The transcendental part of higher Brauer groups in weight 2

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Contents

Zusammenfassung

Die klassische kohomologische Brauergruppe $\mathrm{Br}(X) = \mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m)$ eines glatten Schemas X steht in Verbindung mit vielen tiefergehenden Fragen in der algebraischen Geometrie. So stellt sie beispielsweise eine Obstruktion zur Tate-Vermutung für X in Kodimension 1 dar und hängt mit dem Verhalten der Zeta-Funktion von X bei 1 zusammen. Des Weiteren kann der Fehler des Hasse-Prinzips in manchen Fällen mit Hilfe der Brauergruppe erklärt werden. Es ist daher von Interesse, $\mathrm{Br}(X)$ tatsächlich berechnen zu können. Colliot-Thélène and Skorobogatov zeigten, dass für eine glatte projektive und geometrisch integrale Varietät X über einem Körper k der Charakteristik 0 der Kokern der natürlichen Abbildung $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^{G_k}$ endlich ist.

Die von Bloch konstruierten Zykelkomplexe definieren Komplexe $\mathbb{Z}_X(n)$ étaler Garben auf X und erlauben es, "höhere" Brauergruppen $\mathrm{Br}^r(X) := \mathbb{H}^{2r+1}_{\acute{e}t}(X, \mathbb{Z}_X(r))$ zu definieren. Da $\mathrm{Br}(X)$ und $\mathrm{Br}^1(X)$ isomorph sind, können diese Gruppen $\mathrm{Br}^r(X)$ als natürliche Verallgemeinerung der klassischen Brauergruppe betrachtet werden.

In dieser Dissertation verallgemeinern wir unter einigen weiteren Annahmen das Resultat von Colliot-Thélène und Skorobogatov auf $\mathrm{Br}^2(X)$ und zeigen: Sei X eine glatte, projektive und geometrisch irreduzible Varietät von Dimension höchstens vier über einem Körper k von Charakteristik 0 und kohomologischer Dimension höchstens 2. Ist dann die dritte Betti-Zahl von X gleich 0 und $\mathrm{H}^4_{\acute{e}t}(\overline{X}, \mathbb{Z}_\ell(2))$ torsionsfrei für jede Primzahl ℓ (äquivalent $\mathrm{H}^4_{\mathrm{B}}(X_{\mathbb{C}}, \mathbb{Z}(2))_{\mathrm{tors}} = 0$), so ist der Kokern der natürlichen Abbildung $\mathrm{Br}^2(X) \to \mathrm{Br}^2(\overline{X})^{G_k}$ endlich.

Abstract

The classical cohomological Brauer group $\mathrm{Br}(X) = \mathrm{H}^2_{ét}(X, \mathbb{G}_m)$ of a smooth scheme X is related to many deep questions in algebraic geometry. For example, it yields an obstruction to the Tate conjecture for X in codimension 1 and it is related to the behaviour of the zeta function of X at 1. Furthermore, in some cases, the failure of the local-global Hasse principle can be explained in terms of the Brauer group. Therefore it becomes an interesting question to attempt to actually compute $\mathrm{Br}(X)$. Colliot-Thélène and Skorobogatov showed that for a smooth, projective and geometrically integral variety X over a field k of characteristic 0 the cokernel of the natural map $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^{G_k}$ is finite.

The cycle complexes constructed by Bloch define complexes $\mathbb{Z}_X(r)$ of étale sheaves on X which allow one to define 'higher' Brauer groups $\mathrm{Br}^r(X) := \mathbb{H}^{2r+1}_{ét}(X, \mathbb{Z}_X(r))$. Since $\mathrm{Br}^1(X)$ and $\mathrm{Br}(X)$ are isomorphic, these groups $\mathrm{Br}^r(X)$ can be seen as a natural generalisation of the classical Brauer groups.

In this dissertation we generalise the result of Colliot-Thélène and Skorobogatov to $\mathrm{Br}^2(X)$ under some additional assumptions, i.e. we show: Let X be a smooth, projective and geometrically irreducible variety of dimension at most 4 over a field k of characteristic 0 and cohomological dimension at most 2. If the third Betti number of X is 0 and $\mathrm{H}^4_{ét}(\overline{X}, \mathbb{Z}_\ell(2))$ is torsion free for every prime ℓ (equivalently $\mathrm{H}^4_{\mathrm{B}}(X_{\mathbb{C}}, \mathbb{Z}(2))_{\mathrm{tors}} = 0$), then the cokernel of the natural map $\mathrm{Br}^2(X) \to \mathrm{Br}^2(\overline{X})^{G_k}$ is finite.

1. Introduction

The classical cohomological Brauer group $\mathrm{Br}(X) = \mathrm{H}^2_{ét}(X, \mathbb{G}_m)$ of a regular scheme X plays an important role in several areas of algebraic geometry. For example, it is related to the behaviour of the zeta function of X at 1 and it yields an obstruction to the Tate conjecture for X in codimension 1, if X is a surface over a finite field. Furthermore, the Brauer group $\mathrm{Br}(X)$ gives an obstruction to the Hasse principle for X over a number field and in many cases this obstruction completely explains the failure of the Hasse principle. These roles of the Brauer group as an obstruction motivate the attempt to actually calculate the group $\mathrm{Br}(X)$, or at least to understand its structure.

For regular integral schemes $\mathrm{Br}(X)$ is known to be a torsion group and to be a birational invariant in characteristic zero; however, explicit computations are only known in a few very special cases. One approach to study the Brauer group of a scheme X over a field k is to consider the map which is induced by base change to an algebraic closure $\overline{k}$ of k. This yields a natural map $\alpha : \mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$ whose image is contained in the subgroup of Galois invariants. About this map α Colliot-Thélène and Skorobogatov showed the following:

1.0.1 Theorem (Colliot-Thélène and Skorobogatov, [CTS11, Theorem 2.1])
Let X be a smooth, projective and geometrically integral variety over a field k of characteristic zero with absolute Galois group G_k. Then the cokernel of $\alpha : \mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^{G_k}$ is finite.

To prove this result, Colliot-Thélène and Skorobogatov use that the above map α arises as an edge map in the complex $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^{G_k} \to \mathrm{H}^2(G_k, \mathrm{Pic}(\overline{X}))$ induced by the Hochschild-Serre spectral sequence $\mathrm{H}^p(G_k, \mathrm{H}^q_{ét}(\overline{X}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{ét}(X, \mathbb{G}_m)$. Under the given assumptions, this complex is exact and one can study the cokernel of the first map via the image of the second map. They show further that for every smooth, projective, geometrically integral curve C over k and each morphism $C \to X$ the image of the second map $\mathrm{Br}(\overline{X})^{G_k} \to \mathrm{H}^2(G_k, \mathrm{Pic}(\overline{X}))$ is contained in the kernel of $\mathrm{H}^2(G_k, \mathrm{Pic}(\overline{X})) \to \mathrm{H}^2(G_k, \mathrm{Pic}(\overline{C}))$, and furthermore, that the kernel of the composition $\mathrm{H}^2(G_k, \mathrm{Pic}^0(\overline{X})) \to \mathrm{H}^2(G_k, \mathrm{Pic}(\overline{X})) \to \mathrm{H}^2(G_k, \mathrm{Pic}(\overline{C}))$ has finite exponent. This makes use of the fact that $\mathrm{Pic}^0(\overline{X})$ is an abelian variety and therefore one can use geometric arguments like the Poincaré reducibility theorem. Colliot-Thélène and Skorobogatov show in fact that the cokernel of $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^{G_k}$ has finite exponent, which implies by a formal argument that it is finite.

Lichtenbaum conjectured that there should be a bounded complex $\Gamma(n)$ of étale sheaves satisfying certain properties whose hypercohomology groups generalise the relation between the Brauer group and the behaviour of the zeta function at 1 to arbitrary positive integers $n \geq 1$; in fact, these groups should define motivic cohomology. Bloch's cycle complex $\mathbb{Z}_X(n)$, considered as an (unbounded) complex of étale sheaves on X, is conjectured to be quasi-isomorphic to the complex $\Gamma(n)$ described by Lichtenbaum. Therefore we refer to the hypercohomology groups of $\mathbb{Z}_X(n)$ as Lichtenbaum cohomology; in particular $\mathrm{Br}^n(X) := \mathbb{H}^{2n+1}_{ét}(X, \mathbb{Z}_X(n))$ and $\mathrm{CH}^n_{\mathrm{L}}(X) := \mathbb{H}^{2n}_{ét}(X, \mathbb{Z}_X(n))$ define the higher Brauer and Lichtenbaum-Chow groups respectively. These higher Brauer groups have many properties which are analogous to the properties of the classical Brauer group. For instance, $\mathrm{Br}^n(X)$ yields an obstruction to the Tate conjecture for X in codimension n.

In this dissertation we prove, under some additional assumptions, a generalisation of Theorem 1.0.1 to the higher Brauer group $\mathrm{Br}^2(X)$. Our main result is:

1.0.2 Theorem
Let X be a smooth, projective, geometrically irreducible variety of dimension at most 4 over a field k of characteristic zero with absolute Galois group G_k. Assume further that k has cohomological dimension ≤ 2, the third Betti-number b_3 of X is zero and that the group $H^4_{ét}(\overline{X}, \mathbb{Z}_\ell(2))$ is torsion free for every prime ℓ. Then the cokernel of the map $\mathrm{Br}^2(X) \to \mathrm{Br}^2(\overline{X})^{G_k}$ is finite.

To prove this result we first show that there is a Hochschild-Serre type of spectral sequence for Lichtenbaum cohomology groups, generalising the Hochschild-Serre spectral sequence for $\mathbb{G}_m$. Such a spectral sequence has been used without proof, for example, in [KN14]; however to our knowledge there is no proof of its existence in the literature. This spectral sequence yields a complex analogous to the complex considered by Colliot-Thélène and Skorobogatov. However, in our setting we cannot follow their arguments since in weight 2 the Picard variety $\mathrm{Pic}^0(\overline{X})$ is replaced by the group $\mathrm{CH}^2_{\mathrm{L}}(\overline{X})_{\mathrm{hom}}$, which is not the group of k-points of an abelian variety. Instead we use structure theorems for Lichtenbaum cohomology. Our assumptions on the third Betti number and the torsion in $\mathrm{H}^4_{ét}(\overline{X}, \mathbb{Z}_\ell(2))$ imply that the groups $\mathrm{CH}^2_{\mathrm{L}}(\overline{X})_{\mathrm{hom}}$ are uniquely divisible; we remark that these assumptions are satisfied by many varieties, including examples, where the structure of algebraic cycles is known to be highly nontrivial, such as a product of two $K3$ surfaces. We use the assumption $\dim(X) \leq 4$ to ensure that homological and numerical equivalence on X agree. Finally, the assumption on the field is of a rather technical nature, it simplifies the underlying Hochschild-Serre spectral sequence.

Notations

Let G be an abelian group. For each integer $a \in \mathbb{Z}$ let $G[a]$ be the a-torsion subgroup of G, i.e. the kernel of the multiplication map $m_a : G \to G$ by a. The torsion subgroup $G_{\text{tors}} \subseteq G$ is defined to be the union $G_{\text{tors}} := \bigcup_a G[a]$. Moreover we set $G_{\text{free}} := G/G_{\text{tors}}$. For a prime ℓ we write $G\{\ell\} := \bigcup_{n \in \mathbb{N}} G[\ell^n]$ for the ℓ-primary torsion subgroup of G.

A subgroup $H \subseteq G$ is said to be divisible, if for each $h \in H$ and each positive integer a there is an element $h' \in H$ such that $ah' = h$; if this h' is unique, the subgroup H is called uniquely divisible. We denote by G_{div} the maximal divisible subgroup of G.

If k is a field, we write $\overline{k}$ for an algebraic closure and k_{sep} for a separable closure of k; the absolute Galois group $\text{Gal}(k_{\text{sep}}/k)$ of k is denoted by G_k. If X is a scheme over k, we set $\overline{X} := X \times_k \overline{k}$.

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2. Brauer groups

We define the classical cohomological Brauer group $\mathrm{Br}(X)$ of a scheme X and describe some of its basic properties. In particular, we present Grothendieck's computation of the Brauer group for a smooth projective variety X over an algebraically closed field of characteristic zero, and explain how the Brauer group yields an obstruction to the Hasse local-global principle, as well as to the Tate conjecture for divisors.

2.1. Brauer groups

Let X be a scheme. Recall that the étale sheaf $\mathbb{G}_m$ on X is given by $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^\times$; its cohomology in degree 2 yields the classical cohomological Brauer group of X:

2.1.1 Definition
Let X be a scheme. The (cohomological) Brauer group of X is defined as the étale cohomology group $\mathrm{Br}(X) := \mathrm{H}^2_{ét}(X, \mathbb{G}_m)$.

We remark that there is also the notion of the *algebraic* Brauer group $\mathrm{Br}_{\mathrm{Az}}(X)$ defined as the set of equivalence classes of Azumaya algebras over X, which is a direct generalisation of the Brauer group of a field, see, for example, [Bou58, §10 no. 4]. We will sketch how $\mathrm{Br}_{\mathrm{Az}}(X)$ is related to the cohomological Brauer group $\mathrm{Br}(X)$; for a detailed exposition of this and the construction of $\mathrm{Br}_{\mathrm{Az}}(X)$, we refer to [Mil80, chapter IV].

Let X be a scheme. The algebraic Brauer group $\mathrm{Br}_{\mathrm{Az}}(X)$ can be considered as a subgroup of the cohomological Brauer group $\mathrm{Br}(X)$. More precisely, there is a canonical monomorphism [Mil80, Theorem IV.2.5]

$$\mathrm{Br}_{\mathrm{Az}}(X) \hookrightarrow \mathrm{Br}(X), \tag{1}$$

which is an isomorphism in many cases. For example, Grothendieck [Gro68a, Corollaire 2.2] showed that if X is noetherian of dimension ≤ 1, or regular and noetherian of dimension ≤ 2, then the two groups $\mathrm{Br}_{\mathrm{Az}}(X)$ and $\mathrm{Br}(X)$ are isomorphic.

More recently, it was shown by Gabber and de Jong that if X has an ample invertible sheaf, thus is quasi-compact and separated, then the algebraic Brauer group $\mathrm{Br}_{\mathrm{Az}}(X)$ is isomorphic to the torsion subgroup of the cohomological Brauer group $\mathrm{Br}(X)$ [dJ03]. In particular the monomorphism (1) cannot be surjective, if $\mathrm{Br}(X)$ is not a torsion group; for an example of such a singular scheme, see [Gro68a, Remarques 1.11 b)]. On the other hand, if X is a regular integral scheme, then $\mathrm{Br}(X)$ is torsion (this follows from [Mil80, Corollary IV.2.6]) and the result of Gabber and de Jong shows that for a quasi-projective regular integral scheme over a field, we always have $\mathrm{Br}_{\mathrm{Az}}(X) \cong \mathrm{Br}(X)$.

As a second important result we remark that if X is a regular noetherian scheme, then $\mathrm{Br}(X)$ is an birational invariant in characteristic zero, and for $\dim(X) \leq 2$ in any characteristic [Gro68b, Corollaires 7.2, 7.3].

In general computing Brauer groups is a difficult problem. One of the most important methods in this context is Kummer theory, which we will briefly discuss now: Let a be an integer, which is invertible in $\mathcal{O}_X$, and let μ_a be the sheaf of a-th roots of unity on $X_{ét}$, i.e. $\mu_a(U)$ is the group of a-th roots of unity in $\Gamma(U, \mathcal{O}_X)$; obviously μ_a is a subsheaf of $\mathbb{G}_m$. We have the exact Kummer sequence of étale sheaves

$$1 \to \mu_a \to \mathbb{G}_m \xrightarrow{e_a} \mathbb{G}_m \to 1,$$

where the map $e_a : \mathbb{G}_m \to \mathbb{G}_m$ is given by $e_a(U) : u \mapsto u^a$. Since the cohomology of $\mathbb{G}_m$ in degree 1 is isomorphic to the Picard group $\mathrm{Pic}(X)$ of X, we get from the Kummer sequence the following long exact sequence of cohomology groups

$$\ldots \to \mathrm{H}^1_{ét}(X, \mathbb{G}_m) \xrightarrow{\cdot a} \mathrm{Pic}(X) \to \mathrm{H}^2_{ét}(X, \mu_a) \to \mathrm{H}^2_{ét}(X, \mathbb{G}_m) \xrightarrow{\cdot a} \mathrm{H}^2_{ét}(X, \mathbb{G}_m) \to \ldots.$$

Since, by definition $\mathrm{Br}(X) = \mathrm{H}^2_{ét}(X, \mathbb{G}_m)$, this yields the short exact Kummer sequence

$$0 \to \mathrm{Pic}(X)/a \to \mathrm{H}^2_{ét}(X, \mu_a) \to \mathrm{Br}(X)[a] \to 0. \tag{2}$$

If X is a non-singular quasi-projective variety over a field, we can identify the Picard group $\mathrm{Pic}(X)$ in the above sequence with the Chow group $\mathrm{CH}^1(X)$ of codimension 1 cycles (cf. [Har77, A.2]).

Using these Kummer sequences, Grothendieck has computed the Brauer group of a smooth projective variety over an algebraically closed field of characteristic zero:

2.1.2 Example (Grothendieck, [Gro68b, section 8.])
Let X be a smooth projective variety over an algebraically field of characteristic zero; let ρ be the rank of the Neron-Severi group $\mathrm{NS}(X)$, let b_2 be the second Betti number of X, and let ℓ be a prime. Since over an algebraically closed field the group of points on an abelian variety is divisible, we have $\mathrm{Pic}(X)/\ell^n = \mathrm{NS}(X)/\ell^n$. Thus the short exact Kummer sequence (2) for $a = \ell^n$ has the form

$$0 \to \mathrm{NS}(X)/\ell^n \to \mathrm{H}^2_{ét}(X, \mu_{\ell^n}) \to \mathrm{Br}(X)[\ell^n] \to 0.$$

Passing to the projective limit over all n yields the short exact sequence

$$0 \to \mathrm{NS}(X) \otimes \mathbb{Z}_\ell \to \mathrm{H}^2_{ét}(X, \mathbb{Z}_\ell(1)) \to \mathrm{T}_\ell \,\mathrm{Br}(X) \to 0, \tag{3}$$

where $\mathrm{T}_\ell \mathrm{Br}(X) = \varprojlim_n \mathrm{Br}(X)[\ell^n]$ is the Tate module of $\mathrm{Br}(X)$. Since the Tate module $\mathrm{T}_\ell \mathrm{Br}(X)$ is torsion free, we obtain from (3) an isomorphism $(\mathrm{NS}(X) \otimes \mathbb{Z}_\ell)_{\mathrm{tors}} \cong \mathrm{H}^2_{ét}(X, \mathbb{Z}_\ell(1))_{\mathrm{tors}}$, and the exact sequence of finitely generated free $\mathbb{Z}_\ell$-modules

$$0 \to (\mathrm{NS}(X) \otimes \mathbb{Z}_\ell)_{\mathrm{free}} \to \mathrm{H}^2_{ét}(X, \mathbb{Z}_\ell(1))_{\mathrm{free}} \to \mathrm{T}_\ell \mathrm{Br}(X) \to 0. \tag{4}$$

Grothendieck [Gro68b, 8.1] shows that the maximal divisible subgroup $\mathrm{Br}(X)\{\ell\}_{\mathrm{div}}$ of $\mathrm{Br}(X)\{\ell\}$ is isomorphic to $(\mathrm{T}_\ell \mathrm{Br}(X)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$; thus tensoring the exact sequence (4) with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ and taking the direct sum over all primes ℓ yield the exact sequence

$$0 \to \mathrm{NS}(X)_{\mathrm{free}} \otimes \mathbb{Q}/\mathbb{Z} \to \oplus_\ell \left(\mathrm{H}^2_{et}(X, \mathbb{Z}_\ell(1))_{\mathrm{free}} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell\right) \to \mathrm{Br}(X)_{\mathrm{div}} \to 0$$

and an isomorphism $\mathrm{Br}(X)_{\mathrm{div}} \cong (\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$. Moreover, the non-divisible part of $\mathrm{Br}(X)$ can be described using the fact that $\mathrm{Pic}(X) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is divisible; it follows from (2) that $\mathrm{Br}(X)/\mathrm{Br}(X)_{\mathrm{div}} \cong \oplus_\ell \left(\mathrm{H}^3_{ét}(X, \mathbb{Z}_\ell(1))_{\mathrm{tors}}\right)$, in other words, we have an exact sequence

$$0 \to (\mathbb{Q}/\mathbb{Z})^{b_2-\rho} \to \mathrm{Br}(X) \to \oplus_\ell \left(\mathrm{H}^3_{ét}(X, \mathbb{Z}_\ell(1))_{\mathrm{tors}}\right) \to 0.$$

In particular, the Brauer group $\mathrm{Br}(X)$ of such a variety X is of finite cotype, and $\mathrm{Br}(X)$ is finite if and only if $\rho = b_2$.

2.2. Brauer-Manin obstructions and Tate's conjecture

The task of finding solutions in $\mathbb{Q}$ or $\mathbb{Z}$ of a finite system of polynomial equations

$$\begin{aligned} f_1(X_1, \dots, X_n) &= 0 \\ \vdots \qquad & \quad \vdots \\ f_n(X_1, \dots, X_n) &= 0 \end{aligned} \tag{5}$$

with coefficients in $\mathbb{Z}$ is known as a Diophantine problem. In 1900 Hilbert asked as one of his famous 23 problems if there exists an algorithm deciding whether or not a given Diophantine equation has a solution in the ring of integers. Although in 1970 a negative answer to this question was given by Matiyasevich [Mat70], one may still ask if the problem is more accessible with respect to rational solutions, i.e. finding $\mathbb{Q}$-rational points of the variety given by the system (5).

More generally, the question whether a variety X defined over a number field k has a k-rational point is one of the fundamental question of arithmetic geometry. Let S be the set of places of k, including the archimedean ones. An obvious necessary condition for $X(k)$ to be non-empty is that X has for every $v \in S$ a point over the completion k_v.

Conversely let $\mathcal{C}$ be a class of varieties of k. Then the class $\mathcal{C}$ is said to satisfy the *Hasse principle*, if for every X in $\mathcal{C}$ the following converse holds:

$$X \text{ has a } k_v\text{-rational point for every } v \in S \Rightarrow X \text{ has a } k\text{-rational point.}$$

This local-global principle is of importance, since it can be decided in a finite number of steps, if for a given variety X the set of k_v-rational points is empty. Hasse [Has24] showed that the class of quadratic forms over a number field satisfies this principle; however he was aware that there are classes of varieties which fail to satisfy the Hasse principle. A very simple counterexample is due the Selmer, who proved that the equation $3X_1^3 + 4X_2^3 + 5X_3^3 = 0$ has solutions over $\mathbb{R}$ and over each completion $\mathbb{Q}_p$ of $\mathbb{Q}$, but it has no solution in the rational numbers.

Studying the known counterexamples to the Hasse-principle, Manin [Man71] found that these counterexamples can be explained by an obstruction which is given by the Brauer group of the variety. These obstructions, which will discuss next, are now known as *Brauer-Manin obstructions*; we mainly follow [Poo08]:

Let k be a number field, S the set of all places of k and $\mathbf{A}_k$ be its adéle ring, i.e.

$$\mathbf{A}_k = \left\{ (x_v)_v \in \prod_{v \in S, v \nmid \infty} k_v \mid x_v \in \mathcal{O}_v \text{ for almost all } v \right\} \times \prod_{v \in S, v \mid \infty} k_v.$$

Given a k-variety X, the adelic space $X(\mathbf{A}_k)$ of X is defined by the subset of $\prod_v X(k_v)$ consisting of those $(x_v)_v$ such that $x_v \in X(\mathcal{O}_v)$ for almost all v. Every k-rational point $x \in X(k)$ corresponds to a morphism $x : \mathrm{Spec}(k) \to X$, which induces a morphism on Brauer groups $\mathrm{Br}(X) \xrightarrow{x} \mathrm{Br}(k)$; we call the image of $A \in \mathrm{Br}(X)$ under this morphism $A(x)$. In particular, for a fixed $A \in \mathrm{Br}(X)$ we get a morphism $ev_A : X(k) \to \mathrm{Br}(k)$ by sending $x \in X(k)$ to $\mathrm{Br}(X) \xrightarrow{x} \mathrm{Br}(k)$, followed by evaluation of this morphism at A; analogously we have a morphism $ev_A : X(\mathbf{A}_k) \to \mathrm{Br}(\mathbf{A}_k)$. These morphisms fit into the commutative diagram

$$\begin{array}{ccc} X(k) & \hookrightarrow & X(\mathbf{A}_k) \\ \downarrow{\scriptstyle ev_A} & & \downarrow{\scriptstyle ev_A} \\ \mathrm{Br}(k) & \longrightarrow & \mathrm{Br}(\mathbf{A}_k). \end{array} \tag{6}$$

Let $X(\mathbf{A}_k)^A$ be the set of elements in $X(\mathbf{A}_k)$ whose images under ev_A in $\mathrm{Br}(\mathbf{A}_k)$ lie in the image of the map $\mathrm{Br}(k) \to \mathrm{Br}(\mathbf{A}_k)$. The Brauer set of X is defined as the intersection

$$X(\mathbf{A}_k)^{\mathrm{Br(X)}} := \bigcap_{A \in \mathrm{Br(X)}} X(\mathbf{A}_k)^A.$$

Note that because of the commutative diagram (6), every $X(\mathbf{A}_k)^A$ contains the set of k-points $X(k)$; in particular, we have an inclusion $X(k) \subseteq X(\mathbf{A}_k)^{\mathrm{Br}(X)}$.

2.2.1 Definition
Let k be a number field, let $\mathbf{A}_k$ be its adéle ring and let X be a k-variety. Then there is a *Brauer-Manin obstruction* to the Hasse principle, if $X(\mathbf{A}_k) \neq \emptyset$ but $X(\mathbf{A}_k)^{\mathrm{Br}(X)} = \emptyset$.

Because of the inclusion $X(k) \subseteq X(\mathbf{A}_k)^{\mathrm{Br}(X)}$, the existence of a Brauer-Manin obstruction implies that $X(k) = \emptyset$. Of course this obstruction is only useful, if one can describe the image of the map $\mathrm{Br}(k) \to \mathrm{Br}(\mathbf{A}_k)$, and the obstruction is non-trivial, only if $\mathrm{Br}(k) \to \mathrm{Br}(\mathbf{A}_k)$ is not surjective. We will briefly discuss these issues:

We first recall some facts about the Brauer groups $\mathrm{Br}(k)$ and $\mathrm{Br}(k_v)$ for $v \in S$. The local Brauer groups $\mathrm{Br}(k_v)$ can be described using class field theory [CF67, chapter VI]; more precisely we have the following isomorphisms

$$\mathrm{inv}_v : \mathrm{Br}(k_v) \xrightarrow{\cong} \begin{cases} \mathbb{Q}/\mathbb{Z}, & \text{if } v \nmid \infty \\ \left(\frac{1}{2}\mathbb{Z}\right)/\mathbb{Z} & \text{if } k_v = \mathbb{R} \\ 0 & \text{if } k_v = \mathbb{C}, \end{cases}$$

which are called local invariants. Setting $\mathrm{inv} = \sum_v \mathrm{inv}_v$ we obtain an exact sequence

$$0 \to \mathrm{Br}(k) \to \oplus_v \mathrm{Br}(k_v) \xrightarrow{\mathrm{inv}} \mathbb{Q}/\mathbb{Z} \to 0.$$

Now we fix a number field k and an element $A \in \mathrm{Br}(X)$.

2.2.2 Proposition
Let (x_v) be in $X(\mathbf{A}_k)$. Then $A(x_v) = 0$ for all but finitely many $v \in S$.

Proof. [Poo08, Proposition 8.2.1] There is a finite set of places S' such that X can be spread out to a finite-type $\mathcal{O}_{k,S'}$-scheme X' and A can be spread out to an element $A' \in \mathrm{Br}(X')$. Moreover we can assume that $x_v \in X'(\mathcal{O}_v)$ for each $v \in S'$. Then $A(x_v)$ comes from an element $A'(x_v) \in \mathrm{Br}(\mathcal{O}_v)$; but this group vanishes since there is an isomorphism $\mathrm{Br}(\mathcal{O}_v) \cong \mathrm{Br}(\mathcal{O}_v/\mathfrak{m}_{\mathcal{O}_v})$, and the latter group vanishes as the Brauer group of a finite field [Ser79, p. 162 Example a)]. □

It follows that A induces a map $X(\mathbf{A}_k) \to \mathbb{Q}/\mathbb{Z}$, $(x_v) \mapsto \mathrm{inv}(A(x_v))$, which fits into the following commutative diagram whose bottom row is the short exact sequence coming from class field theory

$$\begin{array}{ccccccccc} & & X(k) & \hookrightarrow & X(\mathbf{A}_k) & & & & \\ & & \downarrow ev_A & & \downarrow ev_A & & & & \\ 0 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & \oplus_v \mathrm{Br}(k_v) & \xrightarrow{\mathrm{inv}} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

In particular, for $x \in X(k) \subseteq X(\mathbf{A}_k)$, we have $\mathrm{inv}(A(x)) = 0$, which implies that

$$X(\mathbf{A}_k)^A = \{(x_v) \in X(\mathbf{A}_k) \mid \mathrm{inv}(A(x)) = 0\}\,.$$

In some cases these obstructions can be computed explicitly. To this end, we have to specify the algebraic and the transcendental part of the Brauer group. Let $\overline{X} = X \times_k \overline{k}$ and let $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^{G_k}$ be the natural map induced by base change.

2.2.3 Definition
Let X be a smooth quasi-projective variety over a field k. Then the kernel (resp. image) of the map $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^{G_k}$ is the algebraic Brauer group $\mathrm{Br}(X)_{\mathrm{alg}}$ (resp. the transcendental Brauer group $\mathrm{Br}(X)_{\mathrm{tr}}$).

Accordingly, we can define the algebraic and transcendental part of the Brauer set

$$X(\mathbf{A}_k)^{\mathrm{Br}(X)_{\mathrm{alg}}} := \left\{(x_v) \in X(\mathbf{A}_k) \mid \mathrm{inv}(A(X)) = 0 \text{ for all } A \in \mathrm{Br}(X)_{\mathrm{alg}}\right\},$$
$$X(\mathbf{A}_k)^{\mathrm{Br}(X)_{\mathrm{tr}}} := \{(x_v) \in X(\mathbf{A}_k) \mid \mathrm{inv}(A(X)) = 0 \text{ for all } A \in \mathrm{Br}(X)_{\mathrm{tr}}\}$$

and we refer to Brauer-Manin obstructions from one of these sets as algebraic and transcendental Brauer-Manin obstructions respectively.

Since elements in $\mathrm{Br}(X)_{\mathrm{tr}}$ are in general more difficult to exhibit than elements in $\mathrm{Br}(X)_{\mathrm{alg}}$, most examples of explicit Brauer-Manin obstructions are given by algebraic elements. Kresch and Tschinkel [KT08] have shown that there is a general procedure for computing the Brauer-Manin obstruction for a smooth projective geometrically irreducible variety X, assuming that $\mathrm{Br}(\overline{X})$ is trivial, i.e. that there are only algebraic Brauer-Manin obstructions, and under the assumption that $\mathrm{Pic}(\overline{X})$ is finitely generated, torsion free and known explicitly by means of cycle representatives with an explicit Galois action; these assumptions hold, for example, for del Pezzo surfaces. For explicit examples of transcendental Brauer-Manin obstructions, using conic bundles over $\mathbb{P}^2_{\mathbb{Q}}$, see Harari [Har96].

When Manin defined the Brauer-Manin obstructions, all counterexamples to the Hasse principle could be explained using these obstructions. So one could ask if the Brauer-Manin obstruction is the only obstruction, i.e. if the following implication

$$X(\mathbf{A}_k)^{\mathrm{Br}(X)} \neq \emptyset \Rightarrow X(k) \neq \emptyset$$

holds. Unfortunately, the answer to this question is no. Skorobogatov [Sko99] gave examples of smooth proper surfaces S over $\mathbb{Q}$, which do not satisfy the Hasse principle but for which the Manin obstruction can not explain this failure; in particular he shows that $S(\mathbb{Q}) = \emptyset$, but $S(\mathbf{A}_k)^{\mathrm{Br}(S)} \neq \emptyset$. Such surfaces are constructed as a quotient of two curves of genus 1 by a certain involution. In his paper he could construct a refinement of the Brauer-Manin obstruction using descent theory, which is the only obstruction to the Hasse principle for this family of surfaces. However, there is another counterexample to the Hasse principle due to Sarnak and Wang [SW95], who showed that the Brauer-Manin obstruction is not the only obstruction to the Hasse principle for smooth hypersurfaces of degree 1130 in $\mathbb{P}^4_{\mathbb{Q}}$, assuming that $X(\mathbb{Q})$ is finite if $X_{\mathbb{C}}$ is hyperbolic. This example can not be explained by the refined Brauer-Manin obstruction of Skorobogatov.

On the other hand, it is known that for certain classes of varieties the Brauer-Manin obstruction is in fact the only obstruction. The following examples involve the Tate-Shafarevich group of an abelian variety A over a number field k, i.e. the abelian group

$$Ш(A) = Ш(A,k) := \bigcap_v \ker\left(\mathrm{H}^1(k,A) \to \mathrm{H}^1(k_v,A)\right).$$

2.2.4 Proposition ([Sko01, Corollary 6.2.5])
Let C be a smooth proper curve defined over k with Jacobian J and suppose that the Tate-Shafarevich group $Ш(J)$ is finite. If C has no k-rational divisor class of degree 1, then the Brauer-Manin obstruction is the only obstruction to the Hasse principle.

Specifying the genus of C we have:

2.2.5 Proposition ([Sko01, p. 114])
Let C be a smooth proper curve of genus 1 over k with Jacobian J and suppose that $Ш(J)$ is finite. Then the Brauer-Manin obstruction is the only obstruction to the Hasse principle.

The Brauer group of a variety Brauer also plays an important role in another part of algebraic and arithmetic geometry, because it provides an obstruction for the Tate conjecture to hold. We will sketch how Tate was led to state his conjecture and we will explain the relation between the Tate conjecture and Brauer groups. We will follow the expositions in [Tat65], [Tat66b], [Tat94] and [Gor79].

Let A be an abelian variety over a number field k, and let S be a finite set of primes of k containing those primes at which A has bad reduction as well as the archimedean primes. Let $d = \dim(A)$. Then the reduction A_v of A at $v \notin S$ is an abelian variety of the same dimension d over the residue field $k(v)$. For every $v \notin S$ there exists a polynomial $P_v(A,T) = \prod_{i=1}^{2d}(1-\alpha_{i,v}T)$ with integral coefficients such that the $\alpha_{i,v}$ have absolute value $|k(v)|^{\frac{1}{2}}$; we denote the value $|k(v)|$ by N_v. Consider the Euler product

$$L_S(A,s) = \prod_{v \notin S} \frac{1}{P_v(A, N_v^{-s})}.$$

It is known that $L_S(A,s)$ is convergent for $\mathrm{Re}(s) > \frac{3}{2}$. It is not known if this Euler product can be continued to an analytic function outside the half-plane $\mathrm{Re}(s) > \frac{3}{2}$ but it is conjectured that it has an analytic continuation to the whole complex plane; this has been proven, for example, for elliptic curves over $\mathbb{Q}$, as a consequence of the Modularity Theorem [BCDT01]. Based on a large set of empirical data for elliptic curves (cf. [BSD63] and [BSD65]) Birch and Swinnerton-Dyer stated a conjecture relating the multiplicity of the zero of $L_S(A,s)$ at $s = 1$ to the rank of the finitely generated abelian group of k-points of the elliptic curve. We state this conjecture in its more general form for abelian varieties as in [Tat66b, p. 416]:

2.2.6 Conjecture
Let A be an abelian variety over a number field k. Then the function $L_S(A,s)$ has a zero of order $r = \mathrm{rk}(A(k))$ at $s = 1$.

Conjecture 2.2.6 implies in particular the existence of a constant $c \in \mathbb{C}$ such that $L_S(s) \sim c(s-1)^r$ as $s \to 1$; however, this c depends on the choice of S, whereas the rank of $A(k)$ does obviously not. So the next natural step is a more profound investigation of the constant c.

For each prime v let μ_v be a Haar measure on k_v such that for almost all v the ring of integers $\mathcal{O}_v$ has measure 1 and let $|\cdot|_v$ be the normed absolute valuation on k_v given by v. The product $\prod_v \mu_v$ gives a measure $|\mu|$ on the compact quotient of the adéle ring $\mathbf{A}_k$ of k by the subfield k. Further, let ω be a non-zero invariant exterior differential form of degree d on A over k; then μ_v and ω define a Haar measure $|\omega|_v \mu_v^d$ on $A(k_v)$. Finally, we call a non-archimedean prime v good, if $\mu_v(\mathcal{O}_V) = 1$ and ω is regular with respect to v and has non-zero reduction at v.

For a finite set S containing all not good primes and all primes where A has degenerate reduction, Birch and Swinnerton-Dyer defined the L-series by the formula

$$L_S^*(A,s) = \frac{|\mu|^d}{\prod_{v \in S} \left(\int_{A(k_v)} |\omega|_v \mu_v^d \right) \prod_{v \notin S} \left(P_v(N_v^{-s}) \right)}.$$

Using a method due to Tamagawa, Birch and Swinnerton-Dyer showed that the asymptotic behaviour for $s \to 1$ is independent of the choice of the set S. This yields a refinement of Conjecture 2.2.6 as follows. Let $A^\vee$ be the dual of the abelian variety A; since A and $A^\vee$ are isogenous over k, the groups $A(k)$ and $A^\vee(k)$ have the same rank. In particular there are bases $a_1, \ldots, a_r$ resp. $a_1^\vee, \ldots, a_r^\vee$ of $A(k)_{\mathrm{free}}$ resp. $A^\vee(k)_{\mathrm{free}}$ of the same length. We write $\langle , \rangle$ for the canonical height pairing on abelian varieties, and $Ш(A)$ for the Tate-Shafarevich group of A. Then the refined conjecture of Birch and Swinnerton-Dyer ([Tat66b, p. 419]) takes the following form:

2.2.7 Conjecture
Let A and S be as above. Then

$$L_S^*(A,s) \sim \frac{|Ш(A)||\det(\langle a_i^\vee, a_j \rangle)|}{|A(k)_{\mathrm{tors}}||A^\vee(k)_{\mathrm{tors}}|}(s-1)^r \quad \text{as } s \to 1.$$

In particular, Conjecture 2.2.7 also assumes that the Tate-Shafarevich group $Ш(A)$ is finite. Whereas it is known that $Ш(A)$ is a torsion group whose p-primary part is of finite corank for each prime p, the finiteness of $Ш(A)$ has been shown only in a few cases such as for some elliptic curves over number fields by Rubin [Rub87] or for a certain class of modular abelian varieties over $\mathbb{Q}$ with real multiplication by Kolyvagin and Logachev [KL89].

Artin realised that there is a relation between the Tate-Shafarevich and the Brauer group in the sense that the finiteness of Ш is in some cases equivalent to the finiteness of the Brauer group. In the following we will sketch this relation and describe how Artin and Tate were led to conjecture a geometric analogue of Conjecture 2.2.7.

Let C be a irreducible smooth curve over a perfect field k. We denote by $k(C)$ the field of rational functions on C and by C° the set of closed points on C; for $y \in C^\circ$ let $k(C)_y$ be the completion of $k(C)$ with respect to the valuation given by y. As a generalisation of the Tate-Shafarevich group defined above, we define for an abelian variety A over $k(C)$

$$Ш(C,A) := \bigcap_{y \in C^\circ} \left(\mathrm{H}^1(k(C),A) \to \mathrm{H}^1(k(C)_y, A) \right).$$

The link between this type of Tate-Shafarevich group and Brauer groups is provided by the following proposition due to Artin:

2.2.8 Proposition ([Gro68b, no. 4])
Let C be as above, let X be a regular surface and let $f : X \to C$ be a proper morphism with fiber dimension 1. *Suppose the generic fiber X_η is smooth and that the geometric fibres are connected. If* $\mathrm{Jac}(X_\eta)$ *is the Jacobian of the generic fibre, and if f admits a section, then there is an exact sequence*

$$0 \to \mathrm{Br}(C) \to \mathrm{Br}(X) \to Ш(C, \mathrm{Jac}(X_\eta)) \to 0.$$

Moreover if C is a complete curve, then $\mathrm{Br}(C) = 0$, *and therefore* $\mathrm{Br}(X) \cong Ш(C, \mathrm{Jac}(X_\eta))$.

Moreover if C is an irreducible algebraic curve over a finite field $\mathbb{F}_q$ and A is an abelian variety over $\mathbb{F}_q(C)$, the group $Ш(A)$ of Conjecture 2.2.7 is equal to $Ш(C,A)$

and following the well-known analogy between number fields and function fields in one variable, it makes sense to state a variant of Conjecture 2.2.7 for A over $\mathbb{F}_q(C)$.

The next step involves zeta functions, we briefly recall the notations: Let X be a scheme of finite type over $\mathbb{Z}$. For $x \in X$ let N_x be the number of elements in the residue field $k(x)$, thus for a closed point $x \in X^\circ$, the number N_x is finite. Then the zeta function of X is defined to be

$$\zeta(X,s) := \prod_{x \in X^\circ} \frac{1}{1 - N_x^{-s}}.$$

The zeta function converges absolutely if the real part of s is greater than the dimension of X and it has an analytic continuation in the complex half-plane where the real part of s is greater than $\dim(X) - \frac{1}{2}$.

Let now X be a smooth projective variety over a finite field $\mathbb{F}_q$ with q elements. The Weil- or Hasse-Weil-zeta function is defined as

$$Z(X,T) := \exp\left(\sum_{m=1}^{\infty} |X(\mathbb{F}_q)| \frac{T^m}{m}\right),$$

and satisfies $Z(X, q^{-s}) = \zeta(X,s)$. In [Wei49] Weil stated his famous conjectures about properties of this function, which led to groundbreaking ideas in algebraic geometry.

2.2.9 Theorem

Let X be a smooth, projective variety of dimension d over a finite field $\mathbb{F}_q$. Then

a) $Z(X,T)$ is a rational function.

b) Let χ be the Euler characteristic of X. Then $Z(X,T)$ satisfies the functional equation

$$Z(X,T) = \pm q^{-\frac{d\chi}{2}} T^{-\chi} Z(X, \frac{1}{q^d T}).$$

c) There are polynomials $P_i(X,T) \in \mathbb{Z}[T]$ for $0 \le i \le 2d$, such that

$$Z(X,T) = \frac{P_1(X,T)P_3(X,T)\cdots P_{2d-1}(X,T)}{P_0(X,T)P_2(X,T)\cdots P_{2d}(X,T)}$$

and $P_i(X,T) = \prod_j^{bi}(1 - \alpha_{i,j}T) \in \mathbb{C}[T]$ with $|\alpha_{i,j}| = q^{\frac{i}{2}}$.

d) If X is the reduction of a smooth, projective variety $\widetilde{X}$ over a number field, then the degree of $P_i(X,T)$ is equal to the i-th Betti number of $\widetilde{X}(\mathbb{C})$.

Proof. Part a) was proven by Dwork [Dwo60] using p-adic methods and independently by Grothendieck [Gro65], who also showed that the $P_i(X,T)$ are the characteristic polynomials of the Frobenius morphism on $\mathrm{H}^i_{ét}(\overline{X}, \mathbb{Q}_\ell)$ and that part b) holds. Finally, c) and d) have been proven by Deligne [Del74]. □

Artin and Tate studied the case where C is an irreducible algebraic curve over a finite field $\mathbb{F}_q$ and $f : X \to C$ is a smooth proper morphism from a surface satisfying the conditions of Proposition 2.2.8. In this case $\text{Ш}(C, \text{Jac}(X_\eta)) \cong \text{Br}(X)$, and using this isomorphism, the function field analogon of Conjecture 2.2.7, and some sophisticated calculations regarding the zeta functions

$$Z(X, q^{-s}) = \frac{P_1(X, q^{-s})P_3(X, q^{-s})}{P_0(X, q^{-s})P_2(X, q^{-s})P_4(X, q^{-s})} \quad \text{and} \quad Z(C, q^{-s}) = \frac{P_1(C, q^{-s})}{P_0(C, q^{-s})P_2(C, q^{-s})}$$

of X and C respectively, Artin and Tate came to the following conjecture about the asymptotic behaviour of the term $P_2(X, q^{-s})$ at $s = 1$.

2.2.10 Conjecture ([Tat66b, p. 426])
Let X be a smooth projective surface over a finite field $\mathbb{F}_q$. Then $\text{Br}(X)$ is finite and we have

$$P_2(X, q^{-s}) \sim \frac{|\operatorname{Br}(X)||\det(D_i \cdot D_j)|}{q^{\alpha(X)} \cdot |\operatorname{NS}(X)_{\text{tors}}|^2}(1 - q^{1-s})^{\rho(X)} \text{ as } s \to 1,$$

where the quantities on the right are defined as follows: $\rho(X)$ is the Picard number of X, i.e. the rank of $\text{NS}(X)_{\text{free}}$, $\{D_1, \ldots, D_{\rho(X)}\}$ is a base for $\text{NS}(X)_{\text{free}}$ and $D_i \cdot D_j$ is the intersection multiplicity of D_i and D_j. Finally, if $\text{Pic}^\circ(X)$ is the Picard variety of X, then $\alpha(X) = \chi(X, \mathcal{O}_X) - 1 + \dim(\text{Pic}^0(X))$.

In addition Artin and Tate supposed that the link between the behaviour of $P_2(X, q^{-s})$ at $s = 1$ and Conjecture 2.2.7 via its function field analogon should hold in greater generality. More precisely, they conjectured the following ([Tat66b, p. 427]): Let $f : X \to C$ be a smooth proper morphism from the surface X onto the curve C such that the geometric fibres are connected and the generic fibre X_η is smooth. Then Conjecture 2.2.7 holds for $\text{Jac}(X_\eta)$ if and only if Conjecture 2.2.10 holds for X. This has been proven by Liu, Lorenzini and Raynaud [LLR05].

Let now k be a field which is finitely generated over its prime field. If V is an irreducible, smooth and projective k-scheme, then there is a smooth, projective morphism $f : X \to Y$ from an irreducible k-scheme X to a regular k-scheme Y such that its general fiber is the morphism $V \to \text{Spec}(k)$. For each closed point $y \in Y$ we denote by X_y the fibre $f^{-1}(y)$, which is a scheme over the finite residue field $k(y)$ of y. As above let N_y be the number of elements of $k(y)$. The zeta function of X can be expressed as the product of the zeta functions of the closed fibres, i.e. we have $\zeta(X, s) = \prod_{y \in Y^\circ} \zeta(X_y, s)$. Furthermore the zeta functions of the fibres can be expressed as rational functions according to Theorem 2.2.9. Thus if we write $\Phi_i(s) := \prod_{y \in Y^\circ} \frac{1}{P_i(X_y, N_y^{-s})}$ for $0 \le i \le 2d$,

$$\zeta(X, s) = \frac{\Phi_0(s)\Phi_2(s) \cdots \Phi_{2d}(s)}{\Phi_1(s)\Phi_3(s) \cdots \Phi_{2d-1}(s)}.$$

If k is any field and V is a k-scheme as above, let $Z^i(V)$ be the free abelian group generated by the irreducible subschemes of codimension i on V. If $\ell \neq \mathrm{char}(k)$ is a prime, there are cycle maps $Z^i(V) \to \mathrm{H}^{2i}_{\acute{e}t}(\overline{V}, \mathbb{Z}_\ell(i))$, which are defined using the fundamental class and cohomology with support. We write $\mathrm{NS}^i(V)$ for the image of this cycle map in codimension i. Motivated by a modification of Conjecture 2.2.6 of Birch and Swinnerton-Dyer, Tate was led to conjecture the following statement relating the rank of $\mathrm{NS}^i(V)$ and the order of the pole of $\Phi_{2i}(s)$ at $s = \dim(Y) + i$.

2.2.11 Conjecture ([Tat65, p. 104])
Let V and $f : X \to Y$ be as above. Then the rank of $\mathrm{NS}^i(V)$ is equal to the order of the pole of $\Phi_{2i}(s)$ at the point $s = \dim(Y) + i$.

The image of the cycle map $Z^i(V) \to \mathrm{H}^{2i}_{\acute{e}t}(\overline{V}, \mathbb{Z}_\ell(i))$ lies in the subgroup of G_k-invariants, where G_k denotes the absolute Galois group of k. If $Z^i(V)_{\mathrm{rat}} \subseteq Z^i(V)$ is the subgroup of cycles which are rationally equivalent to 0, the restriction of the cycle map to $Z^i(V)_{\mathrm{rat}}$ is trivial. Thus there is an induced map on $\mathrm{CH}^i(V) = Z^i(V)/Z^i(V)_{\mathrm{rat}}$, i.e. on the Chow group of codimension i cycles. In particular, there are cycle maps

$$c^i_{\mathbb{Q}_\ell} : \mathrm{CH}^i(V) \otimes \mathbb{Q}_\ell \to \mathrm{H}^{2i}_{\acute{e}t}(\overline{V}, \mathbb{Q}_\ell(i))^{G_k}.$$

Let now k be a finite field. From Theorem 2.2.9 we know that the zeta function $\zeta(V, s)$ has a pole at $s = i$ if and only if $P_{2i}(V, T) = \prod_j \left(1 - \alpha_{2i,j} T\right)$ has a zero at $T = q^{-i}$, and that the order of this pole is equal to the multiplicity of the zero at q^{-i}. On the other hand, we know that P_{2i} is the characteristic polynomial of the Frobenius homomorphism acting on $\mathrm{H}^{2i}_{\acute{e}t}(\overline{V}, \mathbb{Q}_\ell)$, so the multiplicity of the zero of P_{2i} at q^{-i} equals the dimension of the eigenspace corresponding to q^i of the Frobenius homomorphism on $\mathrm{H}^{2i}_{\acute{e}t}(\overline{V}, \mathbb{Q}_\ell)$ (at least if we assume that G_k acts semisimply on $\mathrm{H}^{2i}_{\acute{e}t}(\overline{V}, \mathbb{Q}_\ell)$). Finally, the latter one is – after an appropriate twist – equal to the dimension of $\mathrm{H}^{2i}_{\acute{e}t}(\overline{V}, \mathbb{Q}_\ell(i))^{G_k}$. In particular, if we assume that Conjecture 2.2.11 holds for V in codimension i and G_k acts semisimply on $\mathrm{H}^{2i}_{\acute{e}t}(\overline{V}, \mathbb{Q}_\ell(i))$, then the cycle map $c^i_{\mathbb{Q}_\ell}$ is surjective. Motivated by such arguments, Tate stated the following conjectures:

2.2.12 Conjecture (Semisimplicity of the Galois action)
Let k be a field finitely generated over its prime field, let $\ell \neq \mathrm{char}(k)$ be a prime, and let X be a smooth projective variety over k. Then for each $i \geq 0$ the Galois group G_k acts semisimply on $H^{2i}_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell(i))$.

2.2.13 Conjecture (Tate conjecture)
Let k, ℓ and X be as in 2.2.12. Then for all $i \geq 0$ the cycle map with $\mathbb{Q}_\ell$-coefficients

$$c^i_{\mathbb{Q}_\ell} : \mathrm{CH}^i(X) \otimes \mathbb{Q}_\ell \to H^{2i}_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell(i))^{G_k}$$

is surjective.

In what follows, we will write $\mathrm{TC}^i(X)_{\mathbb{Q}_\ell}$ for the above assertion and refer to it as the Tate conjecture for X in codimension i at the prime ℓ.

2.2.14 Conjecture (Strong Tate conjecture)
Let k, ℓ and X be as in 2.2.12. Then the order of the pole of $\zeta(X,s)$ at $s = i$ is equal to the dimension of the subspace of $H^{2i}_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell(i))$ spanned by the image of the cycle map $c^i_{\mathbb{Q}_\ell}$.

There are various relations between the above conjectures. For example:

2.2.15 Theorem
Let k be a finite field, let X be smooth projective k-variety and let i be a non-negative integer.

a) The Conjectures 2.2.13 and 2.2.12 are independent of ℓ, i.e. if they hold for some prime $\ell \neq \mathrm{char}(k)$, then they hold for any prime different from the characteristic of k.

b) The strong Tate conjecture for X and i holds if and only if the Tate conjecture for X holds in codimension i and $\dim(X) - i$ and the Semisimplicity conjecture holds for i.

c) The Tate conjecture in codimension 1 implies the strong Tate conjecture at $i = 1$.

Proof. See [Tat65, Theorem 2.9]. □

In general, the Tate conjecture is an open problem, in particular in codimension $i \geq 2$. For $i = 1$, i.e. for divisors, it is known, for example, in the following cases:

- All K3-surfaces in characteristic zero [Tat65, Theorem 5.6].
- All K3-surfaces over a finitely generated field of odd characteristic [Per15].
- All abelian varieties, see below.

Tate [Tat66a, Theorem 4] showed that the Tate conjecture for abelian varieties in codimension 1 would follow from the following theorem about homomorphisms between abelian varieties and their Tate vector spaces.

2.2.16 Theorem
Let k be a field finitely generated over its prime field. Let A and B be two abelian varieties over k and let $V_\ell(A) = \mathrm{T}_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ (and analogously for B). Then the natural map

$$\mathrm{Hom}(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \mathrm{Hom}_{G_k}(V_\ell(A), V_\ell(B))$$

is bijective.

Proof. For finite fields this was shown by Tate [Tat66a]; the case of function fields of positive characteristic is due to Zarhin [Zar75]. Faltings [Fal83] proved the assertion for abelian varieties over number fields as part of his proof of the Mordell conjecture. His method can be generalized to arbitrary finitely generated fields. □

One of the very few results establishing Tate's conjecture in higher codimension is due to Soulé [Sou84, Théorème 4]; he shows that for a certain class of varieties (whose motive is closely related to the motive of a product of curves), including abelian varieties and products of smooth projective curves over a finite field, the Tate conjecture for X holds in codimensions 0, 1, $d-1$ and d. Moreover, in these cases the strong Tate conjecture also holds.

Finally, that Brauer groups provide an obstruction to Tate's conjecture, at least in codimension 1 and for surfaces over finite fields, has been shown by Tate, using Kummer theory. More precisely, he proved:

2.2.17 Theorem (Tate, [Tat66b, Theorem 5.2])
Let X be a smooth projective surface over a finite field k and let $\ell \neq \mathrm{char}(k)$ be a prime. Then

$$\mathrm{TC}^1(X)_{\mathbb{Q}_\ell} \Leftrightarrow \mathrm{Br}(X)\{\ell\} < \infty.$$

In fact one can show that $\mathrm{TC}^1(X)_{\mathbb{Q}_\ell}$ holds if and only if there is a prime ℓ' (including $\ell' = \mathrm{char}(k)$) such that $\mathrm{Br}(X)\{\ell'\}$ is finite; moreover $\mathrm{Br}(X)\{\ell'\}$ is finite if and only if $\mathrm{Br}(X)$ is finite, see [Mil75] (for $\mathrm{char}(k) \neq 2$) and [Ulm14] (including $\mathrm{char}(k) = 2$).

We will see in Theorem 3.3.6 that this obstruction can be generalised to higher codimensions for arbitrary smooth, projective and geometrically integral varieties using higher Brauer groups.

We remark that because of the cycle map $\mathrm{CH}^i(X) \otimes \mathbb{Z}_\ell \to \mathrm{H}^{2i}_{\acute{e}t}(\overline{X}, \mathbb{Z}_\ell(i))^{G_k}$ it makes sense to state an 'integral Tate conjecture', claiming that these integral cycle maps are surjective. This is known to be false, counterexamples were given, for example, by Schoen [Sch98] in the number field case, and by Colliot-Thélène and Szamuely [CTS09] in the case of a finite field.

Similar to the Tate conjecture, asserting the surjectivity of a cycle map to étale cohomology, there is the, in fact older, Hodge conjecture, which describes for a smooth projective complex variety the image of the cycle map to singular or Betti cohomology:

2.2.18 Conjecture (Hodge, [Hod52])
Let X be a smooth projective variety over $\mathbb{C}$. Then for all $i \geq 0$ the image of the cycle map

$$c^i_{\mathbb{Q}} : \mathrm{CH}^i(X) \otimes \mathbb{Q} \to H^{2i}_{\mathrm{B}}(X, \mathbb{Q}(i))$$

is the group of Hodge cycles $\mathrm{Hdg}^{2i}(X, \mathbb{Q}) = H^{2i}_{\mathrm{B}}(X, \mathbb{Q}(i)) \cap \mathrm{F}^i H^{2i}_{\mathrm{B}}(X, \mathbb{C})$.

In the following we will write $\mathrm{HC}^i(X)_{\mathbb{Q}}$ for the above assertion. Note that $\mathrm{HC}^1(X)_{\mathbb{Q}}$ follows easily using the exponential sequence and the Lefschetz-(1,1)-theorem; by the hard Lefschetz theorem this also implies $\mathrm{HC}^{d-1}(X)_{\mathbb{Q}}$, where $d = \dim X$.

Both the Tate and the Hodge conjecture predict the existence of sufficiently many subvarieties to obtain the desired image of the cycle map in question. Although Tate could not describe a direct logical connection between the two conjectures, he noted that they 'have an air of compatibility'. In the mean time various implications between these conjectures have been proven, for example, that the Tate conjecture for abelian varieties in characteristic zero implies the Hodge conjecture for abelian varieties (Pyatetskii-Shapiro, [PS71]), or that the Hodge conjecture for abelian varieties of CM-type implies the Tate conjecture for abelian varieties over finite fields (Milne, [Mil99]).

Similar to the Tate conjecture, the Hodge conjecture is known only in very few cases. Again an integral version of the Hodge conjecture can be formulated, but this is also known to be false. The first counterexamples were given by Atiyah and Hirzebruch [AH62]; they constructed a torsion Hodge class, which is not in the image of the cycle map. Moreover, Kollar [Kol92] showed that there are non-torsion cohomology classes which are not in the image of the cycle map but some multiple of them is.

3. Higher Brauer groups

As we have explained in the previous chapter, there is a connection between the conjecture of Birch and Swinnerton-Dyer and the Tate conjecture. When considering this connection, Milne proved further that there is a link between the asymptotic behaviour of the zeta function of a scheme X at the poles $s = 1$ and $s = 2$ and the étale cohomology of the sheaves $\mathbb{Z}$ and $\mathbb{G}_m$ on X respectively. To extend this to poles at $s \geq 3$, Lichtenbaum suggested that there should be not just sheaves but (bounded) complexes $\Gamma(r)$ of étale sheaves for each integer $r > 0$ satisfying certain properties; for details see section 3.1. Although Lichtenbaum has constructed, using algebraic K-theory, a complex of étale sheaves $\Gamma(2)$, which satisfies in special cases many of the desired properties, there is no general construction of $\Gamma(r)$. However, the (unbounded) cycle complex defined by Bloch can be viewed as a complex of étale sheaves $\mathbb{Z}_X(r)_{ét}$ and is conjectured to be quasi-isomorphic to $\Gamma(r)$. We use the complexes $\mathbb{Z}_X(r)_{ét}$ to define higher Brauer groups in section 3.2 and recall some of the known properties of these groups in section 3.3; this generalises, for example, the relation between the Tate conjecture in codimension 1 and the Brauer group to the Tate conjecture in arbitrary codimension and the higher Brauer groups.

3.1. Lichtenbaum's complex

Let X be a smooth, projective variety of dimension d over a finite field $\mathbb{F}_q$. By Theorem 2.2.9 the zeta function of X has the form

$$\zeta(X,s) = \frac{P_1(X,q^{-s}) \cdots P_{2d-1}(X,q^{-s})}{P_0(X,q^{-s})P_2(X,q^{-s}) \cdots P_{2d}(X,q^{-s})},$$

where the $P_i(X,T)$ are polynomials with integral coefficients having reciprocal roots of absolute value $q^{\frac{i}{2}}$. In particular $\zeta(X,s)$ has poles at $s = 0,1,2,\ldots,d$ and the order of the pole at $s = i$ is equal to the multiplicity of q^i as a reciprocal root of $P_{2i}(X,T)$. Thus for each such i, there is an integer ρ_i and a constant c_i with the property that

$$\zeta(X,s) \sim c_i \cdot (1 - q^{i-s})^{-\rho_i} \quad \text{as} \quad s \to i.$$

The conjecture of Birch and Swinnerton-Dyer and Conjecture 2.2.10 of Artin and Tate concern the asymptotic behaviour of the zeta function $\zeta(X,s) = Z(X,q^{-s})$ as $s \to 1$; we first consider the question whether there is a similar description of the behaviour of the zeta function for arbitrary positive integers $s = 0, 1, 2, \ldots$. In this context, Milne [Mil86] made the observation that the behaviour of $\zeta(X,s)$ at $s = 0$ can be described in terms of cohomology of the sheaf $\mathbb{Z}$, while the behaviour at $s = 1$ can – under some additional assumptions – be explained in terms of cohomology of $\mathbb{G}_m$. To be precise, he showed the following:

3.1.1 Theorem ([Mil86, Theorem 0.4])
Let X be as above. Then the following assertions hold.

a)

$$\zeta(X,s) \sim \pm \frac{|H^2_{\acute{e}t}(X,\mathbb{Z})_{\text{cotor}}| \cdot |H^4_{\acute{e}t}(X,\mathbb{Z})| \cdots}{|H^1_{\acute{e}t}(X,\mathbb{Z})| \cdot |H^3_{\acute{e}t}(X,\mathbb{Z})| \cdots} (1-q^{-s})^{-1} \quad as \quad s \to 0$$

b) Assume the Tate conjecture in codimension 1 for X and that $\mathrm{Br}(X)$ is finite. Then

$$\zeta(X,s) \sim \pm \frac{D^1 \cdot |H^1_{\acute{e}t}(X,\mathbb{G}_m)_{\text{tor}}| \cdot |H^3_{\acute{e}t}(X,\mathbb{G}_m)_{\text{cotor}}| \cdots}{|H^0_{\acute{e}t}(X,\mathbb{G}_m)| \cdot |H^2_{\acute{e}t}(X,\mathbb{G}_m)| \cdots} q^{\chi(X,\mathbb{G}_m)} (1-q^{1-s})^{-\rho_1}$$
$$as\ s \to 1,$$

where D^1 is a regulator term (see [Mil86, section 7]), and ρ_1 is the Picard number of X.

We note that the terms on the right hand side of Theorem 3.1.1 a),b) are well-defined: The groups $\mathrm{H}^i_{\acute{e}t}(X,\mathbb{Z})$ and $\mathrm{H}^i_{\acute{e}t}(X,\mathbb{G}_m)$ vanish for large i. Moreover, $\mathrm{H}^i_{\acute{e}t}(X,\mathbb{Z})$ is torsion for $i \neq 0$ and finite for $i \neq 0,2$; for $i = 2$ there is an isomorphism $\mathrm{H}^2_{\acute{e}t}(X,\mathbb{Z}) \cong \mathrm{H}^2_{\acute{e}t}(X,\mathbb{Z})_{\text{cotor}} \oplus \mathbb{Q}/\mathbb{Z}$, where $\mathrm{H}^2_{\acute{e}t}(X,\mathbb{Z})_{\text{cotor}}$ is finite. Also, the groups $\mathrm{H}^i_{\acute{e}t}(X,\mathbb{G}_m)$ are torsion for $i \neq 1$ and finite for $i \neq 1,2,3$. Furthermore $\mathrm{H}^1_{\acute{e}t}(X,\mathbb{G}_m) \cong \mathrm{CH}^1(X)$, which is a finitely generated abelian group, and $\mathrm{H}^3_{\acute{e}t}(X,\mathbb{G}_m)$ is the direct sum of its finite cotorsion subgroup $\mathrm{H}^3_{\acute{e}t}(X,\mathbb{G}_m)_{\text{cotor}}$ and $\mathrm{H}^3_{\acute{e}t}(X,\mathbb{G}_m)_{\text{div}}$.

This raises the question whether there are sheaves generalising the above relation between the zeta function at $s = 0$ and $s = 1$ and the cohomology of the sheaves $\mathbb{Z}$ and $\mathbb{G}_m$ to arbitrary integers $s > 1$. Lichtenbaum [Lic84] conjectured that one could not expect the existence of single sheaves which play an analogous role at positive integers $s > 1$. Instead he suggested that for each integer $r \geq 0$ there should be a bounded complex $\Gamma(r)$ of étale sheaves of abelian groups with certain properties; in fact, the cohomology of these complexes should define motivic cohomology. He stated the following axioms for the $\Gamma(r)$, which are formulated in the derived category of the category of sheaves of abelian groups on the étale site of X, see A.1 for more details.

(L0) $\Gamma(0) = \mathbb{Z}$ and $\Gamma(1) = \mathbb{G}_m[-1]$.

(L1) For $r \geq 1$, the complex $\Gamma(r)$ is acyclic outside $[1, r]$.

(L2) If $\pi : X_{ét} \to X_{Zar}$ is the canonical morphism of sites, then the Zariski sheaf $R^{r+1}\pi_*\Gamma(r) = 0$ (Hilbert's 90th Theorem).

(L3) Let n be a positive integer prime to all residue field characteristics of X. Then there exists an exact triangle

$$\Gamma(r) \xrightarrow{\cdot n} \Gamma(r) \to \mu_n^{\otimes r} \xrightarrow{+1} \Gamma(r)[1].$$

(L4) There exist products $\Gamma(r) \otimes^L \Gamma(s) \to \Gamma(r+s)$ which induce maps on cohomology

$$\mathbb{H}^i_{ét}(X, \Gamma(r)) \otimes \mathbb{H}^j_{ét}(X, \Gamma(s)) \to \mathbb{H}^{i+j}_{ét}(X, \Gamma(r+s)).$$

(L5) The cohomology sheaves $\mathcal{H}^i(X, \Gamma(r))$ are isomorphic to the étale sheaves $\mathrm{gr}^r_\gamma \mathcal{K}_{2r-i}$. Here $\mathcal{K}_j$ is the étale sheaf associated to the presheaf $U \mapsto K_j\Gamma(U, \mathcal{O}_X)$, K_j is the Quillen algebraic K-group, and gr^r_γ is the graded quotient of the γ-filtration (see [Mil88, section 1] and [Sou85]).

(L6) If k is a field, the cohomology groups $\mathbb{H}^r_{ét}(k, \Gamma(r))$ are isomorphic to the Milnor K-groups $K^M_r(k)$ (see [Mil70]).

We remark why axiom **(L2)** is called Hilbert's 90th Theorem: If we specialize to the case when X is the spectrum of a field k and $r = 1$, then **(L2)** says that $\mathbb{H}^2_{ét}(X, \Gamma(1)) = 0$. But by **(L0)** $\mathbb{H}^2_{ét}(X, \Gamma(1))$ is isomorphic to $\mathrm{H}^1_{ét}(k, \mathbb{G}_m)$ and the vanishing of the latter group is Hilbert's 90th Theorem.

In [Lic84, §6] Lichtenbaum showed that the existence of such complexes $\Gamma(r)$ would yield a cohomology theory which has many of the properties expected for motivic cohomology, including, for example, duality theorems.

Milne [Mil88] proved under additional conditions that the expected relation between the behaviour of the zeta function of X at $s \geq 1$ and the cohomology theory defined by the complex $\Gamma(r)$ on X holds. More precisely, Milne proved the following: Let X be a smooth projective variety over a finite field k with q elements and let $p = \mathrm{char}(k)$. One of Milne's assumptions is the existence of natural cycle maps

$$\mathrm{CH}^r(X) \to \mathbb{H}^{2r}_{ét}(X, \Gamma(r)) \tag{7}$$

which are compatible with the étale cycle maps through the maps of the long exact cohomology sequence arising from the Kummer type of exact triangle **(L3)**. Moreover, these cycle maps should be compatible with the product structure given by **(L4)**. Let $\mathbb{H}^i_{ét}(X, (\Gamma/m\Gamma)(r))$:=$\mathrm{H}^i_{ét}(X, \mu_{m_0}^{\otimes r}) \times \mathbb{H}^i_{ét}(X, (\Gamma/p^n\Gamma)(r))$, where $m = m_0 p^n$ and

$\gcd(m_0, p) = 1$, and set $\mathbb{H}^i_{\acute{e}t}(X, \widehat{\Gamma}(r)) := \varprojlim \mathbb{H}^i_{\acute{e}t}(X, (\Gamma/m\Gamma)(r))$. These groups fit into the following short exact sequences (see [Mil88, p. 69 (3.4.3)])

$$0 \to \mathbb{H}^i_{\acute{e}t}(X, \Gamma(r))^\wedge \to \mathbb{H}^i_{\acute{e}t}(X, \widehat{\Gamma}(r)) \to \varprojlim_n \mathbb{H}^{i+1}_{\acute{e}t}(X, \Gamma(r))[n] \to 0. \tag{8}$$

The homomorphism $G_k \to \widehat{\mathbb{Z}}$, which sends the Frobenius element to 1, defines a canonical element in $\mathrm{H}^1(k, \widehat{\mathbb{Z}}) \subseteq \mathrm{H}^1_{\acute{e}t}(X, \widehat{\mathbb{Z}})$. Taking cup products, we obtain for each $r \in \mathbb{Z}$ a map $\epsilon^{2r} : \mathbb{H}^{2r}_{\acute{e}t}(X, \widehat{\Gamma}(r)) \to \mathbb{H}^{2r+1}_{\acute{e}t}(X, \widehat{\Gamma}(r))$. This map ϵ^{2r}, together with the exact sequence (8), yields the following commutative diagram (defining the map δ^r)

$$\begin{array}{ccc} \mathbb{H}^{2r}(X, \Gamma(r))^\wedge & \xrightarrow{\delta^r} & \varprojlim \mathbb{H}^{2r+2}_{\acute{e}t}(X, \Gamma(r))[n] \\ \downarrow & & \uparrow \\ \mathbb{H}^{2r}(X, \widehat{\Gamma}(r)) & \xrightarrow{\epsilon^{2r}} & \mathbb{H}^{2r+1}(X, \widehat{\Gamma}(r)) \end{array}$$

For a group G we denote by G_{nd} the quotient G/G_{div}. We say that the expression

$$\chi'(X, \Gamma(r)) := \frac{|\mathbb{H}^{2r}_{\acute{e}t}(X, \Gamma(r))_{\mathrm{tors}}|}{\det(\delta^r)} \cdot \prod_{i \neq 2r} |\mathbb{H}^i_{\acute{e}t}(X, \Gamma(r))_{\mathrm{nd}}|^{(-1)^i}$$

is defined, if all involved groups are finite and $\det(\delta^r)$ is defined and non-zero.

Moreover, we set $\chi(X, \mathcal{O}_X, r) := \sum_{i=0}^r (r-i)\chi(X, \Omega^i_X)$.

3.1.2 Theorem (Milne, [Mil88, Theorem 4.3])
*Let X be as above and let ρ_r be the rank of the subgroup generated by the image of the cycle map $c^r_{\mathbb{Q}_\ell}$. Assume that $\Gamma(r)$ is a complex satisfying **(L3)** and that there are cycle maps as in (7). Moreover, assume that $\mathbb{H}^{2r+1}_{\acute{e}t}(X, \Gamma(r))_{\mathrm{nd}}$ is torsion and that the strong Tate conjecture holds for r and all ℓ. Then $\chi'(X, \Gamma(r))$ is defined and*

$$\zeta(X, s) \sim \pm \chi'(X, \Gamma(r)) q^{\chi(X, \mathcal{O}_X, r)} (1 - q^{r-s})^{-\rho_r} \quad \text{as } s \to r.$$

Milne also noted that the existence of the complexes $\Gamma(r)$ yields a general obstruction to the Tate conjecture in codimension r, analogous to Theorem 2.2.17 in case $r = 1$.

3.1.3 Theorem ([Mil88, Remark 4.5(g)])
Let X be a smooth projective variety over a finite field $\mathbb{F}_q$ and $\ell \neq \mathrm{char}(\mathbb{F}_q)$ be a prime. Then

$$\mathrm{TC}^r(X)_{\mathbb{Q}_\ell} \Leftrightarrow \mathbb{H}^{2r+1}_{\acute{e}t}(X, \Gamma(r))_{\mathrm{div}} = 0.$$

Milne also showed that for an even dimensional variety X, assuming the existence of the complexes $\Gamma(r)$, finite generation results and the strong Tate conjecture, one can prove a generalisation of the Artin-Tate conjecture 2.2.10 about the behaviour of $P_d(X, q^{-s})$ as $s \to d/2$. Let $A^r(X)$ be the image of $\mathrm{CH}^r(X)$ in $\mathbb{H}^{2r}_{\acute{e}t}(\overline{X}, \widehat{\Gamma}(r))$, and let $\{D_i\}$ be a basis of $A^r(X)_{\mathrm{free}}$.

3.1.4 Theorem (Milne, [Mil88, Theorem 6.6])
Let X be as above of even dimension $d = 2r$. Assume that there is a complex $\Gamma(r)$ satisfying the following conditions:

*i) The axioms **(L3)** and **(L4)**.*

ii) The existence of cycle maps as in (7).

iii) There is a degree map $\mathbb{H}^{2d}_{\acute{e}t}(\overline{X}, \Gamma(d)) \to \mathbb{Z}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{H}^{2d}_{\acute{e}t}(\overline{X}, \Gamma(d)) & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{H}^{2d}_{\acute{e}t}(\overline{X}, (\Gamma/m\Gamma)(r)) & \xrightarrow{\cong} & \mathbb{Z}/m\mathbb{Z} \end{array}$$

iv) The groups $\mathbb{H}^{2r}_{\acute{e}t}(X, \Gamma(r))$ and $\mathbb{H}^{2d-2r}_{\acute{e}t}(X, \Gamma(d-r))$ are finitely generated.

v) The group $\mathbb{H}^{2r+1}_{\acute{e}t}(X, \Gamma(r))$ is torsion.

If the strong Tate conjecture holds for r and all ℓ and the cycle map $\mathrm{CH}^r(X) \to \mathbb{H}^{2r}_{\acute{e}t}(X, \Gamma(r))$ is surjective, then

$$P_{2r}(X, q^{-s}) \sim \pm \frac{|\mathbb{H}^{2r+1}_{\acute{e}t}(X, \Gamma(r))| \cdot \det(D_i \cdot D_j)}{q^{\alpha_r(X)} |A^r(X)_{\mathrm{tors}}|^2} \cdot (1 - q^{r-s})^{\rho_r} \quad \text{as} \quad s \to r,$$

where the value $\alpha_r(X)$ depends on the reciprocal roots of $P_{2r}(X, T)$ and a perfect affine group scheme $\mathcal{H}^i(X, \mathbb{Z}_p(r))$ (see [Mil88, p. 93–94] for an explicit definition of $\alpha_r(X)$).

3.2. Motivic and Lichtenbaum cohomology

Let X be a smooth quasi-projective variety over a field k. In this section we present the construction of the complexes $\mathbb{Z}_X(r)_{\acute{e}t}$ of étale sheaves on X (for $r \geq 0$) given by Bloch [Blo86]; the complexes $\mathbb{Z}_X(r)_{\acute{e}t}$ are conjectured to satisfy the axioms **(L0)** to **(L6)**.

Bloch's complex can be seen as an algebro-geometric analogue to the complex used to define simplicial cohomology in algebraic topology. We recall the explicit definitions.

Let k be a field. The (algebraic) n-simplex over k is defined as the affine k-scheme

$$\Delta^n = \operatorname{Spec}\left(k[t_0,\ldots,t_n]/\left(\sum_{i=0}^{n} t_i - 1\right)\right) \cong \mathbb{A}_k^n$$

Each non-decreasing map $\rho : [0,\ldots,m] \to [0,\ldots,n]$ induces a map of simplices

$$\tilde{\rho} : \Delta^m \to \Delta^n, \quad \tilde{\rho}^*(t_j) = \sum_{\rho(j)=i} t_j,$$

where the sum vanishes if $\phi^{-1}(i) = \emptyset$. If ρ is injective, we have $\tilde{\rho}(\Delta^m) \subseteq \Delta^n$; we call $\tilde{\rho}(\Delta^m)$ a face of Δ^m and $\tilde{\rho}$ a face map. If ρ is surjective, we call $\tilde{\rho}$ a degeneracy map.

Let now $n \geq 0$ be an integer and let $z^n(X \times_k \Delta^m)$ be the group of algebraic cycles on $X \times \Delta^m$ of codimension n, i.e. the free abelian group generated by the irreducible closed subvarieties of $X \times_k \Delta^m$ of codimension n. In the next step we restrict to those elements of $z^n(X \times_k \Delta^m)$ which behave well with respect to the face maps. Recall that two subvarieties Y, Z of a variety X intersect properly, if every irreducible component of $Y \cap Z$ has codimension $\operatorname{codim}(Y) + \operatorname{codim}(Z)$ in X. For $m, n \geq 0$ we denote by $z^n(X, m)$ the subgroup of $z^n(X \times_k \Delta^m)$ generated by the irreducible subvarieties intersecting all faces of $X \times_k \Delta^m$ properly. Let $\partial_i : \Delta^{m-1} \to \Delta^m$ be a face map, i.e. ∂_i is of the form

$$(t_0,\ldots,t_{m-1}) \mapsto (t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{m-1}).$$

Since $T \in z^n(X,m)$ meets all faces of $X \times_k \Delta^m$ properly, ∂_i induces a homomorphism

$$\partial_i^* : z^n(X,m) \to z^n(X,m-1),$$

which maps $T \in z^n(X,m)$ to $T \cap (X \times_k \Delta^{m-1})$ with the appropriate multiplicity.

Let $d^m = \sum_i (-1)^i \partial_i^* : z^n(X,m) \to z^n(X,m-1)$ be the alternating sum of the ∂_i^*. By a standard argument, it is easy to see that the composition $d^{m-1} \circ d^m$ is the zero mapping, hence the groups $z^n(X,m)$ form a complex of abelian groups.

3.2.1 Definition
Let X be a smooth, quasi-projective variety over a field k. For $n \geq 0$ Bloch's cycle complex $z^n(X,\bullet)$ is defined as the unbounded homological complex of abelian groups

$$\cdots \to z^n(X,m) \xrightarrow{d^m} z^n(X,m-1) \xrightarrow{d^{m-1}} \cdots \to z^n(X,0) \to 0.$$

We note that, by construction, the classical Chow groups $\mathrm{CH}^n(X)$ of X can be computed using the complex $z^n(X,\bullet)$, i.e. $\mathrm{CH}^n(X) \cong \mathrm{H}_0(z^n(x,\bullet))$. The higher Chow groups $\mathrm{CH}^n(X,i)$ are defined to be $\mathrm{H}_i(z^n(X,\bullet))$. The complex $z^n(X,\bullet)$ is covariant functorial for proper maps and contravariant functorial for flat maps [Blo86, Proposition 1.3], hence defines a complex of sheaves for the flat topology on X. In particular, the presheaves $z^n(-,m) : U \mapsto z^n(U,m)$ are sheaves for the (small) étale and the Zariski topology on X and $z^n(-,\bullet)$ is a complex of sheaves on the (small) étale and the Zariski site of X [Blo86, section 11].

3.2.2 Definition
Let X be a smooth, quasi-projective variety over a field, and let X_τ ($\tau = \mathrm{Zar}$ or $\tau = \acute{e}t$) be either the small étale or the Zariski site. Define $\mathbb{Z}_X(r)_\tau$ to be the cohomological complex with the sheaf $z^r(-,2r-i)$ in degree i. If A is an abelian group, set

$$A_X(r)_\tau := \mathbb{Z}_X(r)_\tau \otimes A.$$

We will often write $A(r)_\tau$ for the complex $A_X(r)_\tau$; if there is no risk of confusion, we will also write $A(r) = A_X(r)_{\mathrm{Zar}}$ for the complex of Zariski sheaves. The complexes $A_X(r)_\tau$ are by construction only bounded on the right. Note that the differentials of $A_X(r)_\tau$ have degree $+1$, hence unlike the complexes $z^r(-,\bullet)$ they are indeed cohomological complexes. Let $\pi : X_{\acute{e}t} \to X_{Zar}$ be the canonical map. Then

$$\pi_* A_X(r)_{\acute{e}t} = A_X(r)_{Zar}.$$

Bloch conjectures [Blo86, section 11] that the complexes $\mathbb{Z}_X(r)_{\acute{e}t}$ satisfy the axioms **(L0)** to **(L6)** from 3.1, i.e. they provide potential candidates for the complexes $\Gamma(r)$; however, even in weight 2, there is no obvious morphism between the two complexes $\mathbb{Z}_X(2)_{\acute{e}t}$ and $\Gamma(2)$. Bloch showed in [Blo86] that the complexes $\mathbb{Z}_X(r)_{\acute{e}t}$ satisfy the axioms **(L0)** and **(L4)**; moreover Geisser-Levine proved that the $\mathbb{Z}_X(r)_{\acute{e}t}$ also satisfy **(L3)**, cf. Proposition 3.3.1. Axiom **(L6)** follows from the Bloch-Kato conjecture, proven by Rost-Voevodsky [Voe11]: we have $\mathbb{H}^r_{\mathrm{Zar}}(k, \mathbb{Z}(r)_{\mathrm{Zar}}) \cong K^M_r(k)$, and the Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture, thus $\mathbb{H}^r_{\mathrm{Zar}}(k, \mathbb{Z}(r)_{\mathrm{Zar}}) \cong \mathbb{H}^r_{\acute{e}t}(k, \mathbb{Z}(r)_{\acute{e}t})$ (see, for example, [Gei04, Theorem 1.2]), which establishes **(L6)**.

3.2.3 Definition
Let X be a smooth, quasi-projective variety over a field k and let $r \geq 1$ be an integer. Define the motivic respectively étale motivic or Lichtenbaum cohomology of X in degree i and weight r with coefficients in A as the hypercohomology groups

$$\begin{aligned} \mathrm{H}^i_{\mathrm{M}}(X, A(r)) &:= \mathbb{H}^i_{\mathrm{Zar}}(X, A(r)_{\mathrm{Zar}}), \\ \mathrm{H}^i_{\mathrm{L}}(X, A(r)) &:= \mathbb{H}^i_{\acute{e}t}(X, A(r)_{\acute{e}t}), \end{aligned}$$

where $A(r)_\tau$ is the complex from Definition 3.2.2.

We remark that one can recover the Chow group $\mathrm{CH}^r(X)$ either as the 0-th homology of Bloch's cycle complex of abelian groups, or as the Zariski hypercohomology group $\mathrm{H}^{2r}_{\mathrm{M}}(X, \mathbb{Z}(r))$, see [Blo86, p. 269 (iv)].

The complexes $\mathbb{Z}(r)_{ét}$ are conjectured to satisfy the axioms **(L0)** to **(L6)** and since we have seen in Theorem 3.1.3 that there is a relation between the Tate conjecture for X at the prime ℓ in codimension r and the vanishing of $\mathbb{H}^{2r+1}_{ét}(X, \Gamma(r))_{\mathrm{div}}$, we are particularly interested in Lichtenbaum cohomology in the bidegrees $(2r+1, r)$ and $(2r, 2)$. We define the r-th higher Brauer-group of X and the r-th Lichtenbaum-Chow group of X as the corresponding Lichtenbaum cohomology groups in these bidegrees

$$\mathrm{Br}^r(X) := \mathrm{H}^{2r+1}_{\mathrm{L}}(X, \mathbb{Z}(r)) \text{ and } \mathrm{CH}^r_{\mathrm{L}}(X) := \mathrm{H}^{2r}_{\mathrm{L}}(X, \mathbb{Z}(r)).$$

We note that because of the quasi-isomorphism $\mathbb{Z}_X(1)_{ét} \simeq \mathbb{G}_m[-1]$ [Blo86, Corollary 6.4], there are isomorphisms

$$\mathrm{Br}^1(X) = \mathrm{H}^3_{\mathrm{L}}(X, \mathbb{Z}(1)) \cong \mathrm{H}^3_{ét}(X, \mathbb{G}_m[-1]) = \mathrm{Br}(X),$$
$$\mathrm{CH}^1_{\mathrm{L}}(X) = \mathrm{H}^2_{\mathrm{L}}(X, \mathbb{Z}(1)) \cong \mathrm{H}^2_{ét}(X, \mathbb{G}_m[-1]) = \mathrm{CH}^1(X).$$

3.3. Lichtenbaum cohomology and the conjectures of Hodge and Tate

To motivate the study of Lichtenbaum cohomology we state several results, relating the higher Brauer groups to the Hodge and Tate conjectures, which are due to Rosenschon-Srinivas [RS16b]. To formulate étale motivic analogues of the Hodge and Tate conjecture, we need cycle maps from the Lichtenbaum-Chow groups to the corresponding cohomology theories, as for the classical Chow groups.

We note first that Lichtenbaum and motivic cohomology coincide with rational coefficients, since the adjunction associated with $\pi : X_{ét} \to X_{\mathrm{Zar}}$ induces an isomorphism

$$\mathbb{Q}(n) \xrightarrow{\cong} R\pi_*\mathbb{Q}(n)_{ét}$$

in the derived category of sheaves, see, for example, [Kah12, 2.6]. Thus with rational coefficients motivic and Lichtenbaum cohomology coincide and one has the corresponding cycle maps in this setting. However, with integral coefficients motivic and Lichtenbaum cohomology groups differ; thus Lichtenbaum cohomology can be viewed as a different integral structure on rational motivic cohomology. With finite coefficients Geisser-Levine proved that there are the following quasi-isomorphisms, identifying the étale motivic complex with finite coefficients with more familiar étale sheaves.

3.3.1 Proposition
Let X be a smooth quasi-projective variety over a field k, $p = \mathrm{char}(k)$ and $n \in \mathbb{N}$.

a) If m is an integer and p does not divide m, there is a canonical quasi-isomorphism

$$(\mathbb{Z}/m\mathbb{Z})_X(n)_{ét} \xrightarrow{\sim} \mu_m^{\otimes n},$$

where μ_m is the sheaf of m-th roots of unity.

b) *For every $r \geq 1$ there is a canonical quasi-isomorphism*

$$(\mathbb{Z}/p^r\mathbb{Z})_X(n)_{\acute{e}t} \xrightarrow{\sim} \nu_r(n)[-n],$$

where $\nu_r(n)$ is the r-th logarithmic Hodge-Witt sheaf [Blo77], [Ill79].

Proof. [GL01, Theorem 1.5] and [GL00, Theorem 8.5]. □

These quasi-isomorphisms, together with the fact that $\mathbb{Z}_X(n)_{\acute{e}t}$ is a complex of free abelian groups, allow us to construct Kummer type short exact sequences as follows: For each prime power $\ell^r \in \mathbb{Z}$ there are exact sequences of complexes of étale sheaves

$$0 \to \mathbb{Z}(n)_{\acute{e}t} \xrightarrow{\cdot \ell^r} \mathbb{Z}(n)_{\acute{e}t} \to \mathbb{Z}/\ell^r\mathbb{Z}(n)_{\acute{e}t} \to 0, \tag{9}$$

which induce a long exact sequence in cohomology

$$\ldots \to \mathrm{H}^m_{\mathrm{L}}(X, \mathbb{Z}(n)) \xrightarrow{\cdot \ell^r} \mathrm{H}^m_{\mathrm{L}}(X, \mathbb{Z}(n)) \to \mathrm{H}^m_{\mathrm{L}}(X, \mathbb{Z}/\ell^r\mathbb{Z}(n)) \to \mathrm{H}^{m+1}_{\mathrm{L}}(X, \mathbb{Z}(n)) \to \ldots.$$

If $\mathrm{char}(k) \neq \ell$, we may identify $\mathrm{H}^m_{\mathrm{L}}(X, \mathbb{Z}/\ell^r\mathbb{Z}(n)) \cong \mathrm{H}^m_{\acute{e}t}(X, \mu_{\ell^r}^{\otimes n})$ using Proposition 3.3.1 a), to obtain in every bidegree (m, n) short exact sequences of the following form

$$0 \to \mathrm{H}^m_{\mathrm{L}}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}/\ell^r\mathbb{Z} \to \mathrm{H}^m_{\acute{e}t}(X, \mu_{\ell^r}^{\otimes n}) \to \mathrm{H}^{m+1}_{\mathrm{L}}(X, \mathbb{Z}(n))[\ell^r] \to 0. \tag{10}$$

Given these Kummer type short exact sequences (10), one can use a standard specialization argument to determine the torsion and cotorsion in Lichtenbaum cohomology over a separably closed field as in the following proposition.

3.3.2 Proposition (Rosenschon-Srinivas, [RS16b, Proposition 3.1])
Let k be a separably closed field, and let X be a smooth projective k-variety. Assume that ℓ is a prime number different from $\mathrm{char}(k)$*. Then*

a) $H^m_{\mathrm{L}}(X, \mathbb{Z}(n))\{\ell\} \cong H^{m-1}_{\acute{e}t}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ *for* $m \neq 2n+1$.

b) $H^m_{\mathrm{L}}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$ *for* $m \neq 2n$.

In particular, if $\ell \neq$ $\mathrm{char}(k)$ *and $m \neq 2n$, there is a short exact sequence of abelian groups*

$$0 \to H^{m-1}_{\acute{e}t}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \to H^m_{\mathrm{L}}(X, \mathbb{Z}(n)) \to H^m_{\mathrm{L}}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}_\ell \to 0.$$

This proposition shows that the Lichtenbaum cohomology groups are usually divisible with a torsion subgroup of finite cotype; this is known to be false for the motivic cohomology groups in general. We sketch the proof.

Proof. By Proposition 3.3.1, we may identify $\mathrm{H}^m_{\mathrm{L}}(X, \mathbb{Z}/\ell^r\mathbb{Z}(n)) \cong \mathrm{H}^m_{\acute{e}t}(X, \mu_{\ell^r}^{\otimes n})$. The maps $\mathbb{Z}(n)_{\acute{e}t} \to \mathbb{Z}/\ell^r(n)_{\acute{e}t} \to \mu_{\ell^r}^{\otimes n}$ give a map from integral Lichtenbaum cohomology to étale cohomology with finite coefficients. Taking the inverse limit over the maps $\mathrm{H}^m_{\mathrm{L}}(X, \mathbb{Z}(n)) \to \mathrm{H}^m_{\acute{e}t}(X, \mu_{\ell^r}^{\otimes n})$, we obtain a cycle map to ℓ-adic étale cohomology

$$\tilde{c}^{m,n}_{\mathrm{L},\mathbb{Z}_\ell} : \mathrm{H}^m_{\mathrm{L}}(X, \mathbb{Z}(n)) \to \mathrm{H}^m_{\acute{e}t}(X, \mathbb{Z}_\ell(n)). \tag{11}$$

This map fits into the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^{m-1}_{\mathrm{L}}(X, \mathbb{Z}(n))/\ell^r & \longrightarrow & \mathrm{H}^{m-1}_{\mathrm{L}}(X, \mathbb{Z}/\ell^r(n)) & \longrightarrow & \mathrm{H}^m_{\mathrm{L}}(X, \mathbb{Z}(n))[\ell^r] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\cong & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^{m-1}_{\acute{e}t}(X, \mathbb{Z}_\ell(n))/\ell^r & \longrightarrow & \mathrm{H}^{m-1}_{\acute{e}t}(X, \mu_{\ell^r}^{\otimes n}) & \longrightarrow & \mathrm{H}^m_{\acute{e}t}(X, \mathbb{Z}_\ell(n))[\ell^r] & \longrightarrow & 0 \end{array} \tag{12}$$

Taking the direct limit over all powers of ℓ in (12) yields an injective map

$$\mathrm{H}^{m-1}_{\mathrm{L}}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \mathrm{H}^{m-1}_{\acute{e}t}(X, \mathbb{Z}_\ell(n)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell, \tag{13}$$

and the claimed assertions will follow, provided this map is trivial for $m-1 \neq 2n$. For this it suffices to show that the image of the cycle map $\tilde{c}^{m-1,n}_{\mathrm{L},\mathbb{Z}_\ell}$ from (11) is torsion for $m-1 \neq 2n$; then (13) is trivial, since over a separably closed field the torsion subgroup $\mathrm{H}^{m-1}_{\acute{e}t}(X, \mathbb{Z}_\ell(n))_{\mathrm{tors}}$ is a finite group.

Let x be in $\mathrm{H}^{m-1}_{\mathrm{L}}(X, \mathbb{Z}(n))$. There is a regular integral ring R, finitely generated over $\mathbb{Z}$, and a smooth and proper scheme X_0 over R of finite type such that $X = X_0 \times_R k$; moreover there is a $x_0 \in \mathrm{H}^{m-1}_{\mathrm{L}}(X_0, \mathbb{Z}(q))$ which is mapped to x under the canonical map $\mathrm{H}^{m-1}_{\mathrm{L}}(X_0, \mathbb{Z}(q)) \to \mathrm{H}^{m-1}_{\mathrm{L}}(X, \mathbb{Z}(q))$. Let $\mathfrak{p} \subseteq \mathrm{Spec}(R)$ be a maximal ideal such that the residue field $k(\mathfrak{p})$ is finite and has characteristic $\neq \ell$. By smooth and proper base change for étale cohomology [Del77, V Théorème (3.1)], there is an isomorphism $\mathrm{H}^{m-1}_{\acute{e}t}(X, \mathbb{Z}_\ell(n)) \cong \mathrm{H}^{m-1}_{\acute{e}t}(X_0 \times_R \overline{k(\mathfrak{p})}, \mathbb{Z}_\ell(n))$. Since the cycle maps $\tilde{c}^{m-1,n}_{\mathrm{L},\mathbb{Z}_\ell}$ are functorial, we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^{m-1}_{\mathrm{L}}(X_0, \mathbb{Z}(n)) & \xrightarrow{\tilde{c}^{m-1,n}_{\mathrm{L},\mathbb{Z}_\ell}} & \mathrm{H}^{m-1}_{\acute{e}t}(X_0, \mathbb{Z}_\ell(n)) & \longrightarrow & \mathrm{H}^{m-1}_{\acute{e}t}(X, \mathbb{Z}_\ell(n)) \\ \downarrow & & & & \downarrow\cong \\ \mathrm{H}^{m-1}_{\mathrm{L}}(X_0 \times_R \overline{k(\mathfrak{p})}, \mathbb{Z}(n)) & & \xrightarrow{\tilde{c}^{m-1,n}_{\mathrm{L},\mathbb{Z}_\ell}} & & \mathrm{H}^{m-1}_{\acute{e}t}(X_0 \times_R \overline{k(\mathfrak{p})}, \mathbb{Z}_\ell(n)) \end{array}$$

In particular we may assume that k has non-zero characteristic; moreover by compatibility of the cycle maps with base-change, we may assume $R = k(\mathfrak{p})$ and $k = k(\mathfrak{p})_{\mathrm{sep}}$. Because of the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^{m-1}_{\mathrm{L}}(X_0,\mathbb{Z}(n)) & \xrightarrow{\tilde{c}^{m-1,n}_{\mathrm{L},\mathbb{Z}_\ell}} & \mathrm{H}^{m-1}_{\acute{e}t}(X_0,\mathbb{Z}_\ell(n)) \\ \downarrow & & \downarrow \\ \mathrm{H}^{m-1}_{\mathrm{L}}(X,\mathbb{Z}(n)) & \xrightarrow{\tilde{c}^{m-1,n}_{\mathrm{L},\mathbb{Z}_\ell}} & \mathrm{H}^{m-1}_{\acute{e}t}(X,\mathbb{Z}_\ell(n)) \end{array}$$

it suffices to show that the image of $\mathrm{H}^{m-1}_{\acute{e}t}(X_0,\mathbb{Z}_\ell(n))$ is contained in $\mathrm{H}^{m-1}_{\acute{e}t}(X,\mathbb{Z}_\ell(n))_{\mathrm{tors}}$. For this, recall that $X = X_0 \times_{k(\mathfrak{p})} k(\mathfrak{p})_{\mathrm{sep}}$, and consider the composition

$$\mathrm{H}^{m-1}_{\acute{e}t}(X_0,\mathbb{Z}_\ell(n)) \to \mathrm{H}^{m-1}_{\acute{e}t}(X,\mathbb{Z}_\ell(n)) \to \mathrm{H}^{m-1}_{\acute{e}t}(X,\mathbb{Q}_\ell(n)),$$

whose image is contained in the subgroup of Galois-invariants $\mathrm{H}^{m-1}_{\acute{e}t}(X,\mathbb{Q}_\ell(n))^{G_{k(\mathfrak{p})}}$. By the Weil conjectures, cf. Theorem 2.2.9, the absolute values of the eigenvalues of the Frobenius on $\mathrm{H}^{m-1}_{\acute{e}t}(X,\mathbb{Q}_\ell(n))$ are equal to $|k(\mathfrak{p})|^{\frac{m-1}{2}-n}$, which is different from 1 since we assume $m-1 \neq 2n$. In particular the group $\mathrm{H}^{m-1}_{\acute{e}t}(X,\mathbb{Q}_\ell(n))^{G_{k(\mathfrak{p})}}$ vanishes and the image of $\mathrm{H}^{m-1}_{\acute{e}t}(X_0,\mathbb{Z}_\ell(n))$ in $\mathrm{H}^{m-1}_{\acute{e}t}(X,\mathbb{Z}_\ell(n))$ is torsion, which completes the proof. □

A crucial observation here is that for a complex variety X the right hand vertical map in bidegree $(2n,n)$ in (12), together with the usual comparison theorems, yields a surjective map from Lichtenbaum-Chow groups to singular cohomology

$$\mathrm{CH}^n_{\mathrm{L}}(X)_{\mathrm{tors}} \to \mathrm{H}^{2n}_{\mathrm{B}}(X,\mathbb{Z}(n))_{\mathrm{tors}}.$$

Because of the examples of Atiyah-Hirzebruch [AH62] of torsion cohomology classes which are not in the image of the cycle map from Chow groups to singular cohomology, this implies (1) that the Lichtenbaum-Chow groups are a prior 'larger', and (2) this type of cycle map cannot come from the usual construction, using the fundamental class of a cycle on X and cohomology with support. However, given the ℓ-adic cycle maps from (11) for every prime ℓ, together with the cycle class maps with rational coefficients, it follows from a formal argument, that there is a unique integral cycle map

$$c^{m,n}_{\mathrm{L}} : \mathrm{H}^m_{\mathrm{L}}(X,\mathbb{Z}(n)) \to \mathrm{H}^m_{\mathrm{B}}(X,\mathbb{Z}(n))$$

from Lichtenbaum to singular cohomology; for the explicit construction see [RS16b, section 5]. This cycle map extends the usual cycle map and its image $\mathrm{I}^{2n}_{\mathrm{L}}(X) := \mathrm{im}(c^{2n,n}_{\mathrm{L}})$ is contained in the group of integral Hodge cycles $\mathrm{Hdg}^{2n}(X,\mathbb{Z})$. Thus one can state an L-version of the integral Hodge conjecture:

$$\mathrm{HC}^n_{\mathrm{L}}(X)_{\mathbb{Z}} :\Leftrightarrow \mathrm{I}^{2n}_{\mathrm{L}}(X) = \mathrm{Hdg}^{2n}(X,\mathbb{Z})$$

The following theorem shows that the integral structure provided by the Lichtenbaum-Chow groups on the usual Chow groups with rational coefficients is the correct one for an integral Hodge conjecture.

3.3.3 Theorem (Rosenschon-Srinivas, [RS16b, Theorem 1.1])
Let X be a smooth, projective complex variety. Then

a) $\mathrm{HC}^n(X)_{\mathbb{Q}} \Leftrightarrow \mathrm{HC}^n_{\mathrm{L}}(X)_{\mathbb{Z}}$ *for* $n \geq 0$.

b) *The map* $\mathrm{CH}^n_{\mathrm{L}}(X)_{\mathrm{tors}} \to \mathrm{Hdg}^{2n}(X,\mathbb{Z})_{\mathrm{tors}}$ *is surjective for* $n \geq 0$.

Moreover, using an interpretation of the Lichtenbaum cohomology groups as the colimit of the corresponding groups of étale hypercoverings of X, one can show [RS16b, section 4] that for a smooth projective complex variety there is a cycle map from integral Lichtenbaum cohomology to integral Deligne-Beilinson cohomology [EV87]

$$c^{m,n}_{\mathrm{L,D}} : \mathrm{H}^m_{\mathrm{L}}(X,\mathbb{Z}(n)) \to \mathrm{H}^m_{\mathrm{D}}(X,\mathbb{Z}(n)). \tag{14}$$

In the analytic setting, Deligne-Beilinson cohomology is considered to be the absolute cohomology theory and for the 'correct' integral structure on rational motivic cohomology one would expect that the above cycle map is an isomorphism on torsion. This is known to be false for the usual integral motivic cohomology groups. However:

3.3.4 Theorem (Rosenschon-Srinivas, [RS16b, Theorem 1.2])
Let X be a smooth projective complex variety, and let

$$c^{m,n}_{\mathrm{L,D}}\,|_{\mathrm{tors}} : H^m_{\mathrm{L}}(X,\mathbb{Z}(n))_{\mathrm{tors}} \to H^m_{\mathrm{D}}(X,\mathbb{Z}(n))_{\mathrm{tors}}$$

be the restriction of the cycle map (14) to torsion subgroups.

a) *If* $2n - m \geq 0$, *then* $c^{m,n}_{\mathrm{L,D}}\,|_{\mathrm{tors}}$ *is an isomorphism.*

b) *If* $2n - m = -1$, *then the following equivalences hold:*

$$c^{m,n}_{\mathrm{L,D}}\,|_{\mathrm{tors}} \text{ is injective } \Leftrightarrow \mathrm{HC}^n_{\mathrm{L}}(X)_{\mathbb{Z}} \Leftrightarrow \mathrm{HC}^n(X)_{\mathbb{Q}}$$

Let now X be a smooth, projective and geometrically irreducible variety over a field k which is finitely generated over its prime field. For $\ell \neq \mathrm{char}(k)$ there is a cycle map

$$c^{m,n}_{\mathrm{L},\mathbb{Z}_\ell} : \mathrm{H}^m_{\mathrm{L}}(X,\mathbb{Z}(n)) \otimes \mathbb{Z}_\ell \to \mathrm{H}^m_{\acute{e}t}(\overline{X},\mathbb{Z}_\ell(n))^{G_k},$$

which can be constructed as the composition of the cycle map from Lichtenbaum cohomology to continuous étale cohomology (see [Kah02] and [Kah12]), and a map arising from the Hochschild-Serre spectral sequence (see [Jan88]).

Taking the limit over all finite extensions k'/k over the $c^{m,n}_{\mathrm{L},\mathbb{Z}_\ell}$ yields a map

$$\overline{c}^{n}_{\mathrm{L},\mathbb{Z}_\ell} : \mathrm{CH}^n_{\mathrm{L}}(\overline{X}) \otimes \mathbb{Z}_\ell \to \mathrm{Ta}^{2n}(\overline{X},\mathbb{Z}_\ell) := \varinjlim \mathrm{H}^{2n}_{\acute{e}t}(\overline{X},\mathbb{Z}_\ell(n))^{G_{k'}}.$$

The integral L-Tate conjecture $\mathrm{TC}^n_{\mathrm{L}}(\overline{X})_{\mathbb{Z}_\ell}$ is then the assertion that the above map $\overline{c}^n_{\mathrm{L},\mathbb{Z}_\ell}$ is surjective. Similarly, we have a rational version $\mathrm{TC}^n(\overline{X})_{\mathbb{Q}_\ell}$ of this conjecture for the usual Chow groups. As for the Hodge conjecture, the Lichtenbaum-Chow groups have the correct integral structure for an integral Tate conjecture:

3.3.5 Theorem (Rosenschon-Srinivas, [RS16b, Theorem 1.3])
Let $k \subseteq \mathbb{C}$ be a field which is finitely generated over $\mathbb{Q}$, and let X be a smooth projective geometrically integral k-variety. Then for all integers $n \geq 0$ we have an equivalence

$$\mathrm{TC}^n(\overline{X})_{\mathbb{Q}_\ell} \Leftrightarrow \mathrm{TC}^n_{\mathrm{L}}(\overline{X})_{\mathbb{Z}_\ell}.$$

Moreover, $\mathrm{Ta}^{2n}(\overline{X},\mathbb{Z}_\ell)_{\mathrm{tors}}$ *is contained in the image of* $\mathrm{CH}^n_{\mathrm{L}}(X)_{\mathrm{tors}} \otimes \mathbb{Z}_\ell$.

For varieties over finite fields we have:

3.3.6 Theorem (Rosenschon-Srinivas, [RS16b, Theorem 1.4])
Let k be a finite field, and let X be a smooth, projective, geometrically integral k-variety. For every prime number $\ell \neq \mathrm{char}(k)$ and every $n \geq 0$ there are equivalences

$$\mathrm{TC}^n(X)_{\mathbb{Q}_\ell} \Leftrightarrow \mathrm{TC}^n_{\mathrm{L}}(X)_{\mathbb{Z}_\ell} \Leftrightarrow \mathrm{Br}^n(X)\{\ell\} < \infty.$$

Furthermore, the torsion subgroup $H^{2n}_{ét}(\overline{X},\mathbb{Z}_\ell(n))^{G_k}_{\mathrm{tors}}$ is L*-algebraic.*

4. The transcendental part of higher Brauer groups

In this section we give the proof of Theorem 1.0.2. We first recall in section 4.1 the construction of the usual Hochschild-Serre spectral sequence and the resulting complex $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^{G_k} \to \mathrm{H}^2(G_k, \mathrm{Pic}(\overline{X}))$. We construct in section 4.4 analogous complexes for higher Brauer groups coming from a Hochschild-Serre spectral sequence for Lichtenbaum cohomology; this spectral sequence is constructed in section 4.2 and its convergence is shown in section 4.3. In section 4.5 we prove Theorem 1.0.2.

In what follows X will denote a smooth quasi-projective and geometrically integral variety over a field k with absolute Galois group G_k. By $\mathbb{Z}_X(r)$, or simply $\mathbb{Z}(r)$, we mean the complex of Zariski sheaves defined in 3.2; we can view the corresponding étale version of this complex $\mathbb{Z}(r)_{ét} = \pi^*\mathbb{Z}(r)_{Zar}$ as the pullback along the morphism $\pi : X_{ét} \to X_{Zar}$. We use will analogous notations for complexes $A_X(r) = \mathbb{Z}_X(r) \otimes A$, where A is an abelian group.

4.1. The spectral sequence in weight 1

In weight 1 we have $\mathbb{Z}(1)_{ét} \simeq \mathbb{G}_m[-1]$ [Blo86, Corollary 6.4] and may identify the unbounded complex $\mathbb{Z}(1)_{ét}$ with the single étale sheaf $\mathbb{G}_m[-1]$. Consider the Leray spectral sequence with respect to the structure morphism $f : X \to \mathrm{Spec}(k)$ (cf. A.3.7)

$$E_2^{p,q} = \mathrm{H}^p_{ét}\left(\mathrm{Spec}(k), R^q f_* \mathbb{G}_m\right) \Rightarrow \mathrm{H}^{p+q}_{ét}\left(X, \mathbb{G}_m\right). \tag{15}$$

Since $\mathrm{Spec}(k)_{ét}$ can be thought of as the set of all finite, separable extensions of k in a fixed separable closure k_{sep} of k, $R^q f_* \mathbb{G}_m$ is just the sheaf $k' \mapsto \mathrm{H}^q_{ét}\left(X_{k'}, \mathbb{G}_m\right)$ on $\mathrm{Spec}(k)_{ét}$ and using [Tam94, Corollary II.2.20], we obtain for the E_2-terms

$$\mathrm{H}^p_{ét}\left(\mathrm{Spec}(k), R^q f_* \mathbb{G}_m\right) \cong \mathrm{H}^p\left(G_k, \varinjlim_{k'/k} \mathrm{H}^q_{ét}\left(X_{k'}, \mathbb{G}_m\right)\right) = \mathrm{H}^p\left(G_k, \mathrm{H}^q_{ét}(\overline{X}, \mathbb{G}_m)\right),$$

where the right hand side is just Galois cohomology and the direct limit runs over all finite extensions k' of k. Thus the spectral sequence (15) can be written in the form

$$E_2^{p,q} = \mathrm{H}^p(G_k, \mathrm{H}^q_{ét}(\overline{X}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{ét}(X, \mathbb{G}_m). \tag{16}$$

Since Galois cohomology and étale cohomology of a single sheaf both vanish in negative degrees, this is a first quadrant spectral sequence and therefore convergent.

Since $\mathrm{Br}(X) = \mathrm{H}^2_{ét}(X, \mathbb{G}_m)$ and $\mathrm{Pic}(X) = \mathrm{H}^1_{ét}(X, \mathbb{G}_m)$, it is easy to see that the composition of the edge morphism $\mathrm{H}^2 \to E_2^{0,2}$ with the d_2-differential $E_2^{0,2} \to E_2^{2,1}$ yields a complex, which is functorial, since the Leray spectral sequence is. Hence we have:

4.1.1 Proposition
Let X be a smooth quasi-projective k-variety. There is a functorial complex of abelian groups

$$\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^{G_k} \to H^2\left(G_k, \mathrm{Pic}(\overline{X})\right). \tag{17}$$

4.2. The spectral sequence in weight r

We construct a spectral sequence analogous to (16) for Lichtenbaum cohomology. This spectral sequence will arise from a specific double complex, whose associated spectral sequence has the correct E_2-terms and the correct limit term. However, since the complex $\mathbb{Z}(r)_{ét}$ is unbounded and we do not know the vanishing of Lichtenbaum cohomology in negative degrees, this is a priori a right half plane spectral sequence and convergence is an issue. We show first:

4.2.1 Theorem
Let X be a smooth, quasi-projective variety over a field k with absolute Galois group G_k. Then there is a convergent and functorial Hochschild-Serre type spectral sequence of the form

$$E_2^{p,q} = H^p\left(G_k, H^q_{\mathrm{L}}\left(\overline{X}, \mathbb{Z}(r)\right)\right) \Rightarrow H^{p+q}_{\mathrm{L}}\left(X, \mathbb{Z}(r)\right). \tag{18}$$

The proof of Theorem 4.2.1 consists of three parts. First we construct a specific double complex, using that the hypercohomology of unbounded complexes of étale sheaves can be computed using K-injective resolutions. Then we show that the spectral sequence associated with this double complex has the correct E_2-terms and the correct limit terms. Finally we prove that the spectral sequence actually converges.

For the construction of the double complex we need some facts about derived categories and derived functors. Let $\mathcal{A}$ be an abelian category. Recall that the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is constructed by considering chain homotopy classes of cochain complexes of objects of $\mathcal{A}$ followed by localizing the set of quasi-isomorphisms. The morphisms in $\mathcal{D}(\mathcal{A})$ are special roofs $X \leftarrow X' \to Y$; for details see A.1. The hypercohomology $\mathbb{H}^i(X, \mathcal{C}^\bullet)$ of a complex $\mathcal{C}^\bullet$ in $\mathcal{A}$ is usually defined via Cartan-Eilenberg resolutions $I^{\bullet,\bullet}$ of $\mathcal{C}^\bullet$, see A.2. For bounded below complexes, it is also possible to use the total right derived functor $\mathbf{R}F$ of a functor $F : \mathcal{A} \to \mathcal{B}$ between two abelian categories (cf. [Wei95, Corollary 10.5.7]). In particular, if $\mathcal{C}^\bullet$ is a bounded below complex of sheaves on X, then the two definitions agree and we have for every $i \in \mathbb{Z}$

$$\mathbb{H}^i(X, \mathcal{C}^\bullet) = \mathrm{H}^i(\mathbf{R}\Gamma(X, \mathcal{C}^\bullet)),$$

where Γ is the global section functor. In general the existence of the total right derived functor $\mathbf{R}F$ of a functor $F : \mathcal{A} \to \mathcal{B}$ can only be ensured for the subcategory $\mathcal{D}^+(\mathcal{A})$ of bounded below complexes in $\mathcal{D}(\mathcal{A})$, provided $\mathcal{A}$ has enough injectives and the categories of (bounded below) complexes in $\mathcal{A}$ resp. $\mathcal{B}$, $\mathrm{Kom}^+(\mathcal{A})$ and $\mathrm{Kom}(\mathcal{B})$ are triangulated (cf. [Wei95, section 10.5]).

For the construction of the double complex we will use total right derived functors; however, since the complex $\mathbb{Z}(r)_{ét}$ is not bounded below, we need a more sophisticated construction due to Spaltenstein [Spa88] who showed that for a smooth k-scheme X and $\mathcal{A}$ the category of sheaves on the étale site of X, the boundedness condition for the existence of the total right derived functor can be removed using K-injective resolutions. Here a complex $I^\bullet \in \mathcal{D}(\mathcal{A})$ is K-injective if for every acyclic complex $S^\bullet \in \mathcal{D}(\mathcal{A})$ the complex $\mathrm{Hom}^\bullet(I^\bullet, S^\bullet)$ is acyclic, where the complex $\mathrm{Hom}^\bullet(I^\bullet, S^\bullet)$ is given by

$$\mathrm{Hom}^n(I^\bullet, S^\bullet) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}(I^i, S^{i+n}).$$

By definition, a K-injective resolution of a complex $A^\bullet \in \mathcal{D}(\mathcal{A})$ is quasi-isomorphism $A^\bullet \to I^\bullet$ with $I^\bullet$ K-injective. The main property of K-injective resolutions is that they can be used to define (and compute) the total right derived functor of the global section functor, i.e. the hypercohomology of (possibly unbounded) complexes in $\mathcal{A}$. In particular, if $\mathcal{C}^\bullet$ is a complex of sheaves of abelian groups on the étale site of X, then $\mathcal{C}^\bullet$ has a K-injective resolution [Spa88, 4.6] and the hypercohomology groups

$$\mathbb{H}^\bullet(X, \mathcal{C}^\bullet) = \mathrm{H}^\bullet(\mathrm{R}\Gamma(X, \mathcal{C}^\bullet))$$

are well-defined [Spa88, 6.3]. For a more detailed discussion, we refer to [Spa88], see also [Wei96].

To construct the double complex, let $U \in X_{ét}$ and consider the functor $\mathcal{F}(U, -) : \mathbf{Sh}(X_{ét}) \to \mathbf{Ab}$ given by base change to $\overline{k}$, followed by taking global sections

$$\mathcal{F}(U, F) = \Gamma\left(U \times_k \overline{k}, F\right) = \Gamma\left(\overline{U}, F\right).$$

By the above discussion $\mathrm{R}\mathcal{F}(U, -) : \mathcal{D}(\mathbf{Sh}(X_{ét})) \to \mathcal{D}(\mathbf{Ab})$ is well-defined; moreover, given $U \in X_{ét}$ and a complex $\mathcal{C}^\bullet \in \mathcal{D}(\mathbf{Sh}(X_{ét}))$, the cohomology of $\mathrm{R}\mathcal{F}(U, \mathcal{C}^\bullet)$ computes the hypercohomology

$$\mathbb{H}^\bullet(\overline{U}, \mathcal{C}^\bullet) = \mathrm{H}^\bullet\left(\mathrm{R}\mathcal{F}(U, \mathcal{C}^\bullet)\right).$$

We apply this to X and $\mathbb{Z}(r)_{ét}$ to obtain a complex $\mathcal{A}^\bullet = \mathrm{R}\mathcal{F}(X, \mathbb{Z}(r)_{ét})$ such that

$$\mathrm{H}^q_{\mathrm{L}}(\overline{X}, \mathbb{Z}(r)) = \mathrm{H}^q(\mathcal{A}^\bullet) = \mathrm{H}^q(\mathrm{R}\mathcal{F}(X, \mathbb{Z}(r)_{ét}).$$

The complex $\mathcal{A}^\bullet$ is a complex of abelian groups with an action of the Galois group G_k, i.e. it is a complex of Galois-modules. Let $\mathcal{G} : \mathbf{Ab} \to \mathbf{Ab}$ be the functor taking G_k-invariants. Since the category of discrete G_k-modules has enough injectives [NSW13, Lemma 2.6.5], we can choose for each $\mathcal{A}^i$ an injective resolution $\tilde{I}^{i,\bullet}$ of G_k-modules and apply the functor $\mathrm{R}\mathcal{G}$ to these complexes; this defines a double complex $I^{\bullet,\bullet}$ with

$$i\text{-th column of } I^{\bullet,\bullet} = (\mathrm{R}\mathcal{G})(\tilde{I}^{i,\bullet}). \tag{19}$$

The spectral sequence in Theorem 4.2.1 will be one of the spectral sequences associated with this double complex $I^{\bullet,\bullet}$.

Taking first horizontal cohomology and then vertical cohomology of the double complex $I^{\bullet,\bullet}$ defines a spectral sequence (for details, see also A.3.10) with E_2-terms

$${}^{II}E_2^{p,q} = \mathrm{H}^p_{II}\left(\mathrm{H}^{q,\bullet}_I(I^{\bullet,\bullet})\right).$$

By construction, taking the horizontal cohomology of $I^{\bullet,\bullet}$ is equivalent to taking hypercohomology of $\mathbb{Z}(r)_{ét}$ on $\overline{X}$ and taking the vertical cohomology corresponds to taking the Galois cohomology of $\mathrm{H}^\bullet_{\mathrm{L}}(\overline{X}, \mathbb{Z}(r))$. Hence we can write the E_2-terms ${}^{II}E_2^{p,q}$ as

$${}^{II}E_2^{p,q} = \mathrm{H}^p\left(G_k, \mathrm{H}^q_{\mathrm{L}}\left(\overline{X}, \mathbb{Z}(r)\right)\right). \tag{20}$$

The complex $I^{\bullet,\bullet}$ arises from the composition of $\mathrm{R}\mathcal{F}(X,-)$ and $\mathrm{R}\mathcal{G}$, applied to the complex $\mathbb{Z}(r)_{ét}$; thus the limit terms of the associated spectral sequence are the hypercohomology groups

$$\mathrm{H}^{p+q} = \mathbb{H}^{p+q}\left(X, (\mathrm{R}\mathcal{G}) \circ \left(\mathrm{R}\mathcal{F}\left(-, \mathbb{Z}\left(r\right)_{ét}\right)\right)\right) \tag{21}$$

Now we use:

4.2.2 Lemma
Let $\mathrm{R}\mathcal{F}(X,-)$ and $\mathrm{R}\mathcal{G}$ be the functors from above. There is an isomorphism of functors

$$(\mathrm{R}\mathcal{G}) \circ (\mathrm{R}\left(\mathcal{F}(X,-)\right)) \simeq \mathrm{R}\left(\mathcal{G} \circ \mathcal{F}(X,-)\right) : \mathcal{D}(\mathbf{Sh}(X_{ét})) \to \mathcal{D}(\mathbf{Ab}) \tag{22}$$

Proof. This is an application of [Har66, I.5.4]. For ease of notation we abbreviate $\mathrm{R}\mathcal{F}(X,-)$ by $\mathrm{R}\mathcal{F}$. To apply the above result, we need localizing subcategories $\mathbf{K}^+$ and $\mathbf{K}^*$ of the homotopy categories $\mathbf{K}(\mathbf{Sh}(X_{ét}))$ (resp. $\mathbf{K}(\mathbf{Ab})$) of $\mathbf{Sh}(X_{ét})$ (resp. $\mathbf{Ab}$) such that $\mathrm{R}\mathcal{F}(\mathbf{K}^+) \subseteq \mathbf{K}^*$. Furthermore, we need triangulated subcategories $L \subseteq \mathbf{K}^+$ and $M \subseteq \mathbf{K}^*$ such that $\mathrm{R}\mathcal{F}(L) \subseteq M$, every object of $\mathbf{K}^+$ (resp. of $\mathbf{K}^*$) admits a quasi-isomorphism into an object of L (resp. of M), and such that $\mathrm{R}\mathcal{F}$ and $\mathrm{R}\mathcal{G}$ map acyclic objects of L and M respectively to acyclic objects.

For the categories $\mathbf{K}^+$ and $\mathbf{K}^*$ we take the class of quasi-isomorphisms in $\mathbf{K}(\mathbf{Sh}(X_{ét}))$ and $\mathbf{K}(\mathbf{Ab})$ respectively; these classes are localizing by [Har66, I.4.2]. Furthermore $\mathrm{R}\mathcal{F}$ maps quasi-isomorphisms to quasi-isomorphisms. For the triangulated subcategories of $\mathbf{K}^+$ and $\mathbf{K}^*$, we can choose the whole categories. Since $\mathrm{R}\mathcal{F}$ and $\mathrm{R}\mathcal{G}$ map acyclic objects to acyclic objects, (22) follows now from [Har66, I.5.4]. □

Consider $\mathrm{R}\left(\mathcal{G}\circ\mathcal{F}(-,\mathbb{Z}(r)_{\acute{e}t})\right)$; since at the level of complexes we have $\mathbb{Z}(r)_{\acute{e}t}(\overline{X})^{G_k}\cong\mathbb{Z}(r)_{\acute{e}t}(X)$, the cohomology of $\mathrm{R}\left(\mathcal{G}\circ\mathcal{F}(-,\mathbb{Z}(r)_{\acute{e}t})\right)$ coincide with the hypercohomology groups of the complex $\mathbb{Z}(r)_{\acute{e}t}$. Hence, we have from (21) and (22) the isomorphisms

$$\mathrm{H}^{p+q}\cong\mathbb{H}^{p+q}(X,\mathbb{Z}(r)_{et})=\mathrm{H}^{p+q}_{\mathrm{L}}(X,\mathbb{Z}(r)),\tag{23}$$

which shows that the spectral sequence associated with $I^{\bullet,\bullet}$ has the correct limit terms.

4.3. Convergence

The remaining part of the proof of Theorem 4.2.1 is to show that the spectral sequence

$$E_2^{p,q}=\mathrm{H}^p(G_k,\mathrm{H}^q_{\mathrm{L}}(\overline{X},\mathbb{Z}(r)))\Rightarrow\mathrm{H}^{p+q}_{\mathrm{L}}(X,\mathbb{Z}(r))$$

indeed converges and we will do this by showing that this spectral sequence is in fact bounded. We note first that the E_2-terms $E_2^{p,q}=\mathrm{H}^p(G_k,\mathrm{H}^q_{\mathrm{L}}(\overline{X},\mathbb{Z}(r)))$ vanish for $p<0$, because Galois cohomology vanishes in negative degrees. In what follows we will show that $E_2^{p,q}=0$ also for $q<0$ and $p>0$.

Let k be a field, $p=\mathrm{char}(k)$, $\ell\neq p$ a prime, and $n\in\mathbb{N}$. Consider the étale sheaves

$$\mathbb{Q}_\ell/\mathbb{Z}_\ell(n):=\varinjlim_r\mu_{\ell^r}^{\otimes n}\quad\text{and}\quad\mathbb{Q}_p/\mathbb{Z}_p(n):=\varinjlim_r\nu_r(n)[-n],$$

and set $\left(\mathbb{Q}/\mathbb{Z}\right)(n):=\bigoplus_\ell\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$. We consider $\mathbb{Q}/\mathbb{Z}$ as an element of the derived category of the category of étale sheaves of abelian groups on X. By Proposition 3.3.1. we have for every smooth quasi-projective variety X over k the quasi-isomorphism

$$\left(\mathbb{Q}/\mathbb{Z}\right)_X(n)_{\acute{e}t}\xrightarrow{\sim}\left(\mathbb{Q}/\mathbb{Z}\right)(n).\tag{24}$$

The following lemma is a key ingredient for the proof of the convergence of (18).

4.3.1 Lemma
Let X be a smooth, quasi-projective variety over a field k. Then for every $i<0$ and $n\geq 0$

$$H^i_{\mathrm{M}}\left(X,\mathbb{Q}(n)\right)\cong H^i_{\mathrm{L}}\left(X,\mathbb{Z}(n)\right).$$

Proof. In the category $\mathcal{D}(\mathbf{Sh}(X_{\acute{e}t}))$ we have the evident distinguished triangle

$$\mathbb{Z}_X(r)_{\acute{e}t}\to\mathbb{Q}_X(r)_{\acute{e}t}\to(\mathbb{Q}/\mathbb{Z})_X(r)_{\acute{e}t}\xrightarrow{+1}$$

where, using the quasi-isomorphism (24), we may compute the hypercohomology of the complex $(\mathbb{Q}/\mathbb{Z})_X(r)_{\acute{e}t}$ as the cohomology of the étale sheaf $(\mathbb{Q}/\mathbb{Z})(n)_{\acute{e}t}$. Taking cohomology, we obtain the exact sequence of abelian groups

$$\mathrm{H}^{i-1}_{\acute{e}t}(X,\mathbb{Q}/\mathbb{Z}(n))\to\mathrm{H}^i_{\mathrm{L}}(X,\mathbb{Z}(n))\to\mathrm{H}^i_{\mathrm{L}}(X,\mathbb{Q}(n))\to\mathrm{H}^i_{\acute{e}t}(X,\mathbb{Q}/\mathbb{Z}(n)).$$

Since the cohomology of a single étale sheaf vanishes in negative degrees, this implies that $\mathrm{H}^i_{\mathrm{L}}(X, \mathbb{Z}(n)) \cong \mathrm{H}^i_{\mathrm{L}}(X, \mathbb{Q}(n))$ for $i < 0$. Our claim follows now, because with rational coefficients the adjunction associated with $\pi : X_{\acute{e}t} \to X_{Zar}$ induces a quasi-isomorphism $\mathbb{Q}(r)_{\mathrm{Zar}} \xrightarrow{\sim} \mathrm{R}\pi_*\mathbb{Q}(r)_{\acute{e}t}$ [Kah12, 2.6], thus for $i < 0$ we have

$$\mathrm{H}^i_{\mathrm{L}}(X, \mathbb{Z}(r)) \cong \mathrm{H}^i_{\mathrm{L}}(X, \mathbb{Q}(r)) \cong \mathrm{H}^i_{\mathrm{M}}(X, \mathbb{Q}(r)).$$

□

Now we can complete the proof of Theorem 4.2.1.

Proof of Theorem 4.2.1. We briefly recall what we have shown so far. We consider the double complex$I^{\bullet,\bullet}$ from (19) and its associated spectral sequence with E_2-terms ${}^{II}E_2^{p,q} = \mathrm{H}^p_{II}(\mathrm{H}^q_I(I^{\bullet,\bullet}))$. We showed in (20) that the E_2-terms of this spectral sequence can be written as ${}^{II}E_2^{p,q} = \mathrm{H}^p(G_k, \mathrm{H}^q_{\mathrm{L}}(\overline{X}, \mathbb{Z}(r)))$, and in (23) that the limit term H^{p+q} can be identified with $\mathrm{H}^{p+q}_{\mathrm{L}}(X, \mathbb{Z}(r))$. We also have $E_2^{p,q} = 0$ for $p < 0$. For $q < 0$ we have from Lemma 4.3.1 $\mathrm{H}^q_{\mathrm{M}}\left(\overline{X}, \mathbb{Q}(n)\right) \cong \mathrm{H}^q_{\mathrm{L}}\left(\overline{X}, \mathbb{Z}(n)_{\acute{e}t}\right)$; hence in this case $\mathrm{H}^q_{\mathrm{L}}\left(\overline{X}, \mathbb{Z}(n)\right)$ is a $\mathbb{Q}$-vector space and therefore uniquely divisible. But the Galois cohomology $\mathrm{H}^i(G_k, A)$ of a uniquely divisible G_k-module A in degrees $i > 0$ vanishes [NSW13, Proposition (1.6.2)]. Thus $E_2^{pq} = 0$ for $p > 0$ and $q < 0$, i.e. the spectral sequence is indeed bounded and therefore convergent. It is functorial since the underlying complexes are. □

4.4. Higher Brauer groups and Galois cohomology

In this section we construct a complex for higher Brauer groups which is analogous to the complex (17) for the classical Brauer group, using the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p\left(G_k, \mathrm{H}^q_{\mathrm{L}}(\overline{X}, \mathbb{Z}(r))\right) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{L}}(X, \mathbb{Z}(r)),$$

from Theorem 4.2.1.

Note that in the above spectral sequence we have the following terms

$$\mathrm{H}^{2r+1}_{\mathrm{L}}(X, \mathbb{Z}(r)) = \mathrm{Br}^r(X), \; E^{0,2r+1} = \mathrm{Br}^r(\overline{X})^{G_k}, \text{ and } E^{2,2r} = \mathrm{H}^2\left(G_k, \mathrm{CH}^r_{\mathrm{L}}(\overline{X})\right).$$

4.4.1 Theorem
Let X be a smooth, quasi-projective variety over a field k. There is a functorial complex

$$\mathrm{Br}^r(X) \xrightarrow{\alpha} \mathrm{Br}^r(\overline{X})^{G_k} \xrightarrow{\beta} H^2\left(G_k, \mathrm{CH}^r_{\mathrm{L}}(\overline{X})\right), \tag{25}$$

which is exact if and only if $E_3^{0,2r+1} = E_\infty^{0,2r+1}$ in the Hochschild-Serre spectral sequence (18).

We discuss after the proof of Theorem 4.4.1 in Lemma 4.4.2 precise conditions for exactness of the above complex.

Proof of Theorem 4.4.1. Since $E_2^{p,q} = 0$ for $p < 0$ there is an edge morphism $e : \mathrm{H}^{2r+1} \to E_\infty^{0,2r+1}$. By the same vanishing every differential $d_n^{-n,2r+1+n}$ is trivial, hence we have $E_\infty^{0,2r+1} \subseteq E_2^{0,2r+1}$. The map α is then the composition of the edge map with the inclusion

$$\alpha : \mathrm{Br}(X) = \mathrm{H}_{\mathrm{L}}^{2r+1}(X, \mathbb{Z}(r)) \xrightarrow{e} E_\infty^{0,2r+1} \hookrightarrow E_2^{0,2r+1} = \mathrm{Br}^r(\overline{X})^{G_k}.$$

The map β is just the differential

$$\beta : \mathrm{Br}(\overline{X})^{G_k} = E_2^{0,2r+1} \xrightarrow{d_2^{0,2r+1}} E_2^{2,2r} = \mathrm{H}^2\left(G_k, \mathrm{CH}_{\mathrm{L}}^r(X)\right).$$

An easy direct calculation shows that $\mathrm{im}(\alpha) \subseteq E_3^{0,2r+1} = \ker(\beta) \subseteq E_2^{0,2r+1}$, i.e. (25) defines indeed a complex. That this complex is functorial follows from the functoriality of the underlying spectral sequence.

The condition for (25) to be exact is obvious from the construction of α and β: Since α is the composition of the surjective morphism $e : \mathrm{H}^{2r+1} \to E_\infty^{0,2r+1}$ followed by the inclusion $E_\infty^{0,2r+1} \subseteq E_2^{0,2r+1}$, $\mathrm{im}(\alpha)$ can be identified with $E_\infty^{0,2r+1}$. Since (18) is a first quadrant spectral sequence, we have on the other hand

$$E_3^{0,2r+1} = \ker(\beta) / \mathrm{im}(d_2^{-2,2r+2}) = \ker(\beta).$$

Hence the complex (25) is exact if and only if $E_3^{0,2r+1} = E_\infty^{0,2r+1}$. □

4.4.2 Lemma
The complex (25) is exact in the following cases.

a) If $r = 1$, k is a number field and $H_{ét}^0(\overline{X}, \mathbb{G}_m) = \overline{k}^\times$ (for example, if X is projective).

b) If k is a field of cohomological dimension at most 2 .

Proof. In case a) the image of α is $E_\infty^{0,3}$ and one has to show that $E_3^{0,3} = E_5^{0,3} = E_\infty^{0,3}$. This follows since $E_2^{3,1} = \mathrm{H}^3(G_k, \mathrm{H}_{\mathrm{L}}^1(\overline{X}, \mathbb{Z}(1)))$ vanishes, if k is a number field [Maz73, (1.5)], and since the term $E_2^{4,0} = \mathrm{H}^4(G_k, \mathrm{H}_{\mathrm{L}}^0(\overline{X}, \mathbb{Z}(1)))$ vanishes, because $\mathbb{Z}(1)_{ét} \simeq \mathbb{G}_m[-1]$.

For case b) we recall that we have seen in the proof of Theorem 4.4.1 that

$$\ker(\beta) = E_s^{0,2r+1} = \ker\left(d_{s-1}^{0,2r+1} : E_{s-1}^{0,2r+1} \to E_{s-1}^{s-1,2r-s+3}\right).$$

Since the groups $E_{s-1}^{s-1,2r-s+3}$ are subquotients of $E_2^{s-1,2r-s+3}$ for $s \geq 4$, the differentials $d_{s-1}^{0,2r+1}$ vanish for $s \geq 4$, thus we have $E_3^{0,2r+1} = E_\infty^{0,2r+1}$. □

4.5. The transcendental part of higher Brauer groups

Let X be a smooth, quasi-projective variety over a field k and let $\alpha : \mathrm{Br}^r(X) \to \mathrm{Br}^r(\overline{X})^{G_k}$ be the natural map. As in the case of the classical Brauer group, we refer to the kernel of α as the algebraic r-th Brauer group $\mathrm{Br}^r(X)_{\mathrm{alg}}$, and to the image of α as the transcendental r-th Brauer group $\mathrm{Br}^r(X)_{\mathrm{tr}}$.

In this section we prove our main Theorem 1.0.2; we recall the statement.

Theorem
Let X be a smooth, projective, geometrically irreducible variety of dimension at most 4 over a field k of characteristic zero with absolute Galois group G_k. Assume further that k has cohomological dimension ≤ 2, the third Betti-number b_3 of X is zero and that the group $H^4_{\acute{e}t}(\overline{X}, \mathbb{Z}_\ell(2))$ is torsion free for every prime ℓ. Then the cokernel of $\alpha : \mathrm{Br}^2(X) \to \mathrm{Br}^2(\overline{X})^{G_k}$ is finite.

We begin with two lemmas, which show that in case $\dim(X) = 1, 2$ we actually have $\mathrm{Br}^2(\overline{X}) = 0$. Hence in these cases Theorem 1.0.2 holds trivially; in fact we have $\mathrm{coker}(\alpha) = 0$.

4.5.1 Lemma
Let C be a smooth, projective curve over a field of characteristic 0. Then $\mathrm{Br}^2(\overline{C}) = 0$.

Proof. Since $0 \to \mathbb{Z}_C(2)_{\acute{e}t} \to \mathbb{Q}_C(2)_{\acute{e}t} \to (\mathbb{Q}/\mathbb{Z})_C(2)_{\acute{e}t} \to 0$ is an exact sequence of complexes of étale sheaves, we have an exact sequence $\mathrm{H}^4_{\acute{e}t}(\overline{C}, \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{Br}^2(\overline{C}) \to \mathrm{H}^5_{\mathrm{M}}(\overline{C}, \mathbb{Q}(2))$; here we have used the quasi-isomorphism (24), and the fact that Lichtenbaum and motivic cohomology with rational coefficients agree. The term on the left side vanishes, since étale cohomology groups vanish in degrees $\geq 2\dim(C) + 1$, and the term on the right side vanishes by [MVW06, Theorem 19.3]. Thus $\mathrm{Br}^2(\overline{C}) = 0$. □

4.5.2 Lemma
Let S be a smooth projective surface over a field k of characteristic zero. Then $\mathrm{Br}^2(\overline{S}) = 0$.

Proof. The Kummer sequence from (9) induces for every ℓ short exact sequences

$$0 \to \mathrm{CH}^2_{\mathrm{L}}(\overline{S}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \mathrm{H}^4_{\acute{e}t}(\overline{S}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \to \mathrm{Br}^2(\overline{S})\{\ell\} \to 0. \quad (26)$$

Since S is a surface, we have $\mathrm{CH}^2_{\mathrm{L}}(\overline{S}) \otimes \mathbb{Q}/\mathbb{Z} \cong \mathrm{CH}^2(\overline{S}) \otimes \mathbb{Q}/\mathbb{Z}$ (cf. Lemma 4.5.6) as well as $\mathrm{CH}^2(\overline{S}) \otimes \mathbb{Q}/\mathbb{Z} \cong \mathrm{H}^4_{\mathrm{M}}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2))$ (since $\mathrm{H}^5_{\mathrm{M}}(Y, \mathbb{Z}(2)) = 0$ for a smooth variety Y). Taking the direct sum of the sequences (26) over all ℓ and comparing the resulting short exact sequence with the corresponding short exact sequence in motivic cohomology along the change of topology map yields the following commutative diagram

$$\begin{array}{ccccccccc} & & \mathrm{CH}^2(\overline{S}) \otimes \mathbb{Q}/\mathbb{Z} & = & \mathrm{H}^4_{\mathrm{M}}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2)) & & & & \\ & & \| & & \downarrow \psi & & & & \\ 0 & \longrightarrow & \mathrm{CH}^2_{\mathrm{L}}(\overline{S}) \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathrm{H}^4_{\acute{e}t}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2)) & \longrightarrow & \mathrm{Br}^2(\overline{S}) & \longrightarrow & 0 \end{array}$$

Hence $\mathrm{coker}(\psi) = \mathrm{Br}^2(\overline{S})$. On the other hand, for τ either the Zariski or the étale topology, we have the Zariski sheaves $\mathcal{H}^q_\tau(\mathbb{Q}/\mathbb{Z}(n))$ associated to the presheaves $U \mapsto \mathrm{H}^q_\tau(U, \mathbb{Q}/\mathbb{Z}(n))$, which are the coefficients sheaves of the Bloch-Ogus-spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{Zar}}\left(\overline{S}, \mathcal{H}^q_\tau\left(\mathbb{Q}/\mathbb{Z}\left(2\right)\right)\right) \Rightarrow \mathrm{H}^{p+q}_\tau(\overline{S}, \mathbb{Q}/\mathbb{Z}(2)),$$

see A.3.12. For $\tau = \acute{e}t$, this spectral sequence yields the exact sequence

$$\mathrm{H}^0_{\mathrm{Zar}}(\overline{S}, \mathcal{H}^3_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(2))) \to \mathrm{H}^2_{\mathrm{Zar}}(\overline{S}, \mathcal{H}^2_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(2))) \to \mathrm{H}^4_{\acute{e}t}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2)).$$

The Zariski sheaves $\mathcal{H}^q_\tau(\mathbb{Q}/\mathbb{Z}(n))$ have Gersten resolutions of length q, thus in the Bloch-Ogus spectral sequence $E_2^{p,q} = 0$ for $p > q$. Furthermore, by construction of these Gersten resolutions, we have for a surface $E_2^{p,q} = 0$ for $p > 2$. The filtration induced by the Bloch-Ogus spectral sequence is the filtration by coniveau $\mathrm{N}^i\mathrm{H}^n_{\acute{e}t}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2))$. The first arrow in the above exact sequence is the differential $d_2^{0,3} : E_2^{0,3} \to E_2^{2,2}$ whose cokernel is $E_3^{2,2} = E_\infty^{2,2} = \mathrm{N}^2\mathrm{H}^4_{\acute{e}t}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2))$. Since $\overline{S}$ is surface over an algebraically closed field, the sheaf $\mathcal{H}^3_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(n)) = 0$, and we see from the short exact sequence above that

$$\mathrm{N}^2\mathrm{H}^4_{\acute{e}t}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2)) = \mathrm{H}^2_{\mathrm{Zar}}(\overline{S}, \mathcal{H}^2_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(2))) \subseteq \mathrm{H}^4_{\acute{e}t}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2)).$$

Let now $\tau = \mathrm{M}$. By [Kah12, Lemme 2.5] we have $\mathcal{H}^i_{\mathrm{M}}(\mathbb{Q}/\mathbb{Z}(n)) = 0$ for $i > n$; thus in the motivic Bloch-Ogus spectral sequence $E_2^{1,3} = 0 = E_2^{0,4}$, which shows that

$$\mathrm{N}^2\mathrm{H}^4_{\mathrm{M}}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2)) = \mathrm{H}^2_{\mathrm{Zar}}(\overline{S}, \mathcal{H}^2_{\mathrm{M}}(\mathbb{Q}/\mathbb{Z}(2))) \cong \mathrm{H}^4_{\mathrm{M}}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2)).$$

It follows from comparing the Gersten resolutions of the Zariski sheaves $\mathcal{H}^2_{\mathrm{M}}(\mathbb{Q}/\mathbb{Z}(2))$ and $\mathcal{H}^2_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(2))$, using the Merkurjev-Suslin theorem [MS83, 11.5], that these sheaves are isomorphic. In summary, we have a commutative diagram of the following form

$$\begin{array}{ccc} \mathrm{H}^2_{\mathrm{Zar}}(\overline{S}, \mathcal{H}^2_{\mathrm{M}}(\mathbb{Q}/\mathbb{Z}(2))) & \xrightarrow{\cong} & \mathrm{H}^4_{\mathrm{M}}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2)) \\ \downarrow \cong & & \downarrow \psi \\ \mathrm{H}^2_{\mathrm{Zar}}(\overline{S}, \mathcal{H}^2_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(2))) & \hookrightarrow & \mathrm{H}^4_{\acute{e}t}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

We denote $\mathrm{H}^4_{ét}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2))$ by H^4 and $\mathrm{N}^i\mathrm{H}^4_{ét}(\overline{S}, \mathbb{Q}/\mathbb{Z}(2))$ by $\mathrm{N}^i\mathrm{H}^4$ Then we have

$$\mathrm{Br}^2(\overline{S}) \cong \mathrm{coker}(\psi) \cong \mathrm{H}^4/\mathrm{N}^2\mathrm{H}^4,$$

which fits into the evident short exact sequence

$$0 \to \mathrm{gr}^1_{\mathrm{N}}\mathrm{H}^4 \to \mathrm{H}^4/\mathrm{N}^2\mathrm{H}^4 \to \mathrm{H}^4/\mathrm{N}^1\mathrm{H}^4. \tag{27}$$

From the properties of the Bloch-Ogus spectral sequence mentioned above, we see that $\mathrm{gr}^1_{\mathrm{N}}\mathrm{H}^4 = E_2^{1,3}$ and $\mathrm{H}^4/\mathrm{N}^1\mathrm{H}^4 = E_2^{0,4}$. Thus we may rewrite the exact sequence (27) as

$$0 \to \mathrm{H}^1_{\mathrm{Zar}}(\overline{S}, \mathcal{H}^3_{ét}(\mathbb{Q}/\mathbb{Z}(2))) \to \mathrm{Br}^2(\overline{S}) \to \mathrm{H}^0_{\mathrm{Zar}}(\overline{S}, \mathcal{H}^4_{ét}(\mathbb{Q}/\mathbb{Z}(2))).$$

Since on $\overline{S}$ the sheaves $\mathcal{H}^q_{ét}(\mathbb{Q}/\mathbb{Z}(2))$ vanish for $q = 3, 4$, this shows that $\mathrm{Br}^2(\overline{S}) = 0$. □

By Lemma 4.5.1 and Lemma 4.5.2 Theorem 1.0.2 holds in case $\dim(X) = 1, 2$. In what follows we will assume $3 \leq \dim(X) \leq 4$. We show next that it is enough to show that $\mathrm{coker}(\alpha)$ has finite exponent.

4.5.3 Lemma
Let X be a smooth, projective, geometrically irreducible variety over a field of characteristic 0. If the cokernel of the map $\alpha : \mathrm{Br}^2(X) \to \mathrm{Br}^2(\overline{X})^{G_k}$ has finite exponent, it is finite.

Proof. Let ℓ be a prime number. Consider the following commutative diagram, which is obtained by comparing the universal coefficient short exact sequences in Lichtenbaum and ℓ-adic étale cohomology along the cycle maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CH}^2_{\mathrm{L}}(\overline{X}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & \mathrm{H}^4_{\mathrm{L}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) & \longrightarrow & \mathrm{Br}^2(\overline{X})\{\ell\} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow & & \\ 0 & \to & \mathrm{H}^4_{ét}(\overline{X}, \mathbb{Z}_\ell(2)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \to & \mathrm{H}^4_{ét}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) & \to & \mathrm{H}^5_{ét}(\overline{X}, \mathbb{Z}_\ell(2))\{\ell\} & \to & 0 \end{array} \tag{28}$$

Applying the snake lemma to the above diagram (28), and using that $\mathrm{H}^4_{ét}(\overline{X}, \mathbb{Z}_\ell(2))$ is a finitely generated $\mathbb{Z}_\ell$-module, it follows that there is a short exact sequence

$$0 \to (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r \to \mathrm{Br}^2(\overline{X})\{\ell\} \to \mathrm{H}^5_{ét}(\overline{X}, \mathbb{Z}_\ell(2))\{\ell\} \to 0,$$

where r is a non-negative integer and the group $\mathrm{H}^5_{ét}(\overline{X}, \mathbb{Z}_\ell(2))\{\ell\}$ is finite. Hence for each subquotient Q of $\mathrm{Br}^2(\overline{X})$ and every prime power ℓ^n the torsion subgroup $Q[\ell^n]$ is finite. In particular, if Q has finite exponent, then it is finite. □

Let X be as in Theorem 1.0.2. By our assumptions we know from Lemma 4.4.2 that

$$\mathrm{Br}^2(X) \xrightarrow{\alpha} \mathrm{Br}^2(\overline{X})^{G_k} \xrightarrow{\beta} \mathrm{H}^2\left(G_k, \mathrm{CH}^2_{\mathrm{L}}(\overline{X})\right).$$

is exact. In particular we have $\operatorname{coker}(\alpha) = \operatorname{im}(\beta)$.

If S is a smooth projective k-surface and $f : S \to X$ is a morphism, we obtain from the functoriality of the Hochschild-Serre spectral sequence the commutative diagram

$$\begin{array}{ccc} \mathrm{Br}^2(\overline{X})^{G_k} & \xrightarrow{\beta = \beta_{\overline{X}}} & \mathrm{H}^2(G_k, \mathrm{CH}^2_{\mathrm{L}}(\overline{X})) \\ \downarrow & & \downarrow \gamma \\ \mathrm{Br}^2(\overline{S})^{G_k} & \xrightarrow{\beta_{\overline{S}}} & \mathrm{H}^2(G_k, \mathrm{CH}^2_{\mathrm{L}}(\overline{S})) \end{array}$$

By Lemma 4.5.2 the group $\mathrm{Br}^2(\overline{S})^{G_k}$ vanishes and the above diagram shows that

$$\operatorname{coker}(\alpha) = \operatorname{im}(\beta) \subseteq \ker(\gamma) \tag{29}$$

for every such $f : S \to X$. Thus it suffices to show that $\ker(\gamma)$ has finite exponent.

To understand the group $\mathrm{H}^2(G_k, \mathrm{CH}^2_{\mathrm{L}}(\overline{X}))$, we need to study the Lichtenbaum-Chow group $\mathrm{CH}^2_{\mathrm{L}}(\overline{X})$ in more detail. First, define $\mathrm{CH}^n_{\mathrm{L}}(\overline{X})_{\mathrm{hom}}$ as the kernel of the cycle maps

$$\mathrm{CH}^n_{\mathrm{L}}(\overline{X}) \to \oplus_\ell \mathrm{H}^{2n}_{\acute{e}t}(\overline{X}, \mathbb{Z}_\ell(n)).$$

Set $\mathrm{NS}^n_{\mathrm{L}}(\overline{X}) = \mathrm{CH}^n_{\mathrm{L}}(\overline{X}) / \mathrm{CH}^n_{\mathrm{L}}(\overline{X})_{\mathrm{hom}}$.

4.5.4 Lemma
Let X be a smooth, projective variety over a field of characteristic 0 such that $H^3_{\acute{e}t}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)) = 0$. Then $\mathrm{NS}^2_{\mathrm{L}}(\overline{X})$ is a finitely generated free abelian group.

Proof. We show this by comparing with the associated complex variety and singular cohomology. Let $\mathrm{CH}^2_{\mathrm{L}}(X_{\mathbb{C}})'_{\mathrm{hom}} = \ker\left(\mathrm{CH}^2_{\mathrm{L}}(X_{\mathbb{C}}) \to \mathrm{H}^4_{\mathrm{B}}(X_{\mathbb{C}}, \mathbb{Z}(2))\right)$ and set $\mathrm{NS}^2_{\mathrm{L}}(X_{\mathbb{C}})' = \mathrm{CH}^2_{\mathrm{L}}(X_{\mathbb{C}}) / \mathrm{CH}^2_{\mathrm{L}}(X_{\mathbb{C}})'_{\mathrm{hom}} \subseteq \mathrm{H}^4_{\mathrm{B}}(X_{\mathbb{C}}, \mathbb{Z}(2))$. Given our assumption, $\mathrm{NS}^2_{\mathrm{L}}(X_{\mathbb{C}})'$ is a finitely generated free abelian group. Consider the following commutative diagram (where K is defined as the kernel)

$$\begin{array}{ccccccc} 0 \to & \mathrm{CH}^2_{\mathrm{L}}(\overline{X})_{\mathrm{hom}} & \longrightarrow & \mathrm{CH}^2_{\mathrm{L}}(\overline{X}) & \longrightarrow & \mathrm{H}^4_{\acute{e}t}(\overline{X}, \mathbb{Z}_\ell(2)) \\ & \downarrow & & \downarrow & & \downarrow \cong \\ 0 \longrightarrow & K & \longrightarrow & \mathrm{CH}^2_{\mathrm{L}}(X_{\mathbb{C}}) & \to & \mathrm{H}^4_{\mathrm{B}}(X_{\mathbb{C}}, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell \end{array}$$

and where the right vertical isomorphism comes from the usual comparison theorems [Tam94, (11.1.1)]. Since the lower right horizontal map in this diagram is the composite

$$\mathrm{CH}^2_{\mathrm{L}}(X_{\mathbb{C}}) \to \mathrm{H}^4_{\mathrm{B}}(X_{\mathbb{C}}, \mathbb{Z}(2)) \hookrightarrow \mathrm{H}^4_{\mathrm{B}}(X_{\mathbb{C}}, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell,$$

it follows that we have an isomorphism $\mathrm{CH}^2_{\mathrm{L}}(X_{\mathbb{C}})'_{\mathrm{hom}} \cong K$. Hence there is a map $\mathrm{CH}^2_{\mathrm{L}}(\overline{X})_{\mathrm{hom}} \to \mathrm{CH}^2_{\mathrm{L}}(X_{\mathbb{C}})'_{\mathrm{hom}}$, and an induced map $\mathrm{NS}^2_{\mathrm{L}}(\overline{X}) \to \mathrm{NS}^2_{\mathrm{L}}(X_{\mathbb{C}})'$ such that

$$\begin{array}{ccc} \mathrm{NS}^2_{\mathrm{L}}(\overline{X}) \otimes \mathbb{Z}_\ell & \longrightarrow & \mathrm{NS}^2_{\mathrm{L}}(X_{\mathbb{C}})' \otimes \mathbb{Z}_\ell \\ \big\downarrow & & \big\downarrow \\ \mathrm{H}^4_{\acute{e}t}(\overline{X}, \mathbb{Z}_\ell(2)) & \xrightarrow{\cong} & \mathrm{H}^4_{\mathrm{B}}(X_{\mathbb{C}}, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell \end{array}$$

commutes. In particular $\mathrm{NS}^2_{\mathrm{L}}(\overline{X}) \otimes \mathbb{Z}_\ell \to \mathrm{NS}^2_{\mathrm{L}}(X_{\mathbb{C}})' \otimes \mathbb{Z}_\ell$ is injective; since by our assumption $\mathrm{NS}^2_{\mathrm{L}}(\overline{X})$ is torsion free, this proves our claim. ☐

4.5.5 Lemma
Let X be a smooth, projective variety over a field of characteristic 0 such that $H^3_{\acute{e}t}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)) = 0$. Then the abelian group $\mathrm{CH}^2_{\mathrm{L}}(\overline{X})_{\mathrm{hom}}$ is a rational vector space.

Proof. Since there is an isomorphism $\mathrm{CH}^2_{\mathrm{L}}(\overline{X})_{\mathrm{tors}} \cong \mathrm{H}^3_{\acute{e}t}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2))$ (see [RS16b, Remarks 3.2]), we know that $\mathrm{CH}^2_{\mathrm{L}}(\overline{X})_{\mathrm{hom}}$ is torsion free. We show that $\mathrm{CH}^2_{\mathrm{L}}(\overline{X})_{\mathrm{hom}}/\ell^n = 0$ for each prime ℓ and each n; taking the limit over all such powers and the direct sum over all ℓ we see that $\mathrm{CH}^2_{\mathrm{L}}(\overline{X})_{\mathrm{hom}} \to \mathrm{CH}^2_{\mathrm{L}}(\overline{X})_{\mathrm{hom}} \otimes \mathbb{Q}$ is surjective, which implies our claim. By Lemma 4.5.4 $\mathrm{NS}^2_{\mathrm{L}}(\overline{X})$ is a finitely generated free abelian group. Thus, comparing the exact sequences $0 \to \mathrm{CH}^2_{\mathrm{L}}(\overline{X})_{\mathrm{hom}} \to \mathrm{CH}^2_{\mathrm{L}}(\overline{X}) \to \mathrm{NS}^2_{\mathrm{L}}(\overline{X}) \to 0$ along the map multiplication by ℓ^n, it follows from the snake lemma that there is an exact sequence of the form

$$0 \to \mathrm{CH}^2_{\mathrm{L}}(\overline{X})_{\mathrm{hom}}/\ell^n \to \mathrm{CH}^2_{\mathrm{L}}(\overline{X})/\ell^n \to \mathrm{NS}^2_{\mathrm{L}}(\overline{X})/\ell^n \to 0.$$

Hence we need to show that $\mathrm{CH}^2_{\mathrm{L}}(\overline{X})/\ell^n \to \mathrm{NS}^2_{\mathrm{L}}(\overline{X})/\ell^n$ is injective. But the composite

$$\mathrm{CH}^2_{\mathrm{L}}(\overline{X})/\ell^n \to \mathrm{NS}^2_{\mathrm{L}}(\overline{X})/\ell^n \to \mathrm{H}^4_{\acute{e}t}(\overline{X}, \mathbb{Z}_\ell(2))/\ell^n \to \mathrm{H}^4_{\acute{e}t}(\overline{X}, \mu_{\ell^n}^{\otimes 2})$$

coincides with

$$\mathrm{CH}^2_{\mathrm{L}}(\overline{X})/\ell^n \to \mathrm{H}^4_{\mathrm{L}}(\overline{X}, \mathbb{Z}/\ell^n(2)) \xrightarrow{\cong} \mathrm{H}^4_{\acute{e}t}(\overline{X}, \mu_{\ell^n}^{\otimes 2}),$$

where the first map is the injective map from the universal coefficient theorem. This proves our claim. ☐

Since the Galois cohomology of uniquely divisible Galois modules vanishes in positive degrees [NSW13, Proposition (1.6.2)], it follows from Lemma 4.5.5 that for X as in Theorem 1.0.2 we have an isomorphism $\mathrm{H}^2(G_k, \mathrm{CH}^2_{\mathrm{L}}(\overline{X})) \cong \mathrm{H}^2(G_k, \mathrm{NS}^2_{\mathrm{L}}(\overline{X}))$. In particular, for a smooth projective surface $f : S \to X$ as before, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2(G_k, \mathrm{CH}^2_{\mathrm{L}}(\overline{X})) & \xrightarrow{\cong} & \mathrm{H}^2(G_k, \mathrm{NS}^2_{\mathrm{L}}(\overline{X})) \\ \downarrow \gamma & & \overline{\gamma} \downarrow \\ \mathrm{H}^2(G_k, \mathrm{CH}^2_{\mathrm{L}}(\overline{S})) & \longrightarrow & \mathrm{H}^2(G_k, \mathrm{NS}^2_{\mathrm{L}}(\overline{S})) \end{array}$$

Hence $\ker(\gamma) \subseteq \ker(\overline{\gamma})$ and it suffices to show that $\ker(\overline{\gamma})$ has finite exponent.

To understand the kernel of $\overline{\gamma}$, we need to understand the groups $\mathrm{NS}^2_{\mathrm{L}}(\overline{X})$ and $\mathrm{NS}^2_{\mathrm{L}}(\overline{S})$. We first prove a general result about the comparison of Chow groups and Lichtenbaum-Chow groups in codimension $d = \dim(X)$.

4.5.6 Lemma
Let X be a smooth, projective variety of dimension d over a field of characteristic 0. Then $\mathrm{CH}^d(\overline{X}) \cong \mathrm{CH}^d_{\mathrm{L}}(\overline{X})$.

Proof. Let $\pi : \overline{X}_{\acute{e}t} \to \overline{X}_{\mathrm{Zar}}$ be the canonical morphism. By [SV00], [Voe03, 6.6] and [GL01] there is an isomorphism $\mathbb{Z}(d) \simeq \tau_{\leq d+1}\mathrm{R}\pi_*\mathbb{Z}(d)_{\acute{e}t}$ and a distinguished triangle

$$\mathbb{Z}(d) \to \mathrm{R}\pi_*\mathbb{Z}(d)_{\acute{e}t} \to \tau_{\geq d+2}\mathrm{R}\pi_*\mathbb{Z}(d)_{\acute{e}t} \xrightarrow{[+1]} .$$

In particular there is an exact sequence

$$\mathbb{H}^{2d-1}(\overline{X}, \tau_{\geq d+2}\mathrm{R}\pi_*\mathbb{Z}(d)_{\acute{e}t}) \to \mathrm{CH}^d(\overline{X}) \to \mathrm{CH}^d_{\mathrm{L}}(\overline{X}) \to \mathbb{H}^{2d}(\overline{X}, \tau_{\geq d+2}\mathrm{R}\pi_*\mathbb{Z}(d)_{\acute{e}t}). \quad (30)$$

To compute the terms on the left and the right side we use the hypercohomology spectral sequence

$$\mathrm{H}^p(\overline{X}, R^q\tau_{\geq d+2}\mathrm{R}\pi_*\mathbb{Z}(d)_{\acute{e}t}) \Rightarrow \mathbb{H}^{p+q}(\overline{X}, \tau_{\geq d+2}\mathrm{R}\pi_*\mathbb{Z}(d)_{\acute{e}t}).$$

According to [RS16a, (4.4)] we can identify the coefficient sheaves as follows

$$R^q\tau_{\geq d+2}\mathrm{R}\pi_*\mathbb{Z}(d)_{\acute{e}t} = \begin{cases} 0, & \text{if } q \leq d+1, \\ \mathcal{H}^{q-1}_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(d)), & \text{if } q \geq d+2. \end{cases} \quad (31)$$

In our case of interest, we need to consider the case $q \leq 2d-1$ for the term on the left side in (30) and the case $q \leq 2d$ for the term on the right side. For $q \leq d+1$ we have $\mathrm{H}^p(\overline{X}, R^q\tau_{\geq d+2}\mathrm{R}\pi_*\mathbb{Z}(d)_{\acute{e}t}) = 0$ by (31); for $q \geq d+2$ we have $R^q\tau_{\geq d+2}\mathrm{R}\pi_*\mathbb{Z}(d)_{\acute{e}t} = \mathcal{H}^{q-1}_{\acute{e}t}(\mathbb{Q}/\mathbb{Z}(d))$, but these terms vanish, because $q-1 > d$ and because we are working over an algebraically closed field $\overline{k}$. In particular $\mathrm{H}^p(\overline{X}, R^q\tau_{\geq d+2}\mathrm{R}\pi_*\mathbb{Z}(d)_{\acute{e}t}) = 0$ in both cases, and thus $\mathrm{CH}^d(\overline{X}) \cong \mathrm{CH}^d_{\mathrm{L}}(\overline{X})$. □

4.5.7 Lemma
Let S be a smooth projective surface over a field k of characteristic zero. Then $\mathrm{NS}^2_{\mathrm{L}}(\overline{S}) \cong \mathbb{Z}$.

Proof. As for the Lichtenbaum-Chow group, we denote for the Chow group the quotient $\mathrm{CH}^2(\overline{S}) / \mathrm{CH}^2(\overline{S})_{\mathrm{hom}}$ by $\mathrm{NS}^2(\overline{S})$. By Lemma 4.5.6 we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CH}^2(\overline{S})_{\mathrm{hom}} & \longrightarrow & \mathrm{CH}^2(\overline{S}) & \longrightarrow & \mathrm{NS}^2(\overline{S}) & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{CH}^2_{\mathrm{L}}(\overline{S})_{\mathrm{hom}} & \longrightarrow & \mathrm{CH}^2_{\mathrm{L}}(\overline{S}) & \longrightarrow & \mathrm{NS}^2_{\mathrm{L}}(\overline{S}) & \longrightarrow & 0 \end{array}$$

which implies $\mathrm{NS}^2_{\mathrm{L}}(\overline{S}) \cong \mathrm{NS}^2(\overline{S})$. From the degree map we get the exact sequence

$$0 \to \mathrm{CH}^2(\overline{S})_{\mathrm{num}} \to \mathrm{CH}^2(\overline{S}) \to \mathbb{Z} \to 0.$$

Since on the surface $\overline{S}$ homological and numerical equivalence agree [Lie68], we have $\mathrm{NS}^2(\overline{S}) \cong \mathbb{Z}$, which completes the proof. □

Recall that for the Chow groups the composition of the intersection product with the degree map defines a pairing

$$\mathrm{CH}^2(X) \times \mathrm{CH}^{d-2}(X) \to \mathrm{CH}^d(X) \xrightarrow{\deg} \mathbb{Z}$$

whose left and right kernels define the subgroups of cycles numerically equivalent to zero of the respective Chow groups. Since X is of dimension ≤ 4, it follows by the work of Lieberman [Lie68] that homological and numerical equivalence on X agree. In particular we have

$$\mathrm{NS}^2(\overline{X}) = \mathrm{CH}^2(\overline{X}) / \mathrm{CH}^2(\overline{X})_{\mathrm{hom}} = \mathrm{CH}^2(\overline{X}) / \mathrm{CH}^2(\overline{X})_{\mathrm{num}},$$

where the quotient on the right side is a finitely generated free abelian group [Kle68, Theorem 3.5]; a similar argument applies to $\mathrm{NS}^{d-2}(\overline{X})$. In particular, the intersection product composed with the degree map defines a non-degenerate bilinear pairing

$$\mathrm{NS}^2(\overline{X}) \times \mathrm{NS}^{d-2}(\overline{X}) \to \mathrm{CH}^d(\overline{X}) \to \mathbb{Z}. \tag{32}$$

We need a variant of this pairing for Lichtenbaum-Chow groups. From Lemma 4.5.6 we know that $\mathrm{CH}^d(\overline{X}) \cong \mathrm{CH}^d_{\mathrm{L}}(\overline{X})$, in particular, we have a degree map in this setting. This map, together with the cup product defines for $0 \leq n \leq d = \dim(X)$ a pairing

$$\mathrm{H}^{2n}_{\mathrm{L}}(\overline{X}, \mathbb{Z}(n)) \times \mathrm{H}^{2d-2n}_{\mathrm{L}}(\overline{X}, \mathbb{Z}(d-n)) \to \mathrm{H}^{2d}_{\mathrm{L}}(\overline{X}, \mathbb{Z}(d)) \xrightarrow{\deg} \mathbb{Z}.$$

whose left and right kernels we refer to as the subgroups of elements numerically equivalent to zero $\mathrm{H}^{2n}_{\mathrm{L}}(\overline{X}, \mathbb{Z}(n))_{\mathrm{num}}$ and $\mathrm{H}^{2d-2n}_{\mathrm{L}}(\overline{X}, \mathbb{Z}(d-n))_{\mathrm{num}}$ respectively. Since we are assuming $d = 3, 4$, we have for $n = 2$ then $d - 2 = 1, 2$, which implies that

$$\mathrm{CH}^2_{\mathrm{L}}(\overline{X}) \times \mathrm{CH}^{d-2}_{\mathrm{L}}(\overline{X}) \to \mathrm{CH}^d_{\mathrm{L}}(\overline{X}) \to \mathbb{Z},$$

factors through

$$\mathrm{NS}^2_{\mathrm{L}}(\overline{X}) / \mathrm{NS}^2_{\mathrm{L}}(\overline{X})_{\mathrm{tors}} \times \mathrm{NS}^{d-2}_{\mathrm{L}}(\overline{X}) / \mathrm{NS}^{d-2}_{\mathrm{L}}(\overline{X})_{\mathrm{tors}} \to \mathbb{Z}. \tag{33}$$

Furthermore we know that $\mathrm{NS}^2(\overline{X})$ and $\mathrm{NS}^2_{\mathrm{L}}(\overline{X})$ have the same $\mathbb{Z}$-rank (cf. [RS16b, section 5]); this together with Lemma 4.5.4 implies $\mathrm{NS}^2(\overline{X}) \cong \mathrm{NS}^2_{\mathrm{L}}(\overline{X})$. Since always $\mathrm{NS}^1(\overline{X}) = \mathrm{NS}^1_{\mathrm{L}}(\overline{X})$, this implies that the two pairings (32) and (33) in fact agree; in particular both of these pairings are non-degenerate.

We are ready to prove Theorem 1.0.2:

Proof of Theorem 1.0.2. Let m be the rank of $\mathrm{NS}^2_{\mathrm{L}}(\overline{X})$. Choose m irreducible projective surfaces $\overline{S}_i \to \overline{X}$, whose classes in $\mathrm{NS}^{d-2}(\overline{X})$ are such that the intersection pairing with theses classes defines an injective group homomorphism $\mathrm{NS}^2(\overline{X}) \hookrightarrow \mathbb{Z}^m$. Resolving singularities, we might assume that these surfaces $\overline{S}_i$ are smooth. There is a finite field extension k' of k over which these surfaces are defined. It follows from an assertion at the level of cycle complexes [Blo86, Corollary 1.4] that there are restriction and corestriction homomorphisms

$$\mathrm{res}_{k'/k} : \mathrm{Br}^2(X_k) \to \mathrm{Br}^2(X_{k'}) \qquad \text{and} \qquad \mathrm{cores}_{k'/k} : \mathrm{Br}^2(X_{k'}) \to \mathrm{Br}^2(X_k),$$

such that $(\mathrm{cores}_{k'/k} \circ \mathrm{res}_{k'/k})$ is multiplication by $[k' : k]$. These homomorphisms fit into the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Br}^2(X_k) & \xrightarrow{\mathrm{res}_{k'/k}} & \mathrm{Br}^2(X_{k'}) & \xrightarrow{\mathrm{cores}_{k'/k}} & \mathrm{Br}^2(X_k) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_{k'}} & & \downarrow{\scriptstyle \alpha} \\
\mathrm{Br}^2(\overline{X})^{G_k} & \longrightarrow & \mathrm{Br}^2(\overline{X})^{G_{k'}} & \longrightarrow & \mathrm{Br}^2(\overline{X})^{G_k}
\end{array}$$

where $G_{k'}$ is the absolute Galois group of k'. In particular, to show that $\mathrm{coker}(\alpha)$ has finite exponent, we may replace k by a finite extension k'/k.

Because of (29) and the discussion before Lemma 4.5.6, it is sufficient to show that the kernel of the map $\mathrm{H}^2(G_k, \mathrm{NS}^2_{\mathrm{L}}(\overline{X})) \to \mathrm{H}^2(G_k, \oplus_{i=1}^m \mathrm{NS}^2_{\mathrm{L}}(\overline{S_i}))$ has finite exponent.

The pullbacks along the morphisms $S_i \to X$ define a map $\mathrm{NS}^2(\overline{X}) \hookrightarrow \oplus_{i=1}^m \mathrm{NS}^2(\overline{S_i}) = \mathbb{Z}^m$, which can be extended to a map $\mathrm{NS}^2(\overline{X}) \to \oplus_{i=1}^m \mathrm{NS}^2(\overline{S_i}) \to \mathrm{NS}^2(\overline{X})$ such that the composition is multiplication by a positive integer r. In the commutative diagram

$$\begin{array}{ccccc} \mathrm{NS}^2(\overline{X}) & \longrightarrow & \oplus_{i=1}^m \mathrm{NS}^2(\overline{S_i}) & \longrightarrow & \mathrm{NS}^2(\overline{X}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{NS}^2_{\mathrm{L}}(\overline{X}) & \longrightarrow & \oplus_{i=1}^m \mathrm{NS}^2_{\mathrm{L}}(\overline{S_i}) & \dashrightarrow & \mathrm{NS}^2_{\mathrm{L}}(\overline{X}) \end{array}$$

all vertical maps are isomorphisms. Therefore the composition in the lower row $\mathrm{NS}^2_{\mathrm{L}}(\overline{X}) \to \oplus_{i=1}^m \mathrm{NS}^2_{\mathrm{L}}(\overline{S_i}) \to \mathrm{NS}^2_{\mathrm{L}}(\overline{X})$ is also multiplication by r. In particular, the kernel of the map

$$\mathrm{H}^2(G_k, \mathrm{NS}^2_{\mathrm{L}}(\overline{X})) \to \mathrm{H}^2(G_k, \oplus_{i=1}^m \mathrm{NS}^2_{\mathrm{L}}(\overline{S_i}))$$

is contained in the r-torsion subgroup and therefore has finite exponent. □

A. Appendix

A.1. Derived categories

In this section we briefly sketch the construction of derived categories. Due to space limitations we can only give an overview of the methods and must omit most of the proofs; carefully and exhaustively written introductions to this topic can be found in [GM03, III.] and [Wei95, chapter 10], which we follow in our discussion. In the following $\mathcal{A}$ will always be an abelian category.

We denote by $\mathbf{Ch}(\mathcal{A})$ the category of cochain complexes of objects of $\mathcal{A}$, i.e. the objects of $\mathbf{Ch}(\mathcal{A})$ are complexes $C^\bullet := \ldots \to C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \to \ldots$ with differential $d = (d^i)_{i\in\mathbb{Z}}$ such that $d^i d^{i-1} = 0$ for each $i \in \mathbb{Z}$ and the morphisms in $\mathbf{Ch}(\mathcal{A})$ are cochain maps $u : C^\bullet \to D^\bullet$ with $u_i d_C^{i-1} = d_D^{i-1} u_{i-1}$ for each $i \in \mathbb{Z}$; this is again an abelian category [Wei95, Theorem 1.2.3]. A complex $C^\bullet$ in $\mathbf{Ch}(\mathcal{A})$ is said to bounded above or bounded on the right, if there is an integer a such that $C^n = 0$ for $n \geq a$, and it is bounded below or bounded on the left, if there is an integer b such that $C^n = 0$ if $n \leq b$; $C^\bullet$ is bounded, if it is bounded below and bounded above. The categories $\mathbf{Ch}^b(\mathcal{A})$, $\mathbf{Ch}^-(\mathcal{A})$ and $\mathbf{Ch}^+(\mathcal{A})$ are the subcategories of $\mathbf{Ch}(\mathcal{A})$ whose elements are the bounded resp. bounded above resp. bounded below complexes.

There is an important operation on cochain complexes, called shifting or translating: Let $C^\bullet$ be a complex in $\mathbf{Ch}(\mathcal{A})$ with differential map d. For $n \in \mathbb{Z}$ we define the shifted complex $C^\bullet[n]$ to be the complex given by $C^i[n] = C^{i+n}$ in degree i with differential map $(-1)^n d$. If $f : C^\bullet \to D^\bullet$ is a cochain map, then the cone of f is the complex given by $\operatorname{cone}(f)^i = C^\bullet[1]^i \oplus D^i$ with differential $d^i_{\operatorname{cone}(f)}(a,b) = (-d_C(a), f(a) + d_D(b))$ in each degree i. Completely analogous we define the cylinder of f to be the complex $\operatorname{cyl}(f) = C^\bullet \oplus C^\bullet[1] \oplus D^\bullet$ with differential $d^i_{\operatorname{cyl}(f)}(a,b,c) = (d_C(a) - b, -d_C(b), f(b) + d_D(c))$ in degree i.

A morphism $u : C^\bullet \to D^\bullet$ in $\mathbf{Ch}(\mathcal{A})$ is called null homotopic, if there are maps $s_i : C^i \to D^{i-1}$ such that $u_i = s_i d_D^{i-1} + d_C^i s_{i+1}$ for each $i \in \mathbb{Z}$; furthermore cochain maps f and g are called homotopic, if $f - g$ is null homotopic, and $f : C^\bullet \to D^\bullet$ is a cochain homotopy equivalence, if there is a cochain map $g : D^\bullet \to C^\bullet$ such that gf and fg are homotopic to the identity maps on $C^\bullet$ resp. $D^\bullet$. This defines an equivalence relation on the set of cochain maps $C^\bullet \to D^\bullet$.

Using these terms, the homotopy category $\mathbf{K}(\mathcal{A})$ of $\mathcal{A}$ is constructed by defining the objects of $\mathbf{K}(\mathcal{A})$ to be the objects of $\mathbf{Ch}(\mathcal{A})$ and the morphisms of $\mathbf{K}(\mathcal{A})$ to be the sets of equivalence classes of cochain maps $C^\bullet \to D^\bullet$ for objects $C^\bullet, D^\bullet$ of $\mathbf{K}(\mathcal{A})$. The category $\mathbf{K}(\mathcal{A})$ is an additive category and the evident quotient map $\mathbf{Ch}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ is an additive functor. Analogous to the category $\mathbf{Ch}(\mathcal{A})$ we can define the subcategories $\mathbf{K}^b(\mathcal{A})$, $\mathbf{K}^-(\mathcal{A})$ and $\mathbf{K}^+(\mathcal{A})$ of $\mathbf{K}(\mathcal{A})$ to be the categories with objects the bounded, bounded above or bounded below complexes.

Let $C^\bullet$ be a complex in $\mathbf{K}(\mathcal{A})$ with differential d. Of course we can take cohomology of $C^\bullet$ in degree i, i.e. we set $\mathrm{H}^i(C^\bullet) = \ker(d^i)/\operatorname{im}(d^{i-1})$, and this is a well defined functor from $\mathbf{K}(\mathcal{A}) \to \mathcal{A}$. We say that a morphism $f : C^\bullet \to D^\bullet$ is a quasi-isomorphism, if f induces an isomorphism on cohomology groups, i.e. $\mathrm{H}^i(f) : \mathrm{H}^i(C^\bullet) \to \mathrm{H}^i(D^\bullet)$ is an isomorphism for each $i \in \mathbb{Z}$.

A.1.1 Definition ([Wei95, Definition 10.3.1])
Let $\mathcal{C}$ be a category and S be a class of morphisms in $\mathcal{C}$. A localization of $\mathcal{C}$ with respect to S is a category $S^{-1}\mathcal{C}$ together with a functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ such that

a) $q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$ for every s in S.

b) any functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism in $\mathcal{D}$ for every s in S factors uniquely through q.

A.1.2 Example
Let S be the class of homotopy equivalence classes in $\mathbf{Ch}(\mathcal{A})$. Then $\mathbf{K}(\mathcal{A})$ is the localization $S^{-1}\mathbf{Ch}(\mathcal{A})$ (cf. [Wei95, Proposition 10.1.2]).

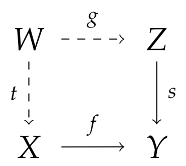
A.1.3 Definition
The derived category $\mathbf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the localization $Q^{-1}\mathbf{K}(\mathcal{A})$ of the homotopy category $\mathbf{K}(\mathcal{A})$ of $\mathcal{A}$ with respect to the class Q of quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$.

A.1.4 Remark
The construction of $\mathbf{D}(\mathcal{A})$ of the derived category of $\mathcal{A}$ given in Definition A.1.3 implies that $\mathbf{D}(\mathcal{A})$ is unique up to equivalence of categories, but it does not ensure that it exists at all! Therefore we will give a description of the derived category, at least for the categories used in this thesis.

A.1.5 Definition
A class S of morphisms in a category $\mathcal{C}$ is called localizing, if it satisfies the following axioms:

1. S is closed under compositions and all identity morphisms of objects in $\mathcal{C}$ are in S.

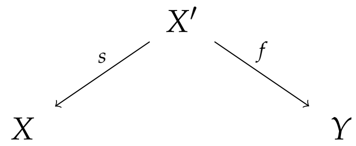
2. For any morphism f in $\mathcal{C}$ and any s in S, there exist a morphism g in $\mathcal{C}$ and a t in S such that the diagram



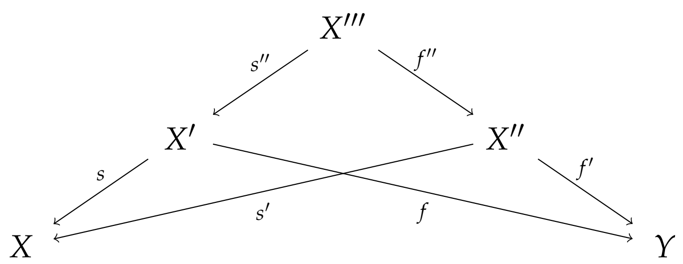
is commutative. Furthermore the symmetric statement, where the directions of the arrows are reversed, is valid.

3. Let f, g be two morphisms between objects X, Y in $\mathcal{C}$. Then the existence of a morphism s in S with $sf = sg$ is equivalent to the existence of a morphism t in S with $ft = gt$.

A.1.6 Remark
Let S be a localizing class in a category $\mathcal{C}$ and consider a 'roof'

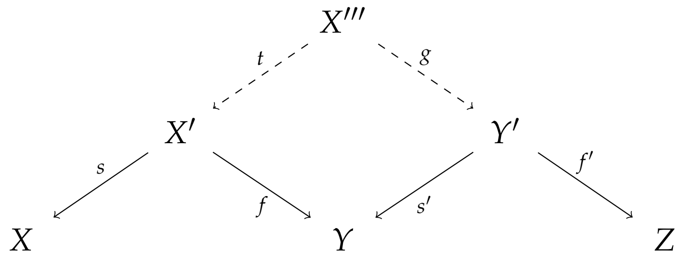


with s in S and f a morphism in $\mathcal{C}$, which we also denote by $X \overset{s}{\leftarrow} X' \overset{f}{\rightarrow} Y$. We say that the roofs $X \overset{s}{\leftarrow} X' \overset{f}{\rightarrow} Y$ and $X \overset{s'}{\leftarrow} X'' \overset{f'}{\rightarrow} Y$ are equivalent if and only if there is a roof $X' \overset{s''}{\longleftarrow} X''' \overset{f''}{\longrightarrow} X''$ such that the following diagram commutes:



This defines an equivalence relation on the class of such roofs.

Now let $X \overset{s}{\leftarrow} X' \overset{f}{\rightarrow} Y$ and $Y \overset{s'}{\leftarrow} Y' \overset{f'}{\rightarrow} Z$ be roofs as defined above. By the second axiom of localizing classes there exist a t in S and a morphism g in $\mathcal{C}$, fitting into the following commutative diagram:



A.1.7 Lemma
Let $\mathcal{C}$ be a category and let S be a localizing class of morphisms in $\mathcal{C}$. Then the localization $S^{-1}\mathcal{C}$ can be described as follows:

a) $\mathrm{Ob}(S^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.

b) A morphism $X \to Y$ in $S^{-1}\mathcal{C}$ is the equivalence class of roofs $X \xleftarrow{s} X' \xrightarrow{f} Y$ as defined in Remark A.1.6. The identity morphism $\mathrm{id} : X \to X$ *is the class of the roof* $X \xleftarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X$.

c) The composition of morphisms $X \xleftarrow{s} X' \xrightarrow{f} Y$ and $Y \xleftarrow{s'} Y' \xrightarrow{f'} Z$ is the equivalence class of the roof $X \xleftarrow{st} X'' \xrightarrow{f'g} Z$ as constructed in Remark A.1.6.

Proof. [GM03, III.2 8.] ☐

A.1.8 Proposition
Let R be a ring and let $\mathcal{A}$ be either the category of R-modules or the category of (pre-)sheaves of R-modules on a topological space X. Then the class of quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ is a localizing class.
Moreover the same assertion holds for the classes of quasi-isomorphisms in the categories $\mathbf{K}^b(\mathcal{A})$, $\mathbf{K}^-(\mathcal{A})$ and $\mathbf{K}^+(\mathcal{A})$.

In particular the derived categories of these categories exist.

Proof. Proofs can be found in [GM03, III.4 4.] or with a thorough discussion of the set theoretical problems in [Wei95, Proposition 10.4.4]. ☐

In what follows we restrict our attention to the abelian categories $\mathcal{A}$ mentioned in Proposition A.1.8.

Although we always start with an abelian category $\mathcal{A}$, the derived category $\mathbf{D}(\mathcal{A})$ will never be abelian. Hence we cannot define exact sequences as usual and therefore we have a priori no tools to deal with the homological properties of the initial category $\mathcal{A}$. This problem is solved by introducing so called distinguished triangles, which can be seen as an analogue of (short) exact sequences in abelian categories, since such distinguished triangles yield exact sequences in cohomology.

A.1.9 Lemma
Let $f : C^\bullet \to D^\bullet$ be a morphism of complexes in $\mathcal{A}$. There exists a commutative diagram in $\mathbf{Ch}(\mathcal{A})$ of the following form, which has exact rows and is functorial in f:

$$\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & D^\bullet & \xrightarrow{\overline{\pi}} & \operatorname{cone}(f) & \xrightarrow{\delta} & C^\bullet[1] & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \alpha} & & \| & & & & \\
0 & \longrightarrow & C^\bullet & \xrightarrow{\overline{f}} & \operatorname{cyl}(f) & \xrightarrow{\pi} & \operatorname{cone}(f) & \longrightarrow & 0 & & \\
 & & \| & & \downarrow{\scriptstyle \beta} & & & & & & \\
 & & C^\bullet & \xrightarrow{f} & D^\bullet & & & & & &
\end{array}$$

Moreover the maps α and β induce a canonical isomorphism between $D^\bullet$ and $\operatorname{cyl}(f)$ in the derived category $\mathbf{D}(\mathcal{A})$.

Proof. The maps π resp. $\overline{\pi}$ and δ are the canonical embeddings resp. projections. The map $\overline{f}$ is given degree-wise by $a \mapsto (a, 0, 0)$, α is the embedding $c \mapsto (0, 0, c)$ in each degree and β is given by $(a, b, c) \mapsto f(a) + c$, also in each degree. Then the commutativity of the diagram is clear and all maps commute with the corresponding differentials of the complexes. Moreover we have $\beta \circ \alpha = \mathrm{id}_D$ and the map $h^i : \operatorname{cyl}(f)^i \to \operatorname{cyl}(f)^{i-1}$, $(a, b, c) \mapsto (0, b, 0)$ yields a homotopy between $\alpha \circ \beta$ and $\mathrm{id}_{\operatorname{cyl}(f)}$. In particular $D^\bullet$ and $\operatorname{cyl}(f)$ are isomorphic in $\mathbf{D}(\mathcal{A})$. $\square$

A.1.10 Definition
Let $\mathcal{C}$ be a category of complexes, e.g. $\mathbf{Ch}(\mathcal{A})$ or $\mathbf{D}(\mathcal{A})$.

a) A triangle in $\mathcal{C}$ is a diagram of the form $C^\bullet \xrightarrow{f} D^\bullet \xrightarrow{g} E^\bullet \xrightarrow{h} C^\bullet[1]$.

b) A morphism of triangles in $\mathcal{C}$ is a commutative diagram

$$\begin{array}{ccccccc}
C^\bullet & \xrightarrow{f} & D^\bullet & \xrightarrow{g} & E^\bullet & \xrightarrow{h} & C^\bullet[1] \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\
\tilde{C}^\bullet & \xrightarrow{\tilde{f}} & \tilde{D}^\bullet & \xrightarrow{\tilde{g}} & \tilde{E}^\bullet & \xrightarrow{\tilde{h}} & \tilde{C}^\bullet[1]
\end{array}$$

It is called an isomorphism if u, v and w are isomorphisms in $\mathcal{C}$.

c) A triangle is distinguished if it is isomorphic to a triangle of the form

$$C^\bullet \xrightarrow{\overline{f}} \operatorname{cyl}(f) \xrightarrow{\pi} \operatorname{cone}(f) \xrightarrow{\delta} C^\bullet[1]$$

arising from a diagram as in Lemma A.1.9.

With respect to cohomological properties, distinguished triangles in $\mathbf{D}(\mathcal{A})$ have basically the same function as short exact sequences in $\mathcal{A}$; to be precise each distinguished triangle gives rise to a long exact sequence in cohomology:

A.1.11 Theorem
Let $C^\bullet \xrightarrow{f} D^\bullet \xrightarrow{g} E^\bullet \xrightarrow{h} C^\bullet[1]$ be a distinguished triangle in $\mathbf{D}(\mathcal{A})$. Then the following sequence is exact:

$$\ldots \to H^i(C^\bullet) \xrightarrow{H^i(f)} H^i(D^\bullet) \xrightarrow{H^i(g)} H^i(E^\bullet) \xrightarrow{H^i(h)} \underbrace{H^i(C^\bullet[1])}_{=H^{i+1}(C^\bullet)} \to \ldots$$

Proof. [GM03, III.4 6.] ☐

A.2. Hypercohomology

The definition of Lichtenbaum cohomology uses hypercohomology of a complex of sheaves. In this section we briefly recall the definition of hypercohomology via Cartan-Eilenberg resolutions.

In the following $\mathcal{A}$ denotes an abelian category with enough injectives.

Let $C^\bullet$ be a cohomological complex in $\mathcal{A}$, i.e. $C^\bullet$ has a differential d of degree $+1$. We denote by $B^\bullet(C^\bullet)$ the complex with $B^i(C^\bullet) = \operatorname{im}(d^{i-1})$; analogous we define the complexes $Z^\bullet(C^\bullet)$ and $H^\bullet(C^\bullet)$ to be the complexes with $Z^i(C^\bullet) = \ker(d^i)$ and $H^i(C^\bullet) = Z^i(C^\bullet)/B^i(C^\bullet)$ respectively. In classical homological algebra hypercohomology is defined using Cartan-Eilenberg resolutions, which we will describe in the following. These resolutions are double complexes in $\mathcal{A}$ and they can be seen as a generalisation of injective resolutions of a single object.

A double complex $M^{\bullet,\bullet}$ in $\mathcal{A}$ is a lattice of objects $\left(M^{i,j}\right)_{i,j\in\mathbb{Z}}$ with differentials $d_I^{i,j} : M^{i,j} \to M^{i+1,j}$ and $d_{II}^{i,j} : M^{i,j} \to M^{i,j+1}$, i.e. $d_I^2 = 0$ and $d_{II}^2 = 0$; moreover the differentials satisfy the equation $d_{II} \circ d_I + d_I \circ d_{II} = 0$. We denote by $M^{\bullet,j}$ the complex

$$\ldots \to M^{i-1,j} \xrightarrow{d_I^{i-1,j}} M^{i,j} \xrightarrow{d_I^{i,q}} M^{i+1,j} \to \ldots$$

and by $M^{i,\bullet}$ the analogous complex for the differential d_{II}. Similar to the case of a complex we define $B^{\bullet,\bullet}(M^{\bullet,\bullet})$ to be the double complex with $B^{\bullet,j}(M^{\bullet,\bullet}) = B^\bullet(M^{\bullet,j})$ and $B^{i,\bullet}(M^{\bullet,\bullet}) = B^\bullet(M^{i,\bullet})$; completely analogous we define the complexes $Z^{\bullet,\bullet}(M^{\bullet,\bullet})$ and $H^{\bullet,\bullet}(M^{\bullet,\bullet})$.

A.2.1 Definition
Let $C^\bullet$ be a complex in $\mathcal{A}$. A Cartan-Eilenberg resolution of $C^\bullet$ is an upper half-plane double complex $I^{\bullet,\bullet}$ of injective objects in $\mathcal{A}$, i.e. $I^{i,j} = 0$ for all $j < 0$ and all $i \in \mathbb{Z}$, together with a map $\epsilon : C^\bullet \to I^{\bullet,0}$ such that:

a) If $C^i = 0$, the column $I^{i,\bullet}$ is trivial, i.e. $I^{i,j} = 0$ for each $j \geq 0$.

b) The complexes $B^{i,\bullet}(I^{\bullet,\bullet})$, $Z^{i,\bullet}(I^{\bullet,\bullet})$ and $H^{i,\bullet}(I^{\bullet,\bullet})$ are injective resolutions of $B^i(C^\bullet)$, $Z^i(C^\bullet)$ and $H^i(C^\bullet)$ respectively.

A.2.2 Proposition
Each complex $C^\bullet$ in $\mathcal{A}$ has a Cartan-Eilenberg resolution $I^{\bullet,\bullet}$.

Proof. The double complex $I^{\bullet,\bullet}$ is constructed inductively. First we set $I^{i,j} = 0$ for $j < 0$. Next we consider the sequences

$$\begin{aligned}
0 \to Z^i(C^\bullet) \to &C^i \to B^{i+1}(C^\bullet) \to 0 \\
0 \to B^{i+1}(C^\bullet) \to &Z^{i+1}(C^\bullet) \to H^{i+1}(C^\bullet) \to 0 \\
0 \to Z^{i+1}(C^\bullet) \to &C^{i+1}(C^\bullet) \to B^{i+2}(C^\bullet) \to 0 \\
0 \to B^{i+2}(C^\bullet) \to &Z^{i+1}(C^\bullet) \to H^{i+1}(C^\bullet) \to 0 \\
&\dots
\end{aligned}$$

and choose an injective resolution $J^{i,\bullet}_{Z^i(C^\bullet)}$ of $Z^i(C^\bullet)$. By the Horseshoe Lemma (cf. [Wei95, Lemma 2.2.8] for a dual version) there are injective resolutions $C^i \to I^{i,\bullet}$ and $B^{i+1}(C^\bullet) \to J^{i+1,\bullet}_{B^{i+1}(C^\bullet)}$, such that there exists a short exact sequence of complexes $0 \to J^{i,\bullet}_{Z^i(C^\bullet)} \to I^{i,\bullet} \to J^{i+1,\bullet}_{B^{i+1}(C^\bullet)} \to 0$ which is compatible with the first sequence $0 \to Z^i(C^\bullet) \to C^i \to B^{i+1}(C^\bullet) \to 0$. Now we choose in the same way resolutions $Z^{i+1}(C^\bullet) \to J^{i+1,\bullet}_{Z^{i+1}(C^\bullet)}$ and $H^{i+1}(C^\bullet) \to J^{i+1,\bullet}_{H^{i+1}(C^\bullet)}$ such that there is a short exact sequence of complexes $0 \to J^{i+1,\bullet}_{B^{i+1}(C^\bullet)} \to J^{i+1,\bullet}_{Z^{i+1}(C^\bullet)} \to J^{i+1,\bullet}_{H^{i+1}(C^\bullet)} \to 0$ compatible with the second sequence $0 \to B^{i+1}(C^\bullet) \to Z^{i+1}(C^\bullet) \to H^{i+1}(C^\bullet) \to 0$. This procedure can be iterated with the sequences above. The Cartan-Eilenberg resolution is then given by the double complex $I^{\bullet,\bullet}$ whose columns are the resolutions $I^{i,\bullet}$; the vertical differentials are differentials of the resolutions $I^{i,\bullet}$ and the horizontal differentials are given by the composition $I^{i,\bullet} \to J^{i+1,\bullet}_{B^{i+1}(C^\bullet)} \to J^{i+1,\bullet}_{Z^{i+1}(C^\bullet)} \to I^{i+1,\bullet}$. □

Now let $C^\bullet$ be a complex in $\mathcal{A}$ and let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $C^\bullet$. If $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor, we can apply F to every object of $I^{\bullet,\bullet}$ and obtain a double complex which we denote by $F(I^{\bullet,\bullet})$. By the total complex $\mathrm{Tot}^{\prod}(F(I^{\bullet,\bullet}))$ we mean the complex with $\mathrm{Tot}^{\prod}(F(I^{\bullet,\bullet}))^r = \prod_{i+j=r} F(I^{i,j})$. If this total complex exists, we define the i-th hyper right derived functor $\mathbb{R}^i F(C^\bullet)$ to be $H^i(\mathrm{Tot}^{\prod}(F(I^{\bullet,\bullet}))$. This yields a functor from $\mathbf{Ch}^+(\mathcal{A})$ to $\mathcal{B}$ and from $\mathbf{Ch}(\mathcal{A})$ to $\mathcal{B}$, if $\mathcal{B}$ is complete.

A.2.3 Definition
Let X be a smooth projective variety and let $\mathcal{F}^\bullet$ be a complex of sheaves on X for any Grothendieck topology τ. The τ-hypercohomology of X in degree i with coefficients in $\mathcal{F}^\bullet$ is defined by

$$\mathbb{H}^i_\tau(X, \mathcal{F}^\bullet) := \mathbb{R}^i\Gamma(\mathcal{F}^\bullet),$$

where Γ is the global section functor.

A.3. Spectral sequences

In this thesis we use various kinds of spectral sequences in order to compute the cohomology of complexes of sheaves. Here we briefly recall the terminology and describe those spectral sequences, which are used in the previous chapters.

In the following $\mathcal{A}$ will denote an abelian category.

A.3.1 Definition
A (cohomology) spectral sequence in $\mathcal{A}$ starting on page $a \in \mathbb{N}$ consists of the following data:

i) A family $\{E_r^{p,q}\}$ of objects of $\mathcal{A}$ for all $p, q \in \mathbb{Z}$ and all $r \geq a$.

ii) Maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ for all $p, q \in \mathbb{Z}$ and $r \geq a$, i.e. the $d_r^{p,q}$ are differentials. We often suppress the superscript (p, q) and simply write d_r.

iii) Isomorphisms between $E_{r+1}^{p,q}$ and the cohomology of $E_r^{*,*}$ at (p, q) with respect to the differentials $d_r^{p,q}$:

$$E_{r+1}^{p,q} \cong \ker\left(d_r^{p,q}\right) / \operatorname{im}\left(d_r^{p-r,q+r-1}\right)$$

Probably the most common way to think about spectral sequences is as a book whose sheets are square numbered by a pair of integers (p, q), such that the object $E_r^{p,q}$ sits on the r-th sheet in position (p, q) and passing to the next page means taking cohomology with respect to the differentials d_r. The differentials on page r go r steps to the right and $r - 1$ steps downwards.

A.3.2 Remark
Let $(E_r^{p,q}, d_r)$ be a spectral sequence starting on page a. Since each $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$, we obtain a nested family of subobjects of $E_a^{p,q}$:

$$0 = B_a^{p,q} \subseteq \cdots \subseteq B_r^{p,q} \subseteq B_{r+1}^{p,q} \subseteq \cdots \subseteq Z_{r+1}^{p,q} \subseteq Z_r^{p,q} \cdots \subseteq Z_a^{p,q} = E_a^{p,q} \tag{34}$$

such that $E_{r+1}^{p,q} = Z_r^{p,q} / B_r^{p,q}$.

A.3.3 Definition
Let $(E_r^{p,q}, d_r)$ be a spectral sequence starting on page a. If the objects $B_\infty^{p,q} = \bigcup_{r=a}^\infty B_r^{p,q}$ and $Z_\infty^{p,q} = \bigcap_{r=a}^\infty Z_r^{p,q}$ exist, we define the limit object at (p,q) to be

$$E_\infty^{p,q} = Z_\infty^{p,q} / B_\infty^{p,q}.$$

A.3.4 Remark
Let $(E_r^{p,q}, d_r)$ be a spectral sequence starting at page a. We say that the object $E_r^{p,q}$ is of total degree $n = p + q$; so the objects of total degree n lie on a line with slope -1 and the differential d_r increases the total degree by 1.

The spectral sequence $(E_r^{p,q}, d_r)$ is said to be bounded, if for each $n \in \mathbb{N}$ there are only finitely many non-zero objects of total degree n in $E_a^{p,q}$. Of course, if the spectral sequence is bounded, the chain (34) of subobjects of $E_a^{p,q}$ is finite and thus the limit objects $E_\infty^{p,q}$ exist.

A special case of a bounded spectral sequence is a so-called first quadrant spectral sequence, i.e. a spectral sequence with $E_r^{p,q} = 0$ whenever $p < 0$ or $q < 0$. The same holds for third quadrant spectral sequences.

The limit terms also exist if for any pair (p,q) there exists a r_0 such that $d_r^{p,q} = 0$ and $d_r^{p-r,q+r-1} = 0$ for each $r \geq r_0$. In this case the isomorphism from A.3.1 iii) identifies $E_r^{p,q}$ with $E_{r+1}^{p,q}$ for each $r \geq r_0$ and we have $E_\infty^{p,q} = E_{r_0}^{p,q}$.

A.3.5 Definition
Let $(E_r^{p,q}, d_r)$ be a spectral sequence starting on page a such that all limit terms $E_\infty^{p,q}$ exist. Further assume that for each $n \in \mathbb{Z}$ there is an object H^n with a decreasing regular[1] filtration

$$\cdots \supseteq F^p H^n \supseteq F^{p+1} H^n \supseteq \cdots.$$

We say that $(E_r^{p,q}, d_r)$ converges to H^* if there are isomorphisms

$$E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$$

for any (p,q). In this case we write

$$E_a^{p,q} \Rightarrow H^{p+q}.$$

A.3.6 Remark
Let $(E_r^{p,q}, d_r)$ be a first quadrant spectral sequence starting at page a which converges to H^*. Then each H^n has a finite filtration

$$0 = F^{n+1} H^n \subseteq F^n H^n \subseteq \cdots \subseteq F^1 H^n \subseteq F^0 H^n = H^n.$$

[1] A decreasing filtration is regular if $\bigcap_p F^p H^n = \{0\}$ and $\bigcup_p F^p H^n = H^n$ for all n.

In particular there are isomorphisms $F^n H^n \cong E_\infty^{n,0}$ and $H^n/F^1H^n \cong E_\infty^{0,n}$. Moreover since each differential $d_r^{n,0}$ is zero, there is a morphism $E_a^{n,0} \twoheadrightarrow E_\infty^{n,0} \subseteq H^n$. Analogous each differential landing in $E_r^{0,n}$ for any $r \geq a$ is zero and thus there is a morphism $H^n \twoheadrightarrow E_\infty^{0,n} \subseteq E_a^{0,n}$. These morphisms are called edge homomorphisms.

Spectral sequences arise in many different situations. In the following we will sketch the construction of those spectral sequences, which are used in this thesis.

A.3.7 Theorem (Leray spectral sequence)
Let X and Y be topological spaces and let $\mathcal{F}$ be a sheaf of abelian groups on X. For any continuous map $f : X \to Y$ there is a spectral sequence starting on page 2

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$

called Leray spectral sequence. So this spectral sequence computes cohomology on X via cohomology on Y.

Proof. [Tam94, (3.7.6)] □

A.3.8 Theorem (Hypercohomology spectral sequence)
Let X be a topological space and $C^\bullet$ be a complex of sheaves of abelian groups on X. Then there is a converging spectral sequence, called hypercohomology spectral sequence, starting on page 2

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(C^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X, C^\bullet),$$

where $\mathcal{H}^q(C^\bullet)$ is the sheaf associated to the presheaf $U \mapsto \frac{\ker\left(\Gamma(U,C^q)\to\Gamma(U,C^{q+1})\right)}{\operatorname{im}\left(\Gamma(U,C^{q-1})\to\Gamma(U,C^q)\right)}$.

Proof. [Wei95, Application 5.7.10] □

Let $K^\bullet$ be a complex in $\mathcal{A}$ with differential d. We say that $K^\bullet$ has a (decreasing) filtration, if each K^n is filtered by subobjects $\cdots \supseteq F^iK^n \supseteq F^{i+1}K^n \supseteq \cdots$ and the differential d respects this filtration, i.e. $d(F^iK^n) \subseteq F^iK^{n+1}$.

A.3.9 Lemma
Let $K^\bullet$ be a filtered complex with differential d. Define for $r \geq 0$

$$Z_r^{p,q} = d^{-1}(F^{p+r}K^{p+q+1}) \cap (F^pK^{p+q})$$

and

$$E_r^{p,q} = Z_r^{p,q} / \left(Z_{r-1}^{p+1,q-1} + d\left(Z_{r-1}^{p-r+1,q+r-2}\right)\right).$$

Then the following assertions hold:

i) The differential d induces a well-defined map $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ *for each* $r \geq 0$.

ii) $d_r \circ d_r = 0$ *for* $r \geq 0$.

iii) The cohomology at $E_r^{p,q}$ *with respect to* d_r *is isomorphic to* $E_{r+1}^{p,q}$.

In particular, $(E_r^{p,q}, d_r)$ *as given above defines a spectral sequence. Moreover if the filtration of* $K^\bullet$ *is finite and regular, this spectral sequence converges.*

Proof. This is the content of [GM03, III.7 5.]. □

Let now $M^{\bullet,\bullet}$ be a double complex in $\mathcal{A}$ with horizontal resp. vertical differentials $d_I^{i,j} : M^{i,j} \to M^{i+1,j}$ and $d_{II}^{i,j} : M^{i,j} \to M^{i,j+1}$; in particular these differentials satisfy $d_I^2 = 0$, $d_{II}^2 = 0$ and $d_i \circ d_{II} + d_{II} \circ d_I = 0$. We set $\mathrm{H}_I^{i,j}(M^{\bullet,j}) = \mathrm{H}_I^i(M^{\bullet,j}) = \ker(d_I^{i,j}) / \operatorname{im}(d_I^{i-1,j})$; then $d_{II}^{i,j}$ induces a differential on the complex

$$\ldots \to \mathrm{H}_I^i(M^{\bullet,j}) \to \mathrm{H}_I^i(M^{\bullet,j+1}) \to \ldots.$$

Denote by $\mathrm{H}_{II}^j\left(\mathrm{H}_I^{i,\bullet}(M^{\bullet,\bullet})\right)$ the cohomology of this complex. Completely analogous we define $\mathrm{H}_I^i\left(\mathrm{H}_{II}^{\bullet,j}(M^{\bullet,\bullet})\right)$.

Moreover let $\mathrm{Tot}(M^{\bullet,\bullet})$ be the total complex of $M^{\bullet,\bullet}$ with differential $d = d_I + d_{II}$, i.e $\mathrm{Tot}(M^{\bullet,\bullet})^n = \prod_{i+j=n} M^{p,q}$. There exist two decreasing filtrations on $\mathrm{Tot}(M^{\bullet,\bullet})$:

$$F_I^p \mathrm{Tot}(M^{\bullet,\bullet})^n = \prod_{i+j=n, i\geq p} M^{i,j} \qquad \text{and} \qquad F_{II}^q \mathrm{Tot}(M^{\bullet,\bullet})^n = \prod_{i+j=n, j\geq q} M^{i,j}. \tag{35}$$

A.3.10 Theorem (Spectral sequence of a double complex)
Let $M^{\bullet,\bullet}$ *be a double complex as above. The filtered complexes given by the filtrations (35) give rise to two spectral sequences* $({}^IE_r^{p,q}, d_{r,I})$ *and* $({}^{II}E_r^{p,q}, d_{r,II})$.
Moreover if $M^{\bullet,\bullet}$ *is a first quadrant double complex, both these spectral sequences converge to* $H^n(\mathrm{Tot}(M^{\bullet,\bullet}))$ *and we have*

$$ {}^IE_2^{p,q} = H_I^p\left(H_{II}^{\bullet,q}(M^{\bullet,\bullet})\right) \Rightarrow H^{p+q}\left(\mathrm{Tot}(M^{\bullet,\bullet})\right)$$
$$ {}^{II}E_2^{p,q} = H_{II}^p\left(H_I^{q,\bullet}(M^{\bullet,\bullet})\right) \Rightarrow H^{p+q}\left(\mathrm{Tot}(M^{\bullet,\bullet})\right),$$

Proof. The existence of the spectral sequences is given by Lemma A.3.9. If $M^{\bullet,\bullet}$ is a first quadrant spectral sequence, the filtrations (35) are finite and regular and thus the spectral sequences converge also by Lemma A.3.9. The computation of the E_2-terms can be found in [GM03, III.7 10.]. □

A.3.11 Remark
The spectral sequence ${}^IE_2^{p,q}$ in Theorem A.3.10 arises by first computing 'vertical cohomology' of the columns of the double complex $M^{\bullet,\bullet}$ and then computing 'horizontal

cohomology' of the rows given by vertical cohomology; the spectral sequence ${}^{II}E_2^{p,q}$ arises the other way round, i.e. by first taking horizontal cohomology and then vertical cohomology. But both spectral sequences converge to the cohomology of the total complex.

Let now X be a equidimensional noetherian scheme of finite type over a field k and let $\mathcal{F}$ be a sheaf of abelian groups on the small étale site $X_{ét}$. For $x \in X$ define $\mathrm{H}^n_{ét,\,x}(X, \mathcal{F}) := \varinjlim_{x \in U} \mathrm{H}^n_{ét,\,\overline{x} \cap U}(U, \mathcal{F})$, where the limit runs through the ordered set of open neighbourhoods of x and $\mathrm{H}^n_{ét,\,\overline{x} \cap U}(U, \mathcal{F})$ is étale cohomology with support in $\overline{x} \cap U$. Further denote by $X^{(i)}$ the points of codimension i in X. Then there is a converging spectral sequence, called coniveau or Bloch-Ogus spectral sequence

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} \mathrm{H}^{p+q}_{ét,\,x}(X, \mathcal{F}) \Rightarrow \mathrm{H}^{p+q}_{ét}(X, \mathcal{F}). \tag{36}$$

For an explicit construction of this spectral sequence see [BO74, section 3] or [CTHK97, Part 1]. Let $Z^i = \{Z \subseteq X \mid Z \text{ closed}, \mathrm{codim}_X(Z) \geq i\}$. Then the filtration of the terms $\mathrm{H}^n_{ét}(X, \mathcal{F})$ is given by

$$N^i \mathrm{H}^n_{ét}(X, \mathcal{F}) = \ker \left(\mathrm{H}^n_{ét}(X, \mathcal{F}) \to \varinjlim_{Z \in Z^i} \mathrm{H}^n_{ét}(X \setminus Z, \mathcal{F}) \right)$$

and it is called coniveau filtration.

A.3.12 Theorem (Bloch-Ogus, [BO74, 6.2, 6.3])
Let $\mathcal{F}$ be an effaceable sheaf of abelian groups on the small étale site of X. Then the E_2-terms of the coniveau spectral sequence (36) are given by

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X, \mathcal{H}^q(\mathcal{F})),$$

where $\mathcal{H}^q(\mathcal{F})$ is the Zariski sheaf associated to the presheaf $U \mapsto H^q_{ét}(X, \mathcal{F})$.
Moreover $E_2^{p,q} = 0$ for $p > q$.

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Eidesstattliche Versicherung

(siehe Promotionsordnung vom 12.07.11, §8, Abs. 2 Pkt. 5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

Götzer, Thomas

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