Topics of Financial Mathematics under Model Uncertainty

Jacopo Mancin

Dissertation an der Fakultät für Mathematik, Informatik und Statistik der Ludwig-Maximilians-Universität München

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, §8, Abs. 2 Pkt. 5)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne enerlaubte Beihilfe angefertigt ist.

Mancin, Jacopo

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Zusammenfassung

Wir behandeln drei klassische Themen der Finanzmathematik im Kontext der Modellunsicherheit, d.h. der Markt ist mit einer Menge von Wahrscheinlichkeiten $\mathscr{P}$ ausgestattet, die möglicherweise zueinander singular sind.
Im ersten Teil untersuchen wir das Problem der mittleren Varianzabsicherung eines Claims f im Rahmen der G-Erwartung und beweisen damit, dass eine direkte Erweiterung der Ergebnisse unter einem festen $\mathbb{P}$ nicht möglich ist. Wenn die zugrundeliegende diskontierte Anlage X ein lokales Martingal ist, dann ist das Problem gleichbedeutend mit der Berechnung der Galtchouck-Kunita-Watanabe-Zerlegung von f, d.h. die Projektion von f auf den geschlossenen Raum der quadratintegrierbaren stochastischen Integrale von X. In unserem Kontext verhindert die intrinsische Nichtlinearität des Modells bereits die Orthogonalität der G-Brown'schen Bewegung B und ihrer quadratischen Kovariation $\langle B \rangle$. Dennoch sind wir in der Lage, eine explizite Charakterisierung des optimalen mittleren Varianz-Portfolios für einige Klassen von bedingten Claims zu finden. Wir entwerfen ein iteratives Lösungsschema, das von der G-Martingal-Darstellung des Claims aus [57] Gebrauch macht. Darüber hinaus leiten wir einige zusätzliche Ergebnisse für das stochastische Kalkül mit G-Brown'scher Bewegung her. Dies ist das erste Ergebnis zur kontinuierlichen quadratischen Absicherung unter Unsicherheit in der Literatur.
Im zweiten Teil stellen wir uns dem allgemeineren Rahmen von [52], der die G-Umgebung verkörpert. Wir betrachten das Problem der dynamischen Superreplikation eines Contingent Claims f. Wir geben einen Prozess Y an, so dass Y_t eine Version des Superreplikationspreises zum Zeitpunkt t von f ist. Dies wird, wie in [50], aus der sublinearen bedingten Erwartung von f unter Verwendung eines Modifikationstheorems für Supermartingale abgeleitet.
Durch die Einführung von $\mathscr{P}_{\text{eq}}(\tau, \mathbb{P})$, d.h. die Menge von Wahrscheinlichkeitsmaßen *äquivalent* zu $\mathbb{P}$ auf $\mathscr{F}_\tau$, zeigen wir eine alternative Repräsentation der bedingten sublinearen Erwartung.
Im letzten Teil entwickeln wir das Konzept der Finanzblase im Rahmen von [52]. Wir definieren den Begriff *robust fundamental value* eines Vermögenswertes so, dass er mit der vorhandenen Literatur vereinbar ist und sich auf den robusten Superreplikationspreis des Vermögenswertes selbst bezieht.
Wir erforschen im Detail die neuen Merkmale unseres robusten Modelles für Blasen durch konkrete Beispiele. Insbesondere sind wir in der Lage, spezifische Anlagedynamiken und eine Reihe von Wahrscheinlichkeitsmaßen, die Blasen hervorbringen, welche unter einigen $\mathbb{P}$-Märkten unmöglich zu erkennen sind. Dies geschieht durch Spezifizieren eines Preisprozesses, der ein strenges lokales $\mathbb{P}$-Martingal für ein $\mathbb{P} \in \mathscr{P}$ und ein echtes $\mathbb{Q}$-Martingal für ein anderes $\mathbb{Q} \in \mathscr{P}$ ist.
Schließlich untersuchen wir das Konzept der *No Dominance*, indem wir ein robu-

stes Gegenstück in der Umgebung von Modellunsicherheit vorschlagen und seine Konsequenzen auf das Konzept der Blase studieren.

Abstract

We address three classical topics of financial mathematics in the context of model uncertainty, i.e. when the market is endowed with a set of probabilities $\mathscr{P}$, possibly mutually singular to each other.
In the first part we study the problem of mean variance hedging of a claim f in the G-expectation framework introduced in [56], proving that a direct extension of the results under a fixed prior $\mathbb{P}$ does not hold. In the classical setting, if the underlying discounted asset X is a local-martingale, the problem is equivalent to retrieve the Galtchouck-Kunita-Watanabe decomposition of f, i.e. to find the projection of f onto the closed space of square integrable stochastic integrals of X. In our context, the intrinsic nonlinearity of the model already prevents the orthogonality of the G-Brownian motion B and its quadratic covariation $\langle B \rangle$. Still we are able to provide an explicit characterization of the optimal mean variance portfolio for some classes of contingent claims. We outline an iterative solution scheme which makes use of the G-martingale representation of the claim obtained in [57]. Moreover we derive some additional results on stochastic calculus with G-Brownian motion. This constitutes the first achievement on continuous-time quadratic hedging under uncertainty in the literature.
In the second part we place ourselves the more general framework outlined in [52], which embodies the G-setting. We consider the problem of dynamic superreplication of a contingent claim f. We provide a process Y such that Y_t is a version of the superreplication price at time t of f. This is derived, as in [50], from the sublinear conditional expectation of f, using a modification theorem for supermartingales. Moreover, by introducing $\mathscr{P}_{\text{eq}}(\tau, \mathbb{P})$, i.e. the set of priors *equivalent* to $\mathbb{P}$ on $\mathscr{F}_\tau$, we show an alternative representation of the conditional sublinear expectation.
In the last part we develop the concept of financial bubble in the setting of [52]. We define the notion of *robust fundamental value* of an asset in a way that it is consistent with the existing literature where only one prior is considered and it is related to the robust superreplication price of the asset itself.
We investigate in detail the new features of our robust model for bubbles through concrete examples. In particular we are able to provide specific asset dynamics and set of priors which originate a bubble impossible to detect under some $\mathbb{P}$-markets. This is done by specifying a price process which is a strict $\mathbb{P}$-local martingale for some $\mathbb{P} \in \mathscr{P}$ and a true $\mathbb{Q}$-martingale for other $\mathbb{Q} \in \mathscr{P}$.
Finally we investigate the concept of no dominance, proposing its robust counterpart in the model uncertainty framework and studying its consequences on the concept of bubble.

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Contents

List of Symbols

$\mathbf{A}^{\times}$	the set of all elements of $\mathbf{M}^2(\mathbb{P})$ strongly orthogonal to each element of some $\mathbf{A} \subseteq \mathbf{M}^2(\mathbb{P})$
$\mathscr{A}_X$	the analytic σ-algebra of the Borel space X
$\mathscr{A}_{t,T}^{\Theta}$	set of all $\mathbb{F}$-adapted Θ-valued processes on $[t,T]$
$\mathscr{B}(\Omega_T)$	the Borel σ-algebra of Ω_T
$B_s^{t,\sigma}$	the process defined by $\int_t^s \sigma_u dW_u$, for $\sigma \in \Theta$ and $s \in [t,T]$
β_t	size of the robust bubble at time t
c	the value $E_G[H] - V_0$
C	the payoff of an European call option with strike D
$C_0([0,T],\mathbf{R})$	the family of all $\mathbf{R}$-valued continuous paths $(\omega_t)_{t\in[0,T]}$ with $\omega_0 = 0$
$C_{l,Lip}(\mathbf{R}^n)$	the set of real-valued, locally Lipschitz functions
$C_t(\Upsilon)$	cost of a trading strategy Υ at time t
$\mathscr{D}$	vector lattice of real valued functions defined on Ω containing 1
$\mathbf{D}$	nonempty, convex and compact subset of $\mathbb{R}^{d\times d}$
ΔB_{t_i}	the increment $B_{t_i} - B_{t_{i-i}}$
$\Delta\langle B\rangle_{t_i}$	the increment $\langle B\rangle_{t_i} - \langle B\rangle_{t_{i-i}}$
Δt_i	the increment $t_i - t_{i-i}$
$E_{\mathbb{P}}[\cdot]$	linear expectation with respect to the probability $\mathbb{P}$
$\mathbb{E}[\cdot]$	sublinear expectation operator
$\mathscr{E}_\tau(\cdot)$	a particular conditional sublinear expectation at time τ
$\mathbb{F} = \{\mathscr{F}_t\}_{t\in[0,T]}$	filtration generated by the canonical process
$\bar{\mathbb{F}}^{\mathbb{P}} := \{\bar{\mathscr{F}}_t^{\mathbb{P}}\}_{t\in[0,T]}$	the $\mathbb{P}$-augmented filtration of $\mathbb{F}$
$\mathscr{F}$	the Borel σ-algebra of Ω_T
$\mathscr{F}_t^{\mathbb{P}}$	completion of the σ-algebra $\mathscr{F}_t$ under $\mathbb{P}$
$\mathscr{F}_t^*$	universal completion of the σ-algebra $\mathscr{F}_t$
$E_G[\cdot]$	G-expectation operator
$E_G[\cdot\vert\mathscr{F}_t]$	G-conditional expectation at time t
$G(\cdot)$	the real function $G(y) = \frac{1}{2}\sup_{\sigma\in\Theta}\left(y\sigma^2\right)$, $y \in \mathbf{R}$
$\mathbb{G}$	the filtration given by $\mathscr{G}_t := \mathscr{F}_t^* \vee \mathscr{N}^{\mathscr{P}}$
γ	discounted, risk free bank account process
$\mathscr{H}$	the collection of $\mathbb{G}$-predictable processes such that $H \cdot S$

	is a $\mathbb{P}$-supermartingale for all $\mathbb{P} \in \mathscr{P}$
$I(\eta)$	the stochastic integral $\int_0^T \eta(s)dB_s$ with respect to the G-Brownian motion
$J_0(V_0,\phi)$	terminal risk associated to the strategy (V_0,ϕ)
$\underline{J}(V_0,\phi)$	lower bound for the terminal risk $J_0(V_0,\phi)$
$\overline{J}(V_0,\phi)$	upper bound for the terminal risk $J_0(V_0,\phi)$
J^*	minimal terminal risk for the claim H
J_n^*	minimal terminal risk for the claim H^n
K	finite variation process associated to the G-martingale decomposition of a random variable
$\mathbb{L}_+^0(\mathscr{F}_t)$	the set of $\mathscr{F}_t$-measurable random variables taking $\mathbb{P}$-a.s. values in $[0,\infty)$
$L_{ip}(\Omega_T)$	the family of cylindrical random variables
$L_G^p(\mathscr{F}_T)$	the completion of $L_{ip}(\Omega_T)$ under the $\|\cdot\|_p$-norm
$\mathfrak{M}$	the subset of $\mathfrak{P}(\Omega)$ given by $\{\mathbb{P} \in \mathfrak{P}(\Omega) :\ B \text{ is a local } \mathbb{P}\text{-martingale}\}$
$\mathfrak{M}_a$	the subset of $\mathfrak{M}$ given by $\{\mathbb{P} \in \mathfrak{M} :\ \langle B\rangle^{\mathbb{P}} \text{ is absolutely continuous } \mathbb{P}\text{-a.s.}\}$
$M_G^{p,0}(0,T)$	the family of processes of the type $\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{1}_{[t_j,t_{j+1})}(t)$ where $\xi_i \in L_G^p(\mathscr{F}_{t_i})$, $i \in 0,\dots,N-1$
$M_G^p(0,T)$	the completion of $M_G^{p,0}(0,T)$ under $\|\cdot\|_{M_G^p}$
$\mathbf{M}^2(\mathbb{P})$	the family of square integrable $\mathbb{P}$-martingales
$\mathscr{M}_{loc}(S)$	the set of all equivalent $\mathbb{P}$-local martingale measures for S
$\mathscr{M}_{UI}(S)$	the subset of those $\mathbb{Q} \in \mathscr{M}_{loc}(S)$ for which S is a uniformly integrable martingale
$\mathscr{M}_{NUI}(S)$	the set given by $\mathscr{M}_{loc}(S) \setminus \mathscr{M}_{UI}(S)$
$\mathbb{N}$	the set of natural numbers
$\mathscr{N}_t^{\mathbb{P}}$	the family $\{N \subseteq \Omega \mid \exists\, C \in \mathscr{F}_t \text{ such that } N \subseteq C \text{ and } \mathbb{P}(C) = 0\}$
$\mathscr{N}^{\mathscr{P}}$	the collection of $(\mathscr{F}_T,\mathbb{P})$-null sets for all $\mathbb{P} \in \mathscr{P}$
$N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$	G-normal distribution with parameters $\underline{\sigma}^2$ and $\overline{\sigma}^2$
$N([\underline{\sigma}^2 t, \overline{\sigma}^2 t] \times \{0\})$	distribution of the quadratic variation at time t of the G-Brownian motion B
$(\omega \otimes_\tau \tilde{\omega})$	the concatenation of the paths ω and $\tilde{\omega}$ at the stopping time τ
$\omega \otimes_\tau A$	the set $\{\omega \otimes_\tau \tilde{\omega} :\ \tilde{\omega} \in A\}$ for $A \in \mathscr{F}$
Ω (or Ω_T)	the set $C_0([0,T],\mathbf{R})$
Ω^t	the t-fold Cartesian product of Ω, when $t \in \mathbb{N}$
P	the payoff of an European put option with strike D
$\mathfrak{P}(\Omega)$	the set of all probability measures defined on $(\Omega,\mathscr{F})$
$\mathbb{P},\mathbb{Q}$	probability measures on $(\Omega,\mathscr{F})$
$\{\mathbb{P}_\tau^\omega\}_{\omega\in\Omega}$	the regular conditional probability distribution given $\mathscr{F}_\tau$ of $\mathbb{P}$

$\mathbb{P}^{\tau,\omega}$ .... the prior given by $\mathbb{P}^{\tau,\omega}(A) := \mathbb{P}^{\omega}_{\tau}(\omega \otimes_{\tau} A)$, for $A \in \mathscr{F}$
$\mathbb{P}^{\sigma}$ .... the image law of the process $(B^{0,\sigma}_t)_{t\in[0,T]}$
$\mathscr{P}, \mathscr{Q}$ .... subsets of $\mathfrak{P}(\Omega)$
$\mathscr{P}_{\mathbf{G}}$ .... the particular $\mathscr{P}$ associated to the G-expectation
$\mathscr{P}(s,\omega)$ .... a subset of $\mathfrak{P}(\Omega)$ satisfying $\mathscr{P}(s,\omega) = \mathscr{P}(s,\tilde{\omega})$ if $\omega|_{[0,s]} = \tilde{\omega}|_{[0,s]}$
$\mathscr{P}(\tau,\mathbb{P})$ .... the following set of probabilities $\{\mathbb{P}' \in \mathscr{P} : \mathbb{P}' = \mathbb{P} \text{ on } \mathscr{F}_{\tau}\}$
$\mathscr{P}_{\text{eq}}(\tau,\mathbb{P})$ .... the following set of probabilities $\{\mathbb{P}' \in \mathscr{P} : \mathbb{P}' \sim \mathbb{P} \text{ on } \mathscr{F}_{\tau}\}$
$\mathscr{P}_{\mathbf{G}}(t,\mathbb{P})$ .... the particular $\mathscr{P}(\tau,\mathbb{P})$ given by $\mathscr{P} = \mathscr{P}_{\mathbf{G}}$ and $\tau = t$
$\mathscr{P}_1$ .... the family of probabilities given by $\{\mathbb{P}^{\sigma} \mid \sigma \in \mathscr{A}^{\Theta}_{0,T}\}$
Φ .... a set of admissible trading strategies
$\mathbb{Q}$ .... the set of rational numbers
$\mathscr{Q}(\tau,\mathbb{Q})$ .... similar to $\mathscr{P}(\tau,\mathbb{P})$
$\mathbb{R}$ .... the set of real numbers
$R_t(\Upsilon)$ .... risk of a trading strategy Υ at time t
$\bar{R}_t(\Upsilon)$ .... a redefinition of $R_t(\Upsilon)$
S^*_t .... robust fundamental value at time t of the price process S
$\mathbb{S}(d)$ .... the space of d-dimensional symmetric matrices
St .... the payoff of a straddle with strike D
T .... deterministic, finite time horizon
Θ .... the closed interval $[\underline{\sigma}^2, \overline{\sigma}^2]$
Υ .... a trading strategy $(\phi_t, \varsigma_t)_{t\in[0,T]}$
$Var_{G,1}(X)$ .... possible definition of variance of the random variable X in the G-setting
$Var_{G,2}(X)$ .... similar to $Var_{G,1}(X)$
$Var_{G,3}(X)$ .... similar to $Var_{G,1}(X)$
$V_t(\Upsilon)$ .... value of the portfolio Υ at time t
(V^*_0, ϕ^*) .... optimal mean-variance strategy associated to a claim H
φ .... the process given by $\theta - \phi X$
$\xi^{\tau,\omega}$ .... the function given by $\xi(\omega \otimes_{\tau} \tilde{\omega})$, $\tilde{\omega} \in \Omega$
W .... a standard Brownian motion
W^*_t .... the the value at time t of the fundamental wealth process
$\|\cdot\|_p$ .... the following norm $E_G(|\cdot|^p))^{\frac{1}{p}}$, for $p \geq 1$
$\|\cdot\|_{M^p_G}$ .... the norm $(E_G \int_0^T |\cdot|^p ds)^{\frac{1}{p}}$, for $p \geq 1$
$\langle X \rangle_t$ .... quadratic variation of a stochastic process X at time t

1 Introduction

1.1 Motivation

When constructing a market model in mathematical finance, the first step is usually fixing some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. This sets at the same time also the null-sets, which are the events deemed to be impossible by the investor. However, it is not evident from the beginning whether $\mathbb{P}$ is the best possible choice and if another probability $\mathbb{P}'$ would be instead more appropriate. Moreover, events which are not likely to occur according to the views of some investor, as the default of a firm or a country, may not be perceived as such by some other agent. Using just one probability measure implies choosing a particular financial model and a perfect understanding of the market. In the present economical situation, it appears although more reasonable to consider a wider set of models and make robust decisions with respect to the scenarios contemplated by all of them.

This type of insights date back to a seminar paper of Knight [41], where the author makes the distinction between *risk* and *uncertainty*. The former notion refers to the situation in which the agent knows the probability of the possible random outcomes of the market, the latter to the opposite case. Thus risk allows for a probabilistic description and therefore appears to be a more favorable situation to face, compared with uncertainty. The empirical existence of model uncertainty was showed through the famous Ellsberg paradox [25], challenging the well accepted hypothesis of [63]. The two settings were later unified in the results of [32], by adopting a framework with multiple probabilities. In fact model uncertainty from a mathematical point of view translates into the adoption of a family of priors, possibly mutually singular to each other. The reason for the introduction of a non-dominated set of priors $\mathscr{P}$ is clearly understood with the following practical example. Suppose to have specified a stochastic volatility model that describes the dynamics of an asset price. As volatility fluctuates reacting to new market information, it seems legitimate to ask whether the agent can be completely confident about the particular process used to model the volatility. Moreover the classical models typically allow for the sole modeling of *drift uncertainty*, in general through an application of the Girsanov formula. By virtue of the previous observations, the possibility to capture volatility uncertainty appears completely natural. If $B = (B_t)_{t \geq 0}$ denotes a Brownian motion on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, while

$\mathbb{P}^{\underline{\sigma}}$ and $\mathbb{P}^{\overline{\sigma}}$ the image laws of the processes $\underline{\sigma}B$ and $\overline{\sigma}B$ respectively, the two probabilities just introduced can be used to model two possible volatility scenarios for a risky asset. The measures $\mathbb{P}^{\underline{\sigma}}$ and $\mathbb{P}^{\overline{\sigma}}$ are mutually singular as

$$\mathbb{P}^{\underline{\sigma}}(\{\langle B\rangle_t = \underline{\sigma}^2 t\}) = \mathbb{P}^{\overline{\sigma}}(\{\langle B\rangle_t = \overline{\sigma}^2 t\}) = 1.$$

When we want to capture an infinite number of such possibilities, the collection $\{\mathbb{P}^{\sigma}\}_{\sigma\in\Theta}$ would not be dominated by any probability. This is the grounding idea of quasi-sure stochastic analysis as introduced in [21]. This concept has been further developed by Peng in 2006 [56], by relying on sublinear expectations, with the introduction of the so called G-setting. The fundamental results of stochastic calculus, such as the Itô formula and the martingale representation theorem, have been revised in this new structure. Stock prices will be driven by a G-Brownian motion, which generalizes the classical Brownian motion by incorporating volatility uncertainty.

All of this opens the door to the redefinition and the study of many of the classical problems of mathematical finance in this more general setting: cornerstones like the concept of no arbitrage and the FTAP have to be tackled with a different approach. The aim of thesis is to study three topics of financial mathematics under model uncertainty, namely quadratic hedging, superreplication and financial bubbles.

This thesis is divided into three chapters, Chapters 2-4, and each one of them is endowed with an introductory session explaining the mathematical and economical motivation of the contributing manuscripts. Chapter 2 deals with quadratic hedging in the context of model uncertainty. We begin by introducing the G-expectation framework and the tools of G-stochastic calculus necessary for every further investigation. These results are exploited also in the next chapters, as we refer several times to the G-setting to construct concrete examples and provide a clearer understanding of more general results. Then in Sections 2.2 and 2.3 we present the market model and obtain the explicit solution of the robust mean-variance hedging problem for some families of contingent claims. Section 2.3.4 in particular outline a stepwise procedure that can be extended numerically to derive the optimal mean-variance hedging portfolio for a general class of claims. We then discuss the scientific placement of the contributing paper by comparing its results with the classic literature in Section 2.3.5. Finally, Section 2.4 includes a study of robust risk minimization in the G-setting. Chapter 3 is dedicated to the analysis of another fundamental problem of mathematical finance in the context of model uncertainty, namely superreplication. We consider a much more general framework than that studied in Chapter 2, which we outline in Section 3.1. We provide an alternative characterization of the conditional sublinear expectation operator in Proposition 3.1.9 and we study a dynamic version of the robust superhedging duality in Section 3.2. These results are used in Chapter 4 for our treatment of financial asset bubbles under model uncertainty. After having discussed the different approaches existing in the literature in Section 4.1.1, we introduce the notion of robust fundamental value studying its consequences in Section 4.1.2. Section 4.1.3 contains several examples of asset bubbles, displaying the particular features of our model. We conclude by extending the concept of no dominance in a market with model uncertainty in Section 4.1.4 and treating the case of an infinite time horizon in Section 4.2.

1.2 Contributing Manuscripts

This thesis is based on the following manuscripts, which were developed by the thesis' author J. Mancin, in cooperation with coauthors:

1. F. Biagini, T. Brandis-Meyer and J. Mancin [2]: *Robust Mean-Variance Hedging via G-Expectation. LMU Mathematics Institute, Preprint, 2015. Available at:* `http://www.fm.mathematik.uni-muenchen.de/publications_new/index.html`

   The results of this paper are the product of a joint work of J. Mancin with two coauthors, Prof. F. Biagini and Prof. T. Brandis-Meyer. It was developed at the LMU Munich. After a suggestion of F. Biagini and T. Brandis-Meyer, J. Mancin started to study the extension of risk-minimization in the model uncertainty framework described by the G-Expectation. In Section 2, the author of the thesis derived some additional achievements on G-stochastic calculus. In joint discussions we decided to move our attention to the problem of robust mean-variance hedging, which appeared more natural in the G-setting. The results achieved in Sections 3, 4.1 and 4.2 were obtained independently by J. Mancin. The investigations in the remaining part of Section 4 are a result of a close cooperation of F. Biagini and J. Mancin. The approximating results presented in Lemmas 4.11, 4.12 and 4.13 were mainly derived by J. Mancin. Finally in Section 5 the author of the thesis derived bounds for the optimal terminal risk, as suggested by F. Biagini.

2. F. Biagini and J. Mancin [7]: *Financial Asset Price Bubbles under Model Uncertainty. LMU Mathematics Institute, Preprint, 2016. Available at:* `http://www.fm.mathematik.uni-muenchen.de/publications_new/index.html`

   This paper, extending for the first time the concept of asset bubble in the context of model uncertainty, emerged by a collaboration of Prof. F. Biagini and J. Mancin. The manuscript was developed at the LMU Munich. Section 2 contains a new representation of a conditional sublinear expectation operator, developed by J. Mancin as suggested by F. Biagini. Sections 4.1 and 4.2, where prerequisites for further examinations are presented, were developed by J. Mancin. In joint discussions we discussed the best way to extend the concept of asset fundamental value within model uncertainty and, after suggestion of F. Biagini, J. Mancin investigated in Section 3 a dynamic version of robust superreplication in the framework introduced in [52]. The remaining part of Section 4, where we provided concrete examples of asset price bubbles and introduced a version of robust no dominance, were developed mainly by J. Mancin. The analysis of the infinite time horizon case, which is contained in Section 5, was performed by J. Mancin with the help of F. Biagini.

The following list indicates how the two manuscripts contribute to each part of the thesis. The formulation of the statements of the propositions, lemmas, theorems, etc. is the same as in the two papers. However, the author, who has been involved in the development of all the results presented in the two articles, provides in the present thesis a more detailed version for most of the proofs, together with some additional findings.

1. Chapter 1 was developed independently by J. Mancin and provides a presentation of the existing mathematical literature on model uncertainty. The chapter concludes with a summary of the thesis.

2. Chapter 2 is based on F. Biagini, T. Brandis-Meyer and J. Mancin [2]. Section 2.4 consists of results obtained during an early stage of the project that led to [2], although without being included in that manuscript.

3. Chapter 3 is based on Sections 2, 3 of F. Biagini and J. Mancin [7].

4. Chapter 4 is mainly based on Sections 4, 5 of [7].

2 Robust Quadratic Hedging via G-Expectation

One fundamental topic of financial mathematics is the pricing and hedging of contingent claims. When a replicating strategy is not available we can consider the following approaches to tackle this problem: utility maximization arguments as in [40], superreplication, whose idea is to obtain the less expensive portfolio dominating the required payoff, as developed in [24], or quadratic hedging as introduced in [29].
A quadratic criterion can be criticized as it assigns equal weight to losses or extra gains. On the other hand, quadratic techniques have been fruitfully exploited by many authors (see [3], [4], [6], [9], [23], [33], [44], [45] and [59]), thanks to their mathematical tractability and the possibility to obtain closed formulas in different practical situations.
The first possible approach within this framework, namely mean-variance hedging, requires the self-financing property of admissible trading strategies. The general idea consists in finding the best approximation, with respect to a quadratic criterion, of a claim H by the terminal value of self-financing portfolios, as first suggested in [12]. In Sections 2.2 and 2.3 we analyze this issue in the G-expectation framework in continuous time. Our study differs from the relating literature on model uncertainty such as the BSDEs approach (see [22]), the parameter uncertainty model (see [71]) or the one period setup of [74], as it makes use of tools provided by the G-calculus.
In the setting we present in Section 2.2, we model the asset $(X_t)_{t\in[0,T]}$ as a symmetric G-martingale (see Definition 2.1.16). We then look at the minimization problem

$$\inf_{(V_0,\phi)\in\mathbb{R}_+\times\Phi} J_0(V_0,\phi) = \inf_{(V_0,\phi)\in\mathbb{R}_+\times\Phi} E_G\left[(H - V_T(V_0,\phi))^2\right], \tag{2.0.1}$$

where Φ denotes a family of admissible portfolios given in Definition 2.2.4 and $V_T(V_0,\phi)$ is the terminal payoff associated to the strategy (V_0,ϕ). The functional (2.0.1) can be seen as the agent determining the optimal portfolio to cope with a financial market realizing the worst possible volatility scenario.

In the classical framework this problem is usually solved by considering a particular probability $\hat{\mathbb{P}}$ (the *variance optimal martingale measure*) and computing

the Galtchouk-Kunita-Watanabe decomposition of the claim H under $\hat{\mathbb{P}}$ (see [64]). This is equivalent to retrieving the projection of H on the space of square integrable stochastic integrals of X.
In the G-setting it is not possible to adopt such technique. In fact, since the framework is intrinsically nonlinear, the G-Brownian motion B and its quadratic covariation $\langle B\rangle$ are not orthogonal (see [34]). This makes more complicated the computation of expressions like

$$E_G\left[\int_0^T \theta_s dB_s \int_0^T \xi_s d\langle B\rangle_s\right],$$

for suitable processes θ and ξ, which are quite common when dealing with the mean-variance hedging problem. Nevertheless G-martingales have been the subject of a lot of research, which allowed to have a deeper understanding of their structure (we refer to [58], [66] and [69]). We base our study on such results and assume $H \in L_G^{2+\varepsilon}$ with representation (2.2.14) to obtain the optimal mean-variance portfolio.
Our most important findings are the explicit characterizations of the optimal mean-variance investment strategies for a wide family of claims. These collections of claims are obtained by enforcing some particular constraints on the process η in the decomposition of H given in (2.2.14). The first step is assuming the process η to be continuous and deterministic, as in Section 2.3.1. The technique adopted for the solution of that particular case allows to overcome the problems presented above, and to obtain a full description of the optimal investment strategy. This solution scheme is then adapted in Section 2.3.2 to the case in which η is depending only on $\langle B\rangle$. We finally study the case in which η is piecewise constant with the form $\eta_s = \sum_{i=0}^{n-1} \eta_{t_i}\mathbf{1}_{(t_i,t_{i+1}]}(s)$ and develop an iterative procedure that we solve completely for $n=2$.

The quadratic hedging approach alternative to mean-variance optimization is risk minimization. This technique insists on the terminal condition $V_T = H$ and as a consequence it implies to drop the self-financing assumption. The risk associated to an an admissible portfolio $\Upsilon = (\phi,\varsigma)$, where ς denotes the investment in the risk-free asset, is a functional of the cost process

$$C_t(\Upsilon) = V_t - \int_0^t \phi_s dX_s, \quad t\in[0,T],$$

that describes the total costs borne by the investor by investing in (ϕ,ς). The value of the risk process at time t is then defined as

$$R_t(\Upsilon) := E_{\mathbb{P}}\left[(C_T(\Upsilon)-C_t(\Upsilon))^2\,|\mathscr{F}_t\right],$$

where $E_{\mathbb{P}}[\cdot]$ is a linear expectation. In Section 2.4 we discuss how to extend the definition of risk process in the G-expectation framework, providing some results regarding the risk minimizing portfolio corresponding for a contingent claim H. As a byproduct we investigate the definition of variance of a random variable within the G-setting.

2.1 G-Setting

In the following section we present an introduction to the G-expectation theory. As the literature on G-Brownian motion is vast and diverse, a comprehensive examination is impossible to be made here. Nevertheless we will outline some important achievements on stochastic calculus for G-Brownian motion (see [20], [55] and [69] for a reference) and present new results providing a deeper understanding of the G-martingale decomposition and the G-convexity property.

2.1.1 The G-Expectation

We proceed as in [55] by first introducing the notion of *sublinear expectation*. Let us fix a set Ω and let $\mathscr{D}$ be a vector lattice of real-valued functions defined on Ω containing 1. $\mathscr{D}$ will serve as a space of random variables. Furthermore given $X_1,\dots,X_n \in \mathscr{D}$, we assume that $\varphi(X_1,\dots,X_n) \in \mathscr{D}$ for any $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, $n \geq 1$, where $\varphi \in C_{l,Lip}(\mathbb{R}^n)$ stands for the set of real-valued functions ψ defined on $\mathbb{R}^n$ such that

$$|\psi(x) - \psi(y)| \leq C(1+|x|^k+|y|^k)|x-y|, \quad \forall x,y \in \mathbb{R}^n,$$

where k and C are integers that depend on the function ψ. A sublinear expectation is defined in the following way.

Definition 2.1.1. A nonlinear expectation $\mathbb{E}$ is a functional $\mathscr{D} \mapsto \mathbb{R}$ satisfying the following properties

1. Monotonicity: If $X,Y \in \mathscr{D}$ and $X \geq Y$ then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.
2. Preserving of constants: $\mathbb{E}[c] = c$, for every $c \in \mathbb{R}$.
3. Sub-additivity:

$$\mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y], \qquad \forall\, X,Y \in \mathscr{D}. \tag{2.1.1}$$

4. Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall\, \lambda \geq 0$, $X \in \mathscr{D}$.
5. Constant translatability. $\mathbb{E}[X+c] = \mathbb{E}[X] + c$.

We call the triple $(\Omega, \mathscr{D}, \mathbb{E})$ a sublinear expectation space.

It is clear from Definition 2.1.1 that sublinear expectation spaces are generalizations of the usual probability spaces. The lack of linearity imposes to provide a definition for the concepts of independence and identical distribution of random variables. Moreover these notions must coincide with their classical counterparts in the case (2.1.1) holds with equality.

Definition 2.1.2. In a sublinear expectation space $(\Omega, \mathscr{D}, \mathbb{E})$ a random variable $Y \in \mathscr{D}$ is said to be independent from another random variable $X \in \mathscr{D}$ under $\mathbb{E}$ if for any test function $\psi \in C_{l,Lip}(\mathbb{R}^2)$ we have

$$\mathbb{E}[\psi(X,Y)] = \mathbb{E}[\mathbb{E}[\psi(x,Y)]_{X=x}],$$

where $\psi(x,Y) \in \mathscr{D}$ for every $x \in \mathbb{R}$, as $\psi(x,\cdot) \in C_{l,Lip}(\mathbb{R})$.

Remark 2.1.3. It is important to notice how in a sublinear expectation space the statement "X is independent to Y" is not equivalent to "Y is independent to X". In fact, as proven in [35], if this happens then X and Y must be *maximally distributed* (see Definition 2.1.14).

Definition 2.1.4. Let X_1 and X_2 be two random variables defined on the sublinear expectation spaces $(\Omega_1, \mathscr{D}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathscr{D}_2, \mathbb{E}_2)$ respectively. They are called identically distributed, denoted by $X_1 \sim X_2$, if

$$\mathbb{E}_1[\psi(X_1)] = \mathbb{E}_2[\psi(X_2)], \qquad \forall \psi \in C_{l,Lip}(\mathbb{R}).$$

We call $\bar{X}$ an independent copy of X if $\bar{X} \sim X$ and $\bar{X}$ is independent from X.

We next define G-distributed random variables in a sublinear expectation space.

Definition 2.1.5. A random variable X on a sublinear expectation space $(\Omega, \mathscr{D}, \mathbb{E})$ is called G-normal distributed if for any $a, b \geq 0$

$$aX + b\bar{X} \sim \sqrt{a^2 + b^2}X,$$

where $\bar{X}$ is an independent copy of X. The letter G denotes the function

$$G(y) := \frac{1}{2}\mathbb{E}[yX^2] : \mathbb{R} \mapsto \mathbb{R}.$$

Such X is symmetric, i.e. $\mathbb{E}[X] = \mathbb{E}[-X] = 0$. In addition we have the following identity

$$G(y) = \frac{1}{2}\overline{\sigma}^2 y^+ - \frac{1}{2}\underline{\sigma}^2 y^-,$$

with $\overline{\sigma}^2 := \mathbb{E}[X^2]$ and $\underline{\sigma}^2 := -\mathbb{E}[-X^2]$. We write X is $N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$ distributed.

Definition 2.1.6. A process $(B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathscr{D}, \mathbb{E})$ is called *G-Brownian* motion if the following properties hold true:

(i) $B_0 = 0$.

(ii) For each $t, s \geq 0$ the increment $B_{t+s} - B_t$ is $N(\{0\} \times [\underline{\sigma}^2 s, \overline{\sigma}^2 s])$ distributed and independent from $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ for any $n \in \mathbb{N}$, $0 \leq t_1 \leq \cdots \leq t_n \leq t$.

It is clear that Definition 2.1.6 preserves the characteristics of the standard Brownian motion and it can be easily seen that, for all $t_0 \geq 0$, $(B_{t+t_0} - B_{t_0})_{t\geq 0}$ is still a G-Brownian motion.
We are now ready to introduce the properties of the G-expectation, i.e. the particular sublinear expectation for which the canonical process is a G-Brownian motion. Let us fix a finite time horizon $T > 0$ and set $\Omega_T := C_0([0,T], \mathbb{R})$, the family of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t\in[0,T]}$ with $\omega_0 = 0$. We denote by $B = (B_t)_{t\in[0,T]}$ the canonical process on Ω_T, i.e. $B_t(\omega) := \omega_t$, $t \in [0,T]$. As in [56] we first consider the family of *cylindrical* random variables:

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \cdots, B_{t_n}) | n \in \mathbb{N}, t_1, \ldots, t_n \in [0,T], \varphi \in C_{l,Lip}(\mathbb{R}^n)\}.$$

The construction of the G-Brownian motion relies on the set $L_{ip}(\Omega_T)$. Let then $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of random variables on a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathscr{D}}, \tilde{E})$ with the property that ξ_i is G-normal distributed and ξ_{i+1} is independent of $(\xi_1, \ldots, \xi_i)$ for each $i \geq 1$. The G-expectation is defined on $L_{ip}(\Omega_T)$ using the following method: for each random variable $X = \varphi(B_{t_1} - B_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}}) \in L_{ip}(\Omega_T)$, where $\varphi \in C_{l,Lip}(\mathbb{R}^n)$ and $t_1, \ldots, t_n \in [0,T]$, let

$$E_G[\varphi(B_{t_1} - B_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}})] := \tilde{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, \ldots, \sqrt{t_n - t_{n-1}}\xi_n)].$$

It is proved in [56] that E_G is a well defined sublinear expectation on $L_{ip}(\Omega_T)$, with the desired characteristic of turning the canonical process B into a G-Brownian motion.

Definition 2.1.7. The sublinear expectation $E_G : L_{ip}(\Omega_T) \mapsto \mathbb{R}$ defined through the above procedure is called *G-expectation*. The canonical process $(B_t)_{t\in[0,T]}$ on such sublinear expectation space $(\Omega_T, L_{ip}(\Omega_T), E_G)$ is a *G-Brownian motion*.

The notion of G-conditional expectation with respect to $\Omega_{t_i} := C_0([0,t_i], \mathbb{R})$ is introduced similarly. Given a random variable $X \in L_{ip}(\Omega_T)$, we define

$$E_G[\varphi(B_{t_1} - B_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}}) | \Omega_{t_i}] := \psi(B_{t_1} - B_{t_0}, \cdots, B_{t_n} - B_{t_{n-1}}),$$

where $\psi(x_1, \ldots, x_i) := \tilde{E}[\varphi(x_1, \ldots, x_i, \sqrt{t_{i+1} - t_i}\xi_{i+1}, \ldots, \sqrt{t_n - t_{n-1}}\xi_n)]$.

Also the classical notion of L^p spaces can be extended to the present setting. Let in fact $\|\xi\|_p := (E_G[|\xi|^p])^{\frac{1}{p}}$ for $\xi \in L_{ip}(\Omega_T)$, $p \geq 1$. The G-conditional expectation $E_G[\cdot|\Omega_t]$, for $t \in [0,T]$, can be extended continuously to $L^p_G(\Omega_T)$, the completion of $L_{ip}(\Omega_T)$ under the $\|\cdot\|_p$-norm. The next property proves to be very useful in many contexts.

Proposition 2.1.8 (Proposition 22 of [56]). *Let $Y \in L^1_G(\Omega_T)$ be such that $E_G[Y] = -E_G[-Y]$. Then we have*

$$E_G[X+Y] = E_G[X] + E_G[Y], \qquad \forall\, X \in L^1_G(\Omega_T).$$

It is well known that a sublinear expectation $\mathbb{E}$ defined on $\mathscr{D}$ can be represented as a “worst case expectation"

$$\mathbb{E}[X] = \sup_{\delta\in\Delta} E_\delta[X], \quad X \in \mathscr{D},$$

for some family of linear expectations $\{E_\delta\}_{\delta\in\Delta}$. This characterization provides useful insights when applied to the G-expectation. To this end we let $\mathscr{F} = \mathscr{B}(\Omega_T)$ be the Borel σ-algebra and fix a probability space $(\Omega_T, \mathscr{F}, \mathbb{P})$. We denote with $W = (W_t)_{t\in[0,T]}$ a classical Brownian motion on $(\Omega_T, \mathscr{F}, \mathbb{P})$. The natural filtration of W is denoted by $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$, where $\mathscr{F}_t := \sigma\{W_s | 0 \leq s \leq t\}$. Let Θ be the interval $\Theta := [\underline{\sigma}, \overline{\sigma}] \subset \mathbb{R}$ such that

$$G(y) = \frac{1}{2}\sup_{\sigma\in\Theta}\left(y\sigma^2\right) = \begin{cases} \frac{1}{2}y\overline{\sigma}^2 & \text{if } y \geq 0, \\ \frac{1}{2}y\underline{\sigma}^2 & \text{if } y < 0, \end{cases}$$

and call $\mathscr{A}^{\Theta}_{t,T}$ the set of all $\mathbb{F}$-adapted Θ-valued processes on $[t,T]$. Given an arbitrary $\sigma = (\sigma_t)_{t\in[0,T]} \in \mathscr{A}^{\Theta}_{t,T}$ and $s \in [t,T]$ we introduce the process

$$B^{t,\sigma}_s := \int_t^s \sigma_u dW_u. \tag{2.1.2}$$

We denote by $\mathbb{P}^\sigma$ the distribution of the process $(B^{0,\sigma}_t)_{t\in[0,T]}$, i.e. $\mathbb{P}^\sigma = \mathbb{P}\circ(B^{0,\sigma})^{-1}$. Define

$$\mathscr{P}_1 := \{\mathbb{P}^\sigma \mid \sigma \in \mathscr{A}^{\Theta}_{0,T}\}, \tag{2.1.3}$$

and $\mathscr{P}_{\mathbf{G}} := \bar{\mathscr{P}}_1$, the closure of $\mathscr{P}_1$ under the weak convergence topology. With these notations at hand we can state the main result of [20]:

Theorem 2.1.9. *For any $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, $n \in \mathbb{N}$, $0 \leq t_1 \leq \cdots \leq t_n \leq T$, we have*

$$\begin{aligned} E_G[\varphi(B_{t_1},\ldots,B_{t_n}-B_{t_{n-1}})] &= \sup_{\sigma\in\mathscr{A}^{\Theta}_{0,T}} E_{\mathbb{P}}[\varphi(B^{0,\sigma}_{t_1},\ldots,B^{t_{n-1},\sigma}_{t_n})] \\ &= \sup_{\sigma\in\mathscr{A}^{\Theta}_{0,T}} E_{\mathbb{P}^{\sigma}}[\varphi(B_{t_1},\ldots,B_{t_n}-B_{t_{n-1}})] \\ &= \sup_{\mathbb{P}^{\sigma}\in\mathscr{P}_1} E_{\mathbb{P}^{\sigma}}[\varphi(B_{t_1},\ldots,B_{t_n}-B_{t_{n-1}})]. \end{aligned}$$

Furthermore,

$$E_G[X] = \sup_{\mathbb{P}\in\mathscr{P}_G} E_{\mathbb{P}}[X], \qquad \forall\, X \in L^1_G(\mathscr{F}_T).$$

Finally we can give the definition of *polar set* in the G-expectation framework.

Definition 2.1.10. A set A is said *polar* if $\mathbb{P}(A) = 0\ \forall\ \mathbb{P} \in \mathscr{P}_{\mathbf{G}}$. A property is said to hold *quasi surely (q.s.)* if it holds outside a polar set.

2.1.2 Stochastic Calculus of Itô type with G-Brownian Motion

One of the most interesting features of the G-expectation framework is that it provides a number of *quasi probabilistic* tools, such as the G-Itô formula or the G-martingale representation theorem, which represent the generalization under uncertainty of many well known probability theory results. In this section we recap some of them using [55] as main reference, while Lemma 2.1.19 is a new result. We start by introducing the stochastic integral with respect to a one dimensional G-Brownian motion. To this end, fixed $p \geq 1$, we need to look at the following class of simple processes: given a partition $\{t_0,\ldots,t_N\}$ of $[0,T]$, $N \in \mathbb{N}$, let

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{1}_{[t_j,t_{j+1})}(t), \tag{2.1.4}$$

where $\xi_i \in L^p_G(\mathscr{F}_{t_i})$, $i \in 0,\ldots,N-1$. We denote by $M^{p,0}_G(0,T)$ the family of such processes. Given $\eta \in M^{p,0}_G(0,T)$, we introduce the norm $\|\eta\|_{M^p_G} := (E_G \int_0^T |\eta_s|^p ds)^{\frac{1}{p}}$ and indicate $M^p_G(0,T)$ the completion of $M^{p,0}_G(0,T)$ under $\|\cdot\|_{M^p_G}$. The stochastic integral with respect to the G-Brownian motion is first defined for processes in $M^{2,0}_G(0,T)$.

Definition 2.1.11. For $\eta \in M^{2,0}_G(0,T)$ with the representation in (2.1.4) we define the integral mapping $I : M^{2,0}_G(0,T) \mapsto L^2_G(\mathscr{F}_T)$ by

$$I(\eta) = \int_0^T \eta(s)dB_s := \sum_{j=0}^{N-1} \eta_j(B_{t_{j+1}} - B_{t_j}).$$

Lemma 2.1.12 (Lemma 30 of [56])**.** *The mapping $I : M_G^{2,0}(0,T) \mapsto L_G^2(\mathscr{F}_T)$ is a linear continuous mapping and thus can be continuously extended to $I : M_G^2(0,T) \mapsto L_G^2(\mathscr{F}_T)$.*

The stochastic integral from Definition 2.1.11 preserves many of the properties we know from the classical Itô case.

Definition 2.1.13. The quadratic variation of the G-Brownian motion is defined as

$$\langle B \rangle_t = B_t^2 - 2\int_0^t B_s dB_s, \qquad \forall\, t \leq T,$$

and it is a continuous increasing process which is absolutely continuous with respect to the Lebesgue measure dt (see Definition 2.2 in [69]).

The process $\langle B \rangle$ completely determines the uncertainty of B and it is not deterministic unless $\overline{\sigma} = \underline{\sigma}$. In particular, for any $s,t \geq 0$, the increment $\langle B \rangle_{s+t} - \langle B \rangle_s$ is independent of $\mathscr{F}_s$ and distributed as $\langle B \rangle_t$. The random variable $\langle B \rangle_t$ is said to be $N([\underline{\sigma}^2 t, \overline{\sigma}^2 t] \times \{0\})$-distributed, i.e., for all $\varphi \in C_{l,Lip}(\mathbb{R})$,

$$E_G[\varphi(\langle B \rangle_t)] = \sup_{\underline{\sigma}^2 \leq v \leq \overline{\sigma}^2} \varphi(vt). \tag{2.1.5}$$

The quadratic variation of the G-Brownian motion then belongs to the following class of random variables.

Definition 2.1.14. An n-dimensional random vector X on a sublinear expectation space $(\Omega, \mathscr{D}, \mathbb{E})$ is called *maximally distributed* if there exists a closed set $\Gamma \subset \mathbb{R}^n$ such that

$$\mathbb{E}(\varphi(X)) = \sup_{x \in \Gamma} \varphi(x),$$

for all $\varphi \in C_{l,Lip}(\mathbb{R}^n)$.

Similarly to the integration with respect to the G-Brownian motion, it is possible to introduce $\int_0^t \eta_s d\langle B \rangle_s$. This is done first by defining the integral for processes $\eta \in M_G^{1,0}(0,T)$, and then, once more by continuity, for every $\eta \in M_G^1(0,T)$.
We now state a version of the Itô formula for G-Brownian motion.

Theorem 2.1.15 (Theorem 6.5 from [55])**.** *Let Φ be a C^2 function on $\mathbb{R}$ such that $\partial_x^2 \Phi$ has polynomial growth. Let α, β and η be bounded processes in $M_G^2(0,T)$. Then, given*

$$X_s = X_0 + \int_0^s \alpha_u du + \int_0^s \beta_u dB_u + \int_0^s \eta_u d\langle B \rangle_u$$

for $s \in [0,T]$*, it holds for each* $t \in [s,T]$

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_x \Phi(X_u)\beta_u dB_u + \int_s^t \partial_x \Phi(X_s)\alpha_u du + \int_s^t \left(\partial_x \Phi(X_u)\eta_u + \frac{1}{2}\partial_x^2 \Phi(X_u)\beta_u^2\right) d\langle B\rangle_u,$$

in $L_G^2(\mathscr{F}_T)$*.*

Definition 2.1.16. A process $M = (M_t)_{t\in[0,T]}$, such that $M_t \in L_G^1(\mathscr{F}_t)$ for any $t \in [0,T]$, is called G-martingale if $E_G[M_t|\mathscr{F}_s] = M_s$ for all $s \le t \le T$. If M and $-M$ are both G-martingales, M is called a symmetric G-martingale.

For any $t \in [0,T]$ and $\mathbb{P} \in \mathscr{P}_\mathbf{G}$ denote

$$\mathscr{P}_\mathbf{G}(t,\mathbb{P}) := \{\mathbb{P}' \in \mathscr{P}_\mathbf{G} : \mathbb{P}' = \mathbb{P} \text{ on } \mathscr{F}_t\}.$$

Similarly as in Theorem 2.1.9, it is possible to retrieve a characterization of the G-conditional expectation as a *worst case risk measure* (see [66] for a reference). In particular, if M is a G-martingale, for all $0 \le s \le t \le T$ and $\mathbb{P} \in \mathscr{P}_\mathbf{G}$ it holds

$$M_s = \operatorname*{ess\,sup}_{\mathbb{Q}' \in \mathscr{P}_\mathbf{G}(s,\mathbb{P})} E_{\mathbb{Q}'}(M_t|\mathscr{F}_s), \qquad \mathbb{P}-a.s. \tag{2.1.6}$$

From (2.1.6) we can infer that every G-martingale is indeed a $\mathbb{P}$-supermartingale for every $\mathbb{P} \in \mathscr{P}_\mathbf{G}$. The next theorem provides more insights on the structure of G-martingales.

Theorem 2.1.17 (Theorem 2.2 of [57])**.** *Let* $H \in L_{ip}(\Omega_T)$*, then for every* $0 \le t \le T$ *we have*

$$E_G[H|\mathscr{F}_t] = E_G[H] + \int_0^t \theta_s dB_s + \int_0^t \eta_s d\langle B\rangle_s - 2\int_0^T G(\eta_s)ds, \tag{2.1.7}$$

where $\theta \in M_G^2(0,T)$ *and* $\eta \in M_G^1(0,T)$*.*

The nonsymmetric component in particular,

$$-K_t := \int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s)ds, \quad t \in [0,T], \tag{2.1.8}$$

is a continuous and non-increasing G-martingale with quadratic variation equal to zero. The representation from Theorem 2.1.17 can be generalized for G-martingales in $L_G^\beta(\mathscr{F}_T)$, with $\beta > 1$.

Theorem 2.1.18 (Theorem 4.5 of [69]). *Let $\beta > 1$ and $H \in L_G^\beta(\mathscr{F}_T)$. Then the G-martingale M with $M_t := E_G[H|\mathscr{F}_t]$, $t \in [0,T]$, has the following representation*

$$M_t = X_0 + \int_0^t \theta_s dB_s - K_t, \tag{2.1.9}$$

where K is a continuous, increasing process with $K_0 = 0$, $K_T \in L_G^\alpha(\mathscr{F}_T)$, $\theta \in M_G^\alpha(0,T)$, $\forall \alpha \in [1,\beta)$, and $-K$ is a G-martingale.

As a byproduct to the previous theorem, it is possible to argue that in the representation of every symmetric G-martingale the process K is quasi surely equal to zero. Otherwise stated, any symmetric G-martingale can be written as the sum of a real number and a stochastic integral with respect to the G-Brownian motion.

We conclude this section by illustrating how the decomposition of the G-martingale $(E_G[H|\mathscr{F}_t])_{t\in[0,T]}$ is connected to the one of $(E_G[-H|\mathscr{F}_t])_{t\in[0,T]}$. In order simplify the computations we restrict our attention to the family of random variables for which the process η in (2.1.8) is of the type

$$\eta_s = \mathbf{1}_{(t,T]}(s)\bar{\eta},$$

where $0 < t < T$, $s \in [0,T]$ and $\bar{\eta} \in L_{ip}(\Omega_t)$. It is then not difficult to generalize the result for the case in which $\eta \in M_G^{1,0}(0,T)$.

Lemma 2.1.19. *Let*

$$H = E_G[H] + \int_0^T \theta_s dB_s + \bar{\eta}(\langle B\rangle_T - \langle B\rangle_t) - 2G(\bar{\eta})(T-t),$$

where $\theta \in M_G^2(0,T)$, and $\bar{\eta} \in L_{ip}(\mathscr{F}_t)$ is such that

$$|\bar{\eta}| = E_G[|\bar{\eta}|] + \int_0^t \mu_s dB_s + \int_0^t \xi_s d\langle B\rangle_s - 2\int_0^t G(\xi_s)ds,$$

for some processes $\mu \in M_G^2(0,t)$ and $\xi \in M_G^1(0,t)$. Then the decomposition of $-H$ is given by

$$-H = E_G[-H] + \int_0^T \bar{\mu}_s dB_s + \int_0^T \bar{\xi}_s d\langle B\rangle_s - 2\int_0^T G(\bar{\xi}_s)ds,$$

where

$$\bar{\mu}_s = \begin{cases} \mu_s(\overline{\sigma}^2 - \underline{\sigma}^2)(T-t) - \theta_s, & \text{if } s \in [0,t], \\ -\theta_s, & \text{if } s \in (t,T], \end{cases}$$

and

$$\bar{\xi}_s = \begin{cases} \xi_s(\overline{\sigma}^2 - \underline{\sigma}^2)(T-t), & \text{if } s \in [0,t], \\ -\bar{\eta}, & \text{if } s \in (t,T]. \end{cases}$$

Proof. The result is achieved by direct computation. First, given $s < t$, we can compute

$$
\begin{aligned}
&E_G\left[-H|\mathscr{F}_s\right] \\
=&E_G\left[-E_G[H]-\int_0^T \theta_u dB_u - \bar\eta(\langle B\rangle_T-\langle B\rangle_t)+2G(\bar\eta)(T-t)\middle|\mathscr{F}_s\right] \\
=&-E_G[H]-\int_0^s \theta_u dB_u + E_G\left[-\bar\eta(\langle B\rangle_T-\langle B\rangle_t)+2G(\bar\eta)(T-t)|\mathscr{F}_s\right] \\
=&-E_G[H]-\int_0^s \theta_u dB_u + \\
&\quad + E_G\left[E_G\left[-\bar\eta(\langle B\rangle_T-\langle B\rangle_t)+2G(\bar\eta)(T-t)|\mathscr{F}_t\right]|\mathscr{F}_s\right] \\
=&-E_G[H]-\int_0^s \theta_u dB_u + (\overline{\sigma}^2-\underline{\sigma}^2)(T-t)E_G\left[|\bar\eta||\mathscr{F}_s\right] \\
=&-E_G[H]+(\overline{\sigma}^2-\underline{\sigma}^2)(T-t)E_G[|\bar\eta|]+\int_0^s \left(\mu_u(\overline{\sigma}^2-\underline{\sigma}^2)(T-t)-\theta_u\right)dB_u \\
&\quad +(\overline{\sigma}^2-\underline{\sigma}^2)(T-t)\int_0^s \xi_u d\langle B\rangle_u - 2\int_0^s G(\xi_u(\overline{\sigma}^2-\underline{\sigma}^2)(T-t))du \\
=&E_G[-H]+\int_0^s \left(\mu_u(\overline{\sigma}^2-\underline{\sigma}^2)(T-t)-\theta_u\right)dB_u + \\
&\quad +(\overline{\sigma}^2-\underline{\sigma}^2)(T-t)\int_0^s \xi_u d\langle B\rangle_u - 2\int_0^s G(\xi_u(\overline{\sigma}^2-\underline{\sigma}^2)(T-t))du,
\end{aligned}
$$

where the last equality comes from

$$E_G[H]+E_G[-H]=E_G[K_T]=E_G\left[-\bar\eta(\langle B\rangle_T-\langle B\rangle_t)+2G(\bar\eta)(T-t)\right].$$

In the case in which $s > t$ it holds

$$
\begin{aligned}
&E_G\left[-\bar\eta(\langle B\rangle_T-\langle B\rangle_t)+2G(\bar\eta)(T-t)|\mathscr{F}_s\right] \\
=&2G(\bar\eta)(T-t)+\bar\eta\langle B\rangle_t+E_G\left[-\bar\eta\langle B\rangle_T|\mathscr{F}_s\right] \\
=&2G(\bar\eta)(T-t)+\bar\eta\langle B\rangle_t+\bar\eta^+\left(E_G\left[-\langle B\rangle_T+\underline{\sigma}^2T\middle|\mathscr{F}_s\right]-\underline{\sigma}^2T\right)+ \\
&\qquad\qquad +\bar\eta^-\left(E_G\left[\langle B\rangle_T-\overline{\sigma}^2T\middle|\mathscr{F}_s\right]+\overline{\sigma}^2T\right) \\
=&2G(\bar\eta)(T-t)+\bar\eta\langle B\rangle_t+\bar\eta^+(-\langle B\rangle_s+\underline{\sigma}^2s-\underline{\sigma}^2T)+\bar\eta^-(\langle B\rangle_s-\overline{\sigma}^2s+\overline{\sigma}^2T) \\
=&2G(\bar\eta)(T-t)+\bar\eta\langle B\rangle_t-\bar\eta\langle B\rangle_s+2G(-\bar\eta)(T-s) \\
=&2G(\bar\eta)(T-t)-\bar\eta(\langle B\rangle_s-\langle B\rangle_t)+2G(-\bar\eta)(T-t)-2G(-\bar\eta)(s-t) \\
=&|\bar\eta|(\overline{\sigma}^2-\underline{\sigma}^2)(T-t)-\bar\eta(\langle B\rangle_s-\langle B\rangle_t)-2G(-\bar\eta)(s-t),
\end{aligned}
$$

where we exploited the equality

$$2G(x)+2G(-x)=|x|(\overline{\sigma}^2-\underline{\sigma}^2) \qquad \forall\, x\in\mathbb{R}.$$

This proves the claim. □

We can obtain some more insights on the link between the representation of H and the one of $-H$ by considering for simplicity $H = \Phi(B_T)$, for some Lipschitz function Φ, and writing down the two system of PDEs which need to be solved to obtain the processes θ, η and $\bar{\theta}$, $\bar{\eta}$ appearing in

$$H = E_G[H] + \int_0^T \theta_s dB_s + \int_0^T \eta_s d\langle B\rangle_s - 2\int_0^T G(\eta_s)ds$$

and

$$-H = E_G[-H] + \int_0^T \bar{\theta}_s dB_s + \int_0^T \bar{\eta}_s d\langle B\rangle_s - 2\int_0^T G(\bar{\eta}_s)ds.$$

The processes θ and η depend respectively only on $\partial_x u(t,x)$ and $\partial_x^2 u(t,x)$, where $u(t,x)$ solves

$$\begin{cases} \partial_t u + \frac{1}{2}\left((\partial_x^2 u)^+ \overline{\sigma}^2 - (\partial_x^2 u)^- \underline{\sigma}^2\right) = 0, \\ u(T,x) = \Phi(x), \end{cases}$$

and a similar dependence holds for $\bar{\theta}$ and $\bar{\eta}$ and the solution $\bar{u}(t,x)$ of

$$\begin{cases} \partial_t \bar{u} + \frac{1}{2}\left((\partial_x^2 \bar{u})^+ \overline{\sigma}^2 - (\partial_x^2 \bar{u})^- \underline{\sigma}^2\right) = 0, \\ \bar{u}(T,x) = -\Phi(x). \end{cases}$$

If we write $u(t,x) = f(t,x,\overline{\sigma}^2,\underline{\sigma}^2)$ then $\bar{u}(t,x) = -f(t,x,\underline{\sigma}^2,\overline{\sigma}^2)$. This means that we can get the decomposition of $-H$ by changing the sign and the dependence from $\overline{\sigma}^2$ to $\underline{\sigma}^2$ and vice versa in the coefficients of the representation of H.

2.1.3 G-Jensen's Inequality

The usual version of Jensen's inequality does not hold true in the G-setting. However a similar result can be achieved by introducing the concept of *G-convexity*. To this end, let us introduce the space of d-dimensional symmetric matrices denoted by $\mathbb{S}(d)$.

Definition 2.1.20. A C^2-function $h : \mathbb{R} \mapsto \mathbb{R}$ is called G-convex if the following condition holds for each $(y,z,A) \in \mathbb{R}^3$:

$$G(h'(y)A + h''(y)zz^\top) - h''(y)G(A) \geq 0,$$

where h' and h'' denote the first and the second derivatives of h, respectively.

Definition 2.1.20 is exploited in Proposition 5.4.6 of [55] to prove the next result.

Proposition 2.1.21. *The following two conditions are equivalent:*

- *The function h is G-convex.*
- *The following Jensen inequality holds:*

$$E_G\left[h(X)|\mathscr{F}_t\right] \geq h(E_G\left[X|\mathscr{F}_t\right]), \quad t \in [0,T],$$

for each $X \in L^1_G(\mathscr{F}_T)$ such that $h(X) \in L^1_G(\mathscr{F}_T)$.

We next show that the function $h(x) = x^2$ is G-convex, which will turn particularly useful when dealing with the mean-variance hedging problem.

Lemma 2.1.22. *The function $x \mapsto x^2$ is G-convex. We show this result for the one dimensional case, but the proof for the general context is similar.*

Proof. We move from Definition 2.1.20 and prove that, for every $(y,z,A) \in \mathbb{R}^3$,

$$G(2yA+2z^2) \geq 2yG(A),$$

which is

$$(yA+z^2)^+\overline{\sigma}^2 - (yA+z^2)^-\underline{\sigma}^2 \geq y(A^+\overline{\sigma}^2 - A^-\underline{\sigma}^2). \tag{2.1.10}$$

We proceed by cases. If A and y are both positive (2.1.10) is clear. When A is greater than zero, but y is not, we only need to focus on the case in which $yA+z^2<0$. In this situation (2.1.10) actually becomes

$$(yA+z^2)\underline{\sigma}^2 \geq yA\overline{\sigma}^2$$
$$yA(\underline{\sigma}^2 - \overline{\sigma}^2) + z^2\underline{\sigma}^2 \geq 0,$$

which is true as $yA(\underline{\sigma}^2 - \overline{\sigma}^2) > 0$. The remaining case where A is negative can be treated analogously. □

2.1.4 Some Estimates

Here we introduce new estimates for the value of the G-expectation of some particular type of random variables. We focus on the evaluation of

$$E_G\left[\int_0^T \theta_t dB_t \int_0^T \eta_t d\langle B\rangle_t\right], \tag{2.1.11}$$

for suitable processes θ and η. The G-expectation of this kind of cross-product between integrals, which is relevant on its own, incurs several times when studying the mean-variance hedging problem. Through the G-Itô formula we see that (2.1.11) is equal to

$$E_G\left[\int_0^T \left(\int_0^s \eta_u d\langle B\rangle_u\right)\theta_s dB_s + \int_0^T \left(\int_0^s \theta dB_u\right)\eta_s d\langle B\rangle_s\right].$$

For the integrability of each term we would need $(\eta_s \int_0^s \theta_u dB_u)_{s\in[0,T]} \in M_G^1(0,T)$ and $(\theta_s \int_0^s \eta_u d\langle B\rangle_u)_{s\in[0,T]} \in M_G^2(0,T)$ which means

$$\int_0^T E_G\left[\left|\eta_s \int_0^s \theta_u dB_u\right|\right] ds < \infty, \tag{2.1.12}$$

$$\int_0^T E_G\left[\left(\theta_s \int_0^s \eta_u d\langle B\rangle_u\right)^2\right] < \infty. \tag{2.1.13}$$

The condition (2.1.12) is implied by $\theta, \eta \in M_G^2(0,T)$. In fact

$$\begin{aligned}\int_0^T E_G\left[\left|\eta_s \int_0^s \theta_u dB_u\right|\right] ds &\leq \int_0^T E_G\left[\eta_s^2\right]^{\frac{1}{2}} E_G\left[\left(\int_0^s \theta_u dB_u\right)^2\right]^{\frac{1}{2}} ds \\ &\leq \overline{\sigma}^2 \int_0^T E_G\left[\eta_s^2\right]^{\frac{1}{2}} E_G\left[\int_0^s \theta_u^2 du\right]^{\frac{1}{2}} ds \\ &\leq \overline{\sigma}^2 \|\theta\|_{M_G^2}^2 \int_0^T E_G\left[\eta_s^2\right]^{\frac{1}{2}} ds \\ &\leq \overline{\sigma}^2 \|\theta\|_{M_G^2}^2 \|\eta\|_{M_G^2}^2,\end{aligned}$$

where the first inequality comes from Proposition 16 in [20] and the second from of Lemma 30 in [56]. To ensure $(\theta_s \int_0^s \eta_u d\langle B\rangle_u)_{s\in[0,T]} \in M_G^2(0,T)$ we could enforce some stronger condition. However for our purpose it is enough to notice that the well-posedness of such stochastic integral is a consequence of the extension of the G-Itô formula to $C^{1,2}$ functions provided in [46].

Proposition 2.1.23. *Let $\theta, \eta \in M_G^2(0,T)$ such that $(\eta_t \int_0^t \theta_s dB_s)_{t\in[0,T]}$ and $(\theta_t \int_0^t \eta_s d\langle B\rangle_s)_{t\in[0,T]}$ both belong to $M_G^2(0,T)$. Then it holds*

$$E_G\left[\int_0^T \theta_t dB_t \int_0^T \eta_t d\langle B\rangle_t\right] \leq E_G\left[\int_0^T 2G(\eta_s \int_0^s \theta_u dB_u) ds\right].$$

Proof. We first apply the Itô formula for G-Brownian motion to get

$$\begin{aligned}&E_G\left[\int_0^T \theta_t dB_t \int_0^T \eta_t d\langle B\rangle_t\right] \\ =&E_G\left[\int_0^T \eta_s \left(\int_0^s \theta_u dB_u\right) d\langle B\rangle_s + \int_0^T \theta_s \left(\int_0^s \eta_u d\langle B\rangle_u\right) dB_s\right] \\ =&E_G\left[\int_0^T \eta_s \left(\int_0^s \theta_u dB_u\right) d\langle B\rangle_s\right].\end{aligned}$$

The proof is then completed by noting that

$$\begin{aligned}
&E_G\left[\int_0^T \eta_s\left(\int_0^s \theta_u dB_u\right) d\langle B\rangle_s\right]\\
=&E_G\Big[\int_0^T \eta_s\left(\int_0^s \theta_u dB_u\right) d\langle B\rangle_s + \int_0^T 2G(\eta_s\int_0^s \theta_u dB_u)ds+\\
&\qquad\qquad - \int_0^T 2G(\eta_s\int_0^s \theta_u dB_u)ds\Big]\\
\leq&E_G\left[\int_0^T 2G(\eta_s\int_0^s \theta_u dB_u)ds\right].
\end{aligned}$$

□

We apply the result of Proposition 2.1.23 to the case in which $\theta \equiv 1 \equiv \eta$ to estimate the value of $E_G[B_t\langle B\rangle_t]$. This is equivalent to the estimation of the precise value of $E_G\left[B_t^3\right]$ as, through an application of the G-Itô formula, we get

$$B_t^3 = 3\int_0^t B_s^2 dB_s + 3\int_0^t B_s d\langle B\rangle_s \tag{2.1.14}$$

and

$$B_t\langle B\rangle_t = \int_0^t B_s d\langle B\rangle_s + \int_0^t \langle B\rangle_s dB_s, \tag{2.1.15}$$

so that

$$E_G\left[B_t^3\right] = 3E_G\left[\int_0^t B_s d\langle B\rangle_s\right] = 3E_G\left[B_t\langle B\rangle_t\right].$$

This is particularly interesting as the exact value of such simple functions of the G-Bronwian motion is not known explicitly. This issue is treated in detail in [34], where the author studies expressions of the type $E_G\left[B_t^{2n+1}\right]$, for $n \in \mathbb{N}$, although without obtaining a close form solution.

Corollary 2.1.24. *For each $t \in [0,T]$ it holds that*

$$E_G[B_t\langle B\rangle_t] \leq E_G\left[\int_0^t 2G(B_s)ds\right] = \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{\sqrt{2\pi}}\overline{\sigma}\frac{2}{3}t^{3/2}. \tag{2.1.16}$$

Proof. Because of Proposition 2.1.23 the only thing to prove is the equality in (2.1.16). To this end we use an approximation argument and let $\{t_i\}_{i=0,\dots,n}$ be a partition of $[0,t]$ with $t_0 = 0$, $t_n = t$ and $t_i - t_{i-1} = \frac{t}{n}$ for each $i = 1,\dots,n$. The simple process $B^n \in M_G^{1,0}(0,t)$, where

$$B_t^n := \sum_{i=1}^n B_{t_{i-1}} \mathbf{1}_{[t_{i-1},t_i)}(t)$$

converges in $M_G^1(0,t)$ to B. In fact, it can be directly verified that

$$E_G\left[\left|\int_0^t (B_s-B_s^n)ds\right|\right] \le E_G\left[\int_0^t |B_s-B_s^n|ds\right] \le n\int_0^{t_1} E_G\left[|B_s|\right]ds$$
$$= n\int_0^{t_1} \frac{\overline{\sigma}\sqrt{2s}}{\sqrt{\pi}}ds = \frac{\overline{\sigma}\sqrt{2}}{\sqrt{\pi}}\frac{2}{3}\left(\frac{t}{n}\right)^{\frac{3}{2}} n \xrightarrow[n\to\infty]{} 0, \qquad (2.1.17)$$

where we exploited the stationarity of the increments of the G-Brownian motion and Example 19 from [56] to deduce $E_G\left[|B_s|\right] = E_{\mathbb{P}}[|N(0,\overline{\sigma}^2 s)|]$. This can be used to obtain the following convergence

$$\sum_{i=1}^n G(B_{t_{i-1}})(t_i-t_{i-1}) \xrightarrow[n\to\infty]{L_G^1(0,t)} \int_0^t G(B_s)ds.$$

Indeed we have

$$E_G\left[\left|\int_0^t G(B_s)ds - \sum_{i=1}^n G(B_{t_{i-1}})(t_i-t_{i-1})\right|\right]$$
$$E_G\left[\left|\int_0^t ((B_s^+\overline{\sigma}^2 - B_s^-\underline{\sigma}^2)ds - \sum_{i=1}^n (B_{t_{i-1}}^+\overline{\sigma}^2 - B_{t_{i-1}}^-\underline{\sigma}^2)(t_i-t_{i-1})\right|\right]$$
$$= E_G\left[\left|\int_0^t \overline{\sigma}^2(B_s^+ - (B_s^n)^+)ds - \int_0^t \underline{\sigma}^2(B_s^- - (B_s^n)^-)ds\right|\right]$$
$$\le E_G\left[\left|\int_0^t \overline{\sigma}^2(B_s^+ - (B_s^n)^+)ds\right|\right] + E_G\left[\left|\int_0^t \underline{\sigma}^2(B_s^- - (B_s^n)^-)ds\right|\right]$$
$$\le E_G\left[\int_0^t \left|\overline{\sigma}^2(B_s^+ - (B_s^n)^+)\right|ds\right] + E_G\left[\int_0^t \left|\underline{\sigma}^2(B_s^- - (B_s^n)^-)\right|ds\right], \qquad (2.1.18)$$

and both terms in (2.1.18) converge to zero as n goes to infinity since $|X^+ - Y^+| \le |X-Y|$, $|X^- - Y^-| \le |X-Y|$ and because of (2.1.17). We next compute

$$E_G\left[\sum_{i=1}^n 2G(B_{t_{i-1}})(t_i-t_{i-1})\right]$$
$$= E_G\left[\sum_{i=0}^{n-1} 2G(B_{t_i-1})(t_i-t_{i-1}) + E_G\left[2G(B_{t_{n-1}})(t_n-t_{n-1})\middle|\mathscr{F}_{t_{n-2}}\right]\right]. \qquad (2.1.19)$$

To evaluate the conditional expectation in (2.1.19) we remark that

$$E_G\left[2G(B_{t_{n-1}})(t_n-t_{n-1})\middle|\mathscr{F}_{t_{n-2}}\right] = f(B_{t_{n-2}}),$$

where

$$\begin{aligned}
f(x) &:= E_G\left[2G(B_{t_{n-1}} - B_{t_{n-2}} + x)(t_n - t_{n-1})\middle|\mathscr{F}_{t_{n-2}}\right] \\
&= E_G\left[2G(B_{t_{n-1}} - B_{t_{n-2}} + x)(t_n - t_{n-1})\right] \\
&= E_{\mathbb{P}^{\overline{\sigma}}}\left[2G(B_{t_{n-1}} - B_{t_{n-2}} + x)(t_n - t_{n-1})\right]
\end{aligned}$$

being $2G(B_{t_{n-1}} - B_{t_{n-2}} + x)(t_n - t_{n-1})$ a convex function of $B_{t_{n-1}} - B_{t_{n-2}}$. We continue by induction and let $n \to \infty$ to obtain

$$\begin{aligned}
E_G&\left[\int_0^t 2G(B_s)ds\right] = E_{\mathbb{P}^{\overline{\sigma}}}\left[\int_0^t 2G(B_s)ds\right] \\
&= \int_0^t \left(\overline{\sigma}^2 E_{\mathbb{P}}\left[\left(\int_0^s \overline{\sigma} dW_u\right)^+\right] - \underline{\sigma}^2 E_{\mathbb{P}}\left[\left(\int_0^s \overline{\sigma} dW_u\right)^-\right]\right) ds \\
&= \int_0^t \left(\overline{\sigma}^2 \left(\overline{\sigma} \int_0^\infty x \frac{1}{\sqrt{2\pi}} \sqrt{s} e^{-\frac{x^2}{2}} dx\right) + \right. \\
&\qquad\qquad \left. + \underline{\sigma}^2 \left(\overline{\sigma} \int_{-\infty}^0 x \frac{1}{\sqrt{2\pi}} \sqrt{s} e^{-\frac{x^2}{2}} dx\right)\right) ds \\
&= \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{\sqrt{2\pi}} \overline{\sigma} \int_0^t \sqrt{s} ds = \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{\sqrt{2\pi}} \overline{\sigma} \frac{2}{3} t^{3/2}.
\end{aligned}$$

To conclude we remark that analogously we can get

$$\begin{aligned}
E_G\left[-B_t \langle B \rangle_t\right] &= E_G\left[-\int_0^t B_s d\langle B \rangle_s\right] \le E_G\left[-\int_0^t G(-B_s)ds\right] \\
&= E_{\mathbb{P}^{\overline{\sigma}}}\left[\int_0^t 2G(-B_s)ds\right] = \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{\sqrt{2\pi}} \overline{\sigma} \frac{2}{3} t^{3/2},
\end{aligned}$$

which is exactly $E_G\left[\int_0^t 2G(B_s)ds\right]$. □

2.2 Robust Mean-Variance Hedging

We outline here our analysis of quadratic hedging techniques in the G-setting by first studying the optimal mean-variance hedging problem. We start by describing the market model and formulating the aim of robust mean-variance hedging. We then provide an upper and lower bound for the objective function as well as the explicit solutions for a broad class of contingent claims.

We place ourselves in the setting outlined in Section 2.1. For the reader's convenience we remind that T is a finite fixed time horizon and the measurable space

$(\Omega, \mathscr{F})$ is such that $\Omega := \{\omega \in C([0,T], \mathbb{R}) : \omega(0) = 0\}$ and $\mathscr{F} = \mathscr{F}_T$, where $\mathbb{F} := \{\mathscr{F}_t\}_{t\in[0,T]}$ is the natural filtration of the canonical process B. Although for simplicity we decided to consider the case $d = 1$ of a one dimensional G-Brownian motion, all the results of this section are valid also for $d > 1$.

Remark 2.2.1. The choice of this particular measurable space is done without loss of generality as explained in the next lemma from [66]. In the following, for any probability measure $\mathbb{P}$ on $(\Omega, \mathscr{F})$, we denote with $\bar{\mathbb{F}}^{\mathbb{P}} := \{\bar{\mathscr{F}}_t^{\mathbb{P}}\}_{t\in[0,T]}$ the $\mathbb{P}$-augmented filtration.

Lemma 2.2.2. *For any $\bar{\mathscr{F}}_t^{\mathbb{P}}$-measurable random variable ξ, there exists a unique ($\mathbb{P}$-a.s.) $\mathscr{F}_t$-measurable random variable $\tilde{\xi}$ such that $\tilde{\xi} = \xi$, $\mathbb{P}$-a.s.. Similarly, for every $\bar{\mathscr{F}}_t^{\mathbb{P}}$-progressively measurable process X, there exists a unique $\mathscr{F}_t$-progressively measurable process $\tilde{X}$ such that $\tilde{X} = X$, $dt \otimes d\mathbb{P}$-a.e.. Moreover, if X is $\mathbb{P}$-almost surely continuous, then one can choose $\tilde{X}$ to be $\mathbb{P}$-almost surely continuous.*

Let $(\gamma_t)_{t\in[0,T]}$ be the discounted risk-free bank account. We take into account the following discounted assets

$$\begin{cases} dX_t = X_t dB_t, & X_0 > 0, \\ d\gamma_t = 0, & \gamma_0 = 1. \end{cases}$$

Remark 2.2.3. The discounted asset X is a symmetric G-martingale. There are different reasons for this choice. In the classical literature on quadratic hedging the market model always satisfies the NFLVR condition. In [73], which was the main reference for arbitrage free pricing in the G-setting at the beginning of this thesis, the author studied a notion of robust no arbitrage, assuming the asset to be a symmetric G-martingale. It was then clear that, in order to be consistent with the existing results on mean-variance hedging, we needed to enforce local martingale dynamics under every prior $\mathbb{P} \in \mathscr{P}_{\mathbf{G}}$. As the concept of stopping times in the G-setting was still not well-studied, the choice of X being a true martingale for every probability measure in $\mathscr{P}_{\mathbf{G}}$ seemed the most suitable for our setting.
Another reason is the fact that, in the classical literature, the analysis of mean-variance hedging and risk minimization starts with the study of the local martingale case. In such context the two quadratic hedging approaches are in fact equivalent. As we aimed at extending the classical results on the G-setting it was then natural to assume X to be a symmetric martingale.

As it is done in the classical case in [64], we consider the following family of trading strategies.

Definition 2.2.4. A trading strategy $\Upsilon = (\phi_t, \varsigma_t)_{t\in[0,T]}$ is called admissible if $(\phi_t)_{t\in[0,T]} \in \Phi$, where

$$\Phi := \left\{ \phi \text{ predictable } \middle| E_G\left[\left(\int_0^T \phi_t X_t dB_t\right)^2\right] < \infty \right\}, \tag{2.2.1}$$

ς is adapted, and it is self-financing, i.e.

$$V_t(\Upsilon) = \varsigma_t \gamma_t + \phi_t X_t = V_0 + \int_0^t \phi_s dX_s, \qquad \forall\, t \in [0,T].$$

For all $t \in [0,T]$ we can write $V_t(\Upsilon) = V_t(V_0, \phi)$, as the couple (V_0, ϕ) determines the value of any of such portfolios Υ at time t.

We study a form of quadratic hedging of claims $H \in L_G^{2+\varepsilon}(\mathscr{F}_T)$, for an $\varepsilon > 0$, by means of admissible portfolios. When a unique probability measure exists $H \in L^2_{\mathbb{P}}$ provides enough tractability to solve the hedging problem in a satisfactory way. In our multiple prior context intuition suggests that some extra regularity is required. The condition $H \in L_G^{2+\varepsilon}(\mathscr{F}_T)$ comes in particular from the need to be able to exploit the G-martingale representation theorem. As a contingent claim can be replicated by means of admissible portfolios if and only if it is symmetric, in the general case the principle of robust mean-variance hedging is to minimize

$$J_0(V_0, \phi) := E_G\left[(H - V_T(V_0, \phi))^2\right] = \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}\left[(H - V_T(V_0, \phi))^2\right] \tag{2.2.2}$$

by the choice of (V_0, ϕ). Alternatively stated we wish to solve

$$\inf_{(V_0,\phi)\in\mathbb{R}_+\times\Phi} J_0(V_0, \phi) = \inf_{(V_0,\phi)\in\mathbb{R}_+\times\Phi} E_G\left[(H - V_T(V_0, \phi))^2\right], \tag{2.2.3}$$

similarly to the treatment of [64] in the classical single prior market. In the case there exists an optimal solution $(V_0^*, \phi^*) \in \mathbb{R}_+ \times \Phi$ for (2.2.3), we name ϕ^* optimal mean-variance strategy associate to the optimal mean-variance portfolio value

$$V_t = V_0^* + \int_0^t \phi_s^* dX_s, \quad t \in [0,T].$$

When $\mathscr{P}_{\mathbf{G}} = \{\mathbb{P}\}$ as in [64], the solution to the mean-variance problem relies on the structure of the space of $L^2_{\mathbb{P}}$-integrable martingales. The *orthogonality* and *stability* properties (see Definition 2.3.19 and Definition 2.3.20) are fundamental and, thanks to the Galtchouk-Kunita-Watanabe decomposition, projecting H onto the linear space $\{x + \int_0^T \phi_s dX_s \mid x \in \mathbb{R} \text{ and } \phi \in \Phi\}$ is enough to solve the problem (for more on this in the classical case we refer again to [64]).
All those characteristics are not preserved in the G-setting. The functional (2.2.3) itself naturally introduces a stochastic game between the agent and the market: the

investor chooses the best possible portfolio while the market displays the worst case scenario. Finally the symmetric criterion (2.2.2) makes the problem equivalent from the buyer and seller perspectives, in a way that the optimal strategies for the hedging of H and $-H$ coincide. From this reasoning we can immediately argue that the G-martingale representation does not provide in general the optimal investment strategy through θ from (2.1.9). This is because, as shown in Lemma 2.1.19, the processes coming from the decomposition of H and $-H$ are generally different. Yet a straightforward application of Theorem 2.1.18 can actually provide some insights.

Lemma 2.2.5. *The initial wealth V_0^* of the optimal mean-variance portfolio lies in the interval $[-E_G[-H], E_G[H]]$.*

Proof. The first step is considering the G-martingale representation of H and $-H$

$$\begin{aligned} H &= E_G[H] + \int_0^T \theta_s dB_s - K_T, \\ -H &= E_G[-H] + \int_0^T \bar{\theta}_s dB_s - \bar{K}_T, \end{aligned} \tag{2.2.4}$$

where θ, $\bar{\theta}$, K and $\bar{K}$ are appropriate processes, as in Theorem 2.1.18. Hence it holds

$$\begin{aligned} &E_G\left[\left(H - V_0 - \int_0^T \phi_s X_s dB_s\right)^2\right] \\ =&E_G\left[\left(E_G[H] - V_0 + \int_0^T (\theta_s - \phi_s X_s) dB_s - K_T\right)^2\right] \\ =&(E_G[H] - V_0)^2 + E_G\Bigg[\left(\int_0^T (\theta_s - \phi_s X_s)\, dB_s - K_T\right)^2 + \\ &\qquad - 2K_T(E_G[H] - V_0)\Bigg], \end{aligned} \tag{2.2.5}$$

and analogously

$$\begin{aligned}
&E_G\left[\left(-H+V_0+\int_0^T \phi_s X_s dB_s\right)^2\right] \\
=&E_G\left[\left(E_G[-H]+V_0+\int_0^T (\bar{\theta}_s+\phi_s X_s) dB_s-\bar{K}_T\right)^2\right] \\
=&(E_G[-H]+V_0)^2+E_G\left[\left(\int_0^T \left(\bar{\theta}_s+\phi_s X_s\right) dB_s-\bar{K}_T\right)^2+\right. \\
&\left. \qquad\qquad -2\bar{K}_T(E_G[-H]+V_0)\right],
\end{aligned}$$

thanks to Proposition 2.1.8 and the characteristics of stochastic integrals with respect to the G-Brownian motion. It is now immediate from the expressions just obtained to argue that, being K_T and $\bar{K}_T$ strictly positive, the optimal initial wealth V_0^* must lie in the closed interval $[-E_G[-H], E_G[H]]$. □

When $E_G[H] = -E_G[-H]$, the result of Lemma 2.2.5 states that $V_0^* = E_G[H]$. This simply derives from the fact that in this case the claim is symmetric and then perfectly replicable, so that $\phi^* X = \theta$ as in the classical case. The statement of Lemma 2.2.5 can be sharpened using the results from [73], which allow us to conclude that, if $-E_G[-H] < E_G[H]$, the optimal initial wealth V_0^* should lay in the *open* interval $(-E_G[-H], E_G[H])$.

A similar argument holds true for the boundedness in the L_G^2-norm of the optimal mean-variance trading strategy.

Lemma 2.2.6. *Let be given a contingent claim $H \in L_G^{2+\varepsilon}(\mathscr{F}_T)$ with*

$$H = E_G[H] + \int_0^T \theta_s dB_s - K_T,$$

for some $\theta \in M_G^2(0,T)$ and $K_T \in L_G^2(\mathscr{F}_T)$. Then there exists a $R \in \mathbb{R}_+$ such that

$$\inf_{(V_0,\phi)\in\mathbb{R}_+\times\Phi} J_0(V_0,\phi) = \inf_{\substack{(V_0,\phi)\in\mathbb{R}_+\times\Phi \\ \|\int_0^T (\theta_s-\phi_s X_s)dB_s\|_2 \le R}} J_0(V_0,\phi).$$

Proof. As a first step we remark that $J(V_0^*,\phi^*)$ is clearly upper bounded as

$$J(V_0^*,\phi^*) \le E_G\left[H^2\right]. \tag{2.2.6}$$

We denote

$$A := E_G[H] - V_0 - K_T,$$
$$D := \int_0^T (\theta_s - \phi_s X_s) dB_s.$$

We can then obtain the following lower bound for $J(V_0, \phi)$

$$\begin{aligned} J(V_0,\phi) &= E_G\left[(A+D)^2\right] = E_G\left[A^2 + D^2 + 2AD\right] \\ &\geq E_G\left[D^2\right] - E_G\left[-A^2\right] - E_G[-2AD] \\ &\geq E_G\left[D^2\right] - E_G\left[-A^2\right] - 2E_G\left[A^2\right]^{\frac{1}{2}} E_G\left[D^2\right]^{\frac{1}{2}}. \end{aligned}$$

This is enough to complete the proof as the lower bound for the terminal risk $J(V_0, \phi)$ is an increasing function of $E_G\left[D^2\right]$. It is therefore clear that a big L_G^2 distance of $\int_0^T \phi_s X_s dB_s$ from $\int_0^T \theta_s dB_s$ prevents $J(V_0, \phi)$ from being smaller than the upper bound in (2.2.6). □

Theorem 2.2.7. *Let be given a claim $H \in L_G^{2+\varepsilon}(\mathscr{F}_T)$ and a sequence of random variables $(H^n)_{n\in\mathbb{N}}$ such that $\|H - H^n\|_{2+\varepsilon} \to 0$ as $n \to \infty$. Then as $n \to \infty$ we have*

$$J_n^* \to J^*,$$

where, for every $n \in \mathbb{N}$,

$$J_n^* := \inf_{(V_0,\phi)\in\mathbb{R}_+\times\Phi} E_G\left[(H^n - V_T(V_0,\phi))^2\right]$$

and

$$J^* := \inf_{(V_0,\phi)\in\mathbb{R}_+\times\Phi} E_G\left[(H - V_T(V_0,\phi))^2\right].$$

Proof. Without loss of generality we make the assumption that H can be represented as in (2.2.4) and that for every $n \in \mathbb{N}$ it holds

$$H^n = E_G[H^n] + \int_0^T \theta_s^n dB_s - K_T^n,$$

for a $\theta^n \in M_G^2(0,T)$ and $K_T^n \in L_G^2(\mathscr{F}_T)$. We then fix an arbitrary trading strategy (V_0, ϕ) and look at the following convergence

$$E_G\left[(H^n - V_T(V_0,\phi))^2\right] \to E_G\left[(H - V_T(V_0,\phi))^2\right]. \tag{2.2.7}$$

The first step of the proof is showing that it is enough to look at the convergence in (2.2.7) for a class of trading strategies which is bounded in L_G^2. As a consequence of Theorem 4.5 in [69], if the sequence $(H^n)_{n\in\mathbb{N}}$ converges in L_G^2 to H then

$$\|\int_0^T (\theta_s^n - \theta_s)\, dB_s\|_2 \to 0$$

and $\|K_T^n - K_T\|_2 \to 0$ as $n \to \infty$. Because of these results, together with Lemma 2.2.5 and Lemma 2.2.6, we can consider an $R \in \mathbb{R}_+$ for which

$$J_n^* = \inf_{\substack{(V_0,\phi)\in\mathbb{R}_+\times\Phi \\ \|V_0+\int_0^T(\theta_s-\phi_s X_s)dB_s\|_2\le R}} E_G\left[(H^n - V_T(V_0,\phi))^2\right]$$
$$J^* = \inf_{\substack{(V_0,\phi)\in\mathbb{R}_+\times\Phi \\ \|V_0+\int_0^T(\theta_s-\phi_s X_s)dB_s\|_2\le R}} E_G\left[(H - V_T(V_0,\phi))^2\right].$$

At the same time we have then established the convergence

$$E_G\left[(H^n - \cdot)^2\right] \to E_G\left[(H - \cdot)^2\right]$$

on the family of portfolios $(V_0,\phi) \in \mathbb{R}_+ \times \Phi$ such that $\|V_0 + \int_0^T (\theta_s - \phi_s X_s) dB_s\|_2 \le R$. If we indicate with $x := V_0 + \int_0^T \phi_s X_s dB_s$ any of such portfolios, for any $\delta > 0$ there exists $\bar{n} \in \mathbb{N}$ such that

$$\begin{aligned}
&\left|E_G\left[(H^n - x)^2\right] - E_G\left[(H-x)^2\right]\right| \le \left|E_G\left[(H^n-x)^2 - (H-x)^2\right]\right| \\
\le& E_G\left[\left|(H^n-x)^2 - (H-x)^2\right|\right] = E_G\left[\left|(H^n - H)(H^n + H - 2x)\right|\right] \\
\le& E_G\left[(H^n - H)^2\right]^{\frac{1}{2}} E_G\left[(H^n + H - 2x)^2\right]^{\frac{1}{2}} \\
\le& E_G\left[(H^n - H)^2\right]^{\frac{1}{2}} \left(E_G\left[(H^n+H)^2\right]^{\frac{1}{2}} + E_G\left[(2x)^2\right]^{\frac{1}{2}}\right) < \delta, \qquad (2.2.8)
\end{aligned}$$

for all $n > \bar{n}$. This is evident once noticing the boundedness of the second term in (2.2.8). We argue that (2.2.8) also proves the uniform convergence, as all the inequalities in (2.2.8) remain valid also upon considering the supremum of x over the set $\|x\|_2 \le R$. We have now the means to prove the main statement. Fixed an arbitrary $\delta > 0$, because of the definition of J^*, we can select a $(\bar{V}_0, \bar{\phi}) \in \mathbb{R}_+ \times \Phi$ such that $\|\bar{V}_0 + \int_0^T (\theta_s - \bar{\phi}_s X_s) dB_s\|_2 \le R$ and

$$J^* + \delta \ge E_G\left[\left(H - \bar{V}_0 - \int_0^T \bar{\phi}_s X_s dB_s\right)^2\right]. \qquad (2.2.9)$$

In addition we can take n big enough to guarantee

$$\left| E_G\left[\left(H-\bar{V}_0-\int_0^T \bar{\phi}_s X_s dB_s\right)^2\right] - E_G\left[\left(H^n-\bar{V}_0-\int_0^T \bar{\phi}_s X_s dB_s\right)^2\right]\right| < \delta, \tag{2.2.10}$$

because of the uniform convergence in (2.2.8). It then follows from (2.2.9) and (2.2.10) that

$$J^* + 2\delta \geq J_n^*. \tag{2.2.11}$$

In the same way we can individuate $(\tilde{V}_0, \tilde{\phi})$ satisfying

$$J_n^* + \delta \geq E_G\left[\left(H^n-\tilde{V}_0-\int_0^T \tilde{\phi}_s X_s dB_s\right)^2\right]$$

and

$$\left| E_G\left[\left(H-\tilde{V}_0-\int_0^T \tilde{\phi}_s X_s dB_s\right)^2\right] - E_G\left[\left(H^n-\tilde{V}_0-\int_0^T \tilde{\phi}_s X_s dB_s\right)^2\right]\right| < \delta,$$

which allows us to conclude that

$$J_n^* \geq J^* - 2\delta. \tag{2.2.12}$$

The expression in (2.2.11) and (2.2.12) complete the proof as δ is arbitrary and

$$J^* - 2\delta \leq J_n^* \leq J^* + 2\delta$$

□

Thanks to the results of Theorem 2.2.7, we start our examination of the mean-variance hedging problem by first focusing on claims belonging to $L_{ip}(\mathscr{F}_T)$. This is reasonable as any $H \in L_G^{2+\varepsilon}(\mathscr{F}_T)$, which is the most general claim we can consider in this setting, can be expressed as the limit in $L_G^{2+\varepsilon}$ of random variables in $L_{ip}(\mathscr{F}_T)$. Furthermore, as reminded in Theorem 2.1.17, this space present the important advantage of providing an additional decomposition of the term K_T coming from their representation since

$$-K_T = \int_0^T \eta_s d\langle B\rangle_s - 2\int_0^T G(\eta_s)ds, \tag{2.2.13}$$

for a process $\eta \in M_G^1(0,T)$. We therefore study claims $H \in L_G^{2+\varepsilon}(\mathscr{F}_T)$ with representation

$$\begin{aligned} H &= E_G[H] + \int_0^T \theta_s dB_s - K_T(\eta) \\ &= E_G[H] + \int_0^T \theta_s dB_s + \int_0^T \eta_s d\langle B\rangle_s - 2\int_0^T G(\eta_s)ds. \end{aligned} \tag{2.2.14}$$

The only assumption we will enforce on θ from (2.2.14) is $M_G^2(0,T)$-integrability, which is the minimum requirement following from the condition $H \in L_G^{2+\varepsilon}(\mathscr{F}_T)$. The tractability of η is more cumbersome and this is why we address the mean-variance hedging problem by steps, by enforcing at each time some additional assumptions. In particular we start by requiring η to be deterministic or maximally distributed. We then proceed treating some cases in which η is a stepwise process, with properties that we specify at each time. In all these cases we provide an explicit solution to (2.2.3). We then conclude by tackling the general case, estimating the minimal terminal risk.

2.3 Explicit Solutions

In the first two cases we address, the process η from (2.2.14) is assumed to be deterministic or a function of $\langle B \rangle$. This turns out to be particularly helpful as it allows to hedge uncertainty away just through the initial endowment V_0, without relying on ϕ. This is a consequence of the fact that the component K_T of the representation of H does not show ambiguity coming from a direct dependence on B. The optimal solutions for this cases are presented in Theorem 2.3.1 and Theorem 2.3.5.

2.3.1 Deterministic η

We address the situation in which the process η from (2.2.14) is deterministic and specify the optimal mean-variance portfolio.

Theorem 2.3.1. *Consider a claim $H \in L_G^{2+\varepsilon}(\mathscr{F}_T)$ of the following form*

$$H = E_G[H] + \int_0^T \theta_s dB_s + \int_0^T \eta_s d\langle B \rangle_s - \int_0^T 2G(\eta_s)ds, \tag{2.3.1}$$

where $\theta \in M_G^2(0,T)$ and $\eta \in M_G^1(0,T)$ is a deterministic process. The optimal mean-variance portfolio is given by

$$\phi_t^* X_t = \theta_t$$

for every $t \in [0,T]$ and

$$V_0^* = \frac{E_G[H] - E_G[-H]}{2}.$$

Proof. The first step is computing the set of all possible values attained by

$$E_G[H] + \int_0^T \eta_s d\langle B \rangle_s - \int_0^T 2G(\eta_s)ds.$$

We claim that the desired span is equal to the interval $[E_G[H]-(\overline{\sigma}^2-\underline{\sigma}^2)\int_0^T|\eta_s|ds, E_G[H]]$. Let us start by studying the upper bound, remarking that for the particular volatility scenario described by

$$\tilde{\sigma}_t = \begin{cases} \overline{\sigma}^2 & \text{if } \eta_t \geq 0, \\ \underline{\sigma}^2 & \text{if } \eta_t < 0, \end{cases}$$

for each $t \in [0,T]$, the expression $\int_0^T \eta_s d\langle B\rangle_s - \int_0^T 2G(\eta_s)ds$ is $\mathbb{P}^{\tilde{\sigma}}$-a.s. equal to zero. Hence it holds $E_{\mathbb{P}^{\tilde{\sigma}}}[H] = E_G[H]$. To obtain the lower bound we look at

$$\tilde{\sigma}'_t = \begin{cases} \overline{\sigma}^2 & \text{if } \eta_t \leq 0, \\ \underline{\sigma}^2 & \text{if } \eta_t > 0, \end{cases}$$

for each $t \in [0,T]$. The volatility $\tilde{\sigma}'$ describes the scenario where $\int_0^T \eta_s d\langle B\rangle_s - \int_0^T 2G(\eta_s)ds$ attains its minimum. As a consequence we have $E_{\mathbb{P}^{\tilde{\sigma}'}}[H] = -E_G[-H]$. In fact, from (2.3.1),

$$\begin{aligned} -H &= -E_G[H] - \int_0^T \theta_s dB_s - \int_0^T \eta_s d\langle B\rangle_s + \int_0^T 2G(\eta_s)ds \\ &= -E_G[H] - \int_0^T \theta_s dB_s - \int_0^T \eta_s d\langle B\rangle_s + \int_0^T 2G(\eta_s)ds \\ &\qquad + \int_0^T 2G(-\eta_s)ds - \int_0^T 2G(-\eta_s)ds \\ &= -E_G[H] + \int_0^T (-\theta_s)dB_s + \int_0^T (-\eta_s)d\langle B\rangle_s - \int_0^T 2G(-\eta_s)ds \\ &\qquad + (\overline{\sigma}^2 - \underline{\sigma}^2)\int_0^T |\eta_s|ds, \end{aligned} \tag{2.3.2}$$

as

$$\int_0^T 2(G(\eta_s)+G(-\eta_s))ds = (\overline{\sigma}^2 - \underline{\sigma}^2)\int_0^T |\eta_s|ds.$$

Notice now how (2.3.2), since η is assumed to be deterministic, yields the G-martingale representation of $-H$. Therefore we can argue

$$-E_G[H] + (\overline{\sigma}^2 - \underline{\sigma}^2)\int_0^T |\eta_s|ds = E_G[-H]. \tag{2.3.3}$$

Then, thanks to Proposition 2.1.21 together with Lemma 2.1.22, we obtain

$$\inf_{(V_0,\phi)} E_G\Big[(E_G[H]-V_0+\int_0^T(\theta_s-\phi_sX_s)dB_s+\int_0^T\eta_s d\langle B\rangle_s+$$
$$-\int_0^T 2G(\eta_s)ds)^2\Big]$$
$$\geq \inf_{(V_0,\phi)}\Bigg(E_G\Big[E_G[H]-V_0+\int_0^T(\theta_s-\phi_sX_s)dB_s+\int_0^T\eta_s d\langle B\rangle_s+$$
$$-\int_0^T 2G(\eta_s)ds\Big]^2\vee E_G\Big[-E_G[H]+V_0-\int_0^T(\theta_s-\phi_sX_s)dB_s+$$
$$-\int_0^T\eta_s d\langle B\rangle_s+\int_0^T 2G(\eta_s)ds\Big]^2\Bigg)$$
$$=\inf_{V_0}\Bigg(E_G\Big[E_G[H]-V_0+\int_0^T\eta_s d\langle B\rangle_s-\int_0^T 2G(\eta_s)ds\Big]^2\ \vee \tag{2.3.4}$$
$$E_G\Big[-E_G[H]+V_0-\int_0^T\eta_s d\langle B\rangle_s+\int_0^T 2G(\eta_s)ds\Big]^2\Bigg)$$
$$=\inf_{V_0}\Bigg(E_G\Big[E_G[H]-V_0+\int_0^T\eta_s d\langle B\rangle_s-\int_0^T 2G(\eta_s)ds\Big]^2\ \vee \tag{2.3.5}$$
$$E_G\Big[E_G[-H]+V_0+\int_0^T(-\eta_s)d\langle B\rangle_s-\int_0^T 2G(-\eta_s)ds\Big]^2\Bigg),$$

where the result of Proposition 2.1.8 was used in (2.3.4) and the expression (2.3.3) in (2.3.5). We can actually reformulate (2.3.5) as

$$\inf_{V_0}\Bigg(E_G\Big[E_G[H]-V_0\Big]^2\ \vee\ E_G\Big[E_G[-H]+V_0\Big]^2\Bigg), \tag{2.3.6}$$

since

$$E_G\Big[a+\int_0^T\xi_s d\langle B\rangle_s-\int_0^T 2G(\xi_s)ds\Big]=$$
$$=\ a+E_G\Big[\int_0^T\xi_s d\langle B\rangle_s-\int_0^T 2G(\xi_s)ds\Big]=a,$$

for $a\in\mathbb{R}$ and $\xi\in M_G^1(0,T)$. The minimum of (2.3.6) is reached when $V_0^*=$

$\frac{E_G[H]-E_G[-H]}{2}$ and it is $\left(\frac{E_G[H]+E_G[-H]}{2}\right)^2$. We complete the proof by showing that

$$E_G\left[\left(E_G[H]-V_0^*+\int_0^T \eta_s d\langle B\rangle_s-\int_0^T 2G(\eta_s)ds\right)^2\right] \\ =\left(\frac{E_G[H]+E_G[-H]}{2}\right)^2.$$

Since the span of

$$E_G[H]+\int_0^T \eta_s d\langle B\rangle_s-\int_0^T 2G(\eta_s)ds$$

is the closed interval with extremes $E_G[H]-(\overline{\sigma}^2-\underline{\sigma}^2)\int_0^T|\eta_s|ds=-E_G[-H]$ and $E_G[H]$, it is evident that $\frac{E_G[H]+E_G[-H]}{2}$ is the maximum of

$$|E_G[H]-V_0^*+\int_0^T \eta_s d\langle B\rangle_s-\int_0^T 2G(\eta_s)ds|$$

under the constraint $V_0^*\in[-E_G[-H],E_G[H]]$. This completes the claim. □

Remark 2.3.2. Notice that the fraction $\phi^*=\frac{\theta}{X}$ which represents the investment strategy of the optimal portfolio is well defined as X is q.s. strictly positive being a geometric G-Brownian motion. In addition note that, since

$$\int_0^T \eta_s d\langle B\rangle_s-\int_0^T 2G(\eta_s)ds=-K_T,$$

we have

$$V_0^*-E_G[H]=\frac{E_G[K_T]}{2},$$

as

$$E_G[K_T]=E_G\left[E_G[H]+\int_0^T \theta_s dB_s-H\right]=E_G[H]+E_G[-H].$$

Remark 2.3.3. Note how Theorem 2.3.1 is coherent with the results on mean-variance hedging in the classical setting. In fact when $\mathscr{P}_\mathbf{G}=\{\mathbb{P}\}$, which means $\overline{\sigma}=\underline{\sigma}$ and $E_G[H]=-E_G[-H]$, the optimal initial wealth is simply $E_\mathbb{P}[H]=E_G[H]$ and the optimal strategy is simply the integrand appearing in the martingale representation.

The class of random variables with decomposition (2.3.1) and η deterministic is significant. In fact given an arbitrary deterministic process $\eta\in M_G^1(0,T)$, any constant $c\in\mathbb{R}$ and any process $\theta\in M_G^2(0,T)$, we can produce the contingent claim

$$H:=c+\int_0^T \theta_s dB_s+\int_0^T \eta_s d\langle B\rangle_s-\int_0^t 2G(\eta_s)ds,$$

which satisfies the assumptions of Theorem 2.3.1. The relevance of such family of claims becomes clear once noticing that its intersection with $L_{ip}(\mathscr{F}_T)$ contains the second degree polynomials in $(B_{t_0}, B_{t_1} - B_{t_0}, \dots, B_{t_n} - B_{t_{n-1}})$, where $\{t_i\}_{i=0}^n$ is a partition of $[0,T]$. To see this we let $n = 1$ and examine the claims which are functions of one single increment of the G-Brownian motion. The processes of the representation of $H = \varphi(B_T - B_0)$ are

$$\eta_t(\omega) = \partial_x^2 u(t,\omega)$$

and

$$\theta_t(\omega) = \partial_x u(t,\omega),$$

where u solves

$$\begin{cases} \partial_t u + G(\partial_x^2 u) = 0, \\ u(T,x) = \varphi(x), \end{cases}$$

for $(t,x) \in [0,T] \times \mathbb{R}$ (see [55] for a reference). When η is deterministic, the partial derivative $\partial_x^2 u(t,\omega)$ depends only on t, i.e. $a(t) := \partial_x^2 u(t,\omega)$. Hence, upon integrating with respect to x, we get that

$$u(t,x) = \frac{a(t)}{2}x^2 + b(t)x + d(t),$$

for some real functions b and d, so that

$$H = \frac{a(T)}{2}B_T^2 + b(T)B_T + d(T).$$

Remark 2.3.4. We can use the results of Theorem 4.1 in [73] to display another family of contingent claims that can be hedged thanks to Theorem 2.3.1. Fixed a real valued Lipschitz function Φ, if we study the case where $H = \Phi(X_T)$, we can argue that (we refer to [73] for more details)

$$\begin{aligned} \Phi(X_T) = E_G\left[\Phi(X_T)\right] + \int_0^T \partial_x u(t,X_t) X_t dB_t \\ + \frac{1}{2}\int_0^T \partial_x^2 u(t,X_t) X_t^2 d\langle B\rangle_t - \int_0^T G(\partial_x^2 u(t,X_t)) X_t^2 dt, \end{aligned}$$

where u is the solution to

$$\begin{cases} \partial_t u + G(x^2 \partial_x^2 u) = 0, \\ u(T,x) = \Phi(x). \end{cases}$$

We can then prove that $\partial_x^2 u(t,X_t)X_t^2$ is a deterministic function for every $t \in [0,T]$ if and only if

$$H = \Phi(X_T) = u(T,X_T) = a(T)\log X_T + b(T)X_T + c(T),$$

for some real functions a, b and c.
In the case the market exhibits another risky asset X', which the agent cannot trade, solving

$$dX'_t = \alpha(X'_t)dB_t, \qquad X'_0 > 0,$$

for some Lipschitz function α, we can modify the previous reasoning and use again Theorem 2.3.1 to optimally hedge every contingent claim $\Phi(X'_T)$, where Φ is a Lipschitz function satisfying

$$\begin{cases} \partial_t u + G(\alpha^2(x)\partial_x^2 u) = 0, \\ u(T,x) = \Phi(x), \end{cases}$$

under the constraint that $\partial_x^2 u(t,x) = \frac{1}{\alpha(x)}$ for every $(t,x) \in [0,T] \times \mathbb{R}$.

Before proceeding with the next results we highlight that Theorem 2.3.1 already shows that not all claims can be perfectly hedged by means of admissible self-financing strategies, thus ruling out the possibility to obtain in the G-setting a chaotic expansion of L_G^2-integrable random variables, as claimed in [13].

2.3.2 Maximally Distributed η

We now move to the second case in which we are able to obtain a complete characterization of the minimal mean-variance portfolio. We assume here that η depends only on the quadratic variation of the G-Brownian motion, i.e. it exhibits only mean uncertainty.

Theorem 2.3.5. *Let $H \in L_G^{2+\varepsilon}(\mathscr{F}_T)$ be of the form*

$$H = E_G[H] + \int_0^T \theta_s dB_s + \int_0^T \psi(\langle B\rangle_s)d\langle B\rangle_s - 2\int_0^T G(\psi(\langle B\rangle_s))ds,$$

where $\theta \in M_G^2(0,T)$ and $\psi : \mathbb{R} \to \mathbb{R}$ is such that there exist $k \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$ for which

$$|\psi(x) - \psi(y)| \le \alpha|x-y|^k,$$

for all $x, y \in \mathbb{R}$. The optimal mean-variance portfolio is given by

$$\phi_t^* X_t = \theta_t$$

for every $t \in [0,T]$ and

$$V_0^* = \frac{E_G[H] - E_G[-H]}{2}.$$

Proof. As in the proof of Theorem 2.3.1, we apply the G-Jensen's inequality to obtain a lower bound for the terminal risk

$$
\begin{aligned}
& E_G\left[\left(c+\int_0^T \varphi_s dB_s+\int_0^T \psi(\langle B\rangle_s)d\langle B\rangle_s-2\int_0^T G(\psi(\langle B\rangle_s))ds\right)^2\right] \\
& \geq E_G\left[c+\int_0^T \varphi_s dB_s+\int_0^T \psi(\langle B\rangle_s)d\langle B\rangle_s-2\int_0^T G(\psi(\langle B\rangle_s))ds\right]^2 \vee \\
& \quad E_G\left[-c-\int_0^T \varphi_s dB_s-\int_0^T \psi(\langle B\rangle_s)d\langle B\rangle_s+2\int_0^T G(\psi(\langle B\rangle_s))ds\right]^2 \\
& = c^2 \vee \left(E_G\left[K_T\right]-c\right)^2,
\end{aligned}
\tag{2.3.7}
$$

where we defined

$$
\begin{aligned}
c &:= E_G\left[H\right]-V_0, \\
\varphi_t &:= \theta_t-\phi_t X_t,
\end{aligned}
\tag{2.3.8}
$$

for all $t \in [0,T]$. The expression (2.3.7) reaches its minimum when $c^* = \frac{E_G[K_T]}{2}$, which is equal to $\left(\frac{E_G[K_T]}{2}\right)^2$. We complete the proof by showing that this minimum is reached when letting $V_0^* = \frac{E_G[H]-E_G[-H]}{2}$ and $\phi_t^* X_t = \theta_t$. To this end we calculate

$$
E_G\left[\Big(\frac{E_G\left[K_T\right]}{2}+\int_0^T \psi(\langle B\rangle_s)d\langle B\rangle_s-2\int_0^T G(\psi(\langle B\rangle_s))ds\Big)^2\right]. \tag{2.3.9}
$$

To do that we use an approximation argument, noticing that

$$
\psi^n(\langle B\rangle) := \sum_{i=0}^{n-1} \psi(\langle B\rangle_{t_i})\mathbf{1}_{[t_i,t_{i+1})} \stackrel{M_G^2(0,T)}{\longrightarrow} \psi(\langle B\rangle), \tag{2.3.10}
$$

where $t_i = \frac{T}{n}i$. In fact it holds

$$
\begin{aligned}
& \int_0^T E_G\left[|\psi(\langle B\rangle_t)-\psi^n(\langle B\rangle_t)|^2\right]dt = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} E_G\left[|\psi(\langle B\rangle_t)-\psi^n(\langle B\rangle_t)|^2\right]dt \\
& \leq \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} E_G\left[|\langle B\rangle_t-\langle B\rangle_{t_i}|^{2k}\right]dt = n\int_0^{t_1} E_G\left[\langle B\rangle_t^{2k}\right]dt = n\int_0^{t_1} t^{2k}dt \\
& = \frac{n}{2k}\left(\frac{T}{n}\right)^{2k+1} \stackrel{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

The convergence $G(\psi^n(\langle B\rangle))$ to $G(\psi(\langle B\rangle))$ can be proved analogously. Therefore the G-expectation in (2.3.9) is the limit value for n going to infinity of

$$
\begin{aligned}
&E_G\left[\Big(\frac{E_G[K_T]}{2}+\sum_{i=0}^{n-1}\psi(\langle B\rangle_{t_i})\Delta\langle B\rangle_{t_{i+1}}-2\sum_{i=0}^{n-1}G(\psi(\langle B\rangle_{t_i}))\Delta t_{i+1}\Big)^2\right]\\
=&E_G\left[\Big(\frac{E_G[K_T]}{2}+\sum_{i=0}^{n-1}\psi\left(\sum_{j=0}^{i}\Delta\langle B\rangle_{t_j}\right)\Delta\langle B\rangle_{t_{i+1}}+\right.\\
&\left.\qquad\qquad -2\sum_{i=0}^{n-1}G\left(\psi\left(\sum_{j=0}^{i}\Delta\langle B\rangle_{t_j}\right)\right)\Delta t_{i+1}\Big)^2\right]\\
=&E_G\left[E_G\left[\Big(\frac{E_G[K_T]}{2}+\sum_{i=0}^{n-1}\psi\left(\sum_{j=0}^{i}\Delta\langle B\rangle_{t_j}\right)\Delta\langle B\rangle_{t_{i+1}}+\right.\right.\\
&\left.\left.\qquad\qquad -2\sum_{i=0}^{n-1}G\left(\psi\left(\sum_{j=0}^{i}\Delta\langle B\rangle_{t_j}\right)\right)\Delta t_{i+1}\Big)^2\middle|\mathscr{F}_{t_{n-1}}\right]\right]\\
=&E_G\left[\sup_{\underline{\sigma}^2\le v_n\le\overline{\sigma}^2}\Big(\frac{E_G[K_T]}{2}+\sum_{i=0}^{n-2}\psi\left(\sum_{j=0}^{i}\Delta\langle B\rangle_{t_j}\right)\Delta\langle B\rangle_{t_{i+1}}+\right.\\
&\left.\quad+\psi\left(\sum_{j=0}^{n-1}\Delta\langle B\rangle_{t_j}\right)v_n\Delta t_n-2\sum_{i=0}^{n-1}G\left(\psi\left(\sum_{j=0}^{i}\Delta\langle B\rangle_{t_j}\right)\right)\Delta t_{i+1}\Big)^2\right]\\
=&E_G\left[E_G\left[\sup_{\underline{\sigma}^2\le v_n\le\overline{\sigma}^2}\Big(\frac{E_G[K_T]}{2}+\sum_{i=0}^{n-2}\psi\left(\sum_{j=0}^{i}\Delta\langle B\rangle_{t_j}\right)\Delta\langle B\rangle_{t_{i+1}}+\right.\right.\\
&\left.\left.+\psi\left(\sum_{j=0}^{n-1}\Delta\langle B\rangle_{t_j}\right)v_n\Delta t_n-2\sum_{i=0}^{n-1}G\left(\psi\left(\sum_{j=0}^{i}\Delta\langle B\rangle_{t_j}\right)\right)\Delta t_{i+1}\Big)^2\middle|\mathscr{F}_{t_{n-2}}\right]\right]\\
=&E_G\left[\sup_{\substack{\underline{\sigma}^2\le v_n\le\overline{\sigma}^2\\ \underline{\sigma}^2\le v_{n-1}\le\overline{\sigma}^2}}\Big(\frac{E_G[K_T]}{2}+\sum_{i=0}^{n-3}\psi\left(\sum_{j=0}^{i}\Delta\langle B\rangle_{t_j}\right)\Delta\langle B\rangle_{t_{i+1}}+\right.\\
&\quad+\psi\left(\sum_{j=0}^{n-2}\Delta\langle B\rangle_{t_j}\right)v_{n-1}\Delta t_{n-1}+\psi\left(\sum_{j=0}^{n-2}\Delta\langle B\rangle_{t_j}+v_{n-1}\Delta t_{n-1}\right)v_n\Delta t_n+\\
&\quad-2G\left(\psi\left(\sum_{j=0}^{n-2}\Delta\langle B\rangle_{t_j}+v_{n-1}\Delta t_{n-1}\right)\right)\Delta t_n+\\
&\left.\quad-2\sum_{i=0}^{n-2}G\left(\psi\left(\sum_{j=0}^{i}\Delta\langle B\rangle_{t_j}\right)\right)\Delta t_{i+1}\Big)^2\right],
\end{aligned}
\tag{2.3.11}
$$

where we make use of the fact that $\Delta\langle B\rangle$ is maximally distributed. By carrying on the iteration (2.3.11) can be proved to be equal to

$$\sup_{\substack{\underline{\sigma}^2 \le v_i \le \overline{\sigma}^2 \\ i=1,\dots,n}} \left(\frac{E_G[K_T]}{2} + \sum_{i=0}^{n-1} \psi\left(\sum_{j=0}^{i} v_j \Delta t_j\right) v_{i+1}\Delta t_{i+1} + \right.$$

$$\left. -2\sum_{i=0}^{n-1} G\left(\psi\left(\sum_{j=0}^{i} v_j \Delta t_j\right)\right)\Delta t_{i+1}\right)^2. \tag{2.3.12}$$

In order to compute the supremum from the last expression, notice that (2.3.12) is a parabola in $(v_i)_{i=1,\dots,n}$, thus its maximum can be reached in the correspondence of two points: when the term depending on $(v_i)_{i=1,\dots,n}$ is minimum, which is equal to zero, or is maximum, which is

$$E_G\left[2\sum_{i=0}^{n-1} G(\psi(\langle B\rangle_{t_i}))\Delta t_{i+1} - \sum_{i=0}^{n-1} \psi(\langle B\rangle_{t_i})\Delta\langle B\rangle_{t_{i+1}}\right].$$

In either situation, when n goes to infinity, thanks to (2.3.10) the value of (2.3.12) converges to $\left(\frac{E_G[K_T]}{2}\right)^2$. □

Because of the symmetry of the mean-variance hedging criterion, the computation of the optimal portfolio for H is not conceptually different from that for $-H$ as the two solutions are equal up to a sign change. In other words, if (V_0^*, ϕ^*) is the strategy that minimally hedges H, $(-V_0^*, -\phi^*)$ is the optimal solution for $-H$. This makes evident how ϕ^* cannot in general be equal to the process θ appearing in the representation of H as of Theorem 2.1.18. The result of Theorem 2.3.5 does not constitute an exception to this intuition.

Remark 2.3.6. Following the proof of Lemma 2.1.19 it is not complicated to show that for random variables of the form

$$H = E_G[H] + \int_0^T \theta_s dB_s + \sum_{i=0}^{n-1} \left(\psi(\langle B\rangle_{t_i})\Delta\langle B\rangle_{t_{i+1}} - 2G(\psi(\langle B\rangle_{t_i}))\Delta t_{i+1}\right),$$

where $\theta \in M_G^2(0,T)$ and ψ is a real continuous function, the G-martingale representation of $-H$ is of the type

$$-H = E_G[-H] + \int_0^T (-\theta_s) dBs - \bar{K}_T,$$

for a suitable process $\bar{K}$ with $\bar{K}_T \in L_G^2(\mathscr{F}_T)$.

We can repeat the reasoning outlined in Remark 2.3.4 to give a characterization of the set of claims that can be represented as in (2.2.14) with a process η that is a function with polynomial growth of $\langle B\rangle$. This family embeds the class of Lipschitz function of $\langle B\rangle$. We can then exploit Theorem 2.3.5 to optimally hedge *volatility swaps*, i.e. $H=\sqrt{\langle B\rangle_T}-K$ with $K\in\mathbb{R}_+$, and other types of volatility derivatives (see [14] for a detailed analysis volatility derivatives). Fixed a Lipschitz function Φ, we can express $\Phi(\langle B\rangle_T)$ as

$$\Phi(\langle B\rangle_T)=E_G\left[\Phi(\langle B\rangle_T)\right]+\int_0^T\partial_x u(s,\langle B\rangle_s)\langle B\rangle_s d\langle B\rangle_s \\ -2\int_0^T G(\partial_x u(s,\langle B\rangle_s))\langle B\rangle_s ds,$$

where $u(t,x)$ is the solution to

$$\begin{cases}\partial_t u+2G(x\partial_x u)=0,\\ u(T,x)=\Phi(x),\end{cases}$$

because of the G-Itô formula (see [56]) and the nonlinear Feynman-Kac formula for G-Brownian motion (see [57]).

2.3.3 G-Martingale Decomposition of Notable Claims

In this section we develop the G-martingale decomposition of call options. We fix $D\in\mathbb{R}_+$ and consider the call option with strike D, whose payoff at time T is given by

$$C:=(X_T-D)_+.$$

It follows from Theorem 4.1 in [73] that C can be written as

$$C=E_G\left[C\right]+\int_0^T X_s\partial_x u(s,X_s)dB_s \\ +\frac{1}{2}\int_0^T\partial_x^2 u(s,X_s)X_s^2 d\langle B\rangle_s-\int_0^T G(\partial_x^2 u(s,X_s))X_s^2 ds,$$

where $u(t,x)=E_G\left[(X_T^{t,x}-D)_+\middle|\mathscr{F}_t\right]$, $X_t^{t,x}=x$. As the payoff of a call option is a convex function of the underlying, we deduce from Corollary 4.3 in [73] that $E_G\left[C\right]=E_{\mathbb{P}^{\overline{\sigma}}}[C]$ and

$$E_G\left[(X_T^{t,x}-D)_+\middle|\mathscr{F}_t\right]=E_{\mathbb{P}^{\overline{\sigma}}}\left[(X_T^{t,x}-D)_+\middle|\mathscr{F}_t\right]. \tag{2.3.13}$$

The value of the linear conditional expectation in (2.3.13) is well known and it is equal to

$$xN(d_1(x))-DN(d_2(x)) \tag{2.3.14}$$

where N stands for the cumulative distribution function of a standard normal random variable, while

$$d_1(x) = \frac{\log(x/D) + \frac{\overline{\sigma}^2}{2}(T-t)}{\overline{\sigma}\sqrt{T-t}} \quad \text{and} \quad d_2(x) = d_1(x) - \overline{\sigma}\sqrt{T-t}.$$

Also the derivative with respect to x of (2.3.14) is known to be equal to $N(d_1(x))$ (see [43]). Some easy computations allow us to obtain also the double derivative of (2.3.14) which is equal to $f(d_1(x))\frac{1}{x\overline{\sigma}\sqrt{T-t}}$, where f here denotes the density function of a standard normal random variable. All in all we obtain

$$\begin{aligned}
C &= E_{\mathbb{P}^{\overline{\sigma}}}[C] + \int_0^T X_s N(d_1(X_s))dB_s + \frac{1}{2}\int_0^T f(d_1(X_s))\frac{1}{X_s\overline{\sigma}\sqrt{T-s}}X_s^2 d\langle B\rangle_s \\
&\quad - \int_0^T G\left(f(d_1(X_s)\frac{1}{X_s\overline{\sigma}\sqrt{T-s}}\right)X_s^2 ds \\
&= E_{\mathbb{P}^{\overline{\sigma}}}[C] + \int_0^T X_s N(d_1(X_s))dB_s + \frac{1}{2}\int_0^T f(d_1(X_s))\frac{1}{\overline{\sigma}\sqrt{T-s}}X_s d\langle B\rangle_s \\
&\quad - \frac{1}{2}\int_0^T \overline{\sigma} f(d_1(X_s))\frac{1}{\sqrt{T-s}}X_s ds.
\end{aligned}$$

A similar argument can be applied to obtain the G-martingale decomposition of a put option or a straddle. We report here only the final result. Denoting

$$\begin{aligned}
P &:= (D - X_T)_+, \\
St &:= |X_T - D|,
\end{aligned}$$

it holds

$$\begin{aligned}
P &= E_{\mathbb{P}^{\overline{\sigma}}}[P] + \int_0^T X_s(N(d_1(X_s)) - 1)dB_s + \frac{1}{2}\int_0^T f(d_1(X_s))\frac{1}{\overline{\sigma}\sqrt{T-s}}X_s d\langle B\rangle_s \\
&\quad - \frac{1}{2}\int_0^T \overline{\sigma} f(d_1(X_s))\frac{1}{\sqrt{T-s}}X_s ds
\end{aligned}$$

and

$$\begin{aligned}
St &= E_{\mathbb{P}^{\overline{\sigma}}}[St] + \int_0^T X_s(2N(d_1(X_s)) - 1)dB_s + \int_0^T f(d_1(X_s))\frac{1}{\overline{\sigma}\sqrt{T-s}}X_s d\langle B\rangle_s \\
&\quad - \int_0^T \overline{\sigma} f(d_1(X_s))\frac{1}{\sqrt{T-s}}X_s ds.
\end{aligned}$$

2.3.4 Piecewise Constant η

In this section we allow the process η to exhibit both mean and volatility uncertainty, thus generalizing the previous results. We focus on piecewise constant processes of the following type

$$\eta_s = \sum_{i=0}^{n-1} \eta_{t_i} \mathbf{1}_{(t_i,t_{i+1}]}(s),$$

for $n \in \mathbb{N}$, where $\{t_i\}_{i=0}^{n}$ is a partition of $[0,T]$, i.e. $0 = t_0 \le t_1 \le \cdots \le t_n = T$, and $\eta_{t_i} \in L_{ip}(\mathscr{F}_{t_i})$ for all $i \in \{0,\dots,n\}$. We will describe an iterative solution scheme, which we solve explicitly till $n=2$. When $n>2$ Theorem 2.3.15 yields an iterative scheme that can be exploited to retrieve numerically the optimal portfolio. In the general case the computational complexity of the problem prevents us from individuating an explicit solution and alternatively we present in Section 2.3.6 an upper and lower bound for the optimal terminal risk.
We start by considering contingent claims that can be written in the following form

$$H = E_G[H] + \theta_{t_1}\Delta B_{t_2} + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2, \tag{2.3.15}$$

where $0 \le t_1 < t_2 \le T$, $\theta_{t_1} \in L_G^2(\mathscr{F}_{t_1})$, $\Delta B_{t_2} := B_{t_2} - B_{t_1}$ and analogously for $\Delta\langle B\rangle_{t_2}$ and Δt_2. The class of investment strategies ϕ is correspondingly assumed to be

$$\phi_t = \phi_{t_1}\mathbf{1}_{(t_1,t_2]},$$

with $\phi_{t_1} \in L_G^2(\mathscr{F}_{t_1})$. By denoting

$$c := E_G[H] - V_0,$$
$$\varphi_t := \theta_t - \phi_t X_t,$$

we can rewrite the risk functional (2.2.2) as

$$\begin{aligned} &E_G\left[(E_G[H] - V_0 + (\theta_{t_1} - \phi_{t_1}X_{t_1})\Delta B_{t_2} + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2)^2\right] \\ =&E_G\left[(c + \varphi_{t_1}\Delta B_{t_2} + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2)^2\right] \\ =&E_G\Big[(c + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2)^2 + \varphi_{t_1}^2\Delta B_{t_2}^2 + \\ &\qquad + 2\varphi_{t_1}\Delta B_{t_2}\eta_{t_1}\Delta\langle B\rangle_{t_2}\Big], \end{aligned} \tag{2.3.16}$$

where Proposition 2.1.8 was used to get the last equality.

Theorem 2.3.7. *Consider a claim $H \in L_G^{2+\varepsilon}(\mathscr{F}_T)$ with decomposition as in* (2.3.15). *The optimal mean-variance portfolio is given by (V_0^*, ϕ^*), where*

$$\phi_t^* X_t = \theta_t$$

for all $t \in [0,T]$ *and* V_0^* *solves*

$$\inf_{V_0} E_G\left[(E_G[H]-V_0)^2 \vee (E_G[H]-V_0-(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|\eta_{t_1}|)^2\right]. \tag{2.3.17}$$

Proof. We first compute

$$\begin{aligned} &E_G\left[\left(c+\eta_{t_1}\Delta\langle B\rangle_{t_2}-2G(\eta_{t_1})\Delta t_2\right)^2\right] \\ =&E_G\left[E_G\left[\left(c+\eta_{t_1}\Delta\langle B\rangle_{t_2}-2G(\eta_{t_1})\Delta t_2\right)^2\middle|\mathscr{F}_{t_1}\right]\right] \\ =&E_G\left[f(\eta_{t_1})\right], \end{aligned} \tag{2.3.18}$$

where

$$f(x)=E_G\left[\left(c+x\Delta\langle B\rangle_{t_2}-2G(x)\Delta t_2\right)^2\right].$$

As $\langle B\rangle$ has the property to be maximally distributed,

$$\begin{aligned} f(x) &= \sup_{\underline{\sigma}^2\le v\le\overline{\sigma}^2}\left(c+xv\Delta t_2-2G(x)\Delta t_2\right)^2 \\ &= \left(c+\overline{\sigma}^2x\Delta t_2-2G(x)\Delta t_2\right)^2 \vee \left(c+\underline{\sigma}^2x\Delta t_2-2G(x)\Delta t_2\right)^2 \\ &= c^2 \vee \left(c-(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|x|\right)^2, \end{aligned}$$

and therefore (2.3.18) is equal to

$$E_G\left[c^2 \vee \left(c-(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|\eta_{t_1}|\right)^2\right]. \tag{2.3.19}$$

From (2.3.19) we can argue that when

$$c^2 \ge \left(c-(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|\eta_{t_1}|\right)^2,$$

which is equivalent to

$$c \ge \frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|\eta_{t_1}|}{2},$$

the worst case scenario over the interval $[t_1,t_2]$ is described by a volatility constantly equal to $\overline{\sigma}^2$. On the other hand, when

$$c \le \frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|\eta_{t_1}|}{2},$$

the worst case scnenario sets the volatility at the constant value $\underline{\sigma}^2$. Therefore, thanks to Proposition 2.1.21, it follows that for every $c \in (0, E_G[H] + E_G[-H])$

$$\begin{aligned}
&\inf_{\varphi} E_G\left[(c + \varphi_{t_1}\Delta B_{t_2} + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2)^2\right] \\
=&\inf_{\varphi} E_G\left[E_G\left[(c + \varphi_{t_1}\Delta B_{t_2} + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2)^2\Big|\mathscr{F}_{t_1}\right]\right] \\
\geq&\inf_{\varphi} E_G\Big[E_G\left[c + \varphi_{t_1}\Delta B_{t_2} + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2|\mathscr{F}_{t_1}\right]^2 \vee \\
&\quad E_G\left[-c - \varphi_{t_1}\Delta B_{t_2} - \eta_{t_1}\Delta\langle B\rangle_{t_2} + 2G(\eta_{t_1})\Delta t_2|\mathscr{F}_{t_1}\right]^2\Big] \\
=&\ E_G\left[c^2 \vee \left(c - (\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2|\eta_{t_1}|\right)^2\right].
\end{aligned} \tag{2.3.20}$$

This completes the proof, as setting $\varphi_{t_1} = 0$ and V_0^* to be the solution of (2.3.17) allows to attain this lower bound. □

Remark 2.3.8. The proof of Theorem 2.3.7 holds true also in the context of a multidimensional G-Brownian motion, taking care of the fact that $|\eta_{t_1}|(\overline{\sigma}^2 - \underline{\sigma}^2)$ must be replaced with the expression $2(\tilde{G}(\eta_{t_1}) + \tilde{G}(-\eta_{t_1}))$, where $\tilde{G}$ is the operator

$$\tilde{G}(A) := \frac{1}{2}\sup_{B\in\Sigma}(A, B), \quad A \in \mathbb{S}(d)$$

and Σ is a bounded, closed and convex subset of $\mathbb{S}_+(d)$. When $d > 1$ it is in fact not possible to obtain a closed form for $2(\tilde{G}(A) + \tilde{G}(-A))$, $A \in \mathbb{S}(d)$, while, for $d = 1$ and $x \in \mathbb{R}$, it holds

$$2(G(x) + G(-x)) = (\overline{\sigma}^2 - \underline{\sigma}^2)|x|.$$

The same argument applies also to the proof of the results in the remaining part of the section.

It is clear from Theorem 2.3.7 that the specification of the optimal initial endowment can be more cumbersome. The characterization stated in Remark 2.3.2 does not hold in general, as we can show in the following counterexample.

Proposition 2.3.9. *Let H be of the form*

$$H = E_G[H] + \theta_{t_1}\Delta B_{t_2} + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2,$$

where $\theta_{t_1} \in L_G^2(\mathscr{F}_{t_1})$ and $\eta_{t_1} = e^{B_{t_1}}$. The optimal initial wealth of the mean-variance portfolio is different from

$$E_G[H] - V_0 = \frac{E_G[2G(\eta_{t_1})\Delta t_2 - \eta_{t_1}\Delta\langle B\rangle_{t_2}]}{2}.$$

Proof. We first calculate $\frac{E_G[H]+E_G[-H]}{2}$. We do that first by conditioning and then exploiting the fact that η_{t_1} is a convex function of the G-Brownian motion (see Proposition 11 in [56]) to get

$$\begin{aligned}E_G[H]+E_G[-H] &= E_G\left[2G(e^{B_{t_1}})\Delta t_2 - e^{B_{t_1}}\Delta\langle B\rangle_{t_2}\right]\\ &= E_G\left[(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2 e^{B_{t_1}}\right]\\ &= E_{\mathbb{P}}[(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2 e^{W_{t_1}\overline{\sigma}}]\\ &= (\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2 e^{\overline{\sigma}^2 t_1/2},\end{aligned}$$

where W denotes a standard Brownian motion for the probability $\mathbb{P}$. We then turn to the minimization over c of

$$\begin{aligned}H(c) :=& E_G\left[c^2\vee\left(c-(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2 e^{B_{t_1}}\right)^2\right]\\ =& E_{\mathbb{P}}\left[c^2\vee\left(c-(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2 e^{W_{t_1}\overline{\sigma}}\right)^2\right]\\ =& E_{\mathbb{P}}\left[\left(\left(e^{W_{t_1}\overline{\sigma}}\Delta t_2(\overline{\sigma}^2-\underline{\sigma}^2)-c\right)^2-c^2\right)^+\right]+c^2\\ =& c^2+E_{\mathbb{P}}\left[e^{W_{t_1}\overline{\sigma}}\Delta t_2(\overline{\sigma}^2-\underline{\sigma}^2)\left(e^{W_{t_1}\overline{\sigma}}\Delta t_2(\overline{\sigma}^2-\underline{\sigma}^2)-2c\right)^+\right]\\ =& c^2+E_{\mathbb{P}}\left[e^{N\sqrt{t_1}\overline{\sigma}}\Delta t_2(\overline{\sigma}^2-\underline{\sigma}^2)\left(e^{N\sqrt{t_1}\overline{\sigma}}\Delta t_2(\overline{\sigma}^2-\underline{\sigma}^2)-2c\right)^+\right],\end{aligned}$$

where N is a standard normal distributed random variable and we made use of the fact that

$$c^2\vee\left(e^{B_{t_1}}\Delta t_2(\overline{\sigma}^2-\underline{\sigma}^2)-c\right)^2$$

is a convex function of B_{t_1}. Denote $y := (\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2$ and

$$\begin{aligned}A(x) :&= \{x\in\mathbb{R} : e^{\overline{\sigma}\sqrt{t_1}x} > \frac{2c}{y}\}\\ &= \left\{x\in\mathbb{R} : x > \frac{\log\left(\frac{2c}{y}\right)}{\overline{\sigma}\sqrt{t_1}}\right\}\\ &= \{x\in\mathbb{R} : x > g(c)\},\end{aligned}$$

with $g(c) := \frac{\log\left(\frac{2c}{y}\right)}{\overline{\sigma}\sqrt{t_1}}$. We use these notations to reformulate $H(c)$ as

$$\begin{aligned}H(c) =& c^2+E_{\mathbb{P}}\left[e^{2\overline{\sigma}N\sqrt{t_1}}y^2\mathbf{1}_{A(N)}\right]-2cyE_{\mathbb{P}}\left[e^{\overline{\sigma}\sqrt{t_1}N}\mathbf{1}_{A(N)}\right]\\ =& c^2+y^2\int_{x>g(c)}e^{2\overline{\sigma}\sqrt{t_1}x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx-2cy\int_{x>g(c)}e^{\overline{\sigma}\sqrt{t_1}x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx.\end{aligned}$$

We now look for the stationary points by computing the derivative with respect to c:

$$H'(c) = 2c - y^2 e^{2\overline{\sigma}\sqrt{t_1}g(c)} \frac{1}{\sqrt{2\pi}} e^{-\frac{g(c)^2}{2}} g'(c) + 2cye^{\overline{\sigma}\sqrt{t_1}g(c)} \frac{1}{\sqrt{2\pi}} e^{-\frac{g(c)^2}{2}} g'(c) + \\ - 2y \int_{x>g(c)} \frac{1}{\sqrt{2\pi}} e^{\overline{\sigma}\sqrt{t_1}x - \frac{x^2}{2}} dx. \tag{2.3.21}$$

Finally, in order to verify whether $c^* = \frac{E_G[H]+E_G[-H]}{2} = \frac{ye^{\frac{\overline{\sigma}^2 t_1}{2}}}{2}$ is a possible minimum, we substitute it into (2.3.21). The result is

$$g(c^*) = \frac{\log\left(\frac{ye^{\frac{\overline{\sigma}^2 t_1}{2}}}{y}\right)}{\overline{\sigma}\sqrt{t_1}} = \frac{1}{2}\overline{\sigma}\sqrt{t_1},$$

hence

$$\begin{aligned}
H'\left(\frac{ye^{\frac{\overline{\sigma}^2 t_1}{2}}}{2}\right) &= ye^{\frac{1}{2}\overline{\sigma}^2 t_1} - y^2 e^{2\overline{\sigma}\sqrt{t_1}\frac{1}{2}\overline{\sigma}\sqrt{t_1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{g(c^*)^2}{2}} g'(c^*) + \\
&\quad + 2\frac{ye^{\frac{1}{2}\overline{\sigma}^2 t_1}}{2} ye^{\overline{\sigma}\sqrt{t_1}\frac{1}{2}\overline{\sigma}\sqrt{t_1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{g(c^*)^2}{2}} g'(c^*) + \\
&\quad - 2y \int_{x>\frac{1}{2}\overline{\sigma}\sqrt{t_1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2\overline{\sigma}\sqrt{t_1}x)} dx \\
&= y\left(e^{\frac{1}{2}\overline{\sigma}^2 t_1} - 2\int_{x>\frac{1}{2}\overline{\sigma}\sqrt{t_1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2\overline{\sigma}\sqrt{t_1}x)} dx\right) \\
&= y\left(e^{\frac{1}{2}\overline{\sigma}^2 t_1} - 2e^{\frac{1}{2}\overline{\sigma}^2 t_1} \int_{z>-\frac{1}{2}\overline{\sigma}\sqrt{t_1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz\right) \\
&= y\left(e^{\frac{1}{2}\overline{\sigma}^2 t_1} - 2e^{\frac{1}{2}\overline{\sigma}^2 t_1} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \right. \\
&\quad \left. - 2e^{\frac{1}{2}\overline{\sigma}^2 t_1} \int_{-\frac{1}{2}\overline{\sigma}\sqrt{t_1}}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz\right) \\
&= -2ye^{\frac{1}{2}\overline{\sigma}^2 t_1} \int_{-\frac{1}{2}\overline{\sigma}\sqrt{t_1}}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz,
\end{aligned}$$

which is not equal to zero. □

We can still obtain the optimal initial endowment for other classes of contingent claims. The next proposition represents also the first iteration of our recursive procedure. We underline that, unlike the cases considered in Sections 4.1 and 4.2, now η incorporates both mean and variance uncertainty.

Proposition 2.3.10. *Consider a claim H of the form*

$$H = E_G[H] + \int_0^{t_2} \theta_s dB_s + \eta_{t_1} \Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2,$$

where $0 = t_0 < t_1 < t_2 = T$, $\theta \in M_G^2(0,t_2)$, $\eta_{t_1} \in L_G^2(\mathscr{F}_{t_1})$ *and*

$$|\eta_{t_1}| = E_G[|\eta_{t_1}|] + \int_0^{t_1} \mu_s dB_s, \tag{2.3.22}$$

for a certain process $\mu \in M_G^2(0,t_1)$. *The optimal mean-variance portfolio is given by*

$$X_t \phi_t^* = \left(\theta_t - \frac{\mu_t(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2}{2}\right)\mathbf{1}_{(t_0,t_1]}(t) + \theta_t \mathbf{1}_{(t_1,t_2]}(t)$$

for $t \in [0,T]$ *and*

$$V_0^* = \frac{E_G[H] - E_G[-H]}{2}.$$

Proof. We proceed as in Theorem 2.3.7 in order to obtain a lower bound for the terminal risk. Let

$$\begin{aligned} \varphi_t &:= \theta_t - \phi_t X_t, \\ c &:= E_G[H] - V_0, \end{aligned}$$

and evaluate

$$\begin{aligned}
&E_G\left[\left(E_G[H] - V_0 + \int_0^{t_2}(\theta_s - \phi_s X_s)dB_s + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2\right)^2\right] \\
&= E_G\left[E_G\left[\left(c + \int_0^{t_2} \varphi_s dB_s + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2\right)^2 \Big| \mathscr{F}_{t_1}\right]\right] \\
&\geq E_G\left[E_G\left[c + \int_0^{t_2} \varphi_s dB_s + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2 \Big| \mathscr{F}_{t_1}\right]^2 \vee \right. \\
&\qquad \left. E_G\left[-c - \int_0^{t_2} \varphi_s dB_s - \eta_{t_1}\Delta\langle B\rangle_{t_2} + 2G(\eta_{t_1})\Delta t_2 \Big| \mathscr{F}_{t_1}\right]^2\right] \\
&= E_G\left[\left(c + \int_0^{t_1} \varphi_s dB_s\right)^2 \vee \left(-c - \int_0^{t_1} \varphi_s dB_s + (\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2 |\eta_{t_1}|\right)^2\right],
\end{aligned} \tag{2.3.23}$$

where we exploited the fact that

$$
\begin{aligned}
&E_G\left[c+\int_0^{t_2}\varphi_s dB_s+\eta_{t_1}\Delta\langle B\rangle_{t_2}-2G(\eta_{t_1})\Delta t_2\middle|\mathscr{F}_{t_1}\right]\\
&=E_G\left[c+\int_0^{t_1}\varphi_s dB_s+\int_{t_1}^{t_2}\varphi_s dB_s+\eta_{t_1}\Delta\langle B\rangle_{t_2}-2G(\eta_{t_1})\Delta t_2\middle|\mathscr{F}_{t_1}\right]\\
&=c+\int_0^{t_1}\varphi_s dB_s,
\end{aligned}
$$

in view Proposition 2.1.8, and analogously

$$
\begin{aligned}
&E_G\left[-c-\int_0^{t_2}\varphi_s dB_s-\eta_{t_1}\Delta\langle B\rangle_{t_2}+2G(\eta_{t_1})\Delta t_2\middle|\mathscr{F}_{t_1}\right]\\
&=-c-\int_0^{t_1}\varphi_s dB_s+(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|\eta_{t_1}|
\end{aligned}
$$

as in (2.3.20). It then follows that on the interval $(t_1,t_2]$ the optimal investment strategy must satisfy $\phi_t^* X_t=\theta_t$. We next reformulate (2.3.23) using (2.3.22) obtaining

$$
\begin{aligned}
E_G\Bigg[\left(c+\int_0^{t_1}\varphi_s dB_s\right)^2\vee\Bigg(&c-(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2 E_G\left[|\eta_{t_1}|\right]+\\
&\int_0^{t_1}\left(\varphi_s-(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2\mu_s\right)dB_s\Bigg)^2\Bigg].
\end{aligned}
\tag{2.3.24}
$$

We now define

$$
\varepsilon:=c-\frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2 E_G\left[|\eta_{t_1}|\right]}{2}
\tag{2.3.25}
$$

and

$$
\psi_s:=\varphi_s-\frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2\mu_s}{2},
\tag{2.3.26}
$$

to further develop (2.3.24) into

$$
\begin{aligned}
&E_G\left[\left(\frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2 E_G\left[|\eta_{t_1}|\right]}{2}+\varepsilon+\int_0^{t_1}\left(\frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2\mu_s}{2}+\psi_s\right)dB_s\right)^2\vee\right.\\
&\left.\left(-\frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2 E_G\left[|\eta_{t_1}|\right]}{2}+\varepsilon+\int_0^{t_1}\left(-\frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2\mu_s}{2}+\psi_s\right)dB_s\right)^2\right]\\
=&E_G\left[\left(\frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|\eta_{t_1}|}{2}+\varepsilon+\int_0^{t_1}\psi_s dB_s\right)^2\vee\right.\\
&\qquad\left.\left(-\frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|\eta_{t_1}|}{2}+\varepsilon+\int_0^{t_1}\psi_s dB_s\right)^2\right]\\
=&E_G\left[\left\{\left(\frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|\eta_{t_1}|}{2}\right)^2+\left(\varepsilon+\int_0^{t_1}\psi_s dB_s\right)^2+\right.\right.\\
&\qquad\left.+2\frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|\eta_{t_1}|}{2}\left(\varepsilon+\int_0^{t_1}\psi_s dB_s\right)\right\}\vee\\
&\left\{\left(\frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|\eta_{t_1}|}{2}\right)^2+\left(\varepsilon+\int_0^{t_1}\psi_s dB_s\right)^2+\right.\\
&\qquad\left.\left.-2\frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|\eta_{t_1}|}{2}\left(\varepsilon+\int_0^{t_1}\psi_s dB_s\right)\right\}\right]\\
=&E_G\left[\left(\frac{(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2|\eta_{t_1}|}{2}+\left|\varepsilon+\int_0^{t_1}\psi_s dB_s\right|\right)^2\right],
\end{aligned}
$$

where the decomposition of $|\eta_{t_1}|$ from (2.3.22) was exploited in the first equality. It is then clear that the optimal solution satisfies $\varepsilon=0$ and $\psi_t=0$ on $(0,t_1]$. □

Definition 2.3.11. The parameter ε in (2.3.25) is called *admissible* if the corresponding value of V_0 is such that $V_0\in(-E_G[-H],E_G[H])$.

To be able to move to the second step of our iterative solution procedure we need to prove some approximation results.

Lemma 2.3.12. *For any $t\in[0,T]$ and any $X\in L_G^p(\mathscr{F}_t)$, with $p\geq 1$ there exists a sequence of random variables of the form*

$$X_n=\sum_{i=0}^{n-1}\mathbf{1}_{A_i}x_i,$$

where $\{A_i\}_{i=0,\dots,n-1}$ *is a partition of* Ω, $A_i \in \mathscr{F}_t$ *and* $x_i \in \mathbb{R}$, *such that*

$$\|X - X_n\|_p \longrightarrow 0, \quad n \to \infty.$$

Proof. For any $N, n \in \mathbb{N}$ let

$$X_n := \sum_{i=0}^{n-1} \frac{N}{n} i \, \mathbf{1}_{\left\{\frac{N}{n} i \le |X| < \frac{N}{n}(i+1)\right\}}.$$

It follows that

$$\begin{aligned} E_G\left[(X - X_n)^p\right] &= E_G\left[X^p \mathbf{1}_{\{|X|>N\}} + \sum_{i=0}^{n-1} (X - \frac{N}{n} i)^p \mathbf{1}_{\left\{\frac{N}{n} i \le |X| < \frac{N}{n}(i+1)\right\}}\right] \\ &\le E_G\left[X^p \mathbf{1}_{\{|X|>N\}}\right] + E_G\left[\sum_{i=0}^{n-1} (X - \frac{N}{n} i)^p \mathbf{1}_{\left\{\frac{N}{n} i \le |X| < \frac{N}{n}(i+1)\right\}}\right] \\ &\le E_G\left[X^p \mathbf{1}_{\{|X|>N\}}\right] + \left(\frac{N}{n}\right)^p E_G\left[\mathbf{1}_{\{|X| \le N\}}\right]. \end{aligned} \tag{2.3.27}$$

We complete the claim by first first letting $n \to \infty$ and then $N \to \infty$ in (2.3.27), as $E_G\left[X^p \mathbf{1}_{|X|>N}\right]$ converges to zero when N tends to infinity in view of Theorem 25 in [20]. □

Lemma 2.3.13. *For any* $t \le T$ *and* $n \in \mathbb{N}$ *let* $\{A_1, \dots, A_n\}$ *be a partition of* Ω *such that* $A_i \in \mathscr{F}_t$ *for every* $i \in \{1, \dots, n\}$. *It holds that*

$$\inf_{\psi \in M_G^2(0,t)} E_{\mathbb{P}}\left[\sum_{i=1}^{n} \mathbf{1}_{A_i} \left(x_i + |\varepsilon + \int_0^t \psi_s dB_s|\right)^2\right] = E_{\mathbb{P}}\left[\sum_{i=1}^{n} \mathbf{1}_{A_i} \left(x_i + |\varepsilon|\right)^2\right],$$

for every $\varepsilon \in \mathbb{R}$, $\mathbb{P} \in \mathscr{P}_G$ *and* $\{x_1, \dots, x_n\} \in \mathbb{R}_+^n$.

Proof. We first notice that without loss of generality the elements of $\{x_1, \dots, x_n\}$ can be assumed to be different from each other and increasingly ordered. We prove the claim proceeding by induction. The base case $n = 1$ is evident. The induction step is shown assuming by contradiction that there exists a process $\bar{\psi} \in M_G^2(0,t)$ such that

$$E_{\mathbb{P}}\left[\sum_{i=1}^{n+1} \mathbf{1}_{A_i} \left(x_i + |\varepsilon + \int_0^t \bar{\psi}_s dB_s|\right)^2\right] < E_{\mathbb{P}}\left[\sum_{i=1}^{n+1} \mathbf{1}_{A_i} \left(x_i + |\varepsilon|\right)^2\right]. \tag{2.3.28}$$

Substitute then x_j, where $j \notin \{1, n+1\}$, with a x_k with $k \in \{1, \dots, n+1\} \setminus j$, to obtain sum of just n distinct terms and follow the the next steps. Notice that

(2.3.28) holds if and only if

$$
\begin{aligned}
E_{\mathbb{P}}\left[\sum_{i=1}^{n+1}\mathbf{1}_{A_i}\left(\tilde{x}_i+|\varepsilon+\int_0^t\bar{\psi}_s dB_s|\right)^2\right] < {} & E_{\mathbb{P}}\left[\sum_{i=1}^{n+1}\mathbf{1}_{A_i}\left(\tilde{x}_i+|\varepsilon|\right)^2\right] \\
& + E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(x+|\varepsilon+\int_0^t\bar{\psi}_s dB_s|\right)^2\right] \\
& - E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(x_j+|\varepsilon+\int_0^t\bar{\psi}_s dB_s|\right)^2\right] \\
& + E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(x_j+|\varepsilon|\right)^2\right] \\
& - E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(x+|\varepsilon|\right)^2\right],
\end{aligned}
\tag{2.3.29}
$$

where $\{\tilde{x}_1,\dots,\tilde{x}_{n+1}\}$ denotes a new sequence in which x_j has been substituted with $x \in \mathbb{R}_+$. To complete the proof we look at

$$
\begin{aligned}
& E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(x+|\varepsilon+\int_0^t\bar{\psi}_s dB_s|\right)^2\right] - E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(x_j+|\varepsilon+\int_0^t\bar{\psi}_s dB_s|\right)^2\right] + \\
& + E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(x_j+|\varepsilon|\right)^2\right] - E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(x+|\varepsilon|\right)^2\right] \\
= & E_{\mathbb{P}}\left[\mathbf{1}_{A_j}(x-x_j)\left(x+x_j+2|\varepsilon+\int_0^t\bar{\psi}_s dB_s|\right)\right] + \\
& - E_{\mathbb{P}}\left[\mathbf{1}_{A_j}(x-x_j)\left(x+x_j+2|\varepsilon|\right)\right] \\
= & E_{\mathbb{P}}\left[2\mathbf{1}_{A_j}(x-x_j)\left(|\varepsilon+\int_0^t\bar{\psi}_s dB_s|-|\varepsilon|\right)\right].
\end{aligned}
\tag{2.3.30}
$$

Now in the case

$$
E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(|\varepsilon+\int_0^t\bar{\psi}_s dB_s|-|\varepsilon|\right)\right] \geq 0
$$

we let $x = x_k$ for any $k \in 1,\dots,j-1$ and obtain for

$$
\{\tilde{A_1},\dots,\tilde{A_n}\} := \{A_1,\dots,A_{k-1},A_k\cup A_j,A_{k+1},\dots,A_{j-1},A_{j+1},\dots,A_{n+1}\} \tag{2.3.31}
$$

and

$$
\{y_1,\dots,y_n\} := \{x_1,\dots,x_{k-1},x_k,x_{k+1},\dots,x_{j-1},x_{j+1},\dots,x_{n+1}\} \tag{2.3.32}
$$

that

$$E_{\mathbb{P}}\left[\sum_{i=1}^{n}\mathbf{1}_{\tilde{A}_i}\left(y_i+|\varepsilon+\int_0^t \bar{\psi}_s dB_s|\right)^2\right]=E_{\mathbb{P}}\left[\sum_{i=1}^{n+1}\mathbf{1}_{A_i}\left(\tilde{x}_i+|\varepsilon+\int_0^t \bar{\psi}_s dB_s|\right)^2\right]$$
$$<E_{\mathbb{P}}\left[\sum_{i=1}^{n+1}\mathbf{1}_{A_i}(\tilde{x}_i+|\varepsilon|)^2\right]=E_{\mathbb{P}}\left[\sum_{i=1}^{n}\mathbf{1}_{\tilde{A}_i}(y_i+|\varepsilon|)^2\right], \tag{2.3.33}$$

which represent a contradiction to the induction hypothesis. On the other hand, in the case

$$E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(|\varepsilon+\int_0^t \bar{\psi}_s dB_s|-|\varepsilon|\right)\right]<0,$$

we get (2.3.33) by letting $x=x_k$ for any $k\in j+1,\dots,n+1$. □

Lemma 2.3.14. *Under the hypothesis of Lemma 2.3.13 and for any $\eta_{t_0}\in\mathbb{R}$ it holds that*

$$E_G\left[\sum_{i=1}^{n}\mathbf{1}_{A_i}(x_i+|\varepsilon+\eta_{t_0}\Delta\langle B\rangle_t-2G(\eta_{t_0})\Delta t|)^2\right]=$$
$$=\sup_{\substack{\sigma\in\mathscr{A}^{\Theta}_{0,t}\\ \sigma\ constant}} E_{\mathbb{P}^{\sigma}}\left[\sum_{i=1}^{n}\mathbf{1}_{A_i}(x_i+|\varepsilon+\eta_{t_0}\Delta\langle B\rangle_t-2G(\eta_{t_0})\Delta t|)^2\right]=$$
$$=E_{\mathbb{P}^{\sigma^*}}\left[\sum_{i=1}^{n}\mathbf{1}_{A_i}(x_i+|\varepsilon+\eta_{t_0}\Delta\langle B\rangle_t-2G(\eta_{t_0})\Delta t|)^2\right],$$

for some $\sigma^*\in[\underline{\sigma},\overline{\sigma}]$.

Proof. We adopt the same notations as in Lemma 2.3.13 and define for simplicity

$$-K_t:=\eta_{t_0}\Delta\langle B\rangle_t-2G(\eta_{t_0})\Delta t.$$

More precisely, also in the present context we make the hypothesis that $\{x_1,\dots,x_n\}$ are distinct real numbers ordered increasingly. We use again an induction argument. The case $n=1$ is trivial in view of (2.1.5), since the increment $\Delta\langle B\rangle_t$ is maximally distributed. We now proceed by contradiction and suppose there exists a $\mathbb{P}\in\mathscr{P}_{\mathbf{G}}$, which does not belong to the set $\{\mathbb{P}^{\sigma},\ \sigma\in[\underline{\sigma},\overline{\sigma}],\ \sigma\text{ constant}\}$, such that

$$E_{\mathbb{P}}\left[\sum_{i=1}^{n+1}\mathbf{1}_{A_i}(x_i+|\varepsilon-K_t|)^2\right]>E_{\mathbb{P}^{\sigma^*}}\left[\sum_{i=1}^{n+1}\mathbf{1}_{A_i}(x_i+|\varepsilon-K_t|)^2\right]. \tag{2.3.34}$$

As a consequence of inequality (2.3.34), there is an index $j \in \{1,\dots,n+1\}$ for which

$$E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(x_j+|\varepsilon-K_t|\right)^2\right] > E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}\left(x_j+|\varepsilon-K_t|\right)^2\right], \tag{2.3.35}$$

which holds if and only if

$$\begin{aligned}\Big(\mathbb{P}(A_j)-\mathbb{P}^{\sigma^*}(A_j)\Big)x_j^2+2x_j\left(E_{\mathbb{P}}\left[\mathbf{1}_{A_j}|\varepsilon-K_t|\right]-E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}|\varepsilon-K_t|\right]\right)+\\ +E_{\mathbb{P}}\left[\mathbf{1}_{A_j}|\varepsilon-K_t|^2\right]-E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}|\varepsilon-K_t|^2\right]>0.\end{aligned} \tag{2.3.36}$$

We underline that, in order for (2.3.35) to be satisfied, it has to hold $\mathbb{P}(A_j)-\mathbb{P}^{\sigma^*}(A_j)>0$. As a result (2.3.36) is a convex function of x_j, going to infinity as x_j goes to infinity. We obtain a contradiction proceeding as in the proof of Lemma 2.3.13, by modifying appropriately (2.3.34) through a substitution of x_j with a suitably chosen value. To this end notice that (2.3.34) holds if and only if

$$\begin{aligned}E_{\mathbb{P}}\left[\sum_{i=1}^{n+1}\mathbf{1}_{A_i}(\tilde{x}_i+|\varepsilon-K_t|)^2\right] >\, & E_{\mathbb{P}^{\sigma^*}}\left[\sum_{i=1}^{n+1}\mathbf{1}_{A_i}(\tilde{x}_i+|\varepsilon-K_t|)^2\right]\\ &+E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}\left(x_j+|\varepsilon-K_t|\right)^2\right]\\ &-E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}(x+|\varepsilon-K_t|)^2\right]\\ &+E_{\mathbb{P}}\left[\mathbf{1}_{A_j}(x+|\varepsilon-K_t|)^2\right]\\ &-E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(x_j+|\varepsilon-K_t|\right)^2\right],\end{aligned} \tag{2.3.37}$$

where $x \in \mathbb{R}$ and $\{\tilde{x}_1,\dots,\tilde{x}_{n+1}\}$ denotes the sequence in which x_j has been substituted with x. To complete the proof, we examine

$$\begin{aligned}&E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}\left(x_j+|\varepsilon-K_t|\right)^2\right]-E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}(x+|\varepsilon-K_t|)^2\right]>\\ &E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(x_j+|\varepsilon-K_t|\right)^2\right]-E_{\mathbb{P}}\left[\mathbf{1}_{A_j}(x+|\varepsilon-K_t|)^2\right],\end{aligned}$$

which holds if and only if

$$\begin{aligned}&E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}(x_j-x)\left(x_j+x+2|\varepsilon-K_t|\right)\right]>\\ &E_{\mathbb{P}}\left[\mathbf{1}_{A_j}(x_j-x)\left(x_j+x+2|\varepsilon-K_t|\right)\right].\end{aligned} \tag{2.3.38}$$

In the case $x>x_j$, (2.3.38) holds true if

$$E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}\left(\frac{x_j+x}{2}+|\varepsilon-K_t|\right)\right] < E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(\frac{x_j+x}{2}+|\varepsilon-K_t|\right)\right],$$

which is equivalent to

$$\left(\mathbb{P}(A_j) - \mathbb{P}^{\sigma^*}(A_j)\right) \frac{x_j + x}{2} > E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}|\varepsilon - K_t|\right] - E_{\mathbb{P}}\left[\mathbf{1}_{A_j}|\varepsilon - K_t|\right]. \quad (2.3.39)$$

In the case we can find $x = x_k$ for which (2.3.39) holds true, with $k \in \{j+1, \dots, n+1\}$, the claim is proved, since it will hold

$$\begin{aligned} E_{\mathbb{P}}\left[\sum_{i=1}^{n} \mathbf{1}_{\tilde{A}_i}(y_i + |\varepsilon - K_t|)^2\right] &= E_{\mathbb{P}}\left[\sum_{i=1}^{n+1} \mathbf{1}_{A_i}(\tilde{x}_i + |\varepsilon - K_t|)^2\right] \\ &> E_{\mathbb{P}^{\sigma^*}}\left[\sum_{i=1}^{n+1} \mathbf{1}_{A_i}(\tilde{x}_i + |\varepsilon - K_t|)^2\right] \\ &= E_{\mathbb{P}^{\sigma^*}}\left[\sum_{i=1}^{n} \mathbf{1}_{\tilde{A}_i}(y_i + |\varepsilon - K_t|)^2\right], \end{aligned}$$

where the partition $\{\tilde{A}_i\}_{i=1,\dots,n}$ and the sequence $\{y_i\}_{i=1,\dots,n}$ have been defined respectively in (2.3.31) and (2.3.32). On the other hand, when this x_k does not exist, which is the case when $j = n+1$ for example, we perform a double substitution. First we replace some x_i with a x_r, with $i \neq r$ and $i, r \in \{1, \dots, n+1\} \setminus j$, as done in (2.3.37), and then we replace x_j with an x big enough to verify

$$\begin{aligned} &E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}\left(x_j + |\varepsilon - K_t|\right)^2\right] - E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}(x + |\varepsilon - K_t|)^2\right] \\ &+ E_{\mathbb{P}}\left[\mathbf{1}_{A_j}(x + |\varepsilon - K_t|)^2\right] - E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(x_j + |\varepsilon - K_t|\right)^2\right] \\ &+ E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_i}(x_i + |\varepsilon - K_t|)^2\right] - E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_i}(x_r + |\varepsilon - K_t|)^2\right] \\ &+ E_{\mathbb{P}}\left[\mathbf{1}_{A_i}(x_r + |\varepsilon - K_t|)^2\right] - E_{\mathbb{P}}\left[\mathbf{1}_{A_i}(x_i + |\varepsilon - K_t|)^2\right] > 0. \end{aligned} \quad (2.3.40)$$

This can be done as

$$\begin{aligned} &E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}\left(x_j + |\varepsilon - K_t|\right)^2\right] - E_{\mathbb{P}^{\sigma^*}}\left[\mathbf{1}_{A_j}(x + |\varepsilon - K_t|)^2\right] \\ &+ E_{\mathbb{P}}\left[\mathbf{1}_{A_j}(x + |\varepsilon - K_t|)^2\right] - E_{\mathbb{P}}\left[\mathbf{1}_{A_j}\left(x_j + |\varepsilon - K_t|\right)^2\right] > 0 \end{aligned}$$

holds if and only if (2.3.39) is verified, and it can take values so large that (2.3.40) is automatically verified by virtue of (2.3.36). □

We can finally prove the major result.

Theorem 2.3.15. *Consider a claim H of the form*

$$H = E_G[H] + \int_0^{t_2} \theta_s dB_s + \eta_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\eta_{t_0})\Delta t_1 + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2,$$

where $0 = t_0 < t_1 < t_2 = T$, $\theta \in M_G^2(0,t_2)$, $\eta_{t_0} \in \mathbb{R}$, $\eta_{t_1} \in L_G^2(\mathscr{F}_{t_1})$ *and*

$$|\eta_{t_1}| = E_G\left[|\eta_{t_1}|\right] + \int_0^{t_1} \mu_s dB_s, \tag{2.3.41}$$

for a certain process $\mu \in M_G^2(0,t_1)$*. The optimal mean-variance portfolio is given by*

$$\phi_t^* X_t = \left(\theta_t - \frac{\mu_t(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2}{2}\right)\mathbf{1}_{(t_0,t_1]}(t) + \theta_t \mathbf{1}_{(t_1,t_2]}(t)$$

for $t \in [0,T]$ *and*

$$V_0^* = E_G[H] - \frac{1}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2 E_G\left[|\eta_{t_1}|\right] - \varepsilon,$$

where $\varepsilon \in \mathbb{R}$ *solves*

$$\inf_{\varepsilon} E_G\left[\left(\frac{|\eta_{t_1}|}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2 + |\varepsilon + \eta_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\eta_{t_0})\Delta t_1|\right)^2\right].$$

Proof. We can exploit the result of Proposition 2.3.10 to argue that

$$\phi_s^* X_s = \theta_s \quad \forall\, s \in (t_1,t_2]$$

and examine the following minimization problem

$$\inf_{\varepsilon,\psi} E_G\left[\left(\frac{|\eta_{t_1}|}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2 + |\varepsilon + \int_0^{t_1} \psi_s dB_s + \eta_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\eta_{t_0})\Delta t_1|\right)^2\right], \tag{2.3.42}$$

where ε and ψ have been defined in (2.3.25) and (2.3.26). To this end we focus on the following auxiliary problem

$$\inf_{\varepsilon,\psi} E_G\left[\left(Y_n + |\varepsilon + \int_0^{t_1} \psi_s dB_s + \eta_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\eta_{t_0})\Delta t_1|\right)^2\right],$$

where $(Y_n)_{n\in\mathbb{N}}$ is a sequence that approximates $\frac{|\eta_{t_1}|}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2$ in $L_G^2(\mathscr{F}_{t_1})$ according to Lemma 2.3.12. In particular the general term Y_n has the form $Y_n = \sum_{i=0}^{n-1} \mathbf{1}_{A_{i,n}} y_{i,n}$, $n \in \mathbb{N}$, where $\{A_{i,n}\}_{i=0,\dots,n-1}$ is a partition of Ω, $A_{i,n} \in \mathscr{F}_t$ and $y_{i,n} \in \mathbb{R}_+$. Given any $n \in \mathbb{N}$ and an admissible ε we obtain the following chain of inequalities

$$
\begin{aligned}
&E_G\left[\left(Y_n+|\varepsilon+\int_0^{t_1}\psi_s dB_s+\eta_{t_0}\Delta\langle B\rangle_{t_1}-2G(\eta_{t_0})\Delta t_1|\right)^2\right]\geq\\
&\geq\sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]}E_{\mathbb{P}^\sigma}\left[\left(Y_n+|\varepsilon+\int_0^{t_1}\psi_s dB_s+\eta_{t_0}\Delta\langle B\rangle_{t_1}-2G(\eta_{t_0})\Delta t_1|\right)^2\right]\\
&\geq\sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]}E_{\mathbb{P}^\sigma}\left[(Y_n+|\varepsilon+\eta_{t_0}\Delta\langle B\rangle_{t_1}-2G(\eta_{t_0})\Delta t_1|)^2\right] \qquad (2.3.43)\\
&=E_G\left[(Y_n+|\varepsilon+\eta_{t_0}\Delta\langle B\rangle_{t_1}-2G(\eta_{t_0})\Delta t_1|)^2\right]. \qquad (2.3.44)
\end{aligned}
$$

The inequality (2.3.43) is evident by virtue of Lemma 2.3.13, since

$$\varepsilon^{P^\sigma}:=\varepsilon+\eta_{t_0}\Delta\langle B\rangle_{t_1}-2G(\eta_{t_0})\Delta t_1$$

is $\mathbb{P}^\sigma$-a.s. constant for any $\sigma\in[\underline{\sigma},\overline{\sigma}]$, as

$$\Delta\langle B\rangle_{t_1}=\sigma^2\Delta t_1 \quad P^\sigma\text{-a.s.}$$

and $y_{i,n}\in\mathbb{R}_+\ \forall\, n,i.$ On the other hand the equality (2.3.44) follows from Lemma 2.3.14. Therefore we can argue that, given any $n\in\mathbb{N}$ and any admissible ε,

$$
\begin{aligned}
&E_G\left[\left(Y_n+|\varepsilon+\int_0^{t_1}\psi_s dB_s+\eta_{t_0}\Delta\langle B\rangle_{t_1}-2G(\eta_{t_0})\Delta t_1|\right)^2\right]\\
&\geq E_G\left[(Y_n+|\varepsilon+\eta_{t_0}\Delta\langle B\rangle_{t_1}-2G(\eta_{t_0})\Delta t_1|)^2\right].
\end{aligned}
\qquad (2.3.45)
$$

From (2.3.45), upon letting $n\to\infty$, we obtain

$$
\begin{aligned}
&E_G\left[\left(\frac{|\eta_{t_1}|}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2+|\varepsilon+\int_0^{t_1}\psi_s dB_s+\eta_{t_0}\Delta\langle B\rangle_{t_1}-2G(\eta_{t_0})\Delta t_1|\right)^2\right]\\
&\geq E_G\left[\left(\frac{|\eta_{t_1}|}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2+|\varepsilon+\eta_{t_0}\Delta\langle B\rangle_{t_1}-2G(\eta_{t_0})\Delta t_1|\right)^2\right],
\end{aligned}
$$

for any admissible ε and any $\psi\in M_G^2(0,t_1)$, thanks to the L_G^2-convergence of Y_n to $\frac{|\eta_{t_1}|}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2$. This yields

$$
\begin{aligned}
&\inf_{\varepsilon,\psi}E_G\left[\left(\frac{|\eta_{t_1}|}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2+|\varepsilon+\int_0^{t_1}\psi_s dB_s+\eta_{t_0}\Delta\langle B\rangle_{t_1}-2G(\eta_{t_0})\Delta t_1|\right)^2\right]\\
&\geq\inf_{\varepsilon}E_G\left[\left(\frac{|\eta_{t_1}|}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2+|\varepsilon+\eta_{t_0}\Delta\langle B\rangle_{t_1}-2G(\eta_{t_0})\Delta t_1|\right)^2\right].
\end{aligned}
$$

□

We now consider a particular claim satisfying the assumptions of Theorem 2.3.15 and compute explicitly its optimal mean-variance portfolio. The relevance of this example is the possibility to obtain a close expression for the optimal initial endowment.

Example 2.3.16. *Consider a claim H of the following form*

$$H = E_G[H] + \int_0^{t_2} \theta_s dB_s + \eta_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\eta_{t_0})\Delta t_1 + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2,$$

where $0 = t_0 < t_1 < t_2 = T$, $\theta \in M_G^2(0,t_2)$, $\eta_{t_0} \in \mathbb{R}_+$, $\eta_{t_1} \in L_G^2(\mathscr{F}_{t_1})$ and

$$|\eta_{t_1}| = \exp\left(B_{t_1} - \frac{1}{2}\langle B\rangle_{t_1}\right) = 1 + \int_0^{t_1} e^{B_s - \frac{1}{2}\langle B\rangle_s} dB_s. \tag{2.3.46}$$

Assume moreover that

$$\frac{1}{2}\Delta t_2 e^{\frac{1}{2}\overline{\sigma}^2\Delta t_1} \geq \eta_{t_0}\Delta t_1 + \frac{1}{2}\Delta t_2. \tag{2.3.47}$$

The optimal mean-variance portfolio is given by

$$X_t\phi_t^* = \left(\theta_t - \frac{e^{B_t - \frac{1}{2}\langle B\rangle_t}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2}{2}\right)\mathbf{1}_{(t_0,t_1]}(t) + \theta_t \mathbf{1}_{(t_1,t_2]}(t)$$

for $t \in [0,T]$ and

$$V_0^* = E_G[H] - \frac{(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2}{2}. \tag{2.3.48}$$

Proof. As a consequence of Theorem 2.3.15 the only thing we have to study is the minimum of

$$E_G\left[\left(\frac{1}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2 e^{B_{t_1} - \frac{1}{2}\langle B\rangle_{t_1}} + |\varepsilon + \eta_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\eta_{t_0})\Delta t_1|\right)^2\right]. \tag{2.3.49}$$

The G-expectation in (2.3.49) can be lower bounded by

$$\begin{aligned} E_G\left[\left(\frac{1}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2 e^{B_{t_1} - \frac{1}{2}\langle B\rangle_{t_1}}\right)^2\right] &= \frac{1}{4}(\overline{\sigma}^2 - \underline{\sigma}^2)^2\Delta t_2^2 E_G\left[e^{2B_{t_1} - \langle B\rangle_{t_1}}\right] \\ &= \frac{1}{4}(\overline{\sigma}^2 - \underline{\sigma}^2)^2\Delta t_2^2 E_{\mathbb{P}^{\overline{\sigma}}}\left[e^{2B_{t_1} - \langle B\rangle_{t_1}}\right] \\ &= \frac{1}{4}(\overline{\sigma}^2 - \underline{\sigma}^2)^2\Delta t_2^2 e^{\overline{\sigma}^2\Delta t_1}. \end{aligned}$$

We then show (2.3.48) by proving that (2.3.49) attains this lower bound when we set $\varepsilon = 0$. We then have to show the following

$$\begin{aligned}
&\sup_{\sigma\in\mathscr{A}^{\Theta}_{0,t_1}} E_{\mathbb{P}}\left[\left(\frac{1}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2 e^{\int_0^{t_1}\sigma_s dW_s-\frac{1}{2}\int_0^{t_1}\sigma_s^2 ds}+\eta_{t_0}\Big|\int_0^{t_1}(\sigma_s^2-\overline{\sigma}^2)ds\Big|\right)^2\right]\\
=&E_G\left[\left(\frac{1}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2 e^{B_{t_1}-\frac{1}{2}\langle B\rangle_{t_1}}+|\eta_{t_0}\Delta\langle B\rangle_{t_1}-2G(\eta_{t_0})\Delta t_1|\right)^2\right]\\
=&E_G\left[\left(\frac{1}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2 e^{B_{t_1}-\frac{1}{2}\langle B\rangle_{t_1}}\right)^2\right],
\end{aligned}$$

where $\mathscr{A}^{\Theta}_{0,t_1}$ indicates the family of $\mathbb{F}$-adapted processes on $[0,t_1]$ with values in $[\underline{\sigma},\overline{\sigma}]$. This is ensured in the case

$$\begin{aligned}
E_{\mathbb{P}}&\left[\left(\frac{1}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2 e^{\int_0^{t_1}\sigma_s dW_s-\frac{1}{2}\int_0^{t_1}\sigma_s^2 ds}+\eta_{t_0}\int_0^{t_1}(\overline{\sigma}^2-\sigma_s^2)ds\right)^2\right]\\
&\qquad\le\frac{1}{4}(\overline{\sigma}^2-\underline{\sigma}^2)^2\Delta t_2^2 e^{\overline{\sigma}^2\Delta t_1}
\end{aligned}\tag{2.3.50}$$

holds true for every $\sigma\in\mathscr{A}^{\Theta}_{0,t_1}$. Since (2.3.50) is equivalent to

$$\begin{aligned}
E_{\mathbb{P}}&\left[\left(\frac{1}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2\left(e^{\int_0^{t_1}\sigma_s dW_s-\frac{1}{2}\int_0^{t_1}\sigma_s^2 ds}+e^{\frac{1}{2}\overline{\sigma}^2\Delta t_1}\right)+\eta_{t_0}\int_0^{t_1}(\overline{\sigma}^2-\sigma_s^2)ds\right)\right.\\
&\left.\cdot\left(\frac{1}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2\left(e^{\int_0^{t_1}\sigma_s dW_s-\frac{1}{2}\int_0^{t_1}\sigma_s^2 ds}-e^{\frac{1}{2}\overline{\sigma}^2\Delta t_1}\right)+\eta_{t_0}\int_0^{t_1}(\overline{\sigma}^2-\sigma_s^2)ds\right)\right]\le 0,
\end{aligned}$$

we achieve the claim by proving that the last expression upper bounded by

$$\begin{aligned}
&\lim_{N\to\infty}C(N)\,E_{\mathbb{P}}\left[\left(\frac{1}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2\left(e^{\int_0^{t_1}\sigma_s dW_s-\frac{1}{2}\int_0^{t_1}\sigma_s^2 ds}-e^{\frac{1}{2}\overline{\sigma}^2\Delta t_1}\right)\right.\right.\\
&\qquad\qquad\left.\left.+\eta_{t_0}\int_0^{t_1}(\overline{\sigma}^2-\sigma_s^2)ds\right)\mathbf{1}_{\{\int_0^{t_1}\sigma_s dW_s<N\}}\right]\\
\le&\lim_{N\to\infty}C(N)\left(\frac{1}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2\left(1-e^{\frac{1}{2}\overline{\sigma}^2\Delta t_1}\right)+\eta_{t_0}E_{\mathbb{P}}\left[\int_0^{t_1}(\overline{\sigma}^2-\sigma_s^2)ds\right]\right)\\
\le&\lim_{N\to\infty}C(N)\left(\frac{1}{2}(\overline{\sigma}^2-\underline{\sigma}^2)\Delta t_2\left(1-e^{\frac{1}{2}\overline{\sigma}^2\Delta t_1}\right)+\eta_{t_0}(\overline{\sigma}^2-\underline{\sigma}^2)\right)<0,
\end{aligned}$$

where the last inequality is a consequence of assumpion (2.3.47) and $C(N)\in\mathbb{R}_+$ for every $N\in\mathbb{N}$. □

In the next theorem we generalize the result of Theorem 2.3.15 by relaxing the assumptions on $|\eta_{t_1}|$, thus achieving the second step of our solution scheme.

Theorem 2.3.17. *Consider a claim H of the form*

$$H = E_G[H] + \int_0^{t_2} \theta_s dB_s + \eta_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\eta_{t_0})\Delta t_1 + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2,$$

where $0 = t_0 < t_1 < t_2 = T$, $\theta \in M_G^2(0,t_2)$, $\eta_{t_0} \in \mathbb{R}$, $\eta_{t_1} \in L_G^2(\mathscr{F}_{t_1})$ and

$$|\eta_{t_1}| = E_G[|\eta_{t_1}|] + \int_0^{t_1} \mu_s dB_s + \xi_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\xi_{t_0})\Delta t_1, \tag{2.3.51}$$

for a certain process $\mu \in M_G^2(0,t_1)$ and $\xi_{t_0} \in \mathbb{R}$. The optimal mean-variance portfolio is given by

$$\phi_t^* X_t = \left(\theta_t - \frac{\mu_t(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2}{2}\right)\mathbf{1}_{(t_0,t_1]}(t) + \theta_t \mathbf{1}_{(t_1,t_2]}(t) \tag{2.3.52}$$

and

$$V_0^* = E_G[H] - \frac{1}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2 E_G[|\eta_{t_1}|] - \varepsilon,$$

where $\varepsilon \in \mathbb{R}$ solves

$$\inf_{\varepsilon} E_G\left[\left(\frac{|\eta_{t_1}|}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_2 + \left|\varepsilon + \left(\eta_{t_0} - \frac{1}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\xi_{t_0}\Delta t_1\right)\Delta\langle B\rangle_{t_1} + \right.\right.\right.$$
$$\left.\left.\left. -2\left(G(\eta_{t_0}) - \frac{1}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_1 G(\xi_{t_0})\right)\Delta t_1\right|\right)^2\right]. \tag{2.3.53}$$

Proof. We omit the proof as it follows almost step by step that of Theorem 2.3.15. □

Remark 2.3.18. We remark that, as it appears for example from (2.3.52), volatility uncertainty always acts against the investor performing mean-variance hedging. This is evident from the fact that at each step the agent has to cope with a factor of the type $(\overline{\sigma}^2 - \underline{\sigma}^2)$. A different behavior is observed in [31] in the context of portfolio optimization with ambiguous correlation, where, for some particular situations, ambiguity can actually be preferred to the classical single prior framework as it can improve the outcome of the investor's decisions (see Corollary 3.7 in [31]).

We now see which are the issues of the solution scheme outlined in this section when the claim is given by

$$H = E_G[H] + \int_0^T \theta_s dB_s + \sum_{i=0}^{2} \left(\eta_{t_i}\Delta\langle B\rangle_{t_{i+1}} - 2G(\eta_{t_i})\Delta t_{i+1}\right). \tag{2.3.54}$$

The assumptions we enforce are analogous to those in Theorem 2.3.15. In particular $0 = t_0 < t_1 < t_2 < t_3 = T$, $\theta \in M_G^2(0,T)$ and $\eta_i \in L_G^2(\mathscr{F}_{t_i})$, for $i \in \{1,2\}$, are such that

$$|\eta_i| = E_G[|\eta_i|] + \int_0^{t_i} \theta_s^i dB_s,$$

for $\theta^i \in M_G^2(0,t_i)$. As a first step we can apply the result of Proposition 2.3.10 to argue that the optimal mean-variance hedging strategy on the interval $(t_2,t_3]$ is given by $\phi_s^* X_s = \theta_s$. We give an heuristic explanation for this result: from time t_2 the uncertainty on the outcome of the claim H depends only on $\Delta\langle B\rangle_{t_3}$ and $\int_{t_2}^{t_3} \theta_s dB_s$, as it can be seen from the decomposition (2.3.54). The ambiguity generated by the quadratic variation of B alone cannot be hedged away by means of dynamic trading, as noticed already in Theorem 2.3.1 and Theorem 2.3.5, but only through the initial wealth. Therefore, on $(t_2,t_3]$, an investor adopting the mean-variance criterion chooses to optimally hedge the component $\int_{t_2}^{t_3} \theta_s dB_s$, while the effects of $\Delta\langle B\rangle_{t_3}$ must be mitigated by the choices made by the agent prior to time t_2. The functional to be minimized with respect to V_0 and $(\phi_s)_{s\in[0,t_2]}$ is then

$$\begin{aligned} E_G\Bigg[&\Big(\frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\Delta t_3 (E_G[|\eta_{t_2}|] + \int_0^{t_2} \theta_s^2 dB_s) + \Big| E_G[H] - V_0 \\ &- \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\Delta t_3 E_G[|\eta_{t_2}|] + \int_0^{t_2} \Big(\theta_s - \phi_s X_s - \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\Delta t_3 \theta_s^2\Big) dB_s \\ &\eta_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\eta_{t_0})\Delta t_1 + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_1)\Delta t_2 \Big|\Big)^2 \Bigg], \end{aligned} \tag{2.3.55}$$

once again according to Proposition 2.3.10. By conditioning the expression above with respect to $\mathscr{F}_{t_1}$ we obtain

$$\begin{aligned} E_G\Bigg[E_G\Bigg[&\Big(\frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\Delta t_3 (E_G[|\eta_{t_2}|] + \int_0^{t_2} \theta_s^2 dB_s) + \Big| E_G[H] - V_0 \\ &- \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\Delta t_3 E_G[|\eta_{t_2}|] + \int_0^{t_2} \Big(\theta_s - \phi_s X_s - \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\Delta t_3 \theta_s^2\Big) dB_s \\ &\eta_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\eta_{t_0})\Delta t_1 + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_1)\Delta t_2 \Big|\Big)^2 \Bigg| \mathscr{F}_{t_1} \Bigg]\Bigg] \end{aligned} \tag{2.3.56}$$

and we can compute the G-conditional expectation by separating the terms which are $\mathscr{F}_{t_1}$-measurable from those which are independent of $\mathscr{F}_{t_1}$ in the following way

$$E_G\left[\Big(a+\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\Delta t_3\int_{t_1}^{t_2}\theta_s^2 dB_s+\Big|b+\int_{t_1}^{t_2}\Big(\theta_s-\phi_s X_s-\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\Delta t_3\theta_s^2\Big)dB_s \right.$$
$$\left.+\eta_{t_1}\Delta\langle B\rangle_{t_2}-2G(\eta_1)\Delta t_2\Big|\Big)^2\right], \tag{2.3.57}$$

where a,b stand for the corresponding terms in (2.3.56) that are $\mathscr{F}_{t_1}$-measurable. We are now in the condition to apply Proposition 2.3.10 to prove that on $(t_1,t_2]$ the optimal strategy is described by

$$\theta_s-\phi_s^* X_s=\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\Delta t_3\theta_s^2. \tag{2.3.58}$$

The intuition behind this result is the following: (2.3.58) allows the agent to optimally protect herself from the fluctuations of $|\eta_{t_2}|$ on the time interval $(t_1,t_2]$ as, once again, the dynamic trading cannot help to hedge the component $\eta_{t_1}\Delta\langle B\rangle_{t_2}-2G(\eta_1)\Delta t_2$ on that time frame. However, also with such characterization of ϕ^*, it is not possible to compute the exact value of the G-conditional expectation in

$$E_G\left[E_G\left[\Big(\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\Delta t_3(E_G[|\eta_{t_2}|]+\int_0^{t_2}\theta_s^2 dB_s)+\Big|E_G[H]-V_0\right.\right.$$
$$-\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\Delta t_3 E_G[|\eta_{t_2}|]+\int_0^{t_1}\Big(\theta_s-\phi_s X_s-\frac{\overline{\sigma}^2-\underline{\sigma}^2}{2}\Delta t_3\theta_s^2\Big)dB_s \tag{2.3.59}$$
$$\left.\left.\eta_{t_0}\Delta\langle B\rangle_{t_1}-2G(\eta_{t_0})\Delta t_1+\eta_{t_1}\Delta\langle B\rangle_{t_2}-2G(\eta_1)\Delta t_2\Big|\Big)^2\right|\mathscr{F}_{t_1}\right]\right],$$

which in turns prevent us from determining ϕ^* on $[0,t_1]$. This is precisely the same obstacle encountered in Theorem 2.3.15 for the computation of V_0^*, and it boils down to the impossibility of computing explicitly expressions of the type

$$E_G\left[\int_0^t\theta_s dB_s\int_0^t\eta_s d\langle B\rangle_s\right],$$

for $\theta\in M_G^2(0,t)$ and $\eta\in M_G^1(0,t)$.
It is possible to enforce some strong assumptions on η_{t_i} to obtain a full characterization of the optimal strategy, extending the previous findings to claims of the

type

$$H = E_G[H] + \int_0^T \theta_s dB_s + \sum_{i=0}^{n-1} \left(\eta_{t_i} \Delta \langle B \rangle_{t_{i+1}} - 2G(\eta_{t_i}) \Delta t_{i+1} \right), \tag{2.3.60}$$

with $n > 2$. A trivial case would result from requiring η_{t_i} to be $\mathscr{F}_{t_1}$-measurable for every $i \in \{1, \dots, n-1\}$. Looking at (2.3.57), this assumption would imply $\theta_s^2 = 0$ on $(t_1, t_2]$, from where the possibility to compute explicitly the expression in (2.3.56) that with these new assumptions becomes

$$\begin{aligned} E_G\Bigg[\Big(& \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2} \Delta t_3 (E_G[|\eta_{t_2}|] + \int_0^{t_1} \theta_s^2 dB_s) + \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2} \Delta t_2 (E_G[|\eta_{t_1}|] + \int_0^{t_1} \theta_s^1 dB_s) \\ & \Big| E_G[H] - V_0 - \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2} \Delta t_3 E_G[|\eta_{t_2}|] - \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2} \Delta t_2 E_G[|\eta_{t_1}|] \\ & + \int_0^{t_1} \left(\theta_s - \phi_s X_s - \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2} \Delta t_3 \theta_s^2 - \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2} \Delta t_2 \theta_s^1 \right) dB_s \\ & + \eta_{t_0} \Delta \langle B \rangle_{t_1} - 2G(\eta_{t_0}) \Delta t_1 \Big| \Big)^2 \Bigg]. \end{aligned}$$

At this point we could simply use Proposition 2.3.10 to obtain ϕ^* on $[0, t_1]$. The same result can be achieved by requiring η_{t_i}, for $i > 1$, to be deterministic or a function of $\langle B \rangle_{t_i}$, thus retrieving the same results of Theorem 2.3.1 and Theorem 2.3.5 respectively.

Another possible assumption for claims of the form (2.3.60) is

$$\eta_{t_i} = \begin{cases} E_G[|\eta_{t_i}|] + \int_{t_{i-1}}^{t_i} \theta_s^i dB_s, & \exists\, k \in \mathbb{N} : i = n - 2k - 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3.61}$$

where each process θ^i belongs to $M_G^2(0,T)$. Going back to the situation in which $n = 3$, we see that (2.3.61) allows to explicitly compute (2.3.59), which becomes

$$\begin{aligned} & E_G\left[\left(\frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2} \Delta t_3 (E_G[|\eta_{t_2}|] + \int_{t_1}^{t_2} \theta_s^2 dB_s) \right)^2 \right] + E_G\Bigg[\Bigg(E_G[H] - V_0 \\ & - \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2} \Delta t_3 E_G[|\eta_{t_2}|] + \int_0^{t_1} \left(\theta_s - \phi_s X_s - \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2} \Delta t_3 \theta_s^2 \right) dB_s \\ & \eta_{t_0} \Delta \langle B \rangle_{t_1} - 2G(\eta_{t_0}) \Delta t_1 \Bigg)^2 \Bigg]. \end{aligned} \tag{2.3.62}$$

As for the previous cases, it is possible to apply Proposition 2.3.10 to the second term in (2.3.62) to complete the analysis of the optimal mean-variance portfolio. It is however impossible to obtain relevant insights from the specification (2.3.61) when n goes to infinity as such sequence of processes converges to 0 in the M_G^2 norm.

Another set of assumptions allowing similar results is the following: let η_{t_i} be defined by

$$\eta_{t_i} = \begin{cases} E_G\left[|\eta_{t_i}|\right] + \sum_{j,i-2j-1\geq 0} \int_{t_{i-2j-1}}^{t_{i-2j}} \theta_s^i dB_s, & \exists\, k \in \mathbb{N} : i = n-2k-1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta^i \in M_G^2(0,T)$ for every i. The solving procedure is similar to the case exposed in (2.3.61) and we will not treat it in full detail here. Also in this context, if there is convergence for n that goes to infinity, the limit is given by the null process.

It would be particularly helpful if the minimization to perform on (2.3.55) have been equivalent to that of

$$\begin{aligned} E_G\Bigg[\Big(&E_G[H] - V_0 - \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\Delta t_3 E_G\left[|\eta_{t_2}|\right] \\ &+ \int_0^{t_2} \Big(\theta_s - \phi_s X_s - \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\Delta t_3 \theta_s^2\Big) dB_s \\ &\eta_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\eta_{t_0})\Delta t_1 + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_1)\Delta t_2\Big)^2\Bigg]. \end{aligned} \tag{2.3.63}$$

If this would be the case, we could apply iteratively Proposition 2.3.10 and complete our solution scheme. However such simplification cannot be done. We treat this argument in detail in Section 2.3.5.

2.3.5 Comparison with the Classical Case

When $\mathscr{P} = \{\mathbb{P}\}$ it is possible to solve the mean-variance hedging problem by steps. This can be easily seen a posteriori, as the structure of the optimal mean-variance hedging portfolio is completely known in the context of a single prior market. To provide a better understanding of this last statement we need to introduce some more notations and preliminary results. Let be fixed a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$, where $\{\mathscr{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions. We denote $\mathbf{M}^2(\mathbb{P})$ the family of square integrable $\mathbb{P}$-martingales.

Definition 2.3.19 (see page 180 in [60]). A closed subspace F of $\mathbf{M}^2(\mathbb{P})$ is called a *stable subspace* if it is stable under stopping (that is, if $M \in F$ and τ is a stopping time, then $M^\tau \in F$).

Definition 2.3.20. Two martingales $M,N \in \mathbf{M}^2(\mathbb{P})$ are said to be *strongly orthogonal* if their product $L = MN$ is a (uniformly integrable) martingale.

Definition 2.3.21. For a subset $\mathbf{A}$ of $\mathbf{M}^2(\mathbb{P})$ we let $\mathbf{A}^\times$ denote the set of all elements of $\mathbf{M}^2(\mathbb{P})$ strongly orthogonal to each element of $\mathbf{A}$.

We can now formulate two important results.

Lemma 2.3.22 (Lemma 2.1 of [64])**.** *Suppose that X is a $\mathbb{P}$-local martingale. For any predictable process θ such that*

$$\|\theta\|_{L^2(X)} := \left(E_{\mathbb{P}}\left[\int_0^T \theta_s^2 d\langle X\rangle_s\right]\right)^{\frac{1}{2}} < \infty,$$

the process $(\int_0^t \theta_s dX_t)_{t\in[0,T]}$ is well defined in the space $\mathbf{M}^2(\mathbb{P})$. Moreover, the space $\mathscr{I}^2(X) := \{\int \theta dX \mid \theta \in L^2(X)\}$ of stochastic integral is a stable subspace of $\mathbf{M}^2(\mathbb{P})$.

Lemma 2.3.23 (see page 183 in [60])**.** *Let $\mathbf{A}$ be a stable subspace of $\mathbf{M}^2(\mathbb{P})$. Then each $M \in \mathbf{M}^2(\mathbb{P})$ has a unique decomposition $M = A + B$, where $A \in \mathbf{A}$ and $B \in \mathbf{A}^\times$.*

It is now clear from Lemma 2.3.22 and Lemma 2.3.23 that if the discounted risky asset X is a $\mathbb{P}$-local martingale, then $H \in L^2_{\mathbb{P}}(\mathscr{F}_T)$ can be uniquely written as

$$H = E_{\mathbb{P}}[H] + \int_0^T \theta_s^H dX_s + L_T^H \quad \mathbb{P}-a.s.$$

where $\|\theta^H\|_{L^2(X)} < \infty$ and L^H is a square integrable martingale which is strongly orthogonal to $\mathscr{I}^2(X)$. As a consequence the optimal mean-variance portfolio in this case is given by $(E_{\mathbb{P}}[H], \theta^H)$.
In this context we can solve the mean-variance problem stepwisely, by considering a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$, first minimizing the terminal risk functional on $(t_{n-1}, t_n]$ and then going backward on the other subintervals. The aim is in fact that to minimize

$$E_{\mathbb{P}}\left[\left(E_{\mathbb{P}}[H] - V_0 + \int_0^T (\theta_s^H - \theta_s) dX_s + L_T^H\right)^2\right] \tag{2.3.64}$$

over $V_0 \in \mathbb{R}_+$ and θ, with $\|\theta\|_{L^2(X)} < \infty$. Fixed any $t \in (0,T)$, we can equivalently rewrite (2.3.64) as

$$
\begin{aligned}
&E_{\mathbb{P}}\Bigg[\Big(E_{\mathbb{P}}[H]-V_0+\int_0^t(\theta_s^H-\theta_s)dX_s+\int_t^T(\theta_s^H-\theta_s)dX_s \\
&\qquad\qquad +L_T^H-L_t^H+L_t^H\Big)^2\Bigg] \\
=&E_{\mathbb{P}}\Bigg[\Big(E_{\mathbb{P}}[H]-V_0+\int_0^t(\theta_s^H-\theta_s)dX_s+L_t^H\Big)^2\Bigg]+ \\
&+2E_{\mathbb{P}}\Bigg[\Big(E_{\mathbb{P}}[H]-V_0+\int_0^t(\theta_s^H-\theta_s)dX_s+L_t^H\Big) \\
&\qquad \Big(\int_t^T(\theta_s^H-\theta_s)dX_s+L_T^H-L_t^H\Big)\Bigg]+ \\
&+E_{\mathbb{P}}\Bigg[\Big(\int_t^T(\theta_s^H-\theta_s)dX_s+L_T^H-L_t^H\Big)^2\Bigg] \qquad (2.3.65) \\
=&E_{\mathbb{P}}\Bigg[\Big(E_{\mathbb{P}}[H]-V_0+\int_0^t(\theta_s^H-\theta_s)dX_s+L_t^H\Big)^2\Bigg]+ \\
&+E_{\mathbb{P}}\Bigg[\Big(\int_t^T(\theta_s^H-\theta_s)dX_s+L_T^H-L_t^H\Big)^2\Bigg],
\end{aligned}
$$

where the last equality follows from the fact that the second term in (2.3.65) is equal to zero, as it can be proved by conditioning with respect to $\mathscr{F}_t$. The mean-variance problem is then separated over two distinct time intervals and the minimization can be performed individually on $[0,t]$ and $(t,T]$. Moreover, as the choice of $t \in (0,T)$ was completely arbitrary, the same argument can be applied to a partition of $[0,T]$ into n subintervals.

As we have remarked in Section 2.3.4, all of this can not be transposed immediately into the G-setting: even focusing on a convenient class of contingent claims it is not possible to disentangle the computation of the optimal portfolio on $(t,T]$ from that on $[0,t]$. Going back to the expression achieved in (2.3.55), we observe that the realization of the process $(K_t)_{t\in[0,T]}$ on $(t_2,t_3]$ intervenes on the determination of the optimal investment strategy also on the time interval $(t_1,t_2]$, which is not the case with $(L_t^H)_{t\in[0,T]}$ in the classical framework. We can also provide a counterexample, showing that the minimization of (2.3.63) and (2.3.55) are not equivalent. This is done by considering a particular claim H satisfying at the same time the hypothesis of Theorem 2.3.5 and Theorem 2.3.17, i.e.

$$H = E_G[H] + \int_0^{t_2} \theta_s dB_s + \eta_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\eta_{t_0})\Delta t_1 + \eta_{t_1}\Delta\langle B\rangle_{t_2} - 2G(\eta_{t_1})\Delta t_2,$$

where $0 = t_0 < t_1 < t_2 = T$, $\theta \in M_G^2(0,t_2)$, $\eta_{t_0} \in \mathbb{R}$, $\eta_{t_1} \in L_G^2(\mathscr{F}_{t_1})$ and

$$|\eta_{t_1}| = E_G[|\eta_{t_1}|] + \xi_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\xi_{t_0})\Delta t_1, \tag{2.3.66}$$

with $\xi_{t_0} \in \mathbb{R}_+$. By Theorem 2.3.5, the optimal mean-variance portfolio for such claim can be computed explicitly. At the same time it is also possible to use our iterative scheme to get the optimal investment strategy and obtain an expression analogue to (2.3.53)

$$\begin{aligned}\inf_{\varepsilon} E_G\Bigg[\Bigg(&\frac{(\overline{\sigma}^2 - \underline{\sigma}^2)}{2}\Delta t_2 \left(E_G[|\eta_{t_1}|] + \xi_{t_0}\Delta\langle B\rangle_{t_1} - 2G(\xi_{t_0})\Delta t_1\right) \\ &+ \Bigg|\varepsilon + \left(\eta_{t_0} - \frac{1}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\xi_{t_0}\Delta t_1\right)\Delta\langle B\rangle_{t_1} + \qquad (2.3.67) \\ &- 2\left(G(\eta_{t_0}) - \frac{1}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\Delta t_1 G(\xi_{t_0})\right)\Delta t_1\Bigg|\Bigg)^2\Bigg]. \qquad (2.3.68)\end{aligned}$$

It is then possible to compute explicitly the optimal initial wealth in (2.3.68), as the only random variable appearing $\Delta\langle B\rangle_{t_1}$ is maximally distributed. This value turns out to be different from the one achieved thanks to Theorem 2.3.5, thus contradicting the general equivalence of (2.3.63) and (2.3.55).

2.3.6 Bounds for the Terminal Risk

As it has been observed, the extension to the general piecewise constant case is much more cumbersome. Thanks to the examination in Section 2.3, it is evident the need to tackle the situation in which the process η presents a direct dependence from the G-Brownian motion B, as

$$|\eta_t| = |\eta_0| + \int_0^t \mu_s dB_s,$$

for every $t \in [0,T]$, with $\mu \in M_G^2[0,T]$. To shed some light on this issue, we obtain in the following a lower and upper bound for the optimal terminal risk.

Lemma 2.3.24. *Consider a claim H of the form*

$$H = E_G[H] + \int_0^T \theta_s dB_s + \int_0^T \eta_s d\langle B\rangle_s - 2\int_0^T G(\eta_s)ds,$$

where $\theta \in M_G^2(0,T)$, $\eta \in M_G^1(0,T)$ *and*

$$|\eta_t| = |\eta_0| + \int_0^t \mu_s dB_s,$$

for a certain process $\mu \in M_G^2(0,T)$, *for every* $t \in [0,T]$. *The optimal terminal risk* (2.2.3) *lies in the closed interval* $[\underline{J}(V_0,\phi), \overline{J}(V_0,\phi)]$, *where*

$$\underline{J}(V_0,\phi) = \left(\frac{E_G\left[-\int_0^T \eta_s d\langle B\rangle_s + 2\int_0^T G(\eta_s)ds \right]}{2} \right)^2,$$

$$\overline{J}(V_0,\phi) = E_G\left[\left(\frac{(\overline{\sigma}^2 - \underline{\sigma}^2)}{2} \int_0^T |\eta|_s ds \right)^2 \right].$$

Proof. Let us first consider the upper bound for $J(V_0,\phi)$:

$$\begin{aligned}
&E_G\left[\left(E_G[H] - V_0 + \int_0^T (\theta_s - \phi_s X_s)\, dB_s + \int_0^T \eta_s d\langle B\rangle_s - 2\int_0^T G(\eta_s)ds \right)^2 \right] \\
&\le E_G\left[\left(E_G[H] - V_0 + \int_0^T (\theta_s - \phi_s X_s)\, dB_s \right)^2 \vee \right. \\
&\qquad \left. \left(E_G[H] - V_0 + \int_0^T (\theta_s - \phi_s X_s)\, dB_s - (\overline{\sigma}^2 - \underline{\sigma}^2) \int_0^T |\eta|_s ds \right)^2 \right]
\end{aligned} \tag{2.3.69}$$

$$\begin{aligned}
&= E_G\left[\left(E_G[H] - V_0 + \int_0^T (\theta_s - \phi_s X_s)\, dB_s \right)^2 \vee \left(E_G[H] - V_0 + \right.\right. \\
&\qquad \left.\left. - |\eta_0|(\overline{\sigma}^2 - \underline{\sigma}^2)T + \int_0^T \left(\theta_s - \phi_s X_s - (T-s)(\overline{\sigma}^2 - \underline{\sigma}^2)\mu_s \right) dB_s \right)^2 \right],
\end{aligned} \tag{2.3.70}$$

where it was used that

$$\int_0^T \eta_s d\langle B\rangle_s - 2\int_0^T G(\eta_s)ds \;\in\; [-(\overline{\sigma}^2 - \underline{\sigma}^2)\int_0^T |\eta|_s ds, 0]$$

in (2.3.69) and that

$$\begin{aligned}\int_0^T |\eta|_s ds &= \int_0^T \left(|\eta_0| + \int_0^s \mu_u dB_u\right) ds \\ &= |\eta_0|T + \int_0^T \int_0^s \mu_u dB_u ds \\ &= |\eta_0|T + T\int_0^T \mu_s dB_s - \int_0^T s\mu_s dB_s \\ &= |\eta_0|T + \int_0^T (T-s)\mu_s dB_s\end{aligned}$$

in (2.3.70). We next exploit the change of variables seen in Proposition 2.3.10 and let

$$\begin{aligned}\varepsilon &:= E_G[H] - V_0 - \frac{T}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)|\eta_0|, \\ \psi_t &:= \theta_t - \phi_t X_t - \frac{(T-s)}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\mu_t,\end{aligned}$$

to riformulate (2.3.70) as

$$\begin{aligned}&E_G\left[\left(\frac{T}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)|\eta_0| + \int_0^T \frac{(T-s)}{2}(\overline{\sigma}^2 - \underline{\sigma}^2)\mu_s dB_s + \left|\varepsilon + \int_0^T \psi_s dB_s\right|\right)^2\right] \\ =&E_G\left[\left(\frac{(\overline{\sigma}^2 - \underline{\sigma}^2)}{2}\int_0^T |\eta|_s ds + \left|\varepsilon + \int_0^T \psi_s dB_s\right|\right)^2\right]\end{aligned}$$

which reaches its minimum if $\varepsilon = 0$ and $\psi \equiv 0$ (see also Proposition 2.3.10). To obtain a lower bound we make again use of the G-Jensen inequality. We obtain as in Theorem 2.3.7 these inequalities

$$\begin{aligned}&E_G\Bigg[\Bigg(E_G[H] - V_0 + \int_0^T (\theta_s - \phi_s X_s)\, dB_s + \\ &\qquad\qquad + \int_0^T \eta_s d\langle B\rangle_s - 2\int_0^T G(\eta_s)ds\Bigg)^2\Bigg] \\ &\geq (E_G[H] - V_0)^2 \vee \left(E_G[H] - V_0 + E_G\left[-\int_0^T \eta_s d\langle B\rangle_s + 2\int_0^T G(\eta_s)ds\right]\right)^2 \qquad (2.3.71) \\ &\geq \left(\frac{E_G\left[-\int_0^T \eta_s d\langle B\rangle_s + 2\int_0^T G(\eta_s)ds\right]}{2}\right)^2, \qquad (2.3.72)\end{aligned}$$

where in (2.3.71) we make use of Proposition 2.1.8 and let

$$\bar{V}_0 = E_G[H] - \frac{E_G\left[-\int_0^T \eta_s d\langle B\rangle_s + 2\int_0^T G(\eta_s)ds\right]}{2}$$

to obtain the minimum over V_0 and get (2.3.72). □

2.4 Robust Risk Minimization

The second main approach of quadratic hedging techniques insists on the perfect replication of the claim, i.e. it requires $H = V(V_0, \phi)$. Therefore, while using the same assumptions and notations of Section 2.2, we need to drop the self-financing constraint and consider a different set of admissible trading strategies. This is evident as the only claims which can be replicated perfectly by means of self-financing strategies are symmetric G-martingales.

In analogy to what it is done in [64] for the martingale case, we take into consideration the space of strategies of the following type.

Definition 2.4.1. A *RM*-strategy is a pair $\Upsilon = (\phi_t, \varsigma_t)_{t\in[0,T]}$ where $(\phi_t)_{t\in[0,T]} \in \Phi$, with

$$\Phi := \left\{ \phi \text{ predictable } | E_G\left[\left(\int_0^T \phi_t X_t dB_t\right)^2\right] < \infty \right\},$$

and ς is an adapted process such that $V_t(\Upsilon) = \phi_t X_t + \varsigma_t$ is right continuous and square integrable (i.e. $V_t(\Upsilon) \in L_G^2(\mathscr{F}_t)$ for all $t \in [0,T]$).

We consider the problem of hedging a contingent claim $H \in L_G^{2+\varepsilon}(\mathscr{F}_T)$, for an $\varepsilon > 0$, using RM-strategies such that $V_T(\Upsilon) = H$. We define the cost of a trading strategy as

$$C_t(\Upsilon) = V_t - \int_0^t \phi_s dX_s,$$

for any $t \in [0,T]$. We can then try to define also the risk process as in the standard case, by substituting the usual linear expectation with the G-expectation:

$$R_t(\Upsilon) = E_G\left[(C_T(\Upsilon) - C_t(\Upsilon))^2 \middle| \mathscr{F}_t\right],$$

for any $t \in [0,T]$. The robust risk minimization problem is then to find a trading strategy Υ^* such that $V_T(\Upsilon^*) = H$ and

$$R_t(\Upsilon^*) \le R_t(\Upsilon),$$

q.s. $\forall t \in [0,T]$ and any RM-trading strategy Υ which is an *admissible continuation of* Υ^* *from* t *on*, in the sense that $V_T(\Upsilon) = V_T(\Upsilon^*)$ q.s, $\phi_s^* = \phi_s$ for $s \le t$ and $\varsigma_s^* = \varsigma_s$ for $s < t$. If such RM-strategy Υ^* exists we call it *R-risk minimizing*.

Remark 2.4.2. The existence of a risk minimizing strategy is ensured for all the cases solved for the mean-variance hedging problem. At time zero the two different quadratic hedging approaches are in fact equivalent. On the other hand, the determination of the optimal risk minimizing strategy at time zero completely solves the hedging problem, as it provides the optimal investment strategy for any time $t \in [0,T]$.

In analogy to the solution of the risk-minimization problem for a singleton $\mathscr{P}_{\mathbf{G}}$, one may think that in the G-setting the cost of a R-risk minimizing strategy should be a G-martingale. However a simple reasoning shows that this generalization is not straightforward.

Lemma 2.4.3. *The cost process of the R-risk minimizing portfolio is a G-martingale if and only if H is symmetric.*

Proof. Any trading strategy Υ able to hedge H is in one-to-one relation with a hedging strategy for $-H$: it just suffices to take $-\Upsilon$. Then the cost $C_t(-\Upsilon)$ of $-\Upsilon$ is equal to $-C_t(\Upsilon)$ as

$$C_t(-\Upsilon) = -V_t - \int_0^t (-\phi_s) dX_s = -\left(V_t - \int_0^t \phi_s dX_s\right).$$

It then follows that the two strategies share the same risk for every $t \in [0,T]$. Therefore, given an optimal solution Υ^* to the risk minimizing problem for H, $(-\Upsilon^*)$ would be optimal for $-H$. If the cost process associated to Υ^* is a G-martingale $(-C_t(\Upsilon^*))_{t\in[0,T]}$ would be a G-martingale as well, thus making $(C(\Upsilon^*))_{t\in[0,T]}$ a symmetric G-martingale. However such property can hold only for symmetric claims, which are perfectly hedgeable through self-financing strategies. □

This is one of the reasons why we could take into consideration this alternative risk definition:

$$\bar{R}_t(\Upsilon) = C_T(\Upsilon)^2 - 2C_t(\Upsilon) E_G\left[C_T(\Upsilon) | \mathscr{F}_t\right] + C_t(\Upsilon)^2,$$

derived just developing the square in the former definition of risk, and splitting the terms as if the G-expectation was linear. We call *$\bar{R}$-risk minimizing* a trading strategy $\phi \in \Phi$ minimizing such functional. This risk definition is well posed as we can prove that $\bar{R}_t(\Upsilon) \geq 0$ for all $t \in [0,T]$ and every RM-strategy Υ. In fact, thanks to Lemma 2.1.22

$$E_G\left[C_T(\Upsilon)^2 \middle| \mathscr{F}_t\right] \geq E_G\left[C_T(\Upsilon) | \mathscr{F}_t\right]^2,$$

hence

$$\bar{R}_t(\Upsilon) \geq \left(E_G\left[C_T(\Upsilon) | \mathscr{F}_t\right] - C_t(\Upsilon)\right)^2 \geq 0.$$

Remark 2.4.4. Note that both risk definitions lead to the standard one established in [64] when $\overline{\sigma} = \underline{\sigma}$.

Remark 2.4.5. We emphasize that this different definition disentangle the connection between H and $-H$, as $\bar{R}(\Upsilon) \neq \bar{R}(-\Upsilon)$.

Using $\bar{R}$ as risk functional we can easily show that optimal strategies have cost processes that are G-martingales.

Proposition 2.4.6. *The cost process associated to a $\bar{R}$-risk minimizing strategy is a G-martingale.*

Proof. We follow here the same technique outlined in [64]: given a strategy Υ, we build $\tilde{\Upsilon}$ by taking $\tilde{\phi} \equiv \phi$ and

$$V_t(\tilde{\Upsilon}) = V_t(\Upsilon)\mathbf{1}_{[0,t_0)}(t) + E_G\left[V_T(\Upsilon) - \int_t^T \phi_s dX_s \middle| \mathscr{F}_t\right]\mathbf{1}_{[t_0,T]}(t).$$

By doing so we have

$$\begin{aligned} V_T(\tilde{\Upsilon}) &= V_T(\Upsilon), \\ C_T(\tilde{\Upsilon}) &= C_T(\Upsilon), \\ C_t(\tilde{\Upsilon}) &= E_G\left[C_T(\tilde{\Upsilon})\middle| \mathscr{F}_t\right]. \end{aligned}$$

If Υ is $\bar{R}$-risk minimizing, then

$$\begin{aligned} \bar{R}_t(\Upsilon) = E_G&\left[C_T(\Upsilon)^2\middle| \mathscr{F}_t\right] - 2C_t(\Upsilon)E_G\left[C_T(\Upsilon)\middle| \mathscr{F}_t\right] + C_t(\Upsilon)^2 \\ &- C_t(\tilde{\Upsilon})^2 + C_t(\tilde{\Upsilon})^2 \\ =\;& \bar{R}_t(\tilde{\Upsilon}) + (C_t(\Upsilon)^2 - 2C_t(\Upsilon)C_t(\tilde{\Upsilon}) + C_t(\tilde{\Upsilon}))^2 \\ =\;& \bar{R}_t(\tilde{\Upsilon}) + (C_t(\Upsilon) - C_t(\tilde{\Upsilon}))^2, \end{aligned}$$

and so $(C_t(\Upsilon) - C_t(\tilde{\Upsilon}))^2$ must be equal to zero, thus making $C_t(\Upsilon)$ a G-martingale. □

It then follows that also $V_t(\Upsilon^*)$ has to be a G-martingale. Being, by definition, $V_T(\Upsilon^*) = H$, from Theorem 2.1.18 we have

$$V_t(\Upsilon^*) = E_G\left[H\middle| \mathscr{F}_t\right] = E_G[H] + \int_0^t \theta_s dB_s - K_t, \tag{2.4.1}$$

for every $t \in [0,T]$, where $(-K_t)_{t\in[0,T]}$ is a continuous, decreasing G-martingale with $K_T \in L_G^2(\mathscr{F}_T)$ and $(\theta_t)_{t\in[0,T]}$ is a process in $M_G^2(0,T)$.

Remark 2.4.7. As the value process of an $\bar{R}$-risk minimizing strategy at any time t is given by the conditional G-expectation $E_G[H|\mathscr{F}_t]$, it suffices to specify ϕ^* in order to describe the optimal portfolio. Therefore, as in the martingale case of [64], finding the optimal risk at time 0 provides the full solution to the problem. This is because if one minimizes $\bar{R}_0(\Upsilon)$ then one finds ϕ_t^* for every $t \in [0,T]$.

With the adoption of $\bar{R}$, the robust risk minimization problem becomes then equivalent to find the portfolio attaining the infimum of

$$\bar{R}_t(\Upsilon) = E_G[C_T(\Upsilon)^2|\mathscr{F}_t] - E_G[C_T(\Upsilon)|\mathscr{F}_t]^2 \tag{2.4.2}$$

over the set of RM-trading strategies such that the corresponding cost process is a G-martingale. To do so we start for simplicity by assuming $E_G[H] = 0$ and $\theta \equiv 0$ in (2.4.1), using only trading strategies with zero initial wealth. With this simplifying assumption the risk minimization problem is the following

$$\inf_{\phi} E_G\left[(-K_T - \int_0^T \phi_t dX_t)^2 \middle| \mathscr{F}_t\right] - E_G\left[-K_T - \int_0^T \phi_t dX_t \middle| \mathscr{F}_t\right]^2. \tag{2.4.3}$$

If we further require that our claim belong to the class $L_{ip}(\mathscr{F}_T)$ we can derive the following result.

Proposition 2.4.8. *Let be given a claim $H \in L_{ip}(\mathscr{F}_T)$ of the form*

$$H = \int_0^T \eta_s d\langle B\rangle_s - \int_0^T 2G(\eta_s)ds,$$

with $\eta \in M_G^1(0,T)$. The optimal solution to the $\bar{R}$-risk minimizing problem over the set of RM-strategies with $V_0 = 0$ is $(0,\phi^*) = (0,0)$.

Proof. By applying the Itô formula for G-Brownian motion we can rewrite (2.4.3) as

$$\inf_{\phi} E_G\Bigg[-\int_t^T 2C_u(\Upsilon)\phi_u dB_u - \int_t^T 4C_u(\Upsilon)G(\eta_u)du + \\ + \int_t^T (2C_u(\Upsilon)\eta_u + \phi_u^2)d\langle B\rangle_u \Bigg| \mathscr{F}_t\Bigg].$$

From this expression we note that the optimal strategy is the one having $\phi \equiv 0$. □

It follows that for a generic claim $H \in L_{ip}(\mathscr{F}_T)$ with $E_G[H] = 0$ of the form

$$H = \int_0^T \theta_s dB_s + \int_0^T \eta_s d\langle B\rangle_s - \int_0^T 2G(\eta_s)ds,$$

the optimal strategy is the one having $\phi_t X_t = \theta_t$ for every $t \in [0,T]$.

Studying the two risk definitions we note that there is no one that strictly dominates the other. We have in fact

$$\begin{aligned} R_t(\Upsilon) &= E_G\left[C_T(\Upsilon)^2 - 2C_t(\Upsilon)C_T(\Upsilon) + C_t(\Upsilon)^2 \middle| \mathscr{F}_t\right] \\ &= E_G\left[C_T(\Upsilon)^2 - 2C_t(\Upsilon)C_T(\Upsilon) \middle| \mathscr{F}_t\right] + C_t(\Upsilon)^2 \\ &\geq E_G\left[C_T(\Upsilon)^2 \middle| \mathscr{F}_t\right] - 2E_G\left[C_t(\Upsilon)C_T(\Upsilon) \middle| \mathscr{F}_t\right] + C_t(\Upsilon)^2 \\ &= E_G\left[C_T(\Upsilon)^2 \middle| \mathscr{F}_t\right] - 2C_t(\Upsilon)E_G\left[C_T(\Upsilon) \middle| \mathscr{F}_t\right] + C_t(\Upsilon)^2 = \bar{R}_t(\Upsilon), \end{aligned} \tag{2.4.4}$$

if $C_t(\Upsilon) \geq 0$, and

$$\begin{aligned} R_t(\Upsilon) &\leq E_G\left[C_T(\Upsilon)^2 \middle| \mathscr{F}_t\right] + 2E_G\left[-C_t(\Upsilon)C_T(\Upsilon) \middle| \mathscr{F}_t\right] + C_t(\Upsilon)^2 \\ &= E_G\left[C_T(\Upsilon)^2 \middle| \mathscr{F}_t\right] - 2C_t(\Upsilon)E_G\left[C_T(\Upsilon) \middle| \mathscr{F}_t\right] + C_t(\Upsilon)^2 = \bar{R}_t(\Upsilon), \end{aligned}$$

if $C_t(\Upsilon) \leq 0$.

2.4.1 Optimal $\bar{R}$-Minimizing Strategy

We extend the result of Proposition 2.4.7 to the case where $H \in L_G^{2+\varepsilon}(\mathscr{F}_T)$. Also in this case the optimal strategy is given by the G-martingale representation theorem. Let us suppose, as done before, that $H = -K_T$, where $-K$ is a G-martingale, that ϕ is the optimal strategy and that there exists at least one $t \in [0,T]$ such that $\phi_t \neq 0$. Our aim is to show that this leads to a contradiction.

Proposition 2.4.9. *Let be given a contingent claim $H \in L_G^{2+\varepsilon}(\mathscr{F}_T)$ of the form $H = -K_T$, where $(-K_t)_{t\in[0,T]}$ is a decreasing G-martingale. The optimal $\bar{R}$-risk minimizing strategy is characterized by $\phi \equiv 0$.*

Proof. We call $(H^n)_{n\in\mathbb{N}}$ a sequence of r.v. in $L_{ip}(\mathscr{F}_T)$ such that H^n converges to H in $L_G^{2+\varepsilon}(\mathscr{F}_T)$. We know from [69] that if

$$H^n = E_G[H^n] + \int_0^T \theta_t^n dB_t - K_T^n,$$

then K^n converges in $L_G^{2+\varepsilon}(\mathscr{F}_T)$ to K. For each $n \in \mathbb{N}$ we then have

$$E_G\left[(H^n - \int_0^T \phi_t X_t dB_t)^2\right] \geq E_G\left[(-K_T^n)^2\right],$$

as $\theta^n \equiv 0$ is the solution to the robust risk minimization problem at time 0 for contingent claims belonging to $L_{ip}(\mathscr{F}_T)$. Passing to the limit we would show that $E_G\left[(H^n - \int_0^T \phi_t X_t dB_t)^2\right]$ and $E_G\left[(-K_T^n)^2\right]$ converge to $E_G\left[(H - \int_0^T \phi_t X_t dB_t)^2\right]$

and $E_G\left[(-K_T)^2\right]$ respectively, thus showing that ϕ must be identically zero. We have in fact

$$\begin{aligned}\lim_{n\to\infty}\sup_{P\in\mathscr{P}_{\mathbf{G}}} E^P\left[(K_T)^2\right]-\sup_{P\in\mathscr{P}_{\mathbf{G}}} E^P\left[(K_T^n)^2\right]\leq\\ &\leq \lim_{n\to\infty}\sup_{P\in\mathscr{P}_{\mathbf{G}}} E^P\left[(K_T)^2\right]-E^P\left[(K_T^n)^2\right]\\ &\leq \lim_{n\to\infty}\sup_{P\in\mathscr{P}_{\mathbf{G}}} E^P\left[|(K_T)^2-(K_T^n)^2|\right]\\ &= \lim_{n\to\infty}\sup_{P\in\mathscr{P}_{\mathbf{G}}} E^P\left[|(K_T-K_T^n)(K_T+K_T^n)|\right]\\ &= \lim_{n\to\infty} E_G\left[|(K_T-K_T^n)(K_T+K_T^n)|\right]\\ &\leq \lim_{n\to\infty} E_G\left[(K_T-K_T^n)^2\right]^{1/2} E_G\left[(K_T+K_T^n)^2\right]^{1/2}.\end{aligned}$$

The term $E_G\left[(K_T-K_T^n)^2\right]^{1/2}$ tends to zero as $L_G^{2+\varepsilon}(\mathscr{F}_T)$ convergence implies $L_G^2(\mathscr{F}_T)$ convergence. The second term is dominated by

$$E_G\left[(K_T)^2\right]^{1/2}+E_G\left[(K_T^n)^2\right]^{1/2}<\infty,$$

as $K\in L_G^2(\mathscr{F}_T)$ and K^n converges to K in $L_G^2(\mathscr{F}_T)$. The same reasoning holds true in the light of the convergence of $E_G\left[(H^n-\int_0^T\phi_t X_t dB_t)^2\right]$ to $E_G\left[(H-\int_0^T\phi_t X_t dB_t)^2\right]$.
□

2.4.2 G-Variance

Despite the results obtained with the risk-definition $\bar{R}_t(\Upsilon)$, its financial interpretation is not completely clear. The equivalent expression we get in the case in which the cost process is a G-martingale

$$E_G\left[C_T(\Upsilon)^2\middle|\mathscr{F}_t\right]-E_G\left[C_T(\Upsilon)\middle|\mathscr{F}_t\right]^2$$

resembles the standard definition of variance, giving us the hope to justify a posteriori the setting of our problem. In this case the risk minimization issue would be equivalent to the minimization of the variance of $C_T(\Upsilon)$.
It should be remembered that, to the best of our knowledge, no definition of variance has been given yet in the G-setting. Scouting the possible definitions of G-variance, we think there are three natural admissible formulations that lead to the classical one as soon as $\overline{\sigma}=\underline{\sigma}$:

1. $\mathrm{Var}_{G,1}(X)=\sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}}E_{\mathbb{P}}\left[(X-E_{\mathbb{P}}[X])^2\right]$,

2. $\mathrm{Var}_{G,2}(X) = E_G\left[(X - E_G[X])^2\right]$,

3. $\mathrm{Var}_{G,3}(X) = E_G\left[X^2\right] - E_G[X]^2$,

where $X \in L_G^2(\mathscr{F}_T)$. These three definitions deliver different values of the variance for the same process. Keeping in mind the interpretation of the G-expectation as a *worst case expectation*, it seems obvious that $\mathrm{Var}_{G,1}$ should be the expression to adopt. Nevertheless it has some drawbacks. In [73] it is said that the quadratic variation of the G-Brownian motion has no variance uncertainty. This is consistent with $\mathrm{Var}_{G,3}$ but if we assume $\mathrm{Var}_{G,1}$ this is not true anymore. A direct calculus shows that

$$\begin{aligned}
\mathrm{Var}_{G,1}(\langle B\rangle_t) &= \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}\left[(\langle B\rangle_t - E_{\mathbb{P}}[\langle B\rangle_t])^2\right] \\
&\geq \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}\left[(\langle B\rangle_t - E_{\mathbb{P}}[\langle B\rangle_t])^2\right] \\
&= \sup_{\sigma\in\mathscr{A}_{0,T}^{\Theta}} E_{\mathbb{P}_0}\left[(\langle B^\sigma\rangle_t - E_{\mathbb{P}_0}[\langle B^\sigma\rangle_t])^2\right] \\
&= \sup_{\sigma\in\mathscr{A}_{0,T}^{\Theta}} E_{\mathbb{P}_0}\left[\left(\int_0^t \sigma_s^2 ds - E_{\mathbb{P}_0}\left[\int_0^t \sigma_s^2 ds\right]\right)^2\right] \\
&= \sup_{\sigma\in\mathscr{A}_{0,T}^{\Theta}} \mathrm{Var}_{\mathbb{P}_0}\left(\int_0^t \sigma_s^2 ds\right) > 0.
\end{aligned}$$

Moreover it is impossible to find any monotonicity relation between $\mathrm{Var}_{G,3}$ and the other variance definitions. For example when $X \geq 0$,

$$\begin{aligned}
\mathrm{Var}_{G,1}(X) = \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}\left[(X - E_{\mathbb{P}}[X])^2\right] &= \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}\left[X^2 - E_{\mathbb{P}}[X]^2\right] \\
&\geq \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}\left[X^2 - \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}}\left(E_{\mathbb{P}}[X]^2\right)\right] \\
&= \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}\left[X^2\right] - \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}}\left(E_{\mathbb{P}}[X]^2\right) \\
&= \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}\left[X^2\right] - \left(\sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}[X]\right)^2 \\
&= E_G\left[X^2\right] - E_G[X]^2 = \mathrm{Var}_{G,3}(X),
\end{aligned}$$

and

$$\mathrm{Var}_{G,3}(X) = E_G\left[X^2\right] - E_G[X]^2 \leq E_G\left[(X - E_G[X])^2\right] = \mathrm{Var}_{G,2}(X),$$

in the same way as in (2.4.4). On the other hand if $X \leq 0$,

$$\begin{aligned}
\mathrm{Var}_{G,1}(X) = \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}\left[(X - E_{\mathbb{P}}[X])^2\right] &= \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}\left[X^2 - E_{\mathbb{P}}[X]^2\right] \\
&\leq \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}\left[X^2 - \inf_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}}\left(E_{\mathbb{P}}[X]^2\right)\right] \\
&= \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}[X^2] - \inf_{P\in\mathscr{P}_{\mathbf{G}}}\left(E_{\mathbb{P}}[X]^2\right) \\
&= \sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}\left[X^2\right] - \left(\sup_{\mathbb{P}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}[X]\right)^2 \\
&= E_G\left[X^2\right] - E_G[X]^2 = \mathrm{Var}_{G,3}(X),
\end{aligned}$$

and

$$\mathrm{Var}_{G,3}(X) = E_G\left[X^2\right] - E_G[X]^2 \geq E_G\left[(X - E_G[X])^2\right] = \mathrm{Var}_{G,2}(X)$$

using a the same approach as above.

3 Robust Superreplication

In this chapter we deal with the framework introduced in [52], which embeds the G-setting described in Chapter 2. This proves to be useful at it allows us to produce concrete examples where uncertainty derives from volatility ambiguity. In addition this model has been adopted by other authors to investigate some of the traditional problems of mathematical finance (see for example [10], [50], [51] and [53]).
The main inspiration for the findings of this chapter is [50], where the author provides an optional decomposition theorem (see Theorem 3.1.13) which is then exploited to obtain a version of superreplication duality under uncertainty. In Theorem 3.2.1 we tackle a dynamic variant of the same problem and show, as suggested in [50], that Theorem 3.1.13 describes the dynamic value of the optimal superreplicating strategy of an upper semianalytic function. This is of major importance for the investigation of financial bubbles under uncertainty that we outline in Chapter 4. In addition we provide an alternative representation of the conditional sublinear expectation from Theorem 2.3. in [52], proving that the essential supremum can be achieved on a bigger collection of probabilities.

3.1 The Setting

We introduce here a generalization of the market model analyzed in Chapter 2, which we are going to use in the following part of the thesis.
Let $\mathscr{P}$ be a set of priors, possibly non-dominated, on $\Omega = C_0(\mathbb{R}_+, \mathbb{R}^d)$, the family of continuous functions $\omega = (\omega_s)_{s\geq 0}$ in $\mathbb{R}^d$ such that $\omega_0 = 0$. We equip this space with the topology of locally uniform convergence and denote with $B = \{B_u(\omega)\}$ the canonical process. Let us then indicate with $\mathscr{F}$ the Borel σ-algebra on Ω. We need to impose some restrictions on the family of probability measures and on the $\mathscr{F}$-measurable functions we are going to take into consideration. The purpose is to be able to study, for some $\mathscr{F}$-measurable function ξ, operators of the type

$$\xi \mapsto \mathscr{E}_0(\xi) := \sup_{\mathbb{P}\in\mathscr{P}} E_{\mathbb{P}}[\xi],$$

which can be extended to time consistent conditional sublinear expectations. Denoting with $\mathbb{F} := \{\mathscr{F}_t\}_{t\geq 0}$ the natural filtration generated by B, for any stopping time τ the most relevant problem is the well-posedness of the following expression

$$\mathscr{E}_\tau(\xi) = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathscr{P}(\tau,\mathbb{P})} E_{\mathbb{P}'}[\xi|\mathscr{F}_\tau] \qquad \mathbb{P}-a.s. \text{ for all } \mathbb{P} \in \mathscr{P}, \tag{3.1.1}$$

where $\mathscr{P}(\tau,\mathbb{P}) = \{\mathbb{P}' \in \mathscr{P} \,:\, \mathbb{P}' = \mathbb{P} \text{ on } \mathscr{F}_\tau\}$. The framework developed in [52], which we adopt, tackles this issue by enforcing some regularity on the random variables and by shrinking the set of probabilities $\mathscr{P}$. Other settings obtain similar results using different assumptions, we refer to [15], [51], [52], [57] and [67] to cite the most important results on this regard.
In addition to the above, the model in [52] has the advantage of ensuring the tractability of stopping times, while incorporating the G-expectation framework.
We begin by introducing the necessary notation, as done in [52], and by clarifying the assumptions we have to enforce on the set of priors $\mathscr{P}$. We start by providing a generalization of the definition of polar set given in Definition 2.1.10.

Definition 3.1.1. A set is said *polar* if it is $(\mathscr{F}_T,\mathbb{P})$-null for all $\mathbb{P} \in \mathscr{P}$. The collection of all polar sets is denoted by $\mathscr{N}^{\mathscr{P}}$.

We need to introduce some results regarding the theory of analytic sets. We start by reminding that a subset of a Polish space is said *analytic* if it can be expressed as the image of a Borel subset of another Polish space for a Borel-measurable function. It then follows as an easy consequence that every Borel set is analytic. The family of analytic sets is generally not stable under complementation, thus preventing it from being a σ-algebra.

Definition 3.1.2. Let X be a Borel space. The *analytic σ-algebra* $\mathscr{A}_X$ is the smallest σ-algebra that contains the analytic subsets of X.

In addition we introduce the notion of *universal completion* of a σ-algebra.

Definition 3.1.3. Given a σ-field $\mathscr{G}$, the universal completion of $\mathscr{G}$ is the σ-field $\mathscr{G}^* = \cap_{\mathbb{P}}\mathscr{G}^{\mathbb{P}}$, where $\mathbb{P}$ ranges over all probability measures on $\mathscr{G}$ and $\mathscr{G}^{\mathbb{P}}$ is the completion of $\mathscr{G}$ under $\mathbb{P}$.

Remark 3.1.4. It follows from Definition 3.1.3 that, for $t \in [0,T]$, $\mathscr{F}_t^*$ does not include all the polar sets. In fact, for any $\mathbb{P} \in \mathscr{P}$, it holds

$$\mathscr{F}_t^{\mathbb{P}} = \mathscr{F}_t \vee \mathscr{N}_t^{\mathbb{P}},$$

where

$$\mathscr{N}_t^{\mathbb{P}} = \{N \subseteq \Omega \mid \exists\, C \in \mathscr{F}_t \text{ such that } N \subseteq C \text{ and } \mathbb{P}(C) = 0\},$$

while $\mathscr{N}^{\mathscr{P}} = \cap_{\mathbb{P}\in\mathscr{P}}\mathscr{N}_T^{\mathbb{P}}$.

For any stopping time τ, we introduce the concatenation of $\omega, \tilde{\omega} \in \Omega$ at τ as the function

$$(\omega \otimes_\tau \tilde{\omega})_u := \omega_u \mathbf{1}_{[0,\tau(\omega))}(u) + (\omega_{\tau(\omega)} + \tilde{\omega}_{u-\tau(\omega)})\mathbf{1}_{[\tau(\omega),\infty)}(u), \quad u \geq 0.$$

For any function ξ on Ω and $\omega \in \Omega$, the function $\xi^{\tau,\omega}$ on Ω is defined by

$$\xi^{\tau,\omega}(\tilde{\omega}) := \xi(\omega \otimes_\tau \tilde{\omega}), \quad \tilde{\omega} \in \Omega.$$

Denote with $\mathfrak{P}(\Omega)$ the family of all probabilities on the space $(\Omega, \mathscr{F})$ endowed with the topology of weak convergence. For any prior $\mathbb{P} \in \mathfrak{P}(\Omega)$ there is a regular conditional probability distribution given $\mathscr{F}_\tau$, denoted $\{\mathbb{P}^\omega_\tau\}_{\omega \in \Omega}$, i.e. $\mathbb{P}^\omega_\tau \in \mathfrak{P}(\Omega)$ for every ω, whilst $\omega \mapsto \mathbb{P}^\omega_\tau(A)$ is $\mathscr{F}_\tau$-measurable for every $A \in \mathscr{F}$ and

$$E_{\mathbb{P}^\omega_\tau}[\xi] = E_{\mathbb{P}}[\xi | \mathscr{F}_\tau](\omega) \quad \text{for} \quad \mathbb{P}-a.e.\ \omega \in \Omega,$$

provided that ξ is bounded and $\mathscr{F}$-measurable. In addition the probability measures $\mathbb{P}^\omega_\tau$ can be assumed to satisfy the following property

$$\mathbb{P}^\omega_\tau\{\omega' \in \Omega : \ \omega' = \omega \text{ on } [0, \tau(\omega)]\} = 1 \quad \text{for all } \omega \in \Omega,$$

so that they assign probability one to the paths coinciding with ω up to $\tau(\omega)$. We next introduce the prior $\mathbb{P}^{\tau,\omega} \in \mathfrak{P}(\Omega)$ by the condition

$$\mathbb{P}^{\tau,\omega}(A) := \mathbb{P}^\omega_\tau(\omega \otimes_\tau A), \quad A \in \mathscr{F}, \quad \text{where } \omega \otimes_\tau A := \{\omega \otimes_\tau \tilde{\omega} : \ \tilde{\omega} \in A\}.$$

It is then possible to derive the following relations

$$E_{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}] = E_{\mathbb{P}^\omega_\tau}[\xi] = E_{\mathbb{P}}[\xi | \mathscr{F}_\tau](\omega) \quad \text{for} \quad \mathbb{P}-a.e.\ \omega \in \Omega.$$

Given any $(s, \omega) \in \mathbb{R}_+ \times \Omega$ we specify a family $\mathscr{P}(s, \omega) \subseteq \mathfrak{P}(\Omega)$. We assume that

$$\mathscr{P}(s,\omega) = \mathscr{P}(s,\tilde{\omega}) \qquad \text{if} \qquad \omega|_{[0,s]} = \tilde{\omega}|_{[0,s]}.$$

We can now outline Assumption 2.1 from [52].

Assumption 3.1.5. *Let $(s, \bar{\omega}) \in \mathbb{R}_+ \times \Omega$, let τ be a stopping time such that $\tau \geq s$ and $\mathbb{P} \in \mathscr{P}(s, \bar{\omega})$. Set $\theta := \tau^{s,\bar{\omega}} - s$.*

(i) Measurability*: The graph $\{(\mathbb{P}', \omega) : \ \omega \in \Omega,\ \mathbb{P}' \in \mathscr{P}(\tau, \omega)\} \subseteq \mathfrak{P}(\Omega) \times \Omega$ is analytic.*

(ii) Invariance*: We have $\mathbb{P}^{\theta,\omega} \in \mathscr{P}(\tau, \bar{\omega} \otimes_s \omega)$ for $\mathbb{P}$-a.e $\omega \in \Omega$.*

(iii) Stability under pasting*: If $\nu : \Omega \to \mathfrak{P}(\Omega)$ is a $\mathscr{F}_\theta$-measurable kernel and $\nu(\omega) \in \mathscr{P}(\tau, \bar{\omega} \otimes_s \omega)$ for $\mathbb{P}$-a.e $\omega \in \Omega$, then the measure defined by*

$$\bar{\mathbb{P}}(A) = \int \int (\mathbf{1}_A)^{\theta,\omega}(\omega')\nu(d\omega';\omega)\mathbb{P}(d\omega), \qquad A \in \mathscr{F} \tag{3.1.2}$$

is an element of $\mathscr{P}(s, \bar{\omega})$.

Under the previous assumptions, Theorem 2.3 in [52] shows the following.

Theorem 3.1.6. *Let $\sigma \leq \tau$ be stopping times and $\xi : \Omega \to \bar{\mathbb{R}}$ be an upper semianalytic function. Then under Assumption 3.1.5 the function*

$$\mathscr{E}_\tau(\xi)(\omega) := \sup_{\mathbb{P} \in \mathscr{P}(\tau,\omega)} E_{\mathbb{P}}[\xi^{\tau,\omega}], \quad \omega \in \Omega$$

is $\mathscr{F}_\tau^*$-measurable and upper semianalytic. Moreover

$$\mathscr{E}_\sigma(\xi)(\omega) = \mathscr{E}_\sigma(\mathscr{E}_\tau(\xi))(\omega) \quad \text{for all} \quad \omega \in \Omega. \tag{3.1.3}$$

Furthermore,

$$\mathscr{E}_\tau(\xi) = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathscr{P}(\tau,\mathbb{P})} E_{\mathbb{P}'}[\xi | \mathscr{F}_\tau] \quad \mathbb{P}-a.s. \quad \text{for all} \quad \mathbb{P} \in \mathscr{P}, \tag{3.1.4}$$

where $\mathscr{P}(\tau, \mathbb{P}) = \{\mathbb{P}' \in \mathscr{P} : \mathbb{P}' = \mathbb{P}$ on $\mathscr{F}_\tau\}$, and in particular

$$\mathscr{E}_\sigma(\xi) = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathscr{P}(\sigma,\mathbb{P})} E_{\mathbb{P}'}[\mathscr{E}_\tau(\xi) | \mathscr{F}_\sigma] \quad \mathbb{P}-a.s. \quad \text{for all} \quad \mathbb{P} \in \mathscr{P}. \tag{3.1.5}$$

Remark 3.1.7. The reason for the need of $\mathscr{E}(\xi)$ to be $\mathbb{F}^*$-adapted in place of $\mathbb{F}$-adapted is treated in detail in [52] and it is deeply related with the use of upper semianalytic random variables. An extensive discussion of this issue is beyond the purposes of this thesis, we just mention that a crucial problem is that $\mathscr{E}_t(\xi)$ does not admit in general a Borel-measurable version (for further details we refer to Section 5.2 in [52]).
We also point out that (3.1.4) does not imply that $\mathscr{E}_\tau(\xi)$ is $\mathscr{F}_\tau$-measurable. The term on the right end side of (3.1.4) is indeed $\mathscr{F}_\tau$-measurable, being the essential supremum of $\mathscr{F}_\tau$-measurable random variables, but $\mathscr{E}_\tau(\xi)$ is simply $\mathbb{P}$-a.s. equal to it for each $\mathbb{P} \in \mathscr{P}$ and therefore does not need to satisfy the same measurability condition.

We denote $\mathscr{P}$-martingale or *robust martingale*, a stochastic process $M = (M_s)_{s \geq 0}$ such that $\mathscr{E}_0(M_t) < \infty$ for every t and

$$M_t = \mathscr{E}_t(M_T) \quad \mathscr{P}-q.s.$$

for any $T \geq t$. Any element of the subclass of $\mathscr{P}$-martingales, for which the robust martingale property holds also for $(-M)$, is called $\mathscr{P}$-symmetric martingale.

It is remarkable that the G-setting can be represented as a particular case of the model just outlined. More in detail, we need to look at the family of martingale measures

$$\mathfrak{M} = \{\mathbb{P} \in \mathfrak{P}(\Omega) : \ B \text{ is a local } \mathbb{P}\text{-martingale}\}$$

and its subfamily

$$\mathfrak{M}_a = \{\mathbb{P} \in \mathfrak{M} : \ \langle B \rangle^{\mathbb{P}} \text{ is absolutely continuous } \mathbb{P}\text{-a.s.}\}.$$

The following result is Proposition 3.1 from [52].

Proposition 3.1.8. *The set*

$$\mathscr{P}_{\boldsymbol{G}} = \{\mathbb{P} \in \mathfrak{M}_a : \ d\langle B \rangle_t^{\mathbb{P}} / dt \in \boldsymbol{D} \ \ \mathbb{P} \times dt - a.e.\},$$

where $\boldsymbol{D}$ is a nonempty, convex and compact subset of $\mathbb{R}^{d \times d}$, satisfies Assumption 3.1.5.

In the one dimensional case, it is a consequence of Theorem 2.1.9 that the operator

$$\mathscr{E}_0^{\mathbf{G}}(\xi) := \sup_{\mathbb{P} \in \mathscr{P}_{\mathbf{G}}} E_{\mathbb{P}}[\xi]$$

induces the G-expectation on the space $L_G^1(\mathscr{F}_T)$. For a general proof in the multi-dimensional case we refer to [52].
We observe that can give a different representation of the operator from (3.1.4). This is done by considering an enlarged set of probabilities with respect to which we consider the essential supremum. To this purpose, fixed arbitrarily a stopping time τ and a prior $\mathbb{P} \in \mathscr{P}$, we define

$$\mathscr{P}_{\text{eq}}(\tau, \mathbb{P}) := \{\mathbb{P}' \in \mathscr{P} : \ \mathbb{P}' \sim \mathbb{P} \text{ on } \mathscr{F}_\tau\} \supseteq \mathscr{P}(\tau, \mathbb{P}), \tag{3.1.6}$$

which is the family of the probabilities in $\mathscr{P}$ which are *equivalent* to $\mathbb{P}$ restricted to $\mathscr{F}_\tau$. In general the set $\mathscr{P}_{\text{eq}}(\tau, \mathbb{P})$ is strictly bigger than $\mathscr{P}(\tau, \mathbb{P})$. One case in which we can observe equality in (3.1.6) is when $\mathscr{P}$ is some collection of martingale measures relative to some process S, each one modeling a complete market.

Proposition 3.1.9. *Let τ be a stopping time and $\xi : \Omega \mapsto \bar{\mathbb{R}}$ an upper semianalytic function. Then under Assumption 3.1.5 for every $\mathbb{P} \in \mathscr{P}$ it holds*

$$\mathscr{E}_\tau(\xi) = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathscr{P}_{\text{eq}}(\tau, \mathbb{P})} E_{\mathbb{P}'}[\xi | \mathscr{F}_\tau] \quad \mathbb{P} - a.s.$$

Proof. We just need to show

$$\underset{\mathbb{P}'\in\mathscr{P}(\tau,\mathbb{P})}{\operatorname{ess\,sup}} E_{\mathbb{P}'}[\xi|\mathscr{F}_\tau] \geq \underset{\mathbb{P}'\in\mathscr{P}_{\text{eq}}(\tau,\mathbb{P})}{\operatorname{ess\,sup}} E_{\mathbb{P}'}[\xi|\mathscr{F}_\tau] \quad \mathbb{P}-a.s. \tag{3.1.7}$$

as the other inequality is clear. Fixed an arbitrary $\mathbb{P}\in\mathscr{P}$, we let

$$A^{\mathbb{P}} := \{\omega\in\Omega \mid \mathscr{E}_\tau(\xi)(\omega) = \underset{\mathbb{P}'\in\mathscr{P}(\tau,\mathbb{P})}{\operatorname{ess\,sup}} E_{\mathbb{P}'}[\xi|\mathscr{F}_\tau](\omega)\}.$$

It is a consequence of Theorem 3.1.6 that $A^{\mathbb{P}}\in\mathscr{F}_\tau^*\subseteq\mathscr{F}_\tau^{\mathbb{P}}$. Therefore there exist $F\in\mathscr{F}_\tau$ and $N\in\mathscr{N}_\tau^{\mathbb{P}}$ ([1] page 18), where

$$\mathscr{N}_\tau^{\mathbb{P}} := \{N\subset\Omega \mid \exists\, C\in\mathscr{F}_\tau \text{ such that } N\subset C \text{ and } \mathbb{P}(C)=0\},$$

such that $A^{\mathbb{P}} = F\cup N$ and for every $\mathbb{Q}\in\mathscr{P}_{\text{eq}}(\tau,\mathbb{P})$

$$1 = \mathbb{P}(A^{\mathbb{P}}) = \mathbb{P}(F) = \mathbb{Q}(F) = \mathbb{Q}(A^{\mathbb{P}}).$$

Hence for every $\mathbb{Q},\mathbb{P}'\in\mathscr{P}_{\text{eq}}(\tau,\mathbb{P})$ it holds

$$\mathscr{E}_\tau(\xi) \geq E_{\mathbb{P}'}[\xi|\mathscr{F}_\tau] \quad \mathbb{Q}-a.s.$$

Being $\mathbb{P}\in\mathscr{P}$ and $\mathbb{Q}\in\mathscr{P}_{\text{eq}}(\tau,\mathbb{P})$ fixed arbitrarily, (3.1.7) is a consequence of the definition of essential supremum. □

Example 3.1.10. It is possible to have a more direct intuition of the result of Proposition 3.1.9 when $\mathscr{F}_\tau$ is generated by a finite partition of Ω. This case is similar to a discrete time financial market, and for simplicity we study a model with terminal time $T=2$. Let be given a Polish space Ω. We denote $\Omega_t := \Omega^t$ to be the t-fold Cartesian product of Ω for $t\in\{0,1,2\}$, assuming that Ω_0 is composed of a unique element. We then set $\mathscr{F}_T = \mathscr{B}(\Omega_T)$ and $\mathscr{F}_1 = \sigma(A_1,\dots,A_n)$, for a partition of Ω given by $(A_i)_{i=1,\dots,n}$, where $n\in\mathbb{N}$.
As in [11], which is the main inspiration for this model, we make the hypothesis that for each $t\in\{0,1\}$ and $\omega\in\Omega_t$ is given a nonempty set $\mathscr{P}_t(\omega)\subseteq\mathfrak{P}(\Omega)$ of priors. We also assume that $\mathscr{P}_t$ is provided with a universally measurable kernel $\mathbb{P}_t : \Omega_t\to\mathfrak{P}(\Omega)$ such that $\mathbb{P}_t(\omega)\in\mathscr{P}_t(\omega)$ for all $\omega\in\Omega_t$. In this way we can consider a family of probabilities $\mathscr{P}$ such that for any $\mathbb{P}\in\mathscr{P}$ and $A\in\mathscr{F}_T$ we have

$$\mathbb{P}(A) = \int_\Omega\int_\Omega \mathbf{1}_A(\omega_1,\omega_2)\mathbb{P}_1(\omega_1;d\omega_2)\mathbb{P}_0(d\omega_1),$$

where $\omega=(\omega_1,\omega_2)$ is an element in Ω_T and $\mathbb{P}_t(\cdot)\in\mathscr{P}_t(\cdot)$, for $t\in\{0,1\}$. In other terms any $\mathbb{P}\in\mathscr{P}$ can be represented as

$$\mathbb{P} = \mathbb{P}_0\otimes\mathbb{P}_1, \tag{3.1.8}$$

for some $\mathbb{P}_0(\cdot) \in \mathscr{P}_0(\cdot)$ and $\mathbb{P}_1(\cdot) \in \mathscr{P}_1(\cdot)$.
Fixed any $\mathscr{F}_T$-measurable function ξ and any probability measure $\mathbb{P} \in \mathscr{P}$ we have

$$E_{\mathbb{P}}[\xi|\mathscr{F}_1] = \sum_{i=1}^{n} \frac{E[\xi \mathbf{1}_{A_i}]}{\mathbb{P}(A_i)} \mathbf{1}_{A_i}. \tag{3.1.9}$$

We can rewrite all the elements on the right hand side of (3.1.9) as

$$\begin{aligned}
\frac{E[\xi \mathbf{1}_{A_i}]}{\mathbb{P}(A_i)} &= \frac{\int_\Omega \int_\Omega \mathbf{1}_{A_i}(\omega_1,\omega_2)\xi(\omega_1,\omega_2)\mathbb{P}_1(\omega_1;d\omega_2)\mathbb{P}_0(d\omega_1)}{\int_\Omega \int_\Omega \mathbf{1}_{A_i}(\omega_1,\omega_2)\mathbb{P}_1(\omega_1;d\omega_2)\mathbb{P}_0(d\omega_1)} \\
&= \frac{\int_{A_i} \int_\Omega \xi(\omega_1,\omega_2)\mathbb{P}_1(\omega_1;d\omega_2)\mathbb{P}_0(d\omega_1)}{\int_{A_i} \mathbb{P}_0(d\omega_1)} \\
&= \frac{\int_{A_i} \xi_{\mathbb{P}_1}^{A_i}(\omega_1)\mathbb{P}_0(d\omega_1)}{\int_{A_i} \mathbb{P}_0(d\omega_1)}
\end{aligned} \tag{3.1.10}$$

where $\xi_{\mathbb{P}_1}^{A_i}(\omega_1) := \int_\Omega \xi(\omega_1,\omega_2)\mathbb{P}_1(\omega_1;d\omega_2)$. Since the value of $\xi_{\mathbb{P}_1}^{A_i}(\omega_1)$ has to be constant on A_i, we indicate it $\xi_{\mathbb{P}_1}^{A_i}$ and it is a consequence of (3.1.10) that

$$\frac{E[\xi \mathbf{1}_{A_i}]}{\mathbb{P}(A_i)} = \frac{\xi_{\mathbb{P}_1}^{A_i} \int_{A_i} \mathbb{P}_0(d\omega_1)}{\int_{A_i} \mathbb{P}_0(d\omega_1)} := \xi_{\mathbb{P}_1}^{A_i}.$$

We finally obtain $E_{\mathbb{P}}[\xi|\mathscr{F}_1] = \sum_{i=1}^n \xi_{\mathbb{P}_1}^{A_i} \mathbf{1}_{A_i}$. By virtue of (3.1.8), it is possible to represent any prior $\mathbb{P}' \in \mathscr{P}$ as $\mathbb{P}' = \mathbb{P}'_0 \otimes \mathbb{P}'_1$. Therefore, in order to guarantee that $\mathbb{P}' \in \mathscr{P}(1,\mathbb{P})$, we must require some particular conditions for the term $\mathbb{P}'_0$, while $\mathbb{P}'_1$ can be arbitrary. We can repeat the same reasoning also for $\bar{\mathbb{P}} \in \mathscr{P}_{eq}(1,\mathbb{P})$. Altogether we have that

$$\operatorname*{ess\,sup}_{\mathbb{P}' \in \mathscr{P}(1,\mathbb{P})} E_{\mathbb{P}'}[\xi|\mathscr{F}_1] = \operatorname*{ess\,sup}_{\bar{\mathbb{P}} \in \mathscr{P}_{\mathrm{eq}}(1,\mathbb{P})} E_{\bar{\mathbb{P}}}[\xi|\mathscr{F}_1] \quad \mathbb{P}-a.s.$$

for all $\mathbb{P} \in \mathscr{P}$.

We conclude this section by introducing the filtrations $\mathbb{G} = (\mathscr{G}_t)_{0 \le t \le T}$, where

$$\mathscr{G}_t := \mathscr{F}_t^* \vee \mathscr{N}^{\mathscr{P}}$$

and $\mathbb{G}_+$, the right-continuous filtration defined by $\mathscr{G}_{t+} = \cap_{s>t} \mathscr{G}_t$. We consider, as done in [50], an asset price process S which is $\mathbf{R}^d$-valued, $\mathbb{G}_+$-adapted with càdlàg paths. We call a $\mathbb{G}$-predictable process an *admissible trading strategy* if $H \cdot S$ is a $\mathbb{P}$-supermartingale for all $\mathbb{P} \in \mathscr{P}$, and we denote by $\mathscr{H}$ the set of all such processes. Finally we define the notion of *saturated* set of priors.

Definition 3.1.11. A set $\mathscr{P}$ of probability measures on $(\Omega, \mathscr{F})$ is called *saturated* if it contains all equivalent sigma martingale measures of its elements

3.1.1 The Filtration

The aim of this section is to provide a clear understanding of the measurability issues that led to the choice of the different filtrations used in [50], when proving the superreplication duality formula

$$\begin{aligned}&\sup_{\mathbb{P}\in\mathscr{P}} E_{\mathbb{P}}[f]\\ &= \min\{x\in\mathbb{R} : \exists\, H\in\mathscr{H} \text{ with } x+H\cdot S\geq f\ \mathbb{P}-a.s. \text{ for all } \mathbb{P}\in\mathscr{P}\},\end{aligned} \tag{3.1.11}$$

where f is an upper semianalytic, $\mathscr{G}_T$-measurable function such that $\mathscr{E}_0(|f|)<\infty$. We start with some preliminary notions.

Definition 3.1.12. A real-valued, $\mathbb{F}$-adapted process with càdlàg paths is called $\mathscr{P}$ local supermartingale if it is a local supermartingale with respect to $(\mathbb{P},\mathbb{F}^{\mathbb{P}}_{+})$ for all $\mathbb{P}\in\mathscr{P}$.

As noticed in [50] the adoption of $\mathbb{F}^{\mathbb{P}}_{+}$ in Definition 3.1.12 is the most general possible. We can prove that a $\mathbb{F}$-adapted and right-continuous process which is a local supermartingale with respect to $(\mathbb{P},\tilde{\mathbb{F}})$, for a filtration $\mathbb{F}\subseteq\tilde{\mathbb{F}}\subseteq\mathbb{F}^{\mathbb{P}}_{+}$, keeps the local supermartingale property with respect to $(\mathbb{P},\mathbb{F}^{\mathbb{P}}_{+})$.
We next report the robust optional decomposition theorem needed to prove (3.1.11).

Theorem 3.1.13 (Theorem 2.4 from [50]). *Let $\mathscr{P}$ be a nonempty, saturated set of local martingale measures for S. If Y is a $\mathscr{P}$ local supermartingale, then there exists an $\mathbb{F}$-predictable process H which is S-integrable for every $\mathbb{P}\in\mathscr{P}$ and such that*

$$Y-H\cdot S \textit{ is increasing } \mathbb{P}\textit{-a.s. for all } \mathbb{P}\in\mathscr{P}. \tag{3.1.12}$$

We point out that the filtration in Theorem 3.1.13 is chosen arbitrary. Nevertheless, as a consequence of (3.1.12), the processes $Y=(Y_t)_{t\in[0,T]}$ and $S=(S_t)_{t\in[0,T]}$ need to satisfy the same measurability condition.
The processes Y derived within the proof of Theorem 3.2 of [50] requires $\mathbb{G}_{+}$-adaptedness. For any $t\in[0,T]$, the function Y_t is obtained from the process Y', which is

$$Y'_t=\limsup_{r\downarrow t,\, r\in\mathbb{Q}}\mathscr{E}_r(f), \qquad \forall\, t\in[0,T], \tag{3.1.13}$$

thanks to a modification theorem for supermartingales (Theorem VI.2 in [19]). It then follows that Y is equal to Y' outside of a polar set N, i.e.

$$Y:=Y'\mathbf{1}_{N^c}. \tag{3.1.14}$$

As $\mathscr{E}_t(f)$ is $\mathscr{F}^*_t$-measurable and a fortiori $\mathscr{G}_t$-measurable for every $t\in[0,T]$, the process Y' is $\mathbb{G}_{+}$-adapted. In addition, as for every $t\in[0,T]$ the σ-algebra $\mathbb{G}_t$

includes all the polar sets, Theorem VI.2 in [19] guarantees that also Y is $\mathbb{G}_+$-adapted. This explains why $\mathbb{G}_+$ is the most suitable filtration to adopt when dealing with the superreplication duality: it is the smallest filtration which is right-continuous and contains the polar sets.
We conclude by recalling that, because of (3.1.12), this also leads to the requirement of the asset price S being $\mathbb{G}_+$-adapted.

3.2 Dynamic Superhedging

In this section we derive the superhedging price at time $t \in (0,T]$ of a $\mathscr{G}_T$-measurable function. We derive a *positive* and a *negative* result: the former by providing the dynamic value of the superhedging portfolio and the latter by proving the impossibility to extend the duality (3.1.11) at general times.

Theorem 3.2.1. *Suppose that $\mathscr{P}$ is a nonempty, saturated set of sigma martingale measures for S satisfying Assumption 3.1.5. Moreover, let f be an upper semianalytic, $\mathscr{G}_T$-measurable function such that $\sup_{\mathbb{P}\in\mathscr{P}} E_P[|f|] < \infty$. Then for the superreplication price π_t at time $t \in (0,T]$ of the contingent claim f given by*

$$\pi_t := \operatorname{ess\,inf}\{c_t \in \mathscr{G}_{t+} \mid \exists\, H \in \mathscr{H} \text{ with } c_t + \int_t^T H_s dS_s \geq f\ \mathbb{P}-a.s. \text{ for all } \mathbb{P} \in \mathscr{P}\}$$

it holds $\pi_t = Y_t$ q.s., where Y is the process defined by (3.1.13) *and* (3.1.14).

Proof. The prove the theorem by showing that both inequalities $\pi_t \leq Y_t$ and $\pi_t \geq Y_t$ hold quasi surely. The former is achieved as in Theorem 3.2 in [50]. By applying Theorem 3.1.13 with the σ-algebra $\mathscr{G}_T$ and the filtration $\mathbb{G}_+$, we obtain a process $H \in \mathscr{H}$ satisfying

$$Y_t - (H\cdot S)_t \geq Y_T - (H\cdot S)_T \quad \mathbb{P}-a.s. \text{ for all } \mathbb{P} \in \mathscr{P}. \tag{3.2.1}$$

As $Y_T = f$ q.s. (see Theorem 3.2 in [50]), from (3.2.1) we derive

$$Y_t + (H\cdot S)_T - (H\cdot S)_t \geq f \quad \mathbb{P}-a.s. \text{ for all } \mathbb{P} \in \mathscr{P}.$$

Hence we argue that $Y_t \geq \pi_t$ q.s. For the converse, consider a $\mathscr{G}_{t+}$-measurable function $\bar{\pi}_t$ and a process $H \in \mathscr{H}$ such that

$$\bar{\pi}_t + \int_t^T H_s dS_s \geq f \quad q.s..$$

For every $r \in [t,T] \cap \mathbb{Q}$ we have that

$$\mathscr{E}_r\left(\bar{\pi}_t + \int_t^T H_s dS_s\right) \geq \mathscr{E}_r(f) \quad q.s. \tag{3.2.2}$$

From the previous expression, as $H \cdot S$ is a supermartingale, we argue that

$$\bar{\pi}_t + \int_t^r H_s dS_s \geq \mathscr{E}_r(f) \quad q.s.$$

Since $H \cdot S$ is right-continuous, we then obtain

$$\bar{\pi}_t = \limsup_{r \downarrow t, r \in \mathbb{Q}} \left(\bar{\pi}_t + \int_t^r H_s dS_s \right) \geq \limsup_{r \downarrow t, r \in \mathbb{Q}} \mathscr{E}_r(f) = Y_t' \quad q.s.$$

and

$$\bar{\pi}_t \geq Y_t' \quad q.s. \tag{3.2.3}$$

We can complete the claim by noticing that (3.2.3) implies also $\bar{\pi}_t \geq Y_t$. □

Remark 3.2.2. We do not consider time 0 in our result as this case has already been tackled in [50]. Moreover, along the proof of Theorem 3.2 in [50], the author solves the issue coming from the necessity of providing an initial portfolio value which is *deterministic* and not only $\mathscr{G}_{0+}$-measurable.

We can obtain a *superhedging duality* in the case of the G-expectation framework. Fixed a $\mathscr{F}_T$-measurable function f, it is possible to prove that $\pi_t = E_G[f|\mathscr{F}_t]$, for every $t \in [0,T]$. Such result depends on the G-martingale representation theorem presented in Theorem 2.1.18, which can be seen as a finer version of the optional decomposition result from Theorem 3.1.13. In that setting the dynamic superreplication price of a contingent claim f can be derived by studying the representation of the G-martingale $(E_G[f|\mathscr{F}_t])_{t\in[0,T]}$. Proving the first inequality, $\pi_t \geq E_G[f|\mathscr{F}_t]$ q.s., is similar to the outline presented in Theorem 3.2.1, while exploiting Theorem 2.1.18 we derive

$$E_G[f|\mathscr{F}_t] = E_G[f] + \int_0^t \theta_s dB_s - K_t,$$

for any $t \in [0,T]$ and appropriate processes θ and K. Being the process K increasing we can easily see that

$$\begin{aligned} E_G[f|\mathscr{F}_t] + \int_t^T \theta_s dB_s &= E_G[f] + \int_0^T \theta_s dB_s - K_t \\ &\geq E_G[f] + \int_0^T \theta_s dB_s - K_T = f, \end{aligned}$$

from which we achieve $\pi_t \leq E_G[f|\mathscr{F}_t]$.
In the final part of this chapter we discuss why we cannot derive an analogous

superhedging duality in the more general setting studied in this section. The main problem is the the possibility to construct a process $H \in \mathscr{H}$ satisfying

$$\mathscr{E}_t(f) + \int_t^T H_s dS_s \geq f \quad q.s.,$$

which is the same as proving that $Y_t \leq \mathscr{E}_t(f)$ $\mathbb{P}$-a.s. for every $\mathbb{P} \in \mathscr{P}$. To this purpose we fix $\mathbb{P} \in \mathscr{P}$. As a consequence of Theorem VI.2 in [19] we have

$$E_{\mathbb{P}}[Y_t|\mathscr{F}_t] \leq \mathscr{E}_t(f) \quad \mathbb{P}-a.s.$$

Hence, by arguing as in Proposition 3.1.9, we derive

$$\operatorname*{ess\,sup}_{\mathbb{P}' \in \mathscr{P}_{\mathrm{eq}}(t,\mathbb{P})} E_{\mathbb{P}'}[Y_t|\mathscr{F}_t] \leq \mathscr{E}_t(f) \quad \mathbb{P}-a.s.$$

Denote then with $\pi_t^{\mathbb{P}}$ be the smallest $\mathscr{F}_t$-measurable function dominating Y_t $\mathbb{P}$-a.s. To proceed as in [50] we need to show that

$$\pi_t^{\mathbb{P}} \leq \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathscr{P}_{\mathrm{eq}}(t,\mathbb{P})} E_{\mathbb{P}'}[Y_t|\mathscr{F}_t] \quad \mathbb{P}-a.s. \tag{3.2.4}$$

However, it follows from Theorem 2.1.6 in [72] that

$$\pi_t^{\mathbb{P}} = \operatorname*{ess\,sup}_{\mathbb{Q}} E_{\mathbb{Q}}[Y_t|\mathscr{F}_t],$$

where the supremum is computed over all the priors $\mathbb{Q}$ on $\mathscr{F}_t$ that are equivalent to $\mathbb{P}$, which is a contradiction to (3.2.4) already when the probability measure is unique and the market is complete.
In order to obtain the equality $Y = \mathscr{E}(f)$ q.s. it would be enough to ensure that the process $\mathscr{E}(f)$ is right-continuous, because of (3.1.13). However such property, whilst holding true in the G-setting, is not guaranteed in a more general context.

4 Financial Asset Bubbles under Model Uncertainty

Asset price bubbles are described in the literature of mathematical finance as the difference between two ingredients: the market price of a financial asset, which stands for the amount that a buyer will have to pay to get ownership of the asset, and its fundamental value. The various approaches in the literature differ for the modeling of the fundamental value, as the market price is often assumed to be exogenous. A first investigation of financial bubbles is done in [38], where the fundamental price is defined as the expected sum of future discounted dividends. This study, which can be seen as the cornerstone of the so called *martingale theory of bubbles*, has the drawback of considering only complete market models, which in turns excludes the possibility of observing the birth of bubbles (see Section 4.1.1). This problem is solved in [39], where the adoption of an incomplete market model, and thus the existence of an infinite number of local martingale measures, allows the authors to develop a *dynamic* setting. This new feature of the framework consists in the possibility of the investor to change her view on the market at a given stopping time or, stated alternatively, to select a pricing measure different from the one previously adopted. This modifies the perception of the fundamental value and can correspond to the birth of a bubble. The martingale theory of bubbles has been adopted by several authors to explain many aspects of this phenomenon. We cite [8], [36], [47] and [61] to mention some of them.

An alternative definition of the concept of fundamental value has been recently suggested in [65]. A possible candidate to model this object could be the superreplication price of the terminal payoff of the asset. The two approaches coincide when applied to the context of complete markets. However it is possible to notice important differences when an infinite number of risk neutral measures exists. Even if the framework of [65] is *static*, i.e. it does not foresee the possibility of the agent to change her pricing measure over time, bubbles can suddenly appear in the price of the asset and disappear at some later time. The reason lies in the properties of the superreplication price: it may happen that at time zero this is equal to the initial price of the asset, while being strictly smaller at some future time.

The objective of this chapter is to introduce a framework for the formation of financial bubbles in a market with model uncertainty. To this end we consider the setting outlined in Chapter 3 to describe a financial market in which the observed price of the asset $S=(S_t)_{t\geq 0}$ is still exogenous. The main challenge consists in defining adequately the notion of *robust fundamental value* $S^*=(S_t^*)_{t\geq 0}$. Then the bubble under uncertainty would be given once more by the process $S-S^*$. It can be useful to remark the parallels with the concept of *robust arbitrage* as stated in [73].

Definition 4.0.3. A robust arbitrage is an admissible portfolio π with initial wealth $y\leq 0$ such that, at some time $T>0$, its value X_T^π satisfies

$$\begin{aligned} X_T^\pi \geq 0 \quad & \mathbb{Q}-a.s. \text{ for all } \mathbb{Q}\in\mathscr{Q} \text{ and} \\ \mathbb{Q}(X_T^\pi>0)>0 \quad & \text{for at least one } \mathbb{Q}\in\mathscr{Q}. \end{aligned}$$

A sensible requirement for a good definition of robust arbitrage is consistency with the classical framework. In addition one notices that a classical arbitrage strategy in one of the underlying $\mathbb{Q}$-markets does not yield automatically a robust arbitrage strategy. This is clear from the definition of robust arbitrage as

$$\begin{aligned} X_T^\pi \geq 0 \quad & \mathbb{Q}-a.s. \\ \mathbb{Q}(X_T^\pi>0)>0 & \end{aligned}$$

for some $\mathbb{Q}\in\mathscr{Q}$ does not guarantee that $X_T^\pi\geq 0$ $\mathbb{Q}'$-a.s. for all $\mathbb{Q}'\in\mathscr{Q}$. Stated otherwise, a robust arbitrage is a trading strategy requiring no initial investment, causing no losses under every scenario $\mathbb{Q}\in\mathscr{Q}$ and providing a positive gain with positive probability for at least one $\mathbb{Q}'\in\mathscr{Q}$.
In a similar way we ask for consistency with the classical literature in the case a unique prior exists. On the other hand a bubble in an underlying $\mathbb{Q}$-market will not determine necessarily the presence of bubble under uncertainty, as it happens with robust arbitrage.
Taking care of this fundamental requirements we decided to model robust fundamental values as superreplication prices, as the framework is intrinsically nonlinear and it does not appear natural to extend here the notion of expected sum of future discounted dividends under a risk neutral measure.

In Section 4.1.2 and Section 4.1.3 we study the characteristics of financial asset bubbles under model uncertainty and present several concrete examples. One of the most interesting feature of our setting is that, when there exists a bubble, this is a $\mathbb{Q}$-local submartingale for every $\mathbb{Q}\in\mathscr{Q}$. The same behavior was observed also in the dynamic model outlined in [5], in correspondence of the *build-up stage* of the bubble, i.e. when the market price deviates from the fundamental value. Therefore one novelty of our framework is the possibility to describe the birth and growth of a bubble in a static model.
In addition, our approach also differentiates itself from the robust model for bubbles introduced in [17], where their existence is linked to the constraints enforced

on the admissible portfolios.
Finally, as opposed to what happens in [65], we produce examples of bubbles originating from price processes that can be true $\mathbb{Q}$-martingales for some $\mathbb{Q} \in \mathscr{Q}$. This means that an investor, trusting $\mathbb{Q}$ to describe the correct probability measure ruling the market, would not detect the bubble.

4.1 Bubbles under Uncertainty

We place ourselves in the market model outlined in Section 3.1. However, to guarantee consistence with the existing literature in the case a unique prior exists, we need to slightly modify our assumptions. Let then be given a discounted asset which is a non-negative, $\mathbb{R}^d$-valued, $\mathbb{F}^*$-adapted and right-continuous process $S = (S_t)_{t\geq 0}$ such that its paths are $\mathscr{P}$-q.s. continuous.
As in [38] we fix a stopping time $\tau > 0$ q.s. modeling the maturity of S and let X_τ be the liquidation value at maturity. We make the hypothesis that X_τ is measurable with respect to the Borel σ-algebra on Ω_T, in order to compute its conditional sublinear expectation. We assume the discounted value of the risk-free asset S^0 to be constantly equal to 1. Owning one unit of the risky asset up to time t entails the investor with the wealth process $W = (W_t)_{t\geq 0}$ defined by

$$W_t := S_t \mathbf{1}_{\{\tau > t\}} + X_\tau \mathbf{1}_{\{\tau \leq t\}}.$$

The martingale theory of bubbles has its roots on the concept of no arbitrage, and more in particular of No Free Launch With Vanishing Risk (NFLVR). In the robust framework the situation is more complicated as there is not yet a robust version of NFLVR. The most important result on no arbitrage in the setting we are taking into consideration involves the concept of arbitrage of the first kind (NA_1). It is a well-known result that NA_1 does not guarantee the existence of an equivalent local martingale measure, but only of a *martingale deflator* or equivalently an absolutely continuous martingale measure (see [30]). A similar result has been provided under model uncertainty by [10]. This comes at the cost of introducing a stopping time ζ that determines a jump to a cemetery state, which is infinite $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathscr{P}$ but might be smaller than infinity for some $\mathbb{Q} \in \mathscr{Q}$, where $\mathscr{Q}$ describes a suitable set of local martingale measures. For this reason, even if it would be possible to model the market through collection $\mathscr{P}$ of objective measures, the results of [10] involve some additional attention to deal with the stopping time ζ. For simplicity we therefore make the following assumption.

Assumption 4.1.1. *We consider a family $\mathscr{Q}$ of probability measures possibly non dominated satisfying Assumption 3.1.5 and such that the wealth process is a $\mathbb{Q}$-local martingale for every $\mathbb{Q} \in \mathscr{Q}$. Thus the set $\mathscr{Q}$ is made of local martingale measures, enforcing NFLVR under all $\mathbb{Q}$-market.*

In this way the wealth W is economically justified under all underlying $\mathbb{Q}$-markets. Moreover, as in the classical setting NFLVR implies NA_1, and the fact robust NA_1 implies NA_1 under all priors included in the uncertainty framework (see [10]), it is reasonable to expect that a robust version of NFLVR will also imply the correspondent robust NA_1. This question, which is interesting and complex on its own, is beyond the aim of this thesis.

As a first step we outline a short summary regarding the modeling of financial asset bubbles in the existing literature. This will be useful to understand more clearly what should be the right notion of robust fundamental value under model uncertainty. We begin by studying the finite time horizon case as the extension to infinite time will be derived easily from it. Fix then $T \in \mathbb{R}_+$ be such that $\tau \leq T$. We remark that $W_t = S_t$ for every $t \in [0,T]$, if $X_\tau = S_\tau$, which will be assumed all over this section.

4.1.1 Classical Setting for Bubble Modeling

We can identify two main standpoints for modeling the fundamental value of a financial asset in the classical setting with one single prior. The martingale theory of bubbles, see [5], [38] and [39] for example, defines the fundamental value as the asset's discounted future payoffs under a local martingale measure. In other words, fixed a probability measure $\mathbb{Q} \in \mathscr{M}_{loc}(S)$, where $\mathscr{M}_{loc}(S)$ stands for the family of all risk neutral measures for S, the fundamental value $S^{*,\mathbb{Q}} = (S_t^{*,\mathbb{Q}})_{t\in[0,T]}$ under $\mathbb{Q}$ is defined to be

$$S_t^{*,\mathbb{Q}} = E_{\mathbb{Q}}[S_T|\mathscr{F}_t]$$

for every $t \in [0,T]$. Thus the *market bubbliness* is built upon the following partition of $\mathscr{M}_{loc}(S)$

$$\mathscr{M}_{loc}(S) = \mathscr{M}_{UI}(S) \cup \mathscr{M}_{NUI}(S),$$

where $\mathscr{M}_{UI}(S)$ represents the set of probabilities $\mathbb{Q} \approx \mathbb{P}$ for which S is a uniformly integrable $\mathbb{Q}$-martingale, while $\mathscr{M}_{NUI}(S) = \mathscr{M}_{loc}(S) \setminus \mathscr{M}_{UI}(S)$.

With this convention the presence of a bubble depends on the beliefs of the investor about the market dynamics. If the agent is adopting a measure $\mathbb{Q} \in \mathscr{M}_{NUI}(W)$, then the risky asset would have a bubble component. On the other hand, if $\mathbb{Q} \in \mathscr{M}_{UI}(W)$ is the pricing measure used by the agent, then she would see no bubble. We can correctly state that the notion of bubble is dynamic: a bubble is born or bursts according to how the agent changes her views regarding the right probability measure ruling the market.

When there exists only one ELMM the situation becomes remarkably simpler. The collection $\mathscr{M}_{loc}(S)$ would then reduce to a singleton ($\mathscr{M}_{loc}(S)$ cannot be empty under the NFLVR assumption) and therefore just two situations are possible: either the exists a bubble from time 0 or there would never be a bubble. The same result can be achieved adopting the approach of [65], where the fundamental price is defined to be superreplication price $\pi = (\pi_t)_{t\in[0,T]}$of the terminal value of S. The duality result (see [42])

$$\pi_t(S) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathscr{M}_{loc}(S)} E_{\mathbb{Q}}[S_T|\mathscr{F}_t], \tag{4.1.1}$$

ensures that the existence of an asset price bubble in a complete market implies the superreplication value at time t to be smaller than the asset price seen in the market.
The two approaches diverge when the market model is incomplete. As a consequence of (4.1.1), if $\mathscr{M}_{UI}(S)$ is not empty then there exists no bubble. As the framework is now static, and not dynamic, the economic interpretation of bubble birth changes. It may happen that the superreplication price equals the market value at initial time, but it can be strictly smaller at some time $t > 0$ (see Example 3.7 in [65]). We say that the bubble is born at time t. Also in this case, even if the agent detects the bubble, she can not make a profit out of it. In fact a portfolio that combines a short position in the risky asset together with a long position in the superreplicating portfolio is not an admissible strategy as it exposes the investor to possible unbounded losses. The second approach to financial bubbles present already an *higher degree of robustness* as, if there is a bubble, this is perceived by any investor, independently from the probability $\mathbb{Q} \in \mathscr{M}_{loc}(S)$ adopted.

It is possible to connect the two settings outlined above. We show that if the set $\mathscr{M}_{UI}(S)$ is empty, which means that all investors agree on the existence of a bubble in the sense of [5] or [39], then there is also a bubble in the setting of [65].

Proposition 4.1.2. *Let $S = (S_t)_{t\in[0,T]}$ be a continuous adapted process in a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ where the filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ satisfies the usual conditions. If $\mathscr{M}_{loc}(S) = \mathscr{M}_{NUI}(S)$ then there is a $t \in [0,T)$ such that with positive probability*

$$S_t > \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathscr{M}_{loc}(S)} E_{\mathbb{Q}}[S_T|\mathscr{F}_t].$$

Proof. We prove the claim arguing by contradiction. Assume that

$$S_t = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathscr{M}_{loc}(S)} E_{\mathbb{Q}}[S_T|\mathscr{F}_t]$$

for every $t \in [0,T]$. The process $\pi(S)$ defined in (4.1.1) is a $\mathbb{Q}$-local martingale for every $\mathbb{Q} \in \mathscr{M}_{loc}(S)$. Hence, because of Theorem 3.1 in [42], the optimal superreplicating strategy $V = (V_t)_{t\in[0,T]}$, where

$$V_t = \sup_{\mathbb{Q}\in\mathscr{M}_{loc}(S)} E_{\mathbb{Q}}[S_T] + \int_0^t H_s dS_s$$

and H is a predictable, S-integrable process, is nonnegative and *self-financing*. Thus that there is $\bar{\mathbb{Q}} \in \mathscr{M}_{loc}(S) \cap \mathscr{M}_{UI}(S)$, in contradiction with the initial assumption. □

Remark 4.1.3. The same result can be achieved also when considering

$$\mathscr{Q} = \{\mathbb{Q} \in \mathfrak{P}(\Omega) \mid \mathbb{Q} \ll \mathbb{P},\ S \text{ is a } \mathbb{Q}\text{-local martingale}\} \tag{4.1.2}$$

in place of $\mathscr{M}_{loc}(S)$. If the set $\mathscr{Q}$ is *m-stable*[1], we can define the fundamental value as

$$S_t^* = \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathscr{Q}} E_{\mathbb{Q}}[S_T | \mathscr{F}_t],$$

for any $t \in [0,T]$ (see [18]). However, since the probabilities in $\mathscr{Q}$ being equivalent to $\mathbb{P}$ form a dense set in $\mathscr{Q}$ (we refer again to [18]), we have

$$S_t^* = \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathscr{Q}} E_{\mathbb{Q}}[S_T | \mathscr{F}_t] = \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathscr{M}_{loc}(S)} E_{\mathbb{Q}}[S_T | \mathscr{F}_t].$$

This means that Proposition 4.1.2 applies also in this context.

4.1.2 Robust Fundamental Value

In this section we introduce the notion of fundamental value in the context of model uncertainty. It is natural that a sensible requirement for a good definition of *robust fundamental value* is consistency with the classical framework. This means that when $\mathscr{Q}$ consists of one single element, the notion must coincide with one of the two approaches discussed in the previous section. As a consequence we can immediately argue that the robust fundamental value need to be defined as a function of some conditional expectation.

A first possible approach is to define a bubble under uncertainty as the situation in which there exists a measure $\mathbb{Q} \in \mathscr{Q}$ for which the fundamental value under $\mathbb{Q}$ is strictly smaller than the market price, while being smaller or equal for all other probability measures. This definition would however turn all classical bubbles into bubbles also in the robust setting. In order to avoid this situation, we could define a '$\mathbb{Q}$-fundamental value' depending on the particular prior $\mathbb{Q}$, as done in [38]. To satisfy our essential requirement, which is consistency with the existing literature, we assume that

$$S_t^{*,\mathbb{Q}} = E_{\mathbb{Q}}[S_T | \mathscr{F}_t] \quad \mathbb{Q} - a.s. \tag{4.1.3}$$

[1] $\mathscr{Q}$ defined in (4.1.2) is m-stable if for elements $\mathbb{Q}^0 \in \mathscr{Q}$, $\mathbb{Q} \in \mathscr{Q}$ such that $\mathbb{Q} \sim \mathbb{P}$, with associate martingales $Z_t^0 = E\left[\frac{d\mathbb{Q}^0}{d\mathbb{P}} | \mathscr{F}_t\right]$ and $Z_t = E\left[\frac{d\mathbb{Q}}{d\mathbb{P}} | \mathscr{F}_t\right]$, and for each stopping time τ, the element L defined as $L_t = Z_t^0$ for $t \le \tau$ and $L_t = Z_T^0 Z_t / Z_T$ for $t \ge \tau$ is a martingale that defines an element in $\mathscr{Q}$. We also assume that every $\mathscr{F}_0$-measurable nonnegative function Z_0 such that $E_{\mathbb{P}}[Z_0] = 1$, defines an element $d\mathbb{Q} = Z_0 d\mathbb{P}$ that is in $\mathscr{Q}$.

for every $\mathbb{Q} \in \mathscr{Q}$. The first drawback of this definition is that such family of $\mathbb{Q}$-fundamental processes is typically not aggregable (this happens already in G-setting, see [66]). Still, we could then define the existence of a bubble under uncertainty as the case where

$$\mathbb{Q}(S_t^{*,\mathbb{Q}} < S_t) > 0 \tag{4.1.4}$$

for every $\mathbb{Q} \in \mathscr{Q}$ and some $t > 0$. By doing that we would be trying to extend the approach of the martingale theory of bubbles to the uncertainty framework. However the incompleteness of the market under uncertainty makes in general impossible to extend the classical concept of risk neutral evaluation of future discounted dividends, as there exists no linear pricing system. In addition, using (4.1.4), the asset price S would have $\mathscr{Q}$-bubble under uncertainty if it has a $\mathbb{Q}$-bubble for every underlying $\mathbb{Q}$-market. This condition seems however too stringent.

Definition 4.1.4. We call *robust fundamental value* the process $S^* = (S_t^*)_{t\in[0,T]}$ where

$$S_t^* = \operatorname*{ess\,sup}_{\mathbb{Q}'\in\mathscr{Q}(t,\mathbb{Q})} E_{\mathbb{Q}'}[S_T|\mathscr{F}_t], \quad \mathbb{Q}-a.s. \tag{4.1.5}$$

for every $\mathbb{Q} \in \mathscr{Q}$, where $\mathscr{Q}(t,\mathbb{Q}) = \{\mathbb{Q}' \in \mathscr{Q} : \mathbb{Q}' = \mathbb{Q} \text{ on } \mathscr{F}_t\}$.

Thanks to the findings in Section 3.2, S^* coincides with the superreplication value in the G-setting. The same interpretation holds for S_0^* if the set of priors $\mathscr{Q}$ is saturated. In that case

$$S_0^* = \inf\{x \in \mathbb{R} \,:\, \exists\, H \in \mathscr{H} \text{ with } x + (H\cdot S)_T \geq S_T \ \mathbb{Q}-a.s. \text{ for all } \mathbb{Q} \in \mathscr{Q}\}$$

and connection with the classical literature now is evident.

Proposition 4.1.5. *Let $\mathscr{P} = \{\mathbb{P}\}$, then the robust fundamental value* (4.1.5) *coincides with the classical superreplication price, i.e.*

$$\operatorname*{ess\,sup}_{\mathbb{Q}\in\mathscr{Q}} E_{\mathbb{Q}}[S_T|\mathscr{F}_t] = \operatorname*{ess\,sup}_{\mathbb{Q}'\in\mathscr{Q}(t,\mathbb{Q})} E_{\mathbb{Q}'}[S_T|\mathscr{F}_t] \quad a.s. \tag{4.1.6}$$

Proof. Notice that in the case $\mathscr{P} = \{\mathbb{P}\}$ the family $\mathscr{Q} = \{\mathbb{Q} \mid \mathbb{Q} \approx \mathbb{P},\ \mathbb{Q}\text{ ELMM}\}$ is composed of probabilities equivalent to each other. We show that

$$\operatorname*{ess\,sup}_{\mathbb{Q}\in\mathscr{Q}(t,\mathbb{Q}^1)} E_{\mathbb{Q}}[S_T|\mathscr{F}_t] = \operatorname*{ess\,sup}_{\mathbb{Q}'\in\mathscr{Q}(t,\mathbb{Q}^2)} E_{\mathbb{Q}'}[S_T|\mathscr{F}_t] \quad a.s. \tag{4.1.7}$$

for each $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathscr{Q}$ using a measure pasting technique analogous to Proposition 9.1 in [18]. This is enough to prove the claim as

$$\operatorname*{ess\,sup}_{\mathbb{Q}\in\mathscr{Q}} E_{\mathbb{Q}}[S_T|\mathscr{F}_t] \geq \operatorname*{ess\,sup}_{\mathbb{Q}'\in\mathscr{Q}(t,\mathbb{Q})} E_{\mathbb{Q}'}[S_T|\mathscr{F}_t],$$

and thanks to (4.1.7) it holds

$$\operatorname*{ess\,sup}_{\mathbb{Q}\in\mathscr{Q}} E_{\mathbb{Q}}[S_T|\mathscr{F}_t] \le \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathscr{Q}} \left\{ \operatorname*{ess\,sup}_{\mathbb{Q}'\in\mathscr{Q}(t,\mathbb{Q})} E_{\mathbb{Q}'}[S_T|\mathscr{F}_t] \right\} = \operatorname*{ess\,sup}_{\mathbb{Q}'\in\mathscr{Q}(t,\mathbb{Q})} E_{\mathbb{Q}'}[S_T|\mathscr{F}_t].$$

Let $\mathbb{Q}^2 \in \mathscr{Q} \setminus \mathscr{Q}(t,\mathbb{Q}^1)$. The alternative situation is trivial. Note that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathscr{F}_t} = \frac{d\mathbb{Q}^i}{d\mathbb{P}}|_{\mathscr{F}_t} := Z_t^i,$$

for each $\mathbb{Q} \in \mathscr{Q}(t,\mathbb{Q}^i)$, $i = 1,2$. This is evident as, for each $A \in \mathscr{F}_t$, we have

$$E_{\mathbb{P}}[Z_t^i \mathbf{1}_A] = E_{\mathbb{Q}^i}[\mathbf{1}_A] = \mathbb{Q}^i(A) = \mathbb{Q}(A) = E_{\mathbb{Q}}[\mathbf{1}_A] = E_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathscr{F}_t}\mathbf{1}_A\right].$$

Set now

$$Z_s := Z_s^1 \mathbf{1}_{\{s\le t\}} + Z_t^1 \frac{Z_s^2}{Z_t^2}\mathbf{1}_{\{t<s\}},$$

the Radon-Nykodim derivative of an equivalent local martingale measure, as shown in Proposition 9.1 in [18]. The probability $\mathbb{Q}'$ we derive from $(Z_s)_{s\in[0,T]}$ is in the set $\mathscr{Q}(t,\mathbb{Q}^1)$ and satisfies

$$\begin{aligned} E_{\mathbb{Q}'}[S_T|\mathscr{F}_t] &= \frac{E_{\mathbb{P}}[S_T Z_T|\mathscr{F}_t]}{Z_t} = \frac{E_{\mathbb{P}}\left[S_T Z_t^1 \frac{Z_T^2}{Z_t^2}|\mathscr{F}_t\right]}{Z_t^1} \\ &= \frac{E_{\mathbb{P}}[S_T Z_T^2|\mathscr{F}_t]}{Z_t^2} = E_{\mathbb{Q}^2}[S_T|\mathscr{F}_t]. \end{aligned}$$

In this way we have proven that for each $\mathbb{Q} \in \mathscr{Q}(t,\mathbb{Q}^2)$ there is a measure $\mathbb{Q}' \in \mathscr{Q}(t,\mathbb{Q}^1)$ such that $E_{\mathbb{Q}'}[S_T|\mathscr{F}_t] = E_{\mathbb{Q}}[S_T|\mathscr{F}_t]$. This is shows (4.1.7) and concludes the proof. □

Remark 4.1.6. We remark that in the proof of Proposition 4.1.5 it is possible to consider almost sure relationships as $\mathscr{P} = \{\mathbb{P}\}$ and we are considering the collection of martingale measures equivalent to $\mathbb{P}$.

We now define the concept of bubble in the uncertainty framework.

Definition 4.1.7. The asset price bubble $\beta = (\beta_t)_{t\in[0,T]}$ for S is given by

$$\beta_t := S_t - S_t^*, \tag{4.1.8}$$

where S^* is defined in (4.1.5).

With this definition there does not have to exist a bubble under every underlying $\mathbb{Q}$-market in order to obtain a bubble in the robust setting. The bubble exists in the case in which there is a stopping time τ for which

$$\mathbb{Q}(S_\tau > S^*_\tau) > 0$$

for a measure $\mathbb{Q} \in \mathscr{Q}$. If this condition is satisfied then also all the measures that coincide with $\bar{\mathbb{Q}}$ on $\mathscr{F}_\tau$ will describe a bubbly market. The similarities with the properties of the robust arbitrage now are clear. Being S is a non-negative $\mathbb{Q}$-local martingale, thus a $\mathbb{Q}$-supermartingale, it holds

$$S_t \geq S^*_t, \qquad \mathbb{Q} - a.s.$$

for each $t \in [0,T]$ and $\mathbb{Q} \in \mathscr{Q}$.
When there is no duality gap, this definition of bubble represents the extension under model uncertainty of the approach outlined in [65]. In all other cases the process S^* can always be interpreted as the worst model price, among all the possible scenarios conceived by the agent.
In the present setting we do not enforce the assumption of saturation, as from Definition 3.1.11. This condition is guaranteed if every prior $\mathbb{Q}$ models a complete market. In full generality the presence of a bubble might be generated either by a difference between the market value and the superhedging price or by a duality gap in

$$\sup_{\mathbb{Q}\in\mathscr{Q}} E_{\mathbb{Q}}[S_T] \leq \inf\{x \in \mathbb{R} : \exists H \in \mathscr{H} \text{ with } x + (H \cdot S)_T \geq S_T \; \mathbb{Q} - a.s. \text{ for all } \mathbb{Q} \in \mathscr{Q}\}.$$

The second possibility is studied in [17] as the reason triggering the birth of a bubble in their model.

4.1.3 Properties and Examples

Lemma 4.1.8. *The bubble β is a non-negative $\mathbb{Q}$-local submartingale for every $\mathbb{Q} \in \mathscr{Q}$, such that $\beta_T = 0$ q.s. Moreover, if there exists a bubble, S is not a $\mathscr{Q}$-martingale.*

Proof. This is a consequence of Definition 4.1.4, since β is the difference between a $\mathbb{Q}$-local martingale and a $\mathbb{Q}$-supermartingale. □

The local submartingale dynamics do not represent a contradiction as it might appear. It is easy to prove that non-negative local submartingales are not necessarily true submartingales, as it happens on the contrary for non-negative local supermartingales. In order to consider a clear example it is enough to focus on the

family of non-negative local martingales: those processes are non-negative local submartingale and supermartingales at the same time. Moreover [26] and [54] display a number of cases of local submartingales with nonstandard characteristics, as decreasing mean for example.
We next outline the first example of bubble by transposing one result [16] in the G-setting.

Remark 4.1.9. Notice that in Example 4.1.11, Example 4.1.12 and Example 4.1.13 the process S is modeled as $\mathbb{Q}$-local martingale for every $\mathbb{Q} \in \mathscr{Q}$ with respect to the completed filtration $\mathbb{F}^{\mathbb{Q}} = \{\mathscr{F}_t^{\mathbb{Q}}\}_{t\geq 0}$. However, because of a result of [70], the asset price is also a $\mathbb{Q}$-local martingale for the filtration $\mathbb{F}^*$. We cite the finding of [70] as it is stated in Theorem 10 in [27].

Theorem 4.1.10. *Let X be a non-negative local martingale for $\mathbb{G}$ and assume that X is adapted to the subfiltration $\mathbb{F}$. Then X is also a local martingale for $\mathbb{F}$.*

Example 4.1.11. We consider the set of priors $\mathscr{Q} = \mathscr{P}_{\mathbf{G}}$ described in Proposition 3.1.8, with $\mathbf{D} = [\underline{\sigma}^2, \overline{\sigma}^2] \subset \mathbb{R}_+ \setminus \{0\}$. We set $S_0 = s > 0$ and

$$S_t = s + \int_0^t \frac{S_u}{\sqrt{T-u}} dB_u, \quad t \in [0,T). \tag{4.1.9}$$

Because of the results in [48], the price process in (4.1.9) is well posed for every $t \in [0, T-\varepsilon]$, with $\varepsilon > 0$. We prove that S has zero terminal value, whilst being a non-negative $\mathbb{Q}$-local martingale for every $\mathbb{Q} \in \mathscr{Q}$. This is enough to guarantee the existence of a bubble. To this end, we consider arbitrarily a probability measure $\mathbb{Q} \in \mathscr{Q}$. It holds

$$S_t = s e^{\int_0^t \varphi_s dB_s - \frac{1}{2}\int_0^t \varphi_s d\langle B\rangle_s}, \quad t \in [0,T).$$

The process $\int_0^\cdot 1/\sqrt{T-s} dB_s$ is a $\mathbb{Q}$-local martingale on $[0,T)$, with quadratic covariation continuous on $[0,T)$ given by

$$\left[\int_0^\cdot 1/\sqrt{T-s} dB_s, \int_0^\cdot 1/\sqrt{T-s} dB_s\right]_u \geq -\underline{\sigma}^2 \ln\left[1-\frac{u}{T}\right].$$

Thanks to the argument outlined in Lemma 5 from [38], which makes use of the Dubins-Schwarz theorem and of the law of the iterated logarithm, we can infer that

$$\lim_{u\to T} S_u = 0 \qquad \mathbb{Q}-a.s. \tag{4.1.10}$$

Therefore we set $S_T = 0$ which implies that S is q.s. continuous on $[0,T]$. This holds true since the family $\{\omega \in \Omega : \lim_{u\to T} S_u \neq 0\}$ constitutes a polar set. If there existed a prior $\mathbb{Q} \in \mathscr{Q}$ for which $\mathbb{Q}(\lim_{u\to T} S_u \neq 0) > 0$, equation (4.1.10) would be false. Therefore S is a strict $\mathbb{Q}$-local martingale for every $\mathbb{Q} \in \mathscr{Q}$ as $E_{\mathbb{Q}}[S_T] = 0 < E_{\mathbb{Q}}[S_0]$, and it is not a robust martingale.

We provide another example by introducing the inverse three dimensional Bessel process under model uncertainty.

Example 4.1.12. Let $\mathscr{Q} = \mathscr{P}_{\mathbf{G}}$, where $\mathbf{D} \subset \mathbb{R}^{3\times 3}$ is the set of matrices $(a_{i,j})_{i,j=1,2,3}$ with $a_{i,j} = 0$ for every $i \neq j$, $a_{1,1} = a_{2,2} = a_{3,3} \in [1,2]$.
We study the price process given by $f(\mathbf{B}) = (f(\mathbf{B}_t))_{t\geq 0}$, where $\mathbf{B}_0 = (1,0,0)$ and $f(\mathbf{x}) = \|\mathbf{x}\|^{-1}$. Since f is a Borel-measurable function, we can consider the sublinear expectation $\mathscr{E}_0(f(\mathbf{B}_t))$ for every $t \geq 0$, because of Theorem 3.1.6. The process $f(\mathbf{B})$ is a strict $\mathbb{Q}$-local martingale for every $\mathbb{Q} \in \mathscr{P}_{\mathbf{G}}$ (see for example [62], Exercise XI.1.16). We prove that $f(\mathbf{B})$ is not a $\mathscr{P}_{\mathbf{G}}$-martingale, thus showing the existence of a bubble. To this end we adapt an argument from [27], which consists in projecting the process $f(\mathbf{B})$ on the filtration generated by one component of the Brownian motion. Theorem 14 from [27] show in detail that

$$E_{\mathbb{P}}\left[\frac{1}{\|\mathbf{B}_t\|}\right] = 2\Phi\left(\frac{1}{\sqrt{t}}\right) - 1, \tag{4.1.11}$$

where Φ represents the distribution function of a random variable $N(0,1)$. It is possible to prove a slight generalization of the preceding result by considering the process $\mathbf{W}^\sigma = \sigma\mathbf{W}$ where $\sigma \in \mathbf{R}$ and $\mathbf{W}$ is a Brownian motion issued at $\mathbf{x} \in \mathbb{R}^3$. Thanks to the invariance by rotation of the Brownian motion we can show that the equality (4.1.11) is a particular case of

$$E_{\mathbb{P}}\left[\frac{1}{\|\mathbf{W}_t^\sigma\|}\right] = \frac{1}{\|\mathbf{x}\|}\left(2\Phi\left(\frac{\|\mathbf{x}\|}{\sigma\sqrt{t}}\right) - 1\right). \tag{4.1.12}$$

Fix then a probability $\mathbb{Q} \in \mathscr{P}_{\mathbf{G}}$ and consider a sequence $\mathbf{B}^n$ of processes given by

$$\mathbf{B}_t^n = \sum_{i=1}^{n} \sigma_{t_{i-1}}(\mathbf{W}_{t_i}^{\mathbb{Q}} - \mathbf{W}_{t_{i-i}}^{\mathbb{Q}}), \tag{4.1.13}$$

where $t_i = Ti/n$, σ_{t_i} is a $\mathscr{F}_{t_i}$-measurable function taking values in $\mathbf{D}$ and $\mathbf{W}^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$, satisfying

$$E_{\mathbb{Q}}\left[\|\mathbf{B}_T - \mathbf{B}_T^n\|_2^2\right] \longrightarrow 0, \quad \text{for } n \longrightarrow \infty. \tag{4.1.14}$$

We can then calculate

$$
\begin{aligned}
E_{\mathbb{Q}}\left[\frac{1}{\|\mathbf{B}_T^n\|}\right] &= E_{\mathbb{Q}}\left[E_{\mathbb{Q}}\left[\frac{1}{\|\mathbf{B}_T^n\|}\Big|\mathscr{F}_{t_{n-1}}\right]\right] = E_{\mathbb{Q}}\left[E_{\mathbb{Q}}\left[\frac{1}{\|\mathbf{B}_T^n-\mathbf{B}_{t_{n-1}}^n+\mathbf{B}_{t_{n-1}}^n\|}\Big|\mathscr{F}_{t_{n-1}}\right]\right]\\
&= E_{\mathbb{Q}}\left[\frac{1}{\|\mathbf{B}_{t_{n-1}}^n\|}\left(2\Phi\left(\frac{\|\mathbf{B}_{t_{n-1}}^n\|}{[\sigma_{t_{n-1}}]_{11}\sqrt{t_n-t_{n-1}}}\right)-1\right)\right]\\
&\leq E_{\mathbb{Q}}\left[\frac{1}{\|\mathbf{B}_{t_{n-1}}^n\|}\left(2\Phi\left(\frac{\|\mathbf{B}_{t_{n-1}}^n\|}{\sqrt{t_n-t_{n-1}}}\right)-1\right)\right]\\
&= E_{\mathbb{Q}}\left[E_{\mathbb{Q}}\left[\frac{1}{\|\tilde{\mathbf{B}}_T^n\|}\Big|\mathscr{F}_{t_{n-1}}\right]\right] = E_{\mathbb{Q}}\left[\frac{1}{\|\tilde{\mathbf{B}}_T^n\|}\right],
\end{aligned}
$$

where $[\sigma_{t_{n-1}}]_{11}$ represents the first component of $\sigma_{t_{n-1}}$, where by construction the entries different from zero are equal to each other and $\tilde{\mathbf{B}}^n$ is defined as in (4.1.13), except that $\sigma_{t_{n-1}}$ is replaced by the identity matrix in $\mathbb{R}^{3\times 3}$. It is then easy to repeat the preceding computation remarking that

$$
\begin{aligned}
E_{\mathbb{Q}}\left[\frac{1}{\|\mathbf{B}_T^n\|}\right] &\leq E_{\mathbb{Q}}\left[\frac{1}{\|\tilde{\mathbf{B}}_T^n\|}\right] = E_{\mathbb{Q}}\left[E_{\mathbb{Q}}\left[\frac{1}{\|\tilde{\mathbf{B}}_T^n\|}\Big|\mathscr{F}_{t_{n-2}}\right]\right]\\
&= E_{\mathbb{Q}}\left[E_{\mathbb{Q}}\left[\frac{1}{\|\mathbf{B}_T^{n-1}+(\mathbf{W}_T-\mathbf{W}_{t_{n-1}})-\mathbf{B}_T^{n-2}+\mathbf{B}_T^{n-2}\|}\Big|\mathscr{F}_{t_{n-2}}\right]\right]\\
&= E_{\mathbb{Q}}\left[\frac{1}{\|\mathbf{B}_{t_{n-2}}^n\|}\left(2\Phi\left(\frac{\|\mathbf{B}_{t_{n-2}}^n\|}{[\sigma_{t_{n-2}}]_{11}\sqrt{t_{n-1}-t_{n-2}}+\sqrt{t_n-t_{n-1}}}\right)-1\right)\right]\\
&\leq E_{\mathbb{Q}}\left[\frac{1}{\|\mathbf{B}_{t_{n-2}}^n\|}\left(2\Phi\left(\frac{\|\mathbf{B}_{t_{n-2}}^n\|}{\sqrt{t_n-t_{n-1}}+\sqrt{t_{n-1}-t_{n-2}}}\right)-1\right)\right].
\end{aligned}
$$

At the n-th iteration we get

$$
E_{\mathbb{Q}}\left[\frac{1}{\|\mathbf{B}_T^n\|}\right] \leq 2\Phi\left(\frac{1}{\sum_{i=1}^n\sqrt{t_i-t_{i-1}}}\right)-1 = 2\Phi\left(\frac{1}{\sqrt{nT}}\right)-1 \longrightarrow 0 \text{ for } n\to\infty. \tag{4.1.15}
$$

We exploit (4.1.15) to argue that

$$
\sup_{\mathbb{Q}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{Q}}\left[\frac{1}{\|\mathbf{B}_T\|}\right] < \sup_{\mathbb{Q}\in\mathscr{P}_{\mathbf{G}}} E_{\mathbb{Q}}\left[\frac{1}{\|\mathbf{B}_0\|}\right] = 1 \tag{4.1.16}
$$

in this way showing that $f(\mathbf{B})$ is not a $\mathscr{Q}$-martingale. This is done by considering a sequence $(f^m)_{m\in\mathbb{N}}$ of bounded and continuous functions converging monotonic increasingly to f, such that

$$
E_{\mathbb{Q}}\left[\frac{1}{f^m(\mathbf{B}_T^n)}\right] \longrightarrow E_{\mathbb{Q}}\left[\frac{1}{f^m(\mathbf{B}_T)}\right] \text{ for } n\to\infty,
$$

thanks to the convergence (4.1.14). In addition, since f^m is dominated by f and because of (4.1.15), $E_{\mathbb{Q}}\left[\frac{1}{f^m(\mathbf{B}_T)}\right] < c$, with $0 < c < 1$, for m sufficiently large. We can then show

$$E_{\mathbb{Q}}\left[\frac{1}{f^m(\mathbf{B}_T)}\right] \longrightarrow E_{\mathbb{Q}}\left[\frac{1}{\|\mathbf{B}_T\|}\right] < c \text{ for } m \to \infty,$$

by virtue of dominated convergence. Since we can repeat the same argument for every $\mathbb{Q} \in \mathscr{P}_{\mathbf{G}}$ we achieve (4.1.16).

In Example 4.1.11 and Example 4.1.12, all market scenarios $\mathbb{Q}$ agree on the existence of a bubble. Alternatively stated the price process we obtained is a strict $\mathbb{Q}$-local martingale for every $\mathbb{Q} \in \mathscr{Q}$.
We can adapt the setting of Example 4.1.12 to construct an asset which is a $\bar{\mathbb{Q}}$-martingale for some $\bar{\mathbb{Q}} \in \mathscr{Q}$. In this way, even if there exists a bubble, an agent not affected by uncertainty, trusting $\bar{\mathbb{Q}}$ to be the right probability ruling the market, would not spot it. This is precisely one of the new features of our setting.

Example 4.1.13. Let $\mathscr{Q} = \mathscr{P}_{\mathbf{G}}$ as in Example 4.1.12 but now we select $a_{1,1} = a_{2,2} = a_{3,3} \in [0,2]$. In this way we allow for the existence of a degenerate case in which there is no randomness. In the same fashion as in Example 4.1.12, the process $f(\mathbf{B})$ is a $\mathbb{Q}$-local martingale for every $\mathbb{Q} \in \mathscr{Q}$. However for the probability measure $\bar{\mathbb{Q}}$, which corresponds to a constant zero volatility, all dynamics are $\bar{\mathbb{Q}}$-a.s. deterministic. As a consequence $f(\mathbf{B}_t) = f(\mathbf{B}_0) = 1$ $\bar{\mathbb{Q}}$-a.s. for all $t \geq 0$, which guarantees that $f(\mathbf{B})$ is a true $\bar{\mathbb{Q}}$-martingale, although being a strict $\mathbb{Q}$-local martingale for every $\mathbb{Q} \in \mathscr{Q} \setminus \{\bar{\mathbb{Q}}\}$.

In the classical literature examples of asset price bubbles are usually provided by constructing a process displaying strict local martingale behavior under some probability and true martingale dynamics for some equivalent prior. In our setting there is one additional degree of freedom, given by the possibility to choose adequately the set of measures describing the market uncertainty.

Example 4.1.14. We consider the setting outlined in [51]. In particular, we focus on the family $\mathscr{Q}_S$ of probabilities

$$\mathbb{Q}^\alpha := \mathbb{Q}_0 \circ (X^\alpha)^{-1}, \quad \text{where} \quad X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s, \quad t \in [0,T]. \qquad (4.1.17)$$

In (4.1.17) $\mathbb{Q}_0$ represents the Wiener measure, while α is any $\mathbb{F}$-progressively measurable processes taking values in $\mathbb{S}_d^+$, such that $\int_0^T |\alpha_s| ds < \infty$ $\mathbb{Q}_0$-a.s. We remind that $\mathbb{S}_d^+ \subset \mathbb{R}^{d\times d}$ stands for the collection of strictly positive definite matrices while the integral in (4.1.17) is the Itô stochastic integral under $\mathbb{Q}_0$. The family $\mathscr{Q}$ is assumed to be stable under pasting, as defined in the following statement.

Definition 4.1.15. The set $\mathscr{Q}$ is stable under $\mathbb{F}$-pasting if for all $\mathbb{Q} \in \mathscr{Q}$, σ stopping time taking finitely many values, $\Lambda \in \mathscr{F}_\sigma$ and $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathscr{Q}(\sigma, \mathbb{Q})$, the measure $\bar{\mathbb{Q}}$ defined by

$$\bar{\mathbb{Q}}(A) := E_{\mathbb{Q}}\left[\mathbb{Q}_1(A|\mathscr{F}_\sigma)\mathbf{1}_\Lambda + \mathbb{Q}_2(A|\mathscr{F}_\sigma)\mathbf{1}_{\Lambda^c}\right], \quad A \in \mathscr{F}_T \tag{4.1.18}$$

is again an element of $\mathscr{Q}$.

Assume then that there exists a $\mathbb{Q}^{\tilde{\alpha}} \in \mathscr{Q}_S$ under which the process S is a strict $\mathbb{Q}^{\tilde{\alpha}}$-local martingale. As a concrete case we can imagine S to be the same as in Example 4.1.11. To see how a bubble can be born in such framework we focus on the subset $\mathscr{Q} \subseteq \mathscr{Q}_S$ described by those $\mathbb{Q}^\alpha \in \mathscr{Q}_S$ for which

$$\alpha_s = \tilde{\alpha}_s \quad \text{for } s \in (t,T] \ \mathbb{Q}_0 - a.s.$$

for some $t \in (0,T)$. This $\mathscr{Q}$ is then stable under pasting, according to Definition 4.1.15, by virtue of Lemma 3.3 in [51]. We are then restricting ourselves to a subfamily of $\mathscr{Q}_S$ where uncertainty disappears after time t, and the volatility of the canonical process on $(t,T]$ determines a strict local martingale dynamics for the asset price. Hence, for each $s > t$,

$$\operatorname*{ess\,sup}_{\mathbb{Q} \in \mathscr{Q}(s, \mathbb{Q}^{\tilde{\alpha}})} E_{\mathbb{Q}}[S_T|\mathscr{F}_s] = E_{\mathbb{Q}^{\tilde{\alpha}}}[S_T|\mathscr{F}_s] < S_s,$$

thus showing the existence of a bubble.

In Section 4.1.2 we discussed how the existence of a bubble is equivalent to the possibility to find $\mathbb{Q} \in \mathscr{Q}$ and $t \in [0,T)$ for which

$$S_t > \operatorname*{ess\,sup}_{\mathbb{Q}' \in \mathscr{Q}(t, \mathbb{Q})} E_{\mathbb{Q}'}[S_T|\mathscr{F}_t].$$

Hence a bubble under uncertainty determines a bubble for all the underlying $\mathbb{Q}'$-markets with $\mathbb{Q}' \in \mathscr{Q}(t, \mathbb{Q})$. This is particularly clear in two cases: when every $\mathbb{Q}$ describes a complete market model or when fundamental values are modeled as in the martingale theory of bubbles. We prove next that that the family of measures $\mathbb{Q} \in \mathscr{Q}$ for which the asset S is a strict local martingale cannot consist of one single element. We do that in the framework described in Example 4.1.14. By doing that we are able to sensibly simplify the calculations and to draw some conclusions regarding the more general setting described in Section 3.1, since the G-setting can be represented using both models.

Proposition 4.1.16. *Consider the financial model introduced in Example 4.1.14. If $\bar{\mathbb{Q}}$ is the pasting of $\mathbb{Q}$, $\mathbb{Q}_1$ and $\mathbb{Q}_2$ at the stopping time σ and $\Lambda \in \mathscr{F}_\sigma$, as in* (4.1.18)*, it holds*

$$E_{\bar{\mathbb{Q}}}[Y|\mathscr{F}_\tau] = E_{\mathbb{Q}}\left[E_{\mathbb{Q}_1}[Y\mathbf{1}_\Lambda|\mathscr{F}_\sigma]|\mathscr{F}_\tau\right] + E_{\mathbb{Q}}\left[E_{\mathbb{Q}_2}[Y\mathbf{1}_{\Lambda^c}|\mathscr{F}_\sigma]|\mathscr{F}_\tau\right] \tag{4.1.19}$$

for any positive $\mathscr{F}_T$-measurable random variable Y and stopping time τ such that $\tau(\omega) \leq \sigma(\omega)$ for every $\omega \in \Omega$.

Proof. We prove the claim by adopting a technique analogue to Lemma 6.40 in [28]. Let Y be a positive $\mathscr{F}_T$-measurable function and τ be a stopping time. Thanks to (4.1.18) we obtain

$$E_{\bar{\mathbb{Q}}}[Y] = E_{\mathbb{Q}}\left[E_{\mathbb{Q}_1}[Y|\mathscr{F}_\sigma]\mathbf{1}_\Lambda + E_{\mathbb{Q}_2}[Y|\mathscr{F}_\sigma]\mathbf{1}_{\Lambda^c}\right].$$

Hence, given any positive $\mathscr{F}_\tau$-measurable function φ, we can compute

$$E_{\bar{\mathbb{Q}}}[Y\varphi\mathbf{1}_{\{\tau\leq\sigma\}}]. \tag{4.1.20}$$

We can express the expectation in (4.1.20) as

$$\begin{aligned}
E_{\bar{\mathbb{Q}}}[Y\varphi\mathbf{1}_{\{\tau\leq\sigma\}}] &= E_{\mathbb{Q}}\left[E_{\mathbb{Q}_1}[Y\varphi\mathbf{1}_{\{\tau\leq\sigma\}}|\mathscr{F}_\sigma]\mathbf{1}_\Lambda + E_{\mathbb{Q}_2}[Y\varphi\mathbf{1}_{\{\tau\leq\sigma\}}|\mathscr{F}_\sigma]\mathbf{1}_{\Lambda^c}\right]\\
&= E_{\mathbb{Q}}\left[E_{\mathbb{Q}_1}[Y\varphi\mathbf{1}_{\{\tau\leq\sigma\}}\mathbf{1}_\Lambda|\mathscr{F}_\sigma] + E_{\mathbb{Q}_2}[Y\varphi\mathbf{1}_{\{\tau\leq\sigma\}}\mathbf{1}_{\Lambda^c}|\mathscr{F}_\sigma]\right]\\
&= E_{\mathbb{Q}}\Big[E_{\mathbb{Q}}\left[E_{\mathbb{Q}_1}[Y\mathbf{1}_\Lambda|\mathscr{F}_\sigma]|\mathscr{F}_\tau\right]\varphi\mathbf{1}_{\{\tau\leq\sigma\}} + E_{\mathbb{Q}}\left[E_{\mathbb{Q}_2}[Y\mathbf{1}_{\Lambda^c}|\mathscr{F}_\sigma]|\mathscr{F}_\tau\right]\varphi\mathbf{1}_{\{\tau\leq\sigma\}}\Big] \qquad (4.1.21)\\
&= E_{\bar{\mathbb{Q}}}\Big[E_{\mathbb{Q}}\left[E_{\mathbb{Q}_1}[Y\mathbf{1}_\Lambda|\mathscr{F}_\sigma]|\mathscr{F}_\tau\right]\varphi\mathbf{1}_{\{\tau\leq\sigma\}} + E_{\mathbb{Q}}\left[E_{\mathbb{Q}_2}[Y\mathbf{1}_{\Lambda^c}|\mathscr{F}_\sigma]|\mathscr{F}_\tau\right]\varphi\mathbf{1}_{\{\tau\leq\sigma\}}\Big]\\
&= E_{\bar{\mathbb{Q}}}\left[\left(E_{\mathbb{Q}}\left[E_{\mathbb{Q}_1}[Y\mathbf{1}_\Lambda|\mathscr{F}_\sigma]|\mathscr{F}_\tau\right] + E_{\mathbb{Q}}\left[E_{\mathbb{Q}_2}[Y\mathbf{1}_{\Lambda^c}|\mathscr{F}_\sigma]|\mathscr{F}_\tau\right]\right)\varphi\mathbf{1}_{\{\tau\leq\sigma\}}\right].
\end{aligned}$$

Therefore if $\tau \leq \sigma$, we can state that (4.1.19) holds. □

Corollary 4.1.17. *Consider $\bar{\mathbb{Q}}$ given by the pasting of $\mathbb{Q}$, $\mathbb{Q}_1$ and $\mathbb{Q}_2$ at the stopping time σ and $\Lambda \in \mathscr{F}_\sigma$, as in* (4.1.18). *If S is a strict $\mathbb{Q}_1$-local martingale, then it is also a strict $\bar{\mathbb{Q}}$-local martingale.*

Proof. Because of Proposition 4.1.16, if S is a strict $\mathbb{Q}_1$-local martingale and σ is such that

$$E_{\mathbb{Q}_1}[S_T|\mathscr{F}_\sigma] < S_\sigma,$$

we have

$$\begin{aligned}
E_{\bar{\mathbb{Q}}}[S_T|\mathscr{F}_\tau] &= E_{\mathbb{Q}}\left[E_{\mathbb{Q}_1}[S_T\mathbf{1}_\Lambda|\mathscr{F}_\sigma]|\mathscr{F}_\tau\right] + E_{\mathbb{Q}}\left[E_{\mathbb{Q}_2}[S_T\mathbf{1}_{\Lambda^c}|\mathscr{F}_\sigma]|\mathscr{F}_\tau\right]\\
&= E_{\mathbb{Q}}\left[E_{\mathbb{Q}_1}[S_T|\mathscr{F}_\sigma]\mathbf{1}_\Lambda|\mathscr{F}_\tau\right] + E_{\mathbb{Q}}\left[E_{\mathbb{Q}_2}[S_T|\mathscr{F}_\sigma]\mathbf{1}_{\Lambda^c}|\mathscr{F}_\tau\right]\\
&< E_{\mathbb{Q}}\left[S_\sigma\mathbf{1}_\Lambda|\mathscr{F}_\tau\right] + E_{\mathbb{Q}}\left[S_\sigma\mathbf{1}_{\Lambda^c}|\mathscr{F}_\tau\right]\\
&\leq S_\tau,
\end{aligned}$$

and therefore S is a strict $\bar{\mathbb{Q}}$-local martingale as well. □

4.1.4 No Dominance

In this section we extend the concept of no dominance under model uncertainty. This concept was introduced in [49], and we report its definition as stated in [37] for the situation in which there is only one probability measure $\mathbb{Q}$.

Definition 4.1.18 (Definition 2.2 from [37])**.** Let be given a financial market with d securities $(S^1,\dots,S^d)$ in a filtered probability space $(\Omega,\mathscr{F},\mathbb{F}=\{\mathscr{F}_t\}_{t\in[0,T]},\mathbb{Q})$. The process $H=(H_t)_{t\in[0,T]}$ is an admissible strategy if it is an $\mathbb{F}$-predictable and S-integrable process such that $H\cdot S\geq -a$, for some $a\in\mathbb{R}_+$. We say that the i-th security S^i is undominated on $[0,T]$ if there is no admissible strategy H such that

$$S_0^i+(H\cdot S)_T\geq S_T^i \ \ \mathbb{Q}-a.s. \quad \text{and} \quad \mathbb{Q}(S_0^i+(H\cdot S)_T>S_T^i)>0.$$

A market satisfies no dominance (ND) on $[0,T]$ if each S^i, $i\in\{1,\dots,d\}$, is undominated on $[0,T]$.

We suggest the following definition to extend this notion to the uncertainty framework.

Definition 4.1.19. Consider a market model under a set of priors $\mathscr{Q}$. The i-th security S^i is *undominated* on $[0,T]$ if there is no admissible strategy $H\in\mathscr{H}$ such that

$$S_0^i+(H\cdot S)_T\geq S_T^i \ \ \mathscr{Q}-q.s. \text{ and there exists a } \mathbb{Q}\in\mathscr{Q} \text{ such that } \mathbb{Q}(S_0^i+(H\cdot S)_T>S_T^i)>0.$$

A market satisfies *robust no dominance* (RND) on $[0,T]$ if each S^i, $i\in\{1,\dots,d\}$, is undominated on $[0,T]$.

Remark 4.1.20. Similarly to what happens in the classical setting, if S^i is undominated on $[0,T]$, then the same holds true on $[0,T']$, for $T'<T$. Let H^i be given by

$$H^i=(0,\dots,0,1,0,\dots,0),$$

with 1 at the i-th entry. As S^i a $\mathbb{Q}$-local martingale for all $\mathbb{Q}\in\mathscr{Q}$, the trading strategy H^i is admissible. If there exists dominating strategy H on $[0,T']$, by adopting the strategy $K=H\mathbf{1}_{\{t\leq T'\}}+H^i\mathbf{1}_{\{t>T'\}}$, we would get

$$S_0^i+(K\cdot S)_T=S_T^i+S_0^i+(H\cdot S)_{T'}-S_{T'}^i\geq S_T^i \quad q.s.,$$

as well as the presence of a probability $\mathbb{Q}\in\mathscr{Q}$ for which

$$\mathbb{Q}(S_0^i+(K\cdot S)_T>S_T^i)>0.$$

No dominance has a fundamental significance in the literature on financial bubbles. When there exists a unique ELMM, the cornerstone of the martingale theory of bubbles outlined in [38] excludes their presence if ND holds. In a similar fashion ND is exactly the component needed to rule out bubbles in the framework of [65], in which fundamental values are defined as superhedging prices. We can derive analogue results also in the uncertainty framework.

Lemma 4.1.21. *Suppose that for each $\mathbb{Q} \in \mathscr{Q}$ the $\mathbb{Q}$-market model is complete. If robust no dominance holds, then there exists no bubble.*

Proof. Note that, if every $\mathbb{Q}$ admits no other ELMM, the duality

$$\sup_{\mathbb{Q}\in\mathscr{Q}} E_{\mathbb{Q}}[S_T] = \inf\{x \in \mathbb{R} : \exists H \in \mathscr{H} \text{ with } x + (H \cdot S)_T \geq S_T \ \mathbb{Q}-a.s. \text{ for all } \mathbb{Q} \in \mathscr{Q}\}, \tag{4.1.22}$$

is ensured by the results of [50]. When there exists a bubble, the superhedging portfolio would dominate S, thus contradicting RND. □

Therefore, in the general context, if RND holds any bubble would come from a duality gap in (4.1.22). This is precisely the situation described in [17].

We conclude this section by noticing how in full generality RND is not enough to ensure NFLVR for every $\mathbb{Q}$-market, $\mathbb{Q} \in \mathscr{Q}$. In fact, while ND is a condition stronger than NFLVR in the classical framework, it is not necessary that RND implies ND for every $\mathbb{Q}$-market.

4.2 Infinite Time Horizon

We extend now our analysis to the case of infinite time horizon. In order to model the impossibility of the agent to benefit from a final payoff at infinite time we need to generalize the definition of robust fundamental value outlined in (4.1.5) by imposing

$$S_t^* = \left(\operatorname*{ess\,sup}_{\mathbb{Q}'\in\mathscr{Q}(t,\mathbb{Q})} E_{\mathbb{Q}'}[S_\tau \mathbf{1}_{\{\tau<\infty\}} | \mathscr{F}_t] \right) \mathbf{1}_{\{t<\tau\}}, \quad \mathbb{Q}-a.s. \tag{4.2.1}$$

for each $t \geq 0$ and $\mathbb{Q} \in \mathscr{Q}$. The fundamental value (4.2.1) is well defined, as we prove in the next proposition, and incorporates the finite time horizon situation (4.1.5).

Proposition 4.2.1. *The fundamental value* (4.2.1) *is well defined. In addition, $S_{t\wedge\tau}$ converges to S_τ q.s. for $t \to \infty$.*

Proof. Consider any probability $\mathbb{Q} \in \mathscr{Q}$. The wealth process W is a $\mathbb{Q}$-supermartingale converging $\mathbb{Q}$-a.s. to S_τ for $t \to \infty$, by virtue of the supermartingale convergence theorem (see [19], V.28 and VI.6). As a consequence $W_t = S_{t\wedge\tau} \to S_\tau$ q.s., because of the same reasoning exploited in Example 4.1.11, and S_τ is Borel measurable. Therefore $S_\tau \mathbf{1}_{\{\tau<\infty\}}$ is a Borel measurable function and it is possible to calculate its sublinear conditional expectation. In addition, since W is a $\mathscr{Q}$-supermartingale, by Fatou's Lemma we get

$$\begin{aligned}
\mathscr{E}_0(S_\tau) = \mathscr{E}_0\left(\liminf_{t\to\infty} S_{t\wedge\tau}\right) &= \sup_{\mathbb{Q}\in\mathscr{Q}} E_{\mathbb{Q}}\left(\liminf_{t\to\infty} S_{t\wedge\tau}\right) \leq \sup_{\mathbb{Q}\in\mathscr{Q}} \liminf_{t\to\infty} E_{\mathbb{Q}}\left(S_{t\wedge\tau}\right) \\
&= \sup_{\mathbb{Q}\in\mathscr{Q}} \liminf_{t\to\infty} E_{\mathbb{Q}}\left(W_t\right) \leq \sup_{\mathbb{Q}\in\mathscr{Q}} E_{\mathbb{Q}}\left(W_0\right) < \infty,
\end{aligned}$$

which ensures $\mathscr{E}_0(S_\tau \mathbf{1}_{\{\tau<\infty\}}) < \infty$. □

We can define the concept of robust fundamental wealth, by introducing the process $W^* = (W^*_t)_{t\geq 0}$, with

$$\begin{aligned}
W^*_t := S^*_t + S_\tau \mathbf{1}_{\{\tau\leq t\}} &= \left(\operatorname*{ess\,sup}_{\mathbb{Q}'\in\mathscr{Q}(t,\mathbb{Q})} E_{\mathbb{Q}'}[S_\tau \mathbf{1}_{\{\tau<\infty\}}|\mathscr{F}_t]\right)\mathbf{1}_{\{t<\tau\}} + S_\tau \mathbf{1}_{\{\tau\leq t\}} \\
&= \operatorname*{ess\,sup}_{\mathbb{Q}'\in\mathscr{Q}(t,\mathbb{Q})} E_{\mathbb{Q}'}[S_\tau \mathbf{1}_{\{\tau<\infty\}}|\mathscr{F}_t], \qquad \mathbb{Q}-a.s. \qquad (4.2.2)
\end{aligned}$$

for every $\mathbb{Q} \in \mathscr{Q}$. Consistently with the setting outlined in 4.1.2, the bubble is defined as

$$\beta_t = S_t - S^*_t = W_t - W^*_t,$$

for every $t \geq 0$. Hence the situation $\tau = \infty$ q.s. determines the existence of a bubble. We describe here Example 2 from [38] to make this point clearer.

Example 4.2.2. Assume $S_t = 1$ for all $t \in \mathbb{R}_+$, i.e. fiat money. As money never matures $\tau = \infty$, $S_\tau = 1$ and $S^*_t = 0$ q.s. for every $t \geq 0$. Since

$$\beta_t = S_t - S^*_t = 1 \quad q.s.$$

this implies that all the value of S is deriving from the bubble.

We summarize these findings in the next statement.

Proposition 4.2.3. *It holds:*

(i) In the case there exists a $\bar{\mathbb{Q}} \in \mathscr{Q}$ and $t \geq 0$ such that $\bar{\mathbb{Q}}'(\tau = \infty) = 1$ for all $\bar{\mathbb{Q}}' \in \mathscr{Q}(t, \bar{\mathbb{Q}})$, there exists a bubble.

(ii) The bubble β is a $\mathbb{Q}$-local submartingale for every $\mathbb{Q} \in \mathscr{Q}$.

(iii) *The wealth process W can be a $\mathscr{Q}$-symmetric martingale also in the presence of a bubble.*

Proof. Statement (i) comes from (4.2.2), remarking that

$$W_t^* = 0 \qquad \bar{\mathbb{Q}} - a.s.,$$

as by assumption

$$S_\tau \mathbf{1}_{\{\tau<\infty\}} = 0 \qquad \bar{\mathbb{Q}}' - a.s.$$

for every $\bar{\mathbb{Q}}' \in \mathscr{Q}(t, \bar{\mathbb{Q}})$. The local submartingale dynamics are a consequence of the definition and of Assumption 4.1.1. The wealth process can display $\mathscr{Q}$-symmetric martingale behavior as shown in Example 4.2.2. □

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