David Mitrouskas

Derivation of Mean Field Equations and their Next-Order Corrections: Bosons and Fermions

Dissertation an der Fakultät für
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Abstract

This thesis is about the derivation of effective mean field equations and their next-order corrections starting from nonrelativistic many-body quantum theory. Mean field equations provide an approximate ansatz for the description of interacting many-particle systems. In this ansatz the interaction between the particles is replaced by a self-consistent external potential leading to an effective one-body description of the many-particle system. Next-order corrections provide an approximation which goes one step further and tries to capture also subleading effects that are not resolved by the mean field ansatz. We present mathematical proofs for the validity of such effective theories for different models that are motivated, e.g., from the theory of ultracold atoms (the bosonic Hartree equation and the corresponding Bogoliubov theory) and from plasma physics (the motion of a tracer particle through a degenerate and dense electron gas). Starting from a many-body Schrödinger equation, our goal is to show that the solutions converge in a particular limit to the solutions to an effective mean field equation and its next-order corrections. After a short introduction and a summary in Chapter one, we present the main part of this work in three self-contained chapters.

In Chapter two we analyze the dynamics of a large number N of nonrelativistic bosons in the weak coupling limit, i.e., for a coupling constant $g_N = N^{-1}$. It is well known that in the limit of infinite particle number, the Hartree equation emerges as an effective one-particle theory of the Bose gas. This is closely related to the remarkable physical phenomenon of Bose-Einstein condensation at low temperature, namely that the majority of particles in a Bose gas occupies the same copy of a single one-particle quantum state. Our emphasis lies in the description of the few particles that fluctuate around the Bose-Einstein condensate. We show convergence of the fully interacting dynamics to an auxiliary time evolution in the norm of the N-particle space. This result allows us to prove several other assertions. Among other things, it is used to derive the Hartree equation with optimal speed of convergence N^{-1} for initial states that are close to ground states of interacting systems and also to prove convergence of the N-particle solution towards a time evolution obtained from the Bogoliubov Hamiltonian on Fock space.

Chapter three is about the low energy properties of the weakly interacting homogeneous Bose gas. Here, we derive a novel estimate for low energy eigenfunctions stating that the probability for finding l particles out of their total number N not in the condensate is exponentially small in the number l. Using this bound, we then prove that the ground state wave function of the microscopic model satisfies certain quasifree type properties. The exponential decay is moreover used to provide an alternative proof for the validity of Bogoliubov's approximation for the low-lying energy eigenvalues. Bogoliubov theory states that the excitation energies of the Bose gas are given by excitations of free quasiparticles obeying an effective energy-momentum dispersion relation which is linear for small momentum. The linearity of the effective dispersion relation is an essential ingredient for the explanation of the superfluid character of the Bose gas.

In Chapter four we study the time evolution of a single particle coupled through a pair potential to a dense and homogeneous ideal Fermi gas in two spatial dimensions. This type of model is well known in plasma physics where it is used to describe the energy loss of ions moving through a dense and degenerate electron gas. We analyze the model for a coupling parameter $g = 1$ and prove closeness of the time evolution to an effective dynamics for large densities of the gas and for long time scales of the order of some power of the density.

The effective dynamics is generated by the free Hamiltonian with a large but constant energy shift. To leading order, this energy shift is given by the spatially homogeneous mean field potential produced by the gas particles, whereas at next-to-leading order, one has to consider an additional correction to the mean field energy which is due to so-called recollision processes.

Zusammenfassung

Die vorliegende Arbeit handelt von der Herleitung effektiver Mean-field-Gleichungen (auch Molekularfeld-Gleichungen genannt) und deren Korrekturen ausgehend von der nichtrelativistischen Vielteilchen-Quantenphysik. Mean-field-Gleichungen stellen einen Ansatz zur näherungsweisen Beschreibung wechselwirkender Vielteilchensysteme dar. Dabei wird die Wechselwirkung zwischen den Teilchen durch selbstkonsistente externe Potentiale ersetzt, was zu einer effektiven Einteilchenbeschreibung des Vielteilchensystems führt. "Korrekturen in nächster Ordnung" bieten eine Näherung, welche einen Schritt weiter geht und Effekte zu beschreiben versucht, die durch die Mean-Field-Näherung nicht erfasst werden. Wir präsentieren rigorose Beweise für die Gültigkeit solcher effektiver Theorien für verschiedene Modelle, die u.a. motiviert sind aus der Theorie der ultrakalten Gase (die bosonische Hartree-Gleichung und die entsprechende Bogoliubov-Theorie), sowie aus der Plasmaphysik (die Bewegung eines Testteilchens durch ein entartetes und dichtes Elektronengas). Ausgehend von einer Vielteilchen-Schrödinger-Gleichung besteht unser Ziel darin, die Konvergenz der Lösungen gegen Lösungen der einfacheren Mean-Field-Gleichung sowie deren Korrekturen nachzuweisen. Nach einer kurzen Einleitung und Zusammenfassung in Kapitel eins, stellen wir den Hauptteil der Arbeit in drei voneinander unabhängigen Kapiteln dar.

In Kapitel zwei analysieren wir die Dynamik einer großen Anzahl N nichtrelativistischer Bosonen im Schwachen-Kopplungs-Limes, sprich, für Kopplungskonstante $g_N = N^{-1}$. Es ist bekannt, dass das Bosegas im Limes unendlich vieler Teilchen durch die effektive Einteilchen-Hartree-Gleichung beschrieben wird. Dies steht in engem Zusammenhang mit dem interessanten physikalischen Phänomen der Bose-Einstein-Kondensation bei niedrigen Temperaturen, nämlich, dass die Mehrheit der Teilchen im Bosegas denselben Einteilchen-Quantenzustand besetzt. Unser Augenmerk liegt auf der Beschreibung der wenigen Teilchen, die sich nicht im Bose-Einstein-Kondensat befinden. Wir zeigen, dass die Lösung der vollwechselwirkenden Theorie in der Norm des N-Teilchen-Hilbertraums gegen eine einfachere Hilfsdynamik konvergiert. Dieses Resultat erlaubt uns, eine Reihe von weiteren Behauptungen zu beweisen. Unter anderem wird es verwendet zur Herleitung der Hartree-Gleichung mit optimaler Konvergenzrate N^{-1} für Anfangszustände, die den Grundzuständen wechselwirkender Systeme ähnlich sind. Außerdem zeigen wir die Konvergez der N-Teilchen-Lösung gegen eine Zeitentwicklung, die durch den Bogoliubov-Hamiltonian auf dem Fockraum beschrieben wird.

Kapitel drei handelt von den Niedrigenergie-Eigenschaften des schwach-wechselwirkenden Bosegases. Wir beweisen eine neuartige Abschätzung für Niedrigenergie-Eigenfunktionen, welche aussagt, dass die Wahrscheinlichkeit dafür, l der N Teilchen nicht im Kondensat zu finden, exponentiell (in der Zahl l) abfällt. Ausgehend von diesem Resultat beweisen wir, dass der Grundzustand des mikroskopischen Modells bestimmte "quasifreie" Eigenschaften erfüllt. Der exponentielle Abfall wird außerdem verwendet, um einen alternativen Beweis für die Richtigkeit der Bogoliubov-Näherung für niedrigliegende Energie-Eigenwerte zu präsentieren. Die Bogoliubov-Theorie besagt, dass die Anregungsenergien des Bosegases durch Anregungen freier Quasiteilchen beschrieben werden, welche eine effektive Energie-Impuls-Dispersionsrelation erfüllen. Die effektive Dispersionsrelation der Quasiteilchen ist für kleine Impulse linear, was als wesentlicher Aspekt bei der Erklärung der suprafluiden Eigenschaften des Bosegases eingeht.

In Kapitel vier analysieren wir die Zeitentwicklung eines einzelnen Teilchens, welches durch ein Paarpotential an ein dichtes, homogenes ideales Fermigas gekoppelt ist. Ähnliche Modelle werden in der Plasmaphysik verwendet, um den Energieverlust von Ionen, die sich durch dichte, entartete Elektronengase bewegen, zu beschreiben. Wir analysieren das Modell für einen Kopplungsparameter $g = 1$ und beweisen die Nähe der Zeitentwicklung zu einer effektiven Dynamik für hohe Dichten des Gases und für lange Zeitskalen, die mit einer bestimmten Potenz der Dichte anwachsen. Die effektive Dynamik wird hierbei durch den freien Hamilton-Operator mit einer großen, aber konstanten Energiekorrektur erzeugt. In führender Ordnung ist die Energiekorrektur durch das homogene Mean-field-Potential der Gasteilchen gegeben, während in der nächsten Ordnung zusätzliche Terme aufgrund von sogenannten Rekollisions-Prozessen berücksichtigt werden müssen.

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Contents

Chapter 1

Introduction

Many interesting physical systems are made up of a large number of particles. Microscopic elements (electrons, atoms or molecules) are the basic building blocks of all states of matter, viz., gases, fluids, plasmas and solids. Their number may range from thousands in a dilute gas up to the Avogadro number $n_{\mathrm{A}} \propto 10^{23}$ in a chemical sample or a solid. On the atomic or molecular level the behavior of the particles is described by a linear many-body Schrödinger equation. When the number of particles is very large, however, direct analytical and numerical attempts for solving the many-body Schrödinger equation are completely impracticable. This forces one to resort to approximate and effective models which are more tractable and thus necessary for predicting and explaining the physical phenomena of interest. Such effective theories are much simpler because they do not describe the motion of every individual particle. Instead, they focus on few collective degrees of freedom which become relevant from a coarse grained perspective when the system is seen as a whole (properties, e.g., like pressure, temperature or coarse grained density). The list of effective theories known for instance in physics and chemistry is incredibly long. The ideal gas equation, the Vlasov and the Boltzmann equations as well as the three laws of thermodynamics, fluid mechanics, plasma equations, Hartree and Hartree-Fock theory (which explain, e.g., many phenomena in the theory of ultracold gases as well as in chemistry), and practically all models used in solid state physics, and many more. From a pragmatic point of view, such effective theories are often presented as the starting point of the analysis. However, one should be aware that there is always a more fundamental theory behind these effective equations. The quest for understanding the relation between the microscopic laws and the effective models which emerge on a macroscopic scale is very old yet still a current field of research. It goes back at least to the works of J. C. Maxwell [90] and L. Boltzmann [23] who argued (in the second half of the 19th century) that gases were made up of microscopic atoms. Even though this idea was not widely accepted at their time, it helped Boltzmann to give a convincing explanation of the second law of thermodynamics as an effective theory which emerges from the Newtonian motion of a large number of atoms. The paradigm of a mathematically rigorous derivation of an effective equation came many decades later and is due to H. Grad [59] and O. E. Lanford [74]. They made the arguments proposed by Boltzmann mathematically precise in a certain dilute limit and proved the validity of the Boltzmann equation as the coarse grained description for the empirical density of a rarified gas of small hard spheres (their derivation holds at least for short times whereas for longer times this is still considered to be an interesting and difficult open problem, see, e.g., [111]). Since then many rigorous results regarding the derivation of all kind of effective equations (e.g., mean field, kinetic, hydrodynamic) have followed, both in classical and quantum mechanics. It has become one of the central themes of statistical physics to provide mathematically precise justifications of macroscopic, effective models starting from a first principle many-body theory.

In this work, we present the rigorous derivation of different bosonic and fermionic mean field models (and their next-order corrections) starting from a nonrelativistic many-particle Schrödinger equation. In the remainder of this chapter, we introduce the basic principles of many-body quantum mechanics, we explain the general idea of the mean field ansatz and summarize the relevant aspects of the models and the physical situations that we have in mind. Eventually, we give an outline of the main part of this thesis including a summary of our main results.

1.1 Many-body quantum mechanics

In nonrelativistic quantum mechanics an N-particle system is described by a complex valued wave function $\Psi = \Psi(x_1, ..., x_N) \in L^2(\Omega^N)$ where L^2 denotes the space of square-integrable functions $\Psi : \Omega^N \to \mathbb{C}$ and $\Omega \subset \mathbb{R}^d$ or $\Omega = \mathbb{R}^d$ for spatial dimension d. Internal degrees of freedom like spin are neglected throughout this work. We abbreviate the scalar product resp. the norm on the Hilbert space $L^2(\Omega^N)$ by

$$\big\langle \Psi, \Phi \big\rangle = \int_{\Omega^N} \overline{\Psi(x_1, ..., x_N)} \Phi(x_1, ..., x_N)\ dx_1...dx_N, \qquad \|\Psi\| = \sqrt{\big\langle \Psi, \Psi \big\rangle}, \tag{1.1}$$

where $\overline{\Psi}$ denotes the complex conjugate of Ψ. It is always assumed that $\|\Psi\| = 1$ such that $|\Psi(x_1, ..., x_N)|^2$ can be interpreted as the probability density of finding particle one at position x_1, particle two at position x_2 and so on. Quantum mechanics postulates a simple rule for the statistical description of outcomes of repetitions of experiments. Most physical quantities of interest, e.g., position, momentum and energy are associated with a self-adjoint operator A (possibly unbounded) on the given Hilbert space. The two most relevant objects to characterize the statistical outcome of an experiment are the *average value* and the variance. For an operator A and a system described by the wave function Ψ, they are defined as[1]

$$\mathbb{E}_\Psi[A] = \big\langle \Psi, A\Psi \big\rangle, \qquad \mathrm{Var}_\Psi[A] = \big\langle \Psi, \Big(A - \mathbb{E}_\Psi[A]\Big)^2 \Psi \big\rangle. \tag{1.3}$$

Relevant operators are, e.g., the differential operator $p_{x_i} = -i\nabla_{x_i}$, i.e., i times the gradient acting on the variable x_i, which is associated with the momentum of the ith particle (throughout this work, we use units such that the Planck constant $\hbar$ and the mass m of the particles satisfy $\hbar = 2m = 1$), the multiplication operator x_i related to the position of the ith particle, real valued and even functions $v : \Omega \to \mathbb{R}$ (also acting as multiplication operators) describing the potential energy between two particles, and many others. A particularly important operator is the *Hamiltonian* H which represents the total energy of

[1]Denoting by $\sum_{a\in\mathcal{A}} P_a$ the orthogonal projector onto the spectral subspace of $L^2(\Omega^N)$ which corresponds to a given subset $\mathcal{A} \subset \mathrm{spec}(A)$ of the spectrum $\mathrm{spec}(A) \subset \mathbb{R}$ of the self-adjoint operator A, one can also define the probability for the event that the quantity associated to the operator A assumes a value which lies in the set $\mathcal{A}$. This probability is given by the "mass" in the normalized wave function Ψ that lies in this spectral subspace, i.e.,

$$\mathbb{P}_\Psi\big(A \in \mathcal{A}\big) = \mathbb{E}_\Psi\Big[\sum_{a\in\mathcal{A}} P_a\Big] = \big\langle \Psi, \Big(\sum_{a\in\mathcal{A}} P_a\Big)\Psi \big\rangle. \tag{1.2}$$

the system. For N nonrelativistic particles (without magnetic potential) it has the general form[2]

$$H = \sum_{i=1}^{N} \Big(-\Delta_{x_i} + W^{\text{ext}}(x_i) \Big) + g_N \sum_{1\leq i<j\leq N} v(x_i - x_j). \tag{1.4}$$

Here, $-\Delta_{x_i} = i\nabla_{x_i} \cdot i\nabla_{x_i}$ denotes the Laplace operator (with appropriate boundary conditions on $\partial\Omega$) and $\sum_i(-\Delta_{x_i})$ corresponds to the total kinetic energy. The function $W^{\text{ext}} : \Omega \to \mathbb{R}$ describes an external potential and $g_N \in \mathbb{R}$ denotes the *coupling constant* which characterizes the strength of a collision between two particles. In all models that we are going to consider, W^{ext} and v are such that the Hamiltonian H is bounded from below and self-adjoint on an appropriate dense subset.

For computing average values like in (1.3), one requires information about the wave function Ψ that describes the system of interest. An important class of wave functions are the so-called *stationary states* for which the expectation value and the variance are time-independent quantities (for time-independent operators). Such stationary states solve the time-independent *Schrödinger eigenvalue equation*,

$$H\Psi = E\Psi, \qquad E \in \mathbb{R}, \qquad \Psi \in L^2(\Omega^N). \tag{1.5}$$

The ground state wave function, e.g., if it exists, is the solution to the eigenvalue equation for the smallest possible value E. It is of particular interest because in many experiments, one starts, e.g., with an ultracold gas of atoms and the idea is that at very low temperature the particles should be correctly described by the wave function corresponding to the lowest possible eigenvalue of H. Stationary states are of course very special and in general, the time evolution of a system is nontrivial. Nonstationary wave functions evolve according to the *time-dependent Schrödinger equation*,

$$i\partial_t \Psi_t = H\Psi_t, \qquad \Psi_{t=0} = \Psi_0 \in L^2(\Omega^N). \tag{1.6}$$

Unique solutions of the time-dependent Schrödinger equation are given via Stones's Theorem by $\Psi_t = U_t\Psi_0$ where $\{U_t\}_{t\in\mathbb{R}}$ is the strongly continuous one-parameter group of unitary operators associated with the self-adjoint operator H. In the case that the Hamiltonian is time-independent, the time-evolution is generated by $U_t = e^{-iHt}$.

For many situations the relevant operators act only on a small subsystem out of the total N particles. In such a case it is useful to introduce so-called *k-particle marginals* $\gamma_\Psi^{(\text{k})}$ of the wave function Ψ (also called k-particle reduced density matrices). $\gamma_\Psi^{(\text{k})} : L^2(\Omega^k) \to L^2(\Omega^k)$, $k \in \mathbb{N}$, is an integral operator which is defined through its kernel

$$\begin{aligned} &\gamma_\Psi^{(k)}(x_1, ..., x_k; y_1, ..., y_k) = \\ &\quad \int_{\Omega^{N-k}} \Psi(x_1, , ...x_k, x_{k+1} ..., x_N)\overline{\Psi(y_1, ..., y_k, x_{k+1} ..., x_N)} dx_{k+1}...dx_N. \end{aligned} \tag{1.7}$$

The average value of a k-particle operator A^k (note that we always use the same notation for A^k and $A^k = A^k \otimes 1^{\otimes N-k}$ where 1 denotes the identity on $L^2(\Omega)$), can be expressed in

[2] In Chapter 4 we study an N+1-body system (one particle plus a fermionic reservoir) where a single tracer particle interacts through a pair potential with N gas particles. There the Hamiltonian has a slightly different form compared to (1.4), see (1.39).

terms of the k-particle marginal,

$$\langle \Psi, A^k \Psi \rangle = \mathrm{Tr}\Big[A^k \gamma_\Psi^{(k)} \Big], \tag{1.8}$$

where $\mathrm{Tr}B^k$ denotes the trace of a trace-class operator B^k, i.e.,

$$\mathrm{Tr}B^k = \mathrm{Tr}_{L^2(\Omega^k)} B^k = \sum_{i\geq 1} \langle \psi_i, B^k \psi_i \rangle_{L^2(\Omega^k)} \tag{1.9}$$

for some orthonormal basis $\{\psi_i\}_{i\geq 1}$ of $L^2(\Omega^k)$. In order to predict the r.h.s. of (1.8), it is now sufficient to have knowledge about $\gamma_\Psi^{(\mathrm{k})}$ instead of the complete wave function Ψ. Our chosen normalization is always such that $\mathrm{Tr}\gamma_\Psi^{(\mathrm{k})} = \|\Psi\| = 1$.

According to a very elementary principle of quantum theory, all particles in nature fall into two distinct groups: *bosons* and *fermions*. The difference between bosonic and fermionic particles is captured in the symmetry property of the corresponding wave function. A wave function Ψ_s describing N indistinguishable bosonic particles is *symmetric* under any permutation of two of the particle coordinates $x_1, ..., x_N$. On the other hand, for indistinguishable fermions, the wave function Ψ_a is required to be totally *antisymmetric* under pairwise permutations of the variables:

$$\Psi_s(x_1, x_2, ..., x_N) = \Psi_s(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(N)}) \quad \forall \sigma \in S_N, \tag{1.10}$$

$$\Psi_a(x_1, x_2, ..., x_N) = (-1)^\sigma \Psi_a(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(N)}) \quad \forall \sigma \in S_N, \tag{1.11}$$

where S_N is the symmetric group of the set $\{1, ..., N\}$ and $(-1)^\sigma$ denotes the sign of the permutation $\sigma \in S_N$. The postulate that fermionic wave functions are antisymmetric is also referred to as Pauli principle or exclusion principle. Below we explain that the Pauli principle can be relevant for the physics of a many-particle fermion system. The correct Hilbert spaces for describing bosonic and fermionic systems are thus the symmetric resp. antisymmetric subspaces $L_s^2(\Omega^N) = \{\Psi \in L^2(\Omega^N) : \Psi \text{ obeys (1.10)}\}$ and $L_a^2(\Omega^N) = \{\Psi \in L^2(\Omega^N) : \Psi \text{ obeys (1.11)}\}$. Everything that has been discussed above is equally true if one replaces $L^2(\Omega^N)$ by its symmetric or antisymmetric subspace. Let us also mention the important fact that the symmetry property of a given initial state Ψ_0 is always preserved under the time evolution in (1.6), i.e., the action of the group $\{U_t\}_{t\in\mathbb{R}}$ leaves the (anti)symmetric subspace of $L^2(\Omega^N)$ invariant. This follows from H being a symmetric expression in the coordinates $x_1, ..., x_N$.

1.2 The mean field approximation

After a brief explanation of the general idea behind the mean field ansatz, we discuss the so-called weak coupling (Hartree) limit for bosons as well as for fermions. The weak coupling limit is defined by the assumption that the strength of the interaction decreases with the number of particles in the system. This type of situation provides the prime example of a many-body model for which an effective mean field description (Hartree theory) emerges in the limit of large particle number (the weak coupling assumption is also meaningful for classical particles where the corresponding mean field theory is given by the Vlasov equation). For fermions, we then introduce another microscopic model for which the mean field ansatz can be expected to make correct predictions. We call this situation the “high density limit for fermions”. Contrary to the weak coupling limit, the interaction does not decrease with the number (or the number per unit volume) of particles in the system. In this model, the Fermi pressure plays a crucial role for the emergence of the effective mean field behavior.

The list of physical models for which the mean field approximation is sensible is of course much longer than the few examples that are discussed or analyzed in this work. Among many others, the most famous ones are the Ising model and many of the related variants of lattice spin systems for which the mean field ansatz provides a main tool of analysis.

1.2.1 The general idea

The major difficulty in solving Schrödinger equations like (1.5) and (1.6) for many particles is the interaction term in the Hamiltonian H which causes the wave function Ψ to be in general a *highly entangled* object, i.e., it does not factorize as a function of the variables $(x_1, ..., x_N) \in \Omega^N$. This makes even numerical attempts for solving the Schrödinger equation impossible because of the large dimension $dN \gg 1$ of the underlying space. The mean field ansatz is a famous approach for an effective and approximate description of interacting many-particle systems. Originally, it has been introduced by P. Curie [32] and P. Weiss [124] in the context of explaining ferromagnetic properties. For bosons it was first used by E. Gross [63] and L. Pitaevskii [105] whereas for fermions it goes even further back to D. Hartree [64], V. Fock [50] and J. Slater [117]. The basic idea behind the mean field approach is to assume that the particles are *independent* of each other and evolve in an *effective*, *self-consistent* external *one-particle potential* (the independence assumption makes sense for bosons but can not hold in the fermionic case for which we explain the necessary generalization below). In this ansatz, the many-body Hamiltonian H is replaced by a set of one-particle Hamiltonians which are in general *nonlinear*. The hope or belief is that the nonlinearity of the effective equation reproduces the relevant effects which are due to the interaction in the original many-body model. The complexity of the problem is significantly reduced by the mean field ansatz and in principle the effective equations are suitable for a numerical or analytical analysis.

Having the underlying N-particle theory in mind, this poses the problem of understanding the *validity* of the mean field approximation. One has to answer the question of how independence between the particles can arise an persist in interacting systems. This certainly happens only under particular physical conditions and on specific length and time scales. For a sharp definition of these conditions and also the correct scales, one studies the interacting many-body model in a particular limit where one or more physical parameters go to infinity (or to zero). One possible such parameter is the particle number N or the average density $\rho = N/|\Omega|$ ($|\Omega|$ being the volume of Ω) tending to ∞. The conditions and the relevant time and length scales for which one may expect, or can even prove, the validity of the mean field ansatz are then formulated in terms of specifying their limiting behavior, the *scaling properties* w.r.t. this limit. One may adjust the mass of the particles, the strength of the interaction, the range of the potential, the volume to which the particles are confined to, the correct time scale, and possibly others. Let us emphasize that it may already be a nontrivial problem to find interesting situations under which a sensible and nontrivial mean field limit can be expected to emerge from the microscopic theory. The mathematical part of the program is to formulate and prove a theorem which assures that under the specified conditions (e.g., an appropriately scaled Hamiltonian and proper initial data) the solution of the many-body Schrödinger equation converges w.r.t. a meaningful notion of distance to the solution of the corresponding effective mean field description. First rigorous results in this regard have been derived by Hepp and Braun [25] and Neunzert and Wick [98] who proved the accurateness of the Vlasov equation starting from Newton's equations for N weakly interacting particles. For quantum systems, the first theorems were proven by Spohn in the bosonic and fermionic case [121] as well as by Narnhofer and Sewell [97] for fermions.

1.2.2 Weak coupling limit for bosons

The idea of independent particles moving through an effective external potential produced by the other particles can only be true, or approximately true, if the strength of the interaction is on the one hand not too strong (single collisions are not allowed to matter as they would destroy independence between the particles) but on the other hand also not too weak (such that the effect due to all other particles sums up to an effective potential). On a heuristic level, this can be understood with the *law of large numbers*. If we assume an exact product wave function $\Psi = \varphi^{\otimes N}$ and a weak coupling constant $g_N = (N-1)^{-1}$, the interaction term in H corresponds to a scaled sum of N independent random variables.[3] The potential acting, e.g., on the first particle at position x, thus satisfies

$$\frac{1}{N-1}\sum_{i=2}^{N} v(x_i - x) \approx \frac{1}{N-1}\sum_{i=2}^{N}\int_\Omega v(x-y)\ |\varphi(y)|^2\ dy = \int_\Omega v(x-y)\ |\varphi(y)|^2\ dy, \quad (1.12)$$

which holds with large probability w.r.t. the distribution $|\varphi^{\otimes N-1}|^2$. The r.h.s. depends now only on the single variable x and the expectation is that one can describe the wave function $\Psi \approx \varphi^{\otimes N}$ in terms of the solution of an effective one-particle equation, the *Hartree equation*,

$$\varepsilon\varphi = \Big(-\Delta_x + W^{\text{ext}}(x) + \big(v * |\varphi|^2\big)(x)\Big)\varphi, \qquad \varepsilon \in \mathbb{R}, \qquad \int_\Omega |\varphi(x)|^2 dx = 1, \qquad (1.13)$$

$$i\partial_t\varphi_t = \Big(-\Delta_x + W^{\text{ext}}(x) + \big(v * |\varphi_t|^2\big)(x)\Big)\varphi_t, \qquad \varphi_{t=0} \in L^2(\Omega), \qquad (1.14)$$

for the stationary as well as the time-dependent case, respectively. By $*$ we denote the convolution of two functions defined on Ω, i.e.,

$$\big(f * g\big)(x) = \int_\Omega f(x-y)g(y)dy. \qquad (1.15)$$

The validity of the mean field ansatz can be studied in the stationary as well as in the time-dependent setup. For the time-independent case, one has to prove that, e.g., the many-body ground state wave function obeys the assumed product structure. The situation is slightly different for the dynamical problem where one wants to show that starting with independent particles, the independence property (the product structure of the wave function) is preserved under the time evolution. The latter is often referred to as propagation of chaos. In both cases, however, one has to be cautious about the meaning of "$\Psi \approx \varphi^{\otimes N}$". One should not expect that for interacting particles, the wave function is close to a product in the sense of the L^2 distance of the full N-particle space. We will later see that this is wrong for stationary wave functions as well as for time-dependent solutions of the Schrödinger equation. Due to the interaction between the particles, the many-body wave function will always develop *weak correlations* which are not described by the Hartree ansatz. The Hartree equation holds rather in a coarse grained sense, namely for the *reduced densities* which have

[3] Note that any coupling g_N that approaches zero as N^{-1} when N tends to ∞ would be equally fine.

been defined in (1.7). Proving the validity of the Hartree equation for the ground state wave function Ψ^0 (and analogously for other low energy eigenfunctions) would consist in a statement of the form

$$\lim_{N\to\infty} \mathrm{Tr}\Big[A^k\Big(\gamma^{(k)}_{\Psi^0} - \gamma^{(k)}_{\varphi_{\mathrm{H}}^{\otimes N}}\Big)\Big] = 0, \tag{1.16}$$

with $k \in \mathbb{N}$, and where A^k belongs to a suitable class of k-particle operators. Ψ_0 is here the (N-dependent) ground state wave function of the many-body Hamiltonian H, and φ_{H} is the ground state solution to the stationary Hartree equation (1.13). Another possible meaning of the mean field approximation for the ground state is a comparison of the energy per particle, i.e., $\lim_{N\to\infty} E^0/N = \varepsilon_{\mathrm{H}}$ for the N-particle ground state energy E^0 and the lowest possible value of the Hartree energy functional,

$$\varepsilon_{\mathrm{H}} = \inf_{\varphi\in L^2(\Omega)} \Big\{\big\langle\varphi, \Big(-\Delta + W^{\mathrm{ext}} + \frac{1}{2}\big(v * |\varphi|^2\big)\Big)\varphi\big\rangle \ : \int_\Omega |\varphi(x)|^2 dx = 1\Big\}. \tag{1.17}$$

Note that (1.13) is the Euler-Langrange equation to the Hartree energy functional $\langle\varphi, \big(-\Delta + W^{\mathrm{ext}} + \frac{1}{2}(v * |\varphi|^2)\big)\varphi\rangle$. For the case that the Hamiltonian H does not posses a ground state, one similarly defines the ground state energy as the infimum of the corresponding energy functional, i.e., $E^0 = \inf_{\|\Psi\|=1} \langle\Psi, H\Psi\rangle$, and one would still expect the comparison between the energies to hold in the limit of large N. Similarly, for the time-dependent setting, a possible criterion for the validity of the mean field ansatz is defined by the convergence of the reduced density at times t, given that the initial state satisfies the same "factorization property", namely

$$\lim_{N\to\infty} \mathrm{Tr}\Big[A^k\Big(\gamma^{(k)}_{\Psi_0} - \gamma^{(k)}_{\varphi_0^{\otimes N}}\Big)\Big] = 0 \quad \Rightarrow \quad \lim_{N\to\infty} \mathrm{Tr}\Big[A^k\Big(\gamma^{(k)}_{\Psi_t} - \gamma^{(k)}_{\varphi_t^{\otimes N}}\Big)\Big] = 0, \tag{1.18}$$

with $k \in \mathbb{N}$, and where Ψ_t solves the time-dependent many-body Schrödinder equation for initial condition Ψ_0, while φ_t is the solution to the time-dependent Hartree equation (1.14) with initial condition φ_0. Assertions like (1.16) and (1.18) mean that the majority of particles in the gas behaves according to the mean field approximation in the sense that one can predict the average values of a suitable class of operators in terms of the product wave function $\varphi^{\otimes N}$, cf. (1.8). This is what one refers to as *condensation* of the Bose gas into the one-particle state φ. Let us stress that the *weak coupling assumption* for bosons is of twofold importance for the mean field approximation. On the one hand, $g_N = (N-1)^{-1}$ is the typical choice for the law of large numbers in (1.12). On the other hand, it guarantees that the *potential energy* of the system is *compatible* with the *kinetic energy*, i.e., of the same order in the particle number N.[4] The kinetic energy is expected to be of order N (due to the N summands in the kinetic term in H), whereas the interaction term in H contains a double sum and thus the potential energy is on average of order

$$g_N \sum_{i<j} \big\langle\varphi^{\otimes N}, v(x_i - x_j)\varphi^{\otimes N}\big\rangle = g_N \frac{N(N-1)}{2}\big\langle\varphi, \big(v * |\varphi|^2\big)\varphi\big\rangle = O(g_N N^2). \tag{1.19}$$

Only for $g_N = O(N^{-1})$, neither the kinetic energy nor the potential energy is dominant in the large N limit. On the level of the mean field equation: a different choice of g_N would imply an additional factor $g_N(N-1)$ in front of the effective potential which could cause

[4] We say that a function $f(N)$ is of order $g(N)$, or in short $f(N) = O(g(N))$, iff there exist two N-independent positive constant C, D such that $Cg(N) \leq f(N) \leq Dg(N)$ for all N larger than some N_0.

the latter to be either subleading or superleading compared to $-\Delta_x$.

The weak coupling (Hartree) model as presented can be used, e.g., to describe bosonic atoms for which the number of electrons is proportional to the nuclear charge [19, 118, 9, 12, 71], and similarly for the description of stars where the Hartree equation is related to the so-called Chandrasekhar Theory describing the collapse of dense stars [88]. More importantly, however, we think of the weak coupling assumption as a meaningful *prototype model* for a more realistic (and also more involved) microscopic theory which is capable to explain the remarkable phenomenon of *Bose-Einstein condensation* in a dilute gas of ultracold atoms. Bose-Einstein condensation is the name for a low temperature phase of a dilute Bose gas in which a macroscopic fraction of the atoms occupies a single one-body wave function. The existence of this phase was experimentally observed for the first time in the 90s in systems of cooled alkali atoms [8, 34]. The theoretical possibility of this effect was already explained about 70 years earlier by A. Einstein [38] (and for massless particles also by S. Bose [24]). Such condensates exhibit many interesting properties that are known to be accurately described by a nonlinear one-particle model which is defined by the so-called *Gross-Pitaevskii* Hamiltonian (first introduced by Gross [63] and Pitaevskii [105]),

$$h_x^{\mathrm{GP},\varphi} = -\Delta_x + W^{\mathrm{ext}}(x) + a|\varphi(x)|^2, \qquad a \in \mathbb{R}. \tag{1.20}$$

Describing the Bose gas in terms of the effective Gross-Pitaevskii Hamiltonian leads to many correct predictions about its nontrivial physical behavior. Among others, the effective description is capable to explain the appearance of vortices in rotating condensates[5] (see, e.g., [31] for the mathematical analysis and [1] for experimental evidence) as well as the time evolution of initially trapped condensates (see, e.g., [73]). Bose-Einstein condensation occurs in dilute gases where the atoms interact strongly but very rarely. Even though this is very different from the physical picture behind the weak coupling limit – a *dense gas* of particles with *weak* and *long-range pair interaction*–, the microscopic interaction results in a similar nonlinear effect (formally, one obtains the Gross-Pitaevskii Hamiltonian from the Hartree equation if one chooses $v(x) = a\delta(x)$). This analogy can be seen as our main motivation for studying the less complicated weak coupling limit of the Bose gas. A more detailed explanation of the connection between the weak coupling limit and the N-particle model that describes a dilute gas of atoms and from which the effective Hamiltonian (1.20) can be derived, is given in Appendix A.

A list of references and remarks about known results for the derivation of the Hartree equation starting from the many-body theory in the weak coupling limit is given in Section 2.1.1 for the time-dependent case and in Section 3.1 for the stationary problem. Let us also mention the excellent summary about recent rigorous results by Lewin in [76], from which some of the ideas explained in this and the following subsection are motivated.

1.2.3 Next-order corrections in the weak coupling limit for bosons

The bosonic mean field ansatz provides also the correct starting point for the explanation of collective phenomena that go beyond the mean field description. E.g., the emergence of interesting effective energy-momentum dispersion relations like the phonon dispersion

[5]Note that in order to describe the appearance of vortices, one needs to include a vector potential in the Gross-Pitaveskii Hamiltonian, i.e., one replaces $-i\nabla_x \mapsto -i\nabla_x + A(x)$ with $A : \mathbb{R}^d \to \mathbb{R}^d$ modeling the rotational forces in the condensate.

relation (for small momentum), or the so-called phonon-maxon-roton dispersion relation[6] (see, e.g., in [122, 70]). Both effects are predicted by means of an effective description of the Bose gas in terms of the quadratic *Bogoliubov Hamiltonian*,

$$\mathbb{H}_{\text{Bog}} = \int_\Omega a_x^* \Big(-\Delta_x + \big(v * |\varphi|^2\big)(x) - \frac{1}{2}\big\langle \varphi, \big(v * |\varphi|^2\big)\varphi \big\rangle \Big) a_x \, dx \qquad (1.21)$$
$$+ \int_\Omega \int_\Omega \varphi(x) v(x-y) \overline{\varphi(y)} a_x^* a_y \, dxdy + \frac{1}{2} \int_\Omega \int_\Omega \Big(\varphi(x) v(x-y) \varphi(y) a_x^* a_y^* + \text{h.c.} \Big) dxdy,$$

acting on the bosonic Fock space over $L^2(\Omega)$ with the Hartree mode φ removed (a_x^* and a_x denote the usual creation and annihilation operators and h.c. stands for the hermitian conjugate). This approximation can be understood as the next-order description of the Hartree equation and was first introduced by Bogoliubov in [22] in order to explain the superfluid property of the Bose-Einstein condensate (which follows essentially from the so-called Landau criterion which is satisfied for a linear dispersion relation, see, e.g., [106, Chapter 6]). Similarly as for the mean field ansatz, it poses the question of understanding its validity and its precise relation to the interacting many-body theory. Bogoliubov theory will be introduced in more detail and analyzed for the time-dependent setup in Chapter 2 resp. for the stationary setting in Chapter 3.

1.2.4 Weak coupling limit for fermions

The basic idea is the same as in the bosonic case. Due to the Pauli principle, however, fermions can not be described by a product (or approximate product) wave function and thus fermionic particles can not be independent of each other. The simplest type of wave functions (which is the correct generalization of the product state since it describes noninteracting particles) is given by an antisymmetric product of N pairwise orthonormal one-particle orbitals $\{\varphi_k\}_{k\geq 0}$, $\varphi_k \in L^2(\Omega)$. We denote the normalized antisymmetric product by

$$\bigwedge_{k=1}^{N} \varphi_k(x_k) = \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} (-1)^\sigma \prod_{k=1}^{N} \varphi_{\sigma(k)}(x_k), \qquad (1.22)$$

where S_N denotes the symmetric group of integers $\{1, ..., N\}$ and $(-1)^\sigma$ is the sign of the permutation $\sigma \in S_N$. Note that the set of N orbitals has to be linearly independent since otherwise the wave function $\bigwedge_{k=1}^N \varphi_k$ would be identically zero. In analogy to the bosonic case, one wants to approximate the solution to the N-body Schrödinger equation by a "product" wave function of the form (1.22) for which the orbitals solve a set of selfconsistent coupled nonlinear mean field equations. The most natural set of such equations for N orbitals are given in the stationary case by the *time-independent fermionic Hartree equations*,[7]

[6] A phonon energy-momentum dispersion relation is characterized by its linearity for small momentum, i.e., the energy depends linearly on the momentum. The phonon-maxon-roton dispersion relation possesses a local maximum in the energy momentum relation which is followed by an additional local minimum. For reasons of illustration, we included some pictures of the excitation spectrum of the Bose gas at the end of Appendix 3.B.

[7] Let us mention that another possible mean field description for fermions are the so-called Hartree-Fock equations for which the effective potential contains an additional exchange term, namely

$$\Big(g_N \sum_{l=1}^N \big(v * |\varphi_l|^2\big)(x)\Big)\varphi_k \mapsto \Big(g_N \sum_{l=1}^N \big(v * |\varphi_l|^2\big)(x)\Big)\varphi_k - \Big(g_N \sum_{l=1}^N \big(v * \overline{\varphi_l}\varphi_k\big)(x)\Big)\varphi_l \qquad (1.23)$$

$$\varepsilon_k \varphi_k = \Big(-\Delta_x + W^{\mathrm{ext}}(x) + g_N \sum_{l=1}^{N} \big(v * |\varphi_l|^2\big)(x) \Big) \varphi_k, \qquad \varepsilon_k \in \mathbb{R}, \qquad \varphi_k \in L^2(\Omega), \tag{1.24}$$

for $\{\varphi_k\}_{k=1}^N$ such that $\langle \varphi_k, \varphi_l \rangle = \delta_{kl}$, and similarly in the time-depending setting by the *time-dependent fermionic Hartree equations*,

$$i\partial_t \varphi_{k,t} = \Big(-\Delta_x + W^{\mathrm{ext}}(x) + g_N \sum_{l=1}^{N} \big(v * |\varphi_{l,t}|^2\big)(x) \Big) \varphi_{k,t}, \qquad \varphi_{k,t=0} \in L^2(\Omega), \tag{1.25}$$

with $\{\varphi_{k,0}\}_{k=1}^N$ pairwise orthonormal (note that the orthonormality of the initial orbitals is preserved which follows readily from $\partial_t \langle \varphi_{k,t}, \varphi_{l,t} \rangle = 0$). The fermionic Hartree and Hartree-Fock equations (and many related variants thereof) have numerous applications in theoretical physics and theoretical chemistry. Most importantly, they are used for the computational analysis of the electronic structure of large atoms and molecules. Moreover, they are widely applied in many related areas as, e.g., in solid state physics, in atomic theory as well as in nuclear physics. Since the number of references that can be easily found with regard to the application of the fermionic Hartree(-Fock) approximation in any of the mentioned fields is immense, we refrain from providing any reference here.

The rigorous derivation of the fermionic Hartree equations in the weak coupling limit (i.e., for $g_N \to 0$ when N tends to ∞) starting from a many-body Hamiltonian H of the form (1.4) is well understood in two interesting regimes (see below and for more details, in Appendix B). The choice of the correct coupling constant g_N for which the kinetic energy and the potential energy are compatible is, however, more subtle than for bosons. This additional difficulty is a consequence of the so-called *kinetic energy inequality* (a variant of the Lieb-Thirring inequality, cf. [84, Chapter 4.2]) for antisymmetric wave functions. The kinetic energy inequality states that for any $\Psi_a \in L^2_a(\Omega^N)$ with $\Omega \subset \mathbb{R}^d$, the average kinetic energy is bounded from below by

$$N \|\nabla \Psi_a\|^2 \geq C_d |\Omega|^{-\frac{2}{d}} N^{1+\frac{2}{d}}, \qquad C_d > 0. \tag{1.26}$$

As an example, think of N particles confined to a region $\Omega \subset \mathbb{R}^3$ with $|\Omega| = 1$. In this case, the kinetic energy of any antisymmetric N-body wave function is necessarily of order $N^{5/3}$ or larger which is much more compared to N bosons having kinetic energy $\propto N$. In order to guarantee that many fermions have compatible energies, one has to choose the coupling strength g_N and also the volume to which the fermions are confined to very carefully. In Appendix B, we give a brief presentation of two different limits for which the kinetic and potential terms scale in the same way and for which the rigorous derivation of the fermionic Hartree equations is well understood. Both models were originally introduced by Narnhofer and Sewell [97] and have been subject also to more recent analysis, e.g., [39, 18, 51] as well as [101, 11, 100].[8]

- The so-called *semiclassical limit* for which the particles are confined to a volume of order one and $g_N = N^{-1}$. Here, one has to replace $-\Delta_{x_i} \mapsto N^{-2/3}\Delta_{x_i}$ and $\partial_t \mapsto$

in (1.24) resp. in (1.25). To our knowledge, however, in the microscopic regimes for which the Hartree-Fock equations have been derived so far, the exchange term turned out to be only a subleading correction to the leading order Hartree description (one may think of this as an interesting open problem, i.e., to find a meaningful limit for which the exchange term in the Hartree-Fock equations is of leading order in the approximation).

[8] Note that the derivation of the fermionic Hartree equations is not further investigated in this thesis.

$N^{-1/3}\partial_t$ (for the time-dependent case) in the microscopic Schrödinger equation as well as in the Hartree equations. This regime corresponds to a semiclassical limit (one can think of the additional prefactors as a small Planck constant $\hbar = \hbar_N \propto N^{-1/3}$), and at leading order, the system can be also approximated by the Vlasov equation (the mean field description of N weakly interacting classical particles). The fermionic Hartree equations provide a subleading correction to the classical approximation. We explain some more details in Appendix B.1.

- Fermions that occupy a region $\Omega \subset \mathbb{R}^3$ of *large volume* $|\Omega| \propto N$ and that interact through a *Coulomb potential* $v(x) \propto |x|^{-1}$ (here, the average density is of order one and the mean field effect is due to the very long range of the interaction). The coupling constant has to be chosen as $g_N = N^{-2/3}$ such that the kinetic energy and the potential energy are compatible, both being proportional to the number of particles N. Even though the potential energy is as large as the kinetic energy, the particles behave approximately freely in this limit because the average forces are of subleading order. The fermionic Hartree equations provide a subleading correction to the free time evolution. For more details, we refer to Appendix B.2.

1.2.5 High density limit for fermions

Another interesting consequence of the antisymmetry of the wave function is the so-called *Fermi pressure*. The Fermi pressure can be interpreted as a force between the particles which is present even for noninteracting fermions and which becomes very strong in a dense Fermi gas. That it may drastically change the statistical properties of the particles can be easily seen already from a very simple example. Think of a one-dimensional unit box with periodic boundary conditions and noninteracting particles in their lowest possible energy state. We ask for the probability for finding the particles on the left side respectively the right side of the middle of the box. For a single particle in the box, the correct wave function is the constant function (a plane wave with zero momentum) and there is no difference between a boson and a fermion. The probability to find the particle on the left or the right side is one half for the boson as well as for the fermion. Putting two identical particles inside the box there is already a significant difference. The bosonic two-particle state is the product of two constant functions which immediately leads to

$$\mathbb{P}_{\Psi_s}(\text{both on the left}) = \mathbb{P}_{\Psi_s}(\text{both on the right}) = 1/4, \tag{1.27}$$

$$\mathbb{P}_{\Psi_s}(\text{one on the left, one on the right}) = 1/2. \tag{1.28}$$

For fermions, the two-particle ground state is more complicated. Due to the antisymmetry, the particles can not be both described by the constant wave function. The wave function corresponding to the lowest energy is given by the antisymmetric product of the constant function and the plane wave with momentum equal to 2π, i.e.,

$$\Psi_a(x_1, x_2) = \frac{1}{\sqrt{2}}\Big(e^{i2\pi x_1} - e^{i2\pi x_2}\Big). \tag{1.29}$$

The corresponding probabilities are given by

$$\mathbb{P}_{\Psi_a}(\text{both on the left}) = \mathbb{P}_{\Psi_a}(\text{both on the right}) = 1/4 - 1/2\pi^2, \tag{1.30}$$

$$\mathbb{P}_{\Psi_a}(\text{one on the left, one on the right}) = 1/2 + 1/\pi^2, \tag{1.31}$$

where we have used that

$$\mathbb{P}_\Psi\big(\text{particle one in } [a,b], \text{ particle two in } [c,d]\big) = \int_a^b dx_1 \int_c^d dx_2 |\Psi(x_1,x_2)|^2. \tag{1.32}$$

The probability of finding both fermions on one side of the box is suppressed compared to the bosonic case and it seems that fermions tend to repel each other. If we proceeded with the indicated example, we would see that the strength of this repulsion becomes stronger when the number of fermions that are concentrated in a given volume increases. It is this effect that causes *dense fermion systems* to be *distributed* much more *rigidly* compared to bosons. In Appendix 4.A, we give an illustrative example of how strong this repulsive effect can become for large densities. For an ideal Fermi gas with average density tending to infinity, we show that a large extent of the particles behaves almost like a rigid body and that there remain only relatively few particles that deviate from to the average distribution. Density *fluctuations* are thus *strongly suppressed* in a dense Fermi gas compared to a Bose gas where the fluctuations behave at best according to the square root of N law of i.i.d. particles (what happens is exactly the same as indicated in (1.30), namely that additional contributions due to the antisymmetry of the wave function diminish the probability for particles randomly building clusters).

In Chapter 4 we present a model of a tracer particle coupled to the ideal Fermi gas in the high density limit. In this model, the mean field description (modified by a comparatively small next-order correction to the mean field energy of the system) provides a good approximation even for a *"strong" coupling constant* $g = 1$. Since the mean field potential (and also the next-order energy correction) is spatially constant, the time evolution of the system is effectively free. In plasma physics, such a behavior is known for degenerate electron gases which for large density lose their ability of stopping ions that move through the electron gas (see, e.g., [33] and for more references, Section 4.1.2). This is very different from the motion of a tracer particle through a dense homogeneous Bose gas in which the free time evolution (the mean field description) would be disturbed already after a very short time $t = t(\rho)$ which approaches zero when the density ρ increases. The reason for this different behavior are the strong fluctuations in the Bose gas for coupling constant $g = 1$.

1.3 Outline and summary

In the remainder of this work, we analyze the weakly interacting Bose gas (in the stationary and also in the time-dependent setting) as well as the dynamics of a tracer particle "strongly" interacting through a pair potential with an ideal Fermi gas in the high density limit. The different chapters are self-contained and can be read independently of each other. Here, we provide a short summary of our main results.

Chapter 2: Time evolution of the Bose gas

We study the Bogoliubov corrections and the trace norm convergence for the Hartree dynamics. The microscopic model is defined by a Hamiltonian H_N of the form (1.4) with $g_N = (N-1)^{-1}$ and v, e.g., the Coulomb potential. Hatree theory states that the many-body solution is approximately given by a product wave function $\Psi_{N,t} \approx \varphi_t^{\otimes N}$ (under the assumption that the initial state has the same product structure) where φ_t solves the nonlinear Hartree equation (1.14). The meaning of approximation is here in the sense of reduced densities like for instance in (1.18). We first show convergence of an auxiliary time evolution to the fully interacting dynamics in the norm of the N-particle space. To this end we introduce an effective N-particle Hamiltonian $\widetilde{H}_N^t$. Denoting the projector onto the one-dimensional subspace spanned by $\varphi_t(x_i)$ as $p_i^t = |\varphi_t(x_i)\rangle\langle\varphi_t(x_i)|$ and its orthogonal complement as $q_i^t = 1 - p_i^t$, one obtains $\widetilde{H}_N^t$ from the original Hamiltonian H_N,

$$H_N = \sum_{i=1}^{N} \Big(- \Delta_{x_i} + W^{\text{ext}}(x_i) \Big) + \frac{1}{N-1} \sum_{1 \le i < j \le N} (p_i^t + q_i^t)(p_j^t + q_j^t) v(x_i - x_j)(p_i^t + q_i^t)(p_j^t + q_j^t), \tag{1.33}$$

by neglecting all terms that contain three or more q^t-projectors. For particular initial conditions (summarized in Assumption 2.2), we prove that the true dynamics is well approximated by the effective time evolution, i.e.,

$$\left\| \Big(e^{-iH_N t} - \mathcal{T} e^{-i \int_0^t \widetilde{H}_N^s ds} \Big) \Psi_{N,t=0} \right\|_{L^2(\mathbb{R}^{3N})} \le \frac{e^{C(1+t)^2}}{\sqrt{N}}, \tag{1.34}$$

where $\mathcal{T}$ is the time-ordering operator, i.e., $\{\mathcal{T} e^{-i \int_0^t \widetilde{H}_N^s ds}\}_{t \in \mathbb{R}}$ stands for the unitary group generated by $\widetilde{H}$. The notion of closeness is here much stronger compared to reduced densities. From (1.34), we derive further results. On the one hand, we show for the reduced one-particle density that

$$\operatorname{Tr} \Big| \gamma^{(1)}_{\Psi_{N,t}} - \gamma^{(1)}_{\varphi_t^{\otimes N}} \Big| \le \frac{e^{C(1+t)^2}}{N}, \tag{1.35}$$

where N^{-1} is the optimal convergence rate, as well as

$$\operatorname{Tr} \Big| \sqrt{1-\Delta} \Big(\gamma^{(1)}_{\Psi_{N,t}} - \gamma^{(1)}_{\varphi_t^{\otimes N}} \Big) \sqrt{1-\Delta} \Big| \le \frac{e^{C(1+t)^2}}{\sqrt{N}}. \tag{1.36}$$

For both estimates, it is relevant to have information about the next-order corrections from (1.34). On the other hand, we show that the fluctuations around the Hartree product in Ψ_t are correctly described by a Bogoliuvov Hamiltonian on Fock space. The main results are stated in Theorems 2.6, 2.7 and 2.9.

Our results in Chapter 2 extend and quantify several previous results regarding the derivation of Hartree and Bogoliubov theory starting from the N-particle Bose gas. A summary of known results together with a list of references is given in Sections 2.1.1 and 2.1.2. We provide the physically important convergence rates, we include time-dependent external fields and also treat singular interactions like Coulomb. Moreover, our results allow more general initial states compared to previous results, e.g., those that are expected to be ground states of interacting systems. The techniques that we employ in this chapter consist to a large extent of a generalization of the method that was introduced by Pickl in [103]. In Section 2.2.4 we use also ideas from Lewin et al. [80].

Chapter 3: Low energy properties of the Bose gas

We analyze the properties of eigenfunctions and eigenvalues of the homogeneous weakly-interacting Bose gas on the unit torus. The Hamiltonian has the same form as in (1.4). However, we set the external potential equal to zero and the pair potential has to satisfy further assumptions. In particular it is required to be of positive type, i.e., $\hat{v} \ge 0$.

Hartree theory describes the leading order contributions of eigenfunctions as well as eigenvalues. E.g., all energy eigenvalues E_N^n close to the ground state energy of H_N satisfiy $\lim_{N\to\infty}(E_N^n/N - \varepsilon_{\mathrm{H}}) = 0$ where $\varepsilon_{\mathrm{H}} = \hat{v}(0)/2$ is the lowest possible Hartree energy. This can be shown to imply that the corresponding eigenfunctions Ψ^n are to leading order described by the Hartree product, namely, $\lim_{N\to\infty}\gamma^{(1)}_{\Psi^n} = |\varphi_{\mathrm{H}}\rangle\langle\varphi_{\mathrm{H}}|$, where φ_{H} is the ground state solution to the stationary Hartree equation. In our main theorem of this chapter, we prove that for any low energy eigenfunction Ψ_N^n of the Hamiltonian H_N, the probability for finding l particles not in the Hartree state is exponentially small in the number l, i.e.,

$$\mathbb{P}_{\Psi_N^n}\big(l \text{ particles not in the Hartree state}\big) \leq Ce^{-Dl}, \tag{1.37}$$

with positive (N-independent) constants C, D. From this bound, we then derive certain quasifree type properties of the ground state Ψ_N^0 of the microscopic system which were so far only known for the corresponding Bogoliubov ground state. Such quasifree type properties are important for the analysis of the time-dependent Schrödinger equation (cf. Chapter 2) as they appear exactly as the required assumptions on the initial wave functions. We also use the exponential decay in order to provide an alternative strategy for proving the validity of Bogoliubov theory for the low energy spectrum of H_N (the original proof was given by Seiringer in [116] and then extended into various directions by Grech and Seiringer [60], by Lewin et al. [81] and by Dereziński and Napiórkowski [37]). Bogoliubov theory states that the low energy excitations of the N-particle system are given by the set of numbers,

$$\Big\{\sum_{i=1}^{j}\sqrt{|k_i|^4 + 2\hat{v}(k_i)|k_i|^2} : k_i \in 2\pi\mathbb{Z}^d\backslash\{0\}, j \in \mathbb{N}\Big\}, \tag{1.38}$$

which can be interpreted as excitation energies of noninteracting quasiparticles that obey an effective energy momentum dispersion relation $e(k) = \sqrt{|k|^4 + 2\hat{v}(k)|k|^2}$. We formulate our main results in Theorem 3.1, Corollary 3.6 and Theorem 3.7.

Chapter 4: Free dynamics of a particle coupled to a dense Fermi gas

The motion of a tracer particle in a dense homogeneous Fermi gas is considered. Here, the problem is simplified by coupling only one particle to a gas of noninteracting fermions. On the other hand, the problem is made harder by considering a thermodynamic limit, without scaling of the interaction parameter. Differently stated, we analyze the model in a regime of "strong" coupling ($g = 1$) where the interaction strength does not become weaker with increase of the density. Due to the antisymmetry of the wave function, however, the interaction of the gas with the tracer particle turns out to be effectively weak, which is in strong contrast to a bosonic or classical gas. The motivation of this model comes from the phenomenon, known in plasma physics, that the ability of a degenerate electron gas to stop ions is decreasing with increasing gas density. This is the opposite of the expectation from classical physics, and also from bosonic systems. There, the higher the density of the gas, the more collisions the ion undergoes and thus the more it is disturbed in its motion. In fermionic systems, it is the Pauli principle or the Fermi pressure that changes the situation drastically by limiting the number of possible interactions and their effective strength. The motion of a tracer particle in a homogeneous ideal electron gas is described by a Hamiltonian

$$H_{N+1} = -\Delta_y - \sum_{i=1}^{N}\Delta_{x_i} + \sum_{i=1}^{N} v(x_i - y), \tag{1.39}$$

where $x_1, ..., x_N$ denote the gas variables and y the tracer particle variable. As initial state we assume a wave function $\varphi \otimes \Omega$ where $\varphi \in L^2([0,L]^d)$ and $\Omega \in L^2([0,L]^{dN})$ is the ground state of the free Fermi gas in the box $[0,L]^d$ with periodic boundary conditions. We prove that for $d = 2$ (in Appendix 4.C, we derive a similar statement also for $d = 1$) and for a certain class of compactly supported potentials $v(x)$,

$$\lim_{\substack{N\to\infty \\ \varrho = N/L^2 = const}} \left\| e^{-iH_{N+1}t}\big(\varphi \otimes \Omega_0\big) - e^{-iH^{\mathrm{mf}}_{N+1}t}\big(\varphi \otimes \Omega_0\big) \right\|_{L^2([0,L]^{2+2N})} \leq \frac{C_\varepsilon (1+t)^{\frac{3}{2}}}{\varrho^{\frac{1}{8}-\varepsilon}}, \quad (1.40)$$

where $\varepsilon > 0$ is a sufficiently small number and C_ε denotes an ε-dependent positive constant. The effective Hamiltonian given by $H^{\mathrm{mf}}_{N+1} = -\Delta_y - \sum_{i=1}^N \Delta_{x_i} + \varrho\hat{v}(0) - E_{re}$ is equivalent to the generator of the free dynamics. Here,

$$\varrho\hat{v}(0) = \sum_{i=1}^{N} \big\langle \Omega_0, v(x_i - y)\Omega_0 \big\rangle_{L^2([0,L]^{2N})} \quad (1.41)$$

is the spatially constant mean field potential felt by the tracer particle due to the gas particles and $E_{re} > 0$ is a next-order energy correction which comes from so-called immediate recollisions diagrams. Eq. (1.40) states that for large densities, the tracer particle moves effectively freely through the electron gas on the relevant time scale. The result is noteworthy because it provides an explicit example which shows that the mean field approximation is valid for fermions far beyond a weak coupling limit $g \to 0$ (note that the inclusion of the next-order correction in the energy is important for the derivation of the stated result; nevertheless, the energy correction does not change the physical behavior since it only provides a constant phase factor). The proven statement can also be interpreted as a derivation of a long-lived resonance of the initial momentum distribution of the tracer particle. We state our exact result in Theorem 4.1.

Chapter 2

Time evolution of the Bose gas

This chapter is organized as follows. In Section 2.1 we introduce the model and explain the Hartree and Bogoliubov ansatz for the description of the weakly interacting Bose gas including a short summary of existing results. Section 2.2 contains our main theorems regarding the derivation of the Hartree and Bogoliubov approximation. All proofs are deferred to Section 2.3.[1]

2.1 Introduction

A system of N spinless bosons in nonrelativistic quantum mechanics is described by a wave function $\Psi_N \in \mathcal{H}_N$ with

$$\mathcal{H}_N := L^2_{\mathrm{s}}(\mathbb{R}^{3N}; dx_1, ..., dx_N), \tag{2.1}$$

the subspace of square integrable functions that are symmetric under permutations of the variables $x_1, ..., x_N \in \mathbb{R}^3$ (we only consider three dimensions here but the analysis is the same for any dimension). We always assume that Ψ is normalized, i.e., $\|\Psi_N\| = 1$, such that $|\Psi_N(x_1, ..., x_N)|^2$ can be interpreted as the probability density of finding the particles at positions $x_1, ..., x_N$. The time evolution of the wave function is governed by the nonrelativistic many-body Schrödinger equation

$$i\partial_t \Psi_{N,t} = H^t_N \Psi_{N,t}, \tag{2.2}$$

where the Hamiltonian operator H^t_N is of the form

$$H^t_N = \sum_{i=1}^{N} h^t_i + \lambda_N \sum_{1\leq i<j\leq N} v(x_i - x_j). \tag{2.3}$$

Here, $h^t_i = -\Delta_i + W^t_i$ denotes a one-particle operator, Δ_i is the Laplacian describing the kinetic energy of the i-th particle and $W^t_i = W^t(x_i)$ a possibly time-dependent external potential. The interaction between the gas particles is described by a real-valued function $v = v(x)$, e.g., the Coulomb potential $v(x) = 1/|x|$. The coupling constant in front of the interaction will be chosen as $\lambda_N = 1/(N-1)$ which ensures that the average interaction

[1]Note: The content of the present chapter is a revised version of the work presented in [92]; the main results being the same as in [92]. While the three authors (Sören Petrat, Peter Pickl and the author of this thesis) contributed equally to obtain the results of this work, its presentation is due to the author of the thesis.

energy is of the same order as the average kinetic energy, namely of order N. In this situation, a nontrivial behavior of the many-body system can be expected for large particle number N.[2]

Our goal in this chapter is to investigate the large N limit of solutions to the Schrödinger equation and, in particular, the corrections to the leading-order mean field component of such solutions.

The physical setting we have in mind is that the gas is initially trapped in a confining potential W^0 and cooled down, such that $\Psi_{N,0}$ is close to the ground state of H_N^0. By removing or changing the external field W^t, the ground state of H_N^0 is in general not an eigenfunction of H_N^t for $t > 0$ anymore, so the time evolution is nontrivial. To our understanding, this is the picture behind the experiments of ultracold gases exhibiting the phenomenon of Bose-Einstein condensation, see, e.g., [20] and references therein.[3]

2.1.1 The Hartree equation

It has been established in many different settings that Hartree theory emerges as the macroscopic description of the low temperature many-body Bose gas in the mean field regime, i.e., for $N \to \infty$, $N\lambda_N \to 1$. Hartree theory is defined by the one-body Hamiltonian

$$h^{t,\varphi} = h^t + v * |\varphi|^2 - \mu^\varphi, \qquad \varphi \in L^2(\mathbb{R}^3), \tag{2.4}$$

where $*$ denotes the convolution of functions on $\mathbb{R}^3$, and the phase factor $\mu^\varphi = \frac{1}{2}\int dx\ (v * |\varphi|^2)(x)|\varphi(x)|^2$ is chosen for later convenience. In order to understand the relation between the microscopic model defined by (2.3) and Hartree theory, think of a completely factorized N-particle wave function $\Psi_N = \varphi^{\otimes N}$ for which the potential term in H_N corresponds to a sum of identically and independently distributed random variables with probability density $|\varphi|^2$. It follows from the law of large numbers that the potential felt by, e.g., the first particle, at position x is given by the average value $(v * |\varphi|^2)(x)$; cf. (1.12). The N-particle Hamiltonian H_N is hence expected to act as a sum of N one-body Hamiltonians each given by (2.4). In more precise terms, the Hartree Hamiltonian governs the leading order dynamics of a wave function which is initially close to a condensate $\varphi_0^{\otimes N}$, e.g., in the sense of

$$\lim_{N\to\infty} \mathrm{Tr}\left|\gamma^{(1)}_{\Psi_{N,0}} - |\varphi_0\rangle\langle\varphi_0|\right| = 0 \quad \Rightarrow \quad \lim_{N\to\infty} \mathrm{Tr}\left|\gamma^{(1)}_{\Psi_{N,t}} - |\varphi_t\rangle\langle\varphi_t|\right| = 0, \tag{2.5}$$

where Tr denotes the trace, and the one-particle state φ_t solves the nonlinear time-dependent Hartree equation

$$i\partial_t\varphi_t = h^{t,\varphi_t}\varphi_t \tag{2.6}$$

with initial condition φ_0. The operator $\gamma^{(1)}_{\Psi_N} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the one-body reduced density matrix of $\Psi_N \in L^2_s(\mathbb{R}^{3N})$, defined by its kernel

$$\gamma^{(1)}_{\Psi_N}(x, y) = \int \Psi_N(x, x_2, ..., x_N)\overline{\Psi_N(y, x_2, ..., x_N)}dx_2...dx_N,$$

[2] This is not necessarily true for fermionic many-body systems. As we will see in Chapter 4, an interesting mean field type behavior can emerge even when the potential and the kinetic energy do not scale in the same way with the relevant parameter which tends to ∞.

[3] Note, however, that for actual experiments the Gross-Pitaevskii limit is more relevant, which is more involved than the mean field limit we are considering in the present work. A brief discussion of the Gross-Pitaevskii limit and its relation to the weak coupling regime can be found in Appendix A.

and $|\varphi_t\rangle\langle\varphi_t|$ is the one-body reduced density w.r.t. to the state $\varphi_t^{\otimes N} \in L^2_{\mathrm{s}}(\mathbb{R}^{3N})$. Implications like (2.5) are referred to as propagation of chaos or persistence of condensation, and have been proven in different and very general settings. Without claim for completeness, let us mention the following results.

- The first rigorous result was derived by Spohn [121] for bounded pair potentials v approaching the question in terms of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy, a set of coupled equations for the k-particle marginals of Ψ_t.

- The same method was then used to derive bosonic mean field equations for singular (Coulomb like) potentials in [14, 48] and for semirelativistic systems in [41].

- In [2, 40, 3, 45, 46, 47], Spohn's approach was applied to the Gross-Pitaevskii regime, i.e., for pair potential $v(x)$ replaced by $v_N(x) = N^3 v(Nx)$, for which the Gross-Pitaevskii equation was derived in the large N-limit.

  Note that the approach using the BBGKY hierarchy is based on an abstract compactness argument and therefore no explicit error bounds are obtained. The speed of convergence in (2.5) is thus not known (this is different in the original work by Spohn where a Duhamel expansion was used in order to obtain an explicit error bound; the requirement that the Duhamel expansion converges, however, restricts the validity of the derivation to short times).

- A different approach, the so-called Hepp method or coherent state approach, goes back to the works by Hepp [66] and also by Ginibre and Velo [56, 57] about the classical limit of bosonic systems. This method makes use of a Fock space representation of the many-body Bose gas. The use of Fock space allows one to analyze coherent states (with nonfixed particle number) which, in turn, facilitates the separation of the mean field component from the microscopic time evolution and allows one to control the fluctuations around the Hartree equation.

- The coherent state approach has inspired many works about the derivation of the Hartree equation, among others the work by Rodnianski and Schlein [115] who first derived the Hartree equation with explicit convergence rate,

  $$\mathrm{Tr}\Big|\gamma^{(1)}_{\Psi_{N,t}} - |\varphi_t\rangle\langle\varphi_t|\Big| \le \frac{e^{Ct}}{\sqrt{N}}, \tag{2.7}$$

  holding for factorized initial data and interaction potentials $v^2 \le C(1-\Delta)$.

- Analogous techniques were used in order to improve the error term to N^{-1} (for initially factorized data) in [29] for the same class of potentials $v^2 \le C(1-\Delta)$ and also in [30] for potentials $v \in L^3(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$.

- The coherent state approach turned out to be fruitful also for a quantitative derivation of the Gross-Pitaevskii equation by Benedikter et al. in [16].

- Starting from initially factorized data, the optimal convergence rate N^{-1} in the Hartree limit was moreover obtained in [44] for bounded potentials and in [75] for the semirelativistic case (i.e., for $-\Delta$ replaced by $\sqrt{1-\Delta}$) of gravitating particles.

- In [55], Fröhlich et al. explain that the mean field limit can be understood as an Ergodov type statement meaning that the time evolution of observables commutes with the Wick quantization up to an error that vanishes in the mean field limit. They prove the accuracy of the time-dependent mean field equation for the Coulomb potential.
- Another recent approach (providing explicit error terms) was introduced by Pickl [103] where the Hartree equation is derived through an ad hoc counting of the number of particles leaving the condensate at a given time. This new technique was applied, e.g., in [72] to derive the Hartree equation for very singular potentials including the critical case $v(x) = |x|^{-2}$ and in [104] for deriving the Gross-Pitaevskii equation with time-dependent external potentials. As one of our main lemmas (cf. Lemma 2.3) is based on a generalization of the method introduced by Pickl, we give a short exposition of its basic ideas below the mentioned lemma.
- In [5, 6] the propagation of Wigner measures was studied in the mean field limit. This idea was used in [4] for deriving the Hartree equation with optimal rate N^{-1} for bounded potentials.

The question whether it is reasonable to assume that the initial condition is factorized in the first place is answered by Hartree theory as well. Indeed, under suitable conditions, the ground state of a weakly interacting Bose gas exhibits Bose-Einstein condensation: The wave function Ψ_N^0 corresponding to the lowest eigenvalue of a Hamiltonian of the form (2.3) factorizes into an N-fold product of a single one-particle wave function φ_{H} which is determined by minimizing the nonlinear Hartree functional

$$\mathcal{E}_{h^{0,\varphi}}(\varphi) = \Big\{ \langle \varphi, h^{0,\varphi}\varphi \rangle : \varphi \in H^1(\mathbb{R}^3), \|\varphi\| = 1 \Big\}. \tag{2.8}$$

The condensation property holds again in the reduced sense (here we assume that a unique minimizer exists), i.e.,

$$\lim_{N\to\infty} \mathrm{Tr}\Big|\gamma^{(1)}_{\Psi_N^0} - |\varphi_{\mathrm{H}}\rangle\langle\varphi_{\mathrm{H}}|\Big| = 0. \tag{2.9}$$

It further holds for a comparison of the energies, i.e., $E_N^0 = N\varepsilon_{\mathrm{H}} + o(N)$ where E_N^0 denotes the infimum of the spectrum $\sigma(H_N^0)$ and ε_{H} the infimum of (2.8) (the symbol $o(A_N)$ stands for terms with $o(A_N)/A_N \to 0$ for $N \to \infty$). The rigorous analysis of this question goes back to [19, 88]. For recent results and an extensive list of references, we refer to [77]. For the special case of a homogeneous Bose gas in the box, we study the stationary problem in Chapter 3. For this case, it is shown in Appendix 3.A that energies close to the ground state energy are given at leading order by $N\varepsilon_{\mathrm{H}}$; the corresponding next-order contribution $o(N)$ is analyzed in detail in Section 3.2.3.

2.1.2 Different notions of distance

The notion of distance in (2.5) and (2.9) is equivalent to convergence of bounded k-particle operators with norm of order one (for fixed k when N tends to ∞). This, in turn, is strong enough to imply a law of large numbers type result for such observables. E.g., in the case $k = 1$ and a one-particle operator A (A_i acting on the ith particle), it implies that for all $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\Psi_{N,t}}\Big(\Big|\frac{1}{N}\sum_{i=1}^N A_i - \langle\varphi_t, A\varphi_t\rangle\Big| \geq \delta\Big) = 0. \tag{2.10}$$

In order to control unbounded observables like energy or momentum, a stronger statement than (2.5) is needed. A suitable generalization is given by

$$\lim_{N\to\infty} \mathrm{Tr}\left|\sqrt{1-\Delta}\left(\gamma^{(1)}_{\Psi_{N,t}} - |\varphi_t\rangle\langle\varphi_t|\right)\sqrt{1-\Delta}\right| = 0, \tag{2.11}$$

i.e., convergence in the so-called energy trace norm. Questions in this direction have been studied in [91, 89] and more recently in [7].

Yet another natural notion of distance, much stronger compared to convergence in terms of reduced densities, is the L^2-norm on the full N-particle space $L^2_s(\mathbb{R}^{3N})$. In the interacting case, i.e., for $v \neq 0$, the ground state is not close to a product of one-particle wave functions, and neither does the initial product structure survive the dynamics in the L^2 sense. If only a single particle is not in the correct condensate wave function, there is no closeness in L^2-norm. On the other hand, condensation is a macroscopic phenomenon, i.e., it still holds, even if a few out of a very large number N of particles are not in the condensate. The property of condensation, and persistence of condensation, is therefore correctly understood in terms of reduced densities, e.g., in the sense explained in (2.5) or (2.11). An approximation in terms of the L^2-distance of $\Psi_{N,t}$ or the ground state Ψ^0_N is nevertheless very interesting. A large N approximation in $L^2(\mathbb{R}^{3N})$ is for instance closely connected to the analysis of low energy excitations which play a crucial role in the explanation of superfluidity and other collective phenomena of Bose-Einstein condensates. Such an approximation can be understood as the next-to-leading order correction to Hartree theory; it is generally known under the name of Bogoliubov theory [22].

Such a norm approximation in terms of Bogoliubov theory was rigorously derived by Lewin et. al. in [80] (and similarly for the NLS equation[4] by Nam and Napiórkowski [95, 96]). Their main result is a full characterization of fluctuations in $\Psi_{N,t}$ around the Hartree product $\varphi_t^{\otimes N}$. It was shown that

$$\lim_{N\to\infty} \left\| \Psi_{N,t} - \sum_{k=0}^{N} \left(\varphi_t^{\otimes(N-k)} \otimes_s \chi_t^{(k)} \right) \right\|_{\mathcal{H}_N} = 0, \tag{2.12}$$

where $\otimes_s$ stands for the normalized symmetric tensor product, cf. (2.47), and the correlation functions $(\chi_t^{(k)})_{k\geq 0}$, $\chi_t^{(k)} \in L^2_s(\mathbb{R}^{3k})$, solve a Schrödinger equation on the bosonic Fock space of excitations with an N-independent, quadratic Hamiltonian.

2.1.3 Objective of this chapter

Our goal in this chapter is to contribute to the understanding of (2.5), (2.11) and (2.12). Besides that, we introduce a first quantized version of Bogoliubov theory.

Our strategy is to first show norm convergence of $\Psi_{N,t}$ towards the solution $\widetilde{\Psi}_{N,t}$ of a Schrödinger equation with a simpler Hamiltonian $\widetilde{H}_{N,t}$ which is quadratic in a sense that we explain below. If we denote by $p_i^t = |\varphi_t(x_i)\rangle\langle\varphi_t(x_i)|$ the orthogonal projector in the variable x_i onto the condensate state φ_t and by $q_i^t = 1 - p_i^t$, then the Hamiltonian $\widetilde{H}_N^t$ is

[4]The NLS or nonlinear Schrödinger limit is defined by H_N as in (2.3) but with the function v replaced by an N-dependent pair potential $v_N(x) = N^{3\beta} v(N^\beta x)$ for $\beta \in [0, 1]$ and $v \in C_0(\mathbb{R}^3)$. For $\beta = 0$, one recovers the Hartree limit for bounded potentials, whereas for $\beta = 1$, it coincides with the Gross-Pitaevskii limit (cf. Appendix A). For $\beta \in (0, 1)$, the NLS limit defines a model between the Hartree and the Gross-Pitaevskii regime for which the limiting equation is given by the Hartree equation with v replaced by $\|v\|_1\delta(x)$.

obtained from the original Hamiltonian (2.3),

$$H_N^t = \sum_{i=1}^N h_i^{t,\varphi_t} + \\ + \lambda_N \sum_{1\le i<j\le N} (p_i^t+q_i^t)(p_j^t+q_j^t)\Big(v_{ij} - v*|\varphi_t|_i^2 - v*|\varphi_t|_j^2 + 2\mu^t\Big)(p_i^t+q_i^t)(p_j^t+q_j^t), \quad (2.13)$$

by discarding all terms that contain three or four q^t's. For wave functions that are, in a suitable sense, sufficiently close to the Hartree product, the Hamiltonian $\widetilde{H}_N^t$ will be related to the usual Bogoliubov Hamiltonian on Fock space (see (2.51)) by a unitary transformation. After showing in Theorem 2.6 that

$$\|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\|_{\mathcal{H}_N} \le \exp\big(C(1+t)^2\big)N^{-1/2}, \quad (2.14)$$

where $\widetilde{\Psi}_{N,t}$ solves the Schrödinger equation with Hamiltonian $\widetilde{H}_N^t$, we derive

1. in Theorem 2.7 that $\gamma^{(1)}_{\Psi_{N,t}}$ converges to $|\varphi_t\rangle\langle\varphi_t|$ in trace norm with optimal rate $1/N$, as well as in energy trace norm with rate $1/\sqrt{N}$,
2. in Theorem 2.9 the approximation of $\Psi_{N,t}$ in terms of correlation functions $(\chi_t^{(k)})_{k\ge 0}$ in the sense of (2.12), with expected optimal rate.

Both points provide extensions of known results. Convergence in trace norm with rate $1/N$ was shown in [44, 29, 30, 75] for initial wave functions that are completely factorized, and in [4] for more general initial conditions but only for bounded pair potential v. It is interesting to note that for our proof, the Hartree approximation is not sufficient to derive (2.5) with optimal rate (this is in agreement with previous results for unbounded v). Instead, one also needs information about higher-order corrections to the Hartree approximation. In [7], convergence in terms of the energy trace norm was derived by means of an abstract argument, without explicit error. The characterization of $\Psi_{N,t}$ in terms of correlation functions $(\chi_t^{(k)})_{k\ge 0}$ was first studied in [80]. Here, we derive the optimal error of this approximation which was not included in the analysis of [80, cf. Remark 3].

From the technical point of view, our approach consists of a generalization of the method that was used by Pickl resp. Knowles and Pickl to derive the Hartree equation in [103, 72]. We expect this approach to turn out stable and versatile and therefore also useful in order to derive similar results for more complicated situations, in particular for the NLS and Gross-Pitaevskii limit.

Next-to-leading order corrections in $\Psi_{N,t}$ have been studied in [95, 96] for the NLS equation with $0 \le \beta < \frac{1}{2}$. Let us note that there are also very strong results about the L^2-approximation for states on Fock space, derived by means of the coherent state method that goes back to Hepp [66]. These results cover the weakly interacting case [62] as well as the NLS limit for all $\beta < 1$ [21] (a detailed list of references can be found in [95, Section 1.2]). The initial states are here coherent states in Fock space, or slight generalizations thereof. These results also give convergence in L^2-norm for initial N-particle states, with a weakened rate of convergence. It is unclear whether the results also imply convergence of initial N-particle ground states, like we consider in this work, or that were considered in [80, 95, 96]. For a detailed comparison of the two different approaches, we refer to [80, Section 3].

2.1.4 Notation

- $|\cdot|$ denotes the standard norm on $\mathbb{C}^d$ for arbitrary dimension d.
- For $k \in \{0, 1, ..., N\}$ we denote the Hilbert space of square integrable wave functions by $\mathcal{H}_k := L^2_s(\mathbb{R}^{3k}; dx_1, ..., dx_k)$. The scalar product on $\mathcal{H}_k$ is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}_k}$ and the norm by $\|\cdot\|_{\mathcal{H}_k}$. We usually omit the subscript $\mathcal{H}_k$ since it is clear from the context which space is meant.
- By H^k_s $(k = 1, 2)$ we denote the symmetric (first resp. second) Sobolev space, i.e., the set of wave functions $\psi \in L^2_s$ for which the Sobolev norm $\|\psi\|_{H^k} = \|(1 + |\cdot|^2)^{\frac{k}{2}} \hat{\psi}\|_{L^2}$ is finite, where $\hat{\psi} \in L^2$ denotes the Fourier transform of ψ.
- For any linear operator $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ we define the operator norm by

$$\|A\|_{op} = \sup_{\|\psi\|=1} \|A\psi\|. \tag{2.15}$$

- The letters C and D are used to denote positive constants with numerical values that may change from one line to the other. We emphasize that all constants denoted by C and D are independent of the relevant parameters N and t.

2.2 Main results

We consider Hamiltonians H^t_N of the form (2.3) with external potential $W^t : \mathbb{R}^3 \to \mathbb{R}$ and pair interaction $v : \mathbb{R}^3 \to \mathbb{R}$ satisfying

Assumptions A.1. (The model)

1. The mapping $t \mapsto W^t$ is $C^1(\mathbb{R}, L^\infty(\mathbb{R}^3))$,
2. $v(x) = v(-x)$ and $v^2 \leq C(1 - \Delta)$ for some $C > 0$.

Note that due to Hardy's inequality, $|\cdot|^{-2} \leq 4(-\Delta)$, this includes the physically important Coulomb interaction $v(x) = |x|^{-1}$.

Next we summarize known results about well-posedness of the many-body Schrödinger and the Hartree equation.

Well-posedness of the N-body time evolution. It follows from the standard Kato-Rellich argument that under Assumptions A.1, the Hamiltonian H^t_N is self-adjoint on $H^2_s(\mathbb{R}^{3N})$. In particular, $H_N \equiv H^t_N - \sum_{i=1}^N W^t(x_i)\Psi_N$ is time-independent and self-adjoint and thus generates the unitary time evolution $e^{-iH_N t}$. Due to A.1.1. we have that the mapping $t \mapsto \sum_{i=1}^N W^t(x_i)\Psi_N$ is Lipschitz continuous for any $\Psi_N \in \mathcal{H}_N$. According to [61, Theorem 2.5], this is sufficient for H^t_N to generate a unitary time evolution in the following sense: For any $T > 0$ there exists a unique two-parameter family of unitary operators $U_N(t, s) : \mathcal{H}_N \to \mathcal{H}_N$, satisfying the following properties:

(a) $U_N(t, s)H^2_s(\mathbb{R}^{3N}) \subset H^2_s(\mathbb{R}^{3N})$ for all $s, t \in [0, T]$, and the map $(0, T) \ni t \mapsto U_N(t, s)\Psi_N$ is continuously differentiable for any $s \in [0, T]$ and $\Psi_N \in H^2_s(\mathbb{R}^{3N})$, with

$$i\partial_t\big(U_N(t, s)\Psi_N\big) = H^t_N\big(U_N(t, s)\Psi_N\big),$$

(b) $U_N(t,t) = 1$ and $U_N(t,r)U_N(r,s) = U_N(t,s)$ for all $r, s, t \in [0,T]$,

(c) $(t,s) \mapsto U_N(t,s)\Psi_N$ is strongly continuous on $[0,T] \times [0,T]$.

Well-posedness of the Hartree time evolution. Given Assumptions A.1, it follows, e.g., by adapting the techniques used in [28] to our setting, that for every wave function $\varphi_0 \in H^1(\mathbb{R}^3)$, the Hartree equation (2.6) admits a unique global solution $\varphi_t \in H^1(\mathbb{R}^3)$ for all $t > 0$. It is also well-known that mass and energy of the Hartree solution are conserved quantities – modulo the change of the time-dependent external potential –, i.e., for any $t \geq 0$, we have

$$\langle \varphi_t, \varphi_t \rangle - \langle \varphi_0, \varphi_0 \rangle = 0, \tag{2.16}$$

$$\langle \varphi_t, h^{t,\varphi_t}\varphi_t \rangle - \langle \varphi_0, h^{0,\varphi_0}\varphi_0 \rangle = \int_0^t \langle \varphi_s, (\partial_s W^s)\varphi_s \rangle ds. \tag{2.17}$$

Moreover, one finds the following bounds for the H^1- and the H^2-norm, respectively,

$$\|\varphi_t\|_{H^1}^2 := \|\nabla\varphi_t\|^2 + \|\varphi_t\|^2 \leq C(1+t), \tag{2.18}$$

$$\|\varphi_t\|_{H^2}^2 := \|\Delta\varphi_t\|^2 + \|\nabla\varphi_t\|^2 + \|\varphi_t\|^2 \leq De^{Ct(1+t)}, \tag{2.19}$$

where $C = C(\|\varphi_0\|_{H^1})$ and $D = D(\|\varphi_0\|_{H^2})$ are time-independent positive constants with $D < \infty$ for $\varphi_0 \in H^2(\mathbb{R}^3)$. In the latter case, (2.19) guarantees that $\varphi_t \in H^2(\mathbb{R}^3)$ for all later times. The H^1-bound is a direct consequence of (2.17), and for details about the derivation of the H^2-bound we refer to [28, Section 3]. Let us also remark that for time-independent external potential $W^t \equiv W$, one obtains the above estimates with the factor $(1+t)$ replaced by one.

Invoking the assumed bound on the pair potential v, together with (2.18), one readily finds that

$$\|v * |\varphi_t|^2\|_\infty \leq C\sqrt{1+t}, \qquad \|v^2 * |\varphi_t|^2\|_\infty \leq C(1+t), \tag{2.20}$$

for some time-independent constant $C = C(\|\varphi_0\|_{H^1})$.

We now define some operators we will use throughout this chapter. Let φ_t denote the solution to the Hartree equation for given initial condition $\varphi_0 \in H^1(\mathbb{R}^3)$ with $\|\varphi_0\| = 1$.

Definition 2.1. For any $1 \leq i \leq N$, we define the time-dependent projectors

$$p_i^t : \mathcal{H}_N \to \mathcal{H}_N, \quad p_i^t\Psi_N(x_1, ..., x_N) = \varphi_t(x_i) \int \overline{\varphi_t(x_i)}\Psi_N(x_1, ..., x_N)dx_i, \tag{2.21}$$

and $q_i^t = 1 - p_i^t$. At some points we use the more convenient braket notation, i.e., $p^t = |\varphi_t\rangle\langle\varphi_t|$ (when acting on the one-particle space) respectively $p_i^t = |\varphi_t\rangle\langle\varphi_t|_i = |\varphi_t(x_i)\rangle\langle\varphi_t(x_i)|$ (when acting on the many-particle space).

Note that p^t and q^t satisfy Heisenberg type equations of motion, namely,

$$i\partial_t p^t = [h^{t,\varphi_t}, p^t], \qquad i\partial_t q^t = [h^{t,\varphi_t}, q^t]. \tag{2.22}$$

Definition 2.2. For every $t \geq 0$ we define an N-particle auxiliary Hamiltonian by

$$\widetilde{H}_N^t = \sum_{i=1}^N h_i^{t,\varphi_t} + \lambda_N \sum_{1\leq i<j\leq N} \Big[\Big(p_i^t q_j^t v_{ij} q_i^t p_j^t + p_i^t p_j^t v_{ij} q_i^t q_j^t \Big) + \text{h.c.} \Big], \tag{2.23}$$

with $v_{ij} = v(x_i - x_j)$, $\lambda_N = (N-1)^{-1}$, and where h.c. stands for the Hermitian conjugate of the preceding expression.

Well-posedness of the auxiliary time evolution. Using (2.20), together with

$$\|vp^t\|_{op}^2 \leq \|p^t v^2 p^t\|_{op} = \|p^t\|_{op} \langle \varphi_t, v^2 \varphi_t \rangle, \tag{2.24}$$

it follows that the time-dependent part in (2.23) is bounded and hence, $\widetilde{H}_N^t$ defines a self-adjoint operator on $H_s^2(\mathbb{R}^{3N})$. Furthermore, one shows with (2.22) and (2.24) that for any $T > 0$, the mapping $(0,T) \ni t \mapsto (\widetilde{H}_N^t + \sum_{i=1}^N \Delta_i)\Psi_N$ is Lipschitz continuous for any $\Psi_N \in H_s^2(\mathbb{R}^{3N})$. We can thus use again [61, Theorem 2.5] implying that for any interval $[0,T]$, there exists a unique group of unitary operators $\widetilde{U}_N(t,s) : \mathcal{H}_N \to \mathcal{H}_N$, such that

(a) $\widetilde{U}_N(t,s) H_s^2(\mathbb{R}^{3N}) \subset H_s^2(\mathbb{R}^{3N})$ for all $s,t \in [0,T]$, and the map $(0,T) \ni t \mapsto \widetilde{U}_N(t,s)\Psi_N$ is continuously differentiable for any $s \in [0,T]$ and $\Psi_N \in H_s^2(\mathbb{R}^{3N})$, with

$$i\partial_t \big(\widetilde{U}_N(t,s)\Psi_N \big) = \widetilde{H}_N^t \big(\widetilde{U}_N(t,s)\Psi_N \big),$$

(b) $\widetilde{U}_N(t,t) = 1$ and $\widetilde{U}_N(t,r)\widetilde{U}_N(r,s) = \widetilde{U}_N(t,s)$ for all $r,s,t \in [0,T]$,

(c) $(t,s) \mapsto U_N(t,s)$ is strongly continuous on $[0,T] \times [0,T]$.

From a straightforward computation, which we postpone to Appendix 2.A, it follows that

$$H_N^t = \widetilde{H}_N^t + \frac{1}{N-1} \sum_{1\leq i<j\leq N} \Big(v_{ij}^{(3q,t)} + v_{ij}^{(4q,t)} \Big), \tag{2.25}$$

with

$$v_{ij}^{(3q,t)} = v_{ji}^{(3q,t)} := q_i^t q_j^t \Big(v_{ij} - \bar{v}_i^t - \bar{v}_j^t + 2\mu^t \Big) \big(q_i^t p_j^t + p_i^t q_j^t \big) + \text{h.c.}, \tag{2.26}$$

$$v_{ij}^{(4q,t)} = v_{ji}^{(4q,t)} := q_i^t q_j^t \Big(v_{ij} - \bar{v}_i^t - \bar{v}_j^t + 2\mu^t \Big) q_i^t q_j^t, \tag{2.27}$$

and $\bar{v}_i^t = \big(v * |\varphi_t|^2\big)(x_i)$ and $\mu^t = \frac{1}{2} \int \big(v * |\varphi_t|^2\big)(x) |\varphi_t(x)|^2 dx$.

Remark 2.1. Note that there are no terms on the r.h.s. in (2.25) that are linear in p^t. The operator $\widetilde{H}_N^t$ contains only quadratic terms while the remainders are cubic or quartic. The reason for the linear terms to not appear is the correct choice of the phase factor μ^t in the Hartree Hamiltonian.

2.2.1 Norm convergence of $\Psi_{N,t}$ towards $\widetilde{\Psi}_{N,t}$

The key ingredients for our first main result are Lemmas 2.3 and 2.4.

Lemma 2.3. *Let φ_t be the unique solution of the Hartree equation (2.6) with initial condition $\varphi_0 \in H^1(\mathbb{R}^3)$, $\|\varphi_0\| = 1$, and let $\Psi_N \in \mathcal{H}_N$, $\|\Psi_N\| = 1$. Then, for $\Phi_{N,t} \in \{U_N(t,0)\Psi_N, \widetilde{U}_N(t,0)\Psi_N\}$ with $t > 0$, and any fixed integer $n \leq N$, there exists a positive constant C_n such that*

$$\Big\langle \Phi_{N,t}, \Big(\prod_{j=1}^{n} q_j^t\Big)\Phi_{N,t}\Big\rangle \leq e^{C(1+t)^{3/2}} \sum_{i=0}^{n} \frac{C_n}{N^{n-i}} \Big\langle \Psi_N, \Big(\prod_{j=1}^{i} q_j^0\Big)\Psi_N\Big\rangle. \tag{2.28}$$

The quantity $\langle \Psi_{N,t}, q_1^t \Psi_{N,t}\rangle \equiv \alpha_N(t)$ for $\Psi_{N,t} = U_N(t,0)\Psi_N$ counts the average relative number of particles outside the condensate in $\Psi_{N,t}$. In [103, 72], it was shown by Pickl resp. Knowles and Pickl that for appropriate initial conditions $\alpha_N(t) \leq e^{Ct}N^{-1}$. This was in turn used to prove $\gamma^{(1)}_{\Psi_{N,t}} \to |\varphi_t\rangle\langle\varphi_t|$ by means of the relation

$$\alpha_N(t) \leq \mathrm{Tr}\Big|\gamma^{(1)}_{\Psi_{N,t}} - |\varphi_t\rangle\langle\varphi_t|\Big| \leq \sqrt{8\alpha_N(t)}, \tag{2.29}$$

which was derived, e.g., in [72, Lemma 2.3]. Since the strategy for proving Lemma 2.3 is the same as in the derivation of the bound for $\alpha_N(t)$, let us explain it in some detail. The idea is to control $\alpha_N(t)$ via a Gronwall argument, i.e., to show that $\partial_t\alpha_N(t) \leq C(\alpha_N(t) + N^{-1})$ from which one infers the desired bound, namely $\alpha_N(t) \leq e^{Ct}(\alpha_N(0) + N^{-1})$.[5] For the simplest case $\Psi_N = \varphi_0^{\otimes N}$, we have $q_1^0\Psi_N = 0$, and thus $\alpha_N(0) = 0$. The time-derivative of $\alpha_N(t)$ has an obvious physical meaning, namely the rate of particles leaving the condensate wave function at time t. It is not difficult to find (for details, we refer to Lemma 2.15) that it is given by

$$\begin{aligned}\partial_t\alpha_N(t) &= 2\,\mathrm{Im}\,\big\langle \Psi_{N,t}, q_1^t p_2^t \Big(v(x_1-x_2) - \big(v * |\varphi_t|^2\big)(x_1)\Big) p_1^t p_2^t \Psi_{N,t}\big\rangle \\ &\quad + 2\,\mathrm{Im}\,\big\langle \Psi_{N,t}, q_1^t q_2^t v(x_1-x_2) p_1^t p_2^t \Psi_{N,t}\big\rangle \\ &\quad + 2\,\mathrm{Im}\,\big\langle \Psi_{N,t}, q_1^t q_2^t \Big(v(x_1-x_2) - \big(v * |\varphi_t|^2\big)(x_1)\Big) p_1^t q_2^t \Psi_{N,t}\big\rangle.\end{aligned} \tag{2.31}$$

The three terms represent the different processes which cause particles to deviate from the mean field evolution φ_t. In the first line ($qpvpp$), two particles both in φ_t collide with each other leaving one particle inside and one particle outside the condensate wave function; in the second line ($qqvpp$), two particles in φ_t collide with a resulting correlated pair. Eventually, in the third line ($qqvpq$), one mean field particle collides with a particle which is already not in the condensate leaving both particles outside φ_t. The imaginary part indicates that also the reverse processes are taken into account. All other possible collisions do not alter the overall number of particles in the condensate and thus do not appear in $\partial_t\alpha_N(t)$. Now, the first term is identically zero because $p_2^t v_{12} p_2^t = (v*|\varphi_t|^2)(x_1)p_2^t$ is cancelled exactly by the mean field potential. The second and third terms are small because of the two respectively three q^t's. In the last line (assuming here $\|v\|_\infty < C$ for simplicity) it follows immediately with Cauchy-Schwarz that

$$\begin{aligned}&\Big|\mathrm{Im}\,\big\langle \Psi_{N,t}, q_1^t q_2^t \Big(v_{12} - v * |\varphi_t|_1^2\Big) p_1^t q_2^t \Psi_{N,t}\big\rangle\Big| \\ &\qquad\qquad \leq C\|q_1^t q_2^t \Psi_{N,t}\|\,\|p_1^t q_2^t \Psi_{N,t}\| \leq C\|q_1^t\Psi_{N,t}\|^2 = C\alpha_N(t),\end{aligned}$$

[5] Gronwall's inequality states that if the derivative of a function $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto f(t)$, satisfies the estimate $\partial_t f(t) \leq C(t)(f(t) + \delta(t))$ for continuous functions $C(t) : \mathbb{R} \to \mathbb{R}^+$ and $\delta(t) : \mathbb{R} \to \mathbb{R}^+$, then $f(t)$ is bounded from above in terms of

$$f(t) \leq \Big(e^{\int_0^t C(s)ds}\Big) f(0) + \int_0^t \delta(s) e^{\int_0^s C(s')ds'} ds. \tag{2.30}$$

as $\|q_1^t\Psi_{N,t}\|^2 = \langle\Psi_{N,t}, q_1^t\Psi_{N,t}\rangle$. In the second term we can not directly apply Cauchy-Schwarz since both q^t's are on one side of the scalar product. This term is indeed not small for all wave functions in $L^2(\mathbb{R}^{3N})$ and the symmetry of $\Psi_{N,t}$ becomes crucial. For symmetric wave functions, one can show that commuting one of the q^t from the left to the right side effectively costs an error proportional to N^{-1}:

$$\begin{aligned}\Big|\operatorname{Im}\langle\Psi_{N,t}, q_1^t q_2^t v_{12} p_1^t p_2^t \Psi_{N,t}\rangle\Big| &= \Big|\frac{1}{N-1}\operatorname{Im}\langle q_1^t\Psi_{N,t}, \Big(\sum_{j=2}^N q_j^t v_{1j} p_j^t\Big) p_1^t \Psi_{N,t}\rangle\Big| \\ &\le \frac{\sqrt{\alpha_N(t)}}{N-1}\Big\|\Big(\sum_{j=2}^N q_j^t v_{1j} p_j^t\Big) p_1^t \Psi_{N,t}\Big\| \\ &\le \frac{\sqrt{\alpha_N(t)}}{N-1} C\Big(\sqrt{N} + N\|q_1^t\Psi_{N,t}\|\Big) \le C\Big(\alpha_N(t) + \frac{1}{N}\Big),\end{aligned}$$

which leads to the bound for $\partial_t\alpha_N(t)$. The proof of Lemma 2.3 is a generalization of this argument.

The derivation of mean field equations via the Grönwall argument for $\alpha_N(t)$ is both very simple and effective. It is comparatively simple because no propagation estimates for the microscopic solution $\Psi_{N,t}$ are needed and thus also no expansion in terms, e.g., of the BBGKY hierarchy appears. It is effective because it yields with very small effort explicit error terms. The latter are important from the physics point of view where N is considered to be large but always finite. Moreover, one can modify the definition of $\alpha_N(t)$ such that the approach is applicable also to more complicated situations. Along with the necessary modifications, this approach has been used to derive the time-dependent Hartree equation for very singular potentials (including the critical case of the Hartree equation $v(x) = |x|^{-2}$) [72, Section 4], the Hartree equation in a large volume mean field limit [35], the NLS equation without positivity condition [102] as well as the Gross-Pitaevskii equation with time-dependent external potentials [104]. It is also applicable for fermions to derive the Hartree and Hartree-Fock equations in the corresponding mean field limit [101, 11, 100] (for a short discussion of the fermionic weak coupling limit, see Section 1.2.4 and Appendix B).

We next analyse the structure of the auxiliary Hamiltonian in more detail. Therefor note that each term in the effective two-body potential in $\widetilde{H}_N^t$ contains exactly two q^t's and two p^t's. This is a crucial property, as it directly implies that the number of particles inside the condensate is changed always in steps of two. More generally it means that under the time evolution generated by the auxiliary Hamiltonian, there is no mass flow between the orthogonal subspaces in $\mathcal{H}_N$ that correspond to an even respectively an odd number of particles outside the Hartree state. In order to make this observation precise, let us introduce the projectors $\widehat{f}^{\varphi_t}_{\mathrm{odd/even}} : \mathcal{H}_N \to \mathcal{H}_N$ by

$$\Psi_N \mapsto \widehat{f}^{\varphi_t}_{\mathrm{odd}}\Psi_N = \sum_{\substack{k=0\\ k \text{ odd}}}^N \Big(\prod_{i=1}^k q_i^t \prod_{j=k+1}^N p_j^t\Big)_{\mathrm{sym}} \Psi_N, \tag{2.32}$$

$$\Psi_N \mapsto \widehat{f}^{\varphi_t}_{\mathrm{even}}\Psi_N = \sum_{\substack{k=0\\ k \text{ even}}}^N \Big(\prod_{i=1}^k q_i^t \prod_{j=k+1}^N p_j^t\Big)_{\mathrm{sym}} \Psi_N, \tag{2.33}$$

where $(\cdot)_{\mathrm{sym}}$ abbreviates the symmetric tensor product, cf. (2.92). It follows directly that

$$1=\widehat{f}^{\varphi_t}_{\mathrm{odd}}+\widehat{f}^{\varphi_t}_{\mathrm{even}},\qquad \widehat{f}^{\varphi_t}_{\mathrm{odd}}\widehat{f}^{\varphi_t}_{\mathrm{even}}=0=\widehat{f}^{\varphi_t}_{\mathrm{even}}\widehat{f}^{\varphi_t}_{\mathrm{odd}},\qquad \widehat{f}^{\varphi_t}_{\mathrm{even}}\widehat{f}^{\varphi_t}_{\mathrm{even}}=\widehat{f}^{\varphi_t}_{\mathrm{even}},$$

and the same also for $\widehat{f}^{\varphi_t}_{\mathrm{odd}}$. The operator $\widehat{f}^{\varphi_t}_{\mathrm{odd/even}}$ projects on the subspace in $\mathcal{H}_N$ with odd/even number of particles that do not occupy the product wave function φ_t. For instance, $\widehat{f}^{\varphi_t}_{\mathrm{even}}\varphi_t^{\otimes N}=\varphi_t^{\otimes N}$, since in $\varphi_t^{\otimes N}$ there is an even number of particles (zero particles) that do not belong to the condensate. We summarize the parity argument that we explained above in the following lemma.

Lemma 2.4. *Let $\varphi_t\in H^1(\mathbb{R}^3)$ be the solution to the Hartree equation (2.6) with initial datum $\varphi_0\in H^1(\mathbb{R}^3)$, and let $\Psi_N\in\mathcal{H}_N$ and $\widetilde{\Psi}_{N,t}=\widetilde{U}_N(t,0)\Psi_N$. It holds that for all $t\geq 0$*

$$\|\widehat{f}^{\varphi_t}_{even}\widetilde{\Psi}_{N,t}\|=\|\widehat{f}^{\varphi_0}_{even}\Psi_N\|,\qquad \|\widehat{f}^{\varphi_t}_{odd}\widetilde{\Psi}_{N,t}\|=\|\widehat{f}^{\varphi_0}_{odd}\Psi_N\|. \tag{2.34}$$

The proof of the lemma is a consequence of the vanishing of the commutators

$$\Big[\widetilde{H}^t_N-\sum_{i=1}^N h_i^{t,\varphi_t},\widehat{f}^{\varphi_t}_{\mathrm{odd}}\Big]=0=\Big[\widetilde{H}^t_N-\sum_{i=1}^N h_i^{t,\varphi_t},\widehat{f}^{\varphi_t}_{\mathrm{even}}\Big]. \tag{2.35}$$

The parity property (2.34) is important for estimating expectation values of operators like $q_1^tA_1p_1^t$, which effectively measure the overlap between even and odd parts in Ψ_N. Starting from wave functions whose mass is initially located in only one of the two sectors, the parity argument turns out to be useful to improve certain estimates. We will use it for instance to get optimal control – optimal in the sense of its N-dependence – of the quantity $|\langle\Psi_{N,t},q_1^t(-\Delta_1)p_1^t\Psi_{N,t}\rangle|$. The latter is important for bounding the kinetic energy of the particles in $\Psi_{N,t}$ which are outside the condensate; see Theorem 2.6 and its proof.

Since we want to obtain an L^2-approximation of $\Psi_{N,t}$, it is necessary to have good control of the behavior of all N particles. This means that also good control of particles outside the condensate is required. For bounded potentials, one can use the fact that the number of such particles is small compared to N, and that they can therefore not disturb the other particles too much. For singular potentials, however, already a few badly behaving particles can in principle cause problems when they come close together and generate a large potential energy. That such behavior is very unlikely is due to energy conservation. In order to deal with singular potentials, the idea is thus to use energy conservation to obtain sufficient control of $\nabla_1q_1^t\Psi_{N,t}$, i.e., to control the regularity of the part of $\Psi_{N,t}$ which describes particles outside the Hartree product. This, in turn, leads to appropriate bounds on the potential energy of these particles, since we assume that the pair potential is dominated by the kinetic energy.

Definition 2.5. We define the energy per particle w.r.t. H_N^t by

$$\mathcal{E}_{N,t}:H^1_{\mathrm{s}}(\mathbb{R}^{3N})\to\mathbb{R},\quad \Psi_N\mapsto\mathcal{E}_{N,t}(\Psi_N):=N^{-1}\big\langle\Psi_N,H_N^t\Psi_N\big\rangle, \tag{2.36}$$

and the Hartree energy by

$$\mathcal{E}_{\mathrm{H},t}:H^1(\mathbb{R}^3)\to\mathbb{R},\quad \varphi\mapsto\mathcal{E}_{\mathrm{H},t}(\varphi):=\big\langle\varphi,h^{t,\varphi}\varphi\big\rangle=\|\nabla\varphi_t\|^2+\big\langle\varphi,W^t\varphi\big\rangle+\mu^t. \tag{2.37}$$

From now on, let $\varphi_t\in H^1(\mathbb{R}^3)$ be the unique solution to the Hartree equation (2.6) with initial condition $\varphi_0\in H^1(\mathbb{R}^3)$, $\|\varphi_0\|=1$, let $\Psi_{N,t}=U_N(t,0)\Psi_N$ and $\widetilde{\Psi}_{N,t}=\widetilde{U}_N(t,0)\Psi_N$ with $\Psi_N\in\mathcal{H}_N$, $\|\Psi_N\|=1$, and let the initial wave functions φ_0, Ψ_N satisfy

Assumptions A.2. (Initial conditions)

1. $\varphi_0 \in H^2(\mathbb{R}^3)$, $\Psi_N \in H^1_{\mathrm{s}}(\mathbb{R}^{3N})$, and $\left|\mathcal{E}_{N,0}(\Psi_N) - \mathcal{E}_{\mathrm{H},0}(\varphi_0)\right| \leq CN^{-1}$,

2. $\left\langle \Psi_N, \left(\prod_{j=1}^n q_j^0\right)\Psi_N\right\rangle \leq CN^{-n}$ for $n = 1,2,3$,

3. $\|\widehat{f}^{\varphi_0}_{\mathrm{odd}}\Psi_N\| \leq CN^{-\frac{1}{2}}$.

Remark 2.2. Instead of Assumption 2.2.3, one could equivalently assume that the even part of the wave function is initially small, i.e., $\|\widehat{f}^{\varphi_0}_{\mathrm{even}}\Psi_N\| \leq CN^{-\frac{1}{2}}$. This would lead to the exact same results with all proofs being completely analogous (therefore we restrict ourselves to Assumption 2.2.3).

A more detailed explanation of Assumptions A.2 is given in Section 2.2.2.

Now we can state our main result. It gives a bound for the difference between $\Psi_{N,t}$ and $\widetilde{\Psi}_{N,t}$ in terms of the L^2-distance of the full N-body space.

Theorem 2.6. *Let φ_0, Ψ_N satisfy Assumptions A.2. Then there exists a time-independent constant $C > 0$ such that for all $t \geq 0$,*

$$\|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\|^2 + \|\nabla_1 q_1^t \Psi_{N,t}\|^2 \leq \frac{\exp\left(C(1+t)^2\right)}{N}. \tag{2.38}$$

Remark 2.3. 1) Going through the proof of the theorem, it seems likely that the N-dependence of the error in (2.38) can not be further improved. Let us also note that in the case of a static external potential, the time-dependent factor $\exp(C(1+t)^2)$ would be replaced by $\exp(C(1+t))$.

2) To obtain convergence of $\Psi_{N,t}$ towards $\widetilde{\Psi}_{N,t}$ for a bounded pair potential v is straightforward. The idea is to apply Duhamel's formula, which leads to

$$\|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\|^2 = -2\int_0^t \mathrm{Im}\left\langle \widetilde{\Psi}_{N,s}, \left(H_N^s - \widetilde{H}_N^s\right)\Psi_{N,s}\right\rangle ds,$$

and then use $\lambda_N = 1/(N-1)$, together with the fact that each term in $H_N^s - \widetilde{H}_N^s$ contains three or four q^s's, cf. (2.25). For a bounded potential, convergence then follows immediately from Lemma 2.3 (however, not directly with the rate above).

3) Similar estimates for the quantity $\|\nabla_1 q_1^t \Psi_{N,t}\|^2$ were derived before, e.g., in [72, 104]. The main difference here is that the explicit error on the r.h.s. is of order $1/N$ instead of $1/\sqrt{N}$ as, e.g., in [72, Lemma 4.6]. This improvement of the N-dependence is due to the control of the norm difference which gives us additional information about the structure of the state $\Psi_{N,t}$. We use in particular the fact that $\widetilde{H}_N^t$ does not couple the odd and even sectors in $\mathcal{H}_N$. Correlations between the two sectors which appear in expressions like $|\langle \Psi_{N,t}, p_1^t(-\Delta_1)q_1^t\Psi_{N,t}\rangle|$ lead to an error proportional $1/\sqrt{N}$ when estimating $\|\nabla_1 q_1^t \Psi_{N,t}\|^2$ like in [72, Lemma 4.6]. That such correlations are surpressed by an additional factor $1/\sqrt{N}$ can of course not directly be inferred from the time evolution of the full Hamiltonian H_N^t.

2.2.2 Discussion of the initial conditions

Before we go on with two applications of Theorem 2.6, we discuss the meaning of Assumptions A.2 in some more detail.

A simple example of an N-particle wave function that satisfies all assumptions is the complete product state $\Psi_N = \varphi_0^{\otimes N}$. Ground states of noninteracting bosons, e.g., are given by such exact product wave functions. For interacting systems, however, it is well understood that the ground state obeys the product structure only approximately since weak correlations between the particles are present. In the sense of the L^2-distance, the true ground state of an interacting system is not close to the product wave function. Therefore, we give some comments on the relevance of Theorem 2.6 for initial states that are given by ground states of interacting, trapped systems, or other systems with suitably attractive external fields, e.g., bosonic atoms.

A.2.1. The first assumption states that the energy per particle w.r.t. the initial state Ψ_N is given by the Hartree energy w.r.t. φ_0, up to an error of order N^{-1}. In other words, the average energy of the system equals N times the Hartree energy plus some N-independent correction at leading order. This next-to-leading order correction is expected to be correctly predicted by a Bogoliubov approximation for the fluctuations around the condensate. For ground states of confined systems, this is known to be true in several cases: Seiringer and Grech have shown in [60] that A.2.1 holds for bounded pair potentials, whereas Lewin et. al. have generalized the statement in [81] to a more general setting, including the Coulomb case $v(x) = 1/|x|$ for confined bosons and also for bosonic atoms.

A.2.2. This inequality is equivalent to the requirement that the expectation value of the first three moments of the number of particles outside the condensate is of order one. Bose-Einstein condensation is usually associated with a bound for the first moment, namely that $1 - \langle \Psi_N, p_1^0 \Psi_N \rangle = O(N^{-1})$, which means that only a finite number – uniform in N – of particles in Ψ_N does not occupy the state φ_0. The constraints on the higher moments can be interpreted as requiring a higher purity of the condensate described by Ψ_N. That such bounds hold for the ground state of a trapped system can be verified, e.g., by applying techniques used in the proofs of the main theorems in [60, 81]. In Chapter 3, we explain that for N bosons in a box, the probability of finding k particles in the ground state outside the condensate is exponentially small. From this, it follows directly that the expectation value of the n-th moment for any $n \geq 1$ is of order $O(N^{-n})$.

A.2.3. Here it would be equally fine to assume $\|\widehat{f}^0_{\text{even}} \Psi_N\| = O(N^{-\frac{1}{2}})$ instead of the odd part being small. Theorem 2.6 holds in this case as well with the corresponding proof being completely analogous.

In order to see that the third assumption is distinct from the previous two, let us consider a simple example: We choose the wave function

$$\mathcal{H}_N \ni \Psi_N = \frac{1}{\sqrt{2}} \varphi_0^{\otimes N} + \varphi_0^{\otimes (N-3)} \otimes_s \chi^{(3)}, \tag{2.39}$$

with $\|\varphi_0\| = 1$, $\chi^{(3)} \in \mathcal{H}_3$, $\|\chi^{(3)}\|^2 = \frac{1}{2}$, and where $\otimes_{\text{s}}$ denotes the normalized, symmetric tensor product, cf. (2.47). If we take $\chi^{(3)}$ sufficiently regular, it is not difficult to verify that Assumption A.2.1 is satisfied. This is because

$$\langle \varphi_0^{\otimes N}, H_N^0 \varphi_0^{\otimes N} \rangle = N \mathcal{E}_{\text{H},0}(\varphi_0).$$

Moreover, Assumption A.2.2 holds also true. Assumption A.2.3, however, does not hold for Ψ_N as in (2.39) since the norm of the odd part of Ψ_N equals $\|\chi^{(3)}\| = 1/\sqrt{2}$, while the norm of the even part is also equal to $1/\sqrt{2}$.

Remark 2.4. In [80], the norm approximation (2.12) was shown for initial N-particle states that are built up from quasifree excitation states on Fock space. This was mainly motivated from the results in [81] where it was shown that in the ground state of a trapped system (and similarly for other systems with appropriate external potentials), the excitations can be approximated by a Bogoliubov ground state which is a quasifree state. Here we focus on initial states satisfying Assumptions A.2. In Corollary 3.6, we show that for the homogeneous bose gas on the unit torus, the ground state satisfies Assumptions A.2.

2.2.3 Trace norm convergence

Using the bound for $\|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\|$ together with Lemmas 2.3 and 2.4, we can show that $\gamma^{(1)}_{\Psi_{N,t}} \to |\varphi_t\rangle\langle\varphi_t|$ in trace norm distance with optimal error which is of order N^{-1}. We emphasize that for our proof, it is crucial to control not only the relative number of particles in the condensate but the full norm approximation, i.e., also the fluctuations around the condensate. Using in addition the estimate for $\|\nabla_1 q_1^t \Psi_{N,t}\|$, we show that $\gamma^{(1)}_{\Psi_{N,t}}$ is close to $|\varphi_t\rangle\langle\varphi_t|$ also in terms of the energy trace distance.

Theorem 2.7. *Let φ_0, Ψ_N satisfy Assumptions A.2. Then there exists a constant C such that for all $t \geq 0$,*

$$\mathrm{Tr}\Big|\gamma^{(1)}_{\Psi_{N,t}} - |\varphi_t\rangle\langle\varphi_t|\Big| \leq \frac{\exp\big(C(1+t)^2\big)}{N}, \tag{2.40}$$

$$\mathrm{Tr}\Big|\sqrt{1-\Delta}\Big(\gamma^{(1)}_{\Psi_{N,t}} - |\varphi_t\rangle\langle\varphi_t|\Big)\sqrt{1-\Delta}\Big| \leq \frac{\exp\big(C(1+t)^2\big)}{\sqrt{N}}. \tag{2.41}$$

Remark 2.5. 1) For a proof of the first statement, we actually require less regularity of φ_0 than stated in Assumptions A.2. To this end note that one finds a similar norm approximation as in Theorem 2.6, using an effective Hamiltonian defined by

$$\widetilde{H}^t_N + \lambda_N \sum_{1\leq i<j\leq N} v_{ij}^{(4q,t)}, \tag{2.42}$$

with $v_{ij}^{(4q,t)}$ defined as in (2.27). In this case, one would not need Assumption A.2.1 to derive a bound for $\|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\|$ analogous to the one in (2.38), and therefore it is sufficient to assume $\varphi_0 \in H^1(\mathbb{R}^3)$. We omit further details since the indicated argument can be readily verified along the steps of the proof of Theorem 2.6 when $\widetilde{H}^t_N$ is replaced by (2.42).

2) The vanishing of the r.h.s. in (2.40) for $N \to \infty$ is a well-known result. For unbounded v, the optimal rate has to our knowledge only been derived so far for initial conditions equal to the full Hartree product [44, 29, 30, 75]. Theorem 2.7 holds for more general wave functions, and in particular, for the ground state of a trapped, interacting system. That it is not possible to improve the error further, can be inferred from (2.29). The l.h.s. in (2.29) converging faster than $1/N$ would imply that the wave function $\Psi_{N,t}$ were close to the state $\varphi_t^{\otimes N}$ in L^2-sense. The latter is false for interacting systems, as can be inferred, e.g., from (2.38). The rate in (2.41) is not expected to be the optimal one. To obtain an error of order N^{-1} also for the energy trace norm will be addressed in a future work.

3) Following the argument from the proof of (2.40), one can show as well that for any fixed integer k, the k-particle reduced density, $\gamma^{(k)}_{\Psi_{N,t}} : \mathcal{H}_k \to \mathcal{H}_k$, defined by its kernel

$$\gamma^{(k)}_{\Psi_{N,t}}(x_1,...,x_k,y_1,...,y_k) = \int \Psi_{N,t}(x_1,...,x_k,x_{k+1}...x_N)\overline{\Psi_{N,t}(y_1,...,y_k,x_{k+1}...x_N)}dx_{k+1}...dx_N, \tag{2.43}$$

converges to the k-fold product of the Hartree density, i.e.,

$$\mathrm{Tr}\left|\gamma^{(k)}_{\Psi_{N,t}} - |\varphi_t\rangle\langle\varphi_t|^{\otimes k}\right| \leq \frac{\exp\left(C_k(1+t)^2\right)}{N}. \tag{2.44}$$

2.2.4 Bogoliubov corrections on Fock space

We define the set of correlation functions $(\widetilde{\chi}^{(k)}_{N,t})^N_{k=0}$, $\widetilde{\chi}^{(k)}_{N,t} \in (q^t_1...q^t_k)\mathcal{H}_k$, by

$$\widetilde{\chi}^{(k)}_{N,t}(x_1,...,x_k) := \sqrt{\binom{N}{k}}\left(\prod_{i=1}^k q^t_i\right)\int\left(\prod_{i=k+1}^N \overline{\varphi_t(x_i)}\right)\widetilde{\Psi}_{N,t}(x_1,...,x_N)\,dx_{k+1}\dots dx_N. \tag{2.45}$$

By means of the partition $1 = \sum_{k=0}^N (q^t_1...q^t_k p^t_{k+1}...p^t_N)_{\mathrm{sym}}$, cf. Definition 2.11, one can show that the following time-dependent decomposition of $\widetilde{\Psi}_{N,t}$, in terms of φ_t and the correlation functions $\widetilde{\chi}^{(k)}_{N,t}$, holds as an identity at each time,

$$\widetilde{\Psi}_{N,t} = \sum_{k=0}^N \varphi_t^{\otimes(N-k)} \otimes_s \widetilde{\chi}^{(k)}_{N,t}. \tag{2.46}$$

Here, $\otimes_s$ stands for the normalized symmetric tensor product between $\psi^{(l)} \in \mathcal{H}_l$ and $\psi^{(k)} \in \mathcal{H}_k$ defined by

$$\psi^{(l)} \otimes_s \psi^{(k)} := \frac{1}{\sqrt{k!l!(k+l)!}}\sum_{\sigma\in P_{k+l}} \psi^{(l)}(x_{\sigma(1)},...,x_{\sigma(l)})\psi^{(k)}(x_{\sigma(l+1)},...,x_{\sigma(l+k)}). \tag{2.47}$$

It follows from (2.45) that the $\widetilde{\chi}^{(k)}_{N,t}$ are orthogonal to φ_t in every coordinate and at all times, as well as that $\|\widetilde{\chi}^{(k)}_{N,t}\|^2$ equals the probability of finding exactly k particles in $\widetilde{\Psi}_{N,t}$ which are not in the condensate wave function. The idea of decomposing an N-particle wave function according to (2.46) and to study the thereby defined k-particle correlation functions was introduced in [81] where it was used to describe the low-energy spectrum – eigenvalues and eigenvectors – of the Bose gas in the mean field limit. The idea was then used to study the time evolution in [80] for the mean field limit and similarly in [95, 96] for the NLS scaling.

We next introduce a hierarchy of equations, called Bogoliubov hierarchy, which determines the time evolution of an infinite set of correlation functions which we denote by $(\chi^{(k)}_t)_{k\geq 0}$, $\chi^{(k)}_t = \chi^{(k)}_t(x_1,...,x_k) \in \mathcal{H}_k$:

$$i\partial_t\chi^{(0)}_t = \frac{1}{\sqrt{2}}\int\int\overline{K^{(2),t}(x,y)}\chi^{(2)}_t(x,y)dxdy, \tag{2.48}$$

$$
\begin{aligned}
i\partial_t \chi_t^{(1)}(x_1) = & \Big(h^{t,\varphi_t}(x_1) + K^{(1),t}(x_1)\Big)\chi_t^{(1)}(x_1) \\
& + \frac{\sqrt{6}}{2}\int\int \overline{K^{(2),t}(x,y)}\chi_t^{(3)}(x_1,x,y)dxdy,
\end{aligned} \tag{2.49}
$$

and for all $k \geq 2$,

$$
\begin{aligned}
i\partial_t \chi_t^{(k)}(x_1,...,x_k) = & \sum_{i=1}^{k}\Big(h^{t,\varphi_t}(x_i) + K^{(1),t}(x_i)\Big)\chi_t^{(k)}(x_1,...,x_k) \\
& + \frac{1}{2\sqrt{k(k-1)}}\sum_{1\leq i<j\leq k} K^{(2),t}(x_i,x_j)\chi_t^{(k-2)}(x_1,...,x_k\backslash x_i\backslash x_j) \\
& + \frac{\sqrt{(k+1)(k+2)}}{2}\int\int \overline{K^{(2),t}(x,y)}\chi_t^{(k+2)}(x_1,...,x_k,x,y)dxdy.
\end{aligned} \tag{2.50}
$$

Here we introduced the following operators: $K^{(1),t}:\mathcal{H}_1 \to \mathcal{H}_1$ is given by $K^{(1),t} = q^t \widetilde{K}^{(1),t} q^t$ where $\widetilde{K}^{(1),t}:\mathcal{H}_1 \to \mathcal{H}_1$ is defined via its integral kernel $\widetilde{K}^{(1),t}(x,y) = \overline{\varphi_t(y)}v(x-y)\varphi_t(x)$. Further, $K^{(2),t}:\mathcal{H}_1\otimes\mathcal{H}_1 \to q^t\mathcal{H}_1\otimes q^t\mathcal{H}_1$ with $K^{(2),t} = q^t\otimes q^t \widetilde{K}^{(2),t}$ where $\widetilde{K}^{(2),t}(x,y) = v(x-y)\varphi_t(x)\varphi_t(y)$.

Remark 2.6. 1) (Equivalence to Bogoliubov theory on Fock space) Interpreting $(\chi_t^{(k)})_{k\geq 0} =: \chi_t$ as an element of the time-dependent Fock space $\mathcal{F}_s(q^t\mathcal{H}_1) = \bigoplus_{k=0}^{\infty}(q_1^t...q_k^t\mathcal{H}_k)$ the above hierarchy (2.48-2.50) is equivalent to the Schrödinger equation

$$
i\partial_t\chi_t = \mathbb{H}^t_{\text{Bog}}\chi_t, \tag{2.51}
$$

where $\mathbb{H}^t_{\text{Bog}}$ is a quadratic, nonparticle conserving Hamiltonian given by

$$
\mathbb{H}^t_{\text{Bog}} = \int a_x^*\Big(h^{t,\varphi_t}(x) + K^{(1),t}(x)\Big)a_x\,dx + \frac{1}{2}\int\int\Big[K^{(2),t}(x,y)a_x^*a_y^* + \text{h.c.}\Big]dxdy.
$$

The operator-valued distributions a_x^*, a_x are defined by

$$
\begin{aligned}
(a_x^*\chi)^{(k)}(x_1,...,x_k) = & \frac{1}{\sqrt{k}}\sum_{i=1}^{k}\delta(x_i - x)\chi^{(k-1)}(x_1,...,x_k\backslash x_i), \\
(a_x\chi)^{(k)}(x_1,...,x_k) = & \sqrt{k+1}\int\chi^{(k+1)}(x_1,...,x_k,x)dx,
\end{aligned}
$$

with $\chi = (\chi^{(k)})_{k\geq 0} \in \mathcal{F}_s$.

2) (Equivalence to Bogoliubov theory for density matrices) Yet another way to understand the Bogoliubov hierarchy was considered, e.g., in [95], motivated by ideas from [62]. If one defines the density matrices $\gamma_t : \mathcal{H}^{(1),t} \to \mathcal{H}^{(1),t}$ and $\alpha_t : \overline{\mathcal{H}^{(1),t}} \to \mathcal{H}^{(1),t}$ by

$$
\langle f, \gamma_t g\rangle = \big\langle \chi_t, a^*(g)a(f)\chi_t\big\rangle_{\mathcal{F}_s}, \quad \langle f, \alpha_t \overline{g}\rangle = \big\langle \chi_t, a(g)a(f)\chi_t\big\rangle_{\mathcal{F}_s},
$$

the hierarchy (2.48-2.50) is equivalent to the pair of coupled equations for γ_t and α_t,

$$
\begin{aligned}
i\partial_t\gamma_t &= \big(h^{t,\varphi_t} + K^{(1),t}\big)\gamma_t - \gamma_t\big(h^{t,\varphi_t} + K^{(1),t}\big) + K^{(2),t}\alpha_t^* - \alpha_t K^{(2),t*}, \\
i\partial_t\alpha_t &= \big(h^{t,\varphi_t} + K^{(1),t}\big)\alpha_t + \alpha_t\big(h^{t,\varphi_t} + K^{(1),t}\big)^{\text{T}} + K^{(2),t} + K^{(2),t}\gamma_t^{\text{T}} + \gamma_t K^{(2),t}.
\end{aligned}
$$

A similar system of Bogoliubov type equations was recently studied also in [10].

3) Well-posedness of the Bogliubov hierarchy (2.48-2.50), or equivalently of the Schrödinger equation (2.51) has been shown in [80, Section 4.3]. The main difficulty is the time-dependence of $\mathbb{H}^t_{\mathrm{Bog}}$, and one essential ingredient is to show that $K^{(2),t}$ is a Hilbert-Schmidt operator. This corresponds to the physical fact that only a finite number of correlations – particles in $\mathcal{F}_{\mathrm{s}}(q^t_1\mathcal{H}_1)$ – is created during time evolution.

Our last goal is to show that the corrections to the Hartree product in $\Psi_{N,t}$ are effectively described by the solutions of the Bogoliubov hierarchy, i.e.,

$$\lim_{N\to\infty} \left\| \Psi_{N,t} - \sum_{k=0}^{N} \left(\varphi_t^{\otimes(N-k)} \otimes_s \chi_t^{(k)} \right) \right\| = 0. \tag{2.52}$$

To this end, it remains to show that $\lim_{N\to\infty} \widetilde{\chi}^{(k)}_{N,t} = \chi^{(k)}_t$.

Lemma 2.8. *Let φ_0, Ψ_N satisfy Assumptions A.2. If $(\widetilde{\chi}^{(k)}_{N,t})_{k=0}^N$ is given by (2.45), and $(\chi^{(k)}_t)_{k\geq 0}$ solves the Bogoliubov hierarchy (2.48) with initial condition $\chi^{(k)}_0 = \widetilde{\chi}^{(k)}_{N,0}$ for all $0 \leq k \leq N$, and $\chi^{(k)}_0 \equiv 0$ for all $k \geq N+1$, then there exists a constant C such that for all $t \geq 0$,*

$$\sum_{k=0}^{N} \|\widetilde{\chi}^{(k)}_{N,t} - \chi^{(k)}_t\|^2 \leq \frac{\exp\left(C(1+t)^2\right)}{N}. \tag{2.53}$$

Remark 2.7. We emphasize that the elements of the tuple $(\widetilde{\chi}^{(k)}_{N,t})_{k=0}^N$ depend explicitly on N whereas the time evolution of the sequence $(\chi^{(k)}_t)_{k\geq 0}$ is N-independent, and the sequence depends on N only through the initial condition.

A quantitative version of (2.52) follows as a corollary of Theorem 2.6 and the previous lemma.

Theorem 2.9. *Let φ_0, Ψ_N satisfy Assumptions A.2. If $(\chi^{(k)}_t)_{k\geq 0}$ solves the Bogoliubov hierarchy (2.48) with initial condition*

$$\chi^{(k)}_0 = \sqrt{\binom{N}{k}} \left(\prod_{i=1}^{k} q^0_i \right) \int \left(\prod_{i=k+1}^{N} \overline{\varphi_0(x_i)} \right) \Psi_N(x_1, ..., x_N)\, dx_{k+1} \dots dx_N$$

for $0 \leq k \leq N$, and $\chi^{(k)}_0 \equiv 0$ for all $k \geq N+1$, then there exists a constant C such that for all $t \geq 0$,

$$\left\| \Psi_{N,t} - \sum_{k=0}^{N} \left(\varphi_t^{\otimes(N-k)} \otimes_s \chi_t^{(k)} \right) \right\|^2 \leq \frac{\exp\left(C(1+t)^2\right)}{N}. \tag{2.54}$$

2.3 Proofs

We first state a technical lemma from which the proofs of the theorems then follow more easily. It can essentially be read as estimates for terms like $|\langle \Phi_N, [q^{\varphi}_1, A_1] \widetilde{\Phi}_N \rangle|$, where A_1 is a one-particle operator and $\Phi_N, \widetilde{\Phi}_N$ wave functions that are symmetric in almost all coordinates. To have control of such terms is important in order to use each of the q^t's that are available in the terms that need to be estimated. We defer its proof to Section 2.3.3.

Lemma 2.10. *Let $\varphi \in \mathcal{H}_1$, $\Phi_N, \widetilde{\Phi}_N \in \mathcal{H}_N$, and $p_i^\varphi = |\varphi(x_i)\rangle\langle\varphi(x_i)|$, $q_i^\varphi = 1 - p_i^\varphi$ as in Definition 2.1. Let further $\widehat{f}_{odd}^\varphi$, $\widehat{f}_{even}^\varphi$ as in (2.32) and (2.33) with φ_t replaced by φ.*

1. *Let A be an operator on $\mathcal{H}_1$ with $\|Ap^\varphi\|_{op}^2 + \|p^\varphi A\|_{op}^2 < \infty$. Then,*

$$\begin{aligned}&\Big|\big\langle \Phi_N, q_1^\varphi A_1 p_1^\varphi \Phi_N\big\rangle + \big\langle \Phi_N, p_1^\varphi A_1 q_1^\varphi \Phi_N\big\rangle\Big| \\ &\quad \le \frac{\|\widehat{f}_{odd}^\varphi \Phi_N\|^2}{10} + 10\Big(\|Ap^\varphi\|_{op}^2 + \|p^\varphi A\|_{op}^2\Big)\Big(2\|q_1^\varphi \Phi_N\|^2 + \frac{1}{N}\Big).\end{aligned} \tag{2.55}$$

2. *Let v be a real-valued and even function satisfying $v^2 \le D(1-\Delta)$ for some $D > 0$. Then,*

$$\begin{aligned}&\Big|\big\langle \widetilde{\Phi}_N, q_1^\varphi q_2^\varphi v(x_1 - x_2) q_1^\varphi q_2^\varphi \Phi_N\big\rangle\Big| \\ &\quad \le C\Big(N\|q_1^\varphi q_2^\varphi q_3^\varphi \widetilde{\Phi}_N\|^2 + \|q_1^\varphi q_2^\varphi \widetilde{\Phi}_N\|^2\Big) + \frac{\|\nabla_1 q_1^\varphi \Phi_N\|^2 + \|q_1^\varphi \Phi_N\|^2}{N}\end{aligned} \tag{2.56}$$

   for some positive constant C.

3. *Let $A_{12} = A_{21}$ be an operator on $\mathcal{H}_2$ with $\|A_{12}p_2^\varphi\|_{op} + \|p_2^\varphi A_{12}\|_{op} < \infty$. Then,*

$$\begin{aligned}&\Big|\big\langle \Phi_N, \Big(q_1^\varphi q_2^\varphi A_{12} q_1^\varphi p_2^\varphi + h.c.\Big)\widetilde{\Phi}_N\big\rangle\Big| \\ &\le \frac{\|\widehat{f}_{odd}^\varphi \Phi_N\|^2 + \|\widehat{f}_{odd}^\varphi \widetilde{\Phi}_N\|^2}{N} + CN\Big(\|A_{12}p_2^\varphi\|_{op}^2 + \|p_2^\varphi A_{12}\|_{op}^2\Big)\Big(\|q_1^\varphi q_2^\varphi q_3^\varphi \Phi_N\|^2 \\ &+ \|q_1^\varphi q_2^\varphi q_3^\varphi \widetilde{\Phi}_N\|^2 + \frac{\|q_1^\varphi q_2^\varphi \Phi_N\|^2 + \|q_1^\varphi q_2^\varphi \widetilde{\Phi}_N\|^2}{N} + \frac{\|q_1^\varphi \Phi_N\|^2 + \|q_1^\varphi \widetilde{\Phi}_N\|^2}{N^2} + \frac{1}{N^3}\Big)\end{aligned} \tag{2.57}$$

2.3.1 Proofs of Theorems 2.6, 2.7 and 2.9

Without further mentioning, we will frequently apply Lemmas 2.3 and 2.4 throughout the following proofs. Together with Assumptions A.2, they imply

$$\big\langle \Phi_{N,t}, \Big(Nq_1^t + N^2 q_1^t q_2^t + N^3 q_1^t q_2^t q_3^t\Big)\Phi_{N,t}\big\rangle \le \exp\big(C(1+t)^{3/2}\big) \tag{2.58}$$

where $\Phi_{N,t} \in \{\Psi_{N,t}, \widetilde{\Psi}_{N,t}\}$, as well as

$$\|\widehat{f}_{\text{odd}}^{\varphi_t} \widetilde{\Psi}_{N,t}\|^2 \le CN^{-1}, \qquad \|\widehat{f}_{\text{odd}}^{\varphi_t} \Psi_{N,t}\|^2 \le 2\|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\|^2 + CN^{-1}. \tag{2.59}$$

Proof of Theorem 2.6. Let $\beta_N(t) := \|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\|^2 + \|\nabla_1 q_1^t \Psi_{N,t}\|^2$. Our strategy is to show that

$$\beta_N(t) \le C\int_0^t \beta_N(s)\, ds + \frac{\exp\big(C(1+t)^2\big)}{N}, \tag{2.60}$$

and then conclude by means of the integral version of Gronwall's inequality. By a standard density argument, we can assume that $\Psi_N = \Psi_{N,0} \in H_s^2(\mathbb{R}^{3N})$.[6]

[6] For given $\Psi_N \in H_s^1(\mathbb{R}^{3N})$, we choose $\Psi_N^{\text{re}} \in H_s^2(\mathbb{R}^{3N})$ with for instance $\|\Psi_N - \Psi_N^{\text{re}}\| \le N^{-2}$. One can now carry out the argument with $\Psi_N^{\text{re}} \in H_s^2(\mathbb{R}^{3N})$ since $U_N(t,0)\Psi_N^{\text{re}}$ and $\widetilde{U}_N(t,0)\Psi_N^{\text{re}}$ are differentiable w.r.t. to t, and then one concludes by means of unitarity that $\|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\| \le \|(U_N(t,0) - \widetilde{U}_N(t,0))\Psi_N^{\text{re}}\| + 2N^{-2}$.

Bound for $\|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\|^2$. Using that H_N^t and $\widetilde{H}_N^t$ are self-adjoint, together with (2.25) and symmetry of the wave functions $\Psi_{N,t}$, $\widetilde{\Psi}_{N,t}$, we find for the time-derivative

$$\begin{aligned}\partial_t\|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\|^2 &= 2\,\mathrm{Im}\,\langle \Psi_{N,t} - \widetilde{\Psi}_{N,t}, H_N^t \Psi_{N,t} - \widetilde{H}_N^t \widetilde{\Psi}_{N,t}\rangle \\ &= -2\,\mathrm{Im}\,\langle \widetilde{\Psi}_{N,t}, \Big(H_N^t - \widetilde{H}_N^t\Big)\Psi_{N,t}\rangle \\ &= -N\,\mathrm{Im}\,\langle \widetilde{\Psi}_{N,t}, v_{12}^{(3q,t)}\Psi_{N,t}\rangle \qquad (2.61)\\ &\qquad - N\,\mathrm{Im}\,\langle \widetilde{\Psi}_{N,t}, v_{12}^{(4q,t)}\Psi_{N,t}\rangle, \qquad (2.62)\end{aligned}$$

with $v_{12}^{(3q,t)}$ and $v_{12}^{(4q,t)}$ defined as in (2.26) resp. (2.27). For the first line, we apply inequality (2.57) with $\|(v_{12} - \bar{v}_1^t - \bar{v}_2^t)p_2^t\|_{op} \le C\sqrt{1+t}$. Then with (2.58) and (2.59), we obtain

$$|(2.61)| \le \|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\|^2 + \frac{e^{C(1+t)^{3/2}}}{N}. \qquad (2.63)$$

In the second line, we find on the one hand that

$$N\Big|\langle \widetilde{\Psi}_{N,t}, q_1^t q_2^t v_{12} q_1^t q_2^t \Psi_{N,t}\rangle\Big| \le \|\nabla_1 q_1^t \Psi_{N,t}\|^2 + \frac{e^{C(1+t)^{3/2}}}{N}, \qquad (2.64)$$

where we used inequality (2.56), and then applied again (2.58). On the other hand, we directly obtain

$$N\Big|\langle \widetilde{\Psi}_{N,t}, q_1^t q_2^t \Big(\bar{v}_1^t + \bar{v}_2^t - 2\mu^t\Big) q_1^t q_2^t \Psi_{N,t}\rangle\Big| \le \frac{e^{C(1+t)^{3/2}}}{N},$$

since $\|\bar{v}^t\|_\infty + \mu^t \le C\sqrt{1+t}$.

Bound for $\|\nabla_1 q_1^t \Psi_{N,t}\|^2$. The argument consists of two steps. First, we show that the "bad part" of the kinetic energy can be bounded as follows:

$$\|\nabla_1 q_1^t \Psi_{N,t}\|^2 \le \frac{1}{5}\|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\|^2 + \big|\widetilde{\mathcal{E}}_{N,t}(\Psi_{N,t}) - \mathcal{E}_{\mathrm{H},t}(\varphi_t)\big| + \frac{e^{C(1+t)^2}}{N}, \qquad (2.65)$$

where

$$\widetilde{\mathcal{E}}_{N,t}(\Psi_{N,t}) := N^{-1}\langle \Psi_{N,t}, \widetilde{H}_N^t \Psi_{N,t}\rangle = \langle \Psi_{N,t}, \Big(h_1^{t,\varphi_t} + \frac{1}{2}v_{12}^{(2\mathrm{q},t)}\Big)\Psi_{N,t}\rangle \qquad (2.66)$$

denotes the energy per particle w.r.t. $\widetilde{H}_N^t$. Here, and also below, we are using the abbreviation

$$v_{12}^{(2q,t)} = v_{21}^{(2q,t)} := \Big(p_1^t q_2^t v_{12} q_1^t p_2^t + p_1^t p_2^t v_{12} q_1^t q_2^t\Big) + \mathrm{h.c.} \qquad (2.67)$$

In the second step, we then use energy conservation of the original Hamiltonian H_N^t – modulo the change due to the external potential W^t – in order to show that the energy difference that appears on the r.h.s. in (2.65) can be approximated in terms of

$$\begin{aligned}&\Big|\widetilde{\mathcal{E}}_{N,t}(\Psi_{N,t}) - \mathcal{E}_{\mathrm{H},t}(\varphi_t)\Big| \\ &\qquad \le \Big|\mathcal{E}_{N,0}(\Psi_N) - \mathcal{E}_{\mathrm{H},0}(\varphi_0)\Big| + \int_0^t \|\Psi_{N,s} - \widetilde{\Psi}_{N,s}\|^2 ds + \frac{e^{C(1+t)^2}}{N}. \qquad (2.68)\end{aligned}$$

To obtain the first inequality, we multiply each of the $\Psi_{N,t}$ in (2.66) by $1 = (p_1^t + q_1^t)(p_2^t + q_2^t)$, and extract the "bad part" of the kinetic energy:

$$\begin{aligned}
\widetilde{\mathcal{E}}_{N,t}(\Psi_{N,t}) - \mathcal{E}_{\mathrm{H},t}(\varphi_t) = \|\nabla_1 q_1^t \Psi_{N,t}\|^2 \\
+ \langle \Psi_{N,t}, q_1^t \Big(W_1^t + \bar{v}_1^t \Big) q_1^t \Psi_{N,t} \rangle \quad (2.69)\\
+ \langle \Psi_{N,t}, \Big(p_1^t h_1^{t,\varphi_t} p_1^t - \langle \varphi_t, h_1^{t,\varphi_t} \varphi_t \rangle \Big) \Psi_{N,t} \rangle \quad (2.70)\\
+ 2\,\mathrm{Re}\, \langle \Psi_{N,t}, \Big(p_1^t h_1^{t,\varphi_t} q_1^t \Big) \Psi_{N,t} \rangle \quad (2.71)\\
+ 2 \langle \Psi_{N,t}, \Big(p_1^t q_2^t v_{12} q_1^t p_2^t \Big) \Psi_{N,t} \rangle \quad (2.72)\\
+ 2\,\mathrm{Re}\, \langle \Psi_{N,t}, \Big(p_1^t p_2^t v_{12} q_1^t q_2^t \Big) \Psi_{N,t} \rangle. \quad (2.73)
\end{aligned}$$

All but the first line on the r.h.s. can be estimated using simple algebra together with (2.58) and (2.59):

$$\begin{aligned}
|(2.69)| &\leq \Big(\|W^t\|_\infty + \|\bar{v}^t\|_\infty \Big) \|q_1^t \Psi_{N,t}\|^2 \leq e^{C(1+t)^{3/2}} N^{-1}, \\
|(2.70)| &= \langle \varphi_t, h_1^{\varphi_t} \varphi_t \rangle \; \|q_1^t \Psi_{N,t}\|^2 \leq e^{C(1+t)^{3/2}} N^{-1}, \\
|(2.72)| &\leq 2 \|v_{12} p_2^t\|_{op} \|q_1^t \Psi_{N,t}\|^2 \leq e^{C(1+t)^{3/2}} N^{-1}, \\
|(2.73)| &\leq 2 \|v_{12} p_2^t\|_{op} \; \|q_1^t q_2^t \Psi_{N,t}\| \leq e^{C(1+t)^{3/2}} N^{-1}.
\end{aligned}$$

For the remaining line, we apply inequality (2.55) with $A = h^{t,\varphi_t}$, and use (2.59):

$$|(2.71)| \leq \frac{1}{5} \|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\|^2 + \frac{C}{5N} + \|h^{t,\varphi_t} p^t\|_{op}^2 \frac{e^{C(1+t)^{3/2}}}{N}. \tag{2.74}$$

Noting that $\|h^{t,\varphi_t} p^t\|_{op} \leq C \|\varphi_t\|_{H^2} \leq \exp(Ct(1+t))$ completes the proof of inequality (2.65).

In the second step, we estimate the energy difference on the r.h.s. of (2.65). For that, we add and subtract $\mathcal{E}_{N,t}(\Psi_{N,t})$, and use the fact that only the time-dependent external potential causes a change in the energy, i.e.,

$$\mathcal{E}_{N,t}(\Psi_{N,t}) - \mathcal{E}_{N,0}(\Psi_N) = \int_0^t \langle \Psi_{N,s}, \Big(\partial_s W^s(x_1) \Big) \Psi_{N,s} \rangle \; ds. \tag{2.75}$$

Together with the analogous relation for the Hartree energy, cf. (2.17), we thus find

$$\begin{aligned}
\Big| \widetilde{\mathcal{E}}_{N,t}(\Psi_{N,t}) - \mathcal{E}_{\mathrm{H},t}(\varphi_t) \Big| &\leq \Big| \mathcal{E}_{N,0}(\Psi_N) - \mathcal{E}_{\mathrm{H},0}(\varphi_0) \Big| + \\
+ \Big| \widetilde{\mathcal{E}}_{N,t}(\Psi_{N,t}) - \mathcal{E}_{N,t}(\Psi_{N,t}) \Big| &+ \int_0^t ds \Big| \langle \Psi_{N,s}, \Big(\dot{W}_1^s - \langle \varphi_s, \dot{W}^s \varphi_s \rangle \Big) \Psi_{N,s} \rangle \Big|, \quad (2.76)
\end{aligned}$$

where $\dot{W}^s \equiv \partial_s W^s$. We proceed with

$$\begin{aligned}
&\Big| \langle \Psi_{N,s}, \Big(\dot{W}_1^s - \langle \varphi_s, \dot{W}^s \varphi_s \rangle \Big) \Psi_{N,s} \rangle \Big| \\
&\leq \Big| \langle \Psi_{N,s}, \Big(p_1^s \dot{W}_1^s p_1^s - \langle \varphi_s \dot{W}_1^s \varphi_s \rangle \Big) \Psi_{N,s} \rangle \Big| + \Big| \langle \Psi_{N,s}, \Big(q_1^s \dot{W}_1^s q_1^s \Big) \Psi_{N,s} \rangle \Big| \\
&+ 2 \Big| \langle \Psi_{N,s}, \Big(p_1^s \dot{W}_1^s q_1^s \Big) \Psi_{N,s} \rangle \Big|
\end{aligned}$$

$$\leq \frac{1}{5}\|\Psi_{N,s} - \widetilde{\Psi}_{N,s}\|^2 + \exp\big(C(1+t)^2\big)N^{-1},$$

where we used that $\dot{W}^s$ is bounded, and applied one more time inequality (2.55). Eventually, we note that

$$\Big|\mathcal{E}_{N,t}(\Psi_{N,t}) - \widetilde{\mathcal{E}}_{N,t}(\Psi_{N,t})\Big| \leq \frac{1}{N}\Big(\|\nabla_1 q_1^t \Psi_{N,t}\|^2 + \frac{e^{C(1+t)^2}}{N}\Big), \tag{2.77}$$

which one verifies following the same steps as in the estimates for (2.61) and (2.62).

Conclusion. Adding everything up, we obtain (2.60) and can apply the Gronwall argument which proves that $\beta_N(t) \leq \exp\big(C(1+t)^2\big)N^{-1}$. □

Proof of Theorem 2.7. **Bound for the trace norm**. We start from the fact that

$$\mathrm{Tr}\Big|\gamma^{(1)}_{\Psi_{N,t}} - |\varphi_t\rangle\langle\varphi_t|\Big| = \sup_{\|A\|\leq 1}\Big|\mathrm{Tr}\Big(A\gamma^{(1)}_{\Psi_{N,t}} - A|\varphi_t\rangle\langle\varphi_t|\Big)\Big| \tag{2.78}$$

where the supremum is taken over compact operators A acting on $\mathcal{H}_1$ with norm smaller or equal to one (the identity follows from duality between trace class operators and compact operators). Inserting $1 = p_1^t + q_1^t$ around A_1, we find

$$\begin{aligned}\mathrm{Tr}\Big(A\gamma^{(1)}_{\Psi_{N,t}} - A|\varphi_t\rangle\langle\varphi_t|\Big) &= \big\langle\Psi_{N,t}, A_1\Psi_{N,t}\big\rangle - \big\langle\varphi_t, A\varphi_t\big\rangle \\ &= \big\langle\Psi_{N,t}, p_1^t A_1 p_1^t\Psi_{N,t}\big\rangle - \big\langle\varphi_t, A_1\varphi_t\big\rangle + \big\langle\Psi_{N,t}, q_1^t A_1 q_1^t\Psi_{N,t}\big\rangle \qquad (2.79)\\ &+ \big\langle\Psi_{N,t}, p_1^t A_1 q_1^t\Psi_{N,t}\big\rangle + \big\langle\Psi_{N,t}, q_1^t A_1 p_1^t\Psi_{N,t}\big\rangle. \qquad (2.80)\end{aligned}$$

The first line is small, since using $p_1^t A_1 p_1^t = \langle\varphi_t, A\varphi_t\rangle p_1^t$, $1 - p_1^t = q_1^t$, $\|A\|_{op} \leq 1$, and (2.58), we obtain

$$|(2.79)| \leq 2\|q_1^t\Psi_{N,t}\|^2 \leq \frac{e^{C(1+t)^{3/2}}}{N}. \tag{2.81}$$

For the second line, we use inequality (2.55) with $\|Ap^t\|_{op} \leq 1$, then Theorem 2.6 together with (2.58) and (2.59) in order to find

$$|(2.80)| \leq \|\Psi_{N,t} - \widetilde{\Psi}_{N,t}\|^2 + \|\widehat{f}^{\varphi_t}_{\mathrm{odd}}\widetilde{\Psi}_{N,t}\|^2 + \Big(\|q_1^t\Psi_{N,t}\|^2 + \frac{1}{N}\Big) \leq \frac{e^{C(1+t)^2}}{N}. \tag{2.82}$$

Bound for the energy trace norm. This part of the theorem follows essentially from the estimate for $\|\nabla_1 q_1^t\Psi_{N,t}\|$ in Theorem 2.6. We start from

$$\begin{aligned}&\mathrm{Tr}\Big|\sqrt{1-\Delta}\Big(\gamma^{(1)}_{\Psi_{N,t}} - |\varphi_t\rangle\langle\varphi_t|\Big)\sqrt{1-\Delta}\Big| \\ &\qquad = \sup_{\|A\|\leq 1}\Big|\mathrm{Tr}\Big(A\sqrt{1-\Delta}\Big(\gamma^{(1)}_{\Psi_{N,t}} - |\varphi_t\rangle\langle\varphi_t|\Big)\sqrt{1-\Delta}\Big)\Big|, \qquad (2.83)\end{aligned}$$

where the supremum is taken again over all compact operators A with norm less or equal than one. Using the abbreviation $B := \sqrt{1-\Delta}A\sqrt{1-\Delta}$, we compute

$$\begin{aligned}&\mathrm{Tr}\Big(A\sqrt{1-\Delta}\Big(\gamma^{(1)}_{\Psi_{N,t}} - |\varphi_t\rangle\langle\varphi_t|\Big)\sqrt{1-\Delta}\Big) \\ &\qquad = \big\langle\Psi_{N,t}, p_1^t B_1 p_1^t\Psi_{N,t}\big\rangle - \langle\varphi_t, B_1\varphi_t\rangle \qquad (2.84)\\ &\qquad + \big\langle\Psi_{N,t}, q_1^t B q_1^t\Psi_{N,t}\big\rangle \qquad (2.85)\end{aligned}$$

$$+ \big\langle \Psi_{N,t}, q_1^t B_1 p_1^t \Psi_{N,t} \big\rangle + \big\langle \Psi_{N,t}, p_1^t B_1 q_1^t \Psi_{N,t} \big\rangle. \tag{2.86}$$

The first line,

$$|(2.84)| = \langle \varphi_t, B\varphi_t \rangle \; \|q_1^t \Psi_{N,t}\|^2 \le \frac{e^{C(1+t)^2}}{N}, \tag{2.87}$$

since $\|\sqrt{1-\Delta}\varphi_t\|^2 = \|\varphi_t\|^2 + \|\nabla\varphi_t\|^2 \le C(1+t)$, cf. (2.18). For the second line, we find

$$|(2.85)| \le \|\sqrt{1-\Delta_1} q_1^t \Psi_{N,t}\|^2 = \|q_1^t \Psi_{N,t}\|^2 + \|\nabla_1 q_1^t \Psi_{N,t}\|^2 \le \frac{e^{C(1+t)^2}}{N}, \tag{2.88}$$

where we used Theorem 2.6 and (2.58). The last line is the one which causes the weaker rate of convergence in (2.41) compared to (2.40). Note here that due to the presence of the gradients in B, one can not use the smallness of the odd part in $\Psi_{N,t}$ any more, and thus one loses a factor $N^{-1/2}$. Using the Cauchy-Schwarz inequality and $\|\sqrt{1-\Delta}p^t\|_{op} \le C(1+t)$, one finds hat

$$|(2.86)| \le 2\|\sqrt{1-\Delta}p^t\|_{op}\Big(\|q_1^t \Psi_{N,t}\| + \|\nabla_1 q_1^t \Psi_{N,t}\|\Big) \le \frac{e^{C(1+t)^2}}{\sqrt{N}}. \tag{2.89}$$

This completes the proof of the theorem. □

Proof of Theorem 2.9. Using the triangle inequality and Theorem 2.6, we know that

$$\Big\|\Psi_{N,t} - \sum_{k=0}^{N}\Big(\varphi_t^{\otimes(N-k)} \otimes_s \chi_t^{(k)}\Big)\Big\| \le \frac{e^{C(1+t)^2}}{\sqrt{N}} + \Big\|\widetilde{\Psi}_{N,t} - \sum_{k=0}^{N}\Big(\varphi_t^{\otimes(N-k)} \otimes_s \chi_t^{(k)}\Big)\Big\|. \tag{2.90}$$

Then, with Lemma 2.8 and $\|\varphi_t\| = 1$, it follows that

$$\Big\|\widetilde{\Psi}_{N,t} - \sum_{k=0}^{N}\Big(\varphi_t^{\otimes(N-k)} \otimes_s \chi_t^{(k)}\Big)\Big\|^2 = \sum_{k=0}^{N} \|\widetilde{\chi}_{N,t}^{(k)} - \chi_t^{(k)}\|^2 \le \frac{\exp\big(C(1+t)^2\big)}{N}. \tag{2.91}$$

□

2.3.2 Preliminaries for proofs of the remaining lemmas

We summarize some necessary definitions and preliminary assertions that are needed to prove the remaining lemmas. Readers familiar with the method that was introduced in [103] may skip this section except for Lemma 2.16.

Let $\varphi \in \mathcal{H}_1$, $p_i^{\varphi} = |\varphi(x_i)\rangle\langle\varphi(x_i)|$ and $q_i^{\varphi} = 1 - p_i^{\varphi}$.

Definition 2.11. Define the family of projectors $(P_{N,k}^{\varphi})_{k=0}^{N}$, $P_{N,k}^{\varphi} : \mathcal{H}_N \to \mathcal{H}_N$ by

$$P_{N,k}^{\varphi} = \Big(\prod_{i=1}^{k} q_i^{\varphi} \prod_{j=k+1}^{N} p_i^{\varphi}\Big)_{sym} = \sum_{a\in\mathcal{A}_k} \prod_{i=1}^{N} (q_i^{\varphi})^{a_i} (p_i^{\varphi})^{1-a_i} \tag{2.92}$$

with

$$\mathcal{A}_k = \Big\{ a = (a_1, ..., a_N) \in \{0,1\}^N \ : \ \sum_{i=1}^{N} a_i = k \Big\}. \tag{2.93}$$

We also set $P^{\varphi}_{N,k} = 0$ for all $k < 0$ and $k > N$. Note the following properties:

1. $P^{\varphi}_{N,k}$ is an orthogonal projector,
2. $P^{\varphi}_{N,k} P^{\varphi}_{N,l} = \delta_{kl} P^{\varphi}_{N,k}$,
3. $1 = \sum_{k=0}^{N} P^{\varphi}_{N,k}$ (this follows from $\cup_{k=0}^{N} \mathcal{A}_k = \{0,1\}^N$),
4. $\left[p^{\varphi}_i, P^{\varphi}_{N,k}\right] = 0 = \left[q^{\varphi}_i, P^{\varphi}_{N,k}\right]$.

Definition 2.12. We call any function $f : \{0, 1, ..., N\} \to \mathbb{R}^+_0$ a weight function (or simply weight) and define the linear combination of weighted projectors w.r.t. the weight f by $\widehat{f}^{\varphi}$,

$$\widehat{f}^{\varphi} : \mathcal{H}_N \to \mathcal{H}_N, \qquad \widehat{f}^{\varphi} \Psi_N = \sum_{k=0}^{N} f(k) P^{\varphi}_{N,k} \Psi_N. \tag{2.94}$$

For any integer $|d| \leq N$, we define the shift operator τ_d by

$$\tau_d f : \{0, 1, ..., N\} \to \mathbb{R}^+_0, \qquad (\tau_d f)(k) = \begin{cases} 0 & \text{for } k + d < 0, \\ f(k+d) & \text{for } \; 0 \leq k + d \leq N, \\ 0 & \text{for } N < k + d. \end{cases}$$

It is straightforward to see (using property 4 resp. 2 below Definition 2.11) that

1. $\|\widehat{f}^{\varphi}\|_{op} \leq \sup_{k \in [0,N]} f(k)$,
2. $[\widehat{f}^{\varphi}, p^{\varphi}_i] = 0 = [\widehat{f}^{\varphi}, q^{\varphi}_i]$, $\quad [\widehat{f}^{\varphi}, P^{\varphi}_{N,k}] = 0$,
3. $$\widehat{g}^{\varphi} \widehat{f}^{\varphi} = \sum_{k,l=0}^{N} f(k) g(l) P^{\varphi}_{N,k} P^{\varphi}_{N,l} = \sum_{k=0}^{N} g(k) f(k) P^{\varphi}_{N,k} = \widehat{(gf)}^{\varphi} = \widehat{(fg)}^{\varphi} = \widehat{f}^{\varphi} \widehat{g}^{\varphi} \tag{2.95}$$

   for any two weights f, g.

We shall make frequent use of the weight functions

$$m(k) = \frac{k}{N}, \qquad n(k) = \sqrt{\frac{k}{N}}. \tag{2.96}$$

They satisfy two important properties, namely

$$\frac{1}{N} \sum_{i=1}^{N} q^{\varphi}_i = \frac{1}{N} \sum_{i=1}^{N} q^{\varphi}_i \Big(\sum_{k=0}^{N} P^{\varphi}_{N,k} \Big) = \frac{1}{N} \sum_{k=1}^{N} \sum_{i=1}^{N} q^{\varphi}_i P^{\varphi}_{N,k} = \frac{1}{N} \sum_{k=1}^{N} k P^{\varphi}_{N,k} = \widehat{m}^{\varphi}, \tag{2.97}$$

and

$$\widehat{\tau_d m}^{\varphi} = \widehat{(\tau_d n)^2}^{\varphi} = (\widehat{\tau_d n}^{\varphi})(\widehat{\tau_d n}^{\varphi}). \tag{2.98}$$

We further introduce the corresponding "inverse" weight functions

$$\mu(k) = \begin{cases} 0 & \text{for } k = 0, \\ \frac{N}{k} & \text{for } 1 \leq k \leq N, \end{cases} \qquad \nu(k) = \begin{cases} 0 & \text{for } k = 0, \\ \sqrt{\frac{N}{k}} & \text{for } 1 \leq k \leq N, \end{cases} \tag{2.99}$$

which satisfy

$$\widehat{m}^{\varphi}\widehat{\mu}^{\varphi} = \widehat{m\mu}^{\varphi} = \sum_{k=0}^{N} m(k)n(k)P_{N,k}^{\varphi} = \sum_{k=1}^{N} P_{N,k}^{\varphi} = 1 - P_{N,0}^{\varphi}, \qquad \widehat{n}^{\varphi}\widehat{\nu}^{\varphi} = 1 - P_{N,0}^{\varphi}, \tag{2.100}$$

and also

$$\widehat{\tau_d\mu}^{\varphi} = \widehat{(\tau_d\nu)^2}^{\varphi} = (\widehat{\tau_d\nu}^{\varphi})(\widehat{\tau_d\nu}^{\varphi}). \tag{2.101}$$

Remark 2.8. The above definition is in agreement with (2.32) and (2.33) if we set

$$f_{\text{even}}(k) = \begin{cases} 1 & \text{for } k \text{ even,} \\ 0 & \text{for } k \text{ odd,} \end{cases} \qquad f_{\text{odd}}(k) = 1 - f_{\text{even}}(k).$$

It follows immediately from the properties of $(P_{N,k}^{\varphi})_{k=0}^{N}$ that $\widehat{f}_{\text{odd}}^{\varphi} + \widehat{f}_{\text{even}}^{\varphi} = 1$, $\widehat{f}_{\text{odd}}^{\varphi}\widehat{f}_{\text{even}}^{\varphi} = \widehat{f}_{\text{even}}^{\varphi}\widehat{f}_{\text{odd}}^{\varphi} = 0$, as well as $\widehat{f}_{\text{even}}^{\varphi}\widehat{f}_{\text{even}}^{\varphi} = \widehat{f}_{\text{even}}^{\varphi}$ and $\widehat{f}_{\text{odd}}^{\varphi}\widehat{f}_{\text{odd}}^{\varphi} = \widehat{f}_{\text{odd}}^{\varphi}$.

Lemma 2.13 (Pull through formula)**.** *Let $Q_{12}^{(0)} = p_1^{\varphi}p_2^{\varphi}$, $Q_{12}^{(1)} = p_1^{\varphi}q_2^{\varphi} + q_1^{\varphi}p_2^{\varphi}$ and $Q_{12}^{(2)} = q_1^{\varphi}q_2^{\varphi}$, and let f be an arbitrary weight function, and A_{12} any operator on $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$. Then the following commutation rule holds for $0 \leq i, j \leq 2$:*

$$Q_{12}^{(i)}A_{12}Q_{12}^{(j)}\widehat{f}^{\varphi} = \widehat{\tau_{j-i}f}^{\varphi}Q_{12}^{(i)}A_{12}Q_{12}^{(j)}. \tag{2.102}$$

Let us explain this for a simple example, $i = 2$, $j = 0$ and $A_{12} = v_{12}$ and $\Psi_N = \varphi^{\otimes N}$:

$$q_1^{\varphi}q_2^{\varphi}v_{12}p_1^{\varphi}p_2^{\varphi}\left(\widehat{f}^{\varphi}\varphi^{\otimes N}\right) = \widehat{\tau_2 f}^{\varphi}\left(q_1^{\varphi}q_2^{\varphi}v_{12}p_1^{\varphi}p_2^{\varphi}\varphi^{\otimes N}\right). \tag{2.103}$$

Proof. For $0 \leq k \leq N$ and $r \in \{0, ..., N-1\}$, we set

$$P_{N,k}^{\varphi,r} = \sum_{a \in \mathcal{A}_k^r} \prod_{i=r+1}^{N} (q_i^{\varphi})^{a_i}(p_i^{\varphi})^{1-a_i} \tag{2.104}$$

with

$$\mathcal{A}_k^r = \left\{ a = (a_1, ..., a_{N-r}) \in \{0,1\}^{N-r} \;:\; \sum_{i=1}^{N-r} a_i = k \right\}. \tag{2.105}$$

For $k < 0$ and $k > N$, let $P_{N,k}^{\varphi,r} = 0$. Then, for $j \in \{0, 1, 2\}$ (note that j is the number of q^{φ}'s in $Q_{12}^{(j)}$),

$$Q_{12}^{(j)}\widehat{f}^{\varphi} = \sum_{k=0}^{N} f(k)Q_{12}^{(j)}P_{N,k}^{\varphi} = \sum_{k=0}^{N} f(k)Q_{12}^{(j)}P_{N,k-j}^{\varphi,2} = \sum_{k=-j}^{N-j} f(k+j)Q_{12}^{(j)}P_{N,k}^{\varphi,2}, \tag{2.106}$$

The pull through formula follows from the fact that $P_{N,k}^{\varphi,2}$ commutes with A_{12}:

$$Q_{12}^{(i)}A_{12}Q_{12}^{(j)}\widehat{f}^{\varphi} = \sum_{k=-j}^{N-j} f(k+j)P_{N,k}^{\varphi,2}Q_{12}^{(i)}A_{12}Q_{12}^{(j)} \tag{2.107}$$

$$
\begin{aligned}
&= \sum_{k=-j}^{N-j} f(k+j) P^{\varphi}_{N,k+i} Q^{(i)}_{12} A_{12} Q^{(j)}_{12} \\
&= \sum_{k=i-j}^{N+i-j} f(k+j-i) P^{\varphi}_{N,k} Q^{(i)}_{12} A_{12} Q^{(j)}_{12} \\
&= \sum_{k=0}^{N} (\tau_{j-i} f)(k) P^{\varphi}_{N,k} Q^{(i)}_{12} A_{12} Q^{(j)}_{12} = \widehat{\tau_{j-i} f}^{\varphi} Q^{(i)}_{12} A_{12} Q^{(j)}_{12}.
\end{aligned}
\tag{2.108}
$$

□

Definition 2.14. We define the so called counting functional w.r.t. φ and with weight f by

$$
\big\langle \cdot, \widehat{f}^{\varphi} \cdot \big\rangle : \mathcal{H}_N \to \mathbb{R}^+_0, \qquad \Psi_N \mapsto \big\langle \Psi_N, \widehat{f}^{\varphi} \Psi_N \big\rangle. \tag{2.109}
$$

For $\varphi_t \in H^1(\mathbb{R}^3)$ the solution of the Hartree equation (2.6), and any $\Psi_N \in \mathcal{H}_N$, the mapping $t \mapsto \langle \Psi_N, \widehat{f}^{\varphi_t} \Psi_N \rangle$ is differentiable with derivative

$$
\partial_t \big\langle \Psi_N, \widehat{f}^{\varphi_t} \Psi_N \big\rangle = -i \big\langle \Psi_N, \Big[\sum_{i=1}^{N} h^{t,\varphi_t}_i, \widehat{f}^{\varphi_t} \Big] \Psi_N \big\rangle. \tag{2.110}
$$

Proof. We recall Definition (2.1) for the set of projectors $(P^{\varphi_t}_{N,k})_{k=0}^N$ and introduce the abbreviation $R^{\varphi_t}_i = (q^t_i)^{a_i} (p^t_i)^{1-a_i}$ such that

$$
P^{\varphi_t}_{N,k} = \sum_{a \in \mathcal{A}_k} \prod_{i=1}^{N} (q^t_i)^{a_i} (p^t_i)^{1-a_i} = \sum_{a \in \mathcal{A}_k} \prod_{i=1}^{N} R^{\varphi_t}_i. \tag{2.111}
$$

Taking the time-derivative, one finds with $\partial_t R^{\varphi_t}_m = -i \big[h^{t,\varphi_t}_m, R^{\varphi_t}_m \big]$,

$$
\begin{aligned}
\partial_t P^{\varphi_t}_{N,k} &= \sum_{a \in \mathcal{A}_k} \partial_t \Big(\prod_{i=1}^{N} R^{\varphi_t}_i \Big) \\
&= \sum_{a \in \mathcal{A}_k} \sum_{m=1}^{N} \Big(\prod_{i=1}^{m-1} R^{\varphi_t}_i \Big) \Big(\partial_t R^{\varphi_t}_m \Big) \Big(\prod_{i=m+1}^{N} R^{\varphi_t}_i \Big) \\
&= -i \sum_{a \in \mathcal{A}_k} \sum_{m=1}^{N} \Big(\prod_{i=1}^{m-1} R^{\varphi_t}_i \Big) \Big[h^{t,\varphi_t}_m, R^{\varphi_t}_m \Big] \Big(\prod_{i=m+1}^{N} R^{\varphi_t}_i \Big) = -i \Big[\sum_{m=1}^{N} h^{t,\varphi_t}_m, P^{\varphi_t}_{N,k} \Big].
\end{aligned}
\tag{2.112}
$$

□

Lemma 2.15. *Let $\varphi_t \in H^1(\mathbb{R}^3)$ be the solution to the Hartree equation (2.6) with $\varphi_0 \in H^1(\mathbb{R}^3)$, $\|\varphi_0\| = 1$. We set*

$$
Z^{\varphi_t}(x_1 - x_2) = \frac{1}{2} \Big(v(x_1 - x_2) - \big(v * |\varphi_t|^2 \big)(x_1) - \big(v * |\varphi_t|^2 \big)(x_2) \Big), \tag{2.113}
$$

and, for any $\Psi \in \mathcal{H}_N$, let

$$(\mathrm{I})^{\varphi_t}_{f,\Psi_N} = 4N\,\mathrm{Im}\,\big\langle \Psi_N, \big(\widehat{f}^{\varphi_t} - \widehat{\tau_{-1}f}^{\varphi_t}\big) q_1^t p_2^t Z^{\varphi_t}(x_1-x_2) p_1^t p_2^t \Psi_N\big\rangle, \tag{2.114}$$
$$(\mathrm{II})^{\varphi_t}_{f,\Psi_N} = 2N\,\mathrm{Im}\,\big\langle \Psi_N, \big(\widehat{f}^{\varphi_t} - \widehat{\tau_{-2}f}^{\varphi_t}\big) q_1^t q_2^t Z^{\varphi_t}(x_1-x_2) p_1^t p_2^t \Psi_N\big\rangle, \tag{2.115}$$
$$(\mathrm{III})^{\varphi_t}_{f,\Psi_N} = 4N\,\mathrm{Im}\,\big\langle \Psi_N, \big(\widehat{f}^{\varphi_t} - \widehat{\tau_{-1}f}^{\varphi_t}\big) q_1^t q_2^t Z^{\varphi_t}(x_1-x_2) p_1^t q_2^t \Psi_N\big\rangle, \tag{2.116}$$

with f being any weight function. Then, with $\Psi_N \in H^2_s(\mathbb{R}^{3N})$, $\|\Psi_N\| = 1$, we have

1. *for $\Psi_{N,t} = U_N(t,0)\Psi_N$,*

$$\partial_t \big\langle \Psi_{N,t}, \widehat{f}^{\varphi_t}\Psi_{N,t}\big\rangle = (\mathrm{I})^{\varphi_t}_{f,\Psi_{N,t}} + (\mathrm{II})^{\varphi_t}_{f,\Psi_{N,t}} + (\mathrm{III})^{\varphi_t}_{f,\Psi_{N,t}}. \tag{2.117}$$

2. *for $\widetilde{\Psi}_{N,t} = \widetilde{U}_N(t,0)\Psi_N$,*

$$\partial_t \big\langle \widetilde{\Psi}_{N,t}, \widehat{f}^{\varphi_t}\widetilde{\Psi}_{N,t}\big\rangle = (\mathrm{II})^{\varphi_t}_{f,\widetilde{\Psi}_{N,t}}. \tag{2.118}$$

Remark 2.9. By the identity in (2.97), it follows that

$$\alpha_N(t) = \big\langle \Psi_{N,t}, q_1^t \Psi_{N,t}\big\rangle = \big\langle \Psi_{N,t}, \widehat{m}^{\varphi_t}\Psi_{N,t}\big\rangle. \tag{2.119}$$

Using $\widehat{m}^{\varphi_t} - \widehat{\tau_{-d}m}^{\varphi_t} = \frac{d}{N}\sum_{k=d}^N P^{\varphi_t}_{N,k}$ (for $d = 1,2$), one can apply Lemma 2.15 in order to obtain (2.31).

Proof of Lemma 2.15. Using (2.110), the fact that $\Psi_{N,t}$ solves the Schrödinger equation and the symmetry of $\Psi_{N,t}$, we find that the time-derivative of the counting functional is given by

$$\begin{aligned}
\partial_t \big\langle \Psi_{N,t}, \widehat{f}^{\varphi_t}\Psi_{N,t}\big\rangle &= i\big\langle \Psi_{N,t}, \Big[H_N^t - \sum_{i=1}^N h_i^{t,\varphi_t}, \widehat{f}^{\varphi_t}\Big]\Psi_{N,t}\big\rangle \\
&= i\big\langle \Psi_{N,t}, \Big[\frac{N}{2}v(x_1-x_2) - \frac{N}{2}\big(v*|\varphi_t|^2\big)(x_1) - \frac{N}{2}\big(v*|\varphi_t|^2\big)(x_2), \widehat{f}^{\varphi_t}\Big]\Psi_{N,t}\big\rangle \\
&= iN\big\langle \Psi_{N,t}, \Big(Z^{\varphi_t}(x_1-x_2)\widehat{f}^{\varphi_t} - \widehat{f}^{\varphi_t}Z^{\varphi_t}(x_1-x_2)\Big)\Psi_{N,t}\big\rangle.
\end{aligned} \tag{2.120}$$

Multiplying both of the $\Psi_{N,t}$ with the identity $1 = (p_1^t + q_1^t)(p_2^t + q_2^t)$, leads to

$$\begin{aligned}
\partial_t \big\langle \Psi_{N,t}, \widehat{f}^{\varphi_t}\Psi_{N,t}\big\rangle &= iN\big\langle \Psi_{N,t}, p_1^t p_2^t\Big(Z^{\varphi_t}(x_1-x_2)\widehat{f}^{\varphi_t} - \widehat{f}^{\varphi_t}Z^{\varphi_t}(x_1-x_2)\Big)p_1^t p_2^t\Psi_{N,t}\big\rangle \\
&+ 2iN\big\langle \Psi_{N,t}, p_1^t q_2^t\Big(Z^{\varphi_t}(x_1-x_2)\widehat{f}^{\varphi_t} - \widehat{f}^{\varphi_t}Z^{\varphi_t}(x_1-x_2)\Big)p_1^t q_2^t\Psi_{N,t}\big\rangle \\
&+ 2iN\big\langle \Psi_{N,t}, p_1^t q_2^t\Big(Z^{\varphi_t}(x_1-x_2)\widehat{f}^{\varphi_t} - \widehat{f}^{\varphi_t}Z^{\varphi_t}(x_1-x_2)\Big)q_1^t p_2^t\Psi_{N,t}\big\rangle \\
&+ iN\big\langle \Psi_{N,t}, q_1^t q_2^t\Big(Z^{\varphi_t}(x_1-x_2)\widehat{f}^{\varphi_t} - \widehat{f}^{\varphi_t}Z^{\varphi_t}(x_1-x_2)\Big)q_1^t q_2^t\Psi_{N,t}\big\rangle \\
&+ 2iN\big\langle \Psi_{N,t}, p_1^t p_2^t\Big(Z^{\varphi_t}(x_1-x_2)\widehat{f}^{\varphi_t} - \widehat{f}^{\varphi_t}Z^{\varphi_t}(x_1-x_2)\Big)q_1^t p_2^t\Psi_{N,t}\big\rangle + \mathrm{c.c.} \\
&+ iN\big\langle \Psi_{N,t}, p_1^t p_2^t\Big(Z^{\varphi_t}(x_1-x_2)\widehat{f}^{\varphi_t} - \widehat{f}^{\varphi_t}Z^{\varphi_t}(x_1-x_2)\Big)q_1^t q_2^t\Psi_{N,t}\big\rangle + \mathrm{c.c.} \\
&+ 2iN\big\langle \Psi_{N,t}, p_1^t q_2^t\Big(Z^{\varphi_t}(x_1-x_2)\widehat{f}^{\varphi_t} - \widehat{f}^{\varphi_t}Z^{\varphi_t}(x_1-x_2)\Big)q_1^t q_2^t\Psi_{N,t}\big\rangle + \mathrm{c.c.}
\end{aligned} \tag{2.121}$$

where c.c. denotes the complex conjugate of the preceding expression. Application of the pull through formula (Lemma 2.13) shows that all but the last three lines are identically zero. In the last three lines, we also use the Pull through formula and then that $Z^{\varphi_t}(x_1-x_2)$ is a symmetric operator.

By the same argument, one finds as well (2.118). The only difference is that (I) and (III) are identically zero due to the definition of the potential in $\widetilde{H}_N^t$. □

Lemma 2.16. *Let $\varphi_t \in H^1(\mathbb{R}^3)$ be the solution to the Hartree equation (2.6) with $\varphi_0 \in H^1(\mathbb{R}^3)$, $\|\varphi_0\| = 1$, and let $m(k) = \frac{k}{N}$ as in (2.96). It holds that for any $\Psi \in \mathcal{H}_N$,*

$$(\mathrm{I})^{\varphi_t}_{m^n,\Psi_N} = 0, \qquad \left|(\mathrm{II})^{\varphi_t}_{m^n,\Psi_N}\right| + \left|(\mathrm{III})^{\varphi_t}_{m^n,\Psi_N}\right| \le C_n \|\varphi_t\|_{H^1} \sum_{l=0}^{n} \frac{\left\langle \Psi_N, (\widehat{m}^{\varphi_t})^l \Psi_N \right\rangle}{N^{n-l}}. \tag{2.122}$$

Proof of Lemma 2.16. **Term (I)**. The first term is identically zero for any n, $(\mathrm{I})^{\varphi_t}_{m^n,\Psi_N} = 0$, because $p_2^t v(x_1-x_2) p_2^t$ in

$$p_2^t Z^{\varphi_t}(x_1-x_2) p_2^t = p_2^t v(x_1-x_2) p_2^t - p_2^t \big(v * |\varphi_t|^2\big)(x_1) p_2^t = 0 \tag{2.123}$$

cancels exactly the mean field potential. It is this term which determines the choice of the effective potential in the Hartree equation.

For the second and third term, we need to compute the difference

$$(\widehat{m}^{\varphi_t})^n - (\widehat{\tau_{-d} m}^{\varphi_t})^n = \sum_{k=0}^{N} \left[\left(\frac{k}{N}\right)^n - \left(\frac{k-d}{N}\right)^n \right] P^{\varphi_t}_{N,k} = \sum_{l=0}^{n-1} \frac{C_{n,l,d}}{N^{n-l}} (\widehat{m}^{\varphi_t})^l, \tag{2.124}$$

where the constants $C_{n,l,d} = \binom{n}{l}(-d)^{n-l}$ are determined by the binomial expansion of $(k-d)^n$.

Term (II). Note that $p_2^t \bar{v}_1^t q_2^t = 0$ and recall the pull through formula as well as the weight functions m, n and μ, ν and also that $q_1^{\varphi} \Psi_N = (1 - P^{\varphi_t}_{N,0}) q_1^{\varphi_t} \Psi_N = (\widehat{\nu}^{\varphi_t} \widehat{n}^{\varphi_t}) q_1^{\varphi_t} \Psi_N$ by means of (2.100). It follows

$$\begin{aligned}
&\left|(\mathrm{II})^{\varphi_t}_{m^n,\Psi_N}\right| \\
&= \Big| \sum_{l=0}^{n-1} \frac{C_{n,l,-2}}{N^{n-l-1}} \left\langle \Psi_N, (\widehat{m}^{\varphi_t})^l \Big(q_1^t q_2^t v_{12} p_1^t p_2^t \Big) \Psi_N \right\rangle \Big| \\
&= \Big| \sum_{l=0}^{n-1} \frac{C_{n,l,-2}}{N^{n-l-1}} \left\langle \Psi_N, (\widehat{m}^{\varphi_t})^{l/2} \widehat{\nu}^{\varphi_t} \Big(q_1^t q_2^t v_{12} p_1^t p_2^t \Big) (\widehat{\tau_{-2} m}^{\varphi_t})^{l/2} \widehat{\tau_{-2} n}^{\varphi_t} \Psi_N \right\rangle \Big| \\
&\le \sum_{l=0}^{n-1} \frac{|C_{n,l,-2}|}{N^{n-l-1}} \|v_{12} p_1^t\|_{op} \, \|q_1^t q_2^t (\widehat{m}^{\varphi_t})^{l/2} \widehat{\nu}^{\varphi_t} \Psi_N\| \, \|(\widehat{\tau_{-2} m}^{\varphi_t})^{(l+1)/2} \Psi_N\| \\
&\le \sum_{l=0}^{n-1} \frac{|C_{n,l,-2}|}{N^{n-l-1}} \|v_{12} p_1^t\|_{op} \sqrt{ \sum_{j=0}^{l+1} \binom{l+1}{j} \left(\frac{2}{N}\right)^{l+1-j} \left\langle \Psi_N (\widehat{m}^{\varphi_t})^j \Psi_N \right\rangle \left\langle \Psi_N, q_1^t q_2^t (\widehat{m}^{\varphi_t})^l \widehat{\mu}^{\varphi_t} \Psi_N \right\rangle }
\end{aligned}$$

$$\leq \sum_{l=0}^{n-1} \frac{|C_{n,l,-2}|}{N^{n-l-1}} \|v_{12} p_1^t\|_{op} \sum_{j=0}^{l+1} \binom{l+1}{j} \left(\frac{2}{N}\right)^{l+1-j} \langle \Psi_N, (\widehat{m}^{\varphi_t})^j \Psi_N \rangle$$

$$\leq C_n \|\varphi_t\|_{H^1} \sum_{l=0}^{n} \frac{\langle \Psi_N, (\widehat{m}^{\varphi_t})^l \Psi_N \rangle}{N^{n-l}} \tag{2.125}$$

The essential ingredient here is the symmetry of the wave function which ensures that not all mass can be located around, e.g., $x_1 \approx x_2$ (for general $\Psi_N \in L^2(\mathbb{R}^{3N})$, the second term would not be necessarily small).

Term (III). Again via the pull through formula, and similarly as in (II),

$$\begin{aligned}
\left|(\mathrm{III})^{\varphi_t}_{m^n,\Psi_N}\right| &= \left| \sum_{l=0}^{n-1} \frac{C_{n,l,-1}}{N^{n-l-1}} \langle \Psi_N, \left(q_1^t q_2^t (v_{12} - \bar{v}_1^t) q_1^t p_2^t\right) (\widehat{m}^{\varphi_t})^l \Psi_N \rangle \right| \\
&= \left| \sum_{l=0}^{n-1} \frac{C_{n,l,-1}}{N^{n-l-1}} \langle (\widehat{\tau_{-1} m}^{\varphi_t})^{l/2} \Psi_N, \left(q_1^t q_2^t (v_{12} - \bar{v}_1^t) q_1^t p_2^t\right) (\widehat{m}^{\varphi_t})^{l/2} \Psi_N \rangle \right| \\
&\leq \sum_{l=0}^{n-1} \frac{|C_{n,l,-1}|}{N^{n-l-1}} \left(\|v_{12} p_2^t\|_{op} + \|\bar{v}^t\|_\infty \right) \|q_1^t (\widehat{m}^{\varphi_t})^{l/2} \Psi_N\| \, \|q_1^t (\widehat{\tau_{-1} m}^{\varphi_t})^{l/2} \Psi_N\| \\
&\leq C_n \|\varphi_t\|_{H^1} \sum_{l=0}^{n} \frac{\langle \Psi_N, (\widehat{m}^{\varphi_t})^l \Psi_N \rangle}{N^{n-l}}.
\end{aligned} \tag{2.126}$$

□

2.3.3 Proofs of Lemmas 2.3, 2.4, 2.8 and 2.10

We begin with

Proof of Lemma 2.10. Let $\Phi_N, \widetilde{\Phi}_N \in \mathcal{H}_N$ and $\varphi \in \mathcal{H}_1$.

1. By means of the pull through formula (2.102) together with the fact that $\widehat{f}^{\varphi}_{\text{even}} \widehat{f}^{\varphi}_{\text{odd}} = 0$, we obtain

$$\widehat{f}^{\varphi}_{\text{odd}} \left(p_1^{\varphi} A_1 q_1^{\varphi} \right) \widehat{f}^{\varphi}_{\text{odd}} = 0 = \widehat{f}^{\varphi}_{\text{even}} \left(p_1^{\varphi} A_1 q_1^{\varphi} \right) \widehat{f}^{\varphi}_{\text{even}}. \tag{2.127}$$

Moreover, it follows from (2.100) and $P^{\varphi}_{N,0} q_1^{\varphi} = 0$ that $q_1^{\varphi} (\widehat{n}^{\varphi} \widehat{\nu}^{\varphi}) \Phi_N = q_1^{\varphi} \Phi_N$. Thus, one finds for the first term on the left side of (2.55),

$$\begin{aligned}
&\left| \langle \Phi_N, \left(p_1^{\varphi} A_1 q_1^{\varphi} \right) \Phi_N \rangle \right| \\
&= \left| \langle \widehat{f}^{\varphi}_{\text{odd}} \Phi_N, \left(p_1^{\varphi} A_1 q_1^{\varphi} \right) \widehat{f}^{\varphi}_{\text{even}} \Phi_N \rangle + \langle \widehat{f}^{\varphi}_{\text{even}} \Phi_N, \left(p_1^{\varphi} A_1 q_1^{\varphi} \right) \left(\widehat{n}^{\varphi} \widehat{\nu}^{\varphi} \right) \widehat{f}^{\varphi}_{\text{odd}} \Phi_N \rangle \right|.
\end{aligned} \tag{2.128}$$

Next, we apply again the pull through formula (2.102) in order to move $\widehat{n}^{\varphi}$ to the left of the scalar product, and then use the Cauchy-Schwarz inequality together with the properties summarized below Definition 2.12. We make use of $q_1^{\varphi} (\widehat{\nu}^{\varphi})^2 = 1$, as expectation values on $\mathcal{H}_N$ (recall that $\widehat{f}^{\varphi}_{\text{odd}} \Phi_N \in \mathcal{H}_N$), and eventually use $(\widehat{\tau_d n}^{\varphi})^2 = \widehat{\tau_d m}^{\varphi} = \frac{d}{N} + \widehat{m}^{\varphi}$ which implies

$$\langle \Phi_N, (\widehat{\tau_d n}^{\varphi})^2 \Phi_N \rangle \leq \frac{|d|}{N} + \|q_1^{\varphi} \Phi_N\|^2, \tag{2.129}$$

Hence, one obtains

$$\begin{aligned}(2.128) &= \Big|\langle \widehat{f}^{\varphi}_{\text{odd}}\Phi_N, \Big(p_1^{\varphi} A_1 q_1^{\varphi}\Big)\widehat{f}^{\varphi}_{\text{even}}\Phi_N\rangle + \langle \widehat{\tau_{-1}n}^{\varphi}\widehat{f}^{\varphi}_{\text{even}}\Phi_N, \Big(p_1^{\varphi} A_1 q_1^{\varphi}\Big)\widehat{\nu}^{\varphi}\widehat{f}^{\varphi}_{\text{odd}}\Phi_N\rangle\Big| \\ &\le \|\widehat{f}^{\varphi}_{\text{odd}}\Phi_N\|\ \|p_1^{\varphi}A_1\|_{op}\ \|q_1^{\varphi}\Phi_N\| + \|\widehat{\tau_{-1}n}^{\varphi}\Phi_N\|\ \|p_1^{\varphi}A_1\|_{op}\ \|q_1^{\varphi}\widehat{\nu}^{\varphi}\widehat{f}^{\varphi}_{\text{odd}}\Phi_N\| \\ &\le 20^{-1}\|\widehat{f}^{\varphi}_{\text{odd}}\Phi_N\|^2 + 10\|p_1^{\varphi}A_1\|^2_{op}\Big(\|q_1^{\varphi}\Phi_N\|^2 + \|\widehat{\tau_{-1}n}^{\varphi}\Phi_N\|^2\Big) \\ &\le 20^{-1}\|\widehat{f}^{\varphi}_{\text{odd}}\widetilde{\Phi}_N\|^2 + 10\|p_1^{\varphi}A_1\|^2_{op}\Big(2\|q_1^{\varphi}\Phi_N\|^2 + N^{-1}\Big). \end{aligned} \tag{2.130}$$

For the second term on the left side of (2.55), we proceed along the same steps which leads to the first statement of the lemma.

2. We use $q_1^{\varphi}\widetilde{\Phi}_N = (\widehat{n}^{\varphi}\widehat{\nu}^{\varphi})q_1^{\varphi}\widetilde{\Phi}_N = q_1^{\varphi}(\widehat{n}^{\varphi}\widehat{\nu}^{\varphi})\widetilde{\Phi}_N$, apply the pull through formula and then the Cauchy-Schwarz inequality,

$$\Big|\langle \Phi_N, \Big(q_1^{\varphi}q_2^{\varphi}v_{12}q_1^{\varphi}q_2^{\varphi}\Big)\Big(\widehat{n}^{\varphi}\widehat{\nu}^{\varphi}\Big)\widetilde{\Phi}_N\rangle\Big| \le \|q_1^{\varphi}q_2^{\varphi}\widehat{n}^{\varphi}\Phi_N\|\ \|v_{12}q_1^{\varphi}q_2^{\varphi}\widehat{\nu}^{\varphi}\widetilde{\Phi}_N\|. \tag{2.131}$$

In the second factor, we invoke the assumed bound on v, i.e., $v^2 \le C(1-\Delta)$,

$$\|v_{12}q_1^{\varphi}q_2^{\varphi}\widehat{\nu}^{\varphi}\widetilde{\Phi}_N\|^2 \le C\Big(\|\nabla_1 q_1^{\varphi}q_2^{\varphi}\widehat{\nu}^{\varphi}\widetilde{\Phi}_N\|^2 + \|q_1^{\varphi}\widetilde{\Phi}_N\|^2\Big), \tag{2.132}$$

and then we use symmetry of $\widetilde{\Phi}_N$ in combination with (2.97), in order to find

$$\langle \widetilde{\Phi}_N, \Big(q_2^{\varphi}\widehat{\nu}^{\varphi}\widehat{\nu}^{\varphi}\Big)q_1^{\varphi}(-\Delta_1)q_1^{\varphi}\widetilde{\Phi}_N\rangle \le \frac{N}{N-1}\langle \widetilde{\Phi}_N, \Big(\widehat{m}^{\varphi}\widehat{\mu}^{\varphi}\Big)q_1^{\varphi}(-\Delta_1)q_1^{\varphi}\widetilde{\Phi}_N\rangle.$$

In the first factor, we make use of $(\widehat{n}^{\varphi})^2 = \widehat{m}^{\varphi}$ and (2.97), and use the symmetry of Φ_N, in order to obtain

$$(2.131) \le C\Big(N\|q_1^{\varphi}q_2^{\varphi}q_3^{\varphi}\Phi_N\|^2 + \|q_1^{\varphi}q_2^{\varphi}\Phi_N\|^2\Big) + \frac{\|\nabla_1 q_1^{\varphi}\widetilde{\Phi}_N\|^2 + \|q_1^{\varphi}\widetilde{\Phi}_N\|^2}{N}. \tag{2.133}$$

3. Following the argument as in part 1, with some obvious modifications, one finds

$$\begin{aligned}&\Big|\langle \Phi_N, \Big(q_1^{\varphi}q_2^{\varphi}A_{12}q_1^{\varphi}p_2^{\varphi}\Big)\widetilde{\Phi}_N\rangle\Big| \\ &= \Big|\langle \widehat{f}^{\varphi}_{\text{odd}}\Phi_N, \widehat{\mu}^{\varphi}\Big(q_1^{\varphi}q_2^{\varphi}A_{12}q_1^{\varphi}p_2^{\varphi}\Big)\widehat{\tau_{-1}m}^{\varphi}\widehat{f}^{\varphi}_{\text{even}}\widetilde{\Phi}_N\rangle \\ &\quad + \langle \widehat{f}^{\varphi}_{\text{even}}\Phi_N, \widehat{\tau_1 n}^{\varphi}\Big(q_1^{\varphi}q_2^{\varphi}A_{12}q_1^{\varphi}p_2^{\varphi}\Big)\widehat{\nu}^{\varphi}\widehat{f}^{\varphi}_{\text{odd}}\widetilde{\Phi}_N\rangle\Big| \\ &\le \|A_{12}p_2^{\varphi}\|_{op}\Big(\|\widehat{\mu}^{\varphi}q_1^{\varphi}q_2^{\varphi}\widehat{f}^{\varphi}_{\text{odd}}\Phi_N\|\ \|q_1^{\varphi}\widehat{\tau_{-1}m}^{\varphi}\widetilde{\Phi}_N\| + \|\widehat{\tau_1 n}^{\varphi}q_1^{\varphi}q_2^{\varphi}\Phi_N\|\ \|q_1^{\varphi}\widehat{\nu}^{\varphi}\widehat{f}^{\varphi}_{\text{odd}}\widetilde{\Phi}_N\|\Big) \\ &\le \frac{\|\widehat{f}^{\varphi}_{\text{odd}}\Phi_N\|^2 + \|\widehat{f}^{\varphi}_{\text{odd}}\widetilde{\Phi}_N\|^2}{2N} + C\|A_{12}p_2^{\varphi}\|^2_{op}N\Big(\|q_1^{\varphi}q_2^{\varphi}q_3^{\varphi}\widetilde{\Phi}_N\|^2 + \|q_1^{\varphi}q_2^{\varphi}q_3^{\varphi}\Phi_N\|^2 + \\ &\qquad + \frac{\|q_1^{\varphi}q_2^{\varphi}\widetilde{\Phi}_N\|^2 + \|q_1^{\varphi}q_2^{\varphi}\Phi_N\|^2}{N} + \frac{\|q_1^{\varphi}\widetilde{\Phi}_N\|^2 + \|q_1^{\varphi}\Phi_N\|^2}{N^2} + \frac{1}{N^3}\Big). \end{aligned} \tag{2.134}$$

The term containing the Hermitian conjugate is estimated in the same manner. □

Proof of Lemma 2.3. First note that

$$\langle \Phi_{N,t}, \Big(\prod_{j=1}^{n} q_j^t\Big)\Phi_{N,t}\rangle \le \langle \Phi_{N,t}, (\widehat{m}^{\varphi_t})^n\Phi_{N,t}\rangle$$

which follows from the symmetry of $\Phi_{N,t}$ together with (2.97) and $(k-i)/(N-i) \le k/N$ for $i \le k \le N$. From Lemmas 2.15 and 2.16 we know that

$$\frac{d}{dt}\big\langle \Phi_{N,t}, (\widehat{m}^{\varphi_t})^n \Phi_{N,t}\big\rangle \le \|\varphi_t\|_{H^1} \sum_{l=0}^{n} \frac{\big\langle \Phi_{N,t}, (\widehat{m}^{\varphi_t})^l \Phi_{N,t}\big\rangle}{N^{n-l}}. \tag{2.135}$$

(note that it is again sufficient to prove the bound for $\Phi_{N,t} \in \{\Psi_{N,t}, \widetilde{\Psi}_{N,t}\}$ with $\Psi_N \in H^2(\mathbb{R}^{3N})$ in order to conclude the statement of the Lemma via a density argument). The remainder of the argument follows by induction. Assume that for all $k \le n-1$,

$$\big\langle \Phi_{N,t}, (\widehat{m}^{\varphi_t})^k \Phi_{N,t}\big\rangle \le e^{C(1+t)^{3/2}} \sum_{l=0}^{k} \frac{C_{n,k}}{N^{l-n}} \big\langle \Psi_N, (\widehat{m}^{\varphi_0})^l \Psi_N\big\rangle. \tag{2.136}$$

By means of (2.135) and $\|\varphi_t\|_{H_1} \le C\sqrt{1+t}$, Gronwall's inequality implies that

$$\big\langle \Phi_{N,t}, (\widehat{m}^{\varphi_t})^n \Phi_{N,t}\big\rangle \le e^{C(1+t)^{3/2}} \sum_{l=0}^{n} \frac{C_{n,l}}{N^{l-n}} \big\langle \Psi_N, (\widehat{m}^{\varphi_0})^l \Psi_N\big\rangle. \tag{2.137}$$

The case $n=1$ follows as well with Gronwall's inequality, cf. Lemmas 2.15 and 2.16,

$$\begin{aligned} &\partial_t \big\langle \Phi_{N,t}, \widehat{m}^{\varphi_t} \Psi_{N,t}\big\rangle \le C\sqrt{1+t}\Big(\big\langle \Phi_{N,t}, \widehat{m}^{\varphi_t}\Phi_{N,t}\big\rangle + \frac{1}{N}\Big) \\ &\qquad\Rightarrow\quad \big\langle \Phi_{N,t}, \widehat{m}^{\varphi_t}\Phi_{N,t}\big\rangle \le e^{C(1+t)^{3/2}}\Big(\big\langle \Psi_N, \widehat{m}^{\varphi_0}\Psi_N\big\rangle + \frac{1}{N}\Big). \end{aligned} \tag{2.138}$$

This completes the proof of the lemma since

$$\big\langle \Psi_N, (\widehat{m}^{\varphi_0})^n \Psi_N\big\rangle \le \sum_{i=1}^{n} \frac{1}{N^{n-i}} \big\langle \Psi_N, \Big(\prod_{j=1}^{i} q_j^0\Big)\Psi_N\big\rangle \tag{2.139}$$

which is verified using again (2.97). □

Proof of Lemma 2.4. The time-derivative of $\|\widehat{f}^{\varphi_t}_{\mathrm{odd}} \widetilde{\Psi}_{N,t}\|^2$ is given by

$$\begin{aligned} \partial_t \big\langle \widetilde{\Psi}_{N,t}, \widehat{f}^{\varphi_t}_{\mathrm{odd}} \widetilde{\Psi}_{N,t}\big\rangle &= (\mathrm{II})^{\varphi_t}_{f_{\mathrm{odd}},\widetilde{\Psi}_{N,t}} \\ &= 2N\,\mathrm{Im}\,\big\langle \widetilde{\Psi}_{N,t}, \big(\widehat{f}^{\varphi_t}_{\mathrm{odd}} - \widehat{\tau_{-2} f}^{\varphi_t}_{\mathrm{odd}}\big) q_1^t q_2^t v_{12} p_1^t p_2^t \widetilde{\Psi}_{N,t}\big\rangle, \end{aligned} \tag{2.140}$$

cf. Lemma 2.15. Recalling the definition of the shifted weight function,

$$\widehat{f}^{\varphi_t}_{\mathrm{odd}} - \widehat{\tau_{-2} f}^{\varphi_t}_{\mathrm{odd}} = f(1) P^{\varphi_t}_{N,1} = P^{\varphi_t}_{N,1}, \tag{2.141}$$

and the fact that

$$q_1^t q_2^t P^{\varphi_t}_{N,1} = 0 \tag{2.142}$$

shows that $\|\widehat{f}^{\varphi_t}_{\mathrm{odd}} \widetilde{\Psi}_{N,t}\| = \|\widehat{f}^{\varphi_0}_{\mathrm{odd}} \widetilde{\Psi}_{N,0}\|$. A similar calculation holds for the even case. □

Proof of Lemma 2.8. Using the decomposition in (2.46), it can be verified by direct calculation that if $\widetilde{\Psi}_{N,t}$ solves the Schrödinger equation $i\partial_t \widetilde{\Psi}_{N,t} = \widetilde{H}^t_N \widetilde{\Psi}_{N,t}$, then the corresponding correlation functions $(\widetilde{\chi}^{(k)}_{N,t})_{k=0}^N$, defined as in (2.45), solve the following system of coupled equations.

$$i\partial_t \widetilde{\chi}^{(0)}_{N,t} = \sqrt{\frac{N}{N-1}} B^{(2\to 0),t} \widetilde{\chi}^{(2)}_{N,t} \tag{2.143}$$

$$i\partial_t \widetilde{\chi}^{(1)}_{N,t} = \left(h^{t,\varphi_t} + K^{(1),t}\right)\widetilde{\chi}^{(1)}_{N,t} + \sqrt{\frac{N-2}{N-1}} B^{(3\to 1),t} \widetilde{\chi}^{(3)}_{N,t}, \tag{2.144}$$

and for all $2 \le k \le N$,

$$\begin{aligned} i\partial_t \widetilde{\chi}^{(k)}_{N,t} = &\sum_{i=1}^k \left(h_i^{t,\varphi_t} + \frac{N-k}{N-1} K_i^{(1),t}\right)\widetilde{\chi}^{(k)}_{N,t} \\ &+ \frac{\sqrt{(N-k+2)(N-k+1)}}{N-1} A^{(k-2\to k),t} \widetilde{\chi}^{(k-2)}_{N,t} \\ &+ \frac{\sqrt{(N-k)(N-k-1)}}{N-1} B^{(k+2\to k),t} \widetilde{\chi}^{(k+2)}_{N,t}. \end{aligned} \tag{2.145}$$

Recall that we are using the convention $\widetilde{\chi}^{(k)}_{N,t} \equiv 0$ for all $k \ge N+1$. Here we have introduced the abbreviations

$$\begin{aligned} A^{(k-2\to k),t} \widetilde{\chi}^{(k-2)}_{N,t} &= \frac{1}{2\sqrt{k(k-1)}} \sum_{1\le i<j\le k} K^{(2),t}(x_i,x_j) \widetilde{\chi}^{(k-2)}_{N,t}(x_1,...,x_k \backslash x_i \backslash x_j), \\ B^{(k+2\to k),t} \widetilde{\chi}^{(k+2)}_{N,t} &= \frac{\sqrt{(k+1)(k+2)}}{2} \int\int \overline{K^{(2),t}(x,y)} \widetilde{\chi}^{(k+2)}_{N,t}(x_1,...,x_k,x,y) dx dy, \end{aligned}$$

with $K^{(1),t}$ and $K^{(2),t}(x,y)$ defined as below (2.50). Let us explain how one arrives at (2.143)-(2.145). Taking the time-derivative of $\widetilde{\chi}^{(k)}_{N,t}$, one finds

$$i\partial_t \widetilde{\chi}^{(k)}_{N,t} = \sum_{i=1}^N h_i^{t,\varphi_t} \widetilde{\chi}^{(k)}_{N,t} + \sqrt{\binom{N}{k}} \Big(\prod_{j=1}^k q_j^t\Big) \langle \varphi_t^{\otimes(N-k)}, \Big(\widetilde{H}^t_N - \sum_{i=1}^k h_i^{t,\varphi_t}\Big)\widetilde{\Psi}_{N,t}\rangle,$$

where the scalar product is taken w.r.t. the coordinates $x_{k+1},...,x_N$. For the term containing the interaction, we show one example, namely

$$\begin{aligned} &\sqrt{\binom{N}{k}} \Big(\prod_{j=1}^k q_j^t\Big) \langle \varphi_t^{\otimes(N-k)}, \Big(\frac{1}{N-1}\sum_{i<j} p_i^t p_j^t v_{ij} q_i^t q_j^t\Big)\widetilde{\Psi}_{N,t}\rangle \\ &= \sqrt{\binom{N}{k}} \frac{(N-k)(N-k-1)}{2(N-1)} \int\int \overline{K^{(2),t}(x,y)} \Big(\prod_{j=1}^{k+2} q_j^t\Big) \langle \varphi_t^{\otimes(N-k-2)}, \widetilde{\Psi}_{N,t}\rangle dx dy \\ &= \frac{\sqrt{(N-k)(N-k-1)}}{N-1} B^{(k+2\to k),t} \widetilde{\chi}^{(k+2)}_{N,t}. \end{aligned}$$

Similarly, one computes also the other terms from $\widetilde{H}^t_N$. Note that for wave functions $\phi^{(k)} \in \mathcal{H}_k$, $\chi^{(k-2)} \in \mathcal{H}_{k-2}$, the operators A and B satisfy the relation

$$\big\langle \phi^{(k)}, A^{(k-2\to k),t} \chi^{(k-2)}\big\rangle = \big\langle B^{(k\to k-2),t} \phi^{(k)}, \chi^{(k-2)}\big\rangle. \tag{2.146}$$

Moreover, one readily finds that

$$\|K^{(1),t}\|_{op} \le C\|\varphi_t\|_{H_1}, \tag{2.147}$$

as well as

$$\|A^{(k-2\to k),t}\|_{op}^2 + \|B^{(k+2\to k),t}\|_{op}^2 \le Ck^2 \|\varphi_t\|_{H^2}^2. \tag{2.148}$$

The equations (2.143–2.145) are almost the same as the ones from the Bogoliubov hierarchy. The differences are the N-dependence of the coefficients and also that the Bogoliubov hierarchy is infinite. The remainder of the proof is to show that the two solutions are close to each other in the sense that

$$g_N(t) := \sum_{k=0}^{\infty} \|\widetilde{\chi}_{N,t}^{(k)} - \chi_t^{(k)}\|^2 \tag{2.149}$$

is small as indicated in the statement of the lemma. We know a priori that $g_N(t)$ is finite for all $t \ge 0$, because $\sum_{k=0}^{\infty} \|\widetilde{\chi}_{N,t}^{(k)}\|^2 = \|\widetilde{\Psi}_{N,t}\|^2 = 1$ and $\sum_{k=0}^{\infty} \|\chi_t^{(k)}\|^2 = 1$. The former follows from unitarity of $\widetilde{U}_N(t,s)$ and $\|\Psi_N\| = 1$. The latter is a consequence of the well-posedness of the Bogoliubov hierarchy (for details about well-posedness of the Bogoliubov hierarchy we refer to [80, Section 4.3]). Moreover, one can rewrite $g_N(t)$ as a finite sum, namely as

$$g_N(t) = 2 - 2\sum_{k=0}^{N} \mathrm{Re}\,\langle \widetilde{\chi}_{N,t}^{(k)}, \chi_t^{(k)}\rangle. \tag{2.150}$$

Next, we compute its time-derivative and then estimate it in order to apply Gronwall's inequality. Using the equations of motion for $\widetilde{\chi}_{N,t}^{(k)}$ and $\chi_t^{(k)}$, we find

$$\frac{d}{dt} g_N(t) = -2\sum_{k=1}^{N} \mathrm{Im}\,\Big\langle \widetilde{\chi}_{N,t}^{(k)}, \sum_{i=1}^{k}\Big(1 - \frac{N-k}{N-1}\Big) K_i^{(1),t} \chi_t^{(k)}\Big\rangle \tag{2.151}$$

$$+ 2\sum_{k=0}^{N-2} \mathrm{Im}\,\Big\langle \frac{\sqrt{(N-k)(N-k-1)}}{N-1} B^{(k+2\to k),t} \widetilde{\chi}_{N,t}^{(k+2)}, \chi_t^{(k)}\Big\rangle \tag{2.152}$$

$$- 2\sum_{k=2}^{N} \mathrm{Im}\,\langle B^{(k\to k-2),t} \widetilde{\chi}_{N,t}^{(k)}, \chi_t^{(k-2)}\rangle \tag{2.153}$$

$$+ 2\sum_{k=2}^{N+2} \mathrm{Im}\,\Big\langle \frac{\sqrt{(N-k+2)(N-k+1)}}{N-1} A^{(k-2\to k),t} \widetilde{\chi}_{N,t}^{(k-2)}, \chi_t^{(k)}\Big\rangle \tag{2.154}$$

$$- 2\sum_{k=0}^{N} \mathrm{Im}\,\langle A^{(k\to k+2),t} \widetilde{\chi}_{N,t}^{(k)}, \chi_t^{(k+2)}\rangle. \tag{2.155}$$

Since $K^{(1),t}$ is self-adjoint, we can replace in the first line $\chi_t^{(k)}$ by $\chi_t^{(k)} - \widetilde{\chi}_{N,t}^{(k)}$. Using the Cauchy-Schwarz inequality as well as the inequality of arithmetic and geometric means, one finds

$$|(2.151)| \le 4\sum_{k=1}^{N} \Big(1 - \frac{N-k}{N-1}\Big)^2 \Big\| \sum_{i=1}^{k} K_i^{(1),t}\Big\|_{op}^2 \|\widetilde{\chi}_{N,t}^{(k)}\|^2 + \sum_{k=1}^{N} \|\widetilde{\chi}_{N,t}^{(k)} - \chi_t^{(k)}\|^2.$$

In the first summand, we use the bound from (2.147), then $(1 - \frac{N-k}{N-1})^2 k^2 \le C\frac{k^4}{N^2}$, and recall the identity $\|\widetilde{\chi}_{N,t}^{(k)}\|^2 = \|P_{N,k}^{\varphi_t} \widetilde{\Psi}_{N,t}\|^2$. With

$$\sum_{k=0}^{N} \frac{k^4}{N^2} \|\widetilde{\chi}_{N,t}^{(k)}\|^2 \le N^2 \sum_{k=0}^{N} \frac{k^3}{N^3} \|P_{N,k}^{\varphi_t} \widetilde{\Psi}_{N,t}\|^2 = N^2 \langle \widetilde{\Psi}_{N,t} (\widehat{m}^{\varphi_t})^3 \widetilde{\Psi}_{N,t}\rangle, \tag{2.156}$$

we can use (2.97) and then (2.58), in order to find

$$|(2.151)| \leq \frac{e^{C(1+t)^{3/2}}}{N} + g_N(t).$$

In (2.153), we substitute the summation index $k \mapsto k+2$ and obtain after adding the second and third line,

$$\begin{aligned} &-\frac{1}{2}\Big((2.152) + (2.153)\Big) \\ &= \sum_{k=0}^{N-2} \Big(1 - \frac{\sqrt{(N-k)(N-k-1)}}{N-1}\Big) \operatorname{Im} \big\langle B^{(k+2\to k),t} \widetilde{\chi}^{(k+2)}_{N,t}, \chi^{(k)}_t - \widetilde{\chi}^{(k)}_{N,t} \big\rangle \qquad (2.157) \\ &+ \sum_{k=0}^{N-2} \Big(1 - \frac{\sqrt{(N-k)(N-k-1)}}{N-1}\Big) \operatorname{Im} \big\langle B^{(k+2\to k),t} \widetilde{\chi}^{(k+2)}_{N,t}, \widetilde{\chi}^{(k)}_{N,t} \big\rangle, \qquad (2.158) \end{aligned}$$

where we have added and subtracted the second line. Similarly, after substituting $k-2 \mapsto k$ in (2.154), we find

$$\begin{aligned} &-\frac{1}{2}\Big((2.154) + (2.155)\Big) \\ &= \sum_{k=0}^{N-2} \Big(1 - \frac{\sqrt{(N-k)(N-k-1)}}{N-1}\Big) \operatorname{Im} \big\langle A^{(k\to k+2),t} \widetilde{\chi}^{(k)}_{N,t}, \chi^{(k+2)}_t - \widetilde{\chi}^{(k+2)}_{N,t} \big\rangle \qquad (2.159) \\ &+ \sum_{k=0}^{N-2} \Big(1 - \frac{\sqrt{(N-k)(N-k-1)}}{N-1}\Big) \operatorname{Im} \big\langle A^{(k\to k+2),t} \widetilde{\chi}^{(k)}_{N,t}, \widetilde{\chi}^{(k+2)}_{N,t} \big\rangle \qquad (2.160) \\ &+ \sum_{k=N-1}^{N} \operatorname{Im} \big\langle A^{(k\to k+2),t} \widetilde{\chi}^{(k)}_{N,t}, \chi^{(k+2)}_t \big\rangle. \qquad (2.161) \end{aligned}$$

By means of (2.146), we have

$$(2.158) + (2.160) = 0.$$

For estimating (2.157), we proceed similarly as in the estimate for (2.151), and find, using (2.148) as well as $(\frac{\sqrt{(N-k)(N-k-1)}}{N-1} - 1)^2 k^2 \leq C\frac{k^3}{N}$, that

$$|(2.157)| \leq C\|\varphi_t\|^2_{H^2} \sum_{k=0}^{N-2} \frac{k^3}{N} \|P^{\varphi_t}_{N,k+2} \widetilde{\Psi}_{N,t}\|^2 + g_N(t). \qquad (2.162)$$

Shifting $k \mapsto k-2$ in the first summand,

$$N^2 \sum_{k=0}^{N-2} \frac{k^3}{N^3} \|P^{\varphi_t}_{N,k+2} \widetilde{\Psi}_{N,t}\|^2 \leq CN^2 \big\langle \widetilde{\Psi}_{N,t}, \Big((\widehat{m}^{\varphi_t})^3 + \frac{(\widehat{m}^{\varphi_t})^2}{N} + \frac{\widehat{m}^{\varphi_t}}{N^2} + \frac{1}{N^3}\Big) \widetilde{\Psi}_{N,t} \big\rangle,$$

and thus, again by (2.58), we find

$$|(2.157)| \leq \frac{e^{C(1+t)^{3/2}}}{N} + g_N(t).$$

Along the same argument, one obtains the same bound also for (2.159). It remains to estimate the last term,

$$\begin{aligned}
|(2.161)| &\leq C\|\varphi_t\|_{H^2} N\Big(\|\widetilde{\chi}_{N,t}^{(N-1)}\|\,\|\chi_t^{(N+1)}\| + \|\widetilde{\chi}_{N,t}^{(N)}\|\,\|\chi_t^{(N+2)}\|\Big) \\
&\leq C\|\varphi_t\|_{H^2} N^2\Big(\|\widetilde{\chi}_{N,t}^{(N-1)}\|^2 + \|\widetilde{\chi}_{N,t}^{(N)}\|^2\Big) + \Big(\|\chi_t^{(N+1)}\|^2 + \|\chi_t^{(N+2)}\|^2\Big) \\
&\leq \frac{e^{C(1+t)^2}}{N} + g_N(t),
\end{aligned} \tag{2.163}$$

where the last step follows from

$$N^2\Big(\|\widetilde{\chi}_{N,t}^{(N-1)}\|^2 + \|\widetilde{\chi}_{N,t}^{(N)}\|^2\Big) \leq 2N^2 \sum_{k=0}^{N} \frac{k^3}{N^3}\|\widetilde{\chi}_{N,t}^{(k)}\|^2 \leq e^{C(1+t)^2} N^{-1}.$$

Altogether, via Gronwall's inequality, we have found

$$g_N(t) \leq e^{C(1+t)^2}\Big(g_N(0) + N^{-1}\Big) = \frac{e^{C(1+t)^2}}{N}, \tag{2.164}$$

since $\widetilde{\chi}_{N,0}^{(k)} = \chi_0^{(k)}$ for all k, i.e., $g_N(0) = 0$. □

Appendices

2.A The difference $H_N^t - \widetilde{H}_N^t$

We verify (2.25): For that, rewrite the Hamiltonian H_N^t as

$$\begin{aligned}
H_N^t &= \sum_{i=1}^N h_i^{t,\varphi_t} - \frac{1}{2(N-1)}\sum_{i\neq j}^N \Big(\bar{v}_i^t + \bar{v}_j^t - 2\mu^t\Big) + \frac{1}{N-1}\sum_{i<j} v_{ij} \\
&= \sum_{i=1}^N h_i^{t,\varphi_t} + \frac{1}{N-1}\sum_{i<j}\Big((p_i^t + q_i^t)(p_j^t + q_j^t)(v_{ij} - \bar{v}_i^t - \bar{v}_j^t + 2\mu^t)(p_i^t + q_i^t)(p_j^t + q_j^t)\Big),
\end{aligned}$$

and compute the different terms with p^t's and q^t's.

1.

$$p_i^t p_j^t (v_{ij} - \bar{v}_i^t - \bar{v}_j^t + 2\mu^t) p_i^t p_j^t = p_i^t p_j^t (2\mu^t - 2\mu^t - 2\mu^t + 2\mu^t) = 0,$$

2.

$$q_i^t p_j^t (v_{ij} - \bar{v}_i^t - \bar{v}_j^t + 2\mu^t) p_i^t p_j^t = (q_i^t \bar{v}_i^t p_i^t) p_j^t - (q_i^t \bar{v}_i^t p_i^t) p_j^t = 0,$$

and similarly for the term with q_i^t and p_j^t reversed on the l.h.s., and also for the Hermitian conjugate of these terms.

3.

$$q_i^t p_j^t (v_{ij} - \bar{v}_i^t - \bar{v}_j^t + 2\mu^t) q_i^t p_j^t = (q_i^t \bar{v}_i^t q_i^t) p_j^t - (q_i^t \bar{v}_i^t q_i^t) p_j^t - (p_j^t \bar{v}_j^t p_j^t) q_i^t + 2\mu^t q_i^t p_j^t = 0,$$

and similarly with q_i^t and p_j^t reversed on both sides.

4.

$$\Big(q_i^t p_j^t (v_{ij} - \bar{v}_i^t - \bar{v}_j^t + 2\mu^t) p_i^t q_j^t + q_i^t q_j^t (v_{ij} - \bar{v}_i^t - \bar{v}_j^t + 2\mu^t) p_i^t p_j^t\Big) + \text{h.c.}$$
$$= \Big(q_i^t p_j^t v_{ij} p_i^t q_j^t + q_i^t q_j^t v_{ij} p_i^t p_j^t\Big) + \text{h.c.} \equiv v_{ij}^{(2q,t)}$$

5.

$$\Big(q_i^t q_j^t (v_{ij} - \bar{v}_i^t - \bar{v}_j^t + 2\mu^t)(p_i^t q_j^t + q_i^t p_j^t) + \text{h.c}\Big) = v_{ij}^{(3q,t)},$$

6.

$$q_i^t q_j^t (v_{ij} - \bar{v}_i^t - \bar{v}_j^t + 2\mu^t) q_i^t q_j^t = v_{ij}^{(4q,t)}.$$

In total, we obtain

$$H_N^t = \sum_{i=1}^N h_i^{t,\varphi_t} + \frac{1}{N-1} \sum_{i<j} \Big(v_{ij}^{(2q,t)} + v_{ij}^{(3q,t)} + v_{ij}^{(4q,t)}\Big),$$

which is the same as (2.25).

Chapter 3

Low energy properties of the homogeneous Bose gas

In this chapter we analyze the low energy spectral properties of the homogeneous weakly interacting Bose gas on the unit torus. After formulating the setup and the problem in the following section, we present our main estimate and two applications thereof in Section 3.2. All proofs are postponed to Section 3.3.

3.1 Setup and problem

The low energy eigenfunctions of the weakly interacting Bose gas are at leading order (in the particle number N) described by mean field or Hartree theory. This is well known and rigorously understood since the works from Benguria and Lieb [19] and Lieb and Yau [88]. The eigenfunctions are, in the sense of reduced densities, approximately equal to an N-fold product of a single one-body state φ_{H}, the so-called condensate wave function which, in turn, is the ground state solution of the Hartree equation (or more generally minimizes the Hartree energy functional). This is the famous phenomenon of Bose-Einstein condensation at low temperature meaning that the overwhelming majority of the N particles in the gas occupies the same copy of a single wave function φ_{H}. The main objective in this chapter is to derive a novel estimate for the probability of finding a given number l of particles not in the state φ_{H}. We show that this probability is exponentially small in the number l. Our analysis is restricted to the case of the homogeneous gas on the torus (we expect a similar result to hold also in the nonhomogeneous case for which the analysis, however, is more complicated). The exponential decay of the probabilities for finding l particles outside the condensate is then applied to show that the fluctuations around the Hartree product in the N-body ground state wave function obey two important properties that are reminiscent of the properties of a quasifree state in Fock space. That the fluctuations can be approximately described by a quasifree state was first observed, among other things, by Lewin et al. in [81] where they analyzed a more general situation (including the homogeneous setup). The quasifree properties for the fluctuations in the microscopic ground state hold asymptotically for large N. Here, we derive error bounds which are expected to be the optimal ones for the homogeneous case.[1] After that we use our main result to characterize the low-lying eigenvalues in terms of Bogoliubov theory. Bogoliubov theory can be seen as the limiting description of the fluctuations around the Hartree product. It states that

[1] The quality of the error bounds is relevant for the derivation of the time-dependent Hartree equation with optimal speed of convergence and initial states close to the true ground state; cf. Remark 3.1.

the low energy excitation spectrum of the N-body system is for large N described to good approximation by the spectrum of noninteracting quasiparticles obeying an effective energy-momentum dispersion relation. It has been derived rigorously first by Seiringer in [116] and since then generalized into various directions [60, 81, 37]. Using the exponential bounds for the probability of finding particles outside the Hartree product, our derivation provides an alternative strategy for the justification of Bogoliubov theory.

We consider an N-particle Hamiltonian of the form

$$H_N = -\sum_{i=1}^{N} \Delta_{x_i} + \frac{1}{N-1} \sum_{1 \le i < j \le N} v(x_i - x_j), \tag{3.1}$$

acting on $L_s^2(\mathbb{T}^{dN})$, $d \in \{1,2,3\}$ being the spatial dimension and $\mathbb{T}$ a one-dimensional torus of length $L = 1$ (the results hold for arbitrary $L < \infty$ but for ease of notation we set $L = 1$). $-\Delta_{x_i}$ is minus the Laplace operator describing the kinetic energy of the ith particle. The factor $(N-1)^{-1}$ denotes the mean field coupling constant and guarantees that on average the interaction energy is of the same order as the kinetic energy, namely $\propto N$. In this scaling regime a nontrivial solution is expected for which neither the kinetic nor the interaction part of the Hamiltonian dominates the physics in the large N limit. The interaction between the particles is modeled by a real valued (N-independent) function $v : \mathbb{T}^d \to \mathbb{R}$, $x \mapsto v(x)$, with Fourier transform defined as

$$\hat{v}(k) = \int_{\mathbb{T}^d} v(x) e^{ikx} dx \quad \text{for all} \quad k \in 2\pi\mathbb{Z}^d. \tag{3.2}$$

The function v is required to satisfy **Assumption 3.1.**

1. $v(x) = v(-x)$, $v(x) \ge 0$ and $\hat{v}(k) \ge 0$,
2. $v \in L^2(\mathbb{T}^d)$ and $\hat{v} \in L^1(2\pi\mathbb{Z}^d)$.

The Hamiltonian H_N is bounded from below and by means of Kato's Theorem it defines a self-adjoint operator on the dense subset $\mathcal{D}(\sum_{i=1}^{N} -\Delta_{x_i}) \subset L_s^2(\mathbb{T}^{dN})$. The system defined by H_N is translation invariant which follows from the chosen boundary conditions and the dependence of the pair potential v on the difference of the particle coordinates $x_i - x_j$ for all $1 \le i,j \le N$. Translation invariance implies that H_N commutes with the total momentum operator $P_N = \sum_{i=1}^{N} (-i\nabla_{x_i})$, i.e., $[H_N, P_N] = 0$. Therefore, there exists a joint spectrum (called energy-momentum spectrum) of the operators H_N and P_N denoted by $\mathrm{spec}(H_N, P_N) \subset \mathbb{R}^{1+d}$. Joint eigenvalues of H_N and P_N are given by all $(E_N, p_N) \in \mathrm{spec}(H_N, P_N)$ which satisfy the eigenvalue equations

$$H_N \Psi_N = E_N \Psi_N \quad \text{and} \quad P_N \Psi_N = p_N \Psi_N \tag{3.3}$$

for joint eigenfunctions $\Psi_N \in L_s^2(\mathbb{T}^{dN})$. In this chapter we study properties of such eigenfunctions and the corresponding energy eigenvalues E_N for the case that E_N is close to the lowest possible eigenvalue E_N^0, the ground state energy. Let us denote the energy eigenvalues larger than E_N^0 by E_N^n ($n \ge 1$) where $E_N^n \le E_N^m \Leftrightarrow n \le m$ (i.e., with increasing order counting multiplicity of degenerate values) and the corresponding eigenfunctions by Ψ_N^n. Eigenvalues and eigenfunctions of a many-body operator like H_N are in general very complicated objects. In the weak coupling regime we consider here, however, the analysis simplifies because for energies not too far from the ground state energy, eigenfunctions Ψ_N^n

are at leading order given by the Hartree product, $\Psi_N^n \approx \varphi_{\mathrm{H}}^{\otimes N}$, in the sense that the energy equals N times the Hartree energy:

$$\lim_{N\to\infty} \frac{E_N^n}{N} = \inf_{\varphi\in L^2(\Omega)} \Big\{ \big\langle \varphi, \Big(-\Delta + \frac{v*|\varphi|^2}{2} \Big) \varphi \big\rangle \ : \ \|\varphi\| = 1 \Big\} \tag{3.4}$$

($*$ denotes the convolution of functions on $\mathbb{T}^d$). In the homogeneous setting, $\varphi_{\mathrm{H}} = 1$ is the constant function, and the minimum of the Hartree energy is given by $\varepsilon_{\mathrm{H}} = \big\langle \varphi_{\mathrm{H}}, \big(-\Delta + \frac{1}{2} v*|\varphi|^2 \big) \varphi_{\mathrm{H}} \big\rangle = \hat{v}(0)/2$. The proof of (3.4) is not very difficult in our setup and we postpone it to Appendix 3.A. In the general case – nonhomogeneous and for more general potentials – it is much more involved and we refer to [77, 78, 79] for recent results and further references. In Appendix 3.A, we explain that it follows from (3.4) that the majority of the N particles in the gas occupies the condensate wave function φ_{H}, i.e., $\lim_{N\to\infty} n_{\mathrm{H}}(N)/N = 1$ with $n_{\mathrm{H}}(N)$ the number of particles in the state φ_{H}. For large but finite particle number N, the probability of having a significant amount of particles outside the condensate wave function is thus necessarily small since otherwise, Eq. (3.4) would be false.

In this chapter we use the same notation as summarized in Section 2.1.4.

3.1.1 Objective of this chapter

Our main goal is to show that the probability to find l out of the N particles described by a low energy eigenfunction Ψ_N^n outside the condensate wave function is exponentially small:

$$\mathbb{P}_{\Psi_N^n}\big(l \text{ particles not in } \varphi_{\mathrm{H}}\big) \le C e^{-Dl} \tag{3.5}$$

for N-independent constants $C, D > 0$. Below Theorem 3.1, we give a detailed explanation of the idea behind the proof of (3.5). Here, we explain it very briefly: If we denote by $P_{N,l}^0$ the projectors on wave functions in $L_s^2(\mathbb{T}^{dN})$ which contain exactly $N-l$ particles in the state φ_{H}, then the probabilities that we want to estimate are given by $\|P_{N,l}^0 \Psi_N^0\|^2$. The exponential decay is inferred from a difference inequality for the discrete function $\|P_{N,l}^0 \Psi_N^0\|^2$ (the integer l being the variable) which is a consequence of the energy eigenvalue equation for Ψ_N^0. This difference inequality turns out to be analogous to the Schrödinger equation for a particle on a one-dimensional lattice in a potential barrier which lies above the energy of the particle: $-\partial_i^2 \psi_i = (E - V)\psi_i$ with $E < V$. In this case the exponential decay of the wave function ψ_i is well known as the tunneling effect.

From (3.5), we then derive

1. in Corollary 3.6 quasifree type properties for the ground state Ψ_N^0,
2. in Theorem 3.7 the validity of Bogoliubov theory for low-lying energies.

It was shown by Lewin et. al. in [81, Theorem A.1] that the ground state of the Bogoliubov Hamiltonian is a quasifree state in Fock space (the Bogoliubov Hamiltonian is introduced in (3.9) and the definition of quasifree states is given in (3.17) and (3.18)). They further proved that the Bogoliubov ground state is related through a partially isometric mapping (see also below) to an N-body wave function which converges in L^2-distance towards the ground state Ψ_N^0. The quasifree properties of the ground state of the Bogoliubov Hamiltonian, however, are not directly transferred to analogous statements about Ψ_N^0. There is further information required about the wave function Ψ_N^0 – e.g., the exponential decay in (3.5) – in order to show that it satisfies similar "quasifree type" properties as the ground state of the

Bogoliubov Hamiltonian. Regarding the second point, the derivation of Bogoliubov theory for the low energy spectrum, our proof offers a different approach to obtain a similar result as by Seiringer in [116]. However, our method does not allow us to cover the same range of energies as in [116].

3.1.2 Bogoliubov approximation

Let us close this section with a short presentation of Bogoliubov's description of the weakly interacting Bose gas. Bogoliubov theory predicts the next-order contribution in $E^n = N\varepsilon_{\mathrm{H}} + o(N)$ where $o(N)$ stands for a function $f(N)$ which is small compared to N in the large N limit: $f(N)/N \to 0$ for $N \to \infty$. It was first introduced in 1947 by Bogoliubov in his famous work on the theory of superfluidity [22]. Application and justification in the context of the ground state have been extensively studied in the mathematical pyhsics literature; e.g., [58, 86, 87, 119, 127, 116, 60, 37, 81] and [107, 108, 109] for a more recent approach using a novel application of the Feshbach-Schur method. That Bogoliubov theory describes also the low energy excitation spectrum has been rigorously shown in [116, 60, 37, 81]. We define the Bogoliubov ground state energy as

$$E_{\mathrm{Bog}} = -\frac{1}{2}\sum_{k\neq 0}\Big(|k|^2 + \hat{v}(k) - e(k)\Big), \tag{3.6}$$

where the elementary excitation energy $e(k)$ is given by

$$e(k) = \sqrt{|k|^4 + 2|k|^2\hat{v}(k)}. \tag{3.7}$$

The sum in (3.6) is meant to run over all values $k \in 2\pi\mathbb{Z}^d\backslash\{0\}$ (this convention will be used throughout). Note that E_{Bog} has a finite value since $e(k) = |k|^2(1 + 2\hat{v}(k)/|k|^2)^{\frac{1}{2}}$ such that the summands in (3.6) behave like $\hat{v}(k)^2/|k|^2$ for large $|k|$ (recall, e.g., that $v \in L^2$ and thus also $\hat{v} \in L^2$). Moreover, we define the Bogoliubov excitation energies by the set of numbers

$$\Big\{\sum_{i=1}^{j} e(k_i) \;:\; k_1, ..., k_j \in 2\pi\mathbb{Z}^d \setminus\{0\}\;,\; j \geq 1\Big\} \tag{3.8}$$

and denote these values by K^n_{Bog} ($n \geq 1$, with increasing order counting multiplicity). In Appendix 3.B we show that the numbers $E^0_{\mathrm{Bog}} = E_{\mathrm{Bog}}$ and $E^n_{\mathrm{Bog}} = E_{\mathrm{Bog}} + K^n_{\mathrm{Bog}}$ ($n \geq 1$) coincide with the eigenvalues of the Bogoliubov Hamiltonian

$$\mathbb{H}_{\mathrm{Bog}} = \sum_{k\neq 0}\Big[k^2 + \frac{\hat{v}(k)}{2}\Big(2a_k^* a_k + a_k^* a_{-k}^* + a_k a_{-k}\Big)\Big], \tag{3.9}$$

which acts on the bosonic Fock space $\mathcal{F}_s$ where a_k and a_k^* satisfy the canonical commutation relations $[a_k, a_{k'}^*] = \delta_{kk'}$, $[a_k^*, a_{k'}^*] = 0 = [a_k, a_{k'}]$. Having the original N-particle system in mind one should think of $\mathcal{F}_s$ as the symmetric Fock space constructed over the one-particle Hilbert space $\{\varphi_{\mathrm{H}}\}^\perp = \{\varphi \in L^2(\mathbb{T}^d) : \langle\varphi, \varphi_{\mathrm{H}}\rangle = 0\}$,

$$\mathcal{F}_s = \bigoplus_{l=0}^{\infty}\bigotimes_{sym.}^{l}\Big(\{\varphi_{\mathrm{H}}\}^\perp\Big). \tag{3.10}$$

The creation and annihilation operators a_k^* and a_k in (3.9) are then defined by adding and removing a plane wave $e^{ikx} \in L^2(\mathbb{T}^d)$ with momentum $k \in 2\pi\mathbb{Z}^d\backslash\{0\}$ to the state $\chi = (\chi_l)_{l\geq 0} \in \mathcal{F}_s$:[2]

$$(a_k\chi)_l = (a_k\chi)_l(x_1, ..., x_l) = \sqrt{l+1} \int_{\mathbb{T}^d} e^{-ikx_{l+1}} \; \chi_{l+1}(x_1, ..., x_{l+1}) \; dx_{l+1}, \tag{3.11}$$

$$(a_k^*\chi)_l = (a_k^*\chi)_l(x_1, ..., x_l) = \frac{1}{\sqrt{l}} \sum_{i=1}^{l} e^{ikx_i} \; \chi_{l-1}(x_1, ..., x_l\backslash x_i). \tag{3.12}$$

It is important to note that a particle in $\mathcal{F}_s$ corresponds to a particle in the Bose gas which does not occupy the state φ_{H}. Without going into much detail at this point, let us mention that the relation between $\mathcal{F}_s$ and $L_s^2(\mathbb{T}^{dN})$ is defined via the partial isometry

$$\chi = (\chi_0, \chi_1, \chi_2, ...) \in \mathcal{F}_s \quad \sim \quad \Phi = \sum_{l=0}^{N} \left(\varphi_{\mathrm{H}}^{\otimes N-l} \otimes_s \chi_l \right) \in L_s^2(\mathbb{T}^{dN}), \tag{3.13}$$

where $\otimes_s$ denotes the normalized symmetric product between two symmetric functions; see (3.116). The vacuum state in $\mathcal{F}_s$ for instance represents the product wave function $\varphi_{\mathrm{H}}^{\otimes N}$ in the N-particle space $L_s^2(\mathbb{T}^{dN})$. It is thus not surprising that $\mathbb{H}_{\mathrm{Bog}}$ is not particle number conserving, as particles in $\mathcal{F}_s$ represent fluctuations around the Hartree product.

We denote the eigenfunctions of the Bogoliubov Hamiltonian by $\chi^n \in \mathcal{F}_s$ ($n \geq 0$), i.e., $\mathbb{H}_{\mathrm{Bog}}\chi^n = E_{\mathrm{Bog}}^n\chi^n$. In [81] it was shown (even in a more general setup) that $\mathbb{H}_{\mathrm{Bog}}$ possesses a unique ground state which is related to the N-particle wave function via

$$\lim_{N\to\infty} \left\| \Psi_N^0 - \sum_{l=0}^{N} \left(\varphi_{\mathrm{H}}^{\otimes N-l} \otimes_s \chi_l^0 \right) \right\|_{L_s^2(\mathbb{T}^{dN})} = 0. \tag{3.14}$$

This implies, e.g., that $\Psi_N^0 \approx \varphi_{\mathrm{H}}^{\otimes N}$ is false in the sense of the full N-particle norm. Note that this is not in contradiction with (3.4) or with $\lim_{N\to\infty} n_{\mathrm{H}}(N)/N = 1$ which are much weaker assertions. The latter follow from $\Psi_N^0 \to \varphi_{\mathrm{H}}^{\otimes N}$ in trace-norm distance of reduced density matrices. E.g,

$$\lim_{N\to\infty} \mathrm{Tr} \left| \gamma_{\Psi_N^0}^{(1)} - \gamma_{\varphi_{\mathrm{H}}^{\otimes N}}^{(1)} \right| = 0, \tag{3.15}$$

where $\mathrm{Tr}A$ stands for the trace of a trace class operator $A : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ and $\gamma_{\Psi_N}^{(1)}$ is the one-particle marginal w.r.t. Ψ_N defined by its integral kernel

$$\gamma_{\Psi_N}^{(1)}(x, y) = \int_{\mathbb{T}^{d(N-1)}} \Psi_N(x, x_2, ..., x_N)\overline{\Psi_N(y, x_2, ..., x_N)} dx_2...dx_N. \tag{3.16}$$

The phenomenon of Bose-Einstein condensation, i.e., the macroscopic occupation of the condensate wave function, is physically well described by the notion of distance in (3.15) whereas in this regard, the norm distance on $L_s^2(\mathbb{T}^{dN})$ is too strong. To see this, recall that even for a single particle in a state $\phi \in \{\varphi_{\mathrm{H}}\}^\perp \subset L^2(\mathbb{T}^d)$, the symmetric N-body state $\varphi_{\mathrm{H}}^{N-1} \otimes_s \phi$ is already orthogonal to the full Hartree product. From the physics point of view, however, it is unreasonable to expect that all N particles occupy the wave function φ_{H}. The

[2]Note that in the present chapter, our choice of notation for indicating the elements from the different sectors in Fock space is different compared to Chapter 2. Here, we use a subscript (instead of a superscript) for indicating the respective sector, i.e., $\chi = (\chi_l)_{l\geq 0}$, $\chi_l \in L_s^2(\mathbb{T}^{dl})$.

distance in (3.14), on the other hand, becomes relevant if one wants to describe also the behavior of the fluctuations around the Hartree product which is, in turn, important for the explanation of interesting phenomena that are not captured by the Hartree approximation (like superfluidity of the Bose gas for instance). It was also shown in [81] that the ground state $\chi^0 \in \mathcal{F}_s$ is quasifree. A quasifree state $\chi \in \mathcal{F}_s$ is defined by the so-called Wick property (see, e.g., [120, Chapter 10]), which states that the expectation value of any product of an odd number of creation/annihilation operators $a_k^\# \in \{a_k, a_k^*\}$ is identically zero whereas the product of any even number of creation/annihilation operators factorizes into all possible pairings or contractions. More precisely, for any integer n, a quasi free state $\chi \in \mathcal{F}_s$ satisfies

$$\langle \chi, \Big(\prod_{i=1}^{2n-1} a_{k_i}^\# \Big) \chi \rangle = 0, \tag{3.17}$$

$$\langle \chi, \Big(\prod_{i=1}^{2n} a_{k_i}^\# \Big) \chi \rangle = \sum_{\sigma \in P_{2n}} \langle \chi, a_{k_{\sigma(1)}}^\# a_{k_{\sigma(2)}}^\# \chi \rangle \cdot \langle \chi, a_{k_{\sigma(3)}}^\# a_{k_{\sigma(4)}}^\# \chi \rangle \cdot ... \cdot \langle \chi, a_{k_{\sigma(2n-1)}}^\# a_{k_{\sigma(2n)}}^\# \chi \rangle, \tag{3.18}$$

where $a_k^\# \in \{a_k, a_k^*\}$, $k_1, ...k_{2n} \in 2\pi\mathbb{Z}^d \backslash \{0\}$, and P_{2n} is the set of pairings (a subset of all permutations of the numbers $\{1, ..., 2n\}$) given by

$$P_{2n} = \Big\{ \sigma \in S_{2n} \ : \sigma(2j-1) < \sigma(2j+1), \ j = 1, ..., n-1, \\ \sigma(2j-1) < \sigma(2j), \ j = 1, ..., n \Big\}. \tag{3.19}$$

Two properties that are reminiscent of the quasifree property are

1. It holds that either

$$\sum_{l \text{ odd}} \|\chi_l\|^2 = 0 \qquad \text{or} \qquad \sum_{l \text{ even}} \|\chi_l\|^2 = 0, \tag{3.20}$$

2. For any $m \geq 1$,

$$\langle \chi, \mathcal{N}^m \chi \rangle \leq C_m \Big(1 + \langle \chi, \mathcal{N} \chi \rangle \Big)^m, \tag{3.21}$$

   where $\mathcal{N}$ is the number operator on $\mathcal{F}_s$, i.e., $\mathcal{N} = \sum_{k \neq 0} a_k^* a_k$.

One the one hand, a state satisfying (3.20) obeys automatically (3.17). On the other hand, that every state satisfying (3.18) obeys (3.21) was shown, e.g., in [95, Lemma 5]. Let us stress that for proving an analogous statement for the true ground state Ψ_N^0 it is not sufficient to use convergence from (3.14). Here, it is required to have additional information about Ψ_N^0. From the exponential decay in (3.5), e.g., an estimate like (3.21) for the true ground state (and other low energy eigenfunctions) can be inferred.

The reason why the energies E_{Bog}^n in (3.8) have this comparatively simple form is that the Bogoliubov Hamiltonian is quadratic in the creation and annihilation operators and can thus be diagonalized by a unitary transformation (the argument of diagonalizing $\mathbb{H}_{\text{Bog}}$ is presented in Appendix 3.B). The excitation energies K_{Bog}^n can be interpreted as sums of one-particle energies $e(k)$ of noninteracting quasiparticles. The interaction effectively results in a new energy-momentum dispersion relation $e(k)$ being linear in k for low momenta (for nonzero $\hat{v}$ when k is small) compared to the free dispersion relation $e^{\text{free}}(k) = |k|^2$.

3.2 Main results

Before stating our results, we introduce an auxiliary Hamiltonian $\widetilde{H}_N$ which provides an intermediate step between H_N and $\mathbb{H}_{\text{Bog}}$. Let $p_i^k = |\varphi^k(x_i)\rangle\langle\varphi^k(x_i)|$, i.e.,

$$p_i^k : L_s^2(\mathbb{T}^{dN}) \to L_s^2(\mathbb{T}^{dN}), \qquad \Psi_N \mapsto \varphi^k(x_i) \int_{\mathbb{T}^d} \overline{\varphi^k(x_i)} \Psi_N(x_1, ..., x_N)\, dx_i, \tag{3.22}$$

the orthogonal projector onto the normalized plane wave $\varphi^k(x_i)$ with momentum $k \in 2\pi\mathbb{Z}^d$ (note that in this notation $\varphi^0 = \varphi_{\text{H}}$), and let $q_i^k = 1 - p_i^k$ the projector onto the corresponding orthogonal complement. Then rewrite the original Hamiltonian H by adding and subtracting the mean field energy $N\hat{v}(0)/2$ and inserting the identity $1 = (p_i^0 + q_i^0)(p_j^0 + q_j^0)$ on the left and right of the two-body potential (we use the abbreviation $v(x_i - x_j) = v_{ij}$),

$$H_N = -\sum_{i=1}^N \Delta_{x_i} + \frac{N\hat{v}(0)}{2} + \frac{1}{N-1} \sum_{1\le i<j\le N} (p_i^0 + q_i^0)(p_j^0 + q_j^0)\Big(v_{ij} - \hat{v}(0)\Big)(p_i^0 + q_i^0)(p_j^0 + q_j^0).$$

From this expression, we define $\widetilde{H}_N$ by discarding all terms that contain three or four q^0's (this is analogous to Definition (2.23)):

$$\widetilde{H}_N = -\sum_{i=1}^N \Delta_{x_i} + \frac{N\hat{v}(0)}{2} + \frac{1}{N-1} \sum_{1\le i<j\le N} \Big[\Big(p_i^0 q_j^0 v_{ij} q_i^0 p_j^0 + p_i^0 p_j^0 v_{ij} q_i^0 q_j^0\Big) + \text{h.c.}\Big]. \tag{3.23}$$

A simple computation (analogous to the one in Appendix (2.A)) shows that the remainder is given by

$$H_N - \widetilde{H}_N = \frac{1}{N-1} \sum_{1\le i<j\le N} \Big[\Big(q_i^0 q_j^0 v_{ij}(q_i^0 p_j^0 + p_i^0 q_j^0) + \text{h.c}\Big) + \Big(q_i^0 q_j^0 (v_{ij} - \hat{v}(0)) q_i^0 q_j^0\Big)\Big]. \tag{3.24}$$

Note that there remain only terms with exactly two p^0's while all contributions with four and three p^0's are cancelled. In Lemma 3.4 we show that for large enough N the operator $\widetilde{H}_N$ possesses a unique ground state $\widetilde{\Psi}_N^0$. This may not be completely obvious. Due to the projectors in the definition of the two-body potential in $\widetilde{H}_N$, the standard technique of showing uniqueness by means of positivity of the ground state wave function may not be applicable (a detailed exposition of the argument for showing uniqueness of the ground state via its positivity is given, e.g., in [123, Section 10.5]). We introduce $\widetilde{H}_N$ for two reasons: On the one hand, it already has the same quadratic structure as $\mathbb{H}_{\text{Bog}}$. This appears here as the fact that $\widetilde{H}_N$ does not contain contributions with an odd number of q^0's. On the other hand, a comparison between H_N and $\widetilde{H}_N$ is straightforward due to (3.24) (the interaction terms in $\widetilde{H}_N$ and $\mathbb{H}_{\text{Bog}}$, e.g., differ by additional combinatorial factors and the operators do not act on the same spaces). We denote eigenvalues and eigenfunctions of $\widetilde{H}_N$ by $\widetilde{E}_N^n$ respectively by $\widetilde{\Psi}_N^n$ ($n \ge 1$ with increasing order and counting multiplicity).

Define the excitation energies of H_N and $\widetilde{H}_N$ by

$$K_N^n = E_N^n - E_N^0 \quad \text{resp.} \quad \widetilde{K}_N^n = \widetilde{E}_N^n - \widetilde{E}_N^0. \tag{3.25}$$

Eventually, let us introduce a set of projectors $(P_{N,l}^0)_{l=0}^N$ with $P_{N,l}^0 : L_s^2(\mathbb{T}^{dN}) \to L_s^2(\mathbb{T}^{dN})$ defined by

$$P_{N,l}^0 = \Big(q_1^0 ... q_l^0 p_{l+1}^0 ... p_N^0\Big)_{sym} \tag{3.26}$$

where $(\cdot)_{sym}$ denotes the symmetric tensor product. The $P^0_{N,l}$ project onto the subspace of $L^2_s(\mathbb{T}^{dN})$ of wave functions which contain exactly l particles outside the Hartree product. The probability of finding l particles not in φ_{H} for a given normalized state Ψ_N is thus equal to $\|P^0_{N,l}\Psi_N\|^2$.

3.2.1 Exponential decay for probabilities $\|P^0_{N,l}\Psi^n\|^2$

Our main result in this chapter is summarized in

Theorem 3.1. *Let $n \geq 0$. There exist positive constants C and D such that*

(a) $\|P^0_{N,l}\Psi^n_N\|^2 \leq Ce^{-Dl}$ *for all* $K^n_N \leq l \leq N$,

(b) $\|P^0_{N,l}\widetilde{\Psi}^n_N\|^2 \leq Ce^{-Dl}$ *for all* $\widetilde{K}^n_N \leq l \leq N$,

and

(c) $\|\chi^n_l\|^2 \leq Ce^{-Dl}$ *for all* $l \geq K^n_{Bog}$.

Let us explain the basic idea behind the proof. We start from the respective eigenvalue equation for Ψ^n_N, $\widetilde{\Psi}^n_N$ or χ^n and arrive at a relation for neighboring values of, e.g., $\|P_{N,l}\Psi^n_N\|^2$ (neighboring in the variable l with l even or l odd, respectively). This relation is then shown to imply the exponential decay in l. We exemplify this for Ψ^0_N: If one takes the scalar product between $H_N\Psi^0_N = E^0_N\Psi^0_N$ and $P_{N,l}\Psi^0_N \equiv \Psi^0_{N,l}$ and uses the upper bound $E^0_N \leq N\hat{v}(0)/2$,[3] one can derive a difference inequality (or recurrence relation) for the values $\{\|\Psi^0_{N,l}\|^2 : l \text{ even}\}$ resp. $\{\|\Psi^0_{N,l}\|^2 : l \text{ odd}\}$, which is similar to

$$\partial_l^2\|\Psi^0_{N,l}\|^2 + 2\|\Psi^0_{N,l}\|^2 \geq \left(\frac{16\pi^2}{\|\hat{v}\|_\infty}\right)\|\Psi^0_{N,l}\|^2. \tag{3.27}$$

Here, $\partial_l^2\|\Psi^0_{N,l}\|^2 = \|\Psi^0_{N,l+2}\|^2 - 2\|\Psi^0_{N,l}\|^2 + \|\Psi^0_{N,l-2}\|^2$. The essential ingredient to find this inequality is the structure of the auxiliary Hamiltonian $\widetilde{H}_N$ together with the fact that $\langle\Psi^0_{N,l}, (H_N - \widetilde{H}_N)\Psi^0_N\rangle$ is comparatively small. One uses in particular that $\widetilde{H}_N$ does not change the number of particles in the condensate by one. The discrete derivative on the l.h.s. comes from the *ppvqq* + h.c. contribution in $\widetilde{H}_N$ which couples $\Psi^0_{N,l}$ with $\Psi^0_{N,l\pm 2}$. Moreover, one uses that the *qpvpq*+h.c. is positive and that $4\pi^2$ is the smallest kinetic energy above zero, i.e., $\langle\Psi^0_{N,l}, -\Delta_1\Psi^0_N\rangle \geq 4\pi^2 l\|\Psi^0_{N,l}\|^2/N$. If we take this difference inequality for granted and assume that $\|\hat{v}\|_\infty < 8\pi^2$, we readily obtain the exponential decay $\|\Psi^0_{N,l}\|^2 \propto e^{-Dl}$ (the inequality is easily solved via an exponential ansatz). Let us emphasize that (3.27) is reminiscent of the time-independent Schrödinger equation of a particle on the one-dimensional lattice with energy equal to 2 inside a potential barrier of height $16\pi^2/\|\hat{v}\|_\infty > 2$. In this situation, the exponential decay of the wave function is well known as the tunneling effect.

This argument is made rigorous in the proof of Theorem 3.1. In particular, we show how the idea can be generalized to $\|\hat{v}\|_\infty \geq 8\pi^2$ and to other low energy eigenfunctions Ψ^n_N, $\widetilde{\Psi}^n_N$ and χ^n_N.

Corollary 3.2. *Let $n \geq 0$ and $m \in \mathbb{N}$ $(m \leq N)$. There exist positive constants C_m and C such that*

[3]The upper bound follows from $E^0_N = \inf_{\|\Psi_N\|=1}\langle\Psi_N, H_N\Psi_N\rangle \leq \langle\varphi_{\mathrm{H}}^{\otimes N}, H\varphi_{\mathrm{H}}^{\otimes N}\rangle = N\hat{v}(0)/2$.

(a)

$$\Big\langle \Psi_N^n, \Big(\prod_{i=1}^m q_i^0 \Big) \Psi_N^n \Big\rangle \le \frac{C_m}{N^m} \Big(1 + (K_N^n)^m \Big), \tag{3.28}$$

(b)

$$\Big\langle \widetilde{\Psi}_N^n, \Big(\prod_{i=1}^m q_i^0 \Big) \widetilde{\Psi}_N^n \Big\rangle \le \frac{C_m}{N^m} \Big(1 + (\widetilde{K}_N^n)^m \Big), \tag{3.29}$$

and

(c)

$$\sum_{l=0}^{\infty} l^m \langle \chi_l^n, \chi_l^n \rangle \le C \Big(1 + (K_{Bog}^n)^m \Big). \tag{3.30}$$

Using (a) and (b) from Corollary 3.2, one can readily derive bounds for the difference of H_N and $\widetilde{H}_N$. For that recall that every term in $H_N - \widetilde{H}_N$ contains three or four q^0's, cf. (3.24).

Corollary 3.3. *Let $n \ge 0$. There is a positive constant C such that*

(a)

$$\Big| \Big\langle \Psi_N^n, (H_N - \widetilde{H}_N) \Psi_N^n \Big\rangle \Big| \le C \Big[\frac{(1 + K_N^n)^{\frac{3}{2}}}{\sqrt{N}} + \frac{(K_N^n)^2}{N} \Big],$$

(b)

$$\Big| \Big\langle \widetilde{\Psi}_N^n, (H_N - \widetilde{H}_N) \widetilde{\Psi}_N^n \Big\rangle \Big| \le C \Big[\frac{(1 + \widetilde{K}_N^n)^{\frac{3}{2}}}{\sqrt{N}} + \frac{(\widetilde{K}_N^n)^2}{N} \Big].$$

3.2.2 Quasifree type properties of the ground state

Let us denote the orthogonal projectors onto the subspaces of $L_s^2(\mathbb{T}^{dN})$ which correspond to an even respectively odd number of particles in the condensate wave function by

$$\widehat{f}^0_{\text{odd}} : L^2(\mathbb{T}^{dN}) \to L^2(\mathbb{T}^{dN}), \quad \Psi_N \mapsto \widehat{f}^0_{\text{odd}} \Psi_N = \sum_{\substack{l=0 \\ l \text{ odd}}}^{N} P_{N,l}^0 \Psi_N, \tag{3.31}$$

$$\widehat{f}^0_{\text{even}} : L^2(\mathbb{T}^{dN}) \to L^2(\mathbb{T}^{dN}), \quad \Psi_N \mapsto \widehat{f}^0_{\text{even}} \Psi_N = \sum_{\substack{l=0 \\ l \text{ even}}}^{N} P_{N,l}^0 \Psi_N, \tag{3.32}$$

They satisfy $\widehat{f}^0_{\text{even}} + \widehat{f}^0_{\text{odd}} = 1$ and $\widehat{f}^0_{\text{even}} \widehat{f}^0_{\text{odd}} = 0$.

Lemma 3.4. *The ground state $\widetilde{\Psi}_N^0$ of $\widetilde{H}_N$ is unique (up to a constant phase factor) and it holds that either*

$$\widehat{f}^0_{even} \widetilde{\Psi}_N^0 = \widetilde{\Psi}_N^0, \quad \widehat{f}^0_{odd} \widetilde{\Psi}_N^0 = 0 \qquad \text{or} \qquad \widehat{f}^0_{even} \widetilde{\Psi}_N^0 = 0, \quad \widehat{f}^0_{odd} \widetilde{\Psi}_N^0 = \widetilde{\Psi}_N^0.$$

Proof. Assuming uniqueness for a moment, the second statement follows from the fact that all interaction terms in $\widetilde{H}_N$ change the number of correlated particles either by zero or by two, as well as that $-\Delta\varphi^0 = 0$. Hence, one finds that

$$\left[\widetilde{H}_N, \widehat{f}^0_{\text{even}}\right] = 0 = \left[\widetilde{H}_N, \widehat{f}^0_{\text{odd}}\right],$$

and thus, knowing that $\widetilde{\Psi}^0_N$ is unique, it follows that all mass is either contained in the even or in the odd sector (otherwise we could construct two ground state eigenfunctions $\widehat{f}^0_{\text{even}}\widetilde{\Psi}^0_N/\|\widehat{f}^0_{\text{even}}\widetilde{\Psi}^0_N\|$ as well as $\widehat{f}^0_{\text{odd}}\widetilde{\Psi}^0_N/\|\widehat{f}^0_{\text{odd}}\widetilde{\Psi}^0_N\|$ contradicting the assumption that the ground state is unique). The proof of uniqueness follows from the following Theorem; see Section 3.3.5 □

Theorem 3.5. *There exists a constant $C > 0$ such that (with appropriately chosen phases; recall that the ground state is defined only up to a constant phase)*

$$\|\Psi^0_N - \widetilde{\Psi}^0_N\| \le \frac{C}{\sqrt{N}}. \tag{3.33}$$

Corollary 3.6. *The ground state wave function Ψ^0_N satisfies the following quasifree type properties:*

1. *It holds that either*

$$\|\widehat{f}^0_{odd}\Psi^0_N\| \le \frac{C}{\sqrt{N}} \qquad or \qquad \|\widehat{f}^0_{even}\Psi^0_N\| \le \frac{C}{\sqrt{N}}. \tag{3.34}$$

2. *For any integer $m \le N$ and some positive constant C_m,*

$$\left\langle \Psi^0_N, \Big(\prod_{i=1}^m q^0_i\Big)\Psi^0_N\right\rangle \le \frac{C_m}{N^m}. \tag{3.35}$$

The second statement follows from Corollary 3.2 for $n = 0$. The first one is an immediate consequence from Theorem 3.5 and Lemma 3.4. The reason why we call this "quasifree type properties" is that they are reminiscent of (3.20) and (3.21), as explained thereafter. For that, recall the relation in (3.13) and note that the operator Nq^0 is the analogue of the number operator $\mathcal{N}$ on the Fock space (3.10). Moreover, it is known, and we recall this in Appendix 3.A, that $\langle\Psi^0_N, (Nq^0_1)\Psi^0_N\rangle \le C$ such that (3.35) expresses a similar "factorization property" as (3.21).

Remark 3.1. In Chapter 2, we have studied the time evolution of the many-body Bose gas by approximating it with the dynamics generated by a Hamiltonian $\widetilde{H}^t_N$ analogous to the one defined in (3.23) (with the obvious modifications for the time-dependent setting and for more general situations including a nonzero external potential). The optimal error term for the time-dependent approximation holds for wave functions satisfying initially certain properties which were summarized in Assumption 2.2. Corollary 3.6 shows that the ground state of the homogeneous gas confined to a box satisfies properties A.2.2 and A.2.3. That it satisfies A.2.1 can be seen from Theorem 3.7.

3.2.3 Low-lying energy eigenvalues

Another consequence of Theorem 3.1 is that the ground state energy E_N^0 and also the excitation energies K_N^n defined in (3.25) converge in the large N limit to the Bogoliubov energy E_{Bog} resp. the Bogoliubov excitations K_{Bog}^n.

Theorem 3.7. *Let E_{Bog} as in (3.6). There is a constant C such that*

$$\Big|E_N^0 - \Big(\frac{N\hat{v}(0)}{2} + E_{Bog}\Big)\Big| \leq \frac{C}{N}. \tag{3.36}$$

Let further $n \geq 1$ and K_{Bog}^n as in (3.8). Then, there exists a constant C such that

$$\Big|K_N^n - K_{Bog}^n\Big| \leq C\Big[\frac{1}{\sqrt{N}} + \frac{(K_{Bog}^n)^{\frac{3}{2}}}{\sqrt{N}} + \frac{K_{Bog}^n}{N} + \frac{(K_{Bog}^n)^2}{N}\Big]. \tag{3.37}$$

The statement of the theorem coincides with some part of [116, Theorem 1]. We emphasize, however, that our statement is valid only for fixed values of n, whereas [116, Theorem 1] proves convergence of excitation energies up to $K^n \ll N$.

Remark 3.2. It is an open problem to derive the Bogoliubov approximation for the excitation spectrum in the thermodynamic limit, i.e., for $N, L \to \infty$, $\rho = (N-1)/L^d = const.$ In this setup the coupling constant $1/(N-1)$ in (3.1) needs to be replaced by $L^d/(N-1) = 1/\rho$ and Bogoliubov theory is expected to become accurate in the large ρ limit. Such a model is of particular interest because the momentum becomes a continuous variable in the thermodynamic limit which is a crucial ingredient in the macroscopic theory of, e.g., superfluidity. We refer the reader for more details to [36, 37]. In [36] Dereziński and Napiórkowski study a so-called mean field large volume limit, namely $N, L \to \infty$ while $L^a \ll \rho$ for some number $a > 0$. This can be considered as an intermediate step towards the true thermodynamic limit. Let us note that at least for the ground state energy, our approach can be applied without major modifications also to this type of limit, leading to a similar result as obtained in [37, Theorem 1.1].

Our strategy for proving Therorem 3.7 is to show via an appropriate comparison of $\mathbb{H}_{\mathrm{Bog}}$ with $\widetilde{H}_N$ that $K_{\mathrm{Bog}}^n \approx \widetilde{K}_N^n$, and then use Corollary 3.2 for proving $\widetilde{K}_N^n \approx K_N^n$. For the first step, we argue that the auxiliary Hamiltonian $\widetilde{H}_N$ is unitarily equivalent to an operator

$$\frac{N\hat{v}(0)}{2} + \widetilde{\mathbb{H}}_N^{\leq N} \tag{3.38}$$

acting on functions in $\mathcal{F}_s^{\leq N} \subset \mathcal{F}_s$ (the first $N+1$ sectors of the Fock space $\mathcal{F}_s$). Extending $\widetilde{\mathbb{H}}_N^{\leq N}$ trivially to the whole Fock space (and denoting this extension by $\widetilde{\mathbb{H}}_N$), we derive a bound similar to

$$-\frac{C}{N}\mathcal{N}^2 + \text{small} \leq \widetilde{\mathbb{H}}_N - \mathbb{H}_{\mathrm{Bog}} \leq \frac{C}{N}\mathcal{N}^2 + \text{small} \tag{3.39}$$

where $\mathcal{N}$ denotes the particle number operator on $\mathcal{F}_s$. We then apply the so-called min-max principle which states that the eigenvalues of a self-adjoint operator A (acting on a Hilbert space $\mathcal{H}$) below the essential spectrum, counted with multiplicity in increasing order, are equal to the min-max values $\{\mu_n(A)\}_{n\geq 0}$ defined through

$$\mu_n(A) = \min_{Y^{n+1}} \max_{\psi \in Y^{n+1},\ \|\psi\|_{\mathcal{H}}=1} \langle \psi, A\psi \rangle_{\mathcal{H}}, \tag{3.40}$$

where the infimum is taken over all $n+1$-dimensional subspaces $Y^{n+1} \subset \mathcal{H}$. Note that in our notation, μ_n is the n+1-th eigenvalue since we start from $n = 0$. If there exists only a finite number M of eigenvalues below the bottom of the essential spectrum, then $\mu_n(A) = \inf \sigma_{ess}(A)$ for all $n > M$. For $A \approx B$ with some other self-adjoint operator B that has eigenvalues $\{\mu_n(B)\}_{n\geq 0}$, one can use (3.40) to find an upper bound for $\mu_n(A)$ in terms of the eigenvalue $\mu_n(B)$ and the explicit error

$$\max_{\psi \in Y_B^{n+1},\ \|\psi\|_{\mathcal{H}}=1} \langle \psi, (A-B)\psi \rangle_{\mathcal{H}}. \tag{3.41}$$

Here $Y_B^{n+1} \subset \mathcal{H}$ is the subspace spanned by the first $n+1$ eigenfunctions of B. Vice versa, one obtains an upper bound for $\mu_n(B)$ in terms of $\mu_n(A)$ + error where the error is given by the same expression as in (3.41) with operators A and B interchanged. Using the same argument in combination with Lemma 3.3, one also proves $\widetilde{K}^n \approx K^n$.

3.3 Proofs

For notational convenience, we omit the subscript N throughout the following sections.

We first note some important properties of the projectors $P^0_{N,l}$ which are easily verified using their definition in (3.26) (cf. Definition 2.11 and Eq. (2.97) with φ_t replaced by φ^0):

1. $P^0_{N,l}$ is an orthogonal projector,
2. $P^0_{N,l}P^0_{N,l'} = \delta_{ll'}P^0_{N,l}$,
3. $1 = \sum_{l=0}^N P^0_{N,l}$,
4. $[p^0_i, P^0_{N,l}] = 0 = [q^0_i, P^0_{N,l}]$,
5. $\frac{1}{N}\sum_{i=1}^N q^0_i = \sum_{l=0}^N \frac{l}{N} P^0_{N,l}$.

From assertions 1, 2 and 5, we directly obtain an important relation that we frequently use throughout the following proofs, namely, that for any symmetric wave function Ψ_N,

$$\|q^0_1(P^0_{N,l}\Psi)\|^2 = \langle P^0_{N,l}\Psi, \Big(\sum_{k=0}^N \frac{k}{N}P^0_{N,k}\Big)P^0_{N,l}\Psi\rangle = \frac{l}{N}\|P^0_{N,l}\Psi\|^2, \tag{3.42}$$

and similarly also for the product $q^0_1 q^0_2$.

3.3.1 Proof of Theorem 3.1

(a) We first show that for some $\delta \in (0,1]$,[4]

$$\|P^0_{N,l}\Psi^n\|^2 \leq Ce^{-Dl} \quad \text{for all} \quad K^n \leq l \leq \lfloor \delta N/2 \rfloor\,. \tag{3.43}$$

Then, we use this estimate to derive the bound also for the remaining values of l:

$$\|P^0_{N,l}\Psi^n\|^2 \leq Ce^{-DN} \quad \text{for all} \quad \lfloor \delta N/2 \rfloor \leq l \leq N. \tag{3.44}$$

[4] By $\lfloor \cdot \rfloor$ we indicate the floor function, i.e., $\lfloor \cdot \rfloor : \mathbb{R}^+_0 \to \mathbb{N}$ with $\lfloor x \rfloor$ being the largest integer less than or equal to the real number x.

Remark 3.3. For the proof of (3.44), we assume in addition that $K^n \equiv K^n_N \ll N$. However, we emphasize that this does not pose a further restriction on the number n, as we consider only fixed n. Going through the part of the proof of Theorem 3.37 where we derive the upper bounds for $\mu_n(\widetilde{H})$ in terms of $\mu_n(\mathbb{H}_{\text{Bog}})$ resp. for $\mu_n(H)$ in terms of $\mu_n(\widetilde{H})$, it can be verified that K^n is of order one w.r.t. N whenever K^n_{Bog} is of order one (which is the case for all fixed n as K^n_{Bog} is per definition N-independent).

In order to show (3.43) let us introduce the abbreviation $\Psi_l \equiv P^0_{N,l}\Psi$ and the constant

$$b_{\delta,l^*} = \frac{4}{\sqrt{l^*}} + \Big(\frac{4\|\hat{v}\|_1}{\|\hat{v}\|_\infty}\Big)\sqrt{\delta} \tag{3.45}$$

for $\delta \in (0,1]$, $l^* \in \mathbb{N}$. Moreover, we need the following two lemmas.

Lemma 3.8. *Let $l^* \leq N$, $\delta \in (0,1]$ and $\Psi \in L^2_s(\mathbb{T}^{dN})$. Then, for all $l^*+2 \leq l \leq \lfloor \delta N \rfloor - 2$,*

$$N\Big|\big\langle \Psi_l, p^0_1 p^0_2 v_{12} q^0_1 q^0_2 \Psi\big\rangle\Big| \leq \frac{N}{2}\sum_{k\neq 0}\hat{v}(k)\Big(\|p^k_1 p^0_2 \Psi_{l+2}\|^2 + \|p^k_1 p^0_2 \Psi_l\|^2\Big) + b_{\delta,l^*}\|\hat{v}\|_\infty\Big(l\|\Psi_l\|^2 + (l+2)\|\Psi_{l+2}\|^2\Big), \tag{3.46}$$

$$N\Big|\big\langle \Psi_l, q^0_1 q^0_2 v_{12} p^0_1 p^0_2 \Psi\big\rangle\Big| \leq \frac{N}{2}\sum_{k\neq 0}\hat{v}(k)\Big(\|p^k_1 p^0_2 \Psi_{l-2}\|^2 + \|p^k_1 p^0_2 \Psi_l\|^2\Big) + b_{\delta,l^*}\|\hat{v}\|_\infty\Big(l\|\Psi_l\|^2 + (l-2)\|\Psi_{l-2}\|^2\Big). \tag{3.47}$$

Lemma 3.9. *Let $\delta \in (0,1]$, $\Psi \in L^2_s(\mathbb{T}^{dN})$ and $\widetilde{V}^{rest} \equiv H - \widetilde{H}$ as in (3.24). Then, for all integers $l \leq \lfloor \delta N \rfloor - 1$,*

$$\Big|\big\langle \Psi_l, \widetilde{V}^{rest}\Psi\big\rangle - \big\langle \Psi_l, q^0_1 q^0_2 v_{12} q^0_1 q^0_2 \Psi\big\rangle\Big| \leq b_{\delta,l^*}\|\hat{v}\|_\infty\Big((l-1)\|\Psi_{l-1}\|^2 + 2l\|\Psi_l\|^2 + (l+1)\|\Psi_{l+1}\|^2\Big). \tag{3.48}$$

Proof of (3.43). We abbreviate $g^n_l \equiv l\|\Psi^n_l\|^2$ and set $\sqrt{\delta} = \min\{\frac{\pi^2}{400\|\hat{v}\|_1}, 1\}$ and

$$l^* = \max\Big\{\Big\lfloor\Big(\frac{200\|\hat{v}\|_\infty}{\pi^2}\Big)^2\Big\rfloor, \Big\lfloor\frac{25K^n}{4\pi^2}\Big\rfloor\Big\}.$$[5]

The argument is divided into three steps: 1) We derive an inequality similar to (3.27). For $\|\hat{v}\|_\infty < 8\pi^2$, it would directly imply (3.27) and we could proceed as explained thereafter. For the general case, we keep additional negative terms on the l.h.s. of (3.27). 2) Those negative terms are used in order to derive a more suitable relation. This new relation, however, does not hold for the numbers $l\|\Psi_l\|^2$ but for appropriately chosen sums of the $l\|\Psi_l\|^2$ (due to the negative terms, this leads to cancellations on the l.h.s. of (3.27) whereas on the r.h.s. there are no cancellations). This procedure leads to a suitable generalization of (3.27), namely the difference inequality (3.65). 3) We solve the difference inequality via an exponential ansatz. Together with the normalization condition $\|\Psi^n\| = 1$, this will imply (3.43).

Step 1. For all integers $l^*+2 \leq l \leq \lfloor \delta N \rfloor - 2$, the following inequality holds:

$$a^n_{\delta,l^*} g^n_l \leq \frac{N}{4}\sum_{k\neq 0}\frac{\hat{v}(k)}{\|\hat{v}\|_\infty}\Big(\|p^k_1 p^0_2 \Psi^n_{l+2}\|^2 + \|p^k_1 p^0_2 \Psi^n_{l-2}\|^2 - 2\|p^k_1 p^0_2 \Psi^n_l\|^2\Big) \tag{3.49}$$

[5]Note that the choices of δ and l^* are not optimal and many other examples would work as well.

$$+ b_{\delta,l^*}\Big(g^n_{l-2} + g^n_{l-1} + g^n_{l+1} + g^n_{l+2}\Big),$$

where

$$a^n_{\delta,l^*} = \frac{1}{\|\hat{v}\|_\infty}\Big(4\pi^2 - \frac{K^n}{l^*}\Big) - 4b_{\delta,l^*}. \tag{3.50}$$

Note that for the chosen values of δ and l^*, the number $a^n_{\delta,l^*} > 0$ is strictly positive. To derive (3.49), we start from the eigenvalue equation for Ψ^n, $H\Psi^n = E^n\Psi^n$, and recall that the energy $E^n = E^0 + K^n$ is bounded from above by $N\hat{v}(0)/2 + K^n$. Taking the scalar product with Ψ^n_l, using $H = \widetilde{H} + \widetilde{V}^{\text{rest}}$, leads to

$$\begin{aligned} &N\big\langle\Psi^n_l, (-\Delta_1)\Psi^n\big\rangle - K^n\|\Psi^n_l\|^2 + N\big\langle\Psi^n_l, p^0_1q^0_2v_{12}q^0_1p^0_2\Psi^n_l\big\rangle \\ &\quad\le -\frac{N}{2}\big\langle\Psi^n_l, p^0_1p^0_2v_{12}q^0_1q^0_2\Psi^n_{l+2}\big\rangle - \frac{N}{2}\big\langle\Psi^n_l, q^0_1q^0_2v_{12}p^0_1p^0_2\Psi^n_{l-2}\big\rangle - \big\langle\Psi^n_l, \widetilde{V}^{\text{rest}}\Psi^n\big\rangle. \end{aligned} \tag{3.51}$$

The upper line can be computed explicitly: Since $p^0_1 + q^0_1 = 1$, $\Delta\varphi^0 = 0$, $q^0_1 = \sum_{k\neq 0}|\varphi^k\rangle\langle\varphi^k|_1$ and $\|q^0_1\Psi^n_l\|^2 = \frac{l}{N}\|\Psi^n_l\|^2$, one obtains for all $l \ge l^*$,

$$N\big\langle\Psi^n_l, (-\Delta_1)\Psi^n\big\rangle - K^n\|\Psi^n_l\|^2 \ge 4\pi^2N\|q^0_1\Psi^n_l\|^2 - K^n\|\Psi^n_l\|^2 \ge \Big(4\pi^2 - \frac{K^n}{l^*}\Big)l\|\Psi^n_l\|^2 > 0, \tag{3.52}$$

and with the Fourier decomposition of the potential, $v(x) = \sum_k \hat{v}(k)e^{ikx}$,

$$\begin{aligned} \big\langle\Psi^n_l, p^0_1q^0_2v_{12}q^0_1p^0_2\Psi^n_l\big\rangle &= \sum_{k\neq 0}\hat{v}(k)\big\langle\Psi^n_l, p^0_1q^0_2e^{ik(x_1-x_2)}q^0_1p^0_2\Psi^n_l\big\rangle \\ &= \sum_{k\neq 0}\hat{v}(k)\big\langle\Psi^n_l, \big(|\varphi^0\rangle\langle\varphi^k|\big)_1\big(|\varphi^k\rangle\langle\varphi^0|\big)_2\Psi^n_l\big\rangle = \sum_{k\neq 0}\hat{v}(k)\|p^k_1p^0_2\Psi^n_l\|^2. \end{aligned} \tag{3.53}$$

In the last step we have used symmetry of Ψ^n_l under permutation of coordinates in order to exchange the integration variables. Since the upper line in (3.51) is thus positive (recall that $\hat{v} \ge 0$), and since also $N\big\langle\Psi^n_l q^0_1q^0_2v_{12}q^0_1q^0_2\Psi^n_l\big\rangle \ge 0$ (since $v \ge 0$), we can apply Lemmas 3.8 and 3.9 in order to bound the lower line and obtain the stated inequality in (3.49).

For $m \in \mathbb{N}$, let $f^n_{j,m}$ denote the arithmetic average of the $2m+1$ numbers $g^n_{j-m}, g^n_{j-m+1}, \ldots, g^n_{j+m}$, i.e.,

$$f^n_{j,m} = \frac{1}{2m+1}\sum_{l=j-m}^{j+m} g^n_l. \tag{3.54}$$

Our next goal is to show that for certain values of j (and for sufficiently large but N-independent m), the $f^n_{j,m}$ satisfy a difference inequality similar to the one in (3.27). This will be used to derive the exponential decay in the third step.

Step 2. Take the sum of both sides in (3.49) for l running from $j-m$ up to $j+m$ ($m \ge 0$ and $l^* + 2 + m \le j \le \lfloor\delta N\rfloor - 2 - m$). The left side gives

$$\sum_{l=j-m}^{j+m}\Big(a^n_{\delta,l^*}g^n_l\Big) = a^n_{\delta,l^*}(2m+1)f^n_{j,m}. \tag{3.55}$$

In the first line on the r.h.s. of (3.49), it is essential that we have a "telescoping sum" and thus cancellations up to the boundary terms. This leads to

$$\sum_{l=j-m}^{j+m}\Big[\frac{N}{4}\sum_{k\neq 0}\frac{\hat v(k)}{\|\hat v\|_\infty}\Big(\|p_1^k p_2^0\Psi_{l+2}^n\|^2-\|p_1^k p_2^0\Psi_l^n\|^2+\|p_1^k p_2^0\Psi_{l-2}^n\|^2-\|p_1^k p_2^0\Psi_l^n\|^2\Big)\Big]$$
$$=\frac{N}{4}\sum_{k\neq 0}\frac{\hat v(k)}{\|\hat v\|_\infty}\Big(\|p_1^k p_2^0\Psi_{j+m+2}^n\|^2-\|p_1^k p_2^0\Psi_{j+m}^n\|^2+\|p_1^k p_2^0\Psi_{j-m-2}^n\|^2-\|p_1^k p_2^0\Psi_{j-m}^n\|^2\Big)$$
$$\leq\frac{N}{4}\Big(\|q_1^0 p_2^0\Psi_{j+m+2}^n\|^2+\|q_1^0 p_2^0\Psi_{j-m-2}^n\|^2\Big)\leq g_{j+m+2}^n+g_{j-m-2}^n, \tag{3.56}$$

where we have discarded the two negative terms from the second line, then used $\sum_{k\neq 0}p_1^k=q_1^0$, and further $\|q_1^0p_2^0\Psi_l^n\|^2\leq\frac{l}{N}\|\Psi_l^n\|^2$. Summing over the second line of the r.h.s. in (3.49), we find

$$\sum_{l=j-m}^{j+m}\Big[b_{\delta,l^*}\Big(g_{l-2}^n+g_{l-1}^n+g_{l+1}^n+g_{l+2}^n\Big)\Big]$$
$$\leq 2b_{\delta,l^*}(2m+1)f_{j,m}^n+b_{\delta,l^*}\Big(g_{j-m-2}^n+g_{j-m-1}^n+g_{j+m+1}^n+g_{j+m+2}^n\Big). \tag{3.57}$$

Together, this leads to the relation

$$\Big(\frac{a_{\delta,l^*}^n-2b_{\delta,l^*}}{1+b_{\delta,l^*}}\Big)(2m+1)f_{j,m}^n\leq\Big(g_{j-m-2}^n+g_{j-m-1}^n+g_{j+m+1}^n+g_{j+m+2}^n\Big), \tag{3.58}$$

being valid for all $l^*+2+m\leq j\leq\lfloor\delta N\rfloor-2-m$ $(m\geq 0)$. It is important to note that the first factor on the l.h.s. is still strictly positive (which can be seen, using, e.g., $a_{\delta,l^*}^n\geq 8\pi^2/(50\|\hat v\|_\infty)$ and $b_{\delta,l^*}\leq 2\pi^2/(50\|\hat v\|_\infty)$):

$$c_{\delta,l^*}^n\equiv\frac{a_{\delta,l^*}^n-2b_{\delta,l^*}}{1+b_{\delta,l^*}}\geq\Big(1+\frac{50\|\hat v\|_\infty}{4\pi^2}\Big)^{-1}>0. \tag{3.59}$$

Let $m^*=8\lfloor 1+50\|\hat v\|_\infty/4\pi^2\rfloor+1$, and compute

$$(2m^*+1)\Big(f_{j-m^*,m^*}^n+f_{j+m^*,m^*}^n\Big)=\sum_{l=j-2m^*}^{j}g_l^n+\sum_{l=j}^{j+2m^*}g_l^n\geq\sum_{l=j-2m^*}^{j-m-1}g_l^n+\sum_{l=j+m^*+1}^{j+2m^*}g_l^n, \tag{3.60}$$

for $l^*+2+2m^*\leq j\leq\lfloor\delta N\rfloor-2-2m^*$. Using (3.58) we obtain a lower bound for the r.h.s. For that, sort all terms into groups of four. E.g., the two first and the two last,

$$\Big(g_{j-2m^*}^n+g_{j-2m^*+1}^n\Big)+\Big(g_{j+2m^*-1}^n+g_{j+2m^*}^n\Big)\geq c_{\delta,l^*}^n(4m^*-3)f_{j,2m^*-2}^n \tag{3.61}$$

following from (3.58) with $m=2m^*-2$. Equivalently, for i even, $2\leq i\leq m^*-2$ (with $m=2m^*-i-2$), one finds

$$\Big(g_{j-2m^*+i}^n+g_{j-2m^*+i+1}^n\Big)+\Big(g_{j+2m^*-i-1}^n+g_{j+2m^*-i}\Big)\geq c_{\delta,l^*}^n(4m^*-2i-3)f_{j,2m^*-i-2}^n. \tag{3.62}$$

Putting everything together, it follows from (3.60) that

$$
\begin{aligned}
f^n_{j-m^*,m^*} + f^n_{j+m^*,m^*} &\geq \frac{1}{2m^*+1}\Big(\sum_{\substack{i=0\\ i \text{ even}}}^{m^*-2} c^n_{\delta,l^*}(4m^*-2i-3) f^n_{j,2m^*-i-2}\Big) \\
&= c^n_{\delta,l^*} \sum_{\substack{i=0,\\ i \text{even}}}^{m^*-2} \Big(\frac{1}{2m^*+1} \sum_{l=j-(2m^*-i-2)}^{j+(2m^*-i-2)} g^n_l\Big) \\
&\geq c^n_{\delta,l^*} \sum_{\substack{i=0\\ i \text{ even}}}^{m^*-2} \Big(\frac{1}{2m^*+1} \sum_{l=j-m^*}^{j+m^*} g^n_l\Big) = c^n_{\delta,l^*}\frac{(m^*-1)}{2} f^n_{j,m^*}.
\end{aligned} \tag{3.63}
$$

The choice of m^* ensures that

$$
c^n_{\delta,l^*}\frac{(m^*-1)}{2} \geq \Big(1+\frac{50\|\hat{v}\|_\infty}{4\pi^2}\Big)^{-1}\frac{(m^*-1)}{2} > 4. \tag{3.64}
$$

In particular, the l.h.s. is larger than 2 uniformly in N. For simplicity we further assume that N is such that the integer $\lfloor \delta N\rfloor - l^* - 4 - 2m^*$ is multiple of m^* (the argument is easily applied to the general case as well). The number m^* then divides the tuple $(g^n_{l^*+2}, ..., g^n_{\lfloor \delta N\rfloor -2})$ into $M+1$ (for some $M \in \mathbb{N}$) partlially overlapping blocks (subtuples), each with length $2m^*+1$ and centered around the values $g^n_{l^*+2+im^*}$, $i = 1, ..., M+1$:

$$
\Big(\overbrace{g^n_{l^*+2}, ..., \underbrace{g^n_{l^*+2+m^*}, ..., g^n_{l^*+2+2m^*}}_{}}^{\text{block 1}}, ..., g^n_{l^*+2+3m^*}, ..., g^n_{l^*+2+4m^*}, \dots \; ..., \overbrace{g^n_{\lfloor\delta N\rfloor-2-2m^*}, ..., g^n_{\lfloor\delta N\rfloor-2}}^{\text{block M+1}}\Big).
$$

(block 2 spans $g^n_{l^*+2+m^*}, ..., g^n_{l^*+2+3m^*}$; block 3 spans $g^n_{l^*+2+2m^*}, ..., g^n_{l^*+2+4m^*}$.)

If we denote the arithmetic average of the elements of each such block by h^n_{i,m^*}, $i = 1, ..., M+1$ (in other words, we set $h^n_{i,m^*} = f^n_{l^*+2+im^*,m^*}$), it follows from (3.63) that the h^n_{i,m^*} satisfy the difference inequality

$$
h^n_{i-1,m^*} + h^n_{i+1,m^*} \geq c^n_{\delta,l^*}\Big(\frac{m^*-1}{2}\Big) h^n_{i,m^*}, \qquad i = 2, ..., M, \tag{3.65}
$$

with $c \equiv c^n_{\delta,l^*}\frac{m^*-1}{2} > 4$. Reading the inequality as $\partial_i^2 h^n_{i,m^*} \geq (c-2)h^n_{i,m^*}$, this should be seen as the correct generalization of (3.27).

Step 3. We now use this relation to derive the exponential decay of the h^n_{i,m^*} as functions of the variable $i \in \{2, ..., M\}$. One solves (3.65) via the exponential ansatz $h^n_{i,m^*} = x^i$, $x > 0$, i.e.,

$$
x^2 - cx + 1 = \Big(x - \frac{c}{2}\Big)^2 - \Big(\frac{c^2}{4} - 1\Big) \geq 0 \quad \Leftrightarrow \quad \Big|x - \frac{c}{2}\Big| \geq \sqrt{\frac{c^2}{4} - 1}, \tag{3.66}
$$

with two possible intervals of solutions, namely

$$
x_+ \geq \frac{c}{2} + \sqrt{\frac{c^2}{4} - 1}, \qquad x_- \leq \frac{c}{2} - \sqrt{\frac{c^2}{4} - 1}, \tag{3.67}
$$

(since $c > 2$, the number below the root is strictly positive). One easily finds that $x_+ > 1$ and $x_- < 1$. Thus, x_+ corresponds to an exponentially growing function whereas x_- describes an exponential decay in the variable i. Here, we are interested only in x_- because the function h^n_{i,m^*} must fulfill the following normalization condition,

$$h^n_{M,m^*} \leq \sum_{i=1}^{M} h^n_{i,m^*} \leq C \sum_{l=0}^{N} l \|\Psi^n_l\|^2 \leq CN. \tag{3.68}$$

For the exponentially increasing solution, we would have that for N-independent positive constants C and D,

$$Ce^{D\delta N} \leq Ce^{DM} \leq h^n_{M,m^*} \leq CN, \tag{3.69}$$

which is false for large N. The solution to the difference inequality (3.65) (satisfying the required normalization condition) is then exponentially bounded from above, namely

$$h^n_{i,m^*} \leq C \left[\frac{c}{2} - \sqrt{\frac{c^2}{4} - 1}\right]^i = Ce^{-Di}, \qquad 2 \leq i \leq M, \tag{3.70}$$

with $D = -\ln\left[\frac{c}{2} - \sqrt{\frac{c^2}{4} - 1}\right] > 0$ since $c > 2$. By definition of the h^n_{i,m^*}, it follows immediately that

$$\|\Psi^n_l\|^2 \leq Ce^{-Dl}, \qquad l^* + 2 + m^* \leq l \leq \lfloor \delta N \rfloor - 2 - m^*. \tag{3.71}$$

Recalling that δ, m^* and l^* are N-independent, this proves (3.43) when N is taken sufficiently large.

Proof of (3.44). Define

$$\Psi^n_r = \sum_{l=\lfloor \frac{\delta N}{2} \rfloor + 1}^{N} P^0_{N,l}\ \Psi^n, \qquad \Psi^n_e = \Psi^n - \Psi^n_r, \qquad \text{and} \qquad \Psi^n_{r,no} = \frac{\Psi^n_r}{\|\Psi^n_r\|} \tag{3.72}$$

(note that whenever $\Psi^n_r \equiv \Psi^n_{N,r} = 0$, there is nothing we need to show and hence we can restrict the argument to a subsequence with $\|\Psi^n_{N,r}\| > 0$). The idea of the proof is to use (3.43) for showing that the product $\|\Psi^n_r\| \langle \Psi^n_{r,no}, (H - E^n)\Psi^n_{r,no} \rangle$ is exponentially small in N. Then, one notes, using the fact that Ψ^n_r contains a nonvanishing fraction of particles outside the condensate φ^0, that the average energy of $\Psi^n_{r,no}$ can not be close to E^n, which is true only for states in which the majority of particles has condensated. Hence, $\|\Psi^n_r\|$ needs to be exponentially small which gives (3.44). Let us explain this in detail.

We start with $n = 0$. It follows from

$$0 = \langle \Psi^0, (H - E^0)\Psi^0 \rangle \geq \langle \Psi^0_r, (H - E^0)\Psi^0_r \rangle + 2\,\mathrm{Re}\,\langle \Psi^0_e, (H - E^0)\Psi^0_r \rangle \tag{3.73}$$

(which is true since $H - E^0 \geq 0$) that

$$\|\Psi^0_r\|^2 \langle \Psi^0_{r,no}, (H - E^0)\Psi^0_{r,no} \rangle \leq Ce^{-DN}. \tag{3.74}$$

To see this note that H couples only the "neighbouring" terms of Ψ_e and Ψ_r, and thus

$$\begin{aligned}\left|\langle \Psi^0_e, (H - E^0)\Psi^0_r \rangle\right| &\leq 2N \left|\langle \Psi^0_e, q^0_1 p^0_2 v_{12} q^0_1 q^0_2 \Psi^0_r \rangle\right| + 2N \left|\langle \Psi^0_e, p^0_1 p^0_2 v_{12} q^0_1 q^0_2 \Psi^0_r \rangle\right| \\ &\leq CN \Big(\|\Psi^0_{\lfloor \frac{\delta N}{2} \rfloor - 1}\| + \|\Psi^0_{\lfloor \frac{\delta N}{2} \rfloor}\| \Big) \|\Psi^0_r\| \leq Ce^{-DN},\end{aligned} \tag{3.75}$$

by means of (3.24), the definitions of Ψ_r^0, Ψ_e^0 and (3.43). The vanishing of the second factor on the left side in (3.74) for large N would imply that $\Psi_{r,no}^0$ were close to the actual ground state which is not correct. In order to show this, we set

$$\Psi_{r,>}^0 = \Psi_{r,no}^0 - \sum_{i=0}^{m} \alpha_r^{i0}\Psi^i, \qquad \alpha_r^{i0} = \langle \Psi^i, \Psi_{r,no}^0\rangle, \tag{3.76}$$

such that $\Psi^i \perp \Psi_{r,>}^0$ for all $0 \le i \le m$. The number m is chosen as the smallest integer such that $E^{m+1} - E^0 \ge \varepsilon > 0$ (for some small $\varepsilon > 0$ and all large N). It then follows that

$$\langle \Psi_{r,no}^0, H\Psi_{r,no}^0\rangle \ge \sum_{i=0}^{m} E^i|\alpha_r^{i0}|^2 + E^{m+1}\|\Psi_{r,>}^0\|^2, \tag{3.77}$$

and thus in particular, using $|\alpha_r^{10}|^2 + \ldots + |\alpha_r^{m0}|^2 + \|\Psi_{r,>}^0\|^2 = 1$,

$$\langle \Psi_{r,no}^0, (H - E^0)\Psi_{r,no}^0\rangle \ge (E^{m+1} - E^0)\|\Psi_{r,>}^0\|^2 \ge \varepsilon\Big\|\Psi_{r,no}^0 - \sum_{i=0}^{m} \alpha_r^{i0}\Psi^i\Big\|^2. \tag{3.78}$$

Recalling the definition of $\Psi_{r,no}^0$ and the identitiy $\sum_{l=0}^{N} P_{N,l}^0 = 1$ with $P_{N,l}^0 P_{N,l'}^0 = P_{N,l}^0\delta_{ll'}$, one also finds

$$\Big\|\Psi_{r,no}^0 - \sum_{i=0}^{m} \alpha_r^{i0}\Psi^i\Big\|^2 \ge \Big\|\sum_{l=0}^{\lfloor \delta N/2\rfloor} P_{N,l}^0\Big(\sum_{i=0}^{m}\alpha_r^{i0}\Psi^i\Big)\Big\|^2 = \sum_{l=0}^{\lfloor \delta N/2\rfloor}\Big\|P_{N,l}^0\Big(\sum_{i=0}^{m}\alpha_r^{i0}\Psi^i\Big)\Big\|^2, \tag{3.79}$$

and, moreover, using the abbreviation $\Phi = \alpha_r^{00}\Psi^0 + \ldots + \alpha_r^{m0}\Psi^m$,

$$(3.79) = 1 - \sum_{l=\lfloor \frac{\delta N}{2}\rfloor + 1}^{N} \|P_{N,l}^0\Phi\|^2 \ge 1 - \frac{2}{\delta}\Big(\sum_{l=0}^{N}\frac{l}{N}\|P_{N,l}^0\Phi\|^2\Big) \ge 1 - \frac{C}{N^a} \tag{3.80}$$

for some $a > 0$. The last step follows from Eq. (3.167), Appendix 3.A and corresponds to the physical fact that the majority of particles in Ψ^m occupies the condensate wave function φ_{H}. This is explained in more detail in Appendix 3.A (here we use that $K^n = O(1)$ for fixed n; cf. Remark (3.3)). Summarizing the different steps, we have found that

$$\langle \Psi_{r,no}^0, (H - E^0)\Psi_{r,no}^0\rangle \ge C\Big(1 - \frac{1}{N^a}\Big), \tag{3.81}$$

which together with (3.74) implies that $\|\Psi_r^0\|^2 = \sum_{l>\delta N/2}^{N} \|P_{N,l}^0\Psi^0\|^2 \le Ce^{-DN}$. This proves (3.44) for $n = 0$.

Next, we fix $n \ge 1$, and assume (3.44) to hold for all $m < n$. We start again from

$$0 = \langle \Psi_r^n, (H - E^n)\Psi_r^n\rangle + \langle \Psi_e^n, (H - E^n)\Psi_e^n\rangle + 2\,\mathrm{Re}\,\langle \Psi_e^n, (H - E^n)\Psi_r^n\rangle \tag{3.82}$$

The bound for $|\langle \Psi_e^n, (H - E^n)\Psi_r^n\rangle|$ is derived as above. However, $H - E^n$ is no more positive such that one needs to control first the negative contributions of the two other terms: Let

$$\Phi_{e,>}^n = \Psi_e^n - \sum_{i=0}^{n} \beta_e^{in}\Psi^i, \qquad \beta_e^{in} = \langle \Psi^i, \Psi_e^n\rangle, \tag{3.83}$$

s.t. $\Psi^i \perp \Phi^n_{e,>}$ for all $0 \le i \le n$. Then,

$$\big\langle \Psi^n_e, (H - E^n)\Psi^n_e \big\rangle \ge \sum_{i=0}^{n-1} |\beta^{in}_e|^2 (E^i - E^n) + \|\Phi^n_{e,>}\|^2 (E^{n+1} - E^n) \ge -Ce^{-DN} \tag{3.84}$$

because $E^i - E^n < 0$ for $i < n$ and (recall $\Psi_e = \sum_{l \le \lfloor \delta N/2 \rfloor} P^0_{N,l}\Psi$ and $\Psi^i_e - \Psi^i = \Psi^i_r$)

$$|\beta^{in}_e|^2 = \big|\big\langle \Psi^i, \Psi^n_e \big\rangle\big|^2 = \big|\big\langle \Psi^i_e, \Psi^n \big\rangle\big|^2 = \big|\big\langle \Psi^i_e - \Psi^i, \Psi^n \big\rangle\big|^2 \le \|\Psi^i_r\|^2 \le Ce^{-DN} \tag{3.85}$$

for all $i < n$. Similarly, for

$$\Phi^n_{r,>} = \Psi^n_r - \sum_{i=0}^{n} \beta^{in}_r \Psi^i, \qquad \beta^{in}_r = \big\langle \Psi^i, \Psi^n_r \big\rangle, \tag{3.86}$$

we find

$$\big\langle \Psi^n_r, (H - E^n)\Psi^n_r \big\rangle \ge \sum_{i=0}^{n-1} |\beta^{in}_r|^2 (E^i - E^n) + \|\Phi^n_{r,>}\|^2 (E^{n+1} - E^n) \ge -Ce^{-DN}, \tag{3.87}$$

because for all $0 \le i < n$ (recall $\Psi_r = \sum_{l > \lfloor \delta N/2 \rfloor} P^0_{N,l}\Psi$),

$$|\beta^{in}_r|^2 = \big|\big\langle \Psi^i, \Psi^n_r \big\rangle\big|^2 = \big|\big\langle \Psi^i_r, \Psi^n_r \big\rangle\big|^2 \le \|\Psi^i_r\|^2 \le Ce^{-DN}. \tag{3.88}$$

Thus, using (3.82), we infer that

$$-Ce^{-DN} \le \big\langle \Psi^n_r, (H - E^n)\Psi^n_r \big\rangle \le Ce^{-DN} - 2\,\mathrm{Re}\big\langle \Psi^n_e, (H - E^n)\Psi^n_r \big\rangle \le Ce^{-DN}, \tag{3.89}$$

and thus

$$\|\Psi^n_r\|^2 \, \Big|\big\langle \Psi^n_{r,no}, (H - E^n)\Psi^n_{r,no} \big\rangle\Big| \le Ce^{-DN}. \tag{3.90}$$

From here, we proceed similar as in the case $n = 0$. Let

$$\Psi^n_{r,>} = \Psi^n_{r,no} - \sum_{i=0}^{m} \alpha^{in}_r \Psi^i, \qquad \alpha^{in}_r = \big\langle \Psi^i, \Psi^n_{r,no} \big\rangle, \tag{3.91}$$

with $m \ge n$ the smallest integer such that $E^{m+1} - E^n \ge \varepsilon > 0$ (for some small ε and all large N). Then,

$$\begin{aligned} &\big\langle \Psi^n_{r,no}, (H - E^n)\Psi^n_{r,no} \big\rangle \\ &\ge \sum_{i=0}^{n-1} |\alpha^{in}_r|^2 (E^i - E^n) + \sum_{i=n+1}^{m} |\alpha^{in}_r|^2 (E^i - E^n) + \|\Psi^n_{r,>}\|^2 (E^{m+1} - E^n) \\ &\ge -Ce^{-DN} + \varepsilon \|\Psi^n_{r,>}\|^2, \end{aligned} \tag{3.92}$$

which follows from $|\alpha^{in}_r|^2 \le \|\Psi^i_r\|^2 \le Ce^{-DN}$ for all $0 \le i < n$ (shown similarly as for the β^{in}_r). Moreover,

$$\|\Psi^n_{r,>}\|^2 \ge \Big\| \sum_{l=0}^{\lfloor \delta N/2 \rfloor} P^0_{N,l} \Big(\sum_{i=0}^{m} \alpha^{in}_r \Psi^i \Big) \Big\|^2 = \sum_{l=0}^{\lfloor \delta N/2 \rfloor} \Big\| P^0_{N,l} \Big(\sum_{i=0}^{m} \alpha^{in}_r \Psi^i \Big) \Big\|^2, \tag{3.93}$$

and, using again the abbreviation $\Phi = \alpha_r^{0n}\Psi^0 + \alpha_r^{1n}\Psi^1 + ... + \alpha_r^{mn}\Psi^m$, we find

$$(3.93) = 1 - \sum_{l=\lfloor\frac{\delta N}{2}\rfloor+1}^{N} \|P_{N,l}^0\Phi\|^2 \geq 1 - \frac{2}{\delta}\Big(\sum_{l=0}^{N}\frac{k}{N}\|P_{N,l}^0\Phi\|^2\Big) \geq 1 - \frac{C}{N^a} \tag{3.94}$$

for some $a > 0$, where the last step follows from Eq. (3.167), Appendix 3.A (here we use again $K^n = O(1)$ for fixed n; cf. Remark (3.3)). Altogether, this implies $\|\Psi_r^n\|^2 \leq Ce^{-DN}$ which proves (3.44) for $n \geq 1$, and hence completes the proof of Theorem 3.1 (a).

(b) The same argument as in the proof of part (a) can be used for proving the statement also for eigenfunctions $\widetilde{\Psi}^n$. The only difference is the absence of $\widetilde{V}^{\text{rest}}$ in (3.51).

Alternatively, one can derive estimates similar to the ones in Lemma 3.8 where the constant b_{δ,l^*} is replaced by a constant $b_{l^*} \propto 1/\sqrt{l^*}$ (in particular, not depending on δ). By means of this new estimate, the argument to prove (3.43) could be employed to derive the exponential bound for $\|\widetilde{\Psi}_l^n\|$ directly for all $\widetilde{K}^n \leq l \leq N$ (in the Proof of (a), to the contrary, it seems necessary to split the argument into two steps, namely (3.43) and (3.44) which is due to the presence of $\widetilde{V}^{\text{rest}}$). This second alternative to prove the statement is more analogous to the proof of (c); see below.

(c) The strategy is the same as in the proof of (3.43). Taking the scalar product between $\mathbb{H}_{\text{Bog}}\chi^n = (E_{\text{Bog}} + K_{\text{Bog}}^n)\chi^n$ with $(0,0,...0,\chi_l^n,0,0,...) \in \mathcal{F}_s$, where $\chi_l^n \in L_s^2(\mathbb{T}^{dl})$ is the lth component of $\chi^n \in \mathcal{F}_s$, we obtain (using $E_{\text{Bog}} < 0$)

$$\begin{aligned}4\pi^2 l\|\chi_l^n\|^2 + K_{\text{Bog}}^n\|\chi_l^n\|^2 + \sum_{k\neq 0}\hat{v}(k)\|a_k\chi_l^n\|^2 \\ \leq \sum_{k\neq 0}\frac{\hat{v}(k)}{2}\Big(\Big|\big\langle\chi_l^n, a_k^*a_{-k}^*\chi_{l-2}^n\big\rangle\Big| + \Big|\big\langle\chi_l^n, a_k a_{-k}\chi_{l+2}^n\big\rangle\Big|\Big).\end{aligned} \tag{3.95}$$

Furthermore, with some elementary algebra using the canonical commutation relations of a_k and a_k^*,

$$\begin{aligned}\Big|\big\langle\chi_l^n, a_k^*a_{-k}^*\chi_{l-2}^n\big\rangle\Big| &\leq \frac{1}{2}\Big(\big\langle\chi_l^n, a_k^*a_k\chi_l^n\big\rangle + \big\langle\chi_{l-2}^n, a_{-k}a_{-k}^*\chi_{l-2}^n\big\rangle\Big) \\ &\leq \frac{1}{2}\Big(\big\langle\chi_l^n, a_k^*a_k\chi_l^n\big\rangle + \big\langle\chi_{l-2}^n, a_{-k}^*a_{-k}\chi_{l-2}^n\big\rangle + \|\chi_{l-2}^n\|^2\Big),\end{aligned} \tag{3.96}$$

$$\Big|\big\langle\chi_l^n, a_k a_{-k}\chi_{l+2}^n\big\rangle\Big| \leq \frac{1}{2}\Big(\big\langle\chi_l^n, a_k^*a_k\chi_l^n\big\rangle + \|\chi_l^n\|^2 + \big\langle\chi_{l+2}^n, a_{-k}^*a_{-k}\chi_{l+2}^n\big\rangle\Big), \tag{3.97}$$

such that for all $l \geq l^*$,

$$\begin{aligned}&\frac{1}{\|\hat{v}\|_\infty}\Big(4\pi^2 - \frac{K_{\text{Bog}}^n}{l^*} - \frac{\|\hat{v}\|_1}{2l^*}\Big)l\|\chi_l^n\|^2 \\ &\leq \frac{1}{4}\sum_{k\neq 0}\frac{\hat{v}(k)}{\|\hat{v}\|_\infty}\Big[\|a_k\chi_{l-2}^n\|^2 + \|a_k\chi_{l+2}^n\|^2 - 2\|a_k\chi_l^n\|^2\Big] + \Big(\frac{\|\hat{v}\|_1}{2(l^*-2)}\Big)(l-2)\|\chi_{l-2}^n\|^2.\end{aligned} \tag{3.98}$$

If we choose $l^* > (K_{\text{Bog}}^n + \|\hat{v}\|_1/2)/(4\pi^2)$, the factor on the l.h.s. is again positive and we can proceed in exact analogy as in the proof of (3.43). However, note the difference that here the inequality holds for all $l \geq l^*$. Going through steps 2 and 3 (with some obvious modifications) one proves the exponential decay, i.e., $\|\chi_l^n\|^2 \leq Ce^{-Dl}$ for all $l \geq K_{\text{Bog}}^n$.

3.3.2 Proofs of Corollaries 3.2 and 3.3

Proof of Corollary 3.2. Note first that for any symmetric wave function Ψ, we have

$$\langle \Psi, \Big(\prod_{i=1}^{m} q_i^0\Big)\Psi\rangle \leq \sum_{l=0}^{N}\Big(\frac{l}{N}\Big)^m \|P^0_{N,l}\Psi\|^2, \tag{3.99}$$

which follows from the identity $\frac{1}{N}\sum_{i=1}^N q_i^0 = \sum_{l=0}^N \frac{l}{N} P^0_{N,l}$ and $\frac{l-1}{N-1} \leq \frac{l}{N}$ for all $l \leq N$. The proof of the corollary is now straightforward:

$$\sum_{l=0}^{K^n} l^m \|P^0_{N,l}\Psi^n\|^2 \leq (K^n)^m \sum_{l=0}^{K^n} \|P^0_{N,l}\Psi^n\|^2 \leq (K^n)^m, \tag{3.100}$$

and

$$\sum_{l=K^n+1}^{N} l^m \|P^0_{N,l}\Psi^n\|^2 \leq C, \tag{3.101}$$

which is true because $\|P^0_{N,l}\Psi^n\|^2 \leq Ce^{-Dl}$. (3.100) and (3.101) also hold for $\widetilde{\Psi}^n$ when K^n is replaced by $\widetilde{K}^n$. □

Proof of Corollary 3.3. We recall $H - \widetilde{H}$ from (3.24), and use the Fourier decomposition of v. With $q_2^0 e^{ikx_2} p_2^0 = |\varphi^{-k}\rangle\langle 1|_2$, $\forall k \in 2\pi\mathbb{Z}^d\backslash\{0\}$, and Corollary 3.2, we find for the terms with three q^0's and one p^0 in (3.24),

$$\begin{aligned} N\Big|\langle \Psi^n, q_1^0 q_2^0 v_{12} q_1^0 p_2^0 \Psi^n\rangle\Big| &\leq N\sum_{k\neq 0}\hat{v}(k)\Big|\langle \Psi^n, q_1^0 |\varphi^{-k}\rangle\langle\varphi^0|_2 q_1^0\Psi^n\rangle\Big| \\ &\leq N\|\hat{v}\|_\infty\Big(\sum_{k\neq 0}\|q_1^0 p_2^{-k}\Psi^n\|\Big)\ \|q_1^0\Psi^n\| \\ &\leq N\|\hat{v}\|_\infty \|q_1^0 q_2^0\Psi^n\|\ \|q_1^0\Psi^n\| \leq \frac{C(1+K^n)^{\frac{3}{2}}}{\sqrt{N}}. \end{aligned} \tag{3.102}$$

For the term with four q^0's in $H - \widetilde{H}$, we obtain

$$\begin{aligned} N\Big|\langle \Psi^n, q_1^0 q_2^0\Big(v_{12} - \hat{v}(0)\Big) q_1^0 q_2^0 \Psi^n\rangle\Big| &\leq N\sum_{k\neq 0}\hat{v}(k)\big|\langle \Psi^n, q_1^0 q_2^0 e^{ik(x_1-x_2)} q_1^0 q_2^0\Psi^n\rangle\big| \\ &\leq N\|\hat{v}\|_1\ \|q_1^0 q_2^0 \Psi^n\|^2 \leq \frac{C(1+K^n)^2}{N}. \end{aligned} \tag{3.103}$$

The same estimates hold again when Ψ^n and K^n are replaced by $\widetilde{\Psi}^n$ and $\widetilde{K}^n$. □

3.3.3 Proofs of Lemma 3.4 and Theorem 3.5

Proof of Lemma 3.4. To obtain uniqueness of the ground state $\widetilde{\Psi}^0$, we show that

$$\|\widetilde{\Psi}^0 - \Psi^0\| \leq CN^{-\frac{1}{4}} \tag{3.104}$$

for $\widetilde{\Psi}^0$, Ψ^0 with appropriately chosen relative phase factor. Uniqueness of $\widetilde{\Psi}^0$ then follows from uniqueness of Ψ^0. That Ψ^0 is nondegenerate follows from standard techniques showing that the ground state of H has to be a positive function (for details, we refer to [113, Section XIII.12]). Assume $\widetilde{\Psi}_1^0 \perp \widetilde{\Psi}_2^0$ two different ground states of $\widetilde{H}$. According to (3.104) we would find that

$$\sqrt{2} = \|\Psi_1^0 - \widetilde{\Psi}_2^0\| \leq \|\widetilde{\Psi}_1^0 - \Psi^0\| + \|\Psi^0 - \widetilde{\Psi}_2^0\| \leq CN^{-\frac{1}{4}} \tag{3.105}$$

which leads to an obvious contradiction for large enough N.

In order to derive (3.104), let $\Psi^0 = \alpha\widetilde{\Psi}^0 + \widetilde{\Psi}^\perp$ with $\widetilde{\Psi}^0 \perp \widetilde{\Psi}^\perp$ and the phase in Ψ^0 such that $\alpha = \langle\widetilde{\Psi}^0, \Psi^0\rangle \in \mathbb{R}$ and $\alpha > 0$. It follows that $1 = \alpha^2 + \|\widetilde{\Psi}^\perp\|^2$ and thus

$$\|\widetilde{\Psi}^0 - \Psi^0\| \leq \|\widetilde{\Psi}^\perp\| + (1-\alpha) \leq \|\widetilde{\Psi}^\perp\| + (1-\alpha^2) = \|\widetilde{\Psi}^\perp\| + \|\widetilde{\Psi}^\perp\|^2 \leq 2\|\widetilde{\Psi}^\perp\|. \tag{3.106}$$

For estimating the norm of $\|\Psi^\perp\|$ we start from

$$\langle\Psi^0, \widetilde{H}\Psi^0\rangle \geq \widetilde{E}^0\alpha^2 + \widetilde{E}^1\|\Psi^\perp\|^2 = \widetilde{E}^0 + (\widetilde{E}^1 - \widetilde{E}^0)\|\widetilde{\Psi}^\perp\|^2. \tag{3.107}$$

With $\widetilde{K}^1 = \widetilde{E}^1 - \widetilde{E}^0$, it follows that

$$\widetilde{K}^1\|\widetilde{\Psi}^\perp\|^2 \leq \langle\Psi^0, \widetilde{H}\Psi^0\rangle - \widetilde{E}^0 \leq \langle\Psi^0, (\widetilde{H} - H)\Psi^0\rangle + \langle\widetilde{\Psi}^0, (H - \widetilde{H})\widetilde{\Psi}^0\rangle \leq \frac{C}{\sqrt{N}}, \tag{3.108}$$

where we have used $\langle\Psi^0, H\Psi^0\rangle \leq \langle\widetilde{\Psi}^0, H\widetilde{\Psi}^0\rangle$ in the second step, and Corollary 3.3 in the third step. That the spectral gap $\widetilde{E}^1 - \widetilde{E}^0 = \widetilde{K}^1 \geq \varepsilon > 0$ is strictly positive is not obvious and we have to take it for granted at this point. It follows for instance from the proof of Theorem 3.7 where we show that $\lim_{N\to\infty} \widetilde{K}^1 = K^1_{\text{Bog}}$, together with the fact that $K^1_{\text{Bog}} \geq \varepsilon > 0$. □

The idea for improving the convergence rate in (3.104) (which is the statement of Theorem 3.5) is to use additionally the property "$\widehat{f}^0_{\text{odd}}\widetilde{\Psi}^0 = 0$ or $\widehat{f}^0_{\text{even}}\widetilde{\Psi}^0 = 0$". To do that we need some technical preparations which are summarized in

Lemma 3.10. *Let $a \in \{\frac{1}{2}, 1\}$, $d \in \{-1, 0, 1\}$, and $(\widehat{\tau_d n}^0)^a$ and $(\widehat{\tau_d \nu}^0)^a$ denote the linear combinations of projectors (cf. Definition (2.12))*

$$(\widehat{\tau_d n}^0)^a = \sum_{k=1-d}^{N-d} \left(\frac{k+d}{N}\right)^a P^0_{N,k}, \qquad (\widehat{\tau_d \nu}^0)^a = \sum_{k=1-d}^{N-d} \left(\frac{N}{k+d}\right)^a P^0_{N,k}. \tag{3.109}$$

Then, for any $\Psi \in L^2_s(\mathbb{T}^{dN})$, the following assertions hold:

(a) $\left[(\widehat{\tau_d m}^0)^a, s_i^0\right] = 0$ *for* $m \in \{n, \nu\}$, $s_i^0 \in \{p_i^0, q_i^0\}$ *and* $1 \leq i \leq N$,

(b) $q_1^0\Psi = (\widehat{\tau_d n}^0)^a(\widehat{\tau_d \nu}^0)^a q_1^0\Psi$ *for* $d \in \{-1, 0, 1\}$,

(c) $q_1^0 q_2^0 v_{12} q_1^0 p_2^0 (\widehat{\tau_d m}^0)^a\Psi = (\widehat{\tau_{d+1} m}^0)^a q_1^0 q_2^0 v_{12} q_1^0 p_2^0\Psi$ *for* $m \in \{n, \nu\}$, $d \in \{-1, 0\}$,

(d) $q_1^0 p_2^0 v_{12} q_1^0 q_2^0 (\widehat{\tau_d m}^0)^a\Psi = (\widehat{\tau_{d-1} m}^0)^a q_1^0 p_2^0 v_{12} q_1^0 q_2^0\Psi$ *for* $m \in \{n, \nu\}$, $d \in \{0, 1\}$,

(e) $\|q_1^0(\widehat{\tau_d n}^0)^{\frac{1}{2}}\Psi\|^2 \leq C\|q_1^0 q_2^0\Psi\|^2$ *and* $\|q_1^0(\widehat{\tau_d n}^0)^1\Psi\|^2 \leq C\|q_1^0 q_2^0 q_3^0\Psi\|^2$ *for* $d \in \{-1, 0, 1\}$,

(f) $\|q_1^0(\widehat{\tau_d \nu}^0)^{\frac{1}{2}}\Psi\|^2 \leq C\|\Psi\|^2$ *and* $\|q_1^0 q_2^0(\widehat{\tau_d \nu}^0)^1\Psi\|^2 \leq C\|\Psi\|^2$ *for* $d \in \{-1, 0, 1\}$.

Proof of Theorem 3.5. Following the same argument as in the proof of Corollary 3.4, it remains to find an improved bound in (3.108):

$$\widetilde{K}^1\|\widetilde{\Psi}^\perp\|^2 \le \big\langle \Psi^0, (\widetilde{H}-H)\Psi^0\big\rangle + \big\langle \widetilde{\Psi}^0, (H-\widetilde{H})\widetilde{\Psi}^0\big\rangle. \tag{3.110}$$

Let us assume $\widehat{f}^0_{\text{odd}}\widetilde{\Psi}^0 = 0$ (the proof would be completely analogous for $\widehat{f}^0_{\text{even}}\widetilde{\Psi}^0{=}0$; Corollary 3.4 states that either of the two is zero). We estimate the terms with three q^0's and one p^0, using that $qqvqp$ only couples the odd with the even part, and vice versa:

$$\Big|\big\langle \widetilde{\Psi}^0, q_1^0q_2^0v_{12}q_1^0p_2^0\widetilde{\Psi}^0\big\rangle\Big| = \Big|\big\langle \widehat{f}^0_{\text{odd}}\widetilde{\Psi}^0, q_1^0q_2^0v_{12}q_1^0p_2^0\widehat{f}^0_{\text{even}}\widetilde{\Psi}^0\big\rangle + \big\langle \widehat{f}^0_{\text{even}}\widetilde{\Psi}^0, q_1^0q_2^0v_{12}q_1^0p_2^0\widehat{f}^0_{\text{odd}}\widetilde{\Psi}^0\big\rangle\Big| = 0.$$

For Ψ_0, we find

$$\Big|\big\langle \Psi^0, q_1^0q_2^0v_{12}q_1^0p_2^0\Psi^0\big\rangle\Big| = \Big|\big\langle \widehat{f}^0_{\text{odd}}\Psi^0, q_1^0q_2^0v_{12}q_1^0p_2^0\widehat{f}^0_{\text{even}}\Psi^0\big\rangle\Big| + \Big|\big\langle \widehat{f}^0_{\text{even}}\Psi^0, q_1^0q_2^0v_{12}q_1^0p_2^0\widehat{f}^0_{\text{odd}}\Psi^0\big\rangle\Big|.$$

Here we proceed with Lemma 3.10, and obtain

$$\begin{aligned}
N\Big|\big\langle \widehat{f}^0_{\text{odd}}\Psi^0, q_1^0q_2^0v_{12}q_1^0p_2^0\widehat{f}^0_{\text{even}}\Psi^0\big\rangle\Big| &= N\Big|\big\langle \widehat{f}^0_{\text{odd}}\Psi^0, (\widehat{\tau_0\nu}^0)^1q_1^0q_2^0v_{12}q_1^0p_2^0(\widehat{\tau_{-1}n}^0)^1\widehat{f}^0_{\text{even}}\Psi^0\big\rangle\Big| \\
&\le N\|q_1^0q_2^0(\widehat{\tau_0\nu}^0)^1\widehat{f}^0_{\text{odd}}\Psi^0\|\;\|v_{12}p_2^0\|_{op}\;\|q_1^0(\widehat{\tau_{-1}n}^0)^1\Psi^0\| \\
&\le CN\|\widehat{f}^0_{\text{odd}}\Psi^0\|\;\|q_1^0q_2^0q_3^0\Psi^0\| \\
&\le \frac{\widetilde{K}^1}{10}\|\widehat{f}^0_{\text{odd}}\Psi^0 - \widehat{f}^0_{\text{odd}}\widetilde{\Psi}^0\|^2 + \frac{CN^2}{\widetilde{K}^1}\|q_1^0q_2^0q_3^0\Psi^0\|^2 \\
&\le \frac{\widetilde{K}^1}{10}\|\Psi^0-\widetilde{\Psi}^0\|^2 + \frac{C}{\widetilde{K}^1N},
\end{aligned} \tag{3.111}$$

where we have also used that $\|\widehat{f}^0_{\text{odd}}\|_{op}\le 1$ as well as $\|v_{12}p_2^0\|_{op} \le \sqrt{\|p_2^0v_{12}^2p_2^0\|_{op}} \le \|v_{12}\|_2 \le C$. Similarly, one finds

$$N\Big|\big\langle \widehat{f}^0_{\text{even}}\Psi^0, q_1^0q_2^0v_{12}q_1^0p_2^0\widehat{f}^0_{\text{odd}}\Psi^0\big\rangle\Big| \le \frac{\widetilde{K}^1}{10}\|\Psi^0-\Psi^0\|^2 + \frac{C}{\widetilde{K}^1N}. \tag{3.112}$$

The term with four q^0's in $H-\widetilde{H}$ has been already estimated in (3.103). Using $K^0 = \widetilde{K}^0 = 0$ completes the proof of Theorem 3.5. □

3.3.4 Proofs of Theorem 3.7

Proof of Theorem 3.7. We first compute the difference between $\widetilde{H}$ and $\mathbb{H}_{\text{Bog}}$. For that, we introduce the unitary mapping

$$\mathbb{U}_{\varphi_{\text{H}}} : L^2_s(\mathbb{T}^{dN}) \to \mathcal{F}_s^{\le N}, \qquad \Psi \mapsto \mathbb{U}_{\varphi_{\text{H}}}\Psi = (\chi_l^\Psi)_{l=0}^N \in \mathcal{F}_s^{\le N}, \tag{3.113}$$

where $\mathcal{F}_s^{\le N} = \bigoplus_{l=0}^N \bigotimes_{sym.}^l \big(\{\varphi_{\text{H}}\}^\perp\big)$. $\mathbb{U}_{\varphi_{\text{H}}}$ is defined by[6]

$$\chi_l^\Psi(x_1,...,x_k) = \sqrt{\binom{N}{l}}\left(\prod_{i=1}^l q_i^0\right)\int\left(\prod_{j=l+1}^N \overline{\varphi^0(x_i)}\right)\Psi(x_1,...,x_N)\,dx_{l+1}\dots dx_N. \tag{3.114}$$

[6]The mapping $\mathbb{U}_{\varphi_{\text{H}}}$ gives the correct relation between the N-particle space and the Fock space of excitations $\mathcal{F}_s^{\le N} \subset \mathcal{F}_s$ which we have mentioned in (3.13). It maps Ψ onto its component in which exactly $N-l$ particles occupy the condensate wave function and then removes the corresponding degrees of freedom. What remains is a symmetric l-particle wave function which is orthogonal to φ_{H} in all coordinates. This mapping was first introduced in [81].

Also note for symmetric Ψ the identity, following from $1 = \sum_{l=0}^{N} P_{N,l}^0$,

$$\Psi = \sum_{l=0}^{N} P_{N,l}^0 \Psi = \sum_{l=0}^{N} \left(\varphi_{\mathrm{H}}^{\otimes N-l} \otimes_s \chi_l^{\Psi} \right), \tag{3.115}$$

where $\otimes_s$ denotes the normalized symmetric product between two symmetric wave functions $\psi_l \in L_s^2(\mathbb{T}^{dl})$, $\psi_k \in L_s^2(\mathbb{T}^{dk})$,

$$\psi_l \otimes_s \psi_k = \frac{1}{\sqrt{k!l!(k+l)!}} \sum_{\sigma \in P_{k+l}} \psi_l(x_{\sigma(1)}, ..., x_{\sigma(l)}) \psi_k(x_{\sigma(l+1)}, ..., x_{\sigma(k+l)}). \tag{3.116}$$

It follows from the definition of $\mathbb{U}_{\varphi_{\mathrm{H}}}$ that for $\Psi, \Phi \in L_s^2(\mathbb{T}^{dN})$,

$$\langle \Psi_l, \Phi_l \rangle = \langle \Psi, P_{N,l}^0 \Phi \rangle = \frac{N!}{(N-l)!l!} \langle \Psi, \left(q_1^0 ... q_l^0 p_{l+1}^0 ... p_N^0 \right) \Psi \rangle = \langle \chi_l^{\Psi}, \chi_l^{\Phi} \rangle. \tag{3.117}$$

Unitarity of $\mathbb{U}_{\varphi_{\mathrm{H}}}$ is then obtained from

$$\langle \Psi, \Phi \rangle = \sum_{l=0}^{N} \langle \Psi_l, \Phi_l \rangle = \sum_{l=0}^{N} \frac{N!}{(N-l)!l!} \langle \Psi, \left(q_1^0 ... q_l^0 p_{l+1}^0 ... p_N^0 \right) \Psi \rangle = \sum_{l=0}^{N} \langle \chi_l^{\Psi}, \chi_l^{\Phi} \rangle.$$

One also finds by direct computation that

$$\mathbb{U}_{\varphi_{\mathrm{H}}} \widetilde{H} \mathbb{U}_{\varphi_{\mathrm{H}}}^{-1} = \frac{N \hat{v}(0)}{2} + \widetilde{\mathbb{H}}^{\leq N}, \tag{3.118}$$

where $\widetilde{\mathbb{H}}^{\leq N} : \mathcal{F}_s^{\leq N} \to \mathcal{F}_s^{\leq N}$ is given by $\widetilde{\mathbb{H}}^{\leq N} = \bigoplus_{l=0}^{N} \widetilde{\mathbb{H}}_l^{\leq N}$ with

$$\widetilde{\mathbb{H}}_l^{\leq N} = \sum_{k \neq 0} |k|^2 a_k^* a_k + \sum_{k \neq 0} \frac{\hat{v}(k)}{2} \left[2 \left(\frac{N-l}{N-1} \right) a_k^* a_k + c_1(N,l) a_k^* a_{-k}^* + c_2(N,l) a_k a_{-k} \right] \tag{3.119}$$

for all $l \leq N-2$, and

$$\widetilde{\mathbb{H}}_l^{\leq N} = \sum_{k \neq 0} |k|^2 a_k^* a_k + \sum_{k \neq 0} \frac{\hat{v}(k)}{2} \left[2 \left(\frac{N-l}{N-1} \right) a_k^* a_k + c_2(N,l) a_k a_{-k} \right] \tag{3.120}$$

for $l \in \{N-1, N\}$. Here, we have used the abbreviations

$$c_1(N,l) = \frac{\sqrt{(N-l+2)(N-l+1)}}{N-1}, \qquad c_2(N,l) = \frac{\sqrt{(N-l)(N-l-1)}}{N-1}. \tag{3.121}$$

We give an example of how one arrives at (3.118): We need to compute

$$\sqrt{\binom{N}{l}} \left(\prod_{i=1}^{l} q_i^0 \right) \langle \varphi_{\mathrm{H}}^{\otimes (N-l)}, \widetilde{H} \Psi \rangle,$$

where the scalar product is taken w.r.t. the coordinates $x_{l+1}, ..., x_N$. The one-particle part of the Hamiltonian leads to

$$\sqrt{\binom{N}{l}} \Big(- \sum_{i=1}^{l} \Delta_i + \frac{N \hat{v}(0)}{2} \Big) \left(\prod_{i=1}^{l} q_i^0 \right) \langle \varphi_{\mathrm{H}}^{\otimes (N-l)}, \Psi \rangle = \Big(\sum_{k \neq 0} |k|^2 \ a_k^* a_k + \frac{N \hat{v}(0)}{2} \Big) \chi_l^{\Psi}.$$

For the $qqvpp$ part of the interaction term in $\widetilde{H}$, one finds

$$\begin{aligned}
&\sqrt{\binom{N}{l}} \left(\prod_{i=1}^{l} q_i^0\right) \langle \varphi_{\mathrm{H}}^{\otimes(N-l)}, \Big(\frac{1}{N-1} \sum_{1\le i<j\le N} q_i^0 q_j^0 v_{ij} p_i^0 p_j^0\Big) \Psi\rangle \\
&= \sqrt{\binom{N}{l}} \sqrt{\binom{N}{l-2}}^{-1} \frac{1}{N-1} \sum_{1\le i<j\le l} q_i^0 q_j^0 v_{ij} \varphi^0(x_i)\varphi^0(x_j)\chi_{l-2}^{\Psi}(x_1,...,x_l\backslash x_i\backslash x_j) \\
&= \sqrt{\binom{N}{l}} \sqrt{\binom{N}{l-2}}^{-1} \frac{1}{N-1} \frac{\sqrt{l(l-1)}}{2} \sum_{k\neq 0} \hat{v}(k) a_k^* a_{-k}^* \chi_{l-2}^{\Psi}(x_1,...,x_{l-2}) \\
&= \frac{\sqrt{(N-l+2)(N-l+1)}}{N-1} \sum_{k\neq 0} \frac{\hat{v}(k)}{2} \Big(a_k^* a_{-k}^* \chi_{l-2}^{\Psi}\Big)(x_1,...,x_l).
\end{aligned}$$

Proceding similar for the other interaction terms in $\widetilde{H}$, one obtained the given expression.

We denote the eigenfunctions of $\widetilde{\mathbb{H}}^{\le N}$ by $\chi^{\widetilde{\Psi}^n} \in \mathcal{F}_s^{\le N}$. Due to unitarity of $\mathbb{U}_{\varphi_{\mathrm{H}}}$, they are given by $\chi^{\widetilde{\Psi}^n} = \mathbb{U}_{\varphi_{\mathrm{H}}} \widetilde{\Psi}^n$. The corresponding eigenvalues equal $\mu_n(\widetilde{\mathbb{H}}^{\le N}) = \widetilde{E}^n - N\hat{v}(0)/2$. Note that the components $\chi_l^{\widetilde{\Psi}^n} \in L_s^2(\mathbb{T}^{dl})$ decay as well exponentially in the number l, i.e.,

$$\|\chi_l^{\widetilde{\Psi}^n}\|^2 \le Ce^{-Dl} \quad \text{for all} \quad l \ge \widetilde{K}^n, \tag{3.122}$$

which follows from $\|\chi_l^{\widetilde{\Psi}^n}\|^2 = \|\widetilde{\Psi}_l^n\|^2$, cf. (3.117), and Theorem 3.1 (b).

Ground state energy. *Lower and upper bound for E^0 in terms of $\widetilde{E}^0$*: We find

$$\Big|E^0 - \widetilde{E}^0\Big| \le \max\Big\{\Big|\big\langle \Psi^0, (H-\widetilde{H})\Psi^0\big\rangle\Big|, \ \Big|\big\langle \widetilde{\Psi}^0, (H-\widetilde{H})\widetilde{\Psi}^0\big\rangle\Big|\Big\} \le \frac{C}{N} \tag{3.123}$$

which follows along the same lines of the proof of Theorem 3.5; cf. the estimates following (3.110).

Next, we want to find upper and lower bounds for $\mu_0(\widetilde{\mathbb{H}}^{\le N})$ in terms of $\mu_0(\mathbb{H}_{\mathrm{Bog}})$. Here, the comparison of $\widetilde{\mathbb{H}}^{\le N}$ and $\mathbb{H}_{\mathrm{Bog}}$ is more tedious because they act on different spaces.

Upper bound for $\mu_0(\widetilde{\mathbb{H}}^{\le N})$ in terms of $\mu_0(\mathbb{H}_{Bog})$: To this end, we introduce

$$\mathbb{H}_{\mathrm{Bog}}^{\le N} = P_{\mathcal{F}_s^{\le N}} \mathbb{H}_{\mathrm{Bog}} P_{\mathcal{F}_s^{\le N}}, \tag{3.124}$$

where $P_{\mathcal{F}_s^{\le N}}$ is the orthogonal projector onto $\mathcal{F}_s^{\le N} \subset \mathcal{F}_s$. Denoting $\chi^{0,\le N} = P_{\mathcal{F}_s^{\le N}} \chi^0$ ($\chi^n \in \mathcal{F}_s$ are the eigenfunctions of $\mathbb{H}_{\mathrm{Bog}}$), we find that

$$\mu_0\big(\widetilde{\mathbb{H}}^{\le N}\big) \le \frac{\big\langle \chi^{0,\le N}, \widetilde{\mathbb{H}}^{\le N} \chi^{0,\le N}\big\rangle_{\mathcal{F}_s^{\le N}}}{\|\chi^{0,\le N}\|^2_{\mathcal{F}_s^{\le N}}}. \tag{3.125}$$

The denominator is easily estimated (using $\|\chi^0\|_{\mathcal{F}_s} = 1$ and the exponential decay of $\|\chi_l^0\|$),

$$\|\chi^{0,\le N}\|^2_{\mathcal{F}_s^{\le N}} = \sum_{l=0}^{N} \|\chi_l^0\|^2 = 1 - \sum_{l=N+1}^{\infty} \|\chi_l^0\|^2 \ge 1 - \frac{C}{N^a} \sum_{l=N+1}^{\infty} l^a e^{-Dl} \ge 1 - \frac{C_a}{N^a} \tag{3.126}$$

for any integer $a > 0$ and some positive constant C_a. Choosing a large enough, one obtains for instance $\|\chi^{0,\le N}\|^{-2}_{\mathcal{F}_s^{\le N}} \le C(1+N^{-2})$. We proceed with

$$\langle \chi^{0,\le N}, \widetilde{\mathbb{H}}^{\le N}\chi^{0,\le N}\rangle_{\mathcal{F}_s^{\le N}} = \langle \chi^{0,\le N}, \mathbb{H}^{\le N}_{\text{Bog}}\chi^{0,\le N}\rangle_{\mathcal{F}_s^{\le N}} \tag{3.127}$$

$$+ \langle \chi^{0,\le N}, (\widetilde{\mathbb{H}}^{\le N} - \mathbb{H}^{\le N}_{\text{Bog}})\chi^{0,\le N}\rangle_{\mathcal{F}_s^{\le N}}, \tag{3.128}$$

where the first line equals

$$(3.127) = \langle \chi^0, \mathbb{H}^{\le N}_{\text{Bog}}\chi^0\rangle_{\mathcal{F}_s} = \mu_0(\mathbb{H}_{\text{Bog}}) + \langle \chi^0, (\mathbb{H}^{\le N}_{\text{Bog}} - \mathbb{H}_{\text{Bog}})\chi^0\rangle_{\mathcal{F}_s}, \tag{3.129}$$

and it remains to compute the difference

$$\begin{aligned}\langle \chi, (\mathbb{H}^{\le N}_{\text{Bog}} - \mathbb{H}_{\text{Bog}})\chi\rangle_{\mathcal{F}_s} \le &- \sum_{l=N+1}^{\infty}\sum_{k\neq 0} k^2\langle \chi_l, a_k^* a_k \chi_l\rangle - \sum_{l=N+1}^{\infty}\sum_{k\neq 0}\hat{v}(k)\langle \chi_l, a_k^* a_k\chi_l\rangle \\ &+ \sum_{l=N-1}^{\infty}\sum_{k\neq 0}\frac{\hat{v}(k)}{2}\Big|\langle \chi_l, a_k a_{-k}\chi_{l+2}\rangle\Big| + \sum_{l=N+1}^{\infty}\sum_{k\neq 0}\frac{\hat{v}(k)}{2}\Big|\langle \chi_l, a_k^* a_{-k}^*\chi_{l-2}\rangle\Big| \\ &\le C\|\hat{v}\|_\infty \sum_{l=N-1}^{\infty} l\|\chi_l\|^2 + C\|\hat{v}\|_1\sum_{l=N-1}^{\infty}\|\chi_l\|^2. \end{aligned}\tag{3.130}$$

Here, we have made use of $\hat{v}(k) \ge 0$ as well as of (3.96) and (3.97) with χ^n replaced by $\chi \in \mathcal{F}_s$. For $\chi = \chi^0$, the last line is small (following again from the exponential decay of $\|\chi^0_l\|$) such that

$$\langle \chi^0, (\mathbb{H}^{\le N}_{\text{Bog}} - \mathbb{H}_{\text{Bog}})\chi^0\rangle_{\mathcal{F}_s} \le \frac{C}{N}. \tag{3.131}$$

Next, we have to estimate the difference $\widetilde{\mathbb{H}}^{\le N} - \mathbb{H}^{\le N}_{\text{Bog}}$. Using again (3.96) and (3.97), and in addition the fact that

$$\Big|1 - \Big(\frac{N-l}{N-1}\Big)\Big| \le C\frac{l}{N}, \qquad \Big|1 - c_1(N,l)\Big| \le C\frac{l}{N}, \qquad \Big|1 - c_2(N,l)\Big| \le C\frac{l}{N}, \tag{3.132}$$

with c_1 and c_2 defined as in (3.121), we obtain

$$\Big|\langle \chi^{0,\le N}, (\widetilde{\mathbb{H}}^{\le N} - \mathbb{H}^{\le N}_{\text{Bog}})\chi^{0,\le N}\rangle_{\mathcal{F}_s^{\le N}}\Big| \le C\|\hat{v}\|_\infty\sum_{l=0}^{N}\frac{l^2}{N}\|\chi_l^{0,\le N}\|^2 + C\|\hat{v}\|_1\sum_{l=0}^{N}\frac{l}{N}\|\chi_l^{0,\le N}\|^2. \tag{3.133}$$

By means of the exponential decay of the $\|\chi_l^{0,\le N}\|$, this leads to

$$\Big|\langle \chi^{0,\le N}, (\widetilde{\mathbb{H}}^{\le N} - \mathbb{H}^{\le N}_{\text{Bog}})\chi^{0,\le N}\rangle_{\mathcal{F}_s^{\le N}}\Big| \le \frac{C}{N}. \tag{3.134}$$

In total, this shows that $\mu_0(\widetilde{\mathbb{H}}^{\le N}) \le \mu_0(\mathbb{H}_{\text{Bog}}) + C/N$, and thus $\widetilde{E}^0 - N\hat{v}(0)/2 - E_{\text{Bog}} \le C/N$.

Lower bound for $\mu_0(\widetilde{\mathbb{H}}^{\le N})$ in terms of $\mu_0(\mathbb{H}_{Bog})$: For the corresponding lower bound, we denote by $\widetilde{\mathbb{H}}$ the trivial extension of $\widetilde{\mathbb{H}}^{\le N}$ to the whole Fock space $\mathcal{F}_s$ and by $\widetilde{\chi}^0 \in \mathcal{F}_s$ the state that equals $\chi^{\widetilde{\Psi}^0}$ on $\mathcal{F}_s^{\le N}$ and is identically zero otherwise (note that $\|\widetilde{\chi}^0\|_{\mathcal{F}_s} = 1$). Then, we proceed in analogy to before,

$$\mu_0(\mathbb{H}_{\text{Bog}}) \le \langle \widetilde{\chi}^0, \mathbb{H}_{\text{Bog}}\widetilde{\chi}^0\rangle_{\mathcal{F}_s} \le \langle \widetilde{\chi}^0, \widetilde{\mathbb{H}}\widetilde{\chi}^0\rangle_{\mathcal{F}_s} + \langle \widetilde{\chi}^0, (\mathbb{H}_{\text{Bog}} - \widetilde{\mathbb{H}})\widetilde{\chi}^0\rangle_{\mathcal{F}_s}. \tag{3.135}$$

The first term is given by

$$\big\langle \widetilde{\chi}^0, \widetilde{\mathbb{H}}\widetilde{\chi}^0 \big\rangle_{\mathcal{F}_s} = \big\langle \chi^{\widetilde{\Psi}^0}, \widetilde{\mathbb{H}}^{\leq N} \chi^{\widetilde{\Psi}^0} \big\rangle_{\mathcal{F}_s^{\leq N}} = \mu_0(\widetilde{\mathbb{H}}^{\leq N}), \tag{3.136}$$

and for the second term, we find similarly as in (3.133),

$$\begin{aligned} \big\langle \widetilde{\chi}^0, \big(\mathbb{H}_{\mathrm{Bog}} - \widetilde{\mathbb{H}}\big)\widetilde{\chi}^0 \big\rangle_{\mathcal{F}_s} &= \big\langle \chi^{\widetilde{\Psi}^0}, \big(\mathbb{H}_{\mathrm{Bog}}^{\leq N} - \widetilde{\mathbb{H}}^{\leq N}\big) \chi^{\widetilde{\Psi}^0} \big\rangle_{\mathcal{F}_s^{\leq N}} \\ &\leq C\|\hat{v}\|_\infty \sum_{l=0}^{N} \frac{l^2}{N} \|\chi_l^{\widetilde{\Psi}^0}\|^2 + C\|\hat{v}\|_1 \sum_{l=0}^{N} \frac{l}{N} \|\chi_l^{\widetilde{\Psi}^0}\|^2. \end{aligned} \tag{3.137}$$

Since $\|\chi_l^{\widetilde{\Psi}^0}\|^2 \propto e^{-Dl}$, cf. (3.122), the last line is again bounded in terms of C/N. This completes the derivation of the lower bound and leads to the claimed estimate for the ground state energy.

Excitation energies. *Upper bound for $\mu_n(\widetilde{\mathbb{H}}^{\leq N})$ in terms of $\mu_n(\mathbb{H}_{Bog})$*: For the upper bound, we use the min-max principle with the subspace

$$Y_{n+1}^{\leq N} = P_{\mathcal{F}_s^{\leq N}} Y_{n+1} \subset \mathcal{F}_s^{\leq N} \tag{3.138}$$

where $Y_{n+1} \subset \mathcal{F}_s$ is the $n+1$-dimensional subspace spanned by the eigenvectors $\{\chi^0, ..., \chi^n\}$. One obtains

$$\mu_n(\widetilde{\mathbb{H}}^{\leq N}) \leq \max_{\substack{\chi^{\leq N} \in Y_{n+1}^{\leq N}, \\ \|\chi^{\leq N}\|_{\mathcal{F}_s^{\leq N}} = 1}} \big\langle \chi^{\leq N}, \mathbb{H}_{\mathrm{Bog}}^{\leq N} \chi^{\leq N} \big\rangle_{\mathcal{F}_s^{\leq N}} + \max_{\substack{\chi^{\leq N} \in Y_{n+1}^{\leq N}, \\ \|\chi^{\leq N}\|_{\mathcal{F}_s^{\leq N}} = 1}} \big\langle \chi^{\leq N}, \big(\widetilde{\mathbb{H}}^{\leq N} - \mathbb{H}_{\mathrm{Bog}}^{\leq N}\big) \chi^{\leq N} \big\rangle_{\mathcal{F}_s^{\leq N}}. \tag{3.139}$$

The difference in the second term is computed similarly as in (3.133), such that

$$\big\langle \chi^{\leq N}, \big(\widetilde{\mathbb{H}}^{\leq N} - \mathbb{H}_{\mathrm{Bog}}^{\leq N}\big) \chi^{\leq N} \big\rangle_{\mathcal{F}_s^{\leq N}} \leq C\|\hat{v}\|_\infty \sum_{l=0}^{N} \frac{l^2}{N} \|\chi_l^{\leq N}\|^2 + C\|\hat{v}\|_1 \sum_{l=0}^{N} \frac{l}{N} \|\chi_l^{\leq N}\|^2. \tag{3.140}$$

Since $\chi^{\leq N}$ lies in $Y_{n+1}^{\leq N}$, we have $\|\chi_l^{\leq N}\| \leq nCe^{-Dl}$ for all $l \geq K_{\mathrm{Bog}}^n$, and thus, we find

$$\max_{\substack{\chi^{\leq N} \in Y_{n+1}^{\leq N}, \\ \|\chi^{\leq N}\|_{\mathcal{F}_s^{\leq N}} = 1}} \big\langle \chi^{\leq N}, \big(\widetilde{\mathbb{H}}^{\leq N} - \mathbb{H}_{\mathrm{Bog}}^{\leq N}\big) \chi^{\leq N} \big\rangle_{\mathcal{F}_s^{\leq N}} \leq C\Big[\frac{1}{N} + \frac{K_{\mathrm{Bog}}^n}{N} + \frac{(K_{\mathrm{Bog}}^n)^2}{N}\Big]. \tag{3.141}$$

For the first term in (3.139), we proceed with

$$\max_{\substack{\chi^{\leq N} \in Y_{n+1}^{\leq N}, \\ \|\chi^{\leq N}\|_{\mathcal{F}_s^{\leq N}} = 1}} \big\langle \chi^{\leq N}, \mathbb{H}_{\mathrm{Bog}}^{\leq N} \chi^{\leq N} \big\rangle_{\mathcal{F}_s^{\leq N}} = \max_{\substack{\chi^{\leq N} \in Y_{n+1}^{\leq N}, \\ \chi^{\leq N} \neq 0}} \frac{\big\langle \chi^{\leq N}, \mathbb{H}_{\mathrm{Bog}}^{\leq N} \chi^{\leq N} \big\rangle_{\mathcal{F}_s^{\leq N}}}{\|\chi^{\leq N}\|^2_{\mathcal{F}_s^{\leq N}}}, \tag{3.142}$$

where we can replace the set over which we take the maximum by $\chi \in Y_{n+1}$, i.e.,

$$(3.142) = \max_{\substack{\chi \in Y_{n+1}, \\ \chi \neq 0}} \frac{\big\langle \chi, \mathbb{H}_{\mathrm{Bog}}^{\leq N} \chi \big\rangle_{\mathcal{F}_s}}{\|P_{\mathcal{F}_s^{\leq N}} \chi\|_{\mathcal{F}_s^2}} = \max_{\substack{\chi \in Y_{n+1}, \\ \chi \neq 0}} \Big[\frac{\big\langle \chi, \mathbb{H}_{\mathrm{Bog}}^{\leq N} \chi \big\rangle_{\mathcal{F}_s}}{\|\chi\|_{\mathcal{F}_s^2}} \frac{\|\chi\|_{\mathcal{F}_s^2}}{\|P_{\mathcal{F}_s^{\leq N}} \chi\|_{\mathcal{F}_s^2}}\Big] \tag{3.143}$$

(note that for all $\chi \in Y_{n+1}$, $\chi \neq 0$, the norm $\|P_{\mathcal{F}_s^{\leq N}}\chi\|_{\mathcal{F}_s^2}$ does not vanish, see below). For the second factor in the brackets, one finds that for all $\chi \in Y_{n+1}$, $\chi \neq 0$, $\chi = \sum_{i=0}^n c_i \chi^i$, it holds that

$$\frac{\|\chi\|^2_{\mathcal{F}_s}}{\|P_{\mathcal{F}_s^{\leq N}}\chi\|_{\mathcal{F}_s^2}} \leq 1 + \frac{C_a}{N^a}, \tag{3.144}$$

where the integer a can be chosen arbitrarily large. To see this, we compute

$$\frac{\|\chi\|^2_{\mathcal{F}_s}}{\|P_{\mathcal{F}_s^{\leq N}}\chi\|_{\mathcal{F}_s^2}} = 1 + \frac{\sum_{i=0}^n |c_i|^2 \sum_{l=N+1}^\infty \|\chi_l^i\|^2}{\sum_{i=0}^n |c_i|^2 \sum_{l=0}^N \|\chi_l^i\|^2}, \tag{3.145}$$

and by means of the exponential decay of $\|\chi_l^i\|^2$ for all $i \in \{0, ..., n\}$, $l \geq K^n_{\text{Bog}}$, the estimate in (3.144) follows. It remains to compute

$$\max_{\substack{\chi \in Y_{n+1},\\ \chi \neq 0}} \frac{\langle \chi, \mathbb{H}^{\leq N}_{\text{Bog}} \chi\rangle_{\mathcal{F}_s}}{\|\chi\|^2_{\mathcal{F}_s}} = \mu_n(\mathbb{H}_{\text{Bog}}) + \max_{\substack{\chi \in Y_{n+1},\\ \chi \neq 0}} \frac{\langle \chi, (\mathbb{H}^{\leq N}_{\text{Bog}} - \mathbb{H}_{\text{Bog}})\chi\rangle_{\mathcal{F}_s}}{\|\chi\|^2_{\mathcal{F}_s}} \tag{3.146}$$

For the last term, we proceed similarly as in (3.130):

$$\max_{\substack{\chi \in Y_{n+1},\\ \chi \neq 0}} \frac{\langle \chi, (\mathbb{H}^{\leq N}_{\text{Bog}} - \mathbb{H}_{\text{Bog}})\chi\rangle_{\mathcal{F}_s}}{\|\chi\|_{\mathcal{F}_s^2}} \leq \max_{\substack{\chi \in Y_{n+1},\\ \|\chi\|_{\mathcal{F}_s}=1}} \Big[C\|\hat{v}\|_\infty \sum_{l=N-1}^\infty l\|\chi_l\|^2 + C\|\hat{v}\|_1 \sum_{l=N-1}^\infty \|\chi_l\|^2\Big] \leq \frac{C}{N}, \tag{3.147}$$

where the bound follows again from the exponential decay of $\|\chi_l\|$. This completes the derivation of the upper bound.

Lower bound for $\mu_n(\widetilde{\mathbb{H}}^{\leq N})$ in terms of $\mu_n(\mathbb{H}_{Bog})$: In order to obtain the lower, we apply the min-max principle with $\widetilde{Y}_{n+1} \subset \mathcal{F}_s$, the subspace spanned by $\{\widetilde{\chi}^1, ..., \widetilde{\chi}^n\}$. One finds

$$\begin{aligned}\mu_n(\mathbb{H}_{\text{Bog}}) &\leq \max_{\widetilde{\chi} \in \widetilde{Y}_{n+1}, \|\widetilde{\chi}\|_{\mathcal{F}_s}=1} \langle \widetilde{\chi}, \widetilde{\mathbb{H}}\widetilde{\chi}\rangle_{\mathcal{F}_s} + \max_{\widetilde{\chi} \in \widetilde{Y}_{n+1}, \|\widetilde{\chi}\|_{\mathcal{F}_s}=1} \langle \widetilde{\chi}, (\mathbb{H}_{\text{Bog}} - \widetilde{\mathbb{H}})\widetilde{\chi}\rangle_{\mathcal{F}_s} \\ &= \mu_n(\widetilde{\mathbb{H}}^{\leq N}) + \max_{\chi^\Psi \in Y^{\widetilde{\Psi}}_{n+1}, \|\chi^\Psi\|_{\mathcal{F}_s^{\leq N}}=1} \langle \chi^\Psi, (\mathbb{H}^{\leq N}_{\text{Bog}} - \widetilde{\mathbb{H}}^{\leq N})\chi^\Psi\rangle_{\mathcal{F}_s^{\leq N}},\end{aligned} \tag{3.148}$$

where $Y^{\widetilde{\Psi}}_{n+1} \subset \mathcal{F}_s^{\leq N}$ denotes the subspace spanned by $\{\chi^{\widetilde{\Psi}^0}, ..., \chi^{\widetilde{\Psi}^n}\}$. The difference in the second term can be computed and estimated similarly as in (3.140). Using the exponential decay of $\|\chi_l^\Psi\|$, one obtains

$$\max_{\chi^\Psi \in Y^{\Psi}_{n+1}, \|\chi^\Psi\|_{\mathcal{F}_s^{\leq N}}=1} \langle \chi^\Psi, (\mathbb{H}^{\leq N}_{\text{Bog}} - \widetilde{\mathbb{H}}^{\leq N})\chi^\Psi\rangle \leq C\Big[\frac{1}{N} + \frac{K^n_{\text{Bog}}}{N} + \frac{(K_{\text{Bog}})^2}{N}\Big]. \tag{3.149}$$

Altogether, recalling $|\widetilde{E}^0 - N\hat{v}(0)/2 - E_{\text{Bog}}| \leq C/N$, we have thus shown that

$$\Big|\widetilde{K}^n - K^n_{\text{Bog}}\Big| \leq C\Big[\frac{1}{N} + \frac{K^n_{\text{Bog}}}{N} + \frac{(K^n_{\text{Bog}})^2}{N}\Big]. \tag{3.150}$$

Lower and upper bound for $\mu_n(H)$ in terms of $\mu_n(\widetilde{H})$: The application of the min-max principle is simpler here because H and $\widetilde{H}$ are defined on the same Hilbert space. In the upper bound, we use $\widetilde{X}_{n+1} \subset L^2_s(\mathbb{T}^{dN})$, the n+1-dimensional subspace spanned by $\{\widetilde{\Psi}^0, ..., \widetilde{\Psi}^n\}$, such that

$$\mu_n(H) \leq \mu_n(\widetilde{H}) + \max_{\widetilde{\Psi}\in\widetilde{X}_{n+1}, \|\widetilde{\Psi}\|=1} \big\langle \widetilde{\Psi}, (H - \widetilde{H})\widetilde{\Psi}\big\rangle. \tag{3.151}$$

Then we use (3.24) together with $\| \sum_{i=0}^n c_i \widetilde{\Psi}^i_l \| \leq nCe^{-Dl}$ for all $l \geq \widetilde{K}^n$, in order to find (the argument is the same as in the proof of Corollary 3.3 with $\widetilde{\Psi}^n$ replaced by $\widetilde{\Psi} \in \widetilde{X}_{n+1}$),

$$\mu_n(H) \leq \mu_n(\widetilde{H}) + C\Big[\frac{1 + (\widetilde{K}^n)^{\frac{3}{2}}}{\sqrt{N}} + \frac{(\widetilde{K}^n)^2}{N}\Big]. \tag{3.152}$$

The lower bound is proven in complete analogy, now taking the subspace spanned by $\{\Psi^0, ..., \Psi^n\}$. Together with $|E^0 - \widetilde{E}^0| \leq C/N$, this leads to

$$\Big|K^n - \widetilde{K}^n\Big| \leq C\Big[\frac{1 + (K^n)^{\frac{3}{2}}}{\sqrt{N}} + \frac{(K^n)^2}{N}\Big]. \tag{3.153}$$

□

3.3.5 Proofs of Lemmas 3.8, 3.9 and 3.10

Proof of Lemma 3.8. We use $p^0_1 e^{ikx_1} q^0_1 = |1\rangle\langle\varphi^k|_1$, $\forall k \in 2\pi\mathbb{Z}^d\backslash\{0\}$, exploit the symmetry of the wave function and apply Cauchy Schwarz,

$$\begin{aligned} N\Big|\big\langle \Psi_l, p^0_1 p^0_2 e^{ik(x_1-x_2)} q^0_1 q^0_2 \Psi_{l+2}\big\rangle\Big| &\leq \Big(\frac{N}{N-1}\Big)\Big\|\Big(\sum_{m=2}^N q^0_m e^{ikx_m} p^0_m\Big)\Psi_l\Big\| \; \|p^k_1 \Psi_{l+2}\| \\ &\leq \Big(1 + \frac{2}{N}\Big)\Big\|\Big(\sum_{m=2}^N q^0_m e^{ikx_m} p^0_m\Big)\Psi_l\Big\| \; \|p^k_1 \Psi_{l+2}\| \end{aligned} \tag{3.154}$$

(note that $\| \, |\varphi^0\rangle\langle\varphi^k|_1 \Psi\| = \|p^k_1 \Psi\|$ for any wave function Ψ). The sum is split into diagonal and off-diagonal contributions, this time using $q^0_2 e^{ikx_2} p^0_2 = |\varphi^{-k}\rangle\langle 1|_2$, $\forall k \in 2\pi\mathbb{Z}^d\backslash\{0\}$, i.e.

$$\begin{aligned} \Big\|\Big(\sum_{m=2}^N q^0_m e^{ikx_m} p^0_m\Big)\Psi_l\Big\|^2 &= N\|q^0_2 e^{ikx_2} p^0_2 \Psi_l\|^2 + N^2\big\langle q^0_2 e^{ikx_2} p^0_2 \Psi_l, q^0_3 e^{ikx_3} p^0_3 \Psi_l\big\rangle \\ &\leq N\|\Psi_l\|^2 + N^2\big\langle |\varphi^{-k}\rangle\langle 1|_2 \Psi_l, |\varphi^{-k}\rangle\langle 1|_3 \Psi_l\big\rangle \\ &= N\|\Psi_l\|^2 + N^2\|p^{-k}_1 p^0_2 \Psi_l\|^2, \end{aligned} \tag{3.155}$$

where we have used symmetry of Ψ_l in order to exchange integration variables. This leads to

$$N\Big|\big\langle \Psi_l, p^0_1 p^0_2 v_{12} q^0_1 q^0_2 \Psi_{l+2}\big\rangle\Big| \leq \sum_{k\neq 0} \hat{v}(k)\Big(\sqrt{N}\|\Psi_l\| \; \|p^k_1 \Psi_{l+2}\| + N\|p^{-k}_1 p^0_2 \Psi_l\| \; \|p^k_1 \Psi_{l+2}\|\Big) \tag{3.156}$$

$$+ \frac{2}{N}\sum_{k\neq 0} \hat{v}(k)\Big(\sqrt{N}\|\Psi_l\| \; \|p^k_1 \Psi_{l+2}\| + N\|p^{-k}_1 p^0_2 \Psi_l\| \; \|p^k_1 \Psi_{l+2}\|\Big). \tag{3.157}$$

Proceeding with the first summand in the first line (using $\varphi^k \perp \varphi^l$ for $l \neq k$),

$$\begin{aligned}\sum_{k\neq 0}\hat{v}(k)\sqrt{N}\|\Psi_l\|\ \|p_1^k\Psi_{l+2}\| &\leq \frac{\|\hat{v}\|_\infty}{2}\sqrt{N}\Big(\|\Psi_l\|\ \|q_1^0\Psi_{l+2}\|\Big)\\ &\leq \frac{\|\hat{v}\|_\infty}{2}\sqrt{l+2}\Big(\|\Psi_l\|\ \|\Psi_{l+2}\|\Big)\\ &\leq \|\hat{v}\|_\infty\frac{2}{\sqrt{l^*}}\Big(l\|\Psi_l\|^2+(l+2)\|\Psi_l\|^2\Big), \qquad (3.158)\end{aligned}$$

where one takes into account that $\frac{\sqrt{l+2}}{l} \leq \frac{4}{\sqrt{l}} \leq \frac{4}{\sqrt{l^*}}$ and further $\|q_1^0\Psi_{l+2}\|^2 = \frac{l+2}{N}\|\Psi_{l+2}\|^2$. In the second summand in (3.156), we insert the identity $1 = p_2^0 + q_2^0$, then use $\hat{v}(k) = \hat{v}(-k)$, and find

$$\begin{aligned}\sum_{k\neq 0}\hat{v}(k)N\|p_1^{-k}p_2^0\Psi_l\|\ \|p_1^k\Psi_{l+2}\| &\leq \frac{N}{2}\sum_{k\neq 0}\hat{v}(k)\Big(\|p_1^{-k}p_2^0\Psi_l\|^2+\|p_1^kp_2^0\Psi_{l+2}\|^2+\|p_1^kq_2^0\Psi_{l+2}\|^2\Big)\\ &\leq \frac{N}{2}\sum_{k\neq 0}\hat{v}(k)\Big(\|p_1^kp_2^0\Psi_l\|^2+\|p_1^kp_2^0\Psi_{l+2}\|^2\Big)+\|\hat{v}\|_\infty\frac{N}{2}\|q_1^0q_2^0\Psi_{l+2}\|^2\\ &\leq \frac{N}{2}\sum_{k\neq 0}\hat{v}(k)\Big(\|p_1^kp_2^0\Psi_l\|^2+\|p_1^kp_2^0\Psi_{l+2}\|^2\Big)+\frac{\|\hat{v}\|_1}{2}\sqrt{\delta}(l+2)\|\Psi_{l+2}\|^2,\end{aligned}$$

where we have used $N\|q_1^0q_2^0\Psi_{l+2}\|^2 = \frac{(l+2)^2}{N}\|\Psi_{l+2}\|^2 \leq \sqrt{\delta}(l+2)\|\Psi_{l+2}\|^2$ and $\|\hat{v}\|_\infty \leq \|\hat{v}\|_1$. Similarly, one finds for the sedond line

$$\begin{aligned}(3.157) &\leq \|\hat{v}\|_\infty\frac{1}{N\sqrt{l^*}}\Big(l\|\Psi_l\|^2+(l+2)\|\Psi_l\|^2\Big)\\ &\quad+\|\hat{v}\|_\infty\frac{1}{N}\Big(l\|\Psi_l\|^2+(l+2)\|\Psi_{l+2}\|^2\Big)+\|\hat{v}\|_1\frac{\sqrt{\delta}}{N}(l+2)\|\Psi_{l+2}\|^2\\ &\leq \|\hat{v}\|_\infty\Big(\frac{2}{\sqrt{l^*}}+\frac{2\|\hat{v}\|_1}{\|\hat{v}\|_\infty}\sqrt{\delta}\Big)\Big(l\|\Psi_l\|^2+(l+2)\|\Psi_{l+2}\|^2\Big). \qquad (3.159)\end{aligned}$$

Together, this leads to (3.46). In close analogy, one derives also the bound in (3.47). □

Proof of Lemma 3.9. We begin by recalling the definition of $\widetilde{V}^{\text{rest}}$,

$$\begin{aligned}&\Big|\big\langle\Psi_l,\widetilde{V}^{\text{rest}}\Psi\big\rangle - N\big\langle\Psi_l,q_1^0q_2^0v_{12}q_1^0q_2^0\Psi_l\big\rangle\Big| \qquad (3.160)\\ &\quad= N\Big|2\big\langle\Psi_l,q_1^0q_2^0v_{12}q_1^0p_2^0\Psi_{l-1}\big\rangle+2\big\langle\Psi_l,q_1^0p_2^0v_{12}q_1^0q_2^0\Psi_{l+1}\big\rangle+\big\langle\Psi_l,q_1^0q_2^0\hat{v}(0)q_1^0q_2^0\Psi_l\big\rangle\Big|.\end{aligned}$$

Let $\delta \in (0,1]$ and $l^*+2 \leq l \leq \lfloor\delta N\rfloor - 1$. For the first term, we find using the Fourier transform of v,

$$\begin{aligned}N\Big|\big\langle\Psi_l,q_1^0q_2^0v_{12}q_1^0p_2^0\Psi_{l-1}\big\rangle\Big| &\leq N\sum_{k\neq 0}\hat{v}(k)\Big|\big\langle q_1^0\Psi_l,|\varphi^k\rangle\langle\varphi^0|_2e^{ikx_1}q_1^0\Psi_{l-1}\big\rangle\Big|\\ &\leq N\sum_{k\neq 0}\hat{v}(k)\|q_1^0p_2^k\Psi_l\|\ \|q_1^0\Psi_{l-1}\|\\ &\leq N\|\hat{v}\|_\infty\frac{l\|\Psi_l\|}{N}\frac{\sqrt{l-1}\|\Psi_{l-1}\|}{\sqrt{N}}\\ &\leq \frac{\sqrt{\delta}}{2}\|\hat{v}\|_\infty\Big(l\|\Psi_l\|^2+(l-1)\|\Psi_{l-1}\|^2\Big). \qquad (3.161)\end{aligned}$$

The same way, one finds

$$N\left|\left\langle\Psi_l, q_1^0 p_2^0 v_{12} q_1^0 q_2^0 \Psi_{l+1}\right\rangle\right| \leq \frac{\sqrt{\delta}}{2}\|\hat{v}\|_\infty \Big(l\|\Psi_l\|^2 + (l+1)\|\Psi_{l+1}\|^2\Big), \tag{3.162}$$

and similarly also

$$N\left|\left\langle\Psi_l, q_1^0 q_2^0 v_{12} q_1^0 q_2^0 \Psi_l\right\rangle\right| = N\sum_{k\neq 0}\hat{v}(k)\left|\left\langle\Psi_l, q_1^0 q_2^0 e^{ik(x_1-x_2)} q_1^0 q_2^0 \Psi_l\right\rangle\right| \leq \delta\|\hat{v}\|_1 l\|\Psi\|^2. \tag{3.163}$$

□

Proof of Lemma 3.10. The lemma can be proven using the relations summarized below Definition (2.12) and also the pull through formula from Lemma 2.13. □

Appendices

3.A Condensation

For the homogeneous Bose gas with pair potential v satisfying Assumption 3.1, it is not difficult to show that the low-energy system condensates at leading order. Therefor note for symmetric wave function Ψ the lower bound for the interaction energy,

$$\begin{aligned}
&\frac{1}{N-1}\sum_{1\leq i<j\leq N}\left\langle\Psi, v(x_i-x_j)\Psi\right\rangle \\
&\quad= \frac{1}{2(N-1)}\sum_{1\leq i\neq j\leq N}\sum_{l,m,n,p}\left\langle\Psi, p_i^l p_j^m v(x_i-x_j) p_i^n p_j^p \Psi\right\rangle \\
&\quad= \frac{1}{2(N-1)}\sum_k \hat{v}(k)\sum_{l,m,n,p}\sum_{1\leq i\neq j\leq N}\left\langle\Psi, p_i^l p_j^m e^{ik(x_i-x_j)} p_i^n p_j^p \Psi\right\rangle \\
&\quad= \frac{N\hat{v}(0)}{2}\|\Psi\|^2 + \frac{1}{2(N-1)}\sum_{k\neq 0}\hat{v}(k)\left\|\sum_{i=1}^N\sum_l |\varphi^l\rangle\langle\varphi^{l+k}|_i\Psi\right\|^2 \\
&\qquad\quad - \frac{N}{2(N-1)}\sum_{k\neq 0}\hat{v}(k)\|\Psi\|^2 \\
&\quad\geq \frac{N\hat{v}(0)}{2}\|\Psi\|^2 - \frac{N}{N-1}\frac{v(0)}{2}\|\Psi\|^2,
\end{aligned} \tag{3.164}$$

where the sums in k, l, m, n, p are meant to run over all values in $2\pi\mathbb{Z}^d$ (recall that $\hat{v}\geq 0$). It then follows for all symmetric wave functions Ψ with $\langle\Psi, H\Psi\rangle/\|\Psi\|^2 = E^0 + K$ that (using the upper bound $E^0 \leq N\hat{v}(0)/2$)

$$\frac{N\hat{v}(0)}{2} - \frac{N}{N-1}\frac{v(0)}{2} \leq \frac{\langle\Psi, H\Psi\rangle}{\|\Psi\|^2} \leq \frac{N\hat{v}(0)}{2} + K. \tag{3.165}$$

For $\Psi = \Psi^n$ and $K = K^n \ll N$, this implies the assertion in (3.4). Furthermore, one finds

$$K \geq \frac{N\langle\Psi, (-\Delta_{x_1})\Psi\rangle}{\|\Psi\|^2} - \frac{N}{N-1}\frac{v(0)}{2}, \tag{3.166}$$

and thus, since $p_1^0 + q_1^0 = 1$ and $\Delta\varphi^0 = 0$,

$$\Big(K + \frac{N}{N-1}\frac{v(0)}{2}\Big)\|\Psi\|^2 \geq N\big\langle\Psi, q_1^0(-\Delta_{x_1})q_1^0\Psi\big\rangle \geq 4\pi^2 N\big\langle\Psi, q_1^0\Psi\big\rangle$$
$$= 4\pi^2 N\Big(\sum_{k=0}^{N}\frac{l}{N}\|P_{N,l}^0\Psi\|^2\Big) \qquad (3.167)$$

by means of the identity $\frac{1}{N}\sum_{i=1}^{N} q_i^0 = \frac{1}{N}\sum_{l=0}^{N} P_{N,l}^0$.

3.B Diagonalizing $\mathbb{H}_{\mathrm{Bog}}$

We briefly present the argument of diagonalizing the Bogoliubov Hamiltonian (3.9). This is a standard argument which can be found in the literature; here we quote the nice and compact exposition from [37, Section 6]. Let

$$A(k) = |k|^2 + \hat{v}(k), \qquad B(k) = \hat{v}(k), \qquad (3.168)$$

and define α_k, β_k, c_k and s_k via

$$\alpha_k = \frac{1}{B_k}\Big(A_k - \sqrt{A_k^2 - B_k^2}\Big) = \tanh(2\beta_k), \qquad (3.169)$$

$$c_k = \frac{1}{\sqrt{1-\alpha_k^2}} = \cosh(2\beta_k), \qquad (3.170)$$

$$s_k = \frac{\alpha_k}{\sqrt{1-\alpha_k^2}} = \sinh(2\beta_k). \qquad (3.171)$$

Moreover, for $S = e^{-X}$ with $X = \sum_{l\neq 0}\beta_l\big(a_l^* a_{-l}^* - a_l a_{-l}\big)$, using the Lie formula and the notation $[X, a_k]_{j+1} = [X, [X, a_k]_j]$, $[X, a_k]_0 = a_k$, one finds that

$$e^{-X}a_k e^X = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\big[X, a_k\big]_j = a_k + 2\beta_k a_{-k}^* + \frac{1}{2}4\beta_k^2 a_k + ... \qquad (3.172)$$

The $j = 1$ term, e.g., follows from

$$-\big[X, a_k\big] = -\sum_{l\neq 0}\beta_l\Big(a_l^*\big[a_{-l}^*, a_k\big] + \big[a_l^*, a_k\big]a_{-l}^*\Big) = \beta_{-k}a_{-k}^* + \beta_k a_{-k}^* = 2\beta_k a_{-k}^*, \qquad (3.173)$$

and similarly for $j \geq 2$. In total, the a_k transform as

$$S a_k S^* = c_k a_k + s_k a_{-k}^*. \qquad (3.174)$$

Application of the unitary transformation e^{-X} (note that X is antihermitian) to the Bogoliubov Hamiltonian then leads to

$$\mathbb{H}_{\mathrm{Bog}} = \sum_{k\neq 0}\frac{1}{2}\Big(A_k(a_k^* a_k + a_{-k}^* a_{-k}) + B_k(a_k^* a_{-k}^* + a_k a_{-k})\Big)$$
$$= -\frac{1}{2}\sum_{k\neq 0}\Big(A_k - \sqrt{A_k^2 - B_k^2}\Big) + \sum_{k\neq 0}\sqrt{A_k^2 - B_k^2}(c_k a_k^* + s_k a_{-k})(c_k a_k + s_k a_{-k}^*)\Big)$$

$$= E_{\text{Bog}} + S\Big(\sum_{k\neq 0} e(k)\ a_k^* a_k\Big) S^*, \tag{3.175}$$

with E_{Bog} and $e(k)$ defined as in (3.6) and (3.7). The spectrum of $\mathbb{H}_{\text{Bog}}$ is thus given by

$$E^0_{\text{Bog}} = E_{\text{Bog}}, \quad \text{and} \quad E^n_{\text{Bog}} = E_{\text{Bog}} + K^n_{\text{Bog}} \ \ (n \geq 1), \tag{3.176}$$

where K^n_{Bog} was defined in (3.8).

Below, we show some examples of the excitation spectrum $(K^n_{\text{Bog}} = K^n_{\text{Bog}}(p), p)$ for the one dimensional Bose gas for different pair potentials $v(x)$. The excitation energies are depicted w.r.t. to the total momentum $p = k_1 + ... + k_j \in (2\pi/L)\mathbb{Z}$ of the respective excitation, cf. (3.8). We show one- (red triangles), two- (blue disks) and three-particle excitations (green disks). The units are chosen such that $\frac{2\pi}{L} = \frac{1}{10}$, and the choice of the pair potential is indicated below the respective figure. Both, the scale and the presented examples are motivated from [37, Section 1] where similar figures of the excitation spectrum were presented.

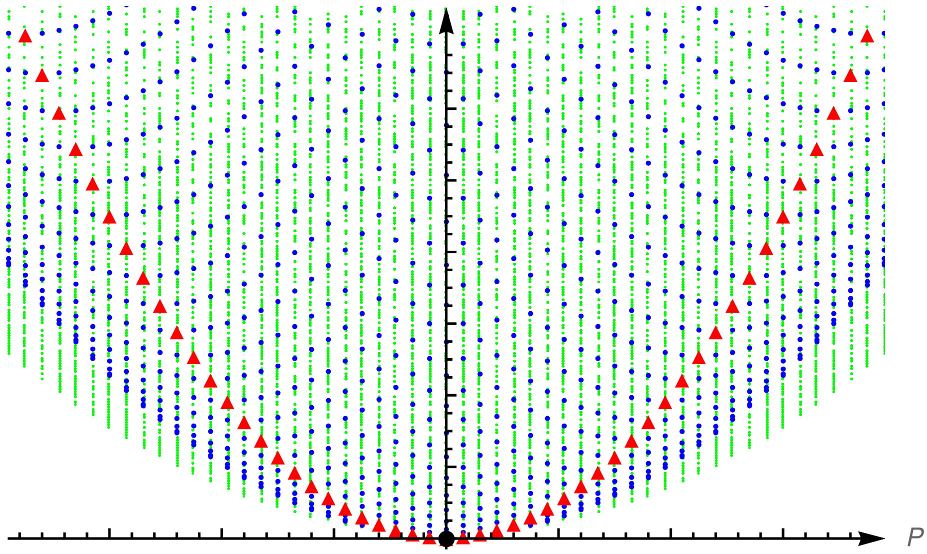

Figure 3.B.1: Excitation spectrum $(K^n_{\text{Bog}}(p), p)$ for pair potential $\hat{v}(k) = \frac{1}{10} e^{-\frac{k^2}{5}}$.

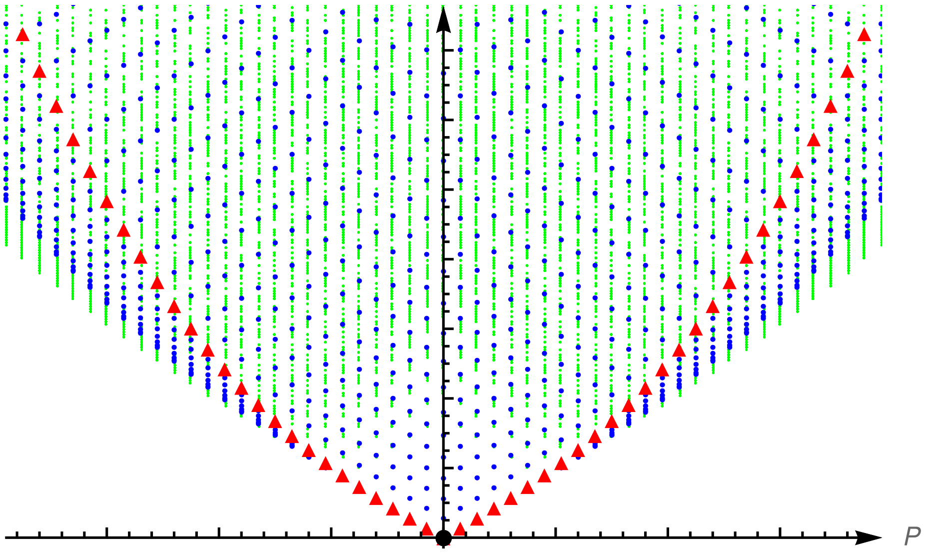

Figure 3.B.2: Excitation spectrum $(K^n_{\text{Bog}}(p), p)$ for pair potential $\hat{v}(k) = \frac{50}{2} e^{-\frac{k^2}{2}}$

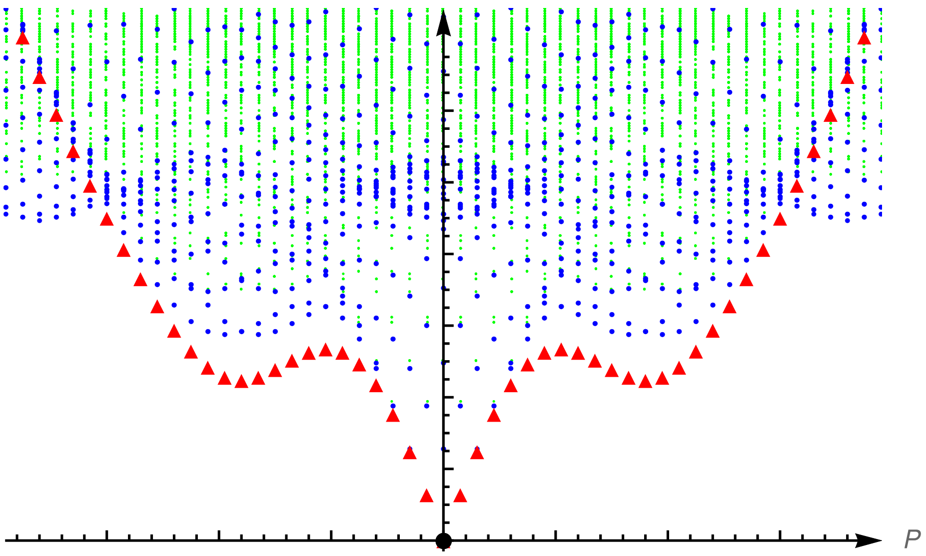

Figure 3.B.3: Excitation spectrum $(K^n_{\text{Bog}}(p), p)$ for pair potential $\hat{v}(k) = \frac{75}{2} e^{-k^2}$

Chapter 4

Free dynamics of a tracer particle coupled to a dense Fermi gas

We derive the effective free time evolution of a tracer particle coupled to the ideal Fermi gas in the high density limit in two spatial dimensions. After presenting the model and our main theorem, we discuss interesting aspects regarding the model and the claimed result. The proof of the main result is provided in Section 4.2 and we close the chapter with a list of four appendices.[1]

4.1 Introduction and main result

In this chapter we consider the dynamics of a tracer particle interacting with a dense and homogeneous two-dimensional fermionic gas. In order to keep the analysis simple we neglect the interaction between the gas particles and focus only on the interaction between tracer particle y and gas particles $x_1, \ldots, x_N$. The general model we wish to study is defined by the Hamiltonian[2]

$$H = -\frac{1}{2m_y}\Delta_y - \sum_{i=1}^{N} \frac{1}{2m_x}\Delta_{x_i} + g\sum_{i=1}^{N} v(x_i - y), \tag{4.1}$$

where $v \in C_0^\infty$ (the space of smooth functions with compact support), and $g > 0$ is a coupling constant. The time evolution of the (N+1)-body wave function $\Psi_t \in \mathcal{H}_y \otimes \mathcal{H}_N = L^2(\mathbb{T}^d) \otimes L^2(\mathbb{T}^{dN})$, where d is the dimension, $\mathbb{T}$ a one-dimensional torus of length $L \in \mathbb{R}$, and L^2 denotes the space of complex square integrable functions (for simplicity, we neglect spin), is given by the Schrödinger equation

$$i\partial_t \Psi_t = H\Psi_t. \tag{4.2}$$

As initial condition we choose a factorized state $\Psi_0 = \varphi_0 \otimes \Omega_0$, where $\varphi_0 \in \mathcal{H}_y$ is the initial wave function of the tracer particle and $\Omega_0 \in \mathcal{H}_N$ is the free fermionic ground state with periodic boundary conditions in the d-dimensional box of side length L. For analyzing Ψ_t we

[1] Note: This chapter was published together with Maximilian Jeblick, Sören Petrat and Peter Pickl in [69]; compared to [69], two new appendices were added. The work is for the most part due to the author of the thesis. It is a continuation of an earlier collaboration with the mentioned coauthors, in particular with Maximilian Jeblick, about the one-dimensional model [67, 68].

[2] For ease of notation we do not explicitly indicate the N and L dependence of the Hamiltonian $H \equiv H_{N,L}$, resp. the wave functions $\Psi \equiv \Psi_{N,L}$ throughout this chapter.

first take the limit $N, L \to \infty$ with $\varrho = N/L^d = const.$ in order to remove finite size effects and then consider large gas densities ϱ. Note that in this situation the average potential energy of the tracer particle is proportional to $g\varrho$. We later choose $g = 1$, such that our analysis is beyond any weak-coupling limit.

We expect that the above model exhibits some interesting phenomena which depend in particular on the time scale that one considers. Here, we consider time scales for which the tracer particle moves in the mean field of the gas particles. Since the mean field potential is spatially homogeneous for the ideal Fermi gas, the effective dynamics is equivalent to the free time evolution. For longer times, we expect that the tracer particle will create electron-hole pairs and eventually lose its energy. The situation may differ depending also on the spatial dimension. For reasons which we explain in Section 4.1.2, we focus on the two dimensional case. Let us also remark that the described model is relevant, e.g., for understanding the motion of ions in a degenerate and dense electron plasma. In this situation it is known that the ability of the plasma to stop ions decreases in the high-density limit; cf. Section 4.1.2.

We prove in this chapter that in the limit $\varrho \to \infty$, the time evolution of the tracer particle is close to the free dynamics on a particularly large time scale, namely for $t \ll \varrho^{1/12}$. Our main result is readily stated:

Theorem 4.1. *Let $d = 2$, the masses $m_x = m_y = 1/2$ and the coupling constant $g = 1$. Let further $\Psi_0 = \varphi_0 \otimes \Omega_0$ where $\varphi_0 \in \mathcal{H}_y$ with $\|\nabla^4 \varphi_0\| \le C$ uniformly in $\varrho = N/L^2$ and Ω_0 is the free fermionic ground state in $\mathbb{T}^2$. Then, for any small enough $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$\lim_{\substack{N,L\to\infty \\ \varrho = N/L^2 = const.}} \left\| e^{-iHt}\Psi_0 - e^{-iH^{\mathrm{mf}}t}\Psi_0 \right\|_{\mathcal{H}_y \otimes \mathcal{H}_N} \le C_\varepsilon (1+t)^{\frac{3}{2}} \varrho^{-\frac{1}{8}+\varepsilon} \tag{4.3}$$

holds for all $t > 0$, where

$$H^{\mathrm{mf}} = -\Delta_y - \sum_{i=1}^{N} \Delta_{x_i} + \varrho \hat{v}(0) - E_{re}(\varrho) \tag{4.4}$$

is the free Hamiltonian with constant mean field $\varrho\hat{v}(0) = \langle \Omega_0, \sum_{i=1}^N v(x_i - y)\Omega_0 \rangle_{\mathcal{H}_N}$ minus a positive ϱ-dependent next-to-leading order energy correction $E_{re}(\varrho)$ which is defined in (4.10).

Let us remark that in Theorem 4.1, we have fixed all scales except for the density ϱ and the time t. The statement is meaningful for all pairs of ϱ and t for which the r.h.s. of (4.3) becomes small compared to one. A more detailed expression for the error term can be inferred from (4.50) in combination with Lemma 4.2. The proof of Theorem 4.1 is given in Section 4.2. Before we discuss the model, the theorem and its application in physics in more detail, let us stress that Theorem 4.1 is nontrivial and might be surprising at first sight:

- Contrast the situation with a tracer particle in a classical or bosonic gas. Since the velocity of the tracer particle is of order one and the interaction proportional to the density ϱ, then after times of $O(1)$, the tracer particle has scattered with $O(\varrho)$ particles in the gas. The expected mean free path of the tracer particle is accordingly small, namely $\propto \varrho^{-\delta}$ for some $\delta > 0$.

- In a fermionic gas, the kinetic energy of the tracer particle can dissipate into its environment by means of particle-hole excitations. One might expect that this kind of friction mechanism would become stronger the larger ϱ. This is the case for a tracer particle in a Bose gas which was shown in the mean field regime on a rigorous level in [53, 52, 15]. For fermions one finds a different behavior: the larger the density, the less the particle is disturbed and, vice versa, disturbs the gas less. As a consequence, the free motion holds on a much larger time scale $t = 0(\varrho^{\delta})$ for some $\delta > 0$; cf. the r.h.s. in (4.3).

Our result follows from a careful analysis of the fluctuations in the gas and their propagation, and relies heavily on the Fermi pressure, i.e., the antisymmetry of the wave function of the fermionic particles. We give a sketch of the proof in Section 4.1.3 and provide a physically more intuitive explanation in Section 4.1.3.

4.1.1 The model in more detail

Let us discuss the considered model and its properties in more detail. First, note that we do not take any internal degrees of freedom such as spin into account. On the level of our main result, we do not expect a qualitative different behavior by doing so. Note also that our focus lies on the analysis of the interaction between the tracer particle and the gas, whereas the mutual interaction of the gas particles is neglected. While this is generally expected to be a reasonable approximation for many situations, its rigorous justification is a very interesting question on its own. Physical units are chosen such that the constant $\hbar$ and the masses of tracer particle and gas particles are dimensionless and $\hbar = 2m_y = 2m_x = 1$.

We model the potential between the tracer particle and each of the gas particles by an infinitely differentiable function with compact support (uniformly in L), i.e., $v \in C_0^\infty(\mathbb{T}^2) \cap C_0^\infty(\mathbb{R}^2)$. Theorem 4.1 holds as well for less regular potentials with fast enough decay at infinity. In order to simplify the proof as much as possible, however, the chosen class of potentials is very convenient. We often abbreviate the total interaction term in H by $V = \sum_{i=1}^N v(x_i - y)$. Since V is bounded, H defines a self-adjoint operator on the second Sobolev space $H^2(\mathbb{T}^{2(N+1)}) \subset \mathcal{H}_y \otimes \mathcal{H}_N$. For the corresponding time evolution, we write $U(t) = e^{-iHt}$, $t \geq 0$.

The initial wave function φ_0 of the tracer particle is restricted to be an element of $H^4(\mathbb{T}^2) \subset \mathcal{H}_y$ with $\|\varphi_0\|_{H^4} < C$ for all values of ϱ. The initial state of the gas is assumed to be given by the ground state of the ideal Fermi gas which is described by the antisymmetric product of N one-particle plane waves,

$$\Omega_0(x_1, ..., x_N) = \frac{1}{\sqrt{N!}} \sum_{\tau \in S_N} (-1)^\tau \prod_{i=1}^N \phi_{p_{\tau(i)}}(x_i), \tag{4.5}$$

with $\phi_p(x) = L^{-1} e^{ip\cdot x} \in L^2(\mathbb{T}^2)$, and $(p_j)_{j=1}^N$ the N pairwise different elements of $(2\pi/L)\mathbb{Z}^2$ with smallest absolute value. S_N denotes the group of permutations of integers $\{1, ..., N\}$ and $(-1)^\tau$ is the sign of the permutation τ. Since the system is defined on a torus of side length L (with periodic boundary conditions), the set of possible momenta in the gas is given by the lattice $(2\pi/L)\mathbb{Z}^2$. We label the momenta such that for $j_1, j_2 \geq 1$ we have $j_1 < j_2 \Leftrightarrow |p_{j_1}| \leq |p_{j_2}|$. The wave function Ω_0 corresponds thus to the lowest possible kinetic energy given by $\sum_{k=1}^N p_k^2$.

It is later very convenient to use fermionic creation and annihilation operators. For wave functions $\Psi \in \mathcal{H}_y \otimes \mathcal{H}_N$ which are antisymmetric in the gas-coordinates, we have

$$a^*(p_l)a(p_k)\Psi(y, x_1, ..., x_N) = \sum_{i=1}^{N} \phi_{p_l}(x_i) \int_{\mathbb{T}^2} dz\, \phi^*_{p_k}(z)\Psi(y, x_1, ..., x_{i-1}, z, x_{i+1}, ...x_N), \quad (4.6)$$

i.e., a particle with momentum $p_k \in (2\pi/L)\mathbb{Z}^2$ is replaced by a particle with momentum $p_l \in (2\pi/L)\mathbb{Z}^2$.

An important quantity that characterizes the state Ω_0 is the Fermi momentum k_F. It is defined as $k_F = |p_N|$ where p_N belongs to the set of momenta $\{p_k \in (2\pi/L)\mathbb{Z}^2 : k = 1, ..., N\}$ which minimizes the kinetic energy $\sum_{i=1}^{N} p_{k_i}^2$. The value k_F defines the so-called Fermi sphere and is related in two space dimensions to the average density ϱ via

$$\varrho = \frac{1}{L^2}\sum_{k=1}^{N} = \int_{|p|\le k_F} \frac{d^2p}{(2\pi)^2} = \frac{k_F^2}{4\pi} \qquad \Leftrightarrow \qquad k_F = \sqrt{4\pi\varrho}. \quad (4.7)$$

We study the model in the thermodynamic limit, i.e., for $N, L \to \infty$, and $\varrho = N/L^2 = const.$ This simplifies the analysis because it allows us to ignore additional effects which are due to the chosen boundary conditions. For very large systems, i.e., in particular for $L/\text{supp}(v) \gg 1$, such boundary effects are not expected to be physically relevant which justifies the analysis in the thermodynamic limit. We emphasize that for the result we are interested in in this work, it is really the parameter $\varrho \gg 1$ which is the physically interesting one. We expect a very similar result to hold if one repeats all estimates for fixed but large values of N and L, and then considers the regime in which $N \gg L$.

Let us next discuss the effective model. The effective dynamics is described by the Schrödinger equation with mean field Hamiltonian H^{mf}. Note that H^{mf} is also self-adjoint on $H^2(\mathbb{T}^{2(N+1)})$ and the corresponding mean field time evolution is denoted as $U^{\text{mf}}(t)$, $t \ge 0$. The average potential w.r.t. Ω_0 that acts at position $y \in \mathbb{T}^2$,

$$E(y) = \langle \Omega_0, V\Omega_0\rangle_{\mathcal{H}_N}(y) = \varrho\hat{v}(0), \quad (4.8)$$

where $\hat{v}$ denotes the Fourier transform, is spatially constant. The homogeneity of $E(y) = E$ is furthermore conserved under the mean field time evolution $U^{\text{mf}}(t)$, i.e.,

$$\langle \Omega_t^f, V\Omega_t^f\rangle_{\mathcal{H}_N} = \langle \Omega_0, V\Omega_0\rangle_{\mathcal{H}_N}, \qquad \Omega_t^f = e^{-iH_N^f t}\Omega_0, \quad (4.9)$$

where $H_N^f = -\sum_{i=1}^{N} \Delta_{x_i}$ denotes the free Hamiltonian of the gas. The Schrödinger equation with Hamiltonian (4.4) defines therefore a self-consistent approximation. The reason why we call H^{mf} a mean field Hamiltonian is that to leading order, it is obtained from H by replacing the potential V by its average value E. The constant $E_{re}(\varrho)$ is due to immediate recollisions between the tracer particle and gas particles. It is given by

$$E_{re}(\varrho) = \lim_{\substack{N,L\to\infty \\ \varrho = N/L^2 = const.}} \frac{1}{L^4}\sum_{k=1}^{N}\sum_{l=N+1}^{\infty} \frac{|\hat{v}(p_k - p_l)|^2}{p_l^2 - p_k^2}\theta\Big(|p_l| - |p_k| - \varrho^{-\frac{1}{2}}\Big), \quad (4.10)$$

where $\theta(x)$ denotes the usual Heaviside step function, i.e., $\theta(x) = 1$ for $x \ge 0$ and zero otherwise. Eq. (4.60) of Lemma 4.3 shows that for any $\varepsilon > 0$ there are positive constants C, C_ε such that

$$C \le E_{re}(\varrho) \le C\varrho^{2\varepsilon} + C_\varepsilon \varrho^{-1/\varepsilon}. \quad (4.11)$$

Since $\varrho\hat{v}(0) - E_{re}(\varrho)$ is constant as a function of the coordinates $y, x_1, ..., x_N$, the time evolution $U^{\mathrm{mf}}(t)$ is physically equivalent to the free dynamics generated by $H_y^f + H_N^f$ (where $H_y^f = -\Delta_y$).

Note that in the rest of this chapter we omit the subscripts $\mathcal{H}_y$, $\mathcal{H}_N$ or $\mathcal{H}_y \otimes \mathcal{H}_N$ on all scalar products and norms, since it is always clear from the argument on which space the scalar product or norm is meant.

4.1.2 Discussion of the main result

We give a list of nonrigorous remarks and assertions about various aspects of the described model and Theorem 4.1.

Spectral properties

H and H^{mf} describe translation invariant systems and therefore the total momentum is conserved by both dynamics, $U(t)$ as well as $U^{\mathrm{mf}}(t)$. In the microscopic model, however, the initial momentum of the tracer particle is not necessarily conserved due to the presence of the interaction. The joint energy-momentum spectrum of $(H, \hat{P}_{\mathrm{tot}})$, $\hat{P}_{\mathrm{tot}} = -i\nabla_y - \sum_{i=1}^N i\nabla_{x_i}$ being the total momentum operator, is thus expected to consist of degenerate values $(E_{\mathrm{tot}}, P_{\mathrm{tot}})$ where the degeneracy results from the different possibilities of splitting the total momentum P_{tot} between the tracer particle and the gas. For $(E_{\mathrm{tot}}, P_{\mathrm{tot}}) = (\langle\Psi_0, H\Psi_0\rangle, \langle\Psi_0, \hat{P}_{\mathrm{tot}}\Psi_0\rangle)$, the kinetic energy of the tracer particle may assume values between $E_y^{\mathrm{kin}} = 0$ and $E_y^{\mathrm{kin}} = P_{\mathrm{tot}}^2$. Note here that the smallest excitation energy of the gas is equal to $4\pi^2/L^2 \ll 1$. It is not difficult to verify that for every value $q^2 \in [0, P_{\mathrm{tot}}^2]$, there exists a wave function $\Psi^q \in \mathcal{H}_y \otimes \mathcal{H}_N$, such that

$$\langle\Psi^q, -\Delta_y\Psi^q\rangle = q^2 + O(L^{-2}), \quad \langle\Psi^q, \Psi^{q'}\rangle = 0 \text{ for } |q - q'| > 4\pi^2/L^2, \tag{4.12}$$

while the Ψ^q are dynamically accessible in the sense that

$$\big\langle\Psi^q, H\Psi^q\big\rangle = \big\langle\Psi_0, H\Psi_0\big\rangle, \quad \big\langle\Psi^q, \hat{P}_{\mathrm{tot}}\Psi^q\big\rangle = \big\langle\Psi_0, \hat{P}_{\mathrm{tot}}\Psi_0\big\rangle. \tag{4.13}$$

This can be seen as follows: let us consider single particle-hole excitations and assume that the tracer particle has initial momentum $\big\langle\varphi_0, (-i\nabla)\varphi_0\big\rangle = \big\langle\Psi_0, \hat{P}_{\mathrm{tot}}\Psi_0\big\rangle = P_0 \neq 0$. From energy conservation it follows that $p_k^2 + P_0^2 = (P_0 - \delta p)^2 + (p_k + \delta p)^2$, where $\delta p = p_\ell - p_k$ is the momentum transfer between the tracer particle and the gas ($|p_k| \leq k_F$, $|p_\ell| > k_F$). This implies the condition $(p_\ell - p_k)(p_\ell - P_0) = 0$. In Figure 4.1 (l.h.s.), the grey disk around P_0 indicates the set of δp that satisfy energy conservation. For each such δp one can find a p_k that also satisfies the above conditions. The grey ring at the circumference of the Fermi sphere shows all such possible momenta p_k for given P_0. All these particle-hole excitations lower the kinetic energy of the tracer particle while they do not change the total energy and the total momentum of the system. On the r.h.s. of Figure 4.1, we depict the resulting kinetic energy spectrum of the tracer particle: for $P_{\mathrm{tot}} \neq 0$ and in the limit $L \to \infty$, our initial wave function Ψ_0 lies on top of a continuous fiber $[0, P_{\mathrm{tot}}^2]$ of dynamically accessible states (in the sense explained above).[3]

Although the rigorous analysis of spectral properties in the thermodynamic limit is very subtle, we expect the above considerations to be true for $d \geq 2$ and $\varrho \gg 1$.

[3] To illustrate the argument more explicitly, let us give a very simple example of a transition that conserves total energy and total momentum, but lowers the kinetic energy of the tracer particle. Suppose, the initial

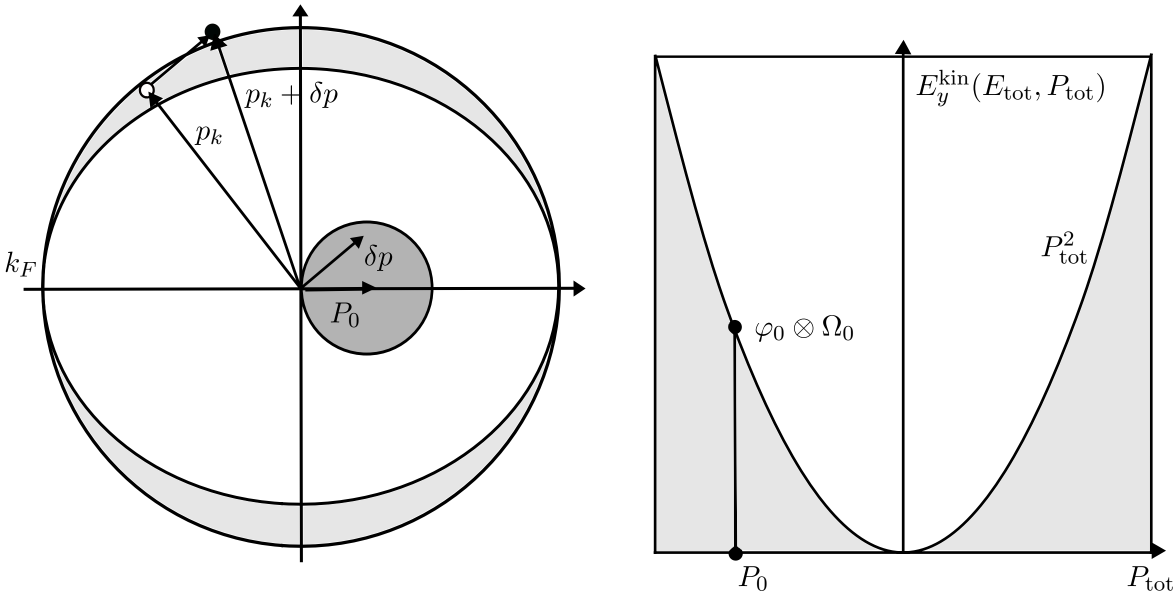

Figure 4.1: L.h.s.: A possible particle-hole excitation in the 2d Fermi sphere which lowers the kinetic energy of the tracer particle from P_0^2 to $(P_0 - \delta p)^2$ while leaving total momentum $P_0 = \left\langle \Psi_0, \hat{P}_{\mathrm{tot}} \Psi_0 \right\rangle$ and total energy $E_0 = \left\langle \Psi_0, H \Psi_0 \right\rangle$ unchanged. R.h.s.: Reduced kinetic energy spectrum of the tracer particle $E_y^{\mathrm{kin}}(E_{\mathrm{tot}}, P_{\mathrm{tot}})$ for given value E_{tot} and as function of P_{tot}. The part below P_{tot}^2 corresponds to dynamically accessible states for which $E_y^{\mathrm{kin}}(E_{\mathrm{tot}}, P_{\mathrm{tot}})$ lies between zero and P_{tot}^2.

Dimension

The spectral properties are very different in one spatial dimension. For $d = 1$, there are no dynamically accessible states, i.e., wave functions with total energy and momentum equal to that of Ψ_0, for which the average kinetic energy of the tracer particle is smaller than its initial value P_{tot}^2. Any nonnegligible momentum transfer from φ_0 to Ω_0 would cause an increase in the energy of the gas proportional to k_F which is due to the quadratic energy dispersion relation.[4] The reduced kinetic energy spectrum of the tracer particle in one dimension and for large k_F is therefore the same as in the free model: $E_y^{\mathrm{kin}} = \langle \Psi_0, \hat{P}_{\mathrm{tot}} \Psi_0 \rangle^2$, with no other values allowed. This makes a result similar to Theorem 4.1 less surprising. A rigorous analysis of the one dimensional model was carried out in [67]. In Appendix 4.C, we write down the theorem for $d = 1$ and give a short sketch of how the the argument we employ to prove Theorem 4.1 is adapted to the one dimensional case.

For $d = 3$, the spectral properties are similar to the case $d = 2$. However, for $d = 3$, it is unclear if a similar result about the dynamics holds; see also Remark 4.1 and Appendix 4.D where we explain the additional difficulties in more detail. This is why we chose to study $d = 2$.

momentum of the tracer particle is given by $P_0 \neq 0$. Let us now consider a particle-hole excitation that absorbs all the momentum of the tracer particle, i.e., the momentum transfer is $\delta p = P_0$, such that the total momentum is conserved. Then the difference in the total energy before and after the collision is

$$\delta E = P_0^2 + p_k^2 - \left(p_k + P_0\right)^2 = -2 p_k \cdot P_0. \tag{4.14}$$

Therefore, energy is conserved in this case if $p_k \cdot P_0 \approx 0$.

[4] This can be seen from (4.14): In one dimension the energy difference is $-2p_k \cdot P_0 \propto -k_F$ for any p_k near the Fermi surface. For $d \geq 2$, this is different as explained above and indicated in Figure 4.1.

Asymptotic energy loss

From the spectral picture that was explained in Section 4.1.2, we expect that eventually, the kinetic energy of the tracer particle dissipates into the gas by means of particle-hole excitations. Recall Figure 4.1 (r.h.s): for $L \to \infty$, the initial wave function Ψ_0 lies above the continuous fiber over $P_0 = \langle \Psi_0, \hat{P}_{\text{tot}} \Psi_0 \rangle$. If the initial momentum is nonzero, the tracer particle occupies an excited state which is coupled to a dispersive medium with a large number of degrees of freedom. In such a situation, one may expect that the excited subsystem approaches asymptotically its lowest energy state. For the Fermi gas, this friction mechanism is suppressed for large values of ϱ. Theorem 4.1 states that Ψ_0, or equivalently, the initial momentum distribution of the tracer particle, is stable on a large time scale, namely at least for $t = o(\varrho^{1/12})$. On some larger time scale the tracer particle is expected to slow down until it reaches its ground state $E_y^{\text{kin}} = 0$.

The rigorous understanding of existence and properties such as lifetime and decay rate of long-lived resonances is, however, very difficult. It needs more refined techniques and perhaps a more general formulation of the model (e.g., defining it directly on $\mathbb{R}^2$) in order to describe the physically correct behavior for $t \to \infty$. In [82], e.g., a similar question was studied on the level of the (fermionic) Hartree equation for which it was shown that a small defect added initially to the translation-invariant homogeneous state disappears for large times due to dispersive effects of the gas.

Norm distance

Since we consider a regime of strong coupling, even a single collision can be enough to disturb the free motion of the system. It is thus necessary to prove that all particles behave according to the mean field equation. This is the reason why the difference in norm between $U(t)\Psi_0$ and $U^{\text{mf}}(t)\Psi_0$ is the right quantity to consider. Note that the situation is different compared to the weak coupling regime where the aim is usually to prove that the relative number of particles which evolve according to the mean field potential is close to one; see, e.g., [13, 39, 54, 18, 101, 11, 100] for the fermionic case.

Fluctuations and mean field regime

The substitution of the potential V in $U(t)\Psi_0$ by its average value E would be easy to justify if fluctuations around E were negligibly small, i.e., if $V \approx E$ would hold with probability close to one (w.r.t. the probability density defined by $|\Omega_t^f|^2$). We show in Lemma 4.3, Eq.(4.57), that this is not the case: while $\lim_{N,L\to\infty,\varrho=const.} \|(V-E)\Omega_t^f\|^2$ is suppressed in the sense that it grows only with $\sqrt{\varrho}$ instead of ϱ (as naively expected from the square root of N law), it still diverges in the limit $\varrho \to \infty$. The reason for the large fluctuations is the strong coupling $g = 1$. If we had assumed a weak coupling, say $g = \varrho^{-1}$, the fluctuations would vanish when ϱ tends to ∞ and an estimate like in Theorem 4.1 would follow almost trivially. We emphasize this because it exemplifies an interesting fact: the mean field regime for fermions does not necessarily coincide with a weak coupling limit $g \to 0$ ($\varrho \to \infty$). For bosons, on the other hand, the mean field regime coincides with the weak coupling limit. In other words, Theorem 4.1 provides an explicit example of a setting where the accuracy of the mean field approximation can be proven for a range of coupling constants g which is much larger compared to the range in the bosonic or classical case. In Section 4.1.3 we give a short explanation of why the mean field description is valid even though the fluctuations can be very large.

Let us also remark that more generally one finds for $d = 1, 2, 3$ spatial dimensions,

$$\lim_{\substack{N,L\to\infty \\ \varrho=N/L^2=const.}} \left\|(V-E)\Omega_t^f\right\|^2 = C_d \varrho^{\frac{d-1}{d}}, \tag{4.15}$$

with d-dependent constants C_d. A similar result about the suppression of fluctuations in a Fermi gas has been mentioned in [27, Eqs. (48)-(50)]. Compared to (4.15), there appears an additional factor $\ln \varrho$ on the r.h.s. which is due to the fact that v was chosen less regular than in our case.

Subleading energy correction $E_{re}(\varrho)$

In Lemma 4.3 we show for $d = 2$ that $C \leq E_{re}(\varrho) \leq C\varrho^{2\varepsilon} + C_\varepsilon \varrho^{-1/\varepsilon}$ for any $\varepsilon > 0$ and positive constants C, C_ε. This means in particular that the claimed estimate in Theorem 4.1 would not be correct without including $E_{re}(\varrho)$ in the definition of H^{mf}. Nevertheless, $E_{re}(\varrho)$ is only a subleading correction to the mean field energy $\varrho\hat{v}(0)$. It arises from so-called immediate recollisions, i.e., collisions of the type where the tracer particle excites a particle-hole pair in the gas and then immediately recollides with the excited particle which recombines with the hole. Such processes appear in the expansion of Ψ_t into the different collision histories that have to be controlled, see the end of Section 4.2.3. Let us remark that if we iterate the Duhamel expansion (4.16) infinitely often, one can identify in each order terms that contain only immediate recollisions. Then one can indeed show that the phase factor $e^{iE_{re}(\varrho)t}$ cancels exactly the leading order contribution of those immediate recollision terms summed up in all orders. We explain this in some more detail in Appendix 4.B. From the definition of $E_{re}(\varrho)$ in (4.10) one can see that only gas particles near the Fermi surface contribute to $E_{re}(\varrho)$.

Application to physics

The presented model is very close to the physically interesting situation of ions moving through a degenerate and dense electron plasma. An understanding of what is often referred to as slowing down of ions in a degenerate plasma has been of interest in the physics literature at least since a work by Fermi and Teller in 1947 [49] (see also [93]). They have pointed out that the efficiency of the gas for slowing down ions with velocities far below the Fermi edge is very low. The same question has later been analyzed explicitly for the high-density case for which the energy loss of the ions was found to be caused mainly by (rare) collisions with other ions instead of interactions with the electrons from the plasma; see, e.g., [114, 33, 125, 126]. These results raised considerable interest in the field of nuclear physics in which it was known that the existence of long-lived ions in the plasma is essential for the occurrence of fusion reactions; e.g., [26, 99]. Let us stress that to our knowledge, the analysis has remained so far on a purely formal level. The rigorous bound we present here (even though for a much simpler model), starting from the microscopic dynamics and taking into account the full strong interaction, seems to be novel.

4.1.3 Sketch of the proof

For deriving Theorem 4.1 we use Duhamel's expansion in order to decompose Ψ_t into different wave functions that correspond to different collision histories of the tracer particle. The main difficulty is to control the interaction with particles occupying momenta close to the Fermi edge. Our main ingredient here is the large shift in the energy and the thereby caused phase cancellation during the scattering with such particles. It turns out to be necessary but also sufficient to use a third order expansion in the difference $H - H^{\mathrm{mf}}$. Let us stress again that $g = 1$. This prevents us from using a straightforward order by order expansion of the time evolution. Thus, after expanding to third order, we have to estimate an error term involving the whole time evolution $U(t)$. In order to convey the main ideas and techniques behind the proof, let us start by expanding

$$U(t)\Psi_0 - U^{\mathrm{mf}}(t)\Psi_0 = -i\int_0^t d\tau_1 U^{\mathrm{mf}}(t-\tau_1)(H - H^{\mathrm{mf}})U^{\mathrm{mf}}(\tau_1)\Psi_0 \\ - i\int_0^t d\tau_1 \Big(U(t-\tau_1) - U^{\mathrm{mf}}(t-\tau_1)\Big)(H - H^{\mathrm{mf}})U^{\mathrm{mf}}(\tau_1)\Psi_0, \tag{4.16}$$

which follows from expanding U around U^{mf} in terms of Duhamel's formula and then splitting $U = U^{\mathrm{mf}} + (U - U^{\mathrm{mf}})$. The first term on the r.h.s. contains deviations from the effective dynamics due to single particle-hole excitations. In order to present the main argument, let us ignore the next-to-leading order energy correction $E_{re}(\varrho)$ in the following. Using some elementary algebra (only momenta inside the Fermi sphere can be annihilated and momenta outside the Fermi sphere created), one readily rewrites

$$(V - E)\Psi_0 = \frac{1}{L^2}\sum_{k=1}^{N}\sum_{l=N+1}^{\infty} \hat{v}(p_l - p_k)\Big(e^{i(p_l-p_k)y}\varphi_0\Big) \otimes a^*(p_l)a(p_k)\Omega_0. \tag{4.17}$$

Abbreviating $k_{kl}(\tau_1) = e^{iH_y^f\tau_1}e^{i(p_l-p_k)y}e^{-iH_y^f\tau_1}$ (which evolves the tracer particle freely to time τ_1 at which its momentum is changed by $p_l - p_k$ and then evolves it back to the initial time), it is also straightforward to arrive at

$$\Big\|\int_0^t d\tau_1 U^{\mathrm{mf}}(-\tau_1)(V - E)U^{\mathrm{mf}}(\tau_1)\Psi_0\Big\|^2 \\ = \underbrace{\frac{1}{L^4}\sum_{k=1}^{N}\sum_{l=N+1}^{\infty}\big|\hat{v}(p_k - p_l)\big|^2}_{=\|(V-E)\Omega_t^f\|^2}\Big\|\int_0^t d\tau_1 e^{i(p_l^2-p_k^2)\tau_1}k_{kl}(\tau_1)\varphi_0\Big\|^2. \tag{4.18}$$

Due to the regularity of the potential v it is unlikely that a single collision causes a large momentum transfer between φ_0 and Ω_0. This is reflected in the fact that the Fourier transform of a smooth and compactly supported function decays faster than any polynomial: for all $p \in \mathbb{N}$ there exists a constant D_p such that

$$\big|\hat{v}(p_k - p_l)\big| \leq \frac{D_p}{(1 + |p_k - p_l|)^p}, \tag{4.19}$$

which follows directly from the Paley-Wiener Theorem; e.g., [112, Theorem XI.11]. At this point it is convenient to introduce the following notation. For $\varepsilon > 0$ we define $v^{\ell,\varepsilon}$ and $v^{s,\varepsilon}$ such that

$$\hat{v}^{\ell,\varepsilon}(p_k - p_l) = \theta\big(|p_k - p_l| - \varrho^{\varepsilon}\big)\hat{v}(p_k - p_l) \tag{4.20}$$

$$\hat{v}^{s,\varepsilon}(p_k - p_l) = \theta\big(\varrho^{\varepsilon} - |p_k - p_l|\big)\hat{v}(p_k - p_l). \tag{4.21}$$

The transition amplitude $|\hat{v}^{\ell,\varepsilon}(p_k - p_l)|^2$ is negligible for $\varrho \gg 1$ which can be inferred from (4.19). What remains to be bounded is the transitions in (4.18) with momentum transfer of order one, i.e.,

$$\frac{1}{L^4}\sum_{k=1}^{N}\sum_{l=N+1}^{\infty}\left|\hat{v}^{s,\varepsilon}(p_k-p_l)\right|^2\left\|\int_0^t d\tau_1 e^{i(p_l^2-p_k^2)\tau_1}k_{kl}(\tau_1)\varphi_0\right\|^2. \tag{4.22}$$

The reason that this term vanishes as well is the oscillation of the integrand $e^{i(p_l^2-p_k^2)\tau_1}k_{kl}(\tau_1)\varphi_0$. Outside a set of critical points of the phase for which $|p_l|-|p_k|\leq\kappa(\varrho)$, for some appropriately small $\kappa(\varrho)\ll 1$, the energy shift grows rapidly: $p_l^2-p_k^2=\big(|p_l|+|p_k|\big)\big(|p_l|-|p_k|\big)\gtrsim\sqrt{\varrho}\kappa(\varrho)\gg 1$. By partial integration, one thus finds that

$$(4.22)\lesssim\frac{t^2}{L^4}\left[\sum_{\substack{k=1\\ \{\text{ stationary points }\}}}^{N}\sum_{l=N+1}^{\infty}+\frac{1}{\varrho\kappa(\varrho)^2}\sum_{k=1}^{N}\sum_{l=N+1}^{\infty}\right]\left|\hat{v}^{s,\varepsilon}(p_k-p_l)\right|^2, \tag{4.23}$$

which will be shown to vanish in the limit $\varrho\to\infty$, see Remarks 4.1 and 4.1 below. This result is the key ingredient to understand the proof of Theorem 4.1. Since the interaction is modeled by a two-body potential, it is reasonable to expect that an appropriate estimate for the higher order terms in (4.16) follows from a bound of the r.h.s. of (4.23). Technically, however, it is more tedious to obtain good control of the higher-order contributions. The difficulty is the appearance of the full time evolution U. Using the Duhamel expansion for $U-U^{\mathrm{mf}}$, one finds an estimate similar to

$$\left\|\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2 U(\tau_2)(V-E)U^{\mathrm{mf}}(\tau_2-\tau_1)(V-E)U^{\mathrm{mf}}(\tau_1)\Psi_0\right\|^2\lesssim t^2\cdot(4.22)\cdot\left\|(V-E)\Omega_t^f\right\|^2,$$

for which the r.h.s., however, is still divergent for $\varrho\to\infty$ (recall that $\lim_{N,L\to\infty,\varrho=const.}\|(V-E)\Omega_t^f\|\to\infty$ when ϱ tends to ∞). Expanding U another time, the main contribution that has to be controlled is given by

$$\left\|\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\int_0^{\tau_2}d\tau_3 U(\tau_3)(V-E)U^{\mathrm{mf}}(\tau_3-\tau_2)(V-E)U^{\mathrm{mf}}(\tau_2-\tau_1)(V-E)U^{\mathrm{mf}}(\tau_1)\Psi_0\right\|.$$

Now one can use the oscillation of the integrand also in the second time-variable. It will be shown in detail that this can be bounded in terms of

$$t^2\lim_{\substack{N,L\to\infty\\ \varrho=N/L^2=const.}}\left(\left\|(V-E)\Omega_t^f\right\|^2\cdot(4.22)^2\right)\to 0\ (\varrho\to\infty). \tag{4.24}$$

This explains why we expand the dynamics up to third order for proving Theorem 4.1.

Let us conclude with some remarks.

Remark 4.1. 1) In Lemma 4.3 we show that $\lim_{N,L\to\infty,\varrho=const.}\|(V-E)\Omega_0\|^2\propto\sqrt{\varrho}$. The term containing the nonstationary terms in (4.23) will therefore be proportional to $\varrho^{-\frac{1}{2}}\kappa(\varrho)^{-2}$. The term containing the stationary points in (4.23) turns out, in $d=2$, to be proportional to $\sqrt{\varrho}\kappa(\varrho)^2$. Thus, one actually needs a finer separation around the stationary points in order to obtain the desired bound in ϱ. The details of this separation are explained in Section 4.2.4.

2) The second summand on the r.h.s. of (4.23) behaves at best like $\|(V-E)\Omega_0\|^2/k_F^2$. Recalling (4.15) as well as $k_F\propto\varrho^{\frac{1}{d}}$ in d dimensions, it is clear that a similar statement as

Theorem 4.1 also holds for $d = 1$. For $d = 3$, on the contrary, the last term is not small, even if one optimizes the separation of the nonstationary points. We provide more details about $d = 1$ and $d = 3$ in Appendices 4.C and 4.D.

3) Some of the techniques we use in the proof of our main result also appear in the proof of quantum diffusion of a particle in an external random potential [42, 43], which is in several respects much more involved. The main difficulty of our problem is that we do not have a small coupling constant, but we need to show that the interaction is effectively small.

Physical picture behind the proof

On the one hand, it is obvious that for $\varrho \gg 1$ the tracer particle can interact only with particles that occupy a momentum close to the Fermi edge. This is due to the exclusion principle and because all momenta smaller than k_F are occupied in Ω_0. Particles with small momentum can simply not be lifted above k_F. In other words, for $\varrho \gg 1$, the Fermi pressure becomes so strong that the particles far inside the Fermi sphere behave very rigidly and are thus hardly disturbed by the presence of the tracer particle (and vice versa). On the other hand, the reason why collisions with particles with momentum close to k_F do not disturb the free motion is that such particles have very large momenta when $\varrho \gg 1$ (i.e., for $m_x = 1/2$, large velocities) and thus interact only on a very short time scale with the tracer particle. Hence, the momentum transfer is effectively small. Let us note that in the limit of very short wave lengths, the particle behavior is dominant, which makes this explanation plausible. In the proof, the high momenta of the gas particles (or, the short time scale of interaction) appear as the factor $\varrho^{-1} \propto k_F^{-2}$ in (4.23).

4.2 Proof of the main result

4.2.1 Notations and definitions

We introduce for any $t \geq 0$, the operators

$$k_{kl}(t) : \mathcal{H}_y \to \mathcal{H}_y, \qquad \varphi \mapsto k_{kl}(t)\varphi = e^{iH_y^f t} e^{-i(p_k - p_l)y} e^{-iH_y^f t} \varphi, \tag{4.25}$$

$$g_{kl}(t) : \mathcal{H}_y \to \mathcal{H}_y, \qquad \varphi \mapsto g_{kl}(t)\varphi = e^{-i(p_k^2 - p_l^2)t} k_{kl}(t)\varphi, \tag{4.26}$$

and

$$D(t) : \mathcal{H}_y \otimes \mathcal{H}_N \to \mathcal{H}_y \otimes \mathcal{H}_N, \qquad \Psi \mapsto D(t)\Psi = U(-t)U^{\mathrm{mf}}(t)\Psi. \tag{4.27}$$

We denote the Fourier transform of the potential v by $\hat{v}$, where $\hat{v}$ is defined such that

$$v(x) = \frac{1}{L^2} \sum_{k=1}^{\infty} \hat{v}(p_k) e^{ip_k x}. \tag{4.28}$$

Moreover, we use the following abbreviations:

- $\hat{v}_{kl} = \hat{v}(p_k - p_l)$,
- $\hat{v}_{kl}^{\ell,\varepsilon} = \theta\big(-\varrho^{\varepsilon} + |p_k - p_l|\big)\hat{v}_{kl}, \quad \hat{v}_{kl}^{s,\varepsilon} = \theta\big(\varrho^{\varepsilon} - |p_k - p_l|\big)\hat{v}_{kl}.$
- $E_k = p_k^2$,
- $\|\cdot\|_{\mathrm{TD}} = \lim_{\mathrm{TD}} \|\cdot\| = \lim_{N,L\to\infty,\varrho=const.} \|\cdot\|$ (note that despite this notation, $\|\cdot\|_{\mathrm{TD}}$ does not define a proper norm since $\|f\|_{\mathrm{TD}}$ may be zero for nonzero f),
- $a^*(p_l)a(p_k)\Omega_0 = \Omega_0^{[l^*k]}$ and all kind of variations thereof.

4.2.2 Collision Histories

Here we introduce the following wave functions. Let us remark that the reason for introducing $\Psi_1, ..., \Psi_4$ as well as $\Psi_{\mathrm{A}}, ..., \Psi_{\mathrm{F}}$ (see below) is not obvious at this point. As we explain hereafter, one can interpret $\Psi_{\mathrm{A}}, ..., \Psi_{\mathrm{F}}$ as different collision histories of the tracer particle. In the proof of Theorem 4.1 in Section 4.2.3 we will see that it is sufficient to control the norm of these collision histories. We set

$$\Psi_1(\tau_2,\tau_1) = \frac{1}{L^4}\sum_{k_1=1}^{N}\sum_{l_1=N+1}^{\infty} |\hat{v}_{k_1 l_1}|^2 \Big(g_{l_1 k_1}(\tau_2) g_{k_1 l_1}(\tau_1)\varphi_0 \Big) \otimes \Omega_0, \tag{4.29}$$

$$\Psi_2(\tau_2,\tau_1) = \frac{1}{L^4}\sum_{k_1,k_2=1}^{N}\sum_{l_1=N+1}^{\infty} \hat{v}_{k_2 k_1}\hat{v}_{k_1 l_1} \Big(g_{k_2 k_1}(\tau_2) g_{k_1 l_1}(\tau_1)\varphi_0 \Big) \otimes \Omega_0^{[k_2 l_1^*]}, \tag{4.30}$$

$$\Psi_3(\tau_2,\tau_1) = \frac{1}{L^4}\sum_{k_1=1}^{N}\sum_{l_1,l_2=N+1}^{\infty} \hat{v}_{l_1 l_2}\hat{v}_{k_1 l_1} \Big(g_{l_1 l_2}(\tau_2) g_{k_1 l_1}(\tau_1)\varphi_0 \Big) \otimes \Omega_0^{[l_2^* k_1]}, \tag{4.31}$$

$$\Psi_4(\tau_2,\tau_1) = \frac{1}{L^4}\sum_{k_1,k_2=1}^{N}\sum_{l_1,l_2=N+1}^{\infty} \hat{v}_{k_2 l_2}\hat{v}_{k_1 l_1} \Big(g_{k_2 l_2}(\tau_2) g_{k_1 l_1}(\tau_1)\varphi_0 \Big) \otimes \Omega_0^{[l_2^* k_2 l_1^* k_1]}, \tag{4.32}$$

which satisfy the equality (this is straightforward to verify)

$$U^{\mathrm{mf}}(-\tau_2)(V-E)U^{\mathrm{mf}}(\tau_2-\tau_1)(V-E)U^{\mathrm{mf}}(\tau_1)\Psi_0 = \Psi_1+\Psi_2+\Psi_3+\Psi_4. \tag{4.33}$$

We further define

$$\Psi_{\mathrm{A}}(t) = E_{re}(\varrho)\int_0^t d\mu_2(\tau) D(\tau_2) U^{\mathrm{mf}}(-\tau_1)(V-E)U^{\mathrm{mf}}(\tau_1)\Psi_0, \tag{4.34}$$

$$\Psi_{\mathrm{B}}(t) = E_{re}(\varrho)\int_0^t d\tau_1 D(\tau_1)\Psi_0 + i\int_0^t d\mu_2(\tau) D(\tau_2)\Psi_1(\tau_2,\tau_1), \tag{4.35}$$

$$\Psi_{\mathrm{C}}(t) = \int_0^t d\mu_2(\tau) D(\tau_2)\Psi_2(\tau_2,\tau_1), \tag{4.36}$$

$$\Psi_{\mathrm{D}}(t) = \int_0^t d\mu_2(\tau) D(\tau_2)\Psi_3(\tau_2,\tau_1), \tag{4.37}$$

$$\Psi_{\mathrm{E}}(t) = E_{re}(\varrho)\int_0^t d\mu_3(\tau) D(\tau_3)\Psi_4(\tau_2,\tau_1), \tag{4.38}$$

$$\Psi_{\mathrm{F}}(t) = \int_0^t d\mu_3(\tau) U(-\tau_3)(V-E)U^{\mathrm{mf}}(\tau_3)\Psi_4(\tau_2,\tau_1), \tag{4.39}$$

where we have introduced the shorthand notation

$$\int_0^t d\mu_n(\tau) = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \,...\int_0^{\tau_{n-1}} d\tau_n. \tag{4.40}$$

The different wave functions $\Psi_{\mathrm{X}}(t)$, $\mathrm{X} \in \{\mathrm{A,B,C,D,E,F}\}$ can be identified with the following collision histories of the tracer particle:

- A: single collisions which cause particle-hole excitations in the Fermi gas.
- B: two collisions with the same particle, removing the particle-hole excitation which was caused in the first collision; the constant $E_{re}(\varrho)$ cancels the contribution in which the second collision follows immediately after the first one.

C: two collisions with the same particle; the second collision scatters the lifted particle into another momentum above the Fermi edge.

D: two collisions; the second collision scatters a particle from below the Fermi edge into the hole that was created in the first collision.

E: two collisions with two different particles; causing two particle-hole excitations.

F: three collisions; three particle-hole excitations but also all possible recollisions with the already scattered particles; the different possibilities are listed in Section 4.1.

4.2.3 Main lemma and proof of Theorem 4.1

The main lemma we have to prove is

Lemma 4.2. *Let $0 < \varepsilon < 1/8$. Under the same assumptions as in Theorem 4.1, there exist positive constants C, C_ε such that*

$$\|\Psi_{\mathrm{A}}(t)\|_{\mathrm{TD}} \leq C(1+t)^2\Big(\varrho^{-\frac{1}{4}+5\varepsilon} + C_\varepsilon \varrho^{-\frac{3}{2\varepsilon}}\Big), \tag{4.41}$$

$$\|\Psi_{\mathrm{B}}(t)\|_{\mathrm{TD}} \leq C(1+t)^2\Big(\varrho^{-\frac{1}{4}+2\varepsilon} + C_\varepsilon \varrho^{-\frac{1}{\varepsilon}}\Big), \tag{4.42}$$

$$\|\Psi_{\mathrm{C}}(t)\|_{\mathrm{TD}} \leq C(1+t)^2\Big(\varrho^{-\frac{1}{4}+2\varepsilon} + C_\varepsilon \varrho^{\varepsilon-\frac{1}{2\varepsilon}}\Big), \tag{4.43}$$

$$\|\Psi_{\mathrm{D}}(t)\|_{\mathrm{TD}} \leq C(1+t)^2\Big(\varrho^{-\frac{1}{4}+2\varepsilon} + C_\varepsilon \varrho^{\varepsilon-\frac{1}{2\varepsilon}}\Big), \tag{4.44}$$

$$\|\Psi_{\mathrm{E}}(t)\|_{\mathrm{TD}} \leq C(1+t)^3\Big(\varrho^{-\frac{1}{2}+8\varepsilon} + C_\varepsilon \varrho^{\frac{1}{4}+6\varepsilon-\frac{1}{2\varepsilon}}\Big), \tag{4.45}$$

$$\|\Psi_{\mathrm{F}}(t)\|_{\mathrm{TD}} \leq C(1+t)^3\Big(\varrho^{-\frac{1}{4}+6\varepsilon} + C_\varepsilon \varrho^{\frac{1}{2}-\frac{1}{2\varepsilon}}\Big), \tag{4.46}$$

hold for all $t > 0$.

We now show that Theorem 4.1 follows from the above bounds.

Proof of Theorem 4.1. We begin with Duhamel's formula,

$$U(t) - U^{\mathrm{mf}}(t) = -i\int_0^t d\tau_1 U^{\mathrm{mf}}(t-\tau_1)\Big(V - E + E_{re}(\varrho)\Big)U(\tau_1), \tag{4.47}$$

then use that $\langle U^{\mathrm{mf}}(\tau_1)\Psi_0, (V-E)U^{\mathrm{mf}}(\tau_1)\Psi_0\rangle = \langle \Psi_0, (V-E)\Psi_0\rangle = 0$, apply Duhamel's formula again, and eventually use the identity in (4.33):

$$\begin{aligned}
&\frac{1}{2}\Big\|U(t)\Psi_0 - U^{\mathrm{mf}}(t)\Psi_0\Big\|^2 = -\operatorname{Re}\big\langle\Big(U(t) - U^{\mathrm{mf}}(t)\Big)\Psi_0, U^{\mathrm{mf}}(t)\Psi_0\big\rangle \\
&= -\operatorname{Re}\big\langle\Psi_0, iE_{re}(\varrho)\int_0^t d\tau_1 U(-\tau_1)U^{\mathrm{mf}}(\tau_1)\Psi_0\big\rangle \\
&\quad + \operatorname{Re}\big\langle i\int_0^t d\tau_1\Big(U(\tau_1) - U^{\mathrm{mf}}(\tau_1)\Big)\Psi_0, \big(V-E\big)U^{\mathrm{mf}}(\tau_1)\Psi_0\big\rangle \\
&= -\operatorname{Re}\big\langle\Psi_0, iE_{re}(\varrho)\int_0^t d\tau_1 U(-\tau_1)U^{\mathrm{mf}}(\tau_1)\Psi_0\big\rangle \\
&\quad + \operatorname{Re}\big\langle\Psi_0, E_{re}(\varrho)\int_0^t d\mu_2(\tau_2)U(-\tau_2)U^{\mathrm{mf}}(\tau_2-\tau_1)\big(V-E\big)U^{\mathrm{mf}}(\tau_1)\Psi_0\big\rangle
\end{aligned}$$

$$+\operatorname{Re}\Big\langle \int_0^t d\mu_2(\tau) U^{\mathrm{mf}}(-\tau_2) U(\tau_2)\Psi_0, \underbrace{U^{\mathrm{mf}}(-\tau_2)(V-E)U^{\mathrm{mf}}(\tau_2-\tau_1)(V-E)U^{\mathrm{mf}}(\tau_1)\Psi_0}_{=\Psi_1+\Psi_2+\Psi_3+\Psi_4} \Big\rangle. \tag{4.48}$$

We proceed with the term that contains Ψ_4. Using $\big\langle \Psi_0, \Psi_4(\tau_2,\tau_1)\big\rangle = 0$ (note that Ψ_4 always contains a particle outside the Fermi sphere), and applying one more time Duhamel's formula, we find

$$\begin{aligned}
&\operatorname{Re}\Big\langle \int_0^t d\mu_2(\tau) U^{\mathrm{mf}}(-\tau_2)\Big(U(\tau_2) - U^{\mathrm{mf}}(\tau_2)\Big)\Psi_0, \Psi_4(\tau_2,\tau_1)\Big\rangle \\
&= \operatorname{Re}\Big\langle \Psi_0, i\int_0^t d\mu_3(\tau) U(-\tau_3)\Big(V - E + E_{re}(\varrho)\Big)U^{\mathrm{mf}}(\tau_3)\Psi_4(\tau_2,\tau_1)\Big\rangle \\
&= \operatorname{Re}\Big\langle \Psi_0, i\Big(\Psi_{\mathrm{E}}(t) + \Psi_{\mathrm{F}}(t)\Big)\Big\rangle.
\end{aligned} \tag{4.49}$$

By means of the triangle inequality and Cauchy-Schwarz, it follows that

$$\begin{aligned}
\Big\| U(t)\Psi_0 - U^{\mathrm{mf}}(t)\Psi_0 \Big\| \le 2\Big(&\sqrt{\|\Psi_{\mathrm{A}}(t)\|} + \sqrt{\|\Psi_{\mathrm{B}}(t)\|} + \sqrt{\|\Psi_{\mathrm{C}}(t)\|} \\
&+ \sqrt{\|\Psi_{\mathrm{D}}(t)\|} + \sqrt{\|\Psi_{\mathrm{E}}(t)\|} + \sqrt{\|\Psi_{\mathrm{F}}(t)\|}\Big).
\end{aligned} \tag{4.50}$$

Application of Lemma 4.2 with $\varepsilon > 0$ sufficiently small then proves the theorem. □

The rest of this section is devoted to the proof of Lemma 4.2.

4.2.4 Proof of Lemma 4.2

Preliminaries

For $\varepsilon > 0$, we define the two-dimensional index set

$$\mathfrak{S}^\varepsilon(N,\varrho) := \Big\{(k,l) \;:\; 1 \le k \le N,\; N+1 \le l,\; |p_k - p_l| < \varrho^\varepsilon\Big\} \subset \mathbb{N}^2, \tag{4.51}$$

and for $M \in \mathbb{N}$ the family of sets

$$\mathfrak{S}_n^{\varepsilon,M}(N,\varrho) := \Big\{(k,l) \in \mathfrak{S}^\varepsilon(N,\varrho) \;:\; \varrho^{-b_n} \le |p_l| - |p_k| < \varrho^{-b_{n+1}}\Big\}, \quad 0 \le n \le M, \tag{4.52}$$

where

$$b_0 = \infty, \qquad b_n = \frac{1}{2} - \frac{n-1}{M}\Big(\frac{1}{2} + \varepsilon\Big), \quad 1 \le n \le M. \tag{4.53}$$

For notational convenience, we omit from now on the N-, ϱ-, ε- and also the M-dependence in the notation: $\mathfrak{S} = \mathfrak{S}^\varepsilon(N,\varrho)$ and $\mathfrak{S}_n = \mathfrak{S}_n^{\varepsilon,M}(N,\varrho)$. The index set $\mathfrak{S}$ corresponds to the transitions that have to be controlled in (4.23), i.e., collisions with momentum transfer smaller than ϱ^ε. The sets of pairs of momenta $\{(p_k,p_l) \in (2\pi/L)^2\mathbb{Z}^4 : (k,l) \in \mathfrak{S}_n\}$ are pairwise disjoint, and

$$\bigcup_{n=0}^{M} \Big\{(p_k,p_l) \in (2\pi/L)^2\mathbb{Z}^4 : (k,l) \in \mathfrak{S}_n\Big\} = \Big\{(p_k,p_l) \in (2\pi/L)^2\mathbb{Z}^4 : (k,l) \in \mathfrak{S}\Big\}. \tag{4.54}$$

The distance of modulus between the occupied momentum p_k and the new momentum state p_l increases in $\mathfrak{S}_n$ for increasing n. With $2c = 1/(\frac{1}{2}+\varepsilon)$,

$$|p_l| - |p_k| \geq \varrho^{-b_n} = \varrho^{-\frac{1}{2}+\frac{n-1}{2cM}} \quad \text{for} \quad (k,l) \in \mathfrak{S}_n,\ 1 \leq n \leq M. \tag{4.55}$$

Hence, also the energy shift increases,

$$E_l - E_k = \Big(|p_l| + |p_k|\Big)\Big(|p_l| - |p_k|\Big) \geq k_F\,\varrho^{-b_n} = C\,\varrho^{\frac{n-1}{2cM}} \quad \text{for} \quad (k,l) \in \mathfrak{S}_n, \tag{4.56}$$

$1 \leq n \leq M$. ϱ^{-b_n} corresponds to the factor $\kappa(\varrho)$ in (4.23).

In the following lemma we state the key estimates that are used in order to prove Lemma 4.2. Recall that $\theta(x) = 1$ for $x \geq 0$ and zero otherwise.

Lemma 4.3. *Assume $0 < \varepsilon < \frac{1}{2}$ and $M, q \in \mathbb{N}$. Let $v(x) \in C_0^\infty(\mathbb{T}^2) \cap C_0^\infty(\mathbb{R}^2)$ and $v^{\ell,\varepsilon}$, $v^{s,\varepsilon}$ defined as in (4.20),(4.21). Then there exist positive constants C, $C_{q,1} < C_{q,2}$, $C_{\varepsilon,q}$ such that*

$$C_{q,1}\varrho^{\frac{1}{2}} \leq \lim_{\text{TD}} \frac{1}{L^4} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \big|\hat{v}(p_k - p_l)\big|^q \leq C_{q,2}\varrho^{\frac{1}{2}}, \tag{4.57}$$

$$\lim_{\text{TD}} \frac{1}{L^4} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \big|\hat{v}^{\ell,\varepsilon}(p_k - p_l)\big|^q \leq C_{\varepsilon,q}\varrho^{-1/\varepsilon}, \tag{4.58}$$

$$\lim_{\text{TD}} \frac{1}{L^4} \sum_{(k,l)\in\mathfrak{S}_n} \leq C\varrho^{\frac{1}{2}+\varepsilon-b_{n+1}}\Big(\varrho^{-b_{n+1}} - \varrho^{-b_n}\Big) \quad \text{for} \quad 0 \leq n \leq M, \tag{4.59}$$

$$C \leq E_{re}(\varrho) = \lim_{\text{TD}} \frac{1}{L^4} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \frac{\big|\hat{v}(p_k - p_l)\big|^2}{E_l - E_k}\theta\Big(|p_l| - |p_k| - \varrho^{-\frac{1}{2}}\Big) \leq C\varrho^{2\varepsilon} + C_\varepsilon\varrho^{-1/\varepsilon}, \tag{4.60}$$

$$\lim_{\text{TD}} \frac{1}{L^2} \sum_{k=1}^{\infty} \big|\hat{v}(p_k - p)\big| \leq C\varrho^{2\varepsilon} + C_\varepsilon\varrho^{-1/\varepsilon} \quad \text{for} \quad p \in (2\pi/L)\mathbb{Z}^2. \tag{4.61}$$

The proof of the lemma is postponed to Section 4.2.5. For notational ease, let us abbreviate the number of possible transitions that correspond to the set $\mathfrak{S}_n$ by

$$\mathcal{V}_n(N,\varrho) := \frac{1}{L^4} \sum_{(k,l)\in\mathfrak{S}_n} = \frac{1}{L^4} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \chi_{\mathfrak{S}_n}\big((k,l)\big), \tag{4.62}$$

$\chi_A : \mathbb{N}^2 \to \{0,1\}$ denoting the characteristic function, i.e., $\chi_A((k,l)) = 1$ whenever $(k,l) \in A \subset \mathbb{N}^2$, otherwise zero. We readily obtain the following

Corollary 4.4. *Given the same assumptions as in Lemma 4.3, and setting $2c = 1/(\frac{1}{2}+\varepsilon)$, there exists a constant $C > 0$ such that*

$$\lim_{\text{TD}} \mathcal{V}_0(N,\varrho) \leq C\varrho^{-\frac{1}{2}+\varepsilon}, \tag{4.63}$$

$$\lim_{\text{TD}} \Big(\varrho^{-(\frac{n-1}{cM})}\mathcal{V}_n(N,\varrho)\Big) \leq C\varrho^{-\frac{1}{2}+\varepsilon}\Big(\varrho^{\frac{1}{cM}} - \varrho^{\frac{1}{2cM}}\Big) \quad \text{for } 1 \leq n \leq M, \tag{4.64}$$

Remark 4.2. The decomposition of $\mathfrak{S}$ into the sets $\mathfrak{S}_n$ is optimal in the sense that the r.h.s. of all estimates in (4.63) and (4.64) behaves asymptotically almost the same, namely $\propto \varrho^{-\frac{1}{2}}$ for small ε and large M.

The corollary follows from Lemma 4.3 together with the definition of the b_n in (4.53). Next, we summarize some bounds which makes the presentation of the proof of Lemma 4.2 more convenient.

Lemma 4.5. *Let $\varepsilon > 0$, $k_{kl}(t)$ as in (4.25), $g_{kl}(t)$ as in (4.26) and $\mathfrak{S} = \mathfrak{S}^\varepsilon(N,\varrho)$ as in (4.51). Given the same assumptions as in Theorem 4.1, the following bounds hold for all $\tau_1, \tau_2 \geq 0$ ($\chi_A : \mathbb{N}^2 \to \{0,1\}$ denotes the characteristic function as explained above),*

$$\chi_{\mathfrak{S}}\big((k,l)\big)\ \|\partial_{\tau_1} k_{kl}(\tau_1)\varphi_0\| \leq C\varrho^{2\varepsilon}, \tag{4.65}$$

$$\chi_{\mathfrak{S}}\big((k,l)\big)\ \big(\|\partial_{\tau_2} k_{kl}(\tau_2) g_{kl}(\tau_1)\varphi_0\| + \|\partial_{\tau_2} k_{lk}(\tau_2) g_{kl}(\tau_1)\varphi_0\|\big) \leq C\varrho^{2\varepsilon}, \tag{4.66}$$

$$\chi_{\mathfrak{S}}\big((k,l)\big)\chi_{\mathfrak{S}}\big((m,n)\big)\ \|\partial_{\tau_2} k_{lk}(\tau_2)\partial_{\tau_1} k_{mn}(\tau_1)\varphi_0\| \leq C\varrho^{4\varepsilon}, \tag{4.67}$$

$$\chi_{\mathfrak{S}}\big((k,l)\big)\chi_{\mathfrak{S}}\big((m,n)\big)\ \|\partial_{\tau_1} k_{kl}(\tau_1) k_{mn}(\tau_1)\varphi_0\| \leq C\varrho^{4\varepsilon}. \tag{4.68}$$

The proof is obtained by means of Stone's theorem and the assumption $\|\nabla^4\varphi_0\| \leq C$ (for more details, see Section 4.2.5). Note that since $\mathfrak{S}_n \subset \mathfrak{S}$, the estimates in (4.65)-(4.68) hold also if $\mathfrak{S}$ is replaced by any of the $\mathfrak{S}_n$ resp. by any pair of $\mathfrak{S}_n$, $\mathfrak{S}_m$ ($0 \leq n, m \leq M$).

From now on, we always assume that $0 < \varepsilon < 1/8$ and denote $2c = (\frac{1}{2} + \varepsilon)^{-1}$. Furthermore, we will equally use the letter τ for indicating the dependence on the variables $\tau = (\tau_1, \tau_2)$ or $\tau = (\tau_1, \tau_2, \tau_3)$. For notational ease, let us also introduce for two real numbers A and B:

$$A \lesssim B \ \Leftrightarrow\ \exists\, C > 0 \ s.t.\ A \leq CB,$$

where the constant C may depend on the supremum of $\hat{v}$ but is independent of any of the relevant parameters (N, L, ϱ, t, ε and M).

Derivation of the bound for $\|\Psi_{\mathrm{A}}(t)\|_{\mathrm{TD}}$

In order to bound $\|\Psi_{\mathrm{A}}(t)\|_{\mathrm{TD}}$, we first split the interaction potential into the contributions coming from small momentum transfer $\hat{v}^{s,\varepsilon}$ and those from large momentum transfer $\hat{v}^{\ell,\varepsilon}$. The $\hat{v}^{\ell,\varepsilon}$ will then be estimated using (4.58). For the $\hat{v}^{s,\varepsilon}$, we seperate the stationary points of the phase which can be estimated using (4.63). For the nonstationary points, we do one partial integration in the time in order to be able to use (4.56) and (4.64).

Let $M \geq 1$, and for $0 \leq n \leq M$,

$$\Psi_{\mathrm{A}}^{s,n}(t) = \frac{1}{L^2} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \hat{v}^{s,\varepsilon}_{k_1 l_1} \int_0^t d\mu_2(\tau) D(\tau_2) \Big(g_{k_1 l_1}(\tau_1)\varphi_0\Big) \otimes \Omega_0^{[l_1^* k_1]}, \tag{4.69}$$

$$\Psi_{\mathrm{A}}^{\ell}(\tau_1) = \frac{1}{L^2} \sum_{k_1=1}^{N} \sum_{l_1=N+1}^{\infty} \hat{v}^{\ell,\varepsilon}_{k_1 l_1} \Big(g_{k_1 l_1}(\tau_1)\varphi_0\Big) \otimes \Omega_0^{[l_1^* k_1]}. \tag{4.70}$$

Using the identity in (4.17) and $v_{k_1 l_1} = v^{s,\varepsilon}_{k_1 l_1} + v^{\ell,\varepsilon}_{k_1 l_1}$, this leads to the following decomposition of $\Psi_{\mathrm{A}}(t)$,

$$\Psi_{\mathrm{A}}(t) = E_{re}(\varrho) \sum_{n=0}^{M} \Psi_{\mathrm{A}}^{s,n}(t) + E_{re}(\varrho) \int_0^t d\mu_2(\tau) D(\tau_2) \Psi_{\mathrm{A}}^{\ell}(\tau_1). \tag{4.71}$$

We emphasize that $\Psi_{\rm A}^{s,n}(t)$ depends on the choice of M through the M-dependence of the sets $\mathfrak{S}_n$, whereas $\Psi_{\rm A}(t)$ and $\Psi_{\rm A}^{\ell}(t)$ are both M-independent. Next, we estimate each term on the r.h.s. of (4.71). In the last one, we find, using $\langle \Omega_0^{[l_1^* k_1]}, \Omega_0^{[n_1^* m_1]}\rangle = \delta_{l_1 n_1}\delta_{k_1 m_1}$ for $k_1, m_1 \le N$, $N+1 \le l_1, n_1$, as well as $\|g_{k_1 l_1}(\tau_1)\varphi_0\| = 1$,

$$\|\Psi_{\rm A}^{\ell}(\tau_1)\|^2 = \frac{1}{L^4}\sum_{k_1=1}^{N}\sum_{l_1=N+1}^{\infty} |\hat{v}_{k_1 l_1}^{\ell,\varepsilon}|^2 \le C_\varepsilon \varrho^{-1/\varepsilon}, \tag{4.72}$$

where the bound has been derived in (4.58). Recalling also (4.60) which states that $E_{re}(\varrho) \lesssim \varrho^{2\varepsilon} + C_\varepsilon \varrho^{-1/\varepsilon}$, and using unitarity of $D(s)$, one obtains for the last term in (4.71)

$$\Big\| E_{re}(\varrho)\int_0^t d\mu_2(\tau) D(\tau_2)\Psi_{\rm A}^{\ell}(\tau_1)\Big\|_{\rm TD} \lesssim t^2\Big(\varrho^{2\varepsilon} + C_\varepsilon \varrho^{-1/\varepsilon}\Big)\varrho^{-1/(2\varepsilon)}. \tag{4.73}$$

In $\Psi_{\rm A}^{s,0}(t)$, we need to estimate the norm

$$\Big\| \frac{1}{L^2}\sum_{(k_1,l_1)\in\mathfrak{S}_0} \hat{v}_{k_1 l_1}^{s,\varepsilon}\Big(g_{k_1 l_1}(\tau_1)\varphi_0\Big)\otimes \Omega_0^{[l_1^* k_1]}\Big\|^2 \lesssim \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_0} = \mathcal{V}_0(N,\rho). \tag{4.74}$$

The remaining expression, i.e., the number of transitions corresponding to the set $\mathfrak{S}_0$, has been estimated in (4.63). Thus,

$$\|\Psi_{\rm A}^{s,0}(t)\|_{\rm TD} \lesssim t^2 \varrho^{-\frac14+\frac12\varepsilon}.$$

Lemma 4.6. *Let $\Psi_{\rm A}^{s,n}(t)$ as in (4.69). Under the same assumptions as in Theorem 4.1,*

$$\|\Psi_{\rm A}^{s,n}(t)\|_{\rm TD} \lesssim (1+t)^2 \varrho^{-\frac14+\frac52\varepsilon}\sqrt{\varrho^{\frac{1}{cM}} - \varrho^{\frac{1}{2cM}}}, \qquad 1\le n\le M, \tag{4.75}$$

holds for all $t>0$.

One can now use that for $M = \lfloor \ln\varrho\rfloor$ (the largest integer smaller than the number $\ln\varrho$),

$$\sum_{n=1}^{M}\|\Psi_{\rm A}^{s,n}(t)\|_{\rm TD} \lesssim (1+t)^2\varrho^{-\frac14+\frac52\varepsilon} M\varrho^{\frac{1}{2cM}} \lesssim (1+t)^2\varrho^{-\frac14+3\varepsilon} \tag{4.76}$$

because $M\varrho^{\frac{1}{2cM}} \le \ln\varrho\cdot e^{\frac{1}{2c}} \lesssim \varrho^{\frac12\varepsilon}$ for any $\varepsilon>0$. This proves the bound for $\|\Psi_{\rm A}(t)\|_{\rm TD}$ in (4.41). That taking first the thermodynamic limit and then $M=\lfloor\ln\varrho\rfloor$ is unproblematic (even when ϱ tends to ∞) is summarized in the following

Remark 4.3. $\Psi_{\rm A}(t)$ as well as $\Psi_{\rm A}^{\ell}(\tau)$ in (4.71) are both M-independent. There is thus no need to interchange the order of the two limits. One first takes the thermodynamic limit on both sides and then passes to the limit of large M. Since only the r.h.s. of (4.71) depends on the choice of M, this provides the desired estimate.

Proof of Lemma 4.6. We first decompose each of the $\Psi_{\rm A}^{s,n}(t)$ via partial integration in τ_1. For that, we recall $g_{k_1 l_1}(\tau_1) = e^{i(E_{l_1}-E_{k_1})\tau_1}k_{k_1 l_1}(\tau_1)$ and rewrite (for $n\ge 1$)

$$\Psi_{\rm A}^{s,n}(t) = \frac{1}{L^2}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\hat{v}_{k_1 l_1}^{s,\varepsilon}\int_0^t d\tau_1\Big(\frac{\partial_{\tau_1} e^{i(E_{l_1}-E_{k_1})\tau_1}}{i(E_{l_1}-E_{k_1})}\Big)\int_0^{\tau_1} d\tau_2 D(\tau_2)\Big(k_{k_1 l_1}(\tau_1)\varphi_0\Big)\otimes\Omega_0^{[l_1^*,k_1]}.$$

Partial integration in τ_1 leads to

$$\Psi_{\mathrm{A}}^{s,n}(t) = \int_0^t d\tau D(\tau_1)\Psi_{\mathrm{A},1}^{s,n}(t,\tau_1) + \int_0^t d\mu_2(\tau)D(\tau_2)\Psi_{\mathrm{A},2}^{s,n}(\tau_1), \tag{4.77}$$

where

$$\Psi_{\mathrm{A},1}^{s,n}(t,\tau_1) = \frac{1}{L^2}\sum_{(k_1,l_1)\in\mathfrak{S}_n} \hat{v}_{k_1l_1}^{s,\varepsilon}\Big(\frac{g_{k_1l_1}(t)-g_{k_1l_1}(\tau_1)}{i(E_{l_1}-E_{k_1})}\varphi_0\Big)\otimes\Omega_0^{[l_1^*,k_1]}, \tag{4.78}$$

$$\Psi_{\mathrm{A},2}^{s,n}(\tau_1) = \frac{1}{L^2}\sum_{(k_1,l_1)\in\mathfrak{S}_n} \hat{v}_{k_1l_1}^{s,\varepsilon}\Big(\frac{e^{i(E_{l_1}-E_{k_1})\tau_1}\partial_{\tau_1}k_{k_1l_1}(\tau_1)}{i(E_{l_1}-E_{k_1})}\varphi_0\Big)\otimes\Omega_0^{[l_1^*,k_1]}. \tag{4.79}$$

Next, we estimate the the norms of $\Psi_{\mathrm{A},1}^{s,n}(\tau)$ and $\Psi_{\mathrm{A},2}^{s,n}(\tau)$:

$$\|\Psi_{\mathrm{A},1}^{s,n}(t,\tau_1)\|^2 \lesssim \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\frac{1}{(E_{l_1}-E_{k_1})^2} \lesssim \Big(\varrho^{-(\frac{n-1}{cM})}\mathcal{V}_n(N,\rho)\Big), \tag{4.80}$$

where we have used (4.56),

$$\|\Psi_{\mathrm{A},2}^{s,n}(\tau_1)\|^2 \lesssim \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\frac{\|\partial_{\tau_1}k_{k_1l_1}(\tau_1)\varphi_0\|^2}{(E_{l_1}-E_{k_1})^2} \lesssim \varrho^{4\varepsilon}\Big(\varrho^{-(\frac{n-1}{cM})}\mathcal{V}_n(N,\rho)\Big), \tag{4.81}$$

where we have made in addition use of (4.65). The remaining expressions have been estimated in (4.64). □

Derivation of the bound for $\|\Psi_{\mathrm{B}}(t)\|_{\mathrm{TD}}$

In the estimate for $\|\Psi_{\mathrm{B}}(t)\|_{\mathrm{TD}}$, we first identify the contribution in $i\int_0^t d\mu_2(\tau)D(\tau_2)\Psi_1(\tau)$ that cancels with the energy correction $E_{re}(\varrho)\int_0^t d\tau_1 D(\tau_1)\Psi_0$. The remaining terms will then be estimated using similar techniques as in Section 4.2.4.

For $0\le n\le M$ ($M\ge 1$), let

$$\Psi_{\mathrm{B}}^{s,0}(t) = \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_0}|\hat{v}_{k_1l_1}^{s,\varepsilon}|^2 i\int_0^t d\mu_2(\tau)D(\tau_2)g_{l_1k_1}(\tau_2)g_{k_1l_1}(\tau_1)\Psi_0, \tag{4.82}$$

$$\Psi_{\mathrm{B},1}^{s,n}(t) = \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}|\hat{v}_{k_1l_1}^{s,\varepsilon}|^2\int_0^t d\tau_2 D(\tau_2)\frac{g_{l_1k_1}(\tau_2)g_{k_1l_1}(t)}{(E_{l_1}-E_{k_1})}\Psi_0, \tag{4.83}$$

$$\Psi_{\mathrm{B},2}^{s,n}(t) = \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}|\hat{v}_{k_1l_1}^{s,\varepsilon}|^2\int_0^t d\tau_2 D(\tau_2)\frac{k_{l_1k_1}(\tau_2)k_{k_1l_1}(\tau_2)}{(E_{l_1}-E_{k_1})}\Psi_0, \tag{4.84}$$

$$\Psi_{\mathrm{B},3}^{s,n}(t) = \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}|\hat{v}_{k_1l_1}^{s,\varepsilon}|^2\int_0^t d\mu_2(\tau)D(\tau_2)\frac{g_{l_1k_1}(\tau_2)e^{i(E_{l_1}-E_{k_1})\tau_1}\partial_{\tau_1}k_{k_1l_1}(\tau_1)}{(E_{l_1}-E_{k_1})}\Psi_0, \tag{4.85}$$

$$\Psi_{\mathrm{B}}^{\ell}(\tau) = \frac{1}{L^4}\sum_{l_1=1}^{N}\sum_{k_1=N+1}^{\infty}|\hat{v}_{k_1l_1}^{\ell,\varepsilon}|^2 g_{l_1k_1}(\tau_2)g_{k_1l_1}(\tau_1)\Psi_0. \tag{4.86}$$

Via partial integration, this leads to the identity

$$\Psi_{\mathrm{B}}(t) = E_{re}(\varrho)\int_0^t d\tau_1 D(\tau_1)\Psi_0 - \sum_{n=1}^{M}\Psi_{\mathrm{B},2}^{s,n}(t)$$

$$+\Psi_{\mathrm{B}}^{s,0}(t)+\sum_{n=1}^{M}\Big[\Psi_{\mathrm{B},1}^{s,n}(t)-\Psi_{\mathrm{B},3}^{s,n}(t)\Big]+i\int_0^t d\mu_2(\tau)D(\tau_2)\Psi_{\mathrm{B}}^{\ell}(\tau). \tag{4.87}$$

The first step in estimating the r.h.s. is to note that for any $M\geq 1$, the thermodynamic limit of the upper line is bounded in terms of

$$\Big\|E_{re}(\varrho)\int_0^t d\tau_1 D(\tau_1)\Psi_0-\sum_{n=1}^{M}\Psi_{\mathrm{B},2}^{s,n}(t)\Big\|_{\mathrm{TD}}\leq C_\varepsilon\varrho^{-1/\varepsilon}. \tag{4.88}$$

In $\Psi_{\mathrm{B},2}^{s,n}(t)$, the fluctuation does not propagate in time (note that $k_{l_1k_1}(\tau_2)k_{k_1l_1}(\tau_2)=1$), and the factor $1/(E_{l_1}-E_{k_1})$ does not make this term small enough. This collision history corresponds to immediate recollisions with the same particle removing the particle-hole excitation which was created in the first scattering. It needs to be canceled directly by the next-to-leading order energy correction in H^{mf}. To see that the above estimate is true, we rewrite

$$\begin{aligned}\sum_{n=1}^{M}\Psi_{\mathrm{B},2}^{s,n}(t)&=\sum_{n=1}^{M}\frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\frac{|\hat{v}_{k_1l_1}^{s,\varepsilon}|^2}{(E_{l_1}-E_{k_1})}\int_0^t d\tau_2 D(\tau_2)\Psi_0 \qquad (4.89)\\ &=\frac{1}{L^4}\sum_{k_1=1}^{N}\sum_{l_1=N+1}^{\infty}\frac{|\hat{v}_{k_1l_1}|^2}{(E_{l_1}-E_{k_1})}\theta\Big(\varrho^\varepsilon-|p_{k_1}-p_{l_1}|\Big)\theta\Big(|p_{l_1}|-|p_{k_1}|-\varrho^{-\frac12}\Big)\int_0^t d\tau_2 D(\tau_2)\Psi_0,\end{aligned}$$

and thus, recalling definition (4.10), we need to estimate

$$\begin{aligned}&\Big|E_{re}(\varrho)-\lim_{\mathrm{TD}}\frac{1}{L^4}\sum_{k_1=1}^{N}\sum_{l_1=N+1}^{\infty}\frac{|\hat{v}_{k_1l_1}|^2}{(E_{l_1}-E_{k_1})}\theta\Big(\varrho^\varepsilon-|p_{k_1}-p_{l_1}|\Big)\theta\Big(|p_{l_1}|-|p_{k_1}|-\varrho^{-\frac12}\Big)\Big|\\ &=\lim_{\mathrm{TD}}\frac{1}{L^4}\sum_{k_1=1}^{N}\sum_{l_1=N+1}^{\infty}\frac{|\hat{v}_{k_1l_1}|^2}{(E_{l_1}-E_{k_1})}\theta\Big(-\varrho^\varepsilon+|p_{k_1}-p_{l_1}|\Big)\theta\Big(|p_{l_1}|-|p_{k_1}|-\varrho^{-\frac12}\Big)\\ &=\lim_{\mathrm{TD}}\frac{1}{L^4}\sum_{k_1=1}^{N}\sum_{l_1=N+1}^{\infty}\frac{|\hat{v}_{k_1l_1}^{\ell,\varepsilon}|^2}{(E_{l_1}-E_{k_1})}\theta\Big(|p_{l_1}|-|p_{k_1}|-\varrho^{-\frac12}\Big)\\ &\lesssim\lim_{\mathrm{TD}}\frac{1}{L^4}\sum_{k_1=1}^{N}\sum_{l_1=N+1}^{\infty}|\hat{v}_{k_1l_1}^{\ell,\varepsilon}|^2.\end{aligned} \tag{4.90}$$

By (4.58), one obtains (4.88). Note that in the last step, we have used $E_{l_1}-E_{k_1}=(|p_{l_1}|-|p_{k_1}|)(|p_{l_1}|+|p_{k_1}|)\geq\varrho^{-\frac12}k_F=C$ (since $|p_{l_1}|\geq k_F$).

It follows with (4.63) that

$$\|\Psi_{\mathrm{B}}^{s,0}(t)\|_{\mathrm{TD}}\lesssim t^2\lim_{\mathrm{TD}}\mathcal{V}_0(N,\rho)\lesssim t^2\varrho^{-\frac12+\varepsilon}, \tag{4.91}$$

as well as with (4.58),

$$\|\Psi_{\mathrm{B}}^{\ell}(\tau)\|_{\mathrm{TD}}\lesssim t^2\lim_{\mathrm{TD}}\frac{1}{L^4}\sum_{k_1=1}^{N}\sum_{l_1=N+1}^{\infty}|\hat{v}_{k_1l_1}^{\ell,\varepsilon}|^2\leq C_\varepsilon t^2\varrho^{-1/\varepsilon}. \tag{4.92}$$

The bounds for the remaining wave functions are summarized in

Lemma 4.7. *Let $\Psi^{s,n}_{B,1}(t)$ and $\Psi^{s,n}_{B,3}(t)$ as in (4.83) and (4.85). Then, under the same assumptions as in Theorem 4.1, there exists a positive constant C_ε such that*

$$\|\Psi^{s,n}_{\mathrm{B},1}(t)\|_{\mathrm{TD}} + \|\Psi^{s,n}_{\mathrm{B},3}(t)\|_{\mathrm{TD}} \lesssim (1+t)^2 \varrho^{-\frac{1}{2}+\varepsilon}\Big(\varrho^{\frac{1}{4}} + C_\varepsilon \varrho^{-1/\varepsilon}\Big)\Big(\varrho^{\frac{1}{cM}} - \varrho^{\frac{1}{2cM}}\Big) \tag{4.93}$$

holds for all $1 \leq n \leq M$ and $t \geq 0$.

Taking the sum of all terms, passing to the thermodynamic limit and then choosing again $M = \lfloor \ln \varrho \rfloor \lesssim \varrho^\varepsilon$, cf. (4.76), one finds

$$\sum_{n=1}^{M} \Big(\|\Psi^{s,n}_{\mathrm{B},1}(t)\|_{\mathrm{TD}} + \|\Psi^{s,n}_{\mathrm{B},3}(t)\|_{\mathrm{TD}}\Big) \lesssim (1+t)^2\Big(\varrho^{-\frac{1}{4}+2\varepsilon} + C_\varepsilon \varrho^{-\frac{1}{2}+2\varepsilon-\frac{1}{\varepsilon}}\Big). \tag{4.94}$$

This proves the bound for $\|\Psi_{\mathrm{B}}(t)\|_{\mathrm{TD}}$ in (4.42) (recall Remark 4.3 and the fact that $\Psi_{\mathrm{B}}(t)$ and $\Psi^{\ell}_{\mathrm{B}}(\tau)$ do both not depend on M).

In order to prove Lemma 4.7, we need the following estimate.

Lemma 4.8. *Let $\psi \in L^2(\mathbb{T}^2)$ and Ω_0 as in (4.5). Then, there exists a positive constant C such that*

$$\Big\|\Big(\partial_\tau D(\tau)\Big)\Big(\psi \otimes \Omega_0\Big)\Big\|^2 \lesssim \|\psi\|^2\Big(E_{re}(\varrho)^2 + \frac{1}{L^4}\sum_{k=1}^{N}\sum_{l=N+1}^{\infty} |\hat{v}_{kl}|^2\Big) \tag{4.95}$$

holds for all $\tau > 0$.

Proof. Using $i\partial_\tau D(\tau) = U(-\tau)(H^{\mathrm{mf}} - H)U^{\mathrm{mf}}(\tau)$, $\|\Omega_0\| = 1$ and some basic algebra similar as in (4.17) and (4.18), it follows directly that

$$\begin{aligned}\Big\|\Big(\partial_\tau D(\tau)\Big)\Big(\psi \otimes \Omega_0\Big)\Big\|^2 &\lesssim \|\psi\|^2\ E_{re}(\varrho)^2 + \Big\|(V-E)\Big(\psi \otimes \Omega_0\Big)\Big\|^2 \\ &\lesssim \|\psi\|^2\ \Big(E_{re}(\varrho)^2 + \frac{1}{L^4}\sum_{k=1}^{N}\sum_{l=N+1}^{\infty}|\hat{v}_{kl}|^2\Big).\end{aligned} \tag{4.96}$$

□

Proof of Lemma 4.7. Let

$$\Psi^{s,n}_{\mathrm{B},11}(t) = \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n} |\hat{v}^{s,\varepsilon}_{k_1 l_1}|^2 \frac{D(t) - g_{l_1k_1}(0)g_{k_1l_1}(t)}{i(E_{l_1}-E_{k_1})^2}\Psi_0, \tag{4.97}$$

$$\Psi^{s,n}_{\mathrm{B},12}(t,\tau_2) = \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n} |\hat{v}^{s,\varepsilon}_{k_1 l_1}|^2 \frac{e^{i(E_{l_1}-E_{k_1})\tau_2}\partial_{\tau_2}\big(D(\tau_2)k_{l_1k_1}(\tau_2)\big)g_{k_1l_1}(t)}{i(E_{l_1}-E_{k_1})^2}\Psi_0, \tag{4.98}$$

$1 \leq n \leq M$. One verifies by means of partial integration that

$$\Psi^{s,n}_{\mathrm{B},1}(t) = \Psi^{s,n}_{\mathrm{B},11}(t) - \int_0^t d\tau_2 \Psi^{s,n}_{\mathrm{B},12}(t,\tau_2). \tag{4.99}$$

By (4.56),

$$\|\Psi^{s,n}_{\mathrm{B},11}(t)\| \lesssim \Big(\varrho^{-(\frac{n-1}{cM})}\mathcal{V}_n(N,\rho)\Big), \tag{4.100}$$

and, by using Lemma 4.8 together with (4.64), (4.66),

$$\|\Psi^{s,n}_{\mathrm{B},12}(t,\tau_2)\| \lesssim \varrho^{-\left(\frac{n-1}{cM}\right)}\frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\Big(\Big\|\Big(\partial_{\tau_2}D(\tau_2)\Big)k_{l_1k_1}(\tau_2)g_{k_1l_1}(t)\Psi_0\Big\|+ \\ +\Big\|\partial_{\tau_2}k_{l_1k_1}(\tau_2)g_{k_1l_1}(t)\varphi_0\Big\|\Big) \\ \lesssim \Big(\varrho^{-\left(\frac{n-1}{cM}\right)}\mathcal{V}_n(N,\rho)\Big)\Big(E_{re}(\varrho)+\Big(\frac{1}{L^4}\sum_{k=1}^{N}\sum_{l=N+1}^{\infty}|\hat{v}_{kl}|^2\Big)^{\frac{1}{2}}+\varrho^{2\varepsilon}\Big). \tag{4.101}$$

Let similarly

$$\Psi^{s,n}_{\mathrm{B},31}(\tau_1) = \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}|\hat{v}^{s,\varepsilon}_{k_1l_1}|^2\frac{\Big(D(\tau_1)g_{l_1k_1}(\tau_1)-g_{l_1k_1}(0)\Big)e^{i(E_{l_1}-E_{k_1})\tau_1}\partial_{\tau_1}k_{k_1l_1}(\tau_1)}{i(E_{l_1}-E_{k_1})^2}\Psi_0,$$

$$\Psi^{s,n}_{\mathrm{B},32}(\tau) = \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}|\hat{v}^{s,\varepsilon}_{k_1l_1}|^2\frac{e^{i(E_{k_1}-E_{l_1})(\tau_2-\tau_1)}\partial_{\tau_2}\Big(D(\tau_2)k_{l_1k_1}(\tau_2)\Big)\partial_{\tau_1}k_{k_1l_1}(\tau_1)}{i(E_{l_1}-E_{k_1})^2}\Psi_0,$$

such that by partial integration in τ_2,

$$\Psi^{s,n}_{\mathrm{B},3}(t) = \int_0^t d\tau_1\Psi^{s,n}_{\mathrm{B},31}(\tau_1) - \int_0^t d\mu_2(\tau)\Psi^{s,n}_{\mathrm{B},32}(\tau). \tag{4.102}$$

From (4.56) together with (4.65), it follows that

$$\|\Psi^{s,n}_{\mathrm{B},31}(\tau)\| \lesssim \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\frac{\|\partial_{\tau_1}k_{k_1l_1}(\tau_1)\varphi_0\|}{(E_{l_1}-E_{k_1})^2} \lesssim \varrho^{2\varepsilon}\Big(\varrho^{-\left(\frac{n-1}{cM}\right)}\mathcal{V}_n(N,\rho)\Big), \tag{4.103}$$

as well as in combination with Lemma 4.8 and (4.66),

$$\|\Psi^{s,n}_{\mathrm{B},32}(t)\| \lesssim \varrho^{-\left(\frac{n-1}{cM}\right)}\frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\Big(\Big\|\Big(\partial_{\tau_2}D(\tau_2)\Big)k_{l_1k_1}(\tau_2)k_{k_1l_1}(\tau_1)\Psi_0\Big\|+\varrho^{2\varepsilon}\Big) \tag{4.104}$$

$$\lesssim \Big(\varrho^{-\left(\frac{n-1}{cM}\right)}\mathcal{V}_n(N,\rho)\Big)\Big(E_{re}(\varrho)+\Big(\frac{1}{L^4}\sum_{k=1}^{N}\sum_{l=N+1}^{\infty}|\hat{v}_{kl}|^2\Big)^{\frac{1}{2}}+\varrho^{2\varepsilon}\Big). \tag{4.105}$$

Lemma 4.7 then follows from (4.57), (4.60) and (4.64). □

Derivation of the bound for $\|\Psi_{\mathrm{C}}(t)\|_{\mathrm{TD}}$

In the estimate for $\|\Psi_{\mathrm{C}}(t)\|_{\mathrm{TD}}$ there appears an additional sum. This can be dealt with by using (4.61).

We define for $0 \le n \le M$,

$$\Psi^{s,n}_{\mathrm{C}}(t) = \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\sum_{k_2=1}^{N}\hat{v}_{k_2k_1}\hat{v}^{s,\varepsilon}_{k_1l_1}\int_0^t d\mu_2(\tau)D(\tau_2)\Big(g_{k_2k_1}(\tau_2)g_{k_1l_1}(\tau_1)\varphi_0\Big)\otimes\Omega_0^{[k_2l_1^*]}, \tag{4.106}$$

$$\Psi^{\ell}_{\mathrm{C}}(\tau) = \frac{1}{L^4}\sum_{l_1=N+1}^{\infty}\sum_{k_1,k_2=1}^{N}\hat{v}_{k_2k_1}\hat{v}^{\ell,\varepsilon}_{k_1l_1}\Big(g_{k_2k_1}(\tau_2)g_{k_1l_1}(\tau_1)\varphi_0\Big)\otimes\Omega_0^{[k_2l_1^*]}, \tag{4.107}$$

such that

$$\Psi_{\mathrm{C}}(t) = \sum_{n=0}^{M} \Psi_{\mathrm{C}}^{s,n}(t) + \int_0^t d\mu_2(\tau) D(\tau_2) \Psi_{\mathrm{C}}^{\ell}(\tau). \tag{4.108}$$

Note that $\langle \Omega_0^{[k_2 l_1^*]}, \Omega_0^{[m_2 n_1^*]} \rangle = \delta_{k_2 m_2} \delta_{l_1 n_1}$ for $k_2, m_2 \leq N$, $N+1 \leq l_1, n_1$. Using this and $|\hat{v}| \leq C$, one finds

$$\begin{aligned} \|\Psi_{\mathrm{C}}^{\ell}(\tau)\|^2 &\lesssim \frac{1}{L^8} \sum_{k_1=1}^{N} \sum_{l_1=N+1}^{\infty} |\hat{v}_{k_1 l_1}^{\ell,\varepsilon}| \sum_{k_2=1}^{N} |\hat{v}_{k_2 k_1}| \sum_{m_1=1}^{N} \sum_{n_1=N+1}^{\infty} |\hat{v}_{m_1 n_1}^{\ell,\varepsilon}| \sum_{m_2=1}^{N} |\hat{v}_{m_2 m_1}| \langle \Omega_0^{[k_2 l_1^*]}, \Omega_0^{[m_2 n_1^*]} \rangle \\ &\lesssim \frac{1}{L^4} \sum_{k_1=1}^{N} \sum_{l_1=N+1}^{\infty} |\hat{v}_{k_1 l_1}^{\ell,\varepsilon}| \frac{1}{L^2} \sum_{k_2=1}^{N} |\hat{v}_{k_2 k_1}| \frac{1}{L^2} \sum_{m_1=1}^{N} |\hat{v}_{m_1 l_1}^{\ell,\varepsilon}|. \end{aligned} \tag{4.109}$$

Then, by means of (4.58) and (4.61), $\|\Psi_{\mathrm{C}}^{\ell}(\tau)\|_{\mathrm{TD}} \leq C_\varepsilon \varrho^{\varepsilon - \frac{1}{\varepsilon}}$, and thus

$$\Big\| \int_0^t d\mu_2(\tau) D(\tau_2) \Psi_{\mathrm{C}}^{\ell}(\tau) \Big\|_{\mathrm{TD}} \leq C_\varepsilon t^2 \varrho^{\varepsilon - \frac{1}{\varepsilon}}. \tag{4.110}$$

Similarly, we can estimate the norm in $\Psi_{\mathrm{C}}^{s,0}(t)$. Using (4.61), we find

$$\begin{aligned} &\Big\| \frac{1}{L^4} \sum_{(k_1,l_1) \in \mathfrak{S}_n} \sum_{k_2=1}^{N} \hat{v}_{k_2 k_1} \hat{v}_{k_1 l_1}^{s,\varepsilon} g_{k_2 k_1}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \otimes \Omega_0^{[k_2 l_1^*]} \Big\|^2 \\ &\qquad \lesssim \frac{1}{L^4} \sum_{(k_1,l_1) \in \mathfrak{S}_0} \frac{1}{L^2} \sum_{k_2=1}^{N} |\hat{v}_{k_2 k_1}| \frac{1}{L^2} \sum_{m_1=1}^{N} |\hat{v}_{m_1 n_1}^{s,\varepsilon}| \lesssim \mathcal{V}_0(N,\rho) \Big(\varrho^{2\varepsilon} + C_\varepsilon \varrho^{-1/\varepsilon} \Big) \varrho^{2\varepsilon}. \end{aligned} \tag{4.111}$$

In combination with (4.63) this gives

$$\|\Psi_{\mathrm{C}}^{s,0}(t)\|_{\mathrm{TD}} \lesssim t^2 \varrho^{-\frac{1}{4} + \frac{3}{2}\varepsilon} \Big(\varrho^{\varepsilon} + C_\varepsilon \varrho^{-1/(2\varepsilon)} \Big). \tag{4.112}$$

Lemma 4.9. *Let $\Psi_C^{s,n}(t)$ as in (4.106). Then, under the same assumptions as in Theorem 4.1, there exists a positive constant C_ε such that*

$$\|\Psi_{\mathrm{C}}^{s,n}(t)\|_{\mathrm{TD}} \lesssim (1+t)^2 \varrho^{-\frac{1}{4} + \frac{7}{2}\varepsilon} \sqrt{\varrho^{\frac{1}{cM}} - \varrho^{\frac{1}{2cM}}} \Big(\varrho^{\varepsilon} + C_\varepsilon \varrho^{-1/(2\varepsilon)} \Big), \qquad 1 \leq n \leq M, \tag{4.113}$$

holds for all $t > 0$.

Here, we can use again that $M \varrho^{\frac{1}{2cM}} \lesssim \varrho^{\frac{1}{2}\varepsilon}$ for $M = \lfloor \ln \varrho \rfloor$, and thus obtain

$$\sum_{n=1}^{M} \|\Psi_{\mathrm{C}}^{s,n}(t)\|_{\mathrm{TD}} \lesssim (1+t)^2 \varrho^{-\frac{1}{4} + 2\varepsilon} \Big(\varrho^{\varepsilon} + C_\varepsilon \varrho^{-1/(2\varepsilon)} \Big). \tag{4.114}$$

This proves the bound for $\|\Psi_{\mathrm{C}}(t)\|_{\mathrm{TD}}$ in (4.43).

Proof of Lemma 4.9. We define for $1 \leq n \leq M$,

$$\Psi_{\mathrm{C},1}^{s,n}(t,\tau_1) = \frac{1}{L^4} \sum_{(k_1,l_1) \in \mathfrak{S}_n} \sum_{k_2=1}^{N} \hat{v}_{k_2 k_1} \hat{v}_{k_1 l_1}^{s,\varepsilon} \Big(g_{k_2 k_1}(\tau_1) \frac{g_{k_1 l_1}(t) - g_{k_1 l_1}(\tau_1)}{i(E_{l_1} - E_{k_1})} \varphi_0 \Big) \otimes \Omega_0^{[k_2 l_1^*]}, \tag{4.115}$$

$$\Psi_{\mathrm{C},2}^{s,n}(\tau) = \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{k_2=1}^{N} \hat{v}_{k_2k_1} \hat{v}_{k_1l_1}^{s,\varepsilon} \Big(g_{k_2k_1}(\tau_2) \frac{e^{i(E_{l_1}-E_{k_1})\tau_1} \partial_{\tau_1} k_{k_1l_1}(\tau_1)}{i(E_{l_1}-E_{k_1})} \varphi_0 \Big) \otimes \Omega_0^{[k_2 l_1^*]}, \tag{4.116}$$

and find via partial integration,

$$\Psi_{\mathrm{C}}^{(n)}(t) = \int_0^t d\tau_1 D(\tau_1) \Psi_{\mathrm{C},1}^{s,n}(t,\tau_1) - \int_0^t d\mu_2(\tau) D(\tau_2) \Psi_{\mathrm{C},2}^{s,n}(\tau). \tag{4.117}$$

For estimating the wave functions on the r.h.s., we use again $\langle \Omega_0^{[k_2 l_1^*]}, \Omega_0^{[m_2 n_1^*]} \rangle = \delta_{k_2 m_2} \delta_{l_1 n_1}$, (4.56) and (4.61),

$$\begin{aligned} \|\Psi_{\mathrm{C},1}^{s,n}(\tau)\|^2 &\lesssim \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \varrho^{-(\frac{n-1}{cM})} \frac{1}{L^2} \sum_{k_2=1}^{N} |\hat{v}_{k_2k_1}| \frac{1}{L^2} \sum_{m_1=1}^{N} |\hat{v}_{m_1l_1}^{s,\varepsilon}| \\ &\lesssim \varrho^{2\varepsilon} \Big(\varrho^{-(\frac{n-1}{cM})} \mathcal{V}_n(N,\rho) \Big) \Big(\varrho^{2\varepsilon} + C_\varepsilon \varrho^{-1/\varepsilon} \Big). \end{aligned} \tag{4.118}$$

Similarly, using in addition (4.65),

$$\begin{aligned} \|\Psi_{\mathrm{C},2}^{s,n}(t,\tau_1)\|^2 &\lesssim \varrho^{4\varepsilon} \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \varrho^{-(\frac{n-1}{cM})} \frac{1}{L^2} \sum_{k_2=1}^{N} |\hat{v}_{k_2k_1}| \frac{1}{L^2} \sum_{m_1=1}^{N} |\hat{v}_{m_1l_1}^{s,\varepsilon}| \\ &\lesssim \varrho^{6\varepsilon} \Big(\varrho^{-(\frac{n-1}{cM})} \mathcal{V}_n(N,\rho) \Big) \Big(\varrho^{2\varepsilon} + C_\varepsilon \varrho^{-1/\varepsilon} \Big). \end{aligned} \tag{4.119}$$

Application of (4.64) completes the proof of the lemma. □

Derivation of the bound for $\|\Psi_{\mathbf{D}}(t)\|_{\mathbf{TD}}$

The term $\Psi_{\mathrm{D}}(t)$ has a similar structure as $\Psi_{\mathrm{C}}(t)$ and can be estimated analogously.

For $0 \le n \le M$, let

$$\Psi_{\mathrm{D}}^{s,n}(t) = \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{l_2=N+1}^{\infty} \hat{v}_{l_1l_2} \hat{v}_{k_1l_1}^{s,\varepsilon} \int_0^t d\mu_2(\tau) D(\tau_2) \Big(g_{l_1l_2}(\tau_2) g_{k_1l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[l_2^* k_1]}, \tag{4.120}$$

$$\Psi_{\mathrm{D}}^{\ell}(\tau) = \frac{1}{L^4} \sum_{k_1=1}^{N} \sum_{l_1,l_2=N+1}^{\infty} \hat{v}_{l_1l_2} \hat{v}_{k_1l_1}^{\ell,\varepsilon} \Big(g_{l_1l_2}(\tau_2) g_{k_1l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[l_2^* k_1]}. \tag{4.121}$$

This leads to the identity

$$\Psi_{\mathrm{D}}(t) = \sum_{n=0}^{M} \Psi_{\mathrm{D}}^{s,n}(t) + \int_0^t d\mu_2(\tau) D(\tau_2) \Psi_{\mathrm{D}}^{\ell}(\tau). \tag{4.122}$$

Using $\langle \Omega_0^{[l_2^* k_1]}, \Omega_0^{[n_2^* m_1]} \rangle = \delta_{l_2 n_2} \delta_{k_1 m_1}$ which holds for $l_2, n_2 \ge N+1$ and $k_1, m_1 \le N$, one finds

$$\|\Psi_{\mathrm{D}}^{\ell}(t)\|^2 \lesssim \frac{1}{L^4} \sum_{k_1=1}^{N} \sum_{l_1=N+1}^{\infty} |\hat{v}_{k_1l_1}^{\ell,\varepsilon}| \frac{1}{L^2} \sum_{l_2=N+1}^{\infty} |\hat{v}_{l_1l_2}| \frac{1}{L^2} \sum_{n_1=N+1}^{\infty} |\hat{v}_{k_1n_1}^{\ell,\varepsilon}|. \tag{4.123}$$

Then, (4.58) in combination with (4.61) leads to $\|\Psi_{\mathrm{D}}^{\ell}(\tau)\|_{\mathrm{TD}} \le C_\varepsilon \varrho^{\varepsilon-\frac{1}{\varepsilon}}$, and hence,

$$\Big\| \int_0^t d\mu_2(\tau) D(\tau_2) \Psi_{\mathrm{D}}^{\ell}(\tau) \Big\|_{\mathrm{TD}} \le C_\varepsilon t^2 \varrho^{\varepsilon-\frac{1}{\varepsilon}}. \tag{4.124}$$

We similarly estimate the norm in $\Psi_{\mathrm{D}}^{s,0}(t)$,

$$\Big\| \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_0} \sum_{l_2=N+1}^{\infty} \hat{v}_{l_1 l_2} \hat{v}^{s,\varepsilon}_{k_1 l_1} \Big(g_{l_1 l_2}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[l_2^* k_1]} \Big\|^2$$
$$\lesssim \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_0} \frac{1}{L^2} \sum_{l_2=N+1}^{\infty} |\hat{v}_{l_1 l_2}| \frac{1}{L^2} \sum_{n_1=N+1}^{\infty} |\hat{v}^{s,\varepsilon}_{k_1 n_1}|. \tag{4.125}$$

With (4.61) and (4.63), it follows that

$$\|\Psi_{\mathrm{D}}^{s,0}(t)\|_{\mathrm{TD}} \lesssim t^2 \varrho^{-\frac{1}{4}+\frac{3}{2}\varepsilon} \Big(\varrho^{\varepsilon} + C_\varepsilon \varrho^{-1/(2\varepsilon)} \Big). \tag{4.126}$$

Lemma 4.10. *Let $\Psi_D^{s,n}(t)$ as in (4.120). Under the same assumptions as in Theorem 4.1, there exists a positive constant C_ε such that*

$$\|\Psi_{\mathrm{D}}^{s,n}(t)\|_{\mathrm{TD}} \lesssim (1+t)^2 \varrho^{-\frac{1}{4}+\frac{7}{2}\varepsilon+\frac{1}{2cM}} \Big(\varrho^{\varepsilon} + C_\varepsilon \varrho^{-1/(2\varepsilon)} \Big), \qquad 1 \le n \le M, \tag{4.127}$$

holds for all $t \ge 0$.

It follows as below Lemma 4.9 that for $M = \lfloor \ln \varrho \rfloor$,

$$\sum_{n=1}^{M} \|\Psi_{\mathrm{D}}^{s,n}(t)\|_{\mathrm{TD}} \lesssim (1+t)^2 \varrho^{-\frac{1}{4}+2\varepsilon} \Big(\varrho^{\varepsilon} + C_\varepsilon \varrho^{-1/(2\varepsilon)} \Big), \tag{4.128}$$

which proves the bound for $\|\Psi_{\mathrm{D}}(t)\|_{\mathrm{TD}}$ in (4.44).

Proof of Lemma 4.10. For $1 \le n \le M$, we set

$$\Psi_{\mathrm{D},1}^{s,n}(t,\tau_1) = \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{l_2=N+1}^{\infty} \hat{v}_{l_1 l_2} \hat{v}^{s,\varepsilon}_{k_1 l_1} \Big(g_{l_1 l_2}(\tau_1) \frac{g_{k_1 l_1}(t) - g_{k_1 l_1}(\tau_1)}{i(E_{l_1} - E_{k_1})} \varphi_0 \Big) \otimes \Omega_0^{[l_2^* k_1]},$$
$$\Psi_{\mathrm{D},2}^{s,n}(\tau) = \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{l_2=N+1}^{\infty} \hat{v}_{l_1 l_2} \hat{v}^{s,\varepsilon}_{k_1 l_1} \Big(g_{l_1 l_2}(\tau_2) \frac{e^{i(E_{l_1}-E_{k_1})\tau_1} \partial_{\tau_1} k_{k_1 l_1}(\tau_1)}{i(E_{l_1} - E_{k_1})} \varphi_0 \Big) \otimes \Omega_0^{[l_2^* k_1]}.$$

Partial integration leads to

$$\Psi_{\mathrm{D}}^{s,n}(t) = \int_0^t d\tau_1 D(\tau_1) \Psi_{\mathrm{D},1}^{s,n}(t,\tau) - \int_0^t d\mu_2(\tau) D(\tau_2) \Psi_{\mathrm{D},2}^{s,n}(\tau). \tag{4.129}$$

It remains to compute the norm of the wave functions on the r.h.s. Using (4.56),

$$\|\Psi_{\mathrm{D},1}^{s,n}(t,\tau_1)\|^2 \lesssim \varrho^{-\left(\frac{n-1}{cM}\right)} \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \frac{1}{L^2} \sum_{l_2=N+1}^{\infty} |\hat{v}_{l_1 l_2}| \frac{1}{L^2} \sum_{n_1=N+1}^{\infty} |\hat{v}^{s,\varepsilon}_{k_1 n_1}|, \tag{4.130}$$

and similarly, using in addition (4.65),

$$\|\Psi_{\mathrm{D},2}^{s,n}(\tau)\|^2 \lesssim \varrho^{4\varepsilon} \varrho^{-\left(\frac{n-1}{cM}\right)} \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \frac{1}{L^2} \sum_{l_2=N+1}^{\infty} |\hat{v}_{l_1 l_2}| \frac{1}{L^2} \sum_{n_1=N+1}^{\infty} |\hat{v}^{s,\varepsilon}_{k_1 n_1}|. \tag{4.131}$$

The lemma then follows from (4.61) together with (4.64). ∎

Derivation of the bound for $\|\Psi_{\mathrm{E}}(t)\|_{\mathrm{TD}}$

The term $\Psi_{\mathrm{E}}(t)$ is more difficult to estimate since it involves four sums. In order to get the desired bound, it is not enough to do one partial integration. We have to split the term more carefully into different contributions and for some of them perform an additional partial integration. This gives an additional phase cancellation which is enough for the desired bound.

For $0 \le n, m \le M$, we define

$$\Psi_{\mathrm{E}}^{s,nm}(t) = \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \hat{v}^{s,\varepsilon}_{k_2l_2} \hat{v}^{s,\varepsilon}_{k_1l_1} \int_0^t d\mu_3(\tau) D(\tau_3) \Big(g_{k_2l_2}(\tau_2) g_{k_1l_1}(\tau_1)\varphi_0 \Big) \otimes \Omega_0^{[l_2^* k_2 l_1^* k_1]}, \tag{4.132}$$

$$\Psi_{\mathrm{E}}^{\ell}(\tau) = \frac{1}{L^4} \sum_{k_1,k_2=1}^{N} \sum_{l_1,l_2=N+1}^{\infty} \hat{w}^{\ell,\varepsilon}_{k_2l_2k_1l_1} \Big(g_{k_2l_2}(\tau_2) g_{k_1l_1}(\tau_1)\varphi_0 \Big) \otimes \Omega_0^{[l_2^* k_2 l_1^* k_1]}, \tag{4.133}$$

where

$$\hat{w}^{\ell,\varepsilon}_{k_2l_2k_1l_1} := \hat{v}^{\ell,\varepsilon}_{k_2l_2} \hat{v}^{s,\varepsilon}_{k_1l_1} + \hat{v}^{s,\varepsilon}_{k_2l_2} \hat{v}^{\ell,\varepsilon}_{k_1l_1} + \hat{v}^{\ell,\varepsilon}_{k_2l_2} \hat{v}^{\ell,\varepsilon}_{k_1l_1}. \tag{4.134}$$

This leads to

$$\Psi_{\mathrm{E}}(t) = E_{re}(\varrho) \sum_{n,m=0}^{M} \Psi_{\mathrm{E}}^{s,nm}(t) + E_{re}(\varrho) \int_0^t d\mu_3(\tau) D(\tau_3) \Psi_{\mathrm{E}}^{\ell}(\tau), \tag{4.135}$$

since $\hat{v}_{k_1l_1}\hat{v}_{k_2l_2} = \hat{v}^{s,\varepsilon}_{k_2l_2}\hat{v}^{s,\varepsilon}_{k_1l_1} + \hat{w}^{\ell,\varepsilon}_{k_2l_2k_1l_1}$. Using that for $k_1, k_2, m_1, m_2 \le N$ and $N+1 \le l_1, l_2, n_1, n_2$,

$$\begin{aligned} \big\langle \Omega_0^{[l_2^* k_2 l_1^* k_1]}, \Omega_0^{[n_2^* m_2 n_1^* m_1]} \big\rangle = {} & (\delta_{k_2m_2}\delta_{k_1m_1} + \delta_{k_2m_1}\delta_{k_1m_2})\delta^{\perp}_{k_2k_1}\delta^{\perp}_{m_2m_1} \times \\ & \times (\delta_{l_2n_2}\delta_{l_1n_1} + \delta_{l_2n_1}\delta_{l_1n_2})\delta^{\perp}_{l_2l_1}\delta^{\perp}_{n_2n_1}, \end{aligned} \tag{4.136}$$

where $\delta^{\perp}_{kl} = 1 - \delta_{kl}$, we find

$$\|\Psi_{\mathrm{E}}^{\ell}(\tau)\|^2 \lesssim \frac{1}{L^8} \sum_{k_1,k_2=1}^{N} \sum_{l_1,l_2=N+1}^{\infty} \Big(|\hat{v}^{\ell,\varepsilon}_{k_2l_2}|\, |\hat{v}^{s,\varepsilon}_{k_1l_1}| + |\hat{v}^{s,\varepsilon}_{k_2l_2}|\, |\hat{v}^{\ell,\varepsilon}_{k_1l_1}| + |\hat{v}^{\ell,\varepsilon}_{k_2l_2}|\, |\hat{v}^{\ell,\varepsilon}_{k_1l_1}| \Big). \tag{4.137}$$

By means of (4.57) and (4.58), $\|\Psi_{\mathrm{E}}^{\ell}(\tau)\|_{\mathrm{TD}} \le C_\varepsilon \varrho^{\frac14 - \frac{1}{2\varepsilon}}$, such that together with (4.60) we find for the last term in (4.135) that

$$\Big\| E_{re}(\varrho) \int_0^t d\mu_3(\tau) D(\tau_3) \Psi_{\mathrm{E}}^{\ell}(\tau) \Big\|_{\mathrm{TD}} \le C_\varepsilon t^3 \varrho^{2\varepsilon + \frac14 - \frac{1}{2\varepsilon}}. \tag{4.138}$$

Similarly, one finds for $\Psi_{\mathrm{E}}^{s,00}(t)$,

$$\Big\| \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_0} \sum_{(k_2,l_2)\in\mathfrak{S}_0} \hat{v}^{s,\varepsilon}_{k_2l_2} \hat{v}^{s,\varepsilon}_{k_1l_1} \Big(g_{k_2l_2}(\tau_2) g_{k_1l_1}(\tau_1)\varphi_0 \Big) \otimes \Omega_0^{[l_2^* k_2 l_1^* k_1]} \Big\|^2 \lesssim \mathcal{V}_0(N,\rho)^2. \tag{4.139}$$

Hence, with (4.63),

$$\|\Psi_{\mathrm{E}}^{s,00}(t)\|_{\mathrm{TD}} \lesssim t^3 \varrho^{-\frac12 + 4\varepsilon}. \tag{4.140}$$

Lemma 4.11. *Let $\Psi_{\mathrm{E}}^{s,nm}(t)$ be defined as in (4.132). Under the same assumptions as in Theorem 4.1,*

$$\|\Psi_{\mathrm{E}}^{s,0n}(t)\|_{\mathrm{TD}} + \|\Psi_{\mathrm{E}}^{s,n0}(t)\|_{\mathrm{TD}} \lesssim (1+t)^3 \varrho^{-\frac{1}{2}+3\varepsilon} \sqrt{\varrho^{\frac{1}{cM}} - \varrho^{\frac{1}{2cM}}}, \quad 1 \le n \le M, \tag{4.141}$$

$$\|\Psi_{\mathrm{E}}^{s,nm}(t)\|_{\mathrm{TD}} \lesssim (1+t)^3 \varrho^{-\frac{1}{2}+5\varepsilon} \Big(\varrho^{\frac{1}{cM}} - \varrho^{\frac{1}{2cM}}\Big), \quad 1 \le n,m \le M, \tag{4.142}$$

holds for all $t>0$.

With $M = \lfloor \ln \varrho \rfloor$ we then obtain that (recall (4.76) and Remark 4.3)

$$\sum_{n=1}^{M} \Big(\|\Psi_{\mathrm{E}}^{s,n0}(t)\|_{\mathrm{TD}} + \|\Psi_{\mathrm{E}}^{s,0n}(t)\|_{\mathrm{TD}} \Big) \lesssim (1+t)^3 \varrho^{-\frac{1}{2}+4\varepsilon}, \tag{4.143}$$

$$\sum_{n=1}^{M} \|\Psi_{\mathrm{E}}^{s,nm}(t)\|_{\mathrm{TD}} \lesssim (1+t)^3 \varrho^{-\frac{1}{2}+6\varepsilon}. \tag{4.144}$$

This proves the bound for $\|\Psi_{\mathrm{E}}(t)\|_{\mathrm{TD}}$ in (4.45).

Proof of Lemma 4.11. We define for $1 \le n \le M$,

$$\Psi_{\mathrm{E},1}^{s,n0}(t,\tau_1) = \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_0} \hat{v}_{k_2l_2}^{s,\varepsilon} \hat{v}_{k_1l_1}^{s,\varepsilon} \Big(g_{k_2l_2}(\tau_1) \frac{g_{k_1l_1}(t) - g_{k_1l_1}(\tau_1)}{i(E_{l_1}-E_{k_1})} \varphi_0 \Big) \otimes \Psi_0^{[l_2^* k_2 l_1^* k_1]},$$

$$\Psi_{\mathrm{E},2}^{s,n0}(\tau) = \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_0} \hat{v}_{k_2l_2}^{s,\varepsilon} \hat{v}_{k_1l_1}^{s,\varepsilon} \Big(g_{k_2l_2}(\tau_2) \frac{e^{i(E_{l_1}-E_{k_1})\tau_1} \partial_{\tau_1} k_{k_1l_1}(\tau_1)}{i(E_{l_1}-E_{k_1})} \varphi_0 \Big) \otimes \Psi_0^{[l_2^* k_2 l_1^* k_1]},$$

$$\Psi_{\mathrm{E},1}^{s,0n}(\tau) = \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_0} \sum_{(k_2,l_2)\in\mathfrak{S}_n} \hat{v}_{k_2l_2}^{s,\varepsilon} \hat{v}_{k_1l_1}^{s,\varepsilon} \Big(\frac{g_{k_2l_2}(\tau_1) - g_{k_2l_2}(\tau_2)}{i(E_{l_2}-E_{k_2})} g_{k_1l_1}(\tau_1) \varphi_0 \Big) \otimes \Psi_0^{[l_2^* k_2 l_1^* k_1]},$$

$$\Psi_{\mathrm{E},2}^{s,0n}(\tau) = \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_0} \sum_{(k_2,l_2)\in\mathfrak{S}_n} \hat{v}_{k_2l_2}^{s,\varepsilon} \hat{v}_{k_1l_1}^{s,\varepsilon} \Big(\frac{e^{i(E_{l_2}-E_{k_2})\tau_2} \partial_{\tau_2} k_{k_2l_2}(\tau_2)}{i(E_{l_2}-E_{k_2})} g_{k_1l_1}(\tau_1) \varphi_0 \Big) \otimes \Psi_0^{[l_2^* k_2 l_1^* k_1]}.$$

By partial integration, this leads to

$$\Psi_{\mathrm{E}}^{s,n0}(t) = E_{re}(\varrho) \int_0^t d\mu_2(\tau) D(\tau_2) \Psi_{\mathrm{E},1}^{s,n0}(t,\tau_1) - E_{re}(\varrho) \int_0^t d\mu_3(\tau) D(\tau_3) \Psi_{\mathrm{E},2}^{s,n0}(\tau), \tag{4.145}$$

$$\Psi_{\mathrm{E}}^{s,0n}(t) = E_{re}(\varrho) \int_0^t d\mu_2(\tau) D(\tau_2) \Psi_{\mathrm{E},1}^{s,0n}(\tau) - E_{re}(\varrho) \int_0^t d\mu_3(\tau) D(\tau_3) \Psi_{\mathrm{E},2}^{s,0n}(\tau). \tag{4.146}$$

Furthermore, for all $1 \le n,m \le M$, and $X \in \{1,2,3\}$, we set

$$\Psi_{\mathrm{E,X}}^{s,nm}(t,\tau) = \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \hat{v}_{k_2l_2}^{s,\varepsilon} \hat{v}_{k_1l_1}^{s,\varepsilon} \Big(G_{k_2l_2k_1l_2}^{(\mathrm{X})}(t,\tau) \varphi_0 \Big) \otimes \Omega_0^{[l_2^* k_2 l_1^* k_1]}, \tag{4.147}$$

where we introduce the operators $G_{k_2l_2k_1l_1}^{(\mathrm{X})}(t,\tau) : \mathcal{H}_y \to \mathcal{H}_y$, defined by

$$G_{k_2l_2k_1l_1}^{(1)}(t,\tau) = \frac{g_{k_2l_2}(t) g_{k_1l_1}(t) - g_{k_2l_2}(\tau_1) g_{k_1l_1}(\tau_1)}{i(E_{l_2}-E_{k_2}) i(E_{l_2}-E_{k_2}+E_{l_1}-E_{k_1})} - \frac{g_{k_2l_2}(\tau_1)\Big(g_{k_1l_1}(t) - g_{k_1l_1}(\tau_1)\Big)}{i(E_{l_2}-E_{k_2}) i(E_{l_1}-E_{k_1})}, \tag{4.148}$$

$$G^{(2)}_{k_2l_2k_1l_1}(t,\tau)=\frac{e^{i(E_{l_1}-E_{k_1}+E_{l_2}-E_{k_2})\tau_1}\partial_{\tau_1}\Big(k_{k_2l_2}(\tau_1)k_{k_1l_1}(\tau_1)\Big)}{i(E_{l_2}-E_{k_2})i(E_{l_2}-E_{k_2}+E_{l_1}-E_{k_1})}$$
$$-\frac{g_{k_2l_2}(\tau_2)e^{i(E_{l_1}-E_{k_1})\tau_1}\Big(\partial_{\tau_1}k_{k_1l_1}(\tau_1)\Big)}{i(E_{l_2}-E_{k_2})i(E_{l_1}-E_{k_1})}$$
$$-\frac{e^{i(E_{l_2}-E_{k_2})\tau_1}\Big(\partial_{\tau_1}k_{k_2l_2}(\tau_1)\Big)\Big(g_{k_1l_1}(t)-g_{k_1l_1}(\tau_1)\Big)}{i(E_{l_2}-E_{k_2})i(E_{l_1}-E_{k_1})}, \tag{4.149}$$
$$G^{(3)}_{k_2l_2k_1l_1}(t,\tau)=\frac{e^{i(E_{l_2}-E_{k_2})\tau_2}\Big(\partial_{\tau_2}k_{k_2l_2}(\tau_2)\Big)e^{i(E_{k_1}-E_{l_1})\tau_1}\Big(\partial_{\tau_1}k_{k_1l_1}(\tau_1)\Big)}{i(E_{k_2}-E_{l_2})i(E_{l_1}-E_{k_1})}. \tag{4.150}$$

With a two-fold partial integration one now finds

$$\Psi^{s,nm}_{\mathrm{E}}(t)=E_{re}(\varrho)\Big[\int_0^t d\tau_1 D(\tau_1)\Psi^{s,nm}_{\mathrm{E},1}(t,\tau)$$
$$-\int_0^t d\mu_2(\tau)D(\tau_2)\Psi^{s,nm}_{\mathrm{E},2}(t,\tau)+\int_0^t d\mu_3(\tau)D(\tau_3)\Psi^{s,nm}_{\mathrm{E},3}(t,\tau)\Big]. \tag{4.151}$$

It remains to compute the norm of the above wave functions. Recalling that the scalar product produces four Kronecker-deltas,

$$\|\Psi^{s,n0}_{\mathrm{E},1}(t,\tau_1)\|^2\lesssim\frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\frac{1}{E_{l_1}-E_{k_1}}\frac{1}{L^4}\sum_{(k_2,l_2)\in\mathfrak{S}_0}\times$$
$$\times\sum_{(m_1,n_1)\in\mathfrak{S}_n}\frac{1}{E_{n_1}-E_{m_1}}\sum_{(m_2,n_2)\in\mathfrak{S}_0}\langle\Omega_0^{[l_2^*k_2l_1^*k_1]},\Omega_0^{[n_2^*m_2n_1^*m_1]}\rangle$$
$$\lesssim\Big(\varrho^{-(\frac{n-1}{cM})}\mathcal{V}_n(N,\rho)\Big)\mathcal{V}_0(N,\rho). \tag{4.152}$$

Using in addition (4.65),

$$\|\Psi^{s,n0}_{\mathrm{E},2}(\tau)\|^2\lesssim\frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\frac{\|\partial_{\tau_1}k_{k_1l_1}(\tau_1)\varphi_0\|}{E_{l_1}-E_{k_1}}\frac{1}{L^4}\sum_{(k_2,l_2)\in\mathfrak{S}_0}\times$$
$$\times\sum_{(m_1,n_1)\in\mathfrak{S}_n}\frac{\|\partial_{\tau_1}k_{m_1n_1}(\tau_1)\varphi_0\|}{E_{n_1}-E_{m_1}}\sum_{(m_2,n_2)\in\mathfrak{S}_0}\langle\Omega_0^{[l_2^*k_2l_1^*k_1]},\Omega_0^{[n_2^*m_2n_1^*m_1]}\rangle$$
$$\lesssim\varrho^{4\varepsilon}\Big(\varrho^{-(\frac{n-1}{cM})}\mathcal{V}_n(N,\rho)\Big)\mathcal{V}_0(N,\rho). \tag{4.153}$$

By means of (4.64), this shows the first part of the first bound in the lemma. We omit the proof for $\Psi^{s,0n}_{\mathrm{E}}(\tau)$ since it works exactly the same way as for $\Psi^{s,n0}_{\mathrm{E}}(\tau)$.

For the second bound of the lemma, note that by using (4.56) and Lemma 4.5, one finds

$$\chi_{\mathfrak{S}_n}\big((k_1,l_1)\big)\chi_{\mathfrak{S}_m}\big((k_2,l_2)\big)\|G^{(1)}_{k_2l_2k_1l_1}(t,\tau)\varphi_0\|\lesssim\varrho^{-(\frac{n-1}{2cM})}\varrho^{-(\frac{m-1}{2cM})}, \tag{4.154}$$
$$\chi_{\mathfrak{S}_n}\big((k_1,l_1)\big)\chi_{\mathfrak{S}_m}\big((k_2,l_2)\big)\|G^{(2)}_{k_2l_2k_1l_1}(t,\tau)\varphi_0\|\lesssim\varrho^{4\varepsilon}\varrho^{-(\frac{n-1}{2cM})}\varrho^{-(\frac{m-1}{2cM})}, \tag{4.155}$$
$$\chi_{\mathfrak{S}_n}\big((k_1,l_1)\big)\chi_{\mathfrak{S}_m}\big((k_2,l_2)\big)\|G^{(3)}_{k_2l_2k_1l_1}(t,\tau)\varphi_0\|\lesssim\varrho^{4\varepsilon}\varrho^{-(\frac{n-1}{2cM})}\varrho^{-(\frac{m-1}{2cM})}. \tag{4.156}$$

Then, for all $1 \le n \le M$ and $X \in \{1,2,3\}$,

$$\begin{aligned}\|\Psi^{s,nm}_{\mathrm{E,X}}(t,\tau)\|^2 \lesssim \frac{1}{L^8} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \sum_{(m_1,n_1)\in\mathfrak{S}_n} \sum_{(m_2,n_2)\in\mathfrak{S}_m} \|G^{(\mathrm{X})}_{k_2l_2k_1l_1}(t,\tau)\varphi_0\| \times \\ \times \|G^{(\mathrm{X})}_{m_2n_2m_1n_1}(t,\tau)\varphi_0\| \left\langle \Omega_0^{[l_2^*k_2l_1^*k_1]}, \Omega_0^{[n_2^*m_2n_1^*m_1]} \right\rangle \\ \lesssim \varrho^{8\varepsilon} \Big(\varrho^{-(\frac{n-1}{cM})}\mathcal{V}_n(N,\rho)\Big)\Big(\varrho^{-(\frac{m-1}{cM})}\mathcal{V}_m(N,\rho)\Big). \qquad (4.157)\end{aligned}$$

The bounds for the remaining expressions have been derived in (4.64). □

Derivation of the bound for $\|\Psi_{\mathrm{F}}(t)\|_{\mathrm{TD}}$

The estimate for $\|\Psi_{\mathrm{F}}(t)\|_{\mathrm{TD}}$ is more tedious, since we have to deal with an additional $(V-E)$, i.e., we have to take into account one more collision, starting from Ψ_4. This leads to many possible collision histories, which we write down in (4.160)-(4.165) below. After that, we use the same techniques as in the previous sections.

We first rewrite the potential

$$(V-E) = \frac{1}{L^2} \sum_{\substack{l_3=1 \\ (l_3\neq k_3)}}^{\infty} \sum_{k_3=1}^{\infty} \hat{v}_{k_3l_3} e^{i(p_{k_3}-p_{l_3})y} a^*(p_{l_3}) a(p_{k_3}) \qquad (4.158)$$

in terms of fermionic creation and annihilation operators, cf. (4.6). This can be used to decompose the wave function $\Psi_{\mathrm{F}}(t)$ (for $M \ge 1$) in terms of

$$\begin{aligned}\Psi_{\mathrm{F}}(t) = \sum_{n,m=0}^{M} \Big(\Psi^{s,nm}_{\mathrm{F},1}(t) + \Psi^{s,nm}_{\mathrm{F},2}(t) + \Psi^{s,nm}_{\mathrm{F},3}(t)\Big) \\ + \int_0^t d\mu_3(\tau) D(\tau_3)\Big(\Psi^{\ell}_{\mathrm{F},1}(\tau) + \Psi^{\ell}_{\mathrm{F},2}(\tau) + \Psi^{\ell}_{\mathrm{F},3}(\tau)\Big), \qquad (4.159)\end{aligned}$$

where we introduce (recall the definition (4.134) for $\hat{w}^{\ell,\varepsilon}$)

$$\begin{aligned}\Psi^{s,nm}_{\mathrm{F},1}(t) = \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \sum_{k_3=1}^{N} \sum_{l_3=N+1}^{\infty} \hat{v}_{k_3l_3} \hat{v}^{s,\varepsilon}_{k_2l_2} \hat{v}^{s,\varepsilon}_{k_1l_1} \times \\ \times \int_0^t d\mu_3(\tau) D(\tau_3) \Big(g_{k_3l_3}(\tau_3) g_{k_2l_2}(\tau_2) g_{k_3l_3}(\tau_1)\varphi_0\Big) \otimes \Omega_0^{[l_3^*k_3l_2^*k_2l_1^*k_1]}, \qquad (4.160)\end{aligned}$$

$$\begin{aligned}\Psi^{s,nm}_{\mathrm{F},2}(t) = \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \sum_{l_3=N+1}^{\infty} \hat{v}_{l_2l_3} \hat{v}^{s,\varepsilon}_{k_2l_2} \hat{v}^{s,\varepsilon}_{k_1l_1} \times \\ \times \int_0^t d\mu_3(\tau) D(\tau_3) \Big(g_{l_2l_3}(\tau_3) g_{k_2l_2}(\tau_2) g_{k_1l_1}(\tau_1)\varphi_0\Big) \otimes \Omega_0^{[l_3^*k_2l_1^*k_1]} \\ + \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \sum_{l_3=N+1}^{\infty} \hat{v}_{l_1l_3} \hat{v}^{s,\varepsilon}_{k_2l_2} \hat{v}^{s,\varepsilon}_{k_1l_1} \times \\ \times \int_0^t d\mu_3(\tau) D(\tau_3) \Big(g_{l_1l_3}(\tau_3) g_{k_2l_2}(\tau_2) g_{k_1l_1}(\tau_1)\varphi_0\Big) \otimes \Omega_0^{[l_3^*l_2^*k_2k_1]} \\ + \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \sum_{k_3=1}^{N} \hat{v}_{k_3k_2} \hat{v}^{s,\varepsilon}_{k_2l_2} \hat{v}^{s,\varepsilon}_{k_1l_1} \times\end{aligned}$$

$$\times \int_0^t d\mu_3(\tau) D(\tau_3) \Big(g_{k_3 k_2}(\tau_3) g_{k_2 l_2}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[k_3 l_2^* l_1^* k_1]}$$

$$+ \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \sum_{k_3=1}^{N} \hat{v}_{k_3 k_1} \hat{v}^{s,\varepsilon}_{k_2 l_2} \hat{v}^{s,\varepsilon}_{k_1 l_1} \times$$

$$\times \int_0^t ds \mu_3(\tau) D(\tau_3) \Big(g_{k_3 k_1}(\tau_3) g_{k_2 l_2}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[k_3 l_2^* k_2 l_1^*]}, \quad (4.161)$$

$$\Psi^{s,nm}_{\mathrm{F},3}(t) = \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \hat{v}_{l_2 k_2} \hat{v}^{s,\varepsilon}_{k_2 l_2} \hat{v}^{s,\varepsilon}_{k_1 l_1} \times$$

$$\times \int_0^t d\mu_3(\tau) D(\tau_3) \Big(g_{l_2 k_2}(\tau_3) g_{k_2 l_2}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[l_1^* k_1]}$$

$$+ \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \hat{v}_{l_1 k_1} \hat{v}^{s,\varepsilon}_{k_2 l_2} \hat{v}^{s,\varepsilon}_{k_1 l_1} \times$$

$$\times \int_0^t d\mu_3(\tau) D(\tau_3) \Big(g_{l_1 k_1}(\tau_3) g_{k_2 l_2}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[l_2^* k_2]}$$

$$+ \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \hat{v}_{l_2 k_1} \hat{v}^{s,\varepsilon}_{k_2 l_2} \hat{v}^{s,\varepsilon}_{k_1 l_1} \times$$

$$\times \int_0^t d\mu_3(\tau) D(\tau_3) \Big(g_{l_2 k_1}(\tau_3) g_{k_2 l_2}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[k_2 l_1^*]}$$

$$+ \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \hat{v}_{l_1 k_2} \hat{v}^{s,\varepsilon}_{k_2 l_2} \hat{v}^{s,\varepsilon}_{k_1 l_1} \times$$

$$\times \int_0^t d\mu_3(s) D(\tau_3) \Big(g_{l_1 k_2}(\tau_3) g_{k_2 l_2}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[l_2^* k_1]}, \quad (4.162)$$

and moreover,

$$\Psi^{\ell}_{\mathrm{F},1}(\tau) = \frac{1}{L^6} \sum_{k_1,k_2,k_3=1}^{N} \sum_{l_1,l_2,l_3=N+1} \hat{v}_{k_3 l_3} \hat{w}^{\ell,\varepsilon}_{k_2 l_2 k_1 l_1} \Big(g_{k_3 l_3}(\tau_3) g_{k_2 l_2}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[l_3^* k_3 l_2^* k_2 l_1^* k_1]}, \quad (4.163)$$

$$\Psi^{\ell}_{\mathrm{F},2}(\tau) = \frac{1}{L^6} \sum_{k_1,k_2=1}^{N} \sum_{l_1,l_2=N+1}^{\infty} \sum_{l_3=N+1}^{\infty} \hat{v}_{l_2 l_3} \hat{w}^{\ell,\varepsilon}_{k_2 l_2 k_1 l_1} \Big(g_{l_2 l_3}(\tau_3) g_{k_2 l_2}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[l_3^* k_2 l_1^* k_1]}$$

$$+ \frac{1}{L^6} \sum_{k_1,k_2=1}^{N} \sum_{l_1,l_2=N+1}^{\infty} \sum_{l_3=N+1}^{\infty} \hat{v}_{l_1 l_3} \hat{w}^{\ell,\varepsilon}_{k_2 l_2 k_1 l_1} \Big(g_{l_1 l_3}(\tau_3) g_{k_2 l_2}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[l_3^* l_2^* k_2 k_1]}$$

$$+ \frac{1}{L^6} \sum_{k_1,k_2=1}^{N} \sum_{l_1,l_2=N+1}^{\infty} \sum_{k_3=1}^{N} \hat{v}_{k_3 k_2} \hat{w}^{\ell,\varepsilon}_{k_2 l_2 k_1 l_1} \Big(g_{k_3 k_2}(\tau_3) g_{k_2 l_2}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[k_3 l_2^* l_1^* k_1]}$$

$$+ \frac{1}{L^6} \sum_{k_1,k_2=1}^{N} \sum_{l_1,l_2=N+1}^{\infty} \sum_{k_3=1}^{N} \hat{v}_{k_3 k_1} \hat{w}^{\ell,\varepsilon}_{k_2 l_2 k_1 l_1} \Big(g_{k_3 k_1}(\tau_3) g_{k_2 l_2}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[k_3 l_2^* k_2 l_1^*]}, \quad (4.164)$$

$$\Psi^{\ell}_{\mathrm{F},3}(\tau) = \frac{1}{L^6} \sum_{k_1,k_2=1}^{N} \sum_{l_1,l_2=N+1}^{\infty} \hat{v}_{l_2 k_2} \hat{w}^{\ell,\varepsilon}_{k_2 l_2 k_1 l_1} \Big(g_{l_2 k_2}(\tau_3) g_{k_2 l_2}(\tau_2) g_{k_1 l_1}(\tau_1) \varphi_0 \Big) \otimes \Omega_0^{[l_1^* k_1]}$$

$$+\frac{1}{L^6}\sum_{k_1,k_2=1}^{N}\sum_{l_1,l_2=N+1}^{\infty}\hat{v}_{l_1k_1}\hat{w}^{\ell,\varepsilon}_{k_2l_2k_1l_1}\Big(g_{l_1k_1}(\tau_3)g_{k_2l_2}(\tau_2)g_{k_1l_1}(\tau_1)\varphi_0\Big)\otimes\Omega^{[l_2^*k_2]}$$
$$+\frac{1}{L^6}\sum_{k_1,k_2=1}^{N}\sum_{l_1,l_2=N+1}^{\infty}\hat{v}_{l_2k_1}\hat{w}^{\ell,\varepsilon}_{k_2l_2k_1l_1}\Big(g_{l_2k_1}(\tau_3)g_{k_2l_2}(\tau_2)g_{k_1l_1}(\tau_1)\varphi_0\Big)\otimes\Omega_0^{[k_2l_1^*]}$$
$$+\frac{1}{L^6}\sum_{k_1,k_2=1}^{N}\sum_{l_1,l_2=N+1}^{\infty}\hat{v}_{l_1k_2}\hat{w}^{\ell,\varepsilon}_{k_2l_2k_1l_1}\Big(g_{l_1k_2}(\tau_3)g_{k_2l_2}(\tau_2)g_{k_1l_1}(\tau_1)\varphi_0\Big)\otimes\Omega_0^{[l_2^*k_1]}. \quad (4.165)$$

The different contributions in $\Psi_{\mathrm{F}}(t)$ correspond to the different collision histories in $(V-E)\Psi_4$.

Bounds for $\|\Psi^{s,nm}_{\mathrm{F},1}(t)\|_{\mathrm{TD}}$ and $\|\Psi^{\ell}_{\mathrm{F},1}(\tau)\|_{\mathrm{TD}}$. We use that for $k_i,m_i\le N$, $N+1\le l_i,n_i$ $(i=1,2,3)$, the scalar product

$$\langle\Omega_0^{[l_3^*k_3l_2^*k_2l_1^*k_1]},\Omega_0^{[n_3^*m_3n_2^*m_2n_1^*m_1]}\rangle=\Big(\sum_{\sigma\in S_3}\delta_{l_3n_{\sigma(3)}}\delta_{l_2n_{\sigma(2)}}\delta_{l_1n_{\sigma(1)}}\Big)\delta^{\perp}_{l_3l_2}\delta^{\perp}_{l_2l_1}\delta^{\perp}_{l_1l_3}\delta^{\perp}_{n_3n_2}\delta^{\perp}_{n_2n_1}\delta^{\perp}_{n_1n_3}\times$$
$$\times\Big(\sum_{\sigma\in S_3}\delta_{k_3m_{\sigma(3)}}\delta_{k_2m_{\sigma(2)}}\delta_{k_1m_{\sigma(1)}}\Big)\delta^{\perp}_{k_3k_2}\delta^{\perp}_{k_2k_1}\delta^{\perp}_{k_1k_3}\delta^{\perp}_{m_3m_2}\delta^{\perp}_{m_2m_1}\delta^{\perp}_{m_1m_3},$$

produces six Kronecker deltas in each summand, in order to find

$$\|\Psi^{\ell}_{\mathrm{F},1}(\tau)\|^2\lesssim\frac{1}{L^8}\sum_{k_1,k_2=1}^{N}\sum_{l_1,l_2=N+1}^{\infty}|\hat{w}^{\ell,\varepsilon}_{k_2l_2k_1l_1}|\frac{1}{L^4}\sum_{k_3=1}^{N}\sum_{l_3=N+1}^{\infty}|\hat{v}_{k_3l_3}|. \quad (4.166)$$

Then, by means of (4.57) and (4.58), $\|\Psi^{\ell}_{\mathrm{F},1}(\tau)\|_{\mathrm{TD}}\le C_\varepsilon\varrho^{\frac14-\frac{1}{2\varepsilon}+\varepsilon}$, which leads to

$$\Big\|\int_0^t d\mu_3(\tau)D(\tau_3)\Psi^{\ell}_{\mathrm{F},1}(\tau)\Big\|_{\mathrm{TD}}\le C_\varepsilon t^3\varrho^{\frac12-\frac{1}{2\varepsilon}}. \quad (4.167)$$

Similarly in $\Psi^{s,00}_{\mathrm{F},1}(t)$, we estimate the norm,

$$\Big\|\frac{1}{L^6}\sum_{(k_1,l_1)\in\mathfrak{S}_0}\sum_{(k_2,l_2)\in\mathfrak{S}_0}\sum_{k_3=1}^{N}\sum_{l_3=N+1}^{\infty}\hat{v}_{k_3l_3}\hat{v}^{s,\varepsilon}_{k_2l_2}\hat{v}^{s,\varepsilon}_{k_1l_1}\times$$
$$\times\Big(g_{k_3l_3}(\tau_3)g_{k_2l_2}(\tau_2)g_{k_1l_1}(\tau_1)\varphi_0\Big)\otimes\Omega_0^{[l_3^*k_3l_2^*k_2l_1^*k_1]}\Big\|^2$$
$$\lesssim\mathcal{V}_0(N,\rho)^2\frac{1}{L^4}\sum_{k_3=1}^{N}\sum_{l_3=N+1}^{\infty}|\hat{v}_{k_3l_3}|. \quad (4.168)$$

Using (4.57) in combination with (4.63), leads to

$$\|\Psi^{s,00}_{\mathrm{F},1}(t)\|_{\mathrm{TD}}\lesssim t^3\varrho^{-\frac14+\varepsilon}. \quad (4.169)$$

Lemma 4.12. *Let $\Psi^{s,nm}_{F,1}(t)$ be defined as in (4.160). Under the same assumptions as in Theorem 4.1,*

$$\|\Psi^{s,0n}_{\mathrm{F},1}(t)\|_{\mathrm{TD}}+\|\Psi^{s,n0}_{\mathrm{F},1}(t)\|_{\mathrm{TD}}\lesssim(1+t)^3\varrho^{-\frac14+3\varepsilon}\sqrt{\varrho^{\frac{1}{cM}}-\varrho^{\frac{1}{2cM}}},\quad 1\le n\le M, \quad (4.170)$$

$$\|\Psi^{s,nm}_{\mathrm{F},1}(t)\|_{\mathrm{TD}} \lesssim (1+t)^3 \varrho^{-\frac{1}{4}+5\varepsilon}\Big(\varrho^{\frac{1}{cM}} - \varrho^{\frac{1}{2cM}}\Big), \qquad 1 \le n,m \le M. \tag{4.171}$$

holds for all $t > 0$.

Next, we set again $M = \lfloor \ln \varrho \rfloor$ and find, using $M\varrho^{\frac{1}{2cM}} \lesssim \varrho^{\varepsilon}$ as well as $M^2\varrho^{\frac{1}{cM}} \lesssim \varrho^{\varepsilon}$,

$$\sum_{n=1}^{M}\Big(\|\Psi^{s,0n}_{\mathrm{F},1}(t)\|_{\mathrm{TD}} + \|\Psi^{s,n0}_{\mathrm{F},1}(t)\|_{\mathrm{TD}}\Big) \lesssim (1+t)^3\varrho^{-\frac{1}{4}+4\varepsilon}, \tag{4.172}$$

$$\sum_{n,m=1}^{M} \|\Psi^{s,nm}_{\mathrm{F},1}(t)\|_{\mathrm{TD}} \lesssim (1+t)^3\varrho^{-\frac{1}{4}+6\varepsilon}. \tag{4.173}$$

Proof of Lemma 4.12. We define for $1 \le n \le M$,

$$\begin{aligned}\Psi^{s,n0}_{\mathrm{F},11}(t,\tau) = \frac{1}{L^6} &\sum_{(k_1,l_1)\in\mathfrak{S}_n}\ \sum_{(k_2,l_2)\in\mathfrak{S}_0}\ \sum_{k_3=1}^{N}\ \sum_{l_3=N+1}^{\infty} \hat{v}_{k_3l_3}\hat{v}^{s,\varepsilon}_{k_2l_2}v^{s,\varepsilon}_{k_1l_1}\times \\ &\times\Big(g_{k_3l_3}(\tau_2)g_{k_2l_2}(\tau_1)\frac{g_{k_1l_1}(t)-g_{k_1l_1}(\tau_1)}{i(E_{l_1}-E_{k_1})}\varphi_0\Big)\otimes\Omega_0^{[l_3^*k_3l_2^*k_2l_1^*k_1]},\end{aligned} \tag{4.174}$$

$$\begin{aligned}\Psi^{s,n0}_{\mathrm{F},12}(\tau) = \frac{1}{L^6} &\sum_{(k_1,l_1)\in\mathfrak{S}_n}\ \sum_{(k_2,l_2)\in\mathfrak{S}_0}\ \sum_{k_3=1}^{N}\ \sum_{l_3=N+1}^{\infty} \hat{v}_{k_3l_3}\hat{v}^{s,\varepsilon}_{k_2l_2}\hat{v}^{s,\varepsilon}_{k_1l_1}\times \\ &\times\Big(g_{k_3l_3}(\tau_3)g_{k_2l_2}(\tau_2)\frac{e^{i(E_{l_1}-E_{k_1})\tau_1}\partial_{\tau_1}k_{k_1l_1}(\tau_1)}{i(E_{l_1}-E_{k_1})}\varphi_0\Big)\otimes\Omega_0^{[l_3^*k_3l_2^*k_2l_1^*k_1]},\end{aligned} \tag{4.175}$$

$$\begin{aligned}\Psi^{s,0n}_{\mathrm{F},11}(\tau) = \frac{1}{L^6} &\sum_{(k_1,l_1)\in\mathfrak{S}_0}\ \sum_{(k_2,l_2)\in\mathfrak{S}_n}\ \sum_{k_3=1}^{N}\ \sum_{l_3=N+1}^{\infty} \hat{v}_{k_3l_3}\hat{v}^{s,\varepsilon}_{k_2l_2}v^{s,\varepsilon}_{k_1l_1}\times \\ &\times\Big(g_{k_3l_3}(\tau_2)\frac{g_{k_2l_2}(\tau_1)-g_{k_2l_2}(\tau_2)}{i(E_{l_2}-E_{k_2})}g_{k_1l_1}(\tau_1)\varphi_0\Big)\otimes\Omega_0^{[l_3^*k_3l_2^*k_2l_1^*k_1]},\end{aligned} \tag{4.176}$$

$$\begin{aligned}\Psi^{s,0n}_{\mathrm{F},12}(\tau) = \frac{1}{L^6} &\sum_{(k_1,l_1)\in\mathfrak{S}_0}\ \sum_{(k_2,l_2)\in\mathfrak{S}_n}\ \sum_{k_3=1}^{N}\ \sum_{l_3=N+1}^{\infty} \hat{v}_{k_3l_3}\hat{v}^{s,\varepsilon}_{k_2l_2}v^{s,\varepsilon}_{k_1l_1}\times \\ &\times\Big(g_{k_3l_3}(\tau_3)\frac{e^{i(E_{l_2}-E_{k_2})\tau_2}\partial_{\tau_2}k_{k_2l_2}(\tau_2)}{i(E_{l_2}-E_{k_2})}g_{k_1l_1}(\tau_1)\varphi_0\Big)\otimes\Omega_0^{[l_3^*k_3l_2^*k_2l_1^*k_1]}.\end{aligned} \tag{4.177}$$

Via partial integration we obtain

$$\Psi^{s,n0}_{\mathrm{F},1}(t) = \int_0^t d\mu_2(\tau)D(\tau_2)\Psi^{s,n0}_{\mathrm{F},11}(t,\tau) - \int_0^t d\mu_3(\tau)D(\tau_3)\Psi^{s,n0}_{\mathrm{F},12}(\tau), \tag{4.178}$$

$$\Psi^{s,0n}_{\mathrm{F},1}(t) = \int_0^t d\mu_2(\tau)D(\tau_2)\Psi^{s,0n}_{\mathrm{F},11}(\tau) - \int_0^t d\mu_3(\tau)D(\tau_3)\Psi^{s,0n}_{\mathrm{F},12}(\tau). \tag{4.179}$$

We further set for $1 \le n,m \le M$, and for $G^{(\mathrm{X})}_{k_2l_2k_1l_1}(t,\tau)$, $X \in \{1,2,3\}$, defined as in (4.148)-(4.150),

$$\Psi^{s,nm}_{\mathrm{F},1\mathrm{X}}(t,\tau) = \frac{1}{L^6}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\ \sum_{(k_2,l_2)\in\mathfrak{S}_m}\ \sum_{k_3=1}^{N}\ \sum_{l_3=N+1}^{\infty}\hat{v}_{k_3l_3}\hat{v}^{s,\varepsilon}_{k_2l_2}\hat{v}^{s,\varepsilon}_{k_1l_1}\Big(g_{k_3l_3}(\tau_{\mathrm{X}})G^{(\mathrm{X})}_{k_2l_2k_1l_1}(t,\tau)\varphi_0\Big) \otimes\Omega_0^{[l_3^*k_3l_2^*k_2l_1^*k_1]}.$$

Partial integration leads again to

$$\Psi^{s,nm}_{\mathrm{F},1}(t) = \int_0^t d\tau_1 D(\tau_1)\Psi^{s,nm}_{\mathrm{F},11}(t,\tau) - \int_0^t d\mu_2(\tau)D(\tau_2)\Psi^{s,nm}_{\mathrm{F},12}(t,\tau) + \int_0^t d\mu_3(\tau)D(\tau_3)\Psi^{s,nm}_{\mathrm{F},13}(t,\tau).$$

Using (4.56) in combination with (4.65), we find for $Y \in \{1,2\}$,

$$\begin{aligned}
\|\Psi^{s,n0}_{\mathrm{F},1\mathrm{Y}}(\tau)\|^2 \lesssim{}& \frac{1}{L^{12}} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \Big(\frac{1+\|\partial_{\tau_1} k_{k_1l_1}(\tau_1)\varphi_0\|}{E_{l_1}-E_{k_1}}\Big) \sum_{(k_2,l_2)\in\mathfrak{S}_0} \sum_{k_3=1}^{N} \sum_{l_3=N+1}^{\infty} |\hat{v}_{k_3l_3}| \times \\
&\times \sum_{(m_1,n_1)\in\mathfrak{S}_n} \Big(\frac{1+\|\partial_{\tau_1} k_{m_1n_1}(\tau_1)\varphi_0\|}{E_{n_1}-E_{m_1}}\Big) \sum_{(m_2,n_2)\in\mathfrak{S}_0} \sum_{m_3=1}^{N} \sum_{n_3=N+1}^{\infty} |\hat{v}_{m_3n_3}| \langle \Omega_0^{[l_3^*k_3l_2^*k_2l_1^*k_1]}, \Omega_0^{[n_3^*m_3n_2^*m_2n_1^*m_1]}\rangle, \\
\lesssim{}& \varrho^{4\varepsilon}\Big(\varrho^{-(\frac{n-1}{cM})}\mathcal{V}_n(N,\rho)\Big)\mathcal{V}_0(N,\rho)\frac{1}{L^4}\sum_{k_3=1}^{N}\sum_{l_3=N+1}^{\infty}|\hat{v}_{k_3l_3}|. \qquad (4.180)
\end{aligned}$$

In complete analogy one finds the same bound for $\Psi^{s,0n}_{\mathrm{F},1\mathrm{Y}}(\tau)$, $Y\in\{1,2\}$. Next, using (4.56) together with (4.154)-(4.156), we find for $X \in \{1,2,3\}$,

$$\begin{aligned}
\|\Psi^{s,nm}_{\mathrm{F},1\mathrm{X}}(\tau)\|^2 \lesssim{}& \frac{1}{L^{12}} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \sum_{k_3=1}^{N} \sum_{l_3=N+1}^{\infty} |\hat{v}_{k_3l_3}| \|G^{(\mathrm{X})}_{k_2l_2k_1l_1}(t,\tau)\varphi_0\| \times \\
&\times \sum_{(m_1,n_1)\in\mathfrak{S}_n} \sum_{(m_2,n_2)\in\mathfrak{S}_m} \sum_{m_3=1}^{N} \sum_{n_3=N+1}^{\infty} |\hat{v}_{m_3n_3}| \|G^{(\mathrm{X})}_{m_2n_2m_1n_1}(t,\tau)\varphi_0\| \, \langle \Omega_0^{[l_3^*k_3l_2^*k_2l_1^*k_1]}, \Omega_0^{[n_3^*m_3n_2^*m_2n_1^*m_1]}\rangle \\
\lesssim{}& \varrho^{8\varepsilon}\Big(\varrho^{-(\frac{n-1}{cM})}\mathcal{V}_n(N,\rho)\Big)\Big(\varrho^{-(\frac{m-1}{cM})}\mathcal{V}_m(N,\rho)\Big)\frac{1}{L^4}\sum_{k_3=1}^{N}\sum_{l_3=N+1}^{\infty}|\hat{v}_{k_3l_3}|^2. \qquad (4.181)
\end{aligned}$$

The stated estimates then follow from (4.57) and Corollary 4.4. □

Bounds for $\|\Psi^{s,nm}_{\mathrm{F},2}(t)\|_{\mathrm{TD}}$ and $\|\Psi^{\ell}_{\mathrm{F},2}(\tau)\|_{\mathrm{TD}}$. In $\Psi^{\ell}_{\mathrm{F},2}(t)$, and similarly in $\Psi^{s,nm}_{\mathrm{F},2}(t)$, we denote the four lines separately by $\Psi^{\ell}_{\mathrm{F},2\mathrm{i}}(t)$, $i=1,2,3,4$. We derive the estimates only for the first line, whereas for $i=2,3,4$ everything works in exact analogy to the case $i=1$. Using the Kronecker deltas in ($k_2,k_1,m_2,m_1 \le N$ and $l_3,l_1,n_3,n_1 \ge N+1$)

$$\begin{aligned}
\langle \Omega_0^{[l_3^*k_2l_1^*k_1]}, \Omega_0^{[n_3^*m_2n_1^*m_1]}\rangle ={}&(\delta_{l_3n_3}\delta_{l_1n_1}+\delta_{l_3n_1}\delta_{l_1n_3})\delta^{\perp}_{l_3l_1}\delta^{\perp}_{n_3n_1}\times \\
&\times(\delta_{k_2m_2}\delta_{k_1m_1}+\delta_{k_2m_1}\delta_{k_1m_2})\delta^{\perp}_{k_1k_2}\delta^{\perp}_{m_1m_2}, \qquad (4.182)
\end{aligned}$$

one finds

$$\|\Psi^{\ell}_{\mathrm{F},21}(\tau)\|^2 \lesssim \frac{1}{L^4}\sum_{k_1,k_2=1}^{N}\sum_{l_1,l_2=N+1}^{\infty}|\hat{w}^{\ell,\varepsilon}_{k_2l_2k_1l_1}|\frac{1}{L^2}\sum_{l_3=N+1}^{\infty}|\hat{v}_{l_2l_3}|\frac{1}{L^2}\sum_{n_2=N+1}^{\infty}(|\hat{v}_{n_2l_3}|+|\hat{v}_{n_2l_1}|). \qquad (4.183)$$

By means of (4.58) and also (4.61), we obtain $\|\Psi^{\ell}_{\mathrm{F},21}(\tau)\|_{\mathrm{TD}} \le C_\varepsilon t^3 \varrho^{\frac{1}{4}+2\varepsilon-\frac{1}{2\varepsilon}}$, and hence,

$$\Big\|\int_0^t d\mu_3(\tau)D(\tau_3)\Psi^{\ell}_{\mathrm{F},21}(\tau)\Big\|_{\mathrm{TD}} \le C_\varepsilon t^3 \varrho^{\frac{1}{4}+2\varepsilon-\frac{1}{2\varepsilon}}. \qquad (4.184)$$

Similarly, we estimate in $\Psi^{s,00}_{\mathrm{F},21}(t)$ the norm

$$\Big\| \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_0} \sum_{(k_2,l_2)\in\mathfrak{S}_0} \sum_{l_3=N+1}^{\infty} \hat{v}_{l_2l_3}\hat{v}^{s,\varepsilon}_{k_2l_2}\hat{v}^{s,\varepsilon}_{k_1l_1}\Big(g_{l_2l_3}(\tau_3)g_{k_2l_2}(\tau_2)g_{k_1l_1}(\tau_1)\varphi_0\Big)\otimes\Omega_0^{[l_3^*k_2l_1^*k_1]}\Big\|^2$$
$$\lesssim \frac{1}{L^4}\sum_{(k_1,l_1)\in\mathfrak{S}_0}\frac{1}{L^4}\sum_{(k_2,l_2)\in\mathfrak{S}_0}\frac{1}{L^2}\sum_{l_3=N+1}^{\infty}|\hat{v}_{l_2l_3}|\frac{1}{L^2}\sum_{n_2=N+1}^{\infty}\Big(|\hat{v}^{s,\varepsilon}_{k_1n_2}|+|\hat{v}^{s,\varepsilon}_{k_2n_2}|\Big). \tag{4.185}$$

Using (4.61) in combination with (4.63), this leads to

$$\|\Psi^{s,00}_{\mathrm{F},21}(t)\|_{\mathrm{TD}} \lesssim t^3\varrho^{-\frac{1}{2}+\varepsilon}\Big(\varrho^{2\varepsilon}+C_\varepsilon\varrho^{-1/\varepsilon}\Big). \tag{4.186}$$

Lemma 4.13. *Let $\Psi^{s,nm}_{F,2}(t)$ be defined as in (4.161). Then, under the same assumptions as in Theorem 4.1, there exists a positive constant C_ε such that*

$$\|\Psi^{s,0n}_{\mathrm{F},2}(t)\|_{\mathrm{TD}}+\|\Psi^{s,n0}_{\mathrm{F},2}(t)\|_{\mathrm{TD}} \lesssim (1+t)^3\varrho^{-\frac{1}{2}+3\varepsilon+\frac{1}{2cM}}\Big(\varrho^{2\varepsilon}+C_\varepsilon\varrho^{-1/\varepsilon}\Big), \tag{4.187}$$
$$\|\Psi^{s,nm}_{\mathrm{F},2}(t)\|_{\mathrm{TD}} \lesssim (1+t)^3\varrho^{-\frac{1}{2}+5\varepsilon+\frac{1}{cM}}\Big(\varrho^{2\varepsilon}+C_\varepsilon\varrho^{-1/\varepsilon}\Big), \tag{4.188}$$

holds for all $1\le n,m\le M$ and $t\ge 0$.

For $M=\lfloor\ln\varrho\rfloor$, one obtains similar as before

$$\sum_{n=1}^{M}\Big(\|\Psi^{s,n0}_{\mathrm{F},2}(t)\|_{\mathrm{TD}}+\|\Psi^{s,0n}_{\mathrm{F},2}(t)\|_{\mathrm{TD}}\Big) \lesssim (1+t)^3\varrho^{-\frac{1}{2}+4\varepsilon}\Big(\varrho^{2\varepsilon}+C_\varepsilon\varrho^{-1/\varepsilon}\Big), \tag{4.189}$$
$$\sum_{n,m=1}^{M}\|\Psi^{s,nm}_{\mathrm{F},2}(t)\|_{\mathrm{TD}} \lesssim (1+t)^3\varrho^{-\frac{1}{2}+6\varepsilon}\Big(\varrho^{2\varepsilon}+C_\varepsilon\varrho^{-1/\varepsilon}\Big). \tag{4.190}$$

Proof of Lemma 4.13. We denote the four lines in $\Psi^{s,nm}_{\mathrm{F},2}(t)$ respectively by $\Psi^{s,nm}_{\mathrm{F},2i}(t)$, $i=1,...,4$. We prove the Lemma for the first line. The same estimates are readily verified for the other three lines as well. Let us define

$$\Psi^{s,n0}_{\mathrm{F},211}(t,\tau)=\frac{1}{L^6}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\sum_{(k_2,l_2)\in\mathfrak{S}_0}\sum_{l_3=N+1}^{\infty}\hat{v}_{l_2l_3}\hat{v}^{s,\varepsilon}_{k_2l_2}\hat{v}^{s,\varepsilon}_{k_1l_1}\times$$
$$\times\Big(g_{l_2l_3}(\tau_2)g_{k_2l_2}(\tau_1)\frac{g_{k_1l_1}(t)-g_{k_1l_1}(\tau_1)}{i(E_{l_1}-E_{k_1})}\varphi_0\Big)\otimes\Omega_0^{[l_3^*k_2l_1^*k_1]}, \tag{4.191}$$

$$\Psi^{s,n0}_{\mathrm{F},212}(\tau)=\frac{1}{L^6}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\sum_{(k_2,l_2)\in\mathfrak{S}_0}\sum_{l_3=N+1}^{\infty}\hat{v}_{l_2l_3}\hat{v}^{s,\varepsilon}_{k_2l_2}\hat{v}^{s,\varepsilon}_{k_1l_1}\times$$
$$\times\Big(g_{l_2l_3}(\tau_3)g_{k_2l_2}(\tau_2)\frac{e^{i(E_{l_1}-E_{k_1})\tau_1}\partial_{\tau_1}k_{k_1l_1}(\tau_1)}{i(E_{l_1}-E_{k_1})}\varphi_0\Big)\otimes\Omega_0^{[l_3^*k_2l_1^*k_1]}, \tag{4.192}$$

$$\Psi^{s,0n}_{\mathrm{F},211}(\tau)=\frac{1}{L^6}\sum_{(k_1,l_1)\in\mathfrak{S}_0}\sum_{(k_2,l_2)\in\mathfrak{S}_n}\sum_{l_3=N+1}^{\infty}\hat{v}_{l_2l_3}\hat{v}^{s,\varepsilon}_{k_2l_2}v^{s,\varepsilon}_{k_1l_1}\times$$
$$\times\Big(g_{l_2l_3}(\tau_2)\frac{g_{k_2l_2}(\tau_1)-g_{k_2l_2}(\tau_2)}{i(E_{l_2}-E_{k_2})}g_{k_1l_1}(\tau_1)\varphi_0\Big)\otimes\Omega_0^{[l_3^*k_2l_1^*k_1]}, \tag{4.193}$$

$$\Psi^{s,0n}_{\mathrm{F},212}(\tau) = \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_0} \sum_{(k_2,l_2)\in\mathfrak{S}_n} \sum_{l_3=N+1}^{\infty} \hat{v}_{l_2 l_3} \hat{v}^{s,\varepsilon}_{k_2 l_2} v^{s,\varepsilon}_{k_1 l_1} \times \\ \times \Big(g_{l_2 l_3}(\tau_3) \frac{e^{i(E_{l_2}-E_{k_2})\tau_2} \partial_{\tau_2} k_{k_2 l_2}(\tau_2)}{i(E_{l_2}-E_{k_2})} g_{k_1 l_1}(\tau_1)\varphi_0 \Big) \otimes \Omega_0^{[l_3^* k_2 l_1^* k_1]}. \tag{4.194}$$

By partial integration,

$$\Psi^{s,n0}_{\mathrm{F},21}(t) = \int_0^t d\mu_2(\tau) D(\tau_2) \Psi^{s,n0}_{\mathrm{F},211}(t,\tau) - \int_0^t d\mu_3(\tau) D(\tau_3) \Psi^{s,n0}_{\mathrm{F},212}(\tau), \tag{4.195}$$

$$\Psi^{s,0n}_{\mathrm{F},21}(t) = \int_0^t d\mu_2(\tau) D(\tau_2) \Psi^{s,0n}_{\mathrm{F},211}(\tau) - \int_0^t d\mu_3(\tau) D(\tau_3) \Psi^{s,0n}_{\mathrm{F},212}(\tau). \tag{4.196}$$

We set further, with $G^{(\mathrm{X})}_{k_2 l_2 k_1 l_1}(t,\tau)$, $X \in \{1,2,3\}$ as in (4.148)-(4.150),

$$\Psi^{s,nm}_{\mathrm{F},21\mathrm{X}}(t,\tau) = \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \sum_{l_3=N+1}^{\infty} \hat{v}_{l_2 l_3} \hat{v}^{s,\varepsilon}_{k_2 l_2} \hat{v}^{s,\varepsilon}_{k_1 l_1} \Big(g_{l_2 l_3}(\tau_{\mathrm{X}}) G^{(\mathrm{X})}_{k_2 l_2 k_1 l_1}(t,\tau) \varphi_0 \Big) \otimes \Omega_0^{[l_3^* k_2 l_1^* k_1]}.$$

By partial integration again,

$$\Psi^{s,nm}_{\mathrm{F},21}(t) = \int_0^t d\tau_1 D(\tau_1) \Psi^{s,nm}_{\mathrm{F},211}(t,\tau) + \int_0^t d\mu_2(\tau) D(\tau_2) \Psi^{s,nm}_{\mathrm{F},212}(t,\tau) + \int_0^t d\mu_3(\tau) D(\tau_3) \Psi^{s,nm}_{\mathrm{F},213}(t,\tau).$$

Next, we compute using (4.65), for $Y \in \{1,2\}$,

$$\|\Psi^{s,n0}_{\mathrm{F},21\mathrm{Y}}(\tau)\|^2 \lesssim \varrho^{4\varepsilon} \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \varrho^{-(\frac{n-1}{cM})} \frac{1}{L^4} \sum_{(k_2,l_2)\in\mathfrak{S}_0} \frac{1}{L^2} \sum_{l_3=N+1}^{\infty} |\hat{v}_{l_3 l_2}| \frac{1}{L^2} \sum_{n_2=N+1}^{\infty} \Big(|\hat{v}_{l_3 n_2}| + |\hat{v}_{l_1 n_2}| \Big).$$

Similarly, one derives the same estimate for $\|\Psi^{s,0n}_{\mathrm{F},21\mathrm{Y}}(\tau)\|$. Furthermore, using (4.154)-(4.156), one finds

$$\|\Psi^{s,nm}_{\mathrm{F},21\mathrm{X}}(\tau)\|^2 \lesssim \varrho^{8\varepsilon} \frac{1}{L^4} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \varrho^{-(\frac{n-1}{cM})} \frac{1}{L^4} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \varrho^{-(\frac{m-1}{cM})} \times \\ \times \frac{1}{L^2} \sum_{l_3=N+1}^{\infty} |\hat{v}_{l_3 l_2}| \frac{1}{L^2} \sum_{n_2=N+1}^{\infty} \Big(|\hat{v}_{l_3 n_2}| + |\hat{v}_{l_1 n_2}| \Big). \tag{4.197}$$

The proof of the lemma then follows from (4.61), (4.63) and (4.64). □

Bounds for $\|\Psi^{s,nm}_{\mathrm{F},3}(t)\|_{\mathrm{TD}}$ and $\|\Psi^{\ell}_{\mathrm{F},3}(\tau)\|_{\mathrm{TD}}$. We denote the four different lines by $\Psi^{s,nm}_{\mathrm{F},3i}(t)$, respectively $\Psi^{\ell}_{\mathrm{F},3i}(\tau)$ and derive the bounds only for $i=1$ since for $i=2,3,4$, the same estimates are derived analogously. Using $\langle \Omega_0^{[l_1^* k_1]}, \Omega_0^{[n_1^* m_1]} \rangle = \delta_{l_1 n_1} \delta_{k_1 m_1}$ for $k_1, m_1 \le N$ and $N+1 \le l_1, n_1$, we find

$$\|\Psi^{\ell}_{\mathrm{F},31}(\tau)\|^2 \lesssim \frac{1}{L^8} \sum_{k_1,k_2=1}^{N} \sum_{l_1,l_2=N+1}^{\infty} |\hat{w}^{\ell,\varepsilon}_{k_2 l_2 k_1 l_1}| \frac{1}{L^4} \sum_{m_2=1}^{N} \sum_{n_2=N+1}^{\infty} |\hat{v}_{n_2 m_2}|. \tag{4.198}$$

By (4.57) and (4.58), it follows that $\|\Psi^{\ell}_{\mathrm{F},31}(\tau)\|_{\mathrm{TD}} \le C_\varepsilon \varrho^{\frac{1}{2}-\frac{1}{2\varepsilon}}$, and thus,

$$\Big\| \int_0^t d\mu_3(\tau) D(\tau_3) \Psi^{\ell}_{\mathrm{F}31}(\tau) \Big\|_{\mathrm{TD}} \le C_\varepsilon t^3 \varrho^{\frac{1}{2}-\frac{1}{2\varepsilon}}. \tag{4.199}$$

Similarly, in $\Psi^{s,00}_{\mathrm{F},31}(t)$, we estimate the norm

$$\Big\| \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_0} \sum_{(k_2,l_2)\in\mathfrak{S}_0} \hat{v}_{l_2k_2}\hat{v}^{s,\varepsilon}_{k_2l_2}\hat{v}^{s,\varepsilon}_{k_1l_1}\Big(g_{l_2k_2}(\tau_3)g_{k_2l_2}(\tau_2)g_{k_1l_1}(\tau_1)\varphi_0\Big)\otimes\Omega_0^{[l_1^*k_1]}\Big\|^2 \lesssim \mathcal{V}_0(N,\rho)^3. \tag{4.200}$$

Hence, by means of (4.63), we find

$$\|\Psi^{s,00}_{\mathrm{F}31}(t)\|_{\mathrm{TD}} \lesssim Ct^3\varrho^{-\frac{3}{4}+\frac{3}{2}\varepsilon}. \tag{4.201}$$

Lemma 4.14. *Let $\Psi^{s,nm}_{F,3}(t)$ be defined as in (4.162). Then, under the same assumptions as in Theorem 4.1, there exists a positive constant C_ε such that*

$$\|\Psi^{s,0n}_{\mathrm{F},3}(t)\|_{\mathrm{TD}} + \|\Psi^{s,n0}_{\mathrm{F},3}(t)\|_{\mathrm{TD}} \lesssim (1+t)^3\varrho^{-\frac{3}{4}+4\varepsilon+\frac{1}{2cM}}\Big(\varrho^{2\varepsilon}+C_\varepsilon\varrho^{-1/\varepsilon}\Big), \tag{4.202}$$

$$\|\Psi^{s,nm}_{\mathrm{F},3}(t)\|_{\mathrm{TD}} \lesssim (1+t)^3\varrho^{-\frac{3}{4}+6\varepsilon+\frac{3}{2cM}}\Big(\varrho^{2\varepsilon}+C_\varepsilon\varrho^{-1/\varepsilon}\Big), \tag{4.203}$$

hold for all $1\le n,m\le M$ and $t\ge 0$.

With $M=\lfloor\ln\varrho\rfloor$, it follows

$$\sum_{n=1}^{M}\Big(\|\Psi^{s,n0}_{\mathrm{F},3}(t)\|_{\mathrm{TD}} + \|\Psi^{s,0n}_{\mathrm{F},3}(t)\|_{\mathrm{TD}}\Big) \lesssim (1+t)^3\varrho^{-\frac{3}{4}+5\varepsilon}\Big(\varrho^{2\varepsilon}+C_\varepsilon\varrho^{-1/\varepsilon}\Big), \tag{4.204}$$

$$\sum_{n,m=1}^{M}\|\Psi^{s,nm}_{\mathrm{F},3}(t)\|_{\mathrm{TD}} \lesssim (1+t)^3\varrho^{-\frac{3}{4}+7\varepsilon}\Big(\varrho^{2\varepsilon}+C_\varepsilon\varrho^{-1/\varepsilon}\Big). \tag{4.205}$$

Proof of Lemma 4.14. Again, we prove the lemma only for the $i=1$ term. Let

$$\Psi^{s,n0}_{\mathrm{F},311}(t,\tau) = \frac{1}{L^6}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\sum_{(k_2,l_2)\in\mathfrak{S}_0}\hat{v}_{l_2k_2}\hat{v}^{s,\varepsilon}_{k_2l_2}\hat{v}^{s,\varepsilon}_{k_1l_1}\times \times\Big(g_{l_2k_2}(\tau_2)g_{k_2l_2}(\tau_1)\frac{g_{k_1l_1}(t)-g_{k_1l_1}(\tau_1)}{i(E_{l_1}-E_{k_1})}\varphi_0\Big)\otimes\Omega_0^{[l_1^*k_1]}, \tag{4.206}$$

$$\Psi^{s,n0}_{\mathrm{F},312}(\tau) = \frac{1}{L^6}\sum_{(k_1,l_1)\in\mathfrak{S}_n}\sum_{(k_2,l_2)\in\mathfrak{S}_0}\hat{v}_{l_2k_2}\hat{v}^{s,\varepsilon}_{k_2l_2}\hat{v}^{s,\varepsilon}_{k_1l_1}\times \times\Big(g_{l_2k_2}(\tau_3)g_{k_2l_2}(\tau_2)\frac{e^{i(E_{l_1}-E_{k_1})\tau_1}\partial_{\tau_1}k_{k_1l_1}(\tau_1)}{i(E_{l_1}-E_{k_1})}\varphi_0\Big)\otimes\Omega_0^{[l_1^*k_1]}, \tag{4.207}$$

$$\Psi^{s,0n}_{\mathrm{F},311}(\tau) = \frac{1}{L^6}\sum_{(k_1,l_1)\in\mathfrak{S}_0}\sum_{(k_2,l_2)\in\mathfrak{S}_n}\hat{v}_{l_2k_2}\hat{v}^{s,\varepsilon}_{k_2l_2}\hat{v}^{s,\varepsilon}_{k_1l_1}\times \times\Big(g_{l_2k_2}(\tau_2)\frac{g_{k_2l_2}(\tau_1)-g_{k_2l_2}(\tau_2)}{i(E_{l_2}-E_{k_2})}g_{k_1l_1}(\tau_1)\varphi_0\Big)\otimes\Omega_0^{[l_1^*k_1]}, \tag{4.208}$$

$$\Psi^{s,0n}_{\mathrm{F},312}(\tau) = \frac{1}{L^6}\sum_{(k_1,l_1)\in\mathfrak{S}_0}\sum_{(k_2,l_2)\in\mathfrak{S}_n}\hat{v}_{l_2k_2}\hat{v}^{s,\varepsilon}_{k_2l_2}\hat{v}^{s,\varepsilon}_{k_1l_1}\times \times\Big(g_{l_2k_2}(\tau_3)\frac{e^{i(E_{l_2}-E_{k_2})\tau_2}\partial_{\tau_2}k_{k_2l_2}(\tau_2)}{i(E_{l_2}-E_{k_2})}g_{k_1l_1}(\tau_1)\Big)\varphi_0\otimes\Omega_0^{[l_1^*k_1]}. \tag{4.209}$$

Via partial integration,

$$\Psi^{s,n0}_{\mathrm{F},31}(t) = \int_0^t d\mu_2(\tau) D(\tau_2) \Psi^{s,n0}_{\mathrm{F},311}(t,\tau) - \int_0^t d\mu_3(\tau) D(\tau_3) \Psi^{s,n0}_{\mathrm{F},312}(\tau), \tag{4.210}$$

$$\Psi^{s,0n}_{\mathrm{F},31}(t) = \int_0^t d\mu_2(\tau) D(\tau_2) \Psi^{s,0n}_{\mathrm{F},311}(\tau) - \int_0^t d\mu_3(\tau) D(\tau_3) \Psi^{s,0n}_{\mathrm{F},312}(\tau). \tag{4.211}$$

We set further, for $G^{(\mathrm{X})}_{k_2 l_2 k_1 l_1}(t,\tau)$, $X \in \{1,2,3\}$ as in (4.148)-(4.150),

$$\Psi^{s,nm}_{\mathrm{F},31\mathrm{X}}(t,\tau) = \frac{1}{L^6} \sum_{(k_1,l_1)\in\mathfrak{S}_n} \sum_{(k_2,l_2)\in\mathfrak{S}_m} \hat{v}_{l_2 k_2} \hat{v}^{s,\varepsilon}_{k_2 l_2} \hat{v}^{s,\varepsilon}_{k_1 l_1} \Big(g_{l_2 k_2}(\tau_{\mathrm{X}}) G^{(\mathrm{X})}_{k_2 l_2 k_1 l_1}(t,\tau) \varphi_0 \Big) \otimes \Omega_0^{[l_1^* k_1]}.$$

Partial integration leads again to

$$\Psi^{s,nm}_{\mathrm{F},31}(t) = \int_0^t d\tau_1 D(\tau_1) \Psi^{s,nm}_{\mathrm{F},311}(t,\tau) - \int_0^t d\mu_2(\tau) D(\tau_2) \Psi^{s,nm}_{\mathrm{F},312}(t,\tau) + \int_0^t d\mu_3(\tau) D(\tau_3) \Psi^{s,nm}_{\mathrm{F},313}(t,\tau).$$

Similar as before, we find for $Y \in \{1,2\}$, using (4.65),

$$\|\Psi^{s,n0}_{\mathrm{F},31\mathrm{Y}}(\tau)\|^2 \lesssim \varrho^{4\varepsilon} \Big(\varrho^{-\left(\frac{n-1}{cM}\right)} \mathcal{V}_n(N,\rho) \Big) \mathcal{V}_0(N,\rho)^2. \tag{4.212}$$

Analogously, one derives the same estimate for $\|\Psi^{s,0n}_{\mathrm{F},31\mathrm{Y}}(\tau)\|$. Furthermore, using (4.154)-(4.156), one finds

$$\|\Psi^{s,nm}_{\mathrm{F},31\mathrm{X}}(t,\tau)\|^2 \lesssim \varrho^{8\varepsilon} \Big(\varrho^{-\left(\frac{n-1}{cM}\right)} \mathcal{V}_n(N,\rho) \Big) \Big(\varrho^{-\left(\frac{m-1}{cM}\right)} \mathcal{V}_m(N,\rho) \Big)^2. \tag{4.213}$$

The stated estimates follow from (4.63) and also (4.64). □

This completes the proof of the bound for $\|\Psi_{\mathrm{F}}(t)\|_{\mathrm{TD}}$ in (4.46).

4.2.5 Proofs of Lemmas 4.3 and 4.5

Proof of Lemma 4.3. Let us first note that the choice $v \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ ensures that the constant D_p in (4.19) is smaller than some $C > 0$ uniformly in the length L of the torus.

We begin with the upper bound in (4.57). Using the Paley-Wiener Theorem, cf. (4.19) with p s.t. $pq > 3$,

$$\begin{aligned}
\lim_{\mathrm{TD}} \frac{1}{L^4} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \big|\hat{v}(p_k - p_l)\big|^q &\le \lim_{\mathrm{TD}} \frac{1}{L^4} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \frac{D_p^q}{(1+|p_k-p_l|)^{qp}} \\
&= \frac{D_p^q}{(2\pi)^4} \int_{|k|\le k_F} d^2k \int_{|l|\ge k_F} d^2l \frac{1}{(1+|k-l|)^{qp}} \\
\Big[|k|+|l| \ge |k+l|\Big] \quad &\le C_q \int_{|k|\le k_F} d^2k \int_{|l|\ge k_F-|k|} d^2l \frac{1}{(1+|l|)^{qp}} \\
&\le C_q \int_{|k|\le k_F} d^2k \left[\frac{-1}{(1+|l|)^{qp-2}} \right]_{k_F-|k|}^{\infty} \\
&\le C_q k_F \left[\frac{1}{(1+k_F-|k|)^{qp-3}} \right]_0^{k_F},
\end{aligned} \tag{4.214}$$

which proves the upper bound.

To show the lower bound in (4.57), we assume for simplicity that $\hat{v}(0) > 0$ (the argument is easily adapted to the general case). Let us denote here $\lim_{\text{TD}} \hat{v} = \hat{v}_{\text{TD}}$ with $v_{\text{TD}} \in \mathcal{C}_0^\infty(\mathbb{R}^2)$. Due to continuity of $\hat{v}_{\text{TD}} : \mathbb{R}^2 \to \mathbb{R}$, there is a nonempty, compact ball of some radius $r > 0$, $\overline{B_r(0)} \subset \mathbb{R}^2$, such that $\hat{v}_{\text{TD}}(k) > 0$ for all $k \in \overline{B_r(0)}$. In particular, for given $l \in \mathbb{R}^2$ with $|l| \in [k_F, k_F + r/10]$, we have $\hat{v}_{\text{TD}}(k-l) > 0$ for all $k \in \overline{B_r(l)}$ with $|k| \leq k_F$. Since the set

$$A = \Big\{(k,l) \in \mathbb{R}^4 \ : \ |l| \in \Big[k_F, k_F + r/10\Big], k \in \overline{B_r(l)}, |k| \leq k_F\Big\} \tag{4.215}$$

is nonempty and compact, there exists a nonzero minimum on A, $m \equiv \min_{(k,l)\in A} \hat{v}_{\text{TD}}(k-l) > 0$. It is then sufficient to consider the transitions corresponding to A in order to obtain the lower bound:

$$\begin{aligned}
\lim_{\text{TD}} \frac{1}{L^4} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \big|\hat{v}(p_k - p_l)\big|^q &= \frac{1}{(2\pi)^4} \int_{|k|\leq k_F} d^2k \int_{|l|\geq k_F} d^2l \ \big|\hat{v}_{\text{TD}}(k-l)\big|^q \\
&\geq \int_{|k|\leq k_F} d^2k \int_{|l|\geq k_F} d^2l \ \big|\hat{v}_{\text{TD}}(k-l)\big|^q \ \chi\big((k,l)\in A\big) \\
&\geq m^q \int_{|k|\leq k_F} d^2k \int_{|l|\geq k_F} d^2l \ \chi\big((k,l)\in A\big) \\
&= m^q \int_{k_F}^{k_F + r/10} d|l| \ |l| \int_{|k|\leq k_F} d^2k \ \chi\big(k \in \overline{B_r(l)}\big) \\
\Big[\text{for sufficiently large } k_F\Big] \qquad &= C_q r^3 k_F.
\end{aligned} \tag{4.216}$$

Remark 4.4. Along the same lines, one verifies (4.15) also for $d = 1$ and $d = 3$.

We next come to (4.58). Let $\varepsilon > 0$. Applying again Paley-Wiener, this time with p s.t. $p/2 - 3 > 0$ and $p > \frac{1}{\varepsilon} + \frac{2}{\varepsilon^2}$, we find

$$\begin{aligned}
\lim_{\text{TD}} \frac{1}{L^4} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \big|\hat{v}^{\ell,\varepsilon}(p_k - p_l)\big| &\leq \lim_{\text{TD}} \frac{1}{L^4} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \frac{D_p \theta\big(|p_k - p_l| - \varrho^\varepsilon\big)}{(1 + |p_k - p_l|)^p} \\
&\leq \frac{D_p}{(2\pi)^4} \int_{|k|\leq k_F} d^2k \int_{|l|\geq k_F} d^2l \frac{\theta(|k-l| - \varrho^\varepsilon)}{(1+|k-l|)^p} \\
&\leq \frac{D_p}{(2\pi)^4 \varrho^{\varepsilon p/2}} \int_{|k|\leq k_F} d^2k \int_{|l|\geq k_F} d^2l \frac{1}{(1+|k-l|)^{\frac{p}{2}}} \\
&\leq \frac{D_p}{(2\pi)^4} \varrho^{-\varepsilon p/2} \varrho^{\frac{1}{2}}
\end{aligned} \tag{4.217}$$

where we have used in the last step the estimate from (4.214). The bound in (4.58) then follows from the choice $p > \frac{1}{\varepsilon} + \frac{2}{\varepsilon^2}$.

To show (4.59), one passes to the thermodynamic limit, and computes by direct integration (for sufficiently large ϱ and $\varepsilon < 1/2$),

$$\lim_{\text{TD}} \frac{1}{L^4} \sum_{(k,l)\in \mathfrak{S}_n} = \lim_{\text{TD}} \frac{1}{L^4} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \theta\Big(\varrho^\varepsilon - |p_k - p_l|\Big) \chi\Big(\varrho^{-b_n} \leq |p_l| - |p_k| < \varrho^{-b_{n+1}}\Big)$$

$$= \frac{1}{(2\pi)^4} \int_{|k|\le k_F} d^2k \int_{|l|\ge k_F} d^2l \; \theta\Big(\varrho^\varepsilon - |k-l|\Big) \; \chi\Big(\varrho^{-b_n} \le |l| - |k| < \varrho^{-b_{n+1}}\Big)$$
$$\le C \varrho^{\frac{1}{2} - b_{n+1}} \varrho^\varepsilon \Big(\varrho^{-b_{n+1}} - \varrho^{-b_n}\Big). \tag{4.218}$$

For the proof of (4.60), we recall the definition in (4.10) and insert $v = v^{s,\varepsilon} + v^{\ell,\varepsilon}$:

$$E_{re}(\varrho) \le \lim_{\text{TD}} \frac{1}{L^4} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \frac{\big|\hat{v}^{s,\varepsilon}(p_k - p_l)\big|^2}{(E_l - E_k)} \theta\Big(|p_l| - |p_k| - \varrho^{-\frac{1}{2}}\Big) \tag{4.219}$$
$$+ \lim_{\text{TD}} \frac{1}{L^4} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \frac{\big|\hat{v}^{\ell,\varepsilon}(p_k - p_l)\big|^2}{(E_l - E_k)} \theta\Big(|p_l| - |p_k| - \varrho^{-\frac{1}{2}}\Big). \tag{4.220}$$

In the first line we proceed with (4.54) and find for any $M \ge 1$,

$$(4.219) = \sum_{n=1}^{M} \lim_{\text{TD}} \frac{1}{L^4} \sum_{(k,l)\in\mathfrak{S}_n} \frac{\big|\hat{v}^{s,\varepsilon}(p_k - p_l)\big|^2}{(E_l - E_k)}$$
$$\le C \sum_{n=1}^{M} \lim_{\text{TD}} \varrho^{-\left(\frac{n-1}{2cM}\right)} \frac{1}{L^4} \sum_{(k,l)\in\mathfrak{S}_n}$$
$$= C \sum_{n=1}^{M} \varrho^{-\left(\frac{n-1}{2cM}\right)} \varrho^{\frac{1}{2}+\varepsilon} \varrho^{-b_{n+1}} \Big(\varrho^{-b_{n+1}} - \varrho^{-b_n}\Big)$$
$$= C \varrho^{-\frac{1}{2}+\varepsilon} \Big(\varrho^{\frac{1}{2cM}} - 1\Big) \sum_{n=1}^{M} \varrho^{\frac{n}{2cM}} \le C \varrho^{-\frac{1}{2}+\varepsilon} \varrho^{\frac{M+1}{2cM}} \le C \varrho^{2\varepsilon}, \tag{4.221}$$

where we have taken the limit $M \to \infty$ and inserted $2c = (\frac{1}{2} + \varepsilon)^{-1}$. The second line (4.220) has been estimated in (4.90). This proves the upper bound in (4.60). For the lower bound, we insert again $v = v^{s,\varepsilon} + v^{\ell,\varepsilon}$,

$$E_{re}(\varrho) \ge \lim_{\text{TD}} \frac{1}{L^4} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \frac{\big|\hat{v}^{s,\varepsilon}(p_k - p_l)\big|^2}{(E_l - E_k)} \theta\Big(|p_l| - |p_k| - \varrho^{-\frac{1}{2}}\Big). \tag{4.222}$$

Since $|p_l - p_k| \le \varrho^\varepsilon$, we find that $|p_l| - |p_k| \le |p_l - p_k| \le \varrho^\varepsilon$ and $|p_l| + |p_k| \le 3\varrho^{\frac{1}{2}}$ (for $\varepsilon \le 1/2$), i.e., $(E_l - E_k)^{-1} \ge \frac{1}{3}\varrho^{-\varepsilon-\frac{1}{2}}$. Furthermore, note that the bound from (4.216) holds also if we replace $\hat{v}$ by $\hat{v}^{s,\varepsilon}$ for any $\varepsilon > 0$. Thus we find

$$(4.222) \ge \frac{1}{3}\varrho^{-\varepsilon-\frac{1}{2}} \lim_{\text{TD}} \frac{1}{L^4} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \big|\hat{v}^{s,\varepsilon}(p_k - p_l)\big|^2 \theta\Big(|p_l| - |p_k| - \varrho^{-\frac{1}{2}}\Big)$$
$$\ge C \varrho^{-\varepsilon-\frac{1}{2}} \Big(\varrho^{\frac{1}{2}} - C\Big)$$
$$\ge C \varrho^{-\varepsilon}. \tag{4.223}$$

Eventually note that here we can pass to the limit $\varepsilon \to 0$ which completes the derivation of the lower bound in (4.60).

The proof of (4.61) follows immediately from the decomposition of the potential; cf. (4.20) and (4.21). □

Proof of Lemma 4.5. We prove only (4.65), since (4.66)-(4.68) are derived in complete analogy.

$$\begin{aligned}\left\|\partial_{\tau_1} k_{kl}(\tau_1)\varphi_0\right\| &= \left\|\big(\partial_\tau e^{iH_y^f\tau_1}\big)e^{i(p_l-p_k)\cdot y}\varphi_{\tau_1}^f + e^{iH_y^f\tau_1}e^{i(p_k-p_l)\cdot y}\partial_{\tau_1}\varphi_{\tau_1}^f\right\| \\ &= \left\||p_k-p_l|^2 e^{iH_y^f\tau_1}e^{i(p_l-p_k)\cdot y}\varphi_{\tau_1}^f - 2(p_l-p_k)\cdot e^{iH_y^f\tau_1}e^{i(p_l-p_k)\cdot y}\nabla_y\varphi_{\tau_1}^f\right\| \\ &\leq |p_k-p_l|^2 + C|p_k-p_l| \end{aligned} \tag{4.224}$$

because $\|\nabla_y\varphi_\tau^f\| = \|\nabla_y\varphi_0\| \leq C$ (uniformly in ϱ). The estimate follows since $|p_k - p_l| \leq \varrho^\varepsilon$ for all $(k,l) \in \mathfrak{S}$. □

Appendices

We close this chapter with four appendices.

4.A Fermi pressure

We have explained in Section 4.1.2 that the fluctuations of the potential around its average value w.r.t. Ω_0 are strongly suppressed (suppressed compared to free bosons where the fluctuations would be proportional to the density ϱ), namely that

$$\|(V-E)\Omega_0\|^2 \leq C_d \varrho^{\frac{d-1}{d}}. \tag{4.225}$$

This is a direct consequence of the Fermi pressure (the antisymmetry of the wave function) which causes many fermions to be distributed much more homogeneously than bosons or classical particles. There is, however, another interesting difference between fermions and bosons due to the Fermi pressure: in a dense Fermi gas, the fluctuations are only caused by momentum modes close to k_F whereas for bosons all modes contribute equally to the fluctuations in the gas. Assume $v(x) = \chi_B(x)$ the characteristic function for some compact ball $B \subset [0, L]^d$ (in this case, the fluctuations of the potential coincide with local density fluctuations since the function $v(x) = \chi_B(x)$ measures the number of particles within the ball B). Let us then rewrite the operator $(V - E)$ in terms of creation and annihilation operators, i.e.,

$$(V-E) = \frac{1}{L}\sum_{k=1}^{\infty}\Big(\frac{1}{L}\sum_{l=1, l\neq k}^{\infty} \hat{v}_{kl} e^{i(p_k-p_k)y} a^*(p_l)a(p_k)\Big) = \frac{1}{L}\sum_{k=1}^{\infty} V^{p_k}, \tag{4.226}$$

with $V^{p_k} = L^{-1}\sum_{l=1,l\neq k}^{\infty} \hat{v}_{kl} e^{i(p_k-p_k)y} a^*(p_l)a(p_k)$. The interesting point about this decomposition is that the $\{V^{p_k}\}_{k\geq 1}$ are centered and uncorrelated (operator valued) random variables w.r.t. to Ω_0, meaning that $\langle \Omega_0, V^{p_k}\Omega_0\rangle = 0$ for all k and $\langle V^{p_k}\Omega_0, V^{p_{k'}}\Omega_0\rangle = 0$ for $k \neq k'$. The variance of the total sum of all V^{p_k} thus equals the sum of all variances of the V^{p_k},

$$\|(V-E)\Omega_0\|^2 = \Big\|\Big(\frac{1}{L}\sum_{k=1}^{N} V^{p_k}\Big)\Omega_0\Big\|^2 = \frac{1}{L^2}\sum_{k=1}^{N}\|V^{p_k}\Omega_0\|^2. \tag{4.227}$$

For $v(x) = \chi_B(x)$ one can associate the operator V^{p_k} with the random variable that describes the local density of all modes carrying momentum k. By means of (4.227) we can look at the local density fluctuations w.r.t. the momentum variable $0 \leq |p_k| \leq k_F$. In Figure 4.A.1 we depict the functions $\{\|V^p\Psi_i\|^2_{\mathrm{TD}} : 0 \leq |p| \leq k_F\}$ for different values of $\varrho = k_F^{1/d}$ and (for comparison) for three different wave functions Ψ_i. The lower line corresponds to $\Psi_1 = \Omega_0$, the ideal Fermi gas. The middle line stands for a nonsymmetric wave function $\Psi_2 = \prod_{i=1}^{N}\phi_i(x_i)$, the product of all one-particle orbitals with momenta inside the Fermi sphere, and $\Psi_3 = \Omega_0^+$ is the bosonic analogue of Ω_0, i.e., Ψ_3 is defined as in (4.5) with $(-1)^\tau$ replaced by $+1$. The four pictures show that, eventually, for high densities, the particles with momentum inside the Fermi sphere tend to be distributed very homogeneously in the case of Ω_0. Here the fluctuations in (4.225) are only caused by modes with very high momentum close to k_F (the lower line in Figure 4.A.1). For Ψ_1 and Ψ_2 this is different as fluctuations are nonzero foll all occupied modes. The small value of $\|V^p\Omega_0\|^2_{\mathrm{TD}}$ for $|p|$ not close to the Fermi momentum is a consequence of the antisymmetry of the wave

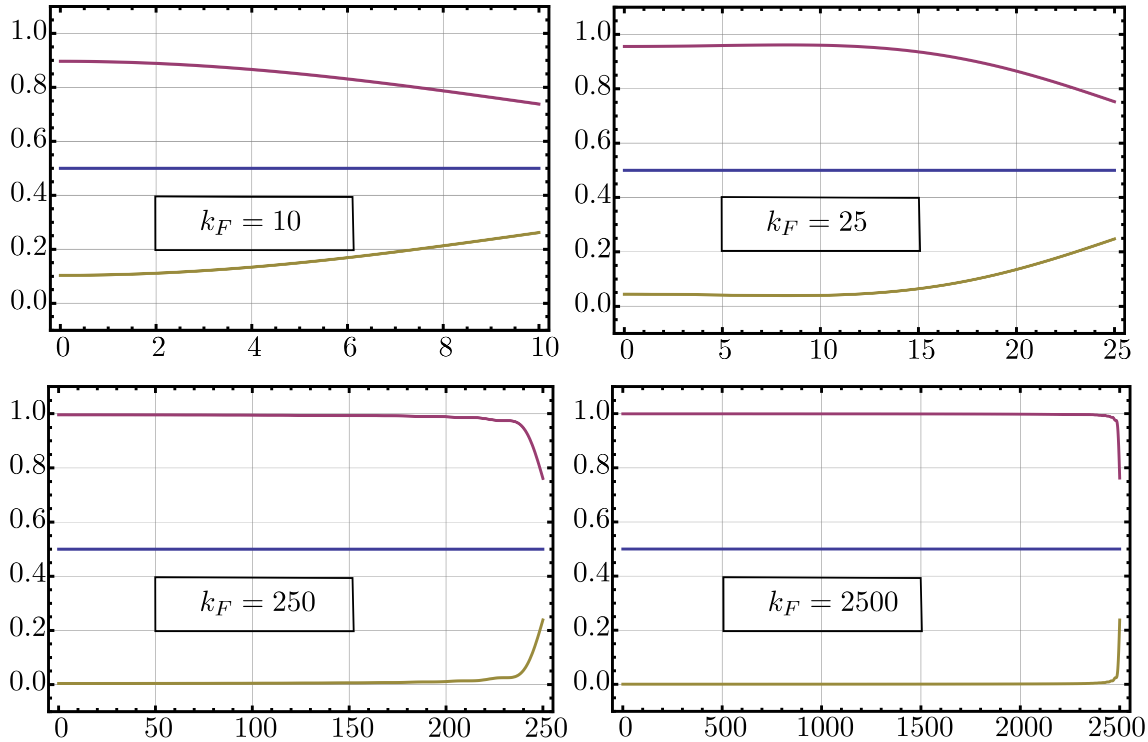

Figure 4.A.1: We show the functions $\{\|V^p\Psi_i\|^2_{\mathrm{TD}} : 0 \le |p| \le k_F\}$ for different values k_F and for three different wave functions Ψ_i (we depict the case $d = 1$; qualitatively, however, the pictures are the same for $d \ge 2$). The lower line corresponds to the ideal Fermi gas $\Psi_1 = \Omega_0$ and shows that the fluctuations are strongly suppressed for modes away from k_F when ϱ becomes large. The middle line indicates the wave function Ψ_2, a nonsymmetric product of plane waves with momenta between zero and k_F. The upper line stands for $\Psi_3 = \Omega_0^+$, the bosonic analogue of (4.5), i.e., with $(-1)^\tau$ replaced by 1.

function in combination with the fast decay of the Fourier transform of v (for that note that $\|V^{p_k}\Omega_0\|^2 = L^{-2}\sum_{l\ge N+1}|\hat{v}(p_k - p_l)|^2$ and $\hat{v}[\chi_B](p_k) \propto 1/p_k^2$ and even faster decay for potentials $v \in C_0^\infty$).

We think that the explained argument provides an interesting heuristic picture of the statistical properties of the dense ideal Fermi gas, namely, that the proability for slow particles in Ω_0 building random clusters approaches zero when ϱ becomes large, whereas for the fast modes in Ω_0, the probability for density fluctuations is nonvanshing but nevertheless much smaller compared to the fluctuations in a dense bosonic gas.

4.B Recollision diagrams

In this appendix we show that the next-order energy correction $E_{re} = E_{re}(\varrho)$ in H^{mf} is due to so-called immediate recollisions that come from all orders in the Duhamel expansion. In the proof of Lemma 4.2 it was not necessary to know that E_{re} has contributions from arbitrary high orders in the expansion since we could identify the correct choice of E_{re} directly from the estimate in (4.90) where it was used to cancel the immediate recollision contribution in Ψ_{B}. However, it is an interesting insight to see exactly where the subleading phase E_{re} comes from. To this end, let us denote by

$$\widetilde{H}^{\mathrm{mf}} = H^{\mathrm{mf}} + E_{re} = -\Delta_y - \sum_{i=1}^{N} \Delta_{x_i} + \varrho\hat{v}(0) \tag{4.228}$$

the mean field Hamiltonian without E_{re}, and then write down the Dyson series of $U(t)\Psi_0$ around $\widetilde{U}^{\mathrm{mf}}(t)$, i.e.,

$$U(t)\Psi_0 = \widetilde{U}^{\mathrm{mf}}(t)\Psi_0 + \widetilde{U}^{\mathrm{mf}}(t)\sum_{n=1}^{\infty}(-i)^n \int_0^t ds_1 ... \int_0^{s_{n-1}} ds_n \Big(\widetilde{U}^{\mathrm{mf}}(-s_1)(V-E)\widetilde{U}^{\mathrm{mf}}(s_1)\Big) ... \times$$
$$\times ... \Big(\widetilde{U}^{\mathrm{mf}}(-s_n)(V-E)\widetilde{U}^{\mathrm{mf}}(s_n)\Big)\Psi_0. \tag{4.229}$$

Comparing the microscopic time evolution with $U^{\mathrm{mf}}(t)\Psi_0 = e^{iE_{re}t}\widetilde{U}^{\mathrm{mf}}(t)\Psi_0$ and using the above expansion for $U(t)\Psi_0$, we find

$$\Big\| U(t)\Psi_0 - e^{iE_{re}t}\widetilde{U}^{\mathrm{mf}}(t)\Psi_0 \Big\|^2 \tag{4.230}$$
$$= 2\,\mathrm{Re}\Big(1 - e^{-iE_{re}t}\big\langle \widetilde{U}^{\mathrm{mf}}(t)\Psi_0, U(t)\Psi_0\big\rangle\Big)$$
$$= 2\,\mathrm{Re}\Big(1 - e^{-iE_{re}t}\Big) - 2\,\mathrm{Re}\, e^{-iE_{re}t}\sum_{n=1}^{\infty}(-i)^n \int_0^t ds_1 ... \int_0^{s_{n-1}} ds_n \times$$
$$\times \big\langle \Psi_0, \Big(\widetilde{U}^{\mathrm{mf}}(-s_1)(V-E)\widetilde{U}^{\mathrm{mf}}(s_1)\Big) ... \Big(\widetilde{U}^{\mathrm{mf}}(-s_n)(V-E)\widetilde{U}^{\mathrm{mf}}(s_n)\Big)\Psi_0\big\rangle,$$

where a nonvanishing contribution remains at zeroth order. This term is of course due to the wrong choice of phase in the expansion in (4.229). Note that the $n=1$ term is exactly zero and we can directly proceed for $n=2$ in order to show how the contributions from immediate recollisions look like. The $n=2$ term is given by

$$-2\,\mathrm{Re}\, e^{-iE_{re}t}(-i)^2 \frac{1}{L^4}\sum_{k=1}^{N}\sum_{l=N+1}^{\infty} |\hat{v}_{kl}|^2 \int_0^t ds_1 \int_0^{s_1} ds_2 e^{-i(E_l-E_k)(s_1-s_2)} f_{kl}(s_1,s_2), \tag{4.231}$$

where we have introduced the abbreviation $f_{kl}(s_1,s_2) = \big\langle \varphi_0, k_{kl}(s_1)k_{lk}(s_2)\varphi_0\big\rangle$. Immediate recollisions are defined as the contributions in such expressions where the second collision happens right after the first, i.e., the ones for $s_1 \approx s_2$. After a partial integration, we find

$$\int_0^t ds_1 \int_0^{s_1} ds_2 e^{-i(E_l-E_k)(s_1-s_2)} f_{kl}(s_1,s_2)$$
$$= \int_0^t ds_1 e^{-i(E_l-E_k)s_1}\Big[\frac{e^{i(E_l-E_k)s_2}}{i(E_l-E_k)} f_{kl}(s_1,s_2)\Big|_{s_2=0}^{s_2=s_1} - \int_0^{s_1} ds_2 \frac{e^{i(E_l-E_k)s_2}}{i(E_l-E_k)}\partial_{s_2} f_{kl}(s_1,s_2)\Big]$$
$$= \int_0^t ds_1 \frac{1}{i(E_l-E_k)} + \text{rest}, \tag{4.232}$$

where in the first term $f_{kl}(s_1,s_1)=1$ and where after a second partial integration,

$$|\text{rest}| \le C\frac{t}{(E_l-E_k)^2} + \text{higher order}. \tag{4.233}$$

Neglecting for simplicity the problem coming from the stationary points $|p_k| \approx |p_l|$ (they can be separated from the sum in (4.231) and then treated exactly as in the proof of Lemma 4.2, cf. Corollary 4.4), one obtains recalling the definition of E_{re} from (4.10),

$$(4.231) = -2\,\mathrm{Re}\, e^{-E_{re}t}(-i)^2(-iE_{re}) + \text{rest}_2, \tag{4.234}$$

where $|\mathrm{rest}_2|$ can be shown to be proportional to t/k_F. The important point is that for all n even, there remains always a term after partial integration which is not small enough. This nonvanishing contribution comes from the boundary terms $s_i = s_{i-1}$ where one can use the cancellation due to the oscillating phase only once (a second partial integration is not possible exactly as in the first term in (4.232)). For $n = 4$, this is the term where two recollisions happen right after each other. The corresponding expression is found to be given by

$$\text{immediate recollisions in the } 4th \text{ term} = -2\,\mathrm{Re}\, e^{-iE_{re}t}(-i)^4 \frac{(-iE_{re}t)^2}{2}, \tag{4.235}$$

and similarly for all n even, it is not difficult to find

$$\text{immediate recollisions in the } nth \text{ term} = -2\,\mathrm{Re}\, e^{-iE_{re}t}(-i)^n \frac{(-iE_{re}t)^{n/2}}{(n/2)!}. \tag{4.236}$$

Summing up all contributions (note that for n odd there are no diagrams that only contain immediate recollisions since there always remains at least a single particle-hole excitation) leads to

$$\begin{aligned}
\text{immediate recollisions from all orders} &= -2\,\mathrm{Re}\, e^{-iE_{re}t} \sum_{n=2, n \text{ even}}^{\infty} (-i)^n \frac{(-iE_{re}t)^{n/2}}{(n/2)!} \\
&= -2\,\mathrm{Re}\, e^{-iE_{re}t} \sum_{n=1}^{\infty} \frac{(iE_{re}t)^n}{n!} \\
&= -2\,\mathrm{Re}\, e^{-iE_{re}t}\left(e^{iE_{re}t} - 1\right) \\
&= -2\,\mathrm{Re}\left(1 - e^{-iE_{re}t}\right).
\end{aligned} \tag{4.237}$$

From the last line, one can now see that the sum of all recollision terms cancels exactly the contribution from the nonvanishing zeroth term in the expansion in (4.230).

4.C The model in one dimension

The main difference in the definition of the model in one spatial dimension is that the possible momenta for $L < \infty$ are now given by $p \in (2\pi/L)\mathbb{Z}$, and that the Fermi momentum $|p_N| = k_F$ is proportional to ϱ. Below, we are going to prove the following theorem which is the analogous statement to Theorem 4.1 (a slightly different statement implying the same result as Theorem 4.15 was derived in [67]).

Theorem 4.15. *Let $d = 1$, the masses $m_x = m_y = 1/2$ and the coupling constant $g = 1$. Let $\Psi_0 = \varphi_0 \otimes \Omega_0$ with $\varphi_0 \in \mathcal{H}_y$ with $\|\nabla^4 \varphi_0\| \leq C$ uniformly in $\varrho = N/L$ and Ω_0 the free fermionic ground state in $\mathbb{T}$. Then, for any small enough $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$\lim_{\substack{N,L\to\infty \\ \varrho = N/L = const.}} \left\| e^{-iHt}\Psi_0 - e^{-iH^{mf}t}\Psi_0 \right\|_{\mathcal{H}_y \otimes \mathcal{H}_N} \leq C_\varepsilon (1+t)^{\frac{3}{2}} \varrho^{-\frac{1}{4}+\varepsilon} \tag{4.238}$$

holds for all $t > 0$, where

$$H^{mf} = -\Delta_y - \sum_{i=1}^{N} \Delta_{x_i} + \varrho \hat{v}(0) \tag{4.239}$$

is the free Hamiltonian with constant mean field $\langle \Omega_0, \sum_{i=1}^{N} v(x_i - y)\Omega_0 \rangle_{\mathcal{H}_N} = \varrho \hat{v}(0)$.

Remark 4.5. 1) Note the two differences compared to Theorem 4.1: the absence of an additional next-to-leading order energy correction in H^{mf} and the better error on the r.h.s.

2) As explained in Section 4.1.2, we expect, and this is in contrast to $d = 2$, that the l.h.s. of (4.238) is small for large ϱ on all time scales. Theorem 4.15 can prove this only to some extent since the error term on the r.h.s. becomes small only as long as $t \ll \varrho^{1/6}$.

One possibility to prove (4.238) is to adapt the proof of Theorem 4.1. For that, note that the argument depends on the dimension essentially through Lemma 4.3 and Corollary 4.4. The corresponding bounds for $d = 1$ are summarized in

Lemma 4.16. *Let $d = 1$, $0 < \varepsilon < 1/2$ and $M, q \in \mathbb{N}$. Let $v(x) \in C_0^\infty(\mathbb{T}) \cap C_0^\infty(\mathbb{R})$ and $v^{\ell,\varepsilon}$, $v^{s,\varepsilon}$ defined as in (4.20),(4.21). Then there exist positive constants C, C_q, $C_{q,\varepsilon}$ such that*

$$\lim_{\mathrm{TD}} \frac{1}{L^2} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \left|\hat{v}(p_k - p_l)\right|^q = C_q, \tag{4.240}$$

$$\lim_{\mathrm{TD}} \frac{1}{L^2} \sum_{k=1}^{N} \sum_{l=N+1}^{\infty} \left|\hat{v}^{\ell,\varepsilon}(p_k - p_l)\right|^q \leq C_{q,\varepsilon} \varrho^{-1/\varepsilon}, \tag{4.241}$$

$$\lim_{\mathrm{TD}} \mathcal{V}_0(N,\rho) \leq C\varrho^{-1}, \tag{4.242}$$

$$\lim_{\mathrm{TD}} \left(\varrho^{-\left(\frac{n-1}{2cM}\right)} \mathcal{V}_n(N,\rho)\right) \leq C\varrho^{-1+\varepsilon}\left(\varrho^{\frac{1}{2cM}} - \varrho^{\frac{1}{cM}}\right), \quad 1 \leq n \leq M \tag{4.243}$$

$$\lim_{\mathrm{TD}} \frac{1}{L} \sum_{k=1}^{N} \left|\hat{v}(p_k - p)\right| \leq C\varrho^{\varepsilon} + C_\varepsilon \varrho^{-1/\varepsilon} \quad \text{for} \quad p \in (2\pi/L)\mathbb{Z}. \tag{4.244}$$

The proof is analogous to the ones for Lemma 4.3 and Corollary 4.4.

The only bound that remains to be shown is the one for $\Psi_{\mathrm{B},2}^{s,n}(t)$, $1 \leq n \leq M$; cf. Section 4.2.4. In the two-dimensional case, this term was directly canceled by $E_{re}^{\varepsilon}(\varrho)$ which is identically zero for $d = 1$. However, one easily verifies, using $E_{l_1} - E_{k_1} \geq Ck_F\varrho^{-b_n} = C\varrho^{-\frac{1}{2}+\frac{n-1}{2cM}}$ (since $k_F \propto \varrho$), that in one dimension,

$$\sum_{n=1}^{M} \|\Psi_{\mathrm{B},2}^{s,n}(t)\|_{\mathrm{TD}} \lesssim (1+t)\varrho^{-\frac{1}{2}+\varepsilon} M \varrho^{\frac{1}{cM}} \lesssim (1+t)\varrho^{-\frac{1}{2}+2\varepsilon}, \tag{4.245}$$

since $M\varrho^{\frac{1}{cM}} \lesssim \varrho^{\varepsilon}$ for $M = \lfloor \ln \varrho \rfloor$. This completes the proof of Theorem 4.15.

4.D The model in three dimensions

We explain why it is not possible to adapt the proof of Theorem 4.1 also to the case $d = 3$. Here, the possible momenta are given by $p \in (2\pi/L)\mathbb{Z}^3$ and $|p_N| = k_F \propto \varrho^{\frac{1}{3}}$. We exemplify this for one particular term, namely

$$\left\| \sum_{n=1}^{M} \Psi_{\mathrm{A}}^{s,n}(t) \right\|^2 = \sum_{n=1}^{M} \|\Psi_{\mathrm{A}}^{s,n}(t)\|_{\mathrm{TD}}^2 \leq \sum_{n=1}^{M} \left[\lim_{\mathrm{TD}} \frac{1}{L^6} \sum_{(k,l)\in\mathfrak{S}_n} \frac{1}{(E_k - E_l)^2} \right], \tag{4.246}$$

which appears at second order in the Duhamel expansion; cf. Section 4.2.4. Here, we have used in addition that the $\Psi_{\mathrm{A}}^{s,n}(t)$ are pairwise orthogonal, and then we applied the first step from (4.80). In order to obtain the optimal bound for the r.h.s., let us be more general as in the case $d = 2$ and define the sets $\mathfrak{S}_n$ with $b_0 = \infty$ and $b_n = b + (n-1)/(2cM)$, for $b > 0$ and $2c = (b+\varepsilon)^{-1}$, $b \in \mathbb{R}$. Using $(E_l - E_k) \geq \varrho^{\frac{1}{3}-b_n}$ for all $(k,l) \in \mathfrak{S}_n$, together with

$$\lim_{\text{TD}} \frac{1}{L^6} \sum_{(k,l)\in\mathfrak{S}_n} \lesssim \varrho^{\frac{2}{3}-b_{n+1}} \varrho^{2\varepsilon} \Big(\varrho^{-b_{n+1}} - \varrho^{-b_n} \Big), \tag{4.247}$$

one finds

$$\frac{1}{L^6} \sum_{(k,l)\in\mathfrak{S}_n} \frac{1}{(E_k - E_l)^2} \lesssim \varrho^{2b_n - 2b_{n+1}} \Big(1 - \varrho^{-b_n + b_{n+1}} \Big) \lesssim \frac{1}{M} \ln \varrho \Big(1 + O(M^{-1} \ln \varrho) \Big), \tag{4.248}$$

$1 \leq n \leq M$. Hence, this way (taking $M \to \infty$) we obtain at best (4.246) $\lesssim \ln \varrho$, which would imply a trivial statement like $\|\Psi_{\text{A}}(t)\|_{\text{TD}} \lesssim \ln \varrho$ already for t of order one.

Appendices

A Gross-Pitaevskii limit for bosons

The N-particle Hamiltonian H_N^{GP} which describes a gas of dilute atoms is defined (for $d = 3$ spatial dimensions) by a Hamiltonian of the form (1.4) but with the pair potential v replaced by an N-dependent pair potential $v_N(x) = N^3 v(Nx)$ with $v \in C_0(\Omega)$ and coupling constant $g_N = 1/(N-1)$. H_N^{GP} looks formally very similar to the weak coupling Hamiltonian for which the coupling is also given by $g_N = 1/(N-1)$ but the pair potential v is N-independent (i.e., of long-range type compared to v_N). Another analogy is that solutions to the microscopic Schrödinger equation $i\partial_t \Psi_{N,t} = H_N^{\mathrm{GP}} \Psi_{N,t}$ (and similarly for the stationary equation) are described in the large N limit by a one-particle nonlinear effective Hamiltonian, the so-called Gross-Pitaevskii Hamiltonian, given by $h_x^{\mathrm{GP},\varphi} = -\Delta_x + a|\varphi(x)|^2$, where a denotes the scattering length of the potential v. It is well known, e.g., that for particular initial conditions,

$$\lim_{N\to\infty} \mathrm{Tr}\left[A^k \left(\gamma^{(k)}_{\Psi_{N,t}} - \gamma^{(k)}_{\varphi_t^{\otimes N}} \right) \right] = 0, \tag{249}$$

where $\Psi_{N,t}$ solves the microscopic Schrödinger equation for appropriate initial conditions and φ_t is the solution to the corresponding Gross-Pitaevskii equation $i\partial_t \varphi_t = h^{\mathrm{GP},\varphi_t} \varphi_t$, see, e.g. [2, 3, 40, 45, 46, 47, 102, 16, 104]. Similar convergence results are known also for the ground state wave function (and ground state energy) for which we refer to the thorough summary in [85]. Despite these close analogies, the Gross-Pitaevskii equation is not a direct consequence of the Hartree equation for $v \propto \delta$. The significant difference is the scattering length a that appears in the effective Hamiltonian. The reason for the scattering length to appear in the effective description is the strongly localized and peaked interaction $v_N(x) = N^3 v(Nx) \to \delta(x)$ $(N \to \infty)$ which causes the N-particle wave function to have a more complicated structure compared to solutions in the weak coupling model. In particular, there emerges another relevant length scale (the range of the interaction) on which a low energy wave function develops pair correlations between the particles. One may think of these correlations as the correct equilibration of Ψ_N having nodes where two particles approach the same point (this reduces the potential energy) and Ψ_N becoming to steep around these nodes (which enhances the kinetic energy). This additional microscopic structure of the wave functions is summarized in the effective coupling a.

The reason why we mention the Gross-Pitaevskii limit is that the N-dependence of H_N^{GP} can be very well motivated.[5] To see this, let us consider N particles in a box of volume

[5] To our knowledge, there is no similarly convincing motivation for the N-dependence of the weak coupling model itself. We rather think of the weak coupling model as a first step towards the physically more realistic Hamiltonian H_N^{GP}. Since both models share some relevant common features, it is helpful to understand the weak coupling model before one approaches the same questions for the much more involved Gross-Pitaevskii limit.

L^3 described by a Hamiltonian $H_N^{\mathrm{GP,unscaled}}$ of the form (1.4) with $W^{\mathrm{ext}} = 0$, $g_N = 1$ and $v \in C_0([0,L]^3)$ (unscaled stands here for N-independent). We assume that the side length of the box increases with the particle number as $L = N$. For a repulsive potential, the particles spread over the whole box at an average distance $L/N^{\frac{1}{3}}$, and thus in particular, the ratio between the support of the interaction and the average distance between the particles approaches zero when N tends to ∞ (which defines the dilute limit of the gas). For reasons of convenience, one now rescales the whole system into a box of volume one meaning that we introduce new spatial coordinates $x \mapsto \widetilde{x} = x/L$. The Hamiltonian $H_{N,x}^{\mathrm{GP,unscaled}}$ is then replaced by

$$\begin{aligned} H_{N,x}^{\mathrm{GP,unscaled}} \mapsto H_{N,\widetilde{x}}^{\mathrm{GP,unscaled}} &= -\frac{1}{L^2}\sum_{i=1}^{N}\Delta_{\widetilde{x}_i} + \sum_{i<j} v\big(L(\widetilde{x}_i - \widetilde{x}_j)\big) \\ &= \frac{1}{N^2}\Big[-\sum_{i=1}^{N}\Delta_{\widetilde{x}_i} + \frac{1}{N}\sum_{i<j} v_N(\widetilde{x}_i - \widetilde{x}_j)\Big] = \frac{1}{N^2}H_{N,\widetilde{x}}^{\mathrm{GP}}, \end{aligned} \tag{250}$$

where $v_N(\widetilde{x}) = N^3 v(N\widetilde{x})$. The so obtained Hamiltonian $H_{N,\widetilde{x}}^{\mathrm{GP}}$ is of the same form as the weak coupling model (with the difference that $v = v_N$ is N-dependent) and the pre factor $1/N^2$ determines the correct energy scale (in the stationary case) resp. the correct time scale (for the time-dependent Schrödinger equation). Replacing v_N by some N-independent function v (e.g., the Coulomb potential), one arrives at the weak coupling model. This step of simplification allows us to analyze the effective "long-range properties" of a microscopic model that describes a realistic Bose-Einstein condensate without taking into account the additional difficulty due to the short-scale structure caused by the strongly peaked pair potential v_N. Many of the results that we derive in the weak coupling limit are expected to be similarly true in the Gross-Pitaevskii regime.

B More about the weak coupling limit for fermions

We briefly present two different models for which the derivation of the time-dependent Hartree equations is well understood (for a recent work on the stationary problem, see, e.g., [51]). Both models were originally introduced by Narnhofer and Sewell [97].

B.1 Semiclassical limit

In the so-called semiclassical limit, one considers N fermionic particles described by the time-dependent Schrödinger equation

$$iN^{-\frac{1}{3}}\partial_t\Psi_{N,t} = \Big(-N^{-\frac{2}{3}}\sum_{i=1}^{N}\Delta_{x_i} + N^{-1}\sum_{1\le i<j\le N} v(x_i - x_j)\Big)\Psi_{N,t}, \tag{251}$$

for which one wants to compare the solution Ψ_t for particular initial conditions (close to a Slater determinant and obeying a certain semiclassical structure, see, e.g. [18, Theorem 2.1]) to the antisymmetric product of orbitals $\{\varphi_{k,t}\}_{k=1}^N$ which solve the fermionic Hartree equations,

$$iN^{-\frac{1}{3}}\partial_t\varphi_{k,t} = \Big(-N^{-\frac{2}{3}}\Delta_x + N^{-1}\sum_{l=1}^{N}\big(v * |\varphi_{l,t}|^2\big)(x)\Big)\varphi_{k,t}. \tag{252}$$

The phisical situation behind this limit is the following: the particles are confined to a volume of order one and thus a priori their kinetic energy for a wave function that is close to the ground state is proportional to $N^{5/3}$, cf. the kinetic energy inequality (1.26). Since the total potential energy is of order N (note the coupling constant $g_N = N^{-1}$), an additional scaling of the kinetic term is required in order to make the potential and kinetic energies compatible. The N-dependent factor in front of the time-derivative adjusts the correct time scale for which the particles (having average velocities $\propto N^{1/3}$) travel a distance of order one. Heuristically, the semiclassical character of this equation can be seen from recasting the additional prefactors into a small Planck constant $\hbar_N = N^{-1/3}$. Starting from (252), the fermionic Hartree equations were derived for a certain class of bounded potentials in [39, 18, 101] (see, e.g., [101, Theorem 2.3]),

$$\mathrm{Tr}\Big|\gamma^{(1)}_{\Psi_{N,t}} - \gamma^{(1)}_{\bigwedge_k \varphi_{k,t}}\Big| \leq C^t\Big(\Big(\mathrm{Tr}\Big|\gamma^{(1)}_{\Psi_{N,t}} - \gamma^{(1)}_{\bigwedge_k \varphi_{k,0}}\Big|\Big)^{\frac{1}{2}} + \frac{1}{\sqrt{N}}\Big), \tag{253}$$

for some time-dependent constant C^t. The derivation for the Coulomb potential is still an open problem for which a partial result was obtained recently in [110]. The semiclassical properties of the microscopic solution can be shown by means of comparing the Wigner distribution w.r.t. $\Psi_{N,t}$ to the solution of a classical Vlasov equation (for recent results, see [17]).

B.2 Large volume limit with Coulomb interaction

Another possible limit for which the fermionic Hartree equations have been derived is defined by the Schrödinger equation

$$i\partial_t \Psi_{N,t} = \Big(-\sum_{i=1}^{N} \Delta_{x_i} + N^{-2/3} \sum_{1\leq i<j\leq N} |x_i - x_j|^{-1}\Big)\Psi_t, \tag{254}$$

and similarly for potentials $v(x) = |x|^{-s}$ and $g_{N,s} = N^{\frac{s}{3}-1}$ for $s \in (0,1)$. Here, the fermions are supposed to be confined to a region $\Omega \subset \mathbb{R}^3$ of large volume $|\Omega| \propto N$ (thus, the average density is of order one) with kinetic energy not larger than CN. The coupling constant is therefore chosen such that the potential energy is also of order N. To see that this is the case for $v(x) = |x|^{-1}$ and $g_N = N^{-2/3}$, let us compute the mean field potential (the direct term) for N plane waves that are confined to a sphere of radius $N^{1/3}$, namely

$$\sum_{l=1}^{N} \Big(|\cdot|^{-1} * |\varphi_l|^2\Big)(x) = 4\pi \int_0^{N^{1/3}} r dr = CN^{2/3}. \tag{255}$$

Note that the exchange term can be easily shown to be subleading compared to the direct term, and thus, for a Slater determinant made up from plane waves, one similarly finds $\langle \bigwedge \varphi, \big(\sum_{i<j} |x_i - x_j|^{-1}\big) \bigwedge \varphi\rangle = O(N^{5/3})$. It has been shown in [11, Theorem II.1] (and similarly also in [101]) that for appropriate initial conditions, the solution to the microscopic Schrödinger equation satisfies

$$\mathrm{Tr}\Big|\gamma^{(1)}_{\Psi_{N,t}} - \gamma^{(1)}_{\bigwedge \varphi_{k,t}}\Big| \leq C^t\Big(\Big(N^{\frac{2}{3}}\mathrm{Tr}\Big|\gamma^{(1)}_{\Psi_{N,0}} - \gamma^{(1)}_{\bigwedge \varphi_{k,0}}\Big|\Big)^{\frac{1}{2}} + N^{-1/6}\Big), \tag{256}$$

where the orbitals solve the fermionic Hartree equations,

$$i\partial_t \varphi_{k,t} = \Big(-\Delta_x + N^{-2/3}\sum_{l=1}^{N} \big(|\cdot|^{-1} * |\varphi_{l,t}|^2\big)(x)\Big)\varphi_{k,t}, \qquad \varphi_{k,t=0} = \varphi_{k,0}. \tag{257}$$

The same result with improved convergence rate was derived more recently in [100] where it was shown moreover that the solution $\Psi_{N,t}$ can be approximately described also by the fermionic Hartree equations with a spatially constant mean field potential. The Hartree equations (257) provide thus a subleading correction to the free time evolution (with appropriately chosen phase). That the dynamics is approximately free can be also inferred from the fact that the average forces produced by the mean field potential in (257) (which is of leading order, i.e., compatible to the kinetic energy) is suppressed by a factor $N^{-1/3}$:

$$\Big|\nabla_x \sum_{l=1}^{N} \Big(|\cdot|^{-1} * |\varphi_l|^2\Big)(x)\Big| \leq \sum_{l=1}^{N} \Big(|\cdot|^{-2} * |\varphi_l|^2\Big)(x) = 4\pi \int_0^{N^{1/3}} dr = CN^{1/3},$$

and thus expected to be subleading compared to the average velocities of the particles.

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, §8, Abs. 2 Pkt. 5)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist. Kapitel 2 und 4 enthalten Resultate, die der Zusammenarbeit mit verschiedenen Koautoren entstammen (für Details siehe den Anfang des entsprechenden Kapitels).

München, den 23.02.2018 David Mitrouskas