# The Chinese String

# Its solution and analogy to the Chinese Rings

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# ZUSAMMENFASSUNG

In der Kognitivpsychologie werden isomorphe mechanische Geschicklichkeitsspiele betrachtet, um herauszufinden, wodurch Probleme ihre Schwierigkeit erhalten. Ein Beispiel für diese Art von Geschicklichkeitsspielen sind die zueinander isomorphen Spiele Turm von Hanoi und Monsters and Globes. Ihre Isomorphie ist durch das Vertauschen der Rollen der beweglichen und der festen Bestandteile gegeben. Ein weiteres Beispiel von zwei Geschicklichkeitsspielen, bei denen die Rollen der festen und beweglichen Bestandteile vertauscht sind, sind Chinesische Ringe und Chinese String. In dieser Dissertation wird der Zusammenhang zwischen Chinesische Ringe und Chinese String genauer untersucht, um herauszufinden, inwiefern eine Isomorphie zwischen diesen beiden gegeben ist.
Zunächst wird eine Einleitung in Chinesische Ringe gegeben. Dabei werden Eigenschaften des zugehörigen Zustandsgraphen und der Zusammenhang zwischen Gros- und Gray-Code angegeben. Darüber hinaus wird die Zugfolge der optimalen Lösung (mit abzählbar unendlich vielen Ringen), auch Gros-Folge genannt, betrachtet und festgestellt, dass diese gleich der "greedy square-free sequence" ist.
Anschließend wird eine mechanische Lösung von Chinese String aufgezeigt. Mithilfe einer imaginären Linie werden Bewegungen definiert und diese Bewegungen werden verwendet, um einen rekursiven Lösungsalgorithmus für eine beliebige Anzahl an Ringen anzugeben. Anhand des Algorithmus werden weitere Eigenschaften der Lösung untersucht und es wird festgestellt, dass diese Lösung zum Lösen von Chinesische Ringe verwendet werden kann, da deren Zugfolgen identisch sind. Neben dem ausführlich behandelten Algorithmus wird eine weitere Lösungsstrategie gleicher Komplexität aufgezeigt, weswegen es im Allgemeinen keine eindeutige optimale Lösung gibt.
Darauf folgend wird eine Analogie zwischen Chinesische Ringe und Chinese String herausgearbeitet. Im Zuge dessen werden für Chinese String diskrete Zustände definiert und Änderungen von regulären Zuständen durch Züge betrachtet. Hierbei wird gezeigt, dass die regulären Zustände und Züge isomorph zu den Zuständen und Zügen von Chinesische Ringe sind. Somit ist der bereits ausführlich behandelte Algorithmus identisch zum optimalen Lösungsalgorithmus von Chinesische Ringe.
Im Anschluss werden verschiedene Änderungen an Chinese String betrachtet und nachgewiesen, dass diese keinen Einfluss auf die Komplexität haben. Hierfür werden zunächst wie bei Kauffman[1] die Ringe gelöst. Anschließend wird dieses Konstrukt nach der Idee von Przytycki und Sikora[2] auf natürliche Weise in eine 3-Sphäre eingebettet und die Fundamentalgruppe des

[1] Louis H. Kauffman. Tangle Complexity and the Topology of the Chinese Rings. In: Jill P. Mesirov, Klaus Schulten und De Witt Summers (Hrsg.), Mathematical Approaches to Biomolecular Structure and Dynamics, Springer, New York, NY, 1996, Seiten 1–10

[2] Józef H. Przytycki und Adam S. Sikora. Topological Insights from the Chinese Rings. Proceedings of the American Mathematical Society, 130: 893–902, 2001.

Komplementes des daraus entstandenen Henkelkörpers betrachtet. Je nach Änderung ergeben sich zwar unterschiedliche Fundamentalgruppen, durch Betrachtung der eingeführten imaginären Linie als Element dieser Fundamentalgruppe wird jedoch nachgewiesen, dass die Änderungen keinen Einfluss auf die Komplexität haben.
Zuletzt wird die Kauffmansche Ring-Vermutung aus dem Jahr 1996 betrachtet. Diese besagt, dass für einen beliebigen Anfangszustand die topologische und die mechanische Austauschzahl identisch sind. Przytycki und Sikora haben hierfür bereits einen Beweis für den gewöhnlichen Anfangszustand in einer speziellen Version von Chinese String geführt. Hier nun wird ein alternativer kombinatorischer Beweis angegeben, der zeigt, dass die Komplexität $2^{n-1}$ beträgt. Hierfür wird der Verlauf der imaginären Linie nach Lösen der Ringe genauer untersucht. Dabei kann festgestellt werden, dass sich der Verlauf der Linie nach dem Lösen eines Ringes aus zwei miteinander verbundenen Kopien des Verlaufs der Linie vor dem Lösen des Ringes ergibt. Daraus resultierend wird ein Satz bewiesen, der besagt, dass für einen beliebigen Anfangs- und Endzustand die Mindestzahl der benötigten Crossings (bei Chinese String) gleich der Mindestzahl der benötigten Züge von Ring 1 (bei Chinesische Ringe) ist. Hieraus ergibt sich somit auch direkt ein Nachweis der Gültigkeit der Aussage der Ring-Vermutung.
Wir erhalten damit, dass Chinese String (in der fixierten Version) und Chinesische Ringe ein Paar isomorpher mechanischer Geschicklichkeitsspiele bilden und somit für weitere Studien der Kognitivpsychologie in Betracht gezogen werden können.

# Abstract

In the field of cognitive psychology one considers isomorphic (mechanical) puzzles to find out what makes problems difficult. One example of this type of puzzles are the Tower of Hanoi and the Monsters and Globes. They are isomorphic, because the roles of the fixed and the moveable components are switched. Another example of two puzzles, where the roles of fixed and moveable components are switched, are the Chinese Rings and the Chinese String. In this dissertation the connection between the Chinese Rings and the Chinese String are analysed to find out in what way we have a mathematical isomorphy between these two puzzles.
At first an introduction to the Chinese Rings is made. Thereby some properties of the corresponding state graph and the relationship between Gros code and Gray code are stated. Furthermore the sequence of moves of the optimal solution (with a countably infinite number of rings), which is also called Gros sequence, is considered and we see, that this sequence is equal to the greedy square-free sequence.
After that a mechanical solution of the Chinese String is presented. With an additional imaginary line movements are defined and these movements are used to state a recursive solving algorithm for an arbitrary number of rings. Based on this algorithm some properties of the solution are considered. One ascertains that this solution can be used for solving the Chinese Rings, because their sequences of moves are identical. Apart from the algorithm described in detail another solving strategy is given, which has the same complexity. Therefore in general the optimal solution is not unique.
Then an analogy between the Chinese Rings and the Chinese String is worked out. In doing so for the Chinese String discrete states are defined and changes of regular states by moves are considered. In the course of this it is proved that the regular states and moves are isomorphic to the states and moves of the Chinese Rings. So the explicitly approached algorithm is identical with the optimal solving algorithm of the Chinese Rings.
Following this, several modifications of the Chinese String are considered and it is verified that they have no influence on the complexity. For this purpose the rings are untangled as Kauffman[3] already did. Then according to Przytycki and Sikora[4] this construction is embedded into a 3-sphere in a natural way and the fundamental group of the complement of the resulting handlebody is studied. Depending on the modification different fundamental groups are given, but by viewing the introduced imaginary line as an element of its respective fundamental group it is shown that the modifications have no influence on the complexity.
Finally Kauffman's Ring conjecture from 1996 is considered. It says that for an arbitrary initial

[3] Louis H. Kauffman. Tangle Complexity and the Topology of the Chinese Rings. In: Jill P. Mesirov, Klaus Schulten und De Witt Summers (editors), Mathematical Approaches to Biomolecular Structure and Dynamics, Springer, New York, NY, 1996, pages 1–10.

[4] Józef H. Przytycki und Adam S. Sikora. Topological Insights from the Chinese Rings. Proceedings of the American Mathematical Society, 130: 893–902, 2001.

situation the topological and the mechanical exchange numbers are equal. Przytycki and Sikora already proved this statement for the canonical initial situation in a special version of the Chinese String. An alternative combinatorial proof is given that shows the complexity to be $2^{n-1}$. For this a closer look at the shape of the imaginary line is done. One can see that the shape of the line after untangling a ring grows out from two connected copies of the shape of the line prior to untangling this ring. Based on this fact a theorem is proved, which says that for an arbitrary initial and final state the minimal number of required crossings (in the Chinese String) is equal to the minimal number of moves of ring 1 (in the Chinese Rings). This verifies directly the validity of the statement of the Ring conjecture.
We obtain that the (fixed) Chinese String and the Chinese Rings are a pair of isomorphic mechanical puzzles and so may be considered for further studies in the field of cognitive psychology.

# Contents

# Chapter 0

# Introduction

At the beginning of the 16th century in *De Viribus Quantitatis*[1] Luca Pacioli wrote[2]:

> Capitolo CVII: Do cavare et mettere una strenghetta salda in al quanti anelli saldi difficil caso.
> Molti hanno certa quantita de annelli saldi messi in certi gambi, quali asettano in una steccha piatta de legno o altro metallo, la quali gambi sonno commo chiuodi o vero aguti ognuno ficto nel suo foro alla fila in ditta stecha, in modo chel capo loro tenga, ch'non posino uscire, et la ponta de ognuno revoltata aluncino ch'tenga ognuna uno anello et ognuno delli anelli ha la ponta de laguto dentro. Et poi, in ditta ponto fermato la nello el chiuodo non po ne avanze per lo capo ne adrieto per la nello ch'sta in la ponta revoltata dentro laltro anello. Et questi anelli possano essere piu de tre quanti te piaci. Ma manco non per chel giuoco non seria bello, et sonno situati uno in laltro commo vedi qui in figura, salvo chel primo diloro non ha niuno dentro.

This old Italian text describes a puzzle, which today is commonly known as the *Chinese Rings*.

## 0.1 Presentation of the Chinese Rings

The *Chinese Rings* (abbreviated as *CR*) are an old disentanglement puzzle, which consists of two components - the first is a loop, normally made of metal, with a, usually wooden, handle on it and the second is a structure containing, usually nine, linked rings of metal (see Figure 0.1). The rings are linked in a way, such that only the rightmost ring and the ring left to the rightmost ring on the loop can be moved. In the initial state all rings are on the loop and it is the aim to get them off the loop by using the mentioned moves. For reasons of simplicity the rings are numbered in ascending order from right to left.
As far as we know today, a solution of the CR was first described by Luca Pacioli in *De Viribus Quantitatis* ([Pac]) at the beginning of the 16th century[3]. In his treatise in the initial state the CR all rings were off the loop and it was the aim to get all rings onto the loop. He presents a

[1] [Pac]
[2] This transcription can be found in [HH], Appendix 1.
[3] see [DH], Section 1.1.

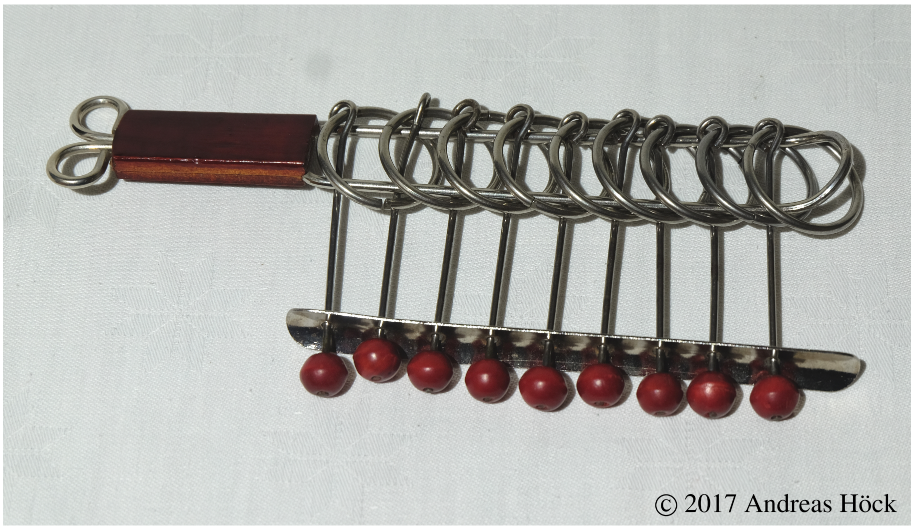


Figure 0.1: The Chinese Rings with nine rings, all on the loop

solution of the CR with seven rings and he explicitly left it to the reader to continue for more rings.
In *De Subtilitate* Gerolamo Cardano describes his solution of the CR with seven rings (see [Car], p.492f.), in 1551[4]. For that he used an algorithm, where the rings 1 and 2 are moved simultaneously if possible and therefore he has a different counting of the number of moves. Since there are less moves than the usual counting has, this solution is also called *accelerated Chinese Rings*. Due to this work, the CR are also known as *Cardano's Rings*[5].
In 1693, in *De Algebra Tractatus*([Wal2][6]) John Wallis used another way to count moves in solving the CR with nine rings. For putting a ring onto the loop he needed two moves, namely lifting the ring above the loop and pushing the loop through the ring. For putting a ring off the loop, first he pulls the loop out of the ring and then the ring can be slipped through it.
The last early treatise about CR to be mentioned here is *Théorie du Baguenodier* by Louis Gros from 1872. In this treatise the author explains the solution as well as the etymology of the *baguenaudier* (the French expression for the Chinese Rings; see [Gro], p.1-5). The counting of moves he used is commonly accepted today.

## 0.2 PRESENTATION OF THE CHINSESE STRING

The *Chinese String* (abbreviated as *CS*) is also a disentanglement puzzle which contains a bottom plate, pegs, rings and a closed rope. The bottom plate is rigid and there are a certain, usually odd, number of rigid pegs in a straight line of ascending length from left to right, which are fixed on the bottom plate. On the tip of each peg a rigid ring is attached, oriented in a way, such that the next longer peg runs through it. The pegs with the rings are numerated in ascending order, beginning with the longest. Usually the bottom plate is made of wood and the pegs and rings

[4] see [HH], Introduction
[5] see [HH], Introduction
[6] Contrary to the Latin version the English version does not contain a section about the CR.

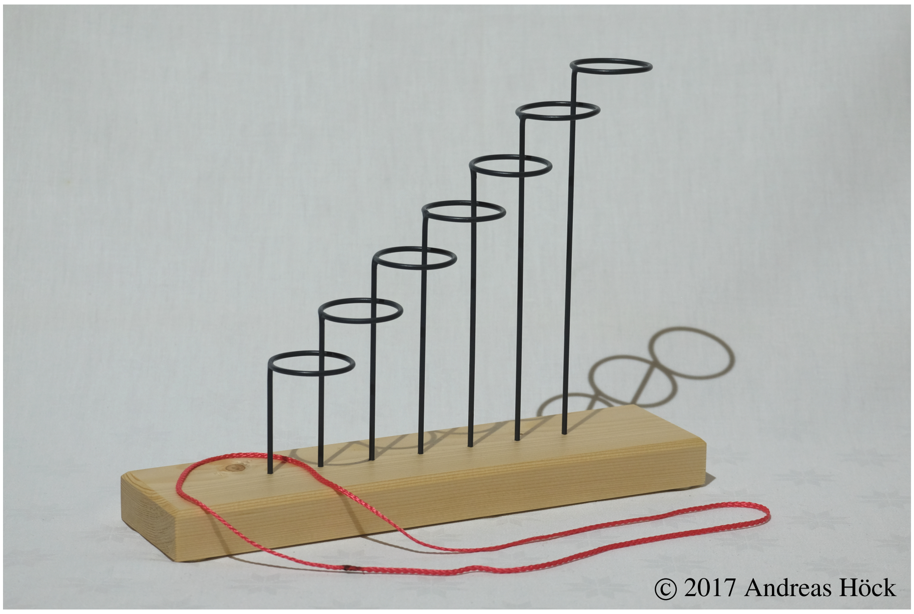


Figure 0.2: The Chinese String with seven rings in the usual initial state

are made of metal. These parts generate the *frame* of the the CS. The CS also contains a flexible closed rope, which is tangled with the frame of the CS in the initial situation. In the usual initial situation the rope runs around the shortest peg exactly once (see Figure 0.2) and it is the aim to separate the rope from the frame. But there can also be given any other initial situation, where the rope is tangled with the frame.
To separate the rope from the frame one has to lead the rope through rings in a special order. The resulting interactions between the rings and the rope are called movements. Some consecutive movements are getting summarized to a move, because they take an effect on the state of the CS. Counting movements in the CS is similar to counting moves in the CR in the way Wallis did and counting moves in the CS is similar to counting moves in the CR in the way Gros did.

There also exist some modifications of the puzzle and two of them are worth mentioning. The first is that the ring on the longest peg is substituted by a ball and the second that the rope is fixed in a point at the bottom plate. In [PS] Przytycki and Sikora considered a version of the puzzle, where the ring on the longest peg is substituted by a ball *and* the rope is fixed at a point at the bottom plate. We call this version *fixed substituted CS*. The goal of all modified versions is usually the same, namely to separate the rope from the arrangements of pegs and rings.

In 1977 Shephard [She] considered a very similar puzzle. In this treatise the author presents a short solution of this puzzle and also mentions that his solution reminds one of Gros's solution for the CR.

## 0.3 The Chinese Rings and the Chinese String

After presenting the CR and the CS at first sight one may be uncertain about the exact relationship between these two puzzles. It is the aim of this thesis to reveal some similarities and differences.

We will take a look at some basic facts about the CR in the following chapter. For this purpose we will also consider the graph corresponding to the CR and use it for determining the *mechanical exchange number* of a state, which is the required number of moves of ring 1 to get all rings off the loop.

After taking a look at the basics of the CR we will come to the CS. In Chapter 2 we will take a look at some interactions of the rope and the frame of the CS and the used movements will be introduced. After that, by considering explicit solutions of the CS with a small number of rings, a recursive algorithm for solving the CS with an arbitrary number of rings will be shown. In doing this we will get two formulas, which allow us to determine the connection between movement and position of the given algorithm. With introducing *moves* it will be shown that the finite sequence of moves the algorithm uses is equal to the finite sequence of moves of the optimal solution of the CR. In analogy to the mechanical exchange number one defines the *topological exchange number*, which is needed for questions about optimality. At the end of this chapter an alternative solving strategy will be presented showing that the given algorithm is not a unique (optimal) solution of the CR. That is why solving the CS and solving the CR are obviously not identical problems.

In Chapter 3 some restrictions on the rope are introduced, such that the two problems become equivalent. In order to work out the necessary restrictions, the state of the CS will be discretized in a way that the solving algorithm of the CR also solves the CS. All passed states are called regular states and we will work out the effect of a movement or move on the state of the CS. If just regular states are allowed, one gets the same rules as for the CR. So in this case the problems are equivalent.

After that, in Chapter 4, we take a look at the modifications on the CS mentioned in 0.2. We need some basics of algebraic topology to show that they all have the same complexity. On the first look this might seem to be obvious, but in fact one has to do some work for that result.

In the penultimate chapter we will prove Kauffman's *Ring conjecture* from 1996, an interesting link between the CR and the CS. It says that for an arbitrary state of the CR and its analogous state of the CS the mechanical and the topological exchange numbers are equal. Kauffman published this conjecture in [Kau], p.8 to find a class of exchange problems with exponentially growing complexity. Based on [Mac] he assumed importance of this phenomenon in untangling biological molecules or polymer chains where entanglements occur[7]. Przytycki and Sikora already proved that the Ring conjecture holds in the very special case of the usual initial situation in the fixed substituted CS. To prove the validity of the Ring conjecture in general we first untangle the pegs and rings and consider the shape of the imaginary line. Therefore we obtain that for solving the CS with $n \in \mathbb{N}$ rings and usual initial state $2^{n-1}$ crossings are required. Using this information we show that for arbitrary initial and final states the minimal number of moves of ring 1 is equal to the minimal number of the required crossings.

At the end we take a brief look at an application in cognitive psychology. For research in the field of cognitive psychology some experiments were performed using the *Tower of Hanoi* and *Monsters and Globes*, a pair of isomorphic[8] puzzles, where the roles of rigid and moveable components are switched. Some results of Hayes and Simon are displayed and it might be an interesting question in what extent the said results also occur considering the CR and CS, which

[7] see [Kau], p.9f.

[8] in the psychological sense

are also, as we will see, isomorphic puzzles where the roles of rigid and moveable components are switched.

# Chapter 1

# Introduction to the Chinese Rings

Now let us take a closer look at the CR. At the beginning all rings are on the loop and the aim is to get them off the loop. To reach this goal there are exactly two possible moves[1] (see Figure 0.1):

- the rightmost ring can be moved (type 0) or
- the ring left to the rightmost ring on the loop can be moved (type 1).

In any situation the CR with $n \in \mathbb{N}_0$ rings can be described by a unique $n$-tuple $s = s_n \ldots s_1 \in B^n = \{0,1\}^n$, where $s_k$ represents the state of ring $k \in [n] = \{1, 2, ..., n\}$ (numbered from right to left). A ring $k \in [n]$ has state $s_k = 1$ if it is on the loop and state $s_k = 0$ if it is off the loop. Therefore $s_n \ldots s_2 s_1$ is transformed into $s_n \ldots s_2 (1 - s_1)$ by a move of type 0 and $s_n \ldots s_k 10^{k-2}$ into $s_n \ldots (1 - s_k) 10^{k-2}$ by a move of type 1 ($k \in [n] \setminus \{1\}$). This shows that two moves in a row of the same type neutralize each other. Since we have only two options and one of these cancels the previous move, the optimal solution just depends on the first move. If the number of rings is odd, one has to start with a move of ring 1 and one has to start with ring 2 otherwise (see [HKMP], Proposition 1.6).

## 1.1 The graph according to the rules of the Chinese Rings

One can visualize the CR with $n \in \mathbb{N}_0$ rings by a labeled graph $R^n$. The set of vertices $V$ is given by all possible states, so $V = B^n$. Two vertices $s, t \in V$ are connected by an edge if and only if there is a legal move to get from $s$ to $t$. Using this we can define the graph $R^n$ formally.

**Definition 1.1.** The graph $R^0$ has just one vertex, labeled by the empty word. For $n \in \mathbb{N}$ the graph $R^n$ according to the rules of the CR is defined by $R^n = (V, E)$, where[2]

$$V = B^n \quad \text{and} \quad E = \left\{\left\{\underline{s}0, \underline{s}1\right\}, \underline{s} \in B^{n-1}\right\} \cup \left\{\left\{\underline{s}010^{r-2}, \underline{s}110^{r-2}\right\}, r \in [n] \setminus \{1\}, \underline{s} \in B^{n-r}\right\}.$$

For $n \in \mathbb{N}$ one can show that the graph $R^n$ is a connected path graph of length $2^n - 1$ and the two vertices with degree 1 are $\alpha^{(n)} := 0^n$ and $\omega^{(n)} := 10^{n-1}$ ([HKMP], p.55; see Figure 1.1 for the case $n = 4$). Since $R^0$ has just one vertex, the empty word $e$, we have $\alpha^{(0)} = e = \omega^{(0)}$. In [HKMP], Proposition 1.6 it is also shown that the CR with $n \in \mathbb{N}_0$ rings have a unique optimal

[1] see [HKMP], p.53f.

[2] from [HKMP], Remark 1.3

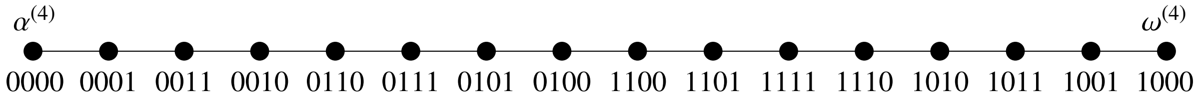


Figure 1.1: The labeled graph $R^4$

solution of length $\left\lceil \frac{2}{3}(2^n - 1) \right\rceil$. The sequence given by the length of the optimal solution is called *Lichtenberg sequence*[3].

Due to the fact that $R^n$ is a connected path graph and its length is exactly 1 smaller than the amount of all possible states, we know that every state must be a vertex on the path and therefore every state can be reached.
For an example of a property of the state graph let us look at a possible *coloring* of $R^n$. Considering Definition 1.1 yields that two adjacent vertices just differ by one bit. So a *vertex coloring* of $R^n$ is given by the function

$$\begin{aligned} V &\to B \\ s_n \dots s_1 &\mapsto \left( \sum_{k=1}^{n} s_k \right) \bmod 2. \end{aligned}$$

For this reason $R^n$ is bipartite. Since in the optimal solution the types of moves alternate, an *edge coloring* is given by the type of move between its vertices.

A more interesting property of the graph $R^n$ is the *distance* $\mathrm{d}(s,t)$ between two vertices $s, t \in V$, which is given by the length of the shortest path from $s$ to $t$. This gives the minimal number of moves required to get from $s$ to $t$. Due to the fact that the diameter of $R^n$ is $2^n - 1$ and the two vertices with degree 1 are $\alpha^{(n)}$ and $\omega^{(n)}$, we get $\mathrm{d}\left(\alpha^{(n)}, \omega^{(n)}\right) = 2^n - 1$. For any vertices $s, t \in V$ we have $\mathrm{d}(s,t) = |\mathrm{d}(s) - \mathrm{d}(t)|$, where $\mathrm{d}(s) := \mathrm{d}\left(s, \alpha^{(n)}\right)$ (see [HKMP], p.58). Since $\mathrm{d}\left(\alpha^{(n)}\right) = 0$ and two adjacent vertices just differ by exactly one bit we have $\mathrm{d}(s) \equiv \left( \sum_{k=1}^{n} s_k \right) \bmod 2$.

To get from $s$ to $t$ on the shortest path one should start with a move of type[4]

$$\begin{cases} \mathrm{d}(s) \bmod 2 = \left( \sum_{k=1}^{n} s_k \right) \bmod 2, & \text{if } \mathrm{d}(s) < \mathrm{d}(t) \\ 1 - \mathrm{d}(s) \bmod 2 = 1 - \left( \sum_{k=1}^{n} s_k \right) \bmod 2, & \text{if } \mathrm{d}(s) > \mathrm{d}(t). \end{cases} \tag{1.1}$$

**Definition 1.2.** In the CR with $n \in \mathbb{N}$ rings the minimal number of moves of type 0 to get from $s \in B^n$ to $t \in B^n$ is denoted by $E_{mech}[s,t]$ and $E_{mech}(s) := E_{mech}\left[s, \alpha^{(n)}\right]$ is called *mechanical exchange number* of $s$.

[3] see [Hin2], p.5
[4] see [HKMP], Remark 1.9

**Lemma 1.3.** *For any state $s \in B^n$ of the CR with $n \in \mathbb{N}$ rings the mechanical exchange number is given by:*

$$E_{mech}(s) = \begin{cases} \frac{1}{2}\,\mathrm{d}(s), & \text{if } \mathrm{d}(s) \equiv 0 \bmod 2 \\ \frac{1}{2}\,(\mathrm{d}(s)+1), & \text{if } \mathrm{d}(s) \equiv 1 \bmod 2. \end{cases}$$

*Proof.* Let $s \in B^n$ and $\mathrm{d}(s) \equiv 0 \bmod 2$. Since every second move is a move of type 0 and $\mathrm{d}(s)$ is even we obtain the statement.
Now let $s \in B^n$ and $\mathrm{d}(s) \equiv 1 \bmod 2$. By (1.1) we get the best first move being a move of type $1 - \mathrm{d}(s) \bmod 2 = 0$. After this move the remaining sequence of moves is even and starts with a move of type 1. The same argument as in the first case yields

$$1 + \frac{\mathrm{d}(s) - 1}{2} = \frac{\mathrm{d}(s) + 1}{2}$$

moves of type 0. □

By this lemma we obtain, that for $n \in \mathbb{N}$ the mechanical exchange number of $\omega^{(n)}$ is $2^{n-1}$.

*Remark.* Considering Figure 1.1 we can observe that the graph $R^n$, $n \in \mathbb{N}$ is reflected in the middle between $0\omega^{(n-1)}$ and $1\omega^{(n-1)}$ (here: between 0100 and 1100) and the states in the left half just differ from the states in the right half in the first bit (see [HKMP], p.55). For this reason in $R^n$, $n \in \mathbb{N}$ we have

$$\mathrm{d}\left(0\omega^{(n-1)}, \omega^{(n)}\right) = 2^{n-1} \quad \text{and}$$
$$\mathrm{d}\left(1\underline{s}\right) = \mathrm{d}\left(1\underline{s}, \alpha^{(n)}\right) = \mathrm{d}\left(0\underline{s}, \omega^{(n)}\right) = \mathrm{d}\left(0\underline{s}, 0\omega^{(n-1)}\right) + 2^{n-1} \tag{1.2}$$

for $\underline{s} \in B^{n-1}$.

## 1.2 The Gros code and the Gray code

We introduced the distance between two states without any indication how to calculate $\mathrm{d}(s)$ for any $s \in B^n$. The following automaton describes a way to determine the distance between a state $s \in B^n$ and $\alpha^{(n)}$ (see Figure 1.2; [HKMP], p.58). The automaton consists of two states A and B.

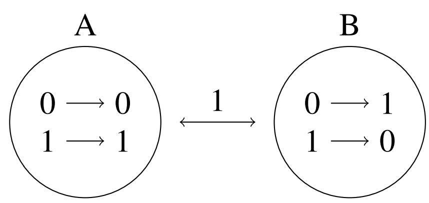


Figure 1.2: Automaton for the Gros code

In state A the input of a bit $s_k \in B$ leads to printing $s_k$ and switching to state B if the bit was 1. In state B the input of a bit $s_k$ leads to printing $1 - s_k$ and switching to state A if the bit was 1.

**Lemma 1.4.** *Let $s = s_n \dots s_1 \in B^n$ be a state of the CR with $n \in \mathbb{N}$ rings. Entering the bits of $s$, from left to right, in the automaton given in Figure 1.2, beginning with $s_n$ in state A, the output, read from left to right, is the binary representation of the value of* $\mathrm{d}(s)$. *Beginning in state B instead of state A gives the binary representation of the length of the path from $s$ to $\omega^{(n)}$.*

*Proof.* [5] Since the state graph is a path graph and the diameter is $2^n - 1$, the path from $s$ to $\omega^{(n)}$ has length $2^n - 1 - \mathrm{d}(s)$.
We prove the statement of the lemma by induction on $n$. The statement obviously holds for $n = 1$. We assume that the statement holds for some $n \in \mathbb{N}$. Let $s \in B^{n+1}$.
If $s = 0\underline{s}$ for some $\underline{s} \in B^n$, then starting in state A leaves the first bit unchanged and one has to continue with the first bit of $\underline{s}$ in state A. Together with the induction hypothesis we obtain $\mathrm{d}(s) = \mathrm{d}\left(\underline{s}\right)$. Starting in state B leads to the first bit being 1 and one continues with the first bit of $\underline{s}$ in state B. By induction hypothesis the automaton returns $2^n + 2^n - 1 - \mathrm{d}\left(\underline{s}\right) = 2^{n+1} - 1 - \mathrm{d}(s)$.
If $s = 1\underline{s}$ for some $\underline{s} \in B^n$, then starting in state A leaves the first bit unchanged and one continues with the first bit of $\underline{s}$ in state B. By induction hypothesis and (1.2) the automaton returns $2^n + \mathrm{d}\left(\underline{s}, \omega^{(n)}\right) = \mathrm{d}(s)$. Starting in state B leads to the first bit being 0 and one continues with the first bit of $\underline{s}$ in state A. By induction hypothesis and (1.2) the automaton returns $\mathrm{d}\left(\underline{s}\right) = 2^n - 1 - \mathrm{d}\left(\underline{s}, \omega^{(n+1)}\right) = 2^n - 1 - (\mathrm{d}(s) - 2^n) = 2^{n+1} - 1 - \mathrm{d}(s)$. □

*Remark.* Let $s \in B^n$ be a state of the CR with $n \in \mathbb{N}$ rings. If the number of bits with value 1 left to $s_i$, $i \in [n]$, is even, then $s_i = br(\mathrm{d}(s))_i$ and $s_i = 1 - br(\mathrm{d}(s))_i$ otherwise, where $br(\mathrm{d}(s))$ is the binary representation of $\mathrm{d}(s)$. This follows immediately from the fact that the state of the automaton is switched if and only if a bit of value 1 is entered.

Another way to calculate the distance from $s \in B^n$ to the final state $\alpha^{(n)}$ of the CR with $n \in \mathbb{N}$ rings is to calculate

$$\bigoplus_{k=0}^{n-1} \mathrm{lsr}(s, k),$$

where $\mathrm{lsr}(s, k)$ is the bit string $s$ logical shifted to the right by $k$ positions and $a \oplus b$ is the binary digital sum of two bit strings $a$ and $b$.[6] For a better understanding of these binary operations, we take a look at the following example.

*Example.* Let $s = 11010101$ be a binary string of length $n = 8$. Then we have $\mathrm{lsr}(s, 1) = 01101010$ and $\mathrm{lsr}(s, 2) = 00110101$ and so on. The binary digital sum $\mathrm{lsr}(s, 0) \oplus \mathrm{lsr}(s, 1) = 1011111$ is given by

$$\begin{array}{c|cccccccc} & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \oplus & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} .$$

[5] see [HKMP], p.58
[6] see [BCG], p.860f.

So from

$$\begin{array}{ccccccccc}
 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\oplus & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}$$

we get $\bigoplus_{k=0}^{7} \operatorname{lsr}(s,k) = 10011001$.

**Lemma 1.5.** *Let $s \in B^n$ be a state of the CR with $n \in \mathbb{N}$ rings. Then $\bigoplus_{k=0}^{n-1} \operatorname{lsr}(s,k)$ is the binary representation of the value of* $\mathrm{d}(s)$.

*Proof.* Let $s_1, \ldots, s_n \in B$ with $s = s_n \ldots s_1$ and set $\tilde{s} := \bigoplus_{k=0}^{n-1} \operatorname{lsr}(s,k)$. For each $i \in [n]$ we have

$$\tilde{s}_i = \left(\sum_{k=0}^{n-i} s_{n-k}\right) \bmod 2 = \left(s_i + \sum_{k=0}^{n-i-1} s_{n-k}\right) \bmod 2 = \begin{cases} s_i, & \text{if } \sum_{k=0}^{n-i-1} s_{n-k} \equiv 0 \bmod 2 \\ 1 - s_i, & \text{if } \sum_{k=0}^{n-i-1} s_{n-k} \equiv 1 \bmod 2. \end{cases}$$

Since $\sum_{k=0}^{n-i-1} s_{n-k}$ is the number of bits of value 1 left to $s_i$, the statement follows by Lemma 1.4 and the remark on page 10. □

After finding a possibility to calculate the distance between two states of the CR, we try to find a way to determine a state with a given distance to $\alpha^{(n)}$. For this purpose we consider the automaton given in Figure 1.3. There are two states A and B, where in state A the input of a bit

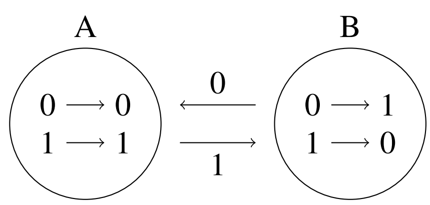


Figure 1.3: Automaton for the Gray code

$s_k$ leads to printing $s_k$ and switching to state B if the bit was 1. In state B the input of a bit $s_k$ leads to printing $1 - s_k$ and switching to state A if the bit was 0. Due to the switching rules, in this automaton a bit gets changed if and only if the previous bit has value 1.

**Lemma 1.6.** *Let $s \in B^n$ be a binary string of length $n \in \mathbb{N}$. After entering $s$, the output of the automaton given in Figure 1.3 is equal to $s \oplus \mathrm{lsr}(s, 1)$.*

*Proof.* Let $s_1, \dots, s_n \in B$ with $s = s_n \dots s_1$ and set $\tilde{s} := s \oplus \mathrm{lsr}(s, 1)$. For $i \in [n]$ we have

$$\tilde{s}_i = s_i + s_{i+1} \bmod 2 = \begin{cases} s_i, & \text{if } s_{i+1} = 0 \\ 1 - s_i, & \text{if } s_{i+1} = 1. \end{cases}$$

So $s_i$ gets changed if and only if $s_{i+1} = 1$. □

**Lemma 1.7.** *Entering the binary representation of a number $d \in [2^n]_0$, $n \in \mathbb{N}$, the output of the automaton given in Figure 1.3 is a binary string, which is the state $s$ of the CR with $n$ rings with distance $d$ to $\alpha^{(n)}$.*

*Proof.* We prove the statement by showing that the map

$$\begin{aligned} st : [2^n]_0 &\to B^n \\ d &\mapsto br(d) \oplus \mathrm{lsr}(br(d), 1) \end{aligned}$$

is inverse to the map

$$\begin{aligned} \mathrm{d} : B^n &\to [2^n]_0 \,, \\ s &\mapsto \mathrm{d}(s) = \left( \bigoplus_{k=0}^{n-1} \mathrm{lsr}(s, k) \right)_2 \end{aligned}$$

where $br(d)$ is the binary representation of $d$.
Let $s \in B^n$ be a state of the CR with $n$ rings. Since $a \oplus a = 0^n$ for all bit strings $a$ of length $n$, we get

$$\begin{aligned} (st \circ \mathrm{d})(s) = st(\mathrm{d}(s)) = st\left( \left( \bigoplus_{k=0}^{n-1} \mathrm{lsr}(s, k) \right)_2 \right) &= \left( \bigoplus_{k=0}^{n-1} \mathrm{lsr}(s, k) \right) \oplus \mathrm{lsr}\left( \bigoplus_{k=0}^{n-1} \mathrm{lsr}(s, k), 1 \right) \\ &= \left( \bigoplus_{k=0}^{n-1} \mathrm{lsr}(s, k) \right) \oplus \left( \bigoplus_{k=0}^{n-2} \mathrm{lsr}(s, k+1) \right) \\ &= s \oplus \left( \bigoplus_{k=1}^{n-1} \mathrm{lsr}(s, k) \right) \oplus \left( \bigoplus_{k=1}^{n-1} \mathrm{lsr}(s, k) \right) = s. \end{aligned}$$

For $d \in [2^n]_0$ we have

$$\begin{aligned} (\mathrm{d} \circ st)(d) = \mathrm{d}(st(d)) = \mathrm{d}(br(d) \oplus \mathrm{lsr}(br(d), 1)) &= \left( \bigoplus_{k=0}^{n-1} \mathrm{lsr}(br(d) \oplus \mathrm{lsr}(br(d), 1), k) \right)_2 \\ &= \left( \left( \bigoplus_{k=0}^{n-1} \mathrm{lsr}(br(d), k) \right) \oplus \left( \bigoplus_{k=0}^{n-2} \mathrm{lsr}(br(d), k+1) \right) \right)_2 \\ &= \left( br(d) \oplus \left( \bigoplus_{k=1}^{n-1} \mathrm{lsr}(br(d), k) \right) \oplus \left( \bigoplus_{k=1}^{n-1} \mathrm{lsr}(br(d), k) \right) \right)_2 = d. \end{aligned}$$

As a consequence we obtain

$$st \circ \mathrm{d} = \mathrm{id}_{B^n} \quad \text{and} \quad \mathrm{d} \circ st = \mathrm{id}_{[2^n]_0}\,. \qquad \square$$

Since the automaton given in Figure 1.2 supplies the bijective map $\mathrm{d} : B^n \to [2^n]_0$, $s \mapsto \mathrm{d}(s)$, it is also a code (injective map). This code is called *Gros code*, named after Louis Gros, who published, as already mentioned in Chapter 0, his theory about the CR in [Gro] in 1872.
Referring to the Hamming Weight, which is the number of symbols of a string different from 0, the image of the Gros Code is also called *Gros Weight*. Since F. Gray holds the patent for the inverse map $st$ since 1953,[7] this code is also known as *Gray Code*. In the paragraph prior to the remark on page 9 the graph $R^n$, $n \in \mathbb{N}$ is reflected in the middle. For that reason the Gray Code is also called *reflected binary code*[8]. To demonstrate the connection between Gros Code and Gray Code one can take a look at Figure 1.4, where $P_{2^3}$ is the path graph with $2^3$ vertices, binarily labeled in ascending order.

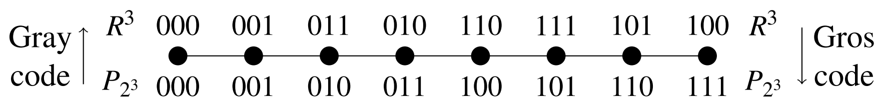


Figure 1.4: The connection between Gros Code and Gray Code

## 1.3 The sequence of moves

After considering the state graph of the CR, we take a brief look at the finite sequence of moves to get from $\alpha^{(n)}$ to a state $s \in B^n$ ($n \in \mathbb{N}$). For this purpose we assume that we have a CR with a countably infinite number of rings, where all rings are off the bar. The sequence of moves to get the rings on the bar is called *Gros sequence*[9]. To determine the $n$-th element $g_n$ of the Gros sequence one has the following recurrence:

$$g_n = \begin{cases} 1, & \text{if } n \equiv 1 \bmod 2 \\ g_{n/2} + 1, & \text{if } n \equiv 0 \bmod 2. \end{cases} \tag{1.3}$$

This recurrence is proven in [HKMP], Proposition 1.10 and we get:

**Lemma 1.8.** *For $n \in \mathbb{N}$ the $n$-th element of the Gros sequence is given by*

$$g_n = k \quad \Leftrightarrow \quad n \equiv 2^{k-1} \bmod 2^k.$$

*Proof.* We prove the lemma by induction on $k$. Directly from the recurrence we obtain that the statement holds for $k = 1$. Now assume that the equivalence $g_n = k \Leftrightarrow n \equiv 2^{k-1} \bmod 2^k$ holds

[7] US-Patent No. 2632058
[8] see [HKMP], p.59
[9] see [HKMP], p.61

for some $k$. By (1.3) we have

$$g_n = k+1 \quad\Leftrightarrow\quad g_{n/2}+1 = k+1 \quad\Leftrightarrow\quad g_{n/2} = k \quad\Leftrightarrow\quad \frac{n}{2} \equiv 2^{k-1} \bmod 2^k$$
$$\Leftrightarrow\quad n \equiv 2^k \bmod 2^{k+1},$$

where in the first equivalence it is used that $n \equiv 0 \bmod 2$ if and only if $g_n \neq 1$. □

A finite subsequence $\underline{s}$ of consecutive elements of a sequence $s$ is called a *square*, if it is of the form $s's'$, where $s'$ is non-empty. If a sequence does not contain any square it is called *square-free*. The *greedy square-free sequence* $(a_n)_{n\in\mathbb{N}}$ is the sequence of natural numbers, where each element $a_n$ is given by the smallest natural number not causing a square in $(a_k)_{k<n}$.

**Lemma 1.9.** *For $k \in \mathbb{N}$, we have $a_n = k$ if and only if $n$ is of the form $2^{k-1} \cdot \nu$, where $\nu$ is odd.*

*Proof.* We prove the statement by complete induction on $k$. Let $k = 1$ and consider the sequence $(a_n)_{n\in\mathbb{N}}$. On account of the greediness the first element of the sequence $(a_n)_{n\in\mathbb{N}}$ is 1. Since $(a_n)_{n\in\mathbb{N}}$ is square-free two adjacent elements must be different. So only every second element may be equal and in connection with the greediness we know that these elements are 1.
Now assume that for all $l \leq k$ we have $a_n = l$ if and only if $n$ is of the form $2^{l-1} \cdot \nu$, where $\nu$ is odd. We consider the subsequence $(a_{2^k\cdot n})_{n\in\mathbb{N}}$. If any element of this sequence is $l \leq k$, we obtain a contradiction to the induction hypothesis. Therefore the smallest element of this subsequence is $k+1$ and if two adjacent elements are equal we get a square. In consequence of the greediness we get that every element in an odd position of this sequence is $k+1$. For this reason we obtain $a_n = k+1$ if and only if $n$ is of the form $2^k \cdot \nu$, where $\nu$ is odd. □

**Theorem 1.10.** *The Gros sequence $(g_n)_{n\in\mathbb{N}}$ and the greedy square-free sequence $(a_n)_{n\in\mathbb{N}}$ are identical.*

*Proof.* By Lemma 1.8 it is sufficient to show

$$a_n = k \quad\Leftrightarrow\quad n \equiv 2^{k-1} \bmod 2^k.$$

From the previous lemma we know that $a_n = k$ if and only if $n$ is of the form $2^{k-1} \cdot \nu$, where $\nu$ is odd. So we get

$$a_n = k \quad\Leftrightarrow\quad n \in \left\{2^{k-1} \cdot \nu \mid \nu \in \mathbb{N} \wedge \nu \equiv 1 \bmod 2\right\} \quad\Leftrightarrow\quad n \in \left\{2^{k-1} + l \cdot 2^k \mid l \in \mathbb{N}_0\right\}$$
$$\Leftrightarrow\quad n \equiv 2^{k-1} \bmod 2^k.$$ □

# CHAPTER 2

# MECHANICAL SOLUTION OF THE CHINESE STRING

In this chapter a mechanical solution of the CS is illustrated, where we do not touch the question of optimality. An answer to this question is given in Chapter 5. Depending on the respective version of the CS there may be other strategies to solve the problem, but only the solving algorithm working on every version is illustrated explicitly. At the end of this chapter we will have a brief look at other possible strategies.

## 2.1 SOME BASICS FOR SOLVING THE CHINESE STRING IN A MECHANICAL WAY

In this section we consider the CS with at least two rings. The CS with fewer rings will be treated in the next section.
Before starting one needs some basics and therefore has to look at the CS with $n \in \mathbb{N}$ rings in the form Kauffman called in his publication [Kau] "an equivalent formulation of the Chinese Rings"[1]. Here we have the original version, with ring 1 and the closed rope is not fixed at any point.
To solve the CS we mark the leftmost ($L$) and the rightmost ($R$) point on the rope imaginarily and we add an imaginary line $A$ from the bottom plate to ring 1 in the same way Kauffman did[2] (see Figure 2.1). In order to describe a solution we define movements.

**Definition 2.1.** A *movement* is an interaction of the rope with the frame, where either the rope is led by a point through a ring or the rope crosses the imaginary line $A$, but not both. The latter interaction is also called *crossing*.

One possible strategy is to use movements to lead the marked point $R$ ring by ring to the right. In doing so we try to avoid to wrap around a ring or peg. This strategy is compatible with all versions mentioned in Chapter 0. The algorithm, presented in this chapter, only needs two types of movements:

a) For $r \in [n]$ the rope is moved at $R$ over the rings $r$ to 1 or 1 to $r$, respectively, such that the part closer to the front is led in front of the rings and the part closer to the rear is led behind the rings symmetrically and the imaginary line $A$ gets crossed once (see Figure 2.2). Since this type of movement does not depend on $r$ it is denoted by $b_1$ and $b_1^{-1}$, respectively and because $A$ gets crossed and we call it *crossing*.

[1] [Kau], p.6
[2] see [Kau], p.8

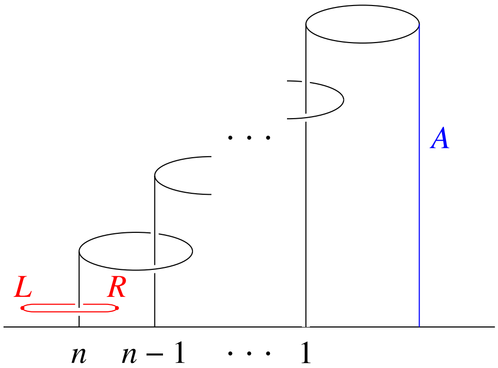


Figure 2.1: The CS containing $n$ pegs with the marked points and imaginary line $A$

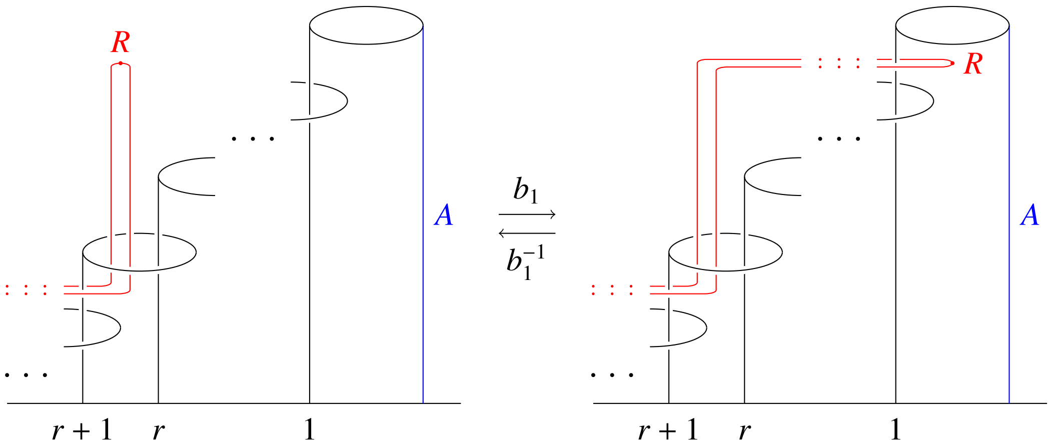


Figure 2.2: The movements $b_1$ and $b_1^{-1}$ (crossings)

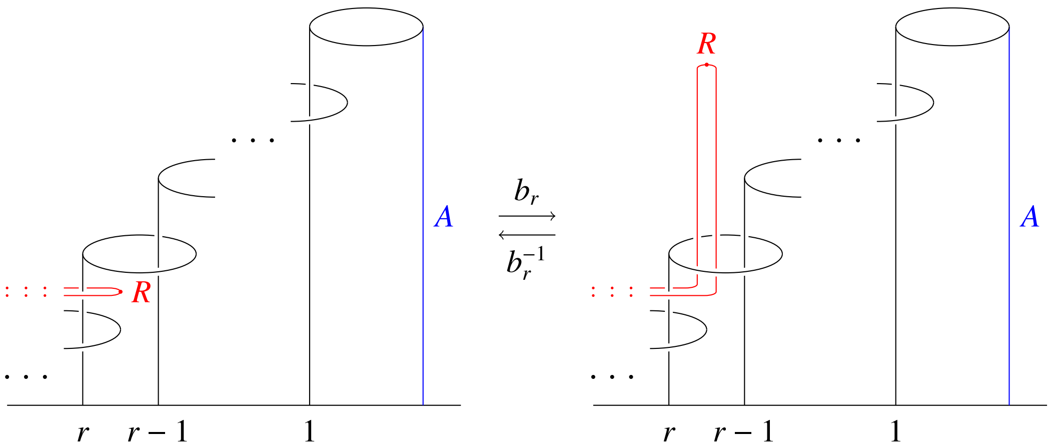


Figure 2.3: The movements $b_r$ and $b_r^{-1}$

b) The rope is moved at $R$ (without twisting) through ring $r \in [n] \setminus \{1\}$ from below or from above. This movement is denoted by $b_r$ and $b_r^{-1}$, respectively (see Figure 2.3).

*Remark.* Since the rope can not be led through ring 2 while it performs a crossing, Cardano's accelerated algorithm can not be adapted to the CS.

## 2.2 An explicit solution of the Chinese String with up to three rings

Now one can take a look at explicit solutions of the CS with a small number of rings. The finite sequence of movements we use to solve the CS is denoted by $B(n)$, where $n \in \mathbb{N}_0$ is the number of rings of the CS.

At first we solve the trivial CS, i.e. the CS with no ring. Here we have nothing to do, because there is no peg where the rope can go around. Therefore the finite sequence of movements $B(0)$ is empty and has length 0.

In the second case we look at the CS with one ring. Here the rope can easily be removed by one crossing $b_1^{-1}$ (see Figure 2.4). So the finite sequence of movements of length 1 is given by

$$B(1) = b_1^{-1}. \tag{2.1}$$

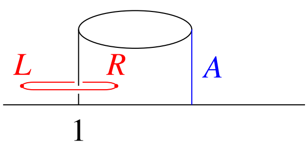


Figure 2.4: The CS with one ring

Now the CS with two rings gets a little bit more interesting. A solution is given by the following procedure (see Figure 2.5):

1. At first lead $R$ through ring 2 from below ($b_2$).
2. Now do a crossing ($b_1$), otherwise one gets a wrap around a peg or ring or undoes the previous movement.
3. One can lead $R$ through ring 2 from above ($b_2^{-1}$).
4. Finally another crossing solves the problem ($b_1^{-1}$).

Summarized for a solution of the CS with two rings we get the finite sequence of movements

$$B(2) = b_2 b_1 b_2^{-1} b_1^{-1}. \tag{2.2}$$

We continue with a solution of the CS with three rings. To reach our goal we conduct the following steps (see Figure 2.6):

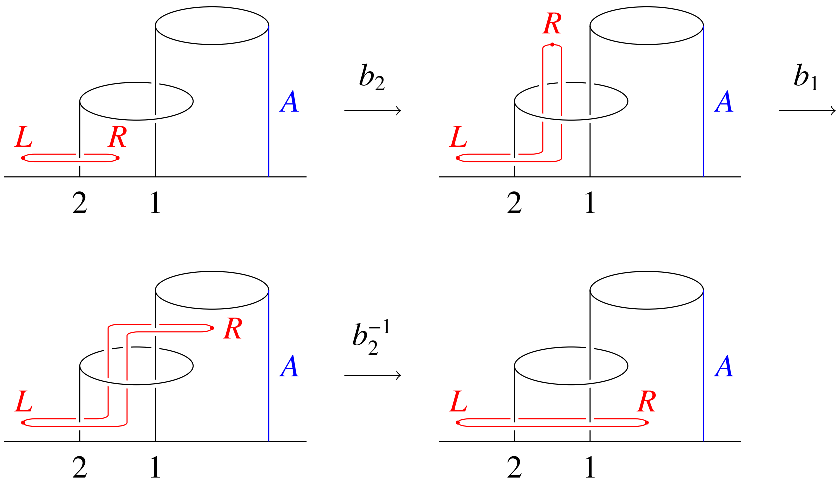


Figure 2.5: A solution for the CS with two rings.

1. One starts by leading $R$ through ring 3 from below ($b_3$).
2. To avoid a wrap around or undoing the previous movement one does a crossing ($b_1$) as in the case of the CS with two rings.
3. Now one can lead $R$ through ring 2 from below ($b_2$).
4. Do a crossing again ($b_1^{-1}$).
5. By leading $R$ through the rings 2 and 3 from above ($b_2^{-1}b_3^{-1}$), the problem of solving the CS with three rings is reduced to the problem of solving the CS with two rings.

So a solution of the CS with three rings is given by the finite sequence of movements

$$B(3) = b_3 b_1 b_2 b_1^{-1} b_2^{-1} b_3^{-1} b_2 b_1 b_2^{-1} b_1^{-1}. \tag{2.3}$$

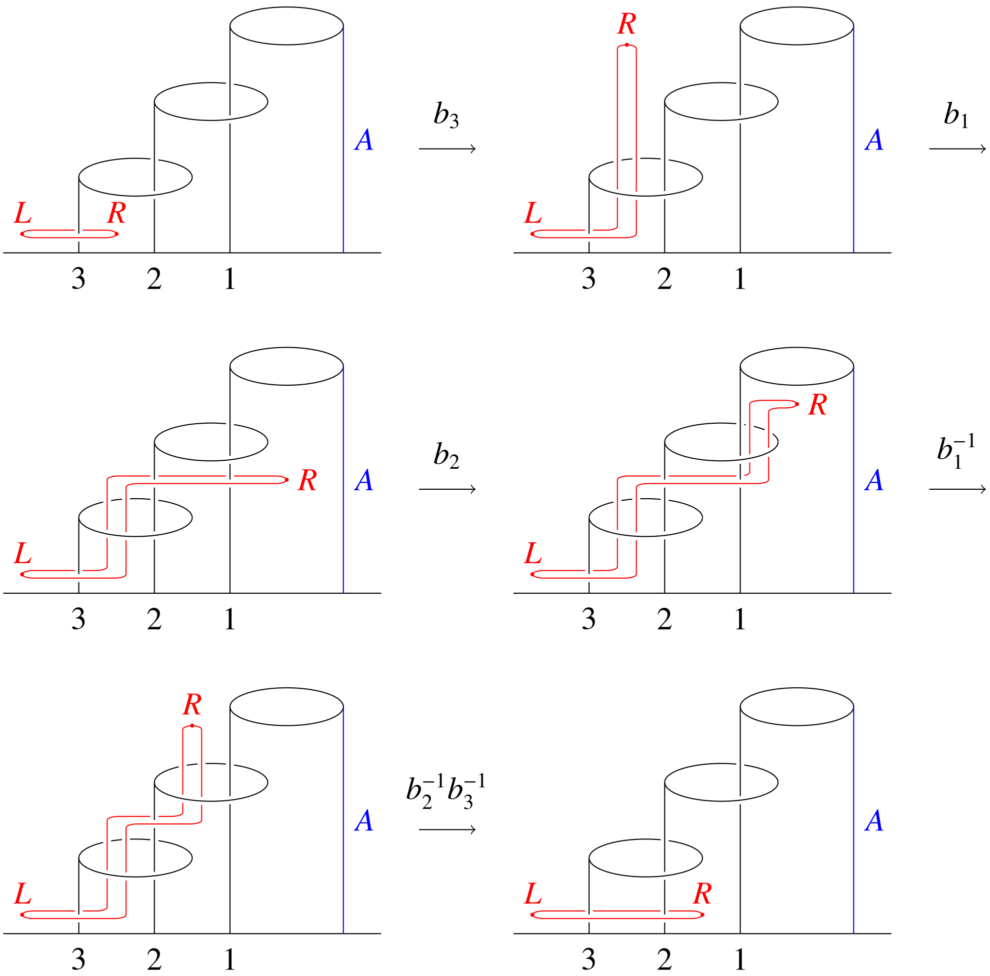


Figure 2.6: A solution for the CS with three rings.

## 2.3 A RECURSIVE SOLUTION FOR THE CHINESE STRING WITH AN ARBITRARY NUMBER OF RINGS

After solving the CS with up to three rings we want to find a solution for an arbitrary number of rings. To solve the CS with $n \geq 3$ rings, a possible strategy is to view rings 1 and 2 as one ring. So we get a CS with $n-1$ rings, where the movements $b_1$ and $b_1^{-1}$ have to be substituted by the solutions of the CS with two rings. Therefore $b_1^{-1}$ has to be replaced by $B(2) = b_2 b_1 b_2^{-1} b_1^{-1}$ and similarly $b_1$ is replaced by $B(2)^{-1} = b_1 b_2 b_1^{-1} b_2^{-1}$. So we get a solution for the CS with $n+1$ rings by the following modifications of the solution of the CS with $n$ rings:
At first for $i \in [n] \setminus \{1\}$ do

$$b_{i+1} \leftarrow b_i \quad \text{and} \quad b_{i+1}^{-1} \leftarrow b_i^{-1}, \tag{2.4}$$

after that do

$$b_1 b_2 b_1^{-1} b_2^{-1} \leftarrow b_1 \quad \text{and} \quad b_2 b_1 b_2^{-1} b_1^{-1} \leftarrow b_1^{-1}, \tag{2.5}$$

where $b \leftarrow a$ means that $a$ is substituted by $b$.
Now let us take a look at the number of movements and crossings the given solution of the CS with $n \in \mathbb{N}$ rings uses. For that reason the number of movements is denoted by $m_n$ and the number of crossings by $cr_n$.
Since the single movement of the CS with one ring is $b_1^{-1}$, i.e. the single movement is a crossing, and by the modification rules (2.5) the number of crossings is doubled by adding a ring, the sequence $(cr_n)_{n\in\mathbb{N}}$ of the number of crossings is defined recursively by

$$cr_{n+1} = 2 \cdot cr_n, \quad cr_1 = 1. \tag{2.6}$$

By modification rule (2.4) we see that every movement which is not a crossing causes exactly one movement, whereas by modification rule (2.5) every crossing generates three additional movements. Therefore we get the following recurrence for the sequence $(m_n)_{n\in\mathbb{N}}$ of the number of movements:

$$m_{n+1} = m_n + 3 \cdot cr_n, \quad m_1 = 1. \tag{2.7}$$

**Lemma 2.2.** *The explicit formulas for the sequence of the number of movements and the sequence of the number of crossings are given by*

$$cr_n = 2^{n-1} \quad \textit{and} \quad m_n = 3 \cdot 2^{n-1} - 2 \quad \textit{for all } n \in \mathbb{N}.$$

*Proof.* We get the equation $cr_n = 2^{n-1}$ for all $n \in \mathbb{N}$ by the recursive definition of the sequence $(cr_n)_{n\in\mathbb{N}}$ with the base case $cr_1 = 1$ (see (2.6)). Now we prove the statement about $m_n$. By (2.7) we get for all natural numbers $n \geq 2$:

$$\begin{aligned} m_n &= m_{n-1} + 3 \cdot cr_{n-1} = m_{n-2} + 3 \cdot cr_{n-2} + 3 \cdot cr_{n-1} = \dots \\ &= m_1 + \sum_{k=1}^{n-1} 3 \cdot cr_k = m_1 + \sum_{k=0}^{n-2} 3 \cdot 2^k = 1 + 3 \cdot \left(2^{n-1} - 1\right) = 3 \cdot 2^{n-1} - 2. \end{aligned}$$ □

Now let us take a closer look at the finite sequence of movements $B(n)$ of the CS with $n \in \mathbb{N}$ rings. For this purpose we split this sequence in blocks of length 6 and the last four movements in one block of length 3 and one block of length 1. For example

$$B(3) = \underbrace{b_3 b_1 b_2 b_1^{-1} b_2^{-1} b_3^{-1}}\, \underbrace{b_2 b_1 b_2^{-1}}\, \underbrace{b_1^{-1}}$$

is the splitting of $B(3)$.

**Lemma 2.3.** *Every block of length 6 is of the form*

$$b_r^{\pm 1} b_1 b_s^{\pm 1} b_1^{-1} b_t^{\pm 1} b_u^{\pm 1},$$

*where* $r, s, t, u \in [n] \setminus \{1\}$. *Every block of length 3 is* $b_2 b_1 b_2^{-1}$ *and the last block (of length 1) is* $b_1^{-1}$.

*Proof.* The statement holds for $n \leq 3$ by (2.1), (2.2) and (2.3). Let $n \geq 3$ be the number of rings of the CS and let all blocks of $B(n)$ fulfil the required conditions.
At first we look at the last block $b_1^{-1}$, the block with length 1. This block turns into $b_2 b_1 b_2^{-1} b_1^{-1}$ by modification rule (2.5). Therefore the last two blocks (one of length 3 and the last of length 1) are of the desired form.
Now we consider the block of length 3, $b_2 b_1 b_2^{-1}$. The modification rules (2.4) and (2.5) lead to this block being $b_3 b_1 b_2 b_1^{-1} b_2^{-1} b_3^{-1}$. This is a block of length 6 fulfilling the required conditions.
Finally we take a look at a block of length 6 of the required form. Let us say this block is $b_r^{\pm 1} b_1 b_s^{\pm 1} b_1^{-1} b_t^{\pm 1} b_u^{\pm 1}$, where $r, s, t, u \in [n] \setminus \{1\}$. By the modification rules (2.4) and (2.5) we get the following substitution:

$$b_{r+1}^{\pm 1} b_1 b_2 b_1^{-1} b_2^{-1} b_{s+1}^{\pm 1} b_2 b_1 b_2^{-1} b_1^{-1} b_{t+1}^{\pm 1} b_{u+1}^{\pm 1} \leftarrow b_r^{\pm 1} b_1 b_s^{\pm 1} b_1^{-1} b_t^{\pm 1} b_u^{\pm 1}. \tag{2.8}$$

So from one block of length 6 we get two blocks of length 6, each fulfilling the required conditions. □

Now we know some facts about the finite sequence of movements $B(n)$ for every $n \in \mathbb{N}$. For $n \leq 3$ we have already seen the explicit solutions. But for an arbitrary $n > 3$ we neither know anything about the $i$-th ($i \in [m_n]$) movement yet nor about the positions in which the movements $b_k$ or $b_k^{-1}$ ($k \in [n]$) were done. For this purpose we take a look at the following lemmas.

**Lemma 2.4.** *Let $1 < n \in \mathbb{N}$ and $k \in [n]$. Then the positions of the movements $b_k$ and $b_k^{-1}$ are given by the following formulas:*

$$B(n)_i = b_k \quad \Leftrightarrow \quad \begin{cases} i \equiv 2 \bmod 6, & \text{for } k = 1 \\ i \equiv 3 \cdot 2^{k-2}, 3 \cdot 2^{k-1} + 1 \bmod 3 \cdot 2^k, & \text{for } 2 \leq k < n \\ i = 1, & \text{for } k = n, \end{cases}$$

$$B(n)_i = b_k^{-1} \quad \Leftrightarrow \quad \begin{cases} i \equiv 4 \bmod 6, & \text{for } k = 1 \\ i \equiv 3 \cdot 2^{k-1} - 1, 9 \cdot 2^{k-2} \bmod 3 \cdot 2^k, & \text{for } 2 \leq k < n \\ i = 3 \cdot 2^{k-2}, & \text{for } k = n. \end{cases}$$

*Proof.* We prove this lemma by induction on the number of rings $n$. For $n = 2, 3$ the statement obviously holds by (2.2) and (2.3).
Let the statement hold for $n \geq 3$. The cases $k = 1$ and $k = 2$ follow immediately from Lemma 2.3 and its proof.
Now consider the case $3 \leq k < n$. Based on the induction hypothesis we may assume that

$$B(n)_i = b_k \quad \Leftrightarrow \quad i \equiv 3 \cdot 2^{k-2}, 3 \cdot 2^{k-1} + 1 \bmod 3 \cdot 2^k \quad \text{and}$$
$$B(n)_i = b_k^{-1} \quad \Leftrightarrow \quad i \equiv 3 \cdot 2^{k-1} - 1, 9 \cdot 2^{k-2} \bmod 3 \cdot 2^k.$$

Due to the modification rules (2.4) and (2.5) it is clear that $b_{k+1}$ and $b_{k+1}^{-1}$ can only be generated by $b_k$ and $b_k^{-1}$, respectively. Substitution (2.8) and the fact that two blocks of length 6 grow out of one block of length 6 lead to

$$B(n)_i = b_k \quad \Leftrightarrow \quad \begin{cases} B(n+1)_{2i} = b_{k+1}, & \text{if } i \equiv 0 \bmod 6 \\ B(n+1)_{2i-1} = b_{k+1}, & \text{if } i \equiv 1 \bmod 6 \end{cases} \quad \text{and}$$

$$B(n)_i = b_k^{-1} \quad \Leftrightarrow \quad \begin{cases} B(n+1)_{2i} = b_{k+1}^{-1}, & \text{if } i \equiv 0 \bmod 6 \\ B(n+1)_{2i+1} = b_{k+1}^{-1}, & \text{if } i \equiv -1 \bmod 6. \end{cases}$$

Therefore we get for all $3 \leq k < n$:

$$B(n+1)_i = b_{k+1} \quad \Leftrightarrow \quad i \equiv 3 \cdot 2^{k-3}, 3 \cdot 2^{k-2} + 1 \bmod 3 \cdot 2^{k-1} \quad \text{and}$$
$$B(n+1)_i = b_{k+1}^{-1} \quad \Leftrightarrow \quad i \equiv 3 \cdot 2^{k-2} - 1, 9 \cdot 2^{k-3} \bmod 3 \cdot 2^{k-1}$$

and so we obtain:

$$B(n+1)_i = b_k \quad \Leftrightarrow \quad i \equiv 3 \cdot 2^{k-2}, 3 \cdot 2^{k-1} + 1 \bmod 3 \cdot 2^k \quad \text{and}$$
$$B(n+1)_i = b_k^{-1} \quad \Leftrightarrow \quad i \equiv 3 \cdot 2^{k-1} - 1, 9 \cdot 2^{k-2} \bmod 3 \cdot 2^k.$$

Finally we have to look at the case $k = n$. From the modification rules (2.4) and (2.5) together with the induction hypothesis

$$B(n)_i = b_n \quad \Leftrightarrow \quad i = 1 \quad \text{and} \quad B(n)_i = b_n^{-1} \quad \Leftrightarrow \quad i = 3 \cdot 2^{n-2}$$

we get

$$B(n+1)_i = b_{n+1} \quad \Leftrightarrow \quad i = 1 \quad \text{and} \quad B(n+1)_i = b_{n+1}^{-1} \quad \Leftrightarrow \quad i = 3 \cdot 2^{n-1}. \qquad \square$$

With this lemma we can determine the $i$-th movement ($i \in [m_n]$) of the solution of the CS with $n \in \mathbb{N}$ rings.

**Lemma 2.5.** *Let* $1 < n \in \mathbb{N}$ *be the number of rings of the CS,* $i \in [m_n]$ *and* $l \in [6]_0$ *with* $i \equiv l \bmod 6$. *Additionally let* $a_{n-2} \ldots a_0 \in B^{n-1}$ *be the binary representation of* $\frac{i-l}{3}$ *and set* $s := \min\{k \in [n-2], a_k = 0\}$. *Moreover for* $i \geq 6$ *set* $r := \min\{k \in [n-2], a_k \neq 0\}$ *and* $r := 0$ *otherwise. Then the i-th movement of the solution of the CS with n rings is given by*

$$B(n)_i = \begin{cases} b_n^{-1}, & \textit{if } i \equiv 0 \bmod 6 \wedge r = n-2 \\ b_{r+2}, & \textit{if } i \equiv 0 \bmod 6 \wedge r \neq n-2 \wedge a_{r+1} = 0 \\ b_{r+2}^{-1}, & \textit{if } i \equiv 0 \bmod 6 \wedge r \neq n-2 \wedge a_{r+1} \neq 0 \\ b_n, & \textit{if } i = 1 \\ b_{r+1}, & \textit{if } i \equiv 1 \bmod 6 \wedge i \neq 1 \\ b_1, & \textit{if } i \equiv 2 \bmod 6 \\ b_2, & \textit{if } i \equiv 3 \bmod 6 \wedge i \equiv 3 \bmod 12 \\ b_2^{-1}, & \textit{if } i \equiv 3 \bmod 6 \wedge i \equiv 9 \bmod 12 \\ b_1^{-1}, & \textit{if } i \equiv 4 \bmod 6 \\ b_{s+1}^{-1}, & \textit{if } i \equiv 5 \bmod 6. \end{cases}$$

*Proof.* From $i \equiv l \bmod 6$ we get $i - l \equiv 0 \bmod 6$ and so $\frac{i-l}{3}$ is a non negative even integer. Since $i \in [m_n]$ one has $i \leq 3 \cdot 2^{n-1} - 2 < 3 \cdot 2^{n-1}$ and therefore the binary representation of $\frac{i-l}{3}$ has at most $n-1$ digits, where $a_0 = 0$. Thus we have

$$i = \sum_{k=1}^{n-2} a_k \cdot 3 \cdot 2^k + l.$$

The set $\{k \in [n-2], a_k = 0\}$ is non-empty, because in case of $\frac{i-l}{3} = 2^{n-1} - 1$ we get by $i - l \equiv 3 \bmod 6$ a contradiction. Now we have to distinguish six cases depending on $l$, the remainder modulo 6.

Case 1: $l = 0$
Here we have to consider three subcases:

Case 1.1: $r = n - 2$
Then we have $i = 3 \cdot 2^{n-2}$ and Lemma 2.4 yields $B(n)_i = b_n^{-1}$.

Case 1.2: $r \neq n - 2 \wedge a_{r+1} = 0$
Then $i \equiv 3 \cdot 2^r \bmod 3 \cdot 2^{r+2}$ holds and by Lemma 2.4 we get $B(n)_i = b_{r+2}$.

Case 1.3: $r \neq n - 2 \wedge a_{r+1} \neq 0$
In this case we get the following equivalence:

$$i \equiv 3 \cdot 2^r + 3 \cdot 2^{r+1} \equiv 3 \cdot 2^r (1 + 2) \equiv 9 \cdot 2^r \bmod 3 \cdot 2^{r+2}.$$

Lemma 2.4 yields $B(n)_i = b_{r+2}^{-1}$.

Case 2: $l = 1$
If $i = 1$ then we immediately get $B(n)_i = b_n$ by Lemma 2.4. If $i \neq 1$ then

$$i - 1 \equiv 3 \cdot 2^r \bmod 3 \cdot 2^{r+1} \quad \Leftrightarrow \quad i \equiv 3 \cdot 2^r + 1 \bmod 3 \cdot 2^{r+1}$$

holds and therefore in accordance with Lemma 2.4 we have $B(n)_i = b_{r+1}$.

Case 3: $l = 2$
This case is trivial and we immediately get $B(n)_i = b_1$ according to Lemma 2.4.

Case 4: $l = 3$
In this case we have to distinguish two subcases:

Case 4.1: $i \equiv 3 \bmod 12$
Now $i \equiv 3 \cdot 2^{2-2} \bmod 3 \cdot 2^{2-2+2}$ holds and as shown in Lemma 2.4 we get $B(n)_i = b_2$.

Case 4.2: $i \equiv 9 \bmod 12$
In this subcase we get $i \equiv 9 \cdot 2^{2-2} \bmod 3 \cdot 2^{2-2+2}$ and by Lemma 2.4 $B(n)_i = b_2^{-1}$ holds.

Case 5: $l = 4$
This case is trivial and Lemma 2.4 yields $B(n)_i = b_1^{-1}$ directly.

Case 6: $l = 5$

Now from Lemma 2.4 together with

$$i - 5 \equiv \sum_{k=1}^{s-1} 3 \cdot 2^k \bmod 3 \cdot 2^{s+1}$$
$$\Leftrightarrow \quad i \equiv \sum_{k=1}^{s-1} 3 \cdot 2^k + 6 - 1 \equiv 3\left(2 + \sum_{k=1}^{s-1} 2^k\right) - 1 \equiv 3 \cdot 2^s - 1 \bmod 3 \cdot 2^{s+1}$$

it can be obtained that $B(n)_i = b_{s+1}^{-1}$.

The case analysis shows that the statement holds. □

By these two lemmas we can see that every block of length 6 of $B(n)$ must be of the form

$$b_{r-1} b_1 b_2^{\pm 1} b_1^{-1} b_{s-1}^{-1} b_s^{\pm 1},$$

where $3 \leq r, s \leq n$. Then a closer look shows that the last two movements of a block of length 6 and the first movement of the following block are always of the form $b_k^{-1} b_{k+1} b_k$ or $b_k^{-1} b_{k+1}^{-1} b_k$, where $k \in [n-1] \setminus \{1\}$ and there are no further blocks of this form in $B(n)$. Now define

$$\tilde{b}_k := b_{k-1}^{-1} b_k b_{k-1} \quad \text{and} \quad \tilde{b}_k^{-1} := b_{k-1}^{-1} b_k^{-1} b_{k-1},$$

where $3 \leq k \leq n$. For the sake of completeness $\tilde{b}_k := b_k$ and $\tilde{b}_k^{-1} = b_k^{-1}$ are also defined for $k \in \{1, 2\}$. The resulting finite sequence is denoted by $\tilde{B}(n)$. Then a brief look shows that for $n \geq 3$ we have $B(n) = b_n \tilde{B}(n)$ and especially all movements of the solution, except the first one, are covered by $\tilde{B}(n)$. For the sake of completeness set $\tilde{B}(1) := b_1^{-1}$ and $\tilde{B}(2) := b_1 b_2^{-1} b_1^{-1}$.

**Definition 2.6.** A *move* is an element of the finite sequence $\tilde{B}(n)$ ($n \in \mathbb{N}$) of the form $\tilde{b}_k$ or $\tilde{b}_k^{-1}$ and referred to simply as $k$. The finite sequence of moves of the solution of the CS with $n$ rings is denoted by $M(n)$.

*Remark.* Taking a closer look at the finite sequence $M(n)$ one can see that in every block of $B(n)$ of length 6 the last two elements and the first element of the following block are getting connected to one element of $M(n)$. So

$$121s \leftarrow b_{r-1}^{\pm 1} b_1 b_2^{\pm 1} b_1^{-1} b_{s-1}^{-1} b_s^{\pm 1}, \tag{2.9}$$

where $3 \leq r, s \leq n$, shows how a block of $M(n)$ results from a block of length 6 of $B(n)$. Therefore from every block of length 6 of $B(n)$ we get a block of length 4 of $M(n)$ and the block of length 3 of $B(n)$ becomes a block of length 2 of $M(n)$. So $B(n)$ contains

$$\frac{m_n - 3 - 1}{6} = \frac{3 \cdot 2^{n-1} - 2 - 4}{6} = 2^{n-2} - 1$$

blocks of length 6. That is why the length of the finite sequence of moves $M(n)$ is

$$4\left(2^{n-2} - 1\right) + 2 + 1 = 2^n - 1.$$

One can see an interesting relationship between the given solution of the CS and the Gros sequence.

**Theorem 2.7.** *The finite sequence of moves* $M(n)$*,* $n \in \mathbb{N}$*, is equal to the first* $2^n - 1$ *elements of the Gros sequence.*

*Proof.* By Lemma 1.8 the $i$-th element, $i \in \mathbb{N}$, of the Gros sequence can be determined by solving the congruence $i \equiv 2^{k-1} \bmod 2^k$. Then $k$ is the $i$-th element of the Gros sequence. For this reason the statement of the theorem is equivalent to

$$M(n)_i = k \quad \Leftrightarrow \quad i \equiv 2^{k-1} \bmod 2^k,$$

where $i \in [2^n - 1]$.
The previous remark, especially (2.9), implies that

$$M(n)_i = 1 \quad \Leftrightarrow \quad i \equiv 1 \bmod 2 \quad \text{and} \quad M(n)_i = 2 \quad \Leftrightarrow \quad i \equiv 2 \bmod 4.$$

So the statement holds in the case $k \in \{1, 2\}$. The previous remark also permits just to consider the last element of every block of length 6, i.e. the positions $j$, where $j \equiv 0 \bmod 6$. This is why we get

$$B(n)_j = b_k^{\pm 1} \quad \Leftrightarrow \quad M(n)_i = M(n)_{\frac{4}{6}j} = k.$$

Due to Lemma 2.4 we have $j \equiv 3 \cdot 2^{k-2} \bmod 3 \cdot 2^k$ for a $k \in [n] \setminus \{1\}$ or $j \equiv 9 \cdot 2^{k-2} \bmod 3 \cdot 2^k$ for a $k \in [n-1] \setminus \{1\}$.
If $j \equiv 3 \cdot 2^{k-2} \bmod 3 \cdot 2^k$ for a $k \in [n] \setminus \{1\}$ then there is a natural number $l$, such that $j = 3 \cdot 2^{k-2} + l \cdot 3 \cdot 2^k$ and if $j \equiv 9 \cdot 2^{k-2} \bmod 3 \cdot 2^k$ for a $k \in [n-1] \setminus \{1\}$ then there is a natural number $m$, such that $j = 9 \cdot 2^{k-2} + m \cdot 3 \cdot 2^k$. Multiplication by $\frac{4}{6}$ leads to

$$\begin{aligned} i &= \frac{4}{6}j = \frac{4}{6}\left(3 \cdot 2^{k-2} + l \cdot 3 \cdot 2^k\right) = 2^{k-1} + l \cdot 2^{k+1} \quad \text{or} \\ i &= \frac{4}{6}j = \frac{4}{6}\left(9 \cdot 2^{k-2} + m \cdot 3 \cdot 2^k\right) = 3 \cdot 2^{k-1} + m \cdot 2^{k+1} = 2^{k-1} + 2^k + m \cdot 2^{k+1}. \end{aligned}$$

Therefore Lemma 2.4 yields $M(n)_i = k \Leftrightarrow i \equiv 2^{k-1} \bmod 2^k$. □

Hence we know that the optimal solution of the CR (see Chapter 1) and the given solution of the CS are provided by the Gros sequence, i.e. one can use the solution of the CR to solve the CS.

An optimal solution of the CR is a solution with the shortest path on the state graph. Since every second move is a move of type 0, one could also define optimality by the minimum of moves of type 0. This is what we do in defining optimality of a solution of the CS.

**Definition 2.8.** An *optimal solution* of the CS is a solution with the minimal number of crossings.

## 2.4 ALTERNATIVE SOLVING STRATEGY

The given algorithm solves the CS with $n$ rings using $m_n = 3 \cdot 2^{n-1} - 2$ movements, $2^{n-1}$ crossings and $2^n - 1$ moves and can also be used if the rope is fixed at any point. For example an alternative strategy is to lead the rope by $L$ at first to the right and then ring by ring to the left (see Figure 2.7) in reverse order as the given solution. This strategy can only be used if the rope is not fixed at

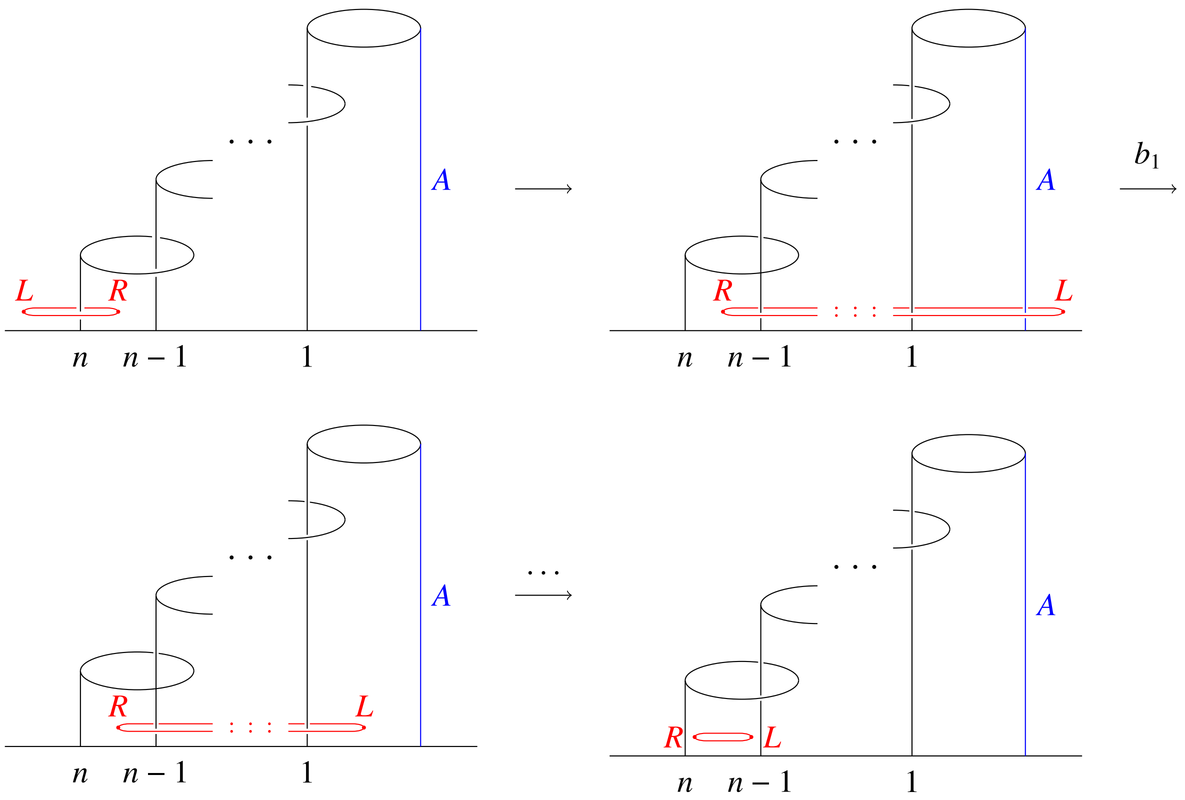


Figure 2.7: Alternative strategy to solve the CS

any point and it can not be adapted to the CR because there is no possible operation analogous to leading $L$ to the right. Since this strategy uses exactly the same movements in reverse order as the given solution, we get $B_A(n) = B(n)^{-1}$ for the finite sequences of movements, where $B_A(n)$ is the finite sequence of movements of the alternative strategy. Without changing the number of movements, crossings and moves one can even switch between both strategies in situations in which the rope does not run trough any ring. In Chapter 5 we will see that these solutions are optimal. Therefore we have at least $2^{n-1}$ optimal solutions of the CS with $n \in \mathbb{N} \setminus \{1\}$ rings and so the optimal solution is not unique. It may be an interesting question whether there are further optimal solutions.

*Remark.* By Theorem 2.7 we can see that this solution of the CS uses exactly the same moves as the optimal solution of the CR does.

# Chapter 3

# Analogy between the Chinese String and the Chinese Rings

In the previous chapter we saw that there are at least $2^{n-1}$ solutions of the CS using $2^n - 1$ moves, while the CR have a unique solution with this length. Since Kauffman called the CS "an equivalent formulation of the Chinese Rings"[1], we try to attach conditions on the rope, such that the two problems, solving the CR and solving the CS, are indeed equivalent.

## 3.1 Discretizing the Chinese String

Since the CR can be described by discrete states, we do this for the CS, too. Similar to the CR we define the state of the CS by the states of its rings. For defining the states we consider the CS with $n \in \mathbb{N}$ rings.
The state of ring 1 is defined by the following:

Ring 1 has state 0, if and only if the rope

a) does not cross the area between peg 1 and the imaginary line $A$ or

b) can be removed from this area without any movement.

Ring 1 has state 1, if and only if the rope

a) (symmetrically) runs in front of and behind peg 1 exactly once and crosses the area between peg 1 and the imaginary line $A$ exactly once or

b) can be brought to this situation without any movement.

For $k \in [n] \setminus \{1\}$ the state of ring $k$ is defined by the following:

Ring $k$ has state 0 (see Figure 3.1), if and only if the rope

a) does not cross the area bounded by ring $k$ and

b) does not cross the area between peg $k-1$ and peg $k$ or

c) can be brought to a situation fulfilling a) and b) without any movement.

Ring $k$ has state 1 (see Figure 3.2), if and only if the rope

[1] in [Kau], p.6

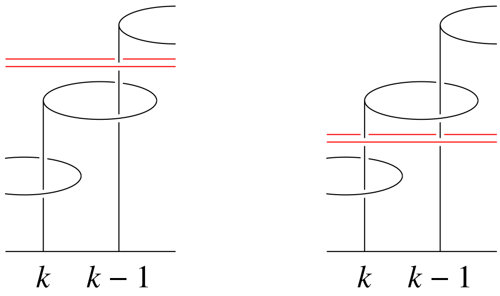


Figure 3.1: Ring $k$ in state 0

a) (symmetrically) runs in front of and behind peg $k$ exactly once and does not cross the area bounded by ring $k$, but the area between peg $k-1$ and peg $k$ exactly once or
b) (symmetrically) runs in front of and behind peg $k$ exactly once and does not cross the area between peg $k-1$ and peg $k$, but the area bounded by ring $k$ exactly twice or
c) can be brought to a situation fulfilling a) or b) without any movement.

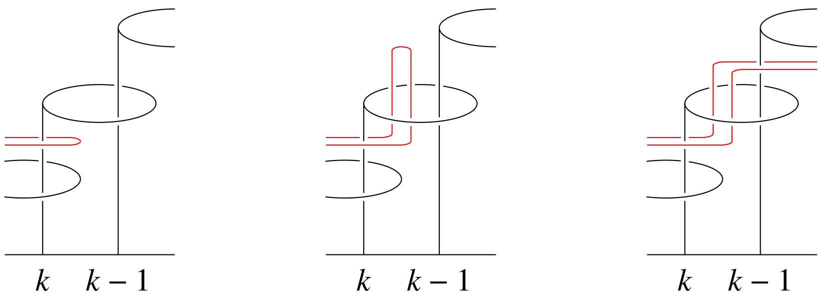


Figure 3.2: Ring $k$ in state 1

If the rope fulfils neither the conditions of state 0 nor the conditions of state 1, we set the state of this ring to $-1$ (see Figure 3.3).

Thereby any situation of the CS with $n \in \mathbb{N}$ rings can be described by an $n$-tuple

$$s = s_n \dots s_1 \in \{-1, 0, 1\}^n ,$$

where $s_k$ corresponds to the state of ring $k$ for $k \in [n]$.

**Definition 3.1.** A ring in state 0 or 1 is called a *regular ring*. If a state of the CS contains only regular rings, it is called a *regular state* and an *irregular state* otherwise.

It is obvious that a state $s$ of the CS with $n \in \mathbb{N}$ rings is regular if and only if $s \in B^n = \{0, 1\}^n$.

Now let us take a brief look at the two familiar situations of the CS with $n \in \mathbb{N}$ rings, the initial and the final situation. The initial situation, where the rope just runs around peg $n$, is described by $10\dots0 = \omega^{(n)} \in B^n$ and the final situation, where the rope is separated from the rings, by $0\dots0 = \alpha^{(n)} \in B^n$.

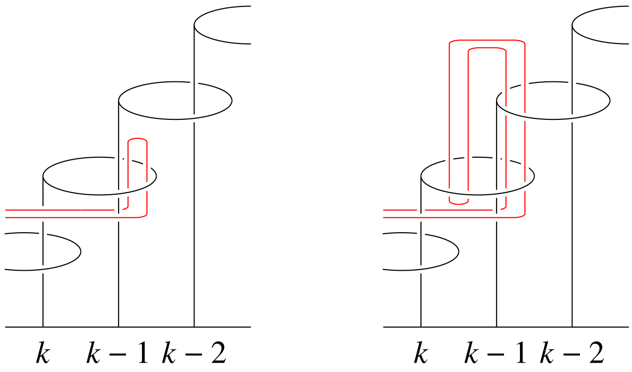


Figure 3.3: Examples for the ring $k$ in the state -1

For any state of the CR one gets the corresponding regular state of the CS with the same number of rings by making the loop flexible and fixing the rings. But also for a given regular state of the CS one gets the corresponding state of the CR by forming the rope into a loop as the CR have (without any movement). Then fix the rope and make the rings moveable as they are in the CR. So the number of states of the CR and the number of regular states of the CS with the same number of rings are equal and we have a bijection between the states of the CR and the regular states of the CS.

*Remark.* In the CS with $n \in \mathbb{N}$ rings the first step of the alternative strategy presented in 2.4 leads to an irregular state, because without any movement the rope can be brought in a situation, where it does not run on both sides of peg $n$ and crosses the area between peg $n$ and peg $n-1$ exactly once.

## 3.2 Changing regular states by a move

After defining the states of the CS we consider how a move can change states. Due to the fact that just regular states correspond to the states of the CR, we look at regular changes of states, i.e. changes between two regular states $s, t \in B^n$ of the CS with $n \in \mathbb{N}$ rings.
At first we consider the state of ring 1. If $s_1 = 0$ then only movement $b_1$ yields $t_1 = 1$ and if $s_1 = 1$ then just movement $b_1^{-1}$ leads to $t_1 = 0$. Any other movement leads to no change of the state of ring 1 or to $t_1 = -1$. So a regular change of ring 1 can only be fulfilled by a crossing.
Now we take a look at an arbitrary ring $k \in [n] \setminus \{1\}$. We want to consider in which way a regular change of ring $k$ can be realised.

**Lemma 3.2.** *In the CS with $n \in \mathbb{N}$ rings the state of ring $k \in [n]$ can neither be changed from $s_k = 1$ to $t_k = 0$ by movement $b_k$ nor from $s_k = 0$ to $t_k = 1$ by movement $b_k^{-1}$.*

*Proof.* For $k = 1$ we already know that the statement holds.
So let $k \neq 1$ and $s_k = 0$. Without any movement the rope can be brought to a situation, where it crosses neither the area bounded by ring $k$ nor the area between peg $k-1$ and peg $k$ (see Figure 3.1). Performing movement $b_k^{-1}$ leads to a situation where the rope runs through ring $k$

top down and therefore has to cross the area between peg $k-1$ and peg $k$ as well. This is an irregular state.
Now let $k \neq 1$ and $s_k = 1$. Then the rope can be brought to situation a) or b) without any movement. In situation a) the rope does not cross the area bounded by ring $k$ but the area between peg $k-1$ and peg $k$ exactly once. Thus movement $b_k$ makes the rope cross the area bounded by ring $k$ exactly twice, but the area between peg $k-1$ and peg $k$ not anymore. We get $t_k = 1$ and no change of states occurs (see Figure 3.2). In situation b) it runs through ring $k$ twice and movement $b_k$ leads to the rope running through ring $k$ more than twice. As a consequence one obtains an irregular state. □

From this lemma it follows that movement $b_k^{-1}$ can lead to a regular change of states only if ring $k$ has state 1 at the beginning of the movement. As a result for $k \geq 3$ the movements $\tilde{b}_k = b_{k-1}^{-1} b_k b_{k-1}$ and $\tilde{b}_k^{-1} = b_{k-1}^{-1} b_k^{-1} b_{k-1}$ are admissible only in the case $s_{k-1} = 1$. Moreover we have the following lemma:

**Lemma 3.3.** *Let the CS contain $n \in \mathbb{N}$ rings and let $k \in [n] \setminus \{1\}$. For a regular change of states by move $k$ one needs $s = s_n \dots s_k \omega^{(k-1)}$.*

*Proof.* Start with the case $k = 2$. Let $s = s_n \dots s_2 0 \in B^n$. If $s_2 = 0$, without any movement the rope can be brought to a situation where it crosses neither the area bounded by ring 2 nor the area between peg 2 and peg 1 nor the area between peg 1 and $A$. In this situation the movements $b_2$ and $b_2^{-1}$ yield an irregular state. If $s_2 = 1$, without any movement the rope can be brought to a situation where it does not cross the area between peg 1 and $A$. Furthermore it crosses either the area bounded by ring 2 exactly twice or the area between peg 2 and peg 1 exactly once, but not both. In both cases the movements $b_2$ and $b_2^{-1}$ do not change the state or give rise to an irregular state.
Now let $k \geq 3$. It is sufficient to consider the case $s_{k-1} = 1$, because movement $b_{k-1}^{-1}$ generates an irregular state otherwise. Let $s \in B^n$ be a regular state where $s_{k-1} = 1$ and $s_i = 1$ for an $i < k-2$. Movement $b_{k-1}^{-1}$ yields an irregular state because in this case the rope has to be led through ring $k-1$ from above and so the rope has to cross the area bounded by ring $k-1$ as well as the area between peg $k-2$ and peg $k-1$. Therefore for a regular change of states we get $s = s_n \dots s_k 1 s_{k-2} \alpha^{(k-3)}$.
It remains to show that ring $k-2$ has to be in state 0. For this purpose we may assume $s = s_n \dots s_k 11 \alpha^{(k-3)}$. Without any movement the rope can be brought to one of the following two situations. In the first situation the rope runs through ring $k-2$ and movement $b_{k-1}^{-1}$ leads to the rope running through ring $k-1$ more than twice (at least twice from below and once from above). This is an irregular state (see Figure 3.4). In the second situation the rope does not cross the area bounded by ring $k-2$ but the area between peg $k-3$ and peg $k-2$ and in case of $k = 3$ the area between peg 1 and $A$. Now after movement $b_{k-1}^{-1}$ the movements $b_k$ and $b_k^{-1}$ each cause an irregular state (see Figure 3.5). □

The proof of the previous lemma also shows that for a regular change of states by the movements $b_k$ and $b_k^{-1}$ the CS has to be in state $s = s_n \dots s_k \omega^{(k-1)}$.

Now let us take a brief look at the consequence of a regular change of states by a move.

**Lemma 3.4.** *In the CS with $n \in \mathbb{N}$ rings after completing move $k \in [n]$ the rope runs through ring $k-1$.*

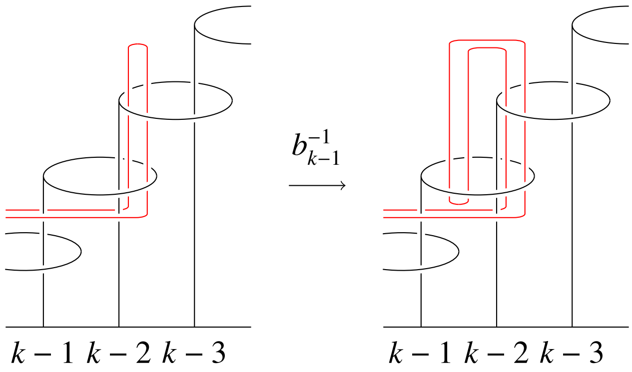


Figure 3.4: The CS in the state $s = s_n \dots s_k 11\alpha^{(k-3)}$

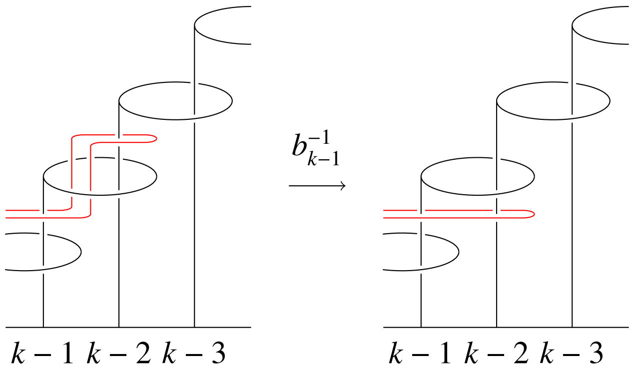


Figure 3.5: The CS in the state $s = s_n \dots s_k 11\alpha^{(k-3)}$

*Proof.* As already shown before, in order to start move $k$, ring $k-1$ has to be in state 1. If the rope does not run through ring $k-1$, movement $b_{k-1}$ does not cause a change of states and furthermore the rope crosses the area bounded by ring $k-1$. If the rope already runs through ring $k-1$ the last movement is not necessary and move $k$ can be started by fulfilling movement $b_{k-1}^{-1}$. Thus the rope does not cross the area bounded by ring $k-1$ anymore but the area between peg $k-2$ and $k-1$ exactly once. Now movement $b_k$ (if $s_k = 0$) or $b_k^{-1}$ (if $s_k = 1$) can be done without causing an irregular state. This leads to a change of the state of ring $k$ but the situation for ring $k-1$ remains unchanged. The last step of the move is movement $b_{k-1}$ and the rope runs through ring $k-1$ again (see Figures 3.6 and 3.7). □

In the initial situation $\omega^{(n)}$ of the CS with $n \in \mathbb{N}$ rings we have to do movement $b_n$ first, then we can start with the first move. If we have any other initial situation and want to start with move $k$ we have to check first whether the rope crosses the area bounded by ring $k-1$. If it does not, the first movement $b_{k-1}^{-1}$ can be skipped and we start with movement $b_k$ and $b_k^{-1}$, respectively.

Now we can set a sufficient as well as necessary condition on a regular change of states.

**Theorem 3.5.** *In the CS with $n \in \mathbb{N}$ rings move $k \in [n]$ fulfils a regular change of states if and only if the CS is in state $s = s_n \dots s_k \omega^{(k-1)}$.*

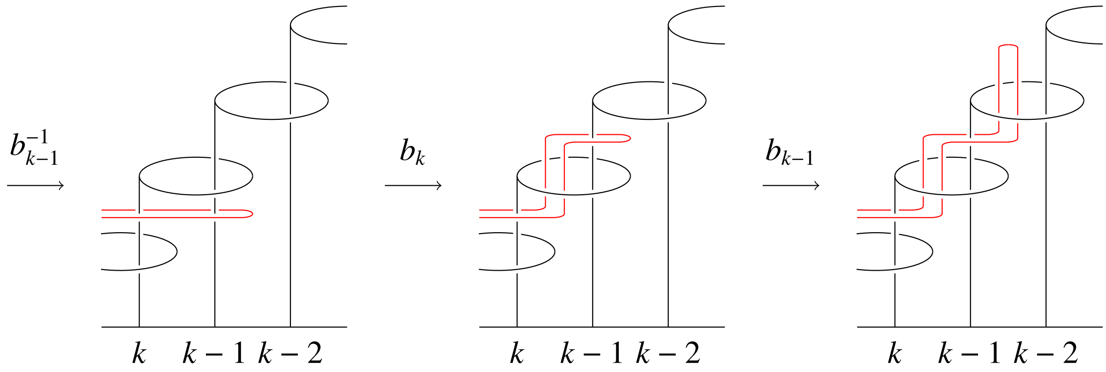


Figure 3.6: The CS during move $k$, if $s_k = 0$

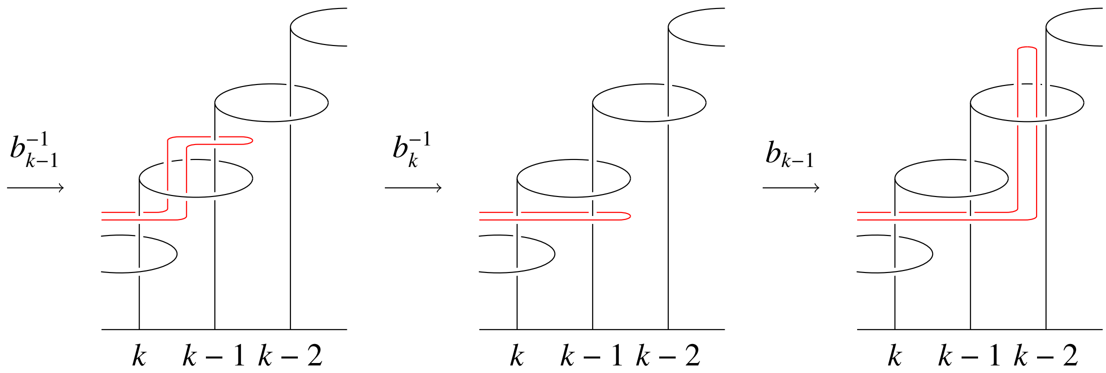


Figure 3.7: The CS during move $k$, if $s_k = 1$

*Proof.* As already shown at the beginning of this section the statement holds for $k = 1$ (see p.29).
So let $k \in [n] \setminus \{1\}$. We assume that move $k$ ($\tilde{b}_k$ or $\tilde{b}_k^{-1}$) changes the state of ring $k$. By Lemma 3.2 we get $s_k = 0$ and $s_k = 1$ and Lemma 3.3 yields $s = s_n \dots s_{k+1} 0 \omega^{(k-1)}$ and $s = s_n \dots s_{k+1} 1 \omega^{(k-1)}$, respectively.
Now let the CS be in the regular state $s \in B^n$, where $s = s_n \dots s_k \omega^{(k-1)}$. Without any movement the rope can be brought to a situation where it crosses either the area bounded by ring $k-1$ exactly twice or the area between peg $k-2$ and peg $k-1$ exactly once, but not both. The first case leads to the second case by doing movement $b_{k-1}^{-1}$. If $s_k = 0$ one continues with $b_k$ and with $b_k^{-1}$ otherwise. In this way one gets a regular change of the state of ring $k$. Now we can go on with $b_{k-1}$ and the rope runs through ring $k-1$ exactly twice (see Figures 3.6 and 3.7). So we obtain the regular state $t \in B^n$, where $t = t_n \dots t_{k+1} (1 - s_k) \omega^{(k-1)}$. □

When allowing only regular states, in every state there are only at most two rings available, whose states can be changed by a move (maybe without changing any state a movement $b_k$, $k \in [n]$ is required first). The state of ring 1 can be changed by a crossing and a change of the state of the ring after the first ring with state 1 is possible. We are familiar with these rules, because they are the same as the rules of the CR (see p.7). So the CR and the CS are

equivalent under the assumption to allow only regular states and the solution given in Chapter 2 is equivalent to the optimal solution of the CR.
In this case an optimal solution of the CS is given by the algorithm shown in Chapter 2. In particular we get the equality of the mechanical and the topological exchange number for any given state $s \in B^n$. Later we will see that the restriction to regular states is not necessary (see Chapter 5).
In consequence of their analogy, the regular states of the CS and the CR have the same state graph. For this reason all properties shown in Chapter 1 hold for the CS, restricted to regular states, too.

In closing this chapter we define the topological exchange number analogously to the mechanical exchange number:

**Definition 3.6.** The minimal number of crossings required to get from state $s \in B^n$ to state $t \in B^n$ of the CS with $n \in \mathbb{N}$ rings is denoted by $E_{top}[s,t]$. The *topological exchange number* of a state $s \in B^n$ is $E_{top}(s) := E_{top}\left[s, \alpha^{(n)}\right]$.

In Chapter 5 we will see that for any given state $s \in B^n$, $n \in \mathbb{N}$, the mechanical exchange number and the topological exchange number are equal.

# Chapter 4

# Modifications of the Chinese String without influence on the complexity

In this chapter we consider two modifications of the CS and we will show that they do not change the *complexity*, which is the minimal number of crossings required when solving the problem with ordinary initial state. Then optimal soultions of the modified CS are also optimal solutions of the original CS.
The first modification is to fix the rope at any point at the bottom plate and the second one is to substitute ring 1 by a ball. To avoid any possibility of confusion the imaginary line $A$ in the CS where ring 1 is substituted by a ball is denoted by $A_b$ ("b" for ball). In the following section we consider the CS with $n \in \mathbb{N} \setminus \{1\}$ rings and initial state $s = \omega^{(n)}$.

To prove that both modifications do not change the complexity, we untangle the rings in the following way[1]. At first one widens ring 2, such that ring 1 can be slipped through it (see Figure 4.1). One continues this procedure to untangle gradually all rings until ring $n-1$ can be slipped

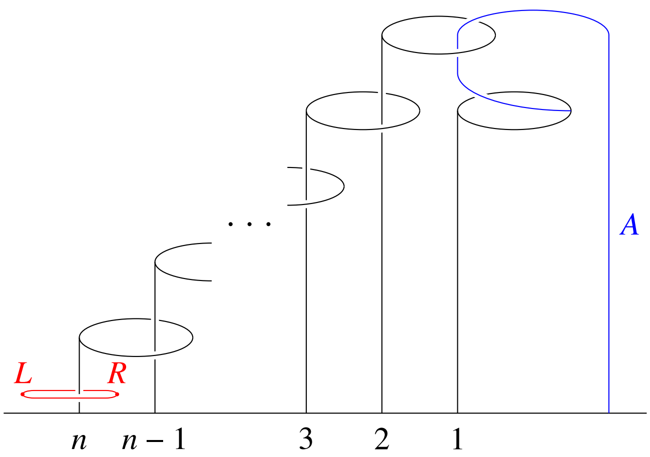


Figure 4.1: The CS after untangling ring 2

through ring $n$. By untangling the rings the shape of $A$ changes, but the shape of the rope does not. The complexity remains unchanged as well. The exact shape of $A$ will be discussed in Chapter 5.

[1] see [Kau], p.9

Also without changing the complexity one can shift the starting point of the line $A$ as shown in Figure 4.2, such that $A$ becomes a closed curve $A'$.

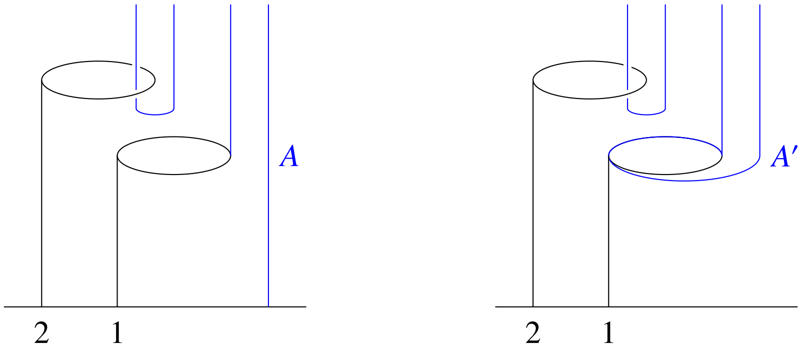


Figure 4.2: The shapes of $A$ and $A'$

For further considerations some basics in algebraic topology are needed.

## 4.1 Selected basics in algebraic topology

**Definition 4.1.** Let $X$ be a topological space, $a, b \in \mathbb{R}$, $a < b$ and $x_0 \in X$. Two paths (continuous maps) $f, g : [a, b] \to X$ with the same starting point $x_0$ and ending point $x_1$ are called *homotopic*, if there exists a *homotopy* (continuous map) $H : [a, b] \times [0, 1] \to X$, such that for all $(s, t) \in [a, b] \times [0, 1]$

$$H(s, 0) = f(s), \quad H(s, 1) = g(s) \quad \text{and} \quad H(a, t) = x_0, \quad H(b, t) = x_1$$

holds. If a path is homotopic to a constant path it is called *null-homotopic*[2].

*Remark.* Homotopy is an equivalence relation on the set containing all paths in a topological space. Therefore all paths in a topological space can be classified into homotopy classes. A homotopy class represented by a path $f$ is denoted by $[f]$. A prove for the property of being an equivalence relation can be found in [Kod], p.166f.

**Definition 4.2.** The set containing all homotopy classes of closed paths in a topological space $X$ with starting and ending point $x_0 \in X$ defines the *fundamental group* $\pi(X, x_0)$. Let $a, b \in \mathbb{R}$, $a < b$ and $f, g : [a, b] \to X$ be two closed paths with starting and ending point $x_0 \in X$, then by

$$\begin{aligned} f \cdot g : [a, b] &\to X \\ t &\mapsto \begin{cases} f(2t - a), & \text{if } a \le t \le \frac{a+b}{2} \\ g(2t - b), & \text{if } \frac{a+b}{2} \le t \le b \end{cases} \end{aligned}$$

a *product* of $f$ and $g$ is defined, which is the group operation of the fundamental group[3].

[2] cf. [Spa], p.23
[3] cf. [Mau], Definition 3.2.2 and Definition 3.2.3

*Remark.* The identity element of the fundamental group of the topological space $X$ is the homotopy class of null-homotopic paths and for any given homotopy class $[f]$ the inverse element $\left[f^{-1}\right]$ is represented by $f^{-1} : [a,b] \to X, t \mapsto f(a+b-t)$. A verification of the group properties can be found in [Wal1], Theorem A-6.

**Lemma 4.3.** *Let $X$ be a pathwise connected topological space and $x_1, x_2 \in X$. Then there exists an isomorphism between the fundamental groups $\pi(X, x_1)$ and $\pi(X, x_2)$.*

*Proof.* [4] Since $X$ is pathwise connected there exists a homotopy class $[\alpha]$, represented by $\alpha : [a,b] \to X$, where $\alpha(a) = x_1$ and $\alpha(b) = x_2$. By

$$\phi : \pi(X, x_1) \to \pi(X, x_2) \qquad \text{and} \qquad \psi : \pi(X, x_2) \to \pi(X, x_1)$$
$$[f] \mapsto \left[\alpha^{-1} \cdot f \cdot \alpha\right] \qquad\qquad [g] \mapsto \left[\alpha \cdot g \cdot \alpha^{-1}\right]$$

homomorphic maps from $\pi(X, x_1)$ to $\pi(X, x_2)$ and from $\pi(X, x_2)$ to $\pi(X, x_1)$, respectively, are given. For each $[f] \in \pi(X, x_1)$ and $[g] \in \pi(X, x_2)$ by

$$(\psi \circ \phi)([f]) = \psi\left(\left[\alpha^{-1} \cdot f \cdot \alpha\right]\right) = \left[\alpha \cdot \alpha^{-1} \cdot f \cdot \alpha \cdot \alpha^{-1}\right] = [f] \quad \text{and}$$
$$(\phi \circ \psi)([g]) = \phi\left(\left[\alpha \cdot g \cdot \alpha^{-1}\right]\right) = \left[\alpha^{-1} \cdot \alpha \cdot g \cdot \alpha^{-1} \cdot \alpha\right] = [g]$$

we have $\psi \circ \phi = \mathrm{id}_{\pi(X,x_1)}$ and $\phi \circ \psi = \mathrm{id}_{\pi(X,x_2)}$. For that reason $\phi$ and $\psi$ are inverse maps and so they are isomorphisms. □

From the lemma it follows that the fundamental group of a pathwise connected topological space is, up to isomorphism, independent of the choice of the starting and ending point. Thus we often omit the starting and ending point $x_0$ of the denotation and use $\pi(X)$ instead.

Now we embed the frame of the CS (bottom plate, pegs and rings) with $n \in \mathbb{N}$ rings in a natural way into the 3-sphere $S^3$, where the point at infinity is neither on the frame of the CS nor in the area between pegs or rings. Then the frame of the CS is homeomorphic to a handlebody $H_n$ of genus $n$. Its complement is also homeomorphic to a handlebody $H'_n = S^3 \setminus H_n$ of the same genus $n$.[5]

Let $k \in \mathbb{N}$ and for all $\alpha \in [k]$ let $G_\alpha$ be a non-trivial group. Consider the set of all words $g_1 g_2 \ldots g_l$ of finite length, where each letter $g_i$ is an element of $G_{\alpha_i} \setminus \left\{\mathrm{id}_{G_{\alpha_i}}\right\}$ and two adjacent letters belong to different $\alpha$s. On this set an operation on two elements $g_1 \ldots g_l$ and $h_1 \ldots h_m$ can be defined by[6]

$$(g_1 \ldots g_l)(h_1 \ldots h_m) = g_1 \ldots g_l h_1 \ldots h_m. \tag{4.1}$$

If $g_l$ and $h_1$ belong to different $\alpha$s the generated word is already *reduced* (i.e. two adjacent letters belong to different $\alpha$s) and if $g_l$ and $h_1$ belong to the same $\alpha$ one has to combine them. One repeats combining letters until the word is reduced. Regarding this operation the inverse element of the word $g_1 \ldots g_l$ is given by $g_l^{-1} \ldots g_1^{-1}$ and the identity is the empty word[7].

[4]from [Rot], p.46
[5]see [PS], p.896
[6]see [Hat], p.41
[7]see [Hat], p.41

**Definition 4.4.** Let $k \in \mathbb{N}$ and for all $\alpha \in [k]$ let $G_\alpha$ be a group. The set of all reduced words with letters in groups $G_\alpha$ together with the operation given in (4.1) is called the *free product of groups* $G_1, \ldots, G_k$ and is denoted by $G_1 * \ldots * G_k$. The free product of $n \in \mathbb{N}$ copies of $\mathbb{Z}$ is called *free group on n generators* and denoted by $F_n$.

*Remark.* The fact that the free product of groups is a group itself can be found in [Hat], p.41f. One can also define a free group with a countably infinite number of copies of $\mathbb{Z}$, which we call *free group on infinite generators*[8].

In order to be able to make any statement about the fundamental group, we look at the following lemmas.

**Lemma 4.5.** *Let X be a topological space and $U, V \subseteq X$, such that U, V and $U \cap V$ are pathwise connected and $U \cup V = X$. If the fundamental group $\pi(U \cap V)$ is the trivial group, then*

$$\pi(X) \cong \pi(U) * \pi(V)$$

*holds.*

A proof of this lemma is provided in [Mas], Theorem 3.1 by using the Seifert-van Kampen theorem and considering commutative diagrams.

**Lemma 4.6.** *Let $H_n$ be a handlebody of genus $n \in \mathbb{N}$ and $x_0$ a point on its boundary $\partial H_n$. Then the fundamental group $\pi(H_n, x_0)$ is isomorphic to a free group $F_n$ on n generators.*

*Proof.* To prove this lemma we use induction. If $n = 1$ the handlebody is a torus. Let $x_0$ be a point on the boundary of the torus $H_1$ and $f$ be a closed path with starting and ending point $x_0$. Then $f$ is either null-homotopic or runs around the hole. All closed paths which run around the hole in an equal number of times and direction are homotopic. So one can describe the homotopy class just by the number of counterclockwise windings. If a path runs counterclockwise as well as clockwise around the hole, the describing integer is the difference between the number of counterclockwise windings and clockwise windings. From this it follows that $\pi(H_1, x_0) \cong \mathbb{Z} \cong F_1$.
Let $\pi(H_n, x_0) \cong F_n$ for some $n \in \mathbb{N}$. Now take a look at the handlebody $H_{n+1}$ of genus $n + 1$. The handlebody $H_{n+1} = U \cup V$ can be splitted in a way that $V$ is homeomorphic to a torus $H_1$, $U$ is homeomorphic to a handlebody $H_n$ of genus $n$ and $U \cap V$ is pathwise connected, contains no further holes and $x_0 \in U \cap V$ (see Figure 4.3). By the induction hypothesis and the base case for the fundamental groups of $U$ and $V$ we get

$$\pi(U) \cong \pi(H_n, x_0) \cong F_n \quad \text{and} \quad \pi(V) \cong \pi(H_1, x_0) \cong F_1.$$

Since all paths in $U \cap V$ are null-homotopic the fundamental group $\pi(U \cap V, x_0)$ is trivial. By the previous lemma we get

$$\pi(H_{n+1}, x_0) \cong \pi(U) * \pi(V) \cong F_n * F_1 \cong F_{n+1}$$

and so the statement holds. □

[8] cf. [Hat], p.42

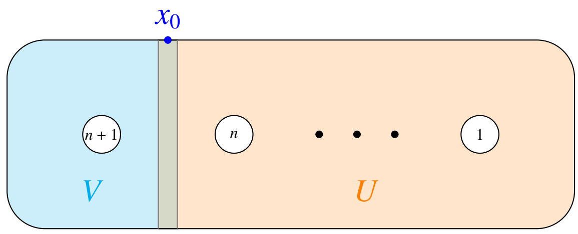


Figure 4.3: Decomposition of $H_{n+1} = U \cup V$

As a result we know that the fundamental groups of the CS with $n \in \mathbb{N}$ rings and its complement are isomorphic to a free group on $n$ generators, respectively.

*Remark.* Since in a handlebody of genus 0 every path is null-homotopic, the fundamental group of the CS with no ring is the trivial group.

Now consider the embedding of the rope, which is a smooth non-self-intersecting closed curve. Movements of the rope correspond to smooth deformations of smooth embeddings. Since in the initial situation the rope does not run through any ring, its embedding does not run around any hole of the embedding of the complement of the frame. For that reason we can see that the CS has a solution (without giving any explicit solving algorithm). In the complement of the *extended CS*, consisting of the frame of the CS united with $A$, the non-self-intersecting curve $A$ intersects the area bounded by the embedding of the rope. If we could smoothly deform the embedding of the rope such that $A$ does not intersect the area bounded by the embedding of the rope anymore, we would be able to solve the CS without any crossings. A verification of the necessity of a crossing will be shown in Chapter 5 by Theorem 5.4.
The embedding of the rope is homeomorphic to a 1-sphere $S^1$ in the complement of the CS. Since a crossing in the proper sense corresponds to a crossing of $A$ and the 1-sphere, the complexity is equal to the minimal number of crossings between $A'$ and the 1-sphere required to make $A'$ null-homotopic in the complement of the union of CS and the rope. For this purpose we take a closer look at the fundamental group of a handlebody after removing a 1-sphere.

*Remark.* To be more precise one should consider smooth non-self-intersecting closed curves and isotopies, which are homotopies $H$, where for each $t \in [0, 1]$ the curve $H(s, t)$ is a smooth embedding.

**Lemma 4.7.** *Let $H_n$ be a handlebody of genus $n \in \mathbb{N}$ and $S \subseteq H_n$ be homeomorphic to a 1-sphere. If $S$ does not run around any hole of the handlebody the fundamental groups of $H_n \setminus S$ and $H_{n+1}$ are isomorphic.*

*Proof.* Let $x_0 \in \partial H_n$ be a point on the boundary of $H_n$. Then $H_n = U \cup V$ can be splitted, such that $U$ contains all holes, $S \subseteq V$, $S$ and $U$ are disjoint and $U \cap (V \setminus S)$ is pathwise connected, contains no hole and $x_0 \in U \cap (V \setminus S)$ (see Figure 4.4).
Therefore $U$ is homeomorphic to a handlebody $H_n$ of genus $n$ and we get for its fundamental group:

$$\pi(U, x_0) \cong \pi(H_n, x_0) \cong F_n,$$

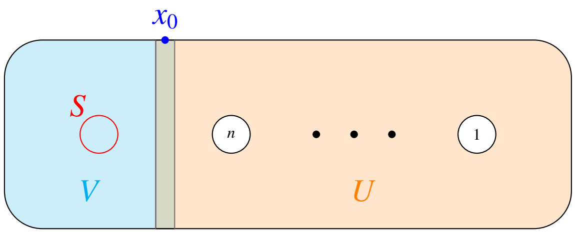


Figure 4.4: Decomposition of $H_n \backslash S^1 = U \cup (V \setminus S)$

where $F_n$ is a free group on $n$ generators.
Since $V \setminus S$ is homeomorphic to a handlebody of genus 1 we obtain

$$\pi(V \setminus S, x_0) \cong \mathbb{Z}.$$

Since every path in $U \cap (V \setminus S)$ is null-homotopic, the fundamental group of $U \cap (V \setminus S)$ is trivial. As a consequence we get by Lemma 4.5:

$$\pi(H_n \setminus S, x_0) \cong \pi(U, x_0) * \pi(V \setminus S, x_0) \cong F_n * \mathbb{Z} \cong F_{n+1}.$$

Due to the fact that the fundamental group is, up to isomorphy, independent of the choice of the starting and ending point $x_0$, we obtain by Lemma 4.6 that the fundamental groups of $H_n \setminus S$ and $H_{n+1}$ are isomorphic. □

With these basics of algebraic topology let us return to the modifications of the CS.

## 4.2 FIXING THE ROPE AT THE BOTTOM PLATE

The first modification we consider is fixing the rope at one point at the bottom plate (see Figure 4.5 ). This modified CS is called *fixed CS*.

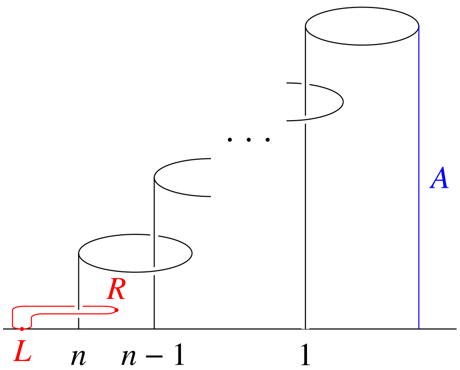


Figure 4.5: The modified CS after fixing the rope at a point $L$ at the base plate

Since $A'$ is a closed path in the complement $H'_n$ of the CS, it is a representative of an element of the fundamental group $\pi(H'_n)$ of $H'_n$ (see Figure 4.6). Using the presented basics of algebraic topology we can show:

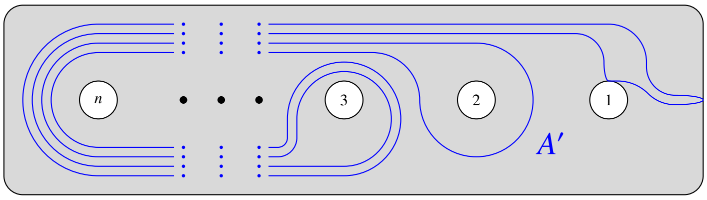


Figure 4.6: The complement of the fixed CS

**Theorem 4.8.** *In the CS with $n \in \mathbb{N}$ rings fixing the rope at one point at the bottom plate does not cause any change of the complexity.*

*Proof.* Consider the complement of the frame of the CS and remove the rope. The resulting figure is denoted by $H_n^*$. The complement of the frame of the CS is homeomorphic to a handlebody $H_n$ of genus $n$ and the rope is homeomorphic to a 1-sphere $S^1$. Since the rope does not cross any area bounded by a ring it does not run around any hole. So we can use Lemma 4.7 to obtain by

$$\pi(H_n^*) \cong \pi\left(H_n \setminus S^1\right) \cong \pi(H_{n+1}) \cong F_{n+1}$$

that the fundamental group of $H_n^*$ is isomorphic to a free group on $n+1$ generators.
Now we take a look at the fixed CS and see that the rope behaves like an additional ring. The complement of the frame of the fixed CS (bottom plate, pegs, rings) unified with the rope is homeomorphic to a handlebody $H_{n+1}$ of genus $n+1$. Therefore the fundamental group of the complement of the fixed CS is also isomorphic to a free group on $n+1$ generators. Since $[A']$ is an element of the fundamental group $\pi(H_n^*)$ and the fundamental groups are isomorphic the statement follows. □

## 4.3 Substituting ring 1 by a ball

Now we take a look at the second modification of the CS with $n \in \mathbb{N}$ rings. In this modification ring 1 is substituted by a ball (see Figure 4.7) and is called *substituted CS* with $n$ rings (despite the fact that the substituted CS just has $n-1$ rings and one ball). In this case the rings can be untangled in the same way as above, again without changing the complexity. The shapes of the lines $A$ and $A_b$ are almost identical and differ just in the ending point. One can contract peg 1 and the ball to a point at the bottom plate and shift the starting point of $A_b$ to this point. As in the case of the fixed CS this procedure does not cause any change of the complexity. We get from $A_b$ a closed curve $A_b'$. Like the original version this version can be embedded in a natural way in the 3-sphere. Then the solid figure is homeomorphic to a handlebody $H_{n-1}$ of genus $n-1$, because there are only $n-1$ rings left. Its complement $H_{n-1}'$ is also homeomorphic to a handlebody of genus $n-1$ and by Lemma 4.6 its fundamental group is isomorphic to a free group $F_{n-1}$ on $n-1$ generators. $A_b'$ is a closed path in the complement of the substituted CS and so it can be seen as a representative of an element of the fundamental group of $H_{n-1}'$ (see Figure 4.8). Now we remove the rope $S \subseteq H_{n-1}'$, which is homeomorphic to the 1-sphere. The

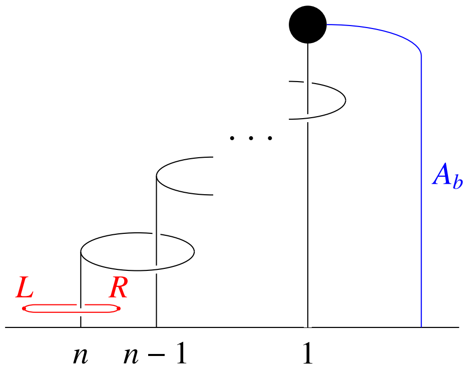


Figure 4.7: The modified CS, where ring 1 is substituted by a ball

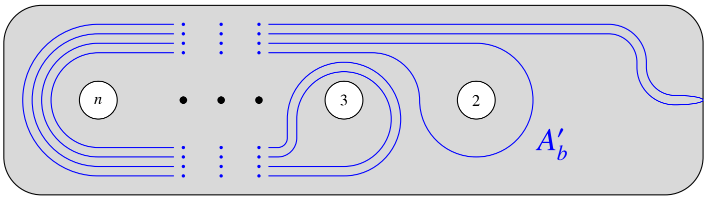


Figure 4.8: The complement of the substituted CS

resulting solid figure is denoted by $H^*_{n-1}$. Since the rope does not run around any hole, we get by Lemma 4.7 that the fundamental groups $\pi\left(H^*_{n-1}\right)$ and $\pi\left(H'_n\right)$ are isomorphic. So $A'_b$ can be seen as a representative of an element of the fundamental group $\pi\left(H'_n\right)$.

Similar to the first modification we can show by algebraic topological basics:

**Theorem 4.9.** *In the CS with $n \in \mathbb{N}$ rings one can substitute ring 1 by a ball without changing the complexity.*

*Proof.* In the CS one can view $A'$ as a representative of an element of the fundamental group $\pi\left(H'_{n+1}\right)$ by Theorem 4.8. Consider just the elements not running around the hole generated by ring 1. These elements generate a subgroup $G$ of $\pi\left(H'_{n+1}\right)$, which is isomorphic to a free group $F_n$ on $n$ generators. Therefore $G$ is isomorphic to $\pi\left(H'_n\right)$ by Lemma 4.6 and so there exists an isomorphism $\phi : G \to \pi\left(H^*_{n-1}\right)$ between $G$ and the fundamental group of the complement of the substituted CS unified with the rope. Because $A$ does not cross the area bounded by ring 1 it is an element of $G$ and due to the positions of $A'$ and $A'_b$ we get $\phi\left([A']\right) = \left[A'_b\right]$. For that reason the complexity of contracting $A'$ in $H'_{n+1}$ is equal to the complexity of contracting $A'_b$ in $H'_n$. So the minimal number of crossings required to solve the CS is equal to the minimal number of crossings required to solve the substituted CS. □

Now we have shown that the two modifications have no influence on the minimal number of crossings required to solve the CS.

The fixed CS with ring 1 substituted by a ball is called *fixed substituted CS*.

**Theorem 4.10.** *The CS and the fixed substituted CS, each with $n \in \mathbb{N}$ rings, have the same complexity.*

Since ring 1 is substituted by a ball, the complement of the frame of the substituted CS is homeomorphic to a handlebody of genus $n - 1$. In the fixed substituted CS the rope behaves like an additional ring and so the complement of the frame of the substituted CS unified with the rope is homeomorphic to a handlebody of genus $n$. With this information the statement of the theorem can be proven in the same way as Theorem 4.8.

So we see that all versions mentioned in Chapter 0 have the same complexity.

# Chapter 5

# Kauffman's Ring conjecture

In [Kau], p.8 Kauffman stated his *Ring conjecture* about the correspondence between the CR and the CS with the same number of rings:

**Conjecture 5.1.** *Let $s$ be a binary string of length $n \in \mathbb{N}$. Then the mechanical exchange number of $s$ and the topological exchange number of $s$ are equal, i.e. $E_{top}(s) = E_{mech}(s)$.*

It is our aim to prove this conjecture at the end of this chapter.

## 5.1 Untangling the Chinese String

To determine the topological exchange number it is helpful to untangle the rings in the same way as we already did in Chapter 4. At first ring 2 is widened, such that ring 1 can be slipped through it. Then we repeat this step ring by ring until all rings are untangled. This version of the CS is called *untangled CS*. As already remarked the shape of $A$ gets changed by untangling the rings, whereas the shape of the rope remains unchanged. To avoid any possibility of confusion we denote the imaginary line of the untangled CS with $n \in \mathbb{N}$ rings by $A_n$.

Now we take a closer look at the path of $A_n$. Based on the paths of the imaginary line in the CS with three (see Figure 5.1), four (see Figure 5.2) and five rings (see Figure 5.3) one can already see some structure.

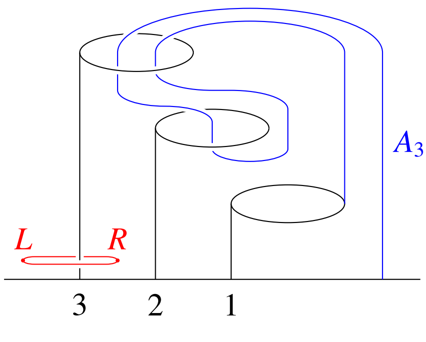


Figure 5.1: The untangled CS with three rings

One can get the shape of the imaginary line (start point at the bottom plate and end point on ring 1) after untangling ring $k+1$ ($k \in [n-1]$) in a recursive way. For that consider the CS after

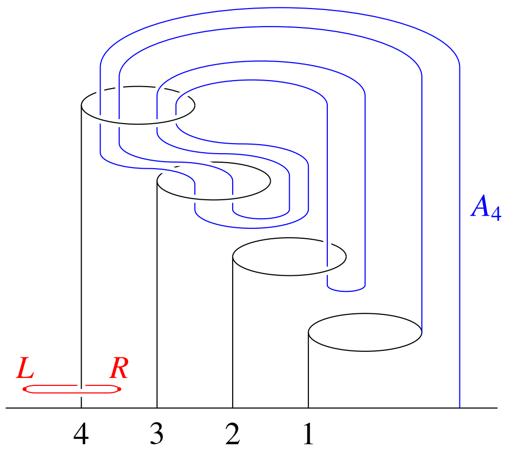


Figure 5.2: The untangled CS with four rings

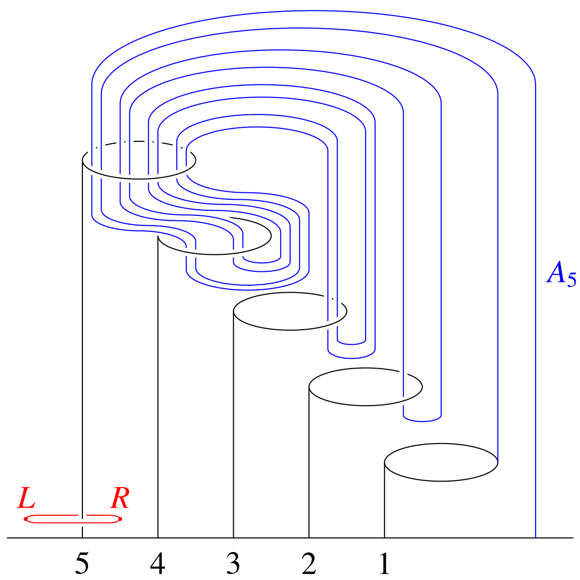


Figure 5.3: The untangled CS with five rings

untangling the first $k$ rings. Untangling ring $k + 1$ causes the following changes on the shape of the imaginary line:

- Instead of running through ring $k$ from above, the line runs through ring $k + 1$ and ring $k$ from above first, then it runs through ring $k + 1$ from below.
- Instead of running through ring $k$ from below, the line runs through ring $k + 1$ from above first, then it runs through ring $k$ and ring $k + 1$ from below.

Now we introduce a notation to describe the shape of $A_n$. The rope running through ring $k$ from above and from below is denoted by $a_k$ and $a_k^{-1}$, respectively. So we can describe the shape of $A_2$ in the untangled CS with two rings by $a_2$. At the CS with three rings the word $a_3a_2a_3^{-1}$ depicts the shape of $A_3$. The changes of the shape of $A$, caused by untangling ring $k+1$, leads to the following modifications on the describing word:

$$a_{k+1}a_ka_{k+1}^{-1} \leftarrow a_k \quad \text{and} \quad a_{k+1}a_k^{-1}a_{k+1}^{-1} \leftarrow a_k^{-1}. \tag{5.1}$$

Using this notation we can show the following:

**Lemma 5.2.** *In the CS with $n \in \mathbb{N}$ rings after untangling ring $k \in [n] \setminus \{1\}$ the letter $a_2$ occurs exactly once in the describing word of the shape of $A$. The part of the word after $a_2$ is inverse to the part prior to $a_2$.*

*Proof.* We prove the statement by induction on $k$. For $k = 2$ the shape of $A$ is described by $a_2$ and the statement obviously holds. Let the statement hold for some $k \in [n] \setminus \{1\}$. By (5.1) untangling ring $k+1$ leads to the substitution of all occurring $a_k$ and $a_k^{-1}$ by $a_{k+1}a_ka_{k+1}^{-1}$ and $a_{k+1}a_k^{-1}a_{k+1}^{-1}$, respectively. So the number of occurring $a_k$ does not increase and since $a_2$ occurs in the induction basis only once, we obtain that the describing word of the shape of $A$ contains $a_2$ exactly once. From the fact that the word $a_{k+1}a_k^{-1}a_{k+1}^{-1}$ is inverse to $a_{k+1}a_ka_{k+1}^{-1}$ and vice versa, we get that the statement holds. □

By this lemma we obtain that the shape of $A_n$ after crossing the area bounded by ring 2 is equal to the shape prior to $a_2$ but in reverse direction. Another implication of the lemma is that we can obtain the shape of $A$ after untangling ring $k+1$, $k \in [n] \setminus \{1\}$, from the shape $A^*$ before untangling ring $k+1$ by the following algorithm:

1. Shift $A^*$ to the left by one ring but the start point remains unchanged (at the bottom plate).
2. Make a copy of the imaginary line $A^*$.
3. Shift the copy to the left, such that two equal homotopic curves occur.
4. Shift the start point of the copy to ring 1 and turn it into the end point. Connect the end point of the copy with the end point of the original to get a loop running through ring 2.

This algorithm allows to make the following statements about the shape of $A_n$ in the untangled CS with $n \geq 3$ rings:

- Untangling the rings does not change the start and end point.
- $A_n$ runs through ring $n$ from below, then runs $2^{k-2}$ times side by side, not intersecting through ring $k$ from above and returns between ring $k-1$ and $k$ on the same way ($k \in [n-2] \setminus \{1\}$).
- $A_n$ runs through ring $n$ from above, then runs $2^{n-3}$ times side by side, not intersecting through ring $n-1$ from above and returns between ring $n-2$ and $n-1$ on the same way.

## 5.2 PROVING KAUFFMAN'S RING CONJECTURE

To prove Conjecture 5.1 we consider the special case of the ordinary initial situation $s$ of the CS with $n \in \mathbb{N}$ rings, i.e. $s = \omega^{(n)}$. As already seen in Chapter 1, p.9, in this case the mechanical exchange number is $E_{mech}\left(\omega^{(n)}\right) = 2^{n-1}$. So for proving the conjecture in the special case it is sufficient to show that the topological exchange number of $\omega^{(n)}$ is also $2^{n-1}$.

Using an additional assumption in [PS] Przytycki and Sikora proved the Ring conjecture in this special case. There it is shown that for the substituted CS in the ordinary initial situation of the CS with $n \in \mathbb{N}$ rings the topological exchange number is $2^{n-1}$. For this purpose they considered the rope as an additional ring and $A_b$ as an element of the fundamental group, which is isomorphic to a free group on $n$ generators. For this reason $A_b$ can be represented as a reduced word (see the paragraph prior to Lemma 5.2). Then they counted the insertions and deletions necessary for deleting the letter representing the additional ring. For this proof they defined a norm and a metric on the set of words and used these tools to prove the statement.
Unfortunately no mention is made of the substitution of ring 1 by a ball nor of any other state than $s = \omega^{(n)}$, $n \in \mathbb{N}$.

In this chapter we want to show another way to prove the conjecture without using further algebraic topological tools. But first we take a look at the behaviour of a crossing.

**Lemma 5.3.** *Consider the CS with $n \in \mathbb{N}$ rings. If immediately after both, the rope and $A_n$ run through a ring a crossing is fulfilled, then the number of crossings required does not decrease.*

*Proof.* To prove the statement we consider all occurring configurations between $A_n$ and the rings. At first we take a look at the rope running through ring $n$. Running through the ring, fulfilling a crossing and pulling back the rope leads to the same situation as doing a crossing without the rope interacting with ring $n$ (see Figures 5.4 and 5.5). So with the assumptions of

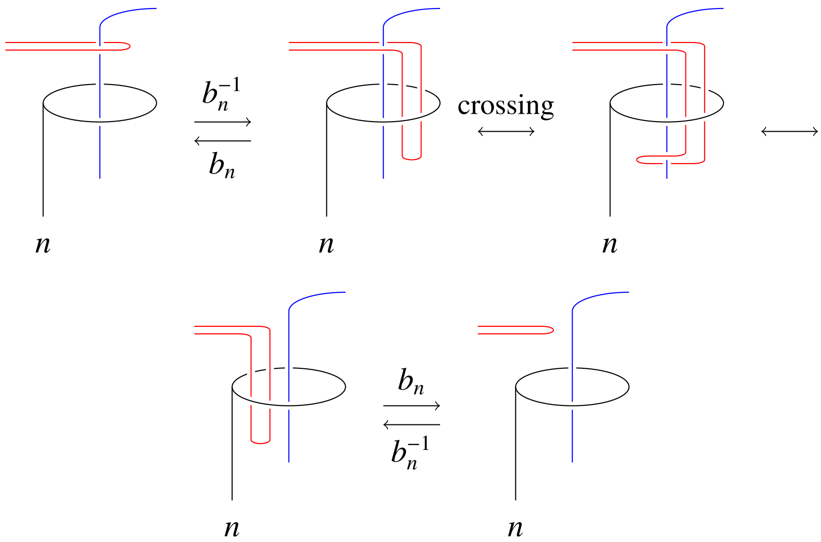


Figure 5.4: The rope runs through ring $n$ from above

the lemma the rope running through ring $n$ can not decrease the number of crossings required.

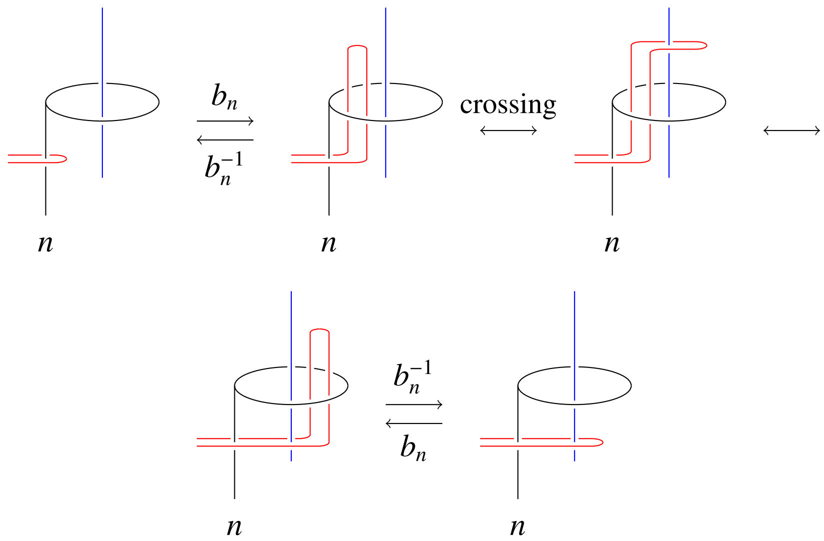


Figure 5.5: The rope runs through ring $n$ from below

Now we consider the rope crossing the area bounded by ring $k \in [n-1] \setminus \{1\}$. In this case due to the paragraph at the end of 5.1 the following four situations for the shape of $A_n$ can occur:

Case 1: The imaginary line $A_n$ runs through ring $n$ first from above, then through ring $n-1$ also from above (see Figure 5.6).

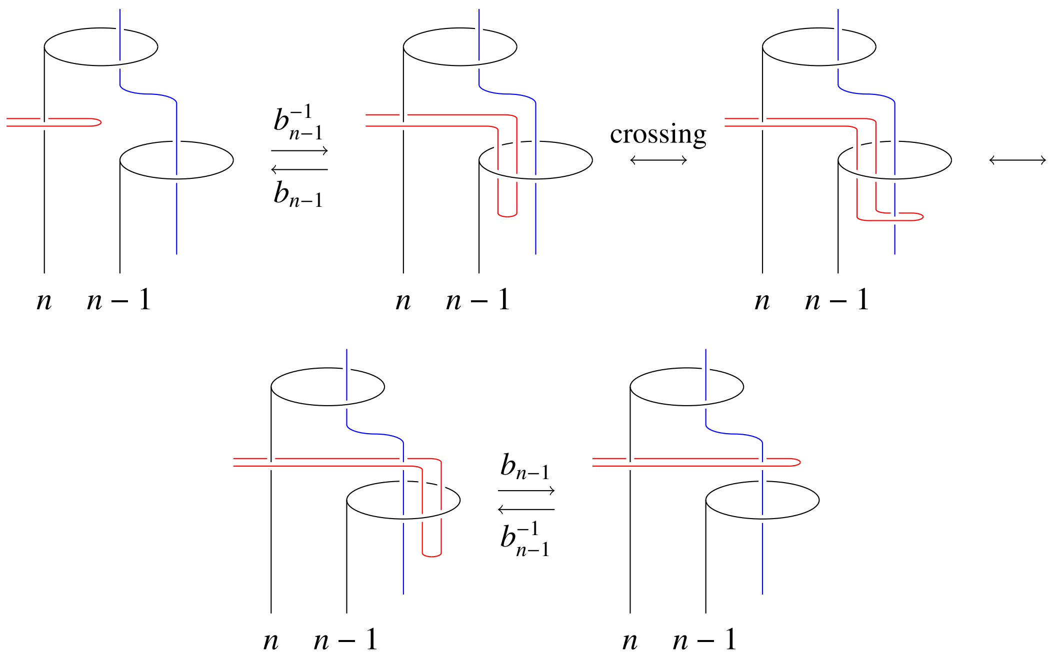


Figure 5.6: Case 1: $A_n$ runs through ring $n$ and ring $n-1$ from above

Case 2: The imaginary line $A_n$ runs through ring $n$ first from above, then through ring $n-1$ from below (see Figure 5.7).

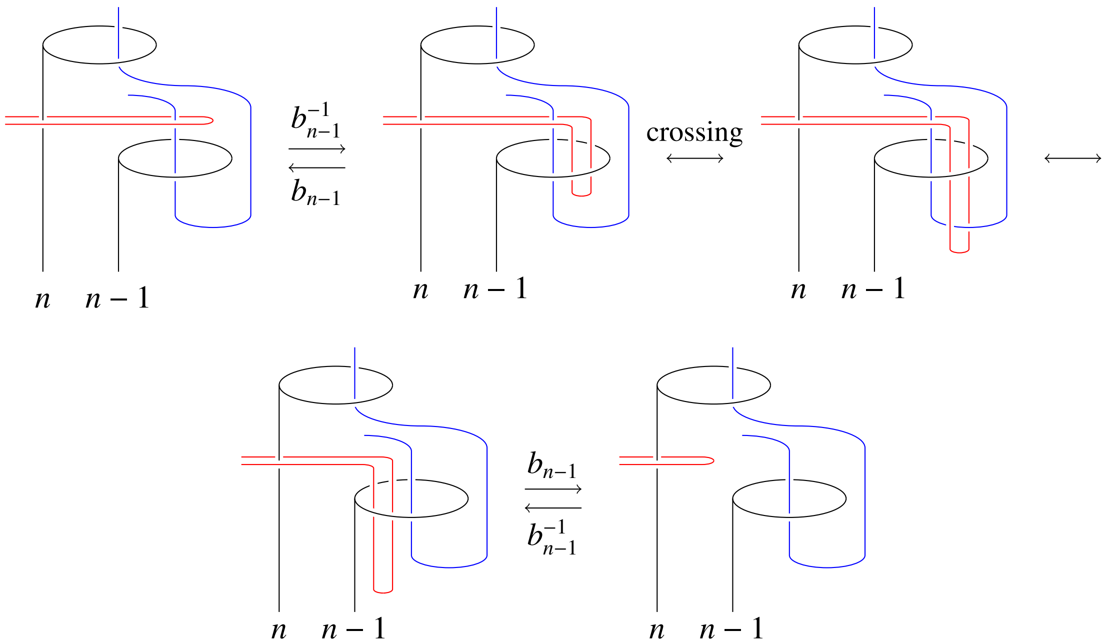


Figure 5.7: Case 2: $A_n$ runs through ring $n$ from above and ring $n-1$ from below

Case 3: The imaginary line $A_n$ runs through ring $n$ first from below, then through ring $k$ also from below (see Figure 5.8).

Case 4: The imaginary line $A_n$ runs through ring $n$ first from below, then through ring $k$ from above (see Figure 5.9).

In this case analysis we consider $A$ in the way that it runs first through ring $n$ and then through ring $k$ and $n-1$, respectively. One could also consider the situation of $A$ in reverse direction, i.e. start with running through ring $k$ and $n-1$, respectively, and after that through ring $n$.
In all of these four cases we see that running through the ring, fulfilling a crossing and pulling back the rope yields the same situation as doing a crossing without the rope interacting with ring $k$. □

From the previous lemma results that the rope can run through several rings in the same order as $A_n$ does and then fulfil a crossing (directly after they run through the last ring together) without decreasing the complexity. To prove that, let the rope run through $l \in \left[2^{n-1}-1\right]$ rings in the same order as $A_n$ does. After a crossing the rope can be pulled back through the last ring and we get the same situation after fulfilling a crossing after the $(l-1)$-th ring.

With these information we can show Kauffman's Ring conjecture in the special case $s = \omega^{(n)}$ using the following theorem.

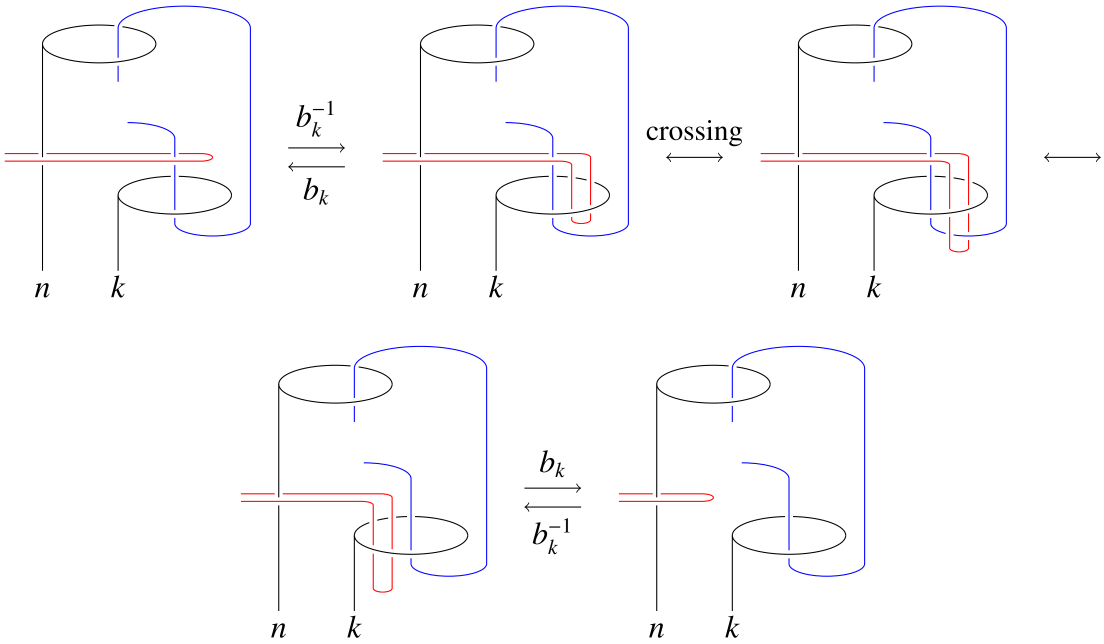


Figure 5.8: Case 3: $A_n$ runs through ring $n$ and ring $k$ from below

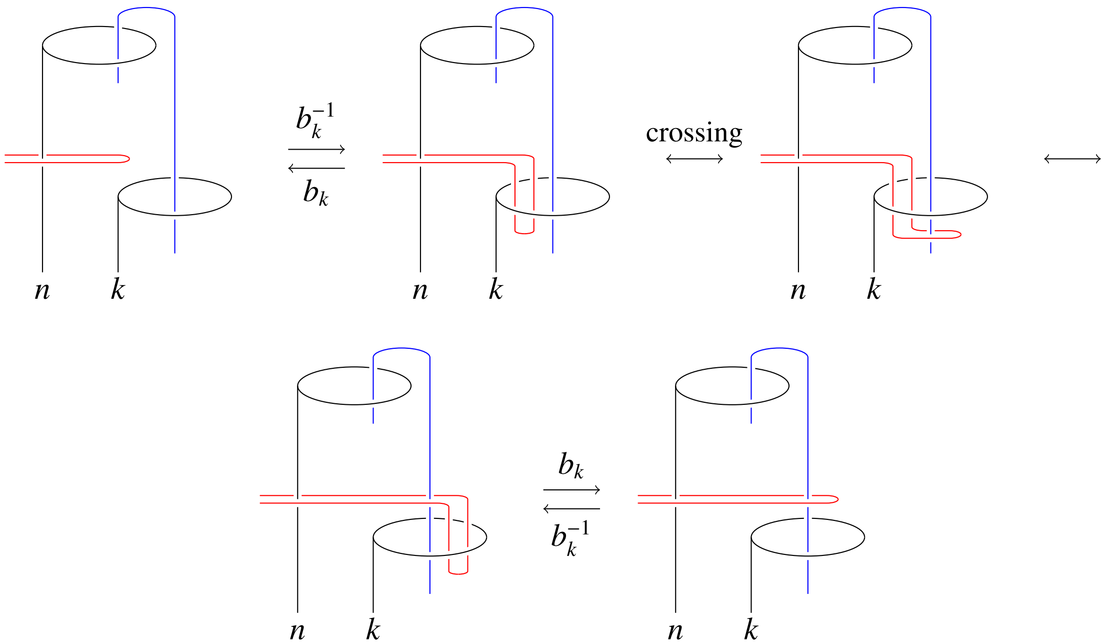


Figure 5.9: Case 4: $A_n$ runs through ring $n$ from below and ring $k$ from above

**Theorem 5.4.** *Solving the CS with $n \in \mathbb{N}$ rings at initial state $\omega^{(n)}$ requires at least $2^{n-1}$ crossings.*

*Proof.* Due to Theorem 4.8 and Theorem 4.9 it is sufficient to prove the statement for the fixed substituted CS. The induction start is the case $n = 1$ (see Figure 5.10). Since the rope can

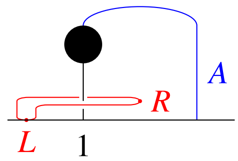


Figure 5.10: The fixed substituted CS with one ring

neither cross the peg nor the ball nor the bottom plate, one crossing is necessary to solve the puzzle.

Now we may assume that the statement holds for some $n \geq 1$. So we consider the untangled fixed substituted CS with $n+1$ rings. According to Lemma 5.2 $A_{n+1}$ is composed of two connected copies of $A_n$. Since by induction hypothesis each copy requires $2^{n-1}$ crossings, one needs $2 \cdot 2^{n-1} = 2^n$ crossings to solve the CS with $n+1$ rings, unless an advantage can be gained by the rope interacting with rings.
If in a solution the rope runs between the two copies while interacting with the rings, then it has to run through the rings in the same order as for $A_{n+1}$. To get out of there it has to perform a crossing. By Lemma 5.3 we obtain another solution which requires the same number of crossings as the previous solution and does not interact with the rings while running between the two copies.
So let us now consider a solution where the rope interacts with the rings, but not while running between the two copies. If this interaction led to a solution with fewer crossings, it would already imply a solution of the CS with $n$ rings requiring less than $2^{n-1}$ crossings. This contradicts the induction hypothesis. □

This theorem shows that the solving algorithm presented in Chapter 2 represents an optimal solution of the puzzle. Another interesting result of this theorem is that for the topological exchange number of $\omega^{(n)} \in B^n$

$$E_{top}\left(\omega^{(n)}\right) = 2^{n-1} \tag{5.2}$$

holds and Kauffman's Ring conjecture is proven in the special case of initial state $s = \omega^{(n)}$.

Finally it remains to show the conjecture in the general case.

**Theorem 5.5.** *Let $s, t \in B^n$ be binary strings of length $n \in \mathbb{N}$. Then*

$$E_{top}[s,t] = E_{mech}[s,t].$$

*In particular, $E_{top}(s) = E_{mech}(s)$.*

*Proof.* Consider the corresponding states of the CS and the corresponding states of the CR, each with $n$ rings. Since every move of the CR can be analogously done on the CS, we get

$$E_{top}[s,t] \leq E_{mech}[s,t] \quad \text{for all } s,t \in B^n,$$

in particular $E_{top}(s) \leq E_{mech}(s)$.
It remains to prove that the reverse inequality also holds. At first we look at the special case $t = \alpha^{(n)}$. For that reason we assume $E_{top}(s) < E_{mech}(s)$ for some $s \in B^n$. Then there is a solution of the CS with $n$ rings and initial state $\omega^{(n)}$, which is analogous to the optimal solution of the CR until reaching the state $s$ and uses $E_{top}(s)$ crossings to get from this state $s$ to the final state $\alpha^{(n)}$. To get from $\omega^{(n)}$ to $s$

$$E_{mech}\left[\omega^{(n)}, s\right] = E_{mech}\left[\omega^{(n)}, \alpha^{(n)}\right] - E_{mech}\left[s, \alpha^{(n)}\right] = 2^{n-1} - E_{mech}(s) < 2^{n-1} - E_{top}(s)$$

crossings are required. By definition $E_{top}(s)$ crossings are required for solving the CS with initial state $s$. Therefore we get a solution of the CS with $n$ rings and initial state $\omega^{(n)}$ using less than $2^{n-1}$ crossings (see Figure 5.11). This is in contradiction to Theorem 5.4 and we get

$$E_{top}(s) = E_{mech}(s)\,. \tag{5.3}$$

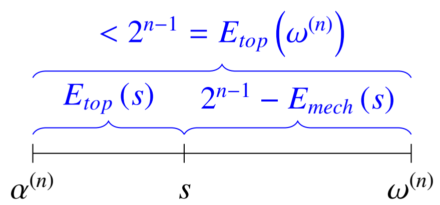


Figure 5.11: Illustration of the contradiction in the special case $t = \alpha^{(n)}$

Now consider the general case and for that reason let $t \in B^n$. Without loss of generality we can suppose that $E_{mech}(s) \leq E_{mech}(t)$. Now we assume that $E_{top}[s,t] < E_{mech}[s,t]$. By the first part of this proof we know to get from $\omega^{(n)}$ to $t$ we need $2^{n-1} - E_{top}(t)$ crossings and from $s$ to $\alpha^{(n)}$ by definition $E_{top}(s)$ crossings are required. Therefore we can get from $\omega^{(n)}$ to $\alpha^{(n)}$ using

$$\begin{aligned} 2^{n-1} - E_{top}(t) + E_{top}[s,t] + E_{top}(s) =& 2^{n-1} - E_{mech}(t) + E_{top}[s,t] + E_{mech}(s) \\ <& 2^{n-1} - E_{mech}(t) + E_{mech}[s,t] + E_{mech}(s) \\ =& 2^{n-1} - E_{mech}(t) + E_{mech}(t) = 2^{n-1} \end{aligned}$$

crossings (see Figure 5.12). According to Theorem 5.4 at least $2^{n-1}$ crossings are required and therefore a contradiction occurs. For this reason we obtain

$$E_{top}[s,t] = E_{mech}[s,t]\,.$$ □

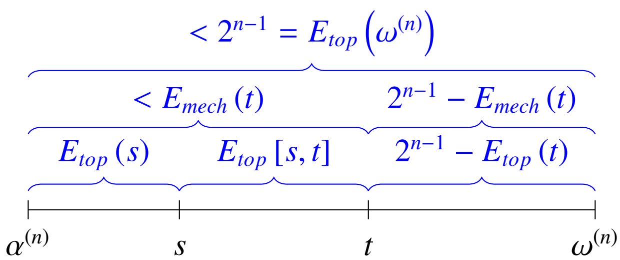


Figure 5.12: Illustration of the contradiction in the general case

# Chapter 6

# Application in cognitive psychology

For some cognitive psychological research in the field of problem solving theory some experiments were performed using isomorphic[1] puzzles. The *Tower of Hanoi* (abbreviated as TH) and the *Monsters and Globes* (abbreviated MaG) are an example of this type of puzzles as well as the CR and the CS.

At first we consider the TH and the MaG. The TH is a puzzle containing three pegs and $n \in \mathbb{N}$ discs. Every disc has a hole in the middle, such that it can be stacked onto one of the three pegs. This is the initial state. The aim is to reach another given state by moving the discs satisfying the following two rules:

1. only one disc may be moved at a time,
2. it is not allowed to put a larger disc on a smaller one (this rule is called the *divine rule*).

For detailed information about the Tower of Hanoi see [HKMP], Chapter 2.

There are several isomorphic versions of the MaG. In each version there are three monsters and three globes, each in three different sizes (small, medium, large). In the initial situation each monster holds one globe and the aim is that each monster holds the globe proportionate to its own size.

In the first version of MaG, the globes were transferred between the monsters satisfying the following three rules[2]:

1. only one globe may be transferred at a time,
2. if a monster is holding more than one globe, only the larger one may be transferred,
3. a globe may not be transferred to a monster holding a larger globe.

Since the globes get transferred it is a *transfer* problem and we call this problem *MaG move problem*[3]. A closer look at the rules shows that the monsters and the globes correspond to the pegs and the discs, respectively, where the size of the globes correspond inversely to the size of the discs (see [Hin1], p.16). The first rule of each puzzle is the same, the second rule of the MaG move problem is given in the TH naturally, because only the topmost disc can be moved

[1] in the psychological sense
[2] from [HS], p.23
[3] in [HS] it is called TA and TA', depending on the initial situation

and the third rule corresponds to the divine rule in the TH. So the MaG move problem and the TH are isomorphic.

In a second version, the size of the globes were changed by the monsters satisfying the following three rules[4]:

1. only one globe may be changed at a time,

2. if two or three globes have the same size, only the globe held by the larger monster may be changed,

3. a globe may not be changed to the same size as the globe of a larger monster.

Since the globes were changed it is a *change* problem and we call this problem *MaG change problem*[5]. This time considering the rules shows that the monsters and the globes correspond to the discs and the pegs, respectively, where the size of the monsters correspond inversely to the size of the discs (see [Hin1], p.16). The three rules of the MaG change problem correspond to the three rules of the MaG move problem and so the MaG change problem and the TH are isomorphic, too.

In [HS] Hayes and Simon found out that there are differences in the solution times of the MaG move problem and the MaG change problem. In the experiment they performed the average time required to solve the MaG change problem was nearly twice the average time required to solve the MaG move problem[6]. As a consequence we see that although the problems are isomorphic, the change problem is more difficult. Hayes and Simon also analysed the data on transfer: subjects, who solved the MaG move problem first have to solve the MaG change problem and vice versa. One could see that the transfer from MaG move problem to MaG change problem was much higher than in the opposite direction (see [HS], p.27).

In [KHS], Table 1 one can see that the average solution time of the TH is much smaller than the average solution time of the MaG move problem. In contrast to the MaG problems the TH has only two explicit rules. Since in the TH for physical reasons only the topmost disc can be moved the second rule of the MaG problems is given by "real world knowledge" in the TH and therefore does not have to be stated explicitly. In [KHS] Kotovsky, Hayes and Simon found out, that "real world knowledge" decreases problem difficulty (see [KHS], p.265), by which the smaller average solution times of the TH can be explained.

Considering the MaG move problem and the MaG change problem one sees that the rigid parts get moveable and the moveable parts get rigid. The same effect can be noticed when comparing the CS with the CR, which are isomorphic, as we have seen in the previous chapters.

In [KS] Kotovsky and Simon found out that solving the CR is much more difficult than solving isomorphic digitized problems. For that purpose they performed an experiment, where the subjects had to solve the CR or an isomorphic digitized problem. By analysing the recorded data they found out that nearly all subjects could not solve the CR within a time limit, whereas the isomorphic problems were solved by many more subjects in time. So they come to the conclusion that the subjects had most problems in moving the rings themselves.

[4]from [HS], p.24

[5]in [HS] it is called CA and CA', depending on the initial situation

[6]see [HS], p.26

Now let us come back to comparing the CR and the CS. On the one hand in the CS the rope can be moved (physically) much easier than the rings in the CR. On the other hand the problem space[7] of the CR is less complex than the problem space of the CS, because in the CR in every situation just two rings can be moved within changing the situation. Therefore it might be an interesting question whether the CR or the CS is more difficult.

As already mentioned in the MaG problems the transfer is asymmetric and depends on the direction. So from the psychological point of view it might also be an interesting question whether there is a transfer between the CS and the CR.

[7] A problem space contains all knowledge in solving the problem. For detailed information see [SN], p.151.

# Symbol Index

| | |
|---|---|
| $A$ | imaginary line of the Chinese String |
| $A_b$ | imaginary line of the substituted Chinese String |
| $A_n$ | imaginary line of the untangled Chinese String with $n$ rings |
| $B$ | set $\{0, 1\}$ |
| $B(n)$ | finite sequence of movements |
| $b^k$ | constant word of length $k$ |
| $b_r, b_r^{-1}$ | movement of rope with ring $r$ |
| $\tilde{b}_r, \tilde{b}_r^{-1}$ | move $r$ |
| $br(n)$ | binary representation of $n$ |
| $\mathrm{d}(s)$ | distance between states $s$ and $\alpha^{(n)}$ |
| $\mathrm{d}(s, t)$ | distance between states $s$ and $t$ |
| $E_{mech}(s)$ | mechanical exchange number of $s$ |
| $E_{mech}[s, t]$ | number of moves of ring 1 between $s$ and $t$ |
| $E_{top}(s)$ | topological exchange number of $s$ |
| $E_{top}[s, t]$ | number of crossings between $s$ and $t$ |
| $F_n$ | free group on $n$ generators |
| $H_n$ | handlebody of genus $n$ |
| $M(n)$ | finite sequence of moves |
| $\mathrm{lsr}(s, k)$ | logical shift to the right by $k$ positions |
| $P_n$ | path graph on $n$ vertices |
| $R^n$ | state graph of the Chinese Rings |
| $S^n$ | $n$-sphere |
| $\alpha^{(n)}$ | word $0^k$ |
| $\pi(X)$ | fundamental group of $X$ |
| $\pi(X, x_0)$ | fundamental group of $X$ at the base point $x_0$ |
| $\omega^{(n)}$ | word $10^{n-1}$ |
| $\mathbb{N}$ | set $\{1, 2, 3, \ldots\}$ |
| $\mathbb{N}_0$ | set $\{0, 1, 2, \ldots\}$ |
| $(a_n)_{n\in\mathbb{N}}$ | greedy square free sequence |
| $(cr_n)_{n\in\mathbb{N}}$ | sequence of number of crossings |
| $(g_n)_{n\in\mathbb{N}}$ | Gros sequence |
| $(m_n)_{n\in\mathbb{N}}$ | sequence of number of movements |
| $[f]$ | homotopy class of $f$ |
| $[n]$ | set $\{1, 2, \ldots, n\}$ |
| $[n]_0$ | set $\{0, 1, \ldots, n-1\}$ |
| $(s)_2$ | $s$ in binary representation |
| $\lceil x \rceil$ | ceiling of x |

| | |
|---|---|
| $\partial H$ | boundary of $H$ |
| $a \oplus b$ | binary digital sum of $a$ and $b$ |
| $G_1 * G_2$ | free product of $G_1$ and $G_2$ |

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# Eidesstattliche Versicherung

(siehe Promotionsordnung vom 12.07.2011, §8, Abs. 2 Pkt. 5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

Höck, Andreas Christopher

München, den 04.05.2017