An agent behavior based model for diffusion price processes

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Eidesstattliche Versicherung

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Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

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Essentially, all models are wrong, but some are useful.

George E. P. Box (1976)

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Abstract

We present an agent behavior based microscopic model for diffusion price processes. After a literature review of the variety of existing models we give a summary for the motivation of this thesis. The general model framework is presented in Chapter 2. We consider a finite market of agents trading a single asset. For the sake of traceability we differentiate the endogenous dynamics of different agents, that is their interaction, and trading mechanisms. To model agents interaction we assign individual characteristics (states) to each agent, which can change over time. Thereby the law for a change of characteristic is dependent on other agents as well as the asset price. To link the endogenous dynamics to the price process we define agent specific trading behavior which influences the asset price by a common pricing rule. First, the induced agent interaction as well as the price process are constructed as Markovian and discrete. In a second step we use exponentially distributed waiting times to embed the two interacting Markov chains in continuous time. Furthermore we state conditions under which in a large market, that is a market with many agents, the agent interaction as well as the price process can be approximated by a diffusion. Within this thesis we consider four examples of using the model framework. The first example aims to make the reader familiar with the terminology. In the second example we show the generality of our model by embedding an existing model in our framework. We model the endogenous agent behavior according to the assumptions made in the original work and asses if we achieve comparable results, that is strong herd behavior of optimists and pessimists leads to phase transitions and oscillations in the price process. Thereby we do not only give more details on the endogenous dynamics than the original work, but also extent the result to a diffusive price process. How this result can be used to model the emergence of a financial guru out of an expert group is shown in the third example. In the last example we prove the flexibility of our model by transferring dynamics observed in quantum mechanics. By modeling the agents behavior similar to an excited quantum system, we explain spikes, jumps and high volatility phases as a result of a hype, which was created by a strong herding behavior.

List of Publications

Parts of this thesis have been published as research papers. Chapter 2 together with Section 3.2 were published under [Hen16]. Section 3.4 was released as [Hen17].

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[Hen17] Christof Henkel. From quantum mechanics to finance: Microfoundations for jumps, spikes and high volatility phases in diffusion price processes. *Physica A: Statistical Mechanics and its Applications*, 469:447 – 458, 2017.

Contents

1 General Introduction

1.1 Literature Review

The foundation for modern financial modeling was set with the French mathematician Louis Bachelier suggesting a Brownian motion based model for describing price fluctuation at the Paris stock exchange in 1900 (see Bachelier [5]). Since then many various probabilistic models, i.e. models in which price processes are modeled as trajectories of stochastic processes, were invented and further developed. Most of these market models derive the dynamics of price processes from the interaction of market participants, the so-called agents. Since the literature not only shows a vast diversity of how the agent interaction is modeled but also how the price process is derived, a short overview of recent models is given in the following. In Föllmer and Schweizer [24] as well as in Horst [33] stock prices are modeled in discrete time as the sequence of temporary equilibria which emerge as a consequence of simultaneous matching of supply and demand of several agents. It is further shown that in a noise trader environment the resulting price process can be approximated by an Ornstein-Uhlenbeck process. Although they already capture some agent interaction and mimic effects, their model is rather simple, which leads to different shortcomings. Firstly, the model lacks feedback effects of the price with respect to the agents behavior, which has been addressed and complementary elaborated in Föllmer, Horst and Kirman [23]. Secondly, simultaneous excess matching seems unrealistic in light of modern financial markets were orders arrive asynchronously in continuous time (see e.g. Bayraktar, Horst and Sircar [10]). To account for this asynchronous order arrival Bayraktar, Horst and Sircar [9],[10] as well as Horst and Rothe [34] use the mathematical framework of queuing theory earlier examined by Mandelbaum, Pats et al. [54] and Mandelbaum, Massey and Reiman [53] for their models. Also the model explained in Lux [48] and Lux [49] takes asynchronous order arrival into consideration by using a so called market maker who matches supply and demand and alters the price accordingly. In order to examine the connection between social economic behavior (e.g. mimic effect represented by herding behavior) and observable price process properties (e.g. volatility clustering, bubbles, crashes) the author differentiates the agents by type and assign specific characteristics. One of the most commonly used characteristics might be the agents opinion. It seems natural to characterize agents by their opinion. Opinion-based models range from binary (e.g. Föllmer [21], Arthur [3], Orléan [57], Latané and Nowak [45], Weisbuch and Boudjema [70] and Sznajd-Weron and Sznajd [64]) to opinions from a continuous spectrum, which are used, for example, to describe large social networks or ratings (see Duffant et al. [17], Gómez-Serrano, Graham and LeBoudec [28] or Weisbuch, Deffuant and Amblard [71]). However the characteristics and the interactions of the agents is described in the respective model, it mostly can be classified in a wider sense as interacting objects with assigned states (respectively phases). Thereby the behavior of an object is modeled by a state transition, where the transition rule might also consider other objects. An early example of binary states is given by the model of Ising [37], originally developed in 1925 to study characteristics of ferro-magnetic material observed in reality. Notably, Ising was able to explain with his model the effect of spontaneous magnetization. Although initially developed for ferro-magnets, the probabilistic framework of Isings model is applicable to other disciplines like for example behavior of binary alloys (Bethe [11]). More recent models considering object interaction were built in diverse contexts. Examples within biology are given by Bramson and Griffeath [12] who examine growth of tumor cells or the model of Kirman [41] explaining herd behavior of ant populations in foraging. For information technology, state transitions can be used to describe TCP connections or HTTP flows (e.g. Baccelli, McDonald and Lelarge [4] or LeBoudec, McDonald and Mundinger [46]). Haken [29] shows the application to laser light fields produced by excited atoms as well as to chemical and biochemical reactions. Weidlich [67] not only examines thermodynamics but also builds the bridge to socio-economics, which is of interest for us hereafter. More precisely he interprets the interacting objects as agents with assigned opinions within a market. The interaction and related dynamics are characterized by agents changing their own opinion according to the predominant opinion in the market. Similar models have been developed by many others. The models of Föllmer [21] as well as Lux [48] might be the most popular. Although the before mentioned models are well suitable to describe social behavior within a market, they lack a sophisticated link to price dynamics. The microscopic model presented in Pakkanen [58] derives the price dynamics of a single asset by interaction between agents. An agent places a buy or sell order in continuous time which is fulfilled by a market maker, who holds a sufficient number of shares in order to match supply and demand instantly. Additionally, the market maker adjusts the asset price according to the current excess demand. Subsequently, the orders not only impact the price of the asset but also the asset price impacts the agents choice to trade as well as the quantity traded. The mathematical framework on which the model is based is quite interesting as it has a lot of advantages. Compared to other models the author allows for a high degree of individualization related to the agents' behavior. The price dynamics are given by a discrete Markov chain which is embedded in continuous time using exponentially distributed waiting times, which results in a price process that is a time homogeneous jump-type Markov process. In a large market the Markov process can then be approximated by a diffusion process. On the other hand, the finite model provides a microscopic foundation for diffusion price processes, heavily used in modern financial mathematics. With Pakkanen's framework diffusion price processes can be broken down to discrete Markov chains, which are often easier to assess analytically, in order to understand phenomena observable in financial markets, . In the primary set-up of Pakkanen's model agent interaction is only taking place via feedback through the asset price, which seems unrealistic. Although the model is partially extended in an example by assigning binary opinions to the agents, a general framework for opinion based studies is missing. Moreover the scaling is restrictively chosen to be proportional to the square root of the number of market participants in order to generate stochastic volatility in the diffusion limit.

1.2 Motivation

The literature review indicates not only a large variety of aspects examined, but also of mathematical frameworks used. A lack of comparability is the result. The aim of this thesis is to develop a model framework, that not only is general enough to entail existing models and to make them comparable, but also flexible enough to incorporate dynamics from other scientific fields like for example physics, chemistry and sociology. We want to further advance the understanding how trader psychology and behavior affects market prices and vice versa. A general market model should not only allow for continuous time trading and a traceable and transparent modeling of agents behavior, but also result in a diffusion price process. Since prices of most financial instruments are modeled in practice as diffusion processes the last is necessary for practical relevance and empirical verifiability. In the following, we provide a model not only containing a convenient framework for describing socio-economic behavior but also a sophisticated link to diffusive price dynamics. Therefore we extend the model elaborated in Pakkanen [58] by providing an additional framework for agent interactions using assigned characteristics. More precisely we assign a state to each agent and measure the endogenous environment by the distribution of all states. We then let the endogenous environment influence each agents' tendency to change his state, thus modeling endogenous interaction. We furthermore allow for interaction between the endogenous environment and the price process leading to feedback effects between agents behavior and asset price which are captured as interacting Markov chains. In contrast to Pakkanen [58] the scaling factor is not fixed in order to be as flexible as possible. See Remark 2.14 in Pakkanen [58] for some more thoughts on the scaling. Although the following chapters are merely an extension of Pakkanen [58], for better readability we state the extended model rather than referencing to the work done there. Moreover as much as possible the same notation is used.

1.3 Computational Implementation

All examples presented in this thesis were implemented in the statistical programming language R (Ihaka and Gentleman [36]) in order to simulate the related distributions and stochastic processes and to illustrate the results. Especially functions which are presented as solutions of ordinary- and stochastic differential equations without provision of a closed analytical form, are illustrated using numerical solvers implemented in the R-packages deSolve (see Soetaert, Petzoldt and Setzer [61]) and sde (Iacus [35]) or self-written implementation of the Euler-Maruyama method[1].

[1] See for example Kloeden [42] Chapter 9.1

2 Microscopic model

2.1 Finite Microscopic Model

In this subsection we set the general finite framework for our microscopic model. We extend the model described in Pakkanen [58] by assuming that the asset price depends on another variable, namely the market character which is defined as the distribution of characteristics of the agents participating in the market. With introducing the market character as a key driver of the asset, we can better separate agents' behavior and price dynamics. Thus the tractability of how the agent, from a rational and psychological point of view, impacts the price is improved. The rather general term of market character does not only include agents opinion, which makes our model comparable with other opinion based models, but also agent type (e.g. noise trader, fundamentalist, guru) or other individual characteristics. We use a Markovian framework, that arose from early models describing phenomena of statistical mechanics and has been the foundation of many models describing interacting objects (see e.g. Kindermann and Snell [40]). Although the Markov assumption is rather restrictive, it makes the model memoryless and hence more simple. Additionally, the assumption is consistent with the property of diffusion processes that are heavily used in financial markets.

We start with the definition of the endogenous environment by specifying heterogeneous market participants, to which we assign states from a fixed finite set. To reduce complexity we introduce a measure for the distribution of states through the agents which leads to the terminology of a market character. Then, we define the occurrence and severeness of interaction between the agents, which is modeled as an influence a possible state transition. This results in a dynamical endogenous Markovian system in which we measure the related aggregated behavior of the agents with the, now dynamic, market character. Next, we specify the individual propensity of the agents to place buy and sell orders and how their actions impact the price. Hereby we explicitly allow for the consideration of external factors in form of random signals. Finally, we embed the price process and the market character memorylessly in continuous time using exponential distributed waiting times for the actions of agents. We close the chapter by summarizing the microscopic model and showing the existence of the underlying probability space in a lemma.

Let $\mathbb{A}_n = \{1, ..., n\}$ be the *set of agents* participating in the market. To classify common or individual characteristics of the agents we assign to each agent a state, which is used later on to model their endogenous interaction.

Definition 2.1 (State- and configuration space)**.**
Let x^a be the *state* of agent $a \in \mathbb{A}_n$ which is an element of the fixed finite *state space* $S = \{s_1, ..., s_m\}, m \in \mathbb{N}$. The vector of all individual states takes values in the compact *configuration space* $C := S^{\mathbb{A}_n} = \{x = (x^a)_{a=1}^n,\ x^a \in S\}$.

During the time $t \in [0, \infty)$ each agent can decide to act (e.g. to change his state). The time of the k-th action is nominated by $T_k \geq 0,\ k \in \mathbb{N}$. The action times are described later in detail, however we use the terminology already to describe the development of the states within discrete time in the following definition.

Definition 2.2 (State process)**.**
The state of agent a at time T_k is defined as $x^a_{T_k} \in S$. We capture the development of agent a's state by the process $(x^a_k)_{k\in\mathbb{N}} := (x^a_{T_k})_{k\in\mathbb{N}}$ and the development of all agents states by the n-dimensional *state process* $(x_k)_{k\in\mathbb{N}} = (x^a_k)_{a\in\mathbb{A}_n, k\in\mathbb{N}}$. We assume that the vector of initial states is distributed following some n-dimensional distribution function. In particular $x_0 \sim F^n_{x_0}$.

In general, the cardinality of the state space S will be much smaller than the one of $\mathbb{A}_n$ (i.e. $m << n$). Moreover, later on we are interested in the development of the market as a whole rather than the development on the level of individual states. Hence it makes sense to coarsen the observable information for the sake of reduced complexity. Rather than the individual states, we consider in the next definition the proportion of all states within the market, representing the overall characteristics of market participants.

Definition 2.3 (Market character)**.**
For each state we measure the proportion of state s_i among the agents at time T_k by

$$M^i_k := n^{-d_1} \sum_{a\in\mathbb{A}_n} \mathbb{1}_{s_i}(x^a_k),\ k \in \mathbb{N},\ d_1 \in \mathbb{Q}^+ \geq 1/2. \tag{2.1}$$

The *market character* at time T_k is is defined as the m-dimensional vector valued process of all state proportions, i.e. $M_k = (M^i_k)_{i=1}^m,\ k \in \mathbb{N}$. Additionally, we denote the initial distribution of the market character resulting from Definition 2.2, that is the m-dimensional probability distribution of M_0 as $F^n_{M_0}$.

Remark 2.4 (Scaling of market character)**.** In other models the scaling of the endogenous environment is fixed to be $1/n$ (e.g. Horst and Rothe [34] or Bayraktar, Horst and Sircar et. al [10]) or $1/\sqrt{n}$ (e.g Pakkanen [58]). As concluded in Pakkanen [58], "Ultimately, the choice of scaling depends on what one wants to model - it seems that $1/\sqrt{n}$ is suited to the study of short-term fluctuations and volatility, whereas $1/n$ is perhaps more appropriate in studies of long-term behavior." We choose a variable scaling factor of $n^{-d_1}, d_1 \in \mathbb{Q}^+ \geq 1/2$, in order to provide a rather general framework, in which we study can study both.[1] Moreover, note that by construction $n^{d_1-1}M_k$ is a probability measure on the configuration space C.

Remark 2.5 (Dimension of market character)**.** While in many opinion-based models only some average-type mood is considered (e.g. Lux [48], Pakkanen [58]), by construction our market character is m-dimensional. Although adding additional complexity, having a rather general market character provides the necessary flexibility to model agents behavior more specifically. Anyhow, if needed, a reduction of the market character information to relevant properties (e.g. average agent state) is still possible.

We assume that any change of the market is a direct consequence of agents' behavior. His behavior is given by so called actions that can either be the change of his state or trading the asset. We index each of these actions by $k \in \mathbb{N}$. All information on actions, that have taken place in the past form the current market history. The k-th action as well as the market history are set more precisely in the next definition.

Definition 2.6 (k-th action, market history)**.**
The *k-th action* is characterized by the tupel $(T_k, A_k, P_k, M_k, B_k), k \in \mathbb{N}$, where T_k is the time when the action occurs, $A_k \in \mathbb{A}_n$ is the acting agent at time T_k and $B_k \in \{0, 1\}$ is an *action indicator* whether the agent trades ($B_k = 1$) or changes his state ($B_k = 0$). P_k is the *price* per share[2] and M_k the above mentioned character of the market. All information is captured in the *market history*, which is given by $\mathcal{G}_k := \sigma(T_i, A_i, P_i, M_i, B_i, i \leq k)$.

Assumption 2.7 (k-th action)**.**
We require that only one agent is acting at a specific point in time as well as that the acting agent either trades or changes his state. Although the first part of the assumption seems rather strong, it is however reasonable as actions are performed in continuous time and are very unlikely to happen at the same point in time. The dichotomous behavior of any agent is assumed mainly for the reason of simplicity, as it leaves the market character and the price process rather separable.

Next, we specify the tendency of each agent to act before characterizing the action and related impact. To determine the likelihood of agent a to be the one who acts at time T_k, we assign to each agent intensity (or rate-) functions and then weight the agents.
The agent specific tendency to act (i.e. to trade or to switch his state) is assumed to be dependent on the price as well as the character of the market. For the propensity to trade we use the trading intensity function defined in Pakkanen [58], but allow additionally for

[1] See introduction of Section 2.2 for additional comments on the scaling.
[2] Note that, the price is not necessarily assumed to be logarithmic as in Pakkanen [58].

dependence on the character of the market. The state transition rate function is defined analogously, depending on both, market character and price. We assume that there is always an agent who wants to act and thus require the trading intensity function as well as the transition rate function to be positive.

Definition 2.8 (Trading intensity, state transition rate, action rate)**.**
Let $\lambda_a\colon \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}_+, a\in\mathbb{A}_n$ be the continuous and bounded *trading intensity function* of agent a. Moreover the *aggregate trading intensity* is defined as the sum of trading intensities of over all agents via $\lambda_{\mathbb{A}_n} := \sum_{a=1}^n \lambda_a$.
Similarly let $\mu_a\colon \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}_+, a\in\mathbb{A}_n$ be the *state transition rate function*, which is assumed to be continuous and bounded and denote the *aggregate state transition rate* by $\mu_{\mathbb{A}_n} := \sum_{a=1}^n \mu_a$. We summarize the intensity of all actions with the *aggregated action rate* $\nu_{\mathbb{A}_n}(x,v) := \lambda_{\mathbb{A}_n}(x,v) + \mu_{\mathbb{A}_n}(x,v)$.

In the next definition we specify the acting probabilities of individual agents. Heuristically we weight the respective intensity or rate function.

Definition 2.9. (Acting probabilities)
The probability, that agent a trades at T_k is defined as

$$\mathbb{P}(A_k = a, B_k = 1|\mathcal{G}_{k-1}) = \frac{\lambda_a(P_{k-1}, M_{k-1})}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})}. \tag{2.2}$$

Similarly, we define the probability, that agent a changes his state at T_k by

$$\mathbb{P}(A_k = a, B_k = 0|\mathcal{G}_{k-1}) = \frac{\mu_a(P_{k-1}, M_{k-1})}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})}. \tag{2.3}$$

Moreover the probability that the k-th action is a state transition is set as

$$\mathbb{P}(B_k = 0|\mathcal{G}_{k-1}) = \sum_{a=1}^{n} \frac{\mu_a(P_{k-1}, M_{k-1})}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})} = \frac{\mu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})}, \tag{2.4}$$

and analogously the probability that the k-th action is a trade is given by

$$\mathbb{P}(B_k = 1|\mathcal{G}_{k-1}) = \frac{\lambda_{\mathbb{A}_n}(P_{k-1}, M_{k-1})}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})} = 1 - \mathbb{P}(B_k = 0|\mathcal{G}_{k-1}). \tag{2.5}$$

The next step is to characterize the state transition laws and consequentially derive the dynamics of the market character. Although the probability is not further determined here, we introduce an extra notation to clarify that we explicitly allow for dependence of individual state transition probabilities on the market character and price.

Definition 2.10 (State transition probability)**.**
We use the following notation for the individual state transition probability, i.e. the probability that agent a changes from s_i to s_j, given that he is the one that wants to change

his state.

$$\Pi^{i,j}_{n,a}(P_{k-1}, M_{k-1}) := \mathbb{P}(x^a_k = s_j | x^a_{k-1} = s_i, B_k = 0, A_k = a) \tag{2.6}$$

We capture all state transition probabilities per agent in a transition matrix, i.e. we define

$$\Pi_{n,a} := (\Pi^{i,j}_{n,a})^m_{i,j=1} \tag{2.7}$$

While Equation (2.6) is quantifying a single movement from state s_i to s_j, we are rather interested in the aggregated dynamics. The aggregated behavior of all agents, i.e. are agents rather joining or leaving a state, is used to describe attractiveness of a state.

Definition 2.11 (Aggregated state transition)**.**
Let $\Pi^{i-}_{n,a}$ describe the aggregated propensity to leave state s_i, i.e. the probability of a state transition from s_i to any other state at time T_k More precisely,

$$\Pi^{i-}_{n,a}(P_{k-1}, M_{k-1}) := n^{d_1-1} M^i_{k-1} \sum_{\substack{j=1,\\ j\neq i}}^{m} \Pi^{i,j}_{n,a}(P_{k-1}, M_{k-1}), \tag{2.8}$$

where the pre-factor $n^{d_1-1}M^i_{k-1}$ is the probability that the acting agent A_k had state s_i, which is well defined as $n^{d_1-1}M_{k-1}$ is a probability measure on C. Analogously, we define the aggregated propensity to switch to state s_i by

$$\Pi^{i+}_{n,a}(P_{k-1}, M_{k-1}) := n^{d_1-1} \sum_{\substack{j=1,\\ j\neq i}}^{m} M^j_{k-1} \Pi^{j,i}_{n,a}(P_{k-1}, M_{k-1}). \tag{2.9}$$

We are now able to derive the dynamics of the market character in the following lemma. As we assume that only one agent can act on each action time T_k the proportion of a state at time T_k can either increase or decrease by n^{-d_1} or stay unchanged.

Lemma 2.12 (Market character dynamics)**.**
The probability that an agent of state s_i switches to a different state s_j and therefore that the proportion of state s_i decreases by n^{-d_1} is given by

$$\mathbb{P}(M^i_k - M^i_{k-1} = -n^{-d_1} | \mathcal{G}_{k-1}) = \frac{\sum_{a=1}^n \mu_a(P_{k-1}, M_{k-1}) \Pi^{i-}_{n,a}(P_{k-1}, M_{k-1})}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})} \tag{2.10}$$

and similarly the probability that the occupancy measure increases by n^{-d_1} is given by

$$\mathbb{P}(M^i_k - M^i_{k-1} = n^{-d_1} | \mathcal{G}_{k-1}) = \frac{\sum_{a=1}^n \mu_a(P_{k-1}, M_{k-1}) \Pi^{i+}_{n,a}(P_{k-1}, M_{k-1})}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})}. \tag{2.11}$$

Proof. See Appendix 5.1. □

After we set the general framework for the endogenous environment we now can describe the impact of the agents behavior on the price and feedback effects. Although the main part of the model consists of endogenous factors (i.e. market character and feedback effects of the price) in order to be flexible as well as to be consistent with Pakkanen [58], we allow the price dynamics to be dependent on a random signal describing exogenous impacts. The volume of shares agent A_k would trade at T_k is quantified by the *excess demand function* defined next.

Definition 2.13 (Excess demand functions, signals).
The *excess demand function* $e_a^n \colon \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{A}_n$ is a measurable function depending on P_{k-1}, M_{k-1} and the random variable ξ_k, which is assumed to be independent of $\mathcal{G}_{k-1}$ and A_k. Additionally, we assume that the *signals* $(\xi_k)_{k=1}^\infty$ are independent and identical distributed (i.i.d.) with cumulative distribution function (cdf) F_ξ.

The price P_k at time T_k will be set by a market maker, which is assumed to handle all trades, and is defined by a pricing rule depending on the excess demand of the acting agent and the old price P_{k-1}.

Definition 2.14 (Pricing rule).
Consider the borel measurable *pricing rule*[3] function $r_n \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ setting the price P_k via

$$P_k = r_n(e_{A_k}^n(P_{k-1}, M_{k-1}, \xi_k), P_{k-1}). \tag{2.12}$$

By construction $(P_k)_{k=0}^\infty$ and $(M_k)_{k=0}^\infty$ now are two interacting Markov chains. In order to embed them homogeneously in continuous time and thus describing the price as well as the character by a time homogeneous Markov process, we further characterize the points in times in which the agents decide to act.

Definition 2.15 (Intra-action times).
The *intra-action times* $(\tau_k)_{k\geq 1}$ are defined as $\tau_k := T_k - T_{k-1}, k \geq 1$.
Since we want the intra-action times to be memory-less for the sake of simplicity, i.e.

$$\mathbb{P}(\tau_k > t + h | \tau_k > h, \mathcal{G}_{k-1}) = \mathbb{P}(\tau_k > t | \mathcal{G}_{k-1}),\ t, h \geq 0. \tag{2.13}$$

the intra-action times are assumed to be exponentially distributed[4]. Heuristically we assume that the rate of the exponential distribution is given by the aggregated action rate, i.e.

$$\mathbb{P}(\tau_k \in [0, t] | \mathcal{G}_{k-1}) = 1 - e^{-\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})t},\ t \geq 0, \tag{2.14}$$

[3] As we could factor out the market character to the excess demand function, where it is already considered, we refrain from including it in the pricing rule.

[4] Note that the exponential distribution is the only continuous memory-less distribution. See e.g. Feller [20] I.3 for a discussion and proof.

More precisely, to ensure a sufficient level of independence between the source of randomization and the price as well as market character we need to assume that the intra-action times $(\tau_k)_{k\geq 1}$ are given by

$$\tau_k := \frac{\gamma_k}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})},\ k \in \mathbb{N}, \tag{2.15}$$

where $(\gamma_k)_{k\geq 1}$ is a sequence of i.i.d. random variables independent of $(P_k, M_k)_{k\geq 0}$ with $\gamma_1 \sim Exp(1)$.

Definition 2.16 (Price process, market character index)**.**
After setting an initial price P_0 which we assume to be distributed according to a cdf F_{P_0} and fixing $T_0 = 0$, we can define the *price process* as

$$X_t^n := \sum_{k=0}^{\infty} P_k \mathbb{1}_{[T_k, T_{k+1})}(t),\ t \geq 0. \tag{2.16}$$

Analogously we introduce the *market character index* via

$$V_t^n := \sum_{k=0}^{\infty} M_k \mathbb{1}_{[T_k, T_{k+1})}(t),\ t \geq 0. \tag{2.17}$$

Note that by construction X_t^n and V_t^n are cádlág and that $F_{M_0}^n$, in contrast to F_{P_0}, is depending on n.
The next lemma now summarizes the construction of the finite microscopic model. It states the existence of a probability space carrying the price process as well as the market character index as time homogeneous Markov processes. Furthermore it gives the rate kernel as the product of action rate and transition kernel. The basis of the lemma builds the synthesis theorem (e.g. Theorem 12.18 of Kallenberg [39]), which embeds a discrete Markov chain into continuous time using exponentially distributed waiting times.

Lemma 2.17 (Existence)**.**
If the preceding Assumptions hold, then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which carries the model such that $(X_t^n, V_t^n)_{t\in[0,\infty)}$ is a time homogeneous pure jump Markov process with rate kernel

$$K_n(x, v, dy, dw) := \nu_{\mathbb{A}_n}(x, v) k_n(x, v, dy, dw), \tag{2.18}$$

where the transition kernel $k_n(x, v, dy, dw)$ is a regular version of the conditional distribution $\mathbb{P}(P_1 - P_0 \in dy, M_1 - M_0 \in dw | P_0 = x, M_0 = v)$.

Proof. By the assumption made in Definition 2.15 and the construction of the Markov chain $(P_k, M_k)_{k=0}^{\infty}$ the synthesis theorem (e.g. Theorem 12.18 of Kallenberg [39]) states that $(X_t^n, V_t^n)_{t\in[0,\infty)}$ is a pure jump-type Markov process and also gives the rate kernel. Time homogeneity is given by the recursive definition of $(P_k, M_k)_{k=0}^{\infty}$ (see e.g. Proposition

8.6 of Kallenberg [39]) as the pricing rule r_n as well as the transition matrix $\Pi_{n,a}$ are independent of the time. □

2.2 Diffusion approximation

In this chapter we elaborate the general conditions under which the indices of our microscopic model can be approximated by a (Itô-) diffusion process $(X_t, V_t)_{t\in[0,\infty)}$, which is described as the solution of the $m+1$ dimensional stochastic differential equation (SDE)

$$d(X_t, V_t) = \hat{b}(X_t, V_t)dt + \hat{\sigma}(X_t, V_t)dB_t, \tag{2.19}$$

where B_t is a $m+1$ dimensional Brownian motion. The functions $\hat{b}(X_t, V_t) \in \mathbb{R}^{m+1}$ and $\hat{\sigma}(X_t, V_t) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$ are called drift- and diffusion coefficient[5]. To make the functions $\hat{b}$ and $\hat{\sigma}$ more clear, one may consider a small time interval Δt. Heuristically, the expected change of (X_t^n, V_t^n) is described by $\hat{b}(X_t^n, V_t^n)\Delta t$, whereas the covariance matrix of a change of (X_t^n, V_t^n) is given by $\hat{\sigma}(X_t^n, V_t^n)^2\Delta t$. In the main theorem of this chapter we not only proof the existence of a limit of $(X_t^n, V_t^n)_{t\in[0,\infty)}$ as $n \to \infty$, but also identify the coefficients $\hat{b}$ and $\hat{\sigma}$ as the limit of the first, respectively second, moment of the finite process $(X_t^n, V_t^n)_{t\in[0,\infty)}$.

To ensure the convergence of the time-homogeneous pure jump process $(X_t^n, V_t^n)_{t\in[0,\infty)}$ to a (continuous) diffusion as $n \to \infty$, we need to make several additional assumptions. Firstly we need to ensure the existence and convergence of the first and second moments of the price process X_t^n as well as of the market character index V_t^n. Secondly, the moments should meet some regularity conditions in the limit. Although convergence to a continuous diffusion can be achieved under different regularity conditions (see e.g. Mao [55] or Xua et al. [73] for more details) we choose the locally Lipschitz and linear growth condition to be consistent with Pakkanen [58]. Finally, we want the action rate function $\nu_{\mathbb{A}_n}$ to converge "nicely" in order to not have jumps in the limit of (X_t^n, V_t^n). Moreover the right scaling of the market character- and price index is of importance. When the number of agents tends to infinity, the ratio of jump size and number of jumps not only determines if the indices converge to a diffusion but also define its first and second moments. The scaling of the indices is of most importance for the jump size. When the scaling is too weak (i.e. d_1, d_2 are small) the first and second moments might not be finite in the limit and hence our indices do not converge to a diffusion. If on the other hand the scaling is too strong (d_1, d_2 is too big) the second moment vanishes (i.e. converges to zero) and our index converges to a "deterministic diffusion" which is characterized as a solution to an ordinary differential equation (ODE). If the scaling is even stronger additionally the first moment vanishes leading to a constant as a limit of the index.

[5] See e.g. Øksendal [56] for an introduction to Itô diffusion processes and basic properties.

To characterize the first moment of X_t^n we introduce a function which quantifies the aggregated expectation in terms of demand and supply in our market in the next definition. The so-called expected aggregate excess demand z_n is the individual expected excess demand of the agents aggregated by weighting their respective trading intensity function and depends on the market character as well as on the price.

Definition 2.18 (Excess demands and pricing rule)**.**
Let the *expected aggregate excess demand* at $(x, v) \in \mathbb{R} \times \mathbb{R}^m$ be defined as

$$z_n(x, v) := n^{-d_2} \sum_{a=1}^{n} \lambda_a(x, v) \mathbb{E}[e_a^n(x, v, \xi_1)], \ d_2 \in \mathbb{Q}^+ \geq 1/2. \tag{2.20}$$

Furthermore we assume the pricing rule r_n is given by

$$r_n(q, x) = x + \alpha n^{-d_2} q + u_n(q, x), \ q, x \in \mathbb{R}, \ d_2 \in \mathbb{Q}^+ \geq 1/2, \tag{2.21}$$

where $\alpha > 0$ and $u_n, n \in \mathbb{N}$ is a borel measurable function such that $\forall \delta > 0 \ \exists C_n^\delta, \ n \in \mathbb{N}$ such that $C_n^\delta = o(n^{-1})$ and $\sup_{|x| \leq \delta} |u_n(q, x)| \leq C_n^\delta |q|, \ \forall q \in \mathbb{R}, \ n \in \mathbb{N}$.

As visible in Equation (2.21) we assume the pricing rule to be nearly affine, that is affine apart from a function u_n which is bounded by a constant C_n^δ that converges to zero when the number of agents tents to infinity. So we ensure that a possible "nice" behavior of excess demands is sufficiently carried to the increments of X_t^n. Thereby we allow for a flexible scaling factor n^{-d_2} following the same rational as stated in Remark 2.4.

Analogously to Equation (2.20) we define the expected aggregated transition related to state s_i by summing the individual agent transitions weighted by the respective transition rate. The resulting expected aggregate state transition then describes the first moments of V_t^n.

Definition 2.19 (Expected aggregated state transition)**.**
We define the *expected aggregate state transition of state* s_i as

$$b_n^i(x, v) := n^{-d_1} \sum_{a=1}^{n} \mu_a(x, v) \left(\Pi_{n,a}^{i+}(x, v) - \Pi_{n,a}^{i-}(x, v) \right). \tag{2.22}$$

In summary, we write the *expected aggregate state transition* as

$$b_n(x, v) := \begin{bmatrix} b_n^1 \\ b_n^2 \\ \vdots \\ b_n^m \end{bmatrix} (x, v) \tag{2.23}$$

.

In the following Definition we quantify the second moments of the price process. In particular we define the *expected trading volume* by aggregating the second moments of the heterogeneous excess demand functions weighted by the respective intensity function.

Definition 2.20 (Trading volume).
The *expected trading volume* at price and character level $(x, v) \in \mathbb{R} \times \mathbb{R}^m$ is defined by

$$\sigma_n(x, v) := \left(n^{-2d_2} \sum_{a=1}^{n} \lambda_a(x, v) \mathbb{E}[e_a^n(x, v, \xi_1)^2] \right)^{1/2} \tag{2.24}$$

Analogously, we describe the second moment of the market character by aggregating the second moments of the individual state transition weighted by the respective rate function. The resulting so-called *transition volume* is described by the variance within states s_i on the one hand and covariances between states s_i and s_j on the other.

Definition 2.21 (Transition volume).
We denote the *transition volume between* s_i *and* s_j with

$$c_n^{i,j}(x, v) := \left(-n^{-(d_1+1)} \sum_{a=1}^{n} \mu_a(x, v)(v_i \Pi_{n,a}^{i,j}(x, v) + v_j \Pi_{n,a}^{j,i}(x, v)) \right)^{1/2}, (x, v) \in \mathbb{R} \times \mathbb{R}^m \tag{2.25}$$

and the *transition volume within* s_i

$$c_n^{i}(x, v) := \left(n^{-2d_1} \sum_{a=1}^{n} \mu_a(x, v)(\Pi_{n,a}^{i+}(x, v) + \Pi_{n,a}^{i-}(x, v)) \right)^{1/2}, (x, v) \in \mathbb{R} \times \mathbb{R}^m. \tag{2.26}$$

In short, we write

$$c_n(x, v) := \begin{bmatrix} c_n^1 & c_n^{1,2} & \dots & c_n^{1,m} \\ c_n^{2,1} & c_n^2 & & \vdots \\ \vdots & & \ddots & \\ c_n^{m,1} & \dots & & c_n^m \end{bmatrix} (x, v) \tag{2.27}$$

and call the function c_n *transition volume.*

Remark 2.22. If $n^{-2d_1} \mu_{\mathbb{A}_n}(x, v) \xrightarrow{n\to\infty} 0$ then $c_n \xrightarrow{n\to\infty} 0$.

Proof. See Appendix 5.2. □

In order to achieve convergence of $(X_t^n, V_t^n)_{t\in[0,\infty)}$ to a continuous diffusion neither jump size nor the intensity should explode. While the jump size of V_t^n is bounded by the construction of the difference $M_k - M_{k-1}$, we need restrictions on the excess demands in order to bound the jump size of X^n_t.

Assumption 2.23 (No explosions)**.**
For every $\delta > 0$,

1. $\limsup_{n\to\infty} \sup_{|(x,v)|\le\delta} |\frac{\lambda_{\mathbb{A}_n}(x,v)}{n}| < \infty$,
2. $\limsup_{n\to\infty} \sup_{|(x,v)|\le\delta} |\frac{\mu_{\mathbb{A}_n}(x,v)}{n^{d_1}}| < \infty$ and
3. $\{e^n_a(x,v,\xi_1)^2 \colon |(x,v)| \le \delta,\ a \in \mathbb{A}_n,\ n \in \mathbb{N}\}$ is uniformly integrable

As mentioned in the introduction of this section we need to require some regularity condition on the limit functions of the first and second moments of our market indices in order to show the existence and describe the diffusion as a unique solution of a SDEs. For readers convenience we restate[6] the regularity condition, which is used in the following theorem, in the next definition.

Definition 2.24 (Locally lipschitz, linear growth)**.** A function f is called locally lipschitz and of linear growth if there exist constants L_1 and $L_{\tilde{n}}$ such that f satisfies the following conditions

1. (locally lipschitz) $\forall \tilde{n} \ge 1,\ \forall (x,v),(y,w) \in \mathbb{R}^{d+1}$ with $\max(|(x,v)|,|(y,w)|) \le \tilde{n}:$
   $|f(x,v) - f(y,w)|^2 \le L_{\tilde{n}}|(x,v) - (y,w)|^2$
2. (linear growth) $\forall (x,v) \in \mathbb{R}^{d+1}:\ |f(x,v)|^2 \le L_1(1 + |(x,v)|^2)$

Now, we are in the position to apply Theorem IX. 4.21 of Jacod and Shiryaev [38], which gives the convergence of the process $(X^n_t, V^n_t)_{t\in[0,\infty)}$ to a diffusion process when the number of market participants tends to infinity. In the large market limit the drift coefficient is determined by the limit of the functions z_n and b_n defined in Definition 2.18 and 2.19, while the diffusion coefficient is given by the limit of functions σ_n and c_n described in Definition 2.20 and 2.21. We summarize the diffusion approximation in the following theorem.

Theorem 2.25 (Diffusion approximation)**.**
We assume that for the functions z_n, b_n, σ_n and c_n there exist continuous functions z, b, σ and c that are locally lipschitz and of linear growth such that $z_n \to z, b_n \to b, \sigma_n \to \sigma$ and $c_n \to c$ uniformly on compact sets (u.o.c.)[7] for $n \to \infty$. If additionally Assumption 2.23 holds and $F^n_{M_0} \xrightarrow{n\to\infty} F_{M_0}$, then

$$(X^n_t, V^n_t)_{t\in[0,\infty)} \xrightarrow{\mathcal{L}} (X_t, V_t)_{t\in[0,\infty)} \ in\ D_{\mathbb{R}^{m+1}}[0,\infty), \tag{2.28}$$

[6] The locally lipschitz and linear growth conditions can be found for example in Jacod and Shiryaev [38] or Mao [55].
[7] See e.g. Remmert [60] §1.3 for the terminology of compact convergence.

where D denotes the Skorokhod space and $(X_t, V_t)_{t\in[0,\infty)}$ is the unique strong solution of the SDEs

$$\begin{cases} dX_t = \alpha z(X_t, V_t)dt + \alpha\sigma(X_t, V_t)dB_t, & X_0 = \zeta \\ dV_t = b(X_t, V_t)dt + c(X_t, V_t)d\mathbf{B}_t, & V_0 = \theta, \end{cases} \tag{2.29}$$

where $(B_t)_{t\in[0,\infty)}$ is a one dimensional standard Brownian motion, $\zeta \sim F_{P_0}$ independent of B_t, and $(\mathbf{B}_t)_{t\in[0,\infty)}$ is a m-dimensional Brownian motion, which is independent of $\theta \sim F_{M_0}$.

Proof. See Appendix 5.3. □

In case the large market limit of the price- and market character index given by Theorem 2.25 is deterministic (i.e. $\sigma_n = 0$ and $c_n = 0$), the rate by which the pure-jump type process $(X_t^n, V_t^n)_{t\in[0,\infty)}$ converges to the limit process $(X_t, V_t)_{t\in[0,\infty)}$ when $n \to \infty$ can be assessed. The following proposition gives particularly the convergence rate as being the speed by which σ_n, respectively c_n, tend to zero.

Proposition 2.26 (Rate of convergence)**.** Assume that $F_{M_0}^n = F_{M_0}$. Let $(a_n)_{n\geq 0}$ be a positive sequence with $a_n \to \infty$ such that

1. $a_n^2\sigma_n^2 \xrightarrow[n\to\infty]{\text{u.o.c.}} \hat{\sigma}^2$ and $a_n^2 c_n^2 \xrightarrow[n\to\infty]{\text{u.o.c.}} \hat{c}^2$ for some continuous functions $\hat{\sigma}, \hat{c}$
2. $\sqrt{n}a_n = O(n^{1/2(d_1+1)} + n^{d_2} + 1/C_n^\delta)$
3. $a_n(z_n - z) \xrightarrow[n\to\infty]{\text{u.o.c.}} 0$ and $a_n(b_n - b) \xrightarrow[n\to\infty]{\text{u.o.c.}} 0$.

Then

$$\sup_{s\leq t} |(X_s^n, V_s^n) - (X_s, V_s)| \leq a_n^{-1} \sup_{s\leq t} |(Y_s, Z_s)|, \;\; \forall t \geq 0, \tag{2.30}$$

where $(Y_t, Z_t)_{t\in[0,\infty)}$ is the solution of the SDE

$$\begin{cases} dY_t = \hat{\sigma}(Y_t, Z_t)dB_t, & Y_0 = 0 \\ dZ_t = \hat{c}(Y_t, Z_t)d\mathbf{B}_t, & Z_0 = 0. \end{cases} \tag{2.31}$$

Proof. See Appendix 5.4. □

3 Examples

3.1 Example 1: A Noise Trader Model

In this section we embed the example of Pakkanen [58] chapter 3.1 in our model. This not only makes the reader familiar with the terminology, but also shows where we extended the model with a more general framework.
In the first sub-section we set the model framework and derive the diffusive limit when the number of market participants tends to infinity. In the second sub-section we reduce the dimension of the market character according to Remark 2.5 and show the consistency of the resulting diffusion.

3.1.1 Model description and diffusion approximation

We consider a finite set of agents $\mathbb{A}_n = \{1, ..., n\},\ n \in \mathbb{N}$. Following Pakkanen's assumptions, we set a dichotomous state space $S = \{-1, 1\}$ representing a pessimistic ($x_k^a = -1$) or an optimistic ($x_k^a = 1$) opinion of agent a at time T_k. We specify the transition matrix to be

$$\Pi_{n,a} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.1}$$

which is assumed to be common to all agents and independent of n. Moreover we choose a scaling factor of $\frac{1}{\sqrt{n}}$ for the market character, i.e. $d_1 = 1/2$. Hence the market character $(M_k)_{k\geq 0} = (M_k^1, M_k^2)_{k\geq 0}$ is a two-dimensional process which represents the distribution of opinions with

$$M_k^1 = \frac{1}{\sqrt{n}} \sum_{a \in \mathbb{A}_n} \mathbb{1}_{\{-1\}}(x_k^a),\ \ k \geq 0 \tag{3.2}$$

$$M_k^2 = \frac{1}{\sqrt{n}} \sum_{a \in \mathbb{A}_n} \mathbb{1}_{\{1\}}(x_k^a),\ \ k \geq 0 \tag{3.3}$$

We assume a common constant state transition rate

$$\mu_a(P_{k-1}, M_{k-1}) = \bar{\mu} \in \mathbb{R}^+, \tag{3.4}$$

whereas the trading intensity function λ_a is as general as in Definition 2.8. Following Pakkanen, the agent buys or sells unit sized shares randomly with equal probability, i.e.

the common excess demand function is given by

$$e_a(P_{k-1}, M_{k-1}, \xi_k) = \xi_k \tag{3.5}$$

with

$$\mathbb{P}(\xi_1 = 1) = \mathbb{P}(\xi_1 = -1) = 1/2 \tag{3.6}$$

The pricing rule is defined as

$$r_n(q, x) = x + \frac{\alpha}{\sqrt{n}} q, \tag{3.7}$$

viz. the pricing rule is affine and the resulting price process is scaled by $d_2 = 1/2$. We set the initial price to be some constant $p_0 \in \mathbb{R}$, and assume that each agent chooses his initial opinion from S independently with equal probabilities, i.e.

$$\mathbb{P}(x_0^a = -1) = \mathbb{P}(x_0^a = 1) = 1/2 \ \forall a \in \mathbb{A}_n. \tag{3.8}$$

Since the excess demand function does not depend on the market character, the symmetric probabilities describing the price development are the same as stated in Pakkanen [58] Equation (3.4), namely

$$\begin{aligned} \mathbb{P}(P_k - P_{k-1} = -\frac{\alpha}{\sqrt{n}} | \mathcal{G}_{k-1}) &= \mathbb{P}(P_k - P_{k-1} = \frac{\alpha}{\sqrt{n}} | \mathcal{G}_{k-1}) \\ &= \frac{1}{2} \frac{\lambda_{\mathbb{A}_n}(P_{k-1}, M_{k-1})}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})} \end{aligned} \tag{3.9}$$

The probabilities describing the market character are given by Lemma 2.12 as

$$\begin{aligned} \mathbb{P}(M_k^1 - M_{k-1}^1 = \frac{1}{\sqrt{n}}) &= \mathbb{P}(M_k^2 - M_{k-1}^2 = -\frac{1}{\sqrt{n}}) \\ &= \frac{\sqrt{n} \bar{\mu} M_{k-1}^2}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \mathbb{P}(M_k^1 - M_{k-1}^1 = -\frac{1}{\sqrt{n}}) &= \mathbb{P}(M_k^2 - M_{k-1}^2 = \frac{1}{\sqrt{n}}) \\ &= \frac{\sqrt{n} \bar{\mu} M_{k-1}^1}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})}, \end{aligned} \tag{3.11}$$

where we used that the aggregated state transition is given by

$$\Pi_n^{1-}(P_{k-1}, M_{k-1}) = \Pi_n^{2+}(P_{k-1}, M_{k-1}) = \frac{1}{\sqrt{n}} M_{k-1}^1, \tag{3.12}$$

$$\Pi_n^{1+}(P_{k-1}, M_{k-1}) = \Pi_n^{2-}(P_{k-1}, M_{k-1}) = \frac{1}{\sqrt{n}} M_{k-1}^2 \tag{3.13}$$

and that $\mu_{\mathbb{A}_n}(P_{k-1}, M_{k-1}) = n\bar{\mu}$.
After noting that the example is now well posed and the existence of a probability space carrying the example is given by Lemma 2.17 we want to apply Theorem 2.25 in order to derive the diffusive limit. Therefore we calculate first and second moments as part of the following proposition.

Proposition 3.1 (First and second moments)**.**
Basic calculations yield

$$z_n(x,v) = 0, \tag{3.14}$$

$$b_n(x,v) = -\Psi v, \tag{3.15}$$

$$\sigma_n(x,v) = \left(\frac{\lambda_{\mathbb{A}_n}}{n}\right)^{1/2} \tag{3.16}$$

and

$$c_n(x,v) = \Psi, \tag{3.17}$$

where

$$\Psi := \begin{pmatrix} \bar{\mu} & -\bar{\mu} \\ -\bar{\mu} & \bar{\mu} \end{pmatrix}. \tag{3.18}$$

Proof. See Appendix 5.5. □

To comply with Assumption 2.23(1) we additionally assume

$$\frac{\lambda_{\mathbb{A}_n}}{n} \xrightarrow[n\to\infty]{\text{u.o.c.}} \bar{\lambda}, \tag{3.19}$$

with $\bar{\lambda}$ being continuous. We summarize the diffusion approximation in the following proposition.

Proposition 3.2 (Diffusion Approximation)**.**

$$(X_t^n, V_t^n)_{t\in[0,\infty)} \xrightarrow{\mathcal{L}} (X_t, V_t)_{t\in[0,\infty)} \ in \ D_{\mathbb{R}\times\mathbb{R}^2}[0,\infty), \tag{3.20}$$

where $(X_t, V_t)_{t\in[0,\infty)}$ is the unique strong solution of the SDE

$$\begin{cases} dX_t = \alpha\bar{\lambda}(X_t, V_t)^{1/2} dB_t, & X_0 = p_0, \\ dV_t = -\Psi V_t dt + \Psi^{1/2} d\mathbf{B}_t, & V_0 = v_0, \end{cases} \tag{3.21}$$

with $(B_t)_{t\in[0,\infty)}$ being a standard Brownian Motion which is independent of the two dimensional Brownian motion $(\mathbf{B}_t)_{t\in[0,\infty)}$, Ψ as defined in Equation (3.18) and $v_0^2 - v_0^1 \sim \mathcal{N}(0,1)$ independent of $\mathbf{B}_t$.

Proof. Note that the distribution of $M_0^2 - M_0^1$ converges to a standard normal distribution when $n \to \infty$ by the de Moivre-Laplace theorem since we assumed Bernulli distributed initial states in Equation (3.6). Now, the proposition follows from application of Theorem 2.25. $\square$

3.1.2 (Mood mean dynamics)

The dynamics stated in Pakkanen [58] Proposition 3.8 are now a direct consequence of Proposition 3.2. Let

$$\widetilde{M}_k = \frac{1}{\sqrt{n}} \sum_{a=1}^{n} x_k^a, \ k \in \mathbb{N}. \tag{3.22}$$

be the one-dimensional opinion index at time T_k as defined in Equation (3.1) of Pakkanen [58]. Note that $M_k^1 = \frac{1}{2}(\sqrt{n} - \widetilde{M}_k)$, $M_k^2 = \frac{1}{2}(\sqrt{n} + \widetilde{M}_k)$ and $\widetilde{M}_k = M_k^2 - M_k^1$. Consequently we re-notate the trading intensity as well as the transition rate to be

$$\widetilde{\lambda}_{\mathbb{A}_n}(P_{k-1}, \widetilde{M}_{k-1}) := \lambda_{\mathbb{A}_n}\left(P_{k-1}, \frac{\sqrt{n} - \widetilde{M}_k}{2}, \frac{\sqrt{n} + \widetilde{M}_k}{2}\right) \tag{3.23}$$

$$\widetilde{\mu}_{\mathbb{A}_n}(P_{k-1}, \widetilde{M}_{k-1}) := \mu_{\mathbb{A}_n}\left(P_{k-1}, \frac{\sqrt{n} - \widetilde{M}_k}{2}, \frac{\sqrt{n} + \widetilde{M}_k}{2}\right) \tag{3.24}$$

Remark 3.3. Following the assumptions made in subsection 3.1.1 we have
$\widetilde{\mu}(P_{k-1}, \widetilde{M}_{k-1}) = \bar{\mu}$ and

$$\frac{\widetilde{\lambda}_{\mathbb{A}_n}}{n} \xrightarrow[n\to\infty]{\text{u.o.c.}} \widehat{\lambda}, \tag{3.25}$$

with

$$\widehat{\lambda}(P_{k-1}, \widetilde{M}_{k-1}) = \bar{\lambda}\left(P_{k-1}, \frac{\sqrt{n} - \widetilde{M}_k}{2}, \frac{\sqrt{n} + \widetilde{M}_k}{2}\right). \tag{3.26}$$

Moreover we set

$$\widetilde{V}_t^n := \sum_{k=0}^{\infty} \widetilde{M}_k \mathbb{1}_{[T_k, T_{k+1})}(t), \ t \geq 0 \tag{3.27}$$

and get analogous to Lemma 2.17 the existence of a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathbb{P})$, such that $(X_t^n, \widetilde{V}_t^n)_{t\in[0,\infty)}$ is a time homogeneous pure-jump type Markov process with rate kernel

$$\widetilde{K}_n(x, \widetilde{v}, dy, d\widetilde{w}) = (\widetilde{\lambda}_{\mathbb{A}_n}(x, \widetilde{v}) + n\bar{\mu})\widetilde{k}_n(x, \widetilde{v}, dy, d\widetilde{w}), \tag{3.28}$$

where $\widetilde{k}_n(x, \widetilde{v}, dy, d\widetilde{w})$ is the regular version of
$\mathbb{P}(P_1 - P_0 \in dy, \widetilde{M}_1 - \widetilde{M}_0 \in d\widetilde{w} | P_0 = x, \widetilde{M}_0 = \widetilde{v})$.
We now can derive Proposition 3.8 of Pakkanen [58], namely

Proposition 3.4.

$$(X^n_t, \widetilde{V}^n_t)_{t\in[0,\infty)} \xrightarrow{\mathcal{L}} (X_t, \widetilde{V}_t)_{t\in[0,\infty)} \ in\ D_{\mathbb{R}^2}[0,\infty), \tag{3.29}$$

where $(X_t, \widetilde{V}_t)_{t\in[0,\infty)}$ is the unique strong solution to the SDE

$$\begin{cases} dX_t = \alpha\widehat{\lambda}(X_t, \widetilde{V}_t)^{1/2} dW_t, & X_0 = p_0, \\ d\widetilde{V}_t = -2\bar{\mu}\widetilde{V}_t dt + 2\bar{\mu}^{1/2} dB_t, & \widetilde{V}_0 = \widetilde{\theta} \end{cases} \tag{3.30}$$

where B_t and W_t are independent Brownian motions and $\widetilde{\theta} \sim \mathcal{N}(0,1)$ independent of B_t.

Proof. All assumptions of Theorem 2.25 are met as

$$\widetilde{M_k} = M^2_k - M^1_k,\ k \geq 0 \tag{3.31}$$

Hence we sorely have to calculate the first and second moments of the rate kernel $\widetilde{K}_n$ in order to identify the drift and diffusion coefficient:

$$\begin{aligned} &\int \widetilde{K}_n(x, \widetilde{v}, dy, d\widetilde{w})\widetilde{w} = \left(\widetilde{\lambda}_{\mathbb{A}_n}(x, \widetilde{v}) + n\bar{\mu}\right) \mathbb{E}[\widetilde{M_1} - \widetilde{M_0} | \widetilde{M_0} = \widetilde{v}, P_0 = x] \\ &= (\lambda_{\mathbb{A}_n}(x, v) + n\bar{\mu})\, \mathbb{E}[\sqrt{n} - 2M^1_1 - \sqrt{n} + 2M^1_0 | M_0 = v, P_0 = x] \\ &= -2\int K_n(x, v, dy, dw) w_1 \\ &= -2\bar{\mu}(v_2 - v_1) \\ &= -2\bar{\mu}\widetilde{v} \end{aligned} \tag{3.32}$$

and analogously

$$\begin{aligned} &\int \widetilde{K}_n(x, \widetilde{v}, dy, d\widetilde{w})\widetilde{w}^2 = \left(\widetilde{\lambda}_{\mathbb{A}_n}(x, \widetilde{v}) + n\bar{\mu}\right) \mathbb{E}[(\widetilde{M_1} - \widetilde{M_0})^2 | \widetilde{M_0} = \widetilde{v}, P_0 = x] \\ &= (\lambda_{\mathbb{A}_n}(x, v) + n\bar{\mu})\, \mathbb{E}[(\sqrt{n} - 2M^1_1 - \sqrt{n} + 2M^1_0)^2 | M_0 = v, P_0 = x] \\ &= 4\int K_n(x, v, dy, dw) w_1^2 \\ &= 4\bar{\mu}. \end{aligned} \tag{3.33}$$

□

3.2 Example 2: Lux's Noise Trader and Fundamentalists Model

3.2.1 Introduction

Inspired by an observation in entomology, in particular related to ant populations and their contagious behavior towards food collection as discussed in Kirman [41], Lux [48] applied a herding mechanism to a fixed number of noise traders to describe an asset market. Thereby the individual noise trader is either optimistic or pessimistic and the rule for opinion change depends on the opinion of the majority as well as price trends. In order to have price changes be determined by a market maker through supply and demand matching, Lux introduces a second type of traders, so called fundamentalists that sell (buy) when the price is above (below) a fundamental value. Lux then used the master equation approach, originated from elementary particle systems in physics (see e.g. Haken [29]), together with methods discussed in Weidlich and Haag [68] to derive the properties of his market and to show its capability of generating bubbles and periodic oscillations. In this chapter we embed the model described in Lux [48] within our framework. We model the endogenous behavior of the agents according to the assumptions made in the model of Lux and assess if we achieve similar results, i.e herd behavior leads to price bubbles represented by temporary equilibria. We structure this section as following. In the first sub-section we establish the endogenous environment using only noise traders, which builds the base of the market character index. We derive its properties and the large markt limit of the market character index, and compare the results to Lux [48] and assess the rate of convergence. In the second sub-section we introduce fundamentalists as an additional group of traders and link the endogenous environment with the price process and vice versa. We again assess the properties, derive the diffusion approximation and make a concluding comparison with Lux [48].

3.2.2 Endogenous dynamics

Finite Model

First we specify all model components that directly affect the market character. We start with a finite set of an even[1] number of agents $\mathbb{A}_n = \{1, ..., n\}, n \in 2\mathbb{N}$. Moreover we consider a state space $S = \{-1, 1\}$ where $s_1 = -1$ represents a pessimistic and $s_2 = 1$ an optimistic opinion. Since no further specifications of the distribution of initial states is given in Lux [48], we assign each agent $a \in \mathbb{A}_n$ an initial state $x_0^a \in S := \{-1, 1\}$ such that the vector of initial states x_0 has some probability distribution $F_{x_0}^n$. In line with Lux [48] we choose the scaling of the market character to be $\frac{1}{n}$, i.e $d_1 = 1$. Following these

[1] We chose the number of agents to be even in order to be consistent with Lux [48].

assumptions, the character of the market at time T_k is given by $M_k = (M_k^1, M_k^2)$ with

$$M_k^1 = \frac{1}{n} \sum_{a \in \mathbb{A}_n} \mathbb{1}_{\{-1\}}(x_k^a), \ k \geq 0, \tag{3.34}$$

$$M_k^2 = \frac{1}{n} \sum_{a \in \mathbb{A}_n} \mathbb{1}_{\{1\}}(x_k^a), \ k \geq 0. \tag{3.35}$$

Since n is constant and the market is dichotomous, the two-dimensional market character is fully described by the one-dimensional *average opinion* defined as

$$\overline{M}_k := \frac{1}{n} \sum_{a \in \mathbb{A}_n} x_k^a, \ k \geq 0 \tag{3.36}$$

by $M_k = \left(\frac{1-\overline{M}_k}{2}, \frac{1+\overline{M}_k}{2} \right)$.

Considering the average opinion not only reduces the dimension and thus simplifies the endogenous dynamics, but also is consistent with the *average opinion* examined in Lux [48]. We denote the initial distribution of $\overline{M}_0$ which results from $F_{x_0}^n$ by Equation (3.36) by $F_{\overline{M}_0}^n$.

Next we define the state transition probability, which describes the likelihood an optimistic, respective pessimistic, agent is to change his opinion. Note that in line with the homogeneity assumption made in Lux [48] the transition probabilities are common to all agents.

Definition 3.5 (Transition probabilities)**.**

The transition probability to switch the state from -1 to 1 is defined as

$$\Pi^{1,2}(\overline{M}_{k-1}) = \beta e^{\gamma \overline{M}_{k-1}} \tag{3.37}$$

and analogous the probability to switch the state from 1 to -1 is defined as

$$\Pi^{2,1}(\overline{M}_{k-1}) = \beta e^{-\gamma \overline{M}_{k-1}}, \tag{3.38}$$

where $\beta, \gamma > 0$ and $\beta < e^{-\gamma}$ and hence the transition matrix, which is assumed to be common to all agents, is given by

$$\Pi_n(\overline{M}_{k-1}) = \begin{pmatrix} 1 - \Pi^{1,2} & \Pi^{1,2} \\ \Pi^{2,1} & 1 - \Pi^{2,1} \end{pmatrix} (\overline{M}_{k-1}). \tag{3.39}$$

Remark 3.6.

The explicit form of transition probabilities presented in Equations (3.37) and (3.38) were chosen by Lux to reflect the following socioeconomic characteristics. Firstly, the transition probability needs to reflect the idea of herding, i.e. the tendency of an agent to change his opinion to be optimistic (pessimistic) is larger when the majority of the traders already has an optimistic (pessimistic) opinion. Moreover the relative change in probability should change linear with the majority's opinion and be symmetric for optimism and pessimism, viz.

$$\frac{\partial \Pi^{1,2}(\bar{v})}{\Pi^{1,2}(\bar{v})} = Cd\bar{v} = -\frac{\partial \Pi^{2,1}(\bar{v})}{\Pi^{2,1}(\bar{v})}, \bar{v} \in [-1, 1], \tag{3.40}$$

for some constant $C \neq 0$. Finally, by definition, the probability needs to be between zero and one. The functional form of Equations (3.37) and (3.38) not only meets the requirements above but also give a good control of the infection by the parameters β and γ. While γ regulates the intensity of the infection and thus herd behavior, β controls the speed of contagion and hence contributes to the time scale.

Since the agents are assumed to behave homogeneously we assume a common state transition rate μ_a. Moreover as we are rather interested in the agent interaction itself and less on the time scale and could factor the transition rate into β anyway, we set

$$\mu_a = 1, \forall a \in \mathbb{A}_n. \tag{3.41}$$

Since the average opinion $\overline{M}_k$ can from time T_k to T_{k+1} either change by $\pm\frac{2}{n}$ or stay unchanged, it has its values on the $n+1$ valued lattice $\mathbb{L}$ from -1 to 1, viz.

$$\overline{M}_k \in \mathbb{L},\ \forall k \geq 0,\ \text{with } \mathbb{L} := \left\{-1, -\frac{n-2}{n}, -\frac{n-4}{n}, \dots, \frac{n-4}{n}, \frac{n-2}{n}, 1\right\}. \tag{3.42}$$

In summary, $(\overline{M}_k)_{k=0}^{\infty}$ is a Markov chain on $\mathbb{L}$ with state dependent transition probabilities, which are by Lemma 2.12 given as

$$\begin{aligned}\mathbb{P}\left(\overline{M}_k - \overline{M}_{k-1} = \frac{2}{n}|\mathcal{G}_{k-1}\right) &= \mathbb{P}\left(M^2_k - M^2_{k-1} = \frac{1}{n}|\mathcal{G}_{k-1}\right) \\ &= M^1_{k-1}\Pi^{1,2}(\overline{M}_{k-1}) \\ &= \frac{1-\overline{M}_{k-1}}{2}\Pi^{1,2}(\overline{M}_{k-1})\end{aligned} \tag{3.43}$$

and

$$\begin{aligned}\mathbb{P}\left(\overline{M}_k - \overline{M}_{k-1} = -\frac{2}{n}|\mathcal{G}_{k-1}\right) &= \mathbb{P}\left(M^2_k - M^2_{k-1} = -\frac{1}{n}|\mathcal{G}_{k-1}\right) \\ &= M^2_{k-1}\Pi^{2,1}(\overline{M}_{k-1}) \\ &= \frac{1+\overline{M}_{k-1}}{2}\Pi^{2,1}(\overline{M}_{k-1})\end{aligned} \tag{3.44}$$

Following Definition 2.16 we embed the Markov chain $(\overline{M}_k)_{k\geq 0}$ in continuous time using the *average opinion index*, defined as

$$\overline{V}_t^n := \sum_{k=0}^{\infty} \overline{M}_k \mathbb{1}_{[T_k, T_{k+1})}(t),\ t \geq 0. \tag{3.45}$$

Note that by Equation (2.14) and (3.41) we have for the intra-action times: $\tau_k \sim Exp(n)$.

Stationary distribution of average opinion

In order to study the stationary behavior of $\overline{V}_t^n$, that is for vanishing time derivatives, we calculate the stationary distribution of the underlying Markov chain $\overline{M}_k$. The result is presented in the next proposition.

Proposition 3.7 (Stationary distribution)**.**
The stationary distribution of $\overline{M}_k$ and $\overline{V}_t^n$ resulting from Equations (3.43), (3.44) and (3.45) is given by

$$\mathbb{P}_{st}(\bar{v}) = \mathbb{P}_{st}(0) \frac{\left(\frac{n}{2}!\right)^2}{n!} \binom{n}{\frac{n(1+\bar{v})}{2}} \exp\left(\frac{\gamma n \bar{v}^2}{2}\right), \bar{v} \in \mathbb{L}, \tag{3.46}$$

where $\mathbb{P}_{st}(0)$ is determined by the normalization condition

$$\sum_{\bar{v} \in \mathbb{L}} \mathbb{P}_{st}(\bar{v}) = 1. \tag{3.47}$$

If

$$\gamma \underset{(<)}{>} \frac{n}{2} \ln\left(\frac{n+2}{n}\right), \tag{3.48}$$

then $\mathbb{P}_{st}$ has a local minimum (maximum) at 0.

Proof. See Appendix 5.6. □

Below, Figure 3.1 shows the stationary distribution exemplary for a market with 20 agents. For a low herding intensity (shown in the right histogram in Figure 3.1 with $\gamma = 0.8$) there exists one maximum at the average opinion of 0, while for a high herding intensity two symmetrical maxima emerge (left histogram with $\gamma 1.2$).
Although the stationary distribution of the Markov chain $(\overline{M}_k)_{k \geq 0}$ respectively the average opinion index $\overline{V}_t^n$ given in Equation (3.46) is analytically exact, due to the binomial coefficients the calculation is numerically intense when the number of market participants is large. Moreover closed form solutions of properties (e.g. maxima and minima) are quite complex to calculate. Therefore in addition to the diffusion approximation given later on we solve the related Fokker-Planck equation[2] in order to approximate the stationary dis-

[2]See e.g. Paul and Baschnagel [59] Chapter 2.2.2 for motivation and mathematical background.

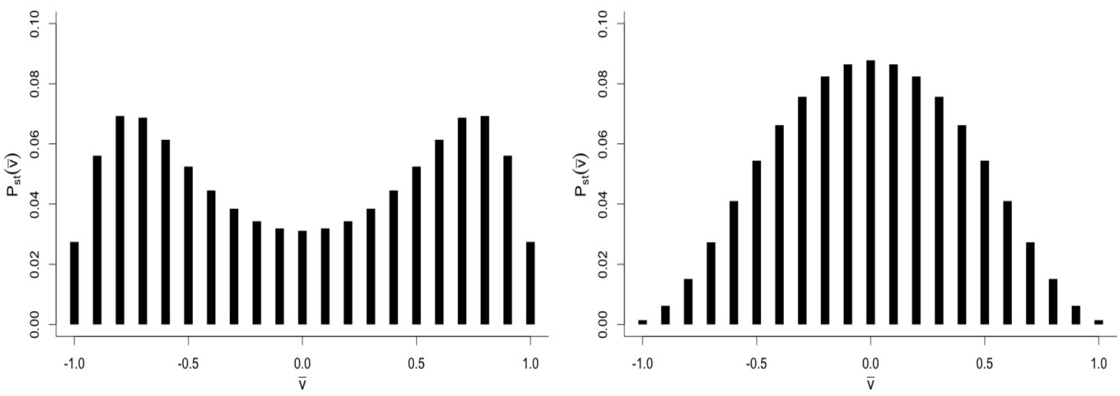

Figure 3.1: Stationary distribution for n = 20 with $\gamma = 0.8$ (right), 1.2 (left)

tribution for a large number of agents[3] and derive some properties. For this purpose, we define the first and second moments of $\overline{M}_k - \overline{M}_{k-1}$ in the next definition, which determine the Fokker-Planck equation. Note that the moments are analogous to the expected aggregated state transition b_n and the transition volume c_n defined in Equation (2.23) and (2.27) and thus enable us to derive the coefficients of the diffusion approximation more easily later on.

Definition 3.8 (Expected aggregated average state transition, transition volume of the average opinion)**.**
We define the *expected aggregated average state transition* as

$$\bar{b}_n(\bar{v}) := \mathbb{E}[\overline{M}_k - \overline{M}_{k-1} | \overline{M}_{k-1} = \bar{v}] \tag{3.49}$$

and the *transition volume of the average opinion* as

$$\bar{c}_n(\bar{v})^2 := \mathbb{E}[(\overline{M}_k - \overline{M}_{k-1})^2 | \overline{M}_{k-1} = \bar{v}]. \tag{3.50}$$

Entering Equations (3.37), (3.38), (3.43) and (3.44) into (3.49) and (3.50) followed by basic calculations yield the specification of $\bar{b}_n$ and $\bar{c}_n$ as part of the following Lemma.

Lemma 3.9 (First and second moment)**.**
The expected aggregated average state transition is given by

$$\bar{b}_n(\bar{v}) = 2\beta[\tanh(\gamma\bar{v}) - \bar{v}]\cosh(\gamma\bar{v}) \tag{3.51}$$

[3]The approximation is indeed already good for small n.

and analogously the transition volume of the average opinion is given by

$$\overline{c}_n(\overline{v})^2 = \frac{4\beta}{n}[1 - \overline{v}\tanh(\gamma\overline{v})]\cosh(\gamma\overline{v}). \tag{3.52}$$

Proof. See Appendix 5.7 □

In the following Proposition we present an continuous distribution function which approximates the discrete stationary distribution of our finite model. Note that in contrast to the large market approximation presented later on it is dependent on n.

Proposition 3.10 (Approximation stationary distribution)**.**
Solving the Fokker-Planck equation corresponding to our endogenous dynamics we can approximate $\mathbb{P}_{st}$ in large markets by the continuous function

$$\widetilde{\mathbb{P}}_{st}(\widetilde{v}) = \frac{\widetilde{\mathbb{P}}_{st}(0)}{(1 - \widetilde{v}\tanh(\gamma\widetilde{v}))\cosh(\gamma\widetilde{v})} \exp\left(n \int_0^{\widetilde{v}} \frac{\tanh(\gamma y) - y}{1 - y\tanh(\gamma y)} dy \right), \tag{3.53}$$

where $\widetilde{\mathbb{P}}_{st}(0)$ follows from the condition

$$\int_{-1}^{1} \widetilde{\mathbb{P}}_{st}(\widetilde{v}) d\widetilde{v} = 1. \tag{3.54}$$

Proof. See Appendix 5.8 □

Below, Figure 3.2 illustrates the approximation for the parameter setting as in Figure 3.1, that is a market with 20 participants and herding intensity of 1.2 (left), respectively 0.8 (right).

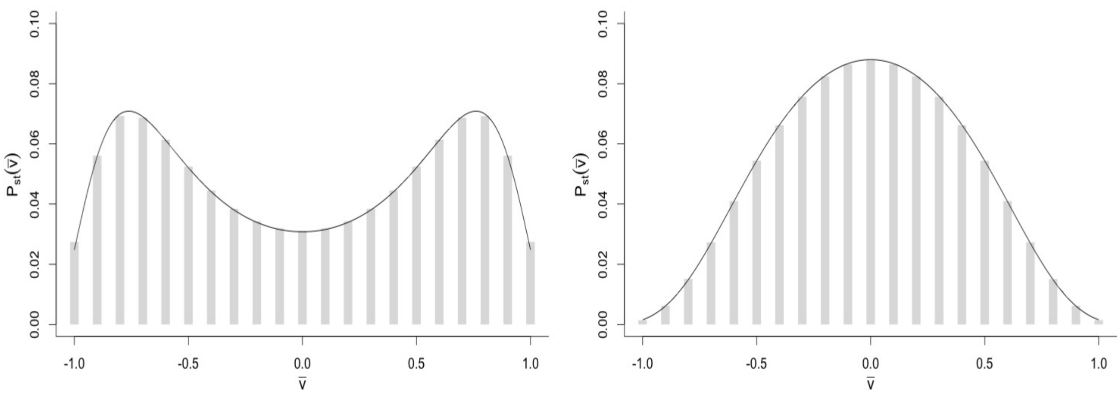

Figure 3.2: Approximation of stationary distribution for n = 20 with γ = 0.8, 1.2

The approximation not only is already quite good for small n, but also enables us to easily determine the stationary maxima as part of the following proposition.

Proposition 3.11 (Stationary bahaviour)**.**
For $\gamma \leq 1$, $\widetilde{\mathbb{P}}_{st}$ has one stationary state at $\bar{v} = 0$. For $\gamma > 1$, $\widetilde{\mathbb{P}}_{st}$ has two symmetrical stationary states $\widetilde{v}_+ = -\widetilde{v}_- \neq 0$, which are determined as the solution of

$$y = \tanh(\gamma y). \tag{3.55}$$

Proof. See Appendix 5.8. □

Note that the stationary states only depend on the herding intensity described by the parameter γ and are especially independent of n. Moreover it is clear that the stationary behavior is also independent of the scaling parameter β. Although modeled within a different mathematical framework, we derived the same properties as in Lux [48]. Namely, herding within a homogeneous population choosing between two symmetric opinions leads to temporary equilibria of two, respectively one, stationary proportion of majority opinion depending on the herding intensity. In case of two equilibria, after a specific time there will occur a equilibrium transition, i.e. the majority will change their opinion symmetrical in the other direction.

Remark 3.12. With the stationary distribution above, for the case of two maxima, the dynamics of $\overline{V}_t^n$ corresponds to the symmetric setup of Kramers problem (See Kramer [43]). Kramers modeled chemical reactions as a one dimensional diffusion process within an energy potential with two meta stable local minima and a local maxima in between. Especially of interest is the transition time, that is the time needed to get from one meta-stable equilibrium to the other, which we discuss next.

As mentioned by Lux the transition time "depends inversely on the number of traders". We elucidate this statement by assessing the equilibria transition time explicitly in the next Proposition.

Proposition 3.13 (Transition time)**.**
Using the approximating stationary distribution defined by Equation (3.53), the transition time τ to switch from stationary state $\widetilde{v}_-$ to $\widetilde{v}_+$ and vice versa is approximately given by

$$\widetilde{\tau} = \frac{\pi \exp\left(n \int_0^{\widetilde{v}_m} \frac{\tanh(\gamma y) - y}{1 - y \tanh(\gamma y)} dy \right)}{\beta \cosh(\gamma \widetilde{v}_m) \sqrt{(1 - \widetilde{v}_m^2)(\gamma - 1)|\gamma(1 - \widetilde{v}_m^2) - 1|}} \tag{3.56}$$

with $\widetilde{v}_m$ being defined as solution to Equation (3.55).

Proof. See Appendix 5.9

□

As comprehensible by Equation (3.56), $\widetilde{\tau}$ is of the form $\widetilde{\tau} = \kappa \exp(n)$, where κ is a constant only depending on β and γ. Hence the transition time increases exponentially fast with the number of market participants.

Before we examine very large markets using the diffusion approximation, we illustrate this sub-chapter within the following figures. In the x-z-plane of Figure 3.3 and 3.4 we present the exact stationary distribution of the average opinion process $\overline{V}_t^n$ defined in Equation (3.45) and it's Fokker-Planck approximation for parameters $\beta = 0.3$ and $\gamma \in \{0.8, 1.2\}$ for 20 and 100 market participants. The figures not only show the difference between a one- and a two-peaked distribution as predicted by Proposition 3.11 where the approximated maxima ($\widetilde{v}_+ = -\widetilde{v}_- \approx 2/3$) are given by Equation (3.55), but also illustrate that the approximation is already quite good for small markets. The x-y-plane of Figure 3.3 and 3.4 on the other hand show the respective trajectories of $\overline{V}_t^n$ for a time horizon of 1000. They not only show that the process makes a transition between the two equilibrium state proportions $\widetilde{v}_+$ and $\widetilde{v}_-$ when γ is large, but also that the respective transition time $\widetilde{\tau}$ is increasing in n.

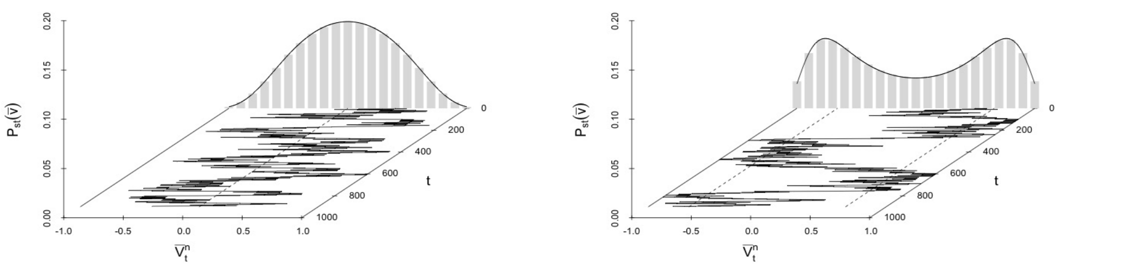

Figure 3.3: $\overline{V}_t^n$ and related stationary distribution for n = 20 with $\gamma = 0.8$ (left), 1.2 (right)

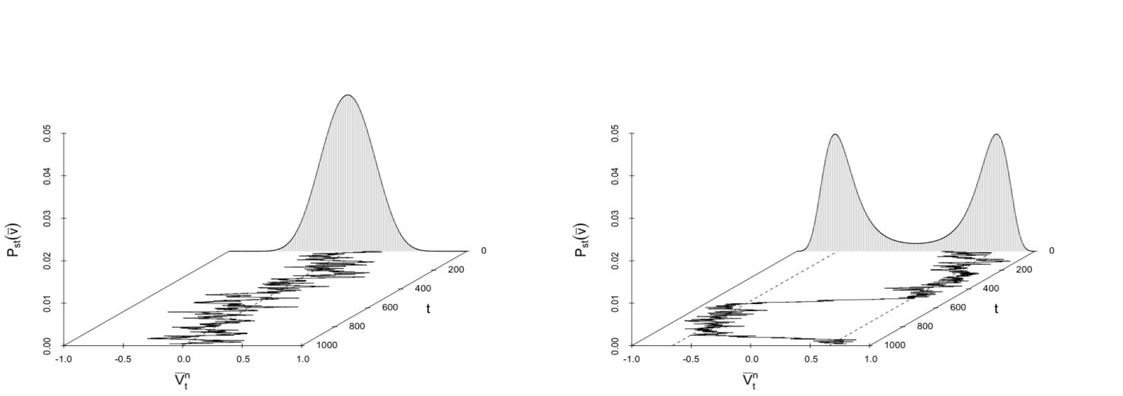

Figure 3.4: $\overline{V}_t^n$ and related stationary distribution for n = 100 with $\gamma = 0.8$ (left), 1.2 (right)

Large market approximation

In this sub-section we state the large market approximation of the *average opinion process* $\overline{V}_t^n$. Some basic properties of the limit are already apparent from the last sub-section. When $n \to \infty$, it is clear that no equilibria transition will occur since $\widetilde{\tau} \to \infty$. Additionally by the form of $\bar{c}_n$ shown in Equation (3.52) it is apparent that the volatility will vanish. Hence the resulting process will be deterministic and driven to the stationary state defined by Proposition 3.11. As soon as the process "hits" the stationary state, the drift coefficient is zero and the process stays constant. Hence in summary the large market limit has different properties than the original market, i.e. no state transition and one, respectively two, absorbing states. The deterministic large market limit has the following form.

Proposition 3.14 (Large Market Approximation)**.**

If the distribution of the initial average mood $F^n_{\overline{M}_0}$ has for $n \to \infty$ a limit $F_{\overline{M}_0}$, then

$$(\overline{V}_t^n)_{t\in[0,\infty)} \xrightarrow{\mathcal{L}} (\overline{V}_t)_{t\in[0,\infty)} \ in \ D_{[-1,1]}[0,\infty), \tag{3.57}$$

where $(\overline{V}_t)_{t\in[0,\infty)}$ is the solution of the ODE

$$d\overline{V}_t = 2\beta \left[\tanh(\gamma \overline{V}_t) - \overline{V}_t\right] \cosh(\gamma \overline{V}_t)dt, \ \overline{V}_0 = \overline{\theta}, \tag{3.58}$$

with $\overline{\theta} \sim F_{\overline{M}_0}$.

Proof. See Appendix 5.10 □

Below we show the solution of the above ODE for different initial values $\overline{\theta}$. The left graph of Figure 3.5 shows $\overline{V}_t$ for $\gamma = 0.8$ and $\beta = 0.3$. Independent of the initial value $\overline{V}_0$, $\overline{V}_t$ converges monotone to 0 for $t \to \infty$. When $\gamma > 1$,as illustrated in the right graph, the solution of Equation (3.58) has three limits for $t \to \infty$ depending on the initial value $\overline{\theta}$. For $\overline{\theta} = 0$, $\overline{V}_t$ is constantly 0. For $\overline{\theta} \underset{(<)}{>} 0$, $\overline{V}_t$ converges monotone to the positive (negative) solution of Equation (3.55).

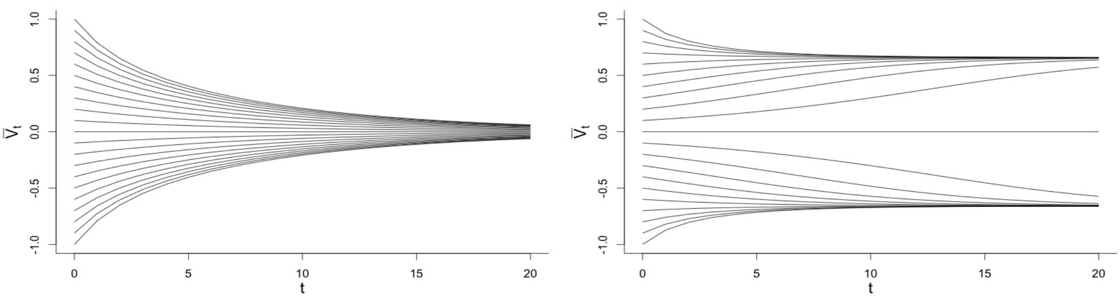

Figure 3.5: $\overline{V}_t$ for $\gamma = 0.8$, 1.2 and $\beta = 0.3$ with different $\overline{V}_0$

Before we link the endogenous model to the price process in the next sub-section we show an application of Proposition 2.26, i.e. we determine the rate of convergence.

Remark 3.15 (Rate of convergence)**.**
If $F^n_{\overline{M}_0} = F_{\overline{M}_0}$, then

$$\sup_{s\leq t} |\overline{V}^n_s - \overline{V}_s| \leq 2\sqrt{\frac{\beta}{n}} \sup_{s\leq t} |Y_s| \leq \frac{C_{\beta,\gamma}}{\sqrt{n}}, \ \forall t \geq 0, \tag{3.59}$$

where Y_t is the unique solution of the SDE

$$dY_t = \sqrt{(1 - Y_t \tanh(\gamma Y_t)) \cosh(\gamma Y_t)} dB_t, \ Y_0 = 0, \tag{3.60}$$

and $C_{\beta,\gamma}$ is a constant depending on β and γ via $C_{\beta,\gamma} = 2\sqrt{\beta}|y^*|$, where y^* is the solution of $1 - y\tanh(\gamma y) = 0$.

Proof. Let $a_n := \sqrt{\frac{n}{4\beta}}$. Moreover define $\hat{c}(y)^2 := (1 - y\tanh(\gamma y))\cosh(\gamma y)$ and $\hat{b} := \overline{b}$. Now, the first inequality in (3.59) simply follows from application of Proposition 2.26. Since $\forall \gamma > 0$, $\hat{c}(y)$ is symmetric to zero from which it falls monotonously to 0, the related process Y_t has less variance the more its distance to zero. When the process Y_t reaches y^*, where y^* is the solution of $\hat{c}(y^*) = 0$, the diffusion coefficient is zero and Y_t stays constant. Hence $\sup_{s\leq t} |Y_s| \leq |y^*|, \ \forall t \geq 0$, which concludes the second inequality. □

Below, in Figure 3.6, we illustrate Remark 3.15 by showing the diffusion coefficient $\hat{c}(\overline{v})$ for $\gamma = 0.8$ (right graph) and a realization of the related solution of SDE (3.60) (left graph).

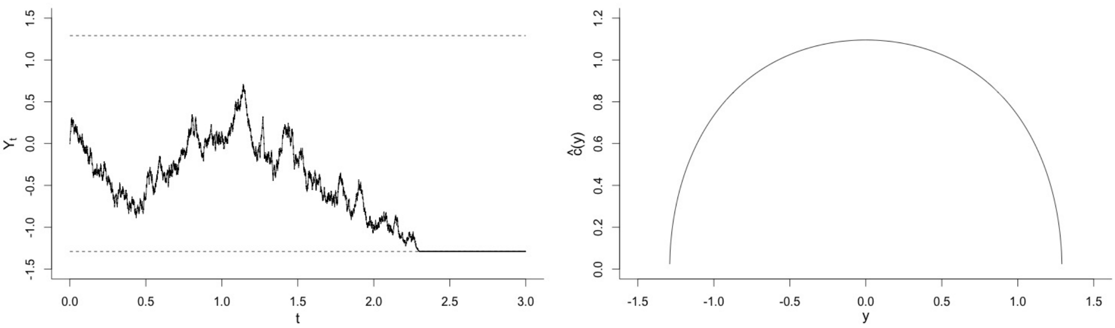

Figure 3.6: Y_t and $\hat{c}(\overline{v})$ for $\gamma = 0.8$

3.2.3 Price dynamics

In this sub-section we extend the endogenous model of the previous sub-section with a link to an asset price. Therefore we characterize the impact of agents' opinion on the asset price and vice versa. Moreover we introduce an additional group of traders called fundamentalists.[4] Last are characterized by basing their behavior on the difference between actual price and a *fundamental value* $F \in \mathbb{R}$. In particular, when the price is below (above) F, they consider the asset cheap (expensive) and want to buy (sell). We assume the fundamentalists are homogeneous, viz. F is common, and the fundamental value is time-invariant. Although we are mostly consistent with Lux [48] in setting our assumptions, we additionally introduce random signals reflected in the agent's trading behavior. Let $k_n \in \mathbb{N}$ denote the number of fundamentalists and $\phi_n = \frac{k_n}{n}$ the proportion of fundamentalists within all agents. Hence $\mathbb{A}_n = \{1, ..., n\}, \;\; n - k_n \in 2\mathbb{N}$ is the set of all agents participating in the market.

Compared to the previous sub-section we extent the state space to be $S = \{-1, 1, 2\}$, where $s_3 = 2$ denotes a fundamentalistic agent. The market character is then given by $M_k = (M_k^1, M_k^2, M_k^3)$, where M_k^1 and M_k^2 is defined in the previous subsection (Equations (3.34) and (3.35)) and

$$M_k^3 = \frac{1}{n} \sum_{a \in \mathbb{A}_n} \mathbb{1}_{\{2\}}(x_k^a), \;\; k \geq 0. \tag{3.61}$$

In line with Lux we define the *average opinion of noise traders* by

$$\overline{M}_k := \left(\frac{1}{1 - \phi_n} \right) (M_k^2 - M_k^1), \;\; k \geq 0. \tag{3.62}$$

Moreover let $\widetilde{F}_{x_0}^n$ denote the initial distribution of states and $\widetilde{F}_{M_0}^n$ the resulting initial distribution of M_0. We assume fundamentalists weight their demand according to a bounded and to F symmetric function $w_2 : \mathbb{R} \to \mathbb{R}$ depending on the current price and F. On the other hand optimists buy a fixed amount of shares (w_1) while pessimists want to sell the same amount. We scale the demand by $1/\sqrt{n}$ and add a random signal ξ_k from the sequence $(\xi_k)_{k \geq 1}$, which is assumed i.i.d. with $\mathbb{E}[\xi_1]$ and $\text{Var}[\xi_1] < \infty$. Let $\widetilde{F}_{P_0}$ be the distribution of the starting price, which for simplicity is assumed independent of n.

Definition 3.16 (Excess demand function)**.**
In summary we assume the following excess demand function

$$e_a^n(P_{k-1}) = \begin{cases} n^{-1/2} w_1 x_k^a + \xi_k, & x_k^a \in \{-1, 1\} \\ n^{-1/2} w_2(P_{k-1}) + \xi_k, & x_k^a = 2 \end{cases} \tag{3.63}$$

Next we define the pricing rule, that is how supply or demands of agents impact the stock

[4]Note that, while in the model of Lux [48] fundamentalists are required to instantly match supply and demand, we could forgo, since in our model orders arrive asynchronously.

price.

Definition 3.17 (Pricing rule)**.**
We assume the pricing rule is given by

$$r_n(q,x) = x + \frac{\alpha}{\sqrt{n}} q \tag{3.64}$$

In Lux's model the price dynamics are an equilibrium result of matching supply and demand of all participating agents.[5] Since we assumed that at a specific point in time only one agent is trading (see Assumption 2.7) and therefore solely impacts the price, the pricing rule in our model largely deviates from Lux by construction. However, we can use the trading intensity functions λ_a in order to "scale" the number of trades and make the models comparable within a fixed time interval. More precisely, following the homogeneity assumption related to our agents, we set

$$\lambda_a(x,v) = \bar{\lambda} \in \mathbb{R}^+. \tag{3.65}$$

Note that for $\bar{\lambda} = n$ the expected net excess demand of all agents for a fixed time interval is the same within our and Lux's model.
Now, also to incorporate a feedback effect from the price to agents behavior we extent the transition probability of the mood-based traders. Therefore we let the transition probability not only depend on the overall mood, but also on the expected price dynamics.

Definition 3.18 (Transition probabilities)**.**
The transition probability to switch the state from -1 to 1 is defined to be

$$\Pi_n^{1,2}(P_{k-1}, M_{k-1}) = \beta e^{\gamma_1 \widehat{z}_n(P_{k-1}, M_{k-1}) + \gamma_2 \overline{M}_{k-1}} \tag{3.66}$$

where

$$\widehat{z}_n(P_{k-1}, M_{k-1}) := \overline{\lambda}[\phi_n w_2(P_{k-1}) + (1-\phi_n) w_1 \overline{M}_{k-1}] \tag{3.67}$$

The transition probability to switch the state from 1 to -1 is defined as

$$\Pi_n^{2,1}(P_{k-1}, M_{k-1}) = \beta e^{-\gamma_1 \widehat{z}_n(P_{k-1}, M_{k-1}) - \gamma_2 \overline{M}_{k-1}}, \tag{3.68}$$

where $\gamma_1, \gamma_2 > 0$. Moreover we assume that fundamentalist can not become optimists/

[5] Since all agents are considered by the supply and demand matching, a transition to an infinite big market (viz. $n \to \infty$) is not possible within Lux's framework.

pessimists and vice versa[6]. Hence the transition matrix is given as

$$\Pi_n(P_{k-1}, M_{k-1}) = \begin{pmatrix} 1-\Pi_n^{1,2} & \Pi_n^{1,2} & 0 \\ \Pi_n^{2,1} & 1-\Pi_n^{2,1} & 0 \\ 0 & 0 & 1 \end{pmatrix} (P_{k-1}, M_{k-1}). \tag{3.69}$$

While $\widehat{z}_n$ measures the expected price dynamics, γ_1 measures the intensity of the price feedback on agents behavior. On the other hand γ_2 describes the herding intensity analogous to the previous chapter.
Moreover since $M_k^3 = \phi_n$ and $M_k^2 = (1-\phi_n) - M_k^1$, the market character is uniquely defined by the average opinion of the noise traders, i.e.

$$M_k = \left(\frac{(1-\phi_n)(1-\overline{M}_k)}{2}, \frac{(1-\phi_n)(1+\overline{M}_k)}{2}, \phi_n \right). \tag{3.70}$$

Now, recall that the price process is defined as

$$X_t^n = \sum_{k=0}^{\infty} P_k \mathbb{1}_{[T_k, T_{k+1})}(t), \ t \geq 0, \tag{3.71}$$

and the market character index as

$$V_t^n = \sum_{k=0}^{\infty} M_k \mathbb{1}_{[T_k, T_{k+1})}(t), \ t \geq 0. \tag{3.72}$$

Moreover, analogous to Equation (3.45) and (3.70) the average opinion index is derived as

$$\overline{V}_t^n = \sum_{k=0}^{\infty} \overline{M}_k \mathbb{1}_{[T_k, T_{k+1})}(t) = \frac{1}{1-\phi}(V_t^{n,2} - V_t^{n,1}), \ t \geq 0. \tag{3.73}$$

Now, to determine the behavior of the exemplary model we again leverage from the results presented in Lux [48]. Although our model is different by construction, the key factors like net-excess demand, weighting of fundamentalists and mood traders, etc. are comparable. In the next remark we state the behavior of our model, which is valid not only for the price process X_t^n and the average opinion index $\overline{V}_t^n$, but also for the underlying Markov chains $(P_k)_{k\geq 0}$ and $(\overline{M}_k)_{k\geq 0}$.

Remark 3.19 (Market behavior)**.**

1. For a high herding intensity γ_2, there exist two equilibria $E_+ = (\widetilde{v}_+, x_+)$ and $E_- = (\widetilde{v}_-, x_-)$, where $\widetilde{v}_+ = -\widetilde{v}_-$ and $w_2(x_-) = -w_2(x_+)$.

[6] Note, that the transition probabilities in Equations (3.66) and (3.68) are not per se well defined. Instead of capping the probabilities at one, we rather use the function w_2 in order to control the impact of large prices P_{k-1}

2. For a small herding intensity γ_2, there is one unique equilibrium $E_0 = (0, F)$. If the intensity of price feedback γ_1 is low,

   a) then E_0 is stable

   b) otherwise E_0 is unstable and there occur periodic cycles.

Since the objective of this section is a high level comparison with the model of Lux [48], we refrain from a more detailed description of the market behavior. Nevertheless, after we state the large limit approximation for the extended example in the next Proposition[7], we illustrate Remark 3.19 by showing trajectories of X_t^n for each case in Figure 3.7 - Figure 3.9.

Proposition 3.20 (Large market approximation)**.**

If $\phi_n \xrightarrow{n\to\infty} \phi$ and $\overline{V}_0^n \xrightarrow{\mathcal{L}} \overline{\theta}$, then

$$(X_t^n, V_t^n)_{t\in[0,\infty)} \xrightarrow{\mathcal{L}} (X_t, V_t)_{t\in[0,\infty)} \text{ in } D_{\mathbb{R}\times[0,1]^3}[0,\infty), \tag{3.74}$$

where $(X_t, V_t)_{t\in[0,\infty)}$ is the unique strong solution of the SDE

$$\begin{cases} dX_t = \alpha z(X_t, \overline{V}_t)dt + \alpha(\bar{\lambda}\mathrm{Var}[\xi_1])^{1/2}dB_t, & X_0 = \zeta \\ d\overline{V}_t = 2\beta\left[\tanh(\gamma_1 z(X_t, \overline{V}_t) + \gamma_2\overline{V}_t) - \overline{V}_t\right]\cosh(\gamma_1 z(X_t, \overline{V}_t) + \gamma_2\overline{V}_t)dt, & \overline{V}_0 = \overline{\theta} \\ dV_t^3 = 0, & V_0^3 = \phi, \end{cases} \tag{3.75}$$

where $\overline{V}_t = \frac{1}{1-\phi}(V_t^2 - V_t^1)$, $z(x,v) := \overline{\lambda}[\phi w_2(x) + (1-\phi)w_1 v]$ and $\zeta \sim \widetilde{F}_{P_0}$.

Proof. See Appendix 5.11 □

[7]Note that, the limit is in contrast to the pure endogenous large market dynamics a diffusion process.

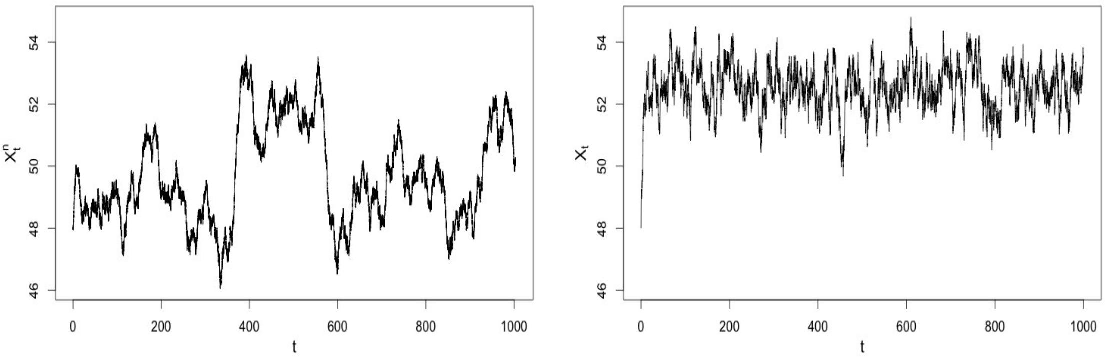

Figure 3.7: X_t^n and X_t for $\gamma_1 = 0.2$, $\gamma_2 = 1.2$, $w_2 = F - x$

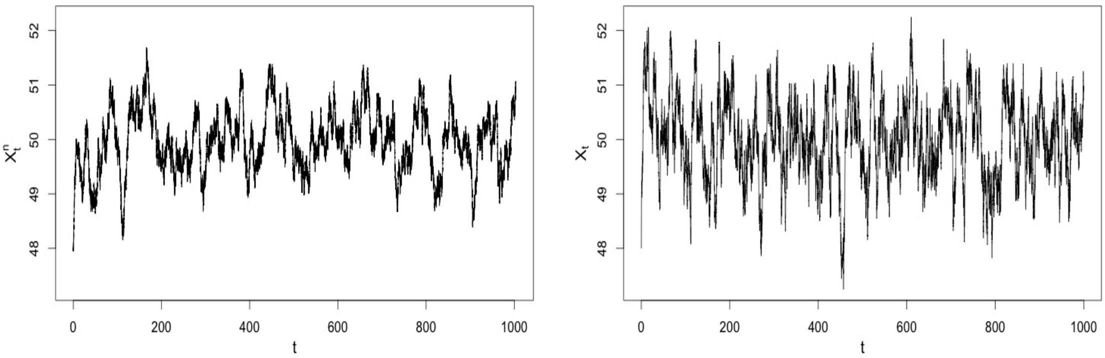

Figure 3.8: X_t^n and X_t for $\gamma_1 = 0.2$, $\gamma_2 = 0.8$, $w_2 = F - x$

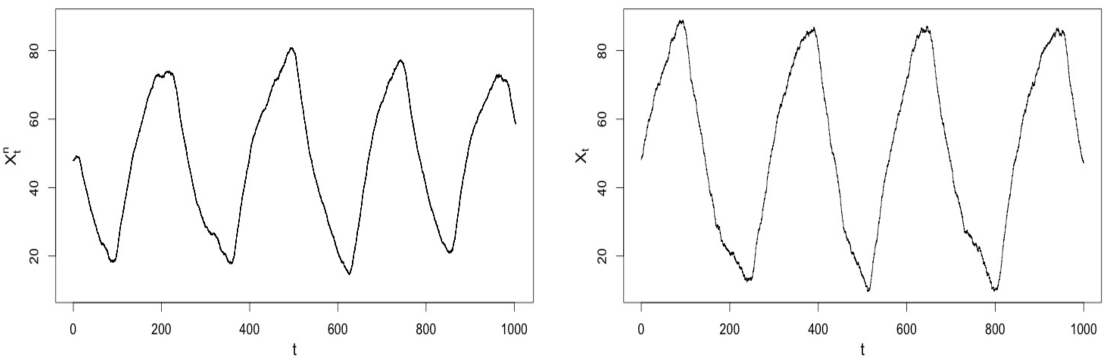

Figure 3.9: X_t^n and X_t for $\gamma_1 = 1.2$, $\gamma_2 = 0.8$, $w_2 = 0.05 * (F - x)$

In Figure 3.7 - Figure 3.9 we show respective trajectories of the solution of Equation (3.75) with $\beta = 0.12$, $\phi = 0.2$, $w_1 = 1$, $P_0 = 48$, $F = 50$ and $\sqrt{\text{Var}[\xi_1]} = 0.2$. We compare X_t^n for $n = 100$ and X_t at 1000 time units. As such we compare a medium large market with the infinite large market, which is used to approximate. In Figure 3.7 we set $\gamma_1 = 0.2$, $\gamma_2 = 1.2$ and $w_2 = F - x$. In line with Remark 3.19 1. we see a regime switch between two equilibria which are symmetric to the fundamental value for X_t^n. The transition also holds true for the diffusion process X_t when $\text{Var}[\xi_1] > 0$, however on a much larger scale than shown in Figure 3.7. For completeness we illustrate the transition of X_t on a 100 times larger scale in Figure 3.10. If we now reduce the herding intensity γ_2 to 0.8, as illustrated in Figure 3.8, X_t^n as well as X_t have an equilibrium at the same point, namely the fundamental value F, which is in accordance with Remark 3.19 2.(a). In Figure 3.9 we illustrate X_t^n and X_t in the case that the intensity of price feedback is high while the influence of fundamentalists is low. Therefore we set γ_1 to 1.2 and w_2 to $0.05 * (F - x)$. Independent of the initial distribution of optimists and pessimists, X_t^n and X_t are then oscillating around F with the same scale, although the amplitude of the diffusion process is slightly higher. In summary, the diffusion process X_t shows the same characteristics as X_t^n and is well suited to be used as large market approximation to examine those. Nevertheless, as apparent from Figure 3.7 the scaling in which the characteristic is displayed might be different.

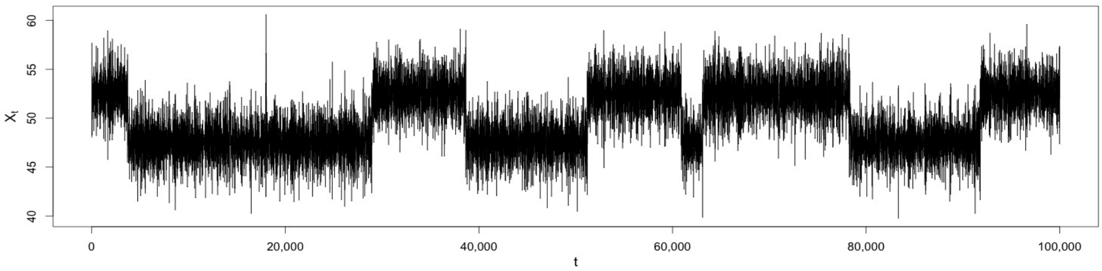

Figure 3.10: X_t for $\gamma_1 = 0.2$, $\gamma_2 = 1.2$, $w_2 = F - x$

In the following, to study the dynamics behind the SDE presented in Equation 3.20, we look at the large market dynamics without random signals. Equation 3.20 then simplifies to the ODE

$$\begin{cases} dX_t = \alpha z(X_t, \overline{V}_t)dt, & X_0 = \zeta \\ d\overline{V}_t = 2\beta \left[\tanh(\gamma_1 z(X_t, \overline{V}_t) + \gamma_2 \overline{V}_t) - \overline{V}_t\right] \cosh(\gamma_1 z(X_t, \overline{V}_t) + \gamma_2 \overline{V}_t)dt, & \overline{V}_0 = \overline{\theta} \\ dV_t^3 = 0, & V_0^3 = \phi, \end{cases} \tag{3.76}$$

where $\overline{V}_t = \frac{1}{1-\phi}(V_t^2 - V_t^1)$, $z(x, v) := \overline{\lambda}[\phi w_2(x) + (1-\phi)w_1 v]$ and $\zeta \sim \widetilde{F}_{P_0}$.

In Figure 3.11 - Figure 3.13 we show several solutions of Equation (3.76) for the different initial values $\overline{\theta}$ for the same set of parameters as in Figure 3.7 - 3.9, that is $\beta = 0.12$, $\phi = 0.2$, $w_1 = 1$ and $P_0 = 48$. Figure 3.11 shows $\overline{V}_t$ and X_t for 100 time units with the same setting as used in Figure 3.7, i.e. $\gamma_1 = 0.2$, $\gamma_2 = 1.2$ and $w_1 = 1$. Depending on the initial value $\overline{\theta}$, $\overline{V}_t$ and X_t converge monotonously to one of two constants, which are symmetrical to 0, respectively F. If we now reduce the herding intensity γ_2 to 0.8, as illustrated in Figure 3.12, $\overline{V}_t$ and X_t converge monotonously to 0, respectively F, independent of the initial value. In Figure 3.13 we illustrate $\overline{V}_t$ and X_t in the case that the intensity of price feedback is high while the influence of fundamentalists is low. Therefore we set γ_1 to 1.2 and w_2 to $0.05 * (F - x)$. Independent of the initial distribution of optimists and pessimists, $\overline{V}_t$ and X_t are then oscillating around 0 , respectively F.
As observable in Figure 3.11 and Figure 3.12 the solution of Equation (3.76) may converge to constants x, $\overline{v}$. In order to specify the constants we require $z(X_t, \overline{V}_t) = 0$ and $d\overline{V}_t = 0$, which is equivalent to

$$\begin{cases} \phi w_2(F, x) + (1-\phi)w_1\overline{v} = 0 \\ \left[\tanh(\gamma_2\overline{v}) - \overline{v}\right] \cosh(\gamma_2\overline{v}) = 0 \end{cases} \tag{3.77}$$

Hence with the parameters above, $x = 50 \pm 4\overline{v}$, where $\overline{v}$ is the solution of $y = \tanh(\gamma_2 y)$, if $\gamma_2 > 1$ and 0 otherwise.

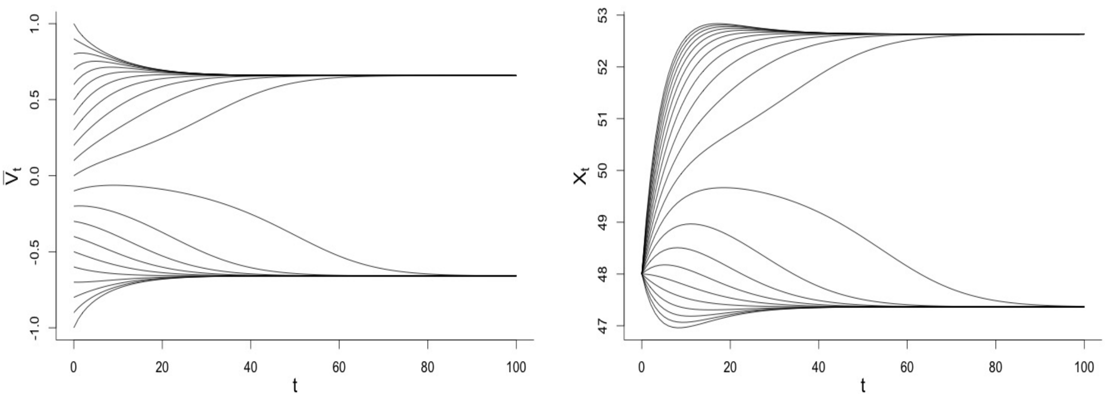

Figure 3.11: $\overline{V}_t$ and X_t for $\gamma_1 = 0.2$, $\gamma_2 = 1.2$, $w_2 = F - x$

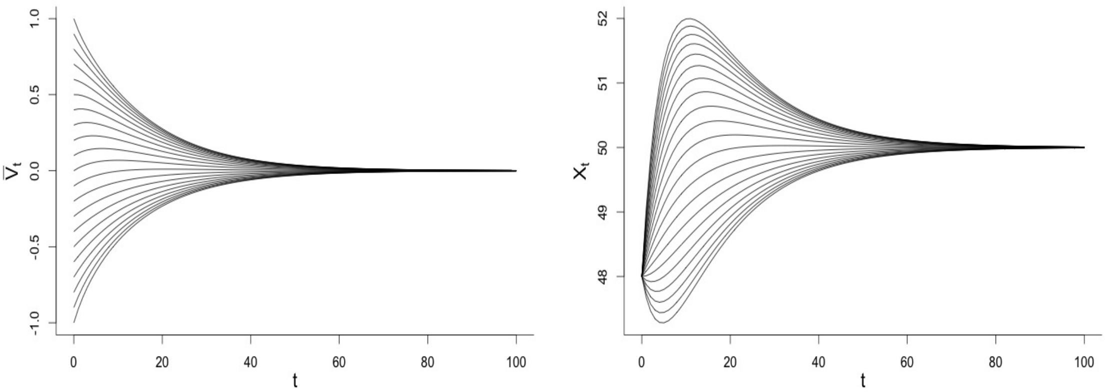

Figure 3.12: $\overline{V}_t$ and X_t for $\gamma_1 = 0.2$, $\gamma_2 = 0.8$, $w_2 = F - x$

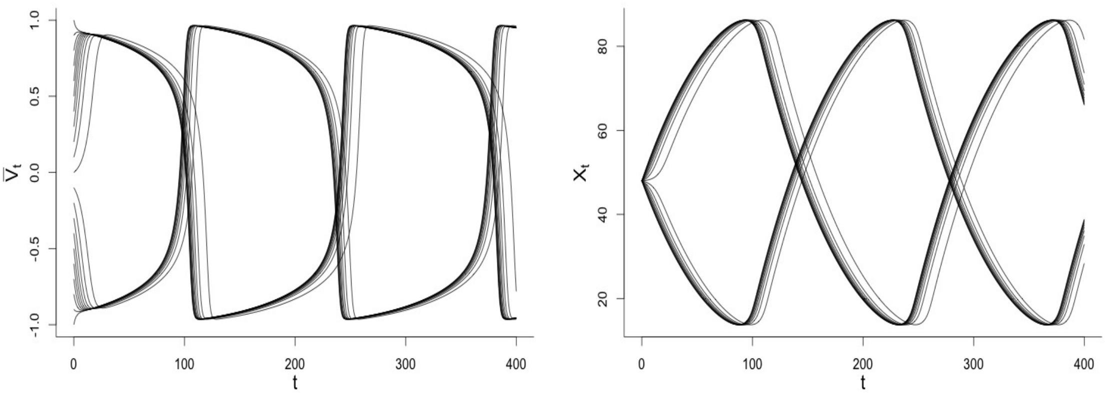

Figure 3.13: $\overline{V}_t$ and X_t for $\gamma_1 = 1.2$, $\gamma_2 = 0.8$, $w_2 = 0.05 * F - x$

3.2.4 Conclusion

To demonstrate the applicability of a seperation of behavior and price process we embedded the model of Lux [48] within our framework. We confirmed, using similar assumption as made in Lux [48], that herding behavior can induce phase transitions and oscillations in the finite-market price process. Introducing random signals, which influence the agents' excess demand function, we were able to extent the result to diffusion price processes, which are the result of a large market limit. Hence we provided an agent behavior based explanation for intrinsic price cycles often seen in financial markets.

3.3 Example 3: A simple periodic guru model

3.3.1 Introduction

In this rather short example we use the basis of Lux's endogenous dynamics (that is, state transition probabilities which model herding behavior) of the previous section to consider herding behavior in a different light. Rather than acting as a homogeneous herd the agents are assumed to follow the actions of individuals, which we call experts, respectively mentors. So the endogenous behavior models a dynamic reputational network in which a financial "guru" emerges spontaneous from a group of experts, gaining a very high reputation and founders when his reputation falls. Note that, since this example is rather illustrative to show other possible applications of Lux's transition probabilities, most of the assumptions made in the following are quite simple.

3.3.2 Finite model and large market approximation

Let $\mathbb{A}_n = \{1, ..., n\}$ be the set of all market participants and $S = \{s_1, ..., s_m\} \subset \mathbb{A}_n$ a subset consisting of mentors. To build the reputational network we assume that every agent can ask one mentor for advice. Mentors can also ask other mentors for advice. So, in this example the assigned characteristic x_k^a for each agent a at time T_k is the index of his adviser, that is $x_k^a \in S$. For simplicity we assume that, if agent a is a mentor, he can also advise himself.
In the following equation we introduce the term of *reputational environment*, which corresponds to the market character of the previous chapters, as the empirical distribution of mentors.

$$M_k = (M_k^i)_{i=1}^m, \tag{3.78}$$

where

$$M_k^i = \frac{1}{n} \sum_{a=1}^{n} \mathbb{1}_{s_i}(x_k^a), k \in \mathbb{N} \tag{3.79}$$

is the *reputation of mentor* s_i defined as the percentage of agents he is advising.
To model herd behavior and to support the emergence of a guru we model the choice of agent a's mentor at time T_k dependent on the reputational environment. In particular, we foster the tendency to choose a mentor with a high reputation. In consequence the reputation grows even more, leading to a guru. However, to also enable the fall of a persisting guru we leave a rest-probability to choose also a new mentor with a lower reputation. In summary we define the transition probability that agent a switches from mentor s_i to mentor s_j as

$$\Pi^{i,j}(P_{k-1}, M_{k-1}) = C_i e^{\gamma(M_{k-1}^j - M_{k-1}^i)}, \tag{3.80}$$

where $\gamma > 0$ is the impact of reputation and C_i is defined by the normalization condition

$$\sum_{j=1}^{m} \Pi^{i,j}(M_{k-1}) = 1, \ i = 1, ..., m. \tag{3.81}$$

Next, we link the reputational environment to the agents trading behavior and further to the price process. We assume a simple market consisting only of one type of traders, namely fundamentalists. We assume that each agent has a fundamental expectation of the assets price F_a and that he bases his individual excess demand on the difference between the actual price and his fundamental value $F_a \in \mathbb{R}$. Analogous to the previous chapter the fundamental agent a considers the asset cheap (expensive) and wants to buy (sell) when the price is below (above) F_a. We assume that the agents buy (sell) a fixed amount of assets $w \in \mathbb{R}^+$ weighted by the difference of the current price and the individual fundamental value. Next, we define how the agent's trading behavior is influenced by his mentor. Therefore we assume that agent a weights his his own fundamental behavior with a constant $\phi \in [0, 1]$ and the behavior of his mentor by $1 - \phi$. Moreover we assume that every mentor is heterogeneously transparent of his fundamental behavior, which we model by the white noise signal ξ_k, which is scaled by $\sqrt{n}$ and multiplied with a mentor specific transparency factor factor $\delta_i \in \mathbb{R}^+, \ i \in \{1, ..., m\}$. We denote the vector of all transparency factors with $\delta = (\delta_1, ..., \delta_m)$, with $F = (F_1, ..., F_m)$ the vector of the mentor's fundamental value and $\overline{F}$ its mean. Moreover let $\overline{F}_a^n = \frac{1}{n} \sum_{a \in \mathbb{A}_n} F_a$ denote the average fundamental value of all agents.

Definition 3.21 (Excess demand function)**.**
In summary we assume the following excess demand function.

$$e_a^n(P_{k-1}, M_{k-1}) := w[\phi F_a + (1 - \phi)(F_{x_{k-1}^a} + \sqrt{n}\delta_{x_{k-1}^a}\xi_k) - P_{k-1}], \tag{3.82}$$

where $(\xi_k)_{k \geq 1}$ is i.i.d. with $\mathbb{E}[\xi_1] = 0$, $\sigma_\xi^2 := \mathbb{E}[\xi_1^2] < \infty$, and $\delta_{x_{k-1}^a} \in \mathbb{R}^+$.

Definition 3.22 (Pricing rule)**.**
We assume the pricing rule is given by

$$r_n(q, x) = x + \frac{\alpha}{n} q \tag{3.83}$$

Again for simplicity reasons we assume that the transition intensities as well as the trading intensities are given by agent common positive constants, i.e. $\mu_a = \bar{\mu} \in \mathbb{R}^+$ and $\lambda_a = \bar{\lambda} \in \mathbb{R}^+$ for all $a \in \mathbb{A}_n$. We follow the general construction of the microscopic model (Section 2.1) that is, we assume initial distributions $P_0 \sim F_{P_0}$, $M_0 \sim F_{M_0^n}$ and embed the Markov chains $(M_k)_{k \geq 0}$ and $(P_k)_{k \geq 0}$ in continuous time using exponentially distributed waiting times with rate $n(\bar{\mu} + \bar{\lambda})$. For the resulting price process $(X_t^n)_{t \geq 0}$ and the reputation process, respectively market character index, $(V_t^n)_{t \geq 0}$, which are well posed by Lemma 2.17, we state the finite market behavior in the following remark in narrative form.

Remark 3.23 (Market behavior).
The market behavior can be distinguished into two general cases, which are independent of the initial values of X_t^n and V_t^n:

1. For small reputation impact γ, there exists one equilibrium at

$$\left((1/m, ..., 1/m), \phi\overline{F}_a^n + (1-\phi)\overline{F}\right) \tag{3.84}$$

2. For large reputation impact γ, there exist m temporary equilibria at

$$\left(V_j^*, \phi\overline{F}_a^n + (1-\phi)V_j^* \circ F\right), \ j = 1, ..., m, \tag{3.85}$$

   where $V^* := \hat{v}\mathbb{1} - e_j\hat{v} + e_j v^*$ with $\mathbb{1}$ being the one-vector, e_j the j-th unit vector and $\hat{v} = \frac{1-v^*}{m-1}$.

Although the determination of v^* and consequently a specification of "small" and "large" γ might be analogous to Section 3.2.2 we refrain from a further analysis. Nevertheless, we discuss the two cases with an illustration of a finite market with three experts, in Figure 3.14 and Figure 3.15. In Figure 3.14 we show the case of a low reputational impact[8], that is $\gamma = 1$, by presenting trajectories of the resulting reputation- and price process. As stated in Remark 3.23 1., all three experts, independent of their initial reputation, develop a rather stable reputation of 1/3. The resulting price process thereby is stable at the weighted fundamental value. The contrast case can be seen in Figure 3.15 when we set a large reputation impact with increasing γ to three, while retaining the other parameters. There we see phases in which one of the experts dominates the reputational process V_t^n with a high reputation v^*. Meanwhile the other experts have an equal low reputation of $\hat{v}$. Notable is not only that the experts take turns in the role of a guru independently of their initial reputation, but also that the resulting temporary equilibrium is the same for all mentors. By construction, the fundamental belief of the predominant expert, i.e. the guru, is transferred to the price process resulting in phases of stable prices.

[8]We set the remaining parameters as $\phi = 0.2, w = 0.5, m = 3, F_a = 40 + 5a$.

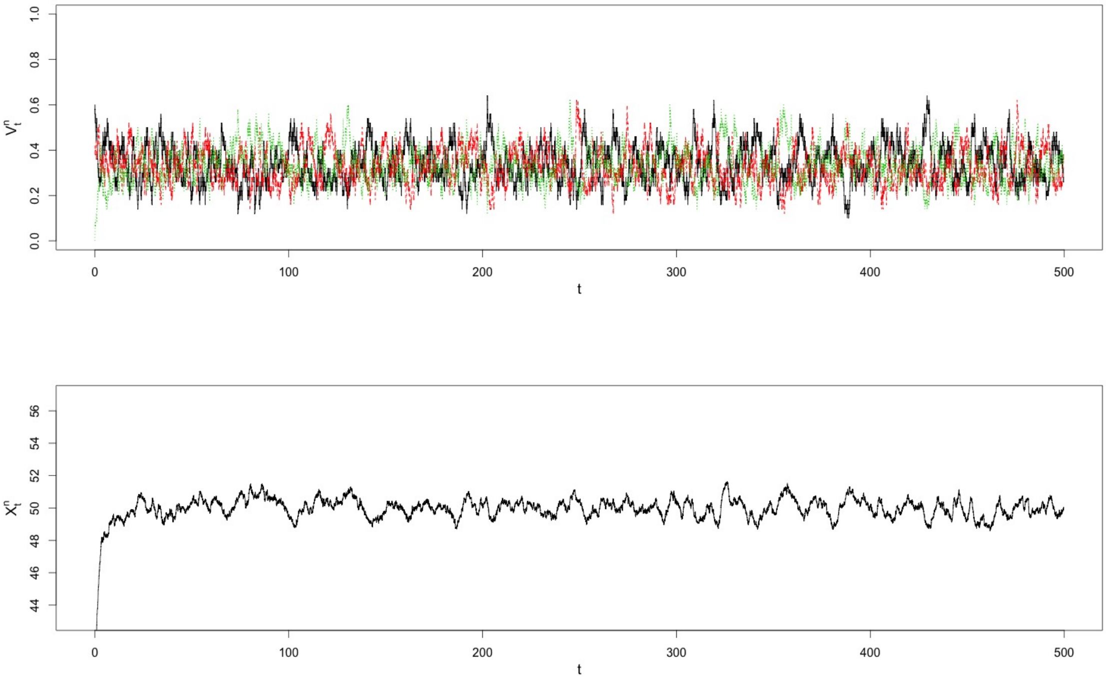

Figure 3.14: Reputational environment V_t^n and Price process X_t^n for $\gamma = 1$

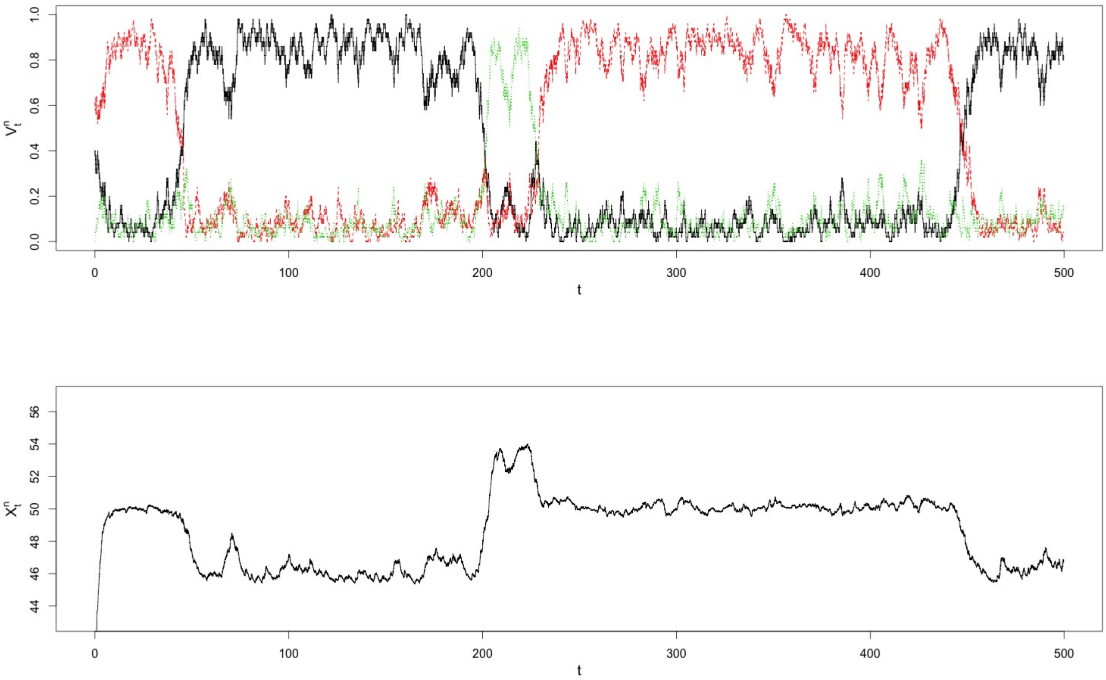

Figure 3.15: Reputational environment V_t^n and Price process X_t^n for $\gamma = 3$

When the market becomes large, the properties of case 2. change, while those of case 1. are the same. Not only does v^* increase with n, but also the predominance of the financial guru persists longer. In the large market limit, which we state in the following proposition, the expert with the highest initial reputation becomes a permanent guru, while all other experts share a low reputation[9]. That also means that the properties of the reputational environment V_t is dependent on its initial values.

Proposition 3.24 (Diffusion approximation)**.**

If $\overline{F}_n \xrightarrow{n\to\infty} \overline{F}$ then

$$(X_t^n, V_t^n)_{t\in[0,\infty)} \xrightarrow{\mathcal{L}} (X_t, V_t)_{t\in[0,\infty)} \text{ in } D_{\mathbb{R}\times[0,1]^m}[0,\infty), \tag{3.86}$$

where $(X_t, V_t)_{t\in[0,\infty)}$ is the unique strong solution of the SDE

$$\begin{cases} dX_t = \alpha\overline{\lambda}w(\phi\overline{F} + (1-\phi)V_t \circ F - X_t)dt + \alpha\sqrt{\overline{\lambda}}(1-\phi)\sigma_\xi\sqrt{V_t \circ \delta^2}dB_t, & X_0 = \zeta \\ dV_t = \overline{\mu}(\Pi^+(X_t, V_t) - \Pi^-(X_t, V_t))dt & V_0 = \theta \end{cases} \tag{3.87}$$

where $F = (F_1, ..., F_m)$, $\Pi^+ = (\Pi^{1+}, ..., \Pi^{m+})$ and $\Pi^- = (\Pi^{1-}, ..., \Pi^{m-})$.

Proof. See Appendix 5.12. □

3.3.3 Conclusion

We used the transition probabilities of the previous section to construct a simple guru model. Therein, when the impact of reputation is large, financial gurus emerge spontaneous from a group of experts. While we refrained from a deeper (technical) analysis we not only showed the applicability of our model to reputational networks, but also presented a first idea how to model guru phenomena.

[9] If several experts have the maximum start reputation, they will share the guru status equally.

3.4 Example 4: Quantum Spikes

3.4.1 Introduction

In this section we show the flexibility of the model by applying dynamics of quantum mechanics to agent's social behavior. To model herd behavior, we transfer dynamics of excitement, originally used in context of a quantum system subject to a thermal bath by Bauer, Bernard and Tilloy [7], respectively Bauer and Bernard [6]. We show, in the large market limit, that spikes and jumps occur in the price process when the herding behavior is intense. As such we provide a microscopic explanation for jumps in asset price processes as oberserved in Aït et al. [1] without using a jump process, as for example employed by Deng [18]. Additionally our dynamics induce high volatility phases that have also been present in various price processes (e.g. Ham [30]).
We structure the content as following. In the first section we specify the general market framework. To describe their interactive behavior we assign each agent heterogeneously an excitement state similar to the excitement of a two level quantum system as in Bauer, Bernard and Tilloy [7] and express the overall market excitement as the distribution of those excitement states. Furthermore we state conditions under which, as a mean-field like result, the overall market excitement can be expressed as a single diffusion process in the large market limit. Then, with defining the agents propensity to trade and by specifying the impact on the asset price, we link the endogenous market dynamics to the asset price movement. We specify conditions under which, in a large market, the asset price development can be approximated by a diffusion price process and conclude with a proposition summarizing the diffusion approximation.

3.4.2 Endogenous dynamics

Before we link the agents behavior to an asset price, we specify all model components, that directly affect the interaction between market participants. Let $\mathbb{A}_n = \{1, ..., n\},\ n \in \mathbb{N}$ be a finite set of agents. Following Bauer, Bernard and Tilloy [7] we consider a state space $S = \{s_1, s_2\} = \{0, 1\}$, where $s_1 = 0$ represents an "unexcited" state and $s_2 = 1$ an "excited" state.

Definition 3.25 (Market excitement)**.**
We measure the proportion of excited agents in the market at time T_k by the *market excitement*

$$\overline{M}_k = \frac{1}{n} \sum_{a=1}^{n} \mathbb{1}_{\{1\}}(x_k^a), k \in \mathbb{N}. \tag{3.88}$$

Additionally, we denote the initial distribution of the market excitement resulting from $F^n_{x_0}$ as $F^n_{\overline{M}_0}$. Note that, by construction, $\overline{M}_k$ is equal to M_k^2 with $d_1 = 1$, the average excitement of all agents and a probability measure on C.

Definition 3.26 (Transition intensity)**.**
In order to heterogeneously specify the agents tendency to consider a state transition we assign to each agent the transition intensity

$$\mu_a = n\gamma_a^2, \tag{3.89}$$

that is an agent dependent constant $\gamma_a^2 \in \mathbb{R}^+$ times the number of agents participating in the market.

Next, we characterize the state transition laws, i.e. the probability that agent a changes from excited to unexcited and vice versa, given that he is the one that considers a state change. We then consequentially derive the dynamics of the market excitement.

Definition 3.27 (Transition probabilities)**.** We use the following individual state transition probabilities.

$$\Pi_{n,a}^{1,2}(\overline{M}_{k-1}) = \beta_a \frac{p_a}{2\gamma_a^2 n} + \frac{\eta_a h^{1,2}(\overline{M}_{k-1})}{2} \tag{3.90}$$

$$\Pi_{n,a}^{2,1}(\overline{M}_{k-1}) = \beta_a \frac{1-p_a}{2\gamma_a^2 n} + \frac{\eta_a h^{2,1}(\overline{M}_{k-1})}{2}, \tag{3.91}$$

where $\beta_a, p_a, \eta_a \in [0,1]$ are agent dependent constants and

$$h^{i,j}(y) = (1-y)^i y^j, \quad y \in [0,1], i \neq j \in \{1,2\}. \tag{3.92}$$

We capture all state transition probabilities per agent in a transition matrix, i.e. we define

$$\Pi_{n,a}(\overline{M}_{k-1}) = \begin{pmatrix} 1-\Pi_{n,a}^{1,2} & \Pi_{n,a}^{1,2} \\ \Pi_{n,a}^{2,1} & 1-\Pi_{n,a}^{2,1} \end{pmatrix} (\overline{M}_{k-1}). \tag{3.93}$$

Remark 3.28. We choose this explicit form of transition probabilities presented in Equations (3.90) and (3.91) for the following reasons. The first part of the sum models individual intrinsic disposition for excitement. Thereby p_a, respectively $1-p_a$, captures the distance between agents a's actual excitement state and his individual preference.[10] So we heuristically reflect a higher drive to transition when the distance to the personal preference is large. Apart from autonomous behavior we also want to model influence of other agents on the individuals excitement state. For this we use the second addend which takes into account the average excitement of all agents and thus models herd behavior. By the choice of the form of $h^{i,j}$ (Equation (3.92)), an unexcited agent ($x_{k-1}^a = 0$) has a higher probability to transition if the market excitement is large. Analogously, the transition probability to become unexcited is bigger when the market excitement is low, that is, if the majority of

[10]Note that $\Pi_{n,a}^{1,2}$ is only relevant for unexcited agents ($x_{k-1}^a = 0$) and analogously $\Pi_{n,a}^{2,1}$ only for agents with $x_{k-1}^a = 1$. Hence the simplified form in Equations (3.90) and (3.91).

agents is unexcited. Besides being simple and symetric, $h^{i,j}$ also induces the same dynamics in the large market limit as a continuous measurement of the quantum system[11], which further supports the choice. We weight the two aspects, that is autonomous behaviour and heteronomy, individually per agent by constants β_a and η_a. Moreover we scale the first part by μ_a to get a well designed probability measure and to model the increasing importance of herding when the market is large.

By construction $\overline{M}_k$ has values on the $n+1$ valued lattice $\mathbb{L}$ from 0 to 1, that is

$$\overline{M}_k \in \mathbb{L},\ \forall k \geq 0,\ \text{with } \mathbb{L} := \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-2}{n}, \frac{n-1}{n}, 1\right\}. \tag{3.94}$$

Definition 3.29 (Endogenous market history)**.** We capture all endogenous information up to $\tilde{T}_k$ in the *endogenous market history*, which is given by the sigma algebra $\tilde{\mathcal{G}}_k := \sigma(\tilde{T}_i, A_i, \overline{M}_i,\ i \leq k) \subset \mathcal{G}_k$. Here the tupel $(T_k, A_k, \overline{M}_k),\ k \in \mathbb{N}$ represents agent A_k who makes a transition at $\tilde{T}_k$ and the resulting market excitement $\overline{M}_k$.

In summary, $(\overline{M}_k)_{k\geq 0}$ is a Markov chain on $\mathbb{L}$ with value dependent transition probabilities, which are stated in the next Lemma.

Lemma 3.30 (Discrete market excitement dynamics)**.**
The transition probabilities of $\overline{M}_k$ are given by

$$\mathbb{P}\left(\overline{M}_k - \overline{M}_{k-1} = -\frac{1}{n}\Big|\tilde{\mathcal{G}}_{k-1}\right) = \frac{\sum_{a=1}^n \left(\beta_a(1-p_a)\overline{M}_{k-1} + n\eta_a\gamma_a^2(1-\overline{M}_{k-1})^2\overline{M}_{k-1}^2\right)}{2n\sum_{a=1}^n \gamma_a^2} \tag{3.95}$$

and

$$\mathbb{P}\left(\overline{M}_k - \overline{M}_{k-1} = \frac{1}{n}\Big|\tilde{\mathcal{G}}_{k-1}\right) = \frac{\sum_{a=1}^n \left(\beta_a p_a(1-\overline{M}_{k-1}) + n\eta_a\gamma_a^2(1-\overline{M}_{k-1})^2\overline{M}_{k-1}^2\right)}{2n\sum_{a=1}^n \gamma_a^2} \tag{3.96}$$

Proof. Application of Lemma 2.12. □

In order to embed the Markov chain $(\overline{M}_k)_{k\geq 0}$ homogeneously in continuous time and thus describing the market excitement by a time homogeneous Markov process, we further characterize the points in times at which the agents decide to make a transition.

Definition 3.31 (Transition waiting times)**.**
The *transition waiting times* $\tilde{\tau} = \tilde{T}_k - \tilde{T}_{k-1},\ k \geq 1$ are assumed to be exponentially

[11] See Bauer, Bernard and Tilloy [7].

distributed with rate $\mu_{\mathbb{A}_n}$, i.e.

$$\mathbb{P}(\tilde{\tau}_k \in [0,t] | \tilde{\mathcal{G}}_{k-1}) = 1 - e^{-nt\sum_{a=1}^{n}\gamma_a^2},\ t \geq 0, \tag{3.97}$$

Definition 3.32 (Market excitement index).
We can define the *market excitement index* via

$$Q_t^n := \sum_{k=0}^{\infty} \overline{M}_k \mathbb{1}_{[T_k, T_{k+1})}(t),\ t \geq 0. \tag{3.98}$$

Note that, by construction, Q_t^n is cádlág and a well defined time homogeneous pure jump type Markov process by Lemma 2.17. We summarize the discrete Markov process Q_t^n in the following lemma.

Lemma 3.33 (Existence).
There exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ in which $(Q_t^n)_{t\in[0,\infty)}$ is a time homogeneous pure jump Markov process with rate kernel

$$K_n(q,s) := n\sum_{a=1}^{n} \gamma_a^2 k_n(q,s), \tag{3.99}$$

The transition kernel $k_n(q,s),\ s \in \left\{-\frac{1}{n}, 0, \frac{1}{n}\right\}$ is a regular version of the conditional distribution $\mathbb{P}(\overline{M}_1 - \overline{M}_0 = s | \overline{M}_0 = q)$, which is given by Lemma 3.30.

Proof. Analogous to Lemma 2.17. □

Although the heterogeneous agents are allowed to have individual parameters the scaled parameters should tend to their mean when the number of market participants goes to infinity. So we ensure a convergence to a mean-field like single equation in the large market limit. This we summarize in the next Assumption.

Assumption 3.34.
We assume

1. $F_{\overline{M}_0}^n \xrightarrow{n\to\infty} F_{\overline{M}_0}$
2. $\sum_{a=1}^{n} \frac{\gamma_a^2 \eta_a}{n} \xrightarrow{n\to\infty} \gamma^2\eta$
3. $\sum_{a=1}^{n} \frac{\beta_a p_a}{2n} \xrightarrow{n\to\infty} \beta p$

for some constants β, γ, η, p and $F_{\overline{M}_0}$ being a probability distribution.

Now, we are ready to state the large market limit for the market excitement index.

Proposition 3.35 (Large market approximation).
If Assumption 3.34 holds, then

$$(Q_t^n)_{t\in[0,\infty)} \xrightarrow{\mathcal{L}} (Q_t)_{t\in[0,\infty)} \ in \ D_{[0,1]}[0,\infty), \tag{3.100}$$

with $(Q_t)_{t\in[0,\infty)}$ being the unique strong solution of the SDE

$$dQ_t = \beta(p - Q_t)dt + \gamma\sqrt{\eta}(1 - Q_t)Q_t dB_t, \quad Q_0 = \theta, \tag{3.101}$$

where $(B_t)_{t\in[0,\infty)}$ is a one dimensional standard Brownian motion, which is independent of $\theta \sim F_{\overline{M}_0}$.

Proof. See Appendix 5.13. □

To illustrate properties of the large market limit Q_t we show two trajectories of the solution of Equation (3.101) for $p = 0.6$, $\eta = \beta = 1$ below in Figure 3.16 and Figure 3.17. The appearance of the process strongly depends on the value of γ. For a small γ, as shown in Figure 3.16 with $\gamma = 1$, the market excitement index moves towards an equilibrium at the constant $p \in [0, 1]$, which is given as the mean of the individual preference level p_a (see Assumption 3.34 3.). Setting a high value of γ (see Figure 3.17, where $\gamma = 10$) the market excitement index is pulled towards the two states $s_1 = 0$ and $s_2 = 1$ with jumps and spikes in between. Although the separation into two cases (i.e. one or two equilibria) by the value of γ is not obvious from the underlying SDE (3.101), it is expected from the microscopic modeling. In the individual transition probabilities (see Equation (3.90) and (3.91)) γ_a scales down the agents individual autonomy and hence represents the exposure to herding. Respectively, γ reflects the average herding intensity. Note that we already showed in Section 3.2, with a similar setup, that minor herding behavior results in a single equilibrium, while strong herd behavior results in two temporary equilibria with phase transitions. However, here the two states s_1 and s_2 serve as the two temporary equilibria, where jumps represent phase transitions and the spikes imply unsucessful jump attempts. Note that the probability to be in the equilibrium s_1 is equal to p (see Bauer, Bernard and Tilloy [7]) and that the behavior is similar to Kramer's double well potential (See Kramer [43]).

Remark 3.36.
Although the SDE presented in Equation (3.101) is exactly the same as in Bauer, Bernard and Tilloy [7], it arises differently. In Bauer, Bernard and Tilloy [7] the SDE is the result of a transiton from discrete to continuous time in the measurement of a single quantum system, while in our model the SDE is induced by the number of interacting objects tending to infinity.

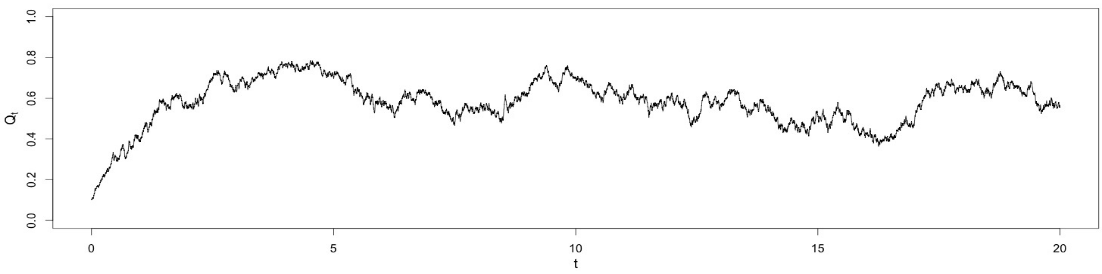

Figure 3.16: Q_t for $\gamma = 1$

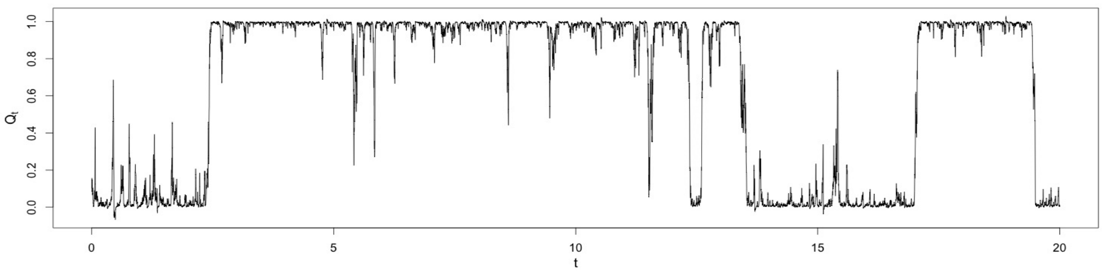

Figure 3.17: Q_t for $\gamma = 10$

3.4.3 Price dynamics

In this section we link the endogenous dynamics of the previous section to the asset price dynamics. Per model construction we assign to each trader an individual trading intensity and an excess demand function. We then define a pricing rule according to which the number of bought or sold shares impacts the price. To further show the flexibility of our model, we introduce fundamentalists as an additional group of traders[12]. We assume the fundamentalists are homogeneous, viz. the fundamental value F is common and time-invariant.

Let $\mathbb{A}_n = \{1, ..., n\},\ n \in \mathbb{N}$ be the set of agents of which a fixed subset $\mathbb{F}_n \subseteq \mathbb{A}_n$ with $|\mathbb{F}_n| = k_n \in \{0, ..., n\}$ are fundamentalists and the rest are noise traders. We denote the portion of fundamentalists with $\phi_n = k_n/n$. We assume that fundamentalists are unexcited and have no desire to change their state, i.e. $\forall a \in \mathbb{F}_n :\ x_0^a = 0,\ \beta_a = \eta_a = 0$. Note that, alternatively we could have introduced fundamentalists as an additional state[13]. However, the current setup illustrates the flexibility arising from heterogeneous transition probabilities. We extent the endogenous market history with the information of P_k and the action indicator B_k and define the market history as $\mathcal{G}_k = \sigma(T_i, A_i, P_i, \overline{M}_i, B_i,\ i \leq k)$.

[12]See Section 3.2.3 for explanation of fundamentalists.
[13]As we did in Section 3.2.3.

Definition 3.37 (Trading intensity, action rate).
We assume that the agents propensity to trade is given by the trading intensity

$$\lambda_a = \bar{\lambda}_a + C_e x^a_{k-1}, \tag{3.102}$$

with $\bar{\lambda}_a \in \mathbb{R}^+$ being an agent dependent basic trading intensity and $C_e \in \mathbb{R}^+$ a positive constant, which reflects the positive impact of excitement on the propensity to trade.

The aggregated action rate is then given as

$$\nu_{\mathbb{A}_n} = \sum_{a=1}^{n} \nu_a = C_e \overline{M}_k + \sum_{a=1}^{n} (n\gamma_a^2 + \bar{\lambda}_a) \tag{3.103}$$

Next we define the traded quantity per agent once he decided to trade. Thereby we differentiate between noise traders and fundamentalists. While fundamentalists base their excess demand on the difference between the last known price and the fundamental value, noise traders trade according to random signals $(\xi_k)_{k\geq 1}$, which are assumed to be i.i.d. with $\mathbb{E}[\xi_1] = 0$ and $\sigma_\xi^2 := \mathbb{E}[\xi_1^2] < \infty$. Note that the variance of the traded quantity is determined by the variance of the market excitement $\overline{M}_k$.

Definition 3.38 (Excess demand function).
In summary we set the following *excess demand function*

$$e_a(P_{k-1}, \overline{M}_{k-1}, \xi_k) = \begin{cases} \frac{1}{\sqrt{n}}(F - P_{k-1}), & a \in \mathbb{F}_n \\ \xi_k \gamma_a^2 \eta_a \overline{M}_{k-1}^2 (1 - \overline{M}_{k-1})^2, & a \notin \mathbb{F}_n. \end{cases} \tag{3.104}$$

After an agent decides to trade, the new price at time T_k is defined by the pricing rule

$$r_n(q, x) = x + \frac{\alpha}{\sqrt{n}} q. \tag{3.105}$$

Recall that the price process is defined as

$$X_t^n := \sum_{k=0}^{\infty} P_k \mathbb{1}_{[T_k, T_{k+1})}(t), \ t \geq 0, \tag{3.106}$$

where $P_0 \sim F_{P_0}$. Further note that the intra-action times are by Definition 2.14 and Equation (3.103) distributed as

$$\mathbb{P}(\tau_k \in [0,t] | \mathcal{G}_{k-1}) = 1 - e^{-t(C_e \overline{M}_k + \sum_{a=1}^{n} (n\gamma_a^2 + \bar{\lambda}_a))}, \ t \geq 0, \tag{3.107}$$

We summarize the finite setup in the following lemma.

Lemma 3.39 (Existence).
There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, in which $(X_t^n, Q_t^n)_{t\in[0,\infty)}$ is a time homogeneous

pure jump Markov process with rate kernel

$$K_n(x, q, dy, s) := \nu_{\mathbb{A}_n} k_n(x, q, dy, s), \tag{3.108}$$

where the transition kernel $k_n(x, q, dy, s)$ is a regular version of the conditional distribution $\mathbb{P}(P_1 - P_0 \in dy, \overline{M}_1 - \overline{M}_0 = s | P_0 = x, \overline{M}_0 = q)$, $s \in \{-\frac{1}{n}, 0, \frac{1}{n}\}$.

Proof. Analogous to Lemma 2.17. □

Before we can state the large market limit for the market excitement index and the price process, we assume some stability of the proportion of fundamentalists. Additionally we require a mean-convergence of the trading intensities of the fundamentalists as well as of the noise traders.

Assumption 3.40.
We assume

1. $\phi_n \xrightarrow{n\to\infty} \phi$
2. $\frac{1}{n}\sum_{a\in\mathbb{F}_n}^n \bar{\lambda}_a \xrightarrow{n\to\infty} \bar{\lambda}_F$
3. $\frac{1}{n}\sum_{a\notin\mathbb{F}_n}^n \bar{\lambda}_a \xrightarrow{n\to\infty} \bar{\lambda}_N$

for some constants $\phi \in [0, 1], \bar{\lambda}_F \in \mathbb{R}^+$ and $\bar{\lambda}_N \in \mathbb{R}^+$.

Next, we state the SDE whose solution approximates the endogenous dynamics and the price process in a large market, that is a market with many participants.

Proposition 3.41 (Diffusion approximation)**.**

If Assumptions 3.34 and 3.40 hold, then

$$(X_t^n, Q_t^n)_{t\in[0,\infty)} \xrightarrow{\mathcal{L}} (X_t, Q_t)_{t\in[0,\infty)} \text{ in } D_{\mathbb{R}\times[0,1]}[0, \infty), \tag{3.109}$$

where $(X_t, Q_t)_{t\in[0,\infty)}$ is the unique strong solution of the SDEs

$$\begin{cases} dQ_t = \beta(p - Q_t)dt + \gamma\sqrt{\eta}(1 - Q_t)Q_t dB_t, & Q_0 = \theta \\ dX_t = \bar{\lambda}_F(F - X_t)dt + \sigma_\xi\sqrt{\bar{\lambda}_N + C_e Q_t}\gamma\sqrt{\eta}(1 - Q_t)Q_t dW_t, & X_0 = \zeta, \end{cases} \tag{3.110}$$

where $(B_t)_{t\in[0,\infty)}$ and $(W_t)_{t\in[0,\infty)}$ are independent one dimensional standard Brownian motions, $\zeta \sim F_{P_0}$ is independent of W_t, and $\theta \sim F_{\overline{M}_0}$ is independent of B_t.

Proof. See Appendix 5.14. □

Equation (3.110) summarizes our model in the large market limit. The endogenous behavior is described by Q_t given by the first SDE, which is not depending on the price process X_t

and is the same as in the previous subsection. On the contrary, X_t depends on Q_t. Not only the volatility coefficient of Q_t reappears in the SDE defining X_t, but Q_t also scales the volatility of X_t with the factor $\bar{\lambda}_N + C_e Q_t$. Notably, last leads to high volatility phases when the majority of agents is excited.

To illustrate the properties of $(X_t, Q_t)_{t\geq 0}$ we show two trajectories. Thereby we repeat the figures of Q_t from the previous section for readers convenience.

In Figure 3.18 and Figure 3.19 we show the first case with a trajectory of X_t with $F = 50$, $\phi = 0.2$, $\delta = 2$ and endogenous dynamics, that is Q_t, with parameters $p = 0.6$, $\eta = \beta = 1$ and $\gamma = 1$. Driven by 20% of the agents being fundamentalists, X_t drifts to the fundamental value. Thereby the volatility is rather stable, since Q_t has a single equilibrium at $p = 0.6$ and the rest of the volatility coefficient of X_t consists of constants.

We illustrate the second case in Figure 3.20 and Figure 3.21 with the same parameters, but setting $\gamma = 10$. There the spikes and phase transitions from Q_t are transferred to the price process and result in spikes and jumps. Moreover, as explained above, the phases of temporary equilibrium of Q_t at $s_1 = 1$ comply with high volatility phases of X_t, since the factor $\bar{\lambda}_N + C_e Q_t$ increases the volatiliy coefficient of X_t when Q_t is large. The intensity of the effect of last is specifically steered by the constant C_e.

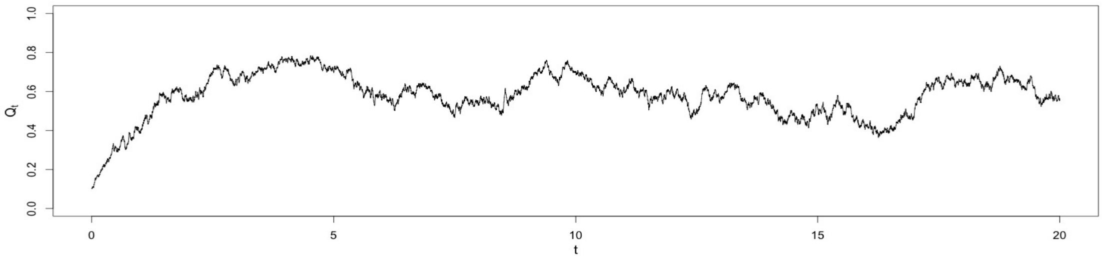

Figure 3.18: Q_t for $\gamma = 1$

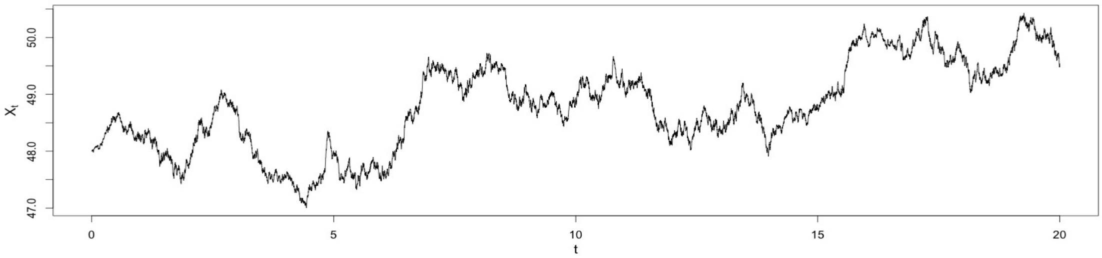

Figure 3.19: X_t for $\gamma = 1$

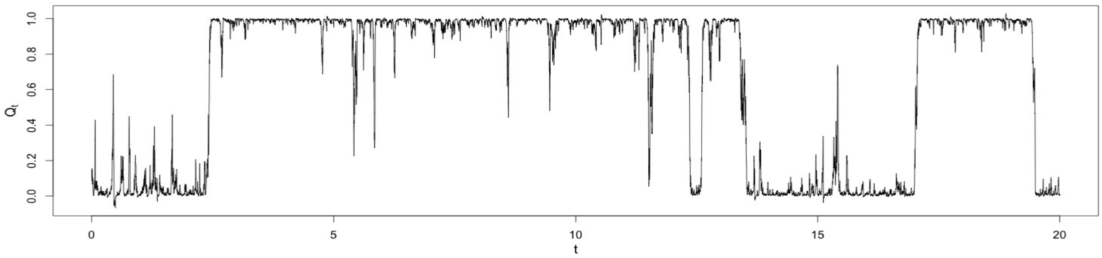

Figure 3.20: Q_t for $\gamma = 10$

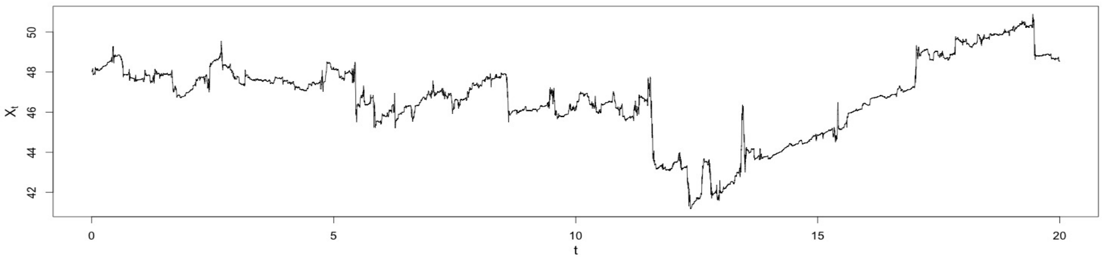

Figure 3.21: X_t for $\gamma = 10$

Remark 3.42 (Approximation with Poisson Jump Process)**.** Since the SDE of Q_t in Equation (3.110) is exactly the same as in Bauer, Bernard and Tilloy [7], we can leverage from a result on statistical properties presented there in Proposition 2. The position of spikes and jumps (that is positions of local maxima and minima) can be approximated, when γ is large, by two Poisson point processes N_t^0 and N_t^1 on $[0,1] \times \mathbb{R}^+$ with intensities

$$\begin{cases} d\Lambda_0 = \beta p dt \left[\delta(1-N_t^0)dN_t^0 + \frac{dN_t^0}{(N_t^0)^2}\right], & N_0^0 = \theta \\ d\Lambda_1 = \beta(1-p)dt \left[\delta(N_t^1)dN_t^1 + \frac{dN_t^1}{(1-N_t^1)^2}\right], & N_0^1 = 1-\theta, \end{cases} \tag{3.111}$$

where δ is the delta function.

Remark 3.43. While the statistics of Q_t are discussed in Bauer, Bernard and Tilloy [8], respectively Tilloy, Bauer and Bernard [66], the statistics of X_t (especially the structure of the spikes) were not studied yet although they are a direct consequence of Q_t. One might use the Fokker Planck approximation to approximate the stationary distribution of X_t^n, respectively X_t, to get more insight. Additionally, in order to study X_t, it might also be worth to study the underlying Markov chain $(P_k)_{k\geq 0}$. However both is out of this thesis scope.

3.4.4 Conclusion

We have proposed microscopic foundations to explain jumps, spikes and high volatility phases in diffusion price processes. The agents endogenous behavior is thereby inspired by the dynamics of excited particles in a quantum system (see Bauer, Bernard and Tilloy [7]). In a second step we linked the endogenous dynamics to the price process by specifying agents individual trading propensity and excess demand functions together with an overall pricing rule. Furthermore, we showed the conditions under which the average agent excitement as well as the price process converge to a diffusion process when the number of market participants tends to infinity. We showed that when herding is negligible, the resulting price process drifts towards the fundamental value with stable volatility. On the contrary, when herding is strong, spikes and jumps occur together with phases of high volatility. Since our model induces large market dynamics that are likewise present in the discussion of quantum systems coupled to a thermal bath with continuous monitoring (see Bauer, Bernard and Tilloy [7]) we build a bridge between quantum mechanics and financial mathematics. So we could leverage from the statistical properties of quantum trajectories and apply a result of Bauer, Bernard and Tilloy [7] to our asset price model by which the occurring jumps and spikes can be approximated by two Poisson processes.

4 Conclusion and Outlook

4.1 Conclusion

We presented a microscopic framework to model diffusion price processes which is based on the behavior of a pool of interacting agents. Thereby we not only allowed for interaction between the agents and price feedback effects, but also for a high level of individualization. We constructed the model starting from a finite discrete Markovian framework, where each agent has specific characteristics and an individual buying behavior. After we quantified the overall market characteristics by a mean-like pooling, which we called market character, we embedded it and the interacting price process in continuous time. We then considered a market with many agents and showed under which conditions the large market can be approximated by a diffusion. To show the usability of our model, we embedded an existing model and not only, using similar assumptions, achieved the same result, but were able to provide more details on the underlying dynamics and extend the result to diffusion price processes. We furthermore provided a short example of how the same dynamics could be used to explain the phenomena of financial gurus and how to examine reputational networks within our framework. Finally, we transferred dynamics observed in the field of quantum mechanics to model the excitement of traders. Assuming a strong herding behavior, we were able to show the emergence of hypes and explain spikes, jumps and high volatility phases of price processes. Although we allowed for a high degree of individualism related to the agents behavior, we made several restrictive assumptions. First, we assumed that only one agent is acting at a time. While this is reasonable in a continuous time framework the assumption was made mainly for simplification and could be loosened. The same is true for the boundedness of the intensity functions. The Markov assumption, however, is critical for the model. Not only is the assumption necessary for the price processes to be diffusive, but it also contributes to the usability of the model since phenomena observed in diffusion price processes can be broken down to Markov chains for easier analysis. Overall, we presented a general and flexible model that has diverse applications and a very good traceability.

4.2 Outlook

For the sake of simplicity several assumptions have been made, that also show limitations of the model. The most crucial, the strong Markov property, seems unrealistic in light of historical data. In order to adress this limitation more complicated microscopic models leading to limits "near" to Markovian, e.g. solutions of stochastic differential delay equations (see Arriojas et al. [2]), could be investigated in the future. However, far less literature in form of limit theorems is available for this case. Further future research might consider making the model even more general. While it seems reasonable that the pricing rule is common to all agents, the affine form is not critical and could be more general. Another possibility would be to consider several assets. Especially, the dependency between assets resulting from a common pool of traders would be of interest. Also this thesis is rather theoretical and lacks a reference to historical data. Quantitative verification of the examples provided within the thesis as well as examination of modern trading techniques (e.g. high frequency trading) would further help to understand how the psychological aspects of agents' behavior influence price processes.

5 Appendix

5.1 Proof of Lemma 2.12

Basic calculations yield

$$
\begin{aligned}
&\mathbb{P}(M_k^i - M_{k-1}^i = -n^{-d_1}|\mathcal{G}_{k-1}) = \mathbb{P}(B_k = 0, x_k^{A_k} \neq s_i, x_{k-1}^{A_k} = s_i|\mathcal{G}_{k-1}) \\
&= \sum_{\substack{j=1,\\ j\neq i}}^{m} \mathbb{P}(B_k = 0, x_k^{A_k} = s_j, x_{k-1}^{A_k} = s_i|\mathcal{G}_{k-1}) \\
&= \sum_{\substack{j=1,\\ j\neq i}}^{m} \sum_{a=1}^{n} \mathbb{P}(B_k = 0, x_k^{A_k} = s_j, x_{k-1}^{A_k} = s_i, A_k = a|\mathcal{G}_{k-1}) \\
&= \sum_{\substack{j=1,\\ j\neq i}}^{m} \sum_{a=1}^{n} \mathbb{P}(x_k^{A_k} = s_j|B_k = 0, x_{k-1}^{A_k} = s_i, A_k = a, \mathcal{G}_{k-1})\mathbb{P}(B_k = 0, x_{k-1}^{A_k} = s_i, A_k = a|\mathcal{G}_{k-1}) \\
&= \sum_{\substack{j=1,\\ j\neq i}}^{m} \sum_{a=1}^{n} \mathbb{P}(x_k^{A_k} = s_j|B_k = 0, x_{k-1}^{A_k} = s_i, A_k = a, \mathcal{G}_{k-1})\mathbb{P}(B_k = 0, A_k = a|\mathcal{G}_{k-1})\mathbb{P}(x_{k-1}^{A_k} = s_i) \\
&= \sum_{\substack{j=1,\\ j\neq i}}^{m} \sum_{a=1}^{n} \Pi_{n,a}^{i,j} \frac{\mu_a(P_{k-1}, M_{k-1})}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})} n^{d_1-1} M_{k-1}^i \\
&= \frac{\sum_{a=1}^{n} \mu_a(P_{k-1}, M_{k-1})\Pi_{n,a}^{i-}(P_{k-1}, M_{k-1})}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})}
\end{aligned}
\tag{5.1}
$$

as well as

$$
\begin{aligned}
&\mathbb{P}(M_k^i - M_{k-1}^i = n^{-d_1} | \mathcal{G}_{k-1}) = \mathbb{P}(B_k = 0, x_k^{A_k} = s_i, x_{k-1}^{A_k} \neq s_i | \mathcal{G}_{k-1}) \\
&= \sum_{\substack{j=1,\\ j\neq i}}^{m} \mathbb{P}(B_k = 0, x_k^{A_k} = s_i, x_{k-1}^{A_k} = s_j | \mathcal{G}_{k-1}) \\
&= \sum_{\substack{j=1,\\ j\neq i}}^{m} \sum_{a=1}^{n} \mathbb{P}(B_k = 0, x_k^{A_k} = s_i, x_{k-1}^{A_k} = s_j, A_k = a | \mathcal{G}_{k-1}) \\
&= \sum_{\substack{j=1,\\ j\neq i}}^{m} \sum_{a=1}^{n} \mathbb{P}(x_k^{A_k} = s_i | B_k = 0, x_{k-1}^{A_k} = s_j, A_k = a, \mathcal{G}_{k-1}) \mathbb{P}(B_k = 0, x_{k-1}^{A_k} = s_j, A_k = a | \mathcal{G}_{k-1}) \\
&= \sum_{\substack{j=1,\\ j\neq i}}^{m} \sum_{a=1}^{n} \mathbb{P}(x_k^{A_k} = s_i | B_k = 0, x_{k-1}^{A_k} = s_j, A_k = a, \mathcal{G}_{k-1}) \mathbb{P}(B_k = 0, A_k = a | \mathcal{G}_{k-1}) \mathbb{P}(x_{k-1}^{A_k} = s_j) \\
&= \sum_{\substack{j=1,\\ j\neq i}}^{m} \sum_{a=1}^{n} \Pi_{n,a}^{j,i} \frac{\mu_a(P_{k-1}, M_{k-1})}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})} n^{d_1 - 1} M_{k-1}^j \\
&= \frac{\sum_{a=1}^{n} \mu_a(P_{k-1}, M_{k-1}) \Pi_{n,a}^{i+}(P_{k-1}, M_{k-1})}{\nu_{\mathbb{A}_n}(P_{k-1}, M_{k-1})}.
\end{aligned}
\tag{5.2}
$$

In the penultimate line we used the representation defined in Equation (2.3) and Equation (2.6) as well as the fact that $n^{d_1-1} M_{k-1}^i$ is a probability measure on C which gives us $\mathbb{P}(x_{k-1}^{A_k} = s_i)$.

5.2 Proof of Remark 2.22

Proof. Let $n^{-2d_1}\mu_{\mathbb{A}_n}(x,v) \xrightarrow{n\to\infty} 0$. Then for all $i \neq j \in \{1,...,m\}$ we have

$$\begin{aligned}
|c_n^{i,j}(x,v)^2| &= \left| -n^{-(d_1+1)} \sum_{a=1}^{n} \mu_a(x,v)(v_i \Pi_{n,a}^{i,j}(x,v) + v_j \Pi_{n,a}^{j,i}(x,v)) \right| \\
&= \left| -n^{-2d_1} \sum_{a=1}^{n} \mu_a(x,v)(n^{d_1-1} v_i \Pi_{n,a}^{i,j}(x,v) + n^{d_1-1} v_j \Pi_{n,a}^{j,i}(x,v)) \right| \\
&\leq \left| -n^{-2d_1} \sum_{a=1}^{n} \mu_a(x,v) 2 \right| \xrightarrow{n\to\infty} 0
\end{aligned} \tag{5.3}$$

and for all $i \in \{1,...,m\}$

$$\begin{aligned}
|c_n^{i}(x,v)^2| &= \left| n^{-2d_1} \sum_{a=1}^{n} \mu_a(x,v)(\Pi_{n,a}^{i+}(x,v) + \Pi_{n,a}^{i-}(x,v)) \right| \\
&\leq \left| -n^{-2d_1} \sum_{a=1}^{n} \mu_a(x,v) 2 \right| \xrightarrow{n\to\infty} 0
\end{aligned} \tag{5.4}$$

□

5.3 Proof of Theorem 2.25

In order to determine the diffusional limit we want to apply Theorem IX. 4.21 of Jacod and Shiryaev [38]. Let us first note that the problem is well defined as the local Lipschitz and linear growth conditions set related to z, b,σ and c imply the existence of a unique solution of Equation (2.29) by Theorem III.2.32 of Jacod and Shiryaev [38]. Moreover IX.4.3. (ii) Jacod and Shiryaev [38] follows from Theorem 21.10 Kallenberg [39].
In order to improve the readability we capture the change of the price by

$$\overline{r}_n(q,x) = r_n(q,x) - x \tag{5.5}$$

Hypothesis (i)

To show Theorem IX. 4.21 (i) of Jacod and Shiryaev [38] we have to calculate the first and second moments of the rate kernel K_n defined in Equation (2.18). We start with the first moment related to the price component. Applying Equations (2.2),(2.18), (2.21) and the desintegration theorem stated in Theorem 6.4 of Kallenberg [39] leads to the following representation.

$$\begin{aligned}\int K_n(x,v,dy,dw)y &= \nu_{\mathbb{A}_n}(x,v)\int k_n(x,v,dy,dw)y \\ &= \nu_{\mathbb{A}_n}(x,v)\mathbb{E}[P_1 - P_0|P_0 = x, M_0 = v] \\ &= \nu_{\mathbb{A}_n}(x,v)\mathbb{E}\left[\sum_{a=1}^{n}\overline{r}_n(e_a^n(P_0,M_0,\xi_1),P_0)\mathbb{1}_{\{A_1=a,B_1=1\}}|P_0 = x, M_0 = v\right] \\ &= \nu_{\mathbb{A}_n}(x,v)\sum_{a=1}^{n}\mathbb{P}(A_1 = a, B_1 = 1|P_0 = x, M_0 = v)\mathbb{E}[\overline{r}_n(e_a^n(x,v,\xi_1),x)] \\ &= \nu_{\mathbb{A}_n}(x,v)\sum_{a=1}^{n}\frac{\lambda_a(x,v)}{\nu_{\mathbb{A}_n}(x,v)}\mathbb{E}\left[\overline{r}_n(e_a^n(x,v,\xi_1),x)\right] \\ &= \sum_{a=1}^{n}\lambda_a(x,v)\mathbb{E}\left[\overline{r}_n(e_a^n(x,v,\xi_1),x)\right] \\ &= \sum_{a=1}^{n}\lambda_a(x,v)\mathbb{E}\left[\alpha n^{-d_2}e_a^n(x,v,\xi_1) + u_n(e_a^n(x,v,\xi_1),x)\right] \\ &= \alpha z_n(x,v) + \sum_{a=1}^{n}\lambda_a(x,v)\mathbb{E}[u_n(e_a^n(x,v,\xi_1),x)]\end{aligned} \tag{5.6}$$

Moreover by Assumption 2.18 for any $\delta > 0$ and $|(x,v)| < \delta$ we have

$$\begin{aligned}
&\left|\sum_{a=1}^{n} \lambda_a(x,v)\mathbb{E}[u_n(e_a^n(x,v,\xi_1),x)]\right| \\
&\le \sup_{|(x,v)|<\delta} \left|\sum_{a=1}^{n} \lambda_a(x,v)\right| \sup_{\substack{|(x,v)|<\delta,\\ a\in\mathbb{A}_n}} \mathbb{E}\left[|u_n(e_a^n(x,v,\xi_1),x)|\right] \\
&\le \sup_{|(x,v)|<\delta} \left|\sum_{a=1}^{n} \lambda_a(x,v)\right| C_\delta^n \sup_{\substack{|(x,v)|<\delta,\\ a\in\mathbb{A}_n}} \mathbb{E}[|e_a^n(x,v,\xi_1)|] \\
&\le nC_n^\delta \sup_{|(x,v)|<\delta} \left|\frac{\lambda_{\mathbb{A}_n}(x,v)}{n}\right| \sup_{\substack{|(x,v)|<\delta,\\ a\in\mathbb{A}_n}} \mathbb{E}[|e_a^n(x,v,\xi_1)|],
\end{aligned} \tag{5.7}$$

where $C_n^\delta = o(n^{-1})$ and the other terms are bounded as assumed in Assumption 2.23. Thus the u.o.c. convergence to zero when $n \to \infty$.
Using the desintegration theorem again, Equation (2.18) and the dynamics stated in Equations (2.10) and (2.11) together with representation (2.22) we get the first moment related to the occupancy measure of the single states s_i via

$$\begin{aligned}
&\int K_n(x,v,dy,dw)w_i = \nu_{\mathbb{A}_n}(x,v) \int k_n(x,v,dy,dw)w_i \\
&= \nu_{\mathbb{A}_n}(x,v)\mathbb{E}[M_1^i - M_0^i | P_0 = x, M_0 = v] \\
&= \frac{\nu_{\mathbb{A}_n}(x,v)}{n^{d_1}} \Big(\mathbb{P}(M_1^i - M_0^i = n^{-d_1} | P_0 = x, M_0 = v) - \mathbb{P}(M_1^i - M_0^i = -n^{-d_1} | P_0 = x, M_0 = v)\Big) \\
&= \frac{\nu_{\mathbb{A}_n}(x,v)}{n^{d_1}} \sum_{a=1}^{n} \frac{\mu_a(x,v)}{\nu_{\mathbb{A}_n}(x,v)} (\Pi_{n,a}^{i+}(x,v) - \Pi_{n,a}^{i-}(x,v)) \\
&= n^{-d_1} \sum_{a=1}^{n} \mu_a(x,v)(\Pi_{n,a}^{i+}(x,v) - \Pi_{n,a}^{i-}(x,v)) \\
&= b_n^i(x,v).
\end{aligned} \tag{5.8}$$

In summary, since we assumed $b_n \to b$, we have

$$\int K_n(x,v,dy,dw)(y,w) \xrightarrow[n\to\infty]{u.o.c.} (\alpha z(x,v), b(x,v)) \tag{5.9}$$

Now let us consider the second moments.

$$\begin{aligned}
&\int K_n(x,v,dy,dw)y^2 = \nu_{\mathbb{A}_n}(x,v)\int k_n(x,v,dy,dw)y^2 \\
&= \nu_{\mathbb{A}_n}(x,v)\mathbb{E}[(P_1-P_0)^2|P_0=x,M_0=v] \\
&= \nu_{\mathbb{A}_n}(x,v)\sum_{a=1}^{n}\mathbb{E}[\overline{r}_n(e_a^n(P_0,M_0,\xi_1),P_0)^2\mathbb{1}_{\{A_1=a,B_1=1\}}|P_0=x,M_0=v] \\
&= \nu_{\mathbb{A}_n}(x,v)\sum_{a=1}^{n}\mathbb{P}(A_1=a,B_1=1|P_0=x,M_0=v)\mathbb{E}[\overline{r}_n(e_a^n(x,v,\xi_1),x)^2] \\
&= \sum_{a=1}^{n}\lambda_a(x,v)\mathbb{E}[\overline{r}_n(e_a^n(x,v,\xi_1),x)^2] \\
&= \alpha^2 n^{-2d_2}\sum_{a=1}^{n}\lambda_a(x,v)\mathbb{E}[e_a^n(x,v,\xi_1)^2]+\rho_n(x,v) \\
&= \alpha^2\sigma_n(x,v)^2+\rho_n(x,v)
\end{aligned} \tag{5.10}$$

where

$$\rho_n(x,v) := \sum_{a=1}^{n}\lambda_a(x,v)\mathbb{E}\left[2\alpha n^{-d_2}e_n^a(x,v,\xi_1)u_n^a(e_n^a(x,v,\xi_1),x)+u_n^a(e_n^a(x,v,\xi_1),x)^2\right] \tag{5.11}$$

Using again Assumption 2.18 we have $\forall\delta>0$ and $|(x,v)|<\delta$

$$\begin{aligned}
|\rho_n(x,v)| &= \left|\sum_{a=1}^{n}\lambda_a(x,v)\mathbb{E}\left[2\alpha n^{-d_2}e_n^a(x,v,\xi_1)u_n^a(e_n^a(x,v,\xi_1),x)+u_n^a(e_n^a(x,v,\xi_1),x)^2\right]\right| \\
&\le \sup_{\substack{|(x,v)|<\delta,\\ a\in\mathbb{A}_n}}\mathbb{E}[(2\alpha n^{-d_2}C_n^\delta+(C_n^\delta)^2)|e_n^a(x,v,\xi_1)|^2]\left|\sum_{a=1}^{n}\lambda_a(x,v)\right| \\
&\le (2\alpha n^{1-d_2}C_n^\delta+n(C_n^\delta)^2)\sup_{\substack{|(x,v)|<\delta,\\ a\in\mathbb{A}_n}}\mathbb{E}[|e_n^a(x,v,\xi_1)|^2]\sup_{|(x,v)|<\delta}\left|\frac{\lambda_{\mathbb{A}_n}(x,v)}{n}\right|.
\end{aligned} \tag{5.12}$$

Now, ρ_n vanishes for $n\to\infty$ as $2\alpha n^{1-d_2}C_n^\delta+n(C_n^\delta)^2$ converges to zero and the other two terms are bounded by Assumption 2.23.

The second moments related to each state are given by

$$\begin{aligned}
&\int K_n(x,v,dy,dw)w_i^2 = \nu_{\mathbb{A}_n}(x,v)\mathbb{E}\left[(M_1^i - M_0^i)^2 | P_0 = x, M_0 = v\right] \\
&= \frac{\nu_{\mathbb{A}_n}(x,v)}{n^{2d_1}} \left(\mathbb{P}(M_1^i - M_0^i = n^{-d_1}|P_0 = x, M_0 = v) + \mathbb{P}(M_1^i - M_0^i = -n^{-d_1}|P_0 = x, M_0 = v)\right) \\
&= n^{-2d_1} \sum_{a=1}^{n} \mu_a(x,v)(\Pi_{n,a}^{i+}(x,v) + \Pi_{n,a}^{i-}(x,v)) \\
&= c_n^i(x,v)^2
\end{aligned} \tag{5.13}$$

which converges to $c^i(x,v)^2$ as assumed in Theorem 2.25.

As we assumed that the agents are not able to change their state and trade at the same time we have

$$\int K_n(x,v,dy,dw)yw_i = 0, \ \forall 1 \leq i \leq m. \tag{5.14}$$

Moreover as only one agent at a time can change his state from s_i to s_j and as only the two respective state occupancy measures are affected by a change we have

$$\begin{aligned}
\int K_n(x,v,dy,dw)w_iw_j &= \nu_{\mathbb{A}_n}(x,v)\mathbb{E}\left[(M_1^i - M_0^i)(M_1^j - M_0^j)|P_0 = x, M_0 = v\right] \\
&= -n^{-2d_1} \sum_{a=1}^{n} \mu_a(x,v)\left(n^{d_1-1}v_i\Pi_{n,a}^{i,j}(x,v) + n^{d_1-1}v_j\Pi_{n,a}^{j,i}(x,v)\right) \\
&= -n^{-(d_1+1)} \sum_{a=1}^{n} \mu_a(x,v)\left(v_i\Pi_{n,a}^{i,j}(x,v) + v_j\Pi_{n,a}^{j,i}(x,v)\right) \\
&= c_n^{i,j}(x,v)^2
\end{aligned} \tag{5.15}$$

which converges to $c^{i,j}(x,v)^2$ for $n \to \infty$ as assumed.
Hence, in summary we have

$$\int K_n(x,v,dy,dw)\begin{bmatrix} y^2 & yw \\ wy & w^2 \end{bmatrix} = \begin{bmatrix} \alpha^2\sigma_n(x,v)^2 + \rho_n(x,v) & 0 \\ 0 & c_n(x,v)^2 \end{bmatrix} \xrightarrow[n\to\infty]{u.o.c.} \begin{bmatrix} \alpha^2 v(x,v)^2 & 0 \\ 0 & c(x,v)^2 \end{bmatrix} \tag{5.16}$$

where

$$\begin{bmatrix} y^2 & yw \\ wy & w^2 \end{bmatrix} := \begin{bmatrix} y^2 & yw_1 & ... & yw_m \\ w_1y & w_1^2 & ... & w_1w_m \\ \vdots & \vdots & \ddots & \vdots \\ w_my & w_mw_1 & ... & w_m^2 \end{bmatrix} \tag{5.17}$$

Hypothesis (ii)

Next we show part (ii) of Theorem IX. 4.21 in Jacod and Shiryaev [38] i.e.

$$\sup_{|(x,v)|<\delta} \int K_n(x,v,dy,dw)|(y,w)|^2 \mathbb{1}_{\{|(y,w)|>\epsilon\}} \xrightarrow{n\to\infty} 0, \ \forall \epsilon > 0 \tag{5.18}$$

Therefore let $\epsilon_1 > 0$ and $\delta > 0$. Thus for $|(x,v)| < \delta$ we have

$$\begin{aligned}
&\sup_{|(x,v)|<\delta} \int K_n(x,v,dy,dw) y^2 \mathbb{1}_{\{|y|>\epsilon_1\}} \\
&= \sup_{|(x,v)|<\delta} \sum_{a=1}^{n} \lambda_a(x,v) \mathbb{E}[\overline{r}_n(e_a^n(x,v,\xi_1),x)^2 \mathbb{1}_{\{|\overline{r}_n(e_a^n(x,v,\xi_1),x)|>\epsilon_1\}}] \\
&\leq \sup_{|(x,v)|<\delta} \sum_{a=1}^{n} \lambda_a(x,v) \sup_{\substack{|(x,v)|<\delta,\\ a\in\mathbb{A}_n}} \mathbb{E}[\overline{r}_n(e_a^n(x,v,\xi_1),x)^2 \mathbb{1}_{\{|\overline{r}_n(e_a^n(x,v,\xi_1),x)|>\epsilon_1\}}] \\
&\leq \sup_{|(x,v)|<\delta} \left| \frac{\lambda_{\mathbb{A}_n}(x,v)}{n} \right| (\alpha n^{1/2-d_2} + \sqrt{n} C_n^\delta)^2 \sup_{\substack{|(x,v)|<\delta,\\ a\in\mathbb{A}_n}} \mathbb{E}[|e_n^a(x,v,\xi_1)|^2 \mathbb{1}_{\{|e_a^n(x,v,\xi_1)|>\hat{\epsilon}_1^n\}}]
\end{aligned} \tag{5.19}$$

where in the last inequality we used

$$\begin{aligned}
\overline{r}_n(q,x) &= \alpha n^{-d_2} q + u_n^a(q,x) \\
&\leq \alpha n^{-d_2}|q| + \sup_{|x|<\delta} u_n^a(q,x) \\
&\leq (\alpha n^{-d_2} + C_n^\delta)|q|
\end{aligned} \tag{5.20}$$

given by Definition 2.18 in the sense that

$$\overline{r}_n(e_a^n(x,v,\xi_1),x)^2 \leq \frac{1}{n}(\alpha n^{1/2-d_2} + \sqrt{n} C_n^\delta)^2 e_n^a(x,v,\xi_1)^2 \tag{5.21}$$

and as a result

$$|\overline{r}_n(e_a^n(x,v,\xi_1),x)| > \epsilon_1 \Leftrightarrow |e_n^a(x,v,\xi_1)| > \hat{\epsilon}_1^n \tag{5.22}$$

with

$$\hat{\epsilon}_1^n := \frac{\sqrt{n}\epsilon_1}{\alpha n^{1/2-d_2} + \sqrt{n} C_n^\delta} \tag{5.23}$$

Now $\sup_{|(x,v)|<\delta} \left| \frac{\lambda_{\mathbb{A}_n}(x,v)}{n} \right|$ is bounded by Assumption 2.23 and $(\alpha n^{1/2-d_2} + \sqrt{n} C_n^\delta)^2$ converges to zero. Moreover $C_n = o(n^{-1})$ and $\sup_{\substack{|(x,v)|<\delta,\\ a\in\mathbb{A}_n}} \mathbb{E}[|e_n^a(x,v,\xi_1)|^2 \mathbb{1}_{\{|e_a^n(x,v,\xi_1)|>\hat{\epsilon}_1^n\}}]$ converges to zero by $\hat{\epsilon}_1^n \xrightarrow{n\to\infty} \infty$ and uniform integrability assumed in Assumption 2.23.

Moreover let $\epsilon_2 > 0$ and $\delta > 0$. For $|(x,v)| < \delta$ we have

$$\begin{aligned}
&\sup_{|(x,v)|<\delta} \int K_n(x,v,dy,dw) w_i^2 \mathbb{1}_{\{|w|>\epsilon_2\}} \\
&= \sup_{|(x,v)|<\delta} n^{-2d_1} \sum_{a=1}^{n} \mu_a(x,v) \left(\Pi_{n,a}^{i+}(x,v) + \Pi_{n,a}^{i-}(x,v)\right) \mathbb{1}_{\{|w|>\epsilon_2\}} \\
&\leq \sup_{|(x,v)|<\delta} n^{-2d_1} \sum_{a=1}^{n} \mu_a(x,v) 2 \mathbb{1}_{\{|w|>\epsilon_2\}} \\
&\leq 2 \sup_{|(x,v)|<\delta} \left| \frac{\mu_{\mathbb{A}_n}(x,v)}{n^{d_1}} \right| \mathbb{1}_{\{|w|>\epsilon_2\}}
\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}
&\sup_{|(x,v)|<\delta} \int K_n(x,v,dy,dw) |w_i w_j| \mathbb{1}_{\{|w|>\epsilon_2\}} \\
&= \sup_{|(x,v)|<\delta} \left| -n^{-2d_1} \sum_{a=1}^{n} \mu_a(x,v) \left(n^{d_1-1} v_i \Pi_{n,a}^{i,j}(x,v) + n^{d_1-1} v_j \Pi_{n,a}^{j,i}(x,v) \right) \right| \mathbb{1}_{\{|w|>\epsilon_2\}} \\
&\leq \sup_{|(x,v)|<\delta} n^{-2d_1} \sum_{a=1}^{n} \mu_a(x,v) 2 \mathbb{1}_{\{|w|>\epsilon_2\}} \\
&\leq 2 \sup_{|(x,v)|<\delta} \left| \frac{\mu_{\mathbb{A}_n}(x,v)}{n^{d_1}} \right| \mathbb{1}_{\{|w|>\epsilon_2\}}
\end{aligned} \tag{5.25}$$

It is clear by Equation (2.10) and (2.11) that

$$R_n := \operatorname{supp}(K_n(\cdot, v, dy, dw)(\cdot, w)) \subseteq \left[-n^{-d_1}, n^{-d_1}\right]^m \tag{5.26}$$

Hence, as $R_n \cap \{|w| \ > \epsilon_2\} \xrightarrow{n\to\infty} \emptyset$, we see that

$$\sup_{|(x,v)|<\delta} \int K_n(x,v,dy,dw) |w|^2 \mathbb{1}_{\{|w|>\epsilon_2\}} \xrightarrow{n\to\infty} 0. \tag{5.27}$$

5.4 Proof of Proposition 2.26

In order to apply Corollary IX.4.28 of Jacod and Shiryaev [38] it is sufficient to show that $\forall \epsilon > 0$ and $\forall \delta > 0$

$$\sup_{|(x,v)|<\delta} a_n^2 \int K_n(x,v,dy,dw)|(y,w)|^2 \mathbb{1}_{\{|y,w|>\epsilon/a_n\}} \xrightarrow{n\to\infty} 0. \tag{5.28}$$

Let $\epsilon_1 > 0$ and $\delta > 0$. Then for $|(x,v)| < \delta$ we have

$$\begin{aligned}
&\sup_{|(x,v)|<\delta} a_n^2 \int K_n(x,v,dy,dw)|y|^2 \mathbb{1}_{\{|y,w|>\epsilon/a_n\}} \\
&= \sup_{|(x,v)|<\delta} a_n^2 \sum_{a=1}^n \lambda_a(x,v)\mathbb{E}[\overline{r}_n(e_a^n(x,v,\xi_1),x)^2 \mathbb{1}_{\{|\overline{r}_n(e_a^n(x,v,\xi_1),x)|>\epsilon_1/a_n\}}] \\
&\leq \sup_{|(x,v)|<\delta} a_n^2 \sum_{a=1}^n \lambda_a(x,v) \sup_{\substack{|(x,v)|<\delta,\\ a\in\mathbb{A}_n}} \mathbb{E}[\overline{r}_n(e_a^n(x,v,\xi_1),x)^2 \mathbb{1}_{\{|\overline{r}_n(e_a^n(x,v,\xi_1),x)|>\epsilon_1/a_n\}}] \\
&\leq \sup_{|(x,v)|<\delta} \left|\frac{\lambda_{\mathbb{A}_n}(x,v)}{n}\right| (a_n(\alpha n^{1/2-d_2} + \sqrt{n}C_n^\delta))^2 \sup_{\substack{|(x,v)|<\delta,\\ a\in\mathbb{A}_n}} \mathbb{E}[|e_n^a(x,v,\xi_1)|^2 \mathbb{1}_{\{|e_a^n(x,v,\xi_1)|>\tilde{\epsilon}_1^n\}}]
\end{aligned} \tag{5.29}$$

with

$$\tilde{\epsilon}_1^n := \frac{\sqrt{n}\epsilon_1}{a_n(\alpha n^{1/2-d_2} + \sqrt{n}C_n^\delta)} \tag{5.30}$$

As we assumed that $\sqrt{n}a_n = O(n^{d_2} + (C_n^\delta)^{-1})$ we get $\tilde{\epsilon}_1^n \xrightarrow{n\to\infty} \infty$ and hence

$$\{|e_a^n(x,v,\xi_1)| > \tilde{\epsilon}_1^n\} \xrightarrow{n\to\infty} \emptyset \tag{5.31}$$

by the uniform integrablity assumed in Assumption 2.23. Moreover $\sup_{|(x,v)|<\delta} \left|\frac{\lambda_{\mathbb{A}_n}(x,v)}{n}\right|$ is bounded by Assumption 2.23 which yields the convergence to zero of Equation (5.29). Moreover let $\epsilon_2 > 0$ and $\delta > 0$. Then for $|(x,v)| < \delta$ we have

$$\begin{aligned}
&\sup_{|(x,v)|<\delta} a_n^2 \int K_n(x,v,dy,dw)|w_i|^2 \mathbb{1}_{\{|w_i|>\epsilon_2/a_n\}} \\
&= \sup_{|(x,v)|<\delta} a_n^2 \sum_{a=1}^n \mu_a(x,v) n^{-2d_1}(\Pi_{n,a}^{i+} + \Pi_{n,a}^{i-})\mathbb{1}_{\{|n^{-d_1}(\Pi_{n,a}^{i+} - \Pi_{n,a}^{i-})|>\epsilon_2/a_n\}} \\
&\leq \sup_{|(x,v)|<\delta} \left|\frac{\mu_{\mathbb{A}_n}(x,v)}{n^{d_1}}\right| 2(a_n n^{-d_1/2})^2 \mathbb{1}_{\{a_n n^{-d_1}>\epsilon_2\}}
\end{aligned} \tag{5.32}$$

Now $\{a_n n^{-d_1} > \epsilon_2\} \xrightarrow{n\to\infty} \emptyset$ and $a_n n^{-d_1/2} < \infty$ since we assumed $a_n\sqrt{n} = O(n^{1/2(d_1+1)})$.

Moreover let $\epsilon_3 > 0$ and $\delta > 0$. Then for $|(x,v)| < \delta$ we have

$$\begin{aligned}
&\sup_{|(x,v)|<\delta} a_n^2 \int K_n(x,v,dy,dw)|w_i w_j| \mathbb{1}_{\{|w_i|>\epsilon_3/a_n\}} \mathbb{1}_{\{|w_j|>\epsilon_3/a_n\}} \\
&= \sup_{|(x,v)|<\delta} a_n^2 \sum_{a=1}^{n} \mu_a(x,v) \left| -n^{-2d_1} \left(n^{d_1-1} v_i \Pi_{n,a}^{i,j}(x,v) + n^{d_1-1} v_j \Pi_{n,a}^{j,i}(x,v) \right) \right| \\
&\cdot \mathbb{1}_{\{|n^{-d_1}(\Pi_{n,a}^{i+}-\Pi_{n,a}^{i-})|>\epsilon_3/a_n\}} \mathbb{1}_{\{|n^{-d_1}(\Pi_{n,a}^{j+}-\Pi_{n,a}^{j-})|>\epsilon_3/a_n\}} \qquad (5.33) \\
&\leq \sup_{|(x,v)|<\delta} a_n^2 n^{-2d_1} \sum_{a=1}^{n} \mu_a(x,v) 2 \mathbb{1}_{\{|n^{-d_1}|>\epsilon_3/a_n\}} \\
&\leq \sup_{|(x,v)|<\delta} \left| \frac{\mu_{\mathbb{A}_n}(x,v)}{n^{d_1}} \right| 2 (a_n n^{d_1/2})^2 \mathbb{1}_{\{a_n n^{-d_1} > \epsilon_3\}}
\end{aligned}$$

Now $\{a_n n^{-d_1} > \epsilon_3\} \xrightarrow{n\to\infty} \emptyset$ and $a_n n^{-d_1/2} < \infty$ since we assumed $a_n \sqrt{n} = O(n^{1/2(d_1+1)})$.

5.5 Proof of Proposition 3.1

Using the definition of z_n (Equation (2.20)) we have

$$\begin{aligned} z_n(x,v) &= n^{-d_2} \sum_{a \in \mathbb{A}_n} \lambda_a(x,v)\mathbb{E}[e_a^n(x,v,\xi_1)] \\ &= 0, \end{aligned} \tag{5.34}$$

since $\mathbb{E}[e_a^n(x,v,\xi_1)] = \mathbb{E}[\xi_1] = 0$.

By the definition of b_n (Equations (2.22) and (2.23)) and the specifications made in Equations (3.4), (3.12) and (3.13) we get

$$\begin{aligned} b^1(x,v) &= \frac{1}{\sqrt{n}} \sum_{a=1}^{n} \mu_a(x,v)[\Pi_n^{1,+}(x,v) - \Pi_n^{1,-}(x,v)] \\ &= \frac{1}{\sqrt{n}} n\bar{\mu}(\frac{1}{\sqrt{n}}v_2 - \frac{1}{\sqrt{n}}v_1) \\ &= \bar{\mu}(v_2 - v_1) \end{aligned} \tag{5.35}$$

and

$$\begin{aligned} b^2(x,v) &= \frac{1}{\sqrt{n}} \sum_{a=1}^{n} \mu_a(x,v)[\Pi_n^{2,+}(x,v) - \Pi_n^{2,-}(x,v)] \\ &= \frac{1}{\sqrt{n}} n\bar{\mu}(\frac{1}{\sqrt{n}}v_1 - \frac{1}{\sqrt{n}}v_2) \\ &= \bar{\mu}(v_1 - v_2). \end{aligned} \tag{5.36}$$

In vector notation

$$b_n(x,v) = -\Psi v \tag{5.37}$$

Moreover

$$\begin{aligned} \sigma_n(x,v)^2 &= n^{-2d_2} \sum_{a \in \mathbb{A}_n} \lambda_a(x,v)\mathbb{E}[e_a^n(x,v,\xi_1)^2] \\ &= n^{-1}\lambda_{\mathbb{A}_n}(x,v)\mathbb{E}[\xi_1^2] \\ &= n^{-1}\lambda_{\mathbb{A}_n}(x,v) \end{aligned} \tag{5.38}$$

Furthermore for $i \neq j \in \{1,2\}$ we have

$$
\begin{aligned}
c_n^i(x,v)^2 &= n^{-2d_1} \sum_{a=1}^{n} \mu_a(x,v)[\Pi_n^{1,+}(x,v) + \Pi_n^{1,-}(x,v)] \\
&= \bar{\mu}\left(\frac{1}{\sqrt{n}}(\sqrt{n} - v_i) + \frac{1}{\sqrt{n}} v_i\right) \\
&= \bar{\mu}
\end{aligned}
\tag{5.39}
$$

and

$$
\begin{aligned}
c_n^{i,j}(x,v)^2 &= -n^{d_1+1} \sum_{a=1}^{n} \mu_a(x,v)(v_i \Pi_{a,n}^{i,j}(x,v) + v_j \Pi_{a,n}^{j,i}(x,v)) \\
&= -\bar{\mu}(\frac{v_i + v_j}{\sqrt{n}}) \\
&= -\bar{\mu}.
\end{aligned}
\tag{5.40}
$$

Hence in summary

$$
c_n(x,v) = \Psi. \tag{5.41}
$$

5.6 Proof of Proposition 3.7

For simplicity we scale the average opinion by $\bar{v}' := \frac{n\bar{v}}{2}$ so it has its values on the lattice $\mathbb{L}' = \frac{n}{2}\mathbb{L} \subset \mathbb{Z}$. Following Weidlich and Haag [68] chapter 2.3.1. the stationary distribution of $\bar{v}'$ is recursively given by

$$\widehat{\mathbb{P}}_{st}(\bar{v}') = \widehat{\mathbb{P}}_{st}(0) \prod_{y=1}^{\bar{v}'} \frac{\left(\frac{n}{2} - (y-1)\right) \Pi^{1,2}\left(\frac{2(y-1)}{n}\right)}{\left(\frac{n}{2} + y\right) \Pi^{2,1}\left(\frac{2y}{n}\right)}, \quad 1 \leq \bar{v}' \leq n/2 \tag{5.42}$$

and

$$\widehat{\mathbb{P}}_{st}(\bar{v}') = \widehat{\mathbb{P}}_{st}(0) \prod_{y=1}^{\bar{v}'} \frac{\left(\frac{n}{2} - (y+1)\right) \Pi^{2,1}\left(\frac{2(y+1)}{n}\right)}{\left(\frac{n}{2} - y\right) \Pi^{1,2}\left(\frac{2y}{n}\right)}, -n/2 \ \leq \bar{v}' \leq -1. \tag{5.43}$$

Inserting Equations (3.37) and (3.38) into (5.42) yields for $1 \leq \bar{v}' \leq n/2$

$$\begin{aligned}
\widehat{\mathbb{P}}_{st}(\bar{v}') &= \widehat{\mathbb{P}}_{st}(0) \prod_{y=1}^{\bar{v}'} \frac{\left(\frac{n}{2} - (y-1)\right) \beta \exp\left(\gamma \frac{2(y-1)}{n}\right)}{\left(\frac{n}{2} + y\right) \beta \exp\left(-\gamma \frac{2y}{n}\right)} \\
&= \widehat{\mathbb{P}}_{st}(0) \prod_{y=1}^{\bar{v}'} \frac{\left(\frac{n}{2} - (y-1)\right)}{\left(\frac{n}{2} + y\right)} \exp\left(\frac{2\gamma}{n}(2y-1)\right) \\
&= \widehat{\mathbb{P}}_{st}(0) \left(\prod_{y=1}^{\bar{v}'} \frac{\left(\frac{n}{2} - (y-1)\right)}{\left(\frac{n}{2} + y\right)}\right) \exp\left(\frac{2\gamma}{n} \sum_{y=1}^{\bar{v}'} 2y - 1\right) \\
&= \widehat{\mathbb{P}}_{st}(0) \frac{\left(\frac{n}{2}!\right)^2}{n!} \binom{n}{\frac{n}{2} + \bar{v}'} \exp\left(\frac{2\gamma}{n} (\bar{v}')^2\right),
\end{aligned} \tag{5.44}$$

where we used that $\sum_{y=1}^{\bar{v}'} 2y - 1 = (\bar{v}')^2$ and $\prod_{y=1}^{\bar{v}'} \frac{\left(\frac{n}{2} - (y-1)\right)}{\left(\frac{n}{2} + y\right)} = \frac{\left(\frac{n}{2}!\right)^2}{n!} \binom{n}{\frac{n}{2} + \bar{v}'}$, which can be shown by induction. By symmetry the last representation in Equation (5.44) is also true for $-n/2 \ \leq \bar{v}' \leq -1$ and thus after re-scaling we have

$$\mathbb{P}_{st}(\bar{v}) = \widehat{\mathbb{P}}_{st}\left(\frac{n\bar{v}}{2}\right) = \mathbb{P}_{st}(0) \frac{\left(\frac{n}{2}!\right)^2}{n!} \binom{n}{\frac{n(1+\bar{v})}{2}} \exp\left(\frac{\gamma n \bar{v}^2}{2}\right), \bar{v} \in \mathbb{L}. \tag{5.45}$$

Next we derive the requirement on γ in order that $\mathbb{P}_{st}$ has a local maximum. By symmetry $\mathbb{P}_{st}$ has a local maximum at 0 if the difference at the next higher lattice point is greater 0, i.e.

$$\mathbb{P}_{st}(2/n) - \mathbb{P}_{st}(0) > 0. \tag{5.46}$$

Inserting representation given in Equation (5.45) yields

$$
\begin{aligned}
&\mathbb{P}_{st}(2/n) - \mathbb{P}_{st}(0) > 0 \\
&\Leftrightarrow \mathbb{P}_{st}(0) \frac{\left(\frac{n}{2}!\right)^2}{n!} \binom{n}{\frac{n+2}{2}} \exp\left(\frac{2\gamma}{n}\right) - \mathbb{P}_{st}(0) > 0 \\
&\Leftrightarrow \frac{\left(\frac{n}{2}!\right)^2}{n!} \binom{n}{\frac{n+2}{2}} \exp\left(\frac{2\gamma}{n}\right) > 1 \\
&\Leftrightarrow \frac{n}{n+2} \exp\left(\frac{2\gamma}{n}\right) > 1 \\
&\Leftrightarrow \gamma > \frac{n}{2} \ln\left(\frac{n+2}{n}\right)
\end{aligned}
\tag{5.47}
$$

And analogously $\mathbb{P}_{st}$ has a local minimum for $\gamma < \frac{n}{2} \ln\left(\frac{n+2}{n}\right)$.

5.7 Proof of Lemma 3.9

Using that $\overline{M}_k = 2M_k^2 - 1 \; \forall k \geq 0$ together with Equations (2.22), respectivly (2.26), (2.8), (2.9), (3.37) and (3.38) basic calculations yield

$$\begin{aligned}
\bar{b}_n(\bar{v}) &= \mathbb{E}[\overline{M}_k - \overline{M}_{k-1} | \overline{M}_{k-1} = \bar{v}] \\
&= 2\mathbb{E}[M_k^2 - M_{k-1}^2 | M_{k-1}^2 = v_2] \\
&= 2b_n^2(v) \\
&= 2n^{-1} \sum_{a=1}^{n} \mu_a(v) \left(\Pi^{2+}(v) - \Pi^{2-}(v) \right) \\
&= 2(v_1 \beta e^{\gamma \bar{v}} - v_2 \beta e^{-\gamma \bar{v}}) \\
&= 2(\frac{1+\bar{v}}{2} \beta e^{\gamma \bar{v}} - \frac{1-\bar{v}}{2} \beta e^{-\gamma \bar{v}}) \\
&= \beta \left(e^{\gamma \bar{v}} - e^{-\gamma \bar{v}} - \bar{v}(e^{\gamma \bar{v}} + e^{-\gamma \bar{v}}) \right) \\
&= 2\beta \left[\sinh(\gamma \bar{v}) - \bar{v} \cosh(\gamma \bar{v}) \right] \\
&= 2\beta \left[\tanh(\gamma \bar{v}) - \bar{v} \right] \cosh(\gamma \bar{v})
\end{aligned} \tag{5.48}$$

$$\begin{aligned}
\bar{c}_n(\bar{v})^2 &= \mathbb{E}[(\overline{M}_k - \overline{M}_{k-1})^2 | \overline{M}_{k-1} = \bar{v}] \\
&= 4\mathbb{E}[(M_k^2 - M_{k-1}^2)^2 | M_{k-1}^2 = v_2] \\
&= 4(c_n^2(v))^2 \\
&= 4n^{-2} \sum_{a=1}^{n} \mu_a(v) \left(\Pi^{2+}(v) + \Pi^{2-}(v) \right) \\
&= \frac{4}{n} \left(\Pi^{1+}(v) + \Pi^{1-}(v) \right) \\
&= \frac{4}{n} \left(v_2 \Pi^{2,1}(v) + v_1 \Pi^{1,2}(v) \right) \\
&= \frac{4}{n} \left(v_2 \beta e^{-\gamma \bar{v}} + v_1 \beta e^{\gamma \bar{v}} \right) \\
&= \frac{4}{n} \left(\frac{1+\bar{v}}{2} \beta e^{-\gamma \bar{v}} + \frac{1-\bar{v}}{2} \beta e^{\gamma \bar{v}} \right) \\
&= \frac{2\beta}{n} \left(e^{\gamma \bar{v}} + e^{-\gamma \bar{v}} + \bar{v} \left(e^{-\gamma \bar{v}} - e^{\gamma \bar{v}} \right) \right) \\
&= \frac{4\beta}{n} \left[\cosh(\gamma \bar{v}) - \bar{v} \sinh(\gamma \bar{v}) \right] \\
&= \frac{4\beta}{n} \left[1 - \bar{v} \tanh(\gamma \bar{v}) \right] \cosh(\gamma \bar{v})
\end{aligned} \tag{5.49}$$

5.8 Proof of Proposition 3.10 and 3.11

In order to approximate the distribution $\mathbb{P}(\overline{v};t)$, which describes the dynamics of the discrete average opinion, we use the Fokker Planck equation in the same way as discussed in Chapter 2.2.2 of Weidlich [68]. Identifying the drift coefficient as $\overline{b}_n(\overline{v})$ and the fluctuation coefficient as $\frac{n}{2}\overline{c}_n(\overline{v})^2$, with $\overline{b}_n$ and $\overline{c}_n$ as defined in equations (3.51) and (3.52), the Fokker Planck equation is derived as

$$\frac{d\widetilde{\mathbb{P}}(\widetilde{v};t)}{dt} = -\frac{d}{d\widetilde{v}}[\overline{b}_n(\widetilde{v})\widetilde{\mathbb{P}}(\widetilde{v};t)] + \frac{1}{2}\frac{d^2}{d\widetilde{v}^2}[\overline{c}_n(\widetilde{v})^2\widetilde{\mathbb{P}}(\widetilde{v};t)], \tag{5.50}$$

where $\widetilde{\mathbb{P}}(\widetilde{v};t)$ is the probability distribution approximating $\mathbb{P}(\overline{v};t)$.
Following Equations (2.57) to (2.60) of Weidlich [68], the exact stationary solution of the Fokker-Planck equation above, which then approximates the stationary distribution $\mathbb{P}_{st}(\overline{v})$, is given by

$$\widetilde{\mathbb{P}}_{st}(\widetilde{v}) = \widetilde{\mathbb{P}}_{st}(0)\frac{\overline{c}_n(0)^2}{\overline{c}_n(\widetilde{v})^2}\exp\left(n\int_0^{\widetilde{v}}\frac{\overline{b}_n(y)}{\frac{n}{2}\overline{c}_n(y)^2}dy\right), \tag{5.51}$$

which simplifies to

$$\widetilde{\mathbb{P}}_{st}(\widetilde{v}) = \frac{\widetilde{\mathbb{P}}_{st}(0)}{(1-\widetilde{v}\tanh(\gamma\widetilde{v}))\cosh(\gamma\widetilde{v})}\exp\left(n\int_0^{\widetilde{v}}\frac{\tanh(\gamma y)-y}{1-y\tanh(\gamma y)}dy\right) \tag{5.52}$$

after inserting the explicit form of $\overline{b}_n$ and $\overline{c}_n$ given by Lemma 3.9.

In order to determine the extrema $\widetilde{v}_m$ of $\widetilde{\mathbb{P}}_{st}$ we claim

$$\frac{d\widetilde{\mathbb{P}}_{st}(\widetilde{v}_m)}{d\widetilde{v}} = 0 \tag{5.53}$$

and

$$\frac{d^2\widetilde{\mathbb{P}}_{st}(\widetilde{v}_m)}{d\widetilde{v}^2} \neq 0, \tag{5.54}$$

which is equivalent to the condition

$$\tanh(\gamma\widetilde{v}_m) - \widetilde{v}_m = 0. \tag{5.55}$$

5.9 Proof of Proposition 3.13

By Equation (2.116) of Weidlich [68] the transition time is given by

$$\widetilde{\tau} = \frac{2\pi}{\frac{n}{2}\overline{c}_n(\widetilde{v}_m)^2\sqrt{\phi''(0)|\phi''(\widetilde{v}_m)|}} \exp\left(\frac{n\phi(\widetilde{v}_m)}{2}\right) \tag{5.56}$$

with

$$\phi(\widetilde{v}) = 2\int_0^{\widetilde{v}} \frac{\overline{b}_n(y)}{\frac{n}{2}\overline{c}_n(y)^2} dy. \tag{5.57}$$

Now by Lemma 3.9 we have

$$\phi(\widetilde{v}) = 2\int_0^{\widetilde{v}} \frac{\tanh(\gamma y) - y}{1 - y\tanh(\gamma y)} dy. \tag{5.58}$$

Moreover

$$\phi''(\widetilde{v}) = \frac{4}{n}\frac{\overline{b}_n'(\widetilde{v})\overline{c}_n(\widetilde{v})^2 - \overline{b}_n(\widetilde{v})(\overline{c}_n^2)'(\widetilde{v})}{\overline{c}_n(\widetilde{v})^4} \tag{5.59}$$

and since $\overline{b}_n(\widetilde{v}_m) = \overline{b}_n(0) = 0$ and $\overline{c}_n(0)^2 = \frac{4\beta}{n}$

$$\phi''(\widetilde{v}_m) = \frac{4\overline{b}_n'(\widetilde{v}_m)}{n\overline{c}_n(\widetilde{v}_m)^2} \tag{5.60}$$

and

$$\phi''(0) = \frac{\overline{b}_n'(0)}{\beta}. \tag{5.61}$$

Moreover

$$\begin{aligned} \overline{b}_n'(\widetilde{v}) &= 2\beta[(\gamma - 1)\cosh(\gamma\widetilde{v}) + \gamma\widetilde{v}\sinh(\gamma\widetilde{v})] \\ &= 2\beta[(\gamma - 1) - \gamma\widetilde{v}\tanh(\gamma\widetilde{v})]\cosh(\gamma\widetilde{v}) \end{aligned} \tag{5.62}$$

and hence because of the identity $\tanh(\gamma\widetilde{v}_m) = \widetilde{v}_m$, Equations (2.27), (5.60) and (5.61) simplify to

$$\overline{c}_n(\widetilde{v}_m)^2 = \frac{4\beta}{n}(1 - \widetilde{v}_m^2)\cosh(\gamma\widetilde{v}_m), \tag{5.63}$$

$$\phi''(\widetilde{v}_m) = 2\frac{\gamma(1 - \widetilde{v}_m^2) - 1}{1 - \widetilde{v}_m^2} \tag{5.64}$$

and

$$\phi''(0) = 2(\gamma - 1). \tag{5.65}$$

Inserting Equation (5.58), (5.63), (5.64) and (5.65) into (5.56) yields the claim.

5.10 Proof of Proposition 3.14

Since $\overline{M}_k = M_k^2 - M_k^1\ \forall k \geq 0$, the aggregated state transition for states $s_1 = -1$ and $s_2 = 1$ are given by Equations (2.8), (2.9), (2.22), (3.37) and (3.38) as

$$\begin{aligned} b_n^1(v) &= n^{-1}\sum_{a=1}^{n}\mu_a(v)\left(\Pi^{1+}(v) - \Pi^{1-}(v)\right) \\ &= v_2\beta e^{-\gamma(v_2-v_1)} - v_1\beta e^{\gamma(v_2-v_1)} \\ &= \frac{1+v_2-v_1}{2}\beta e^{-\gamma(v_2-v_1)} - \frac{1-v_2+v_1}{2}\beta e^{\gamma(v_2-v_1)} \\ &= -\frac{\beta}{2}\left(e^{\gamma(v_2-v_1)} - e^{-\gamma(v_2-v_1)} - (v_2-v_1)(e^{\gamma(v_2-v_1)} + e^{-\gamma(v_2-v_1)})\right) \\ &= -\beta\left[\sinh(\gamma(v_2-v_1)) - (v_2-v_1)\cosh(\gamma(v_2-v_1))\right] \\ &= -\beta\left[\tanh(\gamma(v_2-v_1)) - (v_2-v_1)\right]\cosh(\gamma(v_2-v_1)) \end{aligned} \tag{5.66}$$

and

$$\begin{aligned} b_n^2(v) &= n^{-1}\sum_{a=1}^{n}\mu_a(v)\left(\Pi^{2+}(v) - \Pi^{2-}(v)\right) \\ &= n^{-1}\sum_{a=1}^{n}\mu_a(v)\left(\Pi^{1-}(v) - \Pi^{1+}(v)\right) \\ &= -b_n^1(v) \\ &= \beta\left[\tanh(\gamma(v_2-v_1)) - (v_2-v_1)\right]\cosh(\gamma(v_2-v_1)). \end{aligned} \tag{5.67}$$

Moreover the transition volume is given by the functions

$$\begin{aligned} c_n^i(v) &= n^{-2}\sum_{a=1}^{n}\mu_a(v)\left(\Pi^{i+}(v) + \Pi^{i-}(v)\right) \\ &= \frac{1}{n}\left(\Pi^{1+}(v) + \Pi^{1-}(v)\right) \\ &= \frac{1}{n}\left(v_2\Pi^{2,1}(v) + v_1\Pi^{1,2}(v)\right) \\ &= \frac{1}{n}\left(v_2\beta e^{-\gamma(v_2-v_1)} + v_1\beta e^{\gamma(v_2-v_1)}\right) \\ &= \frac{1}{n}\left(\frac{1+v_2-v_1}{2}\beta e^{-\gamma(v_2-v_1)} + \frac{1-v_2+v_1}{2}\beta e^{\gamma(v_2-v_1)}\right) \\ &= \frac{\beta}{2n}\left(e^{\gamma(v_2-v_1)} + e^{-\gamma(v_2-v_1)} + (v_2-v_1)\left(e^{-\gamma(v_2-v_1)} - e^{\gamma(v_2-v_1)}\right)\right) \\ &= \frac{\beta}{n}\left[\cosh(\gamma(v_2-v_1)) - (v_2-v_1)\sinh(\gamma(v_2-v_1))\right] \\ &= \frac{\beta}{n}\left[1 - (v_2-v_1)\tanh(\gamma(v_2-v_1))\right]\cosh(\gamma(v_2-v_1)) \end{aligned} \tag{5.68}$$

and

$$
\begin{aligned}
c_n^{i,j}(v) &= -n^{-2}\sum_{a=1}^{n}\mu_a(v)\left(v_i\Pi^{i,j}(v)+v_j\Pi^{j,i}(v)\right)\\
&= -\frac{1}{n}\left(v_1\Pi^{1,2}(v)+v_2\Pi^{2,1}(v)\right)\\
&= -c_n^i(v)\\
&= -\frac{\beta}{n}\left[1-(v_2-v_1)\tanh(\gamma(v_2-v_1))\right]\cosh(\gamma(v_2-v_1)).
\end{aligned} \tag{5.69}
$$

As b_n^1 and b_n^2 are independent of n and $|c_n^i(v)| = |c_n^{i,j}(v)| \le 2n^{-1}$ we have

$$
(b_n^1(v), b_n^2(v)) \xrightarrow{n\to\infty} (b^1(v), -b^1(v)) \tag{5.70}
$$

with $b^1(v) = -\beta\left[\tanh(\gamma(v_2-v_1)) - (v_2-v_1)\right]\cosh(\gamma(v_2-v_1))$ and

$$
c_n(v) \xrightarrow{n\to\infty} 0. \tag{5.71}
$$

Now by Theorem 2.25 we get that

$$
(V_t^n)_{t\in[0,\infty)} \xrightarrow{\mathcal{L}} (V_t)_{t\in[0,\infty)} \ in\ D_{[0,1]^2}[0,\infty), \tag{5.72}
$$

where $(V_t)_{t\in[0,\infty)}$ is the unique strong solution of

$$
\begin{cases} dV_t^1 = -\beta\left(\tanh(\gamma(V_t^2-V_t^1)) - V_t^2 + V_t^1\right)\cosh(\gamma(V_t^2-V_t^1))dt \\ dV_t^2 = -dV_t^1 \end{cases} , V_0 = v_0 \tag{5.73}
$$

which is equivalent to $(\overline{V}_t)_{t\in[0,\infty)}$ being the unique strong solution of

$$
d\overline{V}_t = 2\beta\left(\tanh(\gamma\overline{V}_t) - \overline{V}_t\right)\cosh(\gamma\overline{V}_t), V_0 = \overline{\theta}, \tag{5.74}
$$

if we set $\overline{V}_t = V_t^2 - V_t^1$ and $\overline{\theta} = v_0^2 - v_0^1$.

5.11 Proof of Proposition 3.20

To apply Theorem 2.25 we calculate the functions z_n, b_n, σ_n and c_n and show their convergence when $n \to \infty$. Thereby, for simplicity, we substitute $\overline{v} := v_2 - v_1$.
Since

$$\sum_{\substack{a \in \mathbb{A}_n, \\ x_k^a \in \{-1,1\}}}^{n} e_a^n(P_{k-1}, \xi_k) = \sum_{\substack{a \in \mathbb{A}_n, \\ x_k^a \in \{-1,1\}}}^{n} \widetilde{e}_a^n(P_{k-1}, M_{k-1}, \xi_k), \ \forall k \geq 0, \tag{5.75}$$

with

$$\widetilde{e}_a^n(P_{k-1}, M_{k-1}, \xi_k) = \begin{cases} n^{-1/2} w_2(P_{k-1}) + \xi_k, & x_k^a = 2 \\ n^{-1/2} w_1 \overline{M}_{k-1} + \xi_k, & x_k^a \in \{-1, 1\} \end{cases} \tag{5.76}$$

we have

$$\begin{aligned} z_n(x,v) &= n^{-d_2} \sum_{a=1}^{n} \lambda_a(x,v) \mathbb{E}[e_a^n(x,v,\xi_1)] \\ &= \frac{\overline{\lambda}}{\sqrt{n}} \left(\sum_{\substack{a \in \mathbb{A}_n, \\ x_0^a \in \{-1,1\}}}^{n} \mathbb{E}[e_a^n(x,s)] + \sum_{\substack{a \in \mathbb{A}_n, \\ x_0^a = 2}}^{n} \mathbb{E}[e_a^n(x,s)] \right) \\ &= \frac{\overline{\lambda}}{\sqrt{n}} \left(\sum_{\substack{a \in \mathbb{A}_n, \\ x_k^a \in \{-1,1\}}}^{n} \mathbb{E}[\widetilde{e}_a^n(x,v,s)] + \sum_{\substack{a \in \mathbb{A}_n, \\ x_0^a = 2}}^{n} \mathbb{E}[e_a^n(x,s)] \right) \\ &= \frac{\overline{\lambda}}{\sqrt{n}} \left(n(1-\phi_n) n^{-1/2} w_1 \bar{v} + n \phi_n n^{-1/2} w_2(x) \right) \\ &= \overline{\lambda} \left((1-\phi_n) w_1 \bar{v} + \phi_n w_2(x) \right) \end{aligned} \tag{5.77}$$

Moreover

$$\begin{aligned} b_n^1(x,v) &= v_2 \Pi_n^{2,1}(x,v) - v_1 \Pi_n^{1,2}(x,v) \\ &= (1-\phi_n) \frac{1+\bar{v}}{2} \beta e^{-\gamma_1 \widehat{z_n}(x,v) - \gamma_2 \bar{v}} - (1-\phi_n) \frac{1-\bar{v}}{2} \beta e^{\gamma_1 \widehat{z_n}(x,v) - \gamma_2 \bar{v}} \\ &= -(1-\phi_n) \beta \left[\tanh(\gamma_1 \widehat{z_n}(x,v) - \gamma_2 \bar{v}) - \bar{v} \right] \cosh(\gamma_1 \widehat{z_n}(x,v) - \gamma_2 \bar{v}) \end{aligned} \tag{5.78}$$

$$\begin{aligned} b_n^2(x,v) &= v_1 \Pi_n^{1,2}(x,v) - v_2 \Pi_n^{2,1}(x,v) \\ &= -b_n^1(x,v). \end{aligned} \tag{5.79}$$

Furthermore,

$$
\begin{aligned}
\sigma_n(x,v)^2 &= \frac{1}{n}\sum_{a=1}^{n}\bar{\lambda}\mathbb{E}[e_a^n(x,s)^2] \\
&= \frac{\bar{\lambda}}{n}\left(\sum_{\substack{a\in\mathbb{A}_n,\\ x_0^a\in\{-1,1\}}}^{n}\mathbb{E}[e_a^n(x,s)^2] + \sum_{\substack{a\in\mathbb{A}_n,\\ x_0^a=2}}^{n}\mathbb{E}[e_a^n(x,s)^2]\right) \\
&= \frac{\bar{\lambda}}{n}[(n-k_n)n^{-1}w_1^2 + k_n n^{-1}w_2(x)^2 + n\mathrm{Var}[s]] \\
&= \frac{\bar{\lambda}}{n}[(1-\phi_n)w_1^2 + \phi_n w_2(x)^2 + n\mathrm{Var}[s]]
\end{aligned}
\tag{5.80}
$$

At last, we have

$$
\begin{aligned}
c_n^i(x,v)^2 &= n^{-2}\sum_{a=1}^{n}(\Pi_n^{i+}(x,v) + \Pi_n^{i-}(x,v)) \\
&= n^{-1}(\Pi_n^{i+}(x,v) + \Pi_n^{i-}(x,v)) \\
&\le 2/n
\end{aligned}
\tag{5.81}
$$

as well as

$$
\begin{aligned}
c_n^{i,j}(x,v)^2 &= -n^{-2}\sum_{a=1}^{n}(v_i\Pi_n^{i,j}(x,v) + v_j\Pi_n^{j,i}(x,v)) \\
&= -n^{-1}(v_i\Pi_n^{i,j}(x,v) + v_j\Pi_n^{j,i}(x,v)) \\
&\le -2/n.
\end{aligned}
\tag{5.82}
$$

Since we assumed $\phi_n \to \phi$ we have $z_n(x,v) \to z(x,v)$ and $\widehat{z_n}(x,v) \to z(x,v)$ with

$$z(x,v) := \overline{\lambda}\left((1-\phi)w_1\bar{v} + \phi w_2(x)\right) \tag{5.83}$$

Consequently,

$$b_n^1(x,v) = -b_n^2(x,v) \xrightarrow{n\to\infty} -(1-\phi)\beta\left[\tanh(\gamma_1 z(x,v) - \gamma_2\bar{v}) - \bar{v}\right]\cosh(\gamma_1 z(x,v) - \gamma_2\bar{v}) \tag{5.84}$$

and

$$\sigma_n(x,v)^2 \xrightarrow{n\to\infty} \bar{\lambda}\mathrm{Var}[s]. \tag{5.85}$$

Since additionally $c_n \xrightarrow{n\to\infty} 0$ we can apply Theorem 2.25, by which we have

$$(X_t^n, V_t^n)_{t\in[0,\infty)} \xrightarrow{\mathcal{L}} (X_t, V_t)_{t\in[0,\infty)} \ \mathit{in}\ D_{\mathbb{R}\times[0,1]^3}[0,\infty), \tag{5.86}$$

where $(X_t, V_t)_{t\in[0,\infty)}$ is the unique strong solution of

$$\begin{cases} dX_t = \alpha\overline{\lambda}[\phi w_2(X_t) + (1-\phi)w_1\overline{V}_t]dt + \sqrt{\overline{\lambda}\mathrm{Var}[s]}dB_t, & X_0 = \eta \\ d\overline{V}_t = 2\beta\left[\tanh(\gamma_1 dX_t + \gamma_2\overline{V}_t) - \overline{V}_t\right]\cosh(\gamma_1 dX_t + \gamma_2\overline{V}_t)dt, & \overline{V}_0 = \overline{\theta} \\ dV_t^3 = 0, & V_0^3 = \phi, \end{cases} \tag{5.87}$$

when setting $\overline{V}_t = \frac{(V_t^2 - V_t^1)}{1-\phi}$.

5.12 Proof of Proposition 3.24

In order to apply Theorem 2.25 we calculate first and second moments

$$\begin{aligned} z_n(x,v) &= \frac{1}{n}\sum_{a=1}^{n}\lambda_a(x,v)\mathbb{E}[e_a^n(x,v,\xi_1)] \\ &= \overline{\lambda}\left(\phi w\overline{F} - wx + (1-\phi)\frac{1}{n}\sum_{a=1}^{n}\mathbb{E}[wF_{x_{k-1}^a} + \sqrt{n}\delta_{x_{k-1}^a}\xi]\right) \\ &= \overline{\lambda}\left(\phi w\overline{F} - wx + (1-\phi)w\sum_{i=1}^{m}v_iF_i\right) \\ &= \overline{\lambda}w(\phi\overline{F} + (1-\phi)(v\circ F) - x) \end{aligned} \tag{5.88}$$

$$\begin{aligned} \sigma_n(x,v)^2 &= \frac{1}{n^2}\sum_{a=1}^{n}\lambda_a(x,v)\mathbb{E}[e_a^n(x,v,\xi_1)^2] \\ &= \frac{\overline{\lambda}}{n^2}\sum_{a=1}^{n}\mathbb{E}[(\underbrace{\phi wF_a - wx}_{\rho_1} + \underbrace{(1-\phi)wF_{x_{k-1}^a} + \sqrt{n}\delta_{x_{k-1}^a}\xi}_{\rho_2})^2] \\ &= \frac{\overline{\lambda}}{n^2}\sum_{a=1}^{n}\mathbb{E}[\rho_1^2 + 2\rho_1\rho_2 + \rho_2^2] \end{aligned} \tag{5.89}$$

Now,

$$\begin{aligned} &\frac{\overline{\lambda}}{n^2}\sum_{a=1}^{n}\mathbb{E}[\rho_1^2] \\ &= \frac{\overline{\lambda}}{n^2}\sum_{a=1}^{n}\left((\phi wF_a)^2 + 2\phi w^2F_ax + (xw)^2\right) \\ &\leq \frac{\overline{\lambda}}{n^2}\sum_{a=1}^{n}\left(\phi^2w^2\sup_a F_a^2 + 2\phi w^2\sup_a |F_a|x + (xw)^2\right) \\ &= \frac{\overline{\lambda}}{n}\left(\phi^2w^2\sup_a F_a^2 + 2\phi w^2\sup_a |F_a|x + (xw)^2\right) \\ &\xrightarrow{n\to\infty} 0 \end{aligned} \tag{5.90}$$

and

$$\begin{aligned} &\frac{\overline{\lambda}}{n^2}\sum_{a=1}^{n}\mathbb{E}[2\rho_1\rho_2] \\ &= \frac{\overline{\lambda}}{n^2}2(1-\phi)\sum_{a=1}^{n}(\phi wF_a - xw)(wF_{x_{k-1}^a}) \\ &\xrightarrow{n\to\infty} 0 \end{aligned} \tag{5.91}$$

and

$$\begin{aligned}
&\frac{\overline{\lambda}}{n^2}\sum_{a=1}^{n}\mathbb{E}[\rho_2^2] \\
&= \frac{\overline{\lambda}}{n^2}(1-\phi)^2\sum_{a=1}^{n}\mathbb{E}[(wF_{x_{k-1}^a}+\sqrt{n}\delta_{x_{k-1}^a})\xi)^2] \\
&= \frac{\overline{\lambda}}{n^2}(1-\phi)^2\sum_{a=1}^{n}\Big((wF_{x_{k-1}^a})^2+2\mathbb{E}[wF_{x_{k-1}^a}\sqrt{n}\delta_{x_{k-1}^a})\xi]+\mathbb{E}[(\sqrt{n}\delta_{x_{k-1}^a}\xi)^2]\Big) \\
&= \frac{\overline{\lambda}}{n^2}(1-\phi)^2\sum_{a=1}^{n}\Big((wF_{x_{k-1}^a})^2+n\delta^2_{x_{k-1}^a}\mathbb{E}[\xi^2]\Big) \\
&= \overline{\lambda}(1-\phi)^2\left(\sum_{a=1}^{n}\frac{(wF_{x_{k-1}^a})^2}{n^2}+\frac{1}{n}\sum_{a=1}^{n}\delta^2_{x_{k-1}^a}\sigma_\xi^2\right) \\
&= \overline{\lambda}(1-\phi)^2\left(\sum_{a=1}^{n}\frac{(wF_{x_{k-1}^a})^2}{n^2}+\frac{1}{n}\sum_{i=1}^{m}nv_i\delta_i^2\sigma_\xi^2\right) \\
&= \overline{\lambda}(1-\phi)^2\left(\sum_{a=1}^{n}\frac{(wF_{x_{k-1}^a})^2}{n^2}+\frac{1}{n}\sum_{i=1}^{m}nv_i\delta_i^2\sigma_\xi^2\right) \\
&\xrightarrow{n\to\infty}\overline{\lambda}(1-\phi)^2\sigma_\xi^2(v\circ\delta^2)
\end{aligned} \tag{5.92}$$

In summary

$$\sigma_n(x,v)^2 \xrightarrow{n\to\infty} \overline{\lambda}(1-\phi)^2\sigma_\xi^2(v\circ\delta^2) \tag{5.93}$$

$$\begin{aligned}
b_n^i(x,v) &= \frac{1}{n}\sum_{a=1}^{n}\mu_a(x,v)(\Pi^{i+}(x,v)-\Pi^{i-}(x,v)) \\
&= \overline{\mu}(\Pi^{i+}(x,v)-\Pi^{i-}(x,v))
\end{aligned} \tag{5.94}$$

By Remark 2.22 $c_n \to 0$ as $n^{-2d_1}\mu_{\mathbb{A}_n}(x,v)=\frac{\overline{\mu}}{n}\xrightarrow{n\to\infty}0$.

5.13 Proof of Proposition 3.35

We apply Theorem 2.25. Note that $M_k = (1 - \overline{M}_k, \overline{M}_k)$, $V_t^n = (1 - Q_t^n, Q_t^n)$ and $d_1 = 1$. The expected aggregated transition b_n and transition volume c_n necessary to determine the limit of V_t^n, respectively Q_t^n, when $n \to \infty$ are given by

$$\begin{aligned}
b_n^2(x,v) &= \frac{1}{n}\sum_{a=1}^{n} \mu_a(\Pi_{n,a}^{2+}(v) - \Pi_{n,a}^{2-}(v)) \\
&= \frac{1}{n}\sum_{a=1}^{n} \mu_a[(1-v_2)\Pi_{n,a}^{2,1}(v_2) - v_2\Pi_{n,a}^{1,2}(v_2)] \\
&= \frac{1}{n}\sum_{a=1}^{n} \mu_a\left[(1-v_2)(\beta_a\frac{p_a}{2\gamma_a^2 n} + \frac{\eta_a(1-v_2)v_2^2}{2}) - v_2(\beta_a\frac{1-p_a}{2\gamma_a^2 n} + \frac{\eta_a(1-v_2)^2 v_2}{2})\right] \\
&= \frac{1}{n}\sum_{a=1}^{n} \frac{\beta_a}{2}(p_a - v_2) \\
b_n^1(x,1) &= \frac{1}{n}\sum_{a=1}^{n} \mu_a(\Pi_{n,a}^{1+}(v) - \Pi_{n,a}^{1-}(v)) \\
&= -b_n^2(x,v)
\end{aligned} \tag{5.95}$$

$$\begin{aligned}
(c_n^2(x,v))^2 &= \frac{1}{n^2}\sum_{a=1}^{n} \mu_a(\Pi_{n,a}^{2+}(v) + \Pi_{n,a}^{2-}(v)) \\
&= \frac{1}{n^2}\sum_{a=1}^{n} \mu_a[(1-v_2)\Pi_{n,a}^{2,1}(v_2) + v_2\Pi_{n,a}^{1,2}(v_2)] \\
&= \frac{1}{n^2}\sum_{a=1}^{n} \mu_a\left[(1-v_2)(\beta_a\frac{p_a}{2\gamma_a^2 n} + \frac{\eta_a(1-v_2)v_2^2}{2}) + v_2(\beta_a\frac{1-p_a}{2\gamma_a^2 n} + \frac{\eta_a(1-v_2)^2 v_2}{2})\right] \\
&= \frac{1}{n}\sum_{a=1}^{n} \left[\frac{\beta_a(p_a - 2v_2p_a + v_2)}{2n} + \gamma_a^2\eta_a(1-v_2)^2v_2^2\right] \\
(c_n^1(x,v))^2 &= \frac{1}{n^2}\sum_{a=1}^{n} \mu_a(\Pi_{n,a}^{1+}(v) + \Pi_{n,a}^{1-}(v)) \\
&= (c_n^2(x,v))^2 \\
(c_n^{1,2}(x,v))^2 &= (c_n^{2,1}(x,v))^2 \\
&= -(c_n^2(x,v))^2
\end{aligned} \tag{5.96}$$

Now, by Assumption 3.34

$$b_n = (b_n^1, b_n^2) \xrightarrow{n\to\infty} b := \begin{pmatrix} -1 \\ 1 \end{pmatrix} \beta(p - v_2) \tag{5.97}$$

and

$$c_n = \begin{pmatrix} c_n^1 & c_n^{1,2} \\ c_n^{2,1} & c_n^2 \end{pmatrix} \xrightarrow{n\to\infty} c := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \gamma\sqrt{\eta}(1-v_2)v_2. \tag{5.98}$$

So, by Theorem 2.25 $V_t^n \xrightarrow{n\to\infty} V_t$, where V_t is the solution of

$$dV_t = b(V_t)dt + c(V_t)dB_t, V_0 = (1-\theta,\theta) \tag{5.99}$$

and hence $Q_t^n \xrightarrow{n\to\infty} Q_t$, where Q_t is the solution of

$$dQ_t = \beta(p-Q_t)dt + \gamma\sqrt{\eta}(1-Q_t)Q_t dB_t,\ Q_0 = \theta, \tag{5.100}$$

since $Q_t = V_t^2$.

5.14 Proof of Proposition 3.41

Also here we apply Theorem 2.25. Since the dynamics of Q_t^n do not depend on X_t^n, the convergence of Q_t^n and its limit is given by Proposition 3.35. To also show the convergence of X_t^n and to determine the limit when $n\to\infty$, we calculate the expected aggregated excess demand z_n and the trading volume σ_n.

$$\begin{aligned} z_n(x,v) &= n^{-1/2}\sum_{a=1}^{n}\lambda_a\mathbb{E}[e_a^n(x,v,s)] \\ &= \frac{1}{\sqrt{n}}\left(\sum_{a\in\mathbb{F}_n}^{n}\bar{\lambda}_a\mathbb{E}\left[\frac{1}{\sqrt{n}}(F-x)\right] + \sum_{a\notin\mathbb{F}_n}^{n} 0\right) \end{aligned} \tag{5.101}$$

$$\begin{aligned} \sigma_n(x,v)^2 &= \frac{1}{n}\sum_{a=1}^{n}\lambda_a\mathbb{E}[e_a^n(x,v,s)^2] \\ &= \frac{1}{n}\left(\sum_{a\in\mathbb{F}_n}^{n}\bar{\lambda}_a\mathbb{E}\left[\frac{1}{n}(F-x)^2\right] + \sum_{a\notin\mathbb{F}_n}^{n}\lambda_a\sigma_\xi^2\gamma^2\eta q^2(1-q)^2\right) \\ &= \frac{1}{n}\left(\sum_{a\in\mathbb{F}_n}^{n}\bar{\lambda}_a\mathbb{E}\left[\frac{1}{n}(F-x)^2\right] + \sigma_\xi^2\gamma^2\eta q^2(1-q)^2\left[(\sum_{a\notin\mathbb{F}_n}^{n}\bar{\lambda}_a) + \delta nq\right]\right) \end{aligned} \tag{5.102}$$

By Assumption 3.34 and 3.40 we have

$$z_n(x,v) \xrightarrow{n\to\infty} \bar{\lambda}_F(F-x) \tag{5.103}$$

and

$$\sigma_n(x,v)^2 \xrightarrow{n\to\infty} (\bar{\lambda}_N + C_e q)\sigma_\xi^2\gamma^2\eta q^2(1-q)^2 \tag{5.104}$$

After realizing that Assumption 2.23 is fulfilled, we apply Theorem 2.25 and get Proposition 3.41 as a result.

Nomenclature

$\mathbb{A}_n$ Set of agents, page 6

B_k Action indicator, page 7

b_n Expected aggregated state transition, page 13

C Configuration space, page 6

c_n Transition volume, page 14

e_a^n Excess demand function, page 10

$F_{M_0}^n$ Initial distribution of market character, page 6

$F_{x_0}^n$ Initial state distribution, page 6

$\mathcal{G}_k$ Market history, page 7

K_n Rate kernel, page 11

λ_a Trading intensity function, page 8

$\lambda_{\mathbb{A}_n}$ Aggregate trading intensity, page 8

M_k Market character, page 6

μ_a State transition rate function, page 8

$\mu_{\mathbb{A}_n}$ Aggregate state transition rate, page 8

$\nu_{\mathbb{A}_n}$ Aggregated action rate, page 8

P_k Price, page 10

$\Pi_{n,a}^{i+}$ Aggregated state in-transition, page 9

$\Pi_{n,a}^{i-}$ Aggregated state out-transition, page 9

$\Pi_{n,a}$ State transition matrix, page 9

r_n Pricing rule, page 10

S State space, page 6

σ_n Trading volume, page 14

τ_k Intra-action times, page 10

V_t^n Market character index, page 11

x_k State process, page 6

x_k^a State of agent a at time T_k, page 6

X_t^n Price process, page 11

ξ_k Random signal, page 10

z_n Expected aggregated excess demand, page 13

List of Figures

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