative of the projector q^{φ_t} is given by

$$
d_t q_1^{\varphi_t} = -i\left[H_1^{eff}, q_1^{\varphi_t}\right], \qquad (2.76)
$$

where $H_1^{eff} = -\Delta_1 + \Phi_\kappa(x_1,t)$ is the effective Hamiltonian acting on the first variable. This leads to

$$
\begin{aligned}
d_t\beta^a(t) = d_t\left\langle\Psi_{N,t}, q_1^{\varphi_t}\Psi_{N,t}\right\rangle &= \quad i\left\langle\Psi_{N,t}, \left[\left(H_N - H_1^{eff}\right), q_1^{\varphi_t}\right]\Psi_{N,t}\right\rangle \\
&= \quad i\left\langle\Psi_{N,t}, \left[\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right), q_1^{\varphi_t}\right]\Psi_{N,t}\right\rangle \\
&= -2\mathrm{Im}\left\langle\Psi_{N,t}, \left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right) q_1^{\varphi_t}\Psi_{N,t}\right\rangle. \qquad (2.77)
\end{aligned}
$$

We calculate the commutator

$$i\left[H_N,\left(N^{-1/2}a(k)-\alpha_t(k)\right)\right]=-\,i\omega(k)N^{-1/2}a(k)-iN^{-1}\sum_{j=1}^{N}\tilde{\eta}(k)e^{-ikx_j} \tag{2.78}$$

by means of the canonical commutation relations (2.11) and continue with

$$\begin{aligned}
d_t\beta^b(t)=&\int d^3k\, d_t\Big\langle \left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t},\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t}\Big\rangle\\
=&\int d^3k\,\langle i\left[H_N,\left(N^{-1/2}a(k)-\alpha_t(k)\right)\right]\Psi_{N,t},\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t}\rangle\\
&+\int d^3k\,\Big\langle \left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t},i\left[H_N,\left(N^{-1/2}a(k)-\alpha_t(k)\right)\right]\Psi_{N,t}\Big\rangle\\
&-\int d^3k\,\Big\langle \left(\partial_t\alpha_t\right)(k)\Psi_{N,t},\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t}\Big\rangle\\
&-\int d^3k\,\Big\langle \left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t},\left(\partial_t\alpha_t\right)(k)\Psi_{N,t}\Big\rangle\\
=&\,2\int d^3k\,\mathrm{Re}\Big\langle i\left[H_N,\left(N^{-1/2}a(k)-\alpha_t(k)\right)\right]\Psi_{N,t},\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t}\Big\rangle\\
&-2\int d^3k\,\mathrm{Re}\Big\langle \left(\partial_t\alpha_t\right)(k)\Psi_{N,t},\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t}\Big\rangle\\
=&\,2\int d^3k\,\mathrm{Re}\big\{i\omega(k)\Big\langle \left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t},\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t}\Big\rangle\big\}\\
&+2\int d^3k\,\mathrm{Re}\big\{i\Big\langle N^{-1}\sum_{j=1}^{N}\tilde{\eta}(k)e^{-ikx_j}\Psi_{N,t},\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t}\Big\rangle\big\}\\
&-2\int d^3k\,\mathrm{Re}\big\{i\Big\langle (2\pi)^{3/2}\tilde{\eta}(k)\mathcal{FT}[|\varphi_t|^2](k)\Psi_{N,t},\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t}\Big\rangle\big\}.
\end{aligned} \tag{2.79}$$

So if we use the symmetry of the wave function and $\mathrm{Re}\{iz\}=-\mathrm{Im}\{z\}$, we get

$$\begin{aligned}
d_t\beta^b(t)=&-2\int d^3k\,\mathrm{Im}\big\{\omega(k)\Big\langle \left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t},\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t}\Big\rangle\big\}\\
&-2\int d^3k\,\mathrm{Im}\big\{\Big\langle \tilde{\eta}(k)e^{-ikx_1}\Psi_{N,t},\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t}\Big\rangle\big\}\\
&+2\int d^3k\,\mathrm{Im}\big\{\Big\langle (2\pi)^{3/2}\tilde{\eta}(k)\mathcal{FT}[|\varphi_t|^2](k)\Psi_{N,t},\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t}\Big\rangle\big\}\\
=&\,2\mathrm{Im}\big\{\Big\langle\Psi_{N,t},\Big(\int d^3k\,(2\pi)^{3/2}\tilde{\eta}(k)\mathcal{FT}^*[|\varphi_t|^2](k)\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Big)\Psi_{N,t}\Big\rangle\big\}\\
&-2\mathrm{Im}\big\{\Big\langle\Psi_{N,t},\Big(\int d^3k\,\tilde{\eta}(k)e^{ikx_1}\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Big)\Psi_{N,t}\Big\rangle\big\}.
\end{aligned} \tag{2.80}$$

□

Lemma 2.8.4. *Let $\Psi_{N,0}\in\left(L_s^2(\mathbb{R}^{3N})\otimes\mathcal{F}\right)\cap\mathcal{D}(\mathcal{N})\cap\mathcal{D}(\mathcal{N}H_N)$ with $||\Psi_{N,0}||=1$, $(\varphi_0,\alpha_0)\in(H^2(\mathbb{R}^3)\oplus L_1^2(\mathbb{R}^3))$ with $||\varphi_0||=1$ and assume that Conjecture 2.2.1 holds. Let $\Psi_{N,t}$ be the unique solution of (2.6) with initial data $\Psi_{N,0}$ and let (φ_t,α_t) be the unique solution of (2.17) with initial data (φ_0,α_0). Then for any $t\in\mathbb{R}_0^+$ there exists a generic constant C independent of N, Λ and t such that*

$$|d_t\beta\left(\Psi_{N,t},\varphi_t,\alpha_t\right)|\leq C\Lambda^2\left(\beta\left(\Psi_{N,t},\varphi_t,\alpha_t\right)+N^{-1}\right), \tag{2.81}$$

$$\beta\left(\Psi_{N,t},\varphi_t,\alpha_t\right)\leq e^{C\Lambda^2 t}\left(\beta\left(\Psi_{N,0},\varphi_0,\alpha_0\right)+N^{-1}\right). \tag{2.82}$$

Proof. Schwarz's inequality and $ab \leq 1/2(a^2+b^2)$ let us estimate the first line of Lemma 2.8.3 by

$$\begin{aligned} |d_t\beta^a(t)| &\leq 2|\langle \Psi_{N,t}, \left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right) q_1^{\varphi_t}\Psi_{N,t}\rangle| \\ &\leq \left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\Psi_{N,t}\right|\right|^2 + ||q_1^{\varphi_t}\Psi_{N,t}||^2 . \end{aligned} \tag{2.83}$$

By Lemma 2.8.2, we obtain

$$|d_t\beta^a(t)| \leq C\Lambda^2 \left(\beta(t) + N^{-1}\right). \tag{2.84}$$

In order to estimate $d_t\beta^b(t)$ we notice that

$$\begin{aligned} \int d^3k\, \tilde{\eta}(k)e^{ikx_1}\left(N^{-1/2}a(k) - \alpha(k,t)\right) &= \int d^3y\, \eta(x_1-y)\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)(x_1) \\ &= \quad \left(N^{-1/2}\widehat{\Phi}_\kappa^+(x_1) - \Phi_\kappa^+(x_1,t)\right) \end{aligned} \tag{2.85}$$

and

$$\begin{aligned} &\int d^3k\, \tilde{\eta}(k)(2\pi)^{3/2}\mathcal{FT}[|\varphi_t|^2]^*(k)\left(N^{-1/2}a(k) - \alpha_t(k)\right) \\ &= \int d^3y\, \left(\eta * |\varphi_t|^2\right)(y,t)\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right). \end{aligned} \tag{2.86}$$

follow from the convolution theorem. This gives

$$\begin{aligned} d_t\beta^b(t) = &-2\mathrm{Im}\int d^3y\, \langle \Psi_{N,t}, \eta(x_1-y)\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_{N,t}\rangle \\ &+2\mathrm{Im}\int d^3y\, \langle \Psi_{N,t}, \left(\eta * |\varphi_t|^2\right)(y,t)\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_{N,t}\rangle. \end{aligned} \tag{2.87}$$

We see that not only present gauge boson fluctuations around the coherent state lead to a growth in $\beta^b(t)$ but an additional change appears, because the second quantized radiation field couples to the mean particle density of the many-body system while the source of the classical field is given by the density of the condensate wave function. In order to estimate the difference between the densities by the functional $\beta^a(t)$ we insert the identity $1 = p_1^{\varphi_t} + q_1^{\varphi_t}$.

$$\begin{aligned} d_t\beta^b(t) = &-2\mathrm{Im}\int d^3y\, \langle \Psi_{N,t}, p_1^{\varphi_t}\eta(x_1-y)p_1^{\varphi_t}\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_{N,t}\rangle \\ &+2\mathrm{Im}\int d^3y\, \langle \Psi_{N,t}, \left(\eta * |\varphi_t|^2\right)(y,t)\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_{N,t}\rangle \\ &-2\mathrm{Im}\int d^3y\, \langle \Psi_{N,t}, q_1^{\varphi_t}\eta(x_1-y)p_1^{\varphi_t}\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_{N,t}\rangle \\ &-2\mathrm{Im}\int d^3y\, \langle \Psi_{N,t}, \eta(x_1-y)q_1^{\varphi_t}\left(N^{-1/2}\hat{F}_\kappa^+(y) - F_\kappa^+(y,t)\right)\Psi_{N,t}\rangle. \end{aligned} \tag{2.88}$$

The first two lines are the most important. They become small, because the mean particle density of the many-body system is approximately given by the density of the condensate wave function. From $\eta(-x) = \eta(x)$ we conclude

$$p_1^{\varphi_t}\eta(x_1-y)p_1^{\varphi_t} = p_1^{\varphi_t}\int d^3z\, \eta(z-y)|\varphi_t|^2(z,t) = p_1^{\varphi_t}\left(\eta * |\varphi_t|^2\right)(y,t) \tag{2.89}$$

and continue with

$$\begin{aligned}
d_t\beta^b(t) = &-2\mathrm{Im}\int d^3y\,\langle\Psi_{N,t},(p_1^{\varphi_t}-1)\left(\eta*|\varphi_t|^2\right)(y,t)\left(N^{-1/2}\hat{F}_\kappa^+(y)-F_\kappa^+(y,t)\right)\Psi_{N,t}\rangle\\
&-2\mathrm{Im}\big\langle\Psi_{N,t},q_1\int d^3y\,\eta(x_1-y)\left(N^{-1/2}\hat{F}_\kappa^+(y)-F_\kappa^+(y,t)\right)p_1^{\varphi_t}\Psi_{N,t}\big\rangle\\
&-2\mathrm{Im}\big\langle\Psi_{N,t},\int d^3y\,\eta(x_1-y)\left(N^{-1/2}\hat{F}_\kappa^+(y)-F_\kappa^+(y,t)\right)q_1^{\varphi_t}\Psi_{N,t}\big\rangle\\
= &\ 2\mathrm{Im}\int d^3y\,\langle\Psi_{N,t},q_1^{\varphi_t}\left(\eta*|\varphi_t|^2\right)(y,t)\left(N^{-1/2}\hat{F}_\kappa^+(y)-F_\kappa^+(y,t)\right)\Psi_{N,t}\rangle \quad (2.90)\\
&-2\mathrm{Im}\big\langle\Psi_{N,t},q_1^{\varphi_t}\left(N^{-1/2}\widehat{\Phi}_\kappa^+(x_1)-\Phi_\kappa^+(x_1,t)\right)p_1^{\varphi_t}\Psi_{N,t}\big\rangle \quad (2.91)\\
&-2\mathrm{Im}\big\langle\Psi_{N,t},\left(N^{-1/2}\widehat{\Phi}_\kappa^+(x_1)-\Phi_\kappa^+(x_1,t)\right)q_1^{\varphi_t}\Psi_{N,t}\big\rangle. \quad (2.92)
\end{aligned}$$

In the following, we estimate each line separately.

$$\begin{aligned}
|(2.90)|\le 2|&\int d^3y\,\langle\left(\eta*|\varphi_t|^2\right)(y,t)q_1^{\varphi_t}\Psi_{N,t},\left(N^{-1/2}\hat{F}_\kappa^+(y)-F_\kappa^+(y,t)\right)\Psi_{N,t}\rangle|\\
\le\ &\int d^3y\,\langle q_1^{\varphi_t}\Psi_{N,t},|\left(\eta*|\varphi_t|^2\right)(y,t)|^2q_1^{\varphi_t}\Psi_N\rangle\\
+\ &\int d^3y\,\left|\left|\left(N^{-1/2}\hat{F}_\kappa^+(y)-F_\kappa^+(y,t)\right)\Psi_{N,t}\right|\right|^2\\
\le\ &\left|\left|\eta*|\varphi_t|^2\right|\right|_2^2\langle\Psi_{N,t},q_1^{\varphi_t}\Psi_{N,t}\rangle+\beta^b(t)\le C\Lambda^2\beta(t). \quad (2.93)
\end{aligned}$$

Here we have used that

$$\left|\left|\eta*|\varphi_t|^2\right|\right|_2\le||\eta||_2\left|\left||\varphi_t|^2\right|\right|_1=||\eta||_2\,||\varphi_t||_2^2=C\Lambda \quad (2.94)$$

holds due to Young's inequality and (2.65). Lemma 2.8.2 leads to

$$\begin{aligned}
|(2.91)|\le&2|\big\langle q_1^{\varphi_t}\Psi_N,\left(N^{-1/2}\widehat{\Phi}_\kappa^+(x_1)-\Phi_\kappa^+(x_1,t)\right)p_1^{\varphi_t}\Psi_N\big\rangle|\\
\le&\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa^+(x_1)-\Phi_\kappa^+(x_1,t)\right)p_1^{\varphi_t}\Psi_N\right|\right|^2+||q_1^{\varphi_t}\Psi_N||^2\le C\Lambda^2\beta(t) \quad (2.95)
\end{aligned}$$

and

$$\begin{aligned}
|(2.92)|\le&2|\big\langle\left(N^{-1/2}\widehat{\Phi}_\kappa^-(x_1)-\Phi_\kappa^-(x_1,t)\right)\Psi_N,q_1^{\varphi_t}\Psi_N\big\rangle|\\
\le&\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa^-(x_1)-\Phi_\kappa^-(x_1,t)\right)\Psi_N\right|\right|^2+||q_1^{\varphi_t}\Psi_N||^2\\
\le&C\Lambda^2\left(\beta(t)+N^{-1}\right). \quad (2.96)
\end{aligned}$$

In total we have

$$|d_t\beta^b\,(t)\,|\le C\Lambda^2\left(\beta\,(t)+N^{-1}\right). \quad (2.97)$$

Now we can put the terms together to get

$$d_t\beta\,(t)\le|d_t\beta^a\,(t)\,|+|d_t\beta^b\,(t)\,|\le C\Lambda^2\left(\beta\,(t)+N^{-1}\right). \quad (2.98)$$

Gronwall's Lemma 1.3.1 then gives rise to

$$\beta\,(t)\le e^{C\Lambda^2t}\left(\beta\,(0)+N^{-1}\right). \quad (2.99)$$

□

2.8.3 Control of the kinetic energy

In order to prove the convergence of the one-particle reduced density matrix of the charges in Sobolev norm it is necessary to control the kinetic energy of the particles which are not in the condensate (see Section 2.7). To this end we add $\beta^c(\Psi_{N,t},\varphi_t) := \|\nabla_1 q_1^{\varphi_t}\Psi_{N,t}\|^2$ to the functional and perform a Gronwall estimate for the functional $\beta_2(\Psi_{N,t},\varphi_t,\alpha_t)$.

Lemma 2.8.5. *Let $\Psi_{N,0} \in \left(L_s^2(\mathbb{R}^{3N}) \otimes \mathcal{F}\right) \cap \mathcal{D}(\mathcal{N}) \cap \mathcal{D}(\mathcal{N}H_N) \cap \mathcal{D}\left(H_N^2\right)$ with $\|\Psi_{N,0}\| = 1$, $(\varphi_0,\alpha_0) \in (H^4(\mathbb{R}^3) \oplus L_2^2(\mathbb{R}^3))$ with $\|\varphi_0\| = 1$ and assume that Conjecture 2.2.1 holds. Let $\Psi_{N,t}$ be the unique solution of* (2.6) *with initial data $\Psi_{N,0}$ and let (φ_t,α_t) be the unique solution of* (2.17) *with initial data (φ_0,α_0). Then*

$$\begin{aligned} d_t\beta^c(\Psi_{N,t},\varphi_t) = {} & 2\mathrm{Im}\left\langle p_1^{\varphi_t}\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi N,t\right\rangle \\ & - 2\mathrm{Im}\left\langle \left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)p_1^{\varphi_t}\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\ & - 2\mathrm{Im}\left\langle N^{-1/2}\widehat{\Phi}_\kappa(x_1)q_1^{\varphi_t}\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle. \end{aligned} \tag{2.100}$$

Proof. From $(\varphi_0,\alpha_0) \in (H^4(\mathbb{R}^3) \oplus L_2^2(\mathbb{R}^3))$ with $\|\varphi_0\| = 1$ and $\Psi_{N,0} \in \left(L_s^2(\mathbb{R}^{3N}) \otimes \mathcal{F}\right) \cap \mathcal{D}(\mathcal{N}) \cap \mathcal{D}(\mathcal{N}H_N) \cap \mathcal{D}\left(H_N^2\right)$, it follows that $(\varphi_t,\alpha_t) \in (H^4(\mathbb{R}^3) \oplus L_2^2(\mathbb{R}^3))$ with $\|\varphi_t\| = 1$ and $\Psi_{N,t} \in \left(L_s^2(\mathbb{R}^{3N}) \otimes \mathcal{F}\right) \cap \mathcal{D}(\mathcal{N}) \cap \mathcal{D}(\mathcal{N}H_N) \cap \mathcal{D}\left(H_N^2\right)$ for all $t \in \mathbb{R}_0^+$ by Conjecture 2.2.1, Stone's Theorem and Lemma 2.11.3. This ensures that the following expressions are well defined. The derivative of $\beta^c(t)$ is determined by

$$\begin{aligned} d_t\beta^c(t) &= i\left\langle q_1^{\varphi_t}H_N\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle - i\left\langle q_1^{\varphi_t}\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}H_N\Psi_{N,t}\right\rangle \\ &+ i\left\langle \left[H_1^{eff}, q_1^{\varphi_t}\right]\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle - i\left\langle q_1^{\varphi_t}\Psi_{N,t}, (-\Delta_1)\left[H_1^{eff}, q_1^{\varphi_t}\right]\Psi_{N,t}\right\rangle \\ &= i\left\langle q_1^{\varphi_t}H_N\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle - i\left\langle (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}, q_1^{\varphi_t}H_N\Psi_{N,t}\right\rangle \\ &+ i\left\langle \left[H_1^{eff}, q_1^{\varphi_t}\right]\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle - i\left\langle (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}, \left[H_1^{eff}, q_1^{\varphi_t}\right]\Psi_{N,t}\right\rangle \\ &= -2\mathrm{Im}\left\langle q_1^{\varphi_t}H_N\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\ &\quad - 2\mathrm{Im}\left\langle \left[H_1^{eff}, q_1^{\varphi_t}\right]\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle. \end{aligned} \tag{2.101}$$

Since $\left\langle q_1^{\varphi_t}\left(-\Delta_i + N^{-1/2}\widehat{\Phi}_\kappa(x_i)\right)\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle$ and $\left\langle q_1^{\varphi_t}H_f\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle$ are real numbers for $i \in \{2,3,\dots,N\}$ this becomes

$$\begin{aligned} d_t\beta^c(t) = {} & - 2\mathrm{Im}\left\langle q_1^{\varphi_t}\left(-\Delta_1 + N^{-1/2}\widehat{\Phi}_\kappa(x_1)\right)\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\ & + 2\mathrm{Im}\left\langle q_1^{\varphi_t}H_1^{eff}\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\ & - 2\mathrm{Im}\left\langle H_1^{eff}q_1^{\varphi_t}\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\ = {} & - 2\mathrm{Im}\left\langle q_1^{\varphi_t}\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\ & - 2\mathrm{Im}\left\langle \Phi_\kappa(x_1,t)q_1^{\varphi_t}\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\ & - 2\mathrm{Im}\,\|(-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\|^2 \\ = {} & - 2\mathrm{Im}\left\langle q_1^{\varphi_t}\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\ & - 2\mathrm{Im}\left\langle \Phi_\kappa(x_1,t)q_1^{\varphi_t}\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle. \end{aligned} \tag{2.102}$$

The identity $q_1^{\varphi_t}\mathcal{O} = \mathcal{O}p_1^{\varphi_t} + \mathcal{O}q_1^{\varphi_t} - p_1^{\varphi_t}\mathcal{O}$ (for any operator $\mathcal{O}$) and

$$\begin{aligned} -\left\langle \Phi_\kappa(x_1,t)q_1^{\varphi_t}\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle = {} & \left\langle \left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)q_1^{\varphi_t}\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\ & - \left\langle N^{-1/2}\widehat{\Phi}_\kappa(x_1)q_1^{\varphi_t}\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\right\rangle \end{aligned} \tag{2.103}$$

lead to

$$d_t\beta^c(t) = 2\mathrm{Im}\langle p_1^{\varphi_t}\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\rangle \tag{2.104}$$

$$- 2\mathrm{Im}\langle \left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)p_1^{\varphi_t}\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\rangle \tag{2.105}$$

$$- 2\mathrm{Im}\langle N^{-1/2}\widehat{\Phi}_\kappa(x_1)q_1^{\varphi_t}\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\rangle. \tag{2.106}$$

□

Lemma 2.8.6. *Let $\Psi_{N,0} \in \left(L_s^2(\mathbb{R}^{3N})\otimes\mathcal{F}\right)\cap\mathcal{D}(\mathcal{N})\cap\mathcal{D}(\mathcal{N}H_N)\cap\mathcal{D}\left(H_N^2\right)$ with $||\Psi_{N,0}|| = 1$, $(\varphi_0,\alpha_0)\in(H^4(\mathbb{R}^3)\oplus L_2^2(\mathbb{R}^3))$ with $||\varphi_0|| = 1$ and assume that Conjecture 2.2.1 holds. Let $\Psi_{N,t}$ be the unique solution of* (2.6) *with initial data $\Psi_{N,0}$ and let (φ_t,α_t) be the unique solution of* (2.17) *with intial data (φ_0,α_0). Then, there exists a positive monotone increasing function $C(s)$ of the norms $||\alpha_s||_{L^2(\mathbb{R}^3)}$ and $||\varphi_s||_{H^1(\mathbb{R}^3)}$ such that*

$$\begin{aligned}|d_t\beta_2\left(\Psi_{N,t},\varphi_t,\alpha_t\right)| \leq& \Lambda^4 C(t)\left(\beta_2\left(\Psi_{N,t},\varphi_t,\alpha_t\right) + N^{-1}\right),\\ \beta_2\left(\Psi_{N,t},\varphi_t,\alpha_t\right) \leq& e^{\Lambda^4\int_0^t C(s)ds}\left(\beta_2\left(\Psi_{N,0},\varphi_0,\alpha_0\right) + N^{-1}\right)\end{aligned} \tag{2.107}$$

hold for any $t\in\mathbb{R}_0^+$.

Proof. In order to estimate $d_t\beta^c(t)$ by β and $||\nabla_1 q_1^{\varphi_t}\Psi_{N,t}||$ we will integrate by parts and apply Schwarz's inequality. The gradiant will hereby occasionally act on the radiation fields, which will give rise to the vector fields

$$\begin{aligned}(\nabla\widehat{\Phi}_\kappa)(x) =& \int d^3k\,\tilde{\eta}(k)ki\left(e^{ikx}a(k) - e^{-ikx}a^*(k)\right),\\ (\nabla\Phi_\kappa)(x,t) =& \int d^3k\,\tilde{\eta}(k)ki\left(e^{ikx}\alpha_t(k) - e^{-ikx}\alpha_t^*(k)\right).\end{aligned} \tag{2.108}$$

We define the vector field $\tilde{\Theta}(k) := \tilde{\eta}(k)k$ and its Fourier transform Θ with $\sum_{i=1}^3||\Theta^i||_2^2 \leq \Lambda^4/(16\pi^2)$. This allows us to obtain the relation

$$(\nabla\widehat{\Phi}_\kappa^+)(x) = i\left(\Theta * \hat{F}_\kappa^+\right)(x), \quad (\nabla\Phi_\kappa^+)(x,t) = i\left(\Theta * F_\kappa^+\right)(x) \tag{2.109}$$

between the positive frequency part of the vector fields and the auxiliary fields (2.62). In analogy to Lemma 2.8.2 one proves the estimates

$$\begin{aligned}\left|\left|\left(N^{-1/2}(\nabla\widehat{\Phi}_\kappa)(x_1) - (\nabla\Phi_\kappa)(x_1,t)\right)p_1\Psi_N\right|\right|^2 \leq& C\Lambda^4\left(\beta^b(t) + N^{-1}\right),\\ \left|\left|\left(N^{-1/2}(\nabla\widehat{\Phi}_\kappa)(x_1) - (\nabla\Phi_\kappa)(x_1,t)\right)q_1\Psi_N\right|\right|^2 \leq& C\Lambda^4\left(\beta^b(t) + N^{-1}\right),\\ \left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\nabla_1 p_1\Psi_N\right|\right|^2 \leq& C\Lambda^2\,||\nabla\varphi||_2^2\left(\beta^b(t) + N^{-1}\right).\end{aligned} \tag{2.110}$$

The first term of $d_t\beta^c(t)$ is estimated by

$$\begin{aligned}|(2.104)| \leq& 2|\langle p_1^{\varphi_t}\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\Psi_{N,t}, (-\Delta_1)q_1^{\varphi_t}\Psi_{N,t}\rangle|\\ =& 2|\langle\nabla_1 p_1^{\varphi_t}\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\Psi_{N,t}, \nabla_1 q_1^{\varphi_t}\Psi_{N,t}\rangle|\\ \leq& \left|\left|\nabla_1 p_1\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\Psi_N\right|\right|^2 + ||\nabla_1 q_1\Psi_N||^2\\ \leq& ||\nabla\varphi_t||^2\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\Psi_N\right|\right|^2 + ||\nabla_1 q_1\Psi_N||^2.\end{aligned} \tag{2.111}$$

Lemma 2.8.2 gives rise to

$$\begin{aligned}|(2.104)| &\leq C\Lambda^2 \left|\left|\nabla\varphi_t\right|\right|^2 \left(\beta^b + N^{-1}\right) + \left|\left|\nabla_1 q_1 \Psi_N\right|\right|^2 \\ &\leq \Lambda^2 C(\left|\left|\varphi_t\right|\right|_{H^1}) \left(\beta_2(t) + N^{-1}\right). \end{aligned} \tag{2.112}$$

Likewise, we estimate

$$\begin{aligned}|(2.105)| \leq& 2\left|\left\langle \left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right) p_1^{\varphi_t}\Psi_{N,t}, (-\Delta_1) q_1^{\varphi_t}\Psi_{N,t}\right\rangle\right| \\ =& 2\left|\left\langle\nabla_1 \left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right) p_1\Psi_N, \nabla_1 q_1 \Psi_N\right\rangle\right| \\ \leq& \left|\left|\nabla_1 \left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right) p_1\Psi_N\right|\right|^2 + \left|\left|\nabla_1 q_1\Psi_N\right|\right|^2. \end{aligned} \tag{2.113}$$

Due to triangular inequality, $(a+b)^2 \leq 2\left(a^2+b^2\right)$ and (2.110) this becomes

$$\begin{aligned}|(2.105)| \leq& 2\left|\left|\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right)\nabla_1 p_1\Psi_N\right|\right|^2 \\ &+ 2\left|\left|\left(N^{-1/2}(\nabla\widehat{\Phi}_\kappa)(x_1) - (\nabla\Phi_\kappa\right)(x_1)) p_1\Psi_N\right|\right|^2 + \left|\left|\nabla_1 q_1\Psi_N\right|\right|^2 \\ \leq& \Lambda^4 C(\left|\left|\varphi_t\right|\right|_{H^1}) \left(\beta_2(t) + N^{-1}\right). \end{aligned} \tag{2.114}$$

Next, we consider line

$$\begin{aligned}(2.106) =& -2\mathrm{Im}\left\langle\nabla_1 N^{-1/2}\widehat{\Phi}_\kappa(x_1) q_1^{\varphi_t}\Psi_{N,t}, \nabla_1 q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\ =& -2\mathrm{Im}\left\langle N^{-1/2}(\nabla\widehat{\Phi}_\kappa)(x_1) q_1^{\varphi_t}\Psi_{N,t}, \nabla_1 q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\ & -2\mathrm{Im}\left\langle N^{-1/2}\widehat{\Phi}_\kappa(x_1)\nabla_1 q_1^{\varphi_t}\Psi_{N,t}, \nabla_1 q_1^{\varphi_t}\Psi_{N,t}\right\rangle. \end{aligned} \tag{2.115}$$

The scalar product in the last line is easily shown to be real. This yields

$$\begin{aligned}(2.106) =& -2\mathrm{Im}\left\langle N^{-1/2}(\nabla\widehat{\Phi}_\kappa)(x_1) q_1^{\varphi_t}\Psi_{N,t}, \nabla_1 q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\ =& -2\mathrm{Im}\left\langle \left(N^{-1/2}(\nabla\widehat{\Phi}_\kappa)(x_1) - (\nabla\Phi_\kappa)(x_1,t)\right) q_1^{\varphi_t}\Psi_{N,t}, \nabla_1 q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\ & -2\mathrm{Im}\left\langle(\nabla\Phi_\kappa)(x_1,t) q_1^{\varphi_t}\Psi_{N,t}, \nabla_1 q_1^{\varphi_t}\Psi_{N,t}\right\rangle. \end{aligned} \tag{2.116}$$

and allows us to estimate

$$\begin{aligned}|(2.106)| \leq& 2\left|\left\langle \left(N^{-1/2}(\nabla\widehat{\Phi}_\kappa)(x_1) - (\nabla\Phi_\kappa)(x_1,t)\right) q_1^{\varphi_t}\Psi_{N,t}, \nabla_1 q_1^{\varphi_t}\Psi_{N,t}\right\rangle\right| \\ &+ 2\left|\left\langle(\nabla\Phi_\kappa)(x_1,t) q_1^{\varphi_t}\Psi_{N,t}, \nabla_1 q_1^{\varphi_t}\Psi_{N,t}\right\rangle\right| \\ \leq& \left|\left|\left(N^{-1/2}(\nabla\widehat{\Phi}_\kappa)(x_1) - (\nabla\Phi_\kappa)(x_1,t)\right) q_1^{\varphi_t}\Psi_{N,t}\right|\right|^2 + \left|\left|(\nabla\Phi_\kappa)(x_1,t) q_1^{\varphi_t}\Psi_{N,t}\right|\right|^2 \\ &+ 2\left|\left|\nabla_1 q_1^{\varphi_t}\Psi_{N,t}\right|\right|^2 \leq C\Lambda^4\left(\beta^b(t) + N^{-1}\right) + C\Lambda^4 \left|\left|\alpha_t\right|\right|_2^2 \beta^a(t) + 2\beta^c(t) \\ \leq& \Lambda^4 C(\left|\left|\alpha_t\right|\right|_2)\left(\beta_2(t) + N^{-1}\right). \end{aligned} \tag{2.117}$$

Here, we used (2.110) and the fact that

$$\left|\left|\left(\nabla\Phi_\kappa\right)(\cdot,t)\right|\right|_\infty \leq C\Lambda^2 \left|\left|\alpha_t\right|\right|_2 \tag{2.118}$$

holds because of Schwarz's inequality. In total, we have

$$|d_t\beta^c(t)| \leq \Lambda^4 C(\left|\left|\varphi_t\right|\right|_{H^1}, \left|\left|\alpha_t\right|\right|)\left(\beta_2 + N^{-1}\right). \tag{2.119}$$

With Lemma 2.8.4 this implies

$$|d_t\beta_2\left[\Psi_{N,t},\varphi_t,\alpha_t\right]| \leq \Lambda^4 C(||\varphi_t||_{H^1}, ||\alpha_t||)\left(\beta_2\left[\Psi_{N,t},\varphi_t,\alpha_t\right]+N^{-1}\right) \tag{2.120}$$

Using the shorthand notation $C(t) := C(||\varphi_t||_{H^1}, ||\alpha_t||)$ we obtain

$$\beta_2\left[\Psi_{N,t},\varphi_t,\alpha_t\right] \leq e^{\Lambda^4\int_0^t C(s)ds}\left(\beta_2\left[\Psi_{N,0},\varphi_0,\alpha_0\right]+N^{-1}\right) \tag{2.121}$$

by means of Gronwall's inequality 1.3.1.

□

2.9 Initial states

Subsequently, we are concerned with the initial states of Theorem 2.3.1.

Lemma 2.9.1. *Let* $\Psi_{N,0} \in \left(L^2_s(\mathbb{R}^{3N})\otimes\mathcal{F}\right)\cap\mathcal{D}(\mathcal{N})$ *with* $||\Psi_{N,0}|| = 1$ *and* $(\varphi_0,\alpha_0) \in (L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3))$ *with* $||\varphi_0|| = 1$. *Then*

$$\begin{aligned}
&\beta^a(\Psi_{N,0},\varphi_0) \leq Tr_{L^2(\mathbb{R}^3)}|\gamma^{(1,0)}_{N,0}-|\varphi_0\rangle\langle\varphi_0|| = a_N,\\
&\beta^b(\Psi_{N,0},\alpha_0) = N^{-1}\left\langle W^{-1}(\sqrt{N\alpha_0})\Psi_{N,0},\mathcal{N}W^{-1}(\sqrt{N\alpha_0})\Psi_{N,0}\right\rangle = b_N.
\end{aligned} \tag{2.122}$$

Proof. The first inequality is a consequence of Lemma 2.7.1. Before we prove the second relation we justify (2.37). Therefore, is useful to note that the Weyl operator ($f \in L^2(\mathbb{R}^3)$)

$$W(f) = \exp\left(\int d^3k\, f(k)a^*(k) - f^*(k)a(k)\right) \tag{2.123}$$

is unitary

$$W^{-1}(f) = W^*(f) = W(-f) \tag{2.124}$$

and satisfies[16]

$$W^*(f)a(k)W(f) = a(k)+f(k), \quad W^*(f)a^*(k)W(f) = a^*(k)+f^*(k). \tag{2.125}$$

This leads to

$$\begin{aligned}
\beta^b(\Psi_{N,t},\alpha_t) &= \int d^3k\left|\left|\left(N^{-1/2}a(k)-\alpha_t(k)\right)\Psi_{N,t}\right|\right|^2\\
&= \int d^3k\left|\left|W^*(\sqrt{N}\alpha_t)\left(N^{-1/2}a(k)-\alpha_t(k)\right)W(\sqrt{N}\alpha_t)W^*(\sqrt{N}\alpha_t)\Psi_{N,t}\right|\right|^2\\
&= \int d^3k\left|\left|N^{-1/2}a(k)W^*(\sqrt{N}\alpha_t)\Psi_{N,t}\right|\right|^2\\
&= N^{-1}\left\langle W^*(\sqrt{N}\alpha_t)e^{-iH_Nt}\Psi_{N,0},\mathcal{N}W^*(\sqrt{N}\alpha_t)e^{-iH_Nt}\Psi_{N,0}\right\rangle.
\end{aligned} \tag{2.126}$$

Let

$$\mathcal{U}_N(t;0) := W^*(\sqrt{N}\alpha_t)e^{-iH_Nt}W(\sqrt{N}\alpha_0) \tag{2.127}$$

[16] More information is given in [80][p.9]

denote the fluctuation dynamics then

$$\beta^b(\Psi_{N,t},\alpha_t) = N^{-1}\langle \mathcal{U}_N(t;0)W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}, \mathcal{N}\mathcal{U}_N(t;0)W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}\rangle \tag{2.128}$$

follows from the unitarity of the Weyl operator. In particular, we have

$$\beta^b(\Psi_{N,0},\alpha_0) = N^{-1}\langle W^{-1}(\sqrt{N\alpha_0})\Psi_{N,0}, \mathcal{N}W^{-1}(\sqrt{N\alpha_0})\Psi_{N,0}\rangle = b_N. \tag{2.129}$$

□

In the following, we are concerned with initial states of product form (2.15).

Lemma 2.9.2. *Let* $(\varphi_0,\alpha_0) \in (H^2(\mathbb{R}^3) \oplus L^2_1(\mathbb{R}^3))$*with* $||\varphi_0|| = 1$ *and* $\Psi_{N,0} = \varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega$*. Then*

$$\begin{aligned} a_N &= Tr_{L^2(\mathbb{R}^3)}|\gamma^{(1,0)}_{N,0} - |\varphi_0\rangle\langle\varphi_0|| = 0, && (2.130)\\ b_N &= N^{-1}\langle W^{-1}(\sqrt{N\alpha_0})\Psi_{N,0}, \mathcal{N}W^{-1}(\sqrt{N\alpha_0})\Psi_{N,0}\rangle = 0 \textit{ and} && (2.131)\\ \Psi_{N,0} &\in \left(L^2_s(\mathbb{R}^{3N})\otimes\mathcal{F}\right)\cap\mathcal{D}(\mathcal{N})\cap\mathcal{D}(\mathcal{N}H_N). && (2.132)\end{aligned}$$

Let $(\varphi_0,\alpha_0) \in (H^4(\mathbb{R}^3) \oplus L^2_2(\mathbb{R}^3))$*with* $||\varphi_0|| = 1$ *then*

$$\begin{aligned} c_N &= ||\nabla_1 q_1^{\varphi_0}\Psi_{N,0}||^2 = 0 && (2.133)\\ \Psi_{N,0} &\in \left(L^2_s(\mathbb{R}^{3N})\otimes\mathcal{F}\right)\cap\mathcal{D}(\mathcal{N})\cap\mathcal{D}(\mathcal{N}H_N)\cap\mathcal{D}\left(H_N^2\right). && (2.134)\end{aligned}$$

Proof. From the definition of the one-particle reduced density matrix and (2.129) we directly obtain the relations (2.130) and (2.131). Equation (2.133) holds because $\Psi_{N,0}$ is in the kernel of the projector $q_1^{\varphi_0}$. In order to show (2.132) we point out that

$$\Psi^{(n)}_{N,0}(X_N,K_n) = \prod_{i=1}^N \varphi_0(x_i)e^{-N||\alpha_0||^2/2}(n!)^{-1/2}\prod_{j=1}^n (N)^{1/2}\alpha_0(k_j) \tag{2.135}$$

follows from the definition of the the Weyl operators [80][p.8]. A direct calculation gives

$$\sum_{n=1}^\infty n^2 \left|\left|\Psi^{(n)}_{N,0}\right|\right|^2 = N\,||\alpha_0||^2 + N^2\,||\alpha_0||^4. \tag{2.136}$$

Hence, $\Psi^{(n)}_{N,0} \in \mathcal{D}(\mathcal{N})$ (see (2.23)). Moreover, we have $\Psi_{N,0} \in \mathcal{D}(\sum_{i=1}^N -\Delta_i)$ because $\varphi_0 \in H^2(\mathbb{R}^3)$. A straightforward estimate leads to

$$\sum_{n=1}^\infty \int d^{3N}x\, d^{3n}k\, |\sum_{j=1}^n w(k_j)|^2 |\Psi^{(n)}_{N,0}(X_N,K_n)|^2 \leq C(N, ||\alpha_0||_{L^2_1(\mathbb{R}^3)}). \tag{2.137}$$

From (2.12) we then conclude $\Psi^{(n)}_{N,0} \in \mathcal{D}(H_f)$ and $\Psi^{(n)}_{N,0} \in \mathcal{D}(H_N) = \mathcal{D}(\sum_{i=1}^N -\Delta_i)\cap\mathcal{D}(H_f)$. Similarly, one derives

$$\begin{aligned}\sum_{n=1}^N n^2\left|\left|(H_N\Psi_{N,0})^{(n)}\right|\right|^2 &\leq C\sum_{n=1}^\infty n^2\Big(\left|\left|\sum_{j=1}^N \Delta_j\Psi^{(n)}_{N,0}\right|\right|^2 + \left|\left|\sum_{j=1}^N N^{-1/2}(\widehat{\Phi}_\kappa(x_j)\Psi_{N,0})^{(n)}\right|\right|^2\Big)\\ &+ C\sum_{n=1}^\infty n^2\left|\left|(H_f\Psi_{N,0})^{(n)}\right|\right|^2 \leq C(N,\Lambda,||\varphi_0||_{H^2(\mathbb{R}^3)}, ||\alpha_0||_{L^2_1(\mathbb{R}^3)}).\end{aligned} \tag{2.138}$$

and concludes $\Psi_{N,0} \in \mathcal{D}(\mathcal{N}H_N) = \big\{\Psi_N \in \mathcal{D}(H_N) : H_N\Psi_N \in \mathcal{D}(\mathcal{N})\big\}$. In order to show (2.134) we would like to note that $(\varphi_0, \alpha_0) \in (H^4(\mathbb{R}^3), L_2^2(\mathbb{R}^3))$, $|\cdot|^2\tilde{\eta} \in L^2(\mathbb{R}^3)$ and $\tilde{\eta} \in L^2(\mathbb{R}^3)$ imply $H_N\Psi_{N,0} \in \mathcal{D}(\sum_{i=1}^N -\Delta_i)$. By means of the estimate

$$\sum_{n=1}^{\infty} d^{3N}x\, d^{3n}k \,|\sum_{j=1}^{n} w(k_j)|^2 |(H_N\Psi_{N,0})^{(n)}(X_N, K_n)|^2 \leq C(N, \Lambda, ||\varphi_0||_{H^2(\mathbb{R}^3)}, ||\alpha_0||_{L_2^2(\mathbb{R}^3)}) \tag{2.139}$$

one obtains $H_N\Psi_{N,0} \in \mathcal{D}(H_f)$. In total, we have $H_N\Psi_{N,0} \in \mathcal{D}(H_N)$ and $\Psi_{N,0} \in \mathcal{D}(H_N^2)$. □

2.10 Proof of Theorem 2.3.1

In order to finish the proof of Theorem 2.3.1 we remark that Lemma 2.7.1 and Lemma 2.9.1 imply

$$\begin{aligned} \beta(\Psi_{N,0}, \varphi_0, \alpha_0) &\leq a_N + b_N, \\ \beta_2(\Psi_{N,0}, \varphi_0, \alpha_0) &\leq a_N + b_N + c_N. \end{aligned} \tag{2.140}$$

We then choose for a given time $t \in \mathbb{R}_0^+$ the number N of charged particles large enough such that the values of $\beta(\Psi_{N,t}, \varphi_t, \alpha_t)$ in (2.82) and $\beta_2(\Psi_{N,t}, \varphi_t, \alpha_t)$ in (2.107) are smaller than one and derive Theorem 2.3.1 by means of Lemma 2.7.1.

2.11 Appendix: Proof of Lemma 2.8.3

In this section, we derive Lemma 2.8.3 in greater detail. Hereby, we occasionally use the notation $\Psi_{N,t} = \Psi_N(t)$, $\varphi_t = \varphi(t)$ and $\alpha_t = \alpha(t)$. Since the functional $\beta^a(\cdot) : \mathbb{R} \to \mathbb{R}_0^+, t \mapsto \big\langle \Psi_{N,t}, q_1^{\varphi_t}\Psi_{N,t}\big\rangle$ (and likewise β^b) is a real function in t, we can determine its derivative by the quotient

$$d_t\beta^a(t) := \lim_{h\to 0, h\neq 0} \frac{\beta^a(t+h) - \beta^a(t)}{h}. \tag{2.141}$$

Moreover, we would like to note that $\Psi_{N,t} \in \mathcal{D}(H_N)$ and

$$\begin{aligned} &\lim_{h\to 0, h\neq 0} ||\Psi_N(t+h) - \Psi_N(t)|| = 0, \\ &\lim_{h\to 0, h\neq 0} \left|\left|\frac{\Psi_N(t+h) - \Psi_N(t)}{h} + iH_N\Psi_N(t)\right|\right| = 0 \end{aligned} \tag{2.142}$$

follow from $\Psi_{N,0} \in \mathcal{D}(H_N)$ and Stone's Theorem. Let $(\varphi_0, \alpha_0) \in \big(H^2(\mathbb{R}^3) \oplus L_1^2(\mathbb{R}^3)\big)$. According to Conjecture 2.2.1 there exists a strong solution such that $(\varphi_t, \alpha_t) \in \big(H^2(\mathbb{R}^3) \oplus L_1^2(\mathbb{R}^3)\big)$. Furthermore,

$$\begin{aligned} &\lim_{h\to 0, h\neq 0} ||\varphi(t+h) - \varphi(t)|| = 0, \\ &\lim_{h\to 0, h\neq 0} ||\alpha(t+h) - \alpha(t)|| = 0, \\ &\lim_{h\to 0, h\neq 0} \left|\left|\frac{\varphi(t+h) - \varphi(t)}{h} + iH^{eff}\varphi(t)\right|\right| = 0 \quad \text{and} \\ &\lim_{h\to 0, h\neq 0} \left|\left|\frac{\alpha(t+h) - \alpha(t)}{h} + i\Big(\omega\alpha(t) + (2\pi)^{3/2}\tilde{\eta}\mathcal{FT}[|\varphi(t)|^2]\Big)\right|\right| = 0. \end{aligned} \tag{2.143}$$

For ease of notation we define

$$g(t) := \omega\alpha(t) + (2\pi)^{3/2}\tilde{\eta}\mathcal{FT}[|\varphi(t)|^2]. \tag{2.144}$$

The derivative of β^a is given by

Lemma 2.11.1. *Let $\Psi_{N,0} \in \mathcal{D}(H_N)$ with $||\Psi_{N,0}|| = 1$. Let $(\varphi_0, \alpha_0) \in \left(H^2(\mathbb{R}^3) \oplus L_1^2(\mathbb{R}^3)\right)$ with $||\varphi_0|| = 1$ and assume that Conjecture 2.2.1 holds. Then*[17]

$$\begin{aligned} d_t\beta^a(t) &= i\left\langle \Psi_{N,t}, \left[\left(H_N - H_1^{eff}\right), q_1^{\varphi_t}\right] \Psi_{N,t}\right\rangle \\ &= -2\mathrm{Im}\left\langle \Psi_{N,t}, \left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right) q_1^{\varphi_t}\Psi_{N,t}\right\rangle. \end{aligned} \tag{2.145}$$

Proof. We consider

$$\begin{aligned} &\left\langle \Psi_N(t+h), |\varphi(t+h)\rangle\langle\varphi(t+h)|_1\Psi_N(t+h)\right\rangle - \left\langle \Psi_N(t), |\varphi(t)\rangle\langle\varphi(t)|_1\Psi_N(t)\right\rangle \\ &\quad= \mathrm{Re}\left\langle \Psi_N(t+h) - \Psi_N(t), |\varphi(t+h)\rangle\langle\varphi(t+h)|_1\left(\Psi_N(t+h) - \Psi_N(t)\right)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t), |\varphi(t+h) - \varphi(t)\rangle\langle\varphi(t+h)|_1\left(\Psi_N(t+h) - \Psi_N(t)\right)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t), |\varphi(t)\rangle\langle\varphi(t+h) - \varphi(t)|_1\left(\Psi_N(t+h) - \Psi_N(t)\right)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t+h) - \Psi_N(t), |\varphi(t+h) - \varphi(t)\rangle\langle\varphi(t+h)|_1\Psi_N(t)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t+h) - \Psi_N(t), |\varphi(t)\rangle\langle\varphi(t+h) - \varphi(t)|_1\Psi_N(t)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t), |\varphi(t+h) - \varphi(t)\rangle\langle\varphi(t+h) - \varphi(t)|_1\Psi_N(t)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t), |\varphi(t)\rangle\langle\varphi(t)|_1\Psi_N(t+h) - \Psi_N(t)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t+h) - \Psi_N(t), |\varphi(t)\rangle\langle\varphi(t)|_1\Psi_N(t)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t), |\varphi(t+h) - \varphi(t)\rangle\langle\varphi(t)|_1\Psi_N(t)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t), |\varphi(t)\rangle\langle\varphi(t+h) - \varphi(t)|_1\Psi_N(t)\right\rangle. \end{aligned} \tag{2.146}$$

and

$$\begin{aligned} &\mathrm{Re}\left\{i\left\langle \Psi_N(t), \left[\left(H_N - H_1^{eff}\right), p_1^{\varphi(t)}\right]\Psi_N(t)\right\rangle\right\} = \\ &\quad= \mathrm{Re}\left\langle \Psi_N(t), p_1^{\varphi(t)}(-i)H_N\Psi_N(t)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle (-i)H_N\Psi_N(t), p_1^{\varphi(t)}\Psi_N(t)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t), |(-i)H^{eff}\varphi(t)\rangle\langle\varphi(t)|_1\Psi_N(t)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t), |\varphi(t)\rangle\langle(-i)H^{eff}\varphi(t)|_1\Psi_N(t)\right\rangle. \end{aligned} \tag{2.147}$$

This allows us to estimate

$$\begin{aligned} &|h^{-1}(2.146) - (2.147)| \le h^{-1}|\text{ lines 2 till 7 of (2.146)}| \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t), |\varphi(t)\rangle\langle\varphi(t)|_1\frac{\Psi_N(t+h) - \Psi_N(t)}{h} + iH_N\Psi_N(t)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \frac{\Psi_N(t+h) - \Psi_N(t)}{h} + iH_N\Psi_N(t), |\varphi(t)\rangle\langle\varphi(t)|_1\Psi_N(t)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t), |\frac{\varphi(t+h) - \varphi(t)}{h} + iH^{eff}\varphi(t)\rangle\langle\varphi(t)|_1\Psi_N(t)\right\rangle \\ &\quad+ \mathrm{Re}\left\langle \Psi_N(t), |\varphi(t)\rangle\langle\frac{\varphi(t+h) - \varphi(t)}{h} + iH^{eff}\varphi(t)|_1\Psi_N(t)\right\rangle. \end{aligned} \tag{2.148}$$

[17]The commutator is well defined because $\varphi_t \in H^2(\mathbb{R}^3)$ ensure $q_1^{\varphi_t}\Psi_{N,t} = (1 - p_1^{\varphi_t})\Psi_{N,t} \in \mathcal{D}(H_N)$.

By Schwarz's inequality one derives

$$\big\langle \Psi, |\varphi\rangle\langle\xi|_1\chi\big\rangle \leq ||\chi||_{L^2(\mathbb{R}^3)}\, ||\varphi||_{L^2(\mathbb{R}^3)}\, ||\Psi||_{\mathcal{H}^{(N)}}\, ||\xi||_{\mathcal{H}^{(N)}} \tag{2.149}$$

and easily shows that the right hand side converges to zero as $h \to 0$. This proves

$$d_t\big\langle \Psi_{N,t}, p_1^{\varphi_t}\Psi_{N,t}\big\rangle = i\big\langle \Psi_{N,t}, \left[\left(H_N - H_1^{eff}\right), p_1^{\varphi_t}\right]\Psi_{N,t}\big\rangle \tag{2.150}$$

and leads to

$$\begin{aligned}
d_t\big\langle \Psi_{N,t}, q_1^{\varphi_t}\Psi_{N,t}\big\rangle &= d_t\left(1 - \big\langle \Psi_{N,t}, p_1^{\varphi_t}\Psi_{N,t}\big\rangle\right) = -i\big\langle \Psi_{N,t}, \left[\left(H_N - H_1^{eff}\right), p_1^{\varphi_t}\right]\Psi_{N,t}\big\rangle \\
&= i\big\langle \Psi_{N,t}, \left[\left(H_N - H_1^{eff}\right), (1 - p_1^{\varphi_t})\right]\Psi_{N,t}\big\rangle \\
&= i\big\langle \Psi_{N,t}, \left[\left(H_N - H_1^{eff}\right), q_1^{\varphi_t}\right]\Psi_{N,t}\big\rangle \\
&= i\big\langle \Psi_{N,t}, \left[\left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right), q_1^{\varphi_t}\right]\Psi_{N,t}\big\rangle \\
&= -2\mathrm{Im}\big\langle \Psi_{N,t}, \left(N^{-1/2}\widehat{\Phi}_\kappa(x_1) - \Phi_\kappa(x_1,t)\right) q_1^{\varphi_t}\Psi_{N,t}\big\rangle.
\end{aligned} \tag{2.151}$$

□

In the following, we determine the derivative of β^b. This is slightly involved because the creation and annihilation operators are unbounded operators. The best strategy seems to introduce the actual (not pointwise) creation and annihilation operators. For $\{\Psi_N^{(n)}\}_{n\in\mathbb{N}_0} = \Psi_N \in \mathcal{D}\left(\mathcal{N}^{1/2}\right)$ and $f \in L^2(\mathbb{R}^3)$ we define

$$\begin{aligned}
\left(a^*(f)\Psi_N\right)^{(n)}(k_1,\dots,k_n) &:= n^{-1/2}\sum_{j=1}^n f(k_j)\Psi_N^{(n-1)}(k_1,\dots,k_{j-1},k_{j+1},\dots,k_n), \\
\left(a(f)\Psi_N\right)^{(n)}(k_1,\dots,k_n) &:= (n+1)^{1/2}\int d^3k\, f^*(k)\Psi_N^{(n+1)}(k,k_1,\dots,k_n).
\end{aligned} \tag{2.152}$$

They are related to the pointwise creation and annihilation operators (2.10) by

$$a^*(f) = \int d^3k\, f(k)a^*(k), \quad a(f) = \int d^3k\, f^*(k)a(k). \tag{2.153}$$

The functional β^b may then be written as

$$\begin{aligned}
\beta^b(\Psi_{N,t},\alpha_t) &= N^{-1}\big\langle \Psi_{N,t}, \mathcal{N}\Psi_{N,t}\big\rangle + ||\alpha_t||_2^2 \\
&\quad - N^{-1/2}\big\langle \Psi_{N,t}, a^*(\alpha_t)\Psi_{N,t}\big\rangle - N^{-1/2}\big\langle \Psi_{N,t}, a(\alpha_t)\Psi_{N,t}\big\rangle \\
&= \underbrace{N^{-1}\big\langle \Psi_{N,t}, \mathcal{N}\Psi_{N,t}\big\rangle}_{=:\beta_1^b(\Psi_{N,t})} + \underbrace{(-2)N^{-1/2}\mathrm{Re}\big\langle \Psi_{N,t}, a(\alpha_t)\Psi_{N,t}\big\rangle}_{=:\beta_2^b(\Psi_{N,t},\alpha_t)} + \underbrace{||\alpha_t||_2^2}_{=:\beta_3^b(\alpha_t)}\,.
\end{aligned} \tag{2.154}$$

The functional β^b is well defined for $\Psi_{N,t} \in \mathcal{D}\left(\mathcal{N}^{1/2}\right)$ and $\alpha_t \in L^2(\mathbb{R}^3)$. This is seen by the standard inequalities

Lemma 2.11.2. *For $f \in L^2(\mathbb{R}^2)$ and $\Psi_N \in \mathcal{D}(\mathcal{N})$ one has*

$$\begin{aligned}
||a(f)\Psi_N|| &\leq ||f||\left|\left|\mathcal{N}^{1/2}\Psi_N\right|\right|, \\
||a^*(f)\Psi_N|| &\leq ||f||\left|\left|(\mathcal{N}+1)^{1/2}\Psi_N\right|\right|.
\end{aligned} \tag{2.155}$$

Proof. The proof [80][p.7] is a direct application of Schwarz's inequality

$$\begin{aligned} ||a(f)\Psi_N|| &\leq \int d^3k \, |f(k)| \, ||a(k)\Psi_N|| \leq ||f|| \left(\int d^3k \, ||a(k)\Psi_N||^2 \right)^{1/2} \\ &\leq ||f|| \left|\left| \mathcal{N}^{1/2}\Psi_N \right|\right| \end{aligned} \tag{2.156}$$

and the commutation relations (2.11)

$$\begin{aligned} ||a^*(f)\Psi_N||^2 &= \left\langle \Psi_N, a(f)a^*(f)\Psi_N \right\rangle = \left\langle \Psi_N, a^*(f)a(f)\Psi_N \right\rangle + ||f||^2 \, ||\Psi_N||^2 \\ &= ||a(f)\Psi_N||^2 + ||f||^2 \, ||\Psi_N||^2 \leq ||f||^2 \left(\left|\left| \mathcal{N}^{1/2}\Psi_N \right|\right|^2 + ||\Psi_N||^2 \right) \\ &= ||f||^2 \left|\left| (\mathcal{N}+1)^{1/2} \, \Psi_N \right|\right|^2 . \end{aligned} \tag{2.157}$$

□

However, to ensure the well-definedness of $\beta^b(t)$ at any time and compute its derivative it is useful to show the invariance of $\mathcal{D}(\mathcal{N}) \cap \mathcal{D}(\mathcal{N}H_N)$.

Lemma 2.11.3 (Invariance of the domains)**.** *Let $N \in \mathbb{N}$ and $\Psi_{N,0} \in \mathcal{D}(\mathcal{N}) \cap \mathcal{D}(\mathcal{N}H_N)$. Let $\Psi_{N,t}$ be the unique solution of* (2.6) *with initial data $\Psi_{N,0}$. Then $\Psi_{N,t} \in \mathcal{D}(\mathcal{N}) \cap \mathcal{D}(\mathcal{N}H_N)$ for all $t \in \mathbb{R}$. Moreover,*

$$||(\mathcal{N}+2)\, U(t)\Phi_{N,0}|| \leq a(t) \, ||(\mathcal{N}+2)\, \Phi_{N,0}|| \quad \textit{and} \tag{2.158}$$

$$||(\mathcal{N}+2)\, H_N U(t)\chi_{N,0}|| \leq a(t) \, ||(\mathcal{N}+2)\, H_N \chi_{N,0}|| \tag{2.159}$$

is true for all $\Phi_{N,0} \in \mathcal{D}(\mathcal{N})$ and $\chi_{N,0} \in \mathcal{D}(\mathcal{N}H_N)$ with $a(t) = e^{4N^{1/2}||\tilde{\eta}||_2|t|}$.

Proof. Lemma 2.11.3 has been shown in [32, Proposition 4] but we are recalling the proof for sake of completeness. For $f : \mathbb{N}_0 \to \mathbb{C}$ and $\Psi_N \in \mathcal{H}^{(N)}$ we define

$$\begin{cases} (f(\mathcal{N})\,\Psi_N)^{(n)} = f(n)\Psi_N^{(n)}, \\ \mathcal{D}\left((f(\mathcal{N})\right) = \{\Psi_N \in \mathcal{H}^{(N)} : \sum_{n=0}^{\infty} |f(n)|^2 \left|\left|\Psi_N^{(n)}\right|\right|^2_{L^2(\mathbb{R}^{(3N+3n)})} < \infty\}. \end{cases} \tag{2.160}$$

If f is a bounded function, we obtain

$$\begin{cases} ||f(\mathcal{N})\Psi_N|| \leq \sup_{x\in\mathbb{N}_0} |f(x)| \, ||\Psi_N|| \quad \text{for } \Psi_N \in \mathcal{H}^{(N)}, \\ \mathcal{D}\left((f(\mathcal{N})\right) = \mathcal{H}^{(N)}. \end{cases} \tag{2.161}$$

In particularly, we are interested in $h : \mathbb{N}_0 \to \mathbb{R}, n \mapsto h(n) = (n+2)^{-1}$ and the bounded operator $h(\mathcal{N})$. For $\Psi_N \in \mathcal{D}(\widehat{\Phi}_\kappa) \subseteq \mathcal{D}(H_N)$

$$\begin{aligned} h(\mathcal{N})\widehat{\Phi}_\kappa^+\Psi_N &= \widehat{\Phi}_\kappa^+ h(\mathcal{N}-1)\Psi_N, \\ h(\mathcal{N})\widehat{\Phi}_\kappa^-\Psi_N &= \widehat{\Phi}_\kappa^- h(\mathcal{N}+1)\Psi_N, \\ h(\mathcal{N})\widehat{\Phi}_\kappa\Psi_N &= \widehat{\Phi}_\kappa^+ h(\mathcal{N}-1)\Psi_N + \widehat{\Phi}_\kappa^- h(\mathcal{N}+1)\Psi_N. \end{aligned} \tag{2.162}$$

follows from a direct computation. Let $\Psi_{N,0} \in \mathcal{D}(H_N)$ and $\Psi_{N,t} = U(t)\Psi_{N,0}$. In analogy to the derivative of $\beta^a(t)$ one derives

$$d_t \, ||h(\mathcal{N})\Psi_{N,t}||^2 = 2\text{Im}\left\langle h(\mathcal{N})\Psi_{N,t}, \left[h(\mathcal{N}), H^N\right] \Psi_{N,t}\right\rangle. \tag{2.163}$$

To find suitable bounds for the time derivative, we notice that

$$\left|\left|\widehat{\Phi}_{\kappa}^{+}\Psi_N\right|\right| \leq ||\tilde{\eta}||_2 \left|\left|\mathcal{N}^{1/2}\Psi_N\right|\right| \quad \text{and} \quad \left|\left|\widehat{\Phi}_{\kappa}^{-}\Psi_N\right|\right| \leq ||\tilde{\eta}||_2 \left|\left|(\mathcal{N}+1)^{1/2}\,\Psi_N\right|\right|. \tag{2.164}$$

follows from Lemma 2.11.2 and Definition 2.61. We then observe the commutator relation $[h(\mathcal{N}), H_N] = \sum_{j=1}^{N} N^{-1/2}\left[h(\mathcal{N}), \widehat{\Phi}_{\kappa}(x_j)\right]$ and estimate

$$\begin{aligned}
\left|\left|\left[h(\mathcal{N}), H^N\right]\Psi_{N,t}\right|\right| &\leq \sum_{j=1}^{N} N^{-1/2}\left|\left|\widehat{\Phi}_{\kappa}^{+}(x_j)\left(h(\mathcal{N}-1)-h(\mathcal{N})\right)\Psi_{N,t}\right|\right| \\
&+ \sum_{j=1}^{N} N^{-1/2}\left|\left|\widehat{\Phi}_{\kappa}^{-}(x_j)\left(h(\mathcal{N}+1)-h(\mathcal{N})\right)\Psi_{N,t}\right|\right| \\
&\leq N^{1/2}\,||\tilde{\eta}||_2\left|\left|\mathcal{N}^{1/2}\left(h(\mathcal{N}-1)-h(\mathcal{N})\right)\Psi_{N,t}\right|\right| \\
&+ N^{1/2}\,||\tilde{\eta}||_2\left|\left|\mathcal{N}^{1/2}\left(h(\mathcal{N}+1)-h(\mathcal{N})\right)\Psi_{N,t}\right|\right| \\
&= N^{1/2}\,||\tilde{\eta}||_2\left|\left|\mathcal{N}^{1/2}\left(h(\mathcal{N}-1)-h(\mathcal{N})\right)\Psi_{N,t}\right|\right| \\
&+ N^{1/2}\,||\tilde{\eta}||_2\left|\left|\mathcal{N}^{1/2}\left(h(\mathcal{N})-h(\mathcal{N}+1)\right)\Psi_{N,t}\right|\right|.
\end{aligned} \tag{2.165}$$

Due to

$$\begin{aligned}
n^{1/2}\left(h(n-1)-h(n)\right) &= n^{1/2}\left((n+1)^{-1}-(n+2)^{-1}\right) = n^{1/2}\left(\frac{n+2}{n+1}-1\right)(n+2)^{-1} \\
&= \frac{n^{1/2}}{n+1}(n+2)^{-1} \leq (n+2)^{-1}, \\
n^{1/2}\left(h(n)-h(n+1)\right) &= n^{1/2}\left((n+2)^{-1}-(n+3)^{-1}\right) = n^{1/2}\left(1-\frac{n+2}{n+1}\right)(n+2)^{-1} \\
&= \frac{n^{1/2}}{n+3}(n+2)^{-1} \leq (n+2)^{-1}
\end{aligned} \tag{2.166}$$

for all $n \in \mathbb{N}_0$ and (2.160) we have

$$\left|\left|\left[h(\mathcal{N}), H^N\right]\Psi_{N,t}\right|\right| \leq 2N^{1/2}\,||\tilde{\eta}||_2\,||h(\mathcal{N})\Psi_{N,t}|| \tag{2.167}$$

and obtain

$$\begin{aligned}
|d_t\,||h(\mathcal{N})\Psi_{N,t}||^2\,| &\leq 2|\langle h(\mathcal{N})\Psi_{N,t}, \left[h(\mathcal{N}), H^N\right]\Psi_{N,t}\rangle| \\
&\leq 2\,||h(\mathcal{N})\Psi_{N,t}||\,\left|\left|\left[h(\mathcal{N}), H^N\right]\Psi_{N,t}\right|\right| \leq 4N^{1/2}\,||\tilde{\eta}||_2\,||h(\mathcal{N})\Psi_{n,t}||^2.
\end{aligned} \tag{2.168}$$

Then, Gronwall's inequality 1.3.1 leads to

$$\left|\left|(\mathcal{N}+2)^{-1}\,\Psi_{N,t}\right|\right| \leq a(t)\left|\left|(\mathcal{N}+2)^{-1}\,\Psi_{N,0}\right|\right| \quad \text{for all } \Psi_{N,0} \in \mathcal{D}(H_N) \tag{2.169}$$

with $a(t) := e^{4N^{1/2}||\tilde{\eta}||_2|t|}$. Since $\mathcal{D}(H_N)$ is dense in $\mathcal{H}^{(N)}$ and $(\mathcal{N}+2)^{-1}$ is a bounded operator, inequality (2.169) extends to $\mathcal{H}^{(N)}$ by a standard density argument.

Density argument. The operator H_N is self-adjoint and therefore has a dense domain $\mathcal{D}(H_N) \subseteq \mathcal{H}^{(N)}$. This implies that for every vector $\Psi_{N,0} \in \mathcal{H}^{(N)}$ there exists a sequence $\Psi_{N,0}^{(l)} \in \mathcal{D}(H_N)$ such that $\left|\left|\Psi_{N,0} - \Psi_{N,0}^{(l)}\right|\right| \to 0$ as $l \to \infty$. Moreover we have smoothness with respect to the initial data, namely $\left|\left|\Psi_{N,t} - \Psi_{N,t}^{(l)}\right|\right| = \left|\left|\Psi_{N,0} - \Psi_{N,0}^{(l)}\right|\right| \to 0$ as $l \to \infty$ due to the unitarity of $U(t)$. Since $(\mathcal{N}+2)^{-1}$ is a bounded operator we also have $\left|\left|(\mathcal{N}+2)^{-1}\left(\Psi_{N,t} - \Psi_{N,t}^{(l)}\right)\right|\right| \leq \left|\left|(\mathcal{N}+2)^{-1}\right|\right|_{op} \left|\left|\Psi_{N,t} - \Psi_{N,t}^{(l)}\right|\right| \to 0$ as $l \to \infty$ for all $t \in \mathbb{R}$ and obtain

$$\begin{aligned}
\left|\left|(\mathcal{N}+2)^{-1}\Psi_{N,t}\right|\right| &\leq \left|\left|(\mathcal{N}+2)^{-1}\left(\Psi_{N,t} - \Psi_{N,t}^{(l)}\right)\right|\right| + \left|\left|(\mathcal{N}+2)^{-1}\Psi_{N,t}^{(l)}\right|\right| \\
&\leq \left|\left|(\mathcal{N}+2)^{-1}\left(\Psi_{N,t} - \Psi_{N,t}^{(l)}\right)\right|\right| + a(t)\left|\left|(\mathcal{N}+2)^{-1}\Psi_{N,0}^{(l)}\right|\right| \\
&\leq \left|\left|(\mathcal{N}+2)^{-1}\left(\Psi_{N,t} - \Psi_{N,t}^{(l)}\right)\right|\right| + a(t)\left|\left|(\mathcal{N}+2)^{-1}\left(\Psi_{N,0}^{(l)} - \Psi_{N,0}\right)\right|\right| \\
&\quad + a(t)\left|\left|(\mathcal{N}+2)^{-1}\Psi_{N,0}\right|\right|
\end{aligned} \tag{2.170}$$

where we made use of the fact that (2.169) holds for $\Psi_{N,0}^{(l)} \in \mathcal{D}(H_N)$. Taking the limit $l \to \infty$ shows inequality (2.169) for all $\Psi_{N,0} \in \mathcal{H}^{(N)}$. □

In particular this implies

$$\left|\left|(\mathcal{N}+2)^{-1} U(t)\,(\mathcal{N}+2)\,\Psi_{N,0}\right|\right| \leq a(t)\,||\Psi_{N,0}|| \quad \text{for all } \Psi_{N,0} \in \mathcal{D}\big(\mathcal{N}+2\big) \tag{2.171}$$

with $a(t) := e^{4N^{1/2}||\tilde{\eta}||_2|t|}$.
Let $\Phi_{N,0} \in \mathcal{H}^{(N)}$ and $\Psi_{N,0} \in \mathcal{D}\big(\mathcal{N}+2\big)$. The boundedness of $U(t)$ and $(\mathcal{N}+2)^{-1}$ as well as inequality (2.171) let us obtain

$$\begin{aligned}
|\big\langle U(t)\,(\mathcal{N}+2)^{-1}\,\Phi_{N,0}, (\mathcal{N}+2)\,\Psi_{N,0}\big\rangle| &= |\big\langle \Phi_{N,0}, (\mathcal{N}+2)^{-1}\,U(-t)\,(\mathcal{N}+2)\,\Psi_{N,0}\big\rangle| \\
&\leq a(t)\,||\Phi_{N,0}||\,||\Psi_{N,0}||\,.
\end{aligned} \tag{2.172}$$

This shows that the map $\mathcal{D}\big((\mathcal{N}+2)\big) \to \mathbb{C}, \Psi_{N,0} \mapsto \big\langle U(t)\,(\mathcal{N}+2)^{-1}\,\Phi_{N,0}, (\mathcal{N}+2)\,\Psi_{N,0}\big\rangle$ is continuous. Recalling the definition of the domain of an adjoint operator

$$\mathcal{D}(T^*) = \{y \in \mathcal{H} : \tilde{\Psi}_y : \mathcal{D}(T) \to \mathbb{C}, x \mapsto \big\langle y, T(x)\big\rangle \text{ is continuous}\} \tag{2.173}$$

shows that $U(t)\,(\mathcal{N}+2)^{-1}\,\Phi_{N,0} \in \mathcal{D}\left((\mathcal{N}+2)^*\right) = \mathcal{D}\left(\mathcal{N}+2\right)$, because $(\mathcal{N}+2)$ is self-adjoint. This gives

$$\begin{aligned}
a(t)\,||\Phi_{N,0}||\,||\Psi_{N,0}|| &\geq |\big\langle U(t)\,(\mathcal{N}+2)^{-1}\,\Phi_{N,0}, (\mathcal{N}+2)\,\Psi_{N,0}\big\rangle| \\
&= |\big\langle\, (\mathcal{N}+2)\,U(t)\,(\mathcal{N}+2)^{-1}\,\Phi_{N,0}, \Psi_{N,0}\big\rangle|
\end{aligned} \tag{2.174}$$

for $\Phi_{N,0} \in \mathcal{H}^{(N)}$ and $\Psi_{N,0} \in \mathcal{D}\big(\mathcal{N}+2\big)$. By a standard density argument one derives this inequality for all $\Phi_{N,0}, \Psi_{N,0} \in \mathcal{H}^{(N)}$. Choosing $\Psi_{N,0} = (\mathcal{N}+2)\,U(t)\,(\mathcal{N}+2)^{-1}\,\Phi_{N,0}$ leads to

$$\left|\left|(\mathcal{N}+2)\,U(t)\,(\mathcal{N}+2)^{-1}\,\Phi_{N,0}\right|\right| \leq a(t)\,||\Phi_{N,0}|| \quad \text{for all } \Phi_{N,0} \in \mathcal{H}^{(N)} \tag{2.175}$$

and

$$||(\mathcal{N}+2)\,U(t)\Phi_{N,0}|| \leq a(t)\,||(\mathcal{N}+2)\,\Phi_{N,0}|| \quad \text{for all } \Phi_{N,0} \in \mathcal{D}\left(\mathcal{N}+2\right) = \mathcal{D}\left(\mathcal{N}\right). \tag{2.176}$$

This implies

$$||(\mathcal{N}+2)\,H_N U(t)\Phi_{N,0}|| = ||(\mathcal{N}+2)\,U(t)H_N\Phi_{N,0}|| \le a(t)\,||(\mathcal{N}+2)\,H_N\Phi_{N,0}|| \tag{2.177}$$

for all $\Psi_{N,0} \in \mathcal{D}\left((\mathcal{N}+2)\,H_N\right) = \mathcal{D}\left(\mathcal{N}H_N\right) \cap \mathcal{D}\left(H_N\right)$. Hereby, we used the fact, that the unitary time evolution commutes with its generator due to Stone's theorem. The inequalities (2.176) and (2.177) show the invariance of $\mathcal{D}\left(\mathcal{N}\right) \cap \mathcal{D}\left(\mathcal{N}H_N\right) \cap \mathcal{D}\left(H_N\right) = \mathcal{D}\left(\mathcal{N}\right) \cap \mathcal{D}\left(\mathcal{N}H_N\right)$. □

Next, we differentiate the terms in (2.154) and determine the derivative of β^b. The derivative of β^b_1 is given by

Lemma 2.11.4. *Let $\Psi_{N,0} \in \left(L^2_s(\mathbb{R}^{3N}) \otimes \mathcal{F}\right) \cap \mathcal{D}\left(\mathcal{N}\right) \cap \mathcal{D}\left(\mathcal{N}H_N\right)$. Then*

$$d_t\beta^b_1(t) = d_t\big\langle \Psi_{N,t}, \mathcal{N}\Psi_{N,t}\big\rangle = -2N^{-1/2}\mathrm{Im}\big\langle \Psi_{N,t}, \widehat{\Phi}^+_\kappa(x_1)\Psi_{N,t}\big\rangle. \tag{2.178}$$

Proof. From the initial data and Lemma 2.11.3 it follows that $\Psi_N(t) \in \left(L^2_s(\mathbb{R}^{3N}) \otimes \mathcal{F}\right) \cap \mathcal{D}\left(\mathcal{N}\right) \cap \mathcal{D}\left(\mathcal{N}H_N\right)$ for all $t \in \mathbb{R}$. Let $h \in \mathbb{R}$ with $h \neq 0$.

$$\begin{aligned}
\beta^b_1(t+h) - \beta^b_1(t) &= N^{-1}\big\langle \Psi_N(t+h), \mathcal{N}\Psi_N(t+h)\big\rangle - N^{-1}\big\langle \Psi_N(t), \mathcal{N}\Psi_N(t)\big\rangle \\
&= N^{-1}\mathrm{Re}\big\langle \Psi_N(t+h), \mathcal{N}\Psi_N(t+h)\big\rangle - N^{-1}\mathrm{Re}\big\langle \Psi_N(t), \mathcal{N}\Psi_N(t)\big\rangle \\
&= N^{-1}\mathrm{Re}\big\langle \left(\Psi_N(t+h) - \Psi_N(t)\right), \mathcal{N}\Psi_N(t+h)\big\rangle \\
&+ N^{-1}\mathrm{Re}\big\langle \Psi_N(t), \mathcal{N}\Psi_N(t+h)\big\rangle - N^{-1}\mathrm{Re}\big\langle \Psi_N(t), \mathcal{N}\Psi_N(t)\big\rangle \\
&= N^{-1}\mathrm{Re}\big\langle \left(\Psi_N(t+h) - \Psi_N(t)\right), \mathcal{N}\left(\Psi_N(t+h) - \Psi_N(t)\right)\big\rangle \\
&+ N^{-1}\mathrm{Re}\big\langle \left(\Psi_N(t+h) - \Psi_N(t)\right), \mathcal{N}\Psi_N(t)\big\rangle \\
&+ N^{-1}\mathrm{Re}\big\langle \mathcal{N}\Psi_N(t), \left(\Psi_N(t+h) - \Psi_N(t)\right)\big\rangle \\
&= N^{-1}\big\langle \left(\Psi_N(t+h) - \Psi_N(t)\right), \mathcal{N}\left(\Psi_N(t+h) - \Psi_N(t)\right)\big\rangle \\
&+ 2N^{-1}\mathrm{Re}\big\langle \mathcal{N}\Psi_N(t), \left(\Psi_N(t+h) - \Psi_N(t)\right)\big\rangle
\end{aligned} \tag{2.179}$$

This gives

$$\begin{aligned}
&\left|\frac{\beta^b_1(t+h) - \beta^b_1(t)}{h} - 2N^{-1}\mathrm{Re}\big\langle \mathcal{N}\Psi_N(t), -iH_N\Psi_N(t)\big\rangle\right| \le \\
&\le N^{-1}\left|\big\langle \left(\Psi_N(t+h) - \Psi_N(t)\right), \mathcal{N}\frac{\Psi_N(t+h) - \Psi_N(t)}{h}\big\rangle\right| \\
&+ 2N^{-1}\left|\mathrm{Re}\big\langle \mathcal{N}\Psi_N(t), \frac{\Psi_N(t+h) - \Psi_N(t)}{h} + iH_N\Psi_N(t)\big\rangle\right| \\
&\le N^{-1}\left|\big\langle \left(\Psi_N(t+h) - \Psi_N(t)\right), \mathcal{N}\left(\frac{\Psi_N(t+h) - \Psi_N(t)}{h} + iH_N\Psi_N(t) - iH_N\Psi_N(t)\right)\big\rangle\right| \\
&+ 2N^{-1}\left|\mathrm{Re}\big\langle \mathcal{N}\Psi_N(t), \frac{\Psi_N(t+h) - \Psi_N(t)}{h} + iH_N\Psi_N(t)\big\rangle\right| \\
&\le N^{-1}\left|\big\langle \mathcal{N}\left(\Psi_N(t+h) - \Psi_N(t)\right), \frac{\Psi_N(t+h) - \Psi_N(t)}{h} + iH_N\Psi_N(t)\big\rangle\right| \\
&+ N^{-1}\left|\big\langle \left(\Psi_N(t+h) - \Psi_N(t)\right), \mathcal{N}H_N\Psi_N(t)\big\rangle\right| \\
&+ 2N^{-1}\left|\mathrm{Re}\big\langle \mathcal{N}\Psi_N(t), \frac{\Psi_N(t+h) - \Psi_N(t)}{h} + iH_N\Psi_N(t)\big\rangle\right| \\
&\le N^{-1}\left|\left|\frac{\Psi_N(t+h) - \Psi_N(t)}{h} + iH_N\Psi_N(t)\right|\right|\left(3\,||\mathcal{N}\Psi_N(t)|| + ||\mathcal{N}\Psi_N(t+h)||\right) \\
&+ N^{-1}\,||\Psi_N(t+h) - \Psi_N(t)||\,||\mathcal{N}H_N\Psi_N(t)||\,.
\end{aligned} \tag{2.180}$$

Inequality (2.158) ensures the existence of $\lim_{h\to 0, h\neq 0} ||\mathcal{N}\Psi_N(t+h)||$ and (2.143) implies that the right hand side converges to zero when $h \to 0$. This shows

$$\lim_{h\to 0, h\neq 0} \left| \frac{\beta_1^b(t+h) - \beta_1^b(t)}{h} - 2N^{-1}\mathrm{Re}\big\langle \mathcal{N}\Psi_N(t), -iH_N\Psi_N(t)\big\rangle \right| = 0 \tag{2.181}$$

and allows us to compute

$$\begin{aligned}
d_t\beta_1^b(t) &= -2N^{-1}\mathrm{Re}\big\langle \mathcal{N}\Psi_N(t), iH_N\Psi_N(t)\big\rangle \\
&= \quad -N^{-1}\big\langle \mathcal{N}\Psi_N(t), iH_N\Psi_N(t)\big\rangle - N^{-1}\big\langle iH_N\Psi_N(t), \mathcal{N}\Psi_N(t)\big\rangle \\
&= -iN^{-1}\left(\big\langle \mathcal{N}\Psi_N(t), H_f\Psi_N(t)\big\rangle - \big\langle H_f\Psi_N, t, \mathcal{N}\Psi_N(t)\big\rangle\right) && (2.182)\\
&- \qquad i\left(\big\langle \mathcal{N}\Psi_N(t), -\Delta_1\Psi_N(t)\big\rangle - \big\langle -\Delta_1\Psi_N(t), \mathcal{N}\Psi_N(t)\big\rangle\right) && (2.183)\\
&- \quad iN^{-1/2}\big(\big\langle \mathcal{N}\Psi_N(t), \widehat{\Phi}_\kappa(x_1)\Psi_N(t)\big\rangle - \big\langle \widehat{\Phi}_\kappa(x_1)\Psi_N(t), \mathcal{N}\Psi_N(t)\big\rangle\big). && (2.184)
\end{aligned}$$

Line (2.182) $= 0$ because

$$\begin{aligned}
\big\langle \mathcal{N}\Psi_{N,t}, H_f\Psi_{N,t}\big\rangle &= \int d^{3N}x \sum_{n=1}^{\infty} \big\langle \mathcal{N}\Psi_{N,t}^{(n)}, H_f\Psi_{N,t}^{(n)}\big\rangle_{L^2(\mathbb{R}^{3n})}(X_N) \\
&= \int d^{3N}x \sum_{n=1}^{\infty} \big\langle n\Psi_{N,t}^{(n)}, \sum_{j=1}^{n}\omega(k_j)\Psi_{N,t}^{(n)}\big\rangle_{L^2(\mathbb{R}^{3n})}(X_N) \\
&= \int d^{3N}x \sum_{n=1}^{\infty} \big\langle \sum_{j=1}^{n}\omega(k_j)\Psi_{N,t}^{(n)}, n\Psi_{N,t}^{(n)}\big\rangle_{L^2(\mathbb{R}^{3n})}(X_N) \\
&= \int d^{3N}x \sum_{n=1}^{\infty} \big\langle H_f\Psi_{N,t}^{(n)}, \mathcal{N}\Psi_{N,t}^{(n)}\big\rangle_{L^2(\mathbb{R}^{3n})}(X_N) \\
&= \big\langle H_f\Psi_{N,t}, \mathcal{N}\Psi_{N,t}\big\rangle. && (2.185)
\end{aligned}$$

Likewise, we use integration by parts

$$\begin{aligned}
\big\langle \mathcal{N}\Psi_{N,t}, -\Delta_1\Psi_{N,t}\big\rangle &= \sum_{n\geq 0}\int d^{3n}k \int d^{3N}x\, n(\Psi_{N,t}^{(n)})^*(X_N, K_n)(-\Delta_1\Psi_{N,t})^{(n)}(X_N, K_n) \\
&= \sum_{n\geq 0}\int d^{3n}k \int d^{3N}x\, (-\Delta_1\Psi_{N,t}^{(n)})^*(X_N, K_n) n(\Psi_{N,t})^{(n)}(X_N, K_n) \\
&= \big\langle -\Delta_1\Psi_{N,t}, \mathcal{N}\Psi_{N,t}\big\rangle && (2.186)
\end{aligned}$$

and obtain (2.183) $= 0$. A straightforward calculation leads to

$$\big\langle \widehat{\Phi}_\kappa^-(x_1)\Psi_{N,t}, \mathcal{N}\Psi_{N,t}\big\rangle = \big\langle\, (\mathcal{N}+1)\,\Psi_{N,t}, \widehat{\Phi}_\kappa^+(x_1)\Psi_{N,t}\big\rangle \tag{2.187}$$

and allows us to show

$$\begin{aligned}
(2.184) &= -iN^{-1/2}\big(\big\langle \mathcal{N}\Psi_N(t), \widehat{\Phi}_\kappa(x_1)\Psi_N(t)\big\rangle - \big\langle \widehat{\Phi}_\kappa(x_1)\Psi_N(t), \mathcal{N}\Psi_N(t)\big\rangle\big) \\
&= -iN^{-1/2}\big(\big\langle \mathcal{N}\Psi_N(t), \widehat{\Phi}_\kappa^+(x_1)\Psi_N(t)\big\rangle + \big\langle \mathcal{N}\Psi_N(t), \widehat{\Phi}_\kappa^-(x_1)\Psi_N(t)\big\rangle\big) \\
&+ \quad iN^{-1/2}\big(\big\langle \widehat{\Phi}_\kappa^+(x_1)\Psi_N(t), \mathcal{N}\Psi_N(t)\big\rangle + \big\langle \widehat{\Phi}_\kappa^-(x_1)\Psi_N(t), \mathcal{N}\Psi_N(t)\big\rangle\big) \\
&= -iN^{-1/2}\big(\big\langle \mathcal{N}\Psi_N(t), \widehat{\Phi}_\kappa^+(x_1)\tilde{\eta})\Psi_N(t)\big\rangle + \big\langle \widehat{\Phi}_\kappa^+(x_1)\Psi_N(t), (\mathcal{N}+1)\Psi_N(t)\big\rangle\big) \\
&+ \quad iN^{-1/2}\big(\big\langle \widehat{\Phi}_\kappa^+(x_1)\Psi_N(t), \mathcal{N}\Psi_N(t)\big\rangle + \big\langle (\mathcal{N}+1)\Psi_N(t), \widehat{\Phi}_\kappa^+(x_1)\Psi_N(t)\big\rangle\big) \\
&= \quad iN^{-1/2}\big(\big\langle \Psi_N(t), \widehat{\Phi}_\kappa^+(x_1)\Psi_N(t)\big\rangle - \big\langle \widehat{\Phi}_\kappa^+(x_1)\Psi_N(t), \Psi_N(t)\big\rangle\big) \\
&= -2N^{-1/2}\mathrm{Im}\big\langle \Psi_N(t), \widehat{\Phi}_\kappa^+(x_1)\Psi_N(t)\big\rangle. && (2.188)
\end{aligned}$$

In total, this gives

$$d_t\beta_1^b(t) = -2N^{-1/2}\mathrm{Im}\big\langle\Psi_N(t), \widehat{\Phi}_\kappa^+(x_1)\Psi_N(t)\big\rangle$$
$$= -2\mathrm{Im}\big\langle\Psi_N(t), \Big(\int d^3k\,\tilde{\eta}(k)e^{ikx_1}N^{-1/2}a(k)\Big)\Psi_N(t)\big\rangle \tag{2.189}$$

and proves Lemma 2.11.4. □

Now we are interested in the derivative of β_2^b.

Lemma 2.11.5. *Let* $\Psi_{N,0} \in \big(L_s^2(\mathbb{R}^{3N}) \otimes \mathcal{F}\big) \cap \mathcal{D}(\mathcal{N}) \cap \mathcal{D}(\mathcal{N}H_N)$ *with* $||\Psi_{N,0}|| = 1$*. Let* $(\varphi_0, \alpha_0) \in (H^2(\mathbb{R}^3) \oplus L_1^2(\mathbb{R}^3))$ *with* $||\varphi_0|| = 1$ *and assume that Conjecture 2.2.1 holds. Then*

$$d_t\beta_2^b(t) = 2\mathrm{Im}\big\langle\Psi_{N,t}, \Big(\int d^3k\,\tilde{\eta}(k)e^{ikx_1}\alpha_t(k)\Big)\Psi_{N,t}\big\rangle$$
$$+ 2\mathrm{Im}\big\langle\Psi_{N,t}, \Big(\int d^3k\,\tilde{\eta}(k)(2\pi)^{3/2}\mathcal{FT}^*[|\varphi_t|^2](k)N^{-1/2}a(k)\Big)\Psi_{N,t}\big\rangle \tag{2.190}$$

Proof. First, we would like to note that the expressions in (2.190) are well defined because $\tilde{\eta}\mathcal{FT}[|\varphi_t|^2] \in L^2(\mathbb{R}^3)$. This follows from $|\varphi_t|^2 \in L^1(\mathbb{R}^3)$ and $\mathcal{FT}[|\varphi_t|^2] \in L^\infty(\mathbb{R}^3)$. By means of the expansion

$$\big\langle\Psi_N(t+h), a(\alpha(t+h))\Psi_N(t+h)\big\rangle - \big\langle\Psi_N(t), a(\alpha(t))\Psi_N(t)\big\rangle$$
$$= \big\langle\Psi_N(t+h) - \Psi_N(t), a(\alpha(t+h) - \alpha(t))\Psi_N(t+h)\big\rangle$$
$$+ \big\langle\Psi_N(t), a(\alpha(t+h) - \alpha(t))(\Psi_N(t+h) - \Psi_N(t))\big\rangle$$
$$+ \big\langle\Psi_N(t+h) - \Psi_N(t), a(\alpha(t))(\Psi_N(t+h) - \Psi_N(t))\big\rangle$$
$$+ \big\langle\Psi_N(t), a(\alpha(t+h) - \alpha(t))\Psi(t)\big\rangle$$
$$+ \big\langle\Psi_N(t+h) - \Psi_N(t), a(\alpha(t))\Psi_N(t)\big\rangle$$
$$+ \big\langle\Psi_N(t), a(\alpha(t))(\Psi_N(t+h) - \Psi_N(t))\big\rangle \tag{2.191}$$

we estimate

$$\Big|\frac{\beta_2^b(t+h) - \beta_2^b(t)}{h} + 2N^{-1/2}\mathrm{Re}\big\langle\Psi_N(t), a(-ig(t))\big\rangle\Psi_N(t) +$$
$$+ 2N^{-1/2}\mathrm{Re}\big\langle -iH_N\Psi_N(t), a(\alpha(t))\Psi_N(t)\big\rangle + 2N^{-1/2}\mathrm{Re}\big\langle\Psi_N(t), a(\alpha(t)) - iH_N\Psi_N(t)\big\rangle\Big|$$
$$\le 2N^{-1/2}|\big\langle\Psi_N(t+h) - \Psi_N(t), a\Big(\frac{\alpha(t+h) - \alpha(t)}{h}\Big)\Psi_N(t+h)\big\rangle|$$
$$+ 2N^{-1/2}|\big\langle a^*\Big(\frac{\alpha(t+h) - \alpha(t)}{h}\Big)\Psi_N(t), \Psi_N(t+h) - \Psi_N(t)\big\rangle|$$
$$+ 2N^{-1/2}|\big\langle\Psi_N(t+h) - \Psi_N(t), a(\alpha(t))\frac{\Psi_N(t+h) - \Psi_N(t)}{h}\big\rangle| \tag{2.192}$$
$$+ 2N^{-1/2}|\big\langle\Psi_N(t), a\Big(\frac{\alpha(t+h) - \alpha(t)}{h} + ig(t)\Big)\Psi_N(t)\big\rangle|$$
$$+ 2N^{-1/2}|\big\langle\frac{\Psi_N(t+h) - \Psi_N(t)}{h} + iH_N\Psi_N(t), a(\alpha(t))\Psi_N(t)\big\rangle|$$
$$+ 2N^{-1/2}|\big\langle a^*(\alpha(t))\Psi_N(t), \frac{\Psi_N(t+h) - \Psi_N(t)}{h} + iH_N\Psi_N(t)\big\rangle|. \tag{2.193}$$

The third summand can further be estimated by

$$(2.192) \le 2N^{-1/2}|\big\langle a(\alpha^*(t))(\Psi_N(t+h) - \Psi_N(t)), \frac{\Psi_N(t+h) - \Psi_N(t)}{h} + iH_N\Psi_N(t)\big\rangle|$$
$$+ 2N^{-1/2}|\big\langle\Psi_N(t+h) - \Psi_N(t), a(\alpha(t))iH_N\Psi_N(t)\big\rangle|. \tag{2.194}$$

With the help of Schwarz's inequality and the standard estimates of Lemma 2.11.2 one easily shows that the right hand side of (2.193) converges to zero as $h \to 0$. This proves

$$\begin{aligned} d_t\beta_2^b(t) &= 2N^{-1/2}\mathrm{Re}\big\langle iH_N\Psi_{N,t}, a(\alpha_t)\Psi_{N,t}\big\rangle + 2N^{-1/2}\mathrm{Re}\big\langle \Psi_{N,t}, a(\alpha_t)iH_N\Psi_{N,t}\big\rangle \\ &\quad + 2N^{-1/2}\mathrm{Re}\big\langle \Psi_{N,t}, a(ig_t)\Psi_{N,t}\big\rangle \\ &= 2N^{-1/2}\mathrm{Im}\big\{\big\langle H_N\Psi_{N,t}, a(\alpha_t)\Psi_{N,t}\big\rangle - \big\langle a^*(\alpha_t)\Psi_{N,t}, H_N\Psi_{N,t}\big\rangle\big\} \\ &\quad + 2N^{-1/2}\mathrm{Im}\big\langle \Psi_{N,t}, a(g_t)\Psi_{N,t}\big\rangle. \end{aligned} \tag{2.195}$$

Due to the symmetry of the wave function we have

$$\begin{aligned} \big\langle H_N\Psi_{N,t}, a(\alpha(t))\Psi_{N,t}\big\rangle = \quad & N\big\langle(-\Delta_1)\Psi_{N,t}, a(\alpha_t)\Psi_{N,t}\big\rangle + \big\langle H_f\Psi_{N,t}, a(\alpha_t)\Psi_{N,t}\big\rangle \\ & + N^{1/2}\big\langle\widehat{\Phi}_\kappa(x_1)\Psi_{N,t}, a(\alpha_t)\Psi_{N,t}\big\rangle. \end{aligned} \tag{2.196}$$

By means of

$$\begin{aligned} \big\langle(-\Delta_1)\Psi_{N,t}, a(\alpha_t)\Psi_{N,t}\big\rangle &= \big\langle a^*(\alpha_t)\Psi_{N,t}, (-\Delta_1)\Psi_{N,t}\big\rangle, \\ \big\langle H_f\Psi_{N,t}, a(\alpha_t)\Psi_{N,t}\big\rangle &= \big\langle a^*(\alpha_t)\Psi_{N,t}, H_f\Psi_{N,t}\big\rangle - \big\langle \Psi_{N,t}, a(\omega\alpha_t)\Psi_{N,t}\big\rangle, \\ N^{1/2}\big\langle\widehat{\Phi}_\kappa(x_1)\Psi_{N,t}, a(\alpha_t)\Psi_{N,t}\big\rangle &= N^{1/2}\big\langle a^*(\alpha_t)\Psi_{N,t}, \widehat{\Phi}_\kappa(x_1)\Psi_{N,t}\big\rangle \\ &\quad - N^{1/2}\big\langle\Psi_{N,t}, \Big(\int d^3k\, \tilde{\eta}(k)e^{-ikx_1}\alpha_t^*(k)\Big)\Psi_{N,t}\big\rangle. \end{aligned} \tag{2.197}$$

this becomes

$$\begin{aligned} \big\langle H_N\Psi_{N,t}, a(\alpha_t)\Psi_{N,t}\big\rangle &= \big\langle a^*(\alpha_t)\Psi_{N,t}, H_N\Psi_{N,t}\big\rangle - \big\langle \Psi_{N,t}, a(\omega\alpha_t)\Psi_{N,t}\big\rangle \\ &\quad - N^{1/2}\big\langle\Psi_{N,t}, \Big(\int d^3k\tilde{\eta}(k)e^{-ikx_1}\alpha_t^*(k)\Big)\Psi_{N,t}\big\rangle \end{aligned} \tag{2.198}$$

and leads to

$$\begin{aligned} d_t\beta_2^b(t) &= -2N^{-1/2}\mathrm{Im}\big\langle\Psi_{N,t}, a(\omega\alpha_t)\Psi_{N,t}\big\rangle - 2\mathrm{Im}\big\langle\Psi_{N,t}, \Big(\int d^3k\tilde{\eta}(k)e^{-ikx_1}\alpha_t^*(k)\Big)\Psi_{N,t}\big\rangle \\ &\quad + 2N^{-1/2}\mathrm{Im}\big\langle\Psi_{N,t}, a(g_t)\Psi_{N,t}\big\rangle \\ &= -2N^{-1/2}\mathrm{Im}\big\langle\Psi_{N,t}, a(\omega\alpha_t)\Psi_{N,t}\big\rangle - 2\mathrm{Im}\big\langle\Psi_{N,t}, \Big(\int d^3k\tilde{\eta}(k)e^{-ikx_1}\alpha_t^*(k)\Big)\Psi_{N,t}\big\rangle \\ &\quad + \quad 2N^{-1/2}\mathrm{Im}\big\langle\Psi_{N,t}, a(\omega\alpha_t)\Psi_{N,t}\big\rangle \\ &\quad + \qquad 2\mathrm{Im}\big\langle\Psi_{N,t}, \Big(\int d^3k\, \tilde{\eta}(k)(2\pi)^{3/2}\mathcal{F}\mathcal{T}^*[|\varphi_t|^2](k)N^{-1/2}a(k)\Big)\Psi_{N,t}\big\rangle \\ &= 2\mathrm{Im}\big\langle\Psi_{N,t}, \Big(\int d^3k\, \tilde{\eta}(k)e^{ikx_1}\alpha_t(k)\Big)\Psi_{N,t}\big\rangle \\ &\quad + 2\mathrm{Im}\big\langle\Psi_{N,t}, \Big(\int d^3k\, \tilde{\eta}(k)(2\pi)^{3/2}\mathcal{F}\mathcal{T}^*[|\varphi_t|^2](k)N^{-1/2}a(k)\Big)\Psi_{N,t}\big\rangle. \end{aligned} \tag{2.199}$$

□

Lemma 2.11.6. *Let $\Psi_{N,0} \in \big(L_s^2(\mathbb{R}^{3N}) \otimes \mathcal{F}\big)$ with $||\Psi_{N,0}|| = 1$. Let $(\varphi_0, \alpha_0) \in (H^2(\mathbb{R}^3) \oplus L_1^2(\mathbb{R}^3))$ with $||\varphi_0|| = 1$ and assume that Conjecture 2.2.1 holds. Then*

$$d_t\beta_3^b(t) = d_t\,||\alpha_t||_2^2 = -2\mathrm{Im}\big\langle\Psi_{N,t}, \Big(\int d^3k\, \tilde{\eta}(k)(2\pi)^{3/2}\mathcal{F}\mathcal{T}^*[|\varphi_t|^2](k)\alpha_t(k)\Big)\Psi_{N,t}\big\rangle. \tag{2.200}$$

Proof. We observe

$$\big\langle\alpha(t+h),\alpha(t+h)\big\rangle-\big\langle\alpha(t),\alpha(t)\big\rangle=||\alpha(t+h)\alpha(t)||^2+\big\langle\alpha(t+h)-\alpha(t),\alpha(t)\big\rangle$$
$$+\big\langle\alpha(t),\alpha(t+h)-\alpha(t)\big\rangle \qquad (2.201)$$

and estimate

$$\begin{aligned}\Big|\frac{\beta_3^b(t+h)-\beta_3^b(t)}{h}&-\mathrm{Re}\big\langle\alpha(t),-ig(t)\big\rangle-\mathrm{Re}\big\langle-ig(t),\alpha(t)\big\rangle\Big|\le\\
&\le|\mathrm{Re}\big\langle\frac{\alpha(t+h)-\alpha(t)}{h}+ig(t),\alpha(t)\big\rangle|\\
&+|\mathrm{Re}\big\langle\alpha(t),\frac{\alpha(t+h)-\alpha(t)}{h}+ig(t)\big\rangle|\\
&+|\mathrm{Re}\big\langle\frac{\alpha(t+h)-\alpha(t)}{h},\alpha(t+h)-\alpha(t)\big\rangle|, \qquad (2.202)\end{aligned}$$

where g is defined by (2.144). Recalling (2.143) we see that the right hand side converges to zero as $h\to 0$. This shows

$$\begin{aligned}d_t\beta_3^b(t)=\;&\mathrm{Re}\big\langle\alpha_t,-ig_t\big\rangle+\mathrm{Re}\big\langle-ig_t,\alpha_t\big\rangle=\mathrm{Im}\big\{\big\langle\alpha_t,g_t\big\rangle-\big\langle g_t,\alpha_t\big\rangle\big\}\\
=\;&\mathrm{Im}\big\{\big\langle\alpha_t,\omega\alpha_t\big\rangle-\big\langle\omega\alpha_t,\alpha_t\big\rangle\big\}+\mathrm{Im}\big\{\int d^3k\,\tilde{\eta}(k)(2\pi)^{3/2}\mathcal{FT}[|\varphi_t|^2](k)\alpha_t^*(k)\big\}\\
-\;&\mathrm{Im}\big\{\int d^3k\,\tilde{\eta}(k)(2\pi)^{3/2}\mathcal{FT}^*[|\varphi_t|^2](k)\alpha_t(k)\big\}\\
=&-2\mathrm{Im}\big\{\int d^3k\,\tilde{\eta}(k)(2\pi)^{3/2}\mathcal{FT}^*[|\varphi_t|^2](k)\alpha_t(k)\big\}\\
=&-2\mathrm{Im}\big\langle\Psi_{N,t},\big(\int d^3k\,\tilde{\eta}(k)(2\pi)^{3/2}\mathcal{FT}^*[|\varphi_t|^2](k)\alpha_t(k)\big)\Psi_{N,t}\big\rangle. \qquad (2.203)\end{aligned}$$

□

Lemma 2.11.7. *Let* $\Psi_{N,0}\in\big(L_s^2(\mathbb{R}^{3N})\otimes\mathcal{F}\big)\cap\mathcal{D}(\mathcal{N})\cap\mathcal{D}(\mathcal{N}H_N)$ *with* $||\Psi_{N,0}||=1$*. Let* $(\varphi_0,\alpha_0)\in(H^2(\mathbb{R}^3)\oplus L_1^2(\mathbb{R}^3))$ *with* $||\varphi_0||=1$ *and assume that Conjecture 2.2.1 holds. Then*

$$\begin{aligned}d_t\beta^b(t)=\;&2\mathrm{Im}\big\langle\Psi_{N,t},\Big(\int d^3k\,\tilde{\eta}(k)(2\pi)^{3/2}\mathcal{FT}^*[|\varphi_t|^2](k)\Big(N^{-1/2}a(k)-\alpha_t(k)\Big)\Big)\Psi_{N,t}\big\rangle\\
&-2\mathrm{Im}\big\langle\Psi_{N,t},\Big(\int d^3k\,\tilde{\eta}(k)e^{ikx_1}\Big(N^{-1/2}a(k)-\alpha_t(k)\Big)\Big)\Psi_{N,t}\big\rangle. \qquad (2.204)\end{aligned}$$

Proof.

$$\begin{aligned}
d_t\beta^b(t) &= d_t\beta_1^b(t) + d_t\beta_2^b(t) + d_t\beta_3^b(t)\\
&= -2\mathrm{Im}\Big\langle \Psi_{N,t}, \Big(\int d^3k\, \tilde{\eta}(k) e^{ikx_1} N^{-1/2} a(k) \Big) \Psi_{N,t} \Big\rangle\\
&+ \quad 2\mathrm{Im}\Big\langle \Psi_{N,t}, \Big(\int d^3k\, \tilde{\eta}(k) e^{ikx_1} \alpha_t(k) \Big) \Psi_{N,t} \Big\rangle\\
&+ \quad 2\mathrm{Im}\Big\langle \Psi_{N,t}, \Big(\int d^3k\, \tilde{\eta}(k) (2\pi)^{3/2} \mathcal{F}\mathcal{T}^*[|\varphi_t|^2](k) N^{-1/2} a(k) \Big) \Psi_{N,t} \Big\rangle\\
&- \quad 2\mathrm{Im}\Big\langle \Psi_{N,t}, \Big(\int d^3k\, \tilde{\eta}(k) (2\pi)^{3/2} \mathcal{F}\mathcal{T}^*[|\varphi_t|^2](k) \alpha_t(k) \Big) \Psi_{N,t} \Big\rangle\\
&= -2\mathrm{Im}\Big\langle \Psi_{N,t}, \Big(\int d^3k\, \tilde{\eta}(k) e^{ikx_1} \Big(N^{-1/2} a(k) - \alpha_t(k) \Big) \Big) \Psi_{N,t} \Big\rangle\\
&+ \quad 2\mathrm{Im}\Big\langle \Psi_{N,t}, \Big(\int d^3k\, \tilde{\eta}(k) (2\pi)^{3/2} \mathcal{F}\mathcal{T}^*[|\varphi_t|^2](k) \Big(N^{-1/2} a(k) - \alpha_t(k) \Big) \Big) \Psi_{N,t} \Big\rangle.
\end{aligned} \tag{2.205}$$

□

CHAPTER
THREE

DERIVATION OF THE MAXWELL-SCHRÖDINGER EQUATIONS FROM THE PAULI-FIERZ HAMILTONIAN

Abstract We consider the spinless Pauli-Fierz Hamiltonian which describes a quantum system of non-relativistic identical particles coupled to the quantized electromagnetic field. We study the time evolution in a mean-field limit where the number N of charged particles gets large while the coupling to the radiation field is rescaled by $1/\sqrt{N}$. At time zero we assume that almost all charged particles are in the same one-body state (a Bose-Einstein condensate) and we assume also the photons to be close to a coherent state. We show that at later times and in the limit $N \to \infty$ the charged particles as well as the photons exhibit condensation, with the time evolution approximately described by the Maxwell-Schrödinger system, which models the coupling of a non-relativistic particle to the classical electromagnetic field. Our result is obtained by an extension of the „method of counting", introduced in [75], to condensates of charged particles in interaction with their radiation field (see Chapter 2).

Contributions of the author and Acknowledgements This chapter presents joint work with Prof. Dr. Peter Pickl and has with minor modifications already been published as the preprint [54]. Theorem 3.2.1 is formulated in more generality than in [54] and its proof has slightly been changed. In addition, we inserted further references and remarks based on Chapter 2. The preprint was written by me. My contribution to the conceptual ideas is 50% and my share on their technical implementation is 80%. We thank Dr. Dirk André Deckert, Prof. Dr. Jan Dereziński, Prof. Dr. Detlef Dürr, Dr. Marco Falconi, Maximilian Jeblick, Vytautas Matulevičius, and Prof. Dr. Alessandro Michelangeli for many helpful remarks. We are deeply grateful to Vytautas Matulevičius for valuable discussions at the early stage of this project and to Prof. Dr. Alessandro Michelangeli for helpful remarks concerning the Maxwell-Schrödinger system. N.L. gratefully acknowledges support from the Cusanuswerk.

3.1 Setting of the problem

The existence of light quanta, later named photons, was first postulated by Albert Einstein in his renowned paper „On a heuristic point of view about the creation and conversion of light"[25]. This led to the invention of Quantum Electrodynamics and supplemented the nature of light, which was formerly described as a wave in classical electromagnetism, with a particle interpretation. During the last decades the predictions of Quantum Electrodynamics has been tested up to highest accuracy. Nevertheless, in a lot of situations the corpuscular character of light is subordinate and the second-quantized electromagnetic field can be approximated by a classical field satisfying Maxwell's equations. In this paper, the validity of such an approximation is justified in the mean-field regime. More explicitly, we derive the Maxwell-Schrödinger equations from the spinless Pauli-Fierz Hamiltonian. Such a derivation is of great interest to fundamental physics. Moreover, since the applied mean-field approximation reduces the degrees of freedom of the original system tremendously explicit error bounds might also be of interest for numerical simulations. We consider a system, described by a wave function $\Psi_{N,t} \in \mathcal{H}^{(N)}$, of N identical charged bosons in interaction with a photon field. Here,

$$\mathcal{H}^{(N)} := L^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p, \tag{3.1}$$

where the photon field is represented by elements of the Fock space

$$\mathcal{F}_p := \bigoplus_{n \geq 0} \left[L^2(\mathbb{R}^3) \otimes \mathbb{C}^2\right]^{\otimes_s^n}. \tag{3.2}$$

The subscript s indicates symmetry under interchange of variables. The Hilbert space $\mathfrak{h} := L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ consists of wave functions $f(k,\lambda)$, with wave number $k \in \mathbb{R}^3$ and helicity $\lambda = 1, 2$. It is equipped with the inner product

$$\langle f, g \rangle_{\mathfrak{h}} := \sum_{\lambda=1,2} \int d^3k\, f^*(k,\lambda) g(k,\lambda). \tag{3.3}$$

The time evolution of $\Psi_{N,t}$ is governed by the Schrödinger equation

$$i\partial_t \Psi_{N,t} = H_N \Psi_{N,t}, \tag{3.4}$$

where

$$H_N = \sum_{j=1}^{N} \left(-i\nabla_j - \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right)^2 + \frac{1}{N} \sum_{1 \leq j < k \leq N} v(x_j - x_k) + H_f \tag{3.5}$$

denotes the Pauli-Fierz Hamiltonian and

$$\hat{\boldsymbol{A}}_\kappa(x) = \sum_{\lambda=1,2} \int d^3k\, \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) \left(e^{ikx} a(k,\lambda) + e^{-ikx} a^*(k,\lambda)\right) \tag{3.6}$$

the quantized transverse vector potential. The function

$$\tilde{\kappa}(k) = (2\pi)^{-3/2}\, \mathbb{1}_{|k|\leq\Lambda}(k), \quad \text{with } \mathbb{1}_{|k|\leq\Lambda}(k) = \begin{cases} 1 & \text{if } |k| \leq \Lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{3.7}$$

cuts off the high frequency modes of the radiation field. There are two real polarization vectors $\boldsymbol{\epsilon}_1(k)$ and $\boldsymbol{\epsilon}_2(k)$ with

$$|\boldsymbol{\epsilon}_1(k)| = |\boldsymbol{\epsilon}_2(k)| = 1,\ \boldsymbol{\epsilon}_1(k) \cdot k = \boldsymbol{\epsilon}_2(k) \cdot k = \boldsymbol{\epsilon}_1(k) \cdot \boldsymbol{\epsilon}_2(k) = 0. \tag{3.8}$$

The operator valued distributions $a(k,\lambda)$ and $a^*(k,\lambda)$ ($k \in \mathbb{R}^3, \lambda \in \{1,2\}$) are the usual pointwise annihilation and creation operators in $\mathcal{F}_p$, satisfying

$$[a(k,\lambda), a^*(l,\mu)] = \delta_{\lambda,\mu}\delta(k-l), \quad [a(k,\lambda), a(l,\mu)] = [a^*(k,\lambda), a^*(l,\mu)] = 0. \tag{3.9}$$

The energy of the photon field is given by

$$H_f = \sum_{\lambda=1,2} \int d^3k \, |k| a^*(k,\lambda) a(k,\lambda) \tag{3.10}$$

and the potential v describes a direct interaction between the charged particles.

We assume:

(A1) The (repulsive) interaction potential v is a positive, real, and even function satisfying

$$||v||_{L^2+L^\infty} = \inf_{v=v_1+v_2} \{||v_1||_{L^2(\mathbb{R}^3)} + ||v_2||_{L^\infty(\mathbb{R}^3)}\} < \infty \tag{3.11}$$

such that the Pauli-Fierz Hamiltonian H_N is self-adjoint on the domain $\mathcal{D}(H_N) := \mathcal{D}(\sum_{i=1}^N -\Delta_i + H_f)$ (see [45] and [83, p.164]).

The mean-field scaling 1/N in front of the interaction potential and the scaling $1/\sqrt{N}$ in front of the vector potential ensure that the kinetic and potential energy of H_N are of the same order. At first, we are interested in the dynamics generated by H_N for initial conditions of the product form

$$\varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega. \tag{3.12}$$

Here, Ω denotes the vacuum in $\mathcal{F}_p$ and $W(f)$ is the unitary Weyl operator

$$W(f) := \exp\left(\sum_{\lambda=1,2} \int d^3k \, f(k,\lambda) a^*(k,\lambda) - f^*(k,\lambda) a(k,\lambda) \right). \tag{3.13}$$

This choice of initial data corresponds to situations in which both the charged particles and the photons exhibit condensation. Due to different types of interactions, correlations take place and the time evolved state will no longer have an exact product structure. However, for large N and times of order one it can be approximated, in a sense specified below, by a state of the product form $\varphi_t^{\otimes N} \otimes W(\sqrt{N}\alpha_t)\Omega$, where

$$|k|^{1/2}\alpha_t(k,\lambda) = \frac{1}{\sqrt{2}} \boldsymbol{\epsilon}_\lambda(k) \cdot \left(|k| \mathcal{FT}[\boldsymbol{A}](k,t) - i\mathcal{FT}[\boldsymbol{E}](k,t) \right) \tag{3.14}$$

and $(\varphi_t, \boldsymbol{A}(t), \boldsymbol{E}(t))$ solve the Maxwell-Schrödinger system[1]

$$\begin{cases} i\partial_t \varphi_t(x) &= \left((-i\nabla - (\kappa * \boldsymbol{A})(x,t))^2 + (v * |\varphi_t|^2)(x) \right) \varphi_t(x), \\ \nabla \cdot \boldsymbol{A}(x,t) &= 0, \\ \partial_t \boldsymbol{A}(x,t) &= -\boldsymbol{E}(x,t), \\ \partial_t \boldsymbol{E}(x,t) &= (-\Delta \boldsymbol{A})(x,t) - \left(1 - \nabla \mathrm{div} \Delta^{-1}\right) (\kappa * \boldsymbol{j}_t)(x), \\ \boldsymbol{j}_t(x) &= 2 \left(\mathrm{Im}(\varphi_t^* \nabla \varphi_t)(x) - |\varphi_t|^2(x) (\kappa * \boldsymbol{A})(x,t) \right) \end{cases} \tag{3.15}$$

with initial datum

$$\begin{cases} \varphi_0, \\ \boldsymbol{A}(x,0) = (2\pi)^{-3/2} \sum_{\lambda=1,2} \int d^3k \, \frac{1}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) \left(e^{ikx}\alpha_0(k,\lambda) + e^{-ikx}\alpha_0^*(k,\lambda) \right), \\ \boldsymbol{E}(x,0) = (2\pi)^{-3/2} \sum_{\lambda=1,2} \int d^3k \, \sqrt{\frac{|k|}{2}} \boldsymbol{\epsilon}_\lambda(k) i \left(e^{ikx}\alpha_0(k,\lambda) - e^{-ikx}\alpha_0^*(k,\lambda) \right). \end{cases} \tag{3.16}$$

These equations determine the time evolution of a single quantum particle interacting with the classical electromagnetic field it generates. The solution theory of this system is well studied, see [68] and references therein.

[1] Hereby, $(\kappa * \boldsymbol{A})(x,t) = \int d^3k \, e^{ikx} \tilde{\kappa}(k) \boldsymbol{A}(k,t)$.

3.2 Main result

The physical situation we are interested in is the dynamical description of a Bose-Einstein condensate of charged particles. We start with an initial wave function of product form (3.12) (a condition that will be relaxed later) and show that the condensate is stable over time, i.e. correlations are small at later times. Let $\Psi_{N,t} \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{H}^{(N)}$ with $||\Psi_{N,t}|| = 1$. On the Hilbert space $L^2(\mathbb{R}^3)$, define the „one-particle reduced density matrix of the charged particles" by

$$\gamma_{N,t}^{(1,0)} := \mathrm{Tr}_{2,\ldots,N} \otimes \mathrm{Tr}_{\mathcal{F}} |\Psi_{N,t}\rangle\langle\Psi_{N,t}|, \tag{3.17}$$

where $\mathrm{Tr}_{2,\ldots,N}$ denotes the partial trace over the coordinates $x_2, \ldots, x_N$ and $\mathrm{Tr}_{\mathcal{F}}$ the trace over Fock space. The charged particles of the many-body state $\Psi_{N,t}$ are said to exhibit complete asymptotic Bose-Einstein condensation at time t, if there exists $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$, such that

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)} |\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|| \to 0, \tag{3.18}$$

as $N \to \infty$. Such φ_t is called the condensate wave function. For other indicators of condensation and their relation we refer to [65]. Given $\Psi_{N,t} \in \mathcal{D}(H_f)$ with $||\Psi_{N,t}|| = 1$, we introduce the „one-particle reduced energy matrix of the photons " with kernel

$$\gamma_{N,t}^{(0,1)}(k,\lambda;k',\lambda') := N^{-1}|k|^{1/2}|k'|^{1/2}\left\langle\Psi_{N,t}, a^*(k',\lambda')a(k,\lambda)\Psi_{N,t}\right\rangle_{\mathcal{H}^{(N)}}. \tag{3.19}$$

$\gamma_{N,t}^{(0,1)}$ is a positive trace class operator on $\mathfrak{h}$ with $\mathrm{Tr}_{\mathfrak{h}}(\gamma_{N,t}^{(0,1)}) = N^{-1}\left\langle\Psi_{N,t}, H_f\Psi_{N,t}\right\rangle_{\mathcal{H}^{(N)}}$. It is important to note, that (3.19) differs from the usual definition (e.g. [80, p.8]) by the weight factor $|k|^{1/2}|k'|^{1/2}\left\langle\Psi_{N,t}, \mathcal{N}\Psi_{N,t}\right\rangle_{\mathcal{H}^{(N)}}/N$ with $\mathcal{N}$ being the number of photons operator. Our choice ensures that we neglect photons with small energies and measure only deviations from the photon field that are at least of order N. This is reasonable because due to the scaled coupling many photon states with a mean particle number smaller than of order N only have a subleading effect on the dynamics of the charged particles. We say the photons exhibit asymptotic Bose-Einstein condensation, if there exists a state $u_t \in \mathfrak{h}$, such that

$$\mathrm{Tr}_{\mathfrak{h}} |\gamma_{N,t}^{(0,1)} - |u_t\rangle\langle u_t|| \to 0, \tag{3.20}$$

as $N \to \infty$.

In the absence of a cutoff function and v being the Coulomb potential, the Maxwell-Schrödinger system is globally well-posed in the space[2] $C(\mathbb{R}_t, H^3(\mathbb{R}^3) \oplus H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$ [68]. We assume that this also holds in presence of the ultraviolet cutoff $\tilde{\kappa}$ and for potentials of the form (A1). More specific, we choose $\varphi_0 \in H^3(\mathbb{R}^3)$ and $\alpha_0 \in \mathfrak{h}$ such that $(\boldsymbol{A}(0), \boldsymbol{E}(0))$, defined by (3.16), is in $(H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$. Then, we assume

$$\sup_{t\in[0,T]} \{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty \tag{3.21}$$

for any $T \in \mathbb{R}^+$. This ensures (see (3.33)) that u_t, defined by

$$u_t(k,\lambda) := |k|^{1/2}\alpha(k,\lambda) = \frac{1}{\sqrt{2}}\boldsymbol{\epsilon}_\lambda(k) \cdot \left(|k|\mathcal{FT}[\boldsymbol{A}](k,t) - i\mathcal{FT}[\boldsymbol{E}](k,t)\right), \tag{3.22}$$

is an element of the Hilbert space $\mathfrak{h}$.

[2] The direct sum of the Sobolev spaces refers to $(\varphi_t, \boldsymbol{A}(t), \boldsymbol{E}(t))$.

Theorem 3.2.1. *Let v satisfy (A1), $\varphi_0 \in L^2(\mathbb{R}^3)$ with $||\varphi_0|| = 1$, $\alpha_0 \in \mathfrak{h}$ such that $(\boldsymbol{A}(0), \boldsymbol{E}(0)) \in (H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$. Let $\Psi_{N,0} \in \mathcal{D}(H_N) \cap \left(L_s^2(\mathbb{R}^{3N}) \otimes \mathcal{F}_p\right)$ such that*

$$a_N := Tr_{L^2(\mathbb{R}^3)}|\gamma_{N,0}^{(1,0)} - |\varphi_0\rangle\langle\varphi_0|| \to 0, \tag{3.23}$$

$$b_N := N^{-1}\left\langle W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}, H_f W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}\right\rangle_{\mathcal{H}^{(N)}} \to 0 \text{ and} \tag{3.24}$$

$$c_N := \left|\left|\left(N^{-1}H_N - \mathcal{E}_M\left[\varphi_0, \alpha_0\right]\right)\Psi_{N,0}\right|\right|_{\mathcal{H}^{(N)}}^2 \to 0 \tag{3.25}$$

as $N \to \infty$. Here,

$$\mathcal{E}_M\left[\varphi_t, \alpha_t\right] := ||(-i\nabla - \boldsymbol{A}_\kappa(t))\,\varphi_t||^2 + 1/2\left\langle\varphi_t, \left(v * |\varphi_t|^2\right)\varphi_t\right\rangle \tag{3.26}$$

$$+ \sum_{\lambda=1,2}\int d^3k\,|k||\alpha_t(k,\lambda)|^2 \tag{3.27}$$

denotes the energy functional of the Maxwell-Schrödinger system. Let $\Psi_{N,t}$ be the unique solution of (3.4). Let $(\varphi_t, \boldsymbol{A}(t), \boldsymbol{E}(t))$ be the unique solution of (3.15), u_t defined by (3.22) and assume $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T \in \mathbb{R}^+$. Then, there exists a monotone increasing function $C(s)$ of the norms $||\varphi_s||_{H^2(\mathbb{R}^2)}$, $||\nabla\varphi_s||_{L^\infty(\mathbb{R}^3)}$, $||\boldsymbol{A}(s)||_{H^2(\mathbb{R}^3)}$ and $||\boldsymbol{E}(s)||_{L^2(\mathbb{R}^3)}$ such that

$$Tr_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|| \leq \sqrt{a_N + b_N + c_N + N^{-1}}\,\Lambda e^{\Lambda^4\int_0^t ds\,C(s)}, \tag{3.28}$$

$$Tr_{\mathfrak{h}}|\gamma_{N,t}^{(0,1)} - |u_t\rangle\langle u_t|| \leq \sqrt{a_N + b_N + c_N + N^{-1}}\,\Lambda C(t)e^{\Lambda^4\int_0^t ds\,C(s)}. \tag{3.29}$$

for any $t \geq 0$. In particular, for $\Psi_{N,0} = \varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega$ one obtains

$$Tr_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|| \leq N^{-1/2}\Lambda^2 e^{\Lambda^4\int_0^t ds\,C(s)}, \tag{3.30}$$

$$Tr_{\mathfrak{h}}|\gamma_{N,t}^{(0,1)} - |u_t\rangle\langle u_t|| \leq N^{-1/2}\Lambda^2 C(t)e^{\Lambda^4\int_0^t ds\,C(s)}. \tag{3.31}$$

Remark 3.2.2. *Assumption (A1) allows to consider the Coulomb potential $v(x) = |x|^{-1}$. The requirements on the interaction potential can easily be relaxed because our estimates only rely on the finiteness of $\left|\left|v * |\varphi_t|^2\right|\right|_\infty$ and $\left|\left|v^2 * |\varphi_t|^2\right|\right|_\infty$. This is captured by (A1) and $\varphi_t \in H^3(\mathbb{R}^3)$ but also by other means.*

Remark 3.2.3. *For simplicity we apply the mean-field scaling $1/N$ in front of the direct interaction. Using techniques from [76] and [77] it seems possible to treat the direct interaction also in the NLS or Gross-Pitaevskii regime.*

Remark 3.2.4. *The ultraviolet cutoff is essential in our derivation but can be chosen N-dependent.*

3.3 Comparison with the literature

Derivations of classical field equations from Many-body Quantum Dynamics has been established in a series of works: In [39], Ginibre, Nironi and Velo derived the Schrödinger-Klein-Gordon system of equations from the Nelson model with cutoff. They considered a mean-field limit where a finite number of charged particles interacts with a coherent state of gauge bosons whose particle number goes to infinity. Falconi [33] derived the Schrödinger-Klein-Gordon system of equations in a mean-field limit where both the number of the charged particles and the gauge bosons go to infinity. Making use of a Wigner measure approach Ammari and Falconi [1] were able to establish the classical limit of the renormalized Nelson model without cutoff. The replacement of quantized radiation fields by classical interactions has also been justified in other limits. Teufel [86] considered the adiabatic limit of the Nelson model and showed that the interaction mediated by the quantized radiation field is well approximated by a direct Coulomb interaction. In [34] and [35], Frank, Gang and Schlein showed that in the strong coupling limit the dynamics of a polaron is described by an effective equation, in which the phonon field is treated as a classical field. Knowles [52] analyzed a finite number of heavy particles in a strong radiation field and derived the Newton-Maxwell equations from the Pauli Fierz Hamiltonian. In [82] it is shown that the semiclassical set of coupled Maxwell-Schrödinger equations is obtained by neglecting certain terms of the Pauli-Fierz Hamiltonian. To our best knowledge, this is the first rigorous result concerning a mean-field limit of the Pauli-Fierz Hamiltonian. This work continues the master thesis [63].

3.4 Notations

We set Planck's constant $\hbar$, the speed of light c, the charge e, and twice the mass of the particles $2m$ equal to one. Except in definitions, results, and where confusion might be possible, we refrain from indicating the explicit dependence of a quantity on the time t. We use the notations $\varphi(t)$ and φ_t interchangeably to denote a quantity φ at time t. The symbol C is used as a generic positive constant independent of t, N and Λ. We use expressions like $C(||\varphi||_{H^2(\mathbb{R}^3)}, ||\boldsymbol{A}||_{L^2(\mathbb{R}^3)})$ to denote positive monotone increasing function of the norms indicated. Both $\tilde{f}$ and $\mathcal{FT}[f]$ stand for the Fourier transform of f. With a slight abuse of notation $\boldsymbol{A}$ and $\boldsymbol{E}$ denote the vector potential and the electric field, but also their respective Fourier transforms. If we write $\boldsymbol{A}(t)$ or $\boldsymbol{E}(t)$, we always refer to the coordinate representation of the electromagnetic fields. Furthermore, we use the shorthand notation $\boldsymbol{A}_\kappa(x,t) := (\kappa * \boldsymbol{A})(x,t)$.
$H^s(\mathbb{R}^3)$ stands for the Sobolev space with norm $||f||_{H^s(\mathbb{R}^3)} = \left|\left|(1+|k|^2)^{s/2}\tilde{f}\right|\right|_{L^2(\mathbb{R}^3)}$ and $||A||_{HS} = \sqrt{TrA^*A}$ is used for the Hilbert-Schmidt norm. The symbol $\langle\cdot,\cdot\rangle$ denotes the scalar products on $\mathcal{H}^{(N)}$, $L^2(\mathbb{R}^3)$ and $\mathfrak{h}$. Furthermore, we use the shorthand notation $\langle\cdot,\cdot\rangle_{;y} = \int d^3y \langle\cdot,\cdot\rangle$ and $||\cdot||_{;y} = \sqrt{\int d^3y \langle\cdot,\cdot\rangle}$.

3.5 Organization of the proof

The structure of the proof is similar to Chapter 2. However, the interaction between the charges and the radiation field is more singular than in the Nelson model. This causes two major difficulties:

(a) The number of photons with small energies is difficult to control during the time evolution.

(b) There exist additional terms in the time derivative of the functional which can not be controlled with the techniques from Chapter 2.

In order to solve the first problem, we modify the functional β^b from Chapter 2 by a factor of $|k|$ in the integral. In this way we measure the fluctuations of the radiation field but neglect contributions from photons with small energies. The modified functional is well defined on $\mathcal{D}(H_f)$ and we do not need to show the invariance of $\mathcal{D}(\mathcal{N})$ under the time evolution which is generated by the Pauli-Fierz Hamiltonian. The disadvantage of the redefinition is that we only obtain information about the one-particle reduced energy matrix and not the one-particle reduced density matrix of the photons. To overcome the second problem we introduce an additional functional that measures the fluctuations of the energy of the many-body system. This allows us to control many-body states with high energy and to perform a Gronwall estimate. More information is given in Subsection 3.8.1. The proof is organized as follows:

(a) We define a functional β which serves as a measure of condensation. Afterwards, we show that convergence of the functional to zero in the limit $N \to \infty$ implies condensation in terms of reduced density matrices.

(b) In section 3.8 we control the growth of β by means of a Gronwall estimate. To this end, we provide preliminary estimates and control the time derivative of β.

(c) Then, we relate the value of the functional at time zero to the initial data of Theorem 3.2.1.

In our estimates, we need the regularity conditions

$$||\varphi_t||_\infty <\infty, \quad ||\nabla\varphi_t||_\infty < \infty, \quad ||\nabla\varphi_t|| < \infty, \quad ||\Delta\varphi_t|| < \infty, \tag{3.32}$$

$$||\boldsymbol{A}_\kappa(t)||_\infty <\infty, \quad \mathcal{E}_f(t) := \sum_{\lambda=1,2} \int d^3k\, |k||\alpha_t(k,\lambda)|^2 < \infty, \tag{3.33}$$

$$\mathcal{E}_{f^2}(t) := \sum_{\lambda=1,2} \int d^3k\, |k|^2|\alpha_t(k,\lambda)|^2 < \infty. \tag{3.34}$$

Assuming $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T \in \mathbb{R}^+$ the first line follows from Sobolev inequalities. To continue, we define the functions

$$\tilde{\kappa}_<(k) := (2\pi)^{-3/2}\mathbb{1}_{|k|\leq 1}(k), \qquad \tilde{\kappa}_>(k) := (2\pi)^{-3/2}|k|^{-2}\mathbb{1}_{1\leq|k|\leq\Lambda}(k) \tag{3.35}$$

with

$$\begin{aligned}
||\kappa_<||_2^2 =& \langle\tilde{\kappa}_<, \tilde{\kappa}_<\rangle = (2\pi)^{-3}\int_{|k|\leq 1} d^3k = (6\pi^2)^{-1},\\
||\kappa_>||_2^2 =& \langle\tilde{\kappa}_>, \tilde{\kappa}_>\rangle = (2\pi)^{-3}\int_{1\leq|k|\leq\Lambda} d^3k\, |k|^{-4} = (4\pi^2)^{-1}(1-1/\Lambda) \leq (4\pi^2)^{-1}.
\end{aligned} \tag{3.36}$$

This gives

$$\begin{aligned}
\boldsymbol{A}_\kappa(x,t) &= (2\pi)^3 \int d^3k\, e^{ikx}\mathbb{1}_{|k|\leq\Lambda}(k)\boldsymbol{A}(k,t) = (2\pi)^3\int d^3k\, e^{ikx}\mathbb{1}_{|k|\leq 1}(k)\boldsymbol{A}(k,t)\\
&+ (2\pi)^3\int d^3k\, e^{ikx}|k|^{-2}\mathbb{1}_{1\leq|k|\leq\Lambda}(k)|k|^2\boldsymbol{A}(k,t)\\
&= (\kappa_< * \boldsymbol{A})(x,t) - (\kappa_> * \Delta\boldsymbol{A})(x,t).
\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}
||\boldsymbol{A}_\kappa(t)||_\infty &\leq ||(\kappa_< * \boldsymbol{A})(t)||_\infty + ||(\kappa_> * \Delta\boldsymbol{A})(t)||_\infty\\
&\leq ||\kappa_<||\,||\boldsymbol{A}|| + ||\kappa_>||\,||\Delta\boldsymbol{A}|| < ||\boldsymbol{A}(t)||_{H^2(\mathbb{R}^3)}
\end{aligned} \tag{3.38}$$

where we made use of Young's inequality. By means of

$$\sum_{\lambda=1,2} \boldsymbol{\epsilon}_\lambda^i(k)\boldsymbol{\epsilon}_\lambda^j(k) = \delta_{ij} - \frac{k_i k_j}{|k|^2} \tag{3.39}$$

one easily shows

$$\sum_{\lambda=1,2} \int d^3k\, |k||\alpha(k,\lambda)|^2 = 1/2 \int d^3k\, \left(|k|^2 \boldsymbol{A}^2(k,t) + \boldsymbol{E}^2(k,t)\right) \leq ||\boldsymbol{A}(t)||^2_{H^1(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||^2\,,$$

$$\sum_{\lambda=1,2} \int d^3k\, |k|^2|\alpha(k,\lambda)|^2 = 1/2 \int d^3k\, \left(|k|^3 \boldsymbol{A}^2(k,t) + |k|\boldsymbol{E}^2(k,t)\right)$$

$$\leq ||\boldsymbol{A}(t)||^2_{H^1(\mathbb{R}^3)} ||\boldsymbol{A}(t)||_{H^2(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||\, ||\boldsymbol{E}(t)||_{H^1(\mathbb{R}^3)}\,. \tag{3.40}$$

3.6 The counting functional

In this section, we introduce a new indicator of condensation referred to as the „counting functional". Our system under consideration (3.4) describes the interaction of charged particles with a radiation field. Initially, we assume the charges and photons to exhibit condensation and we would like to show that both condensates are stable over time. In case of the charges, this is done by means of a functional, denoted by β^a, which counts for each time t the relative number of charges which are not in the state φ_t. Under suitable conditions on the photon field it is then possible to show that the rate of particles which leave the condensate is small, if initially almost all particles are in the state φ_0. The situation is different for the radiation field because the number of photons is not a conserved quantity. On that account not only existing photons gets correlated but also new photons are created or destroyed. One should note that the high frequency modes of the radiation field do not interact with the charges due to the ultraviolet cutoff (3.7) and evolve according to the free evolution. This is why neither the number of photons changes nor the photon state shows correlations for wave-numbers $|k| \geq \Lambda$. However in the long wave-length sector of $\mathcal{F}_b$ correlations take place and the number of photons varies. To show that the photon field remains coherent we introduce the functional β^b measuring for each time t the fluctuations of the photon field around the classical mode function. An additional factor of $|k|$ in the integral implies that we neglect contributions from photons with small energies. The main difficulties in our derivation arise from the minimal coupling term in the Pauli-Fierz Hamiltonian. On that account we have to control expectation values of certain unbounded operators, see Subsection 3.8.1. This is established by β^c which restricts our consideration to a subspace of many-body states whose energy per particle only fluctuates little around the energy functional of the effective system.
In order to define the counting functional we introduce the projectors $p_j^{\varphi_t}$ and $q_j^{\varphi_t}$.

Definition 3.6.1. *For any $N \in \mathbb{N}$, $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$ and $1 \leq j \leq N$ we define the time-dependent projectors $p_j^{\varphi_t} : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ and $q_j^{\varphi_t} : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ by*

$$p_j^{\varphi_t} f(x_1,\ldots,x_N) := \varphi_t(x_j) \int d^3x_j\, \varphi_t^*(x_j) f(x_1,\ldots,x_N) \quad \textit{for all } f \in L^2(\mathbb{R}^{3N}) \tag{3.41}$$

and $q_j^{\varphi_t} := 1 - p_j^{\varphi_t}$.[3]

[3]For ease of notation we mostly omit the superscript φ_t in the following. Additionally, we use the bra-ket notation $p_j^{\varphi_t} = |\varphi_t(x_j)\rangle\langle\varphi_t(x_j)|$

Moreover, we define the energy functional of the Maxwell-Schrödinger system by

$$\mathcal{E}_M\left[\varphi_t,\alpha_t\right] := \left|\left|\left(-i\nabla - \boldsymbol{A}_\kappa(t)\right)\varphi_t\right|\right|^2 + 1/2\big\langle\varphi_t, \left(v * |\varphi_t|^2\right)\varphi_t\big\rangle \\ + \sum_{\lambda=1,2}\int d^3k\, |k||\alpha_t(k,\lambda)|^2. \tag{3.42}$$

Note that $\mathcal{E}_M[\varphi_t,\alpha_t]$ is finite under assumption (A1) and $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)}+||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T\in\mathbb{R}^+$. The counting functional is defined by

Definition 3.6.2. *Let $\Psi_{N,t}\in\mathcal{D}(H_N)$, $\varphi_t\in H^3(\mathbb{R}^3)$ with $||\varphi_t||=1$ and $\alpha_t\in\mathfrak{h}$ such that $(\boldsymbol{A}(t),\boldsymbol{E}(t))\in(H^3(\mathbb{R}^3)\oplus H^2(\mathbb{R}^3))$. We define*

$$\beta^a(\Psi_{N,t},\varphi_t) := \big\langle\Psi_{N,t}, q_1\otimes\mathbb{1}_{\mathcal{F}_p}\Psi_{N,t}\big\rangle,$$
$$\beta^b(\Psi_{N,t},\alpha_t) := \sum_{\lambda=1,2}\int d^3k\,|k|\Big\langle\left(\frac{a(k,\lambda)}{\sqrt{N}}-\alpha_t(k,\lambda)\right)\Psi_{N,t}, \left(\frac{a(k,\lambda)}{\sqrt{N}}-\alpha_t(k,\lambda)\right)\Psi_{N,t}\Big\rangle,$$
$$\beta^c(\Psi_{N,t},\varphi_t,\alpha_t) := \Big\langle\left(\frac{H_N}{N}-\mathcal{E}_M[\varphi_t,\alpha_t]\right)\Psi_{N,t}, \left(\frac{H_N}{N}-\mathcal{E}_M[\varphi_t,\alpha_t]\right)\Psi_{N,t}\Big\rangle. \tag{3.43}$$

The functional $\beta:\mathcal{D}(H_N)\times H^3(\mathbb{R}^3)\times\mathfrak{h}\to\mathbb{R}_0^+$ is then given by $\beta := \beta^a+\beta^b+\beta^c$.

The functional β^a was already used in [5], [53], [66], [75], [76],[77], [47] and others to derive the Hartree and Gross-Pitaevskii equation, while β^b and β^c are introduced to control the interaction with the radiation field.

3.7 Relation to reduced density matrices

Next, we show that condensation indicated by the counting functional, $\beta\to0$ as $N\to\infty$, implies condensation in terms of reduced density matrices.

Lemma 3.7.1. *Let $\Psi_{N,t}\in\mathcal{D}(H_N)$, $\varphi_t\in L^2(\mathbb{R}^3)$ with $||\varphi_t||=1$, $\alpha_t\in\mathfrak{h}$ such that $||u_t||_{\mathfrak{h}}<\infty$. Then*

$$\beta^a(\Psi_{N,t},\varphi_t) \leq Tr_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|| \leq \sqrt{8\beta^a(\Psi_{N,t},\varphi_t)}, \tag{3.44}$$
$$Tr_{\mathfrak{h}}|\gamma_{N,t}^{(0,1)} - |u_t\rangle\langle u_t|| \leq 3\beta^b(\Psi_{Nt},\alpha_t) + 6\,||u_t||_{\mathfrak{h}}\sqrt{\beta^b(\Psi_{Nt},\alpha_t)}. \tag{3.45}$$

Proof. The first inequality follows from[4]

$$\beta^a = 1-\big\langle\Psi_N,p_1\Psi_N\big\rangle = 1-\big\langle\varphi,\gamma_N^{(1,0)}\varphi\big\rangle = \mathrm{Tr}_{L^2(\mathbb{R}^3)}(|\varphi\rangle\langle\varphi| - |\varphi\rangle\langle\varphi|\gamma_N^{(1,0)}) \\ \leq ||p_1||_{\mathrm{op}}\,\mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi|| = \mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi||. \tag{3.46}$$

In order to proof the remaining inequalities we use

$$\mathrm{Tr}|\gamma-p| \leq 2\,||\gamma-p||_{HS} + \mathrm{Tr}(\gamma-p), \tag{3.47}$$

valid for any one-dimensional projector p and non-negative density matrix γ. The original argument of the proof was first observed by Robert Seiringer, see [80]. We present a version

[4] For ease of notation, we discard the explicit time dependence and write for example Ψ_N instead of $\Psi_{N,t}$.

that is found in [5]: Let $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of eigenvalues of the trace class operator $A := \gamma - p$. Since p is a rank one projection, A has at most one negative eigenvalue. If there is no negative eigenvalue, $\mathrm{Tr}|A| = \mathrm{Tr}(A)$ and (3.47) holds. If there is one negative eigenvalue λ_1, we have $\mathrm{Tr}|A| = |\lambda_1| + \sum_{n\geq 2}\lambda_n = 2|\lambda_1| + \mathrm{Tr}(A)$. Because of $|\lambda_1| \leq ||A||_{\mathrm{op}} \leq ||A||_{HS}$, inequality (3.47) follows.

For the upper bound of (3.44) we notice that $\mathrm{Tr}_{L^2(\mathbb{R}^3)}(\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi|) = 0$. Then, (3.47) reduces to

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi|| \leq 2\left|\left|\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi|\right|\right|_{HS} \tag{3.48}$$

and (3.44) follows from

$$\begin{aligned}\mathrm{Tr}_{L^2(\mathbb{R}^3)}(\gamma_N^{(1,0)} - |\varphi\rangle\langle\varphi|)^2 =& 1 - 2\mathrm{Tr}_{L^2(\mathbb{R}^3)}(|\varphi\rangle\langle\varphi|\gamma_N^{(1,0)}) + \mathrm{Tr}_{L^2(\mathbb{R}^3)}((\gamma_N^{(1,0)})^2)\\ \leq& 2(1 - \mathrm{Tr}_{L^2(\mathbb{R}^3)}(|\varphi\rangle\langle\varphi|\gamma_N^{(1,0)})) = 2\beta^a.\end{aligned} \tag{3.49}$$

To prove inequality (3.45) it is useful to write the kernel of $\gamma_N^{(0,1)} - |u\rangle\langle u|$ as

$$\begin{aligned}(\gamma_N^{(0,1)} - |u\rangle\langle u|)(k,\lambda,l,\mu) &= |k|^{1/2}|l|^{1/2}\left(N^{-1}\langle\Psi_N, a^*(l,\mu)a(k,\lambda)\Psi_N\rangle - \alpha^*(l,\mu)\alpha(k,\lambda)\right)\\ &= |k|^{1/2}|l|^{1/2}\langle\left(N^{-1/2}a(l,\mu) - \alpha(l,\mu)\right)\Psi_N, \left(N^{-1/2}a(k,\lambda) - \alpha(k,\lambda)\right)\Psi_N\rangle\\ &+ |k|^{1/2}|l|^{1/2}\langle\left(N^{-1/2}a(l,\mu) - \alpha(l,\mu)\right)\Psi_N, \Psi_N\rangle\,\alpha(k,\lambda)\\ &+ |k|^{1/2}|l|^{1/2}\langle\Psi_N, \left(N^{-1/2}a(k,\lambda) - \alpha(k,\lambda)\right)\Psi_N\rangle\,\alpha^*(l,\mu).\end{aligned} \tag{3.50}$$

Cauchy-Schwarz inequality gives

$$\begin{aligned}&|(\gamma_N^{(0,1)} - |u\rangle\langle u|)(k,\lambda,l,\mu)|^2\\ &\quad\leq |k||l|\left|\left|\left(N^{-1/2}a(k,\lambda) - \alpha(k,\lambda)\right)\Psi_N\right|\right|^2\left|\left|\left(N^{-1/2}a(l,\mu) - \alpha(l,\mu)\right)\Psi_N\right|\right|^2\\ &\quad+ |k||l|\left|\left|\left(N^{-1/2}a(k,\lambda) - \alpha(k,\lambda)\right)\Psi_N\right|\right|^2|\alpha(l,\mu)|^2\\ &\quad+ |k||l|\left|\left|\left(N^{-1/2}a(l,\mu) - \alpha(l,\mu)\right)\Psi_N\right|\right|^2|\alpha(k,\lambda)|^2\end{aligned} \tag{3.51}$$

and

$$\begin{aligned}\left|\left|\gamma_N^{(0,1)} - |u\rangle\langle u|\right|\right|_{HS}^2 &= \sum_{\lambda,\mu\in\{1,2\}^2}\int\int d^3k d^3l |(\gamma_N^{(0,1)} - |u\rangle\langle u|)(k,\lambda,l,\mu)|^2\\ &\leq (\beta^b)^2 + 2\,||u||_{\mathfrak{h}}^2\,\beta^b\end{aligned} \tag{3.52}$$

follows. Similarly,

$$\begin{aligned}\mathrm{Tr}_{\mathfrak{h}}(\gamma_N^{(0,1)} - |u\rangle\langle u|) &\leq \sum_{\lambda=1,2}\int d^3k|(\gamma_N^{(0,1)} - |u\rangle\langle u|)(k,\lambda,k,\lambda)|\\ &\leq \sum_{\lambda=1,2}\int d^3k|k|\left|\left|\left(N^{-1/2}a(k,\lambda) - \alpha(k,\lambda)\right)\Psi_N\right|\right|^2\\ &+ 2\sum_{\lambda=1,2}\int d^3k|u(k,\lambda)||k|^{1/2}\left|\left|\left(N^{-1/2}a(k,\lambda) - \alpha(k,\lambda)\right)\Psi_N\right|\right|.\end{aligned} \tag{3.53}$$

Applying Schwarz's inequality with respect to the scalar product of $\mathfrak{h}$ yields

$$\begin{aligned}\mathrm{Tr}_{\mathfrak{h}}(\gamma_N^{(0,1)} - |\alpha\rangle\langle\alpha|) &\leq \beta^b + 2\,||u||_{\mathfrak{h}} \Big(\sum_{\lambda=1,2} \int d^3k |k| \left|\left| \Big(N^{-1/2} a(k,\lambda) - \alpha(k,\lambda) \Big) \Psi_N \right|\right|^2 \Big)^{1/2} \\ &\leq \beta^b + 2\,||u||_{\mathfrak{h}} \sqrt{\beta^b}. \end{aligned} \tag{3.54}$$

Monotonicity of the square root and (3.47) give rise to (3.45).

3.8 Estimates on the time derivative

In this section, we control the change of β in time. To this end we separately estimate the time derivative of β^a and β^b. The value of β^c is constant in time because the energies of the many-body system and effective equations are conserved quantities. To control the difference between the quantized and classical vector potential by the functional β^b it is convenient to introduce their positive and negative frequency parts.

$$\begin{aligned}\hat{\boldsymbol{A}}_\kappa^+(x) &:= \sum_{\lambda=1,2} \int d^3k\, \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) e^{ikx} a(k,\lambda), \\ \hat{\boldsymbol{A}}_\kappa^-(x) &:= \sum_{\lambda=1,2} \int d^3k\, \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) e^{-ikx} a^*(k,\lambda), \\ \boldsymbol{A}_\kappa^+(x,t) &:= \sum_{\lambda=1,2} \int d^3k\, \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) e^{ikx} \alpha_t(k,\lambda), \\ \boldsymbol{A}_\kappa^-(x,t) &:= \sum_{\lambda=1,2} \int d^3k\, \tilde{\kappa}(k) \frac{1}{\sqrt{2|k|}} \boldsymbol{\epsilon}_\lambda(k) e^{-ikx} \alpha_t^*(k,\lambda). \end{aligned} \tag{3.55}$$

Moreover, it is helpful to define the positive and negative frequency parts of the quantum mechanical and classical electric field.

$$\begin{aligned}\hat{\boldsymbol{E}}_\kappa^+(x) &:= \sum_{\lambda=1,2} \int d^3k\, \tilde{\kappa}(k) \sqrt{\frac{|k|}{2}} \boldsymbol{\epsilon}_\lambda(k) i e^{ikx} a(k,\lambda), \\ \hat{\boldsymbol{E}}_\kappa^-(x) &:= \sum_{\lambda=1,2} \int d^3k\, \tilde{\kappa}(k) \sqrt{\frac{|k|}{2}} \boldsymbol{\epsilon}_\lambda(k) (-i) e^{-ikx} a^*(k,\lambda), \\ \boldsymbol{E}_\kappa^+(x,t) &:= \sum_{\lambda=1,2} \int d^3k\, \tilde{\kappa}(k) \sqrt{\frac{|k|}{2}} \boldsymbol{\epsilon}_\lambda(k) i e^{ikx} \alpha_t(k,\lambda), \\ \boldsymbol{E}_\kappa^-(x,t) &:= \sum_{\lambda=1,2} \int d^3k\, \tilde{\kappa}(k) \sqrt{\frac{|k|}{2}} \boldsymbol{\epsilon}_\lambda(k) (-i) e^{-ikx} \alpha_t^*(k,\lambda). \end{aligned} \tag{3.56}$$

For $\sharp \in \{\ , +, -\}$, we introduce the shorthand notations

$$\boldsymbol{\mathcal{E}}^\sharp(x,t) := \frac{\hat{\boldsymbol{E}}_\kappa^\sharp(x)}{\sqrt{N}} - \boldsymbol{E}_\kappa^\sharp(x,t), \quad \boldsymbol{\mathcal{A}}^\sharp(x,t) := \frac{\hat{\boldsymbol{A}}_\kappa^\sharp(x)}{\sqrt{N}} - \boldsymbol{A}_\kappa^\sharp(x,t). \tag{3.57}$$

By means of the cutoff function

$$\tilde{\eta}(k) := |k|^{-1} \tilde{\kappa}(k) = (2\pi)^{-\frac{3}{2}} |k|^{-1} \mathbb{1}_{|k| \leq \Lambda}(k) \tag{3.58}$$

we are able to express the vector potential in terms of the electric field.

Lemma 3.8.1. *Let η be the Fourier transform of* (3.58)*, then*

$$\begin{aligned} \hat{\boldsymbol{A}}_\kappa^+(x) &= -i\big(\eta * \hat{\boldsymbol{E}}_\kappa^+\big)(x), & \hat{\boldsymbol{A}}_\kappa^-(x) &= i\big(\eta * \hat{\boldsymbol{E}}_\kappa^-\big)(x), \\ \boldsymbol{A}_\kappa^+(x,t) &= -i\big(\eta * \boldsymbol{E}_\kappa^+\big)(x,t), & \boldsymbol{A}_\kappa^-(x,t) &= i\big(\eta * \boldsymbol{E}_\kappa^-\big)(x,t). \end{aligned} \tag{3.59}$$

Proof. The proof is a simple application of the convolution theorem. □

At various points in our estimates, we replace the vector potential by the electric field and make use of (see Lemma 3.11.1)

$$\int d^3y \, \langle \Psi_{N,t}, \Big(N^{-1/2}\hat{\boldsymbol{E}}_\kappa^-(y) - \boldsymbol{E}_\kappa^-(y,t)\Big) \Big(N^{-1/2}\hat{\boldsymbol{E}}_\kappa^+(y) - \boldsymbol{E}_\kappa^+(y,t)\Big) \Psi_{N,t}\rangle \leq \beta^b(t). \tag{3.60}$$

To obtain proper bounds it is crucial that the L^2-norm of the cutoff functions

$$||\kappa||_2^2 = \Lambda^3/(6\pi^2) \quad \text{and} \quad ||\eta||_2^2 = \Lambda/(2\pi^2) \tag{3.61}$$

is finite.

3.8.1 Preliminary estimates

The minimal coupling term in the Pauli-Fierz Hamiltonian

$$\sum_{j=1}^N (-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j))^2 = \sum_{j=1}^N \Big(-\Delta_1 + 2iN^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\cdot\nabla_1 + N^{-1}\hat{\boldsymbol{A}}_\kappa^2(x_1)\Big) \tag{3.62}$$

contains an interaction that is quadratic in the vector potential. If we want to control the growth of $\beta(t)$ in time this quadratic part (see (3.98)) requires that quantities like $N^{-1}\langle\Psi_{N,t}, q_1\hat{\boldsymbol{A}}_\kappa^2(x_1)q_1\Psi_{N,t}\rangle$ are not only finite but bounded by $\beta(t)$. This holds for every bounded operator B because of

$$\langle\Psi_{N,t}, q_1 B q_1 \Psi_{N,t}\rangle \leq C\,||q_1\Psi_{N,t}||^2 \leq C\beta^a(t) \tag{3.63}$$

but must not be true in general. In case of unbounded operators smallness can sometimes be shown on a subclass of states which have sufficient decay in the occupation of eigenstates. For a self-adjoint operator O with $[O, q_1] \approx 0$ and $c \in \mathbb{R}$ one has

$$\begin{aligned} \langle\Psi_{N,t}, q_1 O q_1 \Psi_{N,t}\rangle \approx& \langle\Psi_{N,t}, q_1 O \Psi_{N,t}\rangle = \langle\Psi_{N,t}, q_1\,(O-c)\,\Psi_{N,t}\rangle + c\langle\Psi_{N,t}, q_1\Psi_{N,t}\rangle \\ \leq& (c+1)\langle\Psi_{N,t}, q_1\Psi_{N,t}\rangle + \langle\Psi_{N,t}, (O-c)^2\,\Psi_{N,t}\rangle. \end{aligned} \tag{3.64}$$

Thus, $\langle\Psi_{N,t}, q_1 O q_1 \Psi_{N,t}\rangle$ is small if $\Psi_{N,t}$ occupies eigenstates of O with eigenvalues $\lambda \neq c$ only with small probability. This is in the spirit of Chebyshev's inequality which is of great use in probability theory. Requiring $\langle\Psi_{N,0}, (O-c)^2\,\Psi_{N,0}\rangle \approx 0$ initially does not imply smallness at later times. However, if we choose for O a conserved quantity its variance is a constant of motion during the time evolution and we only have to restrict our class of initial states. In the following, we consider the variance of the energy per particles of the many-body system (see β^c). Then, we estimate the vector potential and the Laplacian by H_N/N and bound expression like $N^{-1}\langle\Psi_{N,t}, q_1\hat{\boldsymbol{A}}_\kappa^2(x_1)q_1\Psi_{N,t}\rangle$ by the counting functional.

Lemma 3.8.2. *Let $y \in \mathbb{R}^3$ or $y \in \{x_1, \ldots, x_N\}$ and $\Psi_N \in \mathcal{D}(H_N)$. Then*

$$\begin{aligned} \Big|\Big|N^{-1/2}\hat{\boldsymbol{A}}_\kappa^+(y)\Psi_N\Big|\Big|^2 &\leq \Lambda/(2\pi^2)\langle\Psi_N, N^{-1}H_f\Psi_N\rangle, \\ \Big|\Big|N^{-1/2}\hat{\boldsymbol{A}}_\kappa^-(y)\Psi_N\Big|\Big|^2 &\leq \Lambda/(2\pi^2)\langle\Psi_N, N^{-1}H_f\Psi_N\rangle + \Lambda^2/(4\pi^2 N)\,||\Psi_N||^2, \\ \Big|\Big|N^{-1/2}\hat{\boldsymbol{A}}_\kappa(y)\Psi_N\Big|\Big|^2 &\leq 2\Lambda/(\pi^2)\langle\Psi_N, N^{-1}H_f\Psi_N\rangle + \Lambda^2/(2\pi^2 N)\,||\Psi_N||^2. \end{aligned} \tag{3.65}$$

Proof. To ease notation, we define the vector-valued function $\boldsymbol{f}(k,\lambda) := \frac{\tilde{\kappa}(k)}{\sqrt{2|k|}}\boldsymbol{\epsilon}_\lambda(k)$. The first estimate follows from Cauchy-Schwarz inequality

$$\begin{aligned}
&\Big|\Big| \sum_{\lambda=1,2} \int d^3k\, \boldsymbol{f}(k,\lambda) e^{\pm iky} a(k,\lambda)\Psi_N \Big|\Big|^2 \\
&\leq \Big(\sum_{\lambda=1,2} \int d^3k\, |\boldsymbol{f}(k,\lambda)||k|^{-1/2} \Big|\Big| |k|^{1/2} a(k,\lambda)\Psi_N \Big|\Big|^2 \Big)^2 \\
&\leq \Big(\sum_{\lambda=1,2} \int d^3k\, |\boldsymbol{f}(k,\lambda)|^2 |k|^{-1} \Big) \Big(\sum_{\lambda=1,2} \int d^3k\, |k|\, ||a(k,\lambda)\Psi_N||^2 \Big) \\
&= \Lambda/(2\pi^2) \big\langle \Psi_N, H_f \Psi_N \big\rangle.
\end{aligned} \tag{3.66}$$

By use of the canonical commutation relations (3.9), the second bound is obtained via

$$\begin{aligned}
&\Big|\Big| \sum_{\lambda=1,2} \int d^3k\, \boldsymbol{f}(k,\lambda) e^{\pm iky} a^*(k,\lambda)\Psi_N \Big|\Big|^2 = \Big|\Big| \sum_{\lambda=1,2} \int d^3k\, \overline{\boldsymbol{f}(k,\lambda)} e^{\mp iky} a(k,\lambda)\Psi_N \Big|\Big|^2 \\
&+ ||\boldsymbol{f}||_{\mathfrak{h}}^2\, ||\Psi_N||^2 \leq \Lambda^2/(4\pi^2)\, ||\Psi_N||^2 + \Lambda/(2\pi^2) \big\langle \Psi_N, H_f \Psi_N \big\rangle.
\end{aligned} \tag{3.67}$$

The last estimate follows by triangular inequality. □

Lemma 3.8.2 leads to

Corollary 3.8.3. *For $\Psi_N \in \mathcal{D}(H_N)$ we have*

$$\begin{aligned}
\Big|\Big| N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) q_1 \Psi_N \Big|\Big|^2 &\leq 2\Lambda/\pi^2 \big\langle \Psi_N, q_1 N^{-1} H_f q_1 \Psi_N \big\rangle + \Lambda^2/(2\pi^2 N)\beta^a, \\
\Big|\Big| N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) p_1 q_2 \Psi_N \Big|\Big|^2 &\leq 2\Lambda/\pi^2 \big\langle \Psi_N, q_1 N^{-1} H_f q_1 \Psi_N \big\rangle + \Lambda^2/(2\pi^2 N)\beta^a.
\end{aligned} \tag{3.68}$$

Lemma 3.8.4. *Let v satisfy (A1), $\Psi_{N,t} \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H_N)$, $\varphi_t \in H^3(\mathbb{R}^3)$ with $||\varphi_t|| = 1$ and $\alpha_t \in \mathfrak{h}$ such that $(\boldsymbol{A}(t), \boldsymbol{E}(t)) \in (H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$. Then, there exists a monotone increasing function $C(t)$ of $\mathcal{E}_M[\varphi_t, \alpha_t]$, $||\varphi_t||_{H^2(\mathbb{R}^3)}$ and $||\varphi_t||_{L^\infty(\mathbb{R}^3)}$ such that*

$$\big\langle \Psi_{N,t}, q_1^{\varphi_t} N^{-1} H_N q_1^{\varphi_t} \Psi_{N,t} \big\rangle \leq C(t) \left(\beta(t) + \Lambda/N\right). \tag{3.69}$$

Proof. We decompose the Pauli-Fierz Hamiltonian into

$$\big\langle \Psi_N, q_1 N^{-1} H_N q_1 \Psi_N \big\rangle = \big\langle \Psi_N, q_1 N^{-1} \sum_{j=1}^N \Big(-i\nabla_j - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_j) \Big)^2 q_1 \Psi_N \big\rangle \tag{3.70}$$

$$+ \big\langle \Psi_N, q_1 N^{-2} \sum_{1\leq j<k\leq N} v(x_j - x_k) q_1 \Psi_N \big\rangle \tag{3.71}$$

$$+ \big\langle \Psi_N, q_1 N^{-1} H_f q_1 \Psi_N \big\rangle. \tag{3.72}$$

Then, we write the first line as

$$
\begin{aligned}
(3.70) &= \big\langle \Psi_N, q_1 N^{-1} \sum_{j=1}^{N} \Big(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\Big)^2 \Psi_N \big\rangle \\
&\quad + N^{-1} \sum_{j=1}^{N} \big\langle \Psi_N, q_1 \left[\Big(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\Big)^2, q_1 \right] \Psi_N \big\rangle \\
&= \big\langle \Psi_N, q_1 N^{-1} \sum_{j=1}^{N} \Big(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\Big)^2 \Psi_N \big\rangle \\
&\quad + N^{-1} \big\langle \Psi_N, q_1 \left[\Big(-i\nabla_1 - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\Big)^2, q_1 \right] \Psi_N \big\rangle. \qquad (3.73)
\end{aligned}
$$

The second line is given by

$$
\begin{aligned}
(3.71) &= \big\langle \Psi_N, q_1 N^{-2} \sum_{1\le j<k\le N} v(x_j - x_k)\Psi_N \big\rangle + N^{-2} \sum_{1\le j<k\le N} \big\langle \Psi_N, q_1 \left[v(x_j - x_k), q_1 \right] \Psi_N \big\rangle \\
&= \big\langle \Psi_N, q_1 N^{-2} \sum_{1\le j<k\le N} v(x_j - x_k)\Psi_N \big\rangle + (N-1)N^{-2} \big\langle \Psi_N, q_1 \left[v(x_1 - x_2), q_1 \right] \Psi_N \big\rangle. \qquad (3.74)
\end{aligned}
$$

In line (3.72) we use that H_f commutes with operators which only act on the sector of the non-relativistic particles. This leads to

$$
\begin{aligned}
\big\langle \Psi_N, q_1 N^{-1} H_N q_1 \Psi_N \big\rangle &= \big\langle \Psi_N, q_1 N^{-1} H_N \Psi_N \big\rangle \\
&\quad + N^{-1} \big\langle \Psi_N, q_1 \left[\Big(-i\nabla_1 - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\Big)^2, q_1 \right] \Psi_N \big\rangle \qquad (3.75) \\
&\quad + (N-1)N^{-2} \big\langle \Psi_N, q_1 \left[v(x_1 - x_2), q_1 \right] \Psi_N \big\rangle. \qquad (3.76)
\end{aligned}
$$

The first term is estimated by

$$
\begin{aligned}
|(3.75)| &= N^{-1} \big| \big\langle \Psi_N, q_1 \left[\Big(-i\nabla_1 - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\Big)^2, p_1 \right] \Psi_N \big\rangle \big| \\
&= N^{-1} \big| \big\langle \Psi_N, q_1 \Big(-i\nabla_1 - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\Big)^2 p_1 \Psi_N \big\rangle \big| \\
&\le N^{-1} \big| \big\langle q_1 \Psi_N, (-\Delta_1) p_1 \Psi_N \big\rangle \big| \\
&\quad + N^{-1} \big| \big\langle N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) q_1 \Psi_N, \nabla_1 p_1 \Psi_N \big\rangle \big| \\
&\quad + N^{-1} \big| \big\langle N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) q_1 \Psi_N, N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) p_1 \Psi_N \big\rangle \big| \\
&\le N^{-1} \Big(\beta^a + ||\Delta p_1 \Psi_N||^2 + ||\nabla p_1 \Psi_N||^2 \Big) \\
&\quad + N^{-1} \Big(\Big|\Big| N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) q_1 \Psi_N \Big|\Big|^2 + \Big|\Big| N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) p_1 \Psi_N \Big|\Big|^2 \Big). \qquad (3.77)
\end{aligned}
$$

Lemma 3.8.2 and the positivity of the interaction potential v let us continue with

$$
\begin{aligned}
|(3.75)| &\le N^{-1} \Lambda C(||\varphi||_{H^2}) \big(\big\langle \Psi_N, N^{-1} H_f \Psi_N \big\rangle + \Lambda/N \big) \\
&\le N^{-1} \Lambda C(||\varphi||_{H^2}) \big(\big\langle \Psi_N, N^{-1} H_N \Psi_N \big\rangle + \Lambda/N \big) \\
&\le N^{-1} \Lambda C(||\varphi||_{H^2}) \big(\big\langle \Psi_N, (N^{-1} H_N - \mathcal{E}_M) \Psi_N \big\rangle + \mathcal{E}_M \big) \\
&\le N^{-1} \Lambda C(||\varphi||_{H^2}) \big(\big|\big| (N^{-1} H_N - \mathcal{E}_M) \Psi_N \big|\big| + \mathcal{E}_M \big) \\
&\le N^{-1} \Lambda C(||\varphi||_{H^2}) \Big(\sqrt{\beta^c} + \mathcal{E}_M \Big) \le N^{-1} \Lambda C(||\varphi||_{H^2}, \mathcal{E}_M). \qquad (3.78)
\end{aligned}
$$

The second term is bounded by

$$
\begin{aligned}
|(3.76)| &\le N^{-1}|\langle \Psi_N, q_1 \left[v(x_1-x_2), p_1\right] \Psi_N\rangle| = N^{-1}|\langle \Psi_N, q_1 v(x_1-x_2) p_1 \Psi_N\rangle| \\
&\le 1/2\, ||q_1\Psi_N||^2 + 1/(2N^2)\, ||v(x_1-x_1)p_1\Psi_N||^2 \\
&= 1/2\beta^a + 1/(2N^2)\langle \Psi_N, p_1 v^2(x_2-x_1) p_1 \Psi_N\rangle \\
&= 1/2\beta^a + 1/(2N^2)\langle \Psi_N, p_1 \left(v^2 * |\varphi|^2\right)(x_2) \Psi_N\rangle \\
&\le 1/2\beta^a + 1/(2N^2)\, \left|\left|v^2 * |\varphi|^2\right|\right|_\infty .
\end{aligned}
\tag{3.79}
$$

We use assumption (A1) and decompose the interaction potential $v = v_1 + v_2$ into $v_1 \in L^2(\mathbb{R}^3)$ and $v_2 \in L^\infty(\mathbb{R}^3)$. Then, we apply Young's inequality and obtain

$$
\begin{aligned}
\left|\left|v^2 * |\varphi|^2\right|\right|_\infty &\le \left|\left|v_1^2\right|\right|_1 \left|\left||\varphi|^2\right|\right|_\infty + \left|\left|v_2^2\right|\right|_\infty \left|\left||\varphi|^2\right|\right|_1 \\
&= ||v_1||_2^2\, ||\varphi||_\infty^2 + ||v_2||_\infty^2\, ||\varphi||_2^2 \le C(||\varphi||_\infty).
\end{aligned}
\tag{3.80}
$$

Thus,

$$
|(3.75) + (3.76)| \le C(||\varphi||_{H^2}, ||\varphi||_\infty, \mathcal{E}_M)\, (\beta + \Lambda/N) \tag{3.81}
$$

and

$$
\begin{aligned}
\langle \Psi_N, q_1 N^{-1} H_N q_1 \Psi_N\rangle &\le |\langle \Psi_N, q_1 N^{-1} H_N \Psi_N\rangle| + |(3.75) + (3.76)| \\
&\le |\langle \Psi_N, q_1 \left(N^{-1}H_N - \mathcal{E}_M\right) \Psi_N\rangle + \mathcal{E}_M \beta^a| + C(||\varphi||_{H^2}, ||\varphi||_\infty, \mathcal{E}_M)\, (\beta + \Lambda/N) \\
&\le |\langle \Psi_N, q_1 \left(N^{-1}H_N - \mathcal{E}_M\right) \Psi_N\rangle| + C(||\varphi||_{H^2}, ||\varphi||_\infty, \mathcal{E}_M)\, (\beta + \Lambda/N) \\
&\le \langle \Psi_N, \left(N^{-1}H_N - \mathcal{E}_M\right)^2 \Psi_N\rangle + \beta^a + C(||\varphi||_{H^2}, ||\varphi||_\infty, \mathcal{E}_M)\, (\beta + \Lambda/N) \\
&\le C(||\varphi||_{H^2}, ||\varphi||_\infty, \mathcal{E}_M)\, (\beta + \Lambda/N) .
\end{aligned}
\tag{3.82}
$$

□

Lemma 3.8.5. *Let v satisfy (A1), $\Psi_{N,t} \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H_N)$, $\varphi_t \in H^3(\mathbb{R}^3)$ with $||\varphi_t|| = 1$ and $\alpha_t \in \mathfrak{h}$ such that $(\boldsymbol{A}(t), \boldsymbol{E}(t)) \in (H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$. Then, there exists a monotone increasing function $C(t)$ of $\mathcal{E}_M[\varphi_t, \alpha_t]$, $||\varphi_t||_{H^2(\mathbb{R}^3)}$ and $||\varphi_t||_{L^\infty(\mathbb{R}^3)}$ such that*

$$
\begin{aligned}
\left|\left| N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) q_1 \Psi_{N,t} \right|\right|^2 &\le \Lambda C(t)\, (\beta(t) + \Lambda/N) , \\
\left|\left| N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) q_2 \Psi_{N,t} \right|\right|^2 &\le \Lambda C(t)\, (\beta(t) + \Lambda/N) , \\
\left|\left| N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) p_1 q_2 \Psi_N \right|\right|^2 &\le \Lambda C(t)\, (\beta(t) + \Lambda/N) .
\end{aligned}
\tag{3.83}
$$

Proof. We have

$$
\langle \Psi_N, q_1 N^{-1} H_f q_1 \Psi_N\rangle \le \langle \Psi_N, q_1 N^{-1} H_N q_1 \Psi_N\rangle \tag{3.84}
$$

because v is positive. Lemma 3.8.4 and Corollary 3.8.3 then lead to

$$
\langle \Psi_N, q_1 N^{-1} H_f q_1 \Psi_N\rangle \le C(||\varphi||_{H^2}, ||\varphi||_\infty, \mathcal{E}_M)\, (\beta + \Lambda/N) \tag{3.85}
$$

and

$$
\left|\left| N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) q_1 \Psi_N \right|\right|^2 \le \Lambda C(||\varphi||_{H^2}, ||\varphi||_\infty, \mathcal{E}_M)\, (\beta + \Lambda/N) . \tag{3.86}
$$

The other inequalities are shown analogously. □

Lemma 3.8.6. *Let v satisfy (A1), $\Psi_{N,t} \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H_N)$, $\varphi_t \in H^3(\mathbb{R}^3)$ with $||\varphi_t|| = 1$ and $\alpha_t \in \mathfrak{h}$ such that $(\boldsymbol{A}(t), \boldsymbol{E}(t)) \in (H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$. Then, there exists a monotone increasing function $C(t)$ of $\mathcal{E}_M[\varphi_t, \alpha_t]$, $||\varphi_t||_{H^2(\mathbb{R}^3)}$ and $||\varphi_t||_{L^\infty(\mathbb{R}^3)}$ such that*

$$\int d^3y \left|\left|N^{-1} \sum_{j=1}^N q_j \kappa(x_j - y) \left(-i\nabla_j - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_j)\right) \Psi_{N,t}\right|\right|^2 \leq \Lambda^3 C(t) \left(\beta + \Lambda/N\right). \quad (3.87)$$

Proof. We use $\left\langle \cdot, \cdot \right\rangle_{;y} = \int d^3y \left\langle \cdot, \cdot \right\rangle$ and $||\cdot||_{;y} = \sqrt{\int d^3y \left\langle \cdot, \cdot \right\rangle}$ to ease the notation. Then we estimate

$$\begin{aligned}
&\int d^3y \left|\left|N^{-1} \sum_{j=1}^N q_j \kappa(x_j - y) \left(-i\nabla_j - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_j)\right) \Psi_N\right|\right|^2 \\
&= N^{-2} \Big\langle \sum_{i=1}^N q_i \kappa(x_i - y) \left(-i\nabla_i - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_i)\right) \Psi_N, \sum_{j=1}^N q_j \kappa(x_j - y) \left(-i\nabla_j - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_j)\right) \Psi_N \Big\rangle_{;y} \\
&= N^{-1} \big\langle q_1 \kappa(x_1 - y) \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right) \Psi_N, q_1 \kappa(x_1 - y) \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right) \Psi_N \big\rangle_{;y} \\
&+ (N-1)N^{-1} \big\langle q_1 \kappa(x_1 - y) \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right) \Psi_N, q_2 \kappa(x_2 - y) \left(-i\nabla_2 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_2)\right) \Psi_N \big\rangle_{;y} \\
&\leq N^{-1} \left|\left|\kappa(x_1 - y) \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right) \Psi_N\right|\right|_{;y}^2 \\
&+ (N-1)N^{-1} \big\langle \kappa(x_1 - y) \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right) q_2 \Psi_N, \kappa(x_2 - y) \left(-i\nabla_2 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_2)\right) q_1 \Psi_N \big\rangle_{;y} \\
&\leq N^{-1} \left|\left|\kappa(x_1 - y) \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right) \Psi_N\right|\right|_{;y}^2 \\
&+ (N-1)N^{-1} \left|\left|\kappa(x_1 - y) \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right) q_2 \Psi_N\right|\right|_{;y}^2 \\
&= N^{-1} \Big\langle \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right) \Psi_N, \left(\int d^3y \left|\kappa(x_1 - y)\right|^2\right) \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right) \Psi_N \Big\rangle \\
&+ (N-1)N^{-1} \Big\langle \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right) q_2 \Psi_N, \left(\int d^3y \left|\kappa(x_1 - y)\right|^2\right) \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right) q_2 \Psi_N \Big\rangle \\
&= N^{-1} ||\kappa||_2^2 \left\langle \Psi_N, \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right)^2 \Psi_N \right\rangle \\
&+ (N-1)N^{-1} ||\kappa||_2^2 \left\langle \Psi_N, q_2 \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right)^2 q_2 \Psi_N \right\rangle.
\end{aligned} \quad (3.88)$$

So if we insert the identity $1 = p_1 + q_1$ and use the symmetry of the wave function, we get

$$\begin{aligned}
\int d^3y \left|\left|N^{-1} \sum_{j=1}^N q_j \kappa(x_j - y) \left(-i\nabla_j - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_j)\right) \Psi_N\right|\right|^2 \\
&\leq N^{-1} ||\kappa||_2^2 \left\langle \Psi_N, q_1 \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right)^2 q_1 \Psi_N \right\rangle \\
&+ 2N^{-1} ||\kappa||_2^2 \left|\left\langle \Psi_N, q_1 \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right)^2 p_1 \Psi_N \right\rangle\right| \\
&+ N^{-1} ||\kappa||_2^2 \left|\left\langle \Psi_N, p_1 \left(-i\nabla_1 - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1)\right)^2 p_1 \Psi_N \right\rangle\right| \\
&+ N^{-1} ||\kappa||_2^2 \sum_{j=2}^N \left\langle \Psi_N, q_1 \left(-i\nabla_j - N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_j)\right)^2 q_1 \Psi_N \right\rangle.
\end{aligned} \quad (3.89)$$

By adding the lines together this simplifies to

$$\int d^3y \left|\left|N^{-1}\sum_{j=1}^N q_j\kappa(x_j-y)\left(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\right)\Psi_N\right|\right|^2$$
$$= N^{-1}\,||\kappa||_2^2\sum_{j=1}^N \left\langle \Psi_N, q_1\left(-i\nabla_j - N^{1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\right)^2 q_1\Psi_N\right\rangle$$
$$+ 2N^{-1}\,||\kappa||_2^2\,|\left\langle \Psi_N, q_1\left(-i\nabla_1 - N^{-1}\hat{\boldsymbol{A}}_\kappa(x_1)\right)^2 p_1\Psi_N\right\rangle|$$
$$+ N^{-1}\,||\kappa||_2^2\,|\left\langle \Psi_N, p_1\left(-i\nabla_1 - N^{-1}\hat{\boldsymbol{A}}_\kappa(x_1)\right)^2 p_1\Psi_N\right\rangle|. \tag{3.90}$$

Now we estimate the last two lines analogously to (3.75) and obtain

$$\int d^3y \left|\left|N^{-1}\sum_{j=1}^N q_j\kappa(x_j-y)\left(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\right)\Psi_N\right|\right|^2 \le ||\kappa||_2^2\,\Lambda/NC(||\varphi||_{H^2},\mathcal{E}_M)$$
$$+ N^{-1}\,||\kappa||_2^2\sum_{j=1}^N \left\langle \Psi_N, q_1\left(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\right)^2 q_1\Psi_N\right\rangle. \tag{3.91}$$

Because H_f and v are positive operators, this is bounded by

$$||\kappa||_2^2\left\langle \Psi_N, q_1N^{-1}H_Nq_1\Psi_N\right\rangle + ||\kappa||_2^2\,\Lambda/NC(||\varphi||_{H^2},\mathcal{E}_M). \tag{3.92}$$

Then, we apply Lemma 3.8.4 and obtain

$$\int d^3y \left|\left|N^{-1}\sum_{j=1}^N q_j\kappa(x_j-y)\left(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\right)\Psi_N\right|\right|^2 \le ||\kappa||_2^2\,C(t)\left(\beta + \Lambda/N\right)$$
$$\le \Lambda^3C(t)\left(\beta+\Lambda/N\right), \tag{3.93}$$

where $C(t)$ is a monotone increasing function of $\mathcal{E}_M[\varphi_t,\alpha_t]$, $||\varphi_t||_{H^2(\mathbb{R}^3)}$ and $||\varphi_t||_{L^\infty(\mathbb{R}^3)}$. □

3.8.2 Bound on $d_t\beta^a$:

Lemma 3.8.7. *Let v satisfy (A1), $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$, $\alpha_0 \in \mathfrak{h}$ such that $(\boldsymbol{A}(0),\boldsymbol{E}(0)) \in (H^3(\mathbb{R}^3)\oplus H^2(\mathbb{R}^3))$, $\Psi_{N,0} \in \left(L_s^2\left(\mathbb{R}^{3N}\right)\otimes\mathcal{F}_p\right)\cap\mathcal{D}(H_N)$. Let $\Psi_{N,t}$ be the unique solution of (3.4), $(\varphi_t,\boldsymbol{A}(t),\boldsymbol{E}(t))$ be the unique solution of (3.15) and assume $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T\in\mathbb{R}^+$. Then, there exists a monotone increasing function $C(t)$ of $||\boldsymbol{A}_\kappa||_\infty$, $\mathcal{E}_M[\varphi_t,\alpha_t]$, $||\varphi_t||_{H^2(\mathbb{R}^3)}$, $||\varphi_t||_{L^\infty(\mathbb{R}^3)}$ and $||\nabla\varphi_t||_\infty$ such that*

$$|d_t\beta^a(\Psi_{N,t},\varphi_t)| \le \Lambda^2C(t)\left(\beta(\Psi_{N,t},\varphi_t,\alpha_t)+\Lambda/N\right). \tag{3.94}$$

Proof. The time derivative of the projector $q_1^{\varphi_t}$ is given by

$$d_tq_1^{\varphi_t} = -i\left[H_1^{HM}, q_1^{\varphi_t}\right], \tag{3.95}$$

where H_1^{HM} denotes the effective Hamiltonian $H_1^{HM} := (-i\nabla_1 - \boldsymbol{A}_\kappa(x_1,t))^2 + \left(v * |\varphi_t|^2\right)(x_1)$. This allows us to compute the derivative of $\beta^a(t)$ by

$$\begin{aligned}
d_t\beta^a(t) &= d_t\left\langle \Psi_{N,t}, q_1^{\varphi_t}\Psi_{N,t}\right\rangle = i\left\langle \Psi_{N,t}, \left[\left(H_m^N - H_1^{HM}\right), q_1^{\varphi_t}\right]\Psi_{N,t}\right\rangle \\
&= -2\left\langle \Psi_{N,t}, \left[\left(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) - \boldsymbol{A}_\kappa(x_1,t)\right)\cdot\nabla_1, q_1^{\varphi_t}\right]\Psi_{N,t}\right\rangle \\
&\quad + i\left\langle \Psi_{N,t}, \left[\left(N^{-1}\hat{\boldsymbol{A}}_\kappa^2(x_1) - \boldsymbol{A}_\kappa^2(x_1,t)\right), q_1^{\varphi_t}\right]\Psi_{N,t}\right\rangle \\
&\quad + i\left\langle \Psi_{N,t}, \left[\left(N^{-1}\sum_{1\le j<k\le N} v(x_j-x_k) - \left(v*|\varphi_t|^2\right)(x_1)\right), q_1^{\varphi_t}\right]\Psi_{N,t}\right\rangle \\
&= -4\mathrm{Re}\left\langle \Psi_{N,t}, \left(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) - \boldsymbol{A}_\kappa(x_1,t)\right)\cdot\nabla_1 q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\
&\quad - 2\mathrm{Im}\left\langle \Psi_{N,t}, \left(N^{-1}\hat{\boldsymbol{A}}_\kappa^2(x_1) - \boldsymbol{A}_\kappa^2(x_1,t)\right) q_1^{\varphi_t}\Psi_{N,t}\right\rangle \\
&\quad - 2\mathrm{Im}\left\langle \Psi_{N,t}, \left((N-1)/N v(x_1-x_2) - \left(v*|\varphi_t|^2\right)(x_1)\right) q_1^{\varphi_t}\Psi_{N,t}\right\rangle. \qquad (3.96)
\end{aligned}$$

Inserting the identity $1 = p_1 + q_1$ and the relations

$$\mathrm{Re}\left\langle \Psi_N, q_1\left(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) - \boldsymbol{A}_\kappa(x_1,t)\right)\cdot\nabla_1 q_1\Psi_N\right\rangle = 0,$$

$$\mathrm{Im}\left\langle \Psi_N, q_1\left(N^{-1}\hat{\boldsymbol{A}}_\kappa^2(x_1) - \boldsymbol{A}_\kappa^2(x_1,t)\right) q_1\Psi_N\right\rangle = 0,$$

$$\mathrm{Im}\left\langle \Psi_N, q_1\left((N-1)N^{-1}v(x_1-x_2) - \left(v*|\varphi|^2\right)(x_1)\right) q_1\Psi_N\right\rangle = 0,$$

lead to

$$\begin{aligned}
d_t\beta^a =& -4\mathrm{Re}\left\langle \Psi_N, p_1\left(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) - \boldsymbol{A}_\kappa(x_1,t)\right)\cdot\nabla_1 q_1\Psi_N\right\rangle & (3.97)\\
& - 2\mathrm{Im}\left\langle \Psi_N, p_1\left(N^{-1}\hat{\boldsymbol{A}}_\kappa^2(x_1) - \boldsymbol{A}_\kappa^2(x_1,t)\right) q_1\Psi_N\right\rangle & (3.98)\\
& - 2\mathrm{Im}\left\langle \Psi_N, p_1\left((N-1)N^{-1}v(x_1-x_2) - \left(v*|\varphi|^2\right)(x_1,t)\right) q_1\Psi_N\right\rangle. & (3.99)
\end{aligned}$$

In the following, we estimate each line separately. To simplify the presentation we use the shorthand notation (3.57).

Bound on (3.97)**:**

Integration by parts and triangular inequality let us estimate

$$\begin{aligned}
|(3.97)| &\le 4\left|\left\langle \Psi_N, p_1\left(\boldsymbol{\mathcal{A}}^+(x_1,t) + \boldsymbol{\mathcal{A}}^-(x_1,t)\right)\cdot\nabla_1 q_1\Psi_N\right\rangle\right| \\
&\le 4\left|\left\langle \nabla_1 p_1\Psi_N, \boldsymbol{\mathcal{A}}^-(x_1,t) q_1\Psi_N\right\rangle\right| & (3.100)\\
&\quad + 4\left|\left\langle \nabla_1 p_1\Psi_N, \boldsymbol{\mathcal{A}}^+(x_1,t) q_1\Psi_N\right\rangle\right|. & (3.101)
\end{aligned}$$

By means of Lemma 3.8.1, we bound the first line by

$$\begin{aligned}
(3.100) &= 4\left|\left\langle \nabla_1 p_1\Psi_N, \boldsymbol{\mathcal{A}}^-(x_1,t) q_1\Psi_N\right\rangle\right| = 4\left|\left\langle \nabla_1 p_1\Psi_N, \left(\eta * \boldsymbol{\mathcal{E}}^-\right)(x_1,t) q_1\Psi_N\right\rangle\right| \\
&= 4\left|\left\langle \boldsymbol{\mathcal{E}}^+(y,t)\nabla_1 p_1\Psi_N, \eta(x_1-y) q_1\Psi_N\right\rangle_{;y}\right| \\
&\le 4\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\cdot\nabla_1 p_1\Psi_N\right|\right|_{;y}\left|\left|\eta(y-x_1)q_1\Psi_N\right|\right|_{;y} \\
&\le 2\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\cdot\nabla_1 p_1\Psi_N\right|\right|_{;y}^2 + 2\left|\left|\eta\right|\right|_2^2\left|\left|q_1\Psi_N\right|\right|^2 \\
&\le \Lambda\pi^{-2}\beta^a + C(\left|\left|\nabla\varphi\right|\right|_\infty)\beta^b \le \Lambda C(\left|\left|\nabla\varphi\right|\right|_\infty)\beta, & (3.102)
\end{aligned}$$

where we made use of Lemma 3.11.1 and (3.61).
The second term is bounded by

$$
\begin{aligned}
(3.101) &= 4\left|\left\langle \nabla_1 p_1 \Psi_N, \int d^3y\, \eta(x_1-y)\boldsymbol{\mathcal{E}}^+(y,t) q_1 \Psi_N \right\rangle\right| \\
&= 4\left|\left\langle \nabla_1 p_1 \Psi_N, \eta(x_1-y)\boldsymbol{\mathcal{E}}^+(y,t) q_1 \Psi_N \right\rangle_{;y}\right| \\
&= 4\left|\left\langle q_1\eta(x_1-y)\nabla_1 p_1 \Psi_N, \boldsymbol{\mathcal{E}}^+(y,t) \Psi_N \right\rangle_{;y}\right| \\
&= 4\left|\left\langle N^{-1}\sum_{i=1}^N q_i\eta(x_i-y)\nabla_i p_i \Psi_N, \boldsymbol{\mathcal{E}}^+(y,t) \Psi_N \right\rangle_{;y}\right| \\
&\le 2\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right|\right|^2_{;y} + 2\left|\left|N^{-1}\sum_{i=1}^N q_i\eta(x_i-y)\nabla_i p_i\Psi_N\right|\right|^2_{;y}.
\end{aligned} \tag{3.103}
$$

Lemma 3.11.1 and the symmetry of the wave function lead to

$$
\begin{aligned}
(3.101) &\le 2\beta^b + 2N^{-2}\left\langle \sum_{i=1}^N q_i\eta(x_i-y)\nabla_i p_i\Psi_N, \sum_{j=1}^N q_j\eta(x_j-y)\nabla_j p_j\Psi_N \right\rangle_{;y} \\
&\le 2\beta^b + 2N^{-1}\left|\left|q_1\eta(x_1-y)\nabla_1 p_1\Psi_N\right|\right|^2_{;y} \\
&+ 2\left\langle q_1\eta(x_1-y)\nabla_1 p_1\Psi_N, q_2\eta(x_2-y)\nabla_2 p_2\Psi_N\right\rangle_{;y} \\
&\le 2\beta^b + 2N^{-1}\left|\left|\eta(x_1-y)\nabla_1 p_1\Psi_N\right|\right|^2_{;y} \\
&+ 2\left\langle \eta(x_1-y)\nabla_1 p_1 q_2\Psi_N, \eta(x_2-y)\nabla_2 p_2 q_1\Psi_N\right\rangle_{;y} \\
&\le 2\beta^b + 2N^{-1}\left|\left|\eta(x_1-y)\nabla_1 p_1\Psi_N\right|\right|^2_{;y} \\
&+ 2\left|\left|\eta(x_1-y)\nabla_1 p_1 q_2\Psi_N\right|\right|_{;y}\left|\left|\eta(x_2-y)\nabla_2 p_2 q_1\Psi_N\right|\right|_{;y} \\
&\le 2\beta^b + 2N^{-1}\left\langle \eta(x_1-y)\nabla_1 p_1\Psi_N, \eta(x_1-y)\nabla_1 p_1\Psi_N\right\rangle_{;y} \\
&+ 2\left\langle \eta(x_1-y)\nabla_1 p_1 q_2\Psi_N, \eta(x_1-y)\nabla_1 p_1 q_2\Psi_N\right\rangle_{;y}.
\end{aligned} \tag{3.104}
$$

Interchanging the order of integration we have

$$
\begin{aligned}
(3.101) &\le 2N^{-1}\left\langle \nabla_1 p_1\Psi_N, \left(\int d^3y\, |\eta(x_1-y)|^2\right)\nabla_1 p_1\Psi_N\right\rangle + 2\beta^b \\
&+ 2\left\langle \nabla_1 p_1 q_2\Psi_N, \left(\int d^3y\, |\eta(x_1-y)|^2\right)\nabla_1 p_1 q_2\Psi_N\right\rangle \\
&= 2\left|\left|\eta\right|\right|_2^2\left(N^{-1}\left\langle\Psi_N, p_1(-\Delta_1)p_1\Psi_N\right\rangle + \left\langle\Psi_N, q_2 p_1(-\Delta_1)p_1 q_2\Psi_N\right\rangle\right) + 2\beta^b.
\end{aligned} \tag{3.105}
$$

By virtue of $p_1\,(-\Delta)\,p_1 = p_1\,||\nabla\varphi||_2^2$, this becomes

$$
\begin{aligned}
(3.101) &\le 2\left|\left|\eta\right|\right|_2^2\left|\left|\nabla\varphi\right|\right|_2^2\left(N^{-1}\left\langle\Psi_N, p_1\Psi_N\right\rangle + \left\langle\Psi_N, q_2 p_1 q_2\Psi_N\right\rangle\right) + 2\beta^b \\
&\le \left|\left|\eta\right|\right|_2^2 C(||\varphi||_{H^2})\left(\beta^a + \beta^b + N^{-1}\right) \le \Lambda C(||\varphi||_{H^2})\left(\beta + N^{-1}\right)
\end{aligned} \tag{3.106}
$$

and we obtain

$$
|(3.97)| \le \Lambda C(||\varphi||_{H^2}, ||\nabla\varphi||_\infty)\left(\beta + N^{-1}\right). \tag{3.107}
$$

Bound on (3.98)**:**

$$\begin{aligned}
|(3.98)| &\leq 2|\langle \Psi_N, p_1 \left(N^{-1} \hat{\boldsymbol{A}}_\kappa^2(x_1) - \boldsymbol{A}_\kappa^2(x_1,t) \right) q_1 \Psi_N \rangle| \\
&= 2|\langle \Psi_N, p_1 \left(N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) - \boldsymbol{A}_\kappa(x_1,t) \right) \left(N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t) \right) q_1 \Psi_N \rangle| \\
&\leq 2|\langle \Psi_N, p_1 \boldsymbol{\mathcal{A}}^-(x_1,t) \left(N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t) \right) q_1 \Psi_N \rangle| \qquad (3.108) \\
&+ 2|\langle \Psi_N, p_1 \boldsymbol{\mathcal{A}}^+(x_1,t) \left(N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t) \right) q_1 \Psi_N \rangle| \qquad (3.109)
\end{aligned}$$

First, we deal with line (3.108):

$$\begin{aligned}
(3.108) &= 2|\langle \Psi_N, p_1 \left(\eta * \boldsymbol{\mathcal{E}}^- \right)(x_1,t) \left(N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t) \right) q_1 \Psi_N \rangle| \\
&= 2|\langle \boldsymbol{\mathcal{E}}^+(y,t) p_1 \Psi_N, \eta(y-x_1) \left(N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t) \right) q_1 \Psi_N \rangle_{;y}| \\
&\leq \left|\left| \boldsymbol{\mathcal{E}}^+(y,t) p_1 \Psi_N \right|\right|_{;y}^2 + \left|\left| \eta(x_1-y) \left(N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t) \right) q_1 \Psi_N \right|\right|_{;y}^2 \\
&\leq \left|\left| \boldsymbol{\mathcal{E}}^+(y,t) p_1 \Psi_N \right|\right|_{;y}^2 + ||\eta||_2^2 \left|\left| \left(N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t) \right) q_1 \Psi_N \right|\right|^2 . \qquad (3.110)
\end{aligned}$$

Making use of Lemma 3.11.1 and $(a+b) \leq 2(a^2+b^2)$, we obtain

$$(3.108) \leq \beta^b + 2\,||\eta||_2^2 \left(||\boldsymbol{A}_\kappa||_\infty^2 \, ||q_1 \Psi_N||^2 + \left|\left| N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) q_1 \Psi_N \right|\right|^2 \right). \qquad (3.111)$$

By means of (3.61) and Lemma 3.8.5, this becomes

$$(3.108) \leq \Lambda^2 C(||\boldsymbol{A}_\kappa||_\infty, ||\varphi||_{H^2}, ||\varphi||_\infty, \mathcal{E}_M)\,(\beta + \Lambda/N). \qquad (3.112)$$

The second line is bounded by

$$\begin{aligned}
(3.109) &= 2|\langle \Psi_N, p_1 \boldsymbol{\mathcal{A}}^+(x_1,t) \left(N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t) \right) q_1 \Psi_N \rangle| \\
&= 2|\langle \Psi_N, p_1 \left[\left(N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t) \right) \boldsymbol{\mathcal{A}}^+(x_1,t) + \Lambda^2/(4\pi^2 N) \right] q_1 \Psi_N \rangle| \\
&\leq 2|\langle \Psi_N, p_1 \left(N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t) \right) \boldsymbol{\mathcal{A}}^+(x_1,t) q_1 \Psi_N \rangle| \\
&+ 2\Lambda^2/(4\pi^2 N)|\underbrace{\langle \Psi_N, p_1 q_1 \Psi_N \rangle}_{=0}|. \qquad (3.113)
\end{aligned}$$

Here, we have we used the commutation relation

$$\left[\boldsymbol{\mathcal{A}}^+(x_1,t), \left(N^{-1/2} \hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t) \right) \right] = N^{-1} \left[\hat{\boldsymbol{A}}_\kappa^+(x_1), \hat{\boldsymbol{A}}_\kappa^-(x_1) \right] = \Lambda^2/(4\pi^2 N). \qquad (3.114)$$

Lemma 3.8.1 and Lemma 3.11.1 lead to

$$
\begin{aligned}
(3.109) &\leq 2|\Big\langle \Big(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t)\Big)\, p_1\Psi_N, \int d^3y\, \eta(x_1-y)\boldsymbol{\mathcal{E}}^+(y,t)q_1\Psi_N\Big\rangle| \\
&= 2|\Big\langle q_1\eta(x_1-y)\,\Big(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t)\Big)\, p_1\Psi_N, \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\Big\rangle_{;y}| \\
&= 2N^{-1}|\Big\langle \sum_{i=1}^N q_i\eta(x_i-y)\,\Big(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_i) + \boldsymbol{A}_\kappa(x_i,t)\Big)\, p_i\Psi_N, \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\Big\rangle_{;y}| \\
&\leq N^{-2}\Big|\Big|\sum_{i=1}^N q_i\eta(x_i-y)\,\Big(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_i) + \boldsymbol{A}_\kappa(x_i,t)\Big)\, p_i\Psi_N\Big|\Big|_{;y}^2 + \Big|\Big|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\Big|\Big|_{;y}^2 \\
&\leq N^{-2}\Big|\Big|\sum_{i=1}^N q_i\eta(x_i-y)\,\Big(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_i) + \boldsymbol{A}_\kappa(x_i,t)\Big)\, p_i\Psi_N\Big|\Big|_{;y}^2 + \beta^b
\end{aligned}
\tag{3.115}
$$

Similar to the estimate of (3.101) one obtains

$$
\begin{aligned}
(3.109) \leq{}& N^{-1}\left|\left|\eta(x_1-y)\,\Big(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t)\Big)\, p_1\Psi_N\right|\right|_{;y}^2 \\
&+ \left|\left|\eta(x_1-y)\,\Big(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t)\Big)\, p_1q_2\Psi_N\right|\right|_{;y}^2 + \beta^b \\
={}& N^{-1}\,||\eta||_2^2\left|\left|\Big(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t)\Big)\, p_1\Psi_N\right|\right|^2 \\
&+ ||\eta||_2^2\left|\left|\Big(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) + \boldsymbol{A}_\kappa(x_1,t)\Big)\, p_1q_2\Psi_N\right|\right|^2 + \beta^b \\
\leq{}& C\Lambda\left(\beta^b + ||\boldsymbol{A}_\kappa||_\infty^2\,\beta^a + \left|\left|N^{-1/2}\boldsymbol{A}_\kappa(x_1)p_1q_2\Psi_N\right|\right|^2\right) \\
&+ C\Lambda/N\left(||\boldsymbol{A}_\kappa||_\infty^2 + \left|\left|N^{-1/2}\boldsymbol{A}_\kappa(x_1)p_1\Psi_N\right|\right|^2\right).
\end{aligned}
\tag{3.116}
$$

By means of Lemma 3.8.5 this is bounded by

$$
(3.109) \leq \Lambda^2 C(||\boldsymbol{A}_\kappa||_\infty\,, ||\varphi||_{H^2}\,, ||\varphi||_\infty\,, \mathcal{E}_M)\,(\beta + \Lambda/N)\,. \tag{3.117}
$$

In total, we obtain

$$
|(3.98)| \leq (3.108) + (3.109) \leq \Lambda^2 C(||\boldsymbol{A}_\kappa||_\infty\,, ||\varphi||_{H^2}\,, ||\varphi||_\infty\,, \mathcal{E}_M)\,(\beta + \Lambda/N)\,. \tag{3.118}
$$

Bound on (3.99)**:**

Subsequently, we consider the term that arises from the direct interaction. Inserting the identity $1 = p_2 + q_2$ and using the shorthand shorthand notation

$$
Z(x_1,x_2) := (N-1)N^{-1}v(x_1-x_2) - \left(v * |\varphi|^2\right)(x_1) \tag{3.119}
$$

gives

$$
\begin{aligned}
(3.99) ={}& -2\mathrm{Im}\big\langle\Psi_N, p_1Z(x_1,x_2)q_1\Psi_N\big\rangle \\
={}& -2\mathrm{Im}\big\langle\Psi_N, p_1p_2Z(x_1,x_2)q_1p_2\Psi_N\big\rangle \\
&- 2\mathrm{Im}\big\langle\Psi_N, p_1p_2Z(x_1,x_2)q_1q_2\Psi_N\big\rangle \\
&- 2\mathrm{Im}\big\langle\Psi_N, p_1q_2Z(x_1,x_2)q_1p_2\Psi_N\big\rangle \\
&- 2\mathrm{Im}\big\langle\Psi_N, p_1q_2Z(x_1,x_2)q_1q_2\Psi_N\big\rangle.
\end{aligned}
\tag{3.120}
$$

The third term vanishes due to symmetry of the wave function under the interchange of x_1 and x_2 and we are left with

$$|(3.99)| \leq 2|\langle \Psi_N, p_1p_2Z(x_1,x_2)q_1p_2\Psi_N\rangle| \tag{3.121}$$
$$+2|\langle \Psi_N, p_1p_2Z(x_1,x_2)q_1q_2\Psi_N\rangle| \tag{3.122}$$
$$+2|\langle \Psi_N, p_1q_2Z(x_1,x_2)q_1q_2\Psi_N\rangle|. \tag{3.123}$$

The first line is the most important. It is small because the direct interaction of the many-body system is well approximated by the mean-field potential. By means of

$$\begin{aligned} p_2Z(x_1,x_2)p_2 &= p_2\left[(N-1)N^{-1}v(x_1-x_2) - \left(v * |\varphi|^2\right)(x_1)\right]p_2 \\ &= \left[(N-1)N^{-1}-1\right]\left(v*|\varphi|^2\right)(x_1)p_2 = -N^{-1}\left(v*|\varphi|^2\right)(x_1)p_2 \end{aligned} \tag{3.124}$$

one has

$$\begin{aligned} |(3.121)| &\leq 2N^{-1}|\langle \Psi_N, p_1\left(v*|\varphi|^2\right)(x_1)p_2q_1\Psi_N\rangle| \\ &\leq 2N^{-1}\left|\left|\left(v*|\varphi|^2\right)(x_1)p_1\Psi_N\right|\right| \, ||p_2q_1\Psi_N|| \leq 2N^{-1}\left|\left|v*|\varphi|^2\right|\right|_\infty. \end{aligned} \tag{3.125}$$

We decompose the interaction potential $v = v_1 + v_2$ into $v_1 \in L^2(\mathbb{R}^3)$ and $v_2 \in L^\infty(\mathbb{R}^3)$. Then

$$\begin{aligned} \left|\left|v*|\varphi|^2\right|\right|_\infty &\leq \left|\left|v_1*|\varphi|^2\right|\right|_\infty + \left|\left|v_2*|\varphi|^2\right|\right|_\infty \leq ||v_1||_2\left|\left||\varphi|^2\right|\right|_2 + ||v_2||_\infty\left|\left||\varphi|^2\right|\right|_1 \\ &\leq ||v_1||_2\,||\varphi||_\infty\,||\varphi||_2 + ||v_2||_\infty\,||\varphi||_2^2 \leq C(||\varphi||_\infty). \end{aligned} \tag{3.126}$$

holds due to Young's inequality and we obtain

$$|(3.121)| \leq N^{-1}C(||\varphi||_\infty). \tag{3.127}$$

Moreover, we have

$$\begin{aligned} p_1Z^2(x_1,x_2)p_1 &= p_1\langle\varphi_t, \left((N-1)N^{-1}v(x_2-\cdot) - \left(v*|\varphi_t|^2\right)\right)^2\varphi_t\rangle \\ &\leq 2p_1\langle\varphi_t, \left(v^2(x_2-\cdot) + \left(v*|\varphi_t|^2\right)^2\right)\varphi_t\rangle \\ &\leq 2p_1\left(\left|\left|v^2*|\varphi_t|^2\right|\right|_\infty + \left|\left|v*|\varphi_t|^2\right|\right|_\infty^2\right) \leq p_1C(||\varphi||_\infty) \end{aligned} \tag{3.128}$$

because of (3.80) and (3.126). This shows

$$\left|\left|p_1Z^2(x_1,x_2)p_1\right|\right|_{\text{op}} \leq C(||\varphi||_\infty) \tag{3.129}$$

and allows us to estimate

$$\begin{aligned} |(3.122)| &= 2|\langle q_2Z(x_1,x_2)p_1p_2\Psi_N, q_1\Psi_N\rangle| = 2(N-1)^{-1}|\langle \sum_{i=2}^N q_iZ(x_1,x_i)p_1p_i\Psi_N, q_1\Psi_N\rangle| \\ &\leq N^{-1}\left|\left|\sum_{i=2}^N q_iZ(x_1,x_i)p_1p_i\Psi_N\right|\right|^2 + 4\,||q_1\Psi_N||^2 \\ &= N^{-2}\langle \sum_{i=2}^N q_iZ(x_1,x_i)p_1p_i\Psi_N, \sum_{j=2}^N q_jZ(x_1,x_j)p_1p_j\Psi_N\rangle + 4\beta^a \\ &\leq \langle q_2Z(x_1,x_2)p_1p_2\Psi_N, q_3Z(x_1,x_3)p_1p_3\Psi_N\rangle + N^{-1}\,||q_2Z(x_1,x_2)p_1p_2\Psi_N||^2 + 4\beta^a \\ &\leq \langle Z(x_1,x_2)p_1p_2q_3\Psi_N, Z(x_1,x_3)p_1p_3q_2\Psi_N\rangle + N^{-1}\,||Z(x_1,x_2)p_1p_2\Psi_N||^2 + 4\beta^a \\ &\leq ||Z(x_1,x_2)p_1p_2q_3\Psi_N||^2 + N^{-1}\,||Z(x_1,x_2)p_1p_2\Psi_N||^2 + 4\beta^a \\ &\leq \left|\left|p_1Z^2(x_1,x_2)p_1\right|\right|_{op}\left(\beta^a + N^{-1}\right) + 4\beta^a \\ &\leq C(||\varphi||_\infty)\left(\beta + N^{-1}\right). \end{aligned} \tag{3.130}$$

The last term of (3.99) is bounded by

$$\begin{aligned}|(3.123)| =& 2\langle Z(x_1,x_2)p_1q_2\Psi_N, q_1q_2\Psi_N\rangle \\ \leq& \langle \Psi_N, q_2p_1Z^2(x_1,x_2)p_1q_2\Psi_N\rangle + ||q_1q_2\Psi_N||^2 \\ \leq& \left|\left|p_1Z^2(x_1,x_2)p_1\right|\right|_{\mathrm{op}} ||q_2\Psi_N||^2 + \beta^a \leq C(||\varphi||_\infty)\beta. \end{aligned} \tag{3.131}$$

This leads to

$$|(3.99)| \leq C(||\varphi||_\infty)\left(\beta + N^{-1}\right). \tag{3.132}$$

3.8.3 Bound on $d_t\beta^b$:

Lemma 3.8.8. *Let v satisfy (A1), $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$, $\alpha_0 \in \mathfrak{h}$ such that $(\boldsymbol{A}(0), \boldsymbol{E}(0)) \in (H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$, $\Psi_{N,0} \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H_N)$. Let $\Psi_{N,t}$ be the unique solution of* (3.4)*, $(\varphi_t, \boldsymbol{A}(t), \boldsymbol{E}(t))$ be the unique solution of* (3.15) *and assume $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T \in \mathbb{R}^+$. Then, there exists a monotone increasing function $C(t)$ of $\mathcal{E}_M[\varphi_t, \alpha_t]$, $||\varphi_t||_{H^2(\mathbb{R}^3)}$ and $||\varphi_t||_{L^\infty(\mathbb{R}^3)}$ such that*

$$|d_t\beta^b(\Psi_{N,t}, \alpha_t)| \leq \Lambda^4 C(t)\left(\beta(\Psi_{N,t}, \varphi_t, \alpha_t) + \Lambda/N\right). \tag{3.133}$$

Proof. We would like to note that the following calculation can be carried out in more detail. We could for example write β^b as

$$\begin{aligned}\beta^b(\Psi_{N,t}, \alpha_t) =& N^{-1}\langle \Psi_{N,t}, H_f\Psi_{N,t}\rangle + \sum_{\lambda=1,2}\int d^3k\,|k||\alpha(k,\lambda)|^2 \\ &- 2N^{-1/2}\mathrm{Re}\langle \Psi_{N,t}, \big(\sum_{j=1,2}\int d^3k\,|k|\alpha_t(k,\lambda)a^*(k,\lambda)\big)\Psi_{N,t}\rangle \end{aligned} \tag{3.134}$$

and determine its derivative in analogy to Appendix 2.11. This is even easier in the present case because we disregard photons with small energies and β^b is well defined on $\mathcal{D}(H_f) \subset \mathcal{D}(H_N) = \mathcal{D}(\sum_{i=1}^N(-\Delta_i) + H_f)$. This allows us to determine the derivative for many-body wave functions in $\mathcal{D}(H_N^2)$ (which is invariant due to Stone's theorem) and extend the result later to $\mathcal{D}(H_N)$ by a standard density argument.
We compute the commutators

$$\begin{aligned} i\left[H_N, \frac{a(k,\lambda)}{\sqrt{N}}\right] &= -i|k|\frac{a(k,\lambda)}{\sqrt{N}} - \frac{2i}{N}\sum_{j=1}^N \frac{\tilde{\kappa}(k)}{\sqrt{2|k|}}\boldsymbol{\epsilon}_\lambda(k)e^{-ikx_j}\left(i\nabla_j + \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right), \\ i\left[H_N, \frac{a^*(k,\lambda)}{\sqrt{N}}\right] &= \quad i|k|\frac{a^*(k,\lambda)}{\sqrt{N}} + \frac{2i}{N}\sum_{j=1}^N \frac{\tilde{\kappa}(k)}{\sqrt{2|k|}}\boldsymbol{\epsilon}_\lambda(k)e^{ikx_j}\left(i\nabla_j + \frac{\hat{\boldsymbol{A}}_\kappa(x_j)}{\sqrt{N}}\right) \end{aligned} \tag{3.135}$$

by use of the canonical commutation relations (3.9) and observe that the Maxwell-Schrödinger system leads to

$$\partial_t|k|^{1/2}\alpha_t(k,\lambda) = -i|k|^{3/2}\alpha_t(k,\lambda) + \frac{i}{\sqrt{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)(2\pi)^{3/2}\mathcal{FT}[\boldsymbol{j}](k). \tag{3.136}$$

Then, we continue with

$$
\begin{aligned}
d_t\beta^b =& \sum_{\lambda=1,2}\int d^3k\, d_t|k|\big\langle\Psi_N, \Big(N^{-1/2}a^*(k,\lambda)-\alpha_t^*(k,\lambda)\Big)\Big(N^{-1/2}a(k,\lambda)-\alpha_t(k,\lambda)\Big)\Psi_N\big\rangle\\
=& \sum_{\lambda=1,2}\int d^3k\,|k|\big\langle\Psi_N, i\Big[H_N, N^{-1/2}a^*(k,\lambda)\Big]\Big(N^{-1/2}a(k,\lambda)-\alpha_t(k,\lambda)\Big)\Psi_N\big\rangle\\
&+ \sum_{\lambda=1,2}\int d^3k\,|k|\big\langle\Psi_N, \Big(N^{-1/2}a^*(k,\lambda)-\alpha_t^*(k,\lambda)\Big)i\Big[H_N, N^{-1/2}a(k,\lambda)\Big]\Psi_N\big\rangle\\
&- \sum_{\lambda=1,2}\int d^3k\,|k|^{1/2}\big\langle\Psi_N, \Big(\partial_t|k|^{1/2}\alpha_t\Big)^*(k,\lambda)\Big(N^{-1/2}a(k,\lambda)-\alpha_t(k,\lambda)\Big)\Psi_N\big\rangle\\
&- \sum_{\lambda=1,2}\int d^3k\,|k|^{1/2}\big\langle\Psi_N, \Big(N^{-1/2}a^*(k,\lambda)-\alpha_t^*(k,\lambda)\Big)\Big(\partial_t|k|^{1/2}\alpha_t\Big)(k,\lambda)\Psi_N\big\rangle\\
=& \, i\sum_{\lambda=1,2}\int d^3k\,|k|^2\big\langle\Psi_N, \Big(N^{-1/2}a^*(k,\lambda)-\alpha_t^*(k,\lambda)\Big)\Big(N^{-1/2}a(k,\lambda)-\alpha_t(k,\lambda)\Big)\Psi_N\big\rangle\\
&- i\sum_{\lambda=1,2}\int d^3k\,|k|^2\big\langle\Psi_N, \Big(N^{-1/2}a^*(k,\lambda)-\alpha_t^*(k,\lambda)\Big)\Big(N^{-1/2}a(k,\lambda)-\alpha_t(k,\lambda)\Big)\Psi_N\big\rangle\\
&+ 2\sum_{\lambda=1,2}\int d^3k\,\big\langle\Psi_N, i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)e^{ikx_1}\left(i\nabla_1+\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\left(\frac{a(k,\lambda)}{\sqrt{N}}-\alpha_t(k,\lambda)\right)\Psi_N\big\rangle \qquad (3.137)\\
&- 2\sum_{\lambda=1,2}\int d^3k\,\big\langle\Psi_N, \left(\frac{a^*(k,\lambda)}{\sqrt{N}}-\alpha_t^*(k,\lambda)\right)i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)e^{-ikx_1}\left(i\nabla_1+\frac{\hat{\boldsymbol{A}}_\kappa(x_1)}{\sqrt{N}}\right)\Psi_N\big\rangle \qquad (3.138)\\
&+ \sum_{\lambda=1,2}\int d^3k\,\big\langle\Psi_N, i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)(2\pi)^{3/2}\mathcal{FT}[\boldsymbol{j}]^*(k)\left(\frac{a(k,\lambda)}{\sqrt{N}}-\alpha_t(k,\lambda)\right)\Psi_N\big\rangle \qquad (3.139)\\
&- \sum_{\lambda=1,2}\int d^3k\,\big\langle\Psi_N, \left(\frac{a^*(k,\lambda)}{\sqrt{N}}-\alpha_t^*(k,\lambda)\right)i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)(2\pi)^{3/2}\mathcal{FT}[\boldsymbol{j}](k)\Psi_N\big\rangle \qquad (3.140)
\end{aligned}
$$

The first two terms cancel. Moreover, (3.138) = (3.137)* and (3.140) = (3.139)* follows from $\left[\nabla_1, \boldsymbol{\epsilon}_\lambda(k)e^{ikx_1}\right]=0$ (recall Definition (3.8)). This gives rise to

$$
\begin{aligned}
d_t\beta^b =& \, 4\mathrm{Re}\sum_{\lambda=1,2}\int d^3k\,\big\langle\Psi_N, i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)e^{ikx_1}\Big(i\nabla_1+N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\Big)\Big(N^{-1/2}a(k,\lambda)-\alpha_t(k,\lambda)\Big)\Psi_N\big\rangle\\
&+ 2\mathrm{Re}\sum_{\lambda=1,2}\int d^3k\,\big\langle\Psi_N, i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)(2\pi)^{3/2}\mathcal{FT}[\boldsymbol{j}]^*(k)\Big(N^{-1/2}a(k,\lambda)-\alpha_t(k,\lambda)\Big)\Psi_N\big\rangle. \qquad (3.141)
\end{aligned}
$$

Inserting the identity $1 = p_1 + q_1$ and

$$
\sum_{\lambda=1,2}\int d^3k\, i\sqrt{\frac{|k|}{2}}\tilde{\kappa}(k)\boldsymbol{\epsilon}_\lambda(k)e^{ikx_1}\Big(N^{-1/2}a(k,\lambda)-\alpha_t(k,\lambda)\Big) = \big(\kappa * \boldsymbol{\mathcal{E}}^+\big)(x_1,t) \qquad (3.142)
$$

lead to

$$
\begin{aligned}
d_t\beta^b =& +4\mathrm{Re}\big\langle\Psi_N, p_1N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\kappa(y-x_1)p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\big\rangle_{;y}\\
&+2\mathrm{Re}\big\langle\Psi_N, p_1\left(\kappa(y-x_1)i\nabla_1+i\nabla_1\kappa(y-x_1)\right)p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\big\rangle_{;y}\\
&+2\mathrm{Re}\big\langle\Psi_N, \Big(\int d^3z\,\kappa(y-z)\boldsymbol{j}(z)\Big)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\big\rangle_{;y}\\
&+4\mathrm{Re}\big\langle\Psi_N, q_1\kappa(y-x_1)i\nabla_1p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\big\rangle_{;y}\\
&+4\mathrm{Re}\big\langle\Psi_N, q_1N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\kappa(y-x_1)p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\big\rangle_{;y}\\
&+4\mathrm{Re}\big\langle\Psi_N, \Big(i\nabla_1+N^{1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\Big)\kappa(y-x_1)q_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\big\rangle_{;y}. \qquad (3.143)
\end{aligned}
$$

With the relations

$$\begin{aligned}p_1 N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\kappa(y-x_1)p_1 =& p_1\int d^3z\,|\varphi|^2(z)N^{-1/2}\hat{\boldsymbol{A}}_\kappa(z)\kappa(y-z),\\ p_1\left(\kappa(y-x_1)i\nabla_1+i\nabla_1\kappa(y-x_1)\right)p_1 =& -2p_1\int d^3z\,\kappa(y-z)\mathrm{Im}[\varphi^*\nabla\varphi](z),\\ \boldsymbol{j} =& 2\left(\mathrm{Im}(\varphi^*\nabla\varphi)-|\varphi|^2\boldsymbol{A}_\kappa\right)\end{aligned}\tag{3.144}$$

we obtain

$$d_t\beta^b = -4\mathrm{Re}\int d^3z\,|\varphi|^2(z)\left\langle\Psi_N, q_1N^{-1/2}\hat{\boldsymbol{A}}_\kappa(z)\kappa(y-z)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right\rangle_{;y} \tag{3.145}$$

$$+4\mathrm{Re}\int d^3z\,\mathrm{Im}[\varphi^*\nabla\varphi](z)\left\langle\Psi_N, q_1\kappa(y-z)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right\rangle_{;y} \tag{3.146}$$

$$+4\mathrm{Re}\int d^3z\,|\varphi|^2(z)\left\langle\Psi_N, \kappa(y-z)\left(N^{-1/2}\hat{\boldsymbol{A}}_\kappa(z)-\boldsymbol{A}_\kappa(z,t)\right)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right\rangle_{;y} \tag{3.147}$$

$$+4\mathrm{Re}\left\langle\Psi_N, q_1\kappa(y-x_1)i\nabla_1p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right\rangle_{;y} \tag{3.148}$$

$$+4\mathrm{Re}\left\langle\Psi_N, q_1N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\kappa(y-x_1)p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right\rangle_{;y} \tag{3.149}$$

$$+4\mathrm{Re}\left\langle\Psi_N, \left(-i\nabla_1-N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\right)\kappa(y-x_1)q_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right\rangle_{;y}. \tag{3.150}$$

Subsequently, we estimate each line separately:

$$\begin{aligned}|(3.145)| \le& 4\left|\int d^3z\,|\varphi|^2(z)\left\langle\Psi_N, q_1N^{-1/2}\hat{\boldsymbol{A}}_\kappa(z)\kappa(y-z)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right\rangle_{;y}\right|\\ \le& 4\int d^3y\int d^3z\,|\varphi|^2(z)|\kappa(y-z)|\left|\left\langle N^{-1/2}\hat{\boldsymbol{A}}_\kappa(z)q_1\Psi_N, \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right\rangle\right|\\ \le& 4\int d^3y\int d^3z\,|\varphi|^2(z)\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right|\right|\,|\kappa(y-z)|\left|\left|N^{-1/2}\hat{\boldsymbol{A}}_\kappa(z)q_1\Psi_N\right|\right|\\ \le& 2\int d^3y\int d^3z\,|\varphi|^2(z)\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right|\right|^2\\ &+2\int d^3z\,|\varphi|^2(z)\left|\left|N^{-1/2}\hat{\boldsymbol{A}}_\kappa(z)q_1\Psi_N\right|\right|^2\left(\int d^3y\,|\kappa(y-z)|^2\right)\\ =& 2\left\langle\Psi_N, \boldsymbol{\mathcal{E}}^-(y,t)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right\rangle_{;y}\\ &+2\,||\kappa||_2^2\int d^3z\,|\varphi|^2(z)\left|\left|N^{-1/2}\hat{\boldsymbol{A}}_\kappa(z)q_1\Psi_N\right|\right|^2.\end{aligned}\tag{3.151}$$

With the help of Lemma 3.8.5 and (3.61) this becomes

$$|(3.145)| \le \Lambda^4 C(||\varphi||_{H^2}, ||\varphi||_\infty, \mathcal{E}_M)\,(\beta+\Lambda/N)\,. \tag{3.152}$$

Similarly,

$$\begin{aligned}|(3.146)| \le& 4\int d^3y\int d^3z\,|\kappa(y-z)||\varphi(z)||\nabla\varphi(z)|\left|\left\langle q_1\Psi_N, \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right\rangle\right|\\ \le& 4\int d^3y\int d^3z\,|\kappa(y-z)||\varphi(z)||\nabla\varphi(z)|\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right|\right|\,||q_1\Psi_N||\\ \le& 2\int d^3y\int d^3z\,|\nabla\varphi(z)|^2\left|\left|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\right|\right|^2\\ &+2\int d^3z\,|\varphi(z)|^2\,||q_1\Psi_N||^2\left(\int d^3y\,|\kappa(y-z)|^2\right)\\ \le& 2\,||\nabla\varphi||_2^2\,\beta^b+2\,||\kappa||_2^2\,\beta^a \le \Lambda^3C(||\varphi||_{H^2})\beta\end{aligned}\tag{3.153}$$

and

$$\begin{aligned}|(3.147)| \leq & 4\int d^3y \int d^3z\, |\varphi|^2(z)|\kappa(y-z)||\langle \boldsymbol{\mathcal{A}}(z,t)\Psi_N, \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle| \\ \leq & 4\int d^3y \int d^3z\, |\varphi|^2(z)|\kappa(y-z)|\, ||\boldsymbol{\mathcal{A}}(z,t)\Psi_N||\, \big|\big|\boldsymbol{\mathcal{E}}^+(y,t)\big|\big| \\ \leq & 2\int d^3z\, |\varphi|^2(z)\, ||\boldsymbol{\mathcal{A}}(z,t)\Psi_N||^2 \int d^3y\, |\kappa(y-z)|^2 \\ & +2\int d^3z\, |\varphi|^2(z) \int d^3y\, \big|\big|\boldsymbol{\mathcal{E}}^+(y,t)\big|\big|^2 \\ \leq & 2\beta^b + 2\, ||\kappa||_2^2 \int d^3z |\varphi|^2(z)\, ||\boldsymbol{\mathcal{A}}(z,t)\Psi_N||^2 . \end{aligned} \tag{3.154}$$

Linearity and $(a+b)^2 \leq 2(a^2+b^2)$ lead to

$$|(3.147)| \leq 2\beta^b + 4\, ||\kappa||_2^2 \int d^3z |\varphi|^2(z) \left(\big|\big|\boldsymbol{\mathcal{A}}^+(z,t)\Psi_N\big|\big|^2 + \big|\big|\boldsymbol{\mathcal{A}}^-(z,t)\Psi_N\big|\big|^2\right). \tag{3.155}$$

By means of the commutation relation

$$\big[\boldsymbol{\mathcal{A}}^+(z,t), \boldsymbol{\mathcal{A}}^-(z,t)\big] = N^{-1}\left[\hat{\boldsymbol{A}}_\kappa^+(z), \hat{\boldsymbol{A}}_\kappa^-(z)\right] = \Lambda^2/(4\pi^2 N) \tag{3.156}$$

we calculate

$$\begin{aligned}\int d^3z\, |\varphi|^2(z)\, \big|\big|\boldsymbol{\mathcal{A}}^-(z,t)\Psi_N\big|\big|^2 &= \int d^3z |\varphi|^2(z) \langle \Psi_N, \boldsymbol{\mathcal{A}}^+(z,t)\boldsymbol{\mathcal{A}}^-(z,t)\Psi_N\rangle \\ &= \int d^3z\, |\varphi|^2(z) \langle \Psi_N, \left(\boldsymbol{\mathcal{A}}^-(z,t)\boldsymbol{\mathcal{A}}^+(z,t) + \Lambda^2/(4\pi^2N)\right)\Psi_N\rangle \\ &= \int d^3z\, |\varphi|^2(z)\, \big|\big|\boldsymbol{\mathcal{A}}^+(z,t)\Psi_N\big|\big|^2 + \Lambda^2/(4\pi^2N) \end{aligned} \tag{3.157}$$

and obtain

$$|(3.147)| \leq 2\beta^b + ||\kappa||_2^2\, \Lambda^2/(\pi^2 N) + 8\, ||\kappa||_2^2 \int d^3z |\varphi|^2(z)\, \big|\big|\boldsymbol{\mathcal{A}}^+(z,t)\Psi_N\big|\big|^2 . \tag{3.158}$$

Then, we use (3.59) and estimate

$$\begin{aligned}\int d^3z\, |\varphi|^2(z)\, \big|\big|\boldsymbol{\mathcal{A}}^+(z,t)\Psi_N\big|\big|^2 &= \int d^3z\, |\varphi|^2(z) \langle \Psi_N, \boldsymbol{\mathcal{A}}^-(z,t)\boldsymbol{\mathcal{A}}^+(z,t)\Psi_N\rangle \\ &= \int d^3z\, |\varphi|^2(z) \langle \Psi_N, \int d^3y\, \eta(z-y)\boldsymbol{\mathcal{E}}^-(y,t) \int d^3l\, \eta(z-l)\boldsymbol{\mathcal{E}}^+(l,t)\Psi_N\rangle \\ &\leq \int d^3y \int d^3z \int d^3l\, |\varphi|^2(z)|\eta(z-y)||\eta(z-l)||\langle \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N, \boldsymbol{\mathcal{E}}^+(l,t)\Psi_N\rangle| \\ &\leq 1/2 \int d^3z\, |\varphi|^2(z) \int d^3l\, \big|\big|\boldsymbol{\mathcal{E}}^+(l,t)\Psi_N\big|\big|^2 \int d^3y\, |\eta(z-y)|^2 \\ &\quad + 1/2 \int d^3z\, |\varphi|^2(z) \int d^3y\, \big|\big|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\big|\big|^2 \int d^3l\, |\eta(z-l)|^2 \\ &\leq ||\eta||_2^2 \int d^3y\, \big|\big|\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\big|\big|^2 \leq ||\eta||_2^2\, \beta^b \leq C\Lambda\beta^b. \end{aligned} \tag{3.159}$$

This yields

$$|(3.147)| \leq 2\beta^b + ||\kappa||_2^2\, \Lambda^2/(\pi^2 N) + C\Lambda\, ||\kappa||_2^2\, \beta^b \leq C\Lambda^4\, (\beta + \Lambda/N)\, . \tag{3.160}$$

The next terms of $d_t\beta(t)$ are bounded by

$$\begin{aligned}
|(3.148)| &\leq 4|\langle \kappa(y-x_1)q_1\Psi_N, i\nabla_1 p_1 \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle_{;y}| \\
&\leq 2\,||\kappa(y-x_1)q_1\Psi_N||^2_{;y} + 2\,||i\nabla_1 p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N||^2_{;y} \\
&= 2\Big\langle q_1\Psi_N, \Big(\int d^3y\,|\kappa(y-x_1)|^2\Big) q_1\Psi_N\Big\rangle + 2\big\langle \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N, p_1\,(-\Delta_1)\,p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\big\rangle_{;y} \\
&= 2\,||\kappa||_2^2\,\langle\Psi_N, q_1\Psi_N\rangle + 2\,||\nabla\varphi||_2^2\,\langle \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N, p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle_{;y} \\
&\leq 2\,||\kappa||_2^2\,\beta^a + 2\,||\nabla\varphi||_2^2\,\langle\Psi_N, \boldsymbol{\mathcal{E}}^-(y,t)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle_{;y} \leq \Lambda^3 C(||\varphi||_{H^2})\beta
\end{aligned} \tag{3.161}$$

and

$$\begin{aligned}
|(3.149)| &\leq 4|\langle \kappa(y-z_1)N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)q_1\Psi_N, p_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle_{;y}| \\
&\leq 2\,\Big|\Big|\kappa(y-x_1)N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)q_1\Psi_N\Big|\Big|^2_{;y} + 2\,||\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N||^2_{;y} \\
&= 2\Big\langle N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)q_1\Psi_N, \Big(\int d^3y\,|\kappa(y-x_1)|^2\Big)N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)q_1\Psi_N\Big\rangle \\
&+ 2\langle\Psi_N, \boldsymbol{\mathcal{E}}^-(y,t)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\rangle_{;y} \\
&\leq 2\beta^b + 2\,||\kappa||_2^2\,\Big|\Big|N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)q_1\Psi_N\Big|\Big|^2 \\
&\leq \Lambda^4 C(||\varphi||_{H^2}, ||\varphi||_\infty, \mathcal{E}_M)\,(\beta + \Lambda/N)\,.
\end{aligned} \tag{3.162}$$

Here, we made use of Lemma 3.8.5.

$$\begin{aligned}
|(3.150)| &\leq 4|\Big\langle\Psi_N, \Big(-i\nabla_1 - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\Big)\,\kappa(y-x_1)q_1\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\Big\rangle_{;y}| \\
&= 4|\Big\langle N^{-1}\sum_{j=1}^N q_j\kappa(y-x_j)\,\Big(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\Big)\,\Psi_N, \boldsymbol{\mathcal{E}}^+(y,t)\Psi_N\Big\rangle_{;y}| \\
&\leq 2\Big|\Big|N^{-1}\sum_{j=1}^N q_j\kappa(y-x_j)\,\Big(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\Big)\,\Big|\Big|^2_{;y} + 2\,||\boldsymbol{\mathcal{E}}^+(y,t)\Psi_N||^2_{;y}\,.
\end{aligned} \tag{3.163}$$

According to Lemma 3.11.1 and Lemma 3.8.6 this is bounded by

$$|(3.150)| \leq \Lambda^3 C(||\varphi||_{H^2}, ||\varphi||_\infty, \mathcal{E}_M)\,(\beta + \Lambda/N)\,. \tag{3.164}$$

3.8.4 Bound on $d_t\beta$:

The Maxwell-Schrödinger equations are a conserved system and its energy does not change during the time evolution

$$\mathcal{E}_M[\varphi_t, \alpha_t] = \mathcal{E}_M[\varphi_0, \alpha_0]. \tag{3.165}$$

Moreover, β^c is a constant of motion because the self-adjointness of the Pauli-Fierz Hamiltonian implies a strongly continuous unitary group $\{e^{-itH_N}\}_{t\in\mathbb{R}}$ such that $\Psi_{N,t} = e^{-itH_N}\Psi_{N,0}$

and

$$\begin{aligned}\beta^c(\Psi_{N,t},\varphi_t,\alpha_t) &= \left|\left|\left(N^{-1}H_N - \mathcal{E}_M[\varphi_t,\alpha_t]\right)\Psi_{N,t}\right|\right|^2 \\ &= \left|\left|\left(N^{-1}H_N - \mathcal{E}_M[\varphi_0,\alpha_0]\right)e^{-itH_N}\Psi_{N,0}\right|\right|^2 \\ &= \left|\left|e^{-itH_N}\left(N^{-1}H_N - \mathcal{E}_M[\varphi_0,\alpha_0]\right)\Psi_{N,0}\right|\right|^2 = \beta^c(\Psi_{N,0},\varphi_0,\alpha_0). \end{aligned} \tag{3.166}$$

The time derivative of $\beta(t)$ is hence bounded by

Lemma 3.8.9. *Let v satisfy (A1), $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$, $\alpha_0 \in \mathfrak{h}$ such that $(\boldsymbol{A}(0),\boldsymbol{E}(0)) \in (H^3(\mathbb{R}^3)\oplus H^2(\mathbb{R}^3))$, $\Psi_{N,0} \in \left(L_s^2\left(\mathbb{R}^{3N}\right)\otimes\mathcal{F}_p\right)\cap\mathcal{D}(H_N)$. Let $\Psi_{N,t}$ be the unique solution of (3.4), $(\varphi_t,\boldsymbol{A}(t),\boldsymbol{E}(t))$ be the unique solution of (3.15) and assume $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)}+||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)}+||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\}<\infty$ for any $T\in\mathbb{R}^+$. Then, there exists a monotone increasing function $C(t)$ of $||\boldsymbol{A}_\kappa||_\infty$, $\mathcal{E}_M[\varphi_t,\alpha_t]$, $||\varphi_t||_{H^2(\mathbb{R}^3)}$ and $||\varphi_t||_{L^\infty(\mathbb{R}^3)}$ such that*

$$\begin{aligned}|d_t\beta(\Psi_{N,t},\varphi_t,\alpha_t)| &\leq \Lambda^4 C(t)\left(\beta(\Psi_{N,t},\varphi_t,\alpha_t)+\Lambda/N\right), \\ \beta(\Psi_{N,t},\varphi_t,\alpha_t) &\leq e^{\Lambda^4\int_0^t ds C(s)}\left(\beta(\Psi_{N,0},\varphi_0,\alpha_0)+\Lambda/N\right)\end{aligned} \tag{3.167}$$

holds for any $t\geq 0$.

Proof. The first inequality is a direct consequence of Lemma 3.8.7, Lemma 3.8.8 and (3.166). Then, we apply Gronwall's inequality 1.3.1 and obtain

$$\beta(\Psi_{N,t},\varphi_t,\alpha_t) \leq e^{\Lambda^4\int_0^t ds C(s)}\left(\beta(\Psi_{N,0},\varphi_0,\alpha_0)+\Lambda/N\right). \tag{3.168}$$

□

3.9 Initial conditions

In this section we show that $\beta(\Psi_{N,0},\varphi_0,\alpha_0)$ is small for the initial states of Theorem 3.2.1.

Lemma 3.9.1. *Let $\Psi_{N,0}\in\mathcal{D}(H_N)\cap\left(L_s^2(\mathbb{R}^{3N})\otimes\mathcal{F}_p\right)$, $\varphi_0\in H^3(\mathbb{R}^3)$ with $||\varphi_0||=1$ and $\alpha_0\in\mathfrak{h}$ such that $(\boldsymbol{A}(0),\boldsymbol{E}(0))\in(H^3(\mathbb{R}^3)\oplus H^2(\mathbb{R}^3))$. Then*

$$\beta^a(\Psi_{N,0},\varphi_0) \leq Tr_{L^2(\mathbb{R}^3)}|\gamma_{N,0}^{(1,0)} - |\varphi_0\rangle\langle\varphi_0|| = a_N, \tag{3.169}$$

$$\beta^b(\Psi_{N,0},\alpha_0) = N^{-1}\left\langle W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}, H_f W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}\right\rangle = b_N, \tag{3.170}$$

$$\beta^c(\Psi_{N,0},\varphi_0,\alpha_0) = c_N. \tag{3.171}$$

In particular for $\Psi_{N,0}=\varphi_0^{\otimes N}\otimes W(\sqrt{N}\alpha_0)\Omega$ we have

$$\beta[\Psi_{N0},\varphi_0,\alpha_0] \leq C\Lambda^4 N^{-1}. \tag{3.172}$$

Before we prove Lemma 3.9.1 we recall some well known properties of Weyl operators (3.13).

Lemma 3.9.2. *Let $f,g\in\mathfrak{h}=L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$.*

(i) $W(f)$ is a unitary operator and

$$W^*(f)=W^{-1}(f)=W(-f). \tag{3.173}$$

(ii) We have

$$W^*(f)a(k,\lambda)W(f) = a(k,\lambda) + f(k,\lambda),$$
$$W^*(f)a^*(k,\lambda)W(f) = a^*(k,\lambda) + f^*(k,\lambda). \tag{3.174}$$

(iii) From (ii) we see that coherent states are eigenvectors of annihilation operators

$$a(k,\lambda)W(f)\Omega = f(k,\lambda)W(f)\Omega. \tag{3.175}$$

Subsequently, we compute the expectation values of the vector potential, the field energy and higher moments.

Lemma 3.9.3. *Let $\alpha_0 \in \mathfrak{h}$ such that $(\boldsymbol{A}(0), \boldsymbol{E}(0)) \in (H^3(\mathbb{R}^3), H^2(\mathbb{R}^3))$ and $\boldsymbol{E}^+_\kappa(x,t)$, $\mathcal{E}_f(t)$, $\mathcal{E}_{f^2}(t)$ be defined by* (3.56), (3.33), (3.34). *We define*

$$\gamma_{il}^{\perp\Lambda}(x) := \int d^3k\, |\tilde{\kappa}(k)|^2 |k|^{-1} e^{ikx} \left(\delta_{il} - k_i k_l |k|^{-2}\right) \quad \text{with} \quad \left|\left|\gamma_{il}^{\perp\Lambda}\right|\right|_2^2 \leq \Lambda. \tag{3.176}$$

Then

$$\begin{aligned}
\left\langle W(\sqrt{N}\alpha_0)\Omega, N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x) W(\sqrt{N}\alpha_0)\Omega\right\rangle_{\mathcal{F}_p} &= \boldsymbol{A}_\kappa(x,0),\\
\left\langle W(\sqrt{N}\alpha_0)\Omega, N^{-1}\hat{\boldsymbol{A}}^2_\kappa(x) W(\sqrt{N}\alpha_0)\Omega\right\rangle_{\mathcal{F}_p} &= \boldsymbol{A}^2_\kappa(x,0) + \Lambda^2/(4\pi^2 N),\\
\left\langle W(\sqrt{N}\alpha_0)\Omega, N^{-1}\hat{\boldsymbol{A}}^i_\kappa(x)\hat{\boldsymbol{A}}^j_\kappa(y) W(\sqrt{N}\alpha_0)\Omega\right\rangle_{\mathcal{F}_p} &= \boldsymbol{A}^i_\kappa(x,0)\boldsymbol{A}^j_\kappa(y,0) + (2N)^{-1}\gamma_{ij}^{\perp\Lambda}(x-y),\\
\left\langle W(\sqrt{N}\alpha_0)\Omega, N^{-1}H_f W(\sqrt{N}\alpha_0)\Omega\right\rangle_{\mathcal{F}_p} &= \mathcal{E}_f(0),\\
\left\langle W(\sqrt{N}\alpha_0)\Omega, N^{-2}H_f^2 W(\sqrt{N}\alpha_0)\Omega\right\rangle_{\mathcal{F}_p} &= \mathcal{E}^2_f(0) + N^{-1}\mathcal{E}_{f^2}(0),\\
\left\langle W(\sqrt{N}\alpha_0)\Omega, N^{-3/2}\hat{\boldsymbol{A}}_\kappa(x) H_f W(\sqrt{N}\alpha_0)\Omega\right\rangle_{\mathcal{F}_p} &= \boldsymbol{A}_\kappa(x,0)\mathcal{E}_f(0) - iN^{-1}\boldsymbol{E}^+_\kappa(x,0),\\
\left\langle W(\sqrt{N}\alpha_0)\Omega, N^{-2}\hat{\boldsymbol{A}}^2_\kappa(x) H_f W(\sqrt{N}\alpha_0)\Omega\right\rangle_{\mathcal{F}_p} &= \boldsymbol{A}^2_\kappa(x)\mathcal{E}_f(0) + \Lambda^2/(4\pi^2 N)\mathcal{E}_f(0)\\
&\quad + -2iN^{-1}\boldsymbol{A}_\kappa(x,0)\boldsymbol{E}^+_\kappa(x,0).
\end{aligned} \tag{3.177}$$

Proof. The proof is a simple application of the canonical commutation relations (3.9) and part (ii) from Lemma 3.9.2. □

Proof of Lemma 3.9.1. Relation (3.169) directly follows from Lemma 3.7.1. In view of Lemma 3.9.2 we calculate

$$\begin{aligned}
\beta^b(\Psi_{N,0},\alpha_0) &= \sum_{\lambda=1,2}\int d^3k\,|k| \left|\left|\left(N^{-1/2}a(k,\lambda) - \alpha_0(k,\lambda)\right)\Psi_{N,0}\right|\right|^2\\
&= \sum_{\lambda=1,2}\int d^3k\,|k| \left|\left|W^{-1}(\sqrt{N}\alpha_0)\left(N^{-1/2}a(k,\lambda) - \alpha_0(k,\lambda)\right)W(\sqrt{N}\alpha_0)W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}\right|\right|^2\\
&= \sum_{\lambda=1,2}\int d^3k\, \left|\left|N^{-1/2}a(k,\lambda)W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}\right|\right|^2\\
&= N^{-1}\left\langle W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}, H_f W^{-1}(\sqrt{N}\alpha_0)\Psi_{N,0}\right\rangle = b_N.
\end{aligned} \tag{3.178}$$

Equation (3.171) is solely the definition of β^c. In the following, we are interested in initial data $\Psi_{N,0} = \varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega$ of product type. First, we notice that

$$\beta^a(\Psi_{N,0},\varphi_0) = \left\langle \Psi_{N,0}, q_1^{\varphi_0}\Psi_{N,0}\right\rangle = \left\langle\varphi_0,\varphi_0\right\rangle_{L^2(\mathbb{R}^3)} - \left\langle\varphi_0,\varphi_0\right\rangle^2_{L^2(\mathbb{R}^3)} = 0 \tag{3.179}$$

because the scalar product factorizes for product states and q_1 only acts on the Hilbert space of the first charged particle. Then, we follow

$$\beta^b[\Psi_{N,0},\alpha]=0 \tag{3.180}$$

for $\Psi_{N,0}=\varphi_0^{\otimes N}\otimes W(\sqrt{N}\alpha_0)\Omega$ from (3.178). To show that the product structure suppresses the fluctuations of the energy per particle around its mean value is more elaborate. Nevertheless, the idea of the proof simple and in the spirit of the law of large numbers from probability theory. We bound β^c by

$$\begin{aligned}\beta^c(0)&=\left\langle \left(N^{-1}H_N-\mathcal{E}_M\right)\Psi_{N,0},\left(N^{-1}H_N-\mathcal{E}_M\right)\Psi_{N,0}\right\rangle\\&\leq |\langle N^{-1}H_N\Psi_{N,0},N^{-1}H_N\Psi_{N,0}\rangle-\mathcal{E}_M^2|+2\mathcal{E}_M|\mathcal{E}_M-\langle\Psi_{N,0},N^{-1}H_N\Psi_{N,0}\rangle|\end{aligned} \tag{3.181}$$

and show that

(i) $|\langle\Psi_{N,0},N^{-1}H_N\Psi_{N,0}\rangle-\mathcal{E}_M[\varphi_0,\alpha_0]|\leq C\Lambda^2N^{-1}$

(ii) $|\langle N^{-1}H_N\Psi_{N,0},N^{-1}H_N\Psi_{N,0}\rangle-\mathcal{E}_M^2[\varphi_0,\alpha_0]|\leq C\Lambda^4N^{-1}$

holds for states of product type.

(i) The mean value of the energy per particle

For ease of notation we denote $\boldsymbol{A}_\kappa(\cdot,0)$, $\boldsymbol{E}_\kappa^+(\cdot,0)$, $\mathcal{E}_f(0)$, $\mathcal{E}_{f^2}(0)$ by $\boldsymbol{A}_\kappa(\cdot)$, $\boldsymbol{E}_\kappa(\cdot)$, $\mathcal{E}_f$, $\mathcal{E}_{f^2}$ in the following. The mean value of the energy per particle is given by

$$\begin{aligned}\langle\Psi_{N,0},N^{-1}H_N\Psi_{N,0}\rangle&=\left\langle\Psi_{N,0},N^{-1}\sum_{j=1}^N\left(-i\nabla_j-N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\right)^2\Psi_{N,0}\right\rangle\\&\quad+\left\langle\Psi_{N,0},1/(2N^2)\sum_{j\neq k}v(x_j-x_k)\Psi_{N,0}\right\rangle\\&\quad+\langle\Psi_{N,0},N^{-1}H_f\Psi_{N,0}\rangle.\end{aligned} \tag{3.182}$$

Due to symmetry and the product structure of $\Psi_{N,0}$ this becomes

$$\begin{aligned}\langle\Psi_{N,0},N^{-1}H_N\Psi_{N,0}\rangle&=\langle\varphi_0,(-\Delta)\varphi_0\rangle\\&\quad+2i\langle\varphi_0,\langle W(\sqrt{N}\alpha_0)\Omega,N^{-1/2}\hat{\boldsymbol{A}}_\kappa W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p}\cdot\nabla\varphi_0\rangle\\&\quad+\langle\varphi_0,\langle W(\sqrt{N}\alpha_0)\Omega,N^{-1}\hat{\boldsymbol{A}}_\kappa^2W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p}\varphi_0\rangle\\&\quad+(N-1)/(2N)\langle\varphi_0,\left(v*|\varphi_0|^2\right)\varphi_0\rangle\\&\quad+\langle W(\sqrt{N}\alpha_0)\Omega,N^{-1}H_fW(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p}.\end{aligned} \tag{3.183}$$

Lemma 3.9.3 gives

$$\begin{aligned}\langle\Psi_{N,0},N^{-1}H_N\Psi_{N,0}\rangle&=||(-i\nabla-\boldsymbol{A}_\kappa)\varphi_0||^2+1/2\langle\varphi_0,\left(v*|\varphi_0|^2\right)\varphi_0\rangle+\mathcal{E}_f\\&\quad+\Lambda^2/(4\pi^2N)-1/(2N)\langle\varphi_0,\left(v*|\varphi_0|^2\right)\varphi_0\rangle\end{aligned} \tag{3.184}$$

and we obtain

$$\langle\Psi_{N,0},N^{-1}H_N\Psi_{N,0}\rangle=\mathcal{E}_M[\varphi_0,\alpha_0]+\Lambda^2/(4\pi^2N)-1/(2N)\langle\varphi_0,\left(v*|\varphi_0|^2\right)\varphi_0\rangle. \tag{3.185}$$

(ii) The second moment of the energy per particle

Subsequently, we show that the second moment of the energy per particle approximately equals the energy of the effective system squared. We split the double sum, arising from the second moment of the many-body Hamiltonian into its diagonal and off-diagonal part. The diagonal only consists of N constituents and has a subleading contribution for large N. On the contrary, there are N^2 elements from the off-diagonal which give rise to $\mathcal{E}_M^2$. In order to organize the estimate, we decompose the second moment of the energy per particle as well as the effective energy squared into pieces:

$$
\begin{aligned}
&\big\langle N^{-1}H_N\Psi_{N,0}, N^{-1}H_N\Psi_{N,0}\big\rangle = \\
&\quad =N^{-2}\sum_{j,k}\big\langle \Big(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\Big)^2\Psi_{N,0}, \Big(-i\nabla_k - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_k)\Big)^2\Psi_{N,0}\big\rangle && (3.186)\\
&\quad +(4N^4)^{-1}\sum_{i\neq j,k\neq l}\big\langle v(x_i-x_j)\Psi_{N,0}, v(x_k-x_l)\Psi_{N,0}\big\rangle && (3.187)\\
&\quad +N^{-2}\big\langle\Psi_{N,0}, H_f^2\Psi_{N,0}\big\rangle && (3.188)\\
&\quad +N^{-3}\sum_{j,k\neq l}\mathrm{Re}\big\langle \Big(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\Big)^2\Psi_{N,0}, v(x_k-x_l)\Psi_{N,0}\big\rangle && (3.189)\\
&\quad +2N^{-2}\sum_{j}\mathrm{Re}\big\langle \Big(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\Big)^2\Psi_{N,0}, H_f\Psi_{N,0}\big\rangle && (3.190)\\
&\quad +N^{-3}\sum_{j\neq k}\mathrm{Re}\big\langle v(x_j-x_k)\Psi_{N,0}, H_f\Psi_{N,0}\big\rangle && (3.191)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{E}_M^2[\varphi_0,\alpha_0] =&\big\langle\varphi_0, (-i\nabla - \boldsymbol{A}_\kappa)^2\varphi_0\big\rangle^2 && (3.192)\\
&+1/4\big\langle\varphi_0, (v*|\varphi_0|^2)\varphi_0\big\rangle^2 && (3.193)\\
&+\mathcal{E}_f^2 && (3.194)\\
&+\big\langle\varphi_0, \Big((-i\nabla - \boldsymbol{A}_\kappa)^2\Big)\varphi_0\big\rangle\big\langle\varphi_0, (v*|\varphi_0|^2)\varphi_0\big\rangle && (3.195)\\
&+2\big\langle\varphi_0, (-i\nabla - \boldsymbol{A}_\kappa)^2\varphi_0\big\rangle\mathcal{E}_f && (3.196)\\
&+\big\langle\varphi_0, (v*|\varphi_0|^2)\varphi_0\big\rangle\mathcal{E}_f. && (3.197)
\end{aligned}
$$

In the following, we estimate the difference of the corresponding expressions and obtain

$$
|\big\langle N^{-1}H_N\Psi_{N,0}, N^{-1}H_N\Psi_{N,0}\big\rangle - \mathcal{E}_M^2[\varphi_0,\alpha_0]| \leq C\Lambda^4 N^{-1}. \tag{3.198}
$$

$|(3.186) - (3.192)| \leq C\Lambda^4/N$:

The off-diagonal part of (3.186) is given by

$$\begin{aligned}
\Big\langle \Big(-i\nabla_1 - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\Big)^2 \Psi_{N,0}, \Big(-i\nabla_2 - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_2)\Big)^2 \Psi_{N,0}\Big\rangle && (3.199)\\
= \big\langle (-\Delta_1)\,\Psi_{N,0}, (-\Delta_2)\,\Psi_{N,0}\big\rangle && (3.200)\\
+ 2i\big\langle (-\Delta_1)\,\Psi_{N,0}, N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_2)\nabla_2\Psi_{N,0}\big\rangle && (3.201)\\
+ 2i\big\langle N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\nabla_1\Psi_{N,0}, (-\Delta_2)\,\Psi_{N,0}\big\rangle && (3.202)\\
+ \big\langle (-\Delta_1)\,\Psi_{N,0}, N^{-1}\hat{\boldsymbol{A}}_\kappa^2(x_2)\Psi_{N,0}\big\rangle && (3.203)\\
+ \big\langle N^{-1}\hat{\boldsymbol{A}}_\kappa^2(x_1)\Psi_{N,0}, (-\Delta_2)\,\Psi_{N,0}\big\rangle && (3.204)\\
- 4\big\langle N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\nabla_1\Psi_{N,0}, N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_2)\nabla_2\Psi_{N,0}\big\rangle && (3.205)\\
+ 2i\big\langle N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\nabla_1\Psi_{N,0}, N^{-1}\hat{\boldsymbol{A}}_\kappa^2(x_2)\Psi_{N,0}\big\rangle && (3.206)\\
+ 2i\big\langle N^{-1}\hat{\boldsymbol{A}}_\kappa^2(x_1)\Psi_{N,0}, N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_2)\nabla_2\Psi_{N,0}\big\rangle && (3.207)\\
+ \big\langle N^{-1}\hat{\boldsymbol{A}}_\kappa^2(x_1)\Psi_{N,0}, N^{-1}\hat{\boldsymbol{A}}_\kappa^2(x_2)\Psi_{N,0}\big\rangle. && (3.208)
\end{aligned}$$

By means of Lemma 3.9.3 we have

$$\begin{aligned}
(3.200) + (3.201) + (3.202) =& \big\langle \varphi_0, (-\Delta)\,\varphi_0\big\rangle^2 + 4i\big\langle \boldsymbol{A}_\kappa\varphi_0, \nabla\varphi_0\big\rangle\big\langle \varphi_0, (-\Delta)\,\varphi_0\big\rangle,\\
(3.203) + (3.204) =& 2\big\langle \varphi_0, \boldsymbol{A}_\kappa^2\varphi_0\big\rangle\big\langle \varphi_0, (-\Delta)\,\varphi_0\big\rangle + \Lambda^2/(2\pi^2 N)\,||\nabla\varphi_0||^2,\\
(3.205) =& -4\big\langle \varphi_0, \boldsymbol{A}_\kappa\nabla\varphi_0\big\rangle^2 -\\
& -2/N\int d^3x\int d^3y \varphi_0^*(x)\varphi_0^*(y)\gamma_{kl}^{\perp\Lambda}(x-y)\nabla^k\varphi_0(x)\nabla^l\varphi_0(y).
\end{aligned} \tag{3.209}$$

In order to evaluate the last three lines, we use that

$$\begin{aligned}
N^{-3/2}\big\langle W(\sqrt{N}\alpha_0)\Omega, \hat{\boldsymbol{A}}_\kappa^2(x)\hat{\boldsymbol{A}}_\kappa^i(y)W(\sqrt{N}\alpha_0)\Omega\big\rangle_{\mathcal{F}_p} =& \boldsymbol{A}_\kappa^2(x)\boldsymbol{A}_\kappa^i(y) + \Lambda^2/(4\pi^2 N)\boldsymbol{A}_\kappa^i(y)\\
& + N^{-1}\sum_{j=1}^3 \gamma_{ij}^{\perp\Lambda}(x-y)\hat{\boldsymbol{A}}_\kappa^j(x),
\end{aligned} \tag{3.210}$$

and

$$\begin{aligned}
N^{-2}\big\langle W(\sqrt{N}\alpha_0)\Omega, \hat{\boldsymbol{A}}_\kappa^2(x)\hat{\boldsymbol{A}}_\kappa^2(y)W(\sqrt{N}\alpha_0)\Omega\big\rangle_{\mathcal{F}_p} =& \boldsymbol{A}_\kappa^2(x)\boldsymbol{A}_\kappa^2(y) +\\
& + \Lambda^2/(4\pi^2 N)\left(\boldsymbol{A}_\kappa^2(x) + \boldsymbol{A}_\kappa^2(y)\right) + 2/N\sum_{k,l=1}^3 \gamma_{kl}^{\perp\Lambda}(x-y)\boldsymbol{A}_\kappa^k(x)\boldsymbol{A}_\kappa^l(y)\\
& + N^{-2}\Big(\sum_{k,l=1}^3 |\gamma_{kl}^{\perp\Lambda}(x-y)|^2 + \Lambda^4/((2\pi)^4)\Big),
\end{aligned} \tag{3.211}$$

can also be obtained by the canonical commutation relations (3.9) and Lemma 3.9.2.

Consequently, we have

$$\begin{aligned}(3.206)+(3.207) &= 4i\langle\varphi_0,\boldsymbol{A}_\kappa^2\varphi_0\rangle\langle\varphi_0,\boldsymbol{A}_\kappa\nabla\varphi_0\rangle + i\Lambda^2/(\pi^2N)\langle\varphi_0,\boldsymbol{A}_\kappa\nabla\varphi_0\rangle\\ &+4i/N\int d^3x\int d^3y\varphi_0^*(x)\varphi_0^*(y)\gamma_{kl}^{\perp\Lambda}\boldsymbol{A}_\kappa^k(x)\varphi(x)\left(\nabla^l\varphi\right)(y),\\ (3.208) &= \langle\varphi_0,\boldsymbol{A}_\kappa^2\varphi_0\rangle^2+\Lambda^4/(2\pi^2N)\langle\varphi_0,\boldsymbol{A}_\kappa^2\varphi_0\rangle+\Lambda^4/(16\pi^4N)\\ &+N^{-2}\int d^3x\int d^3y\varphi_0^*(x)\varphi_0^*(y)\sum_{k,l}|\gamma_{k,l}^{\perp\Lambda}(x-y)|^2\varphi(x)\varphi(y)\\ &+2/N\int d^3x\int d^3y\varphi_0^*(x)\varphi_0^*(y)\gamma_{kl}^{\perp\Lambda}(x-y)\boldsymbol{A}_\kappa^k(x)\boldsymbol{A}_\kappa^l(y)\varphi(x)\varphi(y).\end{aligned} \tag{3.212}$$

and

$$|(3.199)-\left\langle\varphi_0,(-i\nabla-\boldsymbol{A}_\kappa)^2\varphi_0\right\rangle^2|\le C\Lambda^4/N \tag{3.213}$$

because all error terms are bounded by $C\Lambda^4/N$ with the help of the assumptions of Lemma 3.9.1. Since the diagonal part of (3.186) is of order N^{-1}, this implies

$$|(3.186)-(3.192)|\le C\Lambda^4/N. \tag{3.214}$$

$|(3.187)-(3.193)|\le C/N$:

By virtue of the symmetry of the wave function and $v(-x)=v(x)$ we can write line (3.187) as

$$\begin{aligned}(4N^4)^{-1}\sum_{i\neq j,k\neq l}&\langle v(x_i-x_j)\Psi_{N,0},v(x_k-x_l)\Psi_{N,0}\rangle=\\ &=1/4\langle v(x_1-x_2)\Psi_{N,0},v(x_3-x_4)\Psi_{N,0}\rangle\\ &-(6N^2-11N+6)N^{-3}\langle v(x_1-x_2)\Psi_{N,0},v(x_3-x_4)\Psi_{N,0}\rangle\\ &+(N-1)N^{-3}/2\langle v(x_1-x_2)\Psi_{N,0},v(x_1-x_2)\Psi_{N,0}\rangle\\ &+(N-1)(N-2)N^{-3}\langle v(x_1-x_2)\Psi_{N,0},v(x_1-x_3)\Psi_{N,0}\rangle.\end{aligned} \tag{3.215}$$

The product structure of the initial state gives

$$\begin{aligned}\langle v(x_1-x_2)\Psi_{N,0},v(x_3-x_4)\Psi_{N,0}\rangle &= \left\langle\varphi_0,\left(v*|\varphi_0|^2\right)\varphi_0\right\rangle^2,\\ ||v(x_1-x_2)\Psi_{N,0}||^2 &= \left\langle\varphi_0,\left(v^2*|\varphi_0|^2\right)\varphi_0\right\rangle\end{aligned} \tag{3.216}$$

and we conclude

$$\begin{aligned}|(4N^4)^{-1}\sum_{i\neq j,k\neq l}&\langle v(x_i-x_j)\Psi_{N,0},v(x_k-x_l)\Psi_{N,0}\rangle-1/4\left\langle\varphi_0,\left(v*|\varphi_0|^2\right)\varphi_0\right\rangle^2|\le\\ &\le 6/N|\langle v(x_1-x_2)\Psi_{N,0},v(x_3-x_4)\Psi_{N,0}\rangle|+N^{-1}\,||v(x_1-x_2)\Psi_{N0}||^2\\ &+N^{-1}|\langle v(x_1-x_2)\Psi_{N,0},v(x_1-x_3)\Psi_{N,0}\rangle|\\ &\le 8/N\,||v(x_1-x_2)\Psi_{N,0}||^2=8/N\left\langle\varphi_0,\left(v^2*|\varphi_0|^2\right)\varphi_0\right\rangle.\end{aligned} \tag{3.217}$$

$|(3.188)-(3.194)|\le C/N$:

This bound results from Lemma 3.9.3 because

$$N^{-2}\langle\Psi_{N0},H_f^2\Psi_{N0}\rangle=N^{-2}\langle W(\sqrt{N}\alpha_0)\Omega,H_f^2W(\sqrt{N}\alpha_0)\Omega\rangle_{\mathcal{F}_p}=\mathcal{E}_f^2+N^{-1}\mathcal{E}_{f^2}. \tag{3.218}$$

$|(3.189) - (3.195)| \leq C\Lambda^2/N$:

Line (3.189) simplifies to

$$\begin{aligned}
N^{-3} \sum_{j,k\neq l} \mathrm{Re}\big\langle \Big(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\Big)^2 \Psi_{N,0}, v(x_k - x_l)\Psi_{N,0}\big\rangle \\
&= (N-1)(N-2)N^{-2}\mathrm{Re}\big\langle \Big(-i\nabla_1 - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\Big)^2 \Psi_{N,0}, v(x_2 - x_3)\Psi_{N,0}\big\rangle \\
&+ 2(N-1)N^{-2}\mathrm{Re}\big\langle \Big(-i\nabla_1 - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\Big)^2 \Psi_{N,0}, v(x_2 - x_1)\Psi_{N,0}\big\rangle \\
&= \left(1 - 3(N-2)N^{-2}\right)\left\langle \varphi_0, \left(-i\nabla - \boldsymbol{A}_\kappa\right)^2 \varphi_0\right\rangle\left\langle \varphi_0, \left(v * |\varphi_0|^2\right)\varphi_0\right\rangle \\
&+ (N-1)(N-2)N^{-3}\Lambda^2/(4\pi^2)\left\langle \varphi_0, \left(v * |\varphi_0|^2\right)\varphi_0\right\rangle \\
&+ 2(N-1)N^{-2}\mathrm{Re}\big\langle \Big(-i\nabla_1 - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\Big)^2 \Psi_{N,0}, v(x_2 - x_1)\Psi_{N,0}\big\rangle. \quad (3.219)
\end{aligned}$$

Consequently the estimate follows because $||v(x_1 - x_2)\Psi_{N,0}||^2 = \left\langle \varphi_0, \left(v^2 * |\varphi_0|^2\right)\varphi_0\right\rangle$ and $\left|\left|\Big(-i\nabla_1 - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1)\Big)^2 \Psi_{N,0}\right|\right|$ are finite under the assumptions of Lemma 3.9.1.

$|(3.190) - (3.196)| \leq C\Lambda^2/N$:

Similar to the previous calculations we obtain

$$\begin{aligned}
2N^{-2}\sum_{j=1}^{N} \mathrm{Re}\big\langle \Big(-i\nabla_j - N^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_j)\Big)^2 \Psi_{N,0}, H_f\Psi_{N,0}\big\rangle \\
&=2\mathrm{Re}\big\langle \Big(-\Delta_1 + 2iN^{-1/2}\hat{\boldsymbol{A}}_\kappa(x_1) + N^{-1}\hat{\boldsymbol{A}}^2_\kappa(x_1) + V_{ex}(x_1)\Big)\Psi_{N,0}, N^{-1}H_f\Psi_{N,0}\big\rangle \\
&=2\left\langle \varphi_0, \left(-i\nabla - \boldsymbol{A}_\kappa\right)^2 \varphi_0\right\rangle\mathcal{E}_f \\
&+2N^{-1}\mathrm{Re}\left(\Lambda^2/(4\pi^2)\mathcal{E}_f - 4\left\langle \nabla\varphi_0, \boldsymbol{E}^+_\kappa(x)\varphi_0\right\rangle - 2i\left\langle \boldsymbol{A}_\kappa\varphi_0, \boldsymbol{E}^+_\kappa\varphi_0\right\rangle\right). \quad (3.220)
\end{aligned}$$

By means of

$$\begin{aligned}
|\left\langle \boldsymbol{A}_\kappa\varphi_0, \boldsymbol{E}^+_\kappa\varphi\right\rangle| &\leq ||\boldsymbol{A}_\kappa||_\infty\, ||\varphi_0||_\infty\, \left|\left|\boldsymbol{E}^+_\kappa\right|\right|, \\
|\left\langle \nabla\varphi_0, \boldsymbol{E}^+_\kappa\varphi_0\right\rangle| &\leq ||\nabla\varphi_0||\, ||\varphi_0||_\infty\, \left|\left|\boldsymbol{E}^+_\kappa\right|\right|, \quad (3.221)
\end{aligned}$$

and

$$\left|\left|\boldsymbol{E}^+_\kappa\right|\right|_2^2 = \frac{1}{2}\sum_{\lambda=1,2}\int_{|k|\leq\Lambda} d^3k\, |k||\alpha_0(k,\lambda)|^2 \leq \mathcal{E}_f \quad (3.222)$$

the inequality follows.

$|(3.191) - (3.197)| \leq C/N$:

Making use of symmetry and Lemma 3.9.3 one has

$$\begin{aligned}
N^{-3}\sum_{j\neq k} \mathrm{Re}\left\langle v(x_k - x_k)\Psi_{N,0}, H_f\Psi_{N,0}\right\rangle &= (N-1)N^{-2}\left\langle v(x_1 - x_2)\Psi_{N,0}, H_f\Psi_{N,0}\right\rangle \\
&= \left(1 - N^{-1}\right)\left\langle \varphi_0, \left(v * |\varphi_0|^2\right)\varphi_0\right\rangle\mathcal{E}_f. \quad (3.223)
\end{aligned}$$

This shows the last inequality and altogether we obtain

$$|\left\langle N^{-1}H_N\Psi_{N,0}, N^{-1}H_N\Psi_{N,0}\right\rangle - \mathcal{E}^2_M[\varphi_0, \alpha_0]| \leq C\Lambda^4 N^{-1}, \quad (3.224)$$

which proves Lemma 3.9.1. □

3.10 Proof of Theorem 3.2.1

Let v satisfy (A1), $\varphi_t \in L^2(\mathbb{R}^3)$ with $||\varphi_t|| = 1$, $\alpha_0 \in \mathfrak{h}$ such that $(\boldsymbol{A}(0), \boldsymbol{E}(0)) \in (H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$, $\Psi_{N,0} \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H_N)$. Let $\Psi_{N,t}$ be the unique solution of (3.4), $(\varphi_t, \boldsymbol{A}(t), \boldsymbol{E}(t))$ be the unique solution of (3.15) and assume $\sup_{t\in[0,T]}\{||\varphi_t||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{A}(t)||_{H^3(\mathbb{R}^3)} + ||\boldsymbol{E}(t)||_{H^2(\mathbb{R}^3)}\} < \infty$ for any $T \in \mathbb{R}^+$. According to Lemma 3.8.9 and Lemma 3.9.1 there is a monotone increasing function of $||\boldsymbol{A}_\kappa(s)||_\infty$, $\mathcal{E}_M[\varphi_s, \alpha_s]$, $||\varphi_s||_{H^2(\mathbb{R}^3)}$ and $||\nabla\varphi||_\infty$ such that

$$\beta(\Psi_{N,t}, \varphi_t, \alpha_t) \leq e^{\Lambda^4 \int_0^t ds\, C(s)} \left(a_N + b_N + c_N + \Lambda/N\right). \tag{3.225}$$

The energy $\mathcal{E}_M[\varphi_s, \alpha_s] = \mathcal{E}_M[\varphi_0, \alpha_0]$ is a constant of motion and finite. Moreover, we have $||\boldsymbol{A}_\kappa||_\infty \leq ||\boldsymbol{A}||_{H^2(\mathbb{R}^3)}$. This displays that $C(s)$ only depends on $||\varphi_s||_{H^2(\mathbb{R}^2)}$, $||\nabla\varphi||_\infty$ and $||\boldsymbol{A}||_{H^2(\mathbb{R}^3)}$. We choose for a given time $t \geq 0$ the number N of charges large enough so that $\beta(\Psi_{N,t}, \varphi_t, \alpha_t) \leq 1$ and obtain

$$\begin{aligned} \mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|| &\leq \sqrt{a_N + b_N + c_N + \Lambda/N}\, e^{\Lambda^4 \int_0^t ds\, C(s)} \\ \mathrm{Tr}_{\mathfrak{h}}|\gamma_{N,t}^{(0,1)} - |u_t\rangle\langle u_t|| &\leq \sqrt{a_N + b_N + c_N + \Lambda/N}\, 6(1 + ||u_t||_{\mathfrak{h}})e^{\Lambda^4 \int_0^t ds\, C(s)} \end{aligned} \tag{3.226}$$

by Lemma 3.7.1. Then, we recall (3.33) and derive

$$\begin{aligned} \mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|| &\leq \sqrt{a_N + b_N + c_N + N^{-1}}\, \Lambda e^{\Lambda^4 \int_0^t ds\, C(s)} \\ \mathrm{Tr}_{\mathfrak{h}}|\gamma_{N,t}^{(0,1)} - |u_t\rangle\langle u_t|| &\leq \sqrt{a_N + b_N + c_N + N^{-1}}\, \Lambda C(s)e^{\Lambda^4 \int_0^t ds\, C(s)} \end{aligned} \tag{3.227}$$

where $C(s)$ depends on $||\varphi_s||_{H^2(\mathbb{R}^2)}$, $||\nabla\varphi||_\infty$, $||\boldsymbol{A}||_{H^2(\mathbb{R}^3)}$ and $||\boldsymbol{E}||_{L^2(\mathbb{R}^3)}$. For initial states of product type $\Psi_{N,0} = \varphi_0^{\otimes N} \otimes W(\sqrt{N}\alpha_0)\Omega$ this becomes

$$\begin{aligned} \mathrm{Tr}_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle\varphi_t|| &\leq N^{-1/2}\Lambda^2 e^{\Lambda^4 \int_0^t ds\, C(s)} \\ \mathrm{Tr}_{\mathfrak{h}}|\gamma_{N,t}^{(0,1)} - |u_t\rangle\langle u_t|| &\leq N^{-1/2}\Lambda^2 C(s)e^{\Lambda^4 \int_0^t ds\, C(s)}. \end{aligned} \tag{3.228}$$

3.11 Appendix

Lemma 3.11.1. *Let $\Psi_{N,t} \in \left(L_s^2\left(\mathbb{R}^{3N}\right) \otimes \mathcal{F}_p\right) \cap \mathcal{D}(H_N)$, $\varphi_t \in H^3(\mathbb{R}^3)$ with $||\varphi_t|| = 1$ and $\alpha_t \in \mathfrak{h}$ such that $(\boldsymbol{A}(t), \boldsymbol{E}(t)) \in (H^3(\mathbb{R}^3) \oplus H^2(\mathbb{R}^3))$. Then*

$$\int d^3y \left|\left|\left(N^{-1/2}\hat{\boldsymbol{E}}_\kappa^+(y) - \boldsymbol{E}_\kappa^+(y,t)\right)\Psi_{N,t}\right|\right|^2 = \left\langle \Psi_{N,t}, \boldsymbol{\mathcal{E}}^-(y,t)\boldsymbol{\mathcal{E}}^+(y,t)\Psi_{N,t}\right\rangle_{;y} \leq \beta^b. \tag{3.229}$$

For $\hat{\boldsymbol{G}} \in \{\boldsymbol{\mathcal{A}}^+, \boldsymbol{\mathcal{A}}^-, \boldsymbol{\mathcal{E}}^+, \boldsymbol{\mathcal{E}}^-\}$ one obtains

$$\begin{aligned} \left|\left|\hat{\boldsymbol{G}}(x_1,t)\nabla_1 p_1 \Psi_{N,t}\right|\right|^2 &\leq C\, ||\nabla_1\varphi_t||_{L^\infty(\mathbb{R}^3)}^2 \left\langle \hat{\boldsymbol{G}}(y,t)\Psi_{N,t}, \hat{\boldsymbol{G}}(y,t)\Psi_{N,t}\right\rangle_{;y}, \\ \left|\left|\hat{\boldsymbol{G}}(y,t)\nabla_1 p_1 \Psi_{N,t}\right|\right|_{;y}^2 &\leq C\, ||\nabla_1\varphi_t||_{L^\infty(\mathbb{R}^3)}^2 \left\langle \hat{\boldsymbol{G}}(y,t)\Psi_{N,t}, \hat{\boldsymbol{G}}(y,t)\Psi_{N,t}\right\rangle_{;y}, \\ \left|\left|\hat{\boldsymbol{G}}(x_1,t)p_1\Psi_{N,t}\right|\right|^2 &\leq C\left\langle \hat{\boldsymbol{G}}(y,t)\Psi_{N,t}, \hat{\boldsymbol{G}}(y,t)\Psi_{N,t}\right\rangle_{;y}. \end{aligned} \tag{3.230}$$

Proof. The first inequality is proven by

$$\begin{aligned}
\left|\left|\boldsymbol{\mathcal{E}}^{+}(y,t)\Psi_N\right|\right|^2_{;y} &= 1/2\sum_{\lambda=1,2}\int d^3k\,\tilde{\kappa}(k)|k|^{1/2}\boldsymbol{\epsilon}_\lambda(k)\sum_{\mu=1,2}\int d^3l\,\tilde{\kappa}(l)|l|^{1/2}\boldsymbol{\epsilon}_\mu(l)\int d^3y\,e^{i(l-k)y}\\
&\times\left\langle\left(N^{-1/2}a(k,\lambda)-\alpha_t(k,\lambda)\right)\Psi_N,\left(N^{-1/2}a(l,\mu)-\alpha_t(l,\mu)\right)\Psi_N\right\rangle\\
&=(2\pi)^3/2\int d^3l\,|\tilde{\kappa}(k)|^2|k|\sum_{\lambda,\mu}\boldsymbol{\epsilon}_\lambda(k)\boldsymbol{\epsilon}_\mu(k)\times\\
&\times\left\langle\left(N^{-1/2}a(k,\lambda)-\alpha_t(k,\lambda)\right)\Psi_N,\left(N^{-1/2}a(k,\mu)-\alpha_t(k,\mu)\right)\Psi_N\right\rangle\\
&=1/2\sum_{\lambda=1,2}\int_{|k|\leq\Lambda}d^3k\,|k|\left|\left|\left(N^{-1/2}a(k,\lambda)-\alpha_t(k,\lambda)\right)\Psi_N\right|\right|^2\leq\beta^b. \qquad (3.231)
\end{aligned}$$

To show the second relation we compute

$$\begin{aligned}
\left|\left|\hat{\boldsymbol{G}}(x_1,t)\nabla_1 p_1\Psi_N\right|\right|^2 &=\left\langle\hat{\boldsymbol{G}}(x_1,t)\cdot\nabla_1 p_1\Psi_N,\hat{\boldsymbol{G}}(x_1,t)\cdot\nabla_1 p_1\Psi_N\right\rangle\\
&=\sum_{i,l=1}^{3}\left\langle\hat{\boldsymbol{G}}^i(x_1,t)\nabla_1^i p_1\Psi_N,\hat{\boldsymbol{G}}^l(x_1,t)\nabla_1^l p_1\Psi_N\right\rangle\\
&=\sum_{i,l=1}^{3}\left\langle\Psi_N,p_1\left(-\nabla_1^i\right)\left(\hat{\boldsymbol{G}}^i(x_1,t)\right)^*\hat{\boldsymbol{G}}^l(x_1,t)\nabla_1^l p_1\Psi_N\right\rangle. \qquad (3.232)
\end{aligned}$$

According to

$$\begin{aligned}
p_1\left(-\nabla_1^i\right)\left(\hat{\boldsymbol{G}}^i(x_1,t)\right)^*\hat{\boldsymbol{G}}^l(x_1,t)\nabla_1^l p_1 &= p_1\int d^3y\varphi^*(y)\left(-\nabla_y^i\right)\left(\hat{\boldsymbol{G}}^i(y,t)\right)^*\hat{\boldsymbol{G}}^l(y,t)\ \nabla^l\varphi(y)\\
&=p_1\int d^3y\left(\nabla^i\varphi\right)^*(y)\left(\hat{\boldsymbol{G}}^i(y,t)\right)^*\hat{\boldsymbol{G}}^l(y,t)\nabla^l\varphi(y)
\end{aligned} \qquad (3.233)$$

this becomes

$$\begin{aligned}
\left|\left|\hat{\boldsymbol{G}}(x_1,t)\nabla_1 p_1\Psi_N\right|\right|^2 &= \sum_{i,l=1}^{3}\int d^3y\left(\nabla^i\varphi\right)^*(y)\left(\nabla^l\varphi\right)(y)\left\langle\Psi_N,p_1\left(\hat{\boldsymbol{G}}^i(y,t)\right)^*\hat{\boldsymbol{G}}^l(y,t)\Psi_N\right\rangle\\
&\leq\sum_{i,l=1}^{3}\int d^3y|\nabla^i\varphi||\nabla^l\varphi||\left\langle\hat{\boldsymbol{G}}^i(y,t)\Psi_N,\hat{\boldsymbol{G}}^l(y,t)\Psi_N\right\rangle|\\
&\leq\sum_{i,l=1}^{3}\int d^3y\frac{1}{2}\left(||\nabla\varphi||^2_\infty\left|\left|\hat{\boldsymbol{G}}^l(y,t)\Psi_N\right|\right|^2+||\nabla\varphi||^2_\infty\left|\left|\hat{\boldsymbol{G}}^i(y,t)\Psi_N\right|\right|^2\right)\\
&\leq 3\,||\nabla\varphi||^2_\infty\left\langle\hat{\boldsymbol{G}}(y,t)\Psi_N,\hat{\boldsymbol{G}}(y,t)\Psi_N\right\rangle_{;y}. \qquad (3.234)
\end{aligned}$$

The remaining inequalities are proven analogously. □

CHAPTER

FOUR

DERIVATION OF THE TIME DEPENDENT GROSS-PITAEVSKII EQUATION IN TWO DIMENSIONS

Abstract We present a microscopic derivation of the defocusing two-dimensional cubic nonlinear Schrödinger equation as a mean-field equation starting from an interacting N-particle system of Bosons. We consider the interaction potential to be given either by $W_\beta(x) = N^{-1+2\beta}W(N^\beta x)$, for any $\beta > 0$, or to be given by $V_N(x) = e^{2N}V(e^N x)$, for some spherical symmetric, nonnegative and compactly supported $W, V \in L^\infty(\mathbb{R}^2, \mathbb{R})$. In both cases we prove the convergence of the reduced density matrix corresponding to the exact time evolution to the projector onto the solution of the corresponding nonlinear Schrödinger equation in trace norm. For the latter potential V_N we show that it is crucial to take the microscopic structure of the condensate into account in order to obtain the correct dynamics.

Contributions of the author and Acknowledgements This chapter is a complete copy of a current version of [47]. The paper is joint work with Maximilian Jeblick and Prof. Dr. Peter Pickl. Most of the writing were done by Maximilian Jeblick and Prof. Dr. Peter Pickl. My own contribution to the paper consists of providing technical estimates and a revision of the proof. My share on the originality of the paper is about 25%. We are grateful to Dr. David Mitrouskas for many valuable discussions and would like to thank Dr. Serena Cenatiempo and Phillip Grass for helpful remarks. N.L. gratefully acknowledges financial support from the Cusanuswerk. M.J. gratefully acknowledges financial support from the German National Academic Foundation.

4.1 Introduction

This paper deals with the effective dynamics of a two dimensional condensate of N interacting bosons. Fundamentally, the evolution of the system is described by a time-dependent wave-function $\Psi_t \in L^2_s(\mathbb{R}^{2N}, \mathbb{C}), \|\Psi_t\| = 1$ (Here and below norms without index $\|\cdot\|$ always denote the L^2-norm on the appropriate Hilbert space.). $L^2_s(\mathbb{R}^{2N}, \mathbb{C})$ denotes the set of all $\Psi \in L^2(\mathbb{R}^{2N}, \mathbb{C})$ which are symmetric under pairwise permutations of the variables $x_1, \ldots, x_N \in \mathbb{R}^2$. Assuming that $\Psi_t \in H^2(\mathbb{R}^{2N}, \mathbb{C})$ holds, Ψ_t then solves the N-particle Schrödinger equation

$$\mathrm{i}\partial_t \Psi_t = H_U \Psi_t \tag{4.1}$$

where the (non-relativistic) Hamiltonian $H_U : H^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ is given by

$$H_U = -\sum_{j=1}^{N} \Delta_j + \sum_{1 \le j < k \le N} U(x_j - x_k) + \sum_{j=1}^{N} A_t(x_j) \ . \tag{4.2}$$

In general, even for small particle numbers N, (4.1) cannot be solved neither exactly nor numerically for Ψ_t. Nevertheless, for a certain class of scaled potentials U and certain initial conditions Ψ_0 it is possible to derive an approximate solution of (4.1) in the trace class topology of reduced density matrices. The picture we have in mind is the description of a Bose-Einstein condensate. Initially one starts with the ground state of a trapped, dilute gas and then removes or changes the trap subsequently. In this paper, we will consider two choices for the interaction potential U.

- Let $U(x) = V_N(x) = e^{2N} V(e^N x)$ for a compactly supported, spherically symmetric and nonnegative potential $V \in L^\infty_c(\mathbb{R}^2, \mathbb{R})$. Below, the exponential scaling of V_N will be explained in detail. Note that, in contrast to existing dynamical mean-field results, $\|V_N\|_1 = \mathcal{O}(1)$ does not decay like $1/N$.
- Let, for any fixed $\beta > 0$, $U(x) = W_\beta(x) = N^{-1+2\beta} W(N^\beta x)$ for a compactly supported, spherically symmetric and nonnegative potential $W \in L^\infty_c(\mathbb{R}^2, \mathbb{R})$. This scaling can be motivated by formally imposing that the total potential energy is of the same order as the total kinetic energy, namely of order N, if Ψ_0 is close to the ground state.

Define the one particle reduced density matrix $\gamma^{(1)}_{\Psi_0}$ of Ψ_0 with integral kernel

$$\gamma^{(1)}_{\Psi_0}(x, x') = \int_{\mathbb{R}^{2N-2}} \Psi_0^*(x, x_2, \dots, x_N) \Psi_0(x', x_2, \dots, x_N) d^2x_2 \dots d^2x_N \ .$$

To account for the physical situation of a Bose-Einstein condensate, we assume complete condensation in the limit of large particle number N. This amounts to assume that, for $N \to \infty$, $\gamma^{(1)}_{\Psi_0} \to |\varphi_0\rangle\langle\varphi_0|$ in trace norm for some $\varphi_0 \in L^2(\mathbb{R}^2, \mathbb{C})$, $\|\varphi_0\| = 1$. Our main goal is to show the persistence of condensation over time. This is of particular interest in experiments if one switches off the trapping potential A_t and monitors the expansion of the condensate. We prove that the time evolved reduced density matrix $\gamma^{(1)}_{\Psi_t}$ converges to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm as $N \to \infty$ with convergence rate of order $N^{-\eta}$ for some $\eta > 0$. φ_t then solves the nonlinear Schrödinger equation

$$\mathrm{i}\partial_t \varphi_t = (-\Delta + A_t)\, \varphi_t + b_U |\varphi_t|^2 \varphi_t =: h^{GP}_{b_U}\, \varphi_t \tag{4.3}$$

with initial datum φ_0. Depending on the interaction potential U, we obtain different coupling constants b_U. For $U = W_\beta$, we obtain $b_{W_\beta} = N\|W_\beta\|_1 = \|W\|_1$. This result is already expected from a heuristic law of large numbers argument, see below. In the case $U = V_N$, we have $b_{V_N} = 4\pi$. We like to remark that it is well known that convergence of $\gamma^{(1)}_{\Psi_t}$ to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm is equivalent to the respective convergence in operator norm since $|\varphi_t\rangle\langle\varphi_t|$ is a rank-1-projection, see Remark 1.4. in [80]. Furthermore, the convergence of the one-particle reduced density matrix $\gamma^{(1)}_{\Psi_t} \to |\varphi_t\rangle\langle\varphi_t|$ in trace norm implies convergence of any k-particle reduced density matrix $\gamma^{(k)}_{\Psi_t}$ against $|\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|$ in trace norm as $N \to \infty$ and k fixed, see for example [53].

In the case that the time evolution of Ψ_t is generated by H_{V_N} it is interesting to note that the effective evolution equation of φ_t does not depend on the scattering length a.

This contrasts the three dimensional case, where the correct mean field coupling is given by $8\pi a_{3D}$, a_{3D} denoting the scattering length of the potential in three dimensions. The universal coupling 4π in the case of a positive scattering length is known within the physical literature, see e.g. (30) and (A3) in [22] (note that $\hbar = 1, m = \frac{1}{2}$ in our choice of coordinates). Actually, our dynamical result complements a more general theory describing the ground state properties of dilute Bose gases. It was shown in [60] that for such a gas with repulsive interaction $V \geq 0$, the ground state energy per particle is to leading order given by either the Gross-Pitaevskii energy functional with coupling parameter $8\pi/|\ln(\overline{\rho}a^2)|$ or a Thomas-Fermi type functional. Here, $\overline{\rho}$ denotes the mean density of the gas, see Equation (1.6) in [60] for a precise definition. The authors prove further that only if $N/|\ln(\overline{\rho}a^2)| = \mathcal{O}(1)$ holds, one obtains the Gross-Pitaevskii regime. This directly implies that scattering length of the interaction potential needs to have an exponential decrease in N. In our case, the scattering length of the potential V_N is given by ae^{-N}, a denoting the scattering length of V. The mean density of the system we consider is of order one, i.e. $\overline{\rho} = \mathcal{O}(1)$. This yields $8\pi N/|\ln(\overline{\rho}(e^{-N}a)^2)| \approx 4\pi$ which is in agreement with our findings. It should be pointed out that there has been some debate about the question whether two dimensional Bose-Einstein condensation can be observed experimentally. This amounts to the question whether condensation takes place for temperatures $T > 0$. For an ideal, noninteracting gas in box, the standard grand canonical computation for the critical temperature T_c of a Bose-Einstein condensate shows that there is no condensation for $T > 0$. For trapped, noninteracting Bosons in a confining power-law potential, the findings in [7] however show that in that case $T_c > 0$ holds. Finally, it was proven in [57] that $\gamma_{\Psi}^{(1)}$ converges to $|\varphi\rangle\langle\varphi|$ in trace norm if Ψ the ground state of H_{V_N} and φ is the ground state of the Gross-Pitaevskii energy functional, see (4.5). The assumptions made in the paper are that and the external potential A tends to $+\infty$ as $|x| \to \infty$ and the interaction potential V is nonnegative. It is also remarked that one does not observe 100 % condensation in the ground state of a interacting homogenous system. The emergence of 100 % Bose-Einstein condensation as a ground state phenomena thus highly depends on the particular physical system one considers. Our approach is the following: Initially, we assume the convergence of $\gamma_{\Psi_0}^{(1)}$ to $|\varphi_0\rangle\langle\varphi_0|$. We then show the persistence this condensation for time scales of order one. Our assumption is thus in agreement with the findings in [57]. We like to remark that the two dimensional Thomas-Fermi regime could be observed experimentally [41].

Next, we want to explain how the different coupling constants b_U are obtained in the dynamical setting. For this, we first recall known results from the three dimensional Bose gas. There, one considers the interaction potential to be given by $V_\beta(x) = N^{-1+3\beta}V(N^\beta x)$ for $0 \leq \beta \leq 1$. For $0 < \beta < 1$, one obtains the cubic nonlinear Schrödinger equation with coupling constant $\|V\|_1$. This can be seen as a singular mean-field limit, where the full interaction is replaced by its corresponding mean value $\int_{\mathbb{R}^3} d^3y N^{3\beta}V(N^\beta(x-y))|\varphi_t(y)|^2 \to \|V\|_1|\varphi_t(x)|^2$. For $\beta = 1$, however, the system develops correlations between the particles which cannot be neglected. As already mentioned, the correct mean field coupling is then given by $8\pi a_{3D}$. This is different for a two dimensional condensate. Let us first explain, why the short scale correlation structure is negligible if the potential is given by $W_\beta(x) = N^{-1+2\beta}W(N^\beta x)$ for any $\beta > 0$. Assuming that the energy of Ψ_t is comparable to the ground state energy, the wave function will develop short scale correlations between the particles. One may heuristically think of Ψ_t of Jastrow-type, i.e. $\Psi_t(x_1, \ldots, x_N) \approx \prod_{i<j} F(x_i - x_j) \prod_{k=1}^N \varphi_t(x_k)$.[1] The

[1] One should however note that Ψ_t will not be close to a full product $\prod_{k=1}^N \varphi_t(x_k)$ in norm. For certain types of interactions, it has been shown rigorously that Ψ_t can be approximated by a quasifree state satisfying a Bogoloubov-type dynamics, see [17], [73], [74] and [67] for precise statements.

function F accounts for the pair correlations between the particles at short scales of order $N^{-\beta}$. It is well known that the correlation function F should be described by the zero energy scattering state $j_{N,R}$ of the potential W_β, where $j_{N,R}$ satisfies

$$\begin{cases} \left(-\Delta_x + \frac{1}{2} W_\beta(x)\right) j_{N,R}(x) = 0, \\ j_{N,R}(x) = 1 \text{ for } |x| = R. \end{cases}$$

Here, the boundary radius R is chosen of order $N^{-\beta}$. That is, $F(x_i - x_j) \approx j_{N,R}(x_i - x_j)$ for $|x_i - x_j| = \mathcal{O}(N^{-\beta})$ and $F(x_i - x_j) = 1$ for $|x_i - x_j| \gg \mathcal{O}(N^{-\beta})$. Rescaling to coordinates $y = N^\beta x$, the zero energy scattering state satisfies

$$\left(-\Delta_y + \frac{1}{2} N^{-1} W(y)\right) j_{N,N^\beta R}(y) = 0 \ . \tag{4.4}$$

Due to the factor N^{-1} in front of W, the zero energy scattering equation is almost constant, that is $j_{N,R}(x) \approx 1$, for all $|x| \leq R$. As a consequence, the microscopic structure F, induced by the zero energy scattering state, vanishes for any $\beta > 0$ and does not effect the dynamics of the reduced density matrix $\gamma^{(1)}_{\Psi_t}$. Assuming $\gamma^{(1)}_{\Psi_0} \approx |\varphi_0\rangle\langle\varphi_0|$, one may thus apply a law of large numbers argument and conclude that the interaction on each particle is then approximately given by its mean value

$$\int_{\mathbb{R}^2} d^2y N W_\beta(x-y)|\varphi_t|^2(y) \to \|W\|_1 |\varphi_t|^2(x) \ .$$

This yields to the correct coupling in the effective equation (4.3) in the case $U(x) = W_\beta(x)$. Let us now consider the case for which the dynamics of Ψ_t is generated by the Hamiltonian H_{V_N}. If one would guess the effective coupling of φ_t to be also given by its mean value w.r.t. the distribution $|\varphi_t|^2$, one would end up with the N-dependent equation $\mathrm{i}\partial_t \varphi_t = (-\Delta + A_t)\varphi_t + N \int_{\mathbb{R}^2} d^2x V(x)|\varphi_t|^2 \varphi_t$. Note that the coupling constant of the self interaction differs from its correct value by a factor of $\mathcal{O}(N)$. As in the three dimensional Gross-Pitaevskii regime $\beta = 1$, it is now important to take the correlations explicitly into account. The scaling of the potential yields to $j_{N,R}(x) = j_{0,e^N R}(e^N x)$, which implies that the correlation function will influence the dynamics whenever two particles collide. The coupling parameter can then be inferred from the relation

$$\int_{\mathbb{R}^2} d^2x V_N(x) j_{N,R}(x) = \frac{4\pi}{\ln\left(\frac{R}{ae^{-N}}\right)},$$

where a denotes the scattering length of the potential V. As mentioned, the logarithmic dependence of the integral above on a is special in two dimensions. Since $\frac{4\pi}{\ln\left(\frac{R}{ae^{-N}}\right)} \approx \frac{4\pi}{N}$ holds for $a > 0$, the effective equation for φ_t will not depend on a anymore. Consequently, one obtains as an effective coupling

$$\int_{\mathbb{R}^2} d^2y N V_N(x-y) j_{N,R}(x-y) |\varphi_t|^2(y) \to 4\pi |\varphi_t|^2(x).$$

We like to remark that it is easy to verify that, for any $s > 0$, the potential $V_{sN}(x) = e^{2Ns} V(e^{Ns}x)$ yields to an effective coupling $4\pi/s$. For the sake of simplicity, we will not consider this slight generalization, although our proof is also valid in this case.
The rigorous derivation of effective evolution equations is well known in the literature, see e.g. [17, 12, 27, 28, 29, 30, 53, 73, 74, 75, 76, 77, 80] and references therein. For the

two-dimensional case we consider, it has been proven, for $0 < \beta < 3/4$ and W nonnegative, that $\gamma^{(1)}_{\Psi_t}$ converges to $|\varphi_t\rangle\langle\varphi_t|$ as $N \to \infty$ [48]. For $0 < \beta < 1/6$, it has been established in [20] that the reduced density matrices converge, assuming that the potential W is attractive, i.e. $W \leq 0$. This result was later extended to $0 < \beta < 3/4$, using stability properties of the ground state energy [55].
Another approach which relates more closely to the experimental setup is to consider a three-dimensional gas of Bosons which is strongly confined in one spatial dimension. Then, one obtains an effective two dimensional system in the unconfined directions. We remark that in this dimensional reduction two limits appear, the length scale in the confined direction and the scaling of the interaction in the unconfined directions. Results in this direction can be found in [9] and [49], see also [50]. It is still an open problem to derive our dynamical result starting from a strongly confined three dimensional system. For known results regarding the ground state properties of dilute Bose gases, we refer to the monograph [59], which also summarizes the papers [57], [60] and[62].
Our proof is based on [77], where the emergence of the Gross-Pitaevskii equation was proven by one of us (P.P.) in three dimensions for $\beta = 1$. In particular, we adapt some crucial ideas which allow us to control the microscopic structure of Ψ_t.
We shall shortly discuss the physical relevance of the different scalings. On the first view, the interactions discussed above do look rather unphysical. It is questionable to assume that the coupling constant and/or the range of the interaction change as the particle number increases. Nevertheless, one can think of situations, where for example the support of the interaction is small and the particle number of the system is adjusted accordingly.
The exponential scaling $V_N(x) = e^{2N}V(e^N x)$ is special. In this case it is possible to rescale space- and time-coordinates in such a way that in the new coordinates the interaction is *not* N dependent. Choosing $y = e^N x$ and $\tau = e^{2N}t$ the Schrödinger equation reads

$$\mathrm{i}\frac{d}{d\tau}\Psi_{e^{-2N}\tau} = \Big(-\sum_{j=1}^{N}\Delta_{y_j} + \sum_{1\leq j<k\leq N} V(y_j - y_k) + \sum_{j=1}^{N} A_{e^{-2N}\tau}(e^{-N}y_j)\Big)\Psi_{e^{-2N}\tau} \, .$$

The latter equation thus corresponds to an extremely dilute gas of bosons with density $\sim e^{-2N}$. In order to observe a nontrivial dynamics, this condensate is then monitored over time scales of order $\tau \sim e^{2N}$. Since the trapping potential is adjusted according to the density of the gas in the experiment, the N dependence of $A_{e^{-2N}\tau}(e^{-N}\cdot)$ is reasonable.

4.2 Main result

For the sake of simplicity we will bound expressions which are uniformly bounded in N and t by some constant C. We will not distinguish constants appearing in a sequence of estimates, i.e. in $X \leq CY \leq CZ$ the constants may differ.
For $U \in \{W_\beta, V_N\}$, define the energy functional $\mathcal{E}_U : H^1(\mathbb{R}^{2N}, \mathbb{C}) \to \mathbb{R}$

$$\mathcal{E}_U(\Psi) = N^{-1}\langle\!\langle \Psi, H_U \Psi \rangle\!\rangle \, ,$$

where $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ denotes the scalar product on $L^2(\mathbb{R}^{2N}, \mathbb{C})$. Furthermore, we define the Gross-Pitaevskii energy functional $\mathcal{E}^{GP}_{b_U} : H^1(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}$

$$\mathcal{E}^{GP}_{b_U}(\varphi) = \langle \nabla\varphi, \nabla\varphi \rangle + \langle \varphi, (A_t + \frac{1}{2}b_U|\varphi|^2)\varphi \rangle = \langle \varphi, (h^{GP}_{b_U} - \frac{1}{2}b_U|\varphi|^2)\varphi \rangle \tag{4.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product on $L^2(\mathbb{R}^2,\mathbb{C})$. Note that both $\mathcal{E}_U(\Psi)$ and $\mathcal{E}_{b_U}^{GP}(\varphi)$ depend on t, due to the time varying external potential A_t. For the sake of readability, we will not indicate this time dependence explicitly. We now state our main Theorem:

Theorem 4.2.1. *Let $\Psi_0 \in L_s^2(\mathbb{R}^{2N},\mathbb{C}) \cap H^2(\mathbb{R}^{2N},\mathbb{C})$ with $\|\Psi_0\| = 1$. Let $\varphi_0 \in L^2(\mathbb{R}^2,\mathbb{C})$ with $\|\varphi_0\| = 1$ and assume $\lim_{N\to\infty}\gamma_{\Psi_0}^{(1)} = |\varphi_0\rangle\langle\varphi_0|$ in trace norm. Let the external potential A_t, which is defined in (4.2), satisfy $A_t \in C^1(\mathbb{R}, L^\infty(\mathbb{R}^2,\mathbb{R}))$.*

(a) For any $\beta > 0$, let W_β be given by $W_\beta(x) = N^{-1+2\beta}W(N^\beta x)$, for $W \in L_c^\infty(\mathbb{R}^2,\mathbb{R})$, $W \geq 0$ and W spherically symmetric. Let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{W_\beta}\Psi_t$ with initial datum Ψ_0. Let φ_t the unique solution to $i\partial_t\varphi_t = h_{\|W\|_1}^{GP}\varphi_t$ with initial datum φ_0 and assume that $\varphi_t \in H^3(\mathbb{R}^2,\mathbb{C})$. Let $\lim_{N\to\infty}\left(\mathcal{E}_{W_\beta}(\Psi_0) - \mathcal{E}_{\|W\|_1}^{GP}(\varphi_0)\right) = 0$. Then, for any $\beta > 0$ and for any $t > 0$

$$\lim_{N\to\infty}\gamma_{\Psi_t}^{(1)} = |\varphi_t\rangle\langle\varphi_t| \tag{4.6}$$

in trace norm.

(b) Let V_N be given by $V_N(x) = e^{2N}V(e^N x)$, for $V \in L_c^\infty(\mathbb{R}^2,\mathbb{R})$, $V \geq 0$ and V spherically symmetric. Let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{V_N}\Psi_t$ with initial datum Ψ_0. Let φ_t the unique solution to $i\partial_t\varphi_t = h_{4\pi}^{GP}\varphi_t$ with initial datum φ_0 and assume that $\varphi_t \in H^3(\mathbb{R}^2,\mathbb{C})$. Let $\lim_{N\to\infty}\left(\mathcal{E}_{V_N}(\Psi_0) - \mathcal{E}_{4\pi}^{GP}(\varphi_0)\right) = 0$.

Then, for any $t > 0$

$$\lim_{N\to\infty}\gamma_{\Psi_t}^{(1)} = |\varphi_t\rangle\langle\varphi_t| \tag{4.7}$$

in trace norm.

Remark:

(a) We expect that for regular enough external potentials A_t, the regularity assumption $\varphi_t \in H^3(\mathbb{R}^2,\mathbb{C})$ to follow from regularity assumptions on the initial datum φ_0. In particular, if $\varphi_0 \in \Sigma^3(\mathbb{R}^2,\mathbb{C}) = \{f \in L^2(\mathbb{R}^2,\mathbb{C}) | \sum_{\alpha+\beta\leq 3}\|x^\alpha\partial_x^\beta f\| < \infty\}$ holds, the bound $\|\varphi_t\|_{H^3} < \infty$ has been proven for external potentials which are at most quadratic in space, see [19] and Lemma 4.4.7. In particular, for $\varphi_0 \in \Sigma^3(\mathbb{R}^2,\mathbb{C})$, the bound $\|\varphi_t\|_{H^3} \leq C$ holds if the external potential is not present, i.e. $A_t = 0$.

(b) As already mentioned, the convergence of $\gamma_{\Psi_t}^{(1)}$ to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm is equivalent to convergence in operator norm, since $|\varphi_t\rangle\langle\varphi_t|$ is a rank one projection [80]. Other equivalent definitions of asymptotic 100% condensation can be found in [65].

(c) In our proof we will give explicit error estimates in terms of the particle number N. We shall show that the rate of convergence is of order $N^{-\delta}$ for some $\delta > 0$, assuming that also initially $\gamma_{\Psi_0}^{(1)} \to |\varphi_0\rangle\langle\varphi_0|$ converges in trace norm with rate of at least $N^{-\delta}$.

(d) One can relax the conditions on the initial condition and only require $\Psi_0 \in L_s^2(\mathbb{R}^{2N},\mathbb{C})$ using a standard density argument.

(e) It has been shown that in the limit $N \to \infty$ the energy-difference $\mathcal{E}_{V_N}(\Psi^{gs}) - \mathcal{E}_{4\pi}^{GP}(\varphi^{gs}) \to 0$, where Ψ^{gs} is the ground state of a trapped Bose gas and φ^{gs} the ground state of the respective Gross-Pitaevskii energy functional, see [60], [62].

4.3 Organization of the proof

The method we use in this paper is introduced in detail in [75] and was generalized to derive various mean-field equations. As we have mentioned, our proof is based on [77], which covers the three-dimensional counterpart of our system. Heuristically speaking, the method we are going to employ is based on the idea of counting for each time t the relative number of those particles which are not in the state φ_t. It is then possible to show that the rate of particles which leave the condensate is small, if initially almost all particles are in the state φ_0. In order to compare the exact dynamic, generated by H_U, with the effective dynamic, generated by $h^{GP}_{b_U}$, we define the projectors p_j^φ and q_j^φ.

Definition 4.3.1. *Let $\varphi \in L^2(\mathbb{R}^2, \mathbb{C})$ with $\|\varphi\| = 1$.*

(a) For any $1 \leq j \leq N$ the projectors $p_j^\varphi : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ and $q_j^\varphi : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ are defined as

$$p_j^\varphi \Psi = \varphi(x_j) \int \varphi^*(\tilde{x}_j)\Psi(x_1, \dots, \tilde{x}_j, \dots, x_N) d^2\tilde{x}_j \quad \forall\, \Psi \in L^2(\mathbb{R}^{2N}, \mathbb{C})$$

and $q_j^\varphi = 1 - p_j^\varphi$.
We shall also use, with a slight abuse of notation, the bra-ket notation $p_j^\varphi = |\varphi(x_j)\rangle\langle\varphi(x_j)|$.

(b) For any $0 \leq k \leq N$ we define the set

$$\mathcal{S}_k = \left\{ (s_1, s_2, \dots, s_N) \in \{0,1\}^N \; ; \; \sum_{j=1}^N s_j = k \right\}$$

and the orthogonal projector $P_k^\varphi : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ as

$$P_k^\varphi = \sum_{\vec{a} \in \mathcal{S}_k} \prod_{j=1}^N \left(p_j^\varphi\right)^{1-s_j} \left(q_j^\varphi\right)^{s_j} .$$

For negative k and $k > N$ we set $P_k^\varphi = 0$.

(c) For any function $m : \mathbb{N}_0 \to \mathbb{R}_0^+$ we define the operator $\widehat{m}^\varphi : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ as

$$\widehat{m}^\varphi = \sum_{j=0}^N m(j) P_j^\varphi . \tag{4.8}$$

We also need the shifted operators $\widehat{m}_d^\varphi : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$ given by

$$\widehat{m}_d^\varphi = \sum_{j=-d}^{N-d} m(j+d) P_j^\varphi .$$

Following a general strategy, which is described in detail in [75], we define a functional $\alpha : L^2(\mathbb{R}^{2N}, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}_0^+$ such that

(a) $\frac{\partial}{\partial t}\alpha(\Psi_t, \varphi_t)$ can be estimated by $\alpha(\Psi_t, \varphi_t) + \mathcal{O}(1)$. Using a Grönwall type estimate, it then follows that $\alpha(\Psi_t, \varphi_t) \leq C_1 e^{C_2 t}(\alpha(\Psi_0, \varphi_0) + \mathcal{O}(1))$, for some constants $C_1, C_2 > 0$.

(b) $\alpha(\Psi,\varphi) \to 0$ implies convergence of the reduced one particle density matrix of Ψ to $|\varphi\rangle\langle\varphi|$ in trace norm.

In the case $\beta = 0$ it was shown that the choice

$$\alpha(\Psi,\varphi) = \left\langle\!\!\left\langle \Psi, (\widehat{n}^{\varphi})^j \, \Psi \right\rangle\!\!\right\rangle \;,$$

where $n(k) = \sqrt{k/N}$ and $\langle\!\langle \cdot \rangle\!\rangle$ is scalar product on $L^2(\mathbb{R}^{2N},\mathbb{C})$ fulfills these requirements, for arbitrary $j > 0$, see for example [75] and [53]. For the more involved scaling we consider, it is however necessary to adjust this definition in order to obtain a Grönwall type estimate. Our proof is organized as follows:

(a) In Section 4.4 we recall some important properties of the operator $\widehat{m}$.

(b) For the most difficult scaling given by the potential V_N, it is crucial to take the interaction-induced correlations between the particles into account. In Section 4.5 we provide some estimates on the zero-energy scattering state. Furthermore, we explain how the effective coupling parameter b_{V_N} can be inferred from the microscopic structure.

(c) In Section 4.6 we prove our main Theorem stated above. We first consider the potential W_β and define a counting measure which allows us to establish a Grönwall estimate for all $\beta > 0$. We will explain in detail how one arrives at this Grönwall estimate.

Afterwards, the counting measure is adjusted to the case V_N, taking the microscopic structure $j_{N,R}$ of the wave function into account. We then establish a Grönwall estimate and finally prove the Theorem for V_N.

The needed estimates in Section 4.6 are then proven in Section 4.7.

4.4 Preliminaries

We will first fix the notation we are going to employ during the rest of this chapter.

Notation 4.4.1. *(a) Throughout this chapter hats $\widehat{\cdot}$ will always be used in the sense of Definition 4.3.1 (c). The label n will always be used for the function $n(k) = \sqrt{k/N}$.*

(b) For better readability, we will omit the upper index φ on p_j, q_j, P_j, $P_{j,k}$ and $\widehat{\cdot}$. It will be placed exclusively in a few formulas where their φ-dependence plays an important role.

(c) We will denote the operator norm defined for any linear operator $f : L^2(\mathbb{R}^{2N},\mathbb{C}) \to L^2(\mathbb{R}^{2N},\mathbb{C})$ by

$$\|f\|_{op} = \sup_{\psi \in L^2(\mathbb{R}^{2N},\mathbb{C}), \|\Psi\|=1} \|f\Psi\| \;.$$

(d) We will denote by $\mathcal{K}(\varphi_t, A_t)$ a generic polynomial with finite degree in $\|\varphi_t\|_\infty$, $\|\nabla\varphi_t\|_\infty$, $\|\nabla\varphi_t\|$, $\|\Delta\varphi_t\|$, $\|A_t\|_\infty$, $\int_0^t ds \|\dot{A}_s\|_\infty$ and $\|\dot{A}_t\|_\infty$. Note, in particular, that for a generic constant C the inequality $C \le \mathcal{K}(\varphi_t, A_t)$ holds. The exact form of $\mathcal{K}(\varphi_t, A_t)$ which appears in the final bounds can be reconstructed, collecting all contributions from the different estimates.

(e) *We will denote for any multiplication operator $F : L^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C})$ the corresponding operator*

$$\mathbb{1}^{\otimes(k-1)} \otimes F \otimes \mathbb{1}^{\otimes(N-k)} : L^2(\mathbb{R}^{2N}, \mathbb{C}) \to L^2(\mathbb{R}^{2N}, \mathbb{C})$$

acting on the N-particle Hilbert space by $F(x_k)$. In particular, we will use, for any $\Psi, \Omega \in L^2(\mathbb{R}^{2N}, \mathbb{C})$ the notation

$$\langle\!\langle \Omega, \mathbb{1}^{\otimes(k-1)} \otimes F \otimes \mathbb{1}^{\otimes(N-k)} \Psi \rangle\!\rangle = \langle\!\langle \Omega, F(x_k)\Psi \rangle\!\rangle \; .$$

In analogy, for any two-particle multiplication operator $K : L^2(\mathbb{R}^2, \mathbb{C})^{\otimes 2} \to L^2(\mathbb{R}^2, \mathbb{C})^{\otimes 2}$, we denote the operator acting on any $\Psi \in L^2(\mathbb{R}^{2N}, \mathbb{C})$ by multiplication in the variable x_i and x_j by $K(x_i, x_j)$. In particular, we denote

$$\langle\!\langle \Omega, K(x_i, x_j)\Psi \rangle\!\rangle = \int_{\mathbb{R}^{2N}} K(x_i, x_j)\Omega^*(x_1, \ldots, x_N)\Psi(x_1, \ldots, x_N)d^2x_1 \ldots d^2x_N \; .$$

First we prove some properties of the projectors p_j, q_j, which were defined in Definition 4.3.1.

Lemma 4.4.2. (a) *For any weights $m, r : \mathbb{N}_0 \to \mathbb{R}_0^+$ the commutation relations*

$$\widehat{m}\widehat{r} = \widehat{mr} = \widehat{r}\,\widehat{m} \qquad \widehat{m}p_j = p_j\widehat{m} \qquad \widehat{m}q_j = q_j\widehat{m} \qquad \widehat{m}P_k = P_k\widehat{m}$$

hold.

(b) *Let $n : \mathbb{N}_0 \to \mathbb{R}_0^+$ be given by $n(k) = \sqrt{k/N}$. Then, the square of $\widehat{n}$ equals the relative particle number operator of particles not in the state φ, i.e.*

$$(\widehat{n})^2 = N^{-1} \sum_{j=1}^{N} q_j \; . \tag{4.9}$$

(c) *For any weight $m : \mathbb{N}_0 \to \mathbb{R}_0^+$ and any function $f \in L^\infty\left(\mathbb{R}^4, \mathbb{R}\right)$ and any $j, k = 0, 1, 2$*

$$\widehat{m}Q_j f(x_1, x_2)Q_k = Q_j f(x_1, x_2)\widehat{m}_{j-k}Q_k \; ,$$

where $Q_0 = p_1p_2$, $Q_1 \in \{p_1q_2, q_1p_2\}$ and $Q_2 = q_1q_2$. Furthermore, for $j, k \in \{0, 1\}$

$$\widehat{m}\widetilde{Q}_j\nabla_1\widetilde{Q}_k = \widetilde{Q}_j\nabla_1\widehat{m}_{j-k}\widetilde{Q}_k \; ,$$

where $\widetilde{Q}_0 = p_1$ and $\widetilde{Q}_1 = q_1$.

(d) *For any weight $m : \mathbb{N}_0 \to \mathbb{R}_0^+$ and any function $f \in L^\infty\left(\mathbb{R}^4, \mathbb{C}\right)$*

$$[f(x_1, x_2), \widehat{m}] = [f(x_1, x_2), p_1p_2(\widehat{m} - \widehat{m}_2) + (p_1q_2 + q_1p_2)(\widehat{m} - \widehat{m}_1)] \; .$$

(e) *Let $f \in L^1\left(\mathbb{R}^2, \mathbb{C}\right)$, $g \in L^2\left(\mathbb{R}^2, \mathbb{C}\right)$. Then,*

$$\|p_jf(x_j - x_k)p_j\|_{op} \leq \|f\|_1\|\varphi\|_\infty^2 \; , \tag{4.10}$$
$$\|p_jg^*(x_j - x_k)\|_{op} = \|g(x_j - x_k)p_j\|_{op} \leq \|g\| \; \|\varphi\|_\infty \tag{4.11}$$
$$\||\varphi(x_j)\rangle\langle\nabla_j\varphi(x_j)|h^*(x_j - x_k)\|_{op} = \|h(x_j - x_k)\nabla_jp_j\|_{op} \leq \|h\|\|\nabla\varphi\|_\infty \; . \tag{4.12}$$

Proof. (a) follows immediately from Definition 4.3.1, using that p_j and q_j are orthogonal projectors.

(b) Note that $\cup_{k=0}^{N}\mathcal{S}_k = \{0,1\}^N$, so $1 = \sum_{k=0}^{N} P_k$. Using also $(q_j)^2 = q_j$ and $q_j p_j = 0$ we get

$$\sum_{j=1}^{N} q_j = \sum_{j=1}^{N} q_j \sum_{k=0}^{N} P_k = \sum_{k=0}^{N}\sum_{j=1}^{N} q_j P_k = \sum_{k=0}^{N} k P_k = N\widehat{n^2} = N\widehat{n}^2 \ .$$

(c) Using the definitions above we have

$$\widehat{m} Q_j f(x_1,x_2) Q_k = \sum_{l=0}^{N} m(l) P_l Q_j f(x_1,x_2) Q_k \ .$$

The number of projectors q_j in $P_l Q_j$ in the coordinates $j = 3,\dots,N$ is equal to $l-j$. The p_j and q_j with $j = 3,\dots,N$ commute with $Q_j f(x_1,x_2) Q_k$. Thus $P_l Q_j f(x_1,x_2) Q_k = Q_j f(x_1,x_2) Q_k P_{l-j+k}$ and

$$\begin{aligned}
\widehat{m} Q_j f(x_1,x_2) Q_k =& \sum_{l=0}^{N} m(l) Q_j f(x_1,x_2) Q_k P_{l-j+k} \\
=& \sum_{\widetilde{l}=k-j}^{N+k-j} Q_j f(x_1,x_2) m(\widetilde{l}+j-k) P_{\widetilde{l}} Q_k = Q_j f(x_1,x_2) \widehat{m}_{j-k} Q_k \ .
\end{aligned}$$

Similarly one gets the second formula.

(d) First note that

$$\begin{aligned}
&[f(x_1,x_2),\widehat{m}] - [f(x_1,x_2), p_1p_2(\widehat{m}-\widehat{m}_2) + p_1q_2(\widehat{m}-\widehat{m}_1) + q_1p_2(\widehat{m}-\widehat{m}_1)] \\
&=[f(x_1,x_2), q_1q_2\widehat{m}] + [f(x_1,x_2), p_1p_2\widehat{m}_2 + p_1q_2\widehat{m}_1 + q_1p_2\widehat{m}_1] \ . \qquad (4.13)
\end{aligned}$$

We will show that the right hand side is zero. Multiplying the right hand side with p_1p_2 from the left and using (c) one gets

$$\begin{aligned}
&p_1p_2 f(x_1,x_2) q_1q_2\widehat{m} + p_1p_2 f(x_1,x_2) p_1p_2\widehat{m}_2 - p_1p_2\widehat{m}_2 f(x_1,x_2) \\
&\quad + p_1p_2 f(x_1,x_2) p_1q_2\widehat{m}_1 + p_1p_2 f(x_1,x_2) q_1p_2\widehat{m}_1 \\
&=p_1p_2\widehat{m}_2 f(x_1,x_2) q_1q_2 + p_1p_2\widehat{m}_2 f(x_1,x_2) p_1p_2 - p_1p_2\widehat{m}_2 f(x_1,x_2) \\
&\quad + p_1p_2\widehat{m}_2 f(x_1,x_2) p_1q_2 + p_1p_2\widehat{m}_2 f(x_1,x_2) q_1p_2 \\
&\quad = 0 \ .
\end{aligned}$$

Multiplying (4.13) with p_1q_2 from the left one gets

$$\begin{aligned}
&p_1q_2 f(x_1,x_2) q_1q_2\widehat{m} + p_1q_2 f(x_1,x_2) p_1p_2\widehat{m}_2 + p_1q_2 f(x_1,x_2) p_1q_2\widehat{m}_1 \\
&+ p_1q_2 f(x_1,x_2) q_1p_2\widehat{m}_1 - p_1q_2\widehat{m}_1 f(x_1,x_2) \ .
\end{aligned}$$

Using (c) the latter is zero. Also multiplying with q_1p_2 yields zero due to symmetry in interchanging x_1 with x_2. Multiplying (4.13) with q_1q_2 from the left one gets

$$\begin{aligned}
&q_1q_2 f(x_1,x_2)\widehat{m} q_1q_2 - q_1q_2\widehat{m} f(x_1,x_2) + q_1q_2 f(x_1,x_2) p_1p_2\widehat{m}_2 + \\
&q_1q_2 f(x_1,x_2) p_1q_2\widehat{m}_1 + q_1q_2 f(x_1,x_2) q_1p_2\widehat{m}_1
\end{aligned}$$

which is again zero and so is (4.13).

(e) First note that, for bounded operators A, B, $\|AB\|_{\mathrm{op}} = \|B^* A^*\|_{\mathrm{op}}$ holds, where A^* is the adjoint operator of A. To show (4.10), note that

$$p_j f(x_j - x_k) p_j = p_j (f \star |\varphi|^2)(x_k) \ . \tag{4.14}$$

It follows that

$$\|p_j f(x_j - x_k) p_j\|_{\mathrm{op}} \leq \|f\|_1 \|\varphi\|_\infty^2 \ .$$

For (4.11) we write

$$\begin{aligned}\|g(x_j - x_k) p_j\|_{\mathrm{op}}^2 &= \sup_{\|\Psi\|=1} \|g(x_j - x_k) p_j \Psi\|^2 = \\ &= \sup_{\|\Psi\|=1} \langle\!\langle \Psi, p_j |g(x_j - x_k)|^2 p_j \Psi \rangle\!\rangle \\ &\leq \|p_j |g(x_j - x_k)|^2 p_j\|_{\mathrm{op}} \ .\end{aligned}$$

With (4.10) we get (4.11). For (4.12) we use

$$\begin{aligned}\|g(x_j - x_k) \nabla_j p_j\|_{\mathrm{op}}^2 = \sup_{\|\Psi\|=1} \langle\!\langle \Psi, p_j (|g|^2 * |\nabla\varphi|^2)(x_k) \Psi \rangle\!\rangle \leq \||g|^2 * |\nabla\varphi|^2\|_\infty \\ \leq \|g\|^2 \|\nabla\varphi\|_\infty^2\end{aligned}$$

□

Within our estimates we will encounter wave functions where some of the symmetry is broken (at this point the reader should exemplarily think of the wave function $V_\beta(x_1 - x_2)\Psi$ which is not symmetric under exchange of the variables x_1 and x_3, for example). This leads to the following definition

Definition 4.4.3. *For any finite set $\mathcal{M} \subset \{1, 2, \ldots, N\}$, define the space $\mathcal{H}_\mathcal{M} \subset L^2(\mathbb{R}^{2N}, \mathbb{C})$ as the set of functions which are symmetric in all variables in $\mathcal{M}$*

$$\begin{aligned}\Psi \in \mathcal{H}_\mathcal{M} \Leftrightarrow &\Psi(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_N) = \Psi(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_N) \\ &\textit{for all } j, k \in \mathcal{M} \ .\end{aligned}$$

Based on the combinatorics of the p_j and q_j, we obtain the following

Lemma 4.4.4. *For any $f : \mathbb{N}_0 \to \mathbb{R}_0^+$ and any finite set $\mathcal{M}_a \subset \{1, 2, \ldots, N\}$ with $1 \in \mathcal{M}_a$ and any finite set $\mathcal{M}_b \subset \{1, 2, \ldots, N\}$ with $1, 2 \in \mathcal{M}_b$*

$$\left\|\widehat{f} q_1 \Psi\right\|^2 \leq \frac{N}{|\mathcal{M}_a|} \|\widehat{f}\widehat{n}\Psi\|^2 \qquad \textit{for any } \Psi \in \mathcal{H}_{\mathcal{M}_a}, \tag{4.15}$$

$$\left\|\widehat{f} q_1 q_2 \Psi\right\|^2 \leq \frac{N^2}{|\mathcal{M}_b|(|\mathcal{M}_b| - 1)} \|\widehat{f}(\widehat{n})^2\Psi\|^2 \qquad \textit{for any } \Psi \in \mathcal{H}_{\mathcal{M}_b} \ . \tag{4.16}$$

Proof. Let $\Psi \in \mathcal{H}_{\mathcal{M}_a}$ for some finite set $1 \in \mathcal{M}_a \subset \{1, 2, \ldots, N\}$. By Lemma 4.4.2 (b), (4.15) can be estimated as

$$\begin{aligned}\|\widehat{f}\widehat{n}\Psi\|^2 &= \langle\!\langle \Psi, (\widehat{f})^2 (\widehat{n})^2 \Psi \rangle\!\rangle = N^{-1} \sum_{k=1}^N \langle\!\langle \Psi, (\widehat{f})^2 q_k \Psi \rangle\!\rangle \\ &\geq N^{-1} \sum_{k \in \mathcal{M}_a} \langle\!\langle \Psi, (\widehat{f})^2 q_k \Psi \rangle\!\rangle = \frac{|\mathcal{M}_a|}{N} \langle\!\langle \Psi, (\widehat{f})^2 q_1 \Psi \rangle\!\rangle \\ &= \frac{|\mathcal{M}_a|}{N} \|\widehat{f} q_1 \Psi\|^2 \ .\end{aligned}$$

Similarly, we obtain for $\Psi \in \mathcal{H}_{\mathcal{M}_b}$

$$\begin{aligned}\|\widehat{f}(\widehat{n})^2\Psi\|^2 =&\langle\!\langle \Psi, (\widehat{f})^2(\widehat{n})^4\Psi\rangle\!\rangle \geq N^{-2}\sum_{j,k\in\mathcal{M}_b}\langle\!\langle \Psi, (\widehat{f})^2 q_j q_k\Psi\rangle\!\rangle\\ =&\frac{|\mathcal{M}_b|(|\mathcal{M}_b|-1)}{N^2}\langle\!\langle \Psi, (\widehat{f})^2 q_1 q_2\Psi\rangle\!\rangle + \frac{|\mathcal{M}_b|}{N^2}\langle\!\langle \Psi, (\widehat{f})^2 q_1\Psi\rangle\!\rangle\\ \geq&\frac{|\mathcal{M}_b|(|\mathcal{M}_b|-1)}{N^2}\|\widehat{f}q_1q_2\Psi\|^2\end{aligned}$$

which concludes the Lemma. □

Corollary 4.4.5. *For any weight $m:\mathbb{N}_0\to\mathbb{R}_0^+$*

$$\|\nabla_2\widehat{m}q_2\Psi\| \leq 2\|\widehat{m}\|_{op}\|\nabla_2 q_2\Psi\|, \tag{4.17}$$

$$\|\nabla_2\widehat{m}q_1q_2\Psi\| \leq C\|\widehat{m}\widehat{n}\|_{op}\|\nabla_2 q_2\Psi\|\ . \tag{4.18}$$

Proof. Using $p_2+q_2=1$ and triangle inequality,

$$\|\nabla_2\widehat{m}q_2\Psi\| \leq \|p_2\nabla_2\widehat{m}q_2\Psi\| + \|q_2\nabla_2\widehat{m}q_2\Psi\|, \tag{4.19}$$

$$\|\nabla_2\widehat{m}q_1q_2\Psi\| \leq \|p_2\nabla_2\widehat{m}q_1q_2\Psi\| + \|q_2\nabla_2\widehat{m}q_1q_2\Psi\|\ . \tag{4.20}$$

With Lemma 4.4.2 (c) we get

$$(4.19) = \|\widehat{m}_1 p_2\nabla_2 q_2\Psi\| + \|\widehat{m}q_2\nabla_2 q_2\Psi\| \leq (\|\widehat{m}_1\|_{\mathrm{op}} + \|\widehat{m}\|_{\mathrm{op}})\|\nabla_2 q_2\Psi\|\ .$$

Note that the wave function $p_2\nabla_2 q_2\Psi$ is symmetric under the exchange of any two variables but x_2. Thus we can use Lemma 4.4.4 to get

$$\begin{aligned}(4.20) =& \|q_1\widehat{m}_1 p_2\nabla_2 q_2\Psi\| + \|q_1\widehat{m}q_2\nabla_2 q_2\Psi\|\\ \leq& \frac{N}{N-1}(\|\widehat{m}_1\widehat{n}\|_{\mathrm{op}} + \|\widehat{m}\widehat{n}\|_{\mathrm{op}})\|\nabla_2 q_2\Psi\|\ .\end{aligned}$$

Since $\sqrt{k}\leq\sqrt{k+1}$ for $k\geq 0$ it follows that the latter is bounded by

$$C(\|\widehat{m}_1\widehat{n}_1\|_{\mathrm{op}} + \|\widehat{m}\widehat{n}\|_{\mathrm{op}})\|\nabla_2 q_2\Psi\|\ .$$

Using that $\|\widehat{r}\|_{\mathrm{op}} = \sup_{0\leq k\leq N}\{r(k)\} = \|\widehat{r}_d\|_{\mathrm{op}}$ for any $d\in\mathbb{N}$ and any weight r, the Corollary follows. □

Lemma 4.4.6. *Let $\Omega,\chi\in\mathcal{H}_{\mathcal{M}}$ for some $\mathcal{M}$, let $1\notin\mathcal{M}$ and $2,3\in\mathcal{M}$. Let $O_{j,k}$ be an operator acting on the j^{th} and k^{th} coordinate. Then*

$$|\langle\!\langle \Omega, O_{1,2}\chi\rangle\!\rangle| \leq \|\Omega\|^2 + |\langle\!\langle O_{1,2}\chi, O_{1,3}\chi\rangle\!\rangle| + (|\mathcal{M}|)^{-1}\|O_{1,2}\chi\|^2\ .$$

Proof. Using symmetry and Cauchy Schwarz

$$|\langle\!\langle \Omega, O_{1,2}\chi\rangle\!\rangle| =|\mathcal{M}|^{-1}|\langle\!\langle \Omega, \sum_{j\in\mathcal{M}} O_{1,j}\chi\rangle\!\rangle| \leq |\mathcal{M}|^{-1}\|\Omega\|\ \|\sum_{j\in\mathcal{M}} O_{1,j}\chi\|\ .$$

For the second factor we can write

$$\begin{aligned}\|\sum_{j\in\mathcal{M}} O_{1,j}\chi\|^2 =& \langle\!\langle \sum_{j\in\mathcal{M}} O_{1,j}\chi, \sum_{k\in\mathcal{M}} O_{1,k}\chi\rangle\!\rangle\\ \leq& \sum_{j\in\mathcal{M}}|\langle\!\langle O_{1,j}\chi, O_{1,j}\chi\rangle\!\rangle| + |\sum_{j\neq k\in\mathcal{M}}\langle\!\langle O_{1,j}\chi, O_{1,k}\chi\rangle\!\rangle|\\ \leq& |\mathcal{M}||\langle\!\langle O_{1,2}\chi, O_{1,2}\chi\rangle\!\rangle| + |\mathcal{M}|(|\mathcal{M}|-1)|\langle\!\langle O_{1,2}\chi, O_{1,3}\chi\rangle\!\rangle|\ .\end{aligned}$$

Since $ab\leq 1/2a^2+1/2b^2$ and $(a+b)^2\leq 2a^2+2b^2$ holds for any real numbers a,b, the Lemma follows. □

In our estimates, we need the regularity conditions

$$\|\nabla\varphi_t\|_\infty < \infty, \qquad \|\varphi_t\|_\infty < \infty, \qquad \|\nabla\varphi_t\| < \infty, \qquad \|\Delta\varphi_t\| < \infty\ .$$

That is, we need $\varphi_t \in H^2(\mathbb{R}^2,\mathbb{C})\cap W^{1,\infty}(\mathbb{R}^2,\mathbb{C})$. Then, $\|\Delta|\varphi_t|^2\|$, $\|\Delta|\varphi_t|^2\|_1$ and $\|\varphi_t^2\|$, which also appear in our estimates, can be bounded by

$$\begin{aligned}
\Delta|\varphi_t|^2 =& \varphi_t^*\Delta\varphi_t + \varphi_t\Delta\varphi_t^* + 2(\nabla\varphi_t^*)\cdot(\nabla\varphi_t)\\
\|\Delta|\varphi_t|^2\| \leq& 2\|\Delta\varphi_t\|\|\varphi_t\|_\infty + 2\|\nabla\varphi_t\|\|\nabla\varphi_t\|_\infty\\
\|\Delta|\varphi_t|^2\|_1 \leq& 4\|\Delta\varphi_t\|\\
\|\varphi_t^2\| \leq& \|\varphi_t\|_\infty\|\varphi_t\|\ .
\end{aligned}$$

Recall the Sobolev embedding Theorem, which implies in particular $H^k(\mathbb{R}^2,\mathbb{C}) = W^{k,2}(\mathbb{R}^2,\mathbb{C}) \subset C^{k-2}(\mathbb{R}^2,\mathbb{C})$. If $\varphi \in C^1(\mathbb{R}^2,\mathbb{C})\cap H^1(\mathbb{R}^2,\mathbb{C})$, then $\varphi \in W^{1,\infty}(\mathbb{R}^2,\mathbb{C})$ follows since both φ and $\nabla\varphi$ have to decay at infinity. Thus, $\varphi_t \in H^3(\mathbb{R}^2,\mathbb{C})$ implies $\varphi_t \in H^2(\mathbb{R}^2,\mathbb{C})\cap W^{1,\infty}(\mathbb{R}^2,\mathbb{C})$, which suffices for our estimates. Since φ_t obeys a defocusing nonlinear Schrödinger equation, we expect the regularity of the solution φ_t to follow from the regularity of the initial datum φ_0. For a certain class of external potentials A_t this has been proven in [19]:

Lemma 4.4.7. *Let $\varphi_0 \in \Sigma^k(\mathbb{R}^2,\mathbb{C}) = \{f \in L^2(\mathbb{R}^2,\mathbb{C}) | \sum_{\alpha+\beta\leq k}\|x^\alpha\partial_x^\beta f\| < \infty\}$, for $k \geq 2$. Let, for $b > 0$, φ_t the unique solution to*

$$i\partial_t\varphi_t = (-\Delta + A_t + b|\varphi_t|^2)\varphi_t\ .$$

Let $A_\cdot \in L^\infty_{loc}(\mathbb{R}_t\times\mathbb{R}^2_x,\mathbb{C})$ real valued and smooth with respect to the space variable: for (almost) all $t\in\mathbb{R}$, the map $x\mapsto A_t(x)$ is C^∞. Moreover, A_t is at most quadratic in space, uniformly w.r.t. time t:

$$\forall\alpha\in\mathbb{N}^2, |\alpha|\geq 2, \qquad \partial_x^\alpha A_\cdot \in L^\infty(\mathbb{R}_t\times\mathbb{R}^d_x,\mathbb{C}).$$

In addition, $t\mapsto \sup_{|x|\leq 1}|A_t(x)|$ belongs to $L^\infty(\mathbb{R},\mathbb{C})$. Then

(a) $\varphi_t \in \Sigma^k(\mathbb{R}^2,\mathbb{C})$, which implies $\varphi_t \in H^k(\mathbb{R}^2,\mathbb{C})$.

(b) $\|\varphi_t\| = \|\varphi_0\|$.

(c) Let $\varphi_0 \in \Sigma^3(\mathbb{R}^2,\mathbb{C})$. Assume in addition that $\|A_t\|_\infty < \infty$ and $\|\dot{A}_t\|_\infty < \infty$. Then, for any fixed $t \geq 0$, $\mathcal{K}(\varphi_t, A_t) < \infty$ follows.

Proof. Part (a) is Corollary 1.4. in [19]. We like to remark that $\|\varphi_t\|_{H^k} \leq C$ holds, if $A_t = 0$, see Section 1.2. in [19]. The conditions on A_t are for example satisfied if $A_t \in C_c^\infty(\mathbb{R}^2,\mathbb{R})$ for all $t\in\mathbb{R}$, $A_t(x) = 0$, for all $|t|\geq T$. Part (b) can be verified directly, using the existence of global in time solutions. Part (c) follows from (a) and the embedding $H^3(\mathbb{R}^2,\mathbb{C}) \subset H^2(\mathbb{R}^2,\mathbb{C})\cap W^{1,\infty}(\mathbb{R}^2,\mathbb{C})$.

□

4.5 Microscopic structure in 2 dimensions

4.5.1 The scattering state

In this section we analyze the microscopic structure which is induced by V_N. In particular, we explain why the dynamical properties of the system are determined by the low energy scattering regime.

Definition 4.5.1. *Let $V \in L_c^\infty(\mathbb{R}^2, \mathbb{R})$, $V(x) \geq 0$, V spherically symmetric and let V_N be given by $V_N(x) = e^{2N}V(e^N x)$. For any $R \geq$ diam(supp(V_N)), we define the zero energy scattering state $j_{N,R}$ by*

$$\begin{cases} \left(-\Delta_x + \frac{1}{2}e^{2N}V(e^N x)\right) j_{N,R}(x) = 0, \\ j_{N,R}(x) = 1 \textit{ for } |x| = R\,. \end{cases} \tag{4.21}$$

One may think of R as the mean interparticle distance of the condensate, i.e. $R = \mathcal{O}(N^{-1/2})$. However, one is quite free in choosing R, since the dependence of $j_{N,R}$ on R is only logarithmic (see below).
Next, we want to recall some important properties of the scattering state $j_{N,R}$, see also Appendix C of [59].

Lemma 4.5.2. *Let $V \in L_c^\infty(\mathbb{R}^2, \mathbb{R})$, $V(x) \geq 0$ and spherically symmetric. Define $I_R = \int_{\mathbb{R}^2} d^2x V_N(x) j_{N,R}(x)$. For the scattering state defined previously the following relations hold:*

(a) There exists a nonnegative number a, called scattering length of the potential V, such that

$$I_R = \frac{4\pi}{\ln\left(\frac{e^N R}{a}\right)}.$$

The scattering length a does not depend on R and fulfills $a \leq$ diam(supp(V)). Furthermore, $I_R \geq 0$ holds.

(b) $j_{N,R}$ is a nonnegative function which is spherically symmetric in $|x|$. For $|x| \geq$ diam(supp(V_N)), $j_{N,R}$ is given by

$$j_{N,R}(x) = 1 + \frac{1}{\ln\left(\frac{e^N R}{a}\right)} \ln\left(\frac{|x|}{R}\right).$$

Proof. Rescaling $x \to e^N x = y$, we obtain, setting $\tilde{R} = e^N R$ and $s_{\tilde{R}}(y) = j_{0,e^N R}(y)$, the unscaled scattering equation

$$\begin{cases} \left(-\Delta_y + \frac{1}{2}V(y)\right) s_{\tilde{R}}(y) = 0, \\ s_{\tilde{R}}(y) = 1 \text{ for } |y| = \tilde{R}\,. \end{cases} \tag{4.22}$$

Since we assume V to be nonnegative, one can define the scattering state $s_{\tilde{R}}$ by a variational principle. Theorem C.1 in [59] then implies that $s_{\tilde{R}}$ is a nonnegative, spherically symmetric function in $|y|$. It is then easy to verify that for diam (supp(V)) $\leq |y|$ there exists a number $A \in \mathbb{R}$ such that

$$s_{\tilde{R}}(y) = 1 + \frac{A}{4\pi} \ln\left(\frac{|y|}{\tilde{R}}\right)\,. \tag{4.23}$$

Next, we show that $A = \int_{\mathbb{R}^2} d^2y V(y) s_{\tilde{R}}(y)$. This can be seen by noting that, for $r >$ diam (supp(V)),

$$\begin{aligned} \int_{\mathbb{R}^2} d^2y V(y) s_{\tilde{R}}(y) =& 2\int_{B_r(0)} d^2y \Delta s_{\tilde{R}}(y) = 2\int_{\partial B_r(0)} \nabla s_{\tilde{R}}(y) \cdot ds \\ =& \frac{A}{2\pi} \int_{\partial B_r(0)} \nabla \ln(|y|) \cdot ds = \frac{A}{2\pi} \int_0^{2\pi} \frac{1}{r} r d\varphi \\ =& A\,. \end{aligned}$$

By Theorem C.1 in [59], there exists a number $a \geq 0$, not depending on $\tilde{R}$, such that for all $|y| \geq \text{diam}\ (\text{supp}(V))$

$$s_{\tilde{R}}(y) = \frac{\ln(|y|/a)}{\ln(\tilde{R}/a)} .$$

Comparing this with (4.23), we obtain

$$\int_{\mathbb{R}^2} V(y) s_{\tilde{R}}(y) dy^2 = \frac{4\pi}{\ln\left(\frac{\tilde{R}}{a}\right)} .$$

Since $s_{\tilde{R}}$ is nonnegative, it furthermore follows that $a \leq \text{diam}\ (\text{supp}(V))$. This directly implies $A \geq 0$. By scaling, we obtain

$$I_R = \int_{\mathbb{R}^2} V_N(y) j_{N,R}(y) dy^2 = \int_{\mathbb{R}^2} V(y) s_{\tilde{R}}(y) dy^2 = \frac{4\pi}{\ln\left(\frac{e^N R}{a}\right)} .$$

□

Assuming that the energy per particle $\mathcal{E}_{V_N}(\Psi)$ is of order one, the wave function Ψ will have a microscopic structure near the interactions V_N, given by $j_{N,R}$. The interaction among two particles is then determined by $\frac{4\pi}{N+\ln\left(\frac{R}{a}\right)} \approx \frac{4\pi}{N}$. Keeping in mind that each particle interacts with all other $N-1$ particles, we obtain the effective Gross-Pitaevskii equation, for $\varphi_t \in H^2(\mathbb{R}^2, \mathbb{C})$

$$i\partial_t \varphi_t(x) = (-\Delta + A_t + 4\pi|\varphi_t(x)|^2)\varphi_t(x).$$

Thus, choosing $V_N(x) = e^{2N} V(e^N x)$ leads in our setting to an effective one-particle equation which is determined by the low energy scattering behavior of the particles. We remark that, for any $s > 0$, the potential $e^{2Ns} V(e^{Ns} x)$ yields to the coupling $4\pi/s$.

4.5.2 Properties of the scattering state

Note that the potential V_N is strongly peaked within an exponentially small region. In order to control the short scale structure of Ψ_t, we define, with a slight abuse of notation, a potential M_β with softer scaling behavior in such a way that the potential $V_N - M_\beta$ has scattering length zero. This allows us to "replace" V_N by M_β, which has better scaling behavior and is easier to control. In particular, $\|M_\beta\| \leq CN^{-1+\beta}$ can be controlled for β sufficiently small, while $\|V_N\| = \mathcal{O}(e^N)$ cannot be bounded by any finite polynomial in N. The potential M_β is *not* of the exact scaling $N^{-1+2\beta} M(N^\beta x)$. However, it is in the set $\mathcal{V}_\beta$, which we will define now.

Definition 4.5.3. *For any $\beta > 0$, we define the set of potentials $\mathcal{V}_\beta$ as*

$$\mathcal{V}_\beta = \Big\{ U \in L^2(\mathbb{R}^2, \mathbb{R}) | U(x) \geq 0\ \forall x \in \mathbb{R}^2, \|U\|_1 \leq CN^{-1}, \|U\| \leq CN^{-1+\beta},$$
$$\|U\|_\infty \leq CN^{-1+2\beta}, U(x) = 0\ \forall |x| \geq CN^{-\beta},\ U \textit{ is spherically symmetric} \Big\}.$$

Note that $N^{-1+2\beta} W(N^\beta x) \in \mathcal{V}_\beta$ holds, if W is positive, spherically symmetric and compactly supported.

All relevant estimates in this paper are formulated for $W_\beta \in \mathcal{V}_\beta$.

Definition 4.5.4. *Let $V \in L_c^\infty(\mathbb{R}^2, \mathbb{R})$, $V(x) \geq 0$ and spherically symmetric. For any $\beta > 0$ and any $R_\beta \geq N^{-\beta}$ we define the potential M_β via*

$$M_\beta(x) = \begin{cases} 4\pi N^{-1+2\beta} & \text{if } N^{-\beta} < |x| \leq R_\beta \\ 0 & \text{else} \end{cases} . \tag{4.24}$$

Furthermore, we define the zero energy scattering state f_β of the potential $(V_N - M_\beta)$, that is

$$\begin{cases} \left(-\Delta_x + \frac{1}{2}\left(V_N(x) - M_\beta(x)\right)\right) f_\beta(x) = 0 \\ f_\beta(x) = 1 \text{ for } |x| = R_\beta \end{cases} . \tag{4.25}$$

Note that M_β and f_β depend on R_β. We choose R_β such that the scattering length of the potential $(V_N - M_\beta)$ is zero. This is equivalent to the condition $\int_\mathbb{R} d^2x(V_N(x) - M_\beta(x))f_\beta(x) = 0$.

Lemma 4.5.5. *For the scattering state f_β, defined by* (4.25)*, the following relations hold:*

(a) There exists a minimal value $R_\beta < \infty$ such that $\int_{\mathbb{R}^2} d^2x(V_N(x) - M_\beta(x))f_\beta(x) = 0$.

For the rest of the paper we assume that R_β is chosen such that (a) holds.

(b) There exists $K_\beta \in \mathbb{R}$, $K_\beta > 0$ such that $K_\beta f_\beta(x) = j_{N,R_\beta}(x)\ \forall |x| \leq N^{-\beta}$.

(c) For N sufficiently large the supports of V_N and M_β do not overlap.

(d) f_β is a nonnegative, monotone nondecreasing function in $|x|$.

(e)

$$f_\beta(x) = 1 \text{ for } |x| \geq R_\beta . \tag{4.26}$$

(f)

$$1 \geq K_\beta \geq 1 + \frac{1}{N + \ln\left(\frac{R_\beta}{a}\right)} \ln\left(\frac{N^{-\beta}}{R_\beta}\right) . \tag{4.27}$$

(g) $R_\beta \leq CN^{-\beta}$.

For any fixed $0 < \beta$, N sufficiently large such that V_N and M_β do not overlap, we obtain

(h)

$$|N\|V_N f_\beta\|_1 - 4\pi| = |N\|M_\beta f_\beta\|_1 - 4\pi| \leq C\frac{\ln(N)}{N} .$$

(i) Define

$$g_\beta(x) = 1 - f_\beta(x) .$$

Then,

$$\|g_\beta\|_1 \leq CN^{-1-2\beta} \ln N , \qquad \|g_\beta\| \leq CN^{-1-\beta} \ln N , \qquad \|g_\beta\|_\infty \leq 1 .$$

(j)

$$\left| N \|M_\beta\|_1 - 4\pi \right| \leq C \frac{\ln(N)}{N} \ .$$

(k)

$$M_\beta \in \mathcal{V}_\beta \ , M_\beta f_\beta \in \mathcal{V}_\beta \ .$$

Proof. (a) In the following, we will sometimes denote, with a slight abuse of notation, $f_\beta(x) = f_\beta(r)$ for $r = |x|$ (for this, recall that f_β is radial symmetric).

We first show that $f_\beta(N^{-\beta}) \neq 0$. Assume first that there exists a $x_0, |x_0| \leq N^{-\beta}$ such that $f_\beta(x_0) \neq 0$. We may assume that $f_\beta(x_0) > 0$ (otherwise take $-f_\beta$). By continuity of f_β, there exists a maximal interval $I = [a, b] \subset [0, N^{-\beta}]$ on which $f_\beta(x) \geq 0$ for all $|x| \in I$ holds. Since $M_\beta(x) = 0$ for all x with $|x| \leq N^{-\beta}$, it follows that $\Delta f_\beta = \frac{1}{2} V_N f_\beta$ for all x with $|x| \leq N^{-\beta}$. Using, for $|x| \in I$, $\frac{1}{2} V_N(x) f_\beta(x) \geq 0$, we obtain $\Delta_x f_\beta(x) \geq 0$ for all $|x| \in I$, which implies that f_β is subharmonic on $A = \{x \in \mathbb{R}^2 | a < |x| < b\}$. By the maximum principle, $\max_{x \in \overline{A}}(f_\beta) = \max_{x \in \partial A}(f_\beta)$ holds. If it were now that $\max_{x \in \partial A}(f_\beta) = f_\beta(a) \geq f_\beta(x_0) > 0$, we could conclude for $a > 0$, using continuity of f_β, that there exists an $\epsilon > 0$, such that $f_\beta(x) \geq 0$ for all $a - \epsilon \leq |x| \leq b$. This, however contradicts the assumptions on I. Note that for $a = 0$, we obtain $\partial A = \{x \in \mathbb{R}^2 || x| = b\}$. Thus, we may conclude that $\max_{x \in \overline{A}}(f_\beta) = f_\beta(b)$. Since I is the maximal interval on which f_β is positive, it then follows that $b = N^{-\beta}$. This shows that $f_\beta(N^{-\beta}) \neq 0$, assuming that there exists a $x_0, |x_0| \leq N^{-\beta}$ such that $f_\beta(x_0) \neq 0$.

Assume now that $f_\beta(x) = 0$ for all $|x| \leq N^{-\beta}$. Then, (4.25) is equivalent to

$$\begin{cases} \left(-\Delta_x - \frac{1}{2} M_\beta(x)\right) f_\beta(x) = 0 \\ f_\beta(x) = 1 \text{ for } |x| = R_\beta \\ f_\beta(x) = 0 \text{ for } |x| \leq N^{-\beta} \ . \end{cases}$$

In the following, we show that this equation does not has a solution. We choose a maximal value $r_0 \geq N^{-\beta}$ such that $f(x) = 0$ for all $|x| \leq r_0$ and $r \geq r_0$ arbitrary. Then, we estimate

$$\begin{aligned} \left| \frac{\partial f_\beta}{\partial r} |_{|x|=r} \right| &= \left| \frac{1}{4\pi r} \int_{B_r(0)} d^2x \, M_\beta(|x|) f_\beta(|x|) \right| = \left| \frac{2\pi N^{-1+2\beta}}{r} \int_{r_0}^{r} ds \, f_\beta(s) s \right| \\ &\leq \frac{2\pi N^{-1+2\beta}}{r} \int_{r_0}^{r} ds \sup_{r_0 \leq u \leq r} \left| \frac{\partial f_\beta}{\partial r}(u) \right| (s - r_0) s \\ &\leq 2\pi N^{-1+2\beta} \sup_{r_0 \leq u \leq r} \left| \frac{\partial f_\beta}{\partial r}(u) \right| (r - r_0)^2 \end{aligned}$$

and obtain

$$\sup_{r_0 \leq u \leq r} \left| \frac{\partial f_\beta}{\partial r}(u) \right| \leq 2\pi N^{-1+2\beta} \sup_{r_0 \leq u \leq r} \left| \frac{\partial f_\beta}{\partial r}(u) \right| (r - r_0)^2.$$

However, this inequality only holds for small r if there exists $r_1 \geq r_0$ such that $\frac{\partial f_\beta}{\partial r}(u) = 0$ for all $u \in [r_0, r_1]$. This contradicts our assumption on r_0 and shows that $f_\beta(N^{-\beta}) \neq 0$.

Applying Theorem C.1 in [59] once more, it then follows that either f_β or $-f_\beta$ is a nonnegative, monotone nondecrasing function in $|x|$ for all $|x| \leq N^{-\beta}$.

Using (4.25) and Gauss-theorem,

$$\frac{\partial f_\beta}{\partial r}|_{|x|=r} = \frac{1}{4\pi r}\int_{B_r(0)} d^2x(V_N(x) - M_\beta(x))f_\beta(x)\ . \tag{4.28}$$

Thus, R_β is the minimal value such that $\frac{\partial f_\beta}{\partial r}|_{R_\beta} = 0$. Therefore, f_β or $-f_\beta$ is a nonnegative, monotone nondecreasing continuous function for all $|x| \leq R_\beta$.
Next, we show that $R_\beta < \infty$ by contradiction: Assume $R_\beta = \infty$. Since $f(N^{-\beta}) \neq 0$, we obtain $|\int_{\mathbb{R}^2} d^2x M_\beta(x) f_\beta(x)| \geq |f(N^{-\beta})\int_{\mathbb{R}^2} d^2x M_\beta(x)| = \infty$, which yields to a contradiction since $|\int_{\mathbb{R}^2} d^2x M_\beta(x) f_\beta(x)| = |\int_{\mathbb{R}^2} d^2x V_N(x) f_\beta(x)| < \infty$.

b) Since $f_\beta(N^{-\beta}) \neq 0$, we can define

$$h(x) = f_\beta(x)\frac{j_{N,R_\beta}(N^{-\beta})}{f_\beta(N^{-\beta})}$$

on the compact set $\overline{B_{N^{-\beta}}(0)} = \{x \in \mathbb{R}^2 : |x| \leq N^{-\beta}\}$. It is useful to note that the scattering equations (4.23) and (4.25) have a unique solution on a compact set $\overline{B_{N^{-\beta}}(0)}$. One easily sees that $h(x) = j_{N,R_\beta}(x)$ on $\partial\overline{B_{N^{-\beta}}(0)}$ and satisfies the zero energy scattering equation (4.21). By uniqueness it follows that $h(x) = j_{N,R_\beta}(x)\ \forall x \in \overline{B_{N^{-\beta}}(0)}$. We can conclude that

$$K_\beta = \frac{j_{N,R_\beta}(N^{-\beta})}{f_\beta(N^{-\beta})}. \tag{4.29}$$

Next, we show that the constant K_β is positive. Since $j_{N,R_\beta}(N^{-\beta})$ is positive, it follows from Eq. (4.29) that K_β and $f_\beta(N^{-\beta})$ have equal sign. By (a), the sign of f_β is constant for $|x| \leq R_\beta$. Furthermore, from Gauss-theorem and the scattering equation (4.25) we have

$$\frac{\partial f_\beta}{\partial r} = \frac{1}{4\pi r K_\beta}\int_{B_r(0)} d^2x V_N(x) j_{N,R_\beta}(x) \tag{4.30}$$

for all $r \leq N^{-\beta}$. Since j_{N,R_β} and V_N are nonnegative functions,

$$\mathrm{sgn}\left(\frac{\partial f_\beta}{\partial r}|_{r=N^{-\beta}}\right) = \mathrm{sgn}(K_\beta). \tag{4.31}$$

Recall that R_β is the smallest value such that $\frac{\partial f_\beta}{\partial r}\big|_{r=R_\beta} = 0$. If it were now that K_β is negative, we could conclude from (4.29) and (4.31) that $\frac{\partial f_\beta}{\partial r}|_{r=N^{-\beta}} < 0$ and $f_\beta(N^{-\beta}) < 0$. Since R_β is by definition the smallest value where $\frac{\partial f_\beta}{\partial r} = 0$, we were able to conclude from the continuity of the derivative that $\frac{\partial f_\beta}{\partial r} < 0$ for all $r < R_\beta$ and hence $f(R_\beta) < 0$. However, this were in contradiction to the boundary condition of the zero energy scattering state (see (4.25)) and thus $K_\beta > 0$ follows.

(c) This directly follows from $e^{-N} < CN^{-\beta}$ for N sufficiently large.

(d) From the proof of property b), we see that f_β and its derivative is positive at $N^{-\beta}$. From (4.28), we obtain $\partial_r f_\beta(x) = 0$ for all x with $|x| > R_\beta$. Due to continuity $\partial_r f_\beta(x) > 0$ for all x with $|x| < R_\beta$. Since f_β is continuous, positive at $N^{-\beta}$, and its derivative is a nonnegative function, it follows that f_β is a nonnegative, monotone nondecreasing function in $|x|$.

(e) By definition of R_β, it follows that $\tilde{I} = \int_{\mathbb{R}^2} d^2x(V_N(x) - W_\beta(x))f_\beta(x) = 0$. Therefore, for all $|x| \geq R_\beta$, f_β solves $-\Delta f_\beta(x) = 0$, which has the solution

$$f_\beta(x) = 1 + \frac{\tilde{I}}{4\pi} \ln\left(\frac{|x|}{R_\beta}\right) = 1$$

(f) Since f_β is a positive monotone nondecreasing function in $|x|$, we obtain

$$1 \geq f_\beta(N^{-\beta}) = j_{N,R_\beta}(N^{-\beta})/K_\beta = \left(1 + \frac{1}{N + \ln\left(\frac{R_\beta}{a}\right)} \ln\left(\frac{N^{-\beta}}{R_\beta}\right)\right)/K_\beta$$

We obtain the lower bound

$$K_\beta \geq 1 + \frac{1}{N + \ln\left(\frac{R_\beta}{a}\right)} \ln\left(\frac{N^{-\beta}}{R_\beta}\right).$$

For the upper bound we first prove that $f_\beta(x) \geq j_{N,R_\beta}(x)$ holds for all $|x| \leq R_\beta$. Using the scatting equations (4.23) and (4.25) we obtain

$$\Delta_x(f_\beta(x) - j_{N,R_\beta}(x)) = \frac{1}{2}V_N(x)(f_\beta(x) - j_{N,R_\beta}(x)) - W_\beta(x)f_\beta(x)$$

as well as $f_\beta(R_\beta) - j_{N,R_\beta}(R_\beta) = 0$. Since $W_\beta(x)f_\beta(x) \geq 0$, we obtain that $\Delta_x(f_\beta(x) - j_{N,R_\beta}(x)) \leq 0$ for $N^{-\beta} \leq |x| \leq R_\beta$. That is, $f_\beta(x) - j_{N,R_\beta}(x)$ is superharmonic for $N^{-\beta} < |x| < R_\beta$. Using the minimum principle, we obtain, using that $f_\beta - j_{N,R_\beta}$ is spherically symmetric

$$\min_{N^{-\beta} \leq |x| \leq R_\beta} (f_\beta - j_{N,R_\beta}) = \min_{|x| \in \{N^{-\beta}, R_\beta\}} (f_\beta - j_{N,R_\beta}) \tag{4.32}$$

If it were now that $\min_{|x| \in \{N^{-\beta}, R_\beta\}}(f_\beta - j_{N,R_\beta}) = f_\beta(N^{-\beta}) - j_{N,R_\beta}(N^{-\beta}) \leq f_\beta(R_\beta) - j_{N,R_\beta}(R_\beta) = 0$, we could conclude that $f_\beta(x) - j_{N,R_\beta}(x) \leq 0$ for all $N^{-\beta} \leq |x| \leq R_\beta$. Since $f_\beta(x) - j_{N,R_\beta}(x)$ then obeys

$$\begin{cases} -\Delta(f_\beta(x) - j_{N,R_\beta}(x)) + \frac{1}{2}V_N(x)(f_\beta(x) - j_{N,R_\beta}(x)) = 0 & \text{for } |x| \leq N^{-\beta}, \\ f_\beta(x) - j_{N,R_\beta}(x) \leq 0 & \text{for } |x| = N^{-\beta}, \end{cases}$$

we could then conclude that $f_\beta(x) - j_{N,R_\beta}(x) \leq 0$ for all $|x| \leq R_\beta$. From this, we obtain that $\Delta(f_\beta(x) - j_{N,R_\beta}(x)) \leq 0$ for $|x| \leq R_\beta$. That is, $f_\beta(x) - j_{N,R_\beta}(x)$ is superharmonic for all $|x| \leq R_\beta$. Using the minimum principle once again, we then obtain

$$\min_{\overline{B_{R_\beta}(0)}} (f_\beta - j_{N,R_\beta}) = f_\beta(R_\beta) - j_{N,R_\beta}(R_\beta) = 0$$

which contradicts $f_\beta(x) - j_{N,R_\beta}(x) \leq 0$ for $|x| \leq R_\beta$. Therefore, we can conclude in (4.32) that $\min_{N^{-\beta} \leq |x| \leq R_\beta}(f_\beta - j_{N,R_\beta}) = f_\beta(R_\beta) - j_{N,R_\beta}(R_\beta) = 0$ holds. Then, it follows that $f_\beta(x) - j_{N,R_\beta}(x) \geq 0$ for all $N^{-\beta} \leq |x| \leq R_\beta$. Using the zero energy scattering equation $-\Delta(f_\beta(x) - j_{N,R_\beta}(x)) + \frac{1}{2}V_N(x)(f_\beta(x) - j_{N,R_\beta}(x)) = 0$ for $|x| \leq N^{-\beta}$, we can, together with $f_\beta(N^{-\beta}) - j_{N,R_\beta}(N^{-\beta}) \geq 0$, conclude that $f_\beta(x) - j_{N,R_\beta}(x) \geq 0$ for all $|x| \leq R_\beta$.

As a consequence, we obtain the desired bound $K_\beta = \frac{j_{N,R_\beta}(N^{-\beta})}{f_\beta(N^{-\beta})} \leq 1$.

(g) Since f_β is a nonnegative, monotone nondecreasing function in $|x|$ with $f_\beta(x) = 1$ $\forall |x| \geq R_\beta$, it follows that

$$\begin{aligned} Cf_\beta(N^{-\beta}) =& f_\beta(N^{-\beta}) \int_{\mathbb{R}^2} d^2x V_N(x) \geq \int_{\mathbb{R}^2} d^2x V_N(x) f_\beta(x) \\ =& \int_{\mathbb{R}^2} d^2x M_\beta(x) f_\beta(x) \geq f_\beta(N^{-\beta}) \int_{\mathbb{R}^2} d^2x M_\beta(x) \,. \end{aligned}$$

Therfore, $\int_{\mathbb{R}^2} d^2x M_\beta(x) \leq C$ holds, which implies that $R_\beta \leq CN^{1/2-\beta}$.

From

$$\begin{aligned} \frac{1}{K_\beta} \frac{4\pi}{N + \ln\left(\frac{R_\beta}{a}\right)} =& \frac{1}{K_\beta} \int_{\mathbb{R}^2} d^2x V_N(x) j_{N,R_\beta}(x) = \int_{\mathbb{R}^2} d^2x V_N(x) f_\beta(x) \\ =& \int_{\mathbb{R}^2} d^2x M_\beta(x) f_\beta(x) = 8\pi^2 N^{-1+2\beta} \int_{N^{-\beta}}^{R_\beta} dr r f_\beta(r) \end{aligned}$$

we conclude that

$$\int_{N^{-\beta}}^{R_\beta} dr r f_\beta(r) = \frac{N^{1-2\beta}}{2\pi K_\beta \left(N + \ln\left(\frac{R_\beta}{a}\right)\right)} \,.$$

Since f_β is a nonegative, monotone nondecreasing function in $|x|$,

$$\frac{1}{2}(R_\beta^2 - N^{-2\beta}) \frac{j_{N,R_\beta}(N^{-\beta})}{K_\beta} = \frac{1}{2}(R_\beta^2 - N^{-2\beta}) f_\beta(N^{-\beta}) \leq \int_{N^{-\beta}}^{R_\beta} dr r f_\beta(r)$$

which implies

$$R_\beta^2 N^{2\beta} \leq \frac{N}{\pi \left(N + \ln\left(\frac{R_\beta}{a}\right)\right) j_{N,R_\beta}(N^{-\beta})} + 1$$

Using $R_\beta \leq CN^{1/2-\beta}$, it then follows

$$j_{N,R_\beta}(N^{-\beta}) = 1 + \frac{1}{N + \ln\left(\frac{R_\beta}{a}\right)} \ln\left(\frac{N^{-\beta}}{R_\beta}\right) \geq 1 - \frac{C}{N} \,,$$

which implies $R_\beta \leq CN^{-\beta}$.

(h) Using

$$\|M_\beta f_\beta\|_1 = \|V_N f_\beta\|_1 = K_\beta^{-1} \|V_N j_{N,R_\beta}\|_1 = K_\beta^{-1} \frac{4\pi}{N + \ln\left(\frac{R_\beta}{a}\right)} \,,$$

we obtain

$$\begin{aligned}|N\|V_N f_\beta\|_1 - 4\pi| =& |N\|M_\beta f_\beta\|_1 - 4\pi| = 4\pi\left|K_\beta^{-1}\frac{N}{N+\ln\left(\frac{R_\beta}{a}\right)} - 1\right| \\ =& \frac{4\pi}{K_\beta}\left|\frac{N - NK_\beta + K_\beta\ln\left(\frac{R_\beta}{a}\right)}{N+\ln\left(\frac{R_\beta}{a}\right)}\right| \leq C\frac{\ln(N)}{N}\ .\end{aligned}$$

(i) Using for $|x| \leq R_\beta$ the inequalities $j_{N,R_\beta}(x) \geq 1 + \frac{1}{N+\ln\left(\frac{R_\beta}{a}\right)}\ln\left(\frac{|x|}{R_\beta}\right)$ as well as $1 \geq f_\beta(x) \geq j_{N,R_\beta}(x)$, it follows for $|x| \leq R_\beta$

$$\begin{aligned}0 \leq& g_\beta(x) = 1 - f_\beta(x) \leq 1 - j_{N,R_\beta}(x) \leq -\frac{1}{N+\ln\left(\frac{R_\beta}{a}\right)}\ln\left(\frac{|x|}{R_\beta}\right) \\ \leq& CN^{-1}|\ln(N|x|)|\ .\end{aligned}$$

Since $g_\beta(x) = 0$ for $|x| > R_\beta$, we conclude with $R_\beta \leq CN^{-\beta}$ that

$$\|g_\beta\|_1 \leq \frac{C}{N}\int_0^{R_\beta} drr|\ln(Nr)| \leq CN^{-1-2\beta}\ln N\ ,$$

as well as

$$\begin{aligned}\|g_\beta\|^2 \leq& \frac{C}{N^2}\int_0^{R_\beta} drr\,(\ln(Nr))^2 = CN^{-4}\Big[r^2(2(\ln(r))^2 - 2\ln(r) + 1)\Big]_0^{NR_\beta} \\ \leq& CN^{-2-2\beta}\,(\ln(N))^2\,.\end{aligned}$$

$\|g_\beta\|_\infty = \|1 - f_\beta\|_\infty \leq 1$, since f_β is a nonnegative, monotone nondecreasing function with $f_\beta(x) \leq 1$.

(j) Using (h) and (i), we obtain with $\|M_\beta\|_1 \leq CN^{-1}$

$$\begin{aligned}&|N\|M_\beta\|_1 - 4\pi| \leq |N\|M_\beta f_\beta\|_1 - 4\pi| + N\|M_\beta g_\beta\|_1 \\ &\leq C\left(\frac{\ln(N)}{N} + \|\mathbb{1}_{|\cdot|\geq N^{-\beta}} g_\beta\|_\infty\right)\ .\end{aligned}$$

Since $g_\beta(x)$ is a nonnegative, monotone nonincreasing function, it follows with $K_\beta \leq 1$

$$\begin{aligned}\|\mathbb{1}_{|\cdot|\geq N^{-\beta}} g_\beta\|_\infty =& g_\beta(N^{-\beta}) = 1 - f_\beta(N^{-\beta}) = 1 - \frac{j_{N,R_\beta}(N^{-\beta})}{K_\beta} \\ \leq& 1 - \left(1 + \frac{1}{N+\ln\left(\frac{R_\beta}{a}\right)}\ln\left(\frac{N^{-\beta}}{R_\beta}\right)\right)\ .\end{aligned}$$

and (j) follows.

(k) $M_\beta \in \mathcal{V}_\beta$ follows directly from $R_\beta \leq CN^{-\beta}$. Furthermore, $0 \leq M_\beta(x)f_\beta(x) \leq M_\beta(x)$ implies $M_\beta f_\beta \in \mathcal{V}_\beta$.

□

4.6 Proof of the Theorem

4.6.1 Proof for the potential W_β

Choosing the weight

As we have already mentioned, we define a functional $\alpha : L^2(\mathbb{R}^{2N},\mathbb{C}) \times L^2(\mathbb{R}^2,\mathbb{C}) \to \mathbb{R}_0^+$ such that

(I) $\frac{\partial}{\partial t}\alpha(\Psi_t,\varphi_t)$ can be estimated by $\alpha(\Psi_t,\varphi_t)+\mathcal{O}(1)$, yielding to a bound of $\alpha(\Psi_t,\varphi_t)$ via a Grönwall estimate.

(II) $\alpha(\Psi,\varphi)\to 0$ implies convergence of the reduced one particle density matrix $\gamma_\psi^{(1)}$ to $|\varphi\rangle\langle\varphi|$ in trace norm.

For $\beta>0$, the interaction gets peaked as $N\to\infty$ and one has to use smoothness properties of Ψ_t to be able to control the dynamics of the condensate. For small β and many different choices of the weight, one obtains

$$\alpha(\Psi_t,\varphi_t) \le \alpha(\Psi_0,\varphi_0)$$
$$+\int_0^t ds\left(\mathcal{K}(\varphi_s,A_s)\left(\alpha(\Psi_s,\varphi_s)+\mathcal{O}(1)+\langle\!\langle \Psi_s,\widehat{n}^{\varphi_s}\Psi_s\rangle\!\rangle+\left|\mathcal{E}_{W_\beta}(\Psi_s)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_s)\right|\right)\right) .$$

This enables us to perform an integral type Grönwall estimate if we choose

$$\alpha(\Psi_t,\varphi_t)=\langle\!\langle \Psi_t,\widehat{n}^{\varphi_t}\Psi_t\rangle\!\rangle+\left|\mathcal{E}_{W_\beta}(\Psi_t)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_t)\right| .$$

For large β, however, it is necessary to adjust the weight function for the following reason: Taking the time derivative of $\langle\!\langle \Psi_t,\widehat{n}^{\varphi_t}\Psi_t\rangle\!\rangle$, terms of the form $\widehat{n}-\widehat{n}_1$ and $\widehat{n}-\widehat{n}_2$ appear. The bound $N\|\widehat{n}-\widehat{n}_i\|_{\text{op}}=\mathcal{O}(N^{1/2})$, $i=1,2$ can then be easily verified. For $\beta>1/2$ it is not possible to obtain a sufficient decay in N, see Lemma 4.7.7, part (b). For this reason, it is necessary to choose another weight function $\widehat{m}$ in such a way that $N\|\widehat{m}-\widehat{m}_i\|_{\text{op}}$ is better to control.

Definition 4.6.1. *For $0<\xi<\frac{1}{2}$ define*

$$m(k)=\begin{cases}\sqrt{k/N}, & \text{for } k\ge N^{1-2\xi};\\ 1/2(N^{-1+\xi}k+N^{-\xi}), & \text{else.}\end{cases}$$

and

$$\alpha^<(\Psi,\varphi)=\langle\!\langle \Psi,\widehat{m}^{\varphi}\Psi\rangle\!\rangle+\left|\mathcal{E}_{W_\beta}(\Psi)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)\right| .$$

With this definition, we obtain $N\|\widehat{m}-\widehat{m}_1\|_{\text{op}}\le CN^{\xi}$, see (4.55).

Lemma 4.6.2. *Let $\Psi\in L_s^2(\mathbb{R}^{2N},\mathbb{C})$ and let $\varphi\in L^2(\mathbb{R}^2,\mathbb{C})$. Let $\alpha^<(\Psi,\varphi)$ be defined as above. Then,*

$$\lim_{N\to\infty}\alpha^<(\Psi,\varphi)=0 \quad\Leftrightarrow\quad \lim_{N\to\infty}\gamma_\Psi^{(1)}=|\varphi\rangle\langle\varphi| \text{ in trace norm}$$
$$\text{and } \lim_{N\to\infty}\left(\mathcal{E}_{W_\beta}(\Psi)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)\right)=0 .$$

A proof of this Lemma can be found in [77]. Thus, $\alpha(\Psi_t,\varphi_t)$ satisfies condition (II). To obtain the desired Grönwall estimate, we will calculate $\frac{\partial}{\partial t}\langle\!\langle \Psi,\widehat{m}^{\varphi}\Psi\rangle\!\rangle$ and $\frac{\partial}{\partial t}(\mathcal{E}_{W_\beta}(\Psi_t)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_t))$. For this, define

Definition 4.6.3. *Let $W_\beta \in \mathcal{V}_\beta$. Define*

$$Z_\beta^\varphi(x_j,x_k) = W_\beta(x_j-x_k) - \frac{N\|W_\beta\|_1}{N-1}|\varphi|^2(x_j) - \frac{N\|W_\beta\|_1}{N-1}|\varphi|^2(x_k)\,. \tag{4.33}$$

Note, for $W_\beta(x) = N^{-1+2\beta}W(N^\beta x)$, we have $N\|W_\beta\|_1 = \|W\|_1$. With

$$m^a(k) = m(k) - m(k+1), \qquad m^b(k) = m(k) - m(k+2)$$

and

$$\widehat{r} = \widehat{m}^b p_1p_2 + \widehat{m}^a(p_1q_2 + q_1p_2)\,,$$

we define the functionals $\gamma_{a,b}^< : L^2(\mathbb{R}^{2N},\mathbb{C}) \times L^2(\mathbb{R}^2,\mathbb{C}) \to \mathbb{R}_0^+$ by

$$\gamma_a^<(\Psi,\varphi) = \langle\!\langle \Psi, \dot{A}_t(x_1)\Psi\rangle\!\rangle - \langle \varphi, \dot{A}_t\varphi\rangle \tag{4.34}$$

$$\gamma_b^<(\Psi,\varphi) = N(N-1)\mathrm{Im}\left(\langle\!\langle \Psi, Z_\beta^\varphi(x_1,x_2)\widehat{r}\Psi\rangle\!\rangle\right) \tag{4.35}$$

$$\begin{aligned} &= -2N(N-1)\mathrm{Im}\left(\langle\!\langle \Psi, p_1q_2\widehat{m}^a_{-1}Z_\beta^\varphi(x_1,x_2)p_1p_2\Psi\rangle\!\rangle\right) \\ &\quad -N(N-1)\mathrm{Im}\left(\langle\!\langle \Psi, q_1q_2\widehat{m}^b_{-2}W_\beta(x_1-x_2)p_1p_2\Psi\rangle\!\rangle\right) \\ &\quad -2N(N-1)\mathrm{Im}\left(\langle\!\langle \Psi, q_1q_2\widehat{m}^a_{-1}Z_\beta^\varphi(x_1,x_2)p_1q_2\Psi\rangle\!\rangle\right)\,. \end{aligned} \tag{4.36}$$

Lemma 4.6.4. *Let $W_\beta \in \mathcal{V}_\beta$. Let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{W_\beta}\Psi_t$ with initial datum $\Psi_0 \in L_s^2(\mathbb{R}^{2N},\mathbb{C}) \cap H^2(\mathbb{R}^{2N},\mathbb{C})$, $\|\Psi_0\| = 1$. Let φ_t the unique solution to $i\partial_t\varphi_t = h^{GP}_{N\|W_\beta\|_1}\varphi_t$ with $\varphi_t \in H^3(\mathbb{R}^2,\mathbb{C})$, $\|\varphi_0\| = 1$. Let $\alpha^<(\Psi_t,\varphi_t)$ be defined as in Definition 4.6.1. Then*

$$\alpha^<(\Psi_t,\varphi_t) \le \alpha^<(\Psi_0,\varphi_0) + \int_0^t ds\left(\left|\gamma_a^<(\Psi_s,\varphi_s)\right| + \left|\gamma_b^<(\Psi_s,\varphi_s)\right|\right)\,. \tag{4.37}$$

Proof. For the proof of the Lemma we restore the upper index φ_t in order to pay respect to the time dependence of $\widehat{m}^{\varphi_t}$. The time derivative of φ_t is given by (4.3), i.e. $i\partial_t\varphi_t(x_j) = h^{GP}_{N\|W_\beta\|_1,j}\varphi_t(x_j)$. Here, $h^{GP}_{N\|W_\beta\|_1,j}$ denotes the operator $h^{GP}_{N\|W_\beta\|_1}$ acting on the j^{th} coordinate x_j. We then obtain

$$\begin{aligned} &\frac{d}{dt}\langle\!\langle \Psi_t, \widehat{m}^{\varphi_t}\Psi_t\rangle\!\rangle \\ =&i\langle\!\langle H_{W_\beta}\Psi_t, \widehat{m}^{\varphi_t}\,\Psi_t\rangle\!\rangle - i\langle\!\langle \Psi_t, \widehat{m}^{\varphi_t}\,H_{W_\beta}\Psi_t\rangle\!\rangle - i\langle\!\langle \Psi_t, [\sum_{j=1}^N h^{GP}_{N\|W_\beta\|_1,j}, \widehat{m}^{\varphi_t}\,]\Psi_t\rangle\!\rangle \\ =&i\langle\!\langle \Psi_t, [H_{W_\beta} - \sum_{j=1}^N h^{GP}_{N\|W_\beta\|_1,j}, \widehat{m}^{\varphi_t}\,]\Psi_t\rangle\!\rangle = i\frac{N(N-1)}{2}\langle\!\langle \Psi_t, [Z_\beta^{\varphi_t}(x_1,x_2), \widehat{m}^{\varphi_t}\,]\Psi_t\rangle\!\rangle\,, \end{aligned}$$

where we used symmetry of Ψ_t in the last step. Using Lemma 4.4.2 (d), it follows that the latter equals (dropping the explicit dependence on φ_t from now on)

$$\begin{aligned} &i\frac{N(N-1)}{2}\langle\!\langle \Psi_t, [Z_\beta^{\varphi_t}(x_1,x_2), p_1p_2(\widehat{m}-\widehat{m}_2)]\Psi_t\rangle\!\rangle \\ +&i\frac{N(N-1)}{2}\langle\!\langle \Psi_t, [Z_\beta^{\varphi_t}(x_1,x_2), (p_1q_2+q_1p_2)(\widehat{m}-\widehat{m}_1)]\Psi_t\rangle\!\rangle\,. \end{aligned}$$

Since $Z_\beta^{\varphi_t}$ and $p_1p_2(\widehat{m}-\widehat{m}_2)$ as well as $p_1q_2(\widehat{m}-\widehat{m}_1)$ are selfadjoint, we obtain

$$\frac{d}{dt}\langle\!\langle\Psi_t,\widehat{m}^{\varphi_t}\Psi_t\rangle\!\rangle=-N(N-1)$$
$$\mathrm{Im}\left(\langle\!\langle\Psi_t,(p_1p_2+p_1q_2+q_1p_2+q_1q_2)Z_\beta^{\varphi_t}(x_1,x_2)(\widehat{m}^bp_1p_2+\widehat{m}^a(p_1q_2+q_1p_2))\Psi_t\rangle\!\rangle\right)\ .$$

Note that in view of Lemma 4.4.2 (c) $\widehat{r}Q_jZ_\beta^{\varphi_t}(x_1,x_2)Q_j=Q_jZ_\beta^{\varphi_t}(x_1,x_2)Q_j\widehat{r}$ for any $j\in\{0,1,2\}$ and any weight r. Therefore,

$$\mathrm{Im}\left(\langle\!\langle\Psi_t,p_1p_2Z_\beta^{\varphi_t}(x_1,x_2)\widehat{m}^bp_1p_2\Psi_t\rangle\!\rangle\right)=0$$
$$\mathrm{Im}\left(\langle\!\langle\Psi_t,(p_1q_2+q_1p_2)Z_\beta^{\varphi_t}(x_1,x_2)\widehat{m}^a(p_1q_2+q_1p_2)\Psi_t\rangle\!\rangle\right)=0\ .$$

Using Symmetry and Lemma 4.4.2 (c), we obtain the first line (4.35). Furthermore,

$$\begin{aligned}\frac{d}{dt}\langle\!\langle\Psi_t,\widehat{m}^{\varphi_t}\Psi_t\rangle\!\rangle=&-2N(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t,\widehat{m}^b_{-1}p_1q_2Z_\beta^{\varphi_t}(x_1,x_2)p_1p_2\Psi_t\rangle\!\rangle\right)\\&-N(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t,\widehat{m}^b_{-2}q_1q_2Z_\beta^{\varphi_t}(x_1,x_2)p_1p_2\Psi_t\rangle\!\rangle\right)\\&-2N(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t,p_1p_2Z_\beta^{\varphi_t}(x_1,x_2)\widehat{m}^ap_1q_2\Psi_t\rangle\!\rangle\right)\\&-2N(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t,\widehat{m}^a_{-1}q_1q_2Z_\beta^{\varphi_t}(x_1,x_2)p_1q_2\Psi_t\rangle\!\rangle\right)\ .\end{aligned}$$

Since $p_1p_2|\varphi_t^2|(x_1)q_1q_2=p_1p_2q_2|\varphi_t^2|(x_1)q_1=0=p_1p_2|\varphi_t^2|(x_2)q_1q_2$, we can replace $Z_\beta^{\varphi_t}(x_1,x_2)$ in the second line by $W_\beta(x_1-x_2)$.
The third line equals $2N(N-1)\mathrm{Im}\left(\langle\!\langle\Psi,\widehat{m}^ap_1q_2Z_\beta^{\varphi_t}(x_1,x_2)p_1p_2\Psi\rangle\!\rangle\right)$. Since

$$m(k-1)-m(k+1)-(m(k)-m(k+1))=m(k-1)-m(k)$$

it follows that $\widehat{m}^b_{-1}-\widehat{m}^a=\widehat{m}^a_{-1}$ and we get

$$\begin{aligned}\frac{d}{dt}\langle\!\langle\Psi_t,\widehat{m}^{\varphi_t}\Psi_t\rangle\!\rangle=&-2N(N-1)\mathrm{Im}\left(\langle\!\langle\Psi,p_1q_2\widehat{m}^a_{-1}Z_\beta^{\varphi_t}(x_1,x_2)p_1p_2\Psi\rangle\!\rangle\right)\\&-N(N-1)\mathrm{Im}\left(\langle\!\langle\Psi,q_1q_2\widehat{m}^b_{-2}W_\beta(x_1-x_2)p_1p_2\Psi\rangle\!\rangle\right)\\&-2N(N-1)\mathrm{Im}\left(\langle\!\langle\Psi,q_1q_2\widehat{m}^a_{-1}Z_\beta^{\varphi_t}(x_1,x_2)p_1q_2\Psi\rangle\!\rangle\right)\ .\end{aligned}$$

For the second summand of $\alpha^<(\Psi_t,\varphi_t)$ we have

$$\begin{aligned}&\frac{d}{dt}\left(\mathcal{E}_{W_\beta}(\Psi_t)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_t)\right)=\langle\!\langle\Psi_t,\dot{A}_t(x_1)\Psi_t\rangle\!\rangle-\langle\varphi_t,\dot{A}_t\varphi_t\rangle\\&+i\left\langle\varphi_t,\left[h^{GP}_{N\|W_\beta\|_1},\left(h^{GP}_{N\|W_\beta\|_1}-\frac{N\|W_\beta\|_1}{2}|\varphi_t|^2\right)\right]\varphi_t\right\rangle+\left\langle\varphi_t,\frac{N\|W_\beta\|_1}{2}\left(\frac{d}{dt}|\varphi_t|^2\right)\varphi_t\right\rangle\\&=\langle\!\langle\Psi_t,\dot{A}_t(x_1)\Psi_t\rangle\!\rangle-\langle\varphi_t,\dot{A}_t\varphi_t\rangle+i\left\langle\varphi_t,\left[h^{GP}_{N\|W_\beta\|_1},\frac{N\|W_\beta\|_1}{2}|\varphi_t|^2\right]\varphi_t\right\rangle\\&-i\left\langle\varphi_t,\left[h^{GP}_{N\|W_\beta\|_1},\frac{N\|W_\beta\|_1}{2}|\varphi_t|^2\right]\varphi_t\right\rangle=\gamma_a^<(\Psi_t,\varphi_t)\ .\end{aligned}$$

The Lemma then follows using that $|f(x)|\leq|f(0)|+\int_0^x dy|f'(y)|$ holds for any $f\in C^1(\mathbb{R},\mathbb{C})$. □

Establishing the Grönwall estimate

Lemma 4.6.5. *Let $W_\beta \in \mathcal{V}_\beta$. Let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{W_\beta}\Psi_t$ with initial datum $\Psi_0 \in L^2_s(\mathbb{R}^{2N},\mathbb{C}) \cap H^2(\mathbb{R}^{2N},\mathbb{C})$, $\|\Psi_0\| = 1$. Let φ_t the unique solution to $i\partial_t\varphi_t = h^{GP}_{N\|W_\beta\|_1}\varphi_t$ with $\varphi_t \in H^3(\mathbb{R}^2,\mathbb{C})$. Let $\mathcal{E}_{W_\beta}(\Psi_0) \le C$. Let $\gamma^<_a(\Psi_t,\varphi_t)$ and $\gamma^<_b(\Psi_t,\varphi_t)$ be defined as in Definition (4.6.3). Then, there exists an $\eta > 0$ such that*

$$\gamma^<_a(\Psi_t,\varphi_t) \le C\|\dot{A}_t\|_\infty(\langle\langle\Psi_t,\widehat{n}^{\varphi_t}\Psi_t\rangle\rangle + N^{-\frac{1}{2}}) \tag{4.38}$$

$$\gamma^<_b(\Psi_t,\varphi_t) \le \mathcal{K}(\varphi_t,A_t)\left(\langle\langle\Psi_t,\widehat{n}^{\varphi_t}\Psi_t\rangle\rangle + N^{-\eta} + \left|\mathcal{E}_{W_\beta}(\Psi_t) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_t)\right|\right) \tag{4.39}$$

The proof of this Lemma can be found in Section 4.7.3. Note that

$$|\langle\langle\Psi_t,\widehat{n}^{\varphi_t}\Psi_t\rangle\rangle - \langle\langle\Psi_t,\widehat{m}^{\varphi_t}\Psi_t\rangle\rangle| \le \|\widehat{n}^{\varphi_t} - \widehat{m}^{\varphi_t}\|_{\mathrm{op}} = N^{-\xi}$$

Once we have proven Lemma 4.6.5, we obtain with Lemma 4.6.4, Grönwall's Lemma and the estimate above that

$$\begin{aligned}\alpha^<(\Psi_t,\varphi_t) \le e^{\int_0^t ds\mathcal{K}(\varphi_s,A_s)}\Big(&\alpha^<(\Psi_0,\varphi_0)\\ &+\int_0^t ds\mathcal{K}(\varphi_s,A_s)e^{-\int_0^s d\tau\mathcal{K}(\varphi_\tau,A_\tau)}N^{-\eta}\Big)\,.\end{aligned}$$

Note that under the assumptions $\varphi_t \in H^3(\mathbb{R}^2,\mathbb{C})$ and $A_t \in L^\infty(\mathbb{R}^2,\mathbb{C})$, $\dot{A}_t \in L^\infty(\mathbb{R}^2,\mathbb{C})$ there exists a constant $C_t < \infty$, depending on t, φ_0 and A_t, such that $\int_0^t ds\mathcal{K}(\varphi_s,A_s) \le C_t$, see Lemma 4.4.7. This proves, using Lemma 4.6.2, part (a) of Theorem 4.2.1. If the potential is switched off, one expects that C_t is of order t since in this case $\|\varphi_t\|_\infty$ and $\|\nabla\varphi_t\|_\infty$ are expected to decay like t^{-1}.
We want to explain on a heuristic level why $\gamma^<_b(\Psi_t,\varphi_t)$ is small. The principle argument follows the ideas and estimates of [77]. The first line in (4.36) is the most important one. This expression is only small if the correct coupling parameter $N\|W_\beta\|_1$ is used in the mean-field equation (4.3). Then,

$$Np_1W_\beta(x_1-x_2)p_1 = Np_1W_\beta \star |\varphi|^2(x_2)p_1 \to p_1|\varphi|^2(x_2)\|W\|_1p_1$$

converges against the mean-field potential, and hence the first expression of (4.36) is small. In order to estimate the second and third line of (4.36), one tries to bound $N^2\langle\langle\Psi, q_1q_2\widehat{m}^b_{-2}W_\beta(x_1-x_2)p_1p_2\Psi\rangle\rangle$ and $N^2\langle\langle\Psi, q_1q_2\widehat{m}^a_{-1}Z^\varphi_\beta(x_1-x_2)p_1q_2\Psi\rangle\rangle$ in terms of $\langle\langle\Psi,\widehat{n}\Psi\rangle\rangle + \mathcal{O}(N^{-\eta})$ for some $\eta > 0$. For large β, one needs to use additional smoothness properties of Ψ_t. This explains the appearance of $\left|\mathcal{E}_{W_\beta}(\Psi_t) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_t)\right|$ on the right hand side of (4.39). The concise estimates are quite involved and can be found in Section 4.7.3.

4.6.2 Proof for the exponential scaling V_N

Adapting the weight

For the most involved scaling V_N it is necessary to modify the counting functional $\alpha^<(\Psi,\varphi)$ in order to obtain the desired Grönwall estimate. $\gamma^<_b(\Psi,\varphi)$, which was defined in (4.36), will not be small if we were to replace W_β by V_N. In particular, $\|V_N\| = \mathcal{O}(e^N)$ cannot be bounded by any finite polynomial in $1/N$. In order to control the dynamics of the condensate, one needs to account for the microscopic structure which is induced by V_N, as explained in

Section 4.5. The idea we will employ is the following: For the moment, think of the most simple counting functional, namely $\langle\!\langle \Psi_t, q_1^{\varphi_t}\Psi_t\rangle\!\rangle = 1 - \langle\!\langle \Psi_t, p_1^{\varphi_t}\Psi_t\rangle\!\rangle$. This functional counts the relative number of particles which are not in the state φ_t. Instead of projecting onto φ_t, we now consider the functional

$$1 - \langle\!\langle \Psi_t, \prod_{j=2}^{N} f_\beta(x_1 - x_j) p_1^{\varphi_t} \prod_{j=2}^{N} f_\beta(x_1 - x_j)\Psi_t\rangle\!\rangle \ ,$$

which takes the short scale correlation structure into account. Neglecting all but two-particle interactions, this can be approximated by

$$\begin{aligned} &1 - \langle\!\langle \Psi_t, \left(1 - \sum_{j=2}^{N} g_\beta(x_1 - x_j)\right) p_1^{\varphi_t} \left(1 - \sum_{j=2}^{N} g_\beta(x_1 - x_j)\right) \Psi_t\rangle\!\rangle \\ &\approx \langle\!\langle \Psi_t, q_1^{\varphi_t}\Psi_t\rangle\!\rangle + 2(N-1)\mathrm{Re}\left(\langle\!\langle \Psi_t, g_\beta(x_1 - x_2) p_1^{\varphi_t}\Psi_t\rangle\!\rangle\right) \ . \end{aligned}$$

If we now take the time derivative of this new functional, one gets, among other terms, $2(N-1)\mathrm{Im}\langle\!\langle \Psi_t, [H_{V_N}, f_\beta(x_1 - x_2)] p_1^{\varphi_t}\Psi_t\rangle\!\rangle$. The commutator equals $f_\beta(x_1 - x_2)(V_N(x_1 - x_2) - M_\beta(x_1 - x_2))$ plus mixed derivatives and one sees, that V_N is "replaced" by M_β for the price of new terms that have to be estimated. The strategy we are going to employ is thus to estimate the time derivative of the modified functional and to show that we obtain a Grönwall estimate. Note, that, using Lemma 4.4.2 (e) with Lemma 4.5.5 (i)

$$2(N-1)|\mathrm{Re}\left(\langle \Psi_t, g_\beta(x_1 - x_2) p_1^{\varphi_t}\Psi_t\rangle\right)| \le CN\|\varphi_t\|_\infty \|g_\beta\| \le C\|\varphi_t\|_\infty N^{-\beta}\ln(N)$$

holds. Hence, we obtain the a priori estimate

$$\langle\!\langle \Psi_t, q_1^{\varphi_t}\Psi_t\rangle\!\rangle \le \langle\!\langle \Psi_t, q_1^{\varphi_t}\Psi_t\rangle\!\rangle + 2(N-1)\mathrm{Re}\left(\langle\!\langle \Psi_t, g_\beta(x_1 - x_2) p_1^{\varphi_t}\Psi_t\rangle\!\rangle\right) + C\|\varphi_t\|_\infty N^{-\beta}\ln(N),$$

which explains why the new defined functional implies convergence of the reduced density matrix $\gamma^{(1)}_{\Psi_t}$ to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm. We now adapt the strategy explained above to modify the counting functional $\alpha^<(\Psi, \varphi)$.

Definition 4.6.6. *Let* $\widehat{r} = \widehat{m}^b p_1 p_2 + \widehat{m}^a(p_1 q_2 + q_1 p_2)$.
Let the functional $\alpha : L^2(\mathbb{R}^{2N}, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}_0^+$ *be defined by*

$$\alpha(\Psi, \varphi) = \langle\!\langle \Psi, \widehat{m}\Psi\rangle\!\rangle + \left|\mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi)\right| - N(N-1)\mathrm{Re}\left(\langle\!\langle \Psi, g_\beta(x_1 - x_2)\widehat{r}\Psi\rangle\!\rangle\right) \tag{4.40}$$

and the functional $\gamma : L^2(\mathbb{R}^{2N}, \mathbb{C}) \times L^2(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}$ *be defined by*

$$\gamma(\Psi, \varphi) = |\gamma_a(\Psi, \varphi)| + |\gamma_b(\Psi, \varphi)| + |\gamma_c(\Psi, \varphi)| + |\gamma_d(\Psi, \varphi)| + |\gamma_e(\Psi, \varphi)| + |\gamma_f(\Psi, \varphi)| \ , \tag{4.41}$$

where the different summands are:

(a) The change in the energy-difference

$$\gamma_a(\Psi, \varphi) = \langle\!\langle \Psi, \dot{A}_t(x_1)\Psi\rangle\!\rangle - \langle \varphi, \dot{A}_t\varphi\rangle \ .$$

(b) The new interaction term

$$\begin{aligned} \gamma_b(\Psi, \varphi) = &- N(N-1)\mathrm{Im}\left(\langle\!\langle \Psi, \widetilde{Z}_\beta^\varphi(x_1, x_2)\widehat{r}\,\Psi\rangle\!\rangle\right) \\ &- N(N-1)\mathrm{Im}\left(\langle\!\langle \Psi, g_\beta(x_1 - x_2)\widehat{r}\,\mathcal{Z}^\varphi(x_1, x_2)\Psi\rangle\!\rangle\right), \end{aligned}$$

where, using M_β from Definition 4.5.4,

$$\widetilde{Z}^{\varphi}_{\beta}(x_1,x_2) = \left(M_\beta(x_1-x_2) - 4\pi\frac{|\varphi|^2(x_1)+|\varphi|^2(x_2)}{N-1}\right) f_\beta(x_1-x_2) \tag{4.42}$$
$$\mathcal{Z}^{\varphi}(x_1,x_2) = V_N(x_1-x_2) - \frac{4\pi}{N-1}|\varphi|^2(x_1) - \frac{4\pi}{N-1}|\varphi|^2(x_2)\ .$$

(c) *The mixed derivative term*

$$\gamma_c(\Psi,\varphi) = -\,4N(N-1)\langle\!\langle \Psi, (\nabla_1 g_\beta(x_1-x_2))\nabla_1 \widehat{r}\Psi\rangle\!\rangle\ .$$

(d) *Three particle interactions*

$$\begin{aligned}\gamma_d(\Psi,\varphi) =& 2N(N-1)(N-2)\mathrm{Im}\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\left[V_N(x_1-x_3),\widehat{r}\right]\Psi\rangle\!\rangle\right)\\ &-N(N-1)(N-2)\mathrm{Im}\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\left[4\pi|\varphi|^2(x_3),\widehat{r}\right]\Psi\rangle\!\rangle\right)\ .\end{aligned}$$

(e) *Interaction terms of the correction*

$$\gamma_e(\Psi,\varphi) = \frac{1}{2}N(N-1)(N-2)(N-3)\mathrm{Im}\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\left[V_N(x_3-x_4),\widehat{r}\right]\Psi\rangle\!\rangle\right)\ .$$

(f) *Correction terms of the mean field*

$$\gamma_f(\Psi,\varphi) = -2N(N-2)\mathrm{Im}\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\left[4\pi|\varphi|^2(x_1),\widehat{r}\right]\Psi\rangle\!\rangle\right)\ .$$

Lemma 4.6.7. *Let Ψ_t the unique solution to $i\partial_t\Psi_t = H_{V_N}\Psi_t$ with initial datum $\Psi_0 \in L^2_s(\mathbb{R}^{2N},\mathbb{C}) \cap H^2(\mathbb{R}^{2N},\mathbb{C})$, $\|\Psi_0\| = 1$. Let φ_t the unique solution to $i\partial_t\varphi_t = h^{GP}_{4\pi}\varphi_t$ with $\varphi_t \in H^3(\mathbb{R}^2,\mathbb{C})$, $\|\varphi_0\| = 1$. Let $\alpha(\Psi_t,\varphi_t)$ and $\gamma(\Psi_t,\varphi_t)$ be defined as in* (4.40) *and* (4.41)*. Then*

$$\alpha(\Psi_t,\varphi_t) \le \alpha(\Psi_0,\varphi_0) + \int_0^t ds\gamma(\Psi_s,\varphi_s)\ .$$

Proof. We first calculate

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\langle\!\langle \Psi,\widehat{m}\Psi\rangle\!\rangle - N(N-1)\mathrm{Re}\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\widehat{r}\Psi\rangle\!\rangle\right)\right)\\ &=-\,N(N-1)\mathrm{Im}\left(\langle\!\langle \Psi_t, \mathcal{Z}^{\varphi_t}(x_1,x_2)\widehat{r}\Psi_t\rangle\!\rangle\right)\\ &\quad-\,N(N-1)\mathrm{Re}\left(i\langle\!\langle \Psi_t, g_\beta(x_1-x_2)\left[H_{V_N}-\sum_{i=1}^N h^{GP}_{4\pi,i},\widehat{r}\right]\Psi_t\rangle\!\rangle\right)\\ &\quad-\,N(N-1)\mathrm{Re}\left(i\langle\!\langle \Psi_t, \left[H_{V_N}, g_\beta(x_1-x_2)\right]\widehat{r}\Psi_t\rangle\!\rangle\right)\ .\end{aligned}$$

Using symmetry and $\mathrm{Re}(iz) = -\mathrm{Im}(z)$, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\langle\!\langle \Psi,\widehat{m}\Psi\rangle\!\rangle - N(N-1)\mathrm{Re}\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\widehat{r}\Psi\rangle\!\rangle\right)\right)\\ &=-\,N(N-1)\mathrm{Im}\left(\langle\!\langle \Psi_t, \mathcal{Z}^{\varphi_t}(x_1,x_2)\widehat{r}\Psi_t\rangle\!\rangle\right)\\ &\quad+\,N(N-1)\mathrm{Im}\left(\langle\!\langle \Psi_t, g_\beta(x_1-x_2)\left[\mathcal{Z}^{\varphi_t}(x_1,x_2),\widehat{r}\right]\Psi_t\rangle\!\rangle\right)\\ &\quad+\,2N(N-1)(N-2)\mathrm{Im}\left(\langle\!\langle \Psi_t, g_\beta(x_1-x_2)\left[V_N(x_1-x_3),\widehat{r}\right]\Psi_t\rangle\!\rangle\right)\\ &\quad-\,N(N-1)(N-2)\mathrm{Im}\left(\langle\!\langle \Psi_t, g_\beta(x_1-x_2)\left[4\pi|\varphi_t|^2(x_3),\widehat{r}\right]\Psi_t\rangle\!\rangle\right)\\ &\quad+\,\frac{1}{2}N(N-1)(N-2)(N-3)\mathrm{Im}\left(\langle\!\langle \Psi_t, g_\beta(x_1-x_2)\left[V_N(x_3-x_4),\widehat{r}\right]\Psi_t\rangle\!\rangle\right)\\ &\quad+\,N(N-1)\mathrm{Im}\left(\langle\!\langle \Psi_t, \left[H_{V_N}, g_\beta(x_1-x_2)\right]\widehat{r}\Psi_t\rangle\!\rangle\right)\ .\\ &\quad-\,2N(N-2)\mathrm{Im}\left(\langle\!\langle \Psi_t, g_\beta(x_1-x_2)\left[4\pi|\varphi_t|^2(x_1),\widehat{r}\right]\Psi_t\rangle\!\rangle\right)\ .\end{aligned}$$

The third and fourth lines equal γ_d (recall that Ψ is symmetric), the fifth line equals γ_e and the seventh line equals γ_f. Using that $(1-g_\beta(x_1-x_2))\mathcal{Z}^{\varphi}(x_1,x_2)=\widetilde{Z}^{\varphi}_{\beta}(x_1,x_2)+(V_N(x_1-x_2)-M_\beta(x_1-x_2))f_\beta(x_1-x_2)$ we get

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\langle\!\langle\Psi,\widehat{m}\Psi\rangle\!\rangle-N(N-1)\mathrm{Re}\left(\langle\!\langle\Psi,g_\beta(x_1-x_2)\widehat{r}\Psi\rangle\!\rangle\right)\right)\\
\leq&\gamma_d(\Psi_t,\varphi_t)+\gamma_e(\Psi_t,\varphi_t)+\gamma_f(\Psi_t,\varphi_t)\\
&-N(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t,\widetilde{Z}^{\varphi_t}_{\beta}(x_1,x_2)\widehat{r}\Psi_t\rangle\!\rangle\right)\\
&-N(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t,(V_N(x_1-x_2)-M_{\beta_1}(x_1-x_2))f_\beta(x_1-x_2)\widehat{r}\Psi_t\rangle\!\rangle\right)\\
&-N(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t,g_\beta(x_1-x_2)\widehat{r}\mathcal{Z}^{\varphi_t}(x_1,x_2)\Psi_t\rangle\!\rangle\right)\\
&+N(N-1)\mathrm{Im}\left(\langle\!\langle\Psi_t,[H_{V_N},g_\beta(x_1-x_2)]\,\widehat{r}\Psi_t\rangle\!\rangle\right)\ .
\end{aligned}
\tag{4.43}
$$

The first, second and the fourth line give $\gamma_b+\gamma_d+\gamma_e+\gamma_f$. Using Definition (4.5.4) the commutator in the fifth line equals

$$
\begin{aligned}
[H_{V_N},g_\beta(x_1-x_2)]=&-[H_{V_N},f_\beta(x_1-x_2)]\\
=&[\Delta_1+\Delta_2,f_\beta(x_1-x_2)]\\
=&(\Delta_1+\Delta_2)f_\beta(x_1-x_2)\\
&+(2\nabla_1 f_\beta(x_1-x_2))\nabla_1+(2\nabla_2 f_\beta(x_1-x_2))\nabla_2\\
=&(V_N(x_1-x_2)-M_\beta(x_1-x_2))f_\beta(x_1-x_2)\\
&-(2\nabla_1 g_\beta(x_1-x_2))\nabla_1-(2\nabla_2 g_\beta(x_1-x_2))\nabla_2\ .
\end{aligned}
$$

Using symmetry the third and fifth line in (4.43) give

$$
-4N(N-1)\langle\!\langle\Psi_t,(\nabla_1 g_\beta(x_1-x_2))\nabla_1\widehat{r}\Psi_t\rangle\!\rangle=\gamma_c(\Psi_t,\varphi_t)\ .
$$

Using

$$
\frac{d}{dt}\left(\mathcal{E}_{W_\beta}(\Psi_t)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_t)\right)=\gamma_a(\Psi_t,\varphi_t)\ ,
$$

we obtain the desired result. □

Establishing the Grönwall estimate

Again, we will bound the time derivative of $\alpha(\Psi_t,\varphi_t)$ such that we can employ a Grönwall estimate.

Lemma 4.6.8. *Let Ψ_t the unique solution to $i\partial_t\Psi_t=H_{V_N}\Psi_t$ with initial datum $\Psi_0\in L^2_s(\mathbb{R}^{2N},\mathbb{C})\cap H^2(\mathbb{R}^{2N},\mathbb{C})$, $\|\Psi_0\|=1$. Let φ_t the unique solution to $i\partial_t\varphi_t=h^{GP}_{4\pi}\varphi_t$ with $\varphi_t\in H^3(\mathbb{R}^2,\mathbb{C})$. Let $\mathcal{E}_{V_N}(\Psi_0)\leq C$. Let $\gamma(\Psi_t,\varphi_t)$ be defined as in (4.41). Then, there exists an $\eta>0$ such that*

$$
\gamma(\Psi_t,\varphi_t)\leq\mathcal{K}(\varphi_t,A_t)\left(\langle\!\langle\Psi_t,\widehat{n}\Psi_t\rangle\!\rangle+N^{-\eta}+\left|\mathcal{E}_{V_N(\Psi_0)}-\mathcal{E}^{GP}_{b_{V_N}}(\varphi_0)\right|\right)\ .
\tag{4.44}
$$

A prove of the Lemma can be found in Section 4.7.4.
The most important estimate is the first part of γ_b, which can be estimated in the same way as $\gamma_b^<$. All other estimates are based on the smallness of the L^p-norms of g_β, see Lemma 4.5.5 (i). We now show that Lemma 4.6.8 implies convergence of the reduced density matrix

$\gamma^{(1)}_{\Psi_t}$ to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm. Using $\|\widehat{m}^a\|_{\mathrm{op}} + \|\widehat{m}^b\|_{\mathrm{op}} \leq CN^{-1+\xi}$, see (4.55), together with Equation (4.11) and Lemma 4.5.5 (i), we obtain

$$\begin{aligned}&\|g_\beta(x_1-x_2)\widehat{r}\|_{\mathrm{op}} \leq \|g_\beta(x_1-x_2)p_1(\widehat{m}^b p_2 + \widehat{m}^a q_2)\|_{\mathrm{op}} + \|g_\beta(x_1-x_2)p_2 q_1 \widehat{m}^a\|_{\mathrm{op}} \\ \leq &\mathcal{K}(\varphi, A_t)\|g_\beta\|(\|\widehat{m}^a\|_{\mathrm{op}} + \|\widehat{m}^b\|_{\mathrm{op}}) \leq \mathcal{K}(\varphi, A_t)N^{\xi-2-\beta}\ln(N).\end{aligned}$$

Therefore, we bound

$$N(N-1)|\mathrm{Re}\left(\langle\Psi, g_\beta(x_1-x_2)\widehat{r}\Psi\rangle\right)| \leq \mathcal{K}(\varphi, A_t)N^{-\beta+\xi}\ln(N). \tag{4.45}$$

For β large enough, (4.44) implies together with (4.45) that

$$\gamma(\Psi_t,\varphi_t) \leq \mathcal{K}(\varphi_t, A_t)\left(\alpha(\Psi_t,\varphi_t) + N^{-\eta}\right) ,$$

for some $\eta > 0$. We get with Lemma 4.6.7 and Grönwall's Lemma, using (4.45) again, that

$$\begin{aligned}\alpha^<(\Psi_t,\varphi_t) \leq e^{\int_0^t ds\mathcal{K}(\varphi_s,A_s)}\Big(&\alpha^<(\Psi_0,\varphi_0) \\ &+ \int_0^t ds\mathcal{K}(\varphi_s,A_s)e^{-\int_0^s d\tau\mathcal{K}(\varphi_\tau,A_\tau)}N^{-\eta}\Big) .\end{aligned}$$

Therefore, we obtain part (b) of Theorem 4.2.1.

4.7 Rigorous estimates

4.7.1 Smearing out the potential W_β

In Section 4.5 we have defined the potential M_β to control the strongly peaked potential V_N. We will employ a similar strategy to "smear out" the potential W_β when β is large. For this, we define, for $\beta_1 < \beta$, a potential $U_{\beta_1,\beta} \in \mathcal{V}_{\beta_1}$ such that $\|W_\beta\|_1 = \|U_{\beta_1,\beta}\|_1$. Furthermore, define $h_{\beta_1,\beta}$ by $\Delta h_{\beta_1,\beta} = W_\beta - U_{\beta_1,\beta}$. The function $h_{\beta_1,\beta}$ can be thought as an electrostatic potential which is caused by the charge $W_\beta - U_{\beta_1,\beta}$. It is then possible to rewrite

$$\begin{aligned}&\langle\!\langle\chi, W_\beta(x_1-x_2)\Omega\rangle\!\rangle = \langle\!\langle\chi, U_{\beta_1,\beta}(x_1-x_2)\Omega\rangle\!\rangle \\ &-\langle\!\langle\nabla_1\chi, (\nabla_1 h_{\beta_1,\beta})(x_1-x_2)\Omega\rangle\!\rangle - \langle\!\langle\chi, (\nabla_1 h_{\beta_1,\beta})(x_1-x_2)\nabla_1\Omega\rangle\!\rangle ,\end{aligned}$$

for $\chi,\omega \in L^2_s(\mathbb{R}^{2N},\mathbb{C})$. It is easy to verify that $h_{\beta_1,\beta}$ and $\nabla h_{\beta_1,\beta}$ are faster decaying than the potential W_β. The right hand side of the equation above is hence better to control, if one has additional control of $\nabla_1\Omega$ and $\nabla_1\chi$.

Definition 4.7.1. *For any $0 \leq \beta_1 < \beta$ and any $W_\beta \in \mathcal{V}_\beta$ we define*

$$U_{\beta_1,\beta}(x) = \begin{cases} \frac{4}{\pi}\|W_\beta\|_1 N^{2\beta_1} & \text{for } |x| < 1/2N^{-\beta_1}, \\ 0 & \text{else.}\end{cases}$$

and

$$h_{\beta_1,\beta}(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|(W_\beta(y) - U_{\beta_1,\beta}(y))d^2y . \tag{4.46}$$

Lemma 4.7.2. *For any $0 \leq \beta_1 < \beta$ and any $W_\beta \in \mathcal{V}_\beta$, we obtain with the above definition*

(a)

$$U_{\beta_1,\beta} \in \mathcal{V}_{\beta_1} \ ,$$
$$\Delta h_{\beta_1,\beta} = W_\beta - U_{\beta_1,\beta}.$$

(b) Pointwise estimates

$$|h_{\beta_1,\beta}(x)| \leq CN^{-1}\ln(N), \qquad h_{\beta_1,\beta}(x) = 0 \text{ for } |x| \geq N^{-\beta_1}, \tag{4.47}$$

$$|\nabla h_{\beta_1,\beta}(x)| \leq CN^{-1}\left(|x|^2 + N^{-2\beta}\right)^{-\frac{1}{2}} . \tag{4.48}$$

(c) Norm estimates

$$\begin{aligned} \|h_{\beta_1,\beta}\|_\infty &\leq CN^{-1}\ln(N), \\ \|h_{\beta_1,\beta}\|_\lambda &\leq CN^{-1-\frac{2}{\lambda}\beta_1}\ln(N) \quad \text{for } 1 \leq \lambda \leq \infty, \\ \|\nabla h_{\beta_1,\beta}\|_\lambda &\leq CN^{-1+\beta-\frac{2}{\lambda}\beta_1} \qquad \text{for } 1 \leq \lambda \leq \infty. \end{aligned}$$

Furthermore, for $\lambda = 2$, we obtain the improved bounds

$$\|h_{0,\beta}\| \leq CN^{-1} \text{ for } \beta > 0 \ , \tag{4.49}$$

$$\|\nabla h_{\beta_1,\beta}\| \leq CN^{-1}(\ln(N))^{1/2} \ . \tag{4.50}$$

Proof. (a) $U_{\beta_1,\beta} \in \mathcal{V}_{\beta_1}$ follows directly from the definition of $U_{\beta_1,\beta}$.
The second statement is a well known result of standard electrostatics (therefore recall that the radially symmetric Greens function of the Laplace operator in two dimensions is given by $-\frac{1}{2\pi}\ln|x-y|$). W_β can be understood as a given charge density. $-U_{\beta_1,\beta}$ then corresponds to a smeared out charge density of opposite sign such that the "total charge" is zero. Hence, the "potential" h_{β,β_1} can be chosen to be zero outside the support of the total charge density.[2]

(b) First note that $|h_{\beta_1,\beta}(x)| = 0$ for $|x| \geq 1/2N^{-\beta_1}$, which implies the pointwise estimate

$$\begin{aligned} |h_{\beta_1,\beta}(x)| \leq & \frac{1}{2\pi}\int_{B_{1/2N^{-\beta_1}(0)}} d^2y\, |\ln|x-y| W_\beta(y)| \\ & + \frac{1}{2\pi}\int_{B_{1/2N^{-\beta_1}(0)}} d^2y\, |\ln|x-y| U_{\beta_1,\beta}(y)| \ . \end{aligned}$$

We estimate each term separately. For $RN^{-\beta} < |x|$, we obtain

$$\int_{B_{1/2N^{-\beta_1}(0)}} d^2y |\ln|x-y||W_\beta(y) \leq C\|W_\beta\|_1 |\ln(|x| - RN^{-\beta})|,$$

[2] To see this, recall that the solution of $\Delta h(r) = \rho(r)$ for radially symmetric and regular enough charge density ρ is given by

$$h(r) = \ln(r)\int_0^r r'\rho(r')dr' + \int_r^\infty \ln(r')\rho(r')r'dr' + C,$$

where $C \in \mathbb{R}$. The r.h.s. is zero for $r \notin \operatorname{supp}(\rho)$ when the total charge vanishes $\int_0^\infty r\rho(r)dr = 0$ and C is chosen equal to zero.

which in turn implies

$$\int_{B_{1/2N^{-\beta_1}(0)}} d^2y|\ln|x-y||W_\beta(y) \leq C\|W_\beta\|_1 \ln N^\beta \leq CN^{-1}\ln(N)$$

for all $2RN^{-\beta} \leq |x|$.

Let next $|x| \leq 2RN^{-\beta}$. Note that $|x-y| \leq 1$ in the integral above, using $h_{\beta_1,\beta}(x) = 0$, whenever $|x| > 1/2\beta_1$. This implies $|\ln|x-y|| = -\ln|x-y|$ in the integral. Thus,

$$\begin{aligned}
&\int_{B_{N^{-\beta_1}(0)}} |\ln|x-y||W_\beta(y)d^2y \\
&\leq C\|W_\beta\|_\infty \int_{B_{RN^{-\beta}(0)}} -\ln|x-y|d^2y \\
&\leq CN^{-1+2\beta} \int_{B_{RN^{-\beta}(x)}} -\ln|y|d^2y \\
&\leq CN^{-1+2\beta} \int_{B_{4RN^{-\beta}(0)}} -\ln|y|d^2y \\
&= CN^{-1+2\beta}\Big[-|y|^2(2\ln|y|-1)\Big]_0^{4RN^{-\beta}} \leq CN^{-1}\ln\left(N^\beta\right),
\end{aligned}$$

Repeating these estimates for $U_{\beta_1,\beta}$ proves the first statement.
For the gradient, we estimate the two terms on the r.h.s. of

$$|\nabla h_{\beta_1,\beta}(x)| \leq \frac{1}{2\pi}\int \frac{1}{|x-y|}W_\beta(y)d^2y + \frac{1}{2\pi}\int \frac{1}{|x-y|}U_{\beta,\beta_1}(y)d^2y$$

separately. Let first $2RN^{-\beta} \leq |x|$. Similarly as in the previous argument, one finds

$$\int \frac{1}{|x-y|}W_\beta(y)d^2y \leq \int_{B_{RN^{-\beta}}(0)} \frac{1}{|x-y|}W_\beta(y)d^2y \leq \frac{\|W_\beta\|_1}{|x|-RN^{-\beta}}$$

for $RN^{-\beta} \leq |x|$, which implies that

$$\int \frac{1}{|x-y|}W_\beta(y)d^2y \leq \frac{C\|W_\beta\|_1}{(|x|^2+N^{-2\beta})^{\frac{1}{2}}} \leq \frac{CN^{-1}}{(|x|^2+N^{-2\beta})^{\frac{1}{2}}}$$

for all $2RN^{-\beta} \leq |x|$. For $|x| \leq 2RN^{-\beta}$, we make use of

$$N^\beta \leq \frac{C}{(|x|^2+N^{-2\beta})^{1/2}}$$

and estimate

$$\begin{aligned}
\int \frac{1}{|x-y|}W_\beta(y)d^2y &\leq \|W_\beta\|_\infty \int_{B_{RN^{-\beta}(0)}} \frac{1}{|x-y|}d^2y \\
&\leq CN^{2\beta-1}\int_0^{RN^{-\beta}} d|y| = CN^{-1+\beta} \leq \frac{CN^{-1}}{(|x|^2+N^{-2\beta})^{1/2}}.
\end{aligned}$$

Equivalently, we obtain

$$\int \frac{1}{|x-y|} U_{\beta_1,\beta}(y) d^2y \leq \|U_{\beta_1,\beta}\|_\infty \int_{B_{N^{-\beta_1}(0)}} \frac{1}{|x-y|} d^2y$$
$$= CN^{-1+\beta_1} \leq \frac{CN^{-1}}{(|x|^2 + N^{-2\beta_1})^{1/2}} \leq \frac{CN^{-1}}{(|x|^2 + N^{-2\beta})^{\frac{1}{2}}},$$

for $|x| \leq N^{-\beta_1}$. Since $\nabla h_{\beta_1,\beta}(x) = 0$ for $|x| \geq N^{-\beta_1}$, the second statement of (b) follows.

(c) The first part of (c) follows from (b) and the fact that the support of $h_{\beta 1,\beta}$ and $\nabla h_{\beta_1,\beta}$ has radius $\leq CN^{-\beta_1}$. The bounds on the L^2-norm can be improved by

$$\begin{aligned}\|\nabla h_{\beta_1,\beta}\|_2^2 \leq & C \int_0^{CN^{-\beta_1}} dr r |\nabla h_{\beta_1,\beta}(r)|^2 \leq \frac{C}{N^2} \int_0^{CN^{-\beta_1}} dr \frac{r}{r^2 + N^{-2\beta}} \\ = & \frac{C}{N^2} \ln\left(\frac{N^{-2\beta_1} + N^{-2\beta}}{N^{-2\beta}}\right) \leq \frac{C}{N^2} \ln(N)\end{aligned}$$

Using, for $|x| \geq 2RN^{-\beta}$, the inequality

$$|h_{0,\beta}(x)| \leq CN^{-1} |\ln(|x| - RN^{-\beta})|,$$

we obtain

$$\begin{aligned}\|h_{0,\beta}\|_2^2 = & \int_{\mathbb{R}^2} d^2x \mathbb{1}_{B_{2RN^{-\beta}(0)}}(x) |h_{0,\beta}(x)|^2 + \int_{\mathbb{R}^2} d^2x \mathbb{1}_{B^c_{2RN^{-\beta}(0)}}(x) |h_{0,\beta}(x)|^2 \\ \leq & \|h_{0\beta}\|_\infty^2 |B_{2RN^{-\beta}(0)}| + CN^{-2} \int_{2RN^{-\beta}}^1 dr r |\ln(r - RN^{-\beta})|^2 \\ \leq & C\left(N^{-2-2\beta}(\ln(N))^2 + N^{-2} \int_{RN^{-\beta}}^1 dr (r + RN^{-\beta})(\ln(r))^2\right).\end{aligned}$$

Using

$$\begin{aligned}& \int_{RN^{-\beta}}^1 dr (r + RN^{-\beta})(\ln(r))^2 \\ = & \left(\frac{1}{4} r^2 (2(\ln(r))^2 - 2\ln(r) + 1) + RN^{-\beta} r((\ln(r))^2 - 2\ln(r) + 2)\right)\Big|_{RN^{-\beta}}^1 \\ \leq & C\left(1 + N^{-\beta} + N^{-2\beta}(\ln(N))^2\right),\end{aligned}$$

we obtain, for any $\beta > 0$,

$$\|h_{0,\beta}\|_2^2 \leq CN^{-2}\left(1 + N^{-\beta} + N^{-2\beta}(\ln(N))^2\right) \leq CN^{-2}.$$

□

4.7.2 Estimates on the cutoff

In order to smear out singular potentials as explained in the previous section and to obtain sufficient bounds, it seems at first necessary to show that $\|\nabla_1 q_1 \Psi_t\|$ decays in N. However, this term will in fact not be small for the dynamic generated by V_N. There, we rather expect that $\|\nabla_1 q_1 \Psi_t\| = \mathcal{O}(1)$ holds. It has been shown in [22] and [57] that the interaction energy is purely kinetic in the Gross-Pitaevskii regime, which implies that a relevant part of the kinetic energy is concentrated around the scattering centers. We must thus cutoff the part which is used to form the microscopic structure. For this, we define the set $\overline{\mathcal{A}}_j^{(d)}$ which includes all configurations where the distance between particle x_i and particle x_j , $j \neq i$ is smaller than N^{-d}. It is then possible to prove that the kinetic energy concentrated on the complement of $\overline{\mathcal{A}}_j^{(d)}$, i.e. $\|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|$, is small, see Lemma 4.7.9.

Definition 4.7.3. *For any $j, k = 1, \ldots, N$ and $d > 0$ let*

$$a_{j,k}^{(d)} = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^{2N} : |x_j - x_k| < N^{-d}\} \subseteq \mathbb{R}^{2N} \tag{4.51}$$

$$\overline{\mathcal{A}}_j^{(d)} = \bigcup_{k \neq j} a_{j,k}^{(d)} \qquad \mathcal{A}_j^{(d)} = \mathbb{R}^{2N} \backslash \overline{\mathcal{A}}_j^{(d)} \qquad \overline{\mathcal{B}}_j^{(d)} = \bigcup_{k \neq l \neq j} a_{k,l}^{(d)} \qquad \mathcal{B}_j^{(d)} = \mathbb{R}^{2N} \backslash \overline{\mathcal{B}}_j^{(d)} .$$

Lemma 4.7.4. *Let $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^1(\mathbb{R}^{2N}, \mathbb{C})$ $\|\Psi\| = 1$ and let $\|\nabla_1 \Psi\|$ be uniformly bounded in N. Then, for all $j \neq k$ with $1 \leq j, k \leq N$,*

(a)

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} p_j\|_{op} \leq C\|\varphi\|_\infty N^{1/2-d} ,$$
$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} \nabla_j p_j\|_{op} \leq C\|\nabla\varphi\|_\infty N^{1/2-d} ,$$
$$\|\mathbb{1}_{a_{j,k}^{(d)}} p_j\|_{op} \leq C\|\varphi\|_\infty N^{-d} .$$

(b) For any $1 < p < \infty$

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} \Psi\| \leq C N^{\frac{1-2d}{2} \frac{p-1}{p}} ,$$

which implies that

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} \Psi\| \leq C N^{\frac{1}{2} - d + \epsilon}$$

for any $\epsilon > 0$.

(c)

$$\|\mathbb{1}_{\overline{\mathcal{B}}_j^{(d)}} \Psi\| \leq C N^{1-d+\epsilon}$$

for any $\epsilon > 0$.

(d) For any $k \neq j$

$$\|[\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}}, p_k]\|_{op} = \|[\mathbb{1}_{a_{j,k}^{(d)}}, p_k]\|_{op} = \|[\mathbb{1}_{\mathcal{A}_j^{(d)}}, p_k]\|_{op} \leq C\|\varphi\|_\infty N^{-d} .$$

Proof. (a) First note that the volume of the sets $a_{j,k}^{(d)}$ introduced in Definition 4.7.3 are $|a_{j,k}^{(d)}| = \pi N^{-2d}$.

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} p_j\|_{\text{op}} = \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} p_1\|_{\text{op}} = \|p_1 \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} p_1\|_{\text{op}}^{\frac{1}{2}} \leq \left(\|\varphi\|_\infty^2 \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\|_{1,\infty}\right)^{1/2}$$

where we defined

$$\|f\|_{p,\infty} = \sup_{x_2,\dots,x_N \in \mathbb{R}^2} \left(\int dx_1 |f(x_1,\dots,x_N)|^p\right)^{\frac{1}{p}} .$$

Using $\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \leq \sum_{k=2}^N \mathbb{1}_{a_{1,k}^{(d)}}$ as well as $\left(\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\right)^p = \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}$, we obtain

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\|_{p,\infty} \leq \sup_{x_2,\dots,x_N \in \mathbb{R}^2} \left(\int dx_1 \sum_{k=2}^N \mathbb{1}_{a_{1,k}^{(d)}}\right)^{\frac{1}{p}} \leq (N|a_{1,k}|)^{\frac{1}{p}} \leq CN^{(1-2d)\frac{1}{p}} .$$

This implies

$$\|\mathbb{1}_{\overline{\mathcal{A}}_j^{(d)}} p_j\|_{\text{op}} \leq C\|\varphi\|_\infty N^{\frac{1}{2}-d} .$$

The second statement of (a) can be proven similarly. Analogously, we obtain

$$\|\mathbb{1}_{a_{j,k}^{(d)}} p_j\|_{\text{op}} \leq \|\varphi\|_\infty |a_{j,k}^{(d)}|^{1/2} \leq C\|\varphi\|_\infty N^{-d} .$$

(b) Without loss of generality, we can set $j = 1$. Recall the two-dimensional Sobolev inequality, for $\varrho \in H^1(\mathbb{R}^2, \mathbb{C})$, $\|\varrho\|_m \leq C\|\nabla\varrho\|^{\frac{m-2}{m}} \|\varrho\|^{\frac{2}{m}}$ holds for any $2 < m < \infty$. Using Hölder and Sobolev for the x_1-integration, we get, for $p > 1$

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\Psi\|^2 = \langle\!\langle \Psi, \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\Psi\rangle\!\rangle = \int d^2x_2 \dots d^2x_N \int d^2x_1 |\Psi(x_1,\dots,x_N)|^2 \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}(x_1,\dots,x_N)$$

$$\leq \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\|_{\frac{p}{p-1},\infty} \int d^2x_2 \dots d^2x_N \left(\int d^2x_1 |\Psi(x_1,\dots,x_N)|^{2p}\right)^{1/p}$$

$$\leq CN^{(1-2d)\frac{p-1}{p}} \int d^2x_2 \dots d^2x_N \left(\int d^2x_1 |\nabla_1\Psi(x_1,\dots,x_N)|^2\right)^{\frac{p-1}{p}} \left(\int d^2\tilde{x}_1 |\Psi(\tilde{x}_1,\dots,x_N)|^2\right)^{\frac{1}{p}} .$$

Using Hölder for the $x_2, \dots x_N$-integration with the conjugate pair $r = \frac{p}{p-1}$ and $s = p$, we obtain

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\Psi\|^2 \leq CN^{(1-2d)\frac{p-1}{p}} \|\nabla_1\Psi\|^{2\frac{p-1}{p}} \|\Psi\|^{\frac{2}{p}} .$$

Using $\|\nabla_1\Psi\| < C$, (b) follows.

(c) We use that $\overline{\mathcal{B}}_j^{(d)} \subset \bigcup_{k=1} \overline{\mathcal{A}}_k^{(d)}$. Hence one can find pairwise disjoint sets $\mathcal{C}_k \subset \overline{\mathcal{A}}_k^{(d)}$, $k = 1, \dots, N$ such that $\overline{\mathcal{B}}_j^{(d)} \subset \bigcup_{k=1} \mathcal{C}_k$. Since the sets $\mathcal{C}_k$ are pairwise disjoint, the $\mathbb{1}_{\mathcal{C}_k}\Psi$ are pairwise orthogonal and we get

$$\|\mathbb{1}_{\overline{\mathcal{B}}_j^{(d)}}\Psi\|^2 = \sum_{k=1} \|\mathbb{1}_{\mathcal{C}_k}\Psi\|^2 \leq \sum_{k=1}^N \|\mathbb{1}_{\overline{\mathcal{A}}_k^{(d)}}\Psi\|^2 .$$

(d)

$$\|[\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}, p_2]\|_{\text{op}} \leq \|[\mathbb{1}_{a_{1,2}}, p_2]\|_{\text{op}} \leq \|\mathbb{1}_{a_{1,2}} p_2\|_{\text{op}} + \|p_2 \mathbb{1}_{a_{1,2}}\|_{\text{op}}$$

$$\leq 2\|\varphi\|_\infty |a_{1,2}|^{\frac{1}{2}} \leq C\|\varphi\|_\infty N^{-d} .$$

□

4.7.3 Estimates for the functionals γ_a, $\gamma_a^<$ and $\gamma_b^<$

Control of γ_a and $\gamma_a^<$

Lemma 4.7.5. *For any multiplication operator $B : L^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C})$ and any $\varphi \in L^2(\mathbb{R}^2, \mathbb{C})$ and any $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C})$ we have*

$$|\langle\!\langle \Psi, B(x_1)\Psi\rangle\!\rangle - \langle \varphi, B\varphi\rangle| \leq C\|B\|_\infty(\langle\!\langle \Psi, \widehat{n}^\varphi \Psi\rangle\!\rangle + N^{-\frac{1}{2}}) .$$

Proof. Using $1 = p_1 + q_1$,

$$\begin{aligned}
&\langle\!\langle \Psi, B(x_1)\Psi\rangle\!\rangle - \langle \varphi, B\varphi\rangle \\
=&\langle\!\langle \Psi, p_1 B(x_1) p_1 \Psi\rangle\!\rangle + 2\mathrm{Re}\langle\!\langle \Psi, q_1 B(x_1) p_1 \Psi\rangle\!\rangle + \langle\!\langle \Psi, q_1 B(x_1) q_1 \Psi\rangle\!\rangle - \langle \varphi, B\varphi\rangle \\
\leq&\langle \varphi, B\varphi\rangle(\|p_1\Psi\|^2 - 1) + 2\mathrm{Re}\langle\!\langle \Psi, \widehat{n}^{-1/2} q_1 B(x_1) p_1 \widehat{n}_1^{1/2}\Psi\rangle\!\rangle \\
+&\langle\!\langle \Psi, q_1 B(x_1) q_1 \Psi\rangle\!\rangle
\end{aligned}$$

where we used Lemma 4.4.2 (c). Since $\|p_1\Psi\|^2 - 1 = \|q_1\Psi\|^2$ it follows that

$$\begin{aligned}
|\langle\!\langle \Psi, B(x_1)\Psi\rangle\!\rangle - \langle \varphi, B\varphi\rangle| &\leq C\|B\|_\infty \left(\langle\!\langle \Psi, \widehat{n}^2\Psi\rangle\!\rangle + \langle\!\langle \Psi, \widehat{n}_1\Psi\rangle\!\rangle + \langle\!\langle \Psi, \widehat{n}\Psi\rangle\!\rangle\right) \\
&\leq C\|B\|_\infty(\langle\!\langle \Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-\frac{1}{2}}) . \qquad (4.52)
\end{aligned}$$

□

Using Lemma 4.7.5, setting $B = \dot{A}_t$, we get

$$\gamma_a^<(\Psi_t, \varphi_t) = \gamma_a(\Psi_t, \varphi_t) \leq C\|\dot{A}_t\|_\infty(\langle\!\langle \Psi_t, \widehat{n}^{\varphi_t}\Psi_t\rangle\!\rangle + N^{-\frac{1}{2}}) ,$$

which yields the first bound (4.38) in Lemma 4.6.5.

Control of $\gamma_b^<$ To control $\gamma_b^<$ we will first prove that $\|\nabla_1\Psi_t\|$ is uniformly bounded in N, if initially the energy per particle $\mathcal{E}_U(\Psi_0)$ is of order one.

Lemma 4.7.6. *Let $\Psi_0 \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$ with $\|\Psi_0\| = 1$. For any $U \in L^2(\mathbb{R}^2, \mathbb{R})$, $U(x) \geq 0$, let Ψ_t the unique solution to $i\partial_t\Psi_t = H_U\Psi_t$ with initial datum Ψ_0. Let $\mathcal{E}_U(\Psi_0) \leq C$. Then*

$$\|\nabla_1\Psi_t\| \leq \mathcal{K}(\varphi_t, A_t) .$$

Proof. Using $\frac{d}{dt}\mathcal{E}_U(\Psi_t) \leq \|\dot{A}_t\|_\infty$, we obtain $\mathcal{E}_U(\Psi_t) \leq \mathcal{K}(\varphi_t, A_t)$. This yields

$$\|\nabla_1\Psi_t\|^2 \leq \mathcal{K}(\varphi_t, A_t) - (N-1)\|\sqrt{U}\Psi_t\|^2 + \|A_t\|_\infty \leq \mathcal{K}(\varphi_t, A_t) .$$

□

Next, we control $\widehat{m}^a$ and $\widehat{m}^b$ which were defined in Definition 4.6.1. The difference $m(k) - m(k+1)$ and $m(k) - m(k+2)$ is of leading order given by the derivative of the function $m(k)$ – k understood as real variable – with respect to k. The k-derivative of $m(k)$ equals

$$m(k)' = \begin{cases} 1/(2\sqrt{kN}), & \text{for } k \geq N^{1-2\xi}; \\ 1/2(N^{-1+\xi}), & \text{else.} \end{cases} \qquad (4.53)$$

It is then easy to show that, for any $j \in \mathbb{Z}$, there exists a $C_j < \infty$ such that

$$\widehat{m}_j^x \leq C_j N^{-1} \widehat{n}^{-1} \text{ for } x \in \{a, b\} \tag{4.54}$$
$$\|\widehat{m}_j^x\|_{\mathrm{op}} \leq C_j N^{-1+\xi} \text{ for } x \in \{a, b\} \tag{4.55}$$
$$\|\widehat{n}\widehat{m}_j^x\|_{\mathrm{op}} \leq C_j N^{-1} \text{ for } x \in \{a, b\} \tag{4.56}$$
$$\|\widehat{r}\|_{\mathrm{op}} \leq \|\widehat{m}^a\|_{\mathrm{op}} + \|\widehat{m}^b\|_{\mathrm{op}} \leq C N^{-1+\xi} \,. \tag{4.57}$$

The different terms we have to estimate for $\gamma_b^<$ are found in (4.36). In order to facilitate the notation, let $\widehat{w} \in \{N\widehat{m}^a_{-1}, N\widehat{m}^b_{-2}\}$. Then $w(k) < n(k)^{-1}$ and $\|\widehat{w}\|_{\mathrm{op}} \leq CN^{\xi}$ follows.

Lemma 4.7.7. *Let $\beta > 0$ and $W_\beta \in \mathcal{V}_\beta$. Let $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^2(\mathbb{R}^{2N}, \mathbb{C})$, $\|\Psi\| = 1$ and let $\|\nabla_1 \Psi\| \leq \mathcal{K}(\varphi, A_t)$. Let $w(k) < n(k)^{-1}$ and $\|\widehat{w}\|_{op} \leq CN^{\xi}$ for some $\xi \geq 0$. Then,*

(a)

$$N \left| \langle\!\langle \Psi p_1 p_2 Z_\beta^\varphi(x_1, x_2) q_1 p_2 \widehat{w} \Psi \rangle\!\rangle \right| \leq \mathcal{K}(\varphi, A_t) \left(N^{-1} + N^{-2\beta} \ln(N) \right) \,.$$

(b)

$$\begin{aligned} &N |\langle\!\langle \Psi, p_1 p_2 W_\beta(x_1 - x_2) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle| \\ &\leq \mathcal{K}(\varphi, A_t) \Big(\langle\!\langle \Psi, \widehat{n} \Psi \rangle\!\rangle + \inf_{\eta > 0} \inf_{\beta > \beta_1 > 0} \left(N^{\eta - 2\beta_1} \ln(N)^2 + \|\widehat{w}\|_{op} N^{-1+2\beta_1} + \|\widehat{w}\|_{op}^2 N^{-\eta} \right) \Big) \,. \end{aligned}$$

(c)

$$\begin{aligned} N |\langle\!\langle \Psi p_1 q_2 Z_\beta^\varphi(x_1, x_2) \widehat{w} q_1 q_2 \Psi \rangle\!\rangle| &\leq \mathcal{K}(\varphi, A_t) \Big(\langle\!\langle \Psi, \widehat{n} \Psi \rangle\!\rangle + N^{-1/6} \ln(N) \\ &+ \inf \left\{ \left| \mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi) \right|, \left| \mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}_{N\|W_\beta\|_1}^{GP}(\varphi) \right| + N^{-2\beta} \ln(N) \right\} \Big) \,. \end{aligned}$$

Proof. (a) In view of Lemma 4.4.4, we obtain

$$\begin{aligned} N \left| \langle\!\langle \Psi, p_1 p_2 Z_\beta^\varphi(x_1, x_2) q_1 p_2 \widehat{w} \Psi \rangle\!\rangle \right| &\leq N \|p_1 p_2 Z_\beta^\varphi(x_1, x_2) q_1 p_2\|_{\mathrm{op}} \|\widehat{n} \widehat{w} \Psi\| \\ &\leq C N \|p_1 p_2 Z_\beta^\varphi(x_1, x_2) q_1 p_2\|_{\mathrm{op}} \,. \end{aligned}$$

$\|p_1 p_2 Z_\beta^\varphi(x_1, x_2) q_1 p_2\|_{\mathrm{op}}$ can be estimated using $p_1 q_1 = 0$ and (4.14):

$$\begin{aligned} &N \left\| p_1 p_2 \left(W_\beta(x_1 - x_2) - \frac{N\|W_\beta\|_1}{N-1} |\varphi(x_1)|^2 - \frac{N\|W_\beta\|_1}{N-1} |\varphi(x_2)|^2 \right) q_1 p_2 \right\|_{\mathrm{op}} \\ &\leq \|p_1 p_2 (N W_\beta(x_1 - x_2) - N\|W_\beta\|_1 |\varphi(x_1)|^2) p_2\|_{\mathrm{op}} + C\|\varphi\|_\infty^2 N^{-1} \\ &\leq \|\varphi\|_\infty \, \|N(W_\beta \star |\varphi|^2) - \|N W_\beta\|_1 |\varphi|^2\| + C\|\varphi\|_\infty^2 N^{-1} \,. \end{aligned}$$

Let h be given by

$$h(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} d^2 y \ln|x - y| N W_\beta(y) + \frac{1}{2\pi} \|N W_\beta\|_1 \ln|x| \,,$$

which implies

$$\Delta h(x) = N W_\beta(x) - \|N W_\beta\|_1 \delta(x) \,.$$

As above (see Lemma 4.7.2), we obtain $h(x) = 0$ for $x \notin B_{RN^{-\beta}}(0)$, where $RN^{-\beta}$ is the radius of the support of W_β. Thus,

$$\|h\|_1 \leq \frac{1}{2\pi}\int_{\mathbb{R}^2} d^2x \int_{\mathbb{R}^2} d^2y |\ln|x-y|| \mathbb{1}_{B_{RN^{-\beta}(0)}}(x) N W_\beta(y) \tag{4.58}$$

$$-\frac{1}{2\pi} N \|W_\beta\|_1 \int_{\mathbb{R}^2} d^2x \ln(|x|) \mathbb{1}_{B_{RN^{-\beta}(0)}}(x) \leq C N^{-2\beta}\ln(N) \tag{4.59}$$

Integration by parts and Young's inequality give that

$$\begin{aligned}&\|N(W_\beta \star |\varphi|^2) - \|NW_\beta\|_1 |\varphi|^2\| = \|(\Delta h) \star |\varphi|^2\| \\ \leq &\|h\|_1 \|\Delta|\varphi|^2\|_2 \leq \mathcal{K}(\varphi, A_t) N^{-2\beta}\ln(N) \, .\end{aligned}$$

Thus, we obtain the bound

$$N \left| \langle\!\langle \Psi, p_1 p_2 Z_\beta^\varphi(x_1,x_2) q_1 p_2 \widehat{w}\Psi \rangle\!\rangle \right| \leq \mathcal{K}(\varphi, A_t) \left(N^{-1} + N^{-2\beta}\ln(N) \right) \, , \tag{4.60}$$

which then proves (a).

(b) We will first consider $\beta < 1/2$.
By use of Lemma 4.4.2 (c) and Lemma 4.4.6 with $O_{1,2} = q_2 W_\beta(x_1 - x_2) p_2$, $\Omega = N^{-1/2}(\widehat{w})^{1/2} q_1 \Psi$ and $\chi = N^{1/2} p_1 (\widehat{w}_2)^{1/2}\Psi$ we get

$$\begin{aligned}
&|\langle\!\langle \Psi, p_1p_2 W_\beta(x_1-x_2) q_1 q_2 \widehat{w}\Psi\rangle\!\rangle| \\
=&|\langle\!\langle \Psi, (\widehat{w})^{1/2} q_1 q_2 W_\beta(x_1-x_2) p_1 p_2 (\widehat{w}_2)^{1/2}\Psi\rangle\!\rangle| \\
&\leq N^{-1}\left\|(\widehat{w})^{1/2} q_1 \Psi\right\|^2 + N\big|\langle\!\langle q_2(\widehat{w}_2)^{1/2}\,\Psi, p_1\sqrt{W_\beta}(x_1-x_2) p_3 \sqrt{W_\beta}(x_1-x_3) \\
&\qquad\qquad \sqrt{W_\beta}(x_1-x_2) p_2 \sqrt{W_\beta}(x_1-x_3) p_1 q_3 (\widehat{w}_2)^{1/2}\,\Psi\rangle\!\rangle\big| \\
&+ N(N-1)^{-1}\|q_2 W_\beta(x_1-x_2) p_2 p_1 (\widehat{w}_2)^{1/2}\Psi\|^2 \\
&\leq N^{-1}\left\|(\widehat{w})^{1/2} q_1 \Psi\right\|^2 + N\|\sqrt{W_\beta}(x_1-x_2)p_1\|_{\mathrm{op}}^4 \, \|q_2(\widehat{w}_2)^{1/2}\,\Psi\|^2 \\
&+ 2N(N-1)^{-1}\|p_1 q_2 (\widehat{w}_1)^{1/2} W_\beta(x_1-x_2) p_2 p_1 \Psi\|^2 \\
&+ 2N(N-1)^{-1}\|q_1 q_2 (\widehat{w})^{1/2} W_\beta(x_1-x_2) p_2 p_1 \Psi\|^2 \, .
\end{aligned}$$

With Lemma 4.4.2 (e) we get the bound

$$\begin{aligned}\leq &N^{-1}\|(\widehat{w})^{1/2}\widehat{n}\Psi\|^2 + N\|\varphi\|_\infty^4 \|W_\beta\|_1^2 \, \|\widehat{n}(\widehat{w}_2)^{1/2}\,\Psi\|^2 \\ &+ 2N(N-1)^{-1}\|W_\beta\|^2\|\varphi\|_\infty^2 \left(\|\widehat{w}_1\|_{\mathrm{op}} + \|\widehat{w}\|_{\mathrm{op}}\right) \, .\end{aligned}$$

Note, that $\|W_\beta\|_1 \leq CN^{-1}$, $\|W_\beta\|^2 \leq CN^{-2+2\beta}$. Furthermore, using $\widehat{n} < \widehat{n}_2$, we have under the conditions on $\widehat{w}$

$$\|(\widehat{w})^{1/2}\widehat{n}_2\Psi\| \leq \|(\widehat{w}_2)^{1/2}\widehat{n}_2\Psi\| \leq \|(\widehat{n}_2)^{1/2}\Psi\| \leq \sqrt{\langle \Psi, \widehat{n}\Psi\rangle} + 2N^{-\frac{1}{2}} \, . \tag{4.61}$$

In total, we obtain

$$N|\langle\!\langle \Psi, p_1p_2 W_\beta(x_1-x_2) q_1 q_2 \widehat{w}\Psi\rangle\!\rangle| \leq \mathcal{K}(\varphi, A_t)\left(\langle\!\langle \Psi, \widehat{n}\Psi\rangle\!\rangle + \|\widehat{w}\|_{\mathrm{op}} N^{-1+2\beta}\right)$$

and we get (b) for the case $\beta < 1/2$.

b) for $1/2 \leq \beta$: We use $U_{\beta_1,\beta}$ from Definition 4.7.1 for some $0 < \beta_1 < 1/2$. We then obtain

$$\begin{aligned}
&N\langle\!\langle \Psi, p_1p_2W_\beta(x_1-x_2)\widehat{w}q_1q_2\Psi\rangle\!\rangle \\
=&N\langle\!\langle \Psi, p_1p_2U_{\beta_1,\beta}(x_1-x_2)\widehat{w}q_1q_2\Psi\rangle\!\rangle && (4.62)\\
+&N\langle\!\langle \Psi, p_1p_2\left(W_\beta(x_1-x_2)-U_{\beta_1,\beta}(x_1-x_2)\right)\widehat{w}q_1q_2\Psi\rangle\!\rangle && (4.63)
\end{aligned}$$

Term (4.62) has been controlled above. So we are left to control (4.63).

Let $\Delta h_{\beta_1,\beta} = W_\beta - U_{\beta_1,\beta}$. Integrating by parts and using that $\nabla_1 h_{\beta_1,\beta}(x_1-x_2) = -\nabla_2 h_{\beta_1,\beta}(x_1-x_2)$ gives

$$\begin{aligned}
&N\left|\langle\!\langle \Psi, p_1p_2\left(W_\beta(x_1-x_2)-U_{\beta_1,\beta}(x_1-x_2)\right)\widehat{w}q_1q_2\Psi\rangle\!\rangle\right| \\
&\leq N\left|\langle\!\langle \nabla_1p_1\Psi, p_2\nabla_2h_{\beta_1,\beta}(x_1-x_2)\widehat{w}q_1q_2\Psi\rangle\!\rangle\right| && (4.64)\\
&+N\left|\langle\!\langle \Psi, p_1p_2\nabla_2h_{\beta_1,\beta}(x_1-x_2)\nabla_1\widehat{w}q_1q_2\Psi\rangle\!\rangle\right| . && (4.65)
\end{aligned}$$

Let $t_1 \in \{p_1, \nabla_1p_1\}$ and let $\Gamma \in \{\widehat{w}q_1\Psi, \nabla_1\widehat{w}q_1\Psi\}$.

For both (4.64) and (4.65), we use Lemma 4.4.6 with $O_{1,2} = N^{1+\eta/2}q_2\nabla_2h_{\beta_1,\beta}(x_1-x_2)p_2$, $\chi = t_1\Psi$ and $\Omega = N^{-\eta/2}\Gamma$. This yields

$$\begin{aligned}
&(4.64)+(4.65) \leq 2\sup_{t_1\in\{p_1,\nabla_1p_1\},\Gamma\in\{\widehat{w}q_1\Psi,\nabla_1\widehat{w}q_1\Psi\}}\Big(N^{-\eta}\|\Gamma\|^2 && (4.66)\\
&+\frac{N^{2+\eta}}{N-1}\|q_2\nabla_2h_{\beta_1,\beta}(x_1-x_2)t_1p_2\Psi\|^2 && (4.67)\\
&+N^{2+\eta}\left|\langle\!\langle \Psi, t_1p_2q_3\nabla_2h_{\beta_1,\beta}(x_1-x_2)\nabla_3h_{\beta_1,\beta}(x_1-x_3)t_1q_2p_3\Psi\rangle\!\rangle\right|\Big) . && (4.68)
\end{aligned}$$

The first term can be bounded using Corrolary 4.4.5 by

$$\begin{aligned}
N^{-\eta}\|\nabla_1\widehat{w}q_1\Psi\|^2 \leq& N^{-\eta}\|\widehat{w}\|_{\mathrm{op}}^2\|\nabla_1q_1\Psi\|^2\\
N^{-\eta}\|\widehat{w}q_1\Psi\|^2 \leq& CN^{-\eta} .
\end{aligned}$$

Thus $(4.66) \leq \mathcal{K}(\varphi, A_t)N^{-\eta}\|\widehat{w}\|_{\mathrm{op}}^2$ using that $\|\nabla_1q_1\Psi\| \leq \mathcal{K}(\varphi, A_t)$. By $\|t_1\Psi\|^2 \leq \mathcal{K}(\varphi, A_t)$, we obtain

$$\begin{aligned}
(4.67) \leq& \mathcal{K}(\varphi, A_t)\frac{N^{2+\eta}}{N-1}\|\nabla_2h_{\beta_1,\beta}(x_1-x_2)p_2\|_{\mathrm{op}}^2 \leq \mathcal{K}(\varphi, A_t)\frac{N^{2+\eta}}{N-1}\|\varphi\|_\infty^2\|\nabla h_{\beta_1,\beta}\|^2\\
\leq& \mathcal{K}(\varphi, A_t)N^{\eta-1}\ln(N) ,
\end{aligned}$$

where we used Lemma 4.7.2 in the last step.

Next, we estimate

$$\begin{aligned}
(4.68) \leq& N^{2+\eta}\|p_2\nabla_2h_{\beta_1,\beta}(x_1-x_2)t_1q_2\Psi\|^2\\
\leq& 2N^{2+\eta}\|p_2h_{\beta_1,\beta}(x_1-x_2)t_1\nabla_2q_2\Psi\|^2\\
&+2N^{2+\eta}\||\varphi(x_2)\rangle\langle\nabla\varphi(x_2)|h_{\beta_1,\beta}(x_1-x_2)t_1q_2\Psi\|^2\\
\leq& 2N^{2+\eta}\|p_2h_{\beta_1,\beta}(x_1-x_2)\|_{\mathrm{op}}^2\|t_1\nabla_2q_2\Psi\|^2\\
&+2N^{2+\eta}\||\varphi(x_2)\rangle\langle\nabla\varphi(x_2)|h_{\beta_1,\beta}(x_1-x_2)\|_{\mathrm{op}}^2\|t_1q_2\Psi\|^2\\
\leq& \mathcal{K}(\varphi, A_t)N^{2+\eta}\|h_{\beta_1,\beta}\|^2\\
\leq& \mathcal{K}(\varphi, A_t)N^{\eta-2\beta_1}\ln(N)^2 .
\end{aligned}$$

Thus, for all $\eta \in \mathbb{R}$

$$\begin{aligned}&N\langle\!\langle \Psi, p_1p_2\left(W_\beta(x_1-x_2)-U_{\beta_1,\beta}(x_1-x_2)\right)\widehat{w}q_1q_2\Psi\rangle\!\rangle\\ \le&\mathcal{K}(\varphi,A_t)\left(\|\widehat{w}\|_{\mathrm{op}}^2N^{-\eta}+N^{\eta-1}\ln(N)+N^{\eta-2\beta_1}\ln(N)^2\right).\end{aligned}$$

Combining both estimates for $\beta<1/2$ and $\beta\ge 1/2$, we obtain, using $N^{\eta-1}\ln(N)<N^{\eta-2\beta_1}\ln(N)$,

$$\begin{aligned}&N\langle\!\langle \Psi, p_1p_2W_\beta(x_1-x_2)\widehat{w}q_1q_2\Psi\rangle\!\rangle\\ &\le \mathcal{K}(\varphi,A_t)\left(\langle\!\langle\Psi,\widehat{n}\Psi\rangle\!\rangle+\inf_{\eta>0}\inf_{\beta_1>0}\left(N^{\eta-2\beta_1}\ln(N)^2+N^{-1+2\beta_1}+\|\widehat{w}\|_{\mathrm{op}}^2N^{-\eta}\right)\right).\end{aligned}$$

and we get (b) in full generality.

(c) We first estimate, noting that $q_1p_2|\varphi|^2(x_1)q_1q_2=0$,

$$\begin{aligned}&N\left|\langle\!\langle\Psi,q_1p_2\frac{N\|W_\beta\|_1}{N-1}|\varphi|^2(x_2)\widehat{w}q_1q_2\Psi\rangle\!\rangle\right|\le C\|\varphi\|_\infty^2\|\widehat{w}q_2\|_{\mathrm{op}}\|q_1\Psi\|^2\\ &\le \mathcal{K}(\varphi,A_t)\langle\!\langle\Psi,\widehat{n}\Psi\rangle\!\rangle\,.\end{aligned}$$

It is left to estimate $N\,|\langle\!\langle\Psi,q_1p_2W_\beta(x_1-x_2)\widehat{w}q_1q_2\Psi\rangle\!\rangle|$. Let $U_{0,\beta}$ be given as in Definition 4.7.1. Using Lemma 4.4.2 (c) and integrating by parts we get

$$\begin{aligned}&N\,|\langle\!\langle\Psi,q_1p_2W_\beta(x_1-x_2)\widehat{w}q_1q_2\Psi\rangle\!\rangle|\\ \le&N\,|\langle\!\langle\Psi,q_1p_2U_{0,\beta}(x_1-x_2)q_1q_2\widehat{w}\Psi\rangle\!\rangle|+N\,|\langle\!\langle\Psi,q_1p_2(\Delta_1h_{0,\beta}(x_1-x_2))q_1q_2\widehat{w}\Psi\rangle\!\rangle|\\ \le&\|U_{0,\beta}\|_\infty N\|q_1\Psi\|\,\|\widehat{w}q_1q_2\Psi\|\\ &+N\,|\langle\!\langle\nabla_1q_1p_2\Psi,(\nabla_1h_{0,\beta}(x_1-x_2))\widehat{w}q_1q_2\Psi\rangle\!\rangle|\\ &+N\,|\langle\!\langle\Psi,\widehat{w}_1q_1p_2(\nabla_1h_{0,\beta}(x_1-x_2))\nabla_1q_1q_2\Psi\rangle\!\rangle|\\ \le&N\|U_{0,\beta}\|_\infty\|q_1\Psi\|\,\|\widehat{w}q_1q_2\Psi\| &(4.69)\\ &+N\left|\langle\!\langle\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1q_1\Psi,p_2(\nabla_1h_{0,\beta}(x_1-x_2))\widehat{w}q_1q_2\Psi\rangle\!\rangle\right| &(4.70)\\ &+N\left|\langle\!\langle\nabla_1q_1\Psi,\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}p_2(\nabla_1h_{0,\beta}(x_1-x_2))q_1q_2\widehat{w}\Psi\rangle\!\rangle\right| &(4.71)\\ &+N\left|\langle\!\langle\Psi,\widehat{w}_1q_1p_2(\nabla_1h_{0,\beta}(x_1-x_2))q_2\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1q_1\Psi\rangle\!\rangle\right| &(4.72)\\ &+N\left|\langle\!\langle\Psi,\widehat{w}_1q_1p_2(\nabla_1h_{0,\beta}(x_1-x_2))q_2\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1q_1\Psi\rangle\!\rangle\right|\,. &(4.73)\end{aligned}$$

Lemma 4.4.4 and Lemma 4.7.2 (a) yields the bound

$$(4.69)\le C\langle\!\langle\Psi,\widehat{n}\Psi\rangle\!\rangle\,.$$

For (4.71) and (4.73) we use Cauchy Schwarz and then Sobolev inequality as in Lemma 4.7.4 to get, for any $p>1$,

$$\begin{aligned}&(4.71)+(4.73)\le N\,\|\nabla_1q_1\Psi\|\,\left\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}p_2(\nabla_1h_{0,\beta}(x_1-x_2))q_1q_2\widehat{w}\Psi\right\|\\ &+N\,\|\nabla_1q_1\Psi\|\,\left\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}q_2(\nabla_1h_{0,\beta}(x_1-x_2))q_1p_2\widehat{w}_1\Psi\right\|\\ \le&CN\|\nabla_1q_1\Psi\|\,N^{\frac{1-2d}{2}\frac{p-1}{p}}\|\nabla_1p_2(\nabla_1h_{0,\beta}(x_1-x_2))q_1q_2\widehat{w}\Psi\|^{\frac{p-1}{p}}\|p_2(\nabla_1h_{0,\beta}(x_1-x_2))q_1q_2\widehat{w}\Psi\|^{1/p}\\ +&CN\|\nabla_1q_1\Psi\|\,N^{\frac{1-2d}{2}\frac{p-1}{p}}\|\nabla_1q_2(\nabla_1h_{0,\beta}(x_1-x_2))q_1p_2\widehat{w}_1\Psi\|^{\frac{p-1}{p}}\|q_2(\nabla_1h_{0,\beta}(x_1-x_2))q_1p_2\widehat{w}_1\Psi\|^{1/p}\end{aligned}$$

Using Lemma 4.4.2, Lemma 4.4.4, Corollary 4.4.5 and Lemma 4.7.2, we obtain

$$\begin{aligned}\|\nabla_1 p_2(\nabla_1 h_{0,\beta}(x_1-x_2))q_1q_2\widehat{w}\Psi\| \leq&\|p_2(\Delta_1 h_{0,\beta}(x_1-x_2))q_1q_2\widehat{w}\Psi\|\\&+\|p_2(\nabla_1 h_{0,\beta}(x_1-x_2))\nabla_1 q_1q_2\widehat{w}\Psi\|\\ \leq& C\left(\|p_2(W_\beta-U_{0,\beta})(x_1-x_2)\|_{\mathrm{op}}+\|p_2\nabla_1 h_{0,\beta}(x_1-x_2))\|_{\mathrm{op}}\right)\\ \leq& C\|\varphi\|_\infty\left(N^{-1+\beta}+N^{-1}(\ln(N))^{1/2}\right)\,,\end{aligned}$$

and similarly

$$\begin{aligned}\|\nabla_1 q_2(\nabla_1 h_{0,\beta}(x_1-x_2))q_1p_2\widehat{w}_1\Psi\| \leq&\|q_2(\Delta_1 h_{0,\beta}(x_1-x_2))q_1p_2\widehat{w}_1\Psi\|\\&+\|q_2(\nabla_1 h_{0,\beta}(x_1-x_2))\nabla_1 q_1p_2\widehat{w}_1\Psi\|\\ \leq& C\left(\|p_2(W_\beta-U_{0,\beta})(x_1-x_2)\|_{\mathrm{op}}+\|\widehat{w}_1\|_{\mathrm{op}}\|p_2\nabla_1 h_{0,\beta}(x_1-x_2))\|_{\mathrm{op}}\right)\\ \leq& C\|\varphi\|_\infty\left(N^{-1+\beta}+\|\widehat{w}\|_{\mathrm{op}}N^{-1}(\ln(N))^{1/2}\right)\,.\end{aligned}$$

Moreover, we estimate

$$\begin{aligned}\|p_2(\nabla_1 h_{0,\beta}(x_1-x_2))q_1q_2\widehat{w}\Psi\| &\leq C\|\varphi\|_\infty\|\nabla_1 h_{0,\beta}\|_2\leq C\|\varphi\|_\infty N^{-1}(\ln(N))^{1/2}\\ \|q_2(\nabla_1 h_{0,\beta}(x_1-x_2))q_1p_2\widehat{w}\Psi\| &\leq C\|\varphi\|_\infty\|\nabla_1 h_{0,\beta}\|_2\leq C\|\varphi\|_\infty N^{-1}(\ln(N))^{1/2}\,.\end{aligned}$$

Hence, we obtain, for any $p>1$,

$$(4.71)+(4.73)\leq C\|\varphi\|_\infty N^{1+\frac{1-2d}{2}\frac{p-1}{p}}\left(N^{-1+\beta}+\|\widehat{w}\|_{\mathrm{op}}N^{-1}(\ln(N))^{1/2}\right)^{\frac{p-1}{p}}\left(N^{-1}(\ln(N))^{1/2}\right)^{1/p}.$$

For d large enough, the right hand side can be bounded by N^{-1}, that is

$$(4.71)+(4.73)\leq C\|\varphi\|_\infty N^{-1}\,.$$

For (4.70) we use that $\nabla_2 h_{0,\beta}(x_1-x_2)=-\nabla_1 h_{0,\beta}(x_1-x_2)$, Cauchy Schwarz and $ab\leq a^2+b^2$ and get

$$(4.70)\leq\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2+N^2\|p_2(\nabla_2 h_{0,\beta}(x_1-x_2))\widehat{w}q_1q_2\Psi\|^2\,. \tag{4.74}$$

$\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2$ can be bounded using Lemma 4.7.9.

Integration by parts and Lemma 4.4.2 (c) as well as $(a+b)^2\leq 2a^2+2b^2$ gives for the second summand

$$\begin{aligned}&N^2\|p_1(\nabla_1 h_{0,\beta}(x_1-x_2))q_1q_2\widehat{w}\Psi\|^2\leq 2N^2\|p_1h_{0,\beta}(x_1-x_2)\nabla_1 q_1q_2\widehat{w}\Psi\|^2\\&+2N^2\||\varphi(x_1)\rangle\langle\nabla_1\varphi(x_1)|h_{0,\beta}(x_1-x_2)q_1q_2\widehat{w}\Psi\|^2\\&\leq 2N^2\|p_1h_{0,\beta}(x_1-x_2)q_2(p_1\widehat{w}_1+q_1\widehat{w})\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2 &(4.75)\\&+2N^2\|p_1h_{0,\beta}(x_1-x_2)q_2p_1\widehat{w}_1\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 q_1\Psi\|^2 &(4.76)\\&+2N^2\|p_1h_{0,\beta}(x_1-x_2)q_2q_1\widehat{w}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 q_1\Psi\|^2 &(4.77)\\&+2N^2\||\varphi(x_1)\rangle\langle\nabla_1\varphi(x_1)|h_{0,\beta}(x_1-x_2)q_1q_2\widehat{w}\Psi\|^2\,. &(4.78)\end{aligned}$$

For (4.75) we use Lemma 4.4.4, Lemma 4.4.2 (e) with Lemma 4.7.2 (c) and then Lemma 4.7.9.

$$\begin{aligned}(4.75)&\leq CN^2\|p_1h_{0,\beta}(x_1-x_2)\|_{\mathrm{op}}^2\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2\\&\leq\mathcal{K}(\varphi,A_t)\Big(\langle\Psi,\widehat{n}^\varphi\Psi\rangle+N^{-1/6}\ln(N)\\&+\inf\left\{\left|\mathcal{E}_{V_N}(\Psi)-\mathcal{E}_{4\pi}^{GP}(\varphi)\right|,\left|\mathcal{E}_{W_\beta}(\Psi)-\mathcal{E}_{N\|W_\beta\|_1}^{GP}(\varphi)\right|+N^{-2\beta}\ln(N)\right\}\Big)\,.\end{aligned}$$

Let $s_1 \in \{p_1, q_1\}$ and let $\widehat{d} \in \{\widehat{w}, \widehat{w}_1\}$. Note that $\|\widehat{d}\|_{\text{op}} = \|\widehat{w}\|_{\text{op}}$. Then, (4.76) and (4.77) can be estimated as

$$
\begin{aligned}
&(4.76), (4.77) \leq \\
&\leq 2N^2 \|\nabla_1 q_1 \Psi\| \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \widehat{d} s_1 q_2 h_{0,\beta}(x_1 - x_2) p_1 h_{0,\beta}(x_1 - x_2) q_2 s_1 \widehat{d} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 q_1 \Psi\| \\
&\leq CN^{2+\frac{1-2d}{2}\frac{p-1}{p}} \|\nabla_1 q_1 \Psi\| \|\nabla_1 \widehat{d} s_1 q_2 h_{0,\beta}(x_1 - x_2) p_1 h_{0,\beta}(x_1 - x_2) q_2 s_1 \widehat{d} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 q_1 \Psi\|^{\frac{p-1}{p}} \\
&\times \|\widehat{d} s_1 q_2 h_{0,\beta}(x_1 - x_2) p_1 h_{0,\beta}(x_1 - x_2) q_2 s_1 \widehat{d} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 q_1 \Psi\|^{\frac{1}{p}} \\
&\leq CN^{2+\frac{1-2d}{2}\frac{p-1}{p}} \|\nabla_1 q_1 \Psi\| \|\widehat{w}\|_{\text{op}} \|p_1 h_{0,\beta}(x_1 - x_2)\|_{\text{op}} \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 q_1 \Psi\| \\
&\times \|\nabla_1 \widehat{d} s_1 q_2 h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}}^{\frac{p-1}{p}} \|\widehat{d} s_1 q_2 h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}}^{\frac{1}{p}} \\
&\leq \mathcal{K}(\varphi, A_t) N^{1+\frac{1-2d}{2}\frac{p-1}{p}} \|\widehat{w}\|_{\text{op}}^2 \|\nabla_1 s_1 h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}}^{\frac{p-1}{p}} \|h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}}^{\frac{1}{p}} \\
&\leq \mathcal{K}(\varphi, A_t) N^{1+\frac{1-2d}{2}\frac{p-1}{p}} \|\widehat{w}\|_{\text{op}}^2 \left(\|\nabla\varphi\| \|\nabla_1 h_{0,\beta}\| + \|h_{0,\beta}\|\right)^{\frac{p-1}{p}} \|h_{0,\beta}\|^{\frac{1}{p}} \\
&\leq \mathcal{K}(\varphi, A_t) \|\widehat{w}\|_{\text{op}}^2 (1 + \ln(N))^{\frac{p-1}{2p}} N^{\frac{1-2d}{2}\frac{p-1}{p}} \ .
\end{aligned}
$$

Here, we used, for $s_1 \in \{p_1, 1 - p_1\}$,

$$
\begin{aligned}
\|\nabla_1 s_1 h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}} &\leq \|\nabla_1 p_1 h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}} + \|\nabla_1 h_{0,\beta}(x_1 - x_2) p_1\|_{\text{op}} \\
&\leq \|\varphi\|_\infty \left(\|\nabla\varphi\| \|h_{0,\beta}\| + \|\nabla h_{0,\beta}\|\right)
\end{aligned}
$$

and then applied Lemma 4.4.2 (e).

For d large enough, we obtain

$$
(4.76) + (4.77) \leq \mathcal{K}(\varphi, A_t) N^{-2} \ .
$$

Line (4.78) can be bounded by

$$
\begin{aligned}
(4.78) \leq& N^2 \|h_{0,\beta}(x_1 - x_2) \nabla_1 p_1\|_{\text{op}}^2 \ \|q_1 q_2 \widehat{w} \Psi\|^2 \leq N^2 \|h_{0,\beta}\|^2 \|\nabla\varphi\|_\infty^2 \|q_1 \widehat{w}\|_{\text{op}}^2 \|q_1 \Psi\|^2 \\
\leq& C \|\nabla\varphi\|_\infty^2 \langle\langle \Psi \widehat{n} \Psi \rangle\rangle \ .
\end{aligned}
$$

For (4.72) we use Lemma 4.4.6 with $\Omega = \mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi$, $O_{1,2} = N q_2 (\nabla_2 h_{0,\beta}(x_1 - x_2)) p_2$ and $\chi = \widehat{w} q_1 \Psi$.

$$
\begin{aligned}
(4.72) \leq& \|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 &(4.79)\\
&+ 2N \|q_2 (\nabla_2 h_{0,\beta}(x_1 - x_2)) \widehat{w} q_1 p_2 \Psi\|^2 &(4.80)\\
&+ N^2 \left| \langle\langle \Psi, q_1 q_3 \widehat{w} (\nabla_2 h_{0,\beta}(x_1 - x_2)) p_2 p_3 (\nabla_3 h_{0,\beta}(x_1 - x_3)) \widehat{w} q_1 q_2 \Psi \rangle\rangle \right| \ . &(4.81)
\end{aligned}
$$

Line (4.80) is bounded by

$$
\begin{aligned}
(4.80) \leq& C \|\varphi\|_\infty^2 N \|(\nabla_2 h_{0,\beta}(x_1 - x_2)) p_2\|_{\text{op}}^2 \|\widehat{w} q_1\|_{\text{op}}^2 \\
\leq& C \|\varphi\|_\infty^2 N \|\nabla_2 h_{0,\beta}(x_1 - x_2)\|^2 \leq C \|\varphi\|_\infty^2 N^{-1} \ln(N) \ .
\end{aligned}
$$

(4.79)+(4.81) is bounded by

$$
\|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 + N^2 \|p_2 (\nabla_2 h_{0,\beta}(x_1 - x_2)) \widehat{w} q_1 q_2 \Psi\|^2 \ .
$$

Both terms have been controlled above (see (4.74)). In total, we obtain

$$N|\langle\langle \Psi p_1 q_2 Z_\beta^\varphi(x_1,x_2)\widehat{w} q_1 q_2 \Psi\rangle\rangle| \le \mathcal{K}(\varphi,A_t)\Big(\langle\langle \Psi,\widehat{n}\Psi\rangle\rangle + N^{-1/6}\ln(N)$$
$$+\inf\left\{\left|\mathcal{E}_{V_N}(\Psi)-\mathcal{E}^{GP}_{4\pi}(\varphi)\right|,\left|\mathcal{E}_{W_\beta}(\Psi)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)\right| + N^{-2\beta}\ln(N)\right\}\Big)\ .$$

□

Using this Lemma, it follows that there exists an $\eta>0$ such that

$$\gamma_b^<(\Psi_t,\varphi_t) \le \mathcal{K}(\varphi_t,A_t)\left(\langle\langle \Psi_t,\widehat{n}^{\varphi_t}\Psi_t\rangle\rangle + N^{-\eta} + \left|\mathcal{E}_{W_\beta}(\Psi_0)-\mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi_0)\right|\right)\ .$$

This proves Lemma 4.6.5.

4.7.4 Estimates for the functional γ

For the most involved scaling which is induced by V_N, we need to control $\|p_1 V_N\Psi\|$.

Lemma 4.7.8. *Let $\Psi\in L^2_s(\mathbb{R}^{2N},\mathbb{C})$ and let $\mathcal{E}_{V_N}(\Psi)\le C$. Then*

$$\|p_1 V_N\Psi\| \le \mathcal{K}(\varphi,A_t)N^{-\frac{1}{2}}\ . \tag{4.82}$$

Proof. We estimate

$$\|p_1 V_N(x_1-x_2)\Psi\| = \|p_1 \mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_2)V_N(x_1-x_2)\Psi\|$$
$$\le \|p_1 \mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_2)\|_{\mathrm{op}}\|V_N(x_1-x_2)\Psi\|\ .$$

We have

$$\|p_1 \mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_2)\|^2_{\mathrm{op}} \le \|\varphi\|^2_\infty \|\mathbb{1}_{\mathrm{supp}(V_N)}\|_1 \le C\|\varphi\|^2_\infty e^{-2N}\ .$$

Using

$$C\ge \mathcal{E}_{V_N}(\Psi) = \|\nabla\Psi\|^2 + (N-1)\|\sqrt{V_N}(x_1-x_2)\Psi\|^2 + \langle\langle \Psi, A_t(x_1)\Psi\rangle\rangle$$

as well as

$$\|V_N(x_1-x_2)\Psi\|^2 = \|\sqrt{V_N}(x_1-x_2)\sqrt{V_N}(x_1-x_2)\Psi\|^2 \le \|\sqrt{V_N}\|^2_\infty\|\sqrt{V_N}(x_1-x_2)\Psi\|^2$$
$$\le Ce^{2N}\frac{\mathcal{E}_{V_N}(\Psi)+\|A_t\|_\infty}{N} \le C(1+\|A_t\|_\infty)\frac{e^{2N}}{N}\ ,$$

we obtain

$$\|p_1 V_N\Psi\| \le \mathcal{K}(\varphi,A_t)N^{-\frac{1}{2}}\ .$$

□

Control of γ_b Recall that

$$\begin{aligned}\gamma_b(\Psi,\varphi) = &-N(N-1)\mathrm{Im}\left(\langle\!\langle \Psi, \widetilde{Z}^{\varphi}_{\beta}(x_1,x_2)\widehat{r}\,\Psi\rangle\!\rangle\right)\\ &- N(N-1)\mathrm{Im}\left(\langle\!\langle \Psi, g_\beta(x_1-x_2)\widehat{r}\,\mathcal{Z}^{\varphi}(x_1,x_2)\Psi\rangle\!\rangle\right) \ .\end{aligned}$$

Estimate (4.82) yields to the bound $\|p_1\mathcal{Z}^{\varphi}(x_1,x_2)\Psi\| \leq \mathcal{K}(\varphi,A_t)N^{-1/2}$. Therefore, the second line of γ_b is controlled by

$$\begin{aligned}&N^2\|g_\beta(x_1-x_2)p_1\|_{\mathrm{op}}\|\widehat{r}\|_{\mathrm{op}}\|p_1\mathcal{Z}^{\varphi}(x_1,x_2)\Psi\|\\ \leq&\mathcal{K}(\varphi,A_t)N^{3/2}\|g_\beta\|\|\widehat{r}\|_{\mathrm{op}} \leq \mathcal{K}(\varphi,A_t)N^{\xi-1/2-\beta/2}\ln(N) \ .\end{aligned}$$

The first line of γ_b can be bounded with (4.42) and $f_\beta = 1-g_\beta$ by

$$N(N-1)|\mathrm{Im}\left(\langle\!\langle \Psi, \widetilde{Z}^{\varphi}_{\beta}(x_1,x_2)\widehat{r}\,\Psi\rangle\!\rangle\right)|$$

$$\leq N^2|\mathrm{Im}\big(\langle\!\langle\Psi, \big(M_\beta(x_1-x_2)f_\beta(x_1-x_2) - \frac{N}{N-1}\big(\|M_\beta f_\beta\|_1|\varphi(x_1)|^2 + \|M_\beta f_\beta\|_1|\varphi(x_2)|^2\big)\big)\widehat{r}\Psi\rangle\!\rangle\big)| \tag{4.83}$$

$$+\frac{N^2}{N-1}|\langle\!\langle\Psi, \left(\|NM_\beta f_\beta\|_1 - 4\pi\right)\left(|\varphi(x_1)|^2+|\varphi(x_2)|^2\right)\widehat{r}\Psi\rangle\!\rangle| \tag{4.84}$$

$$+\frac{N^2}{N-1}|\langle\!\langle\Psi, \left(4\pi|\varphi(x_1)|^2+4\pi|\varphi(x_2)|^2\right)g_\beta(x_1-x_2)\widehat{r}\Psi\rangle\!\rangle| \ . \tag{4.85}$$

Since $M_\beta f_\beta \in \mathcal{V}_\beta$, (4.83) is of the same form as $\gamma_b^<(\Psi,\varphi)$. Using Lemma 4.5.5 (h), the second term is controlled by

$$(4.84) \leq C\|\varphi\|_\infty^2 N\left(N\|M_\beta f_\beta\|_1 - 4\pi\right)\|\hat{r}\|_{\mathrm{op}} \leq C\|\varphi\|_\infty^2 N^{-1+\xi}\ln(N) \ .$$

The last term is controlled by

$$(4.85) \leq CN\|\varphi\|_\infty^2\|g_\beta(x_1-x_2)p_1\|_{\mathrm{op}}\|\widehat{r}\|_{\mathrm{op}} \leq C\|\varphi\|_\infty^3 N^{-1-\beta+\xi}\ln(N) \ .$$

and we get

$$|\gamma_b(\Psi,\varphi)| \leq \mathcal{K}(\varphi,A_t)\left(\langle\!\langle\Psi,\widehat{m}\Psi\rangle\!\rangle + |\mathcal{E}_{V_N}(\Psi) - \mathcal{E}^{GP}_{4\pi}(\varphi)| + N^{-\eta}\right)$$

for some $\eta > 0$.

Control of γ_c Recall that

$$\gamma_c(\Psi,\varphi) = -\,4N(N-1)\langle\!\langle\Psi, (\nabla_1 g_\beta(x_1-x_2))\nabla_1\widehat{r}\Psi\rangle\!\rangle \ .$$

Using $\widehat{r} = (p_2+q_2)\widehat{r} = p_2\widehat{r} + p_1q_2\widehat{m}^a$ and $\nabla_1 g_\beta(x_1-x_2) = -\nabla_2 g_\beta(x_1-x_2)$, integration by parts yields to

$$|\gamma_c(\Psi,\varphi)| \leq 4N^2|\langle\!\langle\Psi, g_\beta(x_1-x_2)\nabla_1\nabla_2(p_2\widehat{r} + p_1q_2\widehat{m}^a)\Psi\rangle\!\rangle| \tag{4.86}$$

$$+\,4N^2|\langle\!\langle\nabla_2\Psi, g_\beta(x_1-x_2)\nabla_1 p_2\widehat{r}\Psi\rangle\!\rangle| \tag{4.87}$$

$$+\,4N^2|\langle\!\langle\nabla_2\Psi, g_\beta(x_1-x_2)\nabla_1 p_1q_2\widehat{m}^a\Psi\rangle\!\rangle| \ . \tag{4.88}$$

We begin with

$$\begin{aligned}(4.86) \leq&CN^2\|g_\beta\|\|\nabla\varphi\|_\infty\left(\|\nabla_1\widehat{r}\psi\| + \|\nabla_2 q_2\widehat{m}^a\Psi\|\right)\\ \leq&CN^{1-\beta}\ln(N)\|\nabla\varphi\|_\infty\left(\|\nabla_1\widehat{r}\psi\| + \|\nabla_2 q_2\widehat{m}^a\Psi\|\right) \ .\end{aligned}$$

Let $s_1, t_1 \in \{p_1, q_1\}, \quad s_2, t_2 \in \{p_2, q_2\}$. Inserting the identity $1 = (p_1 + q_1)(p_2 + q_2)$, we obtain, for $a \in \{-2, -1, 0, 1, 2\}$,

$$\begin{aligned}\|\nabla_1 \widehat{r}\Psi\| \leq & C \sup_{s_1,s_2,t_1,t_2,a} \|\widehat{r}_a s_1 s_2 \nabla_1 t_1 t_2 \Psi\| \leq C \sup_{t_1,a} \|\widehat{r}_a\|_{\mathrm{op}} \|\nabla_1 t_1 \Psi\| \\ \leq & C N^{-1+\xi} \ .\end{aligned}$$

In analogy $\|\nabla_2 q_2 \widehat{m}^a \Psi\| \leq C \|\widehat{m}^a\|_{\mathrm{op}} \leq C N^{-1+\xi}$. This yields the bound

$$(4.86) \leq \mathcal{K}(\varphi, A_t) N^{-\beta+\xi} \ln(N) \ .$$

Furthermore, (4.87) is bounded by

$$(4.87) \leq 4N^2 \|\nabla_2 \Psi\| \; \|g_\beta\| \; \|\nabla\varphi\|_\infty \|\nabla_1 \widehat{r}\Psi\| \leq C \|\nabla\varphi\|_\infty \; N^{\xi-\beta} \ln(N) \ . \tag{4.89}$$

Similarly, we obtain

$$(4.88) \leq 4N^2 \|\nabla_2 \Psi\| \; \|g_\beta\| \; \|\nabla\varphi\|_\infty \|q_2 \widehat{m}^a \Psi\| \leq C \|\nabla\varphi\|_\infty \; N^{\xi-\beta} \ln(N) \ .$$

It follows that $|\gamma_c(\Psi, \varphi)| \leq \mathcal{K}(\varphi, A_t) N^{\xi-\beta} \ln(N)$.

Control of γ_d To control γ_d and γ_e we will use the notation

$$\begin{aligned} m^c(k) &= m^a(k) - m^a(k+1) \quad & m^d(k) &= m^a(k) - m^a(k+2) \\ m^e(k) &= m^b(k) - m^b(k+1) \quad & m^f(k) &= m^b(k) - m^b(k+2) \ . \end{aligned} \tag{4.90}$$

Since the second k-derivative of m is given by (see (4.53) for the first derivative)

$$m(k)'' = \begin{cases} -1/(4\sqrt{k^3 N}), & \text{for } k \geq N^{1-2\xi}; \\ 0, & \text{else.} \end{cases}$$

it is easy to verify that

$$\|\widehat{m}_j^x\|_{\mathrm{op}} \leq C N^{-2+3\xi} \text{ for } x \in \{c, d, e, f\} \ . \tag{4.91}$$

Recall that

$$\begin{aligned}\gamma_d(\Psi, \varphi) = & 2N(N-1)(N-2) \mathrm{Im} \left(\langle\!\langle \Psi, g_\beta(x_1 - x_2) \left[V_N(x_1 - x_3), \widehat{r} \right] \Psi \rangle\!\rangle \right) \\ & N(N-1)(N-2) \mathrm{Im} \left(\langle\!\langle \Psi, g_\beta(x_1 - x_2) \left[4\pi |\varphi|^2(x_3), \widehat{r} \right] \Psi \rangle\!\rangle \right) \ .\end{aligned}$$

Since $p_j + q_j = 1$, we can rewrite $\widehat{r}$ as

$$\widehat{r} = \widehat{m}^b p_1 p_2 + \widehat{m}^a (p_1 q_2 + q_1 p_2) = (\widehat{m}^b - 2\widehat{m}^a) p_1 p_2 + \widehat{m}^a (p_1 + p_2) \ .$$

Thus,

$$\begin{aligned}|\gamma_d(\Psi, \varphi)| \leq & C N^3 \left| \langle\!\langle \Psi, g_\beta(x_1 - x_2) \left[V_N(x_1 - x_3), (\widehat{m}^b - 2\widehat{m}^a) p_1 p_2 + \widehat{m}^a (p_1 + p_2) \right] \Psi \rangle\!\rangle \right| \\ & + C N^3 \left| \langle\!\langle \Psi, g_\beta(x_1 - x_2) \left[4\pi |\varphi|^2(x_3), \widehat{r} \right] \Psi \rangle\!\rangle \right| \\ \leq & C N^3 \left| \langle\!\langle \Psi, g_\beta(x_1 - x_2) p_2 \left[V_N(x_1 - x_3), \widehat{m}^a \right] \Psi \rangle\!\rangle \right| & (4.92) \\ & + C N^3 \left| \langle\!\langle \Psi, g_\beta(x_1 - x_2) V_N(x_1 - x_3) (\widehat{m}^b - 2\widehat{m}^a) p_1 p_2 \Psi \rangle\!\rangle \right| & (4.93) \\ & + C N^3 \left| \langle\!\langle \Psi, g_\beta(x_1 - x_2) (\widehat{m}^b - 2\widehat{m}^a) p_1 p_2 V_N(x_1 - x_3) \Psi \rangle\!\rangle \right| & (4.94) \\ & + C N^3 \left| \langle\!\langle \Psi, g_\beta(x_1 - x_2) \widehat{m}^a p_1 V_N(x_1 - x_3) \Psi \rangle\!\rangle \right| & (4.95) \\ & + C N^3 \left| \langle\!\langle \Psi, g_\beta(x_1 - x_2) V_N(x_1 - x_3) \widehat{m}^a p_1 \Psi \rangle\!\rangle \right| & (4.96) \\ & + C N^3 \left| \langle\!\langle \Psi, g_\beta(x_1 - x_2) \left[4\pi |\varphi|^2(x_3), \widehat{r} \right] \Psi \rangle\!\rangle \right| \ . & (4.97)\end{aligned}$$

Using Lemma 4.4.2 (d), we obtain the following estimate:

$$\begin{aligned}
(4.92) =&CN^3 \left|\langle\!\langle \Psi, g_\beta(x_1-x_2)p_2 \left[V_N(x_1-x_3), p_1p_3\widehat{m}^d + p_1q_3\widehat{m}^c + q_1p_3\widehat{m}^c\right] \Psi\rangle\!\rangle\right| \\
\le&CN^3 \big|\langle\!\langle \Psi, V_N(x_1-x_3)g_\beta(x_1-x_2)p_2\mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_3) \\
&\qquad\qquad\qquad \left(p_1p_3\widehat{m}^d + p_1q_3\widehat{m}^c + q_1p_3\widehat{m}^c\right) \Psi\rangle\!\rangle\big| \\
&+ CN^3 \left|\langle\!\langle \Psi, g_\beta(x_1-x_2)p_2 \left(p_1p_3\widehat{m}^d + p_1q_3\widehat{m}^c + q_1p_3\widehat{m}^c\right) V_N(x_1-x_3)\Psi\rangle\!\rangle\right| .
\end{aligned}$$

Both lines are bounded by

$$\begin{aligned}
&CN^3 \|V_N(x_1-x_3)\Psi\| \, \|g_\beta(x_1-x_2)p_2\|_{\mathrm{op}} \\
&\left(2\|\mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_3)p_1\|_{\mathrm{op}} + \|\mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_3)p_3\|_{\mathrm{op}}\right) \left(\|\widehat{m}^d\|_{\mathrm{op}} + \|\widehat{m}^c\|_{\mathrm{op}}\right) .
\end{aligned}$$

In view of Lemma 4.4.2 (e) with Lemma 4.5.5 (i), $\|g_\beta(x_1-x_2)p_2\|_{\mathrm{op}} \le \|\varphi\|_\infty\|g_\beta\| \le C\|\varphi\|_\infty N^{-1-\beta}\ln(N)$. Using (4.91), together with $\|\mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_3)p_1\|_{\mathrm{op}}\|V_N(x_1-x_3)\Psi\| \le \mathcal{K}(\varphi, A_t)N^{-1/2}$, we obtain, using $\xi < 1/2$,

$$(4.92) \le \mathcal{K}(\varphi, A_t)N^{-1/2+3\xi-\beta}\ln(N) \le \mathcal{K}(\varphi, A_t)N^{1/2+\xi-\beta}\ln(N) .$$

We continue with

$$\begin{aligned}
&(4.93) + (4.94) + (4.95) \\
\le&CN^3\|V_N(x_1-x_3)\Psi\|\|g_\beta(x_1-x_2)p_2\|_{\mathrm{op}} \\
&\times \|\mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_3)p_1\|_{\mathrm{op}}\|(\widehat{m}^b - 2\widehat{m}^a)\|_{\mathrm{op}} \\
&+ CN^3\|g_\beta(x_1-x_2)p_2\|_{\mathrm{op}}\|\widehat{m}^b - 2\widehat{m}^a\|_{\mathrm{op}}\|p_1V_N(x_1-x_3)\Psi\| \\
&+ CN^3\|g_\beta(x_1-x_2)p_1\|_{\mathrm{op}}\|\widehat{m}^a\|_{\mathrm{op}}\|p_1V_N(x_1-x_3)\Psi\| \\
\le&\mathcal{K}(\varphi, A_t)N^{1/2+\xi-\beta}\ln(N) .
\end{aligned}$$

Next, we estimate (4.96). The support of the function $g_\beta(x_1-x_2)V_N(x_1-x_3)$ is such that $|x_1-x_2| \le CN^{-\beta}$, as well as $|x_1-x_3| \le Ce^{-N}$. Therefore, $g_\beta(x_1-x_2)V_N(x_1-x_3) \neq 0$ implies $|x_2-x_3| \le CN^{-\beta}$. We estimate

$$\begin{aligned}
(4.96) =&CN^3\left|\langle\!\langle \Psi, g_\beta(x_1-x_2)V_N(x_1-x_3)p_1\mathbb{1}_{B_{CN^{-\beta}}(0)}(x_2-x_3)\widehat{m}^a\Psi\rangle\!\rangle\right| \\
\le&CN^3\|p_1V_N(x_1-x_3)g_\beta(x_1-x_2)\Psi\|\|\mathbb{1}_{B_{CN^{-\beta}}(0)}(x_2-x_3)\widehat{m}^a\Psi\| \\
\le&CN^3\|p_1\mathbb{1}_{\mathrm{supp}(V_N)}(x_1-x_3)\|_{\mathrm{op}}\|g_\beta(x_1-x_2)V_N(x_1-x_3)\Psi\|\|\mathbb{1}_{B_{CN^{-\beta}}(0)}(x_2-x_3)\widehat{m}^a\Psi\| \\
\le&CN^{5/2}\|g_\beta\|_\infty\|\mathbb{1}_{B_{CN^{-\beta}}(0)}\|^{\frac{1}{2}}_{\frac{p}{p-1}}\|\nabla_1\widehat{m}^a\Psi\|^{\frac{p-1}{p}}\|\widehat{m}^a\Psi\|^{\frac{1}{p}} \\
\le&CN^{5/2}\|g_\beta\|_\infty N^{-\beta/2}\|\nabla_1\widehat{m}^a\Psi\|^{1/2}\|\widehat{m}^a\Psi\|^{1/2} \\
\le&CN^{3/2+\xi-\beta/2} .
\end{aligned}$$

In the fourth line, we applied Sobolev inequality as in the proof of Lemma 4.7.4, then setting $p = 2$. Furthermore, we used $\|\nabla_1\widehat{m}^a\Psi\|^{1/2}\|\widehat{m}^a\Psi\|^{1/2} \le CN^{-1+\xi}$, as well as $\|g_\beta\|_\infty \le C$, see Lemma 4.5.5. Using Lemma 4.4.2 (d), (4.97) can be bounded by

$$\begin{aligned}
&CN^3\left|\langle\!\langle \Psi, g_\beta(x_1-x_2)\left[4\pi|\varphi|^2(x_3), p_1p_2(\widehat{r}-\widehat{r}_2) + (p_1q_2+q_1p_2)(\widehat{r}-\widehat{r}_1)\right]\Psi\rangle\!\rangle\right| \\
&\le CN^3\|\varphi\|_\infty^2\left(\|\widehat{r}-\widehat{r}_2\|_{\mathrm{op}} + \|\widehat{r}-\widehat{r}_1\|_{\mathrm{op}}\right)\|g_\beta(x_1-x_2)p_2\|_{\mathrm{op}} .
\end{aligned}$$

Note that $\|\widehat{r}-\widehat{r}_2\|_{\text{op}}+\|\widehat{r}-\widehat{r}_1\|_{\text{op}}\leq\sum_{j\in\{c,d,e,f\}}\|\widehat{m}^j\|_{\text{op}}\leq CN^{-2+3\xi}$ holds. With $\|g_\beta(x_1-x_2)p_2\|_{\text{op}}\leq CN^{-1-\beta}\ln(N)$, it then follows that

$$|(4.97)|\leq C\|\varphi\|_\infty^2 N^{3\xi-\beta}\ln(N)\ .$$

In total, we obtain

$$|\gamma_d(\Psi,\varphi)|\leq\mathcal{K}(\varphi,A_t)\left(N^{3/2+\xi-\beta/2}+N^{1/2+3\xi-\beta}\ln(N)\right)\ .$$

Control of γ_e Recall that

$$\begin{aligned}\gamma_e(\Psi,\varphi)=&-\frac{1}{2}N(N-1)(N-2)(N-3)\\&\operatorname{Im}\left(\langle\!\langle\Psi,g_\beta(x_1-x_2)\left[V_N(x_3-x_4),\widehat{r}\right]\Psi\rangle\!\rangle\right)\ .\end{aligned}$$

Using symmetry, Lemma 4.4.2 (d) and notation (4.90), γ_e is bounded by

$$\begin{aligned}\gamma_e(\Psi,\varphi)\leq&N^4\Big|\langle\!\langle\Psi,g_\beta(x_1-x_2)\big[V_N(x_3-x_4),\widehat{m}^cp_1p_2p_3p_4+2\widehat{m}^dp_1p_2p_3q_4\\&\qquad+2\widehat{m}^ep_1q_2p_3p_4+4\widehat{m}^fp_1q_2p_3q_4\big]\Psi\rangle\!\rangle\Big|\\\leq&4N^4\|V_N(x_3-x_4)\Psi\|\|\mathbb{1}_{\text{supp}(V_N)}(x_3-x_4)p_3\|_{\text{op}}\|g_\beta(x_1-x_2)p_1\|_{\text{op}}\\&(\|\widehat{m}^c\|_{\text{op}}+\|\widehat{m}^d\|_{\text{op}}+\|\widehat{m}^e\|_{\text{op}}+\|\widehat{m}^f\|_{\text{op}})\ .\end{aligned}$$

We get with (4.91), Lemma 4.5.5 and Lemma 4.4.2 that

$$|\gamma_e(\Psi,\varphi)|\leq\mathcal{K}(\varphi,A_t)N^{1/2+3\xi-\beta}\ln(N)\ .$$

Control of γ_f Recall that

$$\gamma_f(\Psi,\varphi)=2N(N-1)\frac{N-2}{N-1}\operatorname{Im}\left(\langle\!\langle\Psi,g_\beta(x_1-x_2)\left[4\pi|\varphi|^2(x_1),\widehat{r}\right]\Psi\rangle\!\rangle\right)\ .$$

We obtain the estimate

$$|\gamma_f(\Psi,\varphi)|\leq\mathcal{K}(\varphi,A_t)N^2\|g_\beta\|\|\widehat{r}\|_{\text{op}}\leq\mathcal{K}(\varphi,A_t)N^{\xi-\beta}\ln(N)\ .$$

Collecting all estimates, we get with $\xi<1/2$

$$|\gamma_c(\Psi,\varphi)|+|\gamma_d(\Psi,\varphi)|+|\gamma_e(\Psi,\varphi)|+|\gamma_f(\Psi,\varphi)|\leq\mathcal{K}(\varphi,A_t)N^{2-\beta/2}\ln(N)\ .$$

Choosing β sufficiently large, we obtain the desired decay and hence Lemma 4.6.8.

4.7.5 Energy estimates

Lemma 4.7.9. *Let $\Psi\in L_s^2(\mathbb{R}^{2N},\mathbb{C})\cap H^1(\mathbb{R}^{2N},\mathbb{C})$, $\|\Psi\|=1$ with $\|\nabla_1\Psi\|\leq\mathcal{K}(\varphi,A_t)$. Let $\varphi\in H^3(\mathbb{R}^2,\mathbb{C})$, $\|\varphi\|=1$. Define the sets $\mathcal{A}_1^{(d)},\overline{\mathcal{B}}_1^{(d)}$ as in Definition 4.7.3. Then, for d large enough,*

$$\begin{aligned}&\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1q_1\Psi\|^2+\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\nabla_1q_1\Psi\|^2\leq\mathcal{K}(\varphi,A_t)\Big(\langle\!\langle\Psi,\widehat{n}^\varphi\Psi\rangle\!\rangle+N^{-1/6}\ln(N)\\&+\inf\left\{\left|\mathcal{E}_{V_N}(\Psi)-\mathcal{E}_{4\pi}^{GP}(\varphi)\right|,\left|\mathcal{E}_{W_\beta}(\Psi)-\mathcal{E}_{N\|W_\beta\|_1}^{GP}(\varphi)\right|+N^{-2\beta}\ln(N)\right\}\Big)\ .\end{aligned}$$

Proof. We start with expanding $\mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)$. This yields

$$\begin{aligned}\mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi) &= \|\nabla_1\Psi\|^2 + \frac{N-1}{2}\|\sqrt{W_\beta}(x_1-x_2)\Psi\|^2 \\ &- \|\nabla\varphi\|^2 - \frac{1}{2}N\|W_\beta\|_1\|\varphi^2\|^2 + \langle\!\langle\Psi, A_t(x_1)\Psi\rangle\!\rangle - \langle\varphi, A_t\varphi\rangle \\ =&\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{A}_1^{(d)}}}\nabla_1\Psi\|^2 + M(\Psi,\varphi) + Q_\beta(\Psi,\varphi)\ ,\end{aligned}$$

where we have defined

$$\begin{aligned}M(\Psi,\varphi) =&2\mathrm{Re}\left(\langle\!\langle\nabla_1 q_1\Psi, \mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 p_1\Psi\rangle\!\rangle\right) &(4.98)\\ &+ \|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 p_1\Psi\|^2 - \|\nabla\varphi\|^2 &(4.99)\\ &+ \langle\!\langle\Psi, A_t(x_1)\Psi\rangle\!\rangle - \langle\varphi, A_t\varphi\rangle\ , &(4.100)\\ Q_\beta(\Psi,\varphi) =&\|\mathbb{1}_{\mathcal{B}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1\Psi\|^2 \\ &+\frac{N-1}{2}\langle\!\langle\Psi, (1-p_1p_2)W_\beta(x_1-x_2)(1-p_1p_2)\Psi\rangle\!\rangle \\ &+\frac{N-1}{2}\langle\!\langle\Psi, p_1p_2W_\beta(x_1-x_2)p_1p_2\Psi\rangle\!\rangle - \frac{1}{2}N\|W_\beta\|_1\|\varphi^2\|^2 \\ &+(N-1)\mathrm{Re}\langle\!\langle\Psi, (1-p_1p_2)W_\beta(x_1-x_2)p_1p_2\Psi\rangle\!\rangle\ .\end{aligned}$$

Notice that the first two terms in $Q_\beta(\Psi,\varphi)$ are nonnegative. This yields to the bound

$$\begin{aligned}S_\beta(\Psi,\varphi) =&(N-1)|\langle\!\langle\Psi, (1-p_1p_2)W_\beta(x_1-x_2)p_1p_2\Psi\rangle\!\rangle| &(4.101)\\ &+\left|\frac{N-1}{2}\langle\!\langle\Psi, p_1p_2W_\beta(x_1-x_2)p_1p_2\Psi\rangle\!\rangle - \frac{1}{2}N\|W_\beta\|_1\|\varphi^2\|^2\right| &(4.102)\\ &\geq -Q_\beta(\Psi,\varphi)\ .\end{aligned}$$

In total, we obtain

$$|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 q_1\Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\nabla_1 q_1\Psi\|^2 \leq M(\Psi,\varphi) + S_\beta(\Psi,\varphi) + \left|\mathcal{E}_{W_\beta}(\Psi) - \mathcal{E}^{GP}_{N\|W_\beta\|_1}(\varphi)\right|\ .$$

Next, we split up the energy difference $\mathcal{E}_{V_N}(\Psi) - \mathcal{E}^{GP}_{4\pi}(\varphi)$,

$$\begin{aligned}\mathcal{E}_{V_N}(\Psi) - \mathcal{E}^{GP}_{4\pi}(\varphi) = \|\nabla_1\Psi\|^2 + \frac{N-1}{2}\|\sqrt{V_N}(x_1-x_2)\Psi\|^2 - \|\nabla\varphi\|^2 \\ - 2\pi\|\varphi^2\|^2 + \langle\!\langle\Psi, A_t(x_1)\Psi\rangle\!\rangle - \langle\varphi, A_t\varphi\rangle\ .\end{aligned}$$

In order to better estimate the terms corresponding to the two-particle interactions, we introduce, for $\mu > d$, the potential $M_\mu(x)$, defined in Definition 4.5.4, and continue with

$$\begin{aligned}\mathcal{E}_{V_N}(\Psi) - \mathcal{E}^{GP}_{4\pi}(\varphi) =&\|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1\Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{A}^{(d)}}_1}\nabla_1\Psi\|^2 + \|\mathbb{1}_{\mathcal{B}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1\Psi\|^2 \\ &+ \frac{N-1}{2}\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\sqrt{V_N}(x_1-x_2)\Psi\|^2 \\ &+ \frac{1}{2}\langle\!\langle\Psi, \sum_{j\neq 1}\mathbb{1}_{\mathcal{B}_1^{(d)}}\left(V_N - M_\mu\right)(x_1-x_j)\Psi\rangle\!\rangle \\ &+ \frac{1}{2}\langle\!\langle\Psi, \sum_{j\neq 1}\mathbb{1}_{\mathcal{B}_1^{(d)}}M_\mu(x_1-x_j)\Psi\rangle\!\rangle - \|\nabla\varphi\|^2 - 2\pi\|\varphi^2\|^2 \\ &+ \langle\!\langle\Psi, A_t(x_1)\Psi\rangle\!\rangle - \langle\varphi, A_t\varphi\rangle\ .\end{aligned}$$

Using that $q_1 = 1 - p_1$ and symmetry gives (after reordering)

$$
\begin{aligned}
&\mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi) \\
=&\|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \mathbb{1}_{\overline{\mathcal{A}^{(d)}}_1} \nabla_1 \Psi\|^2 + \frac{N-1}{2} \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \sqrt{V_N}(x_1 - x_2)\Psi\|^2 \\
&+ \frac{N-1}{2} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} (1 - p_1 p_2) M_\mu(x_1 - x_2)(1 - p_1 p_2) \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \\
&+ \|\mathbb{1}_{\mathcal{B}_1^{(d)}} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle\!\langle \Psi, \sum_{j \neq 1} \mathbb{1}_{\mathcal{B}_1^{(d)}} \left(V_N - M_\mu\right)(x_1 - x_j)\Psi \rangle\!\rangle \\
&+ \frac{N-1}{2} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 M_\mu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle - 2\pi \|\varphi^2\|^2 \\
&+ 2\mathrm{Re} \left(\langle\!\langle \nabla_1 q_1 \Psi, \mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 p_1 \Psi \rangle\!\rangle \right) \\
&+ (N-1)\mathrm{Re} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} (1 - p_1 p_2) M_\mu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \\
&+ \|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 p_1 \Psi\|^2 - \|\nabla \varphi\|^2 \\
&+ \langle\!\langle \Psi, A_t(x_1)\Psi \rangle\!\rangle - \langle \varphi, A_t \varphi \rangle \\
=&\|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \nabla_1 \Psi\|^2 + M(\Psi, \varphi) + \tilde{Q}_\mu(\Psi, \varphi) \, .
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{Q}_\mu(\Psi, \varphi) =& \frac{N-1}{2} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} (1 - p_1 p_2) M_\mu(x_1 - x_2)(1 - p_1 p_2) \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \\
&+ \frac{N-1}{2} \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \sqrt{V_N}(x_1 - x_2)\Psi\|^2 \\
&+ \|\mathbb{1}_{\mathcal{B}_1^{(d)}} \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}} \nabla_1 \Psi\|^2 + \frac{1}{2} \langle\!\langle \Psi, \sum_{j \neq 1} \mathbb{1}_{\mathcal{B}_1^{(d)}} \left(V_N - M_\mu\right)(x_1 - x_j)\Psi \rangle\!\rangle \qquad (4.103) \\
&+ (N-1)\mathrm{Re} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} (1 - p_1 p_2) M_\mu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \\
&+ \frac{N-1}{2} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 M_\mu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle - 2\pi \|\varphi^2\|^2 \, .
\end{aligned}
$$

The first two terms in $\tilde{Q}_\mu(\Psi, \varphi)$ are nonnegative. For $\mu > d$ Lemma 4.7.10 below shows that (4.103) is also nonnegative. Thus, for $\mu > d$, we obtain the bound

$$
\begin{aligned}
\tilde{S}_\mu(\Psi, \varphi) =&(N-1) \left| \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} (1 - p_1 p_2) M_\mu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \right| \qquad (4.104) \\
&+ \left| \frac{N-1}{2} \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 M_\mu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle - 2\pi \|\varphi^2\|^2 \right| \qquad (4.105) \\
\geq& - \tilde{Q}_\mu(\Psi, \varphi) \, .
\end{aligned}
$$

In total, we obtain

$$
\|\mathbb{1}_{\mathcal{A}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}} \nabla_1 q_1 \Psi\|^2 \leq |M(\Psi, \varphi)| + \tilde{S}_\mu(\Psi, \varphi) + \left| \mathcal{E}_{V_N}(\Psi) - \mathcal{E}_{4\pi}^{GP}(\varphi) \right| \, .
$$

Next, we will estimate $M(\Psi, \varphi), S_\beta(\Psi, \varphi)$ and $\tilde{S}_\mu(\Psi, \varphi)$.

- Estimate of $S_\beta(\Psi, \varphi)$ and $\tilde{S}_\mu(\Psi, \varphi)$.
  We first estimate (4.105), using the same estimate as in (4.58). Note that

$$
\langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 M_\mu(x_1 - x_2) p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle = \langle \varphi, M_\mu \star |\varphi|^2 \varphi \rangle \langle\!\langle \Psi, \mathbb{1}_{\mathcal{B}_1^{(d)}} p_1 p_2 \mathbb{1}_{\mathcal{B}_1^{(d)}} \Psi \rangle\!\rangle \, .
$$

Using $\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi\| \leq CN^{1-d+\epsilon}$, for any $\epsilon > 0$, (see Lemma 4.7.3) we obtain, together with $\|p_1p_2\Psi\|^2 = 1 + 2\|p_1q_2\Psi\|^2 + \|q_1q_2\Psi\|^2$

$$\begin{aligned}|(4.105)| \leq& 3\|q_1\Psi\|^2 \\ &+C\left(N^{1-d+\epsilon} + N^{2-2d+2\epsilon}\right) + \frac{1}{2}|N\langle\varphi, M_\mu \star |\varphi|^2\varphi\rangle - N\|M_\mu\|_1\|\varphi^2\|^2| \\ &+\frac{1}{2}|4\pi - N\|M_\mu\|_1|\|\varphi^2\|^2 + \frac{1}{2}\langle\varphi, M_\mu \star |\varphi|^2\varphi\rangle \ .\end{aligned}$$

Note that, using Young's inequality and (4.58)

$$\begin{aligned}&|\langle\varphi, NM_\mu \star |\varphi|^2\varphi\rangle - N\|M_\mu\|_1\|\varphi^2\|^2| \\ =&\left|\int_{\mathbb{R}^2} d^2x|\varphi(x)|^2\left(N(M_\mu \star |\varphi|^2)(x) - N\|M_\mu\|_1|\varphi(x)|^2\right)\right| \\ \leq&\|\varphi\|_\infty^2\|N(M_\mu \star |\varphi|^2) - \|NM_\mu\|_1|\varphi|^2\|_1 \leq C\|\varphi\|_\infty^2\|\Delta|\varphi|^2\|_1 N^{-2\mu}\ln(N) \\ \leq&\mathcal{K}(\varphi, A_t)N^{-2\mu}\ln(N) \ .\end{aligned}$$

Since $|N\|M_\mu\|_1 - 4\pi| \leq C\frac{\ln(N)}{N}$ (see Lemma 4.5.5) and $\langle\varphi, M_\mu\star|\varphi|^2\varphi\rangle \leq \|\varphi\|_\infty^4\|M_\mu\|_1 \leq C\|\varphi\|_\infty^4 N^{-1}$, it follows that

$$\begin{aligned}|(4.105)| \leq&\mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}^\varphi\Psi\rangle\!\rangle + N^{1-d+\epsilon} + N^{2-2d+2\epsilon} + N^{-2\mu}\ln(N) + N^{-1}\ln(N)\right) \\ \leq&\mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}^\varphi\Psi\rangle\!\rangle + N^{-1}\ln(N)\right) \ ,\end{aligned} \tag{4.106}$$

where the last inequality holds for d large enough (recall that we chose $\mu > d$).
Using the same estimates, we obtain

$$(4.102) \leq \mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}^\varphi\Psi\rangle\!\rangle + N^{-2\beta}\ln(N) + N^{-1}\ln(N)\right) \ .$$

Line (4.104) and line (4.101) are controlled by Lemma 4.7.11, which is stated below.

$$(4.101), (4.104) \leq \mathcal{K}(\varphi, A_t)(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1/6}\ln(N)) \ .$$

In total, we obtain, for any $\mu > d \geq 1$, the bound

$$\begin{aligned}S_\beta(\Psi, \varphi) \leq&\mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-2\beta}\ln(N) + N^{-1/6}\ln(N)\right) \\ \tilde{S}_\mu(\Psi, \varphi) \leq&\mathcal{K}(\varphi, A_t)\left(\langle\!\langle\Psi, \widehat{n}\Psi\rangle\!\rangle + N^{-1/6}\ln(N)\right) \ .\end{aligned}$$

- Estimate of $M(\Psi, \varphi)$.
  First, we estimate (4.98).

$$\begin{aligned}|(4.98)| \leq& 2|\langle\!\langle\nabla_1 q_1\Psi, \mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 p_1\Psi\rangle\!\rangle| + 2|\langle\!\langle\nabla_1 q_1\Psi, \nabla_1 p_1\Psi\rangle\!\rangle| \\ &\leq 2\|\nabla_1 q_1\Psi\|\ \|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 p_1\|_{\mathrm{op}} + 2|\langle\!\langle\widehat{n}^{-1/2}q_1\Psi, \Delta_1 p_1\widehat{n}_1^{1/2}\Psi\rangle\!\rangle| \ .\end{aligned}$$

By Lemma 4.7.3, we obtain $\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\nabla_1 p_1\|_{\mathrm{op}} \leq C\|\nabla\varphi\|_\infty N^{1/2-d}$.
Furthermore, we use $\|\nabla_1 q_1\Psi\| \leq \|\nabla_1\Psi\| + \|\nabla_1 p_1\Psi\| \leq \mathcal{K}(\varphi, A_t)$ (see also Lemma 4.7.6)

and $|\langle\langle \widehat{n}^{-1/2} q_1\Psi, \Delta_1 p_1 \widehat{n}_1^{1/2}\Psi\rangle\rangle| \leq \mathcal{K}(\varphi, A_t)\|\widehat{n}_1^{1/2}\Psi\|\|\widehat{n}^{1/2}\Psi\| \leq \mathcal{K}(\varphi, A_t)(\langle\langle\Psi, \widehat{n}\Psi\rangle\rangle + N^{-1})$. Hence, for d large enough,

$$|(4.98)| \leq \mathcal{K}(\varphi, A_t)(\langle\langle\Psi, \widehat{n}\Psi\rangle\rangle + N^{\frac{1}{2}-d} + N^{-1}) \leq \mathcal{K}(\varphi, A_t)(\langle\langle\Psi, \widehat{n}\Psi\rangle\rangle + N^{-1}) .$$

Line (4.99) is estimated for d large enough, noting that $\|\nabla_1 p_1\Psi\|^2 = \|\nabla\varphi\|^2\|p_1\Psi\|^2$, by

$$\begin{aligned}(4.99) =& \|\mathbb{1}_{\mathcal{A}_1^{(d)}}\nabla_1 p_1\Psi\|^2 - \|\nabla\varphi\|^2 \\ \leq& |\|\nabla_1 p_1\Psi\|^2 - \|\nabla\varphi\|^2| + \|\mathbb{1}_{\overline{\mathcal{A}_1^{(d)}}}\nabla_1 p_1\Psi\|^2 \\ \leq& C\left(\|\nabla\varphi\|^2\langle\langle\Psi, q_1\Psi\rangle\rangle + \|\nabla\varphi\|_\infty^2 N^{1-2d}\right) \\ \leq& \mathcal{K}(\varphi, A_t)\langle\langle\Psi\widehat{n}, \Psi\rangle\rangle .\end{aligned}$$

For line (4.100), we use Lemma 4.7.5 to obtain

$$(4.100) \leq C\|A_t\|_\infty\left(\langle\langle\Psi, \widehat{n}\Psi\rangle\rangle + N^{-1/2}\right) .$$

In total, we obtain

$$M(\Psi, \varphi) \leq \mathcal{K}(\varphi, A_t)\left(\langle\langle\Psi, \widehat{n}\Psi\rangle\rangle + N^{-1/2}\right) .$$

□

Lemma 4.7.10.

(a) Let R_β and M_β be defined as in Lemma 4.5.4. Then, for any $\Psi \in H^1(\mathbb{R}^{2N}, \mathbb{C})$

$$\|\mathbb{1}_{|x_1-x_2|\leq R_\beta}\nabla_1\Psi\|^2 + \frac{1}{2}\langle\langle\Psi, (V_N - M_\beta)(x_1 - x_2)\Psi\rangle\rangle \geq 0 .$$

(b) Let M_β be defined as in Lemma 4.5.4. Let $\Psi \in L_s^2(\mathbb{R}^{2N}, \mathbb{C}) \cap H^1(\mathbb{R}^{2N}, \mathbb{C})$. Then, for sufficiently large N and for $\beta > d$,

$$\|\mathbb{1}_{\mathcal{B}_1^{(d)}}\mathbb{1}_{\overline{\mathcal{A}_1^{(d)}}}\nabla_1\Psi\|^2 + \frac{1}{2}\langle\langle\Psi, \sum_{j\neq 1}\mathbb{1}_{\mathcal{B}_1^{(d)}}(V_N - M_\beta)(x_1 - x_j)\Psi\rangle\rangle \geq 0 .$$

Proof. (a) We first show nonnegativity of the one-particle operator $H^{Z_n} : H^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C})$ given by

$$H^{Z_n} = -\Delta + \frac{1}{2}\sum_{z_k\in Z_n}(V_N(\cdot - z_k) - M_\beta(\cdot - z_k))$$

for any $n \in \mathbb{N}$ and any n-elemental subset $Z_n \subset \mathbb{R}^2$ which is such that the supports of the potentials $M_\beta(\cdot - z_k)$ are pairwise disjoint for any two $z_k \in Z_n$. Since $f_\beta(\cdot - z_k)$ is the the zero energy scattering state of the potential $1/2V_N(\cdot - z_k) - 1/2W_\beta(\cdot - z_k)$, it follows that

$$F_\beta^{Z_n} = \prod_{z_k\in Z_n} f_\beta(\cdot - z_k) .$$

fulfills $H^{Z_n} F_\beta^{Z_n} = 0$ for any such Z_n. By construction f_β is a positive function, so is $F_\beta^{Z_n}$. Since $\frac{1}{2}\sum_{z_k \in Z_n}(V_N(\cdot - z_k) - M_\beta(\cdot - z_k)) \in L^\infty(\mathbb{R}^2, \mathbb{C})$, this potential is a infinitesimal perturbation of $-\Delta$, thus $\sigma_{\text{ess}}(H^{Z_n}) = [0, \infty)$. Assume now that H^{Z_n} is not nonnegative. Then, there exists a ground state $\Psi_G \in H^2(\mathbb{R}^2, \mathbb{C})$ of H^{Z_n} of negative energy $E < 0$. The phase of the ground state can be chosen such that the ground state is real and positive (see e.g. [85], Theorem 10.12.). Since such a ground state of negative energy decays exponentially, that is $\Psi_G(x) \leq C_1 e^{-C_2|x|}, C_1, C_2 > 0$, the following scalar product is well defined (although $F_\beta^{Z_n} \notin L^2(\mathbb{R}^2, \mathbb{C})$).

$$\langle F_\beta^{Z_n}, H^{Z_n}\Psi_G\rangle = \langle F_\beta^{Z_n}, E\Psi_G\rangle < 0 \, . \tag{4.107}$$

On the other hand we have since $F_{\beta_1,\beta}^{X_n}$ is the zero energy scattering state

$$\langle F_\beta^{Z_n}, H^{Z_n}\Psi_G\rangle = \langle H^{Z_n} F_\beta^{Z_n}, \Psi_G\rangle = 0 \, .$$

This contradicts (4.107) and the nonnegativity of H^{Z_n} follows.

Now, assume that there exists a $\psi \in H^2(\mathbb{R}^2, \mathbb{C})$ such that the quadratic form

$$Q(\psi) = \|\mathbb{1}_{|\cdot| \leq R_\beta} \nabla\psi\|^2 + \frac{1}{2}\langle\psi, (V_N(\cdot) - M_\beta(\cdot))\psi\rangle < 0 \, .$$

Since V_{β_1} and M_{β_1} are spherically symmetric we can assume that ψ is spherically symmetric. Subsituting $\psi \to a\psi,\ a \in \mathbb{R}$, we can furthermore assume that, for all $|x| = R_\beta$, $\psi(x) = 1 - \epsilon$ for $\epsilon > 0$.

Define $\tilde\psi$ such that $\tilde\psi(x) = \psi(x)$ for $|x| \leq R_\beta$ and $\tilde\psi(x) = 1$ for $|x| > R_\beta + \epsilon$ and $\epsilon > 0$. Furthermore, $\tilde\psi$ can be constructed such that $\|\mathbb{1}_{|\cdot| \geq R_\beta}\nabla\tilde\psi\|^2 \leq C(\epsilon + \epsilon^2)$.

Then $Q(\tilde\psi) = Q(\psi) < 0$ holds, because the operator associated with the quadratic form is supported inside the ball $B_0(R_\beta)$.

Using $\tilde\psi$, we can construct a set of points Z_n and a $\chi \in H^2(\mathbb{R}^2, \mathbb{C})$ such that $\langle\chi, H^{Z_n}\chi\rangle < 0$, contradicting to nonnegativity of H^{Z_n}.

For $R > 1$ let

$$\xi_R(x) = \begin{cases} R^2/x^2, & \text{for } |x| > R; \\ 1, & \text{else.} \end{cases}$$

Let now Z_n be a subset $Z_n \subset \mathbb{R}^2$ with $|Z_n| = n$ which is such that the supports of the potentials $M_\beta(\cdot - z_k)$ lie within the Ball around zero with radius R and are pairwise disjoint for any two $z_k \in Z_n$. Since we are in two dimensions we can choose a n which is of order R^2.

Let now $\chi_R(x) = \xi_R(x)\prod_{z_k \in Z_n}\tilde\psi(x - z_k)$. By construction, there exists a $D = \mathcal{O}(1)$ such that $\chi_R(x) = \tilde\psi(x - z_k)$ for $|x - z_k| \leq D$. From this, we obtain

$$\begin{aligned}
\langle\chi_R, H^{Z_n}\chi_R\rangle =& \|\nabla\chi_R\|^2 + n\frac{1}{2}\langle\psi, (V_N(\cdot) - M_\beta(\cdot))\psi\rangle \\
=& nQ(\psi) + \sum_{z_k \in Z_n}\|\mathbb{1}_{|\cdot - z_k| \geq R_\beta}\nabla\chi_R\|^2 \\
\leq& nQ(\psi) + Cn(\epsilon + \epsilon^2) + \|\nabla\xi_R\|^2 \\
=& nQ(\psi) + Cn(\epsilon + \epsilon^2) + C \, .
\end{aligned}$$

Choosing R and hence n large enough and ϵ small, we can find a Z_n such that $\langle \chi_R, H^{Z_n}\chi_R\rangle$ is negative, contradicting nonnegativity of H^{Z_n}.
Now, we can prove that

$$\|\mathbb{1}_{|x_1-x_2|\leq R_{\beta_1}}\nabla_1\Psi\|^2+\frac{1}{2}\langle\!\langle\Psi,(V_N-M_\beta)(x_1-x_2)\Psi\rangle\!\rangle\geq 0\,. \tag{4.108}$$

holds for any $\Psi\in H^2(\mathbb{R}^{2N},\mathbb{C})$. Using the coordinate transformation $\tilde{x}_1=x_1-x_2$, $\tilde{x}_i=x_i\ \forall i\geq 2$, we have $\nabla_{x_1}=\nabla_{\tilde{x}_1}$. Thus (4.108) is equivalent to $\|\mathbb{1}_{|x_1|\leq R_{\beta_1}}\nabla_1\Psi\|^2+\frac{1}{2}\langle\!\langle\Psi,(V_N-M_\beta)(x_1)\Psi\rangle\!\rangle\geq 0\ \forall\Psi\in H^2(\mathbb{R}^{2N},\mathbb{C})$ which follows directly from $Q(\psi)\geq 0$ for all $\psi\in H^2(\mathbb{R}^2,\mathbb{C})$. By a standard density argument, we can conclude that $Q(\Psi)\geq 0$ $\forall\Psi\in H^1(\mathbb{R}^{2N},\mathbb{C})$.

(b) Define $c_k=\{(x_1,\dots,x_N)\in\mathbb{R}^{2N}||x_1-x_k|\leq R_\beta\}$ and $\mathcal{C}_1=\cup_{k=2}^N c_k$. For $(x_1,\dots,x_N)\in\mathcal{B}_1^{(d)}$ it holds that $|x_i-x_j|\geq N^{-d}$ for $2\leq i,j\leq N$. Let $\beta>d$. Assume that $N^{-d}>2R_\beta$, which hold for N sufficiently large, since $R_\beta\leq CN^{-\beta}$. Then, it follows that, for $i\neq j$, $\left(c_i\cap\mathcal{B}_1^{(d)}\right)\cap\left(c_j\cap\mathcal{B}_1^{(d)}\right)=\emptyset$. Under the same conditions, we also have $\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\geq\mathbb{1}_{\mathcal{C}_1}$. Therefore

$$\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\mathbb{1}_{\mathcal{B}_1^{(d)}}\geq\mathbb{1}_{\mathcal{C}_1}\mathbb{1}_{\mathcal{B}_1^{(d)}}=\mathbb{1}_{\mathcal{C}_1\cap\mathcal{B}_1^{(d)}}=\mathbb{1}_{\cup_{k=2}^N\left(c_k\cap\mathcal{B}_1^{(d)}\right)}=\sum_{k=2}^N\mathbb{1}_{c_k\cap\mathcal{B}_1^{(d)}}=\mathbb{1}_{\mathcal{B}_1^{(d)}}\sum_{k=2}^N\mathbb{1}_{c_k}\,.$$

Note that $\mathbb{1}_{\mathcal{B}_1^{(d)}}$ depends only on $x_2,\dots,x_N$. By this

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1^{(d)}}\mathbb{1}_{\mathcal{B}_1^{(d)}}\nabla_1\Psi\|^2\geq\sum_{k=2}^N\|\mathbb{1}_{c_k}\nabla_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2=(N-1)\|\mathbb{1}_{|x_1-x_2|\leq R_\beta}\nabla_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2\,.$$

This yields

$$\begin{aligned}(4.103)\geq{}&(N-1)\big(\|\mathbb{1}_{|x_1-x_2|\leq R_\beta}\nabla_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2+\frac{1}{2}\langle\!\langle\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi,(V_N-M_\beta)(x_1-x_2)\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle\big)\\ \geq{}&0\,.\end{aligned}$$

where the last inequality follows from (a)

□

Lemma 4.7.11. *Let $W_\beta\in\mathcal{V}_\beta$. Let $\Psi\in L_s^2(\mathbb{R}^{2N},\mathbb{C})\cap H^1(\mathbb{R}^{2N},\mathbb{C})$ and $\|\nabla_1\Psi\|$ be bounded uniformly in N. Let d in Definition 4.7.3 of $\mathbb{1}_{\mathcal{B}_1^{(d)}}$ sufficiently large. Let $\Gamma\in\{\Psi,\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\}$. Then, for all $\beta>0$,*

(a)

$$N\,|\langle\!\langle\Gamma,q_1p_2W_\beta(x_1-x_2)p_1p_2\Gamma\rangle\!\rangle|\leq C\|\varphi\|_\infty^2\langle\!\langle\Psi,\hat{n}\Psi\rangle\!\rangle\,.$$

(b)

$$N|\langle\!\langle\Gamma,p_1p_2W_\beta(x_1-x_2)q_1q_2\Gamma\rangle\!\rangle|\leq\mathcal{K}(\varphi,A_t)\left(\langle\!\langle\Psi,\widehat{n}\Psi\rangle\!\rangle+N^{-1/6}\ln(N)\right)\,.$$

(c)

$$N|\langle\!\langle\Gamma,(1-p_1p_2)W_\beta(x_1-x_2)p_1p_2\Gamma\rangle\!\rangle|\leq\mathcal{K}(\varphi,A_t)\left(\langle\!\langle\Psi,\widehat{n}\Psi\rangle\!\rangle+N^{-1/6}\ln(N)\right)\,.$$

Proof. (a) Let first $\Gamma = \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi$. Then,

$$
\begin{aligned}
&N\left|\langle\!\langle \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, q_1p_2W_\beta(x_1-x_2)p_1p_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle\right| \\
\le&N\left|\langle\!\langle \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi, q_1p_2W_\beta(x_1-x_2)p_1p_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle\right| \qquad (4.109)\\
+&N\left|\langle\!\langle \Psi, q_1p_2W_\beta(x_1-x_2)p_1p_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle\right| \ . \qquad (4.110)
\end{aligned}
$$

Using Lemma 4.7.4 together with $\|p_2W_\beta(x_1-x_2)p_2\|_{\mathrm{op}} \le \|\varphi\|_\infty^2\|W_\beta\|_1$, the first line can be bounded, for any $\epsilon > 0$, by

$$
(4.109) \le \mathcal{K}(\varphi, A_t)N\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi\|\|W_\beta\|_1 \le \mathcal{K}(\varphi, A_t)N^{1-d+\epsilon}. \qquad (4.111)
$$

The second term is bounded by

$$
\begin{aligned}
(4.110) =&N\left|\langle\!\langle \sqrt{W_\beta(x_1-x_2)}q_1p_2(\hat{n})^{-\frac{1}{2}}\Psi, \sqrt{W_\beta(x_1-x_2)}p_1p_2\hat{n}_1^{\frac{1}{2}}\mathbb{1}_{\mathcal{B}^{(d)}{}_1}\Psi\rangle\!\rangle\right| \\
\le&CN\|\sqrt{W_\beta(x_1-x_2)}p_2\|_{\mathrm{op}}^2\left(\|q_1(\hat{n})^{-\frac{1}{2}}\Psi\|^2 + \|\hat{n}_1^{\frac{1}{2}}\mathbb{1}_{\mathcal{B}^{(d)}{}_1}\Psi\|^2\right) \\
\le&CN\|\sqrt{W_\beta(x_1-x_2)}p_2\|_{\mathrm{op}}^2\left(\langle\!\langle\Psi, \hat{n}\Psi\rangle\!\rangle + \|\hat{n}_1^{\frac{1}{2}}\Psi\|^2 + \|\hat{n}_1^{\frac{1}{2}}\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi\|^2\right) \\
\le&CN\|W_\beta\|_1\|\varphi\|_\infty^2\left(\langle\!\langle\Psi, \hat{n}\Psi\rangle\!\rangle + \|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi\|^2\right) \\
\le&C\|\varphi\|_\infty^2\left(\langle\!\langle\Psi, \hat{n}\Psi\rangle\!\rangle + N^{1-d+\epsilon}\right).
\end{aligned}
$$

Choosing d large enough, $N^{1-d+\epsilon}$ is smaller than $\langle\!\langle\Psi, \hat{n}\Psi\rangle\!\rangle$. This yields (a) in the case $\Gamma = \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi$. The inequality (a) can be proven analogously for $\Gamma = \Psi$.

(b) Let $\Gamma = \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi$. We first consider (b) for potentials with $\beta < 1/4$. We have to estimate

$$
\begin{aligned}
&N|\langle\!\langle \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\!\rangle| \le N|\langle\!\langle\Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\Psi\rangle\!\rangle| \\
+&N|\langle\!\langle \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\Psi\rangle\!\rangle| + N|\langle\!\langle\Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\mathbb{1}_{\overline{\mathcal{B}_1^{(d)}}}\Psi\rangle\!\rangle| \\
+&N|\langle\!\langle \mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\mathbb{1}_{\overline{\mathcal{B}_1^{(d)}}}\Psi\rangle\!\rangle| \\
\le&N|\langle\!\langle\Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\Psi\rangle\!\rangle| \qquad (4.112)\\
+&CN\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi\|\|W_\beta\|_\infty \ . \qquad (4.113)
\end{aligned}
$$

The last term is bounded, for any $\epsilon > 0$, by

$$
(4.113) \le CNN^{1-d+\epsilon}N^{-1+2\beta} \le N^{-2} \ ,
$$

where the last inequality holds choosing d large enough. Using Lemma 4.4.2 (c) and Lemma 4.4.6 with $O_{1,2} = q_2W_\beta(x_1-x_2)p_2$, $\Omega = N^{-1/2}q_1\Psi$ and $\chi = N^{1/2}p_1\Psi$ we get

$$
\begin{aligned}
(4.112) \le& \|q_1\Psi\|^2 + N^2\big|\langle\!\langle q_2\,\Psi, p_1\sqrt{W_\beta}(x_1-x_2)p_3\sqrt{W_\beta}(x_1-x_3) \\
&\qquad \sqrt{W_\beta}(x_1-x_2)p_2\sqrt{W_\beta}(x_1-x_3)p_1q_3\,\Psi\rangle\!\rangle\big| \\
&+ N^2(N-1)^{-1}\|q_2W_\beta(x_1-x_2)p_2p_1\Psi\|^2 \\
&\le \|q_1\Psi\|^2 + N^2\|\sqrt{W_\beta}(x_1-x_2)p_1\|_{\mathrm{op}}^4\,\|q_2\,\Psi\|^2 \\
&+ CN\|W_\beta(x_1-x_2)p_2\|_{\mathrm{op}}^2 \ .
\end{aligned}
$$

With Lemma 4.4.2 (e) we get the bound

$$(4.112) \leq \|q_1\Psi\|^2 + N^2\|\varphi\|_\infty^4\|W_\beta\|_1^2\,\|q_1\Psi\|^2 + CN\|W_\beta\|^2\|\varphi\|_\infty^2\,.$$

Note, that $\|W_\beta\|_1 \leq CN^{-1}$, $\|W_\beta\|^2 \leq CN^{-2+2\beta}$ Hence

$$(4.112) \leq C\left(\langle\langle\Psi, q_1\Psi\rangle\rangle + \mathcal{K}(\varphi)N^{-1+2\beta}\right)\,.$$

Note that, for $\beta < 1/4$, $N^{-1+2\beta} \leq N^{-1/6}\ln(N)$. Using the same bounds for $\Gamma = \Psi$, we obtain (b) for the case $\beta < 1/4$.

b) for $1/4 \leq \beta$:
We use $U_{\beta_1,\beta}$ from Definition 4.7.1 for some $0 < \beta_1 < 1/4$.
$Z_\beta^\varphi(x_1,x_2) - W_\beta + U_{\beta_1,\beta}$ has the form of $Z_{\beta_1}^\varphi(x_1,x_2)$ which has been controlled above. It is left to control

$$N\left|\langle\langle \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, p_1p_2\left(W_\beta(x_1-x_2) - U_{\beta_1,\beta}(x_1-x_2)\right)q_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\rangle\right|\,.$$

Let $\Delta h_{\beta_1,\beta} = W_\beta - U_{\beta_1,\beta}$. Integrating by parts and using that $\nabla_1 h_{\beta_1,\beta}(x_1-x_2) = -\nabla_2 h_{\beta_1,\beta}(x_1-x_2)$ gives

$$N\left|\langle\langle \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, p_1p_2\left(W_\beta(x_1-x_2) - U_{\beta_1,\beta}(x_1-x_2)\right)q_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\rangle\right|$$
$$= N\left|\langle\langle \nabla_1 p_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, p_2\nabla_2 h_{\beta_1,\beta}(x_1-x_2)q_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\rangle\right| \tag{4.114}$$
$$+ N\left|\langle\langle \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, p_1p_2\nabla_2 h_{\beta_1,\beta}(x_1-x_2)\nabla_1 q_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\rangle\right|\,. \tag{4.115}$$

Let $(a_1,b_1) = (q_1,\nabla p_1)$ or $(a_1,b_1) = (\nabla q_1, p_1)$. Then, both terms can be estimated as follows:
We use Lemma 4.4.6 with $\Omega = N^{-\eta/2}a_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi$, $O_{1,2} = N^{1+\eta/2}q_2\nabla_2 h_{\beta_1,\beta}(x_1-x_2)p_2$ and $\chi = b_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi$. We choose $\eta < 2\beta_1$.

$$N\left|\langle\langle \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, a_1p_2\nabla_2 h_{\beta_1,\beta}(x_1-x_2)b_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\rangle\right|$$
$$\leq N^{-\eta}\|a_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2 \tag{4.116}$$
$$+ \frac{N^{2+\eta}}{N-1}\|q_2\nabla_2 h_{\beta_1,\beta}(x_1-x_2)b_1p_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2 \tag{4.117}$$
$$+ N^{2+\eta}\left|\langle\langle \mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, b_1p_2q_3\nabla_2 h_{\beta_1,\beta}(x_1-x_2)\nabla_3 h_{\beta_1,\beta}(x_1-x_3)b_1q_2p_3\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\rangle\rangle\right|^{1/2}\,. \tag{4.118}$$

We obtain (note that $\mathbb{1}_{\mathcal{B}_1^{(d)}}$ does not depend on x_1)

$$(4.116) \leq N^{-\eta}\|a_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2 = N^{-\eta}\|\mathbb{1}_{\mathcal{B}_1^{(d)}}a_1\Psi\|^2 \leq \mathcal{K}(\varphi, A_t)N^{-\eta}\,.$$

since both $\|\nabla q_1\Psi\|$ and $\|q_1\Psi\|$ are bounded uniformly in N. Since q_2 is a projector it follows that

$$(4.117) \leq \frac{N^{2+\eta}}{N-1}\|\nabla_2 h_{\beta_1,\beta}(x_1-x_2)p_2\|_{\text{op}}^2\|b_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2$$
$$\leq C\frac{N^{2+\eta}}{N-1}\|\varphi\|_\infty^2\|\nabla h_{\beta_1,\beta}\|^2\|b_1\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2 \leq \mathcal{K}(\varphi, A_t)N^{\eta-1}\ln(N)\|\varphi\|_\infty^2\,,$$

where we used Lemma 4.7.2 in the last step.

Next, we estimate

$$
\begin{aligned}
(4.118) \leq & N^{2+\eta}\|p_2\nabla_2 h_{\beta_1,\beta}(x_1-x_2)b_1q_2\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi\|^2 \\
\leq & 2N^{2+\eta}\|p_2\nabla_2 h_{\beta_1,\beta}(x_1-x_2)b_1q_2\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi\|^2 \qquad (4.119)\\
& +2N^{2+\eta}\|p_2\nabla_2 h_{\beta_1,\beta}(x_1-x_2)b_1q_2\Psi\|^2 . \qquad (4.120)
\end{aligned}
$$

The first term can be estimated as

$$
\begin{aligned}
(4.119) \leq & CN^{2+\eta}\|\nabla_2 h_{\beta_1,\beta}(x_1-x_2)b_1\|_{\mathrm{op}}^2\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi\|^2 \\
\leq & CN^{2+\eta}\|\nabla_2 h_{\beta_1,\beta}\|^2(\|\varphi\|_\infty^2+\|\nabla\varphi\|_\infty^2)\|\mathbb{1}_{\overline{\mathcal{B}}_1^{(d)}}\Psi\|^2 \\
\leq & \mathcal{K}(\varphi,A_t)N^{2+\eta}N^{-2}\ln(N)N^{2-2d+2\epsilon} = \mathcal{K}(\varphi,A_t)N^{2-2d+2\epsilon+\eta}\ln(N) ,
\end{aligned}
$$

for any $\epsilon > 0$. For d large enough, this term is subleading. The last term can be estimated as

$$
\begin{aligned}
(4.120) \leq & 2N^{2+\eta}\|p_2 h_{\beta_1,\beta}(x_1-x_2)b_1\nabla_2 q_2\Psi\|^2 \\
& +2N^{2+\eta}\||\varphi(x_2)\rangle\langle\nabla\varphi(x_2)|h_{\beta_1,\beta}(x_1-x_2)b_1q_2\Psi\|^2 \\
\leq & CN^{2+\eta}\|p_2 h_{\beta_1,\beta}(x_1-x_2)\|_{\mathrm{op}}^2\|b_1\nabla_2 q_2\Psi\|^2 \\
& +CN^{2+\eta}\||\varphi(x_2)\rangle\langle\nabla\varphi(x_2)|h_{\beta_1,\beta}(x_1-x_2)\|_{\mathrm{op}}^2\|b_1q_2\Psi\|^2 \\
\leq & CN^{2+\eta}\left(\|\nabla\varphi\|_\infty^2+\|\varphi\|_\infty^2\right)\|h_{\beta_1,\beta}\|^2(1+\|\nabla\varphi\|^2) \\
\leq & \mathcal{K}(\varphi,A_t)N^{\eta-2\beta_1}\ln(N)^2 .
\end{aligned}
$$

Combining both estimates we obtain, for any $\beta > 1$,

$$
\begin{aligned}
& N\left|\langle\!\langle\mathbb{1}_{\mathcal{B}_1^{(d)}}\Psi, p_1p_2W_\beta(x_1-x_2)q_1q_2\mathbb{1}_{{\mathcal{B}^{(d)}}_1}\Psi\rangle\!\rangle\right| \\
& \leq \inf_{\eta>0}\inf_{0<\mu<1/4}\left(\mathcal{K}(\varphi,A_t)\left(\langle\!\langle\Psi,\widehat{n}\Psi\rangle\!\rangle+N^{-1+2\mu}+N^{-\eta}+N^{\eta-1}\ln(N)+N^{\eta-2\mu}\ln(N)\right)\right) \\
& \leq \mathcal{K}(\varphi,A_t)\left(\langle\!\langle\Psi,\widehat{n}\Psi\rangle\!\rangle+N^{-1/6}\ln(N)\right) .
\end{aligned}
$$

where the last inequality comes from choosing $\eta = 1/3$ and $\mu = 1/4$. For $\Gamma = \Psi$, (b) can be estimated the same way, yielding the same bound.

(c) This follows from (a) and (b), using that $1 - p_1p_2 = q_1q_2 + p_1q_2 + q_1p_2$.

□

CHAPTER
FIVE

OUTLOOK

In this thesis we have studied the interaction between charged bosons and quantized radiation fields. We extended the "method of counting" and showed that condensates of charges create coherent states which behave like classical fields. Our findings can be improved in many respects.

(a) The rate of convergence in Theorem 2.3.1 and Theorem 3.2.1 is known to be unideal. However, it seems promising to obtain a rate of order N^{-1} if one combines the strategy of Chapter 2 with ideas from [67] and regards fluctuations around the mean-field dynamics.

(b) Moreover, one could consider charges with relativistic dispersion. An appropriate adaption of the proof is straightforward in case of the Nelson model but looks more elaborate for the Pauli-Fierz Hamiltonian.

(c) The restriction to bosonic charges was purely technical. Especially from a physical point of view it would be more interesting to study the interaction between electrons and photons.

(d) Ammari and Falconi [1] investigated the mean-field limit of the renormalized Nelson model without cutoff. Their derivation by means of Wigner measures does not provide quantitave bounds on the rate of convergence which might be obtained by other techniques.

Furthermore, we have derived the defocusing cubic nonlinear Schrödinger equation and the Gross-Pitaevksii equation in two dimensions from the many-body Schrödinger equation. This results can be generalized in the following ways.

(a) From an experimental point of view it would be more rigorous to start our derivation from a three-dimemsional gas of bosons which is strongly confined in one direction. Results in this matter were obtained in [9, 49, 50]. However, it is still an open problem to derive the two-dimensional Gross-Pitaevskii equation from a strongly confined three dimensional system.

(b) Additionally, one could try to extend our results to attractive interaction potentials. The difficulty here is to control the kinetic energy of the particles which are not in the condensate. Progress on this issue was recently made in [46].

NOTATION

In this thesis we choose units in which Planck's constant $\hbar$, the speed of light c, the elementary charge e and twice the mass of the particles $2m$ is equal to one. We use the notations $\varphi(t)$ and φ_t interchangeably to denote a quantity φ at time t. Occasionally, we refrain from indicating the explicit dependence of a quantity on the time t. The chapters slightly differ in notation. However, all variations made are stated explicitly. The following list contains symbols commonly used in this work.

List of Symbols

C	a generic constant that might depend on fixed parameters
$C(\|\|f\|\|_{L^2}, \|\|g\|\|_{H^1})$	a positive increasing function of the norms indicated
$L^p(\mathbb{R}^d, \mathbb{C})$	the space of complex-valued Lebesgue-measurable functions f such that $\|\|f\|\|_{L^p(\mathbb{R}^d)} < \infty$, where $\|\|f\|\|_{L^p(\mathbb{R}^d)} = \|\|f\|\|_{L^p} := \begin{cases} \left(\int_{\mathbb{R}^d} dx\, \|f(x)\|^p \right)^{1/p} & \text{if } 0 < p < \infty \\ \operatorname{ess\,sup}_{x \in \mathbb{R}^d} \|f(x)\| & \text{if } p = \infty \end{cases}$
$L^p(\mathbb{R}^d) = L^p$	$L^p(\mathbb{R}^d, \mathbb{C})$
$L^p(\mathbb{R}^d, \mathbb{R})$	the space of real-valued Lebesgue-measurable functions f such that $\|\|f\|\|_{L^p(\mathbb{R}^d)} < \infty$.
$\|\|f\|\|_p$	$\|\|f\|\|_{L^p(\mathbb{R}^d)}$
$L^2_m(\mathbb{R}^3)$	the weighted L^2 spaces on $\mathbb{R}^3$ such that $\|\|f\|\|_{L^2_m(\mathbb{R}^3)} < \infty$, where $\|\|f\|\|_{L^2_m(\mathbb{R}^3)} = \|\|f\|\|_{L^2_m} := \left(\int_{\mathbb{R}^3} dk\, (1+k^2)^m \|f(k)\|^2 \right)^{1/2}$
$\mathcal{FT}(f) = \tilde{f}$	the Fourier transform of f, defined by $\tilde{f}(k) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} dx\, f(x) e^{-ikx}$
$H^s(\mathbb{R}^d) = H^s$	the Sobolev space of functions on $\mathbb{R}^d$ such that $\|\|f\|\|_{H^s(\mathbb{R}^d)} < \infty$, where $\|\|f\|\|_{H^s(\mathbb{R}^d)} = \|\|f\|\|_{H^s} := \left(\int_{\mathbb{R}^d} dx (1 + \|k\|^2)^{s/2} \|\tilde{f}(k)\|^2 \right)^{1/2}$
$f * g$	the convolution $(f * g)(x) := \int_{\mathbb{R}^d} dy f(x-y) g(y)$
∇	the differential operator $\left(\frac{\partial}{\partial x^1}, \dots \frac{\partial}{\partial x^d} \right)$
Δ	the Laplacian $\sum_{i=1}^d \frac{\partial^2}{(\partial x^i)^2}$
$\langle \cdot, \cdot \rangle$	the scalar product of a Hilbert space

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, §8, Abs. 2 Pkt. 5)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist. Kapitel 3 enthält Resultate, die aus einer Zusammenarbeit mit Prof. Dr. Peter Pickl stammen. Kapitel 4 besteht aus einer gemeinsamen Veröffentlichung mit Prof. Dr. Peter Pickl und Maximilian Jeblick. Die von mir zu den Veröffentlichungen geleisteten Beiträge werden detailliert am Anfang der entsprechenden Kapitel beschrieben.

München, den 10.03.18 Nikolai Konstantin Leopold