Distances and automatic sequences in distinguished variants of Hanoi graphs

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Abstract

In this thesis three open problems concerning Hanoi-type graphs are addressed. I prove a theorem to determine all shortest paths between two arbitrary vertices s and t in the general Sierpiński graph S_p^n with base $p \geq 3$ and exponent $n \geq 0$ and find an algorithm based on this theorem which gives us the index of the potential auxiliary subgraph, the distance between s and t and the best first move(s). Using the isomorphism between S_3^n and the Hanoi graphs H_3^n, this algorithm also determines the shortest paths in H_3^n. The results are also used in order to simplify proofs of already known metric properties of S_p^n. Additionally, I compute the average number of input pairs (s_i, t_i) for $i \in \{1, \ldots, n\}$ to be read by the algorithm. The theorem and the algorithm for S_p^n are modified for the Sierpiński triangle graphs, which are deeply connected to the well-known Sierpiński triangle and the Sierpiński graphs, with the result that the shortest paths in the Sierpiński triangle graphs can be determined for the first time.
The Hanoi graphs H_3^n are then considered as directed graphs by differentiating the directions of the disc moves between the pegs of the corresponding Tower of Hanoi. For the problem to transfer a tower from one peg to another peg there are five different solvable variants. Here, the variants $TH(C_3^+)$ and $TH(K_3^-)$ are discussed concerning the infinite sequences of moves which arise from the solutions as n tends to infinity. The Allouche-Sapir Conjecture says that these sequences are not d-automatic for any d. I prove this for the $TH(C_3^+)$ sequence with the aid of the frequency of a letter and its rationality in automatic sequences. For the $TH(K_3^-)$ sequence I employ Cobham's Theorem about multiplicative independence, automatic sequences and ultimate periodicity. I show that this sequence is the image, under a 1-uniform morphism, of an iterative fixed point of a primitive prolongable endomorphism. F. Durand's method[a] is then used for the decision about the question whether the sequence is ultimately periodic. The method of I. V. Mitrofanov[b], which works with subword schemata, is applied to the problem as well. Using the theory of recognisable sets, a sufficient condition for deciding the question about the automaticity of the $TH(K_3^-)$ sequence is deduced.
Finally, a yet not studied distance problem on the so-called Star Tower of Hanoi, which is based on the star graph $St(4)$, is considered. Assuming that the Frame-Stewart type strategy is optimal, a recurrence for the length of the resulting paths is deduced and solved up to $n = 12$.

[a] F. Durand, HD0L ω-equivalence and periodicity problems in the primitive case (to the memory of G. Rauzy). Journal of Uniform Distribution Theory, 7(1):199-215, 2012

[b] I. V. Mitrofanov, Periodicity of Morphic Words, Journal of Mathematical Sciences, 206(6):679-687, 2015

Zusammenfassung

Ich beweise ein Theorem zur Bestimmung aller kürzesten Wege zwischen zwei beliebigen Ecken s und t in den allgemeinen Sierpiński-Graphen S_p^n mit Basis $p \geq 3$ und Exponent $n \geq 0$ und erstelle auf diesem Theorem beruhend einen Algorithmus, der den Index des allfälligen Hilfsuntergraphen, den Abstand zwischen s und t und einen besten ersten Schritt liefert. Unter Verwendung des Isomorphismus zwischen S_3^n und den Hanoi-Graphen H_3^n bestimmt dieser Algorithmus auch die kürzesten Wege in H_3^n. Die Ergebnisse werden benutzt, um Beweise bereits bekannter metrischer Eigenschaften der S_p^n zu vereinfachen. Zusätzlich berechne ich die durchschnittlich benötigte Anzahl von Eingabepaaren (s_i, t_i) für $i \in \{1, \ldots, n\}$ in den Algorithmus. Das Theorem und der Algorithmus für S_p^n werden für die Klasse der Sierpiński-Dreiecksgraphen, welche in direktem Zusammenhang mit dem berühmten Sierpiński-Dreieck und den Sierpiński-Graphen stehen, modifiziert, sodass erstmals auch die kürzesten Wege in diesen Graphen bestimmt werden können.
Die Hanoi-Graphen H_3^n werden dann als gerichtete Graphen betrachtet, indem man die Richtungen der Bewegungen zwischen den Stäben des entsprechenden Turms von Hanoi differenziert. Für das Problem des Versetzens eines Turms von einem Stab auf einen anderen gibt es fünf verschiedene lösbare Varianten. Die Varianten $TH(C_3^+)$ und $TH(K_3^-)$ werden bezüglich der unendlichen Folgen von Bewegungen betrachtet, die sich durch die Lösung für n gegen Unendlich strebend ergeben. Die Allouche-Sapir-Vermutung besagt, dass für kein d diese Folgen d-automatisch erzeugt sind. Ich beweise dies für die $TH(C_3^+)$ Folge mit Hilfe der Theorie über die Häufigkeit eines Buchstabens und deren Rationalität in automatisch erzeugten Folgen. Für die $TH(K_3^-)$ Folge wird Cobhams Theorem über multiplikative Unabhängigkeit, automatisch erzeugte Folgen und ultimative Periodizität verwendet. Ich zeige, dass diese Folge das Bild, unter einem 1-uniformen Morphismus, eines iterativen Fixpunktes eines primitiven verlängerbaren Endomorphismus ist. Die Methode von F. Durand[a] wird dann für die Entscheidung über die Frage, ob die Folge ultimativ periodisch ist, verwendet. Ebenso wird die Methode von I. V. Mitrofanov[b], welche mit Teilwortschemata arbeitet, auf das Problem angewandt. Unter Verwendung der Theorie über erkennbare Mengen wird eine hinreichende Bedingung für die Frage der Automatizität der $TH(K_3^-)$ Folge hergeleitet.
Zuletzt wird ein bislang nicht untersuchtes Abstandsproblem im sogenannten Stern-Turm-von-Hanoi betrachtet, welcher auf dem Stern-Graphen $St(4)$ beruht. Unter der Annahme, dass die Frame-Stewart-Strategie optimal sei, wird eine Rekursionsvorschrift für die Länge der so gewonnenen Wege entwickelt und bis $n = 12$ gelöst.

[a] F. Durand, HD0L ω-equivalence and periodicity problems in the primitive case (to the memory of G. Rauzy). Journal of Uniform Distribution Theory, 7(1):199-215, 2012

[b] I. V. Mitrofanov, Periodicity of Morphic Words, Journal of Mathematical Sciences, 206(6):679-687, 2015

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Weyarn, August 2016 Caroline Holz auf der Heide

Contents

Symbol Index

η	morphism corresponding to $\mathbf{w}$, 66
κ	coding corresponding to $\mathbf{w}$, 66
λ_r	Perron-Frobenius eigenvalue, 61
$\rho(M)$	spectral radius of matrix M, 96
Σ^*	set of finite words on alphabet Σ, 56
Σ_{C+}	set $\{x, r, z, t, u, s, a, b, c, \bar{a}\}$, 66
Σ_{K-}	set $\left\{x, y, z, d, e, f, a, b, c, \bar{a}, \bar{b}\right\}$, 73
τ	coding corresponding to $\mathbf{u}$, 73
ϕ	morphism corresponding to $\mathbf{u}$, 73
$\psi^\infty(a)$	(iterative) fixed point of morphism ψ on letter a, 58
$\binom{n}{k}$	combinatorial number, 19
$\binom{S}{k}$	set of all subsets of S of size k, 19
$\|\|G\|\|$	size of set $E(G)$ for graph G, 18
$i \vartriangle j$	binary operation defined on T, 10
$\hat{k}$	primitive vertex in ST_p^n, 39
$\|M\|$	size of set M, 11
$\mathbb{N}$	set $\{1, 2, 3, \ldots\}$, 3
$\mathbb{N}_k$	set $\{k, k+1, \ldots\}$, 3
$[n]_0$	set $\{0, \ldots, n-1\}$, 7
$[n]$	set $\{1, \ldots, n\}$, 7
$[n]_2$	set $\{2, \ldots, n\}$, 7
$\|s\|$	length of word s, 56
$\|s\|_a$	number of occurrences of letter a in word s, 56
$(s \vartriangle j)$	best first move vector, 10
$[\mathcal{S}]$	binary truth value of statement $\mathcal{S}$, 10
$\mathcal{S}(S, T)$	subword scheme for ordered sets of finite words S and T, 92

Chapter 0

Introduction

It was in 1883 when François Édouard Anatole Lucas introduced a new puzzle[1] motivated by the following legend which was developed by H. de Parville [22] and translated into English by W. W. R. Ball ([11, p.228 f]):

> "In the great temple at Benares, beneath the dome which marks the centre of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four discs of pure gold, the largest disc resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the Tower of Bramah. Day and night unceasingly the priests transfer the discs from one diamond needle to another according to the fixed and immutable laws of Bramah, which require that the priest on duty must not move more than one disc at a time and that he must place this disc on a needle so that there is no smaller disc below it. When the sixty-four discs shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple, and Brahmins alike will crumble into dust, and with a thunderclap the world will vanish."

Lucas marketed the game with only eight discs under the pseudonym *N. Claus de Siam* which is an anagram of *Lucas d'Amiens*. As name for his new puzzle he chose "La Tour d'Hanoï" ("The Tower of Hanoi"), since at this time Hanoi was in the headlines of French newspapers. Between 1883 and 1885 the Sino-French war took place, among others in Tongkin, the northernmost part of what is now Vietnam. The name Tongkin is the corruption of Đông Kinh, the name of Hanoi during the Lê dynasty. Under French influence Hanoi was made the capital of this region. Maybe Lucas chose the name "The Tower of Hanoi" to promote the selling of his game. Lucas was born on 4 April 1842 in Amiens, in the north of France. Besides the Tower of Hanoi, he presented many

[1] According to Lucas, it was published in 1882 by himself, but one cannot find an evidence for this ([33]).

other puzzles like the Chinese Rings in his publications about Recreational Mathematics [60]. But he is also known for results in number theory, for instance the Lucas numbers[2] and the proof that the Mersenne number $2^{127} - 1$ is prime. In 1891, Lucas participated in a banquet of the congress of the “Association française pour l’avancement des sciences", of which he was an active member. During this event, a servant dropped a pile of plates and a piece of porcelain flew up and hit the cheek of Lucas which caused a deep wound. A few days later, on 3 October 1891, he died as a result of this injury. The tomb can be found on the Montmartre cemetery of Paris ([36]).

Figure 0.1: The tomb of Édouard Lucas on the Montmartre cemetary, Paris: 770cp1857, 23rd division, 8th row, 27 Avenue des Carrieres

Following the legend, the *Tower of Hanoi* consists of three needles, afterwards called pegs and numbered with $0, 1, 2$ such that $T = \{0, 1, 2\}$ is the set of pegs, and a number of discs of increasing size. To transfer one disc, you have to follow the divine rule that you “must place this disc on a needle so that there is no smaller disc below it”. Lucas also stated the famous recursive solution of the puzzle for an arbitrary number of discs ([33]). We explain it by an example. Assuming one can solve the puzzle for four discs, then one can solve it for five discs as well since one transfers first the upper four discs to the non-goal peg, then the fifth disc to the goal peg and finally the four discs from the non-goal peg to the goal peg. The recursive solution needs $2^n - 1$ moves to transfer a tower of n discs ([33]). For the number of discs in the legend, namely 64, we would need

[2]The Lucas numbers $2, 1, 3, 4, 7, 11, 18, \ldots$ (OEIS AE000032) use the same recurrence relation as the famous Fibonacci numbers (OEIS AE000045) (but starting with 2 and 1) and are closely related to these. Lucas studied both of them.

incredible 18 446 744 073 709 551 615 moves. Surprisingly, the minimality proof of the recursive algorithm was lacking until 1981, when D. Wood [81, Theorem] gave one (cf. [33, pp. 44-45] for a deeper discussion). R. Olive found an algorithm which gives the moving disc in an optimal solution, i.e., the solution with the minimum number of moves, of the Tower of Hanoi for the case that we transfer a tower from one peg to another ([33, pp. 74–75]). The tasks on the tower were later categorised in three classes of problems, first the case that we transfer a tower from one peg to another ($P0$), second the case that we transfer discs from an arbitrary regular state to a selected peg ($P1$) and last the case that we transfer discs from an arbitrary regular state to another regular state ($P2$). To study the last class, use was made of the *Hanoi graphs* H_3^n, first considered by Scorer et al. [75], for n discs and 3 pegs. Initially, a distance on these graphs was determined for the second class of problems [34]. It turned out that the diameter of H_3^n is equal to the distance between two perfect vertices. For the moves of the largest disc the following result, named the boxer rule [33], was proved. If on a shortest path between two vertices the largest disc once moves away from a peg, it will not return to the same peg.

D. Romik [70] then gave an algorithm, based on what has become known as "Romik's automaton", to determine the distance in the third class of problems. He used for the proof another class of graphs, the *Sierpiński graphs*. This was possible as the Sierpiński graphs S_3^n for 3 pegs are a variant of the Hanoi graphs H_3^n, more precisely they are isomorphic ([52], [70]). This variant of the Hanoi graphs is named after the famous Sierpiński triangle because of their deep connection which was found by A. M. Hinz and A. Schief ([33, Section 4.3], [42]). These graphs can be interpreted as state graphs of a variant of the Tower of Hanoi puzzle, the *Switching Tower of Hanoi* ([52]). S. Klavžar and U. Milutinović generalised the graphs in [52] to the Sierpiński graphs S_p^n with base $p \in \mathbb{N}$ and exponent $n \in \mathbb{N}_0$. Here, we introduce the notation $\mathbb{N}_k$ for all whole numbers k such that $\mathbb{N}_k$ is the set of all whole numbers greater than or equal to k. Then $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_1 = \{1, 2, 3, \ldots\}$, whereby we refer to $\mathbb{N}_1$ by $\mathbb{N}$. The *general Sierpiński graphs* were studied, inter alia, concerning their metric properties in [68], their distances between special vertices in [52], [54], and [82] and their average eccentricity in [41]. Some properties of the S_3^n or H_3^n, like the boxer rule, and the fact that for every non-extreme vertex of the graph there is a vertex such there are two shortest paths between them, could be carried over to S_p^n.

The book "The Tower of Hanoi —Myths and Maths" [33] by Hinz, Klavžar, Milutinović and C. Petr, published in 2013, was the first to summarise the known results about the Tower of Hanoi and related topics. In the last chapter of this book [33, Chapter 9] open problems were stated. One of these problems was to "design an automaton analogous to Romik's automaton for a "$P2$ task" in S_p^n, $p \geq 4$"[3]. Solving this problem is part of this thesis. We prove a new theorem about the determination of the shortest path(s) for arbitrary vertices s and t in S_p^n and design an algorithm which gives us the possible "detour" peg, the minimal distance and the best first move(s). Most of the results in an alternative representation were published by Hinz and the author of the present thesis in [37]. An overview on the Sierpiński graphs was given by Hinz, Klavžar and S. S. Zemljič [40]. This survey paper also covers the *Sierpiński triangle graphs* known from L. L. Cristea and B. Steinsky [20], further studied by M. Jakovac [50] and again treated in [43]. The theorem as well as the algorithm for the Sierpiński graphs can be extended to these new graphs which will be done in the present work.

[3] [33, p. 262]

One further problem listed in the last chapter of the book [33] was the subsequent conjecture about special variants of the Tower of Hanoi.

Conjecture 0.1 (Allouche-Sapir Conjecture)**.** *The two solvable Tower of Hanoi [variants] with oriented disc moves on three pegs that are not the Classical, the Cyclic, and the Linear Tower of Hanoi, are not d-automatic for any d.*[4]

Let us describe what this means. The technical details, written in italics, will be explained in the later chapters. We consider problems of $P0$ type. There are altogether five *solvable* variants of the Tower of Hanoi (TH) with 3 pegs, where one restricts the allowed orientation for disc moves. These are the Classical ($TH(K_3)$), the Linear ($TH(L_3)$), the Cyclic TH ($TH(C_3)$), $TH(C_3^+)$, and $TH(K_3^-)$, respectively. The state graphs of these variants are again variants of the Hanoi graphs, but with directed edges or arcs. The infinite sequence obtained from the moves of the solution of the classical case as n goes to infinity was proved to be 2-*automatic* by J.-P. Allouche and F. Dress [5]. The Linear TH was mentioned for the first time in [75] by Scorer et al. and studied in detail in [32] by H. Hering. The 3-automaticity of the infinite sequence corresponding to it was then shown by Allouche and A. Sapir [7]. Both proofs of automaticity used the algorithm which was given by Sapir [74] for solving recursively all these variants with the minimum number of moves. The Cyclic TH appeared in 1981 in [9], where M. D. Atkinson proposed an algorithm to transfer a perfect tower from peg 0 to peg 1. Other algorithms were given by different authors, for which we refer to [33, Chapter 8.2]. In 1994, Allouche [2] proved that the Cyclic TH sequence is not d-automatic for any $d \in \mathbb{N}_2$. For the $TH(C_3^+)$ and the $TH(K_3^-)$ *sequence* it was shown in [7] that they are *morphic*. But the question whether they are automatic remained open. The book [8] gives an overview on automatic sequences. In 1972, A. Cobham [18] proved that the *frequency of a letter* in an automatic sequence, if it exists, is rational. Michel [61] showed in 1975 that if a morphic sequence is *primitive*, then the frequency of all letters exists. K. Saari [71] then stated that the frequency of letters exists for every *pure* binary *morphic* sequence. Later, in [72], he found a necessary and sufficient criterion for the existence and the value of the frequency of letters in a morphic sequence. Furthermore, he gave an if-and-only-if condition that all frequencies do in fact exist. These results, which were not at the disposal of Allouche and Sapir, can be used to show that certain sequences are not d-automatic for any $d \in \mathbb{N}$ as it is successfully done for the $TH(C_3^+)$ sequence in the present thesis.
For the $TH(K_3^-)$ sequence we will make use of another theorem, known as Cobham's Theorem [17], in the way that we want to prove whether the sequence is not *ultimately periodic* in order to answer the question of automaticity. The idea for this approach was already outlined in [7] and again discussed during a private communication between Allouche and the author of the present work [3]. Morphic sequences are known in the theory of L-systems as *HD0L sequences*. Hence the problem to decide whether a morphic sequence is ultimately periodic is called the *HD0L ultimate periodicity problem*. The decidability for *D0L sequences* was shown by J.-J. Pansiot [67] and T. Harju and M. Linna [31] and for automatic sequences by J. Honkala [45]. Later, Allouche, N. Rampersad, and J. Shallit [6] presented a simpler proof for automatic sequences. F. Durand answered the question for primitive HD0L sequences in [25]. An equivalent formulation of the HD0L ultimate periodicity problem in terms of *recognisable sets* and abstract numeration systems

[4] ibid

was given by Honkala and M. Rigo in [47]. For this formulation Honkala already showed in [45] the decidability in the restricted case of usual integer bases, i.e., for d-automatic sequences. Durand succeeded then in giving a positive answer to the general problem in [26]. At around the same time, I.V. Mitrofanov gave another solution, first put on ArXiv [62]. Later, a simpler proof was published by him [63]. We will use both proofs, Durand's as well as Mitrofanov's, in order to try to determine whether the $TH(K_3^-)$ sequence is ultimately periodic.

The first extension of the Tower of Hanoi to four pegs was introduced by H. E. Dudeney [23]. He described the problem in a story about the Reve, some pilgrims and a number of cheeses of varying size. Therefore, this puzzle is known as The Reve's puzzle. A presumed optimal solution to transfer a tower from one peg to another peg was given independently by B. M. Frame and J. S. Stewart [76] in 1941. The claim that the Frame-Stewart algorithm is optimal is known as the Frame-Stewart Conjecture. It was open until T. Bousch proved in 2014 the optimality of the algorithm [13]. In The Reve's puzzle itself the orientations of the disc moves are not restricted. But according to [30, p. 218] there exist 83 non-isomorphic strong digraphs, which represent the allowed moves, for variants of the TH with 4 pegs. In 1994, P. K. Stockmeyer [78] introduced one of these variants and called it the *Star Tower of Hanoi* (Star TH) because of the structure of the corresponding directed graph. It turned out that this puzzle is related to the Linear TH in the way that this is the "Star TH" with three pegs. He gave a presumed optimal algorithm with the minimising value to transfer a tower from an external peg to another external peg. Chappelon and Matsuura [15] extended this problem to more than four pegs by adding external pegs and generalised the algorithm. They showed that realising the presumed optimal algorithm the minimal number of moves is given by the generalised Frame-Stewart numbers. Stockmeyer conjectured that his algorithm "makes the smallest number of moves among all procedures that solve the Star puzzle", which is as Stockmeyer's Conjecture also part of the list in [33, Chapter 9]. A further object of this thesis is to find a presumed optimal algorithm and the number of moves for the transfer of a tower from the central peg to an external peg for the case of four pegs.

This dissertation comprises three chapters. In Chapter 1 we focus on the $P2$ decision algorithm for Sierpiński graphs with base $p \in \mathbb{N}$. First, we define the Sierpiński graphs S_3^n with base 3 and exponent $n \in \mathbb{N}_0$ and give the isomorphism between the Hanoi graphs H_3^n and S_3^n. The theorem and the automaton of Romik, which can be adapted for the use for H_3^n by employing the isomorphism, is then presented for S_3^n. Furthermore, we generalise the Sierpiński graphs to the base $p \in \mathbb{N}$ and give some statements concerning them, in particular the definition of the distance function and that there are at most two shortest paths between any two vertices in S_p^n. In the next section we formulate the main theorem of this chapter about shortest paths between arbitrary vertices in S_p^n and find an algorithm with some conclusions as well as an analysis of the necessary pairs of input for a decision. Then an application of our algorithm to Sierpiński triangle graphs is given.
Chapter 2 mainly covers the variants of the TH with 3 pegs which are determined by the digraphs C_3^+ and K_3^- and the question whether the corresponding sequences are d-automatic for any $d \in \mathbb{N}$. At the beginning, we give a summary of the terminology of finite and infinite words or sequences used in this work and introduce morphic and automatic sequences. The first approach to disprove the automaticity consists of the application of [18, Theorem 6] about automaticity and the rationality of the frequency of a letter. For that purpose we define the frequency of a letter and describe the steps to the theorem which states an if-and-only-if condition for the existence of the frequency

and gives the value of the frequency if it exists. These steps are portrayed for the $TH(C_3^+)$ sequence and culminate in the theorem that this sequence is not d-automatic by showing that the frequencies for all letters are irrational. The same approach is then used to attack the $TH(K_3^-)$ sequence. But it turns out that we are unable to determine whether the frequency is rational. Hence we choose in the following section another approach which makes use of Cobham's Theorem about the ultimate periodicity and automatic sequences. We present the results of Durand's work about the decidability of the ultimately periodicity problem for morphic and primitive morphic sequences as far as they are useful for us and apply them to our problem. We find a primitive morphism and a coding such that the $TH(K_3^-)$ sequence is a primitive morphic sequence and give an algorithm for the determination of a constant which we need to decide on the ultimate periodicity. In the last subsection we introduce subword schemes, explain an algorithm which decides whether a sequence is ultimately periodic with the aid of subword schemes, and apply the results on our sequence. Further, we analyse these subword schemes for our sequence and give a sufficient condition for the d-automaticity of the $TH(K_3^-)$ sequence using the theory of recognisable sets.

In Chapter 3 we investigate the problem of finding a pattern in the number of moves for an increasing number of discs in the Star TH. First, we describe the presumed optimal solution for the $P0$ task from an external peg to another external peg by the Frame-Stewart strategy. After that we analyse the two different approaches of Frame and Stewart for a solution and the sequence of moves for the problem to transfer a tower from the central peg to an external peg. It will turn out that both strategies give the same formula for the number of moves. We analyse the number of moves for different values of splitting and give the first values for the number of moves resulting from the algorithm with the minimising value of splitting.

Chapter 1

A $P2$ decision algorithm for Sierpiński graphs with base $p \in \mathbb{N}_3$

One very well-known variant of the Tower of Hanoi is the *Switching Tower of Hanoi (Switching TH)* with three pegs, labeled with 0, 1, and 2. Assume that we have n discs. Then a *state* $s \in T^n$ is a legal distribution of n discs and written as $s = s_n \ldots s_1$, where $s_i \in T$ means that the i-th disc lies on peg s_i. A state is *regular* if no larger disc lies on a smaller one. A regular state such that all discs lie on a single peg is called *perfect*.

The problem to transfer a perfect tower from one peg to another peg or to go from a perfect state to a perfect state, respectively, in a minimum number of moves is called the problem of $P0$ *type*. If we want to go from a regular state to a perfect state, the problem is of $P1$ *type*. Consequently, the problem to go from a regular state to another regular state is called a problem of $P2$ *type*.

For $n \in \mathbb{N}$ we introduce the notation $[n]_0$ for the set $\{0, \ldots, n-1\}$, $[n]$ for the set $\{1, \ldots, n\}$, and $[n]_2$ for the set $\{2, \ldots, n\}$. The allowed moves to transfer the discs in the Switching TH with $n \in \mathbb{N}_0$ discs are as follows. Assume we have a regular state in which the $d-1 \in [n]_0$ topmost discs on peg i are the $d-1$ smallest ones. Then we can switch the d-th smallest disc on peg $j \neq i$ with the $d-1$ discs on i. That $d-1 = 0$ is included, means that arbitrary moves of the smallest disc are allowed.

If we interpret the states as vertices and the moves as edges, we can define a graph G consisting of $V(G)$ and $E(G)$, where $V(G)$ is the vertex set and $E(G)$ is the edge set. The graph associated with the Switching TH is the *Sierpiński graph* S_3^n *with base* $p = 3$ *and exponent* $n \in \mathbb{N}_0$. Its vertex set is $V(S_3^n) = T^n$. The edges of S_3^n are of the form $\left\{\underline{s}ij^{d-1}, \underline{s}ji^{d-1}\right\}$, where $i, j \in T, i \neq j, d \in [n]$, and $\underline{s} \in T^{n-d}$. Hence the edge set is

$$E(S_3^n) = \left\{\left\{\underline{s}ji^{d-1}, \underline{s}ij^{d-1}\right\} \;\middle|\; i, j \in T, i \neq j, d \in [n], \underline{s} \in T^{n-d}\right\}. \tag{1.1}$$

The disc d is here the moving single disc in the move as described above with any distribution $\underline{s}$ of discs larger than d.

There exists another definition of the edge set using the fact that S_3^{n+1} contains three subgraphs iS_3^n, which are generated by the shift of the labels of one copy of S_3^n (see Figure 1.1 for $n = 0, 1, 2$). Then

$$\begin{aligned} & E(S_3^0) &=& \emptyset, \\ \forall n \in \mathbb{N}_0 : \ & E(S_3^{n+1}) &=& \left\{\{is, is'\} \,\middle|\, i \in T, \{s, s'\} \in E(S_3^n)\right\} \\ &&& \cup \{\{ij^n, ji^n\} \mid i, j \in T, i \neq j\}. \end{aligned} \tag{1.2}$$

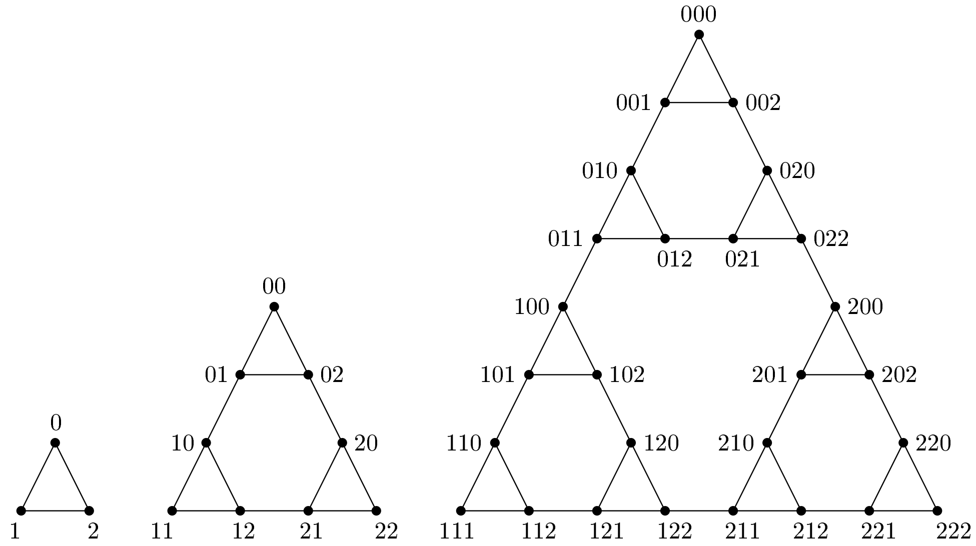

Figure 1.1: Sierpiński graphs S_3^1, S_3^2, and S_3^3

A comparison of Figures 1.1 and 1.2 imposes the question whether S_3^n and the Hanoi graph H_3^n are isomorphic for fixed $n \in \mathbb{N}$. We will now clarify this.

1.1 The isomorphism between the Hanoi graphs H_3^n and the Sierpiński graphs S_3^n

In this section we want to show the existence of an isomorphism between the Sierpiński graphs S_3^n and the Hanoi graphs H_3^n and some results about H_3^n which can be transferred to Sierpiński graphs S_3^n with the aid of the isomorphism. At first we will describe the vertex set $V(H_3^n)$ of the Hanoi graph with base $p = 3$ and exponent $n \in \mathbb{N}_0$ and the edge set $E(H_3^n)$:

$$\begin{aligned} V(H_3^n) &=& T^n, \\ E(H_3^n) &=& \left\{\left\{\underline{s}i(3-i-j)^{d-1}, \underline{s}j(3-i-j)^{d-1}\right\} \,\middle|\, i, j \in T, i \neq j, d \in [n], \underline{s} \in T^{n-d}\right\} \end{aligned} \tag{1.3}$$

For further reading about the Hanoi graphs we refer to the book [33].

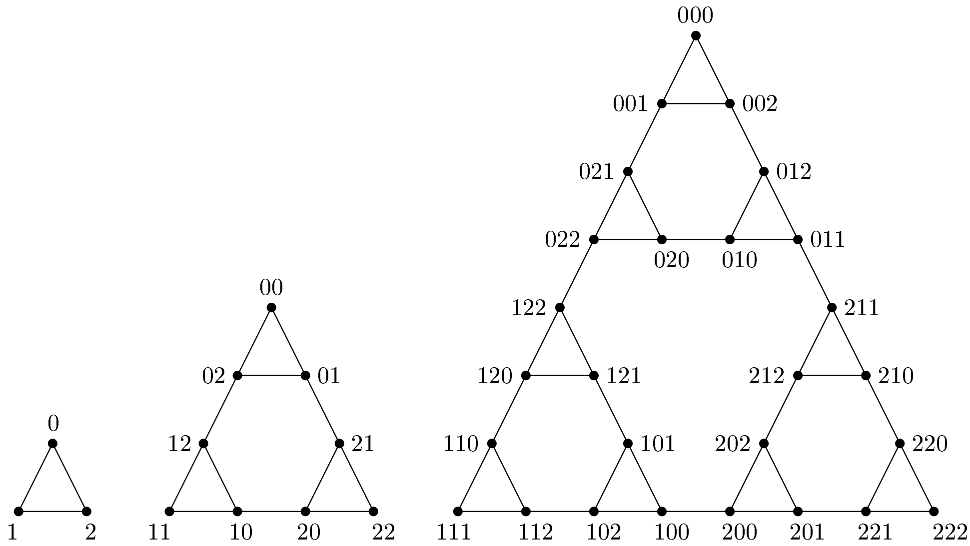

Figure 1.2: Hanoi graphs H_3^1, H_3^2, and H_3^3

As we can deduce from Figures 1.1 and 1.2, both types of graphs, namely H_3^n and S_3^n, have the vertex subset $\overline{V} = \{0^n, 1^n, 2^n\}$ in common. These vertices are the only ones with degree two, while the other vertices in both graphs have degree three. Hence any isomorphism from H_3^n onto S_3^n has to map $\overline{V}$ onto itself. The elements of $\overline{V}$ are called *extreme vertices* in S_3^n and *perfect vertices* in H_3^n (see Section 1.3 for an explanation of the different appellations).

We present a proof for the existence of an isomorphism between H_3^n and S_3^n.

Theorem 1.1 ([52, Theorem 2]). *For any $n \in \mathbb{N}$ the graph S_3^n is isomorphic to the graph H_3^n.*

Proof. By induction on n we define isomorphisms $\theta_n : S_3^n \to H_3^n$. For the base case $n = 1$ we see that both graphs are complete graphs on three vertices and θ_1 is the identity map. Let $n \in \mathbb{N}_2$. We partition the vertex set $V(S_3^n)$ into three subsets V_0, V_1, and V_2, where the elements of V_i are the vertices which begin with i ($i \in T$). Every V_i is connected with every V_j ($i \neq j$) by exactly one edge, namely $\{ijjj\ldots j, jiii\ldots i\}$. Now we obtain a partition of $V(H_3^n)$ into sets W_0, W_1, and W_2. The elements of W_i are the vertices beginning with i ($i \in T$). Then for any i and j ($i \neq j$) there is exactly one edge between W_i and W_j, namely $\{jk\ldots kk, ik\ldots kk\}$ ($i \neq k \neq j$). We consider a suitable automorphism of H_3^{n-1} which maps the ends of the connecting edges onto the corresponding ends of the connecting edges. This automorphism is induced by a permutation of $\{0, 1, 2\}$. Applying the induction hypothesis, we can now map V_i onto W_i using the isomorphism θ_{n-1} and the above-mentioned automorphism. Taking the maps for all $i \in T$ as one map, we get the map θ_n from $V(S_3^n)$ onto $V(H_3^n)$. □

The isomorphism θ is explicitly given by

$$\forall s \in T^n \quad \forall d \in [n] : \quad \theta(s)_d = s_n \vartriangle \ldots \vartriangle s_d, \tag{1.4}$$

where $\triangle$ is a binary operation defined on T. The operation is determined by Table 1.1.

Table 1.1: Cayley table for $\triangle$ on T

$\triangle$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

We remark that the operation $\triangle$ is commutative, since its Cayley table is symmetric along the diagonal axis. But it is *not* associative. For example we calculate $(1 \triangle 0) \triangle 0 \neq 1 \triangle (0 \triangle 0)$. Hence we must evaluate expressions like in (1.4) strictly from the right. An automaton which realises the isomorphism between H_3^n and S_3^n can be found in [33, p.145] or in [70] (here the evaluation starts from the left!).

For the sequel we define in H_3^n for the state $s \in T^n$, $j \in T$, and $d \in [n]$

$$(s \triangle j)_d = s_{d+1} \triangle \cdots \triangle s_n \triangle j. \tag{1.5}$$

Note that $(s \triangle j)_d$ is the "first goal" of disc d to make the best first move possible in the optimal solution to get from state s to the perfect state j^n.

Furthermore, we introduce *Iverson's convention* which says that $[\mathcal{S}] = 1$, if statement $\mathcal{S}$ is true, and $[\mathcal{S}] = 0$, if $\mathcal{S}$ is false.

Then the distance formula from the state $s \in T^n$ to the perfect state j^n $(j \in T)$, a problem of $P1$ *type*, in the Hanoi graph H_3^n (see [33, Equation (2.8)]) is

$$\mathrm{d}_H(s, j^n) = \sum_{d=1}^{n} [s_d \neq (s \triangle j)_d] \cdot 2^{d-1}, \tag{1.6}$$

where we denote the distance function by d .

Remark. For a $P0$ type, i.e., from a perfect state i^n to another perfect state j^n $(i, j \in T,\ i \neq j)$, we get $\mathrm{d}_H(i^n, j^n) = 2^n - 1$.

For $i \in T$ let φ_i be the permutation on T which has the single fixed point i. Then $i \triangle j = \varphi_i(j)$ for each $j \in T$. For Formula (1.4) we get

$$\forall s \in T^n \quad \forall d \in [n]: \quad \theta(s)_d = \varphi_{s_n} \circ \cdots \circ \varphi_{s_{d+1}}(s_d).$$

In a similar way we can write for Formula (1.5)

$$\forall j \in T \quad \forall s \in T^n \quad \forall d \in [n] : (s \triangle j)_d = \varphi_{s_{d+1}} \circ \cdots \circ \varphi_{s_n}(j).$$

Noting that $\varphi_k^{-1} = \varphi_k$, we can deduce that

$$\forall j \in T \quad \forall s \in T^n \quad \forall d \in [n] : s_d = (s \triangle j)_d \Leftrightarrow \theta(s)_d = j.$$

Together with Formula (1.6) and the definition of the isomorphism θ, we get now for all $s \in T^n$ and $j \in T$

$$\mathrm{d}_S(\theta(s), j^n) = \mathrm{d}_H(s, j^n) = \sum_{d=1}^{n} [s_d \neq (s \triangle j)_d] \cdot 2^{d-1} = \sum_{d=1}^{n} [\theta(s)_d \neq j] \cdot 2^{d-1}.$$

It follows that the distance d_S in S_3^n, again from a state $s \in T^n$ to the perfect state j^n ($j \in T$), is

$$\forall s \in T^n \quad \forall j \in T: \quad \mathrm{d}_S(s, j^n) = \sum_{d=1}^{n} [s_d \neq j] \cdot 2^{d-1}. \tag{1.7}$$

We refer to [33] for further reading.

There are some more properties of the Sierpiński graphs S_3^n or Hanoi graphs H_3^n, respectively. We define by $\beta(n)$ the number of 1s in the binary expansion of n. Further, we say that a set M is finite and has *size* $|M| := m \in \mathbb{N}_0$ if there is a bijection from M to $[m]$. The subsequent results can be found in [33, Proposition 2.13], [33, Proposition 2.16] and [34, Proposition 5].

Proposition 1.2. *a) For any $s \in T^n$,*

$$\mathrm{d}(s, 0^n) + \mathrm{d}(s, 1^n) + \mathrm{d}(s, 2^n) = 2 \cdot (2^n - 1).$$

b) Fix a perfect state j^n with $j \in T$. Then for $\mu \in [2^n]_0$:

$$|\{s \in T^n \mid \mathrm{d}(s, j^n) = \mu\}| = 2^{\beta(\mu)}.$$

We will later see that this proposition can be extended to the so-called general Sierpiński graphs.

1.2 A $P2$ decision automaton for S_3^n - D. Romik's Automaton

We want to get a deeper understanding of distances in Sierpiński graphs with base 3. Therefore, we first take a closer look on Hanoi graphs. In the sequel we will label the largest disc as the disc with number $(n+1)$. One main result for H_3^{n+1} is the so-called *boxer rule*.

Lemma 1.3 ([33, Lemma 2.26])**.** *If on a shortest path between two vertices in H_3^{n+1} ($n \in \mathbb{N}_0$) the largest disc is moved away from a peg, it will never come back to the same peg.*

With the boxer rule we get

Lemma 1.4 ([34, Lemma 1])**.** *On a shortest path between two vertices in H_3^{n+1} the largest disc moves at most twice.*

The shortest paths in H_3^{n+1} can be categorised into three cases:

- A disc $(n+1)$ moves only once;
- C disc $(n+1)$ moves *necessarily* twice;
- B both strategies are optimal;

as the subsequent theorem shows.

Theorem 1.5 ([34, Theorem 4], [33, Theorem 2.32]). *Let $s, t \in T^{n+1}$. Then there exist at most two shortest paths between s and t. If there are two, the number of the moves of the largest disc $d \in [n+1]$ for which $s_d \neq t_d$, makes the difference. It can be one or two.*

How do these two paths look like? Assume we want to go from is to jt (a $P2$ type problem) in H_3^{n+1} with $s, t \in T^n$, $i, j \in T$, and $i \neq j$. (In the case that $i = j$, i.e., that we go from is to it, we can reduce this problem to a problem in H_3^n, where we want to go from s to t.) Let $k = 3 - i - j$. If the largest disc moves once, the path is $is \to ik^n \to jk^n \to jt$ with length

$$\mathrm{d}_1(is, jt) := \mathrm{d}(s, k^n) + 1 + \mathrm{d}(t, k^n),$$

otherwise the path is $is \to ij^n \to kj^n \to ki^n \to ji^n \to jt$ with length

$$\begin{aligned} \mathrm{d}_2(is, jt) &:= \mathrm{d}(s, j^n) + 1 + 2^n - 1 + 1 + \mathrm{d}(t, i^n) \\ &= \mathrm{d}(s, j^n) + 1 + 2^n + \mathrm{d}(t, i^n). \end{aligned}$$

It is an interesting question whether for a given vertex is in H_3^{n+1} there always exists a vertex such that we are in case B, i.e., that both paths have equal length.

Lemma 1.6 ([33, Proposition 2.33]). *For every vertex $is \in T^{n+1} \setminus \{0^{n+1}, 1^{n+1}, 2^{n+1}\}, n \in \mathbb{N}$, it can be found a $jt \in T^{n+1}$ $(i \neq j)$ such that there are two shortest paths between these vertices in H_3^{n+1}.*

Remark. In [38, Corollary 3.7], it is stated that the perfect states are the only ones such that for any other regular state there is a unique shortest path with at most one move of the largest disc.

With these results we can now construct an algorithm which tells us, depending on the given vertices, whether we are in case A, C, or B. After the evaluation of the algorithm, we know consequently whether we need one or two moves of the largest disc (called *LDM*) or both strategies are optimal.

The underlying theorem, as well as the algorithm, is due to D. Romik [70]. At first he states the theorem for S_3^n together with a "machine" which realises the statements of the theorem. Additionally, he presents a "machine" similar to the one in [33, p.145] for the isomorphism θ between H_3^n and S_3^n. The two "machines" running in parallel even give us the possibility not only to determine the number of largest disc moves but also to calculate the distance between vertices in H_3^n. A theorem and an algorithm which leads us directly to the decision between A, C, and B can be found in [33, Section 2.4.3].

But we are mainly interested in the theorem for S_3^{n+1}, where we have the choice between the paths $is \to ij^n \to ji^n \to jt$ and $is \to ik^n \to ki^n \to kj^n \to jk^n \to jt$ with $s, t \in T^n$, $i, j \in T$, $i \neq j$, and $k = 3 - i - j$. If $\overline{s} = s_{d-1} \dots s_1 \in T^{d-1}$, we define $\overline{s}' = s_{d-2} \dots s_1 \in T^{d-2}$ with $d \in [n+1]_2$.

Theorem 1.7 ([70, Theorem 1])**.** *Let $s = \underline{s}i\bar{s}, t = \underline{s}j\bar{t} \in T^{n+1}$, $n \in \mathbb{N}$, with $i \neq j$ and $\bar{s}, \bar{t} \in T^{b-1}, b \in [n+1]$. We define the functions with $k = 3 - i - j$*

$$
\begin{aligned}
A\left(\bar{s},\bar{t}\right) &= \min\left\{\mathrm{d}\left(\bar{s}, j^{b-1}\right) + \mathrm{d}\left(\bar{t}, i^{b-1}\right), 2^{b-1} + \mathrm{d}\left(\bar{s}, k^{b-1}\right) + \mathrm{d}\left(\bar{t}, k^{b-1}\right)\right\},\\
B\left(\bar{s},\bar{t}\right) &= \min\left\{\mathrm{d}\left(\bar{s}, j^{b-1}\right) + \mathrm{d}\left(\bar{t}, i^{b-1}\right), \mathrm{d}\left(\bar{s}, k^{b-1}\right) + \mathrm{d}\left(\bar{t}, k^{b-1}\right)\right\},\\
C\left(\bar{s},\bar{t}\right) &= \min\left\{2^{b-1} + \mathrm{d}\left(\bar{s}, j^{b-1}\right) + \mathrm{d}\left(\bar{t}, i^{b-1}\right), \mathrm{d}\left(\bar{s}, k^{b-1}\right) + \mathrm{d}\left(\bar{t}, k^{b-1}\right)\right\}.
\end{aligned}
$$

Then we have the equations

$$
A\left(\bar{s},\bar{t}\right) = \begin{cases}
\mathrm{d}\left(\bar{s}, j^{b-2}\right) + d\left(\bar{t}, i^{b-2}\right) & (s_{b-2}, t_{b-2}) \begin{array}{ll} = (i,j) & \mathit{or} \\ = (k,i) & \mathit{or} \\ = (i,i) & \mathit{or} \\ = (j,k) & \mathit{or} \\ = (j,j) & \mathit{or} \\ = (j,i) & \end{array} \quad \text{path with one LDM} \\[2ex]
2^{b-1} + A\left(\bar{s}', \bar{t}'\right) & (s_{b-1}, t_{b-1}) \begin{array}{ll} = (i,k) & \mathit{or} \\ = (k,j) & \end{array} \\[2ex]
2^{b-1} + B\left(\bar{s}', \bar{t}'\right) & (s_{b-1}, t_{b-1}) = (k,k)
\end{cases}
$$

$$
B\left(\bar{s},\bar{t}\right) = \begin{cases}
\mathrm{d}\left(\bar{s}, j^{b-1}\right) + \mathrm{d}\left(\bar{t}, i^{b-1}\right) & (s_{b-1}, t_{b-1}) = (j,i) \quad \text{path with one LDM} \\[2ex]
\mathrm{d}\left(\bar{s}, k^{b-1}\right) + \mathrm{d}\left(\bar{t}, k^{b-1}\right) & (s_{b-1}, t_{b-1}) = (k,k) \quad \text{path with two LDMs} \\[2ex]
2^{b-2} + A\left(\bar{s}', \bar{t}'\right) & (s_{b-1}, t_{b-1}) \begin{array}{ll} = (i,i) & \mathit{or} \\ = (j,j) & \end{array} \\[2ex]
2^{b-2} + C\left(\bar{s}', \bar{t}'\right) & (s_{b-1}, t_{b-1}) \begin{array}{ll} = (k,j) & \mathit{or} \\ = (i,k) & \end{array} \\[2ex]
2^{b-2} + B\left(\bar{s}', \bar{t}'\right) & (s_{b-1}, t_{b-1}) \begin{array}{ll} = (k,i) & \mathit{or} \\ = (j,k) & \end{array} \\[2ex]
2^{b-1} + B\left(\bar{s}', \bar{t}'\right) & (s_{b-1}, t_{b-1}) = (i,j)
\end{cases}
$$

$$C\left(\overline{s},\overline{t}\right) = \begin{cases} & = (k,k) \quad \textit{or} \\ & = (k,i) \quad \textit{or} \\ & = (k,j) \quad \textit{or} \\ \mathrm{d}\left(\overline{s},k^{b-1}\right) + \mathrm{d}\left(\overline{t},k^{b-1}\right) \quad (s_{b-1},t_{b-1}) & = (i,j) \quad \textit{or} \quad \text{path with two LDMs} \\ & = (j,k) \quad \textit{or} \\ & = (i,k) \\ 2^{b-1} + B\left(\overline{s}',\overline{t}'\right) \quad (s_{b-1},t_{b-1}) & = (j,i) \\ 2^{b-1} + C\left(\overline{s}',\overline{t}'\right) \quad (s_{b-1},t_{b-1}) & = (j,j) \quad \textit{or} \\ & = (i,i). \end{cases}$$

These functions stand for the three possibilities "path with one LDM", "both paths are equal", *and* "path with two LDMs" *at the end of the reading of all pairs.*

Using this theorem, we can construct a "machine" or *automaton* which decides between the strategies with one and with two LDMs (see Figure 1.3). This leads to Algorithm 1.

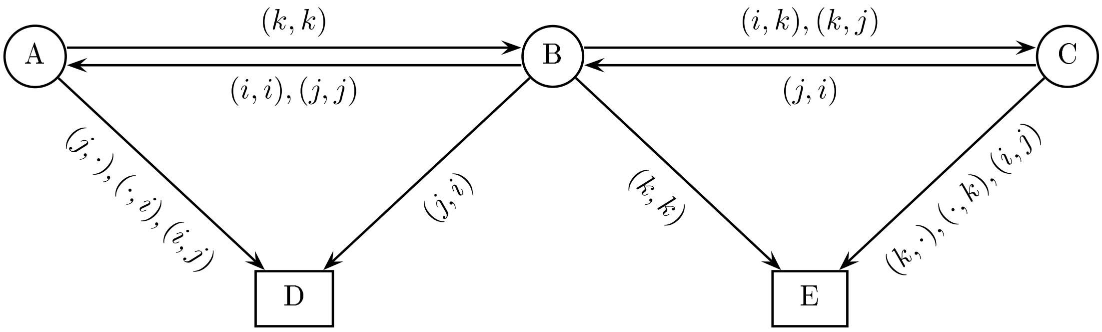

Figure 1.3: Romik's automaton for S_3^n

With the algorithm we can now prove an analog to Lemma 1.6 for S_3^{n+1}.

Corollary 1.8 ([38, Corollary 3.6], [33, Proposition 4.2]). *Let* $T = \{i, j, k\}$. *For every vertex* $is \in T^{n+1} \setminus \{0^{n+1}, 1^{n+1}, 2^{n+1}\}$, $n \in \mathbb{N}$, *it can be found a* $jt \in T^{n+1}$ $(i \neq j)$ *such that there are two shortest paths between these vertices in* S_3^{n+1}.

Proof. Every non-extreme vertex $is \in T^{n+1}$ is of the form $i^{1+n-d}k\overline{s}$ with $\overline{s} \in T^{d-1}$ and $d \in [n]$. Assuming that we go from is to jt, we want to find out which form the vertex jt must have such that we stay in state B of the automaton in Figure 1.3. Looking at the figure, we recognise that for every $m \in T$ only the input $(m, k \vartriangle (i \vartriangle m))$ leads thereto that B is not left anymore. Define the vertex $jt\,(i \neq j)$ as $jk^{n-d+1}\overline{t}$ with $\forall \delta \in [d-1] : t_\delta = k \vartriangle (i \vartriangle s_\delta)$. Now we evaluate the pair (is, jt)

Algorithm 1 The $P2$ decision algorithm for S_3^{n+1}

```
procedure P2S(n, s, t)
    parameter n : number of discs minus 1 (n ∈ ℕ)
    parameter s : initial configuration (s ∈ T^{n+1})
    parameter t : goal configuration (t ∈ T^{n+1}, s_{n+1} ≠ t_{n+1} )
    i ← s_{n+1}
    j ← t_{n+1}
    start in state A of the automaton
    δ ← n
    while δ > 0 do
        apply automaton to pair (s_δ, t_δ)      ▹ algorithm STOPs if automaton
                                                  reaches state D or E
        δ ← δ − 1
    end while
end procedure
```

in the automaton. The first input pair (i, j) determines the automaton as in Figure 1.3. Then the following $n - d$ pairs (i, k) keep it in state A. With the pair (k, k) we move to state B. Then using our above analysis and the definition of $\bar{t}$ we will not leave the state B anymore to the end. □

It is of interest how long the average running time of the automaton is in which it is decided, which of the possibilities is optimal in the problem of $P2$ type. We will deduce this using the theory of Markov chains. For the basic theory we refer to [77]. We number the states A, B, C, D, and E with $1, 2, 3, 4$, and 5. Let us consider the automaton as a Markov chain with states $1, 2, 3, 4$, and 5, where we start in 1 and move from one state to another with a certain probability. The *transition matrix* P of the automaton gives these probabilities.

$$P = \frac{1}{9}\begin{pmatrix} 2 & 1 & 0 & 6 & 0 \\ 2 & 3 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Looking at the states 4 and 5, we see that they are a bit special in the following way. An *absorbing state* is a state that, once entered, cannot be left. Then a Markov chain is called *absorbing* if the process has absorbing states. Because of the two absorbing states 4 and 5, our Markov chain is absorbing. The matrix P is of the form

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}.$$

Q is the part of the matrix which describes the transition probabilities from some *transient state* to another, whereas R gives the transition probabilities from transient to absorbing states. I is the identity matrix and 0 stands for the zero matrix. In an absorbing Markov chain, $Q^n \to 0$ when $n \to \infty$ and $I - Q$ has an inverse

$$M = (I - Q)^{-1} = \sum_{n=0}^{\infty} Q^n.$$

M_{uv} is the expected number of visits which the chain made to state v provided that it has started in state u.

We get

$$M = \left(\frac{1}{9}\begin{pmatrix} 7 & -1 & 0 \\ -2 & 6 & -2 \\ 0 & -1 & 7 \end{pmatrix}\right)^{-1} = \begin{pmatrix} \frac{180}{133} & \frac{9}{38} & \frac{9}{133} \\ \frac{9}{19} & \frac{63}{38} & \frac{9}{19} \\ \frac{9}{133} & \frac{9}{38} & \frac{180}{133} \end{pmatrix}.$$

As we start in state 1, we get as expected time we will be in one of the states $1, 2$ or 3 the sum of the first row

$$\frac{180}{133} + \frac{9}{38} + \frac{9}{133} = \frac{63}{38}.$$

It follows the subsequent theorem due to Romik [70]:

Theorem 1.9. *The average number of disc pairs evaluated by Romik's automaton is bounded above by and converges, as $n \to \infty$, to $\frac{63}{38}$.*

Since in any case we also have to read the pair of largest discs (s_{n+1}, t_{n+1}) in addition, we calculate altogether $1 + \frac{63}{38} = \frac{101}{38}$ as the number of pairs which have to be evaluated in average till the decision problem is solved. It can also be seen that in any case at least two pairs of input data have to be checked by the algorithm.

Remark. According to [33, pp.147], we can even reduce the number of input pairs further. We observe that the input of a pair with j as first component in A of the automaton in Figure 1.3 will always lead to D. In this case we need only half a pair of input, and we have to check in A $\frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot 1 = \frac{5}{6}$ pairs. In addition, we notice that this is also possible for C with k as first component. Using the above analysis, we accordingly need only

$$\frac{5}{6} \cdot \frac{180}{133} + \frac{9}{38} + \frac{5}{6} \cdot \frac{9}{133} = \frac{27}{19}$$

pairs of input. Together with the pair of largest discs (s_{n+1}, t_{n+1}), we get $\frac{46}{19}$ pairs which have to be checked in average. Still at least two pairs of input data have to be checked in any case.

1.3 Sierpiński graphs S_p^n with base $p \in \mathbb{N}$ and exponent $n \in \mathbb{N}_0$

In the last sections we have used the set $T = \{0, 1, 2\}$ as the set of pegs. In the following we extend now the set of pegs to $[p]_0$ to define the general Sierpiński graphs with base p and exponent n.

Definition 1.10. The *(general) Sierpiński graphs S_p^n with base $p \in \mathbb{N}$ and exponent $n \in \mathbb{N}_0$* are defined by the vertex set $V(S_p^n) = [p]_0^n$ and the edge set

$$\begin{aligned} & E(S_p^0) &=& \emptyset, \\ \forall n \in \mathbb{N}_0 : \quad & E(S_p^{n+1}) &=& \left\{\{is, is'\} \;\middle|\; i \in [p]_0, \{s, s'\} \in E(S_p^n)\right\} \\ & & & \cup \{\{ij^n, ji^n\} \mid i, j \in [p]_0, i \neq j\}\,. \end{aligned} \tag{1.8}$$

Hence two vertices $s, t \in [p]_0^n$ with $s = s_n \dots s_1$ and $t = t_n \dots t_1$ are adjacent in S_p^n if and only if there exists a $d \in [n]$ such that

a) $\forall k \in [n] \setminus [d] : s_k = t_k$,

b) $s_d \neq t_d$,

c) $\forall k \in [d-1] : s_k = t_d \wedge t_k = s_d$.

If $d = 1$, Condition c) is void. The same follows for Condition a) in the case $d = n$. This was the first definition of Sierpiński graphs by S. Klavžar and U. Milutinović [52]. (Note that in [52] states are given by $s_1 \dots s_n$.)

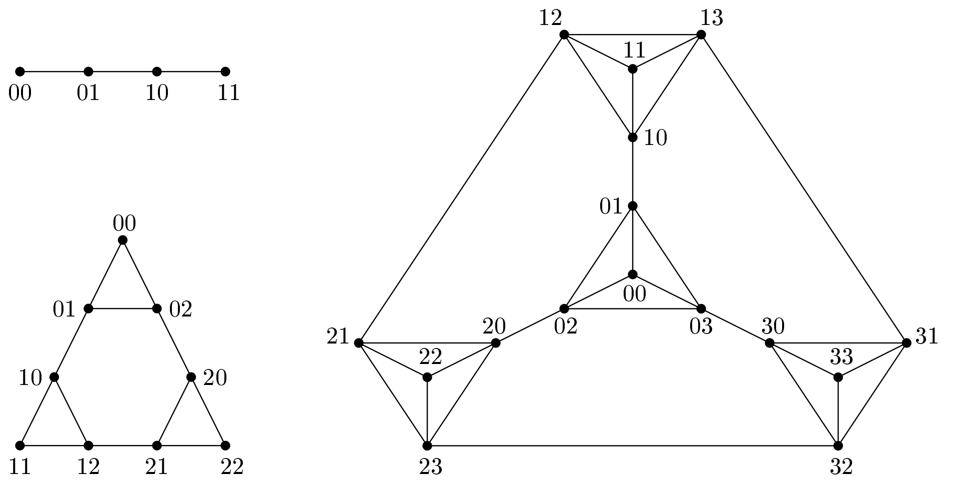

Figure 1.4: The Sierpiński graphs S_2^2 (top left), S_3^2 (bottom left), and S_4^2 (right)

One basic example for Sierpiński graphs is the graph S_p^1 for any $p \in \mathbb{N}$, which is the complete graph K_p on p vertices. For any $n \in \mathbb{N}$, S_2^n is isomorphic to the path P_{2^n} on 2^n vertices (for S_2^2 see Figure 1.4).

Similar to (1.1) for S_3^n, we can define the edge set by

$$E(S_p^n) = \left\{ \left\{ \underline{s}ij^{d-1}, \underline{s}ji^{d-1} \right\} \;\middle|\; i, j \in [p]_0, i \neq j, d \in [n], \underline{s} \in [p]_0^{n-d} \right\}.$$

It is obvious that the number of vertices, namely $|V(S_p^n)|$, is equal to p^n. Following [68], we calculate the number of edges $\|S_p^n\| = |E(S_p^n)|$. We see that there are exactly p extreme vertices of degree $p-1$. The $p^n - p$ inner vertices have degree p. As a result, we get

$$\|S_p^n\| = \frac{1}{2}\left(p(p-1) + (p^n - p)p\right) = \frac{p}{2}\left(p^n - 1\right).$$

Additionally, we can observe for any $n \in \mathbb{N}$ and any $p \in \mathbb{N}_3$ that for every pair of vertices in S_p^n there is a path between these two vertices with the property that every vertex of S_p^n is visited exactly once on this path, called a *Hamiltonian path*.

Lemma 1.11 ([52, Proposition 3]). *For any $n \in \mathbb{N}$ and any $p \in \mathbb{N}_3$ the graph S_p^n is hamiltonian.*

The graphs are connected based on Definition 1.8 or on the last lemma. Hence we can define a distance function, denoted by d, on these graphs with base $p \in \mathbb{N}$. This was done for the first time in [52, Lemma 4] by Klavžar and Milutinović.

Theorem 1.12 ([33, Proposition 4.5.], [52, Lemma 4]). *For any $j \in [p]_0$ and for any vertex $s = s_n \ldots s_1$ of S_p^n there is exactly one shortest path between j^n and s, and*

$$\mathrm{d}(j^n, s) = \sum_{d=1}^{n} [s_d \neq j] \cdot 2^{d-1}. \tag{1.9}$$

It follows that for any $i, j \in [p]_0$ $(i \neq j)$ the distance between the extreme vertices i^n and j^n is $\mathrm{d}(i^n, j^n) = 2^n - 1$.

Proof. We prove the theorem by induction on n. If $n = 0$, the statement is clearly true. Let $n \in \mathbb{N}_0$. For $p = 1$ it is obvious that the theorem holds. Now let $p \in \mathbb{N}_2$ and $s = s_{n+1}\overline{s}$, $\overline{s} \in [p]_0^n$. We distinguish two cases, namely $s_{n+1} = j$ and $s_{n+1} \neq j$. If $s_{n+1} = j$, then we can take the shortest path from j^n to $\overline{s}$ in S_p^n and add a j in front of each vertex. It follows that

$$\mathrm{d}(j^{n+1}, s) \leq \sum_{d=1}^{n} [s_d \neq j] \cdot 2^{d-1} = \sum_{d=1}^{n+1} [s_d \neq j] \cdot 2^{d-1}.$$

If $s_{n+1} \neq j$, we find a path from j^{n+1} to s by going from j^{n+1} to js_{n+1}^n in $2^n - 1$ steps, then moving to $s_{n+1}j^n$ in one step, and finally from here to $s_{n+1}\overline{s}$ on a (shortest) path of length $\leq \sum_{d=1}^{n} [s_d \neq j] \cdot 2^{d-1}$. Hence

$$\begin{aligned} \mathrm{d}(j^{n+1}, s) &\leq (2^n - 1) + 1 + \textstyle\sum_{d=1}^{n} [s_d \neq j] \cdot 2^{d-1} \\ &\leq 2^{n+1} - 1 = \textstyle\sum_{d=1}^{n+1} [s_d \neq j] \cdot 2^{d-1}. \end{aligned}$$

We prove now that this is the unique shortest path by obtaining that no optimal path can touch a subgraph kS_p^n for $s_{n+1} \neq k \neq j$. Assume that there is a shorter path with the property that kS_p^n is the first subgraph which is touched by the path.

Then the path contains the following parts:

- a path from j^{n+1} to jk^n,
- the edge $\{jk^n, kj^n\}$,
- a path from kj^n to ki^n, $j \neq i \neq k$, and
- one edge in order to leave the subgraph kS^n_p at ki^n.

If we now employ the induction assumption, we add the length of this path to altogether at least $(2^n - 1) + 1 + (2^n - 1) + 1 = 2^{n+1}$. But we already found a shorter one above such that this path cannot be the shortest. □

D. Parisse showed some further properties of Sierpiński graphs in [68] which correspond to Proposition 1.2 for Sierpiński graphs S^n_3.

Proposition 1.13. *Let $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$.*

a) For any $s \in [p]^n_0$,

$$\sum_{l=0}^{p-1} \mathrm{d}(s, l^n) = (p-1) \cdot (2^n - 1).$$

b) Fix an extreme vertex j^n. Then for $\mu \in [2^n]_0$:

$$|\{s \in [p]^n_0 \mid \ \mathrm{d}(s, j^n) = \mu\}| = (p-1)^{\beta(\mu)}$$

and

$$\sum_{\mu=0}^{2^n-1} (p-1)^{\beta(\mu)} = p^n,$$

where $\beta(\mu)$ is number of 1*s in the binary expansion of μ.*

Proof. In [68, Proposition 2.5], [68, Corollary 2.4], and [33, Corollary 4.6], one can find the proofs for the statements. □

We introduce the *combinatorial number*. Let $k \in \mathbb{N}_0$ and $n \in \mathbb{N}_k$. Then the combinatorial number is defined by

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

Let S be a set of size n. We define $\binom{S}{k}$ to be the set of all subsets of size k of S. Formally, $\binom{S}{k} = \{T \subseteq S \mid |T| = k\}$. Furthermore, for all $k \in [n+1]_0$ the size $\left|\binom{S}{k}\right| = \binom{n}{k}$; see [33, p.15].

Let G be a graph. The *eccentricity of a vertex* $s \in V(G)$, denoted by $\varepsilon_G(s)$, gives the maximal distance between the vertex s and all the other vertices in $V(G)$. According to [68], this integer for a vertex $s \in [p]_0^n$ in S_p^n is given by the maximum of the distances between s and the extreme vertices j^n. The *average eccentricity of* G is the arithmetic mean of all eccentricities, i.e.,

$$\overline{\varepsilon}(G) := \frac{1}{|G|} \sum_{s \in V(G)} \varepsilon_G(s).$$

For $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$, the average eccentricity of S_p^n is

$$\overline{\varepsilon}(S_p^n) = \left(1 - \binom{2p}{p-1}^{-1}\right) 2^n - \frac{p-1}{p} - \sum_{k=0}^{p-2} (-1)^{p-k} \frac{p-1-k}{2p-k} \binom{p}{k} \left(\frac{k}{p}\right)^n ,$$

which was shown in [41]. Furthermore, we define that the *diameter of* G is

$$\mathrm{diam}(G) := \max\{\varepsilon_G(s) \mid s \in V(G)\} = \max\{\mathrm{d}(s,t) \mid s,t \in V(G)\}.$$

It was proved in [68] that for all $n \in \mathbb{N}$ and $p \in \mathbb{N}_2$ the diameter of S_p^n is equal to $2^n - 1$ using the result [52, Lemma 4]. This value $2^n - 1$ is especially attained by the calculation of the distance between the extreme vertices. Therefore, we called them extreme. In the case of Hanoi graphs H_p^n, this is not true for the perfect vertices. R. Korf found out that for $p = 4$ and $n = 15$ (but not for smaller n) the eccentricity of a perfect state and, consequently, the diameter of H_4^{15} is strictly larger than the distance between two perfect states. In the calculations of the eccentricities of perfect states in H_4^n for $n \leq 22$ in [55] and [56], it was shown that Korf's phenomenon also occurs for $n = 20$ to 22.

There exists an analogue to the boxer rule in Lemma 1.3 for the S_p^{n+1}.

Lemma 1.14 ([33, Lemma 4.7])**.** *If on a shortest path between two vertices in S_p^{n+1} ($n \in \mathbb{N}_0$) the largest disc is moved away from a peg, it will not return to the same peg.*

Proof. We prove this by contradiction. Assume that disc $(n+1)$ leaves iS_p^n and returns there on a shortest path. Then this shortest path must contain a path $ij^n \to P' \to ik^n$. Since there is only one edge between two subgraphs iS_p^n and jS_p^n, the path P' is thereby a ji^n, ki^n-path with $|\{k,i,j\}| = 3$. This path P' must contain a ji^n, jl^n-path ($i \neq l \neq j$) such that the length of P' is greater than or equal to $2^n - 1$ according to Theorem 1.12. Hence the length of the path $ij^n \to P' \to ik^n$ is greater than 2^n. But by the same theorem we also know that $\mathrm{d}(ij^n, ik^n) < 2^n$. Therefore, this path cannot be contained in a shortest path. □

In Corollary 1.8 we stated that for every non-extreme vertex of S_3^n there is a vertex such that there are two shortest paths between them.

For S_p^n we can find a similar statement in [82].

Proposition 1.15 ([82, Corollary 3.5])**.** *Let $n \in \mathbb{N}_2$ and $p \in \mathbb{N}_3$. Further, let s be any non-extreme vertex in S_p^n. Then there exists a $t \in [p]_0^n$ such that there are two shortest s,t-paths.*

As we saw, there is a correspondence of statements between the Hanoi graphs H_3^n and the general Sierpiński graphs S_p^n. Hinz, Klavžar, and Zemljič even showed that an isomorphic copy of S_p^n is a spanning subgraph[5] of H_p^n (i.e., an *isomorphic embedding* exists,) if and only if p is odd or if $n = 1$.

Theorem 1.16 ([39, Theorem 3.1])**.** *Let $p, n \in \mathbb{N}$. Then S_p^n can be embedded isomorphically into H_p^n if and only if p is odd or $n = 1$.*

In the following theorem we characterise the distance between two arbitrary vertices more closely. Let $n \in \mathbb{N}_0$, $f \in [p]_0$, and $V_f = \left\{fs \mid s \in [p]_0^n\right\}$ be the set of vertices in S_p^{n+1} consisting of all vertices beginning with f.

Theorem 1.17 ([52, Theorem 5], [33, Theorem 4.8])**.** *Assume $p \in \mathbb{N}_3$. Let $s = \underline{s}i\overline{s}, t = \underline{s}j\overline{t} \in [p]_0^{n+1}$ be two vertices with $i \neq j$, $\overline{s}, \overline{t} \in [p]_0^d$ for $d \in [n]$ and $\underline{s} \in [p]_0^{n-d}$. Then*

$$\begin{aligned} \mathrm{d}(s,t) = \quad & \min\Big\{\mathrm{d}\left(\overline{s}, j^d\right) + 1 + \mathrm{d}\left(\overline{t}, i^d\right), \\ & \mathrm{d}\left(\overline{s}, k^d\right) + 1 + 2^d + \mathrm{d}\left(\overline{t}, k^d\right) \mid k \in [p]_0 \setminus \{i, j\}\Big\} \end{aligned}$$

Proof. By induction on n. If $n = 0$, we get $d = 0$ and $\mathrm{d}(s,t) = 1$. We note that it is sufficient to consider only paths in the subgraph of S_p^{n+1} whose vertices start with $\underline{s}$. Hence we can assume that $d = n$. Let $n \in \mathbb{N}$. Let P be a shortest path among the paths between s and t which have vertices from $V_i \cup V_j$. Since P must contain the edge between V_i and V_j, namely $\{ij^n, ji^n\}$, we obtain by Theorem 1.12

$$|P| = \mathrm{d}\left(\overline{s}, j^n\right) + 1 + \mathrm{d}\left(\overline{t}, i^n\right)$$

and the uniqueness of the so-called *direct path*. Now we consider a shortest path P' between s and t with vertices only from $V_i \cup V_k \cup V_j$, where $k \in [p]_0 \setminus \{i, j\}$ and $P' \cap V_k \neq \emptyset$. This path must contain the edges $\{ik^n, ki^n\}$ and $\{kj^n, jk^n\}$ such that we get again by Theorem 1.12

$$|P'| = \mathrm{d}\left(\overline{s}, k^n\right) + 1 + (2^n - 1) + 1 + \mathrm{d}\left(\overline{t}, k^n\right).$$

For fixed k this so-called V_k*-path* is unique by the same theorem. The length of the direct path P is obviously strictly less than 2^{n+1}. If we consider a path which contains also vertices from a subgraph V_l with $|\{i, j, k, l\}| = 4$, its length is at least $1 + (2^n - 1) + 1 + (2^n - 1) + 1 = 2^{n+1} + 1$. Hence the theorem follows. □

We know even more about shortest paths in S_p^n.

Theorem 1.18 ([52, Theorem 6])**.** *There are at most two shortest paths between any two vertices of S_p^n.*

[5] Let $G = (V(G), E(G))$ be a graph. A graph $H = (V(H), E(H))$ is called a *subgraph* of the graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *spanning subgraph* of G is a subgraph of G which contains every vertex of G.

In [54], Klavžar and Zemljič introduced a new kind of vertices, called *almost-extreme vertices,* which are either of the form ij^n or $i^n j$, where $i \neq j$. The almost-extreme vertices are obviously the immediate neighbours of extreme vertices. For these vertices we can calculate the distance to arbitrary vertices in S_p^{n+1} with base $p \in \mathbb{N}_2$.

Proposition 1.19 ([54, Proposition 4]). *Let $m \in [n+1]_0$, $p \in \mathbb{N}_2$, $i, j \in [p]_0$, $i \neq j$, $l \in [p]_0 \setminus \{i\}$, and $s \in [p]_0^m$. Then*

$$\mathrm{d}(i^{n-m} js, i^n l) = \mathrm{d}(s, i^m) + 2^m - [j = l] = \sum_{d=1}^{m} [s_d \neq i] \cdot 2^{d-1} + 2^m - [j = l].$$

For the almost-extreme vertices $s = ij^n$ or $i^n j$ with $\{i, j\} \in \binom{[p]_0}{2}$, $p \in \mathbb{N}_3$, and $n \in \mathbb{N}$ we can also determine the set of vertices to (or from) which two shortest paths lead. We will denote this set of vertices by B_s.

Proposition 1.20 ([82, Theorem 3.1]). *For any almost-extreme vertex $s = ij^n$ in S_p^{n+1}, we have*

$$B_s = \left\{ \underline{t} j i^{d-1} \mid d \in [n], \underline{t} \in ([p]_0 \setminus \{i, j\})^{n-d+1} \right\}.$$

Proposition 1.21 ([82, Theorem 3.3]). *For any almost-extreme vertex $s = i^n j$ in S_p^{n+1}, we have*

$$B_s = \left\{ i^{n-d} t j^d \mid d \in [n], t \in [p]_0 \setminus \{i, j\} \right\}.$$

A class of almost-extreme vertices are the "special" vertices. We call a vertex $s \in [p]_0^{n+1}$ *special* with $i, j, k \in [p]_0$ and $|\{i, j, k\}| = 3$, if there is a $\delta \in [n+1]$ such that $s = \underline{s} k \overline{s}$ with $\underline{s} \in ([p] \setminus \{j, k\})^{n+1-\delta}$ and $\overline{s} \in [p]_0^{\delta-1}$.

Proposition 1.22 ([54, Proposition 7]). *Let $m \in [n]$, $p \in \mathbb{N}_3$, $i, j, k \in [p]_0$, $|\{i, j, k\}| = 3$, and $s \in [p]_0^m$. Then*

$$\mathrm{d}(i^{n-m+1} s, i^{n-m} jk^m) = \begin{cases} \mathrm{d}(s, k^m) + 2^m + 1, & \text{if } s \text{ is special,} \\ \mathrm{d}(s, j^m) + 2^m - [i = k](2^m - 1), & \text{otherwise.} \end{cases}$$

1.4 A *P2* decision algorithm for Sierpiński graphs S_p^n with base $p \in \mathbb{N}_3$ and exponent $n \in \mathbb{N}_0$

We have seen in Section 1.2 that there exists an automaton for the Sierpiński graphs S_3^n or Hanoi graphs H_3^n, respectively, with which we can decide whether we need one LDM, two LDMs or whether both shortest paths are of equal length. The aim of this section is to find a similar automaton for the Sierpiński graphs S_p^{n+1} with base $p \in \mathbb{N}_3$ and exponent $n \in \mathbb{N}_0$. Since there is no decision necessary for the cases $p = 1$ and $p = 2$, we omit them in the following. An alternative representation of the material in this section can be largely found in the author's article [37] with A. M. Hinz.

1.4.1 The underlying principle

From the last section we already know the basic properties of the Sierpiński graphs, which we keep in mind. In Lemma 1.14 we stated that the largest disc once removed from a peg will never return there. Furthermore, we know that the diameter of S_p^{n+1} is $2^{n+1}-1$ and analysed that in S_p^{n+1} the passage through two "detour" subgraphs take at least $2^{n+1}+1$ moves.

We recall Theorem 1.17 about distances in S_p^{n+1} and the notation $V_f = \left\{fs \mid s \in [p]_0^n\right\}$ for the set of vertices in S_p^{n+1} consisting of all vertices beginning with $f \in [p]_0$ for $n \in \mathbb{N}_0$.

Theorem 1.23 ([37, Lemma 1.1]). *Let $s = \underline{s}i\overline{s}, t = \underline{s}j\overline{t} \in [p]_0^{n+1}$ be two vertices with $i, j \in [p]_0$, $i \neq j$, $\overline{s}, \overline{t} \in [p]_0^d$ for $d \in [n]$ and $\underline{s} \in [p]_0^{n-d}$. Then*

$$\mathrm{d}(s,t) = \min\left\{\mathrm{d}_k\left(i\overline{s}, j\overline{t}\right) \mid k \in [p+1]_0 \setminus \{i,j\}\right\}.$$

Thereby the distance $\mathrm{d}_k\left(i\overline{s}, j\overline{t}\right) := \mathrm{d}\left(\overline{s}, k^d\right) + 1 + 2^d + \mathrm{d}\left(\overline{t}, k^d\right)$ for $k \in [p]_0$ gives the number of moves of the unique V_k-path realised by $i\overline{s} \to ik^d \to ki^d \to kj^d \to jk^d \to j\overline{t}$. The distance $\mathrm{d}_p\left(i\overline{s}, j\overline{t}\right) := \mathrm{d}\left(\overline{s}, j^d\right) + 1 + \mathrm{d}\left(\overline{t}, i^d\right)$ is realised by the path $i\overline{s} \to ij^d \to ji^d \to j\overline{t}$. We omitted the prefix $\underline{s}$ which remains constant throughout the paths.

We consider the Sierpiński graphs S_p^{N+1} with $N \in \mathbb{N}_0$ in the following theorem, which is also stated in [37] in another version. The exponent is changed to $N+1$ in order to make later direct use of Formula (1.9) together with the results which arises from the application of the theorem. If $\overline{s} = s_{d-1}\ldots s_1 \in [p]_0^{d-1}$, we define $\overline{s}' = s_{d-2}\ldots s_1 \in [p]_0^{d-2}$ with $d \in [N+1]_2$.

Theorem 1.24. *Let $s = \underline{s}is_{n-1}\overline{s}, t = \underline{s}jt_{n-1}\overline{t} \in [p]_0^{N+1}$, $N \in \mathbb{N}_0$, with $i \neq j$ and $n \in [N+1]$, $\overline{s}, \overline{t} \in [p]_0^{n-2}$, and further $\underline{s} \in [p]_0^{N+1-n}$. If either $s_{n-1} \neq i, j$ or $t_{n-1} \neq i, j$, we set the value s_{n-1} or t_{n-1}, respectively, equal to h. If both are not equal to i and j and $s_{n-1} \neq t_{n-1}$, we set s_{n-1} equal to g and t_{n-1} equal to h, otherwise we set both equal to h. We define the functions with $k \in \{g,h\}$*

$$\begin{aligned}
A'\left(\overline{s},\overline{t}\right) &= \min\left\{\mathrm{d}\left(\overline{s}, j^{n-2}\right) + \mathrm{d}\left(\overline{t}, i^{n-2}\right), 2^{n-2} + \mathrm{d}\left(\overline{s}, k^{n-2}\right) + \mathrm{d}\left(\overline{t}, k^{n-2}\right)\right\}\\
A\left(\overline{s},\overline{t}\right) &= \min\left\{\mathrm{d}\left(\overline{s}, j^{n-2}\right) + \mathrm{d}\left(\overline{t}, i^{n-2}\right), 2^{n-2} + \mathrm{d}\left(\overline{s}, k^{n-2}\right) + \mathrm{d}\left(\overline{t}, k^{n-2}\right)\right\}\\
B\left(\overline{s},\overline{t}\right) &= \min\left\{\mathrm{d}\left(\overline{s}, j^{n-2}\right) + \mathrm{d}\left(\overline{t}, i^{n-2}\right), \mathrm{d}\left(\overline{s}, k^{n-2}\right) + \mathrm{d}\left(\overline{t}, k^{n-2}\right)\right\}\\
C\left(\overline{s},\overline{t}\right) &= \min\left\{2^{n-2} + \mathrm{d}\left(\overline{s}, j^{n-2}\right) + \mathrm{d}\left(\overline{t}, i^{n-2}\right), \mathrm{d}\left(\overline{s}, k^{n-2}\right) + \mathrm{d}\left(\overline{t}, k^{n-2}\right)\right\}.
\end{aligned}$$

Then we have four cases for $\mathrm{d}(s,t)$ depending on the pair (s_{n-1}, t_{n-1})

$$\mathrm{d}(s,t) = \begin{cases}
 & \qquad\qquad\qquad\qquad\qquad\qquad = (i,j) \quad or \\
2^{n-1} + 1 + \mathrm{d}\left(\overline{s}, j^{n-2}\right) + \mathrm{d}\left(\overline{t}, i^{n-2}\right) & (s_{n-1}, t_{n-1}) = (j,\cdot) \quad or \quad \text{direct path}\\
 & \qquad\qquad\qquad\qquad\qquad\qquad = (\cdot, i)\\
2^{n-1} + 1 + A'\left(\overline{s},\overline{t}\right) & (s_{n-1}, t_{n-1}) = (g,h)\\
2^{n-1} + 1 + A\left(\overline{s},\overline{t}\right) & (s_{n-1}, t_{n-1}) = (i,h) \quad or\\
 & \qquad\qquad\qquad\qquad\qquad\qquad = (h,j)\\
2^{n-1} + 1 + B\left(\overline{s},\overline{t}\right) & (s_{n-1}, t_{n-1}) = (h,h)
\end{cases}$$

In the second case we set $k = g$ if only g occurs in (s_{n-2}, t_{n-2}), and $k = h$ if only h occurs in this pair, and in the third and fourth case always $k = h$ is used. Then we get

$$
A'\left(\bar{s},\bar{t}\right) = \begin{cases}
\mathrm{d}\left(\bar{s}, j^{n-2}\right) + \mathrm{d}\left(\bar{t}, i^{n-2}\right) & (s_{n-2}, t_{n-2}) \begin{array}{l} = (i,j) \\ = (j,\cdot) \\ = (\cdot,i) \\ = (i,l') \\ = (l',j) \\ = (l',m') \end{array} \begin{array}{l} \textit{or} \\ \textit{or} \\ \textit{or} \\ \textit{or} \\ \textit{or} \\ \end{array} \quad \text{direct path} \\[2ex]
2^{n-2} + A'\left(\bar{s}',\bar{t}'\right) & (s_{n-2}, t_{n-2}) \begin{array}{l} = (g,h) \\ = (h,g) \end{array} \begin{array}{l} \textit{or} \\ \end{array} \\[2ex]
2^{n-2} + A\left(\bar{s}',\bar{t}'\right) & (s_{n-2}, t_{n-2}) \begin{array}{l} = (u',k) \\ = (k,v') \end{array} \begin{array}{l} \textit{or} \\ \end{array} \\[2ex]
2^{n-2} + B\left(\bar{s}',\bar{t}'\right) & (s_{n-2}, t_{n-2}) = (k,k)
\end{cases}
$$

$$
A\left(\bar{s},\bar{t}\right) = \begin{cases}
\mathrm{d}\left(\bar{s}, j^{n-2}\right) + \mathrm{d}\left(\bar{t}, i^{n-2}\right) & (s_{n-2}, t_{n-2}) \begin{array}{l} = (i,j) \\ = (j,\cdot) \\ = (\cdot,i) \\ = (l,j) \\ = (i,l) \\ = (l,m) \end{array} \begin{array}{l} \textit{or} \\ \textit{or} \\ \textit{or} \\ \textit{or} \\ \textit{or} \\ \end{array} \quad \text{direct path} \\[2ex]
2^{n-2} + A\left(\bar{s}',\bar{t}'\right) & (s_{n-2}, t_{n-2}) = \textit{all other cases} \\[2ex]
2^{n-2} + B\left(\bar{s}',\bar{t}'\right) & (s_{n-2}, t_{n-2}) = (k,k)
\end{cases}
$$

$$
B\left(\bar{s},\bar{t}\right) = \begin{cases}
\mathrm{d}\left(\bar{s}, j^{n-2}\right) + \mathrm{d}\left(\bar{t}, i^{n-2}\right) & (s_{n-2}, t_{n-2}) = (j,i) \quad \text{direct path} \\[2ex]
\mathrm{d}\left(\bar{s}, k^{n-2}\right) + \mathrm{d}\left(\bar{t}, k^{n-2}\right) & (s_{n-2}, t_{n-2}) = (k,k) \quad V_k\text{-path} \\[2ex]
2^{n-3} + A\left(\bar{s}',\bar{t}'\right) & (s_{n-2}, t_{n-2}) \begin{array}{l} = (u,i) \\ = (j,v) \end{array} \begin{array}{l} \textit{or} \\ \end{array} \\[2ex]
2^{n-3} + C\left(\bar{s}',\bar{t}'\right) & (s_{n-2}, t_{n-2}) \begin{array}{l} = (k,v) \\ = (u,k) \end{array} \begin{array}{l} \textit{or} \\ \end{array} \\[2ex]
2^{n-3} + B\left(\bar{s}',\bar{t}'\right) & (s_{n-2}, t_{n-2}) \begin{array}{l} = (k,i) \\ = (j,k) \end{array} \begin{array}{l} \textit{or} \\ \end{array} \\[2ex]
2^{n-2} + B\left(\bar{s}',\bar{t}'\right) & (s_{n-2}, t_{n-2}) = \textit{all other cases}
\end{cases}
$$

$$C\left(\bar{s},\bar{t}\right) = \begin{cases} & = (k,\cdot) & \textit{or} \\ \mathrm{d}\left(\bar{s},k^{n-2}\right)+\mathrm{d}\left(\bar{t},k^{n-2}\right) \quad (s_{n-2},t_{n-2}) & = (\cdot,k) & \textit{or} \quad V_k\text{-path} \\ & = (u,v) & \\ 2^{n-2}+B\left(\bar{s}',\bar{t}'\right) \quad (s_{n-2},t_{n-2}) & = (j,i) & \\ 2^{n-2}+C\left(\bar{s}',\bar{t}'\right) \quad (s_{n-2},t_{n-2}) & = \textit{all other cases} & \end{cases}$$

with $u \in [p]_0\backslash\{k,j\}$, $u' \in [p]_0\backslash\{g,h,j\}$, $v \in [p]_0\backslash\{k,i\}$, $v' \in [p]_0\backslash\{g,h,i\}$, $l,\ m \in [p]_0 \setminus \{k,i,j\}$, and $l',m' \in [p]_0\backslash\{g,h,i,j\}$. These functions stand for the three possibilities "direct path", "both paths", and "V_k-path" at the end of the input of all pairs.

Proof. It is sufficient to assume $s = is_{n-1}\dots s_1$ and $t = jt_{n-1}\dots t_1$. If either $s_{n-1} \neq i,j$ or $t_{n-1} \neq i,j$, we set the value s_{n-1} or t_{n-1}, respectively, equal to h. If both are not equal to i and j and $s_{n-1} \neq t_{n-1}$, we set s_{n-1} equal to g and t_{n-1} equal to h, otherwise both equal to h.
According to Theorem 1.23, the length of the direct path between s and t is

$$\mathrm{d}\left(s_{n-1}\dots s_1, j^{n-1}\right)+1+\mathrm{d}\left(t_{n-1}\dots t_1, i^{n-1}\right),$$

while the length of the V_f-path is

$$\mathrm{d}\left(s_{n-1}\dots s_1, f^{n-1}\right)+1+2^{n-1}+\mathrm{d}\left(t_{n-1}\dots t_1, f^{n-1}\right)$$

for any $f \in [p]_0 \setminus \{i,j\}$. We define by

$$\begin{aligned}\Delta_f :=& 2^{n-1}+\mathrm{d}\left(s_{n-1},\dots s_1, f^{n-1}\right)-\mathrm{d}\left(s_{d-1},\dots s_1, f^{d-1}\right)+\mathrm{d}\left(t_{n-1},\dots t_1, f^{n-1}\right)\\ &-\mathrm{d}\left(t_{d-1},\dots t_1, f^{d-1}\right)-\mathrm{d}\left(s_{n-1},\dots s_1, j^{n-1}\right)+\mathrm{d}\left(s_{d-1},\dots s_1, j^{d-1}\right)-\mathrm{d}\left(t_{n-1},\dots t_1, i^{n-1}\right)\\ &+\mathrm{d}\left(t_{d-1},\dots t_1, i^{d-1}\right),\end{aligned}$$

the difference between the V_f-path and the direct path down to the position $d \in [n]$ for any $f \in [p]_0 \setminus \{i,j\}$. We will see that there are five possibilities for Δ_f :

D	$\Delta_f \geq 2^d$	*direct path*
$A\,(/A')$	$\Delta_f = 2^{d-1}$	I LDM
B	$\Delta_f = 0$	I/II LDM(s)
C	$\Delta_f = -2^{d-1}$	II LDMs
E	$\Delta_f \leq -2^d$	*V_f-path*,

which provides us information about the length of the path. Furthermore, it will emerge that the final decision between A, B, or C cannot be made until all pairs are read. To find the shortest path(s), we distinguish several cases.

1. Case: $s = iis_{n-2}\ldots s_1, t = jjt_{n-2}\ldots t_1$
   We calculate for the length of the direct path
   $$\mathrm{d}\left(i\ldots s_1, j^{n-1}\right) + 1 + \mathrm{d}\left(j\ldots t_1, i^{n-1}\right) =$$
   $$2^{n-1} + 1 + \mathrm{d}\left(s_{n-2}\ldots s_1, j^{n-2}\right) + \mathrm{d}\left(t_{n-2}\ldots t_1, i^{n-2}\right)$$
   and for the length of the V_f-path for any $f \in [p]_0 \setminus \{i, j\}$
   $$\mathrm{d}\left(i\ldots s_1, f^{n-1}\right) + 1 + 2^{n-1} + \mathrm{d}\left(j\ldots t_1, f^{n-1}\right) =$$
   $$2^{n-1} + 1 + 2^{n-1} + \mathrm{d}\left(s_{n-2}\ldots s_1, f^{n-2}\right) + \mathrm{d}\left(t_{n-2}\ldots t_1, f^{n-2}\right).$$
   It follows that the direct path is always the shortest as the length is at most $2^n - 1$. We notice that $\Delta_f = 2^d$ with $d = n-1$ and we are in D.

2. Case: $s = ijs_{n-2}\ldots s_1, t = jt_{n-1}\ldots t_1$
   The length of the direct path is at most $(2^{n-2}-1)+1+(2^{n-1}-1)$, while the V_f-path is at least $2^{n-2} + 2^{n-1} + 1$ for any $f \in [p]_0 \setminus \{i, j\}$. So the direct path is again shorter. Additionally, we remark that we are in D. The case $s = is_{n-1}\ldots s_1$ and $t = jit_{n-2}\ldots t_1$ is treated analogously.

3. Case: $s = iis_{n-2}\ldots s_1, t = jht_{n-2}\ldots t_1$
   Let $f \in [p]_0 \setminus \{i, j, h\}$. Then the length of the V_f-path is
   $$\mathrm{d}\left(i\ldots s_1, f^{n-1}\right) + 1 + 2^{n-1} + \mathrm{d}\left(h\ldots t_1, f^{n-1}\right) \geq 2^n + 1$$
   and cannot be the shortest one. The possible shortest paths are the direct and the V_h-path.

   3.1 Subcase: $s_{n-2} = i, t_{n-2} = j$
   It follows that we are again in D, since $\Delta_h = 2^d$ with $d = n-2$. Therefore, the direct path is shorter.

   3.2 Subcase:

   a) $s_{n-2} = j, t_{n-2} = o$ $(o \in [p]_0 \setminus \{h\})$
   We arrive at D with $\Delta_h \geq 2^d$ $(d = n-2)$.

   b) $s_{n-2} = j, t_{n-2} = h$
   The length of the direct path is calculated to be
   $$\mathrm{d}\left(ij\ldots s_1, j^{n-1}\right) + 1 + \mathrm{d}\left(hh\ldots t_1, i^{n-1}\right) =$$
   $$2^{n-1} + 1 + 2^{n-3} + \mathrm{d}\left(s_{n-3}\ldots s_1, j^{n-3}\right) + \mathrm{d}\left(t_{n-3}\ldots t_1, i^{n-3}\right),$$
   while the length of the V_h-path is
   $$\mathrm{d}\left(ij\ldots s_1, h^{n-1}\right) + 1 + 2^{n-1} + \mathrm{d}\left(hh\ldots t_1, h^{n-1}\right) =$$
   $$2^{n-1} + 1 + 2^{n-2} + 2^{n-3} + \mathrm{d}\left(s_{n-3}\ldots s_1, h^{n-3}\right) + \mathrm{d}\left(t_{n-3}\ldots t_1, h^{n-3}\right).$$

Therefore, $\Delta_h = 2^d$ with $d = n-2$, and we are in D. Then even in the worst case for the length of the direct path, namely that all following pairs are (h, h), the direct path is the shortest as it would have only the length $2^{n-1} + 2^{n-2} + 2^{n-3} - 1$ compared to the length of the V_h-path $2^{n-1} + 2^{n-2} + 2^{n-3} + 1$.

The case $t_{n-2} = i$ is treated analogously.

3.3 Subcase:

a) $s_{n-2} = i, t_{n-2} = h$
The difference between both paths is $\Delta_h = 2^{d-1}$ with $d = n-2$, and we arrive at A. Now we look at the case A. For the cases $(s_{n-3}, t_{n-3}) = (i, h), (h, j), (h, o), (o, h)$ with $o \in [p]_0 \setminus \{h\}$ we have $\Delta_h = 2^{d-1}$ for $d = n-3$, and we stay in A. If $(s_{n-3}, t_{n-3}) = (h, h)$ we see that the direct path has length $2^{n-1} + 2^{n-2} + 2^{n-3} + 1 + \mathrm{d}\left(s_{n-4} \ldots s_1, j^{n-4}\right) + \mathrm{d}\left(t_{n-4} \ldots t_1, i^{n-4}\right)$ and the V_h-path has length $2^{n-1} + 2^{n-2} + 2^{n-3} + 1 + \mathrm{d}\left(s_{n-4} \ldots s_1, h^{n-4}\right) + \mathrm{d}\left(t_{n-4} \ldots t_1, h^{n-4}\right)$. Hence both paths may be shortest ones (state B) and we analyse (s_{n-4}, t_{n-4}) as in 3.4 a). In all other cases $\Delta_h = 2^d$ with $d = n-3$, and we are in D.

b) $s_{n-2} = i, t_{n-2} = u$ $(u \in [p]_0 \setminus \{h, j\})$
This is again D as $\Delta_h \geq 2^d$ with $d = n-2$.

The cases $(s_{n-2}, t_{n-2}) = (h, j)$ and (v, j) with $v \in [p]_0 \setminus \{h, i\}$ are treated analogously.

3.4 Subcase:

a) $s_{n-2} = h = t_{n-2}$
We are in B, since both paths have equal length. Hence we look at the next pair (s_{n-3}, t_{n-3}). In the cases $(l, m), (i, o), (o, j), (h, i), (j, h)$ with $l, m \in [p]_0 \setminus \{h, i, j\}$ and $o \in [p]_0 \setminus \{h\}$ they still have equal length, and we stay in B. For the cases (h, v) and (u, h) with $u \in [p]_0 \setminus \{h, j\}$ and $v \in [p]_0 \setminus \{h, i\}$ we see that the length of the V_h-path is $2^{n-1} + 2^{n-2} + 2^{n-4} + 1 + \mathrm{d}\left(s_{n-4} \ldots s_1, h^{n-4}\right) + \mathrm{d}\left(t_{n-4} \ldots t_1, h^{n-4}\right)$ and of the direct path is $2^{n-1} + 2^{n-2} + 2^{n-3} + 1 + \mathrm{d}\left(s_{n-4} \ldots s_1, j^{n-4}\right) + \mathrm{d}\left(t_{n-4} \ldots t_1, i^{n-4}\right)$. So $\Delta_h = -2^{d-1}$ with $d = n-3$, and we go to C. The state C will be analysed in Case 4.3 a). The cases (u, i) and (j, v) (u, v as above) are treated as 3.3 a) as $\Delta_h = 2^{d-1}$ with $d = n-3$. If $(s_{n-3}, t_{n-3}) = (h, h)$, the V_h-path is the shorter one with $\Delta_h = -2^d$ $(d = n-3)$ and we arrive at E (see Case 4.4). For the case (j, i) the direct path is the shortest with $\Delta_h = 2^d$ $(d = n-3)$ as we will see in Case 4.2 b) and we reach D.

b) $s_{n-2} = o = t_{n-2}$ $(o \in [p]_0 \setminus \{h\})$
At this point $\Delta_h \geq 2^d (d = n-2)$ and we are in D.

3.5 Subcase:

a) $s_{n-2} = h, t_{n-2} = l$ $(l \in [p]_0 \setminus \{h, i, j\})$
Here $\Delta_h = 2^{d-1}$ with $d = n-2$ and this is A as in 3.3 a). The case $s_{n-2} = k$ and

$t_{n-2} = h$ is treated analogously.

b) $s_{n-2} = l, t_{n-2} = m$ $(l, m \in [p]_0 \setminus \{h, i, j\}$ and $l \neq m)$
We are again in D.

We notice that the same is true for $s = ihs_{n-2} \dots s_1$ and $t = jjt_{n-2} \dots t_1$.

4. Case: $s = ihs_{n-2} \dots s_1, t = jht_{n-2} \dots t_1$
Let $f \in [p]_0 \setminus \{i, j, h\}$. We must only look at the length of the direct and the V_h-path, since the V_f-path has length at least $2^n + 1$ as in Case 3.

4.1 Subcase: $s_{n-2} = i, t_{n-2} = j$
Both paths have equal length (state B), and we can treat the subcase as in 3.4 a).

4.2 Subcase:

a) $s_{n-2} = j, t_{n-2} = v$ $(v \in [p]_0 \setminus \{h, i\})$
It follows that $\Delta_h = 2^{d-1}$ with $d = n - 3$. We arrive at A (see Subcase 3.3 a)).

b) $s_{n-2} = j, t_{n-2} = i$
We get $\Delta_h = 2^d$ with $d = n - 2$ and state D.

c) $s_{n-2} = j, t_{n-2} = h$
Both paths have equal length (state B). So we go back to 3.4 a).

The cases $s_{n-2} \in [p]_0 \setminus \{h, j\}$ or $s_{n-2} = h$ and $t_{n-2} = i$ are treated analogously.

4.3 Subcase:

a) $s_{n-2} = i, t_{n-2} = h$
The length of the direct path is $2^{n-1}+1+2^{n-2}+\mathrm{d}\left(s_{n-3} \dots s_1, j^{n-3}\right)+\mathrm{d}\left(t_{n-3} \dots t_1, i^{n-3}\right)$, while the V_h-path is $2^{n-1} + 1 + 2^{n-3} + \mathrm{d}\left(s_{n-3} \dots s_1, h^{n-3}\right) + \mathrm{d}\left(t_{n-3} \dots t_1, h^{n-3}\right)$. Thus $\Delta_h = -2^{d-1}$ $(d = n-2)$, and we are in C. The V_h-path seems to be the shortest. But if $(s_{n-3}, t_{n-3}) = (j, i)$, we are in state B and look at Subcase 3.4a). If we have the pairs $(s_{n-3}, t_{n-3}) = (j, v)$ and (u, i) with $u \in [p]_0 \setminus \{h, j\}$ and $v \in [p]_0 \setminus \{h, i\}$), the V_h-path seems again to be the shortest one, since $\Delta_h = -2^{d-1}$ $(d = n - 3)$, and as before we have to analyse the next pair. If $(s_{n-3}, t_{n-3}) = (h, \cdot), (\cdot, h), (u, v)$ $(u, v$ as above) the V_h-path is the shortest and $\Delta_h \leq -2^d$ $(d = n - 3)$. We will show this for the case (j, h), in which the V_h-path has length $2^{n-1} + 2^{n-3} + 2^{n-4} + 1 + \mathrm{d}\left(s_{n-4} \dots s_1, h^{n-4}\right) + \mathrm{d}\left(t_{n-4} \dots t_1, h^{n-4}\right)$ and the direct path is $2^{n-1} + 2^{n-2} + 2^{n-4} + 1 + \mathrm{d}\left(s_{n-4} \dots s_1, j^{n-4}\right) + \mathrm{d}\left(t_{n-4} \dots t_1, i^{n-4}\right)$ long. But even in the worst case that all subsequent pairs are (j, i), the V_h-path is still shorter. We reach state E.

b) $s_{n-2} = i, t_{n-2} = l$ $(l \in [p]_0 \setminus \{h, i, j\})$
In this subcase $\Delta_h = 0$ (state B) and we analyse it as Case 3.4 a).

The cases $s_{n-2} = h$ or $s_{n-2} = l$ and $t_{n-2} = j$ are treated analogously.

4.4 Subcase: $s_{n-2} = h = t_{n-2}$
Here Δ_h is equal to -2^d with $d = n-2$, and we are in state E.

4.5 Subcase:

a) $s_{n-2} = l, t_{n-2} = m$ $(l, m \in [p]_0 \setminus \{h, i, j\})$
Both paths have equal length. Thus we are in B and must analyse (s_{n-3}, t_{n-3}) as in 3.4 a).

b) $s_{n-2} = h, t_{n-2} = v$ $(v \in [p]_0 \setminus \{h, i\})$
Δ_h is equal to -2^{d-1} $(d = n-2)$. We are in state C and must evaluate (s_{n-3}, t_{n-3}) as in 4.3a).

c) $s_{n-2} = u, t_{n-2} = h$ $(u \in [p]_0 \setminus \{h, j\})$
Again Δ_h is equal to -2^{d-1} $(d = n-2)$, and we are in state C.

5. Case: $s = igs_{n-2} \dots s_1, t = jht_{n-2} \dots t_1$ $(g \in [p]_0 \setminus \{i, j, h\})$
Let $f \in [p]_0 \setminus \{i, j, g, h\}$. Then as in the previous two cases we see that the V_f-path cannot be the shortest. Therefore, we only look at the direct, the V_h- and the V_g-path.

5.1 Subcase: $s_{n-2} = i, t_{n-2} = j$
Δ_h as well as Δ_g is equal to 2^d $(d = n-2)$, and we are in D.

5.2 Subcase:

a) $s_{n-2} = j, t_{n-2} = o'$ $(o' \in [p]_0 \setminus \{g, h\})$
At this point Δ_h or Δ_g, respectively, is greater and equal to 2^d for $d = n-2$ (state D).

b) $s_{n-2} = j, t_{n-2} = g$
The direct path has length $2^{n-1} + 2^{n-3} + 1 + \mathrm{d}\left(s_{n-3} \dots s_1, j^{n-3}\right) + \mathrm{d}\left(t_{n-3} \dots t_1, i^{n-3}\right)$ and the V_g-path $2^{n-1} + 2^{n-2} + 2^{n-3} + 1 + \mathrm{d}\left(s_{n-3} \dots s_1, g^{n-3}\right) + \mathrm{d}\left(t_{n-3} \dots t_1, g^{n-3}\right)$, while the V_h–path has length $\geq 2^n + 1$. Hence $\Delta_g = 2^d$ $(d = n-2)$, and we go to D. In the case with $t_{n-2} = h$ we get the same only with the V_h-path instead of the V_g-path.

We can treat the cases $(s_{n-2}, i), (h, i)$, and (g, i) $(s_{n-2} \in [p]_0 \setminus \{h, g\})$ analogously.

5.3 Subcase:

a) $s_{n-2} = i, t_{n-2} = h$
The V_g-path is longer than $2^n + 1$. The difference Δ_h between the direct and the V_h-path is 2^{d-1} (state A), and we proceed as in 3.3 a). The case that $s_{n-2} = i$ and $t_{n-2} = g$, is treated like Subcase 3.3 a) with g instead of h as $\Delta_g = 2^{d-1}$.

b) $s_{n-2} = i, t_{n-2} = l'$ $(l' \in [p]_0 \setminus \{g, h, i, j\})$
The direct path is the shortest because of $\Delta_h = 2^d$ and $\Delta_g = 2^d$ for $d = n-2$.

The same applies to the pairs $(s_{n-2}, t_{n-2}) = (h, j), (g, j)$, and (l', j) $(l' \in [p]_0 \setminus \{g, h, i, j\})$.

5.4 Subcase: $s_{n-2} = h = t_{n-2}$
The V_g-path is once more too long. The other two are of equal length ($\Delta_h = 0$ and state B), and we look back to 3.4 a). If $s_{n-2} = g = t_{n-2}$, the same is true with g instead of h.

5.5 Subcase:

a) $s_{n-2} = l', t_{n-2} = m'$ $(l', m' \in [p]_0 \setminus \{g, h, i, j\})$
Again the direct path is the shortest with $\Delta_h = 2^d$ and $\Delta_g = 2^d$ $(d = n-2)$. We reach D.

b) $s_{n-2} = h, t_{n-2} = v'$ $(v' \in [p]_0 \setminus \{g, h, i\})$
Δ_h is equal to $2^{d-1}(d = n-2)$, and we are in state A. The same follows for $s_{n-2} = g$ instead of h.

c) $s_{n-2} = u', t_{n-2} = h$ $(u' \in [p]_0 \setminus \{g, h, j\})$
Again Δ_h is equal to $2^{d-1}(d = n-2)$, and we are in state A. As above we can substitute h with g.

5.6 Subcase: $s_{n-2} = g, t_{n-2} = h$
The direct path is $2^{n-1} + 2^{n-2} + 1 + \mathrm{d}\left(s_{n-3} \dots s_1, j^{n-3}\right) + \mathrm{d}\left(t_{n-3} \dots t_1, i^{n-3}\right)$ long. The V_h-path (resp. V_g-path) has the length $2^{n-1} + 2^{n-2} + 2^{n-3} + 1 + \mathrm{d}\left(s_{n-3} \dots s_1, h^{n-3}\right) + \mathrm{d}\left(t_{n-3} \dots t_1, h^{n-3}\right)$ (with g instead of h). It follows that both Δ_h and Δ_g are 2^{d-1} for $d = n-2$. First, the direct path seems to be shorter than the other two paths, but we have to analyse the next pair (s_{n-3}, t_{n-3}). We call this new state A' because of its special property and similarity to the state A. If $(s_{n-3}, t_{n-3}) \in \{(g, h), (h, g)\}$, we start anew (state A') as before and analyse the following pair on the occurrence of g or h. Otherwise we treat this case analogously to 3.3 a) according to g or h, resp. to V_g-path or V_h-path. We notice that the same applies to $(s_{n-2}, t_{n-2}) = (h, g)$. □

Using the previous theorem, the question, whether we have to use one ($LDM = 1$) or two ($LDM = 2$) largest disc moves for an optimal solution or if both possibilities lead to an optimal path ($LDM = 0$), is decided. The theorem also gives the "detour" peg k for $LDM \neq 1$. Hence we can calculate the distance $\mathrm{d}(\underline{s}is, \underline{s}jt)$ together with Formula (1.9). We get for $LDM < 2$

$$\begin{aligned} \mathrm{d}(is, jt) &= \textstyle\sum_{d=1}^{n}(1 - [s_d = j]) \cdot 2^{d-1} + 1 + \sum_{d=1}^{n}(1 - [t_d = i]) \cdot 2^{d-1} \\ &= 2^{n+1} - 1 - \textstyle\sum_{d=1}^{n}([s_d = j] + [t_d = i]) \cdot 2^{d-1}, \end{aligned} \tag{1.10}$$

and for $LDM = 2$

$$\begin{aligned} \mathrm{d}(is, jt) &= \textstyle\sum_{d=1}^{n}(1 - [s_d = k]) \cdot 2^{d-1} + 2^n + 1 + \sum_{d=1}^{n}(1 - [t_d = k]) \cdot 2^{d-1} \\ &= 3 \cdot 2^n - 1 - \textstyle\sum_{d=1}^{n}([s_d = k] + [t_d = k]) \cdot 2^{d-1}. \end{aligned} \tag{1.11}$$

Note that for the calculation of the distance all pairs must be read, whereas for the decision only a possibly smaller number of pairs has to be read. Since we know now the shortest path(s), we can also determine the best first move(s) to get from $\underline{s}is$ to $\underline{s}jt$ with the following lemma.

Lemma 1.25 ([52, p.103], [37, Lemma 1.2]). *With the same assumptions for $\underline{s}\overline{s} := \underline{s}is$ and j as in Theorem 1.23 the optimal first move from $\underline{s}\overline{s}$ to $\underline{s}ji^d$ in S_p^{n+1} is from $\underline{\underline{s}}\overline{s}_b j^{b-1}$ to $\underline{\underline{s}}j\overline{s}_b^{b-1}$, where $b \in [d+1]$, $\underline{\underline{s}} = \underline{s}\overline{s}_{d+1} \ldots \overline{s}_{b+1} \in [p]_0^{n-b+1}$, and $\overline{s}_b \neq j$, i.e., $b = \min\{\delta \in [d+1] \mid \overline{s}_\delta \neq j\}$.*

The next section makes use of the last results. We will apply them in order to construct an algorithm which gives us the "detour" peg k, the distance, and the best first move(s) for an arbitrary pair s and t in S_p^{N+1}.

1.4.2 The $P2$ decision algorithm

We construct an algorithm, the $P2$ decision algorithm, which gives us the length of the shortest path(s) between $s \in V(S_p^{N+1})$ and $t \in V(S_p^{N+1})$ for $p \in \mathbb{N}_3$ and the best first move(s). Starting with the preprocessing algorithm, we filter out the case of $s = t$ and reduce the problem of determining the optimal path(s) between $s = \underline{s}\overline{s}$ and $t = \underline{s}\overline{t}$ to find it between $\overline{s}$ and $\overline{t}$. Then we use an algorithm to set g and h following the rules of Theorem 1.24. In Algorithm 4 we find the "detour" peg k and decide whether we are in the case "*direct path*", "*both paths*" or "V_k*-path*". Using the results of the algorithms, we can go to the postprocessing algorithm, where we calculate the distance and give the best first move(s). Note that for the case "*both paths*" there are two different best first moves.

Algorithm 2 The P2 decision algorithm for S_p^{N+1}

procedure P2S(N, p, s, t)
    parameter N : number of discs minus 1 $\{N \in \mathbb{N}_0\}$
    parameter p : $\{p \in \mathbb{N}_3\}$
    parameter s : initial configuration $\{s \in [p]_0^{N+1}\}$
    parameter t : goal configuration $\{t \in [p]_0^{N+1}\}$
    Pre(N, p, s, t)
    Post(s, t, i, j, LDM, k, n)
end procedure

We start with $N + 1$ discs in the $P2$ decision algorithm. After the preprocessing, we know $n + 1$, which is the maximal position, in which s and t differ. In the Detour peg algorithm, we start with $\delta = n$. If the algorithm stops in D or in A (when the input data have run out), we set LDM equal to 1. If we end in E or C, we set $LDM = 2$. For the case of B as ending state we set $LDM = 0$. At the end of this algorithm, we know LDM and the "detour" peg $k = \diamond$ and can use this information for the postprocessing algorithm. The states D and A stand for the class I task (one move of the largest disc, $LDM = 1$), while E and C is the class II task (two moves, $LDM = 2$). The class I/II task (equality between one or two moves, $LDM = 0$) is represented by the state B.

Algorithm 3 The Preprocessing algorithm for S_p^{N+1}

procedure PRE(N, p, s, t)
  parameter N : number of discs minus 1 $\{N \in \mathbb{N}_0\}$
  parameter p : $\{p \in \mathbb{N}_3\}$
  parameter s : initial configuration $\{s \in [p]_0^{N+1}\}$
  parameter t : goal configuration $\{t \in [p]_0^{N+1}\}$
  $d \leftarrow N + 1$
  while $d > 0$ **and** $s_d = t_d$ **do**
    $d \leftarrow d - 1$
  end while
  if $d = 0$ **then**
    no move
  else
    $n \leftarrow d - 1$
    DPS(n, p, s, t)
  end if
end procedure

Algorithm 4 Detour peg decision algorithm for S_p^n

procedure DPS(n, p, s, t)
  parameter n : resulting number of discs after preprocessing minus 1 $\{n \in \mathbb{N}_0\}$
  parameter p : $\{p \in \mathbb{N}_3\}$
  parameter s : initial configuration
  parameter t : goal configuration
  $i \leftarrow s_{n+1}, j \leftarrow t_{n+1}$
  $\diamond \leftarrow p$
  SETH(n, p, s, t)
  start in state $START$ of $P2$ automaton
  $\delta \leftarrow n$
  while $\delta > 0$ **do**
    apply automaton to pair (s_δ, t_δ) ▹ if automaton returns A and $\diamond = p$ then
      if ($g = s_{\delta-1}$ or $g = t_{\delta-1}$) and
        ($h = s_{\delta-1}$ or $h = t_{\delta-1}$) then continue
      if $h = s_{\delta-1}$ or $h = t_{\delta-1}$ then $\diamond \leftarrow h$
      if $g = s_{\delta-1}$ or $g = t_{\delta-1}$ then $\diamond \leftarrow g$
    ▹ algorithm STOPs if automaton reaches D or E
    $\delta \leftarrow \delta - 1$
  end while
end procedure

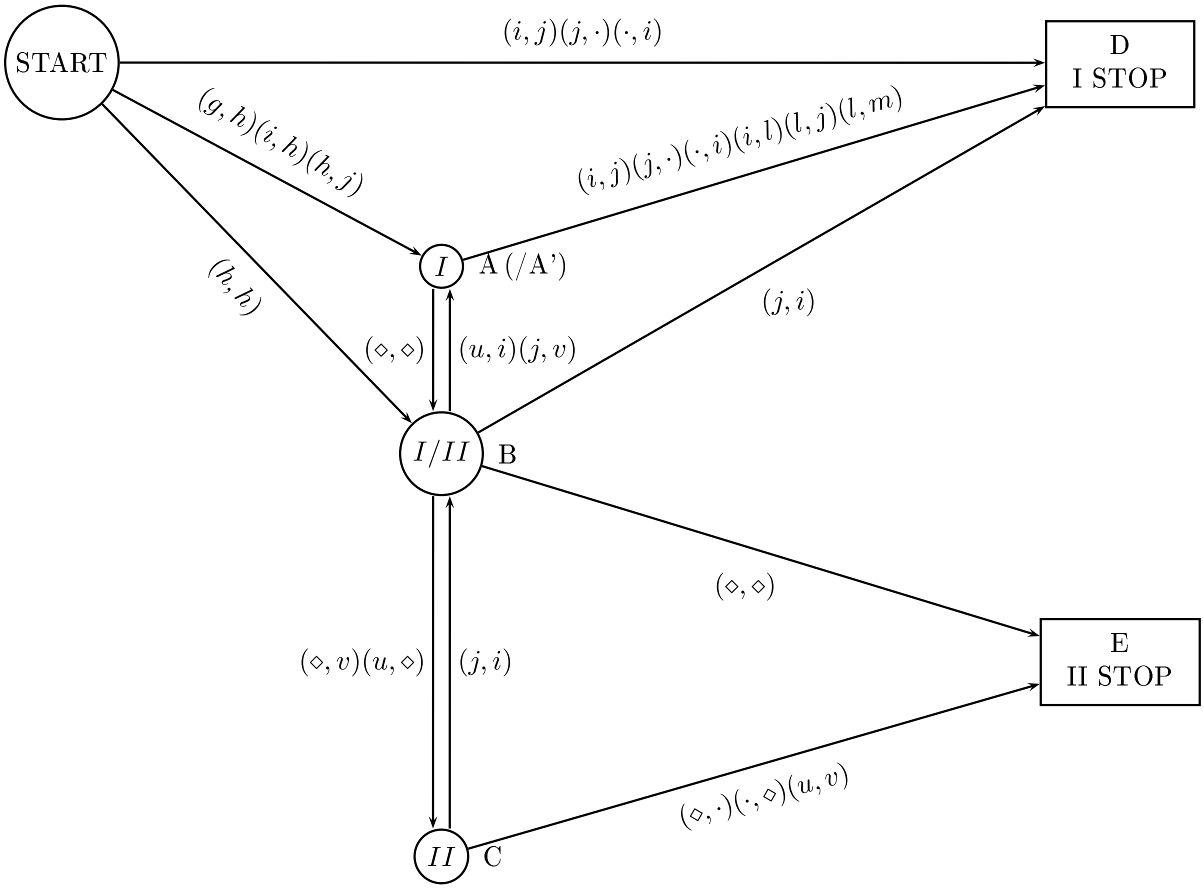

Figure 1.5: Automaton for the $P2$ decision problem with $u \in [p]_0 \setminus \{\diamond, j\}, v \in [p]_0 \setminus \{\diamond, i\}$, and $l, m \in [p]_0 \setminus \{\{i,j\} \cup K\}$, where $K = \{\diamond\}$ if $\diamond$ is already defined and $K = \{g, h\}$ otherwise

1.4.3 Some applications

In Section 1.3 we stated some propositions about distances for vertices of a special form in S_p^{N+1}. In the sequel we will show that several ones can be proved and extended easily with our algorithm. We recall Proposition 1.19 and even extend the statement.

Proposition 1.26 ([54, Proposition 4], [82, Theorem 3.3], [37, Proposition 2.3])**.** *Let $n \in [N]$, $i, j \in [p]_0$, $i \neq j$, $l \in [p]_0 \setminus \{i\}$, and $s \in [p]_0^n$. Then for all problems from $i^N l$ to $i^{N-n} js$ (or from $i^{N-n} js$ to $i^N l$) we get $LDM < 2$ and $LDM = 0$ if and only if $l \neq j$ and $s = l^n$.*

Proof. After the preprocessing, we can assume that $N - n = 0$. If $n = 1$, we end in A or D with pair (l, s_1), except when $s_1 = l \neq j$, in which case we stop in B. Let now $n \in \mathbb{N}_2$. If the second input pair is (i,i) or (i,j), we go directly to D. Otherwise, i.e., for (i, s_n) with $s_n \neq i, j$ we set $s_n = h = \diamond$

Algorithm 5 Set g and h algorithm

```
procedure SETH(n, p, s, t)
    parameter n : resulting number of discs after preprocessing minus 1 {n ∈ ℕ_0}
    parameter p : {p ∈ ℕ_3}
    parameter s : initial configuration
    parameter t : goal configuration
    i ← s_{n+1}, j ← t_{n+1}
    g ← p, h ← p
    if s_n ≠ i, j then
        if t_n ≠ i, j and s_n ≠ t_n then
            g ← s_n, h ← t_n
        else
            h ← s_n
            ◇ ← h
        end if
    else
        if t_n ≠ i, j then
            h ← t_n
            ◇ ← h
        end if
    end if
end procedure
```

and move to A. We move to B in the last step with (l, s_1) only if $s_d = h = l$ for all $d \in [n]$. Hence we will never end in C or E. We omit the postprocessing, since we do not want to determine the distance exactly. □

In Proposition 1.22 we made, inter alia, a statement about distances to special vertices.

Proposition 1.27. *Let $n \in [N]$, $i, j, k \in [p]_0$, $|\{i, j, k\}| = 3$, and $s \in [p]_0^n$ be special. Then*

$$\mathrm{d}(i^{N-n+1}s, i^{N-n}jk^n) = \mathrm{d}(s, k^n) + 2^n + 1.$$

Proof. After the preprocessing, the problem is reduced to determining the distance between $i\underline{s}k\overline{s}$ and jk^n with $\underline{s} \in ([p]_0 \setminus \{j,k\})^{n-\delta}$ and $\overline{s} \in [p]_0^{\delta-1}$ for a $\delta \in [n]$. The second input pair can be either (i,k) or (g,k) with $g \in [p]_0 \setminus \{k\}$. In the first case, we set $h = k = \diamond$ and go to A. Then we stay in A for the pairs $(\underline{s}_d, k)$ from $n-1$ to $\delta+1$ and move to B with (k,k) and stay there or move to E or C for the remaining pairs. In the second case, we go to A and must then distinguish two cases. The first one has (g,k) as third input pair, such that we stay in A until we get a pair (l,k) with $l \in [p]_0 \setminus \{g,k,j\}$. Then we set $\diamond = k$ and stay again in A until the input pair (k,k), which let us move to B, where we again stay or move to E or C for the remaining cases. The second case has immediately (l,k) with $l \in [p]_0 \setminus \{g,k,j\}$ as input pair. Hence we proceed as above. It follows that we are either in $LDM = 0$ or $LDM = 2$. Hence we calculate the distance to the desired value by the postprocessing. □

Algorithm 6 The Postprocessing algorithm for S_p^{n+1}

procedure $\textsc{Post}(s, t, i, j, LDM, k, n)$
  parameter s : initial configuration $\{s \in [p]_0^{n+1}\}$
  parameter t : goal configuration $\{t \in [p]_0^{n+1}\}$
  parameter i : initial peg
  parameter j : goal peg
  parameter LDM : times of Largest Disc Moves $\{b \in \{0, 1, 2\}\}$
  parameter k : $k \in [p + 1]_0$
  parameter n : resulting number of discs after preprocessing minus $1\{n \in \mathbb{N}_0\}$
  $\delta \leftarrow 1$
  $b \leftarrow LDM$
  if $b = 0$ **or** $b = 1$ **then**
    while $\delta < n + 1$ **and** $s_\delta = j$ **do**
      $\delta \leftarrow \delta + 1$
    end while
    if $\delta = n + 1$ **then**
      Best First Move from $\underline{s}ij^n$ to $\underline{s}ji^n$
    else
      Best First Move from $\underline{s}s_\delta j^{\delta-1}$ to $\underline{s}js_\delta^{\delta-1}$
    end if
    distance is $d(s, t) = d(s_n \ldots s_1, j^n) + 1 + d(t_n \ldots t_1, i^n)$
  end if
  if $b = 0$ **or** $b = 2$ **then**
    while $\delta < n + 1$ **and** $s_\delta = k$ **do**
      $\delta \leftarrow \delta + 1$
    end while
    if $\delta = n + 1$ **then**
      Best First Move from $\underline{s}ik^n$ to $\underline{s}ki^n$
    else
      Best First Move from $\underline{s}s_\delta k^{\delta-1}$ to $\underline{s}ks_\delta^{\delta-1}$
    end if
    distance is $d(s, t) = d(s_n \ldots s_1, k^n) + 1 + 2^n + d(t_n \ldots t_1, k^n)$
  end if
end procedure

These pairs of vertices are not the only ones which can have two optimal paths. We already stated in Proposition 1.15 that for all non-extreme vertices there is a vertex such that there are two optimal paths between them.

Proposition 1.28. *Let $p \in \mathbb{N}_3$. For all non-extreme vertices $s \in [p]_0^{N+1}$ there is a vertex $t \in [p]_0^{N+1}$ such that $LDM = 0$. Moreover, the vertex t can be chosen in such a way that $s_{N+1} \neq t_{N+1}$, i.e., the largest disc is $N + 1$.*

Proof. We know that s is non-extreme. Hence it follows that there is some $\{i, h\} \in \binom{[p]_0}{2}$, $n \in [N]$, and $\overline{s} \in [p]_0^{n-1}$ such that $s = i^{N-n+1}h\overline{s}$. Now we choose $j \in [p]_0 \setminus \{i, h\}$ and set $t = jh^{N-n+1}\overline{t}$ with, for $d \in [n-1]$,

$$\begin{aligned} t_d &= h, && \text{if} \quad s_d = j, \\ t_d &= i, && \text{if} \quad s_d = h, \\ t_d &= j && \text{otherwise.} \end{aligned}$$

Since no preprocessing is necessary, the first pair is (i, j). Then the pair (i, h) occurs $N - n$ times such that we first move to A with $\diamond = h$ and stay there until the pair (h, h). With this pair we go to B. The remaining $n - 1$ input pairs are of the form (j, h), (h, i) or (l, j) with $l \in [p]_0 \setminus \{j, h\}$. Therefore, state B will not be left anymore. □

There are some other properties which can be shown with the aid of the algorithm for the general Sierpiński graphs S_p^n and, consequently, also for the Hanoi graphs H_3^n. Employing the approach of Theorem 1.24 on Sierpiński triangle graphs, we can also make a statement about distances for this class of graphs as we will see in Section 1.5.

1.4.4 The average number of pairs to be read to solve the problem

It is of interest to us how many disc pairs must be read to solve the $P2$ decision problem for S_p^{N+1}. As might be expected, this number depends on p. We already analysed this for $p = 3$ in Section 1.2 with the aid of the theory of Markov chains. In the following section we will divide the state A of the automaton into A' for the case $|\{i, j, g, h\}| = 4$ and A otherwise as in Theorem 1.24. This helps us to distinguish easier between these two cases. We assume that $s = i\overline{s}$ and $t = j\overline{t}$ with $i, j \in [p]_0$, $i \neq j$, $n \in [N]$, and $\overline{s}, \overline{t} \in [p]_0^n$.
In order to find the average number of pairs, we want to consider the automaton in Figure 1.5 as a Markov chain with six states in which the process starts in A', A, or in B and move from state to state with a certain probability. We can assume that we start in A', A, or in B, since we know that the first two pairs must always be read and have therefore no direct influence on the average number. The probabilities for the six states A', A, B, C, D, and E are shown in the transition matrix of the automaton

$$P = \frac{1}{p^2}\begin{pmatrix} 2 & 4p-12 & 2 & 0 & p^2-4p+8 & 0 \\ 0 & 2p-4 & 1 & 0 & p^2-2p+3 & 0 \\ 0 & 2p-4 & p^2-4p+6 & 2p-4 & 1 & 1 \\ 0 & 0 & 1 & 2p-4 & 0 & p^2-2p+3 \\ 0 & 0 & 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & p^2 \end{pmatrix}.$$

Because of the two absorbing states in our Markov chain, this Markov chain is again absorbing. The matrix P is of the form

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix},$$

where Q is again the part of the matrix which describes the transition probabilities from some transient state to another, R gives the transition probabilities from transient to absorbing states, and I is the identity matrix. In an absorbing Markov chain, $Q^n \to 0$ when $n \to \infty$ and $I - Q$ has an inverse

$$M = (I - Q)^{-1} = \sum_{n=0}^{\infty} Q^n.$$

M_{uv} is the expected number of visits which the chain made to state v provided that it has started in state u.

$$M = (I - Q)^{-1} = \begin{pmatrix} \frac{(p^2-2)}{p^2} & -\frac{4(p-3)}{p^2} & -\frac{2}{p^2} & 0 \\ 0 & \frac{(p^2-2p+4)}{p^2} & -\frac{1}{p^2} & 0 \\ 0 & -\frac{2(p-2)}{p^2} & \frac{2(2p-3)}{p^2} & -\frac{2(p-2)}{p^2} \\ 0 & 0 & -\frac{1}{p^2} & \frac{(p^2-2p+4)}{p^2} \end{pmatrix}^{-1}$$

$$= p^2 \begin{pmatrix} \frac{1}{(p^2-2)} & \frac{2(4p^4-25p^3+64p^2-90p+52)}{(2p^7-11p^6+30p^5-38p^4-4p^3+88p^2-128p+64)} & \frac{1}{(2p^3-7p^2+12p-8)} & \frac{2(p-2)}{(2p^5-11p^4+34p^3-60p^2+64p-32)} \\ 0 & \frac{(2p^3-7p^2+13p-10)}{(2p^5-11p^4+34p^3-60p^2+64p-32)} & \frac{1}{2(2p^3-7p^2+12p-8)} & \frac{(p-2)}{(2p^5-11p^4+34p^3-60p^2+64p-32)} \\ 0 & \frac{(p-2)}{(2p^3-7p^2+12p-8)} & \frac{(p^2-2p+4)}{2(2p^3-7p^2+12p-8)} & \frac{(p-2)}{(2p^3-7p^2+12p-8)} \\ 0 & \frac{(p-2)}{(2p^5-11p^4+34p^3-60p^2+64p-32)} & \frac{1}{2(2p^3-7p^2+12p-8)} & \frac{(2p^3-7p^2+13p-10)}{(2p^5-11p^4+34p^3-60p^2+64p-32)} \end{pmatrix}$$

(This calculation was done with the computer algebra system *Sage* 5.10.) If we started in A', we get the expected number of times we will be in A', A, B, or C by the sum of the first row. The same applies to the states A, B, and C. Thus we get the (4×1)-vector r with the sums of the rows. Since we start at the states A', A, and B of the automaton in Figure 1.5 with a certain probability, we must multiply the (1×4)-vector t of the a-priori probabilities with the vector r

$$t \cdot r = \begin{pmatrix} \frac{(p-2)(p-3)}{p^2} & \frac{2p-4}{p^2} & \frac{p-2}{p^2} & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{2p^2(p^3+p^2-10p+9)}{(2p^5-7p^4+8p^3+6p^2-24p+16)} \\ \frac{p^2(4p-5)}{2(2p^3-7p^2+12p-8)} \\ \frac{p^2(p^2+2p-4)}{2(2p^3-7p^2+12p-8)} \\ \frac{p^2(4p-5)}{2(2p^3-7p^2+12p-8)} \end{pmatrix}$$

to get the expected number of times

$$AN_p = \frac{5p^5 - 8p^4 - 72p^3 + 272p^2 - 352p + 160}{2(2p^5 - 7p^4 + 8p^3 + 6p^2 - 24p + 16)} \tag{1.12}$$

we will be in one of the states A', A, B, or C depending on p.

Theorem 1.29. *The average number of pairs checked by the P2 decision automaton for S_p^{N+1} after preprocessing, when the largest disc is $n+1 \in [N+1]$ is bounded above by and converges, as $n \to \infty$, to $2 + AN_p$.*

We have to add two pairs as we always have to evaluate the pairs (s_{n+1}, t_{n+1}) and (s_n, t_n) in advance.

In the cases $p = 3, 4, 5,$ and 6 we have the values

$$\begin{aligned}
p = 3 \quad & AN_3 = \tfrac{25}{38}, \\
p = 4 \quad & AN_4 = 1, \\
p = 5 \quad & AN_5 = \tfrac{6825}{5842}, \\
p = 6 \quad & AN_6 = \tfrac{1300}{1037},
\end{aligned}$$

for example.

We see that for S_3^{n+1} the number $2 + AN_3 = \frac{101}{38}$ is equal to the one we calculated in Section 1.2.

Remark. By taking a closer look on the automaton in Figure 1.5, one can reduce the number of input pairs further. We notice that the input of j as first component of a pair in A' as well as in A is always followed by a move to D. Hence we have to evaluate only half a pair in these cases. In A' and A we just need to check $\frac{p}{p^2} \cdot \frac{1}{2} + \frac{(p^2-p)}{p^2} \cdot 1 = \frac{(2p^2-p)}{2p^2}$ pairs, in other words, save p out of $2p^2$ input data. Looking at the state C, we see that this also applies to the pairs with $\diamond$ as first input in C. Using these results, we have to check only

$$AN_p^{\text{red}} = \frac{5\,p^6 - 10\,p^5 - 69\,p^4 + 307\,p^3 - 486\,p^2 + 336\,p - 80}{2\,(2\,p^6 - 7\,p^5 + 8\,p^4 + 6\,p^3 - 24\,p^2 + 16\,p)}$$

pairs of input. For instance, we get for $p = 3$ the number $AN_3^{\text{red}} = 67/114$, for $p = 4$ the number $AN_4^{\text{red}} = 51/56$, for $p = 5$ the number $AN_5^{\text{red}} = 6315/5842$, and for $p = 6$ the number $AN_6^{\text{red}} = 7303/6222$. For $p = 3, 4, 5, 6$ the numbers AN_p^{red} are all strictly less than AN_p.

1.5 An algorithm to determine the shortest paths in Sierpiński triangle graphs ST_p^n with base $p \in \mathbb{N}_3$ and exponent $n \in \mathbb{N}_0$

The Section 1.4.3 worked with applications of our algorithm on Sierpiński graphs themselves. In this section we will see that our algorithm can even be applied on another class of graphs, the Sierpiński triangle graphs. These graphs have their name from the famous Sierpiński triangle (or gasket), which Wacław Sierpiński designed in 1915. He started from a closed equilateral triangle and successively removed open middle triangles such that in the step of order $n \in \mathbb{N}_0$ there are triangular gaps of n different sizes. We can interpret the corners of the subtriangles as vertices and their sides as edges to get the class of Sierpiński triangle graphs ST_3^n with base 3 and exponent n. Since there is a connection between the Sierpiński triangle and the Sierpiński graphs (cf. [42], [33, Section 4.3]), it is not surprising that there is one between our new class of graphs and the Sierpiński graphs S_p^n. This relation was used by M. Jakovac in [50] to generalise the Sierpiński triangle graphs to $p \in \mathbb{N}_3$ in the following way. Let sij^δ and sji^δ with $i, j \in [p]_0$, $i \neq j$, $\delta \in [n]$, and $s \in [p]_0^{n-\delta}$ be two vertices of the graph S_p^{n+1}. In ST_p^n we identify them in one vertex, named by $s\{i, j\}$. The vertices of the form k^{n+1} in S_p^{n+1} correspond to the so-called *primitive vertices* in ST_p^n, written $\hat{k}$. Since the vertices $\{i, j\} \in \binom{[p]_0}{2}$ will be important for our further study, we give them a name, namely *corner vertices*. Then

Definition 1.30. The *Sierpiński triangle graphs* are defined by

$$V(ST_p^n) = \left\{\hat{k} \,\middle|\, k \in [p]_0\right\} \cup \left\{s\{i, j\} \,\middle|\, s \in [p]_0^{n-\delta}, \delta \in [n], \{i, j\} \in \binom{[p]_0}{2}\right\},$$

where $s\{i, j\}$ stands for the contracted edge $\{sij^\delta, sji^\delta\} \in E(S_p^{n+1})$.

An explicit description of the adjacency of the vertices can be found in [50, Proposition 2.1] and [43, Definition 4]. Since S_p^{n+1} can be built from p copies of S_p^n, the graph ST_p^n consists also of p copies of ST_p^{n-1}. Each pair of these copies share one vertex, in fact the corresponding corner vertex, for instance the vertex $\{i, j\}$ for iST_p^{n-1} and jST_p^{n-1} ($i \neq j$). There is also a recursive definition for the class ST_p^n similar to the one for ST_3^n ([43]). Starting from $ST_p^0 \cong K_p$ with $V(ST_p^0) = \{\hat{k} \mid k \in [p]_0\}$, we take p copies of ST_p^n, characterised by $k \in [p]_0$, whereby we change the labelling of all primitive vertices $\hat{l}$ except $\hat{k}$ to $\{k, l\}$ and add a k to the left of all other vertices. Then the p copies are linked by identifying equal vertices.

We obtain easily that the primitive vertices have degree $p-1$ and all others have $2(p-1)$, and that the graphs ST_p^n are connected.

Accounted for by the connection to Sierpiński graphs, there are several other consequences we can state.

Proposition 1.31 ([43], [50, Proposition 2.3])**.** *The graphs ST_p^n have $\frac{p}{2}(p^n + 1)$ vertices and $\frac{p-1}{2} \cdot p^{n+1}$ edges.*

Proof. There are two ways to prove this proposition. We will begin with the easiest way. The vertex set $V(ST_p^n)$ contains, apart from the p primitive vertices $\hat{k}$, exactly one half of the non-extreme vertices of S_p^{n+1}. Hence we get

$$|V(ST_p^n)| = p + \frac{1}{2}\left(|S_p^{n+1}| - p\right) = p + \frac{1}{2}\left(p^{n+1} - p\right) = \frac{p}{2}(p^n + 1).$$

A more direct way uses the fact that every vertex in ST_p^n is of the form $s_{n-\delta} \ldots s_1\{i, j\}$. We have p choices for every s_k, $k \in [n-\delta]$, and $\binom{p}{2}$ choices for the pair $\{i, j\}$. Together with the p primitive vertices, we have

$$|V(ST_p^n)| = p + \sum_{k=0}^{n-1} p^k \cdot \binom{p}{2} = \frac{p}{2}(p^n + 1).$$

We determine the number of edges by:

$$
\begin{aligned}
\|ST_p^n\| &= \frac{1}{2} \sum_{v \in ST_p^n} \deg(v) \\
&= \frac{1}{2}\left(p\,(p-1) + \left(\frac{p}{2}\,(p^n+1) - p\right) \cdot 2(p-1)\right) \\
&= \frac{p-1}{2} p^{n+1}.
\end{aligned}
$$

□

A property which the Sierpiński and the Sierpiński triangle graphs have in common is the hamiltonicity of the graphs (see Lemma 1.11).

Theorem 1.32 ([50, Theorem 3.2])**.** *For any $n \in \mathbb{N}_1$ and $p \in \mathbb{N}_3$, the graphs ST_p^n are hamiltonian.*

We want to take a closer look at the distance d_n on the graphs ST_p^n in the next two statements.

Lemma 1.33 ([43, Proposition 1 for $p = 3$ and p.9])**.** *For $n \in \mathbb{N}_0$, $\nu \in [n]_0$, and $u, w \in V(ST_p^\nu)$*

$$\mathrm{d}_n(u, w) = 2^{n-\nu}\, \mathrm{d}_\nu(u, w), \tag{1.13}$$

and the number of shortest paths between u and w in ST_p^n is the same as in ST_p^ν.

For the distances to the primitive vertices $\hat{k}$ we get p-independent formulas

Theorem 1.34 ([43, Theorem 1 for $p = 3$ and p.9])**.** *For $k, l \in [p]_0$, there is a unique shortest path between $\hat{k}$ and $\hat{l}$ in ST_p^0, which has length*

$$\mathrm{d}_0(\hat{k}, \hat{l}) = [k \neq l]. \tag{1.14}$$

For $\nu \in \mathbb{N}$ and $u = s\{i, j\}$ with $s \in [p]_0^{\nu-1}$ we have

$$\mathrm{d}_\nu(u, \hat{k}) = \mathrm{d}_\nu(s\{i, j\}, \hat{k}) = 1 + [i \neq k][j \neq k] + \sum_{d=1}^{\nu-1} [s_d \neq k] \cdot 2^d \in [2^\nu], \tag{1.15}$$

and there are $1 + [i \neq k][j \neq k]$ shortest paths between u and $\hat{k}$ in ST_p^ν.

Hence $\mathrm{diam}(ST_p^n) = 2^n$ (cf. [20, Proposition 1] for $p = 3$ and [43]) and

$$\forall u \in V(ST_p^n) : \sum_{k=0}^{p-1} \mathrm{d}_n(u, \hat{k}) = (p-1) \cdot 2^n$$

(cf. [20, Proposition 2] for $p = 3$ and [43]). As in the case of Sierpiński graphs, we ask how many optimal paths between a vertex and a corner vertex or between two arbitrary vertices, respectively, exist. According to the recursive definition of ST_p^n, we already know that $\mathrm{d}_n(\{l, m\}, \{l', m'\}) = 2^{n-1}$

if $\{l,m\},\{l',m'\} \in \binom{[p]_0}{2}$ and $\{l,m\} \neq \{l',m'\}$. Together with Formulas (1.13), (1.14), and (1.15), we have now determined the distances in ST_p^n for $n = 0$ and $n = 1$. We will consider a path in ST_p^n from a vertex $u \in iST_p^{n-1}$ to a vertex $w \in jST_p^{n-1}$ as different from another u,w-path only if we use other subgraphs kST_p^{n-1} with $i \neq k \neq j$ on the way from the initial vertex to the final vertex. It will turn out that there are two possible types for shortest u,w-paths, namely the "direct type" and the "detour type". However, both types of paths can have equal length. We will see that in general there exist at most two shortest paths, except in a special case, where even three shortest paths of equal length, one of the "direct type" and two of the "detour type", exist.

As in the previous section, we are again interested in an algorithm to determine the shortest paths in this new class of graphs. As Sierpiński triangle graphs with base $p = 3$ play sometimes a special role, we will start with these simpler graphs in the first subsection. In the second one we find a general algorithm for all $p \in \mathbb{N}_3$.

1.5.1 A $P2$ decision algorithm for Sierpiński triangle graphs ST_3^n

In this subsection we consider the Sierpiński triangle graphs ST_3^n with base 3 (see Figure 1.6 for ST_3^3). From the introduction we already know that there are three primitive vertices $\hat{0}$, $\hat{1}$, and $\hat{2}$ of degree 2, whereas the other vertices have degree 4. With Proposition 1.31 we get

$$|V(ST_3^n)| = \frac{3}{2}(3^n + 1) \quad \text{and} \quad |E(ST_3^n)| = 3^{n+1}.$$

Now we want to determine the distance d_n in ST_3^n. For these graphs there also exists another labelling[6] which is different from the notation of Jakovac (see [43]). For a better comparison of the automata for ST_3^n and ST_p^n we will maintain the previous notation. We know two statements about the distances so far. For $\nu \in [n]_0$ and $u, w \in V(ST_3^\nu)$ the formula

$$\mathrm{d}_n(u,w) = 2^{n-\nu}\,\mathrm{d}_\nu(u,w) \tag{1.16}$$

holds. Similarly, we can apply the formula for the distance between a primitive vertex and an arbitrary vertex. We recall that for $k, l \in T$

$$\mathrm{d}_0(\hat{k},\hat{l}) = [k \neq l] \tag{1.17}$$

and for $\nu \in \mathbb{N}$ and $u = s\{i,j\} \in ST_3^\nu$ with $s \in T^{\nu-1}$

$$\mathrm{d}_\nu(s\{i,j\},\hat{k}) = 1 + [i \neq k][j \neq k] + \sum_{d=1}^{\nu-1}[s_d \neq k] \cdot 2^d. \tag{1.18}$$

Additionally, we know that $d_n(\{l,m\},\{l',m'\}) = 2^{n-1}$ if $\{l,m\},\{l',m'\} \in \binom{T}{2}$ and $\{l,m\} \neq \{l',m'\}$. As explained in the introduction of this section, we have now determined the distances in ST_3^n for $n = 0$ and $n = 1$. Let $n \in \mathbb{N}_2$. We note that in ST_3^n with $i, c \in T$, $\{l,m\} \in \binom{T}{2}$, $\nu \in [n]_2$, and $\underline{s} \in T^{\nu-2}$

$$\mathrm{d}_n\left(i\underline{s}\{l,m\},\{c,i\}\right) = \mathrm{d}_n\left(i\underline{s}\{l,m\},\hat{c}\right) - 2^{n-1}. \tag{1.19}$$

[6] A *labelling* is a characterisation of a representative from a class of isomorphic graphs, i.e., a specific graph of that class.

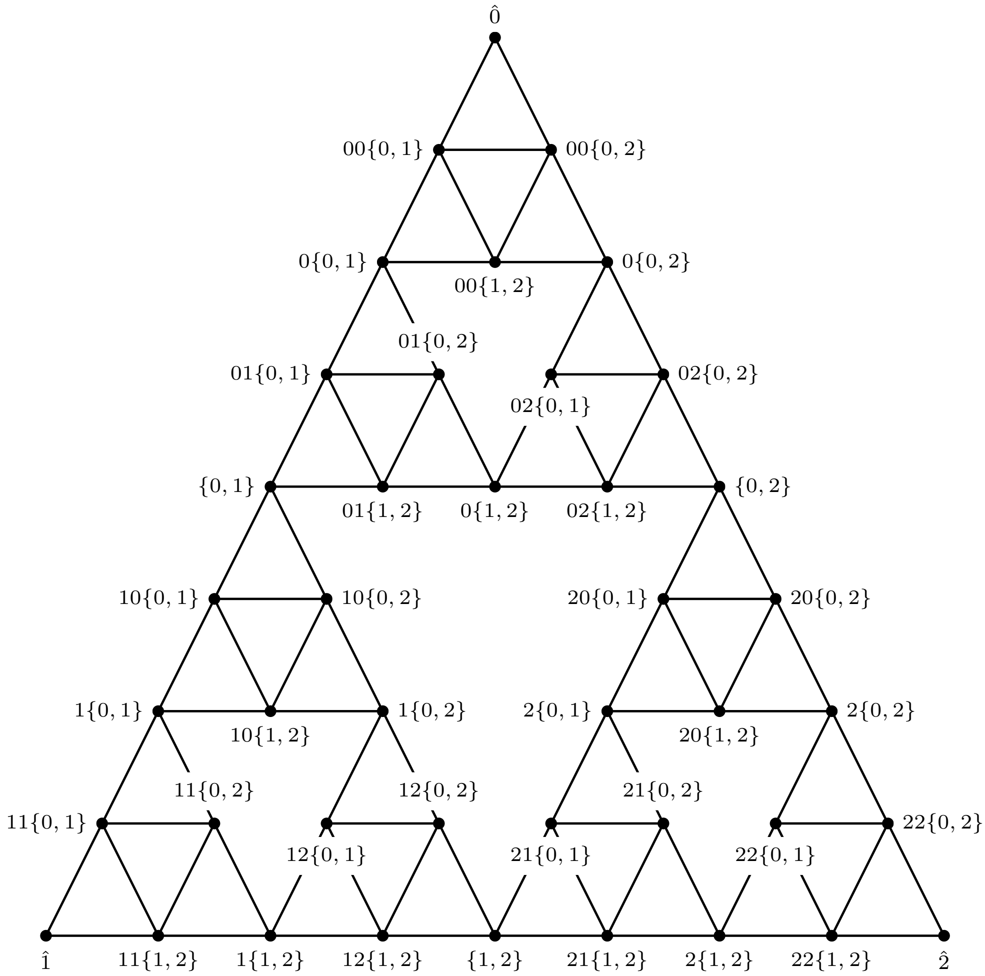

Figure 1.6: The Sierpiński triangle graph ST_3^3

Looking at Formula (1.18), this can easily be seen as $\mathrm{d}_n(\{c, i\}, \hat{c}) = 2^{n-1}$. Let $\{j, k\} \in \binom{T\setminus\{i\}}{2}$. We get for the distance between an arbitrary vertex and the corner vertex $\{j, k\}$

$$\begin{aligned}
\mathrm{d}_n(i\underline{s}\{l, m\}, \{j, k\}) &= \min\left\{\mathrm{d}_n(i\underline{s}\{l, m\}, \{i, j\}), \mathrm{d}_n(i\underline{s}\{l, m\}, \{i, k\})\right\} + 2^{n-1} \\
&= \min\left\{\mathrm{d}_n(i\underline{s}\{l, m\}, \hat{j}), \mathrm{d}_n(i\underline{s}\{l, m\}, \hat{k})\right\} - 2^{n-1} + 2^{n-1} \\
&= 2^{n-\nu} \min\left\{\mathrm{d}_\nu(i\underline{s}\{l, m\}, \hat{j}), \mathrm{d}_\nu(i\underline{s}\{l, m\}, \hat{k})\right\} \qquad (1.20)
\end{aligned}$$

with $i \in T$, $\{l, m\} \in \binom{T}{2}$, $\nu \in [n]_2$, and $\underline{s} \in T^{\nu-2}$. Then

$$\mathrm{d}_\nu(i\underline{s}\{l, m\}, \hat{j}) = \mathrm{d}_\nu(i\underline{s}\{l, m\}, \hat{k})$$

if $\underline{s}_d = i$ for all $d \in [\nu - 2]$ and $\{l, m\} = \{j, k\}$. The minimum is attained by $\mathrm{d}_\nu(i\underline{s}\{l, m\}, \hat{j})$ if either $\underline{s}_d = j$ for a $d \in [\nu - 2]$ and $\underline{s}_{d'} = i$ for all $d' \in [\nu - 2] \setminus [d]$ or $\underline{s}_d = i$ for all $d \in [\nu - 2]$ and $\{l, m\} = \{i, j\}$, whereas it is attained by $\mathrm{d}_\nu(i\underline{s}\{l, m\}, \hat{k})$ if transposing k and j one of the previous conditions holds. One can understand this for the case ST_3^3 by looking at Figure 1.6.

For the task $i\underline{s}\{l,m\} \to j\underline{t}\{l',m'\}$ in ST_3^n with $i, j \in T$, $i \neq j$, $\nu \in [n]_2$, $\mu \in [\nu-1]_0$, $\underline{s} \in T^{n-\mu-2}$, $\underline{t} \in T^{\nu-\mu-2}$, and $\{l,m\},\{l',m'\} \in \binom{T}{2}$ there are two possible types of paths, the path of *direct type* and the path of *detour type*. For the direct type we use the iST_3^{n-1}- and the jST_3^{n-1}-subgraphs of ST_3^n. Corresponding to our understanding of difference between paths, the path of detour type must additionally use the kST_3^{n-1}-subgraph ($k = 3-i-j$). The paths of both types can have equal length; see for instance the task $10\{1,2\} \to 20\{0,2\}$ in ST_3^3 in Figure 1.6. Hence in ST_3^n there exist at most two shortest paths.
Since the ST_3^n arises by contracting the edges of S_3^{n+1}, the only vertex which two subgraphs iST_3^{n-1} and jST_3^{n-1} ($i, j \in T$, $i \neq j$) have in common is $\{i, j\}$. Hence the direct path has length

$$\mathrm{d}_n\left(i\underline{s}\{l,m\},\{i,j\}\right) + \mathrm{d}_n\left(j\underline{t}\{l',m'\},\{i,j\}\right).$$

Using $d_n(\{i,k\},\{j,k\}) = 2^{n-1}$, the detour path has length

$$\mathrm{d}_n\left(i\underline{s}\{l,m\},\{i,k\}\right) + 2^{n-1} + \mathrm{d}_n\left(j\underline{t}\{l',m'\},\{j,k\}\right).$$

With Formula (1.19) we get for the direct path

$$\mathrm{d}_n\left(i\underline{s}\{l,m\},\hat{j}\right) + \mathrm{d}_n\left(j\underline{t}\{l',m'\},\hat{i}\right) - 2^n$$

and for the detour path

$$\mathrm{d}_n\left(i\underline{s}\{l,m\},\hat{k}\right) + \mathrm{d}_n\left(j\underline{t}\{l',m'\},\hat{k}\right) - 2^{n-1}.$$

Now we look at the difference between both paths and analyse this similarly to the case in S_p^n in [37]. With $\square \in \{>,<,=\}$ we deduce that

$$\begin{aligned}
\mathrm{d}_n\left(i\underline{s}\{l,m\},\hat{j}\right) - \mathrm{d}_n\left(i\underline{s}\{l,m\},\hat{k}\right) + \mathrm{d}_n\left(j\underline{t}\{l',m'\},\hat{i}\right) - \mathrm{d}_n\left(j\underline{t}\{l',m'\},\hat{k}\right) &\quad \square \quad 2^{n-1}\\
2^{\mu}\left[\mathrm{d}_{n-\mu}\left(i\underline{s}\{l,m\},\hat{j}\right) - \mathrm{d}_{n-\mu}\left(i\underline{s}\{l,m\},\hat{k}\right)\right] + 2^{n-\nu+\mu}\left[\mathrm{d}_{\nu-\mu}\left(j\underline{t}\{l',m'\},\hat{i}\right) - \mathrm{d}_{\nu-\mu}\left(j\underline{t}\{l',m'\},\hat{k}\right)\right] &\quad \square \quad 2^{n-1}.
\end{aligned}$$

This can be summarised to

$$\begin{aligned}
&2^{\mu}\left\{[l \neq j][m \neq j] - [l \neq k][m \neq k] + \sum_{d=1}^{n-\mu-2}\left([\underline{s}_d \neq j] - [\underline{s}_d \neq k]\right)\cdot 2^d\right\} +\\
&2^{n-\nu+\mu}\left\{[l' \neq i][m' \neq i] - [l' \neq k][m' \neq k] + \sum_{d=1}^{\nu-\mu-2}\left([\underline{t}_d \neq i] - [\underline{t}_d \neq k]\right)\cdot 2^d\right\} \qquad \square \quad 2^{n-1}.
\end{aligned}$$

If $\nu = n$ and $\mu = 0$, we simplify this to

$$\begin{aligned}
&[l \neq j][m \neq j] - [l \neq k][m \neq k] + [l' \neq i][m' \neq i] - [l' \neq k][m' \neq k] +\\
&\qquad \sum_{d=1}^{n-2}\underbrace{\left([\underline{s}_d = k] - [\underline{s}_d = j] + [\underline{t}_d = k] - [\underline{t}_d = i]\right)}_{\in\{-2,-1,0,1,2\}}\cdot 2^d \quad \square \quad 2^{n-1}.
\end{aligned}$$

We can follow the lines of the proofs of Proposition 1.0 and 1.2 (here no use of Proposition 1.1 is necessary) in [37] or use the proof of Theorem 1.7 in [70] up to the last position $(\{l,m\},\{l',m'\})$.

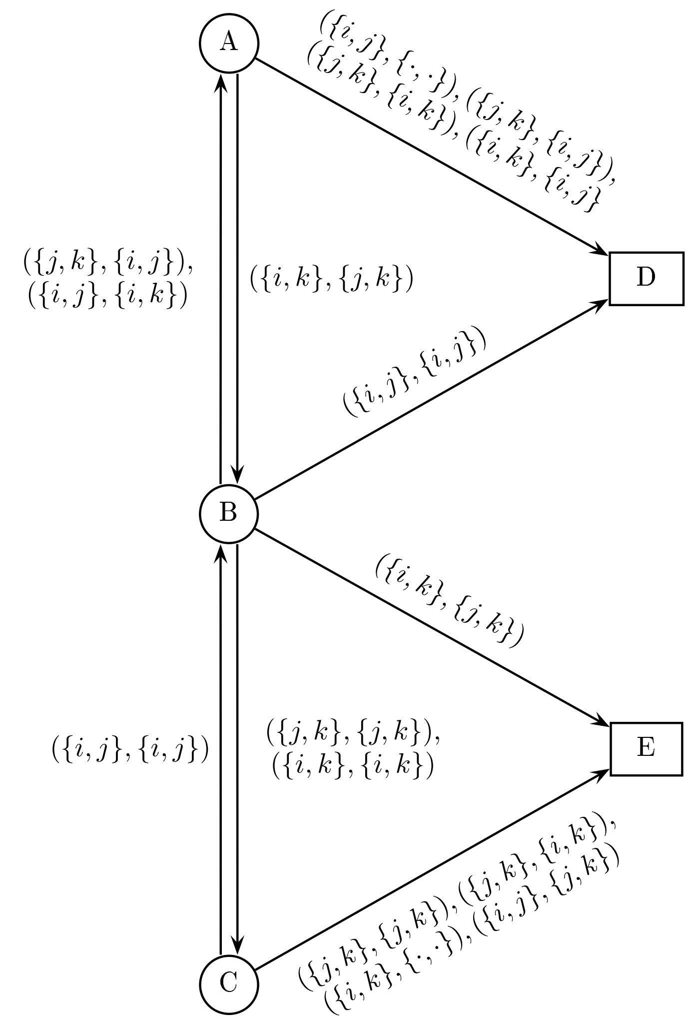

Figure 1.7: Automaton for ST_3^n in the case that $\nu = n$ and $\mu = 0$. For $(\{i,k\},\{i,k\}), (\{j,k\},\{j,k\})$ in A, for $(\{i,j\},\{j,k\}), (\{i,k\},\{i,j\}), (\{j,k\},\{i,k\})$ in B and for $(\{j,k\},\{i,j\}), (\{i,j\},\{i,k\})$ in C, the automaton stops in the respective state.

Reaching the last position, we have to modify the automaton in Figure 1.3 or [37, Fig. 4]. We obtain the automaton in Figure 1.7.

If $\nu \in [n]_2$, $\mu \in [\nu-1]_0$, and $i \neq j$, we can adapt the technique of the article [43] to our notation of the Sierpiński triangle graphs following Jakovac's.
We define

$$\tilde{s} = \underline{s}_{n-\mu-2} \dots \underline{s}_1 \{\tilde{l}, \tilde{m}\}$$

with $3 - ((3-l-m) \vartriangle i) = \tilde{l} + \tilde{m}$ and $(\tilde{l}, \tilde{m}) \in \binom{T}{2}$ and

$$\tilde{t} = t_{\nu-\mu-2} \dots t_1 t_0 (\tilde{s}_{n-\nu-1} \vartriangle k) \dots (\tilde{s}_1 \vartriangle k) \{\tilde{l'}, \tilde{m'}\}$$

with $t_0 = ((3 - l' - m') \vartriangle j)$ and $3 - ((3 - \tilde{l} - \tilde{m}) \vartriangle k) = \tilde{l'} + \tilde{m'}$ with $(\tilde{l'}, \tilde{m'}) \in \binom{T}{2}$.

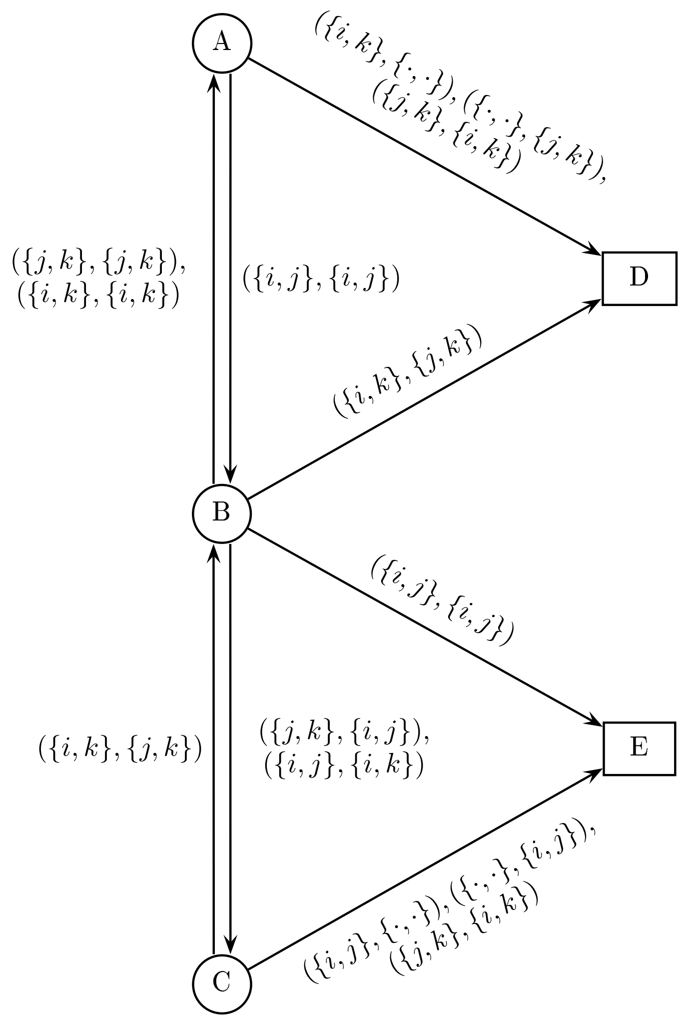

Figure 1.8: Automaton for ST_3^n in the case that $\nu \in [n]_2$ and $\mu \in [\nu - 1]_0$

Now we must only change the labels on the automaton in Figure 1.3 for the last position to get the automaton in Figure 1.8.

1.5.2 A $P2$ decision algorithm for Sierpiński triangle graphs ST_p^n with base $p \in \mathbb{N}_3$

After we determined the shortest paths in ST_3^n in the last subsection, we will now take a closer look on ST_p^n with $p \in \mathbb{N}_3$. We defined the Sierpiński triangle graphs ST_p^n for general p in Definition 1.30. For instance, we get for $p = 4$ and $n = 2$ the Sierpiński triangle graph ST_4^2 in Figure 1.9.

In the sequel we consider two arbitrary vertices in ST_p^n and determine the distances between them. The cases $n = 0$ and $n = 1$ were already done in the introduction of this section, so let $n \in \mathbb{N}_2$.

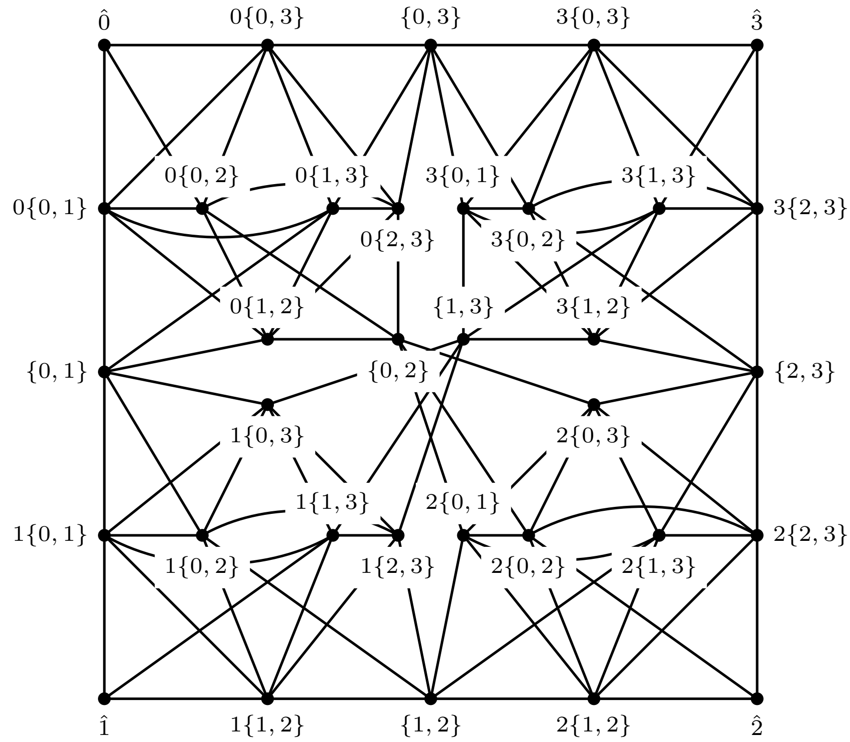

Figure 1.9: The Sierpiński triangle graph ST_4^2

Let $u = i\underline{s}\{l, m\}, w = j\underline{t}\{l', m'\} \in V(ST_p^n)$ with $i, j \in [p]_0$, $i \neq j$, $\nu \in [n]_2$, $\mu \in [\nu - 1]_0$, $\underline{s} \in [p]_0^{n-\mu-2}$, $\underline{t} \in [p]_0^{\nu-\mu-2}$, and $\{l, m\}, \{l', m'\} \in \binom{[p]_0}{2}$. Any u, w-path with only one pass through a corner vertex must lead from u to $\{i, j\}$ and then to w. Its length is

$$\begin{aligned} &\mathrm{d}_n\left(i\underline{s}\{l, m\}, \{i, j\}\right) + \mathrm{d}_n\left(j\underline{t}\{l', m'\}, \{i, j\}\right) \\ &= 2^{\mu}\,\mathrm{d}_{n-\mu}\left(i\underline{s}\{l, m\}, \{i, j\}\right) + 2^{n-\nu+\mu}\,\mathrm{d}_{\nu-\mu}\left(j\underline{t}\{l', m'\}, \{i, j\}\right) \\ &\leq 2^{n-1} + 2^{n-1} = 2^n. \end{aligned} \tag{1.21}$$

Another option for a u, w-path is the path through one detour subgraph, i.e., through the corner vertices $\{i, k\}$ and $\{j, k\}$ with $k \in [p]_0$ and $|\{i, j, k\}| = 3$. Since $\mathrm{d}_n(\{i, k\}, \{j, k\}) = 2^{n-1}$, we see that the length of this path is

$$\mathrm{d}_n\left(i\underline{s}\{l, m\}, \{i, k\}\right) + 2^{n-1} + \mathrm{d}_n\left(j\underline{t}\{l', m'\}, \{j, k\}\right). \tag{1.22}$$

Now the question arises: can a path running through more than one detour subgraph be minimal? No, it cannot. Already in the case of two subgraphs we need more than 2^n moves and, consequently, this cannot be minimal.
First, we will consider the distances to corner vertices as in Formula (1.19) and (1.20) for ST_3^n. We get for any $c \in [p]_0 \setminus \{i\}$

$$\mathrm{d}_n\left(i\underline{s}\{l, m\}, \{i, c\}\right) = \mathrm{d}_n\left(i\underline{s}\{l, m\}, \hat{c}\right) - 2^{n-1}.$$

The distance between a vertex $i\underline{s}\{l,m\}$ in ST_p^n with $i \in [p]_0$, $\{l,m\} \in \binom{[p]_0}{2}$, $\nu \in [n]_2$, and $\underline{s} \in [p]_0^{\nu-2}$ and an arbitrary corner vertex $\{c,c'\} \in \binom{[p]_0\setminus\{i\}}{2}$ can be determined as:

$$\begin{aligned}
&\mathrm{d}_n\left(i\underline{s}\{l,m\},\{c,c'\}\right)\\
&= \min\left\{\mathrm{d}_n\left(i\underline{s}\{l,m\},\{i,c\}\right)+2^{n-1},\mathrm{d}_n\left(i\underline{s}\{l,m\},\{i,c'\}\right)+2^{n-1}\right\}\\
&= \min\left\{2^{n-\nu}\,\mathrm{d}_\nu\left(i\underline{s}\{l,m\},\hat{c}\right),2^{n-\nu}\,\mathrm{d}_\nu\left(i\underline{s}\{l,m\},\hat{c'}\right)\right\}\\
&= 2^{n-\nu}\min\left\{\mathrm{d}_\nu\left(i\underline{s}\{l,m\},\hat{c}\right),\mathrm{d}_\nu\left(i\underline{s}\{l,m\},\hat{c'}\right)\right\}.
\end{aligned}$$

The path through $\{i,k\}, c \neq k \neq c'$ cannot be minimal, since we would have to go through $i\underline{s}\{l,m\} \to \{i,k\} \to \{k,c\} \to \{c,c'\}$ and this would need more than 2^n moves. The distances $\mathrm{d}_\nu\left(i\underline{s}\{l,m\},\hat{c}\right)$ and $\mathrm{d}_\nu\left(i\underline{s}\{l,m\},\hat{c'}\right)$ are equal if $\underline{s}_d \neq c,c'$ for all $d \in [\nu-2]$ and the special positions are either $\{l,m\}=\{c,c'\}$ or $\{l,m\}=\{l',m'\}$ with $\{l',m'\} \in \binom{[p]_0\setminus\{c,c'\}}{2}$; as for instance the task $1\{1,2\}$ to $\{0,3\}$ in ST_4^2 in Figure 1.9. The minimum is attained by the first distance if either $\underline{s}_d = c$ for a $d \in [\nu-2]$ and $\underline{s}_{d'} \neq c,c'$ for all $d' \in [\nu-2]\setminus[d]$ or $\underline{s}_d \neq c,c'$ for all $d \in [\nu-2]$ and $\{l,m\}=\{l',c\}$ with $l' \in [p]_0\setminus\{c'\}$, whereas it is attained by the second one if transposing c and c' one of the previous conditions holds.

In the following we look at the distance between two arbitrary vertices in ST_p^n. For $\nu \in [n]_2$, $\mu \in [\nu-1]_0$, $i,j \in [p]_0$, and $i \neq j$, let $u = is_{n-\mu-2}\ldots s_1\{l,m\}$ and $w = jt_{\nu-\mu-2}\ldots t_1\{l',m'\}$ be vertices of ST_p^n with $\{l,m\},\{l',m'\} \in \binom{[p]_0}{2}$. As in the last subsection, we analyse the length difference between the two types of paths similarly to the case of S_p^n in [37]. We already determined the distances in Formulas (1.21) and (1.22). For the difference between the distances of both possible types of paths between u and w we get with $\square \in \{>,<,=\}$:

$$\begin{aligned}
&[l\neq j][m\neq j]-[l\neq k][m\neq k]+\sum_{d=1}^{n-\nu-1}([s_d=k]-[s_d=j])\cdot 2^d+\\
&([s_{n-\nu}=k]-[s_{n-\nu}=j]+[l'\neq i][m'\neq i]-[l'\neq k][m'\neq k])\cdot 2^{n-\nu}+\\
&\sum_{d=n-\nu+1}^{n-\mu-2}([s_d=k]-[s_d=j]+[t_{\nu-n+d}=k]-[t_{\nu-n+d}=i])\cdot 2^d \quad \square \quad 2^{n-1}.
\end{aligned}$$

With $g := s_{n-\mu-2}$ and $h := t_{\nu-\mu-2}$ we can deduce that

$$\begin{aligned}
&[l\neq j][m\neq j]-[l\neq k][m\neq k]+\sum_{d=1}^{n-\nu-1}([s_d=k]-[s_d=j])\cdot 2^d+\\
&([s_{n-\nu}=k]-[s_{n-\nu}=j]+[l'\neq i][m'\neq i]-[l'\neq k][m'\neq k])\cdot 2^{n-\nu}+\\
&\sum_{d=n-\nu+1}^{n-\mu-3}([s_d=k]-[s_d=j]+[t_{\nu-n+d}=k]-[t_{\nu-n+d}=i])\cdot 2^d\\
&\square \quad 2^{n-\mu-2}\cdot\left(2^{\mu+1}-([g=k]-[g=j]+[h=k]-[h=i])\right).
\end{aligned}$$

We proceed on the same lines as in [37] with respect to the special positions $\{l,m\}$ and $\{l',m'\}$. If $\nu < n$, we can stop after the reading of $(s_{n-\nu},\{l',m'\})$ except in the case where we are in B. Then we define for $d \in [n-\nu-1]$

$$\begin{aligned}\tilde{t}_d &= i, \quad \text{if } s_d = j;\\ &= k, \quad \text{if } s_d = k;\\ &= j, \quad \text{if } s_d = x, \quad x \in [p]_0 \setminus \{k, j\};\end{aligned}$$

and

$$\begin{aligned}\tilde{\tilde{t}} &= i, \quad \text{if } \{l,m\} = \{j,u\}, \quad u \in [p]_0 \setminus \{k\};\\ &= k, \quad \text{if } \{l,m\} = \{k,w\}, \quad w \in [p]_0 \setminus \{j\};\\ &= j, \quad \text{in all other cases.}\end{aligned}$$

These pairs will be evaluated in a special automaton, namely Automaton 3.

We update the old Automata 0, 1, and 2 of [37] with the new types of pairs and get the new three Automata 0,1, and 2.

The four Automata 0, 1, 2, and 3 in Figures 1.11, 1.12, 1.13, and 1.10 together can be used in the same way as in [37] to determine the distances between arbitrary vertices in ST_p^n. Applying our automata, we can carry Algorithm 1 in [37] over to the Sierpiński triangle graphs ST_p^n with base $p \in \mathbb{N}_3$, where we have to keep in mind that we can stop the evaluation with the pair $(s_{n-\nu}, \{l', m'\})$ except in the case where we are in B and have to apply Automaton 3 in addition. According to our understanding of difference between paths, we see that there are at most two different shortest paths between arbitrary vertices with the exception of the following cases. For $i, j, g, h \in [p]_0$ and $|\{i, j, g, h\}| = 4$ we consider the tasks from $is^\delta\{g,h\}$ to $jt^\delta\{g,h\}$ with $\delta \in [n-1]_0$ and $(s_d, t_d) \in \{(g,h),(h,g)\}$ for all $d \in [\delta]$ and from $i\{l,m\}$ to $j\{l,m\}$ with $\{l,m\} \in \binom{[p]_0 \setminus \{j,i\}}{2}$ in ST_p^n. Note that for these cases p must be in $\mathbb{N}_4$. Using Algorithm 1 in [37] together with our automata, we see that we end in state B for both tasks, but with two possible ks, namely $k = g$ or h for the first case and $k = l$ or m in the second one. Hence we have altogether three shortest paths, one of the direct type and two of the detour type; see for instance from $1\{0,3\}$ to $2\{0,3\}$ in the graph ST_4^2 in Figure 1.9.

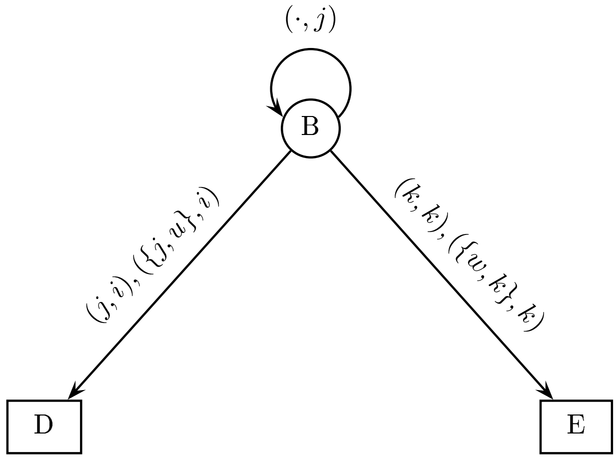

Figure 1.10: Automaton 3

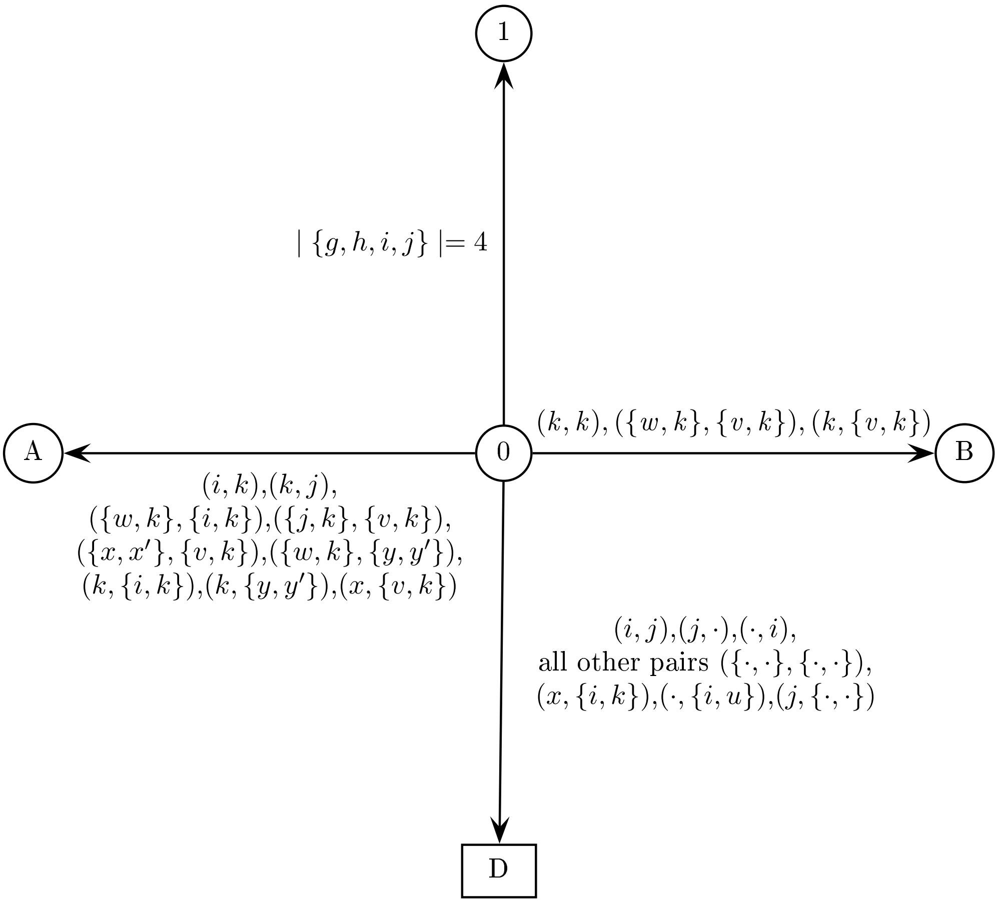

Figure 1.11: Automaton 0 with $u \in [p]_0 \setminus \{k\}$, $v \in [p]_0 \setminus \{i\}$, $w \in [p]_0 \setminus \{j\}$, $x, x' \in [p]_0 \setminus \{k, j\}$, and $y, y' \in [p]_0 \setminus \{k, i\}$, where $x \neq x'$ and $y \neq y'$

1.6 Conclusion and Outlook

In this chapter we proved a theorem which determines all shortest paths between two arbitrary vertices $\underline{s}is$ and $\underline{s}jt$ in the Sierpiński graphs S_p^{N+1} with base $p \in \mathbb{N}_3$ and exponent $N+1 \in \mathbb{N}_1$. We gave a $P2$ decision algorithm with four included algorithms and an automaton which we need in order to determine the distance and the best first move(s). In this decision algorithm we first filter out the cases where the vertices are equal and omit $\underline{s}$, fix the values of the second input pair in the next step, then run through an automaton in order to find the index of the possible “detour” subgraph, and determine in the last algorithm the best first move(s) and calculate the distance. We applied our results on already known metric properties of Sierpiński graphs in order to simplify their proofs. Moreover, we calculated the average number of pairs which have to be read to get a decision.

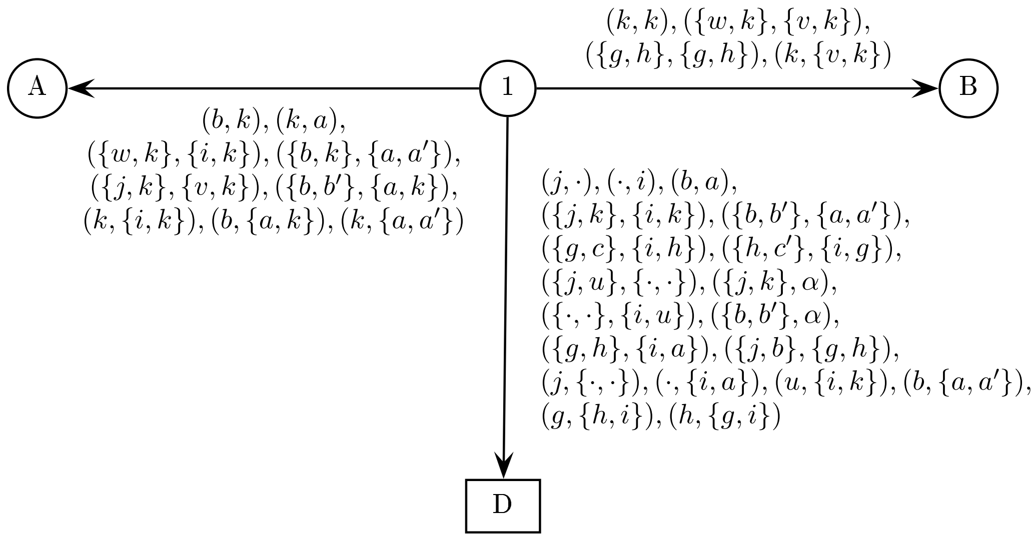

Figure 1.12: Automaton 1 with $u \in [p]_0 \setminus \{k\}, v \in [p]_0 \setminus \{i\}, w \in [p]_0 \setminus \{j\}, c \in [p]_0 \setminus \{h\}$, $c' \in [p]_0 \setminus \{g\}$, $a, a' \in [p]_0 \setminus \{i, g, h\}$, and $b, b' \in [p]_0 \setminus \{j, g, h\}$, where $a \neq a'$ and $b \neq b'$. For the input $\{\cdot,\cdot\}$ we choose $\alpha \in \binom{[p]_0}{2} \setminus \{\{v,k\},\{g,h\}\}$. Note that we stay in state 1 for the pairs (g,h), (h,g), $(\{b,b'\},\{g,h\})$, $(\{g,h\},\{a,a'\})$, $(\{b,h\},\{a,g\})$, $(\{b,g\},\{a,h\})$, $(g,\{h,a\})$, $(h,\{g,a\})$, and $(b,\{g,h\})$.

Romik provided in his paper [70] in addition to his decision automaton for S_3^n an automaton (cf. [70, Fig. 3.2]) which computes the distance for arbitrary vertices in S_3^n directly by running through the automaton. This could be a development of the here stated automaton for the general S_p^{N+1} by adding additional counters for the distance and the variable N. Our algorithm can be useful in the further study of the metric structure of the Hanoi graphs, since according to Theorem 1.16 S_p^{N+1} is embedded as a spanning subgraph into H_p^{N+1} for odd p or for $N = 1$. At the moment the Hanoi graph H_4^{15} is analysed concerning its diameter $\mathrm{diam}(H_4^{15})$ on the SuperMUC which is the high-end supercomputer at the Leibniz-Rechenzentrum (Leibniz Supercomputing Centre) in Garching near Munich. The project is named HLRB Project pr87mo ([44]).

In the last subsection we modified the $P2$ decision algorithm for Sierpiński triangle graphs. We determined for the first time all shortest paths in this new class of graphs. One interesting use of this could be the further analysis of the Sierpiński triangle graphs concerning their metric properties like the eccentricity and the average eccentricity.

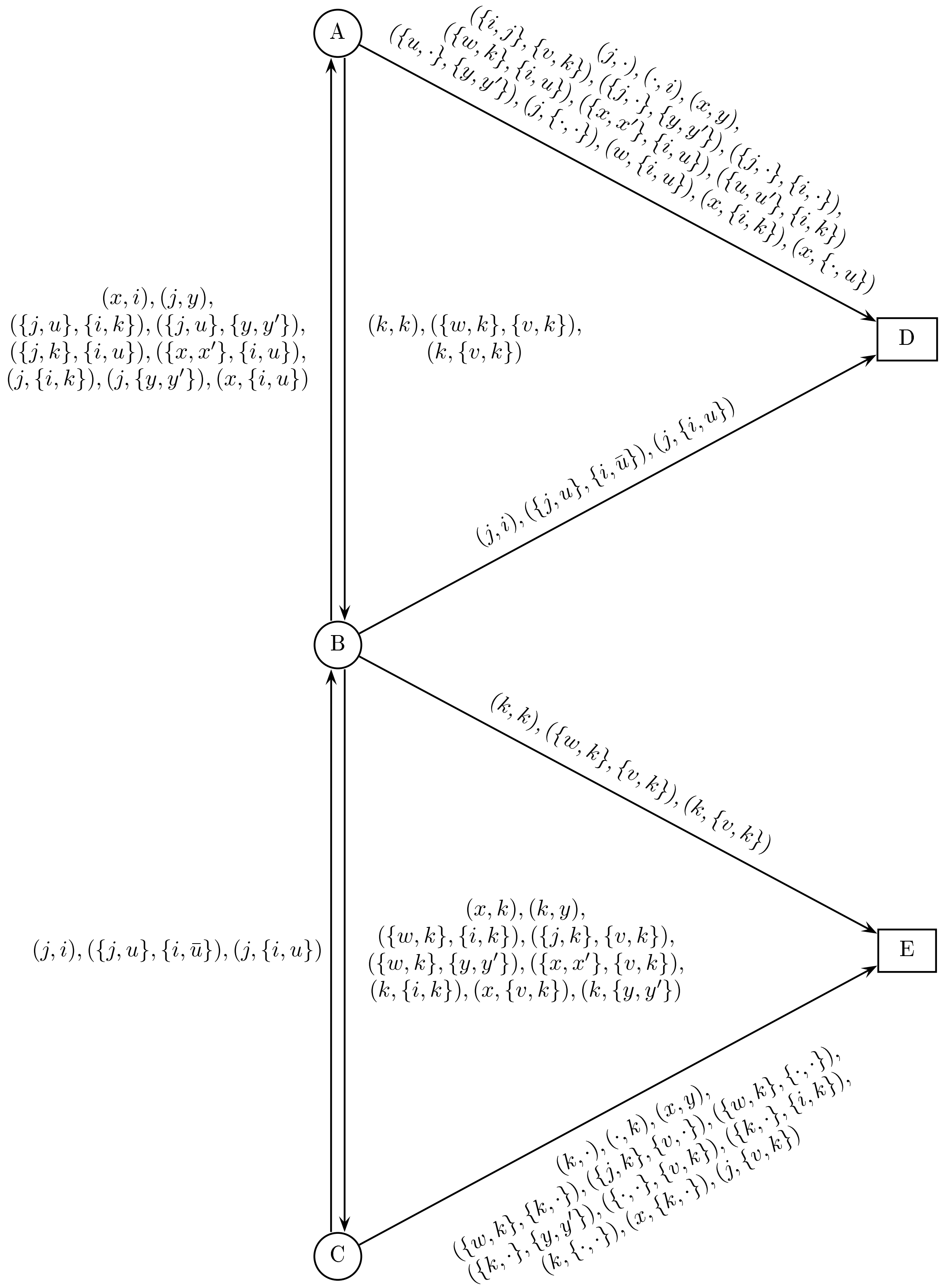

Figure 1.13: Automaton 2 with $u, u', \bar{u} \in [p]_0 \setminus \{k\}$, $v \in [p]_0 \setminus \{i\}$, $w \in [p]_0 \setminus \{j\}$, $x, x' \in [p]_0 \setminus \{k, j\}$, and $y, y' \in [p]_0 \setminus \{k, i\}$, where $x \neq x'$, $y \neq y'$, and $u \neq u'$.

Chapter 2

Variations on the Tower of Hanoi with 3 pegs

There are many variations known on the classical Tower of Hanoi with three pegs. One can modify the colours of the discs and the rules for the moves (see [33, Chapter 6] for further reading) or just restrict the allowed moves regarding their orientations. The new versions with excluded moves can be described by digraphs $D = (V(D), A(D))$ whose vertices $v \in V(D)$ are the pegs and whose arcs $(i, j) \in A(D)$ mean that a disc may move from peg i to peg j. The variant of the Tower of Hanoi where the oriented disc moves are defined by the digraph D will be named $TH(D)$.

Remark. In several articles these digraphs D are called Hanoi graphs; see for instance [10], [58], and [57]. But we will use the term Hanoi graph as defined in the book [33].

A $TH(D)$ is *solvable* if for any choice of source and goal peg and for every number of discs there exists a sequence of legal moves to transfer the tower of discs from the source peg to the goal peg.

We know from [33, Theorem 8.4] that for every digraph $D = (V(D), A(D))$ with at least three vertices the $TH(D)$ is solvable if and only if D is strongly connected, i.e., if for any pair of distinct vertices $v, w \in V(D)$ there exists a directed path from v to w and a directed path from w to v. For three pegs we have five strongly connected digraphs which are not isomorphic (see [74]). These digraphs are depicted in Figure 2.1.

In the upper row of Figure 2.1 we see from left to right the digraph C_3 of the Cyclic TH, L_3 of the Linear TH, and C_3^+ of the $TH(C_3^+)$. Underneath them we have the digraph K_3^- of the $TH(K_3^-)$ and the (di)graph K_3 of the well-known $TH(K_3)$, whose state graph is the classical Hanoi graph H_3^n. The state graphs of the other variants arise from the H_3^n by transforming the H_3^n into directed graphs considering the allowed moves between the pegs in the different cases. In Figure 2.2 the directed graphs for $TH(C_3^+)$ and $TH(K_3^-)$ are shown.

In the sequel we focus on problems of $P0$ type (i.e., to go from a perfect state to a perfect state) on the above-mentioned towers and the minimal number of moves for their solution. For solving the problems there exists one algorithm for all these variants which is due to A. Sapir [74]. For some variants there already existed algorithms before. For the classical $TH(K_3)$ an algorithm was

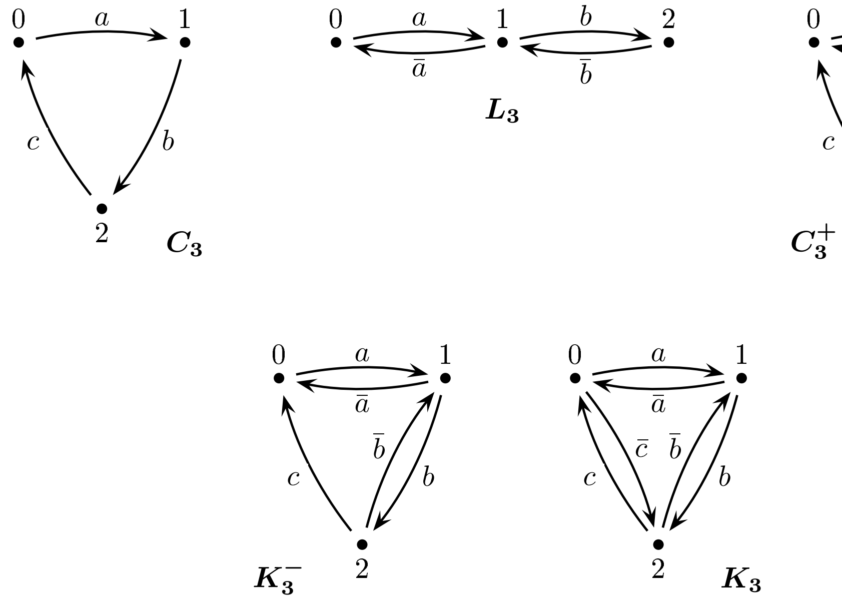

Figure 2.1: The strongly connected digraphs on three vertices

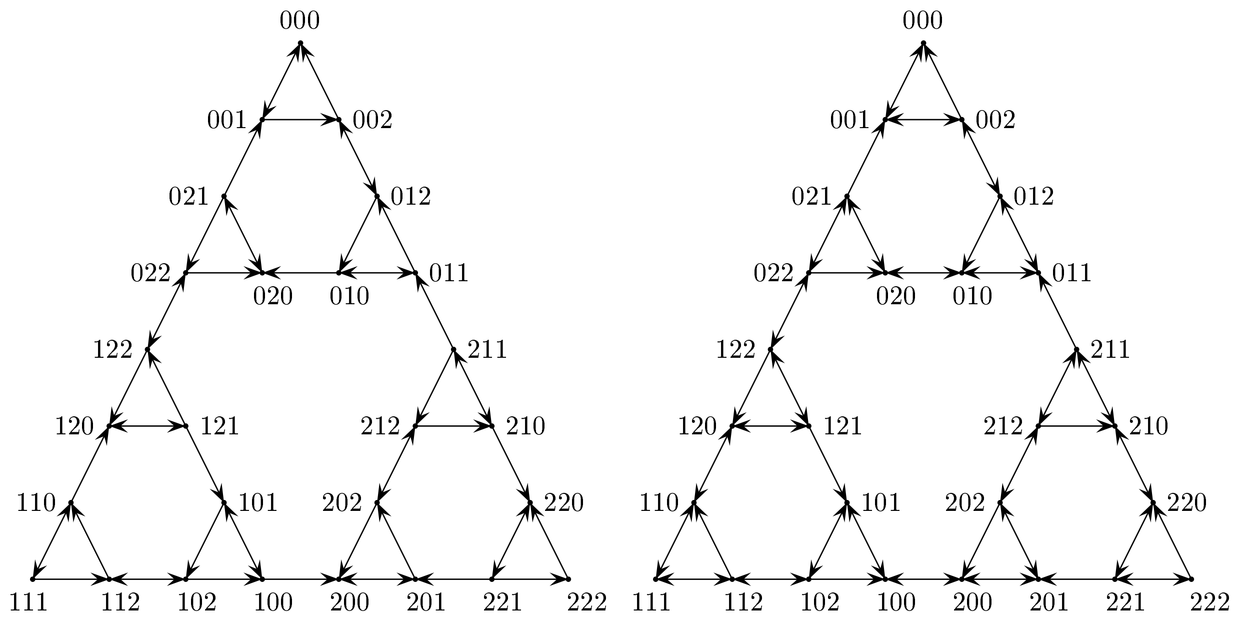

Figure 2.2: The directed state graphs of $TH(C_3^+)$ and $TH(K_3^-)$ for $n = 3$

given by Lucas's nephew Raoul Olive. Hence this algorithm is named Olive's algorithm in the literature (see [33, pp. 74–75]). The Linear TH (or three-in-a-row Hanoi) appeared first in 1944 in [75, p. 99]. H. Hering took then a closer look at this variant in [32]. An algorithm for the optimal solution can be found in [33, pp. 242–244]). In 1981, M. D. Atkinson described the Cyclic TH and also presented an algorithm for its solution in [9]. With Algorithm 7 all these variants can be uniquely solved with a minimum number of moves, as stated in the following theorem.

Algorithm 7 The Sapir Algorithm

```
procedure SAP(D, n, i, j)
    parameter D : strongly connected digraph with V(D) = T
    parameter n : number of discs {n ∈ ℕ_0}
    parameter i : source peg {i ∈ T}
    parameter j : goal peg {j ∈ T}
    if n ≥ 1 and i ≠ j then
        k ← 3 − i − j
        if there is an arc from i to j then
            SAP(D, n − 1, i, k)
            move disc n from i to j
            SAP(D, n − 1, k, j)
        else
            SAP(D, n − 1, i, j)
            move disc n from i to k
            SAP(D, n − 1, j, i)
            move disc n from k to j
            SAP(D, n − 1, i, j)
        end if
    end if
end procedure
```

Theorem 2.1. *For every strongly connected digraph D with three vertices / pegs the $TH(D)$ can be uniquely solved by Algorithm 7 with the minimum number of moves.*

Proof. A proof of this theorem can be found in [33, p. 245f] or [74, Theorem 1] or for the Cyclic TH in [9]. □

The Sapir Algorithm 7 gives rise to an infinite sequence of moves for each $TH(D)$, with D as in Figure 2.1, which is obtained as the limit of the finite sequences of moves for n discs as n goes to infinity. More details on these sequences will be given in the later sections. The sequences are defined on the basic alphabet $\{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$ excluding the forbidden moves in each special case. We illustrate the labelling in Figure 2.1. Now our special interest lies in the sequences for $TH(K_3^-)$ and $TH(C_3^+)$. From J.-P. Allouche [7] we know that both are morphic sequences. We want to investigate whether they are automatic.

For the other three variants this is already known. Allouche and F. Dress showed in [5] that the $TH(K_3)$ sequence is 2-automatic. Additionally, they gave an automaton with which we can find the n-th move in $TH(K_3)$ with n as binary number (evaluating from right to left and $n \in \mathbb{N}$). Allouche and Sapir [7] proved using Algorithm 7 that the Linear TH sequence is 3-automatic. In contrast, the Cyclic TH sequence is not d-automatic for any d, as it was shown in [2] by Allouche. In [7], Allouche and Sapir conjectured that the only Hanoi sequences with restricted moves which are d-automatic for some d are the $TH(K_3)$ sequence and the Linear TH sequence. This was formulated as a conjecture in [33, Chapter 9] (cf. Chapter 0).

2.1 Words, morphisms, and sequences

We will outline the most important definitions and theorems for our further considerations, especially concerning morphic and automatic sequences. For the statements concerning matrix analysis one may look into some standard source for the theory, as for instance [48].

Let Σ be a finite set of letters. We call Σ an *alphabet*. Further, let $n \in \mathbb{N}_0$. Then a *finite word* s is a map from $[n]_0$ to Σ and is written as $s = s_0 s_1 \ldots s_{n-1}$. In the case of $n = 0$ the word is the *empty word*, denoted by ϵ. The set Σ^* is the set of finite words on Σ.

Example 2.2. Let $\Sigma = \{0, 1\}$. Then $\Sigma^* = \{\epsilon, 0, 1, 00, 01, 11, 10, 000, 001, \ldots\}$.

If $s \in \Sigma^*$, then its *length* $|s|$ is the value n of the corresponding map between $[n]_0$ and Σ, i.e., the number of letters in s. Note that $|\epsilon| = 0$. Assume the letter a is an element of Σ, then the number of its occurrences in s is denoted by $|s|_a$. The set Σ is a subset of Σ^* in the sense that a letter $a \in \Sigma$ can be identified with a word of length 1.

Let $s, t \in \Sigma^*$. The operation st is the *concatenation* of s and t which is the consecutive writing of the words s and t. In general, the concatenation is not commutative, whereas it is associative as $r(st) = (rs)t$ for all finite words r, s, and t. This operation together with the set Σ^* is a monoid[7], where the empty word is the identity element.
We say that $s' \neq \epsilon$ is a *prefix* of s if there exists an $s'' \in \Sigma^*$ such that $s = s's''$. The word $s' \neq \epsilon$ is a *strict prefix* of s if $s'' \neq \epsilon$. We say that a word s is a *subword* or (*factor*) of a word w if there exist words r and t such that $w = rst$.
Similarly to the definition of a finite word, an *infinite word* can be viewed as a map from $\mathbb{N}_0$ to Σ. In our text we will use the term *sequence* for an infinite word. Let $\mathbf{s} = s_0 s_1 s_2 \ldots$ be a sequence. For $i \in \mathbb{N}_0$ we set $\mathbf{s}_{[i]} = s_i$ and for $i \in \mathbb{N}_0$ and $j \geq i$ we define $\mathbf{s}_{[i \ldots j]} = s_i s_{i+1} \ldots s_j$.

Example 2.3. As an example we give the *Prouhet-Thue-Morse sequence*

$$\mathbf{t} = 0110100110010110 \ldots .$$

The i-th element $\mathbf{t}_{[i]} = t_i$ of $\mathbf{t}$ is equal to 0, if the number of 1s in the binary expansion of i is even, and equal to 1, if it is odd; see for instance [1].

A basic tool in working with alphabets and words is the *morphism* (or *homomorphism*). Let Δ be another alphabet. A *morphism* is a map $\psi : \Sigma^* \to \Delta^*$ which satisfies $\psi(st) = \psi(s)\psi(t)$ for all words $s, t \in \Sigma^*$. If $\Sigma = \Delta$, we call the map an *endomorphism* and can iterate the application of ψ. Then $\psi^0(a) = a$ and $\psi^n(a) = \psi(\psi^{n-1}(a))$ for all $a \in \Sigma$ and $n \in \mathbb{N}$.

Morphisms can have several properties. A morphism $\psi : \Sigma^* \to \Delta^*$ is *k-uniform* for $k \in \mathbb{N}$ if $|\psi(a)| = k$ for all $a \in \Sigma$. A 1-uniform morphism is a *coding*.

[7] A *monoid* S is a set that is closed under an associative binary operation and has an identity element $i \in S$ such that, for all $a \in S$, $ia = a = ai$; see for instance [49].

Example 2.4. The Prouhet-Thue-Morse sequence $\mathbf{t}$ of Example 2.3 is generated by the 2-uniform morphism $\psi : \{0,1\}^* \to \{0,1\}^*$ which is given by

$$\begin{aligned} 0 &\to 01, \\ 1 &\to 10. \end{aligned}$$

For a proof we refer to [1, Proposition 3].

If $\psi(a) \neq \epsilon$ for all $a \in \Sigma$, then ψ is *non-erasing*. If $\psi(a) = \epsilon$ for all $a \in \Sigma$, then we say that ψ is *trivial*.

One very important type of morphisms is the primitive endomorphism because of its special property. An endomorphism $\psi : \Sigma^* \to \Sigma^*$ is *primitive* if there exists an $n \in \mathbb{N}$ such that for all $a, b \in \Sigma$ the letter a occurs in $\psi^n(b)$. The name is connected with the primitivity of the so-called incidence matrix of ψ.
Let $\Sigma = \{a_1, a_2, \dots a_m\}$, $\Delta = \{b_1, b_2, \dots b_n\}$, and $\psi : \Sigma^* \to \Delta^*$. The *incidence matrix* (or *transition matrix* or *substitution matrix*) $M(\psi)$ of the morphism ψ is

$$M(\psi) = (m_{i,j})_{\substack{i\in[n]\\ j\in[m]}}$$

where $m_{i,j} = |\psi(a_j)|_{b_i}$, i.e., $m_{i,j}$ is the number of occurrences of b_i in $\psi(a_j)$. For $n = m$ we define that $M^0 = I$ and for all $k \in \mathbb{N} : M^{k+1} = M^k \cdot M$. An $(n \times n)$-matrix $M = (m_{i,j})_{i,j\in[n]}$ is non-negative, denoted by $M \geq 0$, if $m_{i,j} \geq 0$ for every $i, j \in [n]$, and is positive, written $M > 0$, if $m_{i,j} > 0$ for every $i, j \in [n]$. A non-negative matrix M is *primitive* if there exists a $k \in \mathbb{N}$ such that all entries of M^k are positive. We see that an endomorphism is primitive if and only if its incidence matrix is primitive.

A non-negative matrix is *reducible* if there exists a permutation matrix[8] P such that

$$P^{-1}MP = \begin{pmatrix} A_1 & R \\ 0 & A_2 \end{pmatrix} \quad \text{or} \quad M = P\begin{pmatrix} A_1 & R \\ 0 & A_2 \end{pmatrix}P^{-1} \tag{2.1}$$

where A_1, A_2 are square matrices, and $0, R$ are rectangular matrices such that the dimensions match. If a non-negative matrix is not reducible, we say that it is *irreducible*.

Lemma 2.5. *Every primitive matrix is irreducible.*

Proof. This statement is a direct consequence of [8, Proposition 8.3.4 (e)], which says: Let $M = (m_{i,j})_{i,j\in[n]}$ be a non-negative $(n \times n)$-matrix. The matrix M is irreducible if and only if for each i, j there exists a $k = k(i, j)$ such that $m^k_{i,j}$ is positive (and there is such a k with $k \in [n]$). □

Lemma 2.6 ([48, Theorem 8.5.6])**.** *Let P be a primitive $(n \times n)$-matrix. Then there exists an $s \leq (n-1)\, n^n$ such that $P^s > 0$.*

[8] Let $n \in \mathbb{N}$. An $(n \times n)$-matrix P is a *permutation matrix* if exactly one entry in each row and column is equal to 1 and all other entries are 0.

In order to explain what morphic and automatic sequences are, we return to endomorphisms. Let $\psi : \Sigma^* \to \Sigma^*$ be an endomorphism. If there exist a letter $a \in \Sigma$ and a word $s \in \Sigma^*$ such that $\psi(a) = as$ and $\psi^n(s) \neq \epsilon$ for each $n \in \mathbb{N}$, we say that the endomorphism ψ is *prolongable on* $a \in \Sigma$. Then we see that $\psi^{n+1}(a) = \psi^n(\psi(a)) = \psi^n(as) = \psi^n(a)\psi^n(s)$, and as a consequence that $\psi^n(a)$ is a strict prefix of $\psi^{n+1}(a)$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \psi^n(a)$ exists and is infinite. This limit is denoted by

$$\psi^\infty(a) := \lim_{n\to\infty} \psi^n(a) = as\psi(s)\psi^2(s)\psi^3(s)\ldots$$

which is a fixed point of (the extension by continuity of) the endomorphism ψ (to infinite sequences), i.e., $\psi(\psi^\infty(a)) = \psi^\infty(a)$. Often we will refer to such a sequence as an *(iterative) fixed point of a morphism.*

Definition 2.7. If a sequence $\mathbf{s}$ is an iterative fixed point of an endomorphism ψ, more precisely $\mathbf{s} = \psi^\infty(a)$, it is *pure morphic.* If there exists a coding $\tau : \Sigma^* \to \Delta^*$ and $\mathbf{s} = \tau(\psi^\infty(a))$, we call the sequence *morphic.*

Example 2.8. Recall Examples 2.3 and 2.4. Then $\mathbf{t} = \psi^\infty(0)$ (cf. [1, Proposition 3]).

Example 2.9. Recall the Cyclic TH sequence mentioned above. Its alphabet is $\{a, b, c\}$. This sequence is the image, under the coding $\tau : \{f, g, h, u, v, w\}^* \to \{a, b, c\}^*$, of the fixed point of the endomorphism ψ on $\{f, g, h, u, v, w\}^*$ where

$$\tau(f) = a = \tau(w), \quad \tau(g) = b = \tau(u), \quad \tau(h) = c = \tau(v),$$

and

$$\begin{array}{lllllll} \psi(f) & = & fvf, & \quad & \psi(g) & = & gwg, \\ \psi(h) & = & huh, & & \psi(u) & = & fg, \\ \psi(v) & = & gh, & & \psi(w) & = & hf. \end{array}$$

The proof can be found in [5, Section 5] and [7, Theorem 1]. Obviously, it is a morphic sequence.

We keep in mind that a (pure) morphic sequence can be an iterative fixed point of two different morphisms which are not powers of the same morphism.

The following type of endomorphisms plays an important role in our investigation.

Definition 2.10. Let $d \in \mathbb{N}_2$. A sequence $\mathbf{s}$ is *d-automatic* if it is the image, under a coding, of an iterative fixed point of a d-uniform morphism.

Example 2.11. As examples we will take a closer look on the $TH(K_3)$ sequence, or (classical) Hanoi sequence, and the Linear TH sequence. The first one is defined on the basic alphabet $\{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$. Its prefixes of length $2^N - 1$ give the minimal sequence of moves in the classical Hanoi game to transfer N discs from peg 0 to peg 1 if N is odd and from peg 0 to peg 2 if N is even. The sequence is the iterative fixed point of the 2-uniform morphism, given on the basic alphabet by

$$\begin{array}{lll} a \to a\bar{c}, & \quad & \bar{a} \to ac, \\ b \to c\bar{b}, & & \bar{b} \to cb, \\ c \to b\bar{a}, & & \bar{c} \to ba. \end{array}$$

The second one, sometimes also called the three-in-a-row Hanoi sequence, on $\{a, b, \bar{a}, \bar{b}\}$ is the iterative fixed point of the 3-uniform morphism

$$\begin{aligned} a &\to aba, & \bar{a} &\to ab\bar{a}, \\ b &\to \bar{b}\bar{a}b, & \bar{b} &\to \bar{b}\bar{a}\bar{b}. \end{aligned}$$

We obtain that both sequences are automatic. The classical Hanoi sequence is 2-automatic and the Linear TH sequence is 3-automatic. These results can also be found in [7] and especially for the classical case in [4] and [5]. For the $TH(K_3)$ sequence Hinz even showed that the basic alphabet can be reduced to a 5-letter alphabet. Grouping the elements of the sequence by triples, we see that only the five triples $a\bar{c}b, ac\bar{b}, \bar{a}cb, acb$, and $\bar{a}c\bar{b}$ occur. Starting with the $TH(K_3)$ sequence, this is used to construct a square-free[9] sequence on an alphabet consisting of these different triples ([35, Theorem 2]).

Example 2.12. In Example 2.9 we looked at the Cyclic TH sequence. In [2], it was proved by Allouche that for any $d \in \mathbb{N}_2$ this sequence is not d-automatic.

There are many alternative definitions of d-automatic sequences. One of them explains why they are called automatic. We can say that a sequence is d-automatic if it can be generated by a deterministic finite automaton with output on the input alphabet $[d]_0$, called a d-DFAO; see for instance [8, Definition 5.1.1]. Another definition works with formal power series and is mentioned in the literature as Cristol's Theorem ([8, Theorem 12.2.5]). We use our definition, which is equivalent to the other ones for $d \in \mathbb{N}_2$ by a theorem of Cobham ([8, Theorem 6.3.2]) as it the most useful one for our purpose. For more details on automatic sequences we refer to the book [8] by Allouche and J. Shallit.

Remark. Usually, automatic sequences are defined only for $d \in \mathbb{N}_2$. But we can extend the concept for $d = 1$. A sequence is 1-automatic if it is generated by an 1-DFAO with $\Sigma = \{1\}$ (or $\Sigma = \{|\}$). Thereby the n-th term of the sequence depends on the unary notation for n, e.g., $(1)^n$ (or $|^n$). More details can be found in [8, Section 5.7].

2.2 An approach to disprove the automaticity of the $TH(C_3^+)$ and the $TH(K_3^-)$ sequences based on the frequency of a letter

In this approach to decide about the automaticity of our sequences we will use the frequency of a letter. Consider a sequence $\mathbf{s} = s_0 s_1 s_2 \ldots$ over a finite alphabet Σ, where $s_k \in \Sigma$ ($k \in \mathbb{N}_0$). Let s be a prefix of $\mathbf{s}$ and $|s|$ its length. Then the *frequency of the letter $a \in \Sigma$ in s* is defined by $\frac{|s|_a}{|s|}$. The *frequency* $\mathrm{Freq}_a(\mathbf{s})$ *of the letter $a \in \Sigma$ in the sequence* $\mathbf{s}$ is defined as the limit

$$\mathrm{Freq}_a(\mathbf{s}) = \lim_{n\to\infty} \frac{|\mathbf{s}_{[0\ldots n-1]}|_a}{n}, \quad \text{if it exists.}$$

[9] A *square* is a word of the form tt, where t is a subword of the word. If $\mathbf{s}$ is a sequence which contains no non-empty subword of this form, it is called *square-free*.

The following theorem says that, assuming the automaticity of a sequence, the frequency of a letter, if it exists, is a rational number.

Theorem 2.13. *Let* $\mathbf{s} = (s_n)_{n\in\mathbb{N}_0}$ *be an automatic sequence. If the frequency of a letter exists, then it is a rational number.*

Proof. See [8, Theorem 8.4.5] or [18, Theorem 6]. □

If we could prove in each case that the frequency of a letter in our sequences is not rational, we could conclude that the sequences themselves cannot be automatic.

Example 2.14. The non-automaticity of the Cyclic TH sequence was proved in [2], where use has been made of Theorem 2.13 as well. Unfortunately, the frequencies of the three letters a, b, and c are all rational. Since Theorem 2.13 is no if-and-only-if statement, this is not sufficient to state the automaticity of this sequence.
But Allouche found a factor, namely *aba*, of the sequence with irrational frequency whence he could deduce the non-automaticity (cf. [2, Proposition 3.5]).

One very helpful theorem to determine the rationality could be [8, Theorem 8.4.7]. It says that, if we have a primitive morphic sequence, then the frequencies of all letters exist and are non-zero. Furthermore, the vector of frequencies of the letters occurring in the sequence is the positive normalised vector associated with the so-called Perron-Frobenius eigenvalue of the corresponding incidence matrix. Consequently, we can calculate the frequencies and prove the rationality. However, we cannot employ this theorem on our cases, since the conditions of the theorem ask for a primitive morphic sequence. We do not know so far whether the $TH(C_3^+)$ and $TH(K_3^-)$ sequences are primitive.
In Sections 2.2.2 and 2.2.3 we will see that our morphisms to generate the sequences are not primitive which does, however, not mean that no primitive ones exist. For the $TH(K_3^-)$ sequence we later show in Theorem 2.42 that it is a morphic sequence with respect to a primitive prolongable morphism. Unfortunately, we then still cannot use the theorem (see the remark after Theorem 2.42).

Note that most of the calculations in the following sections are done with the computer algebra system (CAS) *Sage* 5.10.

2.2.1 Necessary tools

In this section we introduce some definitions and the theorems and lemmata which we need particularly for our approach.

The following is part of the so-called Perron-Frobenius Theory (see [8]). We define the Perron-Frobenius eigenvalue. Here, $|\cdot|$ stands for the absolute value.

Theorem 2.15 ([8, Theorem 8.3.11]). *Let M be a non-negative $(m \times m)$-matrix. Then there exists a $\lambda_r \in \mathbb{R}_0^+$ such that:*

a) λ_r is an eigenvalue of M, and every eigenvalue λ of M satisfies $|\lambda| \leq \lambda_r$.

b) There exists a non-negative eigenvector corresponding to the eigenvalue λ_r.

c) There exists a positive integer h such that every eigenvalue λ of M with $|\lambda| = \lambda_r$ fulfills $\lambda^h = \lambda_r^h$.

The number λ_r is called the Perron-Frobenius eigenvalue.

We note that, for a given morphic sequence $\mathbf{s}$, the number λ_r is not unique, since, if λ_r is the Perron-Frobenius eigenvalue of $M(\psi)$, then λ_r^n is the Perron-Frobenius eigenvalue of $M(\psi^n)$, but ψ as well as ψ^n generates the same sequence $\mathbf{s}$.

Let M be an irreducible matrix that has precisely h eigenvalues (counted with multiplicities) whose absolute values equal the Perron-Frobenius eigenvalue. Then we call the integer h the *index of M*.

Theorem 2.16 ([8, Theorem 8.3.10]). *Let M be a non-negative $(m \times m)$-matrix.*

a) If M is irreducible, and if λ_r is its Perron-Frobenius eigenvalue and its index equals h, then the h eigenvalues of M whose absolute values equal λ_r are the numbers $\lambda_r \mathrm{e}^{2il\pi/h}$, where $l \in [h]_0$. All these eigenvalues are simple roots of the characteristic polynomial of M.

b) The matrix M is primitive if and only if it is irreducible and its index is equal to 1.

It remains to define a dominating Jordan block and a simple generator. Recall that two matrices A and B are similar, denoted by $A \sim B$, if there exists an invertible matrix P such that $A = P^{-1}BP$. We know that every $(m \times m)$-matrix M is similar to a matrix in Jordan form. This means

$$M \sim J = \begin{pmatrix} J_1 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{k-1} & 0 \\ 0 & 0 & \cdots & 0 & J_k \end{pmatrix},$$

where J_i are Jordan blocks. These Jordan blocks are of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{pmatrix} \tag{2.2}$$

with dimension $m_i \times m_i$ and, where λ_i is a complex number.

Let M be a non-negative square matrix similar to J and again λ_r the Perron-Frobenius eigenvalue of M. Then a Jordan block J_i is called a *dominating Jordan block of J* if

- $\lambda_i = \lambda_r$, and
- for all $j \in [k]$,
$$\lambda_j = \lambda_r \Rightarrow m_i \geq m_j.$$

The dominating Jordan block of a non-negative square matrix is well-defined, since the Jordan blocks are unique. Note that the matrix in Jordan form itself is not unique.

Let $\psi : \Sigma^* \to \Sigma^*$ be an endomorphism which is prolongable on $a \in \Sigma$ and $\tau : \Sigma^* \to \Delta^*$ a coding. We denote by $M(\psi)$ the incidence matrix of ψ, which is non-negative, and by $\lambda_r(\psi)$ the corresponding Perron-Frobenius eigenvalue.

Definition 2.17. The morphism ψ is a *simple generator* of the sequence $\tau(\psi^\infty(a))$ if

(G1) the set of letters occurring in $\psi(a)$ is Σ,

(G2) if λ is an eigenvalue of $M(\psi)$ and the absolute value of λ equals $\lambda_r(\psi)$, then λ equals the Perron-Frobenius eigenvalue.

The following theorem deals with morphic sequences and the existence of a simple generator for them. The proof of the theorem also describes how to construct a simple generator of a morphic sequence by modifying the initial map.

Theorem 2.18 ([72, Lemma 2])**.** *Any morphic sequence has a simple generator.*

Proof. Let $\mathbf{s} = \tau(\psi^\infty(a))$ be a morphic sequence with the endomorphism $\psi : \Sigma^* \to \Sigma^*$ and the coding $\tau : \Sigma^* \to \Delta^*$, where Σ and Δ are again finite alphabets and $a \in \Sigma$. We define $\overline{\Sigma} \subseteq \Sigma$ to be the set of letters which occur in $\psi^\infty(a)$. This implies $\psi\left(\overline{\Sigma}\right) \subset \overline{\Sigma}^*$, and we restrict ψ to $\overline{\Sigma}^*$. As $\mathbf{s}$ is a morphic sequence, ψ is prolongable on a and, therefore, there exists an integer $l \in \mathbb{N}$ such that all letters of $\overline{\Sigma}$ already occur in $\psi^l(a)$. By iteration of Equation (2.1), the incidence matrix $M(\psi)$ is similar to a matrix with the following structure

$$M(\psi) \sim \begin{pmatrix} A_{1,1} & B_{1,2} & \dots & B_{1,n} \\ 0 & A_{2,2} & \dots & B_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{n,n} \end{pmatrix}.$$

Each entry $A_{i,i}$ on the diagonal is either an irreducible square matrix or a zero square matrix, $B_{i,j}$ are rectangular matrices such that the dimensions match, and the entries below the diagonal are all 0. If $M(\psi)$ itself is irreducible, we see that $n = 1$ and $M(\psi) = A_{1,1}$. Let h_i be the index of $A_{i,i}$ for $i \in [n]$. If $A_{i,i}$ is a zero matrix, we let $h_i = 1$. Then we set $h = \operatorname{lcm}(h_1, \dots, h_n)$. It follows from Theorem 2.16 that all eigenvalues λ of $M(\psi)$ with $|\lambda| = \lambda_r(\psi)$ satisfy the equation $\lambda^h = \lambda_r(\psi)^h$.

Now we have the tools to create a simple generator. The morphism $\psi' : \overline{\Sigma}^* \to \overline{\Sigma}^*$ is determined by the condition $\psi'(o) = \psi^{hl}(o)$ for all letters $o \in \overline{\Sigma}$ and is now shown to be the wanted one.

We check if ψ' fulfills the conditions (G1) and (G2). It is apparent that $\mathbf{s} = \tau(\psi'^{\infty}(a))$. The condition (G1) is satisfied, since $\psi^l(a)$ is a prefix of $\psi'(a) = \psi^{hl}(a)$ and thus all the letters of $\overline{\Sigma}$ occur in $\psi'(a)$. Now we have a look at (G2). Suppose λ is an eigenvalue of $M(\psi')$. Since $M(\psi') = M(\psi)^{hl}$, we get as a consequence

$$\det(M(\psi') - \lambda I) = \det(M(\psi) - x_1 I) \cdots \det(M(\psi) - x_{hl} I),$$

where the x_is are the hl-th roots of λ. Hence λ is an eigenvalue of $M(\psi')$ if and only if some of its hl-th roots, here x_i, is an eigenvalue of $M(\psi)$. Using this observation in the case when $\lambda = \lambda_r(\psi')$, we obtain $\lambda_r(\psi') = \lambda_r(\psi)^{hl}$. Now we assume that $\lambda \neq \lambda_r(\psi')$ and $|\lambda| = \lambda_r(\psi')$. As $\lambda = x_i^{hl}$ and $x_i^h = \lambda_r(\psi)^h$, we get $\lambda = \lambda_r(\psi)^{hl} = \lambda_r(\psi')$, which is a contradiction. Thus ψ' satisfies (G2) and altogether we can conclude that ψ' is a simple generator. □

From now on we assume that $\mathbf{s} = \tau(\psi^{\infty}(a))$ is a morphic sequence where $\psi : \Sigma^* \to \Sigma^*$ is a simple generator and $\tau : \Sigma^* \to \Delta^*$ is a coding. Further, we note that the Perron-Frobenius eigenvalue $\lambda_r(\psi)$ of the incidence matrix of a prolongable morphism ψ is always at least 1. Otherwise, i.e., if $0 \leq \lambda_r(\psi) < 1$, $M(\psi^n)$ would tend to 0 as $n \to \infty$. This is a contradiction as ψ is prolongable. Recall that ψ is prolongable on $a \in \Sigma$ if $\psi(a) = as$ for some $s \in \Sigma^*$ such that $\psi^n(s) \neq \epsilon$ for all $n \in \mathbb{N}$.

In our further text a limit matrix is always obtained in the following way. Let $A = (a_{i,j})_{i,j \in [m]}$ be a square matrix. Then

$$\lim_{n\to\infty} A^n = B = (b_{i,j})_{i,j\in[m]}, \text{ where } b_{i,j} = \lim_{n\to\infty} a_{i,j}^n \text{ for every } i, j \in [m].$$

So we define the limit elementwise.

The following lemma is the first step to determine the frequency of a letter. Recall the common notation that for two functions $f, g : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$ and $x \to x_0$ we denote $f(x) = O(g(x))$ if there exist a $C > 0$ and a $\delta > 0$ such that

$$|f(x)| \leq C \cdot |g(x)| \text{ for all } x \text{ with } |x - x_0| < \delta.$$

Lemma 2.19 ([72, Lemma 3]). *Let $j+1$ be the size of the dominating Jordan block in $J(\psi)$, where $J(\psi)$ is the Jordan form of $M(\psi)$. Then the limit matrix*

$$E(\psi) := \lim_{n\to\infty} \frac{M(\psi)^n}{n^j \lambda_r(\psi)^n} \tag{2.3}$$

exists, and has the following properties:

- *The rank of $E(\psi)$ equals the number of occurrences of the dominating Jordan block in $J(\psi)$.*

- *For all $o, p \in \Sigma$ and $n \in \mathbb{N}_0$*

$$|\psi^n(o)|_p = e_{p,o} n^j \lambda_r^n + O(n^{j-1}\lambda_r^n),$$

where $e_{p,o}$ is the entry of $E(\psi)$ at position (p, o).

- *The column $E_a = (e_{p,a})_{p\in\Sigma}$ is non-zero.*

Proof. We will prove the existence of $E(\psi)$ and the first property of the limit matrix, for the other statements see [8, Theorem 8.3.14 a)] and [72, Lemma 3]. Set $\lambda_r(\psi) = r$ and denote by t the number of occurrences of the dominating Jordan block in $J(\psi) = (J_1, \ldots, J_l)$. The matrices J_i are Jordan blocks of the form (2.2). We can see that for all $n \in \mathbb{N}_0$

$$J_i^n = \begin{pmatrix} \lambda_i^n & \binom{n}{1}\lambda_i^{n-1} & \ldots & \binom{n}{m_i-1}\lambda_i^{n-m_i+1} \\ 0 & \lambda_i^n & \ldots & \binom{n}{m_i-2}\lambda_i^{n-m_i+2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_i^n \end{pmatrix}.$$

Note also that

$$\binom{n}{j} r^{n-j} = \underbrace{\frac{1}{j!r^j}}_{>0} n^j r^n + O(n^{j-1}r^n).$$

We know that ψ is a simple generator. Therefore, the equality $|\lambda_i| = r$ implies $\lambda_i = r$ and if either $\lambda_i \neq r$ or $p < j$, then it must follow that

$$\binom{n}{p}\lambda_i^{n-p} = O(n^{j-1}r^n).$$

Altogether we get the existence of the limit

$$D := \lim_{n\to\infty} \frac{(J(\psi))^n}{n^j r^n}.$$

Each occurrence of the dominating Jordan block contributes exactly one positive entry, namely $\frac{1}{j!r^j}$, in D, while all the other entries are 0. The matrix D has rank t as all these positive entries lie in different rows and columns. Let Q be a matrix with inverse Q^{-1} such that $M(\psi) = QJ(\psi)Q^{-1}$. By $(M(\psi))^n = Q(J(\psi))^n Q^{-1}$ we can conclude that

$$\lim_{n\to\infty} \frac{(M(\psi))^n}{n^j r^n} = QDQ^{-1}. \tag{2.4}$$

This matrix is non-negative, since the limit of a sequence of non-negative matrices is non-negative. We set $E(\psi) = (e_{p,o})_{p,o\in\Sigma}$ as this limit matrix. The rank of $E(\psi)$ equals that of D, namely t, since $E(\psi)$ is obviously similar to D. □

From the last lemma we deduce the following corollary. We let $M_b = (m_{p,b})_{p\in\Delta}$ denote the column of the matrix $M = (m_{p,o})_{\substack{p\in\Delta\\ o\in\Sigma}}$ that corresponds to $b \in \Sigma$.

Corollary 2.20 ([72, Corollary 1])**.** *Again, $j+1$ is the size of the dominating Jordan block in $J(\psi)$. Define $W(\tau,\psi) := M(\tau)E(\psi)$, where $M(\tau)$ is the incidence matrix of the coding τ and $E(\psi)$ is the matrix from (2.3). Then*

- *For all $o \in \Sigma$, $p \in \Delta$ and $n \in \mathbb{N}_0$,*

$$|\tau(\psi^n(o))|_p = w_{p,o} n^j \lambda_r^n + O(n^{j-1}\lambda_r^n),$$

where $w_{p,o}$ is the entry of $W(\tau,\psi)$ at position (p,o).

- *The column W_a is non-zero.*

The following theorem guarantees the existence of the frequency of a letter.

Theorem 2.21. *Let $\mathbf{s} = \tau(\psi^\infty(a))$ be a morphic sequence, where $a \in \Sigma$, $\psi : \Sigma^* \to \Sigma^*$ is a simple generator and $\tau : \Sigma^* \to \Delta^*$ is a coding. We will denote the matrix $M(\tau)E(\psi)$ by $W(\tau,\psi)$. The frequency of the letter $q \in \Delta$ in $\mathbf{s}$ exists if and only if*

$$\frac{w_{q,b}}{\sum_{p\in\Delta} w_{p,b}} = \frac{w_{q,a}}{\sum_{p\in\Delta} w_{p,a}} \tag{2.5}$$

for all non-zero columns W_b of $W(\tau,\psi)$. If it exists, the frequency of $q \in \Delta$ is the value in (2.5).

Proof. See [72, Theorem 2]. □

The next corollary arises from the last theorem and gives a necessary and sufficient condition for the existence of the frequencies of all letters in morphic sequences.

Corollary 2.22 ([72, Corollary 2])**.** *Under the hypotheses of Theorem 2.21, the frequencies exist for all letters in $\tau(\psi^\infty(a))$ if and only if the rank of the matrix $W(\tau,\psi)$ equals one.*

Proof. The frequencies of all letters exist if and only if (2.5) holds for every $q \in \Delta$ and for all letters $b \in \Sigma$ for which W_b is non-zero. We infer that this applies if and only if

$$W_b = \frac{\sum_{p\in\Delta} w_{p,b}}{\sum_{p\in\Delta} w_{p,a}} W_a$$

for all non-zero columns W_b. (In [72], this is stated differently.) Since W_a is non-zero, this is equivalent to the condition that the rank of $W(\tau,\psi)$ is one and the proof is done. □

This completes the collection of the general tools we will need, and we can now proceed with our special sequences.

2.2.2 The $TH(C_3^+)$ sequence

The permitted moves in $TH(C_3^+)$ are the elements of $\Delta_{C+} := \{a, b, c, \bar{a}\}$. In [7], it is described that the $TH(C_3^+)$ sequence **w** is obtained as the image, under the coding κ^T, of the iterative fixed point of the morphism η^T. Here the superscript T stands for temporary, since we will modify them below. Let Σ_{C+}^T be the set $\{x, r, z, t, u, s, a, b, c, \bar{a}, \bar{b}\}$. The map $\eta^T : \Sigma_{C+}^{T,*} \to \Sigma_{C+}^{T,*}$ is defined by

$$\begin{array}{lllllllll}
\eta^T(x) & = & rbasa, & \eta^T(r) & = & rbaubr, & \eta^T(z) & = & t\bar{a}u, \\
\eta^T(t) & = & zbrb, & \eta^T(u) & = & saczc, & \eta^T(s) & = & sactas, \\
\eta^T(a) & = & a, & \eta^T(b) & = & b, & \eta^T(c) & = & c, \\
\eta^T(\bar{a}) & = & \bar{a}, & \eta^T(\bar{b}) & = & \bar{b}, & & &
\end{array}$$

and the coding $\kappa^T : \Sigma_{C+}^{T,*} \to \Delta_{C+}^{T,*}$ is given by

$$\begin{array}{lllllllll}
\kappa^T(x) & = & a, & \kappa^T(r) & = & a, & \kappa^T(z) & = & \bar{a}, \\
\kappa^T(t) & = & b, & \kappa^T(u) & = & c, & \kappa^T(s) & = & c, \\
\kappa^T(a) & = & a, & \kappa^T(b) & = & b, & \kappa^T(c) & = & c, \\
\kappa^T(\bar{a}) & = & \bar{a}, & \kappa^T(\bar{b}) & = & \bar{b} & & &
\end{array}$$

with $\Delta_{C+}^T := \{a, b, c, \bar{a}, \bar{b}\}$. We see that the letter $\bar{b}$ is not an element of Δ_{C+} so that we delete it in Σ_{C+}^T and Δ_{C+}^T. Referring to the proof of the next theorem, we will make another emendation. The new morphism is called η and the new coding κ with the sets Δ_{C+} and Σ_{C+}.

Theorem 2.23. *Let the map $\eta : \Sigma_{C+}^* \to \Sigma_{C+}^*$ be defined by*

$$\begin{array}{lllllllll}
\eta(x) & = & rbasa, & \eta(r) & = & rbaubr, & \eta(z) & = & t\bar{a}u, \\
\eta(t) & = & zbrb, & \eta(u) & = & sacz, & \eta(s) & = & sactas, \\
\eta(a) & = & a, & \eta(b) & = & b, & \eta(c) & = & c, \\
\eta(\bar{a}) & = & \bar{a} & & & & & &
\end{array}$$

and the coding $\kappa : \Sigma_{C+}^* \to \Delta_{C+}^*$ be defined by

$$\begin{array}{lllllllll}
\kappa(x) & = & a, & \kappa(r) & = & a, & \kappa(z) & = & \bar{a}, \\
\kappa(t) & = & b, & \kappa(u) & = & c, & \kappa(s) & = & c, \\
\kappa(a) & = & a, & \kappa(b) & = & b, & \kappa(c) & = & c, \\
\kappa(\bar{a}) & = & \bar{a} & & & & & &
\end{array}$$

with $\Sigma_{C+} := \{x, r, z, t, u, s, a, b, c, \bar{a}\}$ and $\Delta_{C+} := \{a, b, c, \bar{a}\}$. Then the $TH(C_3^+)$ sequence **w** *is the image, under the coding κ, of the iterative fixed point of the morphism η. In fact,* $\mathbf{w} = \kappa(\eta^\infty(r))$.

Proof. We define the following words on Δ_{C+} given by The Sapir Algorithm 7:

- X_n is the word to transfer n discs from peg 0 to peg 1,
- Y_n is the word to transfer n discs from peg 0 to peg 2,

- Z_n is the word to transfer n discs from peg 1 to peg 0,
- T_n is the word to transfer n discs from peg 1 to peg 2,
- U_n is the word to transfer n discs from peg 2 to peg 0,
- V_n is the word to transfer n discs from peg 2 to peg 1

with $X_1 = a$, $Y_1 = ab$, $Z_1 = \bar{a}$, $T_1 = b$, $U_1 = c$, and $V_1 = ca$ as initial conditions. Using again Algorithm 7, we obtain:

$$\begin{array}{rcl} X_{n+1} & = & Y_n a V_n \\ Y_{n+1} & = & Y_n a U_n b Y_n \\ Z_{n+1} & = & T_n \bar{a} U_n \\ T_{n+1} & = & Z_n b Y_n \\ U_{n+1} & = & V_n c Z_n \\ V_{n+1} & = & V_n c T_n a V_n. \end{array}$$

We easily see that Y_n ends in b for any $n \in \mathbb{N}$ and V_n ends in a for any $n \in \mathbb{N}$. Thus we define R_n by $Y_n = R_n b$ with $R_1 = a$ and S_n by $V_n = S_n a$ with $S_1 = c$. The relations above are then:

$$\begin{array}{rcl} X_{n+1} & = & R_n b a S_n a \\ R_{n+1} & = & R_n b a U_n b R_n \\ Z_{n+1} & = & T_n \bar{a} U_n \\ T_{n+1} & = & Z_n b R_n b \\ U_{n+1} & = & S_n a c Z_n \\ S_{n+1} & = & S_n a c T_n a S_n. \end{array}$$

Hence there exist infinite sequences X_∞, Z_∞, and S_∞ consisting of elements of Δ_{C+} such that

$$\begin{array}{rcccl} \lim_{n\to\infty} X_n & = & X_\infty & = & \lim_{n\to\infty} R_n \\ \lim_{n\to\infty} Z_n & = & Z_\infty & = & \lim_{n\to\infty} T_n \\ \lim_{n\to\infty} U_n & = & S_\infty & = & \lim_{n\to\infty} S_n. \end{array}$$

Using the morphism η and the coding κ, we get by induction that $\kappa(\eta^{n-1}(o)) = O_n$ for all elements o of $\{x, r, z, t, u, s\}$ and the corresponding capital letters O from $\{X, R, Z, T, U, S\}$. Then the sequence is $\mathbf{w} = \kappa(\eta^\infty(r)) = X_\infty = R_\infty$. □

We can verify the correctness of these maps by following the moves of $\kappa(\eta^2(o))$, where o is an element of $\{x, y, z, t, u, v\}$, in the directed state graph of $TH(C_3^+)$ in Figure 2.2.

By analysing the occurrences of each letter $a_i \in \Sigma_{C+}$ in each $\eta(a_j)$ with $i, j \in [10]$, we set the incidence matrix $M(\eta)$

$$M(\eta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & & & & \\ 1 & 2 & 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 0 & 1 & 1 & 0 & & & & \\ 0 & 0 & 1 & 0 & 0 & 1 & & & & \\ 0 & 1 & 1 & 0 & 0 & 0 & & & \mathbf{0} & \\ 1 & 0 & 0 & 0 & 1 & 2 & & & & \\ 2 & 1 & 0 & 0 & 1 & 2 & 1 & & & \\ 1 & 2 & 0 & 2 & 0 & 0 & & 1 & & \\ 0 & 0 & 0 & 0 & 1 & 1 & & & 1 & \\ 0 & 0 & 1 & 0 & 0 & 0 & & \mathbf{0} & & 1 \end{pmatrix}.$$

Its characteristic polynomial is calculated to be $y(y-1)^7(y^2-y-4)$. Therefore, the eigenvalues are $\lambda_1 = 0$ with multiplicity 1, $\lambda_{2,3,4,5,6,7,8} = 1$ with multiplicity 7 and the two roots λ_9 and λ_{10} of (y^2-y-4), namely $\frac{1\pm\sqrt{17}}{2}$, as simple ones. The maximum of the absolute values of all eigenvalues of $M(\eta)$ is $\lambda_{10} = \frac{1+\sqrt{17}}{2}$, which is also the Perron-Frobenius eigenvalue of $M(\eta)$ with the non-negative eigenvector

$$v_{10} = \begin{pmatrix} 0 & 1 & \frac{5-\sqrt{17}}{2} & \frac{-3+\sqrt{17}}{2} & \frac{-3+\sqrt{17}}{2} & 1 & \frac{5+\sqrt{17}}{4} & 2 & 1 & \frac{-3+\sqrt{17}}{4} \end{pmatrix}$$

In addition, it should be mentioned that the matrix is reducible and hence not primitive (see Section 2.3.1 and Lemma 2.5).

With the aid of the subsequent lemma of [48, Corollary 8.5.8] or [80, pp. 647-648] the reader can also check for primitivity.

Lemma 2.24. *The non-negative $(n \times n)$-matrix M is primitive if and only if $M^{n^2-2n+2} > 0$.*

In the next theorem we specify the simple generator. We denote by $\overline{\Sigma}_{C+} \subset \Sigma_{C+}$ the set of the letters occurring in $\eta^\infty(r)$. Then $\overline{\Sigma}_{C+} = \{r, z, t, u, s, a, b, c, \bar{a}\}$ as x does not occur in $\eta^\infty(r)$.

Theorem 2.25. *The morphism $f : \overline{\Sigma}^*_{C+} \to \overline{\Sigma}^*_{C+}$ with $f(o) = \eta^3(o)$ for all $o \in \overline{\Sigma}_{C+}$ is the simple generator of the $TH(C_3^+)$ sequence* $\mathbf{w}$.

Proof. The incidence matrix $M(\eta)$ for $\eta : \overline{\Sigma}^*_{C+} \to \overline{\Sigma}^*_{C+}$ is similar to

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = PM(\eta)P^{-1}$$

by

$$P = \frac{1}{16}\begin{pmatrix} 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & 18 & -11 & -11 & -9 & 16 & 0 & -16 & 32 \\ -17 & -14 & -3 & -3 & -17 & 0 & 16 & 16 & 16 \\ -4 & -8 & 4 & 4 & -4 & 0 & 0 & 16 & 0 \\ 1 & -2 & 3 & 3 & 1 & 0 & 0 & 16 & -16 \\ 16 & 16 & 0 & 0 & 16 & 16 & -16 & -48 & 16 \\ -6 & -4 & 6 & -2 & -6 & -8 & 8 & 16 & -8 \\ -12 & -24 & -4 & 12 & 4 & -32 & 16 & 64 & -32 \\ 8 & 32 & 8 & 8 & 8 & 16 & -16 & -48 & 16 \end{pmatrix}.$$

The submatrices $A_{2,2} = A_{3,3} = A_{4,4} = A_{5,5} = \begin{pmatrix}1\end{pmatrix}$ of $PM(\eta)P^{-1}$, named as in the proof of Theorem 2.18, are irreducible, as is

$$A_{1,1} = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

The square matrix $A_{1,1}$ is even primitive as we calculate $A_{1,1}^4 > 0$.

The integer h from the proof of Theorem 2.18, which is the least common multiple of all indices of the $A_{i,i}$s, is 1, since the characteristic polynomial of $A_{1,1}$ is $(y-1)^3(y^2-y-4)$ and thus $h_1 = 1$. We define the morphism $f : \overline{\Sigma}_{C+}^* \to \overline{\Sigma}_{C+}^*$ from the condition $f(r) = \eta^{hl}(r) = \eta^l(r)$. To find l, we apply η on r until every letter of $\overline{\Sigma}_{C+}$ occurs in the result. We get $l = 3$ and, consequently, $f = \eta^3(r)$. Thus (G1) is satisfied. We determine the mapping rules of the letters under f.

We find that

$$\begin{aligned}
f(r) &= \eta^3(r) = rbaubrbasaczbrbaubrbasactasact\bar{a}u \\
&\qquad brbaubrbasaczbrbaubr, \\
f(z) &= \eta^3(z) = t\bar{a}ubrbaubrb\bar{a}sactasact\bar{a}u, \\
f(t) &= \eta^3(t) = zbrb\bar{a}saczbrbaubrbasaczbrbaubrb, \\
f(u) &= \eta^3(u) = sactasaczbrbasactasaczbrb\bar{a}sacz, \\
f(s) &= \eta^3(s) = sactasaczbrbasactasact\bar{a}ubrbaubr \\
&\qquad basactasaczbrbasactas, \\
f(a) &= \eta^3(a) = a, \qquad f(c) = \eta^3(c) = c, \\
f(b) &= \eta^3(b) = b, \qquad f(\bar{a}) = \eta^3(\bar{a}) = \bar{a}.
\end{aligned}$$

Since η is a morphism, f is also a morphism.

As incidence matrix of f we get

$$M(f) = \begin{pmatrix} 8 & 2 & 5 & 2 & 4 & & & & \\ 2 & 0 & 3 & 3 & 2 & & & & \\ 2 & 3 & 0 & 2 & 5 & & & & \\ 5 & 3 & 2 & 0 & 2 & & & \text{\huge 0} & \\ 4 & 2 & 2 & 5 & 8 & & & & \\ 12 & 4 & 5 & 8 & 15 & 1 & & & \\ 14 & 4 & 10 & 4 & 8 & & 1 & & \\ 4 & 2 & 2 & 5 & 7 & & & 1 & \\ 1 & 3 & 1 & 1 & 1 & & \text{\huge 0} & & 1 \end{pmatrix}.$$

The characteristic polynomial equals $(t-1)^7(t^2 - 13t - 64)$. This gives nine eigenvalues, namely $\lambda_{1,2,3,4,5,6,7} = 1$ with multiplicity 7 and $\lambda_8 = \frac{13-5\sqrt{17}}{2}$ and $\lambda_9 = \frac{13+5\sqrt{17}}{2}$ both with multiplicity one. The eigenvalue with the maximal absolute value is the Perron-Frobenius eigenvalue, here λ_9, with the non-negative eigenvector

$$v_9 = \begin{pmatrix} 1 & \frac{5-\sqrt{17}}{2} & \frac{-3+\sqrt{17}}{2} & \frac{-3+\sqrt{17}}{2} & 1 & \frac{5+\sqrt{17}}{4} & 2 & 1 & \frac{-3+\sqrt{17}}{4} \end{pmatrix}.$$

All other eigenvalues of $M(f)$ have absolute values which are not equal to λ_9. Hence (G2) is fulfilled. □

The Jordan block corresponding to λ_9 occurs only once in the Jordan form of $M(f)$, since

$$Q^{-1}M(f)Q = \begin{pmatrix} \lambda_8 & & & & & & & & \\ & 1 & 1 & & & & & & \\ & 0 & 1 & & & & & & \\ & & & 1 & 1 & & & & \\ & & & 0 & 1 & & & \text{\huge 0} & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & \text{\huge 0} & & & & & 1 & \\ & & & & & & & & \lambda_9 \end{pmatrix},$$

where Q is an invertible matrix. The Jordan blocks are $J_1 = (\lambda_8)$, $J_4 = J_5 = J_6 = (1)$, $J_7 = (\lambda_9)$, and

$$J_2 = J_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This clearly implies $j = 0$, since $j + 1$ is the size of the dominating Jordan block J_6. We calculate the matrix Q, which is the one which has the eigenvectors of λ_8 and λ_9 as first and last column, respectively, such that

$$Q = \begin{pmatrix} 1 & 3 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \mathrm{a} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \mathrm{e} \\ \mathrm{b} & -3 & 0 & 0 & 0 & -1 & 0 & 0 & \mathrm{f} \\ \mathrm{b} & 3 & 0 & 0 & 1 & 1 & 0 & 0 & \mathrm{f} \\ 1 & -3 & -1 & 0 & -1 & -1 & 0 & 0 & 1 \\ \mathrm{c} & -3 & 0 & -3 & 0 & 0 & 1 & 0 & \mathrm{g} \\ 2 & 6 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \mathrm{d} & 0 & 0 & 3 & 0 & 0 & 0 & 0 & \mathrm{h} \end{pmatrix},$$

where the roman letters in Q stand for

$$\begin{array}{rclrcl} \mathrm{a} & = & \frac{\sqrt{17}+5}{2}, & \mathrm{e} & = & -\frac{\sqrt{17}-5}{2}, \\ \mathrm{b} & = & -\frac{\sqrt{17}+3}{2}, & \mathrm{f} & = & \frac{\sqrt{17}-3}{2}, \\ \mathrm{c} & = & -\frac{\sqrt{17}-5}{4}, & \mathrm{g} & = & \frac{\sqrt{17}+5}{4}, \\ \mathrm{d} & = & -\frac{\sqrt{17}+3}{4}, & \mathrm{h} & = & \frac{\sqrt{17}-3}{4}. \end{array}$$

With the last results we compute the matrix $E(f)$ from Lemma 2.19. The limit

$$E(f) = \lim_{n\to\infty} \frac{(M(f))^n}{\lambda_9^n} = \lim_{n\to\infty} Q \underbrace{\left(\frac{J(f)}{\lambda_9}\right)^n}_{=:B} Q^{-1}$$

exists with

$$B^n = \frac{1}{\lambda_9^n} \begin{pmatrix} \lambda_8^n & & & & & & & & \\ & 1^n & \binom{n}{1}1^{n-1} & & & & & & \\ & 0 & 1^n & & & & & & \\ & & & 1^n & \binom{n}{1}1^{n-1} & & & \mathbf{0} & \\ & & & 0 & 1^n & & & & \\ & & & & & 1^n & & & \\ & & & & & & 1^n & & \\ & \mathbf{0} & & & & & & 1^n & \\ & & & & & & & & \lambda_9^n \end{pmatrix}.$$

This yields $E(f) = Q\begin{pmatrix} \mathbf{0} & \\ & 1 \end{pmatrix}Q^{-1}$, where the (1)-matrix has dimension 1×1 and lies on the diagonal. The incidence matrix of the coding κ is given by

$$M(\kappa) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

From what has already been proved, we calculate the (4×9)-matrix

$$W(\kappa, f) = M(\kappa)E(f). \tag{2.6}$$

With (2.6) we obtain

Theorem 2.26. *The frequency of the letter a in* **w** *exists and has the value*

$$\mathrm{Freq}_a(\mathbf{w}) = \frac{17 + 9\sqrt{17}}{68}.$$

Proof. Recall that f is prolongable on the letter r of $\overline{\Sigma}_{C+}$. We calculate

$$\begin{aligned} F := \frac{w_{a,r}}{\sum_{p \in \Delta_{C+}} w_{p,r}} &= \frac{\mathrm{g}+1}{\mathrm{e}+2\mathrm{f}+\mathrm{g}+\mathrm{h}+5} \\ &= \frac{\frac{1}{4}\left(\sqrt{17}+9\right)}{\sqrt{17}} \\ &= \frac{17+9\sqrt{17}}{68} \end{aligned} \tag{2.7}$$

as the candidate for the frequency of $a \in \Delta_{C+}$, according to Theorem 2.21. By the same theorem, it remains to prove that

$$\frac{w_{a,k}}{\sum_{p \in \Delta_{C+}} w_{p,k}} = \frac{w_{a,r}}{\sum_{p \in \Delta_{C+}} w_{p,r}}$$

for all non-zero columns W_k of $W(\kappa, f)$. We have

$$\frac{w_{a,k}}{\sum_{p \in \Delta_{C+}} w_{p,k}} = \frac{\mathrm{g}+1}{\sigma}$$

for all $k \in \{z, t, u, s\}$ and $\sigma := \mathrm{e} + 2\mathrm{f} + \mathrm{g} + \mathrm{h} + 5$. All these numbers are equal to F of (2.7) so that $\mathrm{Freq}_a(\mathbf{w}) = F$. □

By virtue of Corollary 2.22, the frequencies of all letters in Δ_{C+} actually exist, since the rank of $W(\kappa, f)$ equals one. A trivial calculation with *Sage* shows that

$$\begin{aligned} \mathrm{Freq}_b(\mathbf{w}) &= \frac{\mathrm{f}+2}{\sigma} = \frac{\sqrt{17}+1}{2\sqrt{17}} = \frac{\sqrt{17}+17}{34}, \\ \mathrm{Freq}_c(\mathbf{w}) &= \frac{\mathrm{f}+2}{\sigma} = \frac{\sqrt{17}+17}{34}, \\ \mathrm{Freq}_{\bar{a}}(\mathbf{w}) &= \frac{\mathrm{e}+\mathrm{h}}{\sigma} = \frac{-\sqrt{17}+7}{4\sqrt{17}} = \frac{7\sqrt{17}-17}{68}. \end{aligned}$$

All these frequencies of letters in Δ_{C+} are not rational. From Theorem 2.13 we know that if a sequence is d-automatic for any d, then, assuming its existence, the frequency of a letter in this sequence has to be rational . Hence we proved now that

Theorem 2.27. *For any $d \in \mathbb{N}_2$ the $TH(C_3^+)$ sequence* **w** *is not d-automatic.*

2.2.3 The $TH(K_3^-)$ sequence

With a view to Figure 2.1, we get the allowed moves in $TH(K_3^-)$. These moves can be summarised in the set $\Delta_{K^-} := \{a, b, c, \bar{a}, \bar{b}\}$. By definition, a sequence is morphic if it can be obtained as the image, under a coding, of an iterative fixed point of a morphism. For the $TH(K_3^-)$ sequence $\mathbf{u}$ a morphism $\phi : \{x, y, z, d, e, f, a, b, c, \bar{a}, \bar{b}\} \to \{x, y, z, d, e, f, a, b, c, \bar{a}, \bar{b}\}$ is defined by

$$\begin{array}{lllllllll}
\phi(x) & = & ybaf, & \phi(y) & = & ybaeby, & \phi(z) & = & d\bar{a}e, \\
\phi(d) & = & zbyb, & \phi(e) & = & fcz, & \phi(f) & = & e\bar{b}x, \\
\phi(a) & = & a, & \phi(b) & = & b, & \phi(c) & = & c, \\
\phi(\bar{a}) & = & \bar{a}, & \phi(\bar{b}) & = & \bar{b}, & & &
\end{array}$$

and the corresponding coding, under the map

$$\tau : \{x, y, z, d, e, f, a, b, c, \bar{a}, \bar{b}\} \to \Delta_{K^-},$$

is given by

$$\begin{array}{lllllllll}
\tau(x) & = & a, & \tau(y) & = & a, & \tau(z) & = & \bar{a}, \\
\tau(d) & = & b, & \tau(e) & = & c, & \tau(f) & = & \bar{b}, \\
\tau(a) & = & a, & \tau(b) & = & b, & \tau(c) & = & c, \\
\tau(\bar{a}) & = & \bar{a}, & \tau(\bar{b}) & = & \bar{b}. & & &
\end{array}$$

The finite alphabet $\{x, y, z, d, e, f, a, b, c, \bar{a}, \bar{b}\}$ is called Σ_{K^-}.

Theorem 2.28 ([7, Theorem 2]). *The $TH(K_3^-)$ sequence $\mathbf{u}$ is the image, under the coding τ, of the iterative fixed point of the morphism ϕ, in particular $\mathbf{u} = \tau(\phi^\infty(y))$.*

Proof. We define the following words on Δ_{K^-} given by The Sapir Algorithm 7:

- X_n is the word to transfer n discs from peg 0 to peg 1,
- S_n is the word to transfer n discs from peg 0 to peg 2,
- Z_n is the word to transfer n discs from peg 1 to peg 0,
- D_n is the word to transfer n discs from peg 1 to peg 2,
- E_n is the word to transfer n discs from peg 2 to peg 0,
- F_n is the word to transfer n discs from peg 2 to peg 1

with $X_1 = a$, $S_1 = ab$, $Z_1 = \bar{a}$, $D_1 = b$, $E_1 = c$, and $F_1 = \bar{b}$ as initial conditions. Using again Algorithm 7, we obtain:

$$\begin{array}{lll}
X_{n+1} & = & S_n a F_n \\
S_{n+1} & = & S_n a E_n b S_n \\
Z_{n+1} & = & D_n \bar{a} E_n \\
D_{n+1} & = & Z_n b S_n \\
E_{n+1} & = & F_n c Z_n \\
F_{n+1} & = & E_n \bar{b} X_n.
\end{array}$$

We easily see that S_n ends in b for any $n \in \mathbb{N}$. Thus we define Y_n by $S_n = Y_n b$ with $Y_1 = a$. The relations above are then:

$$\begin{array}{rcl}
X_{n+1} & = & Y_n ba F_n \\
Y_{n+1} & = & Y_n ba E_n b Y_n \\
Z_{n+1} & = & D_n \bar{a} E_n \\
D_{n+1} & = & Z_n b Y_n b \\
E_{n+1} & = & F_n c Z_n \\
F_{n+1} & = & E_n \bar{b} X_n .
\end{array}$$

Hence there exist infinite sequences X_∞, Z_∞, and E_∞ consisting of elements of Δ_{K-} such that

$$\begin{array}{rcccccc}
\lim_{n\to\infty} X_n & = & X_\infty & = & \lim_{n\to\infty} S_n & = & \lim_{n\to\infty} Y_n \\
\lim_{n\to\infty} Z_n & = & Z_\infty & = & \lim_{n\to\infty} D_n & & \\
\lim_{n\to\infty} E_n & = & E_\infty & = & \lim_{n\to\infty} F_n . & &
\end{array}$$

Using the morphism ϕ and the coding τ, we get by induction that $\tau(\phi^{n-1}(o)) = O_n$ for all elements o of $\{x, y, z, d, e, f\}$ and corresponding capital letters O in $\{X, Y, Z, D, E, F\}$. Then the sequence is $\mathbf{u} = \tau(\phi^\infty(y)) = X_\infty$. $\square$

Again, we can verify the correctness of these maps by following the moves of $\tau(\phi^2(o))$, where the letter o is an element of $\{x, s, z, d, e, f\}$ in the directed state graph of $TH(K_3^-)$ in Figure 2.2.

For the incidence matrix of ϕ we get

$$M(\phi) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & & & & & \\
1 & 2 & 0 & 1 & 0 & 0 & & & & & \\
0 & 0 & 0 & 1 & 1 & 0 & & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & & & \mathbf{0} & & \\
0 & 1 & 1 & 0 & 0 & 1 & & & & & \\
1 & 0 & 0 & 0 & 1 & 0 & & & & & \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & & & & \\
1 & 2 & 0 & 2 & 0 & 0 & & 1 & & & \\
0 & 0 & 0 & 0 & 1 & 0 & & & 1 & & \\
0 & 0 & 1 & 0 & 0 & 0 & & \mathbf{0} & & 1 & \\
0 & 0 & 0 & 0 & 0 & 1 & & & & & 1
\end{pmatrix}. \tag{2.8}$$

Its characteristic polynomial is given by $(t+1)(t-1)^7(t^3 - t^2 - 4t + 2)$. Hence the eigenvalues are $\lambda_1 = -1$ with multiplicity 1, $\lambda_{2,3,4,5,6,7,8} = 1$ with multiplicity 7 and the three roots $\lambda_9, \lambda_{10}, \lambda_{11}$ (ordered by ascending absolute value) of $(t^3 - t^2 - 4t + 2)$ as simple ones. The maximum of the absolute values of all eigenvalues of $M(\phi)$ is the maximal root λ_{11} of $(t^3 - t^2 - 4t + 2)$. This root is therefore also the Perron-Frobenius eigenvalue of $M(\phi)$.

Note that this matrix is again reducible and not primitive (see Section 2.3.1 and Lemma 2.24).

The following theorem specifies the simple generator for **u**. The set $\overline{\Sigma}_{K-}$ is the set of letters which occur in $\eta^\infty(y)$. Here, $\overline{\Sigma}_{K-}$ is equal to Σ_{K-}.

Theorem 2.29. *The morphism $g : \Sigma_{K-}^* \to \Sigma_{K-}^*$ with $g(o) = \phi^3(o)$ for all $o \in \Sigma_{K-}$ is a simple generator for the $TH(K_3^-)$ sequence* **u**.

Proof. The matrix $M(\phi)$ can be transformed in the following way

$$M(\phi) \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & & & & & 1 & & & & \\ & & & & & & & 1 & & & \\ & & \mathbf{0} & & & & & & 1 & & \\ & & & & & & & & & 1 & \\ & & & & & & & & & & 1 \end{pmatrix},$$

since $M(\phi)$ is similar to its transpose.

We show briefly that every square matrix A is similar to its transpose. Let $J(A)$ be the matrix in Jordan form of A. Thus $Q^{-1}AQ = J(A)$ for an invertible matrix Q and

$$J(A)^T = Q^T A^T (Q^{-1})^T = Q^T A^T (Q^T)^{-1}.$$

It follows that $J(A)^T$ is similar to A^T. Using the transitive property of similarity, it is sufficient to prove that $J(A)^T$ is similar to $J(A)$. Because of the block decomposition of $J(A)$, we have only to show that any Jordan block $J(A)_i$ is similar to its transpose. This can be seen easily. In fact, if P is the permutation matrix with a line of 1s from top right to bottom left, then we get $PJ(A)_iP^{-1} = (J(A)_i)^T$.

The submatrices $A_{2,2} = A_{3,3} = A_{4,4} = A_{5,5} = A_{6,6} = \begin{pmatrix} 1 \end{pmatrix}$ are irreducible and so is

$$A_{1,1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

since $A_{1,1}^6 > 0$. The characteristic polynomial of $A_{1,1}$ is $(t+1)(t-1)^2(t^3 - t^2 - 4t + 2)$. It follows that the index of $A_{1,1}$ equals 1. Therefore, the integer h from the proof of Theorem 2.18 is

$h = \text{lcm}(1, 1, \ldots, 1) = 1$. Now we are able to find a morphism $g : \Sigma_{K-}^* \to \Sigma_{K-}^*$ with the condition $g(y) = \phi^{hl}(y) = \phi^l(y)$. But what is $l \in \mathbb{N}$? Every letter of Σ_{K-} has to occur in $\phi^l(y)$. By an easy calculation, we see that $l = 3$ and, consequently, $g = \phi^3(y)$. We determine the new mapping rules under g for every letter of Σ_{K-}

$$\begin{array}{lllll}
g(y) = \phi^3(y) = & ybaebybafczbybaebybae\bar{b}xcd\bar{a}e & & & \\
 & bybaebybafczbybaeby, & & & \\
g(x) = \phi^3(x) = & ybaebybafczbybaebybafcz\bar{b}ybaf, & & & \\
g(z) = \phi^3(z) = & d\bar{a}ebybaebyb\bar{a}e\bar{b}xcd\bar{a}e, & & & \\
g(d) = \phi^3(d) = & zbyb\bar{a}fczbybaebybafczbybaebyb, & & & \\
g(e) = \phi^3(e) = & fcz\bar{b}ybafczbyb\bar{a}fcz, & & & \\
g(f) = \phi^3(f) = & e\bar{b}xcd\bar{a}e\bar{b}ybaebybae\bar{b}x, & & & \\
g(a) = \phi^3(a) = & a, & & g(\bar{a}) = \phi^3(\bar{a}) = & \bar{a}, \\
g(b) = \phi^3(b) = & b, & & g(\bar{b}) = \phi^3(\bar{b}) = & \bar{b}, \\
g(c) = \phi^3(c) = & c. & & &
\end{array}$$

It is obvious that g is a morphism and that (G1) is satisfied.

We must again specify an incidence matrix, but this time for the simple generator g. It is

$$M(g) = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 2 & & & & & \\
5 & 8 & 2 & 5 & 2 & 2 & & & & & \\
2 & 2 & 0 & 3 & 3 & 0 & & & & & \\
0 & 1 & 2 & 0 & 0 & 1 & & \mathbf{0} & & & \\
2 & 6 & 4 & 2 & 0 & 4 & & & & & \\
3 & 2 & 0 & 2 & 3 & 0 & & & & & \\
5 & 7 & 1 & 3 & 1 & 2 & 1 & & & & \\
8 & 14 & 4 & 10 & 3 & 3 & & 1 & & & \\
2 & 3 & 1 & 2 & 3 & 1 & & & 1 & & \\
0 & 1 & 3 & 1 & 1 & 1 & & \mathbf{0} & & 1 & \\
1 & 1 & 1 & 0 & 1 & 3 & & & & & 1
\end{pmatrix}.$$

As the characteristic polynomial of $M(g)$ we get

$$\begin{aligned}
&t^{11} - 13t^{10} - 20t^9 + 352t^8 - 1014t^7 + 1190t^6 \\
&\quad -224t^5 - 960t^4 + 1133t^3 - 561t^2 + 124t - 8 \\
&= (t-1)^7(t+1)(t^3 - 7t^2 - 76t + 8).
\end{aligned}$$

We immediately see that 1 and -1 are the eigenvalues with multiplicity 7 and 1, respectively, but these are not all. We further detect the three roots of $t^3 - 7t^2 - 76t + 8 = at^3 + bx^2 + ct + d$ by Cardano's method ([19]). From

$$\begin{aligned}
p &= -\frac{b^2}{3a^2} + \frac{c}{a} = -\frac{277}{3} \\
q &= \frac{d}{a} - \frac{bc}{3a^2} + \frac{2b^3}{27a^3} = -\frac{5258}{27}
\end{aligned}$$

we calculate the discriminant $D = \frac{p^3}{27} + \frac{q^2}{4} = -\frac{531196}{27} < 0$. Hence there are three real roots and we are in the "casus irreducibilis". With [19, p.78 ff.] the roots are

$$\begin{aligned}
t_1 &= 2\sqrt{\tfrac{277}{9}}\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{47322}{14958}\sqrt{\tfrac{9}{277}} - \tfrac{2\pi}{3}\right)\right) + \tfrac{7}{3},\\
t_2 &= 2\sqrt{\tfrac{277}{9}}\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{47322}{14958}\sqrt{\tfrac{9}{277}} - 2\tfrac{2\pi}{3}\right)\right) + \tfrac{7}{3},\\
t_3 &= 2\sqrt{\tfrac{277}{9}}\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{47322}{14958}\sqrt{\tfrac{9}{277}}\right)\right) + \tfrac{7}{3},
\end{aligned}$$

It follows that t_3 is the Perron-Frobenius eigenvalue of $M(g)$. The condition (G2) is fulfilled as t_3 itself is the only eigenvalue of $M(g)$ whose absolute value equals t_3. □

We are interested in finding the Jordan matrix decomposition of $M(g)$. There exists an invertible square matrix Q such that $M(g) = QJ(g)Q^{-1}$, where

$$J(g) = \begin{pmatrix}
t_1 & & & & & & & & & & & \\
 & -1 & & & & & & & & & & \\
 & & t_2 & & & & & & \mathbf{0} & & & \\
 & & & 1 & 1 & & & & & & & \\
 & & & 0 & 1 & & & & & & & \\
 & & & & & 1 & 1 & & & & & \\
 & & & & & 0 & 1 & & & & & \\
 & & & & & & & 1 & & & & \\
 & & \mathbf{0} & & & & & & 1 & & & \\
 & & & & & & & & & 1 & & \\
 & & & & & & & & & & t_3
\end{pmatrix}$$

is the matrix in Jordan form. We have nine Jordan blocks, namely

$$J_1 = (t_1), \quad J_2 = (-1), \quad J_3 = (t_2), \quad J_4 = J_5 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad J_6 = J_7 = J_8 = (1), \quad \text{and } J_9 = (t_3).$$

It is easily seen that the dominating Jordan block of $M(g)$ is J_9, which implies $j+1 = 1$ as its size. By the definition of $E(g)$ in Lemma 2.19, this yields

$$E(g) = \lim_{n\to\infty} \frac{(M(g))^n}{t_3^n} = \lim_{n\to\infty} Q\underbrace{\left(\frac{J(g)}{t_3}\right)}_{=:B}^{n} Q^{-1} = \lim_{n\to\infty} QB^nQ^{-1}.$$

The task is to find B^n in order to evaluate the limit. B^n is given by

$$B^n = \frac{1}{t_3^n}\begin{pmatrix} t_1^n & & & & & & & & & & & \\ & (-1)^n & & & & & & & & & & \\ & & t_2^n & & & & & \mathbf{0} & & & & \\ & & & 1^n & \binom{n}{1}1^{n-1} & & & & & & & \\ & & & 0 & 1^n & & & & & & & \\ & & & & & 1^n & \binom{n}{1}1^{n-1} & & & & & \\ & & & & & 0 & 1^n & & & & & \\ & & & & & & & 1^n & & & & \\ & & & & & & & & 1^n & & & \\ & & & \mathbf{0} & & & & & & 1^n & & \\ & & & & & & & & & & 1^n & \\ & & & & & & & & & & & t_3^n \end{pmatrix}.$$

We thus get

$$E(g) = Q\left(\lim_{n\to\infty} B^n\right)Q^{-1} = Q\begin{pmatrix} \mathbf{0} & \\ & 1 \end{pmatrix}Q^{-1},$$

where the (1)-matrix has dimension 1×1 and lies on the diagonal.

The invertible matrix Q has the form $Q = [v_1 \dots v_2 \dots v_3]$ where v_i are the eigenvectors of t_i $(i = 1, 2, 3)$. We calculate Q with the algebra system *Sage* and see that

$$Q = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \mathrm{a} & 0 & \mathrm{h} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \mathrm{o} \\ \mathrm{b} & 1 & \mathrm{i} & 0 & -1 & 0 & -1 & 0 & 0 & 0 & \mathrm{p} \\ 1 & -1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ \mathrm{c} & 0 & \mathrm{j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{q} \\ \mathrm{b} & -1 & \mathrm{i} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \mathrm{p} \\ \mathrm{d} & -\frac{1}{2} & \mathrm{k} & 3 & 0 & 3 & 0 & 1 & 0 & 0 & \mathrm{r} \\ \mathrm{e} & \frac{1}{2} & \mathrm{l} & -3 & 0 & 0 & 0 & 0 & 1 & 0 & \mathrm{s} \\ \mathrm{f} & 0 & \mathrm{m} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathrm{t} \\ \mathrm{g} & -\frac{1}{2} & \mathrm{n} & -3 & 0 & -3 & 0 & 0 & 0 & 0 & \mathrm{u} \\ \mathrm{g} & \frac{1}{2} & \mathrm{n} & 3 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{u} \end{pmatrix}. \tag{2.9}$$

The first column is v_1, the third one v_2 and the last one v_3. Note that the roman letters in this matrix are not related to the elements of Σ_{K-}. By now, there is only one part missing to construct the matrix $W(\tau, g)$ of Corollary 2.20 and consequently the frequency of a letter of Σ_{K-}, namely to find the incidence matrix of the coding τ. We see that

$$M(\tau) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We are now in a position to determine $W(\tau, g)$. By definition, the matrix $W(\tau, g)$ is equal to the (5×11)-matrix $M(\tau)E(g)$. From this we get

Theorem 2.30. *The frequency of the letter $a \in \Delta_{K-}$ in* **u** *exists and is equal to*

$$\mathrm{Freq}_a(\mathbf{u}) = \frac{(\mathrm{o} + \mathrm{r} + 1)}{(\mathrm{o} + 2\mathrm{p} + \mathrm{q} + \mathrm{r} + \mathrm{s} + \mathrm{t} + 2\mathrm{u} + 2)},$$

where the roman letters are the numbers as defined in the matrix Q in (2.9).

Proof. Recall that g is prolongable on the letter y of Σ_{K-}. As candidate for the frequency of the letter $a \in \Delta_{K-}$ we obtain

$$G := \frac{w_{a,y}}{\sum_{p\in\Delta_{K-}} w_{p,y}} = \frac{(\mathrm{o+r+1})}{(\mathrm{o+2p+q+r+s+t+2u+2})} \tag{2.10}$$

according to Theorem 2.21. We see that

$$\frac{w_{a,y}}{\sum_{p\in\Delta_{K-}} w_{p,y}} = \frac{w_{a,k}}{\sum_{p\in\Delta_{K-}} w_{p,k}}$$

for all non-zero columns W_k of $W(\tau, \phi)$. Finally, we conclude that the frequency of the letter $a \in \Delta_{K-}$ in $\tau(g^\infty(y))$ exists and is equal to G. □

It remains to see whether the number G is rational or not. In the case of rationality we could conclude that the sequence **u** is possibly d-automatic for some d, while in the opposite case it cannot be d-automatic for any d. Unfortunately, we cannot determine whether the expression (2.10) is irrational because of its complexity. Hence we will try another approach to disprove the automaticity of this sequence in the next section.

2.3 A new approach to disprove the automaticity of the $TH(K_3^-)$ sequence

As in the previous section, we define the set of allowed moves in $TH(K_3^-)$ as $\Delta_{K-} := \{a, b, c, \bar{a}, \bar{b}\}$. Then the $TH(K_3^-)$ sequence **u** is the image, under the coding

$$\tau : \{x, y, z, d, e, f, a, b, c, \bar{a}, \bar{b}\} \to \{a, b, c, \bar{a}, \bar{b}\}$$

given by

$$\begin{array}{lllllllll}
\tau(x) & = & a, & \tau(y) & = & a, & \tau(z) & = & \bar{a}, \\
\tau(d) & = & b, & \tau(e) & = & c, & \tau(f) & = & \bar{b}, \\
\tau(a) & = & a, & \tau(b) & = & b, & \tau(c) & = & c, \\
\tau(\bar{a}) & = & \bar{a}, & \tau(\bar{b}) & = & \bar{b}, & & &
\end{array}$$

of the fixed point of the morphism $\phi : \Sigma_{K-}^* \to \Sigma_{k-}^*$ defined by

$$\begin{array}{lllllllll}
\phi(x) & = & ybaf, & \phi(y) & = & ybaeby, & \phi(z) & = & d\bar{a}e, \\
\phi(d) & = & zbyb, & \phi(e) & = & fcz, & \phi(f) & = & e\bar{b}x, \\
\phi(a) & = & a, & \phi(b) & = & b, & \phi(c) & = & c, \\
\phi(\bar{a}) & = & \bar{a}, & \phi(\bar{b}) & = & \bar{b}. & & &
\end{array}$$

The set $\Sigma_{K-} := \{x, y, z, d, e, f, a, b, c, \bar{a}, \bar{b}\}$ is a finite alphabet. Then our sequence is $\mathbf{u} = \tau(\phi^\infty(y))$.

The new approach to decide whether the sequence **u** is automatic or not is the use of Cobham's Theorem [17], which makes a statement about the ultimate periodicity of sequences under certain conditions. The idea of the application of this theorem was already outlined in [7] and again discussed during a private communication between Allouche and the author of the present thesis [3]. For the formulation of that theorem we need some preparations. Let x and y be finite words. Then we call an infinite sequence **s** of the form xy^∞, where $y \neq \epsilon$ and $y^\infty = yyy\ldots$, *ultimately periodic*[10]. If x is the empty word, the sequence is called *periodic*.

One calls two integers $p, q \in \mathbb{N}_1$ *multiplicatively independent* if no positive integers α and β exist such that $p^\alpha = q^\beta$. Otherwise p and q are *multiplicatively dependent*. The next lemma characterises this property.

Lemma 2.31 ([8, Theorem 2.5.7])**.** *Let $p, q \in \mathbb{N}_2$. The following statements are equivalent:*

a) p and q are multiplicatively dependent.

b) $\log_p q$ is rational.

c) $\log_q p$ is rational.

d) There exist an integer $n \in \mathbb{N}_2$ and two integers $x, y \in \mathbb{N}$ such that $p = n^x$ and $q = n^y$.

Theorem 2.32 (Cobham's Theorem, [17])**.** *Let p and q be multiplicatively independent integers and let* **s** *be a sequence which is both p- and q-automatic. Then* **s** *is ultimately periodic.*

Next, we show that if the sequence **u** is not ultimately periodic, then we can conclude that **u** is not d-automatic for any $d \in \mathbb{N}_2$.

Why can we conclude this? [7, Proposition 1] tells us that the Perron-Frobenius eigenvalue of the incidence matrix $M(\phi)$ of the $TH(K_3^-)$ sequence and any integer $d \in \mathbb{N}$ are multiplicatively independent. We prove this statement more extensively in the next lemma with the aid of Lemma 2.31 and an algebraic argument (for the theory about field extensions; see for instance [29]).

Lemma 2.33. *Let λ_r ($\in \mathbb{N}_1$) be the maximal root of the polynomial ($t^3 - t^2 - 4t + 2$). Then for any $d \in \mathbb{N}$ the real number $\log_d \lambda_r = \dfrac{\ln \lambda_r}{\ln d}$ is irrational.*

[10] Recall the remark in Section 2.1. A sequence is 1-automatic if and only if it is ultimately periodic (see [8, Theorem 5.7.1]).

Proof. We assume that $\log_d \lambda_r = \frac{n}{m}$ with positive integers m, n. Then $d^{\frac{n}{m}} = \lambda_r$, which implies $d^n = \lambda_r^m$. Since d^n is an integer, it is sufficient to show that for any $m \in \mathbb{N}$ the numbers λ_r^m are not integers. For this purpose we use the following argument. Suppose $\alpha := \lambda_r^m$ is an integer. We can easily see that λ_r is algebraic over $\mathbb{Q}$. Now let I be the set of all polynomials P over $\mathbb{Q}$ such that $P(\lambda_r) = 0$. I is an ideal of $\mathbb{Q}[T]$, where $\mathbb{Q}[T]$ is the polynomial ring over $\mathbb{Q}$. As $\mathbb{Q}$ is a field, $\mathbb{Q}[T]$ is a principal ideal domain and, therefore, I consists of all polynomial multiples over $\mathbb{Q}$ of some $m(t) \in \mathbb{Q}[T]$. Since $(t^3 - t^2 - 4t + 2)$ has no rational root, it is irreducible over $\mathbb{Q}$, and it follows that $m(t) = t^3 - t^2 - 4t + 2$. Then, since λ_r is also a root of $(t^m - \alpha)$ and $(t^m - \alpha) \in \mathbb{Q}[T]$, the polynomial $(t^3 - t^2 - 4t + 2)$ divides $(t^m - \alpha)$, i.e., there exists a polynomial $Q(t) = a_{m-3}t^{m-3} + \ldots + a_1 t + a_0 \in \mathbb{Q}[T]$ such that

$$(t^3 - t^2 - 4t + 2)Q(t) = t^m - \alpha. \tag{2.11}$$

If $a_i = r_i/s_i$ for $r_i, s_i \in \mathbb{Z}$ with $i \in [m-2]_0$ and if we let $s = \mathrm{lcm}(s_0, \ldots, s_{m-3})$, then we can write $Q(t) = s^{-1}Q'(t)$ where $Q'(t) \in \mathbb{Z}[T]$. Hence we can assume that $Q(t)$ is a polynomial with integer coefficients. We reduce Equation (2.11) modulo 2

$$t^2(t-1)Q(t) \equiv t^m - \alpha \mod 2.$$

Then, if we set $t \mod 2 = 0$, the integer $\alpha \mod 2 = 0$. However, if we set $t \mod 2 = 1$, the integer $\alpha \mod 2 = 1$, which is a contradiction. □

Assuming that **u** is not ultimately periodic, it would follow with the previous lemma that for any $d \in \mathbb{N}$ the sequence **u** is not d-automatic. Hence it remains to prove whether the sequence is not ultimately periodic. The decidability of the question whether an infinite sequence is ultimately periodic, called the *HD0L*[11] *ultimate periodicity problem*, has been an open problem for a long time.

As described in the introduction, first steps were done for special types of sequences. F. Durand [25] solved the problem in the case that the morphism is primitive in 2012. Shortly thereafter, he proved the decidability of the whole problem [26].

At around the same time, I. V. Mitrofanov presented two other proofs of the decidability of the HD0L ultimate periodicity problem. The first proof was published as a preprint [62] and according to Mitrofanov it is based on the paper [51]. In the sequel we will make use of the second proof in [63].

In the first subsection we will employ the proof in F. Durand's paper [26] and continue after the appearance of difficulties in the exact calculation with the one of I. V. Mitrofanov [63] in the second subsection.

[11] A *D0L system* is a triple $G = (A, \psi, a)$ where A is a finite alphabet, $\psi : A^* \to A^*$ is an endomorphism and a is a word in A^*. An *HD0L system* is a 5-tuple $G = (A, B, \psi, \tau, a)$ where (A, ψ, a) is a D0L system, B is a finite alphabet and $\tau : A^* \to B^*$ is a morphism. If $\tau(\psi^n(a))$ converges (for the usual product topology on $A^{\mathbb{N}}$) as n tends to infinity, we call the limit a *HD0L sequence*. For further reading about HD0L sequences we refer to [25].

2.3.1 F. Durand's method about the ultimate periodicity of (primitive) sequences

We formulate the problem explicitly: given two finite alphabets A and B, a morphism $\psi : A^* \to A^*$, a word $a \in A^*$, and a morphism $\tau : A^* \to B^*$, is the sequence $\tau(\psi^\infty(a))$ ultimately periodic?

From the last section we know that $M(\phi)$ is not primitive and, accordingly, ϕ is not primitive either. Hence we apply the results of the article [26]. Our procedure is to display statements of [26] successively, immediately deducing the derivable results for our sequence.

We will state the first proposition, which is based on [26, Corollary 3.3] or [27, Proposition 5].

The alphabet A, over which ψ is defined, can be partitioned such that a power of the incidence matrix $M(\psi)$ has a special form which is described in the subsequent proposition.

Proposition 2.34. *Let $P = (p_{i,j})_{i,j\in A}$ be a matrix with non-negative coefficients. Then there exist three positive integers r, s, and t, where $s \le t-1$, and a partition $\left\{A_j : j \in [t]\right\}$ of A such that*

$$P^r = \begin{array}{cc} \begin{array}{cccccccc} A_1 & A_2 & \dots & A_s & A_{s+1} & A_{s+2} & \dots & A_t \end{array} & \\ \begin{pmatrix} P_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ P_{2,1} & P_2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{s,1} & P_{s,2} & \dots & P_s & 0 & 0 & \dots & 0 \\ P_{s+1,1} & P_{s+1,2} & \dots & P_{s+1,s} & P_{s+1} & 0 & \dots & 0 \\ P_{s+2,1} & P_{s+2,2} & \dots & P_{s+2,s} & 0 & P_{s+2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{t,1} & P_{t,2} & \dots & P_{t,s} & 0 & 0 & \dots & P_t \end{pmatrix} & \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_s \\ A_{s+1} \\ A_{s+2} \\ \vdots \\ A_t \end{array} \end{array}$$

and the submatrices P_j, corresponding to the subsets A_j of A, have only positive entries or are the zero matrix.

With Lemma 2.24 and [59, Sections 4.4 and 4.5] about the irreducible components of non-negative matrices and the cyclic structures of irreducible matrices, the partition and r can be computed algorithmically.

We demonstrate the procedure as applied in our context. Recall $M(\phi)$ of (2.8). There is a multidigraph $G(\phi)$ associated with $M(\phi)$ whose vertex set is Σ_{K-} and edge set consists of $M(\phi)_{IJ}$ distinct arcs with initial vertex I and terminal vertex J. We say that $I \sim J$ if there exists a path in $G(\phi)$ from I to J. Then I *communicates with* J if $I \sim J$ and $J \sim I$. This is an equivalence relation. (The proof is left to the reader.) The communicating classes partition the set Σ_{K-} as these are the maximal sets of vertices such that each one communicates with all the others in the class.

By drawing the associated graph, we get six communicating classes $A_1 = \{x, y, z, d, e, f\}$ and A_2, A_3, A_4, A_5, A_6 with one element of Δ_{K-} each.

If we read these classes again as vertices and draw an arc from class A_i to class A_j if and only if $A_i \neq A_j$ and there is an arc in $G(\phi)$ with some element in A_i as initial vertex and some element in

A_j as terminal vertex, we get a graph H. Note that H is a digraph. This graph of communicating classes has only one *sink*, i.e., a vertex A_v such that no arc leaves A_v. By removing this sink, we can order the classes such that there can be a path in H from A_i to A_j only if $j > i$. When we arrange the vertices in the corresponding order, we get the block triangular form

$$M(\phi) = \begin{pmatrix} M_1 & 0 & 0 & 0 & 0 & 0 \\ * & M_2 & 0 & 0 & 0 & 0 \\ * & * & M_3 & 0 & 0 & 0 \\ * & * & * & M_4 & 0 & 0 \\ * & * & * & * & M_5 & 0 \\ * & * & * & * & * & M_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & & & & & \\ 1 & 2 & 0 & 1 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 1 & 1 & 0 & & & & & \\ 0 & 0 & 1 & 0 & 0 & 0 & & & \mathbf{0} & & \\ 0 & 1 & 1 & 0 & 0 & 1 & & & & & \\ 1 & 0 & 0 & 0 & 1 & 0 & & & & & \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & & & & \\ 1 & 2 & 0 & 2 & 0 & 0 & & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & 0 & & & 1 & & \\ 0 & 0 & 1 & 0 & 0 & 0 & & \mathbf{0} & & 1 & \\ 0 & 0 & 0 & 0 & 0 & 1 & & & & & 1 \end{pmatrix},$$

where $*$ are submatrices. We denote by G_i the subgraph of G with vertex set A_i and whose arcs are those of G whose initial and terminal vertices are in A_i. These graphs G_i associated with the matrices M_i are called the irreducible components of $G(\phi)$. They are irreducible as they are respectively strongly connected.

Remark. A matrix A is irreducible if and only if the multidigraph $G(A)$ associated with A is *strongly connected*, i.e., if for any pair of distinct vertices a and b there exists a directed path from a to b and a directed path from b to a. We give only a sketch of the proof. Assume that A is reducible, then there exists a permutation matrix P such that

$$A = P\begin{pmatrix} A_1 & R \\ 0 & A_2 \end{pmatrix}P^{-1} = P\bar{A}P^{-1}.$$

The multidigraph associated with $\bar{A}$ cannot be strongly connected, since the vertices corresponding to the lower left part of $\bar{A}$ are not connected. As the graphs $G(A)$ and $G(\bar{A})$ are isomorphic, $G(A)$ is not strongly connected. Now suppose that $G(A)$ is not strongly connected, i.e., there exist non-empty sets S and S' such that no directed path from one vertex of S to one of S' exists. Relabel the vertices of $G(A)$ as elements of S and S' and permute the matrix A in the same manner to get $\bar{A}$. The new graph is $G(\bar{A})$. As there are no directed paths from one vertex in S to one in S', we deduce that $\bar{A}$ must have a block of zeros in the lower left corner. Hence $\bar{A}$ is reducible. Since A and $\bar{A}$ are permutationally similar, we see that A is irreducible.

We will see that each irreducible matrix has a characteristic period p.
Let A be a non-negative matrix. The set of vertices in the graph associated with A is equal to the set of *states* of A. The *period* $\operatorname{per}(I)$ of a state I is, by definition, the greatest common divisor of all those integers $m \in \mathbb{N}$ for which $(A^m)_{II} > 0$. In the case that no such integer exists, we set $\operatorname{per}(I) = \infty$. The *period* $\operatorname{per}(A)$ of a matrix A is the greatest common divisor of all finite numbers $\operatorname{per}(I)$, or is ∞ if $\operatorname{per}(I) = \infty$ for all I. We call a matrix A aperiodic if $\operatorname{per}(A) = 1$.

Lemma 2.35 ([59, Lemma 4.5.3])**.** *Let A be an irreducible matrix. Then all states have the same period. Moreover,* $\operatorname{per}(A)$ *is the period of any of its states.*

Thus the matrix M_1 is irreducible as well as aperiodic. By the subsequent lemma, we conclude that M_1 is primitive.

Lemma 2.36 ([59, Theorem 4.5.8])**.** *Let A be a non-negative matrix. The following are equivalent:*

a) A is primitive.

b) A is irreducible and aperiodic.

c) $A^n > 0$ for all sufficiently large n.

With $r = 6$, $s = 1$, and $t = 6$ the new matrix, which has the required form of Proposition 2.34, is

$$M(\phi)^6 = \begin{pmatrix}
13 & 14 & 2 & 12 & 11 & 2 & & & & & \\
54 & 94 & 39 & 54 & 28 & 39 & & & & & \\
16 & 39 & 24 & 16 & 4 & 23 & & & & & \\
12 & 14 & 2 & 13 & 11 & 2 & \multicolumn{5}{c}{\text{\Large 0}} \\
50 & 68 & 18 & 50 & 36 & 18 & & & & & \\
16 & 39 & 23 & 16 & 4 & 24 & & & & & \\
50 & 83 & 30 & 47 & 24 & 33 & 1 & & & & \\
101 & 176 & 72 & 104 & 52 & 69 & 0 & 1 & & & \\
28 & 53 & 25 & 28 & 15 & 25 & 0 & 0 & 1 & & \\
16 & 24 & 11 & 19 & 19 & 8 & 0 & 0 & 0 & 1 & \\
19 & 24 & 8 & 16 & 15 & 11 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

The next corollaries are consequences of Proposition 2.34.

Corollary 2.37 ([26, Corollary 3.4])**.** *Let $\psi : A^* \to A^*$ be an endomorphism, whose incidence matrix has the form of P^r in Proposition 2.34. Then for all $b \in A$ and all $i \in \mathbb{N}$ the letters having an occurrence in $\psi^{|A|^i}(b)$ or $\psi^{|A|^{i+1}}(b)$ are the same.*

We call a letter $a \in A$ *growing (with respect to ψ)* if $\lim_{n\to\infty} |\psi^n(a)| = \infty$. A morphism ψ is *growing* if all letters of A are growing. We define that a morphism is *erasing* if there exists a letter $a \in A$ such that $\psi(a)$ is the empty word.

Corollary 2.38 ([26, Corollary 3.5])**.** *The problem whether an endomorphism has non-growing letters is decidable.*

Proof. Let ψ be an endomorphism. We use the notation of Proposition 2.34. We write $A^{(\text{zero})}$ (resp. $A^{(\text{one})}$) for the set of all letters $a \in A_j$ where P_j is the zero matrix (resp. the matrix (1)). The letters belonging to $A \setminus \left(A^{(\text{zero})} \cup A^{(\text{one})}\right)$ are growing according to the definition of an incidence matrix

and the concept of growing.
Let $a \in A_j \cap A^{(\text{one})}$ for some j. From the proof of Corollary 2.37 (i.e., [26, Corollary 3.4]), we conclude that a is non-growing if and only if all letters occurring in $\psi^{r|A|}(a)$, except a, are erasing w.r.t. $\psi^{r|A|}$. We call the set of such non-growing letters $\bar{A}$.
Let $a \in A_i \cap A^{(\text{zero})}$ for some i. The letter a is non-growing if and only if all letters occurring in $\psi^{r|A|}(a)$, except a, are erasing w.r.t. $\psi^{r|A|}$ or belong to $\bar{A}$. With Proposition 2.34 and Corollary 2.37, we are able to decide whether a letter is erasing w.r.t. $\psi^{r|A|}$. □

In our case, the sets are $\Sigma_{K-}^{(\text{zero})} = \emptyset$ and $\Sigma_{K-}^{(\text{one})} = \{a, b, c, \bar{a}, \bar{b}\}$. The letters of $\Sigma_{K-} \setminus \Sigma_{K-}^{(\text{one})}$ are therefore growing. By calculating $M(\phi)^{r|\Sigma_{K-}|} = M(\phi)^{66}$, it is easy to check that the letters $a, b, c, \bar{a}$, and $\bar{b}$ are non-growing. Consequently, ϕ is non-growing.

In what follows, we assume that the sequence $\mathbf{s} = \tau(\psi^\infty(a))$ fulfills these four conditions:

(C1) ψ is non-erasing,

(C2) the incidence matrix $M(\psi)$ of ψ has the form $M(\psi)^r$ in Proposition 2.34,

(C3) for all $b \in A$ and all $i \in \mathbb{N}$ the sets of letters having an occurrence in $\psi^i(b)$ or $\psi^{i+1}(b)$ are the same,

(C4) the set of letters occurring in $\psi^\infty(a)$ is A.

Obviously, the sequence $\mathbf{u}$ satisfies the first and the last condition. For the second statement, we have to take a power of ϕ, here $r = 6$. Since we additionally require (C3), we consider $\phi^{|\Sigma_{K-}|}$ instead of ϕ, here $\phi^{|\Sigma_{K-}|} = \phi^{11}$. Note that this is possible without changing $\mathbf{u}$ or $\phi^\infty(y)$ as ϕ is a prolongable morphism.

With the following theorem we decide whether we can redefine our sequence as a morphic sequence w.r.t. a growing prolongable endomorphism.

Theorem 2.39 ([26, Lemma 3.15], [66, Théorème 4.1])**.** *The prolongable endomorphism ψ satisfies exactly one of the following statements:*

a) There exists a growing letter $b \in A$, occurring in $\psi^\infty(a)$, with the property that $\psi(b) = \alpha b \beta$ (or $\beta b \alpha$) with $\alpha \in A^*$ and $\beta \in B^* \setminus \{\epsilon\}$, where B is the set of non-growing letters.

b) The length of subwords of $\psi^\infty(a)$ consisting of non-growing letters is bounded.

In the second case, the sequence $\psi^\infty(a)$ can be defined algorithmically as a morphic sequence w.r.t. a growing prolongable endomorphism. Moreover, it is decidable which statement ψ satisfies.

We check that the first statement is not fulfilled by ϕ^{11} with the aid of trivial calculations, whence we are in b). Let B be the set of non-growing letters and $C = A \setminus A^{(\text{one})}$ the set of growing letters. The second part in the proof of our Theorem 2.39 in [66] gives us an approach for an algorithm with which we can construct the new morphism. First, we describe the general case.

We know that the length of words occurring in $S = \psi^\infty(a)$ and consisting of elements in B is bounded. Let $\psi^\infty(a) = c_0 b_0 c_1 \ldots$ with $c_0, c_1 \in C$ and $b_0 \in B^*$. For the elements of the alphabet Y we take the symbols $[cbc']$, where cbc' is a factor of S with $b \in B^*$ and $c, c' \in C$. Since the subwords of S are of bounded length, Y is finite. We define the morphism $\psi' : Y^* \to Y^*$ by

$$\psi'([cbc']) = [z_1 b_1 z_2][z_2 b_2 z_3] \ldots [z_n b_n z_{n+1}], \tag{2.12}$$

where $\psi(cb) = b_0 z_1 b_1 z_2 \ldots b_{n-1} z_n b_n'$, the image $\psi(c')$ begins with $b_n'' z_{n+1}$, and $b_n = b_n' b_n''$ with $b_j \in B^*, z_j \in C$. The morphism ψ' is growing as the growth order[12] of the factor $[cbc']$ w.r.t. ψ' is the same as the one of c w.r.t. ψ. Given the map h from Y^* to A^* defined by $[cbc'] \mapsto cb$, we see that $\psi^i(c_0)$ is a prefix of $h(\psi'^i([c_0 b_0 c_1]))$ for $i \in \mathbb{N}_1$, and $S = h(\psi'^\infty([c_0 b_0 c_1]))$.

In our case, the sequence $\phi^\infty(y)$ starts with $ybae$, where y and e are elements of $C = \{x, y, z, d, e, f\}$ and $ba \in B^* = \{a, b, c, \bar{a}, \bar{b}\}^*$. Our next step is the definition of the alphabet Y and the proof that the elements of Y are the only ones in $\phi^\infty(y)$ of the form $[cbc']$ with $c, c' \in C$ and $b \in B^*$.

Theorem 2.40. *The set of factors in $\phi^\infty(y)$ of the form $[cbc']$ with $c, c' \in C$ and $b \in B^*$ is*

$$\begin{aligned} Y \;=\; & \Big\{[ybae], [eby], [ybaf], [fcz], [zby], [e\bar{b}x], [xcd], [d\bar{a}e], \\ & [yb\bar{a}e], [yb\bar{a}f], [z\bar{b}y], [e\bar{b}y]\Big\}. \end{aligned}$$

Proof. By definition of ϕ,

$$[ybae], [eby], [ybaf], [fcz], [zby], [e\bar{b}x], \text{ and } [d\bar{a}e]$$

are clearly elements of Y. The other elements of Y are factors in $\phi^\infty(y)$ as they are images under ϕ. We show this, for instance, for $[yb\bar{a}e]$. This factor $yb\bar{a}e$ can be the image under ϕ of $d\bar{a}f$ or $yb\bar{a}f$. Since in turn no image of ϕ ends in d, the first case is excluded. In the second case, we have $d\bar{a}e$ and $yb\bar{a}e$ as possible images. The latter factor is excluded, since we started with this factor, and the other is $\phi(z) = d\bar{a}e$. The elements of Y are the only ones of the form $[cbc']$ with $c, c' \in C$ and $b \in B^*$ as the other possibilities are excluded. As an example, we prove this for the factors with fa as a prefix. They must be images of factors starting with xa, but these are in turn images of the factors with prefix fa. Hence all factors with fa as a prefix are excluded. This completes the proof, the detailed verification for the other elements of Y being left to the reader. □

Obviously, the set Y is finite. The application of (2.12) gives the new morphism $\phi' : Y^* \to Y^*$ defined by

$$\begin{aligned} &\phi'([ybae]) = [ybae][eby][ybaf], && \phi'([eby]) = [fcz][zby], \\ &\phi'([ybaf]) = [ybae][eby][ybae], && \phi'([fcz]) = [e\bar{b}x][xcd], \\ &\phi'([zby]) = [d\bar{a}e][eby], && \phi'([e\bar{b}x]) = [fcz][z\bar{b}y], \\ &\phi'([xcd]) = [ybaf][fcz], && \phi'([d\bar{a}e]) = [zby][yb\bar{a}f], \\ &\phi'([yb\bar{a}e]) = [ybae][eby][yb\bar{a}f], && \phi'([yb\bar{a}f]) = [ybae]][eby][yb\bar{a}e], \\ &\phi'([z\bar{b}y]) = [d\bar{a}e][e\bar{b}y], && \phi'([e\bar{b}y]) = [fcz][z\bar{b}y]. \end{aligned}$$

[12]If $|\psi^n(a)|$ for $a \in A$ grows as $n^{\alpha_a}\beta_a^n =: go_a$, we call go_a the *growth order of a* for an integer $\alpha_a \geq 0$ and a number $\beta_a \geq 1$. For more details see [73].

Then $h(\phi'^{\infty}([ybae]) = \phi^{\infty}(y)$ with $h : Y^* \to \Sigma^*_{K-}$, $[cbc'] \mapsto cb$, and the $TH(K_3^-)$ sequence $\mathbf{u}$ is $\tau(h(\phi'^{\infty}([ybae]))$. Corresponding to the new map ϕ', we define a new morphism $\tau' : Y^* \to \Delta^*_{K-}$ under the condition that $\tau'(\phi'^{\infty}([ybae]) = \tau(\phi^{\infty}(y))$. As mapping rules we get

$$\begin{array}{ll}
\tau'([ybae]) = aba, & \tau'([eby]) = cb, \\
\tau'([ybaf]) = aba, & \tau'([fcz]) = \bar{b}c, \\
\tau'([zby]) = \bar{a}b, & \tau'([e\bar{b}x]) = c\bar{b}, \\
\tau'([xcd]) = ac, & \tau'([d\bar{a}e]) = b\bar{a}, \\
\tau'([yb\bar{a}e]) = ab\bar{a}, & \tau'([yb\bar{a}f]) = ab\bar{a}, \\
\tau'([z\bar{b}y]) = \bar{a}\bar{b}, & \tau'([e\bar{b}y]) = c\bar{b}.
\end{array}$$

With the order of the elements of Y as in Theorem 2.40 the incidence matrix of ϕ' is

$$M(\phi') = \begin{pmatrix}
1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$

By analysing the graph $G(\phi')$, we get the irreducibility of $M(\phi')$ and by Lemma 2.35 the aperiodicity. Additionally, $M(\phi')^8 > 0$. We can again apply Proposition 2.34 and get $r = 8$, $s = 1$, and $t = 0$. Since the matrix is now primitive, we can employ the statements for the primitive case in Durand's paper [25] in the sequel.

By the above, we know that τ' is not a coding. Therefore, $\tau'(\phi'^{\infty}([ybae]))$ is not a morphic sequence yet. But there exists a theorem which says that, if a sequence is an image, under a morphism, of a fixed point of a primitive morphism, it is a morphic sequence w.r.t. a primitive prolongable endomorphism. This result can be found in [14, Théorème 4], [24, Proposition 3.1], [25, Proposition 17], [46], and [65, Théorème 1.1].

Theorem 2.41. *Let $\psi : A^* \to A^*$ be a primitive morphism which generates the sequence $S = \psi^{\infty}(a)$ for some $a \in A$ and let $\tau : A^* \to B^*$ be a morphism such that $\mathbf{s} = \tau(S)$ where A and B are two finite alphabets. Then $\mathbf{s}$ is a morphic sequence with respect to a primitive prolongable endomorphism.*

Proof. We set $m = |A|$. Then Lemma 2.6 says that there exists an $l \le (m-1)^m$ such that every letter of A has an occurrence in all images of ψ^l. Consequently, no letter is erasing under the morphism $\sigma = \tau\psi^l$ and $\mathbf{s} = \sigma(S)$.

Define $D = \{(a, n) \mid a \in A,\ n \in [|\sigma(a)|]_0\}$ and the morphism $\rho : A^* \to D^*$ by

$$\rho(a) = (a, 0) \ldots (a, |\sigma(a)| - 1).$$

There exists an integer ω ($\leq \max_{a \in A} |\sigma(a)|$) with $|\psi^\omega(a)| \geq |\sigma(a)|$ for all $a \in A$. We will denote by μ the endomorphism on D^* given by

$$\begin{aligned} \mu\left((a, n)\right) &= \rho\left(\psi^\omega(a)_{[n]}\right) && \text{if } n \in [|\sigma(a)| - 1]_0, \\ \mu((a, |\sigma(a)| - 1)) &= \rho\left(\psi^\omega(a)_{[|\sigma(a)|-1, |\psi^\omega(a)|-1]}\right), && \text{otherwise.} \end{aligned}$$

We get for all $a \in A$

$$\begin{aligned} \mu(\rho(a)) &= \mu((a, 0) \ldots (a, |\sigma(a)| - 1)) \\ &= \rho\left(\psi^\omega(a)_{[0]}\right) \ldots \rho\left(\psi^\omega(a)_{[|\psi^\omega(a)|-1]}\right) = \rho(\psi^\omega(a)). \end{aligned}$$

It follows that $\mu(\rho(S)) = \rho(\psi^\omega(S)) = \rho(S)$. Hence $\rho(S)$ is a fixed point of μ starting with some letter $(a_0, 0)$ and $\mu\rho = \rho\psi^\omega$.
Let $\chi : D^* \to B^*$ be the coding defined by $\chi((a, n)) = \sigma(a)_{[n]}$ for all $(a, n) \in D$. By the above, we get for all $a \in A$

$$\chi(\rho(a)) = \chi((a, 0) \ldots (a, |\sigma(a)| - 1)) = \sigma(a).$$

Consequently, $\chi(\rho(S)) = \sigma(S)$. From $\mu\rho = \rho\psi^\omega$ we deduce that for all $\nu \in \mathbb{N}$ the identity $\mu^\nu \rho = \rho\psi^{\nu\omega}$ is fulfilled. Since we assumed that ψ is primitive, we can infer the primitivity of μ. □

Coming back to our sequence **u**, we summarise what we already know. All images of ϕ'^8 have an occurrence of all letters of Y, because of the primitivity of $M(\phi')$, and $M(\phi')^8 > 0$. The morphism τ' is not a coding. Hence all assumptions of Theorem 2.41 are satisfied, and we can employ it in our case.
Set $\sigma = \tau'\phi'^8$ and

$$\begin{aligned} D = \Big\{ & ([ybae], 0), \ldots, ([ybae], 3461), ([eby], 0), \ldots, ([eby], 1183), \\ & ([ybaf], 0), \ldots, ([ybaf], 3461), ([fcz], 0), \ldots, ([fcz], 1394), \\ & ([zby], 0), \ldots, ([zby], 1394), ([e\bar{b}x], 0), \ldots, ([e\bar{b}x], 1183), \\ & ([xcd], 0), \ldots, ([xcd], 2067), ([d\bar{a}e], 0), \ldots, ([d\bar{a}e], 2067), \\ & ([yb\bar{a}e], 0), \ldots, ([yb\bar{a}e], 3461), ([yb\bar{a}f], 0), \ldots, ([yb\bar{a}f], 3461), \\ & ([z\bar{b}y], 0), \ldots, ([z\bar{b}y], 1394), ([e\bar{b}y], 0), \ldots, ([e\bar{b}y], 1183) \Big\}. \end{aligned}$$

We calculated the length of $\sigma([w])$ for each $[w] \in Y$ with the aid of the following matrix multiplication. The incidence matrix of the coding τ' is

$$\begin{pmatrix} 2 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and for the endomorphism ϕ'^8 it is

$$\begin{pmatrix}
283 & 109 & 282 & 100 & 100 & 109 & 182 & 178 & 282 & 282 & 100 & 109 \\
282 & 110 & 282 & 100 & 100 & 109 & 178 & 182 & 282 & 282 & 100 & 109 \\
164 & 36 & 165 & 80 & 76 & 36 & 84 & 84 & 164 & 164 & 76 & 36 \\
211 & 40 & 211 & 110 & 109 & 40 & 102 & 102 & 211 & 211 & 109 & 40 \\
164 & 36 & 164 & 76 & 81 & 36 & 84 & 84 & 164 & 164 & 80 & 36 \\
82 & 47 & 82 & 20 & 20 & 47 & 63 & 62 & 82 & 82 & 20 & 47 \\
82 & 47 & 82 & 20 & 20 & 47 & 63 & 62 & 82 & 82 & 20 & 47 \\
82 & 47 & 82 & 20 & 20 & 47 & 62 & 63 & 82 & 82 & 20 & 47 \\
16 & 15 & 16 & 2 & 2 & 15 & 14 & 18 & 17 & 16 & 2 & 15 \\
47 & 4 & 47 & 29 & 33 & 4 & 18 & 18 & 47 & 48 & 33 & 4 \\
47 & 4 & 47 & 33 & 29 & 4 & 18 & 18 & 47 & 47 & 30 & 4 \\
16 & 15 & 16 & 2 & 2 & 16 & 17 & 14 & 16 & 16 & 2 & 16
\end{pmatrix},$$

with the result that the incidence matrix of σ is

$$\begin{aligned}
M(\sigma) &= M(\tau')M(\phi')^8 \\
&= \begin{pmatrix}
1039 & 356 & 1039 & 411 & 407 & 356 & 627 & 622 & 1038 & 1038 & 407 & 356 \\
1038 & 357 & 1038 & 407 & 412 & 356 & 622 & 627 & 1038 & 1038 & 411 & 356 \\
673 & 259 & 673 & 252 & 251 & 259 & 423 & 422 & 673 & 673 & 251 & 259 \\
356 & 106 & 356 & 160 & 165 & 106 & 196 & 201 & 357 & 357 & 165 & 106 \\
356 & 106 & 356 & 165 & 160 & 107 & 200 & 196 & 356 & 356 & 161 & 107
\end{pmatrix}.
\end{aligned}$$

By $\rho([w]) = ([w], 0) \ldots ([w], |\sigma([w])| - 1)$ we define a map from Y^* to D^*.

The integer $\omega = 9$ satisfies the conditions posed in the proof of Theorem 2.41. It is smaller than $\max_{[w]\in Y} |\sigma([w])|$ and $|\phi'^9([w])| = |\sigma([w])|$ for all $[w] \in Y$. The morphism $\mu : D^* \to D^*$ is defined by

$$\begin{aligned}
\mu(([w], n)) &= \rho(\phi'^9([w])_{[n]}) && \text{if } n \in [|\sigma([w])| - 1]_0, \\
\mu(([w], |\sigma([w])| - 1)) &= \rho(\phi'^9([w])_{[|\sigma([w])|-1]}) && \text{otherwise.}
\end{aligned}$$

For the alphabet D we also need a new coding $\chi : D^* \to \Delta^*_{K-}$, where the image $\chi(([w], n))$ with $([w], n) \in D$ is the nth letter of $\sigma([w])$.
It is clear that $\chi(\rho(U)) = \sigma(U)$, where $U = \phi'^\infty([ybae])$ is the sequence which is generated by ϕ'. We have thus proved the following theorem.

Theorem 2.42. *The $TH(K_3^-)$ sequence* $\mathbf{u}$ *is the image, under a coding, of a fixed point of a primitive prolongable endomorphism.*

Remark. This causes the question whether we can now use [8, Theorem 8.4.7]. We know that $|D| = 25721$. Consequently, the incidence matrices of μ and χ are very big and the necessary calculations in the application of the theorem would be very complex.

We know that μ is primitive, i.e., there exists an integer k such that $M(\mu)^k > 0$, and χ is a coding. Hence we are in a position to use the following theorem and continue as in its proof.

Theorem 2.43. *With the assumptions that the incidence matrix of μ has strictly positive entries and that the morphism χ is a coding, it is decidable whether the infinite sequence* **u** *is ultimately periodic.*

This theorem and the proof can be found in a more general form in [25, Theorem 26]. Following the proof of [25, Theorem 26], we have to determine the constant K as a limit for our further calculation. For any endomorphism $\psi : A^* \to A^*$ we define $||\psi|| = \max_{a\in A} |\psi(a)|$.

Definition 2.44. For this subsection we say that *occurrence* in $\mathbf{s}$ of a word u is every integer i such that $\mathbf{s}_{[i,i+|u|-1]} = u$. Let us denote by R_ψ the maximal difference between two successive occurrences of a word of length 2 in any fixed point of the primitive prolongable morphism ψ.

As a consequence of Lemma 2.6, this number is bounded by $2||\psi||^{(m-1)m^m}$ with $m = |A|$.

Lemma 2.45 ([25, Lemma 7])**.** *Set* $Q_\psi = \max\left\{\max_{k\in[((m-1)m^m+2)]_0} \frac{||\psi^k||}{\min_{a\in A}|\psi^k(a)|}, ||\psi||\right\}$ *for a primitive prolongable endomorphism ψ defined on A with $m = |A|$. Then for each such ψ and all n :*

$$||\psi^n|| \leq Q_\psi \min_{a\in A} |\psi^n(a)|.$$

With $K_\psi = Q_\psi R_\psi ||\psi||$ (see [25, Theorem 8]), we have according to [25, Section 5]

$$K = (4K_\psi^3)^{||\psi||K_\psi^2+1}(K_\psi + 1)^{K_\psi^2} \tag{2.13}$$

if the incidence matrix of ψ has strictly positive entries.

For our sequence **u** with the primitive prolongable endomorphism μ over D and the coding χ we get $M := ||\mu|| = 3462$ as the maximal length of the function values over all elements of D. We present an algorithm for the calculation of Q_μ with $m = |D| = 25721$. Since moreover $\min_{a\in D} |\mu(a)| = 1184$, we have as a first possible candidate for the constant Q_μ the quotient $3462/1184 = 1731/592$. For the real implementation of Algorithm 8 it would be useful to migrate to calculations with integers instead of the here used fractions.

The number $L = (m-1)m^m + 1$ is very large, since $(m-1)m^{900} + 1$ is already about $4.67 \cdot 10^{3973}$. Additionally, the incidence matrix $M(\mu)$ is a (25721×25721)-matrix. Hence the calculation will be very complex. For this reason we will try another approach for checking whether the sequence is ultimately periodic.

2.3.2 Ultimate periodicity and subword schemes using I. V. Mitrofanov's method

We will now make use of the proof of the decidability of the HD0L ultimate periodicity problem of I. V. Mitrofanov in [63]. The general idea is similar to Durand's one. At first he proves the decidability for the primitive case and in the second step he reduces the general case to the

Algorithm 8 The calculation of Q_μ

```
procedure CALC(m, M, M(μ), L)
    parameter m : number of elements in alphabet D
    parameter M : maximal length of the function values over all elements of D
    parameter M(μ) : incidence matrix of μ
    parameter L : the value L = ((m − 1)m^m + 1)
    k ← 1
    A^e ← M(μ)
    div ← 3465/1184
    Q ← 0
    while k ≤ L do
        A^e ← A^e ∗ M(μ)
        add all entries of every column of A^e
        max ← maximal value of the sums over the columns
        min ← minimal value of the sums over the columns
        d ← max / min
        if d > div and d > M then
            div ← d
            k ← k + 1
        else
            k ← k + 1
        end if
    end while
    if div = 3465/1184 then
        Q ← M
    else
        Q ← div
    end if
end procedure
```

primitive one. We can immediately go to the primitive case because of our work in the previous subsection.

We have already converted our original prolongable endomorphism ϕ and coding τ into the primitive prolongable endomorphism μ and the coding χ. But we will go back one step to the primitive morphism ϕ' and the morphism τ' defined on the alphabet Y.

The problem which is solved in the sequel: Given two finite alphabets A and B, a primitive morphism $\psi : A^* \to A^*$ which is prolongable on the letter $a \in A$, and a non-erasing morphism $\tau : A^* \to B^*$. Is the word $\tau(\psi^\infty(a))$ periodic?

Remark. According to [64, Proposition 3], a sequence generated by a primitive growing endomorphism ψ and a morphism τ is uniformly recurrent (see below for a definition). If this sequence is ultimately periodic, then it is periodic. This result is [24, Lemma 2.7]. Hence in our case there is no difference between "ultimately periodic" and "periodic".

We give some basic definitions. Recall that a word t is called a subword of s if $s = s_0ts_1$ for some words s_0 and s_1. It is called a *beginning* or an *ending* of s if s_0 or s_1 is empty, respectively. For what follows, we will also need the definition of a subword scheme. We think of two words $s_0 = abcc$ and $s_1 = cbbc$. They are located in $s_2 = cbbcabccbbc$ as $t_0 = aba$ and $t_1 = abc$ are located in $t_2 = abcababc$. There are two occurrences of s_1, one at the beginning and one at the end of s_2. The subword s_0 occurs only once in the middle and overlaps with the second occurrence of s_1. The prior statements are also true if we substitute s_0, s_1, and s_2 with t_0, t_1, and t_2. In this way $(s_2; s_0, s_1)$ and $(t_2; t_0, t_1)$ have the same subword scheme.

Definition 2.46. Let s be a finite word with length n. A *node* of s is one of $n + 1$ positions: the beginning of s (the *beginning* node), the end of s (the *ending* node) or one of the $n - 1$ positions between two neighbouring letters (an *ordinary* node).

Any pair of nodes fixes a subword in s. Note that it can also be the empty word.

From all nodes of a subword we can filter out the interesting ones.

Definition 2.47. Suppose $S = (s_0, s_1, \dots, s_n)$ and $T = (t_0, t_1, \dots, t_m)$ are two ordered[13] sets of finite words defined on the same alphabet. For any $s_j \in S$, we call the following nodes *interesting nodes*:

- the beginning node,
- the ending node,
- every node that is a beginning or an ending of any occurrence of a word of T.

Definition 2.48. A *subword scheme* $\mathcal{S}(S, T)$ for S and T consists of the ordered pair:

1. the ordered set of n numbers $z_0, z_1, \dots z_{n-1}$, where z_j is the number of interesting nodes in S_j,
2. a table of size $n \times m$.

The elements of the table are sets of ordered pairs of positive integers. In the i-th row and j-th column you find all occurrences of t_j in s_i denoted by pairs, where the first element is the beginning of an occurrence and the second element is the end of the same occurrence. The interesting nodes in s_j are numbered from 1 to z_j.

The theory of subword schemes is connected with the determination of sequences by morphisms and codings. At first we define the set of generating words of a sequence.

Definition 2.49. An ordered set G of subwords of length one or two of $\psi^\infty(a)$ is called the set of *generating words*.

[13]Hereafter, by “ordered" is meant that we keep the original arbitrary order of the elements of the sets unchanged.

We fix an arbitrary order on G. Then

$$T^r := \{\tau(\psi^r(g)) \mid g \in G\},$$

where the elements of each T^r are arranged according to the order in G.

Now we use this for our sequence.

Theorem 2.50. *The set G of generating words of $\phi'^{\infty}([ybae])$ is*

$$\begin{aligned}
&Y \cup \Big\{[ybae][eby], [eby][ybaf], [fcz][zby], [eby][ybae],\\
&[e\bar{b}x][xcd], [d\bar{a}e][eby], [fcz][z\bar{b}y], [ybaf][fcz], [zby][yb\bar{a}f],\\
&[eby][yb\bar{a}f], [eby][yb\bar{a}e], [d\bar{a}e][e\bar{b}y]\Big\}\\
&\cup \Big\{[ybae][e\bar{b}x], [zby][ybae], [xcd][d\bar{a}e], [yb\bar{a}e][e\bar{b}x],\\
&[yb\bar{a}f][fcz], [z\bar{b}y][ybae], [e\bar{b}y][ybae]\Big\}.
\end{aligned}$$

Proof. It is obvious that G contains the set Y. The second subset is contained because of the mapping rules of ϕ'. We get the elements of the last remaining subset by an analysis similar to that in the proof of Theorem 2.40. Note that here the form is only restricted by the maximal word length of two. □

We employ our new knowledge on the construction of subword schemes. With the aid of the computer algebra system *Sage*, we can determine the set T^2 in order to construct $\mathcal{S}(T^2, T^1)$. Since $|G| = 31$, we will get a table of size 31×31. For instance, we look for the interesting nodes of $\tau(\phi'^2([ybae])) = abacbaba\bar{b}c\bar{a}babacbaba \in T^2$ with respect to

$$\begin{aligned}
T^1 \quad = \quad &\Big\{abacbaba, \bar{b}c\bar{a}b, abacbaba, c\bar{b}ac, b\bar{a}cb, \bar{b}c\bar{a}\bar{b}, aba\bar{b}c, \bar{a}bab\bar{a},\\
&abacbab\bar{a}, abacbab\bar{a}, b\bar{a}c\bar{b}, \bar{b}x\bar{a}\bar{b}, \ldots\Big\}.
\end{aligned}$$

We see that

$$\tau(\phi'^2([ybae])) = {}_1abacb_2aba_3\bar{b}c_4\bar{a}b_5abacbaba_6,$$

where the subscripts give the interesting nodes. Hence we have six interesting nodes and the first entry in the vector of the subword scheme is 6. By this analysis, we also get the entries of the first row of the table (see Tables 2.1 and 2.2).

The subword scheme $\mathcal{S}(T^2, T^1)$ consists of the vector

$$(6, 3, 6, 3, 3, 3, 3, 3, 6, 6, 3, 3, 8, 8, 7, 8, 5, 7, 7, 10, 8, 8, 8, 5, 8, 8, 5, 8, 8, 8, 8)$$

and the table consisting of Tables 2.1 and 2.2 (read them consecutively).

The *size* of a subword scheme is the largest number of pairs in a cell of its table. In our case, it is equal to 2.

Table 2.1: Table corresponding to $\mathcal{S}(T^2, T^1)$ (Part 1)

(1,3)	(3,5)	(1,3)				(2,4)						(1,5)	(3,6)
			(1,2)	(2,3)									
(1,3)	(3,5)	(1,3)				(2,4)						(1,5)	(3,6)
					(1,2)	(2,3)					(1,2)		
	(2,3)						(1,2)						
			(1,3)							(2,3)			
(1,2)		(1,2)	(2,3)										
				(1,2)				(2,3)	(2,3)				
(1,3)	(3,5)	(1,3)				(2,4)		(5,6)	(5,6)			(1,5)	
(1,3)	(3,5)	(1,3)				(2,4)		(5,6)	(5,6)			(1,5)	
					(2,3)		(1,2)				(2,3)		
			(1,3)							(2,3)			
(1,3)	(3,5)	(1,3)	(6,7)	(7,8)		(2,4)						(1,5)	(3,6)
(3,5)	(5,7)	(3,5)	(1,2)	(2,3)		(4,6)						(3,7)	(5,8)
	(3,5),(6,7)				(1,2)	(2,4)	(4,6)				(1,2)		
(3,5)	(5,7)	(3,5)	(1,2)	(2,3)		(4,6)						(3,7)	(5,8)
(3,4)		(3,4)	(1,2),(4,5)							(2,3)			
			(4,6)	(1,2),(6,7)				(2,4)	(2,4)	(3,5)			
					(1,2),(6,7)	(2,4)	(4,6)				(1,2),(6,7)		
(1,3),(5,7)	(3,5)	(1,3)(5,7)			(7,9)	(2,4),(6,8)					(7,9)		(3,7)
(3,5)	(2,3),(5,7)	(3,5)		(4,6)	(1,2)	(7,8)					(1,2)		(2,5)
(3,5)	(5,7)	(3,5)	(1,2)	(2,3)		(4,6)		(7,8)	(7,8)			(3,7)	
(3,5)	(5,7)	(3,5)	(1,2)	(2,3)		(4,6)		(7,8)	(7,8)			(3,7)	
			(3,4)	(1,2)				(2,3)	(2,3)	(4,5)			
(1,3),(5,6)	(3,5)	(1,3)(5,6)	(6,7)			(2,4)				(7,8)		(1,5)	(3,6)
(3,5),(7,8)	(5,7),(2,3)	(3,5)(7,8)				(4,6)	(1,2)					(3,7)	(2,5)
(1,2)		(1,2)	(2,3)	(3,4)				(4,5)	(4,5)				
(1,3)	(3,5)	(1,3)	(6,7)			(2,4)		(5,6)	(5,6)	(7,8)		(1,5)	
(1,3)	(3,5)	(1,3)			(6,7)	(2,4),(7,8)		(5,6)	(5,6)		(6,7)	(1,5)	
(3,5),(7,8)	(5,7)	(3,5)(7,8)			(2,3)	(4,6)	(1,2)				(2,3)	(3,7)	(5,8)
(3,5),(7,8)	(5,7)	(3,5)(7,8)	(1,2)			(4,6)				(2,3)		(3,7)	(5,8)

Table 2.2: Table corresponding to $\mathcal{S}(T^2, T^1)$ (Part 2)

	(3,6)															
(1,3)																
	(3,6)															
		(1,3)														
			(1,3)													
				(1,3)												
					(1,3)											
						(1,3)										
							(3,6)	(3,6)								
							(3,6)	(3,6)								
									(1,3)							
				(1,3)												
(6,8)	(3,6)				(5,7)											
(1,3)	(5,8)															
		(1,4)	(4,7)									(2,6)				
(1,3)	(5,8)															
				(1,3)	(3,5)										(2,4)	
(4,7)						(1,4)								(2,6)		
		(1,4)							(4,7)			(2,6)				
	(3,7)	(7,10)								(5,9)						
	(2,5)		(1,3)				(5,8)	(5,8)								
(1,3)							(5,8)	(5,8)								
(1,3)							(5,8)	(5,8)								
				(3,5)		(1,3)								(2,4)		
	(3,6)			(6,8)	(5,7)											(3,6)
	(2,5)		(1,3)													
(2,4)					(1,3)	(3,5)										
				(6,8)			(3,6)	(3,6)						(5,7)		
		(6,8)					(3,6)	(3,6)					(5,7)			
	(5,8)								(1,3)							(2,5)
	(5,8)			(1,3)											(2,5)	

For the further procedure, we need the following lemma which makes a statement concerning subword schemes of increasing order.

Lemma 2.51 ([63, Theorem 3.8])**.** *Given a prolongable endomorphism and a coding, we can algorithmically determine a number d such that from $n_1 + d < m_1$, $n_2 + d < m_2$ and from $\mathcal{S}(T^{m_1}, T^{n_1}) = \mathcal{S}(T^{m_2}, T^{n_2})$ it follows that $\mathcal{S}(T^{m_1+1}, T^{n_1+1}) = \mathcal{S}(T^{m_2+1}, T^{n_2+1})$.*

Two subword schemes are equal if the tables coincide. The necessary lemmas for an algorithm to determine the subword schemes can be found in [63].

Lemma 2.52. *There exists a number K, called the* uniform recurrence constant*, such that, if s_1 and s_2 are subwords of $\tau(\psi^\infty(a))$ and $|s_2| > K|s_1|$, then s_1 is a subword of s_2.*

This property of $\tau(\psi^\infty(a))$ is called *linear repetition* (see [21]) and is linked to the linear recurrence. We say a sequence $\mathbf{s}$ is *uniformly recurrent* if, for every subword s, there exists an integer t such that every subword of length t of $\mathbf{s}$ contains s. In [27] and [28], a sequence is then considered to be *linearly recurrent for the constant K* if it is uniformly recurrent and the difference between two successive occurrences of any word s_1 is bounded by $K|s_1|$.

If a sequence is linearly recurrent for K, it is also linearly repetitive for K as we show in the sequel. Let n be a positive integer and s_2 a subword of $\tau(\psi^\infty(a))$ of length greater than Kn. Let s_1 be a subword of length n. The difference between two successive occurrences of s_1 is less than Kn. Consequently, s_2 has at least one occurrence of s_1.

Lemma 2.53 ([27, Proposition 19])**.** *Let $\mathbf{s} = \psi^\infty(a)$ where ψ is a primitive growing endomorphism prolongable on a. Then $\mathbf{s}$ is linearly recurrent for the constant $K = Q_\psi R_\psi \|\psi\|$, where R_ψ is the number from Definition 2.44 and Q_ψ is the constant from Lemma 2.45.*

Lemma 2.54. *There exist real numbers $L_1 < L_2$ and $\lambda \in \mathbb{N}_2$ such that, for any r and a_i, the following is satisfied:*

$$L_1\lambda^r < |\tau(\psi^r(a_i))| < L_2\lambda^r.$$

To specify the numbers L_1, L_2, and λ, we use the three lemmas below.

Let M be a square matrix. The *spectrum of M*, denoted by $\operatorname{Spec}(M)$, is the set of its eigenvalues. The *spectral radius of M* is the real number

$$\rho(M) = \max\{|\lambda| \mid \lambda \in \operatorname{Spec}(M)\}.$$

Lemma 2.55 ([27, Lemma 15],[48, Corollary 8.1.33])**.** *Let $M = (m_{i,j})_{i,j\in[n]}$ be a non-negative $(n \times n)$-matrix. If M has a positive eigenvector v, then for all $r \in \mathbb{N}$ and all $i \in [n]$ we have*

$$\left(\frac{\min_{k\in[n]} v_k}{\max_{k\in[n]} v_k}\right)\rho(M)^r \le \sum_{j=1}^{n} m_{ij}^r \le \left(\frac{\max_{k\in[n]} v_k}{\min_{k\in[n]} v_k}\right)\rho(M)^r, \tag{2.14}$$

where $\rho(M)$ is the spectral radius of M.

Definition 2.56. Let $\psi : A^* \to A^*$ be a non-erasing endomorphism. For all $a \in A$, if there exists a pair $(d(a), \lambda(a))$ satisfying

$$\lim_{n\to\infty} \frac{|\psi^n(a)|}{c(a)n^{d(a)}\lambda(a)^n} = 1$$

for some constant $c(a)$, then the pair $(d(a), \lambda(a))$ is the *growth type* of a w.r.t. ψ.

Lemma 2.57 ([16, Proposition 34]). *Let $\psi : A^* \to A^*$ be a morphism prolongable on the letter a and $M(\psi)$ its incidence matrix. If all letters of A occur in $\psi^\infty(a)$, then $\lambda(a) = \rho(M(\psi))$.*

Lemma 2.58 ([27, Lemma 17]). *Let ψ be an endomorphism and $\lambda \in \mathbb{N}_2$ such that all letters of A have growth type $(0, \lambda)$. Then there exists a computable constant P_ψ such that, for all r, the following holds:*

$$\left(\frac{1}{P_\psi}\right)\lambda^r \leq \min_j \sum_i m_{ij}^r \leq \max_j \sum_i m_{ij}^r \leq P_\psi \lambda^r. \tag{2.15}$$

Additionally, we observe that with $M(\psi)^r = (m_{i,j})_{i,j\in A}$

$$|\psi^r(a)| = \sum_{b\in A} |\psi^r(a)|_b = \sum_{b\in A} m_{b,a}.$$

If the endomorphism ψ is primitive, we can determine the values L_1, L_2, and λ with the help of Theorem 2.15, Lemma 2.55, Lemma 2.57, and Lemma 2.58 keeping in mind that τ is a coding.

Let us find an explicit N such that

$$\frac{2L_2K\lambda^d}{L_1} < N \tag{2.16}$$

where L_1, L_2, K, λ, and d are the numbers from the above lemmas.

By Algorithm 9, we can figure out whether a sequence generated by a primitive prolongable endomorphism and a coding is periodic.

If the algorithm yields the periodicity of the sequence, we additionally know the period length which is less than or equal to $\max_{t\in T^{n_0+i}} |t|$, where i is the number which led to the stop of the algorithm. A proof of the correctness of the algorithm was given by Mitrofanov (see [63]). As we can build only a finite number of different schemes with size $< N$, one of the conditions in Algorithm 9, namely either that two subword schemes coincide or that the size of the subword scheme is greater than N, will be true at some step. Therefore, we get a unique decision about the periodicity by this algorithm.

If the sequence is non-periodic, we can conclude that the $TH(K_3^-)$ sequence is not d-automatic for any d. If the sequence is periodic, we obtain that the $TH(K_3^-)$ sequence is d-automatic for all $d \in \mathbb{N}_2$ by another theorem. This theorem is proved using the theory of d-recognisable sets. At first we will give some basic definitions and results. For further reading we refer to [69], [12], and [8].

Algorithm 9 The decision whether the sequence $\tau(\psi^\infty(a))$ is periodic

```
procedure DecPer(τ(ψ^∞(a)), N, d)
    parameter τ(ψ^∞(a)) : sequence
    parameter N : number from (2.16)
    parameter d : number from Lemma 2.51
    Choose an arbitrary n_0
    Build S(T^{n_0+1+d}, T^{n_0+1})
    i ← 2
    break ← false
    while break is false do
        Build S(T^{n_0+i+d}, T^{n_0+i})
        if S(T^{n_0+i+d}, T^{n_0+i}) coincides with any previous scheme then
            τ(ψ^∞(a)) is non-periodic
            break ← true
        else if size of S(T^{n_0+i+d}, T^{n_0+i}) > N then
            τ(ψ^∞(a)) is periodic
            break ← true
        else
            i ← i + 1
        end if
    end while
end procedure
```

Definition 2.59. a) Let A be an alphabet. A subset L of A^* is a *language*.

b) A *deterministic finite automaton* $\mathcal{A}$, called DFA, is defined to be a 5-tuple $\mathcal{A} = (Q, \Sigma, \delta, q_o, E)$, where Q is a finite set of states, Σ is the finite input alphabet, $\delta : Q \times \Sigma \to Q$ is the transition function, $q_0 \in Q$ is the initial state, and $E \subseteq Q$ is the set of accepting states.

c) A *regular language* L is a language accepted by some DFA $\mathcal{A}$, i.e.,

$$L = L(\mathcal{A}) = \{w \in \Sigma^* \mid \delta(q_0, w) \in E\}\,^{14}.$$

d) A set $X \subseteq \mathbb{N}$ of integers is *d-recognisable*, if the language

$$\mathrm{rep}_d(X) = \{\mathrm{rep}_d(n) \mid n \in X\}$$

is regular, where $\mathrm{rep}_d(n)$ is the base-d expansion of n.

Definition 2.60. a) The *characteristic sequence* $\mathbf{s}$ of a set $X \subseteq \mathbb{N}$ is defined by $s_n = [n \in X]$.

b) A set $X \subseteq \mathbb{N}$ is *ultimately periodic* if its characteristic sequence is ultimately periodic.

[14] We extend the domain of δ to $Q \times \Sigma^*$ in this way that we define $\delta(q, \epsilon) = q$ for all $q \in Q$, and $\delta(q, sa) = \delta(\delta(q, s), a)$ for all $q \in Q$, $s \in \Sigma^*$, and $a \in \Sigma$.

A connection between the d-recognisable sets and the d-automatic sequences is given by the subsequent lemma.

Lemma 2.61 ([69, Proposition 1.37]). *Let $d \in \mathbb{N}_2$. A set $X \subseteq \mathbb{N}$ is d-recognisable if and only if its characteristic sequence is d-automatic.*

Example 2.62. A well-known example of a 2-recognisable set is the *Prouhet-Thue-Morse set*

$$\{n \in \mathbb{N} \mid \operatorname{rep}_2(n) \text{ contains an odd number of 1s}\}.$$

Its characteristic sequence is the *Prouhet-Thue-Morse sequence*

$$\mathbf{t} = 0110100110010110\ldots.$$

According to Examples 2.3 and 2.4, the Prouhet-Thue-Morse sequence is 2-automatic. One can find more details concerning this sequence in [1].

We make use of the following proposition to state the theorem about ultimately periodic sequences.

Proposition 2.63 ([12, Proposition 1.5.3]). *Let $d \in \mathbb{N}_2$. Any ultimately periodic set $X \subseteq \mathbb{N}$ is d-recognisable.*

Remark. We can also formulate Cobham's Theorem 2.32 in the field of recognisable sets. Let p and q be multiplicatively independent integers and let $X \subseteq \mathbb{N}$ be both p- and q-recognisable. Then X is ultimately periodic.

With the proposition we can now deduce our theorem.

Theorem 2.64. *Any ultimately periodic sequence is d-automatic for all $d \in \mathbb{N}_2$.*

An alternative proof of this theorem was given in [8, Theorem 5.4.2].
By the decision between periodicity and non-periodicity, we can deduce either the non-automaticity of the sequence $\mathbf{u}$ for any $d \in \mathbb{N}_2$ or the automaticity for all $d \in \mathbb{N}_2$.

2.4 Conclusion and Outlook

In this chapter we disproved the automaticity of the $TH(C_3^+)$ sequence $\mathbf{w}$. We used a theorem of Cobham which says that in an automatic sequence the frequency of a letter is rational, if it exists. By showing that the frequencies of all letters exist and are irrational, we could conclude that, for any d, $\mathbf{w}$ is not d-automatic. This confirms the Allouche-Sapir Conjecture concerning the $TH(C_3^+)$ sequence.

Since we could not ascertain whether the frequency of a letter in the $TH(K_3^-)$ sequence $\mathbf{u}$ is irrational, we chose a new approach where we made use of another theorem of Cobham. It says that, if one has two multiplicatively independent numbers k and l and a sequence is both k- and l-automatic, then the sequence is ultimately periodic. By disproving the ultimate periodicity, we would be able to conclude that the sequence is not automatic. For this purpose, we had so far two proofs on hand which show the decidability of the question on the ultimate periodicity, one of F. Durand and one of I. V. Mitrofanov. First, we proved that $\mathbf{u}$ is the image, under a coding, of an iterative fixed point of a primitive prolongable endomorphism. According to Durand, we can then decide about the ultimate periodicity of this sequence. But it turned out that the necessary calculations are very complex.

For Mitrofanov's proof we needed the set of generating words and the subword schemes of the sequence. We determined this set and a subword scheme. Hereafter, an algorithm for the decision about the periodicity was given. We formulated a sufficient condition for the automaticity of the $TH(K_3^-)$ sequence. But for the final decision about the periodicity a few numbers remained to ascertain. In both approaches, first, one will have to calculate the numbers R_ν from Definition 2.44 and Q_ν from Lemma 2.45 with the aid of Algorithm 8. Depending on the method, one will have then to determine K_ν and K of (2.13) to apply Theorem 2.43 or L_1, L_2, and λ as we saw in this section to find N and proceed with Algorithm 9. Since the calculations are going to be very complex, as it was already predicted for the general case in [26, Chapter 1.3], one will have to make use of the capabilities of high performance computing. But these efforts are worth it, since the proof of ultimate periodicity provides a unique decision whether $\mathbf{u}$ is d-automatic or not.

Chapter 3

The Star Tower of Hanoi: a variant of the Tower of Hanoi with 4 pegs

In the previous chapter we considered variants of the Hanoi graphs which were directed graphs. Now we take a closer look at directed graphs to specify then a special form of them. Recall that a directed graph (or digraph) with vertex set V and arc set A is a graph in which all arcs are directed. In a digraph the in-degree of a vertex $v \in V$ is the number of arcs with v as terminal vertex. Similarly, the out-degree of a vertex $v \in V$ is given by the number of arcs which leave the vertex v. Then we define

Definition 3.1. A *(directed) star* $St(p)$ depending on $p \in \mathbb{N}$ is a directed graph with vertex set $V = [p]_0$ and arc set $A = \{(0, i), (i, 0) : i \in [p-1]\}$.

It is easily seen that every (directed) star has $p-1$ vertices with [in-degree, out-degree] $= [1, 1]$ and one vertex, namely the center, with $[p-1, p-1]$. A (directed) star is, of course, also a simple, undirected graph. This structure can be used to find new variants of the Tower of Hanoi. The number p stands here for the p pegs, where the vertex 0 or peg 0 is always the central peg. One of the simplest examples for these variants is the Linear TH, here called the $TH(St(3))$, on three pegs, which we already know from the previous chapter. The version with four pegs, namely $TH(St(4))$, is of special interest, since it is similar to the unrestricted The Reve's puzzle.

In 1994, Paul K. Stockmeyer introduced in [78] this new variant. According to the underlying directed star $St(4)$ on four pegs, it consists of three pegs, labeled 1, 2, and 3, arranged in an equilateral triangle, and one peg, labeled 0, in the center. The only allowed moves are between the central peg and the external ones. The move from 0 to 1 is called $\bar{a}$ and in the opposite direction a, b goes from 0 to 2 and $\bar{b}$ back and c goes from 0 to 3 and $\bar{c}$ back (see Figure 3.1). We will shortly see why we chose this labelling.

In the book [33] we can find this puzzle as well. Both sources looked at the problem to transfer a perfect tower from an external peg to another external peg. They determined the minimum number of moves for this question among the algorithms which apply the Frame-Stewart-type strategy. The Frame-Stewart-type strategy is based on the two different algorithms to solve the

The Reve's puzzle by B. M. Stewart and J. S. Frame. For the The Reve's puzzle, it turned out that the presumed minimum numbers of moves arising from the respective algorithms are the same for both strategies ([33, Proposition 5.3]). The same was shown for $n \in \mathbb{N}$ and $p \in \mathbb{N}_3$, i.e., for the Multi-Peg Tower of Hanoi, in [53]. Hence we call it the *Frame-Stewart-type strategy*. In the editorial note by O. Dunkel to [76], it was pointed out that the proof that these algorithms are optimal was still lacking. This problem was known in the literature as the Frame-Stewart Conjecture.

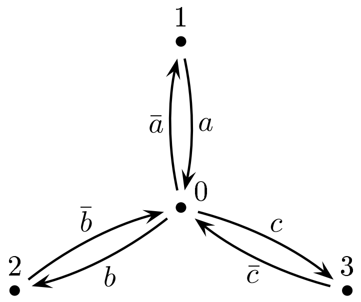

Figure 3.1: The directed star $St(4)$ on four pegs

We come back to our puzzle. To analyse the solution, use was made of the already known optimal algorithms for the Linear TH on three pegs in [33, Chapter 8.3]. We chose the above labelling of the moves in the Star TH in view of the similarity to the Linear TH, for instance if we omit peg 3 and the corresponding moves. The algorithm to solve the Star TH problem with n discs for the case "external peg to another external peg" is described in the sequel. To this end, let $n \in \mathbb{N}$, $m \in [n]$ and $k \in [3]$.

1. Recursively move the smallest $n - m$ discs from the source peg to the non-goal peg k;

2. Avoiding peg k, move the largest m discs from the source peg to the goal peg;

3. Recursively move the smallest $n - m$ discs from peg k to the goal peg.

If $n = 0$, we fix $m = 0$. Obviously, the second part is solved by the same algorithm as the Linear TH, where we want to move the discs from the non-central peg to the other non-central peg. In [33, Chapter 8], this algorithm is called Linear-02(m). Algorithm 10, as well as the later algorithm for the Linear TH, arises from the application of The Sapir Algorithm 7 in Chapter 2 on the digraph of the Linear TH.

We know by [33, Section 8.1] and [78, Section 3] that we need $3^m - 1$ moves to transfer the perfect tower in the Linear TH with Linear-02(m). Hence, we set for the Star TH

$$St_{12}^0 = 0; \quad \forall n \in \mathbb{N} : St_{12}^n = \min\{2St_{12}^{n-m} + 3^m - 1 \mid m \in [n]\} \tag{3.1}$$

Algorithm 10 Linear: from the non-central peg to the other non-central peg

> **procedure** LINEAR-02(n)
>   **parameter** n : number of discs $\{n \in \mathbb{N}_0\}$
>   **if** $n \neq 0$ **then**
>   transfer $n-1$ smallest discs from the non-central source peg to non-central goal peg
>   move disc n from non-central source peg to central peg
>   transfer $n-1$ smallest discs from non-central goal peg to non-central source peg
>   move disc n from central peg to non-central goal peg
>   transfer $n-1$ smallest discs from non-central source peg to non-central goal peg
>   **end if**
> **end procedure**

For abbreviation, we wrote St^n instead of $St^n(4)$. For $n = 1, 2$ we calculate that $m = 1$ is optimal, but for $n = 3, 4, 5$ and 6 we get $m = 2$ and finally $m = 3$ or more for $n \geq 7$. In the next theorem we specify this in general.

Theorem 3.2 ([78, Theorem 2], [33, Theorem 8.8])**.** *The value*

$$m = \left\lfloor \frac{\ln(a_n^{(3)})}{\ln(3)} \right\rfloor + 1,$$

where $a_n^{(3)}$ is n-th element of the 3-smooth sequence[15]*, is the unique value that defines St_{12}^n in (3.1). In the resulting algorithm, for every $i \in [n]$, there is a disc which makes exactly $2a_i^{(3)}$ moves, i.e., altogether*

$$St_{12}^n = 2\sum_{i=1}^{n} a_i^{(3)}$$

for the minimum number of moves using the Frame-Stewart-type strategy.

This method followed the lines of B. M. Stewart. But there is another possible approach. We can look at the halfway situation before the only move of the largest disc as well. This procedure is due to J. S. Frame. If we try to apply Frame's idea, we would get a new algorithm with $n \in \mathbb{N}$, $m \in [n]_0$ and $k \in [3]$:

1. Move the smallest $n - m - 1$ discs from the source peg to the non-goal k;
2. Avoiding peg k, move m discs from the source peg to the goal peg;
3. Move the largest disc from the source peg to the central peg;
4. Avoiding peg k, move m discs from the goal peg to the source peg;

[15] The 3-smooth sequence $\left(a_n^{(3)}\right)_{n\in\mathbb{N}}$ consists of the 3-smooth numbers $2^j \cdot 3^k$, $j, k \in \mathbb{N}_0$ ordered increasingly: $a^{(3)} = (1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27 \ldots)$

5. Move the largest disc from the central peg to the goal peg;

6. Avoiding peg k, move m discs from the source peg to the goal peg;

7. Move the smallest $n - m - 1$ discs from peg k to the goal peg.

For $n = 0$ we set $m = 0$ and use only the third and the fifth step. But this does not really follow Frame as we have two moves of the largest disc. Here, it is not possible to move the largest disc in one move from the source peg to the (non-central) goal peg. Indeed, only Stewart's approach is applicable, as we already stated in Theorem 3.2.

But what happens if it is our aim to transfer the perfect tower with $n \in \mathbb{N}$ discs from the central peg 0 to an external peg or in the opposite direction? First, we use Stewart's approach. Let $n \in \mathbb{N}$, $m \in [n]$ and again $k \in [3]$. Then this algorithm solves the problem $0^n \to 1^n/2^n/3^n$:

1. Recursively move the smallest $n - m$ discs from the central peg 0 to a non-goal peg k;

2. Avoiding peg k, move the m largest discs from central peg 0 to the goal peg;

3. Recursively move the smallest $n - m$ discs from peg k to the goal peg

or in the opposite direction $1^n/2^n/3^n \to 0^n$:

1. Recursively move the smallest $n - m$ discs from the source peg to a non-goal peg k;

2. Avoiding peg k, move the largest m discs from the source peg to the central (goal) peg;

3. Recursively move the smallest $n - m$ discs from peg k to the central (goal) peg.

For the case $n = 0$ we fix $m = 0$.

Algorithm 11 Linear: from source peg to central peg or from central peg to goal peg

```
procedure LINEAR-01(n)
    parameter n : number of discs {n ∈ ℕ_0}
    if n ≠ 0 then
        transfer n − 1 smallest discs from source peg to non-central, non-goal peg
        move disc n from source peg to goal peg
        transfer n − 1 smallest discs from non-central, non-goal peg to goal peg
    end if
end procedure
```

The second part of both is solved by the algorithm Linear-01(m), which needs $\frac{1}{2}(3^m - 1)$ moves, since every disc $d \in [n] \setminus [n - m]$ moves 3^{n-d} times in the solution.

Being interested in the minimum number of moves for the "central peg to external peg" problem, we set

$$St_{01}^{0} = 0; \quad \forall n \in \mathbb{N} : St_{01}^{n} = \min\left\{ St_{01}^{n-m} + \frac{1}{2}\left(3^{m} - 1\right) + \underbrace{St_{12}^{n-m}}_{=2\sum_{i=1}^{n-m} a_i^{(3)}} \mid m \in [n] \right\}. \tag{3.2}$$

Again, we can calculate the first few values of m. We see that

$$\begin{aligned} m &= 1 \quad \text{for} \quad n = 1, 2 \\ m &= 2 \quad \text{for} \quad n = 3, 4, 5 \\ m &= 3 \quad \text{for} \quad n = 5, 6, 7, 8, 9 \\ m &\geq 4 \quad \text{for} \quad n = 10, 11, 12, \ldots. \end{aligned}$$

It is conspicuous that we have two options for $n = 5$. Now we are able to find the sequences of disc moves for the first first few values of n, as it is done in Tables 3.1 and 3.2. It will turn out that for $n = 5, 8$ and 12 we have two different sequences of disc moves whereby the total number of moves are equal for both possibilities. This phenomenon will presumably occur for greater n again and again, since it arises from the recursive structure of the calculation of the total number of moves with Formula (3.2).

We can also summarise the total number of moves for the first values of n in a sequence:

$$St_{01} = (1, 4, 7, 14, 23, 32, 47, 68, 93, 120, 153, 198, \ldots) \tag{3.3}$$

Further elements of this sequence can be found again by Formula (3.2) and the comparison of the total number of moves for different m. Here, too, we can make use of the recursive structure of the calculations in Tables 3.1 and 3.2.

After rewriting the expression St_{01}^{n} the number is equal to St_{10}^{n}, which is the recursion for the minimum number of moves for the "external peg to central peg" problem, since

$$St_{01}^{n} = \min\left\{ St_{01}^{m'} + \frac{1}{2}\left(3^{n-m'} - 1\right) + St_{12}^{m'} \mid m' \in [n]_0 \right\}$$

and

$$St_{10}^{n} = \min\left\{ St_{12}^{m'} + \frac{1}{2}\left(3^{n-m'} - 1\right) + St_{10}^{m'} \mid m' \in [n]_0 \right\}.$$

Now we have to look at Frame's approach and the questions whether it is applicable in this case and if there is a difference to the number of moves resulting from Stewart's way. First, we find the algorithms for the two directions and see that the largest disc moves only once. Corresponding to the above recursions, we call the recursion for the presumed minimum number of moves using Frame $\overline{St_{01}^{n+1}}$ and $\overline{St_{10}^{n+1}}$.

	disc *d*	*nom*		disc *d*	*nom*
n=1,m=1:	1	1			
n=2,m=1:	1	3			
	2	1			
n=3,m=2:	1	3			
	2	3			
	3	1			
n=4,m=2:	1	7			
	2	3			
	3	3			
	4	1			
n=5,m=2:	1	7	m=3:	1	7
	2	9		2	3
	3	3		3	9
	4	3		4	3
	5	1		5	1
n=6,m=3:	1	7			
	2	9			
	3	3			
	4	9			
	5	3			
	6	1			
n=7,m=3:	1	15			
	2	7			
	3	9			
	4	3			
	5	9			
	6	3			
	7	1			
n=8,m=3:	for n-m=5:				
m'=2:	1	15	m'=3:	1	15
	2	21		2	15
	3	7		3	13
	4	9		4	9
	5	3		5	3
	6	9		6	9
	7	3		7	3
	8	1		8	1

	disc *d*	*nom*
n=9,m=3:	1	23
	2	17
	3	15
	4	13
	5	9
	6	3
	7	9
	8	3
	9	1
n=10,m=4:	1	23
	2	17
	3	15
	4	13
	5	9
	6	3
	7	27
	8	9
	9	3
	10	1
n=11,m=4:	1	31
	2	15
	3	21
	4	7
	5	27
	6	9
	7	3
	8	27
	9	9
	10	3
	11	1

Table 3.1: Calculations of the number of moves (*nom*) of each disc for $n = 1, \dots, 11$ for the minimising value m

	disc d	*nom*		disc d	*nom*
n=12,m=4:	for n-m=8:				
m'=2:	1	31	m'=3:	1	31
	2	45		2	39
	3	15		3	21
	4	21		4	21
	5	7		5	7
	6	27		6	27
	7	9		7	9
	8	3		8	3
	9	27		9	27
	10	9		10	9
	11	3		11	3
	12	1		12	1

Table 3.2: Calculations of the number of moves (*nom*) of each disc for $n = 12$ for the minimising value m

Again, let $n \in \mathbb{N}_0$ and $k \in [3]$. If we go from $0^{n+1} \to 1^{n+1}/2^{n+1}/3^{n+1}$, we use the following algorithm with $m' \in [n+1]_0$ and $k' \in [3] \setminus \{k\}$:

1. Move m' discs from the central peg to the non-goal peg k;
2. Avoiding peg k, move $n - m'$ discs from the central peg to the non-goal peg k';
3. Move the largest disc from the central to the goal peg;
4. Avoiding peg k, move $n - m'$ discs from the non-goal peg k' to the goal peg;
5. Move m' discs from peg k to the goal peg

Fo the case that we want to go from $1^{n+1}/2^{n+1}/3^{n+1} \to 0^{n+1}$, we apply this algorithm, again with $m' \in [n+1]_0$ and $k' \in [3] \setminus \{k\}$:

1. Move m' discs from the source peg to the non-goal, non-central peg k;
2. Avoiding peg k, move $n - m'$ discs from the source peg to the non-goal peg k';
3. Move the largest disc from the source peg to the central peg;
4. Avoiding peg k, move $n - m'$ discs from peg k' to the central peg;
5. Move m' discs from peg k to the central peg.

We simplify the formulas of St_{01}^{n+1} and St_{10}^{n+1} to compare them with the numbers $\overline{St_{01}^{n+1}}$ and $\overline{St_{10}^{n+1}}$.

$$St_{01}^{n+1} = \min\left\{St_{01}^{m'} + \frac{1}{2}\left(3^{n-m'+1} - 1\right) + St_{12}^{m'} \mid m' \in [n+1]_0\right\}$$
$$St_{10}^{n+1} = \min\left\{St_{12}^{m'} + \frac{1}{2}\left(3^{n-m'+1} - 1\right) + St_{10}^{m'} \mid m' \in [n+1]_0\right\}.$$

The corresponding numbers in Frame's version are

$$\begin{aligned}\overline{St_{01}^{n+1}} &= \min\left\{\overline{St_{01}^{m'}} + \frac{1}{2}\left(3^{n-m'} - 1\right) + 1 + \left(3^{n-m'} - 1\right) + St_{12}^{m'} \mid m' \in [n+1]_0\right\}\\ &= \min\left\{\overline{St_{01}^{m'}} + \frac{1}{2}\left(3^{n-m'+1} - 1\right) + St_{12}^{m'} \mid m' \in [n+1]_0\right\}\end{aligned}$$

and

$$\begin{aligned}\overline{St_{10}^{n+1}} &= \min\left\{St_{12}^{m'} + \left(3^{n-m'} - 1\right) + 1 + \frac{1}{2}\left(3^{n-m'} - 1\right) + \overline{St_{10}^{m'}} \mid m' \in [n+1]_0\right\}\\ &= \min\left\{St_{12}^{m'} + \frac{1}{2}\left(3^{n-m'+1} - 1\right) + \overline{St_{10}^{m'}} \mid m' \in [n+1]_0\right\}.\end{aligned}$$

By comparing the numbers, we obtain that the respective recursions for Stewart's and Frame's strategy coincide. We conclude that it is enough to look at Stewart's version for all transfers of a perfect tower from the central peg to an arbitrary external peg in the Star TH. Hence we can also use the common term Frame-Stewart-type strategy.

The question whether the Frame-Stewart-type strategy is optimal for the problems to transfer a perfect tower from an external peg to another external peg or from the central peg to an arbitrary external peg is still open. One very helpful step could be the proof the Frame-Stewart Conjecture for four pegs, which was found by T. Bousch [13] in 2014. Stockmeyer [79] has already done exhaustive computer searches on the "external peg to external peg" problem and the "central peg to external peg" problem in the Star TH with four pegs. For the first problem, he got for the total required number of moves

$$2, 6, 12, 20, 32, 48, 66, 90, 122, 158, 206, 260, 324, 396, 492, \text{ and } 600 \text{ for } n = 1, 2, \ldots, 16.$$

These are exactly the values of St_{12}^{n} in Theorem 3.2 for $n = 1, \ldots, 16$. For the second problem, he calculated

$$1, 4, 7, 14, 23, 32, 47, 68, 93, 120, 153, 198, 255, 318, \text{ and } 399 \text{ for } n = 1, 2, \ldots, 15.$$

The beginning of this sequence corresponds to the sequence of (3.3), which we calculated here using the Frame-Stewart-type strategy. With the aid of the *OEIS* (The on-line encyclopedia of integer sequences) and by generating more data, one could probably find a system in the sequence of numbers for which we have two different sequences of disc moves or for which we have two

different possible ms (For instance, the phenomenon of $n = 5$ occurs again at $n = 8, 12, 15$). But the main interest beside these sequences is to search for a regularity in the sequence of ms for the presumed minimal number of moves St^n_{01}, as it was found for the sequence of values m for the presumed minimal number of moves St^n_{12} in the use of the 3-smooth sequence $\left(a^{(3)}_n\right)_{n\in\mathbb{N}}$ (OEIS A003586) by Stockmeyer.
Obviously, this puzzle can be extended to $p \in \mathbb{N}_3$ by considering the directed star $St(p)$. We apply again the Frame-Stewart-type strategy on the "external peg to external peg" problem and get as algorithm for $n \in \mathbb{N}$, $m \in [n]$, and $k \in [p-1]$:

1) Recursively move the smallest $n - m$ discs from the source peg to the non-goal peg k,

2) Avoiding peg k, move the largest m discs from the source peg to the goal peg,

3) Recursively move the smallest $n - m$ discs from peg k to the goal peg.

If $n = 0$, we set $m = 0$. The first and the third step must be solved in the Star TH with p pegs for the "external peg to external peg" problem, whereas the second one in the Star TH with only $(p - 1)$ pegs. Stockmeyer [79] found out that for a very large number of discs for $k \in \mathbb{N}$ there are $\binom{k+p-5}{p-4} = \binom{k+p-5}{k-1}$ discs that each make exactly $2^k \cdot 3^j$ moves for all $j \in \mathbb{N}_0$. Considering the "central peg to external peg" problem, we get as algorithm for $n \in \mathbb{N}$, $m \in [n]$, and $k \in [p-1]$:

1) Recursively move the smallest $n - m$ discs from the central peg to the non-goal peg k,

2) Avoiding peg k, move the largest m discs from the central peg to the goal peg,

3) Recursively move the smallest $n - m$ discs from peg k to the goal peg.

Again, if $n = 0$, we fix $m = 0$. Here we have twice the Star TH with p pegs for the "central peg to external peg" problem or the "external peg to external peg" problem, respectively, and once with $(p - 1)$ pegs for the "central peg to external peg" problem. For a further analysis one could look again at Frame and his approach in [76]. But it would turn out that his induction proof is here not applicable as we do not know enough about the Star TH for the "external peg to external peg" problem yet. The problem for p pegs requires further research. One aim would be to find the minimum number of moves or at least a similar statement for the disc moves as for the "external peg to external peg" problem.

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Eidesstattliche Erklärung

(Siehe Promotionsordnung vom 12.07.11, § 8, Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

Holz auf der Heide, Caroline

Name, Vorname

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Ort, Datum Unterschrift Doktorand/in