Invertible Objects in Motivic Homotopy Theory

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Abstract

If X is a (reasonable) base scheme then there are the categories of interest in stable motivic homotopy theory $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$, constructed by Morel-Voevodsky and others. These should be thought of as generalisations respectively of the stable homotopy category $\mathbf{SH}$ and the derived category of abelian groups $D(Ab)$, which are studied in classical topology, to the "world of smooth schemes over X".

Just like in topology, the categories $\mathbf{SH}(X), \mathbf{DM}(X)$ are symmetric monoidal: there is a bifunctor $(E, F) \mapsto E \otimes F$ satisfying certain properties; in particular there is a *unit* $\mathbb{1}$ satisfying $E \otimes \mathbb{1} \simeq \mathbb{1} \otimes E \simeq E$ for all E. In any symmetric monoidal category $\mathcal{C}$ an object E is called *invertible* if there is an object F such that $E \otimes F \simeq \mathbb{1}$. Modulo set theoretic problems (which do not occur in practice) the isomorphism classes of invertible objects of a symmetric monoidal category $\mathcal{C}$ form an abelian group $Pic(\mathcal{C})$ called the *Picard group of* $\mathcal{C}$.

The aim of this work is to study $Pic(\mathbf{SH}(X)), Pic(\mathbf{DM}(X))$ and relations between these various groups. A complete computation seems out of reach at the moment. We can show that (in good cases) the natural homomorphism $Pic(\mathbf{SH}(X)) \to \prod_{x \in X} Pic(\mathbf{SH}(x))$ coming from pull back to points has as kernel the *locally trivial invertible spectra* $Pic^0(\mathbf{SH}(X))$. (For $\mathbf{DM}$, this homomorphism is injective.) Moreover if $x = Spec(k)$ is a point, then (again in good cases) the homomorphism $Pic(\mathbf{SH}(x)) \to Pic(\mathbf{DM}(x))$ is injective. This reduces (in some sense) the study of $Pic(\mathbf{SH}(X))$ to the study of the Picard groups of $\mathbf{DM}$ over fields, and the latter category is much better understood.

We then show that, for example, the reduced motive of a smooth affine quadric is invertible in $\mathbf{DM}(k)$. By the previous results, it follows that affine quadric bundles over X yield (in good cases) invertible objects in $\mathbf{SH}(X)$. This is related to a conjecture of Po Hu.

Zusammenfassung

Ist X ein (nicht zu exotisches) Basis-Schema dann sind die Kategorien $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$, welche für die stabile motivische Homotopietheorie von interesse sind, von Morel-Voevodsky und anderen konstruiert worden. Man sollte sich diese Kategorien respektive als Verallgemeinerungen der stabilen Homotopiekategorie $\mathbf{SH}$ und der derivierten Kategorie abelscher Gruppen $D(Ab)$ vorstellen, welche in der klassischen Topologie studiert werden.

Genau wie in der Topologie sind die Kategorien $\mathbf{SH}(X), \mathbf{DM}(X)$ symmetrisch monoidal: es gibt einen Bifunktor $(E, F) \mapsto E \otimes F$ der gewisse Bedingungen erfüllt; insbesondere gibt es eine *Einheit* mit der Eigenschaft, dass $E \otimes \mathbb{1} \simeq \mathbb{1} \otimes E \simeq E$ für alle E. In einer beliebigen symmetrisch monoidalen Kategorie heißt ein Objekt E *invertierbar* wenn es ein Objekt F gibt, so dass $E \otimes F \simeq \mathbb{1}$. Unter Vernachlässigung mengentheoretischer Probleme (die in der Praxis nicht auftreten) bilden die Isomorphieklassen der invertierbaren Objekte einer symmetrisch moidalen Kategorie eine abelsche Gruppe $Pic(\mathcal{C})$, genannt die *Picard-Gruppe von* $\mathcal{C}$.

Das Ziel dieser Arbeit ist es, die Gruppen $Pic(\mathbf{SH}(X))$ und $Pic(\mathbf{DM}(X))$ zu studieren. Eine vollständige Berechnung erscheint derzeit nicht durchführbar. Wir können zeigen, dass der Kern des natürlichen Homomorphismus $Pic(\mathbf{SH}(X)) \to \prod_{x \in X} Pic(\mathbf{SH}(x))$ (in guten Fällen) genau aus den *lokal trivialen invertierbaren Spektra* besteht. (Im Falle von $\mathbf{DM}$ ist dieser Homomorphismus injektiv.) Darüber hinaus zeigen wir, dass wenn $x = Spec(k)$, dann ist der Homomorphismus $Pic(\mathbf{SH}(x)) \to Pic(\mathbf{DM}(x))$ (wiederum in guten Fällen) injektiv. Das reduziert (in gewissem Sinne) das Studium von $Pic(\mathbf{SH}(X))$ auf das studium der Picard-Gruppen von $\mathbf{DM}$ über Körpern, und die letztere Kategorie ist sehr viel besser verstanden.

Wir zeigen dann, zum Beispiel, dass das reduzierte Motiv einer glatten, affinen Quadrik in $\mathbf{DM}(k)$ invertierbar ist. Es folgt aus den vorherigen Ergebnissen, dass Bündel von affinen Quadriken über X (in guten Fällen) Beispiele von invertierbare Objekte in $\mathbf{SH}(X)$ liefern. Dies hängt zusammen mit einer Vermutung von Po Hu.

Contents

Chapter 1

Introduction

In Section 1.1, we set the stage: we introduce symmetric monoidal categories and their Picard groups, the stable homotopy category **SH**, the motivic stable homotopy category $\mathbf{SH}(X)$ and the triangulated categories of motives. In Section 1.2 we explain previous Picard group computations. This is for two reasons: (1) it illustrates the kind of results we hope to establish, and (2) we will actually take fairly concrete inspiration from the methods outlined there. The impatient reader may wish to jump straight into Section 1.3 where we provide an overview of the main results and organisation of this thesis.

1.1 Background

1.1.1 Stable Homotopy Theory

The object of algebraic topology is to study (often geometrical) problems by assigning algebraic invariants to topological spaces. These invariants have to be functorial, so are preserved under homeomorphisms. However one usually finds that the algebraic invariants one writes down are preserved under a much weaker notion of equivalence, called *homotopy equivalence*. When thinking about spaces up to homotopy equivalence, we talk about the "homotopy theory of spaces". In fact just as spaces with their (continuous) maps form a category, so do spaces with homotopy classes of maps as morphisms. If X and Y are spaces we write $[X, Y]$ for the morphisms in this category. If X and Y are sufficiently nice, these are just the continuous maps $X \to Y$, up to homotopy. Write $X \simeq Y$ if X and Y are isomorphic in this category. (Then we say that X and Y are *weakly (homotopy) equivalent*.)

It is a long-standing tradition to describe sufficiently nice spaces by simplicial complexes. The more modern notion is that of *simplicial sets*. The category *sSet* of simplicial sets is just the functor category $Fun(\Delta^{op}, Set)$ where Δ is the simplex category, which is the category of finite ordered sets with order preserving maps. As suggested by the analogy with spaces, simplicial sets also have a homotopy theory, and in fact the homotopy theory of simplicial sets is equivalent to the homotopy theory of topological spaces. The algebraically or combinatorially minded person wishing to study the homotopy theory of spaces may thus content herself with studying simplicial sets. This result was first established in full generality by Quillen. For the purpose of this introduction we shall not make a big distinction between topological spaces and simplicial sets and refer to either of them as just *spaces*.

In order to rightfully call our subject *algebraic* topology, we have to find algebraic invariants of spaces. There are some very classical examples: given a pointed space X, an abelian group A and an integer n, we have the *(singular) cohomology group* $\tilde{H}^n(X, A)$. These groups of course satisfy many well-known properties. We want to recall the *suspension isomorphism* (see for example [40, Exercise 2.3.3]). Thus let X be a pointed space and write $\Sigma X = X \times I/(X \times 0 \cup * \times I) \simeq X \times I/(X \times 0)$ for the reduced suspension, where I denotes the unit interval (or the simplicial set Δ^1). Then there is a natural homomorphism $\tilde{H}^n(X, A) \to \tilde{H}^{n+1}(\Sigma X, A)$ which is always an isomorphism.

We can do something similar with homotopy classes of morphisms of spaces. Given a pointed space X, put $\pi_n(X) = [S^n, X]$. Here we mean pointed homotopy classes of maps, and S^n denotes the n-sphere. (It turns out that for $n > 0$ the set $\pi_n(X)$ is actually a group, abelian for $n > 1$, but we do not need this.) There is then a natural suspension map $\pi_n(X) \to \pi_{n+1}(\Sigma X)$ (essentially because Σ is a functor and $\Sigma S^n \simeq S^{n+1}$) but it need not in general be an isomorphism! There is some hope, however. The Freudenthal suspension theorem (see e.g. [40, Corollary 4.24]) says that the groups

$$\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X) \to \dots$$

eventually stabilise, at least if X is a finite CW complex.

This suggests that, in order to simplify the homotopy theory, it would be useful to pass to a category where $\mathrm{Hom}(X, Y) = \mathrm{colim}_n[\Sigma^n X, \Sigma^n Y]$. Doing this naively will not yield a category with good properties. It is however possible to do this in a non-naive way and construct a very good category, denoted **SH** and called the *stable homotopy category*. It comes with a functor $\Sigma^\infty : Spc_* \to \mathbf{SH}$ which has the property that if X, Y are sufficiently nice pointed spaces (e.g. finite CW complexes), then $[\Sigma^\infty X, \Sigma^\infty Y] = \mathrm{colim}_n[\Sigma^n X, \Sigma^n Y]$. The category **SH** is the main object of study in the subject of stable homotopy theory.

Just as the homotopy category of spaces can be obtained from several "point-set level" categories of spaces, the stable homotopy category **SH** can be obtained from several categories of "point-set level objects" called *spectra* by passing to an appropriate equivalence relation on maps, also called weak equivalence.

As we have alluded to above the category **SH** has many good properties. For example it is triangulated and has arbitrary products and coproducts, and it is compactly generated.

1.1.2 Symmetric Monoidal Categories and *Pic*

A sufficiently good category of unpointed spaces is cartesian closed: finite products exist, and moreover for a space X the functor "product with X" has a right adjoint which is often denoted $Y \mapsto Y^X$. This means that for spaces X, Y, Z there is a natural isomorphism $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, Z^Y)$. We will mostly write $\underline{\mathrm{Hom}}(Y, Z)$ for Z^Y.

A similar structure exists on the homotopy category of spaces.

When considering pointed spaces, their cartesian product is again pointed. It is however often more natural to study a different kind of "product", the *smash product*. Given pointed spaces X, Y we put $X \wedge Y := X \times Y/(* \times Y \cup X \times *)$. For example, $S^1 \wedge X = \Sigma X$. The bifunctor $X, Y \mapsto X \wedge Y$ is no longer determined purely formally by the category of pointed spaces itself but rather constitutes extra data which we choose to keep track of. The categorical formalisation of such a situation is called a *(symmetric) monoidal* or *tensor* category (see for example [22, Section 1]).

Essentially, a monoidal category $\mathcal{C}$ consists of a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ written $X, Y \mapsto X \otimes Y$ and certain extra data, satisfying certain compatibility conditions. In particular, there are a *unit object* $\mathbb{1} \in \mathcal{C}$ and natural isomorphisms $\mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1}$. Moreover, given $X, Y, Z \in \mathcal{C}$ there exist natural *associativity isomorphisms* $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$. The most important upshot of the additional compatibilities is that given any finite set $X_1, X_2, \dots, X_n$ of objects of $\mathcal{C}$, we can make unambiguous sense of the tensor product $X_1 \otimes X_2 \otimes \dots \otimes X_n$. That is to say for any two ways of building the n-fold tensor product out of binary tensor products by introducing brackets, any two isomorphisms between these two ways of bracketing obtained by repeatedly applying the associativity isomorphism are the same.

A monoidal category is called closed if the functor $\bullet \otimes Y$ always has a right adjoint. We denote this adjoint by $\underline{\mathrm{Hom}}(Y, \bullet)$. The monoidal category is called symmetric if we are supplied with for each $X, Y \in \mathcal{C}$ a natural isomorphism $X \otimes Y \cong Y \otimes X$, satisfying certain further compatibilities. A functor between (symmetric) monoidal categories is called (symmetric) monoidal if it preserves all the extra data, in an appropriate way.

Any (closed) category with finite products is (closed) symmetric monoidal, with monoidal operation given by the categorical product and the unit given by the final object. The associativity and commutativity isomorphisms come from the universal property. In particular a good category of spaces is closed symmetric monoidal, and so is the homotopy category of spaces. A good category of pointed spaces is closed symmetric monoidal with monoidal operation the smash product $\wedge$ and unit the two-point space, i.e. S^0. The same is true for the homotopy category of pointed spaces.

Suppose now that $\mathcal{C}$ is a symmetric monoidal category and we are given $T \in \mathcal{C}$. One might wish to pass to a category $\mathcal{C}'$ where $\mathrm{Hom}_{\mathcal{C}'}(X,Y) = \mathrm{colim}_n \mathrm{Hom}_{\mathcal{C}}(X \otimes T^{\otimes n}, Y \otimes T^{\otimes n})$. For example this is how we naively tried to construct the stable homotopy category. It is easy to see that this defines a category. One may next ask if, or to what extent, the monoidal operation defines extra structure on $\mathcal{C}'$. It turns out that as long as the cyclic permutation $T \otimes T \otimes T \cong T \otimes T \otimes T$ is the identity, the category $\mathcal{C}'$ is actually symmetric monoidal again, and the functor $\mathcal{C} \to \mathcal{C}'$ is also symmetric monoidal. This may have first been observed by Voevodsky.

This suggests that the category **SH** should be symmetric monoidal, and since we said that we employ a complicated non-naive construction in order for **SH** to have good formal properties, it should be in fact closed symmetric monoidal. This is indeed true, and good categories of spectra are also closed symmetric monoidal. This is a hard result for **SH** probably first written up by Adams [1]. This is an even harder result for categories of spectra, see [29] for a treatment in the language of topological spaces and [49] for a treatment in terms of simplicial sets.

Let us now go back to an arbitrary symmetric monoidal category $\mathcal{C}$. An object $E \in \mathcal{C}$ is called *invertible* if there exists an object $F \in \mathcal{C}$ such that $E \otimes F \cong \mathbb{1}$. If the class of invertible objects forms a set, then the set of equivalence classes of invertible objects actually forms an abelian group. This is clearly an invariant of $\mathcal{C}$, called the *Picard group* of $\mathcal{C}$ and denoted $Pic(\mathcal{C})$. It seems plausible that whenever we wish to seriously study a symmetric monoidal category $\mathcal{C}$, we should also study $Pic(\mathcal{C})$. Examples of why this is useful will be given in the next section. Let us point out that if $X, Y \in \mathcal{C}$ and $L \in \mathcal{C}$ is invertible, then $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(X \otimes L, Y \otimes L)$ is an isomorphism, as one readily sees by tensoring with the inverse of L.

For now, let us investigate the Picard groups of the symmetric monoidal categories we have mentioned so far. It is very easy to check that the Picard groups of the point-set level categories of spaces, pointed or not, are trivial, so we concentrate on homotopy categories. If $X \times Y$ is terminal in some category $\mathcal{C}$, then X has to be terminal. It follows that the Picard group of any cartesian monoidal category (i.e. a symmetric monoidal category in which the monoidal operation coincides with the categorical product) is trivial. So the Picard group of the homotopy category of spaces is trivial. One may also check that the Picard group of the homotopy category of pointed spaces is trivial, but that is not so formal.

The story changes with **SH**. Essentially by design, in **SH** we have $[E,F] = [E \wedge S^1, F \wedge S^1]$. The easiest way to explain this phenomenon would be if there was some object $S^{-1} \in \mathbf{SH}$ with $S^1 \wedge S^{-1} \simeq \mathbb{1}$. In fact this is essentially how the stable homotopy category is constructed. We have thus found invertible elements S^n for each $n \in \mathbb{Z}$ and have in fact constructed a homomorphism $\mathbb{Z} \to Pic(\mathbf{SH})$. This homomorphism is injective. Indeed, it is well known that in the homotopy category of pointed spaces, $[S^n, S^m] = *$ for $m > n$. It follows that the same is true in **SH** (because the colimit of a sequence of one-point sets is the one-point set). Moreover one may prove that $[S^n, S^n] = \mathbb{Z}$ (for $n > 0$) and so the same holds in **SH** (for all n). Consequently S^m and S^n define distinct objects of $Pic(\mathbf{SH})$ if $m \neq n$.

One may prove that $Pic(\mathbf{SH}) = \mathbb{Z}$ is precisely the group we have identified: every invertible object is weakly equivalent to S^n for some (unique) $n \in \mathbb{Z}$. The most efficient way to do this is probably to observe that there is a functor $C_* : \mathbf{SH} \to D(Ab)$, where $D(Ab)$ denotes the derived category of abelian groups and C_* is essentially the singular chain complex. One checks that the functor C_* is symmetric monoidal and induces an injection on Picard groups (we say that C_* is Pic-injective). The category $D(Ab)$ is much more amenable to computation and the result $Pic(D(Ab)) = \mathbb{Z}$ is easily obtained (see also next section). Thus $Pic(\mathbf{SH}) = \mathbb{Z}$ as well.

1.1.3 Motives and Motivic Homotopy Theory

It is a deep insight of 20th century mathematics that a convenient way of doing *geometry* is by considering spaces with extra structure, and maps between these spaces which preserves this structure. Examples that come to mind are smooth manifolds with smooth maps between them, and complex analytic manifolds with holomorphic maps between them. Arguably the most restrictive possible class of spaces and maps are the algebraic varieties with (locally) polynomial maps between them, and they are the subject of study in the field of *algebraic geometry.* In fact, as soon as we decide to work with things basically built from polynomial equations, we can dispense with the underlying topological space and just consider the subject as a flavour of pure algebra: we need not just consider the algebraic varieties over $\mathbb{C}$, but we can in fact consider all schemes.

Of course, the business of classifying spaces with extra structure is typically even harder than classifying spaces without extra structure, so classifying algebraic varieties up to isomorphism should be expected to be very hard, and this is indeed so. Following the picture outlined earlier, it seems like a reasonable idea to study algebraic varieties up to algebraic homotopy. We thus say that given a morphism of varieties $\phi : \mathbb{A}^1 \times V \to W$, where $\mathbb{A}^1$ is the affine line, the two restrictions $\phi|_{0\times V} : V \to V$ and $\phi|_{1\times V} : V \to W$ are *naively* $\mathbb{A}^1$*-homotopic.*

In a way, trying to pass from the category of algebraic varieties to the homotopy category of algebraic varieties poses similar problems to passing from the homotopy category of spaces to $\mathbf{SH}$. There is a naive way of doing it, but the category produced in that way is not quite right, does not have good formal properties, and computing anything in it is hopeless.

Still, there is a way of building a reasonable homotopy theory of algebraic varieties [83]. One starts with a base scheme S and considers the category $Sm(S)$ of schemes *smooth over* S. Then, through a highly difficult process recalled in chapter 2, one may build a homotopy category $Ho(\mathcal{H}(S))$ with a functor $Sm(S) \to Ho(\mathcal{H}(S))$ which is universal in a precise sense. It is called the (unstable, unpointed) motivic homotopy category. It certainly inverts the naive $\mathbb{A}^1$-homotopy equivalences, but in general it does rather more.

Keeping in line with the story from algebraic topology, we should next stabilise the pointed homotopy category of varieties by inverting an appropriate "sphere". It turns out that a good choice of "sphere" is the projective line (Riemann sphere) $\mathbb{P}^1$. We shall not try to explain the significance of this choice, but let us point out that the *real projective line* $\mathbb{P}^1(\mathbb{R})$ is homeomorphic to S^1, and the *complex projective line* $\mathbb{P}^1(\mathbb{C})$ is homeomorphic to S^2, both of which are spheres. The resulting category is denoted $\mathbf{SH}(S)$ and called the *stable motivic homotopy category.* This is a triangulated symmetric monoidal category, so we can ask about its Picard group $Pic(\mathbf{SH}(S))$. This is the main object of investigation of this thesis.

As we have seen in the previous section, the study of spaces (and, by extension, $\mathbf{SH}$) can be facilitated by considering their singular cohomology groups H^* and, by extension, the functor $C_* : \mathbf{SH} \to D(Ab)$. It would be desirable to have a replacement for H^*, C_* and $D(Ab)$ in our motivic setting. Arguably the most classical replacement of H^* are the motivic cohomology groups $H^{*,*}$, see e.g. [75, Lecture 3]. In this case the analogue for $D(Ab)$ is a triangulated symmetric monoidal category $\mathbf{DM}(S)$, called the category of motives (with transfers). If S is the spectrum of a perfect field then (a version of) $\mathbf{DM}(S)$ was first constructed and studied by Voevodsky [107]. If S is more complicated it is very hard to make sense of $\mathbf{DM}(S)$, although nowadays in many cases we have a satisfactory definition.

There is a different approach. The category $D(Ab)$ is built in essentially the same way as $\mathbf{SH}$, but replacing at the very beginning the category of sets by the category of abelian groups: we start with the category of simplicial abelian groups, which is equivalent to the category chain complexes in non-negative degrees. We turn weak equivalences, which correspond to quasi-isomorphisms, into isomorphisms, and we invert the analogue of S^1, which is just $\mathbb{Z}[1]$, i.e. we pass to complexes in all degrees. This suggests a way building our analogue of $D(Ab)$: the construction of $\mathbf{SH}(S)$ begins with the category $sPre(Sm(S))$ of simplicial presheaves on $Sm(S)$ and then performs various formal manipulations. Thus one might start with the category of presheaves of simplicial abelian groups on $Sm(S)$ and perform similar manipulations. One arrives at a category $D_{\mathbb{A}^1}(S)$ which is defined for all S. See for example [80, Section 5.2]. This category has many good formal properties, but unfortunately it is (so far) much less well understood than $\mathbf{DM}(S)$. For this reason, in this thesis, we will focus on studying $\mathbf{DM}$ instead of studying $D_{\mathbb{A}^1}(S)$, at least to the extent that we actually want to study $\mathbf{SH}(S)$ and wish to simplify the problem. Our results and techniques however make it very clear that a better understanding of $D_{\mathbb{A}^1}(S)$ would be highly desirable for studying $Pic(\mathbf{SH}(S))$.

The goal of this thesis can now be stated rather clearly: to investigate in any way possible the groups $Pic(\mathbf{SH}(X))$, presumably by also studying the groups $Pic(\mathbf{DM}(X))$, and the relationships among these groups.

1.2 Prior and Analogous Work

In the first two subsections we describe computations of Picard groups of certain symmetric monoidal categories which have been influential in some our own computations. In the last subsection we describe prior work on (conjecturally) invertible motives which has been a starting point in our attempts at construct invertible objects.

1.2.1 Picard Groups of Derived Categories

Here is one of the first non-trivial Picard group calculations one might attempt: Let X be a scheme. What is $Pic(D(X))$? Here we write $D(X)$ for the unbounded derived category of quasi-coherent sheaves on X. Write $C(X)$ for the abelian group of continuous functions on X with values in the discrete abelian group $\mathbb{Z}$. Then Fausk proves that there is a split short exact sequence [30]

$$0 \to Pic(X) \to Pic(D(X)) \to C(X) \to 0.$$

Here $Pic(X)$ just denotes the group of line bundles on X, i.e. the invertible objects in the category of quasi-coherent sheaves on X (which is the heart of $D(X)$). Note that for reasonable X, we just have $C(X) = \mathbb{Z}^{\pi_0(X)}$, where $\pi_0(X)$ denotes the set of connected components of X. So what the theorem says is that an invertible object of $D(X)$ is specified by giving a line bundle on X and an integer for every connected component of X. If X is connected, that means that every element of $Pic(D(X))$ is (uniquely) of the form $\mathcal{L}[i]$ for some (unique) $\mathcal{L} \in Pic(X)$ and $i \in \mathbb{Z}$. In general, an invertible object is of this form on every connected component of X.

With some amount of hindsight, we can explain the result as follows. For $x \in X$ any point, write $i_x : \{x\} \to X$ for the inclusion. Then we get a pull-back functor $i_x^* : D(X) \to D(x)$. Of course $x = Spec(k)$ for some field k, and thus $D(x) = D(k)$ is a very well-understood category (it is just the category of graded k-vector spaces). Let us write $D^c(X)$ for the subcategory of compact objects. In reasonable situations (X quasi-compact quasi-separated), the unit is compact, and so all invertible objects are. In fact in reasonable situations the compact objects are precisely those complexes which are locally on X quasi-equivalent to a bounded complex of vector bundles [85, Theorem 63]. In particular such objects have only finitely many non-vanishing cohomology sheaves.

Suppose now that $E \in D(X)$ has $H^iE = 0$ for $i > 0$, and $x \in X$. Then by considering a projective resolution of $E|_{Spec(\mathcal{O}_{X,x})}$ one easily finds that $H^i i_x^* E = 0$ for $i > 0$ and $H^0 i_x^* E = i_x^* H^0 E$. In particular if $H^0 E$ is coherent (e.g. E a perfect complex), then $H^0 E = 0$ if and only if $H^0 i_x^* E = 0$ for all closed points x (use Nakayama's lemma).

Now on $D^c(X)$ there is a duality functor $E \mapsto DE := \underline{\mathrm{Hom}}(E, \mathcal{O})$. It has the property that if $H^iE = 0$ for $i > 0$ then $H^i DE = 0$ for $i < 0$. (Indeed locally on X such E admits a resolution by vector bundles in degrees ≤ 0, and so DE is locally quasi-isomorphic to a complex of vector bundles in degrees ≥ 0.) We have thus almost proved the following result.

Proposition 1.1. *Let $E \in D^c(X)$ and assume that there are $m, n \in \mathbb{Z}$ such that for every $x \in X$ we have $H^i i_x^* E = 0$ for $i \notin [m, n]$. Then $H^i E = 0$ for $i \notin [m, n]$.*

The converse also holds.

Proof. We know that $H^iE = 0$ for $i \notin [m_0, n_0]$ for some m_0, n_0, by quasi-compactness of X and perfectness of E. The reasoning above allows to conclude that $H^iE = 0$ for $i > n$. Now we can apply the same reasoning to prove that $H^iDE = 0$ for $i > -m_0$, and hence $H^iE = H^iDDE = 0$ for $i < m_0$.

The converse statement is proved by observing that $i_x^*DE = Di_x^*E$. □

Consequently we find that the sequence

$$0 \to Pic(X) \to Pic(D(X)) \xrightarrow{\psi} C^\delta(X)$$

is exact, where $C^\delta(X)$ denotes the set of all $\mathbb{Z}$-valued functions on X, and $\psi(E)$ is defined as follows: for $x \in X$, we know that $i_x^*E \in D(k(x))$ must be invertible, so $i_x^*E \simeq k[n]$ for some unique n, and we put $\psi(E)(x) = n$.

To obtain the rest of Fausk's result it suffices to show that $\psi(E)$ is continuous (because we have already seen that $Pic(D(X)) \to C(X)$ is surjective). This is not so relevant for our work, but here is a proof. Note that in the proof of the proposition, we only needed to consider closed points. So we know that $Pic(D(X)) \to C^\delta(X^{cl})$ is $Pic(X)$, where X^{cl} denotes the set of closed points. If X is local then $Pic(X) = 0$ and so we conclude that $Pic(D(X)) = \mathbb{Z}$. If X is arbitrary again and $x, y \in X$ with $y \in \overline{\{x\}}$, then $y \in Spec(\mathcal{O}_{X,x})$. But $E|_{Spec(\mathcal{O}_{X,x})} \simeq \mathcal{O}_{X,x}[n]$ for some n by what we just said, and so $\psi(E)(x) = \psi(E)(y)$. It follows that $\psi(E)$ is locally constant and thus in particular continuous.

Let us point out in particular that this proves the claim from the last section, that $Pic(D(Ab)) = \mathbb{Z}$. (Assuming that we know that $Pic(Ab) = 0$, which follows for example from the structure theory of finitely generated abelian groups.)

1.2.2 Picard Groups in Equivariant Stable Homotopy Theory

We now consider a more topological example, that of stable homotopy theory equivariant with respect to some group G. This is developed in quite some level of generality, but for simplicity we shall assume that G is finite. A reference is [31]. There is then a category of pointed G-spaces. A map between such spaces is called a (equivariant) weak equivalence if it induces a weak equivalence on all the spaces of H-fixed points, where $H \subset G$ is a subgroup. Turning equivariant weak equivalences into isomorphisms produces the pointed equivariant homotopy category. This category comes with a smash product. It contains the ordinary sphere S^1, with the trivial action, and the first reflex in building a stable equivariant homotopy category would be to invert S^1 with respect to the smash product. It turns out that this is not quite the right thing to do. For example, this naive construction does not give a satisfactory theory of duality. (These reasons are very similar to why in stable motivic homotopy theory we need to invert not just S^1, but $\mathbb{P}^1$.) If V is a (real) representation of G, then the one-point compactification S^V is a space with the property that for every subgroup $H \subset G$, the fixed points $(S^V)^H$ are an ordinary sphere. This makes it quite plausible that all the "spheres" S^V should be inverted. It turns out that this is the same as inverting the one-point compactification of the regular representation. The resulting category is called the (genuine) equivariant stable homotopy category and denoted $\mathbf{SH}(G)$. Given an un-pointed G-space X, we write $\Sigma^\infty X_+ \in \mathbf{SH}(G)$ for the stabilisation of the pointed space X_+, consisting of X and a disjoint base point.

$\mathbf{SH}(G)$ is a symmetric monoidal triangulated category. It is generated by the objects $\Sigma^\infty(G/H)_+$, where $H \subset G$ is a subgroup and G/H denotes the finite G-space of orbits of H in G. It turns out that there is a symmetric monoidal triangulated functor $\Phi^G : \mathbf{SH}(G) \to \mathbf{SH}$ called the *geometric fixed points*, with the property that $\Phi^G(\Sigma^\infty(G/H)_+) = 0$ for $H \neq G$ and $\Phi^G(\Sigma^\infty(G/G)_+) \simeq S^0$. In fact, for any pointed G-space X, one has $\Phi^G(\Sigma^\infty X) \simeq \Sigma^\infty X^G$, explaining the reference to fixed points in the name.

If $H \subset G$ is a subgroup, there is an obvious functor $\mathbf{SH}(G) \to \mathbf{SH}(H)$ coming from restricting the group actions. The composite $\mathbf{SH}(G) \to \mathbf{SH}(H) \xrightarrow{\Phi^H} \mathbf{SH}$ is also denoted Φ^H.

Now if $E \in Pic(\mathbf{SH}(G))$ and $H \subset G$ is a subgroup, then $\Phi^H(E) \in \mathbf{SH}$ must be invertible (since Φ^H is a monoidal functor), so $\Phi^H(E) \simeq S[n]$ for some (unique) $n \in \mathbb{Z}$. Write $C(G)$ for the set of functions from subgroups of G to $\mathbb{Z}$. Then we have produced a homomorphism $\psi : Pic(\mathbf{SH}(G)) \to C(G)$, where $\psi(E)(H)$ is defined by $\Phi^H(E) \simeq S[\psi(E)(H)]$. We have deliberately set this up analogously to the previous subsection. The game now is to identify the kernel of ψ.

There is actually a fairly obvious source of elements in the kernel. Indeed let $\mathcal{C}$ be any additive, symmetric monoidal category which is Karoubi-closed, meaning that whenever $X \in \mathcal{C}$ and $e \in End(X)$ with $e^2 = e$, then there is a corresponding decomposition $X \cong X_1 \oplus X_2$ such that e is the composite $X \to X_1 \to X$. Now write $A = End(\mathbb{1})$. This is a commutative ring. If $\mathcal{P}(A)$ is the category of finitely generated projective A-modules, i.e. summands of free A-modules of finite rank, then there is an embedding $\mathcal{P}(A) \hookrightarrow \mathcal{C}$. It is obtained by writing an element $P \in \mathcal{P}(A)$ as a summand of A^n and then taking the corresponding summand of $\mathbb{1}^{\oplus n}$ in $\mathcal{C}$. In particular, there is always an injection $Pic(A) \hookrightarrow Pic(\mathcal{C})$.

The category $\mathbf{SH}(G)$ is Karoubi-closed (this is one of the "good properties" we alluded to earlier that would follow from a good construction of stable homotopy categories), and in fact the above construction identifies all of the elements in the kernel of ψ:

Theorem 1.2 (Fausk-Lewis-May, [31])**.** *There is an exact sequence*

$$0 \to Pic(End(\mathbb{1}_{\mathbf{SH}(G)})) \to Pic(\mathbf{SH}(G)) \xrightarrow{\psi} C(G).$$

There are a number of obvious follow-up questions. Surely $End(\mathbb{1})$ should be determined; after that figuring out $Pic(End(\mathbb{1}))$ could perhaps be left to the algebraists. This has been done, but the specific computations are not very relevant to our work. Another question is to determine the image of ψ. Recall that in the previous subsection, we first built a homomorphism into the group $C^\delta(X)$ of all functions on X, and then later argued that the image actually consists of the continuous functions.

Something similar can be done here. Firstly if H_1, H_2 are conjugate subgroups of G, then $\Phi^{H_1}(E) \simeq \Phi^{H_2}(E)$, essentially because for a G-space X one has $X^{H_1} = gX^{H_2}$ whenever $H_1 = gH_2g^{-1}$. Thus the image of ψ consist of functions which are constant on conjugacy classes. If G is a non-trivial topological group, then the image of ψ will consist of continuous functions in an appropriate sense, but this condition is vacuous for G finite. Unfortunately, these conditions together still do not identify the image of ψ. For finite groups G, the image of ψ can be specified by complicated congruence conditions known as Borel-Smith conditions [10].

1.2.3 Hu's Conjectures on Pfister Quadrics

We now come to a somewhat different topic. If we want to study Picard groups in motivic homotopy theory, what could be possible examples of invertible objects? Of course in any triangulated symmetric monoidal category we have $\mathbb{1}[1]$, and also basically by design we have $\mathbb{P}^1$. As explained in the previous subsection, we also always see $Pic(End(\mathbb{1}))$. One might guess that there are no other examples of invertible objects, but this is false in general. Thinking back to classical stable homotopy theory, the invertible objects come from the spheres, i.e. the solution sets of equations of the form

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

This suggests looking at affine varieties defined by quadratic equations like this. Such varieties are known as (affine) quadrics. These are defined by equations like

$$a_1 x_1^2 + \cdots + a_n x_n^2 = b.$$

(Here $a_i \neq 0$ for all i, and $b \neq 0$.)

Over $\mathbb{R}$, every a_1 can be chosen to be ± 1, and it turns out that the homotopy type only depends on the number of plus coefficients (and we may assume without loss of generality that $b = 1$, of course). In motivic homotopy theory over general fields, we have no obvious reason to expect such collapsing of homotopy types, so we deal with general quadrics.

The "best" Quadrics are known as Pfister quadrics. They are denoted

$$\langle\langle a_1, \ldots, a_n \rangle\rangle = \otimes_{i=1}^n (x^2 - a_i y^2).$$

This is a quadric in 2^n variables. For example $\langle\langle a_1 \rangle\rangle = x^2 - a_1 y^2$ and $\langle\langle a_1, a_2 \rangle\rangle = x^2 - a_1 y^2 - a_2 z^2 + a_1 a_2 w^2$. Write $U^b_{\langle\langle a_1,\ldots,a_n\rangle\rangle}$ for the affine variety defined by

$$\langle\langle a_1, \ldots, a_n \rangle\rangle = b.$$

Note that the first term in any Pfister quadric is x^2 (i.e. coefficient 1), so if $b = 1$ then $U^1_{\langle\langle a_1,\ldots,a_n\rangle\rangle}$ has a canonical point, and we can consider the suspension spectrum $\Sigma^\infty U^1_{\langle\langle a_1,\ldots,a_n\rangle\rangle}$. For general b there is no base point, but we can consider the reduced suspension $\tilde{U}^b_{\langle\langle a_1,\ldots,a_n\rangle\rangle}$. This is a pointed space and so we can consider its suspension spectrum.

By a very sophisticated analogy with equivariant homotopy theory, Po Hu has arrived at the following conjecture.

Conjecture 1.3 (Po Hu [53]). *For $a_1, \ldots, a_n, b \in k^\times$ ($n \geq 0$) there is an isomorphism in $\mathbf{SH}(k)$*

$$\Sigma^\infty U^1_{\langle\langle a_1,\ldots,a_n,b\rangle\rangle} \wedge \Sigma^\infty \tilde{U}^b_{\langle\langle a_1,\ldots,a_n\rangle\rangle} \simeq \Sigma^\infty U^1_{\langle\langle a_1,\ldots,a_n\rangle\rangle} \wedge (\mathbb{P}^1)^{\wedge 2^n}.$$

For $n = 0$ this conjecture directly implies that the terms on the left hand side are both invertible, and then for $n > 0$ it inductively implies that all of the (reduced) spectra of Pfister quadrics are invertible. In fact the cases $n = 0, 1$ are dealt with in loc. cit., so these provide examples of "exotic" invertible objects! (One may check that they are not in the subgroup we described so far, as also done in loc. cit.)

1.3 Overview of the Main Results

The first three subsections provide a tour of the main results of this thesis, arranged in an order that the author believes is helpful for understanding them in a larger context. This is not the order of the most efficient proofs, and as such not the order in which the results appear in this thesis. The actual structure of the thesis is described in the last (fourth) subsection.

1.3.1 Invertible Motives over a Field

As we have seen in the previous section, computing Picard groups can be seriously difficult. As such we should try to start with the simplest case. Certainly homology is easier than homotopy, so we should investigate **DM** before investigating **SH**. Next spectra of fields (i.e. points) are clearly simpler than more general base schemes, so we shall concentrate on $\mathbf{DM}(k, A)$ for some ring of coefficients A. In fact we are mostly interested in the case where A is $\mathbb{Z}$ or a ring closely related to it, like $\mathbb{Q}, \mathbb{Z}[1/n], \mathbb{Z}/p$.

Recall that $M(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{1}\{1\}$, defining the second summand. Here $\mathbb{1}\{1\}$ is an invertible object known as the Lefschetz motive. The notation $\mathbb{1}\{1\} = \mathbb{1}(1)[2]$ is often used. We write $M\{n\} = M \otimes \mathbb{1}\{1\}^{\otimes n}$ and similarly for $M(n)$.

Conditional Computation. At least with rational coefficients and assuming the standard conjectures, over a field of characteristic zero, it turns out to be possible to give a complete calculation:

Theorem 1.4 (See Corollary 5.72.).

1. *Let k be a field and l/k a quadratic (Galois) extension. Then there is a splitting $M_\mathbb{Q}(Spec(l)) \cong \mathbb{1} \oplus \tilde{M}_\mathbb{Q}(l) \in \mathbf{DM}(k, \mathbb{Q})$ (defining the second summand). Here $\tilde{M}(l)$ is not isomorphic to $\mathbb{1}$, but $\tilde{M}(l)^{\otimes 2} \simeq \mathbb{1}$.*

2. *Now let k be a field of characteristic zero and assume that the standard conjectures, the Beilinson conjectures, and the Hodge conjecture hold (over $\mathbb{C}$). Then every element of $Pic(\mathbf{DM}(k, \mathbb{Q}))$ is either of the form $\mathbb{1}[m]\{n\}$ or of the form $\tilde{M}(l)[m]\{n\}$, for unique $m, n \in \mathbb{Z}$ and a unique quadratic Galois extension l/k. In other words*

$$Pic(\mathbf{DM}(k, \mathbb{Q})) = \mathrm{Hom}_{cts}(Gal(k), \mathbb{Z}/2) \oplus \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}/2[k^\times/2] \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

It is probably possible to prove a similar result for fields in positive characteristic, replacing the Hodge conjecture by the Tate conjecture, but we are more interested in proving unconditional results, and are looking at conjectural results only for inspiration. It would be much more interesting to get a conjectural computation with integral coefficients, but the author does not know how to do this.

Unable to compute the whole of $Pic(\mathbf{DM}(k))$ unconditionally, the author then attempted to tackle easier sub-problems. The general idea is this: fix some set S of smooth projective varieties over k and consider the subcategory of $\mathbf{DM}(k)$ generated by the motives of varieties in S. Of course this depends on the notion of "generated", but it should be rather clear what we have in mind: the smallest subcategory of $\mathbf{DM}(k)$ containing the motives of all the varieties in S which is closed under tensor product, sums, summands, triangles, and isomorphisms.

It turns out that in favourable cases (i.e. for "good" choices of S), the strategy from the previous section can actually be repeated. That is to say we manage to find a collection of functors $\Phi^l : \langle S \rangle \to \mathcal{C}$, where each Φ^l is a triangulated, symmetric monoidal functor, the category $\mathcal{C}$ is easy to understand (in particular we can compute its Picard group), and the action of Φ^l on the motives of S is also reasonably explicit. The l here is some essentially arbitrary index, but usually we index on finitely generated field extensions of k. Now of course not just any family $\{\Phi^l\}_l$ will do; we need to also be able to identify the kernel of $Pic(\langle S \rangle) \to \prod_l Pic(\mathcal{C})$.

In analogy with the situation in equivariant stable homotopy theory, we call the functors Φ^l *generalised geometric fixed point functors*.

The way we cook up such functors is using *weight structures*. This is an algebraic notion with a definition that looks very much like t-structures, but behaves very differently in some regards. Weight structures were invented and applied to motives by Bondarko [13]. They may have been known under a different name to algebraists earlier than that [88].

Artin Motives. Instead of going into the details of weight structures (for this see Section 5.2), we shall present some theorems that can be proved with this strategy. The easiest case is perhaps the *Artin Motives*. The category $\mathbf{DAM}(k)$ is obtained as $\langle S \rangle$, where S is the set of smooth zero-dimensional schemes, i.e. the spectra of finite separable field extensions of k.

The category $\mathbf{DAM}(k, \mathbb{Z})$ cannot directly be studied using generalised fixed point functors. However, if k is a p-special field, i.e. all finite separable extensions of k have degree a power of p, then $\mathbf{DAM}(k, \mathbb{Z}/p)$ *can* be studied using fixed points functors. In fact the functor $\Phi^k : \mathbf{DAM}(k, \mathbb{Z}/p) \to D(\mathbb{Z}/p)$ has the property that $\Phi^k(M(Spec(k))) = \mathbb{Z}/p[0]$, and $\Phi^k(M(Spec(l))) = 0$ for non-trivial field extensions l/k. If l/k is a finite (separable) extension then Φ^l is obtained as the composite with base change, i.e. $\mathbf{DAM}(k, \mathbb{Z}/p) \to \mathbf{DAM}(l, \mathbb{Z}/p) \to D(\mathbb{Z}/p)$. Note the very close analogy with equivariant homotopy theory, both in the action of Φ^k on the generators, and in the definition of Φ^l for l/k an extension! Now if we go back to a general base field k and integral coefficients, then for each prime p we can choose a p-special extension k_p/k and then for every finite separable extension l/k_p we get a functor $\Phi^l : \mathbf{DAM}(k, \mathbb{Z}) \to \mathbf{DAM}(k, \mathbb{Z}/p) \to \mathbf{DAM}(k_p, \mathbb{Z}/p) \xrightarrow{\Phi^l} D(Ab)$. If $E \in Pic(\mathbf{DAM}(k, \mathbb{Z}))$ then $\Phi^l(E) = \mathbb{Z}/p[\phi(E)(l)]$, defining a function $\phi(E) \in C(k)$, where $C(k)$ is an appropriate product of copies of $\mathbb{Z}$.

Theorem 1.5 (See Proposition 5.45 and Corollary 5.46.)**.** *The homomorphism*

$$Pic(\mathbf{DAM}(k, \mathbb{Z})) \to C(k)$$

is injective. In particular $Pic(\mathbf{DAM}(k, \mathbb{Z}))$ *is a torsion-free group.*

It is important here to use integral coefficients, otherwise the Picard group acquires torsion. For a typical example, let l/k be a quadratic (Galois) extension. We can consider the reduced motive $\tilde{M}(Spec(l))$, fitting into the distinguished triangle $\tilde{M}(Spec(l)) \to M(Spec(l)) \to M(Spec(k))$. One may prove that $\tilde{M}(Spec(l))$ is invertible and that $\phi(\tilde{M}(Spec(l)))(l_2) \neq 0$. Thus $\tilde{M}(Spec(l))$ is of infinite order. But already the image of $\tilde{M}(Spec(l))$ in $\mathbf{DAM}(k, \mathbb{Z}[1/2])$ is 2-torsion.

The next step would be to investigate the image of ϕ in $C(k)$. For general S (assuming our method works for S), this seems even more hopeless than in the case of equivariant stable homotopy theory. Even for Artin motives, the author does not know a complete answer. We prove in subsection 5.4.3 that the functions $\phi(E)$ satisfy certain congruence conditions quite similar to the Borel-Smith conditions. Write $C(k)^{BS}$ for the subgroup of $C(k)$ consisting of functions satisfying these conditions.

In subsection 5.4.4 we investigate a functor from equivariant stable homotopy theory to derived Artin motives. It was first defined by Po Hu in [52]. We show show that a cotorsion subgroup $JO(k) \subset C(k)^{BS}$ is contained in the image of ϕ, and actually has been studied before (in a different guise). The author does not know if $JO(k) = im(\phi)$ but contends that this is not the case.

Artin-Tate Motives. As a next step, we investigate Artin-Tate motives. Thus we add $\mathbb{1}\{\pm 1\}$ to our set S. Write $\mathbf{DATM}(k)$ for this category. We can give a reasonably satisfying computation of $Pic(\mathbf{DATM}(k))$, allowing for the fact that $\mathbf{DAM}(k)$ tends to be complicated. Again weight structures play a crucial role in the proof.

Theorem 1.6 (See Theorem 5.57.). *Every object in $Pic(\mathbf{DATM}(k))$ is (uniquely) of the form $E\{n\}$, for some $E \in Pic(\mathbf{DAM}(k))$, $n \in \mathbb{Z}$.*

Quadrics. We now investigate a somewhat different class of motives, by taking S to be the set of smooth projective quadrics. Write $\mathbf{DQM}(k, A)$ for the resulting category. We note that $\mathbb{1}\{1\} \in \mathbf{DQM}(k, A)$ because split quadrics are Tate, and that $\mathbf{DQM}(k, A)$ also contains the motives of smooth *affine* quadrics, by considering the Gysin triangle. Again we cannot construct fixed point functors on $\mathbf{DQM}(k, \mathbb{Z})$, but there *are* helpful functors on $\mathbf{DQM}(k, \mathbb{Z}/2)$. Write $Tate(\mathbb{Z}/2)$ for the category of finite-dimensional, graded $\mathbb{Z}/2$-vector spaces, with graded homomorphisms. Then there is a symmetric monoidal triangulated functor $\Phi^k : \mathbf{DQM}(k, \mathbb{Z}/2) \to D^b(Tate(\mathbb{Z}/2))$. If X is the motive of a projective quadric (or a product thereof), then one may write $X \cong T \oplus X'$, where T is a Tate motive (a sum of $\mathbb{1}\{n\}$ terms) and X' affords no Tate summands (other than zero). The functor Φ^k has the property that $\Phi^k(X) = T$ (note that $Tate(\mathbb{Z}/2)$ is equivalent to the category of Tate motives with $\mathbb{Z}/2$ coefficients). Thus Φ^k "detects Tate summands".

We may again define more general functors Φ^l by base change. There is a further functor $\Psi : \mathbf{DQM}(k, \mathbb{Z}) \to D(Tate(\mathbb{Z}))$ which is obtained essentially by geometric base change (note that all motives of quadrics are geometrically Tate). It turns out that these functors together again induce an injection on Picard groups:

Theorem 1.7 (See Theorem 5.31.). *The family of functors $\{\Phi^l\}_l \cup \{\Psi\}$ is conservative on compact objects. The induced homomorphism*

$$Pic(\mathbf{DQM}(k, \mathbb{Z})) \to Pic(D(Tate(\mathbb{Z}))) \times \prod_l Pic(D(Tate(\mathbb{Z}/2)))$$

is injective.

Note that $Pic(D(Tate(A))) = \mathbb{Z} \oplus \mathbb{Z}$ generated by $A\{1\}$ and $A[1]$, for any PID A (say). In particular $Pic(\mathbf{DQM}(k, \mathbb{Z}))$ is again torsion-free. As before, as soon as we consider $\mathbf{DQM}(k, A)$ with $1/2 \in A$ this story changes and in fact we just get $Pic(\mathbf{DQM}(k, \mathbb{Z}[1/2])) = \mathbb{Z}/2[k^\times/2] \oplus \mathbb{Z} \oplus \mathbb{Z}$, i.e. the same result as we expect rationally.

The family of functors being conservative means that we can detect invertible objects. Since they are also very computable, this opens to door to proving Po Hu's conjecture (or at least its analogue for $\mathbf{DM}$). In fact we obtain the following.

Theorem 1.8 (See Theorems 5.34 and 5.38.). *The Hu-Conjecture 1.3 holds in $\mathbf{DM}(k)$.*

Additionally, if X is the motive of a smooth affine quadric, then $\tilde{M}(X) \in \mathbf{DM}(k)$ is invertible.

(Here $\tilde{M}(X)$ denotes the reduced motive of X, i.e. the homotopy fibre of the structural morphism $MX \to M(Spec(k))$.) We recall that the Hu-conjecture implies the invertibility of affine Pfister quadrics, so the second part of the theorem is a considerable generalisation of that statement.

Applicability of the Method. We have seen a number of cases where we used weight structures to cook up "fixed point functors" with good properties. It seems reasonable to ask if this can one day be extended to cover all of **DM**. The answer to that is probably "no". The reason is that in order for a hypothetical functor $\Phi : \mathbf{DM}(k) \to D(Tate)$ detecting Tate motives to be symmetric monoidal (which is surely necessary to study Picard groups) we need that if X, Y are motives and X does not afford Tate summands, then neither does $X \otimes Y$. Put differently, we need the Tate objects to form a tensor ideal. This is essentially impossible unless S consists of varieties the motives of which are geometrically Tate (consider Hodge structures).

Are all invertibles Artin-Tate? The result from Theorem 1.6 that all elements of $Pic(\mathbf{DATM}(k))$ are twists of invertible Artin motives might lead one to guess that in fact all elements of $Pic(\mathbf{DM}(k))$ are twists of Artin motives. This is also compatible with the conditional computation from Theorem 1.4.

Unfortunately this is false. The reduced motives of affine quadrics are counterexamples, in general. Indeed if Q is a positive-dimensional affine quadric then the only Artin-Tate motive which MQ could be isomorphic to is $\mathbb{Z}$, as one may check by employing the functor $L\pi_0$ from [5]. But in general $\tilde{M}Q \neq \mathbb{Z}$ as one sees from the Pic-injectivity theorem above.

1.3.2 Invertible Motivic Spectra over General Bases

Having got a feel for the problem by considering concrete and accessible cases, let us ask the most general question possible: what is $Pic(\mathbf{SH}(X))$ for any (sufficiently nice) scheme X? Since we can barely control the simplest possible situations in the previous subsection, this question seems hopeless. A rather better question is then: how can we relate the groups $Pic(\mathbf{SH}(X))$ for various X?

The first guess might be that the assignment $X \mapsto Pic(\mathbf{SH}(X))$ is a sheaf (in the Nisnevich topology, say). This is true (but not obvious) for $X \mapsto Pic(\mathbf{DM}(X))$ (see the proof of Theorem 3.23). But it is not true for $X \mapsto \mathbf{SH}(X)$. This is not very surprising. The assignment $X \mapsto Pic(X)$ is also not a sheaf, since indeed all invertible sheaves are line bundles, and so locally trivial! Of course we know that $Pic(X) = H^1(X, \mathcal{O}^\times)$, so again there is a complete classification. We can also deal with $X \mapsto Pic(D(X))$. Write F for the sheaf associated to the presheaf $X \mapsto Pic(D(X))$ (in the Zariski topology). By our computation in Subsection 1.2.1, this is just the constant sheaf $\underline{\mathbb{Z}}$. There is of course a homomorphism $Pic(D(X)) \to F(X)$ and Fausk's computation implies that its kernel is $Pic(X)$. We have thus found a two-step filtration on $Pic(D(X))$ with subquotients $F(X) = H^0(X, Pic(D(\bullet)))$ and $Pic(X) = H^1(X, \mathcal{O}^\times)$.

Descent Spectral Sequences. This suggests that perhaps we can find a filtration of $Pic(\mathbf{SH}(X))$ with subquotients we can understand. Now in appropriate circles "filtrations" immediately means "spectral sequences", and the best way to get spectral sequences is from filtering a homotopy type. This idea can be implemented, but it brings in a whole lot of additional technical sophistication.

We view $\mathbf{SH}(\bullet)$ as a kind of presheaf of model categories on the site $Ft(k)$ of schemes of finite type over a fixed field k, say. Of course there are lots of things wrong with the above statement, but for the purpose of this introduction it is good enough.

Now for any monoidal model category $\mathcal{M}$, there is a space $PIC(\mathcal{M})$ which is obtained by taking the nerve of the category $P \subset \mathcal{M}$, where the objects of P are those whose image in $Ho(\mathcal{M})$ is invertible, and where the morphisms are the weak equivalences. By design, $\pi_0(PIC(\mathcal{M})) = Pic(Ho(\mathcal{M}))$. Also by design $PIC(\mathcal{M})$ is an H-group, so in particular all connected components are equivalent. It follows from classical results of Dwyer-Kan that $\pi_1(PIC(\mathcal{M})) = [\mathbb{1}, \mathbb{1}]^\times$ and $\pi_{i+1}(PIC(\mathcal{M})) = [\mathbb{1}[i], \mathbb{1}]$ for $i > 0$ [26].

The idea now is that the assignment $X \mapsto \mathbf{SH}(X)$ should be a "sheaf" (of model categories) in an appropriate sense, and that as a consequence of this the assignment $X \mapsto PIC(\mathbf{SH}(X))$ is a "sheaf" (of spaces) in an appropriate sense. One way of making sense of these statements is using the theory of ∞-categories. In this language, the implication "$\mathcal{F}$ a sheaf of monoidal (higher) categories $\Rightarrow PIC(\mathcal{F})$ a sheaf of spaces" is literally true [72, Proposition 2.2.3]. Moreover the fact that $X \mapsto \mathbf{SH}(X)$ is a sheaf is also known in this language, see e.g. [51, Proposition 4.8].

In this thesis we use the language of model categories instead of ∞-categories, and in this language setting up a way to prove that $X \mapsto PIC(\mathbf{SH}(X))$ is a sheaf is quite some work. The upshot, one way or another, is that the assignment yields a simplicial presheaf which is fibrant in the local model structure (more precisely, globally weakly equivalent to its fibrant replacement, i.e. satisfying descent). One thus obtains a spectral sequence

$$E_2^{pq} = H^p(X, \underline{\pi}_{-q}PIC(\mathbf{SH}(?))) \mapsto \pi_{-p-q}PIC(\mathbf{SH}(X)).$$

This gives the desired filtration on $PIC(\mathbf{SH}(X))$. A similar spectral sequence exists for $X \mapsto \mathbf{DM}(X)$ or $X \mapsto D(X)$. In the former case we have $\pi_i(PIC(\mathbf{DM}(X))) = 0$ for $i > 1$ and $\pi_1(PIC(\mathbf{DM}(X))) = \mathbb{Z}^\times = \mathbb{Z}/2$. Since constant sheaves have vanishing higher Nisnevich topology, the spectral sequence collapses at the E_2 page and the filtration is actually a one-step filtration, i.e. $X \mapsto Pic(\mathbf{DM}(X))$ is a sheaf.

For $X \mapsto D(X)$ one has $\pi_0(PIC(D(X))) = Pic(D(X))$, and the sheaf associated with that is $\underline{\mathbb{Z}}$. We also have $\pi_1(PIC(D(X))) = \mathcal{O}(X)^\times$, and $\pi_i(PIC(D(X))) = 0$ for $i > 1$. Thus we get back the motivating two-step filtration.

Locally Trivial Objects. The first step of the filtration on $Pic(\mathbf{SH}(X))$ comes from the homomorphism $Pic(\mathbf{SH}(X)) \to (a_{Nis}Pic)(X)$, where $a_{Nis}Pic$ denotes the Nisnevich sheaf associated with the presheaf $X \mapsto Pic(\mathbf{SH}(X))$. In other words, the kernel of this homomorphism is precisely the group $Pic^0(\mathbf{SH}(X))$ of Nisnevich-locally trivial invertible spectra.

One may show that if $E \in \mathbf{SH}(X)$ is such that there is a Nisnevich covering $f : U \to X$ and $f^*E \simeq \mathbb{1}$, then E is invertible. Thus $Pic^0(\mathbf{SH}(X))$ is just the set of equivalence classes of locally trivial spectra. Studying locally trivial objects of various kinds has a long-standing tradition in topology. Let us write $\mathcal{H}(X)$ for the unstable motivic homotopy category and $\mathcal{H}_*(X)$ for the unstable pointed motivic homotopy category. Write $LOC(\mathcal{H}_*, (\mathbb{P}^1)^{\wedge n})(X)$ for the space of objects in $\mathcal{H}_*(X)$ which are Nisnevich locally weakly equivalent to $(\mathbb{P}^1)^{\wedge n}$. Given such a space $T \in \mathcal{H}_*(X)$, the desuspended suspension spectrum $\Sigma^\infty(T) \wedge (\mathbb{P}^1)^{\wedge -n}$ is locally equivalent to $\mathbb{1}$, i.e. defines an element of $Pic^0(\mathbf{SH}(X))$, by our previous remarks. This defines (more or less) a morphism $LOC(\mathcal{H}_*, (\mathbb{P}^1)^{\wedge n}) \to PIC^0(\mathbf{SH}(X))$. This way we can relate the stable problem of finding locally trivial invertible spectra to the unstable problem of finding spaces locally weakly equivalent to our spheres $(\mathbb{P}^1)^{\wedge n}$.

We can make the relationship precise: smashing with $\mathbb{P}^1$ defines a map $LOC(\mathcal{H}_*, (\mathbb{P}^1)^{\wedge n})(X) \to LOC(\mathcal{H}_*, (\mathbb{P}^1)^{\wedge n+1})(X)$ and one may prove that this induced a map

$$\operatorname{colim}_n LOC(\mathcal{H}_*, (\mathbb{P}^1)^{\wedge n})(X) \to PIC^0(\mathbf{SH}(X))$$

which is in fact a weak equivalence, see Proposition 2.33.

The assignment $X \mapsto LOC(\mathcal{H}_*, T)(X)$ is a simplicial presheaf, which is in fact fibrant in the local model structure, and in good cases even $\mathbb{A}^1$-local. It follows in particular that the assignment $X \mapsto \pi_0(LOC(\mathcal{H}_*, T)(X))$, i.e. the functor of equivalence classes of locally trivial objects over X, is representable in $\mathcal{H}_*(k)$! This representability problem has been studied before, if in a slightly different guise, by Wendt [114, Section 5]. He shows that the representing space (i.e. $LOC(\mathcal{H}_*, T)$) is weakly equivalent in the local model structure to $B\underline{Aut}^h(T)$, the monoid of homotopy self-equivalences of T. We reprove Wendt's result by completely different means, see Corollary 2.14.

1.3.3 Bridging the Gap

We have studied (to some extent) the two extremes in the spectrum of possible problems about Picard groups in motivic homotopy theory, i.e. both $Pic(\mathbf{SH}(X))$ for general X, and $Pic(\mathbf{DM}(k))$ for fields k. It seems reasonable to ask how we can related the two problems. There are two reductions: from general bases to fields, and from $\mathbf{SH}$ to $\mathbf{DM}$.

From General Bases to Fields When studying $Pic(D(X))$, we have seen that the natural inclusions of points $i_x : \{x\} \to X$ yield functors $i_x^* : Pic(D(X)) \to Pic(D(x))$, and that the kernel consists precisely of those objects which are locally on X trivial (equivalent to $\mathcal{O}[0]$). When studying $\mathbf{SH}(X)$, the inclusion i_x still induces $i_x^* : \mathbf{SH}(X) \to \mathbf{SH}(x)$. It turns out that in favourable cases, we can prove a result completely analogous to the situation with $D(X)$.

Theorem 1.9 (See Theorem 3.22.)**.** *Let k be a field of characteristic zero containing $\sqrt{-1}$ and let X/k be a smooth variety. The kernel of the homomorphism*

$$Pic(\mathbf{SH}(X)) \to \prod_{x \in X} Pic(\mathbf{SH}(x))$$

consists precisely of the Nisnevich-locally trivial invertible spectra, i.e. the subgroup $Pic^0(\mathbf{SH}(X))$.

Some comments are in order. The product is not only over closed points, but over all points. This is because the result is proved using a technique called recollement (or gluing). The assumption of characteristic zero is necessary because we use resolution of singularities. The author believes that it should be possible to remove it in an appropriate sense, but this is likely quite hard. The assumption that k contains $\sqrt{-1}$ is a technical one that can likely be dispensed with more easily. (See also the comments after the proof of Theorem 3.22.)

Let us point out that in general, $Pic^0(\mathbf{SH}(X))$ is non-trivial. There is a homomorphism $Pic(X) \to Pic^0(\mathbf{SH}(X))$, coming from the invertibility of Thom spaces of vector bundles over X. There is also a homomorphism $Pic^0(\mathbf{SH}(X)) \to H^1(X, \underline{GW}^\times)$ which one can get from the spectral sequence. Here $\underline{GW}$ is the sheaf of unramified Grothendieck-Witt theory, which is closely related to endomorphisms of the sphere spectrum. The composite $Pic(X) = H^1(X, \mathcal{O}^\times) \to H^1(X, \underline{GW}^\times)$ is just the homomorphism induced by functoriality from $\mathcal{O}^\times \to \underline{GW}, a \mapsto \langle a \rangle$. It thus suffices to exhibit a space X and an element of $Pic(X)$ which is not killed by this homomorphism.

We take $X = \mathbb{P}^1$. The Mayer-Vietoris long exact sequence for the cover $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$ together with homotopy invariance of $\underline{GW}^\times, \mathcal{O}^\times$ yields identifications $H^1(\mathbb{P}^1, \mathcal{O}^\times) = \mathcal{O}^\times(\mathbb{G}_m)/\mathcal{O}^\times(*) = \mathbb{Z}$ and $H^1(\mathbb{P}^1, \underline{GW}^\times) = \underline{GW}(\mathbb{G}_m)/\underline{GW}(*)$. Since $\underline{GW}$ is unramified we get an injection $H^1(\mathbb{P}^1, \underline{GW}^\times) \hookrightarrow \underline{GW}(k(X))/\underline{GW}(k)$. Using the interpretation of $\underline{GW}(K)$ for fields as the Grothendieck ring of isomorphism classes of bilinear spaces over K, we finally find that $H^1(\mathbb{P}^1, \mathcal{O}^\times) \to H^1(\mathbb{P}^1, \underline{GW}^\times)$ is not the zero map (the tautological element on the left is not killed).

From $\mathbf{SH}(k)$ to $\mathbf{DM}(k)$. The second step is to reduce from homotopy to homology. Actually this is essentially always possible, in an easily controlled way, when using the "right" notion of homology. In our situation this means working with $D_{\mathbb{A}^1}(k)$. Indeed one may fairly easily prove: the functor $M' : \mathbf{SH}(k) \to D_{\mathbb{A}^1}(k)$ is conservative on connective objects, and induces an injection on Picard groups. This is completely analogous to the situation with $C_* : \mathbf{SH} \to D(Ab)$ in ordinary stable homotopy theory.

However, we do not want to use $D_{\mathbb{A}^1}(k)$ but $\mathbf{DM}(k)$. This makes the situation much harder. We manage to prove the following result.

Theorem 1.10 (See Theorems 4.20 and 4.18.)**.** *Let k be a perfect field of finite 2-étale cohomological dimension and exponential characteristic e. The functor $M : \mathbf{SH}(k)_e \to \mathbf{DM}(k, \mathbb{Z}[1/e])$ is conservative on compact objects, and induces an injection on Picard groups.*

Here $\mathbf{SH}(k)_e$ denotes the category of e-local objects, i.e. those $E \in \mathbf{SH}(k)$ such that $E \xrightarrow{e} E$ is an isomorphism. Recall that if k is of characteristic zero then $e = 1$, whereas if k is of characteristic $p > 0$ then $e = p$. Contrary to the previous theorem, the assumption of finite 2-étale cohomological dimension is essential in this one.

As a corollary, we find that the Hu-conjecture 1.3 is true for fields as in the theorem.

1.3.4 Organisation of this Thesis

As mentioned before, the structure of this work does not follow the path described in the previous subsections. Let us be more specific. Principally, the thesis is split into a main body and an appendix.

The main purpose of the appendix is to develop in the language of model categories the background to give meaning to the statement "$X \mapsto \mathbf{SH}(X)$ is a sheaf of model categories" and to prove "hence $X \mapsto PIC(\mathbf{SH}(X))$ is a fibrant simplicial presheaf (in the Nisnevich topology)". This is quite a lot of work, but we judge it to be not very interesting (and thus relegate it to an appendix), since it is essentially known already in the language of ∞-categories.

Since all the background material is in appendix A, in chapter 2 we can jump right in. We introduce formally the categories $\mathbf{SH}(X)$ and their model categories, and prove that we get a sheaf of model categories (in the sense of the appendix). Hence we get essentially for free the descent spectral sequence for classifying spaces of invertible (or locally trivial) objects. We also study the relationship between stable and unstable classifying spaces of locally trivial objects, and between pointed and unpointed locally trivial objects. Finally we show that objects which are locally trivial in the cdh topology are actually already locally trivial in the Nisnevich topology. While this result may seems slightly esoteric, we make good use of it in the next chapter.

Chapter 3 mostly leaves the world of model categories. We study invertible objects in $\mathbf{SH}(X)$ at the level of the triangulated category. Recall that the assignment $X \mapsto \mathbf{SH}(X)$ satisfies the "six functors formalism" [18, Introduction]. In particular, if $j : U \subset X$ is an open subscheme with reduced, closed complement $i : Z \subset X$, then there exist so called gluing triangles. One speaks of a *recollement* or *gluing* of the triangulated categories $\mathbf{SH}(X), \mathbf{SH}(U)$ and $\mathbf{SH}(Z)$. This is of course classical. It is well-known that in such a situation, given $E_1 \in \mathbf{SH}(U)$ and $E_2 \in \mathbf{SH}(Z)$ the objects $E \in \mathbf{SH}(X)$ with $i^*E \simeq E_2$ and $j^*E \simeq E_1$ can be classified. We want to study the kernel of $Pic(\mathbf{SH}(X)) \to Pic(\mathbf{SH}(U)) \times Pic(\mathbf{SH}(Z))$. This does not follow immediately from the classification alluded to above, because $i^*E \simeq \mathbb{1}$ and $j^*E \simeq \mathbb{1}$ does *not* imply that E is invertible! (For example consider $i_*\mathbb{1} \oplus j_!\mathbb{1}$.) We prove that the kernel of $Pic(\mathbf{SH}(X)) \to Pic(\mathbf{SH}(U)) \times Pic(\mathbf{SH}(Z))$ can nonetheless be classified in a very similar manner.

In the rest of the chapter, we exploit this result on gluing of symmetric monoidal categories and their Picard groups to prove that pointwise trivial invertible spectra are locally trivial, and that pointwise trivial invertible motives are in fact trivial. The proof uses resolution of singularities in an essential way, and so only works in characteristic zero.

Chapter 4 is dedicated to what we call the *Motivic Hurewicz Theorem*. This has to do with the functor $M : \mathbf{SH}(k) \to \mathbf{DM}(k)$. More specifically, each of the categories $\mathbf{SH}(k), \mathbf{DM}(k)$ admits a t-structure. Let us write $\underline{\pi}_i(E)$ for the homotopy objects of $E \in \mathbf{SH}(k)$, and $\underline{h}_i(F)$ for the homotopy objects of $F \in \mathbf{DM}(k)$. We write $\mathbf{SH}(k)^{\heartsuit}, \mathbf{DM}(k)^{\heartsuit}$ for the hearts, so $\underline{\pi}_i(E) \in \mathbf{SH}(k)^{\heartsuit}$ and $\underline{h}_i(F) \in \mathbf{DM}(k)^{\heartsuit}$. The motivic Hurewicz theorem says that if $\underline{\pi}_i(E) = 0$ for $i < 0$, then $\underline{h}_i(ME) = 0$ for $i < 0$ and that $\underline{h}_0(E)$ can be determined solely from knowing $\underline{\pi}_0(E)$ (in a specific way which we do not explain here). The principal application is to prove that if k is a field of finite 2-étale cohomological dimension, then the functor M is conservative on compact objects, and induces an injection on Picard groups. This is proved by using Levine's work on Voevodsky's slice filtration.

Finally in chapter 5 we go back to doing concrete computations. We introduce in detail the construction of the category $\mathbf{DM}(k, A)$ and prove some results about the behaviour of the Picard group under changing the coefficients (A) or the base field (k). We then introduce Bondarko's weight structures and prove our general abstract fixed point functors theorem. Most of the rest of the chapter consists of applications of this theorem combined with the general remarks about change of coefficients and base. We prove the invertibility of affine quadrics and establish the "$\mathbf{DM}(k)$ version" of Hu's conjecture. Then we study Artin and Artin-Tate motives, and finally Tate spectra (i.e. the triangulated, symmetric monoidal, thick subcategory of $\mathbf{SH}(k)$ generated by S^1 and $\Sigma^\infty(\mathbb{P}^1)$). In the last section we explain the conditional computation of $Pic(\mathbf{DM}(k, \mathbb{Q}))$ assuming the standard conjectures.

More details can be found in the chapter introductions.

1.4 Notations and Conventions

Smallness Issues. Recall that a (locally small) category $\mathcal{C}$ is defined as a *class* $Ob(\mathcal{C})$ of objects together with for each $X, Y \in Ob(\mathcal{C})$ a *set* $\mathrm{Hom}(X, Y)$, and composition operations, satisfying certain conditions. The category $\mathcal{C}$ is called *small* if $Ob(\mathcal{C})$ is a *set* (i.e. not a proper class).

In this case it is possible to consider the *nerve* $N(\mathcal{C})$ (see Section A.1.4). This is a simplicial *set*. An issue arises when we wish to talk about nerves of categories that are not small, and perhaps not even essentially small. In this case $N(\mathcal{C})$ would be a "simplicial class", and it is not an object of the category of simplicial sets.

We elect to *ignore* this problem. The reason is that there is a standard method of fixing this problem, and we feel that employing it systematically would mostly serve to obscure notation. The most correct way of treating the problem is by considering *universes*. This is exemplified in [94, Chapter 5]. Essentially whenever we perform any constructions, we perform them inside a "very large set" U. So for example, a U-small category $\mathcal{C}$ is one such that $Ob(\mathcal{C}) \in U$. Then one may (say) form the category $\mathcal{C}' = Fun(\mathcal{C}^{op}, U - Set)$. This is no longer U-small, in general. However there exists a larger universe $U \in V$ such that $\mathcal{C}'$ is V-small. It follows that $N(\mathcal{C}')$ is a simplicial V-set, and so on.

In many cases, the situation becomes simpler when considering homotopy. For example if $\mathcal{C}, \mathcal{D}$ are equivalent V-categories (say), then $N(\mathcal{C})$ and $N(\mathcal{D})$ are homotopy equivalent V-simplicial sets. Now suppose that $\mathcal{C}_0$ is a U-category which is equivalent to $\mathcal{C}$ as a V-category. There is an embedding of simplicial U-sets into simplicial V-sets, and similarly on the level of homotopy categories: $Ho(sSet_U) \subset Ho(sSet_V)$. Consequently $N(\mathcal{C}) \in Ho(sSet_V)$ is in the essential image of $Ho(sSet_U)$, being equivalent to $N(\mathcal{C}_0)$. Such categories $\mathcal{C}$ are called *essentially U-small*, and as we have seen we can treat their nerves as existing up to homotopy in the universe U.

Terminology. By a Karoubi-closed category we mean an additive category such that every idempotent endomorphism splits off a summand.

Notations. See Table 1.1 for some of the notations that we frequently use. If appropriate, the third column references a section where the notation is further explained.

We warn that we sometimes (slightly) abuse notation by starting with $E, F \in Ho(\mathcal{C})$ and then writing $E \simeq F$ or $[E, F]$ where we should really be writing $E \cong F$ and $\mathrm{Hom}(E, F)$. This should not cause confusion because $Ho(Ho(\mathcal{C}))$ does not make sense.

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notation	explanation	section
$F : \mathcal{C} \leftrightarrows \mathcal{D} : U$	adjunction between categories $\mathcal{C}, \mathcal{D}$ with left adjoint F and right adjoint U	
$\mathbb{1}, \mathbb{1}_{\mathcal{C}}$	unit of a monoidal category $\mathcal{C}$	
$\otimes, \wedge$	product in a monoidal category	
$Ho(\mathcal{C})$	homotopy category of model category $\mathcal{C}$	
$E \cong F$	isomorphism between E and F	
$E \simeq F$	weak equivalence between E and F	
$\mathrm{Hom}(E, F)$	set of maps from E to F	
$Map(E, F)$	space (simplicial set) of maps from E to F	
$[E, F]$	maps from E to F in the homotopy category	
MX	motive of a scheme or motivic spectrum X	5.2.2, 5.1.1, 4.2
$\tilde{M}X$	reduced motive	
$M\{n\}$	Tate twist of motives	5.2.2
$\underline{\pi}_*(E)_*, \underline{h}_*(E)_*$	homotopy/homology sheaves of motivic spectrum/motive E	4.2
$\mathbf{SH}(X)$	stable motivic homotopy category over X	2.2
$\mathbf{DM}(X)$	triangulated category of motives over X	

Table 1.1: Notations and symbols employed.

Chapter 2

Classifying Spaces of Invertible Objects

We now begin in earnest our study of invertible objects in motivic homotopy theory. More specifically, in this chapter we shall study the classifying spaces of invertible (or locally trivial) objects.

This chapter is somewhat technical, in particular since it relies heavily on the appendix. For this reason we begin in the next section with a gentle introduction to the main ideas and results. For an overview of the organisation of the rest of this chapter, see Subsection 2.1.3.

2.1 Introduction to Chapter Two

2.1.1 Results in Concrete Terms

Let X be a smooth scheme over a (perfect) field k. We write $\mathbf{SH}(X)$ for the motivic stable homotopy category over X. This is defined in more detail later, but basically it is the category obtained from simplicial presheaves on $Sm(X)$ by working Nisnevich-locally, contracting the affine line, and inverting the Riemann sphere $\mathbb{P}^1$. This is a symmetric monoidal, triangulated category. If $f : X \to Y$ is a morphism of smooth schemes, there is a pull-back functor $f^* : \mathbf{SH}(Y) \to \mathbf{SH}(X)$.

An object $E \in \mathbf{SH}(X)$ is called *invertible* if there is an object F such that $E \wedge F \simeq S$. Here $\wedge$ denotes the monoidal operation ("smash product") and S the monoidal unit (we often also also write $\otimes$ and $\mathbb{1}$, respectively). The isomorphism classes of invertible objects form an abelian group (under smash product) denoted $Pic(\mathbf{SH}(X))$.

An invertible object $E \in \mathbf{SH}(X)$ is called *locally trivial* if there is a Nisnevich cover $f : U \to X$ such that $f^*(E) \simeq S \in \mathbf{SH}(U)$. (In fact any locally trivial object is already invertible.) We write $Pic^0(\mathbf{SH}(X)) \subset Pic(\mathbf{SH}(X))$ for the subgroup of locally trivial objects.

One of the main aims of this chapter is to study the presheaf $Pic^0(\mathbf{SH}) : X \mapsto Pic^0(\mathbf{SH}(X))$. As already mentioned in the introduction, this presheaf is *not* a sheaf in the Nisnevich topology, and this is not very surprising. In order to explain the descent properties of the Picard group, it turns out that it is actually better to consider the *Picard space*. For $X \in Sm(k)$ there is a simplicial set $PIC^0(\mathbf{SH}(X))$ which we call the Picard space. One has $\pi_0(PIC^0(\mathbf{SH}(X))) = Pic^0(\mathbf{SH}(X))$, but there are also higher homotopy groups. In fact $\pi_1(PIC^0(\mathbf{SH}(X)))$ is the group of *automorphisms* of $S \in \mathbf{SH}(X)$, and for $i > 0$ we have $\pi_{i+1}(PIC^0(\mathbf{SH}(X))) = [S[i], S]$. So this space incorporates the automorphisms and higher homotopy groups of the tensor unit.

Moreover one can set things up in such a way that the assignment $PIC^0(\mathbf{SH}) : X \mapsto PIC^0(\mathbf{SH}(X))$ is a presheaf of simplicial sets on $Sm(k)$. Moreover it is a "derived sheaf" of simplicial sets, in a precise sense. Recall that the category $sPre(Sm(k))$ affords both global and local model structures. The homotopy category $Ho(sPre(Sm(k))_{gl})$ consists of "homotopical presheaves" on $Sm(k)$, and the homotopy category $Ho(sPre(Sm(k))_{Nis}) \subset Ho(sPre(Sm(k))_{gl})$ consists of "homotopical sheaves". (Such homotopical sheaves are characterised among homotopical presheaves by a similar but more complicated descent condition than ordinary sheaves, see Theorem A.35.) When saying that $PIC^0(\mathbf{SH})$ is a "derived sheaf", we mean that the image of $PIC^0(\mathbf{SH})$ in $Ho(sPre(Sm(k))_{gl})$ does in fact lie in the subcategory $Ho(sPre(Sm(k))_{Nis})$ of homotopical sheaves.

We can describe the space $PIC^0(\mathbf{SH}(X)) \in Ho(sPre(Sm(k))_{Nis})$ more directly. To do so, recall that $\mathbf{SH}(X)$ is a simplicial model category, for every X. Let $s_X : X \to Spec(k)$ be the structural map. Choose a cofibrant-fibrant replacement $\tilde{S}$ of $S \in \mathbf{SH}(k)$. Then $s_X^* \tilde{S}$ is still cofibrant-fibrant, and so a replacement of $S \in \mathbf{SH}(X)$. There is the simplicial mapping space $Map(s_X^* \tilde{S}, s_X^* \tilde{S})$. We have $\pi_0(Map(s_X^* \tilde{S}, s_X^*)) = [S, S]$. By retaining only those components of the space corresponding to invertible endomorphisms, we obtain a new space $Aut^h(s_X^* \tilde{S})$. Then $\pi_i(Aut^h(s_X^* \tilde{S})) = \pi_{i+1} PIC^0(\mathbf{SH}(X))$ for all $i > 0$, so this suggests that we are on the right track. In fact $Aut^h(s_X^* \tilde{S})$ is a simplicial *monoid*, and so we can apply the classical nerve (or classifying space) construction to obtain a delooping $BAut^h(s_X^* \tilde{S})$.

All of these delooped spaces $BAut^h(s_X^* \tilde{S})$ are functorial in X, so fit together to yield a simplicial presheaf $B\underline{Aut}^h(\tilde{S}) \in sPre(Sm(k))$. One may then show that $PIC^0(\mathbf{SH})$ is "the associated homotopical sheaf" of $B\underline{Aut}^h(\tilde{S})$. In particular

$$[X, B\underline{Aut}^h(\tilde{S})]_{Ho(sPre(Sm(k))_{Nis})} = Pic^0(\mathbf{SH}(X)).$$

What we have described so far are some of the ingredients of the proof of Corollary 2.32. It should be mentioned that Matthias Wendt has obtained essentially the same result by a very different method [114, Section 5].

Having set up this machinery we are able to prove several interesting related results. (See Subsection 2.1.3 for an overview.)

2.1.2 The Language of Quillen Sheaves

The most difficult part of the argument outlined in the previous subsection is to show that $PIC^0(\mathbf{SH})$ is a homotopical sheaf. We do this by arguing that the assignment $X \mapsto \mathbf{SH}(X)$ is a "homotopical sheaf of categories" and that the functor PIC^0 preserves the sheaf condition, in an appropriate sense. These ideas pull in the technical complications.

In order to make sense of a homotopical (pre)sheaf of categories, we have the notion of a τ-Quillen presheaf, see section A.8. What this means is that we have a category $\mathcal{C}$, in our case $Sm(k)$, together with a notion "τ" of *covering family*, in our case $\tau = Nis$. A τ-Quillen presheaf $\mathcal{M}$ on $\mathcal{C}$ then consists of for each $X \in \mathcal{C}$ a Quillen model category $\mathcal{M}(X)$ and for each $f : X \to Y \in \mathcal{C}$ a pullback functor $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$. These pullbacks are always required to be *left Quillen* functors. If moreover f occurs in a covering family, then f^* is also required to be *right Quillen*. In our example, we have for each $X \in Sm(k)$ the model category $\mathcal{SH}(X)$. The pullback functors are always left Quillen, and it is well known that pullback along smooth morphisms is right Quillen. The morphisms of schemes occurring in covering families are precisely the étale morphisms, which are smooth, so we do get the left adjoints the definition of τ-Quillen presheaf asks for.

We often need to specify additional attributes on model categories, like simplicial, monoidal, proper etc. If $\mathcal{M}$ is a τ-Quillen presheaf, we usually say that $\mathcal{M}$ has a certain attribute if each $\mathcal{M}(X)$ has that attribute. Sometimes we ask for more, for example for a monoidal τ-Quillen presheaf we require that f^* be a monoidal functor, and so on.

One crucial property of a τ-Quillen presheaf is being a *sheaf*. We do not explain here what this means, see section 2.2 for a reminder or section A.8 for the details. Most of the appendix is dedicated to developing techniques to show that $X \mapsto \mathcal{SH}(X)$ is a τ-Quillen sheaf, and that PIC^0 of a monoidal τ-Quillen sheaf is a homotopical sheaf (of spaces).

2.1.3 Overview of the Chapter

In Section 2.2 we recall some notions from the appendix. After that we explain the construction of the model categories $\mathcal{H}(X), \mathcal{H}_*(X), \mathcal{SH}(X)$ and some variants. The main aim is to use results from the appendix to ensure that these are sheaves of model categories, more specifically τ-Quillen sheaves.

In Section 2.3 we construct the classifying spaces we are interested in. These come in two variants. In the first subsection we construct the presheaf of monoids of homotopy automorphisms $\underline{Aut}^h(T)$ of an object T, and its classifying presheaf $B\underline{Aut}^h(T)$. This has the appeal of being reasonably straightforward to define, and is also very similar to classifying space constructions in topology. Unfortunately these spaces are hard to manipulate. In the second subsection we explain two alternative constructions using nerves of model categories. The first is denoted $LOC(\mathcal{M},T)$ and is the classifying space of objects locally weakly equivalent to T. The second is denoted $GLOB(\mathcal{M},T)$ and is the space of objects globally weakly equivalent to T. Here $\mathcal{M}$ is a presheaf of model categories on some site $(\mathcal{C},\tau)$ with final object $*$, $T \in \mathcal{M}(*)$, and we say that $E \in \mathcal{M}$ is locally weakly equivalent to T if there exists a cover $U_\bullet \to X$ such that $E|_{U_i}$ is weakly equivalent to the pullback of T for each i. We say that E is globally weakly equivalent to T if it is weakly equivalent to the pullback of T.

In Section 2.4 we compare the various classifying spaces. We show that $B\underline{Aut}^h(T)$ is weakly equivalent in the global model structure on simplicial presheaves to the space $GLOB(\mathcal{M},T)$. Moreover the space $GLOB(\mathcal{M},T)$ is weakly equivalent to $LOC(\mathcal{M},T)$ in the local model structure, and the latter space satisfies descent. In particular, $[X, B\underline{Aut}^h(T)]_\tau = \pi_0 LOC(\mathcal{M},T)$, and we get a descent spectral sequence. We also investigate what happens with these classifying spaces when passing from a model category $\mathcal{M}$ to its pointed version $\mathcal{M}_*$ (if $T_* \in \mathcal{M}_*$ is highly connected, then so is the map $B\underline{Aut}^h_*(T_*) \to B\underline{Aut}^h(T)$, where T denotes the unpointed object underlying T_*) and when passing from a monoidal model category $\mathcal{M}$ to a stabilisation $Stab(\mathcal{M},P)$ (in good cases the classifying space of $\Sigma^\infty(T)$ is the colimit of the classifying spaces of $T \otimes P^{\otimes n}$). The final subsection illustrates these rather abstract results by applying them in concrete terms to our favourite sheaves of model categories $\mathcal{SH}, \mathcal{H}_*, \mathcal{H}$.

In the final Section 2.5 we prove a comparison result between the classifying space of locally trivial objects in $\mathcal{SH}$ and in $\underline{\mathcal{SH}}$. Here by $\underline{\mathcal{SH}}$ we mean the "big" version of $\mathcal{SH}$, built by starting with all varieties over the base and the cdh topology, instead of smooth varieties and the Nisnevich topology.

2.2 The τ-Quillen Sheaves of Interest

We rapidly recall some definitions. See the appendix for more details, in particular Section A.8.

Recall that a *τ-Quillen presheaf* $\mathcal{M}$ on a Verdier site $(\mathcal{C},\tau)$ consists of a pseudofunctor $\mathcal{M} : \mathcal{C}^{op} \to MCat^L$, satisfying certain properties. This means in particular that for every $X \in \mathcal{C}$ we are given a model category $\mathcal{M}(X)$, for every morphism $f : X \to Y \in \mathcal{C}$ we are given a left Quillen functor $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$, and that the functor f^* is also right Quillen whenever the morphism f is *basal*, which essentially means "part of a covering family". The left adjoint to f^* is denoted $f_\#$, the right adjoint f_*.

A τ-Quillen presheaf $\mathcal{M}$ is called proper, combinatorial etc. if each $\mathcal{M}(X)$ has this property. It is called simplicial or monoidal if each model category $\mathcal{M}(X)$ has this property, and so does each restriction functor f^*. For $\mathcal{M}$ to be simplicial we moreover ask that each of $f_*, f_\#$ is simplicial and so is the adjunction. (But we do not ask for $f_\#, f_*$ to be monoidal in order to call $\mathcal{M}$ monoidal.)

If $\mathcal{M}$ is a monoidal τ-Quillen presheaf, then for every basal map $f : X \to Y$ and every $E \in \mathcal{M}(Y), F \in \mathcal{M}(X)$ there is (by adjunction) a natural map $f_\#(F \otimes f^*E) \to f_\#(F) \otimes F$. We say that $\mathcal{M}$ *satisfies the projection formula* if this map is always an *isomorphism*. (This notion is introduced in Section A.8. Its main utility is seen when considering spectra in Subsection A.9.3.)

A τ-Quillen presheaf is called a *sheaf* or is said to *satisfy descent* if $\mathcal{C}$ is *suitable* (this is a technical condition which is almost always fulfilled in practice), $\mathcal{M}$ is combinatorial and left proper, if for every internal hypercover $\phi : U_\bullet \to X \in \mathcal{C}$, the natural right Quillen functor

$$\phi^* : \mathcal{M}(X) \to \operatorname{holim}_\Delta \mathcal{M}(U_\bullet)$$

is a Quillen equivalence, and if $\mathcal{M}(\cup_i X_i) \to \prod_i \mathcal{M}(X_i)$ is a Quillen equivalence. See Definition

A.84 for the details. Homotopy limits of diagrams of combinatorial, left proper model categories are defined in section A.7. (The definition uses Bousfield localisation, which is why we need properness and combinatoriality.)

Theorem A.88 together with the results in section A.9 provides a plethora of τ-Quillen sheaves to work with. We always start with some base site $(\mathcal{S}, \tau)$, where typically $\mathcal{S}$ is a category of schemes, say smooth over a fixed field k, or of finite type over k, etc. We now want to construct a τ-Quillen (pre)sheaf on $\mathcal{S}$. For this it is convenient to start with a τ-fibred Verdier site $\mathcal{C}$ on $\mathcal{S}$. This term is defined precisely right before Theorem A.88,but it roughly means that for every $X \in \mathcal{C}$ we are given a Verdier site $(\mathcal{C}(X), \tau_X)$, for every morphism $f : X \to Y \in \mathcal{S}$ we are given a restriction functor $f^* : \mathcal{C}(Y) \to \mathcal{C}(X)$ and, if f is basal, then f^* has a left adjoint $f_\#$. The various topologies have to be compatible in appropriate ways (in particular f^* and $f_\#$ preserve covering families), and if all of these conditions are fulfilled, Theorem A.88 provides us with a τ-Quillen sheaf

$$sPre(\mathcal{C}) : \mathcal{S} \ni X \mapsto sPre(\mathcal{C}(X))_{\tau_X, proj}$$

with good properties. The right hand side denotes the projective τ_X-local model structure on the category of simplicial presheaves on $\mathcal{C}(X)$, see Section A.2. We think of $sPre(\mathcal{C})(X)$ as a model category of *spaces* over X.

We want to modify these model categories of spaces, to obtain more specialised categories. This involves three steps: pointing, $\mathbb{A}^1$-localisation, and $\mathbb{P}^1$-stabilisation. These can actually be done in any order, except that pointing should come before stabilisation.

Consider first $\mathbb{A}^1$-localisation. For this, let us fix a cartesian section

$$A : \mathcal{S} \ni X \mapsto A(X) \in \mathcal{C}(X).$$

This just means that for $f : X \to Y \in \mathcal{S}$ we have that $A(X) \cong f^*A(Y)$. Denote by $L_{A(X)}sPre(\mathcal{C})(X)$ the Bousfield localisation at the set of maps $R_{T \otimes A(X)} \to R_T$, where $T \in \mathcal{C}(X)$ and R_T denotes the presheaf represented by T. Then we obtain a τ-Quillen presheaf

$$L_A sPre(\mathcal{C}) : X \mapsto L_{A(X)} sPre(\mathcal{C})(X).$$

Theorem A.97 shows that this construction has good properties. As pointed out there, it is not entirely formal that monoidal model categories are preserved under localisation. This is not a problem in our situation:

Lemma 2.1. *The assignment $L_A sPre(\mathcal{C})$ is a simplicial, tractable, proper, monoidal τ-Quillen sheaf which satisfies the projection formula.*

Proof. We note that $sPre(\mathcal{C})$ is monoidal and satisfies the projection formula, by Theorem A.88. From this it follows easily that Theorem A.97 applies (use that representable objects are cofibrant, see Theorem A.23). We thus need only show that $(L_A sPre(\mathcal{C}))(X)$ is monoidal. For this we use Theorem A.11. The representable presheaves form small homotopy generators. The condition in the theorem then boils down to the observation that for any $T, U \in \mathcal{C}(X)$ the map $R_T \times R_{U \times A} \to R_T \times R_U$ is an A-weak equivalence, which is clear. □

The next step is usually to pass to *pointed spaces*. Typically $\mathcal{S}$ has a final object $*$ and each $\mathcal{C}(X)$ has finite products and in particular a final object which is preserved under the restrictions f^*. We write

$$sPre(\mathcal{C})_* : X \mapsto */sPre(\mathcal{C})(X)$$

for the pseudo-presheaf which has as sections the pointed versions of the categories $sPre(\mathcal{C})(X)$. This just means that cofibrations, fibrations and weak equivalences are detected under the forgetful functor $sPre(\mathcal{C})_*(X) \to sPre(\mathcal{C})(X)$. Theorem A.93 shows that this construction has good properties.

We can similarly point the localised τ-Quillen sheaf $L_A sPre(\mathcal{C})$ to obtain $L_A sPre(\mathcal{C})_*$. (Localisation and pointing commute in a precise sense, so we shall not notationally make the order of the two processes explicit.)

Next we need to perform $\mathbb{P}^1$-stabilisation. For this, we choose a cofibrant object $P \in sPre(\mathcal{C})_*(*)$ and put $P(X) = (X \to *)^* P \in sPre(\mathcal{C})_*(X)$. Then each $P(X)$ is cofibrant, and we wish to form a category of $P(X)$-spectra. More generally, given a monoidal model category $\mathcal{M}$ and an object $P \in \mathcal{M}$, there are (at least) two ways of going about forming model categories of spectra. There is the model category $Stab(\mathcal{M}, P)$ consisting of *naive spectra*, i.e. sequences $(X_1, X_2, \dots)$ with bonding maps $X_i \otimes P \to X_{i+1}$. This model category is *not* monoidal. There is also the category $Stab^\Sigma(\mathcal{M}, P)$ of *symmetric spectra*. These have a more complicated definition, but the advantage is that the category of symmetric spectra *is* monoidal. In good cases, the model categories of spectra and symmetric spectra are related by a zig-zag of Quillen equivalences. See Section A.9.3 for more details.

Getting back to our τ-Quillen sheaf story, both the $Stab(\mathcal{M}, P)$ and the $Stab^\Sigma(\mathcal{M}, P)$ construction are functorial in $\mathcal{M}$ in an appropriate sense, so it is quite plausible that we can define τ-Quillen presheaves

$$Stab(L_A sPre(\mathcal{C})_*, P) : X \mapsto Stab(L_A sPre(\mathcal{C})_*(X), P(X)),$$
$$Stab^\Sigma(L_A sPre(\mathcal{C})_*, P) : X \mapsto Stab^\Sigma(L_A sPre(\mathcal{C})_*(X), P(X)).$$

Theorems A.102 and A.108 establish good properties of these τ-Quillen presheaves. Namely, both are left proper, tractable, simplicial τ-Quillen presheaves. $Stab(L_A sPre(\mathcal{C})_*, P)$ is in fact a τ-Quillen sheaf, and $Stab^\Sigma(L_A sPre(\mathcal{C})_*, P)$ is symmetric monoidal and satisfies the projection formula; however the author cannot directly prove that it is a sheaf. Corollary A.112 and the paragraph thereafter explain that in good cases, the τ-Quillen presheaf $Stab^\Sigma(L_A sPre(\mathcal{C})_*, P)$ *is* a sheaf, essentially because it is related to $Stab(L_A sPre(\mathcal{C})_*, P)$ by a zig-zag of Quillen equivalences.

Now let us be a bit more specific. We use $\mathcal{S} = Ft(k)$, the category of schemes of finite type over some field k. (This is not the most general choice possible, but general enough for what we intend to do.) We let τ denote one of $\{Zar, Nis, et\}$, i.e. either the Zariski, Nisnevich or étale topology. We put $\mathcal{C}(X) = Sm(X)_\tau$ or $\mathcal{C}(X) = Ft(X)_{cdh}$, where cdh stands for the cdh topology. (We always use the "same" topology on $\mathcal{S}$ as on $\mathcal{C}$.) Of course we choose $A = \mathbb{A}^1$ and for P we use an appropriate model of $\mathbb{P}^1$ pointed at one, such as $\mathbb{P}^1$, T or $S^1 \wedge \mathbb{G}_m$.

Definition 2.2. *We use the following notation.*

$$\mathcal{H}_\tau = L_{\mathbb{A}^1} sPre(Sm, \tau)$$
$$\mathcal{H}_{*,\tau} = L_{\mathbb{A}^1} sPre(Sm, \tau)_*$$
$$\mathcal{SH}_\tau = Stab^\Sigma(L_{\mathbb{A}^1} sPre(Sm, \tau)_*, P)$$

$$\underline{\mathcal{H}}_{cdh} = L_{\mathbb{A}^1} sPre(Ft, cdh)$$
$$\underline{\mathcal{H}}_{*,cdh} = L_{\mathbb{A}^1} sPre(Ft, cdh)_*$$
$$\underline{\mathcal{SH}}_{cdh} = Stab^\Sigma(L_{\mathbb{A}^1} sPre(Ft, cdh)_*, P).$$

When omitting τ, we mean the Nisnevich topology:

$$\mathcal{H} = \mathcal{H}_{Nis}$$
$$\mathcal{H}_* = \mathcal{H}_{*,Nis}$$
$$\mathcal{SH} = \mathcal{SH}_{Nis}.$$

We will not actually use the étale versions of these categories, except for illustrative purposes. The "big" version $\underline{\mathcal{SH}}_{cdh}$ will only be used in Section 2.5. We can then summarise the above discussion as follows.

Theorem 2.3. *Each of the above twelve assignments defines a closed symmetric monoidal, combinatorial, left proper, tractable, simplicial Quillen sheaf (in the respective topology) on $Ft(k)$.*

The unstable model categories in the Zariski, Nisnevich and cdh topologies are almost finitely generated (see Section A.4).

Proof. To only thing that remains to be proved is the almost finite generation. This follows easily from Corollary A.42 and the remark thereafter. □

Remark 1. It is possible, but harder, to repeat this story with **DM** in place of **SH**. Using the results of [20] and [19] one may define similar τ-Quillen presheaves $\mathcal{DM}_\tau, \underline{\mathcal{DM}}_{cdh}$ and show that they are sheaves. The results we are going to prove for **SH** are not very interesting in the case of **DM** as explained in Chapter 3. Thus we shall not construct $\mathcal{DM}$ in detail here and just use it for illustrative purposes.

Remark 2. There is an $\mathbb{A}^1$-weak equivalence $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$. Consequently inverting $\mathbb{P}^1$ also inverts S^1, which implies that the homotopy categories $\mathbf{SH}(X) = Ho(\mathcal{SH}(X))$ etc. are triangulated. One may also invert *only* S^1, obtaining τ-Quillen sheaves

$$\mathcal{SH}^{S^1}(X) = Stab^{\Sigma}(L_{\mathbb{A}^1} sPre(Sm, Nis), S^1)(X),$$

etc. These categories have less geometric significance than their $\mathbb{P}^1$ counterparts but they have triangulated homotopy categories and so are sometimes easier to handle than the unstable categories.

2.3 The Classifying Spaces of Interest

Having constructed our τ-Quillen (pre)sheaves, we now need to extract classifying spaces from them. As explained in the chapter introduction, there are two related ways of going about this. In subsection 2.3.1 we introduce the monoids of homotopy automorphisms and perform the bar construction on them, yielding spaces which have clear geometric significance. In subsection 2.3.2 we introduce a different classifying space construction using nerves of categories of weak equivalences. These are somewhat more arcane, but turn out to have good properties which are easy to establish.

2.3.1 Monoids of Homotopy Automorphisms

Let $(\mathcal{C}, \tau)$ be a Verdier site with final object $*$ and $\mathcal{M}$ a simplicial τ-Quillen presheaf on $\mathcal{C}$. Write $\mathcal{P} = \mathcal{P}_{\mathcal{M}}$ for the subcategory of $\mathcal{C}$ consisting of those objects $X \in \mathcal{C}$ such that $(X \to *)^* : \mathcal{M}(*) \to \mathcal{M}(X)$ is right Quillen. Of course $\mathcal{P}$ contains all objects occurring in coverings of $*$, but frequently it is larger. In fact if $(\mathcal{C}, \tau) = (Ft/S, \tau)$ with $\tau \in \{Zar, Nis, et\}$ then $\mathcal{P}$ is $Sm(S)$, and if $(\mathcal{C}, \tau) = (Ft, cdh)$ then $\mathcal{P} = \mathcal{C}$.

For reasons of convenience we shall now define certain classifying spaces as presheaves of simplicial sets on $\mathcal{P}$. We will see later in this chapter that restricting to $\mathcal{P}$ is not really necessary.

Thus, fix $E \in \mathcal{M}(*)$ which is fibrant and cofibrant. For $X \in \mathcal{P}$, write $E_X = (X \to *)^* E$. Since $(X \to *)^*$ is bi-Quillen by assumption, E_X is also cofibrant and fibrant. We would like to define an object $\underline{\mathrm{Hom}}'_{\mathcal{P}}(E, E) \in sPre(\mathcal{P})$ by

$$\underline{\mathrm{Hom}}'_{\mathcal{P}}(E, E)(X) = Map(E_X, E_X),$$

where Map denotes the simplicial mapping space in $\mathcal{M}(X)$. This has the correct homotopy type because E_X is fibrant and cofibrant. There is the small problem that $\mathcal{M} : \mathcal{C} \to MCat^R$ is not a *functor*, but (in practice) just a *pseudo-functor*. See section A.6 for a review of these objects. For now we recall some basics. $\mathcal{M}$ being a pseudofunctor means in particular that given $X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathcal{C}$ we do not have (in general) $f^* g^* = (f \circ g)^*$, but only a natural isomorphism $f^* g^* \cong (f \circ g)^*$. Consequently $\underline{\mathrm{Hom}}'_{\mathcal{P}}(E, E)$ as currently defined is not a (simplicial) presheaf, because the functoriality in $\mathcal{C}$ only holds "up to natural isomorphism" (essentially, up to homotopy).

Fortunately there is an easy way out. Given a pseudofunctor $\mathcal{M}$, there is always its *rectification* or *strictification* $\mathcal{M} \to \mathcal{M}^r$, see Theorem A.49 and the paragraph thereafter. What is important for us is that $\mathcal{M}^r$ is a *strict* functor on $\mathcal{C}$ such that $\mathcal{M}(X) \cong \mathcal{M}^r(X)$ are equivalent categories. In particular $\mathcal{M}^r$ is also a τ-Quillen presheaf and inherits essentially all properties (simpliciality, monoidality, being a sheaf, etc.) that $\mathcal{M}$ may have. We now define

$$\underline{\mathrm{Hom}}_{\mathcal{P}}(E,E)(X) := Map_{\mathcal{M}^r(X)}(\tilde{E}_X, \tilde{E}_X).$$

Here $\tilde{E} \in \mathcal{M}^r(*)$ is the image of $E \in \mathcal{M}(*)$; it is cofibrant and fibrant. Now $\underline{\mathrm{Hom}}_{\mathcal{P}}(E,E)$ is a true simplicial presheaf. We can form the associated $\underline{\pi}_0$ presheaf

$$(\underline{\pi}_0 \underline{\mathrm{Hom}}_{\mathcal{P}}(E,E))(X) := \pi_0(\underline{\mathrm{Hom}}_{\mathcal{P}}(E,E)(X)).$$

Because of the assumption that E is fibrant-cofibrant we have

$$\underline{\pi}_0 \underline{\mathrm{Hom}}_{\mathcal{P}}(E,E)(X) = \pi_0 Map^d(E_X, E_X) = [E_X, E_X]_{Ho(\mathcal{M}(X))}.$$

We let $\underline{\pi}_0(\underline{\mathrm{Hom}}_{\mathcal{P}}(E,E))^\times$ denote the sub-presheaf which has as sections over X those elements of $[E_X, E_X]$ which are *invertible* in $Ho(\mathcal{M}(X))$. (This *is* a presheaf because the induced restriction maps $f^* : \pi_0 Map(E_X, E_X) \to \pi_0 Map(E_Y, E_Y)$ coincide with the derived restrictions $Rf^* : Ho(\mathcal{M}(X)) \to Ho(\mathcal{M}(Y))$, again because E_X is cofibrant and fibrant.)

Definition 2.4. *The* presheaf of homotopy automorphisms *of E in $\mathcal{M}$ is the pullback*

$$\begin{array}{ccc} \underline{Aut}^h_{\mathcal{P}}(E) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{P}}(E,E) \\ \downarrow & & \downarrow \\ \underline{\pi}_0(\underline{\mathrm{Hom}}_{\mathcal{P}}(E,E))^\times & \longrightarrow & \underline{\pi}_0(\underline{\mathrm{Hom}}_{\mathcal{P}}(E,E)). \end{array}$$

It is an element of $sPre(\mathcal{P})$.

The classifying space of homotopy automorphisms *of E in $\mathcal{M}$ is the sectionwise bar construction*

$$(B\underline{Aut}^h_{\mathcal{P}}(E))(X) := B(\underline{Aut}^h_{\mathcal{P}}(E)(X)).$$

Recall that the *bar construction* B or *nerve* N is a functor from simplicial categories to simplicial sets [27, 1.4(vi)]. Applied to ordinary categories one obtains the usual nerve as recalled in A.1.4 (and in this situation usually the letter N is used). Applied to simplicial monoids such as $\underline{\mathrm{Hom}}_{\mathcal{P}}(E,E)(X)$ one obtains a generalisation of the simplicial construction of the classifying space of a group (and in this situation usually the letter B is used).

We again end this subsection with the concrete examples we are interested in. It is fair to say that the main object of study of this chapter is

$$B\underline{Aut}^h(\mathbb{1}_{\mathcal{SH}}) := B\underline{Aut}^h_{P_{\mathcal{SH}}}(\tilde{\mathbb{1}}) \in sPre(Sm(k)),$$

where $\tilde{\mathbb{1}} \in \mathcal{SH}(k)$ is a cofibrant-fibrant replacement of the tensor unit. We will also be interested in

$$B\underline{Aut}^h_*((\mathbb{P}^1)^{\wedge n}) := B\underline{Aut}^h_{P_{\mathcal{H}_*}}(P^{\wedge n})$$

and in

$$B\underline{Aut}^h((\mathbb{P}^1)^{\wedge n}) := B\underline{Aut}^h_{P_{\mathcal{H}}}(P^{\wedge n}).$$

We refer to these as classifying spaces of pointed and unpointed homotopy automorphisms of $(\mathbb{P}^1)^{\wedge n}$, respectively. Here $P^{\wedge n}$ again denotes a suitable cofibrant-fibrant replacement.

Remark 1. As our somewhat sloppy notation suggests, $B\underline{Aut}^h(X)$ is actually a weak equivalence invariant of X, so that for example the exact choice of cofibrant-fibrant replacement does not matter. These and many other properties can be proved without difficulty directly, but they will also follow more elegantly from the alternative construction to be given in the next subsection. See Section 2.4.

Remark 2. Some of our classifying spaces $B\underline{Aut}^h$ coincide (in a somewhat roundabout way) with those studied by Wendt [115, Section 5.3]. We will derive from our results in Section 2.4 an alternative proof of his main theorem.

2.3.2 Nerves of Categories of Locally Trivial Objects

We again start with a Verdier site $(\mathcal{C}, \tau)$ and a τ-Quillen presheaf $\mathcal{M}$ on $\mathcal{C}$. For now it is not necessary to assume that $\mathcal{M}$ is simplicial. We want to define the classifying space of weak equivalences in $\mathcal{M}$. For this, given $X \in \mathcal{C}$ we consider the category $\mathcal{M}(X)^c_w$ consisting of the cofibrant objects in $\mathcal{M}(X)$, with morphisms the weak equivalences. This is just an ordinary category, and we are interested in its nerve $N(\mathcal{M}(X)^c_w)$. See Section A.1.4 for a review of some of the properties of this construction (and see Section 1.4 for some comments about smallness issues). Certainly the "set" $\pi_0(N(\mathcal{M}(X)^c_w))$ is the "set" of weak equivalence classes of objects in $\mathcal{M}(X)$, or equivalently the isomorphism classes of objects in $Ho(\mathcal{M}(X))$. It follows that $\pi_0(N(\mathcal{M}(X)^c_w))$ actually defines a presheaf on $\mathcal{C}$. Again this is not true for $X \mapsto N(\mathcal{M}(X)^c_w)$, because $\mathcal{M}$ need not be a strict functor. However we may replace $\mathcal{M}$ by $\mathcal{M}^r$; it follows from (a very special case of) Lemma A.15 that $N(\mathcal{M}(X)^c_w) \simeq N(\mathcal{M}^r(X)^c_w)$ and so up to homotopy this is just as good. Note that each of the restrictions $f^* : \mathcal{M}(X) \to \mathcal{M}(Y)$ for $f : Y \to X \in \mathcal{C}$ are left Quillen, so preserve cofibrant objects and weak equivalences between cofibrant objects. Thus we indeed get a presheaf.

Definition 2.5. *The simplicial presheaf*

$$X \mapsto N(\mathcal{M}^r(X)^c_w)$$

is denoted $CORE(\mathcal{M}) \in sPre(\mathcal{C})$.

Suppose that $\mathcal{C}$ has a final object $*$ and pick $F \in Ho(\mathcal{M}(*))$. Write $F_X = L(X \to *)^*F$. Define sub-presheaves $L(F)$ and $G(F)$ of $\underline{\pi}_0 CORE(\mathcal{M})$ via

$$G(F)(X) = \{[F_X]\} \subset \pi_0(CORE(\mathcal{M})(X)) = Ob(Ho(\mathcal{M}(X)))/\simeq$$

(so G is a constant one-point presheaf) and

$$L(F)(X) = \{[T] \in Ob(Ho(\mathcal{M}(X)))/\simeq \,|T \; \tau\text{-locally equivalent to } F\}.$$

Here we say that T is τ-locally equivalent to F if there exists a covering $\phi_\bullet : U_\bullet \to X$ such that $L\phi_i^* T \simeq F_{U_i}$ for each i.

Definition 2.6. *In the above situation we define objects* $LOC(\mathcal{M}, F), GLOB(\mathcal{M}, F) \in sPre(\mathcal{C})$ *by the pullbacks*

$$\begin{array}{ccc} LOC(\mathcal{M}, F) & \longrightarrow & CORE(\mathcal{M}, F) \\ \downarrow & & \downarrow \\ L(F) & \longrightarrow & \underline{\pi}_0 CORE(\mathcal{M}, F) \end{array}$$

and

$$\begin{array}{ccc} GLOB(\mathcal{M}, F) & \longrightarrow & CORE(\mathcal{M}, F) \\ \downarrow & & \downarrow \\ G(F) & \longrightarrow & \underline{\pi}_0 CORE(\mathcal{M}, F). \end{array}$$

We call these the classifying spaces of objects in $\mathcal{M}$ locally (respectively globally) weakly equivalent to F.

Let us point out that $CORE(\mathcal{M}, F)$ and $LOC(\mathcal{M}, F)$ are "large" simplicial presheaves, in the sense of Section 1.4. This is not really a problem. First of all we will only use $CORE$ for notational convenience, we actually want to study LOC. This is still a large simplicial presheaf. However it is not hard to see that $L(F)$ is a presheaf of *small* sets, and as explained in Section A.1.4 it follows that $LOC(\mathcal{M}, F)(X)$ is homotopy equivalent to a small set.

Our main reason for studying τ-Quillen sheaves is the following result.

Theorem 2.7. *Let $(\mathcal{C}, \tau)$ be a suitable Verdier site and $\mathcal{M}$ a simplicial τ-Quillen sheaf on $\mathcal{C}$. The object $CORE(\mathcal{M}) \in sPre(\mathcal{C})$ satisfies descent in the τ-topology. If $\mathcal{C}$ has a final object $*$ and we fix $F \in Ho(\mathcal{M}(*))$ then the object $LOC(\mathcal{M}, F)$ also has descent in the τ-topology.*

Proof. The first statement follows from Theorem A.80 and the definitions of a τ-Quillen sheaf and descent, i.e. Theorem A.35 and Definition A.84.

We now deal with descent for LOC. For a presheaf of *sets* (i.e. discrete simplicial sets) P let us write aP for the associated sheaf in the τ-topology. As a first step, I claim that the following diagram is a pullback:

$$\begin{array}{ccc} LOC(\mathcal{M}, F)(X) & \longrightarrow & CORE(\mathcal{M})(X) \\ \downarrow & & \downarrow \\ (aL(F))(X) & \longrightarrow & (a\pi_0(CORE(\mathcal{M})))(X). \end{array}$$

By the pasting law for pullback diagrams it suffices to show that

$$\begin{array}{ccc} L(F)(X) & \longrightarrow & \pi_0(CORE(\mathcal{M}))(X) \\ \downarrow & & \downarrow \\ (aL(F))(X) & \longrightarrow & (a\pi_0(CORE(\mathcal{M})))(X) \end{array}$$

is a pullback diagram. This is an easy consequence of the local nature of the definition of $L(F)$.

The result now follows from the next lemma, the fact that sheaves of (discrete) sets satisfy descent, being fibrant in a local model structure [60, Lemma 5.11], and the fact that all our pullback diagrams are sectionwise homotopy pullbacks since any map of sets is a fibration of simplicial sets and the model structure on simplicial sets is proper. □

Lemma 2.8. *Let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

be a diagram of simplicial presheaves which is a sectionwise homotopy pullback. Suppose that B, C, D satisfy descent. Then A satisfies descent.

Proof. Let $U_\bullet \to X$ be a hypercover. Since homotopy limits commute we get

$$\begin{aligned} \operatorname{holim}_n A(U_n) &\simeq \operatorname{holim}_n B(U_n) \times^h_{D(U_n)} C(U_n) \\ &\simeq \operatorname{holim}_n B(U_n) \times^h_{\operatorname{holim}_n D(U_n)} \operatorname{holim}_n C(U_n) \\ &\simeq B(X) \times^h_{D(X)} C(X) \\ &\simeq A(X). \end{aligned}$$

□

Before getting back to concrete examples, let us point out the following further abstract result. In the special case that $\mathcal{M}$ is a monoidal τ-Quillen presheaf, we write

$$PIC^0(\mathcal{M}) := LOC(\mathcal{M}, \mathbb{1}).$$

Proposition 2.9. *Let $\mathcal{M}$ be a simplicial, closed symmetric monoidal τ-Quillen sheaf satisfying the projection formula, and assume that the tensor units are cofibrant. Then $PIC^0(\mathcal{M})$ is a presheaf of infinite loop spaces in a natural way; in particular there exists a presheaf of spectra $\mathbb{P}IC^0(\mathcal{M}) \in Spt^\Sigma(sPre(\mathcal{C}))$ with a global weak equivalence $\Omega^\infty \mathbb{P}IC^0(\mathcal{M}) \simeq PIC(\mathcal{M})$.*

Proof. This follows from infinite loop space machine theory [73]. Namely, there is a functor $K : SymMon \to Spt$, where $SymMon$ is the category of symmetric monoidal categories, and Spt is a good category of spectra. It has the property that $K(\mathcal{D})$ is a connective spectrum such that $\Omega^\infty K(\mathcal{D}) \simeq N(\mathcal{D})'$, functorially in $\mathcal{D}$. Here N is the ordinary nerve, and $'$ denotes group completion.

In our case we let $\mathcal{D}(X)$ be the subcategory of $\mathcal{M}^r(X)$ consisting of those cofibrant objects locally weakly equivalent to $\mathbb{1}$, and the weak equivalences between them. Note that this defines a strict functor with values in symmetric monoidal categories and monoidal functors. Then $PIC^0(\mathcal{M})(X) = N(\mathcal{D}(X))$ and hence we can just put

$$\mathbb{P}IC^0(\mathcal{M})(X) = K(\mathcal{D}(X)).$$

This works because $\pi_0(PIC^0(\mathcal{M})(X))$ is already a group. Indeed if $f : U \to X$ is basal and $A, B \in Ho(\mathcal{M}(X))$ then $f^*R\underline{Hom}(A,B) \simeq R\underline{Hom}(f^*A, f^*B)$ as one checks using adjunction and the projection formula. Consequently if $E \in \pi_0(PIC^0(\mathcal{M}(X)))$ then $DE := R\underline{Hom}(E, \mathbb{1})$ satisfies $E \otimes DE \simeq \mathbb{1}$ and $DE \in \pi_0(PIC^0(\mathcal{M}(X)))$ (use that pullback along a cover is conservative for a sheaf). □

Corollary 2.10. *There exists a spectral sequence*

$$E_2^{pq} = H^p_\tau(X, \underline{\pi}_{-q}\mathbb{1}).$$

If the cohomological dimensions are uniformly bounded, the spectral sequence converges strongly to $[\Sigma^\infty X_+[-p-q], \mathbb{P}IC^0(\mathcal{M})]_\tau$*. For* $-p-q \geq 0$ *this group coincides with* $\pi_{-p-q}(PIC^0(\mathcal{M})(X))$*. In particular for* $-p-q = 0$ *it is the group of objects in* $Ho(\mathcal{M}(X))$ *locally weakly equivalent to* $\mathbb{1}$*.*

Proof. The spectral sequence and convergence condition come from A.44. By adjunction for $-p-q \geq 0$ we have $[\Sigma^\infty X_+[-p-q], \mathbb{P}IC^0(\mathcal{M})] = [X_+ \wedge S^{-p-q}, \Omega^\infty \mathbb{P}IC^0(\mathcal{M})]$. Since $\Omega^\infty \mathbb{P}IC^0(\mathcal{M}) = PIC^0(\mathcal{M})$ by the previous proposition it remains to show that $[X_+ \wedge S^{-p-q}, PIC^0(\mathcal{M})] = \pi_{-p-q}PIC^0(\mathcal{M})(X)$. This is follows from Theorem A.35 and the fact that $PIC^0(\mathcal{M})$ satisfies descent. □

Remark 1. The existence of classifying spectra is not important to the existence of the spectral sequence. Suitably interpreted, there are spectral sequences

$$H^p(X, \underline{\pi}_{-q}LOC(\mathcal{M}, F)) \Rightarrow \pi_{-p-q}LOC(\mathcal{M})(X)$$

in great generality. There are some issues regarding the fact that not all terms here are groups, but they can be dealt with. We mostly do not need this generality so avoid the complications.

Remark 2. For a symmetric monoidal τ-Quillen presheaf $\mathcal{M}$ one may define a simplicial presheaf $PIC(\mathcal{M})$ such that $PIC(\mathcal{M})(X)$ is the classifying space (or even spectrum) of all invertible objects in $Ho(\mathcal{M}(X))$, not just those locally equivalent to $\mathbb{1}$. This space also satisfies descent, and there is a similar spectral sequence.

Examples. The spaces $PIC^0(\mathcal{SH}), PIC^0(\underline{\mathcal{SH}}), PIC^0(\mathcal{DM})$ as well as variants like $PIC(\mathcal{SH})$, $LOC(\mathcal{H}_*, (\mathbb{P}^1)^{\wedge n})$ are important for us. By the above results they all satisfy descent and so we get descent spectral sequences.

Proposition 2.11. *1. For* $X \in Ft(k)$ *there is a strongly convergent spectral sequence*

$$H^p_{Nis}(X, \underline{\pi}_{-q}PIC^0(\mathcal{SH})) \Rightarrow [X[-p-q], \mathbb{P}IC^0(\mathcal{SH})]_{Nis}.$$

Here $[\bullet, \bullet]_{Nis}$ *denotes morphisms in the* S^1*-stable homotopy category. As before, for* $-p-q \geq 0$ *these coincide with homotopy groups of* $PIC^0(\mathcal{SH})(X)$*.*

2. *For $X \in Ft(k)$ there is a strongly convergent spectral sequence*

$$H^p_{cdh}(X, \underline{\pi}_{-q} PIC^0(\underline{\mathcal{SH}})) \Rightarrow [X[-p-q], \mathbb{P}IC^0(\underline{\mathcal{SH}})]_{cdh}.$$

Here $[\bullet, \bullet]_{cdh}$ denotes morphisms in the S^1-stable homotopy category. As before, for $-p-q \geq 0$ these coincide with homotopy groups of $PIC^0(\underline{\mathcal{SH}})(X)$.

2.4 Relations Between Classifying Spaces

2.4.1 Comparing $BAut^h$ and LOC

Recall that associated with the τ-Quillen presheaf $\mathcal{M}$ on the Verdier site $(\mathcal{C}, \tau)$ with final object $*$ we have the full subcategory $\mathcal{P} \subset \mathcal{C}$ of those objects X such that $(X \to *)^*$ is right Quillen, not just left Quillen. Then for $F \in \mathcal{M}(*)$ cofibrant-fibrant we defined $B\underline{Aut}^h(F) \in sPre(\mathcal{P})$. We also have $GLOB(\mathcal{M}, F), LOC(\mathcal{M}, F) \in sPre(\mathcal{C})$. We shall write $GLOB(\mathcal{M}, F)|_{\mathcal{P}}$ for the *restriction* to $\mathcal{P}$, and similarly for LOC. The following result is a consequence of work of Dwyer-Kan on simplicial localisation.

Theorem 2.12. *Let $\mathcal{M}$ be a simplicial τ-Quillen presheaf on the Verdier site $(\mathcal{C}, \tau)$ with final object $*$, and fix $F \in \mathcal{M}(*)$ which is cofibrant and fibrant.*

The objects $B\underline{Aut}^h_{\mathcal{P}}(F)$ and $GLOB(\mathcal{M}, F)|_{\mathcal{P}}$ are related by a string of global weak equivalences, and thus define isomorphic objects of $Ho(sPre(\mathcal{P})_{gl})$.

Proof. Using rectification, we may and will assume that $\mathcal{M}$ is a strict functor.

For $X \in \mathcal{P}$ let $A_*(X)$ be the one-object simplicial category which is the subcategory of the simplicial category $\mathcal{M}(X)$ on the object F_X, with morphisms the weak equivalences. Thus $A_*(X)$ is $\underline{Aut}^h(F)$ considered as a simplicial category. Write $W^c_*(X)$ for the many-object simplicial category which is the subcategory of $\mathcal{M}(X)$ on all cofibrant objects weakly equivalent to F_X, with morphisms again the weak equivalences. Write $W^c(X)$ for the ordinary category which has as objects those cofibrant objects of $\mathcal{M}(X)$ weakly equivalent to F_X, and as morphisms the weak equivalences. Then A_*, W^c_* and W^c are strict presheaves of (simplicial) categories on $\mathcal{P}$. We have $BA = B\underline{Aut}^h(F)$ and $BW^c = GLOB(\mathcal{M}, F)|_{\mathcal{P}}$.

Write $diagL^H W^c_*(X)$ for the diagonal of the *Hammock localisation* of the simplicial category $W^c_*(X)$ [25]. This is still a (strict) presheaf of simplicial categories. We have morphisms of presheaves

$$\begin{array}{ccccc} A & \longrightarrow & W^c_* & \longleftarrow & W^c \\ & & \downarrow & & \\ & & diagL^H W_*. & & \end{array}$$

The composites $A \to diagL^H W^c_*$ and $W^c \to diagL^H W^c_*$ induce sectionwise weak equivalences on (appropriately defined) nerves by [26, Corollary 4.7], [27, Paragraphs 1.4(vii), 5.5(ii)] and [25, Proposition 2.2]. (Observe that all of these nerves are connected.) The result follows. □

We also have the following more elementary observation.

Proposition 2.13. *Let $\mathcal{M}$ be a τ-Quillen presheaf on the Verdier site $(\mathcal{C}, \tau)$ with final object $*$, and fix $F \in Ho(\mathcal{M}(*))$. The canonical morphism $GLOB(\mathcal{M}, F) \to LOC(\mathcal{M}, F)$ is a τ-local weak equivalence in $sPre(\mathcal{C})$.*

Proof. Since $GLOB(\mathcal{M}, F)(X) \subset LOC(\mathcal{M}, F)(X)$ is the inclusion of a connected component, it follows easily from the definition of τ-local weak equivalences in terms of associated homotopy sheaves (Definition A.22) that is suffices to show that the map $a_\tau \underline{\pi}_0 GLOB(\mathcal{M}, F) \to a_\tau \underline{\pi}_0 LOC(\mathcal{M}, F)$ is an isomorphism (of sheaves).

Now $\pi_0(GLOB(\mathcal{M}, F)(X)) = *$, so we need to show that $a_\tau \underline{\pi}_0 LOC(\mathcal{M}, F) = *$ (the constant sheaf associated with the one-point set). This is essentially clear: for every $[T] \in \pi_0 LOC(\mathcal{M}, F)(X)$ there exists a τ-cover $U_\bullet \to X$ such that $[T]|U_i = [F]$. Consequently the map $\underline{\pi}_0 LOC(\mathcal{M}, F) \to a_\tau \underline{\pi}_0 LOC(\mathcal{M}, F)$ factors through $\underline{\pi}_0 LOC(\mathcal{M}, F) \to *$. The result follows. □

Combining the above two results with the descent theorem from the last section, we obtain the following combined result, which basically says that the space $B\underline{Aut}^h(F)$ as an object of $Ho(sPre(\mathcal{P})_\tau)$ classifies locally trivial fibrations with fibre F. It should be compared with [114, Theorem 5.10], which seems to be a special case of our result.

Corollary 2.14. *Let $\mathcal{M}$ be a simplicial τ-Quillen sheaf on the suitable Verdier site $(\mathcal{C}, \tau)$. Fix $F \in \mathcal{M}(*)$ which is both fibrant and cofibrant. Then for $X \in \mathcal{P}$ there is a natural weak equivalence*

$$Map^d_\tau(X, B\underline{Aut}^h(F)) \simeq LOC(\mathcal{M}, F)(X).$$

Here Map^d_τ denotes the derived mapping space, computed in the τ-local model structure on $sPre(\mathcal{P})$.

In particular

$$[X, B\underline{Aut}^h(F)]_\tau = \pi_0(LOC(\mathcal{M}, F)(X)).$$

Here $[\bullet, \bullet]_\tau$ denotes morphisms in $Ho(sPre(\mathcal{P})_\tau)$.

Proof. We know that $B\underline{Aut}^h(F) \simeq GLOB(\mathcal{M}, F)_\mathcal{P} \to LOC(\mathcal{M}, F)_\mathcal{P}$ is a string of τ-local weak equivalences by Theorem 2.12 and Proposition 2.13. We also know that $LOC(\mathcal{M}, F)$ satisfies descent by Theorem 2.7 and hence is τ-local by Theorem A.35. It is thus sectionwise weakly equivalent to its τ-local fibrant replacement, and the first claim follows. The second is a direct consequence. □

Let us also point out the following essentially obvious facts.

Corollary 2.15. *Let $(\mathcal{C}, \tau)$ be a Verdier site, $\mathcal{M}$ a simplicial τ-Quillen presheaf and $F \in Ho(\mathcal{M}(*))$. Then the homotopy presheaves of $GLOB(\mathcal{M}, F)$ are given by*

$$\pi_0(GLOB(\mathcal{M}, F)) = *$$

$$\pi_1(GLOB(\mathcal{M}, F)(X)) = [F_X, F_X]^\times$$

$$\pi_{i+1}(GLOB(\mathcal{M}, F)) = [F_X \wedge S^i, F].$$

Here $F_X = (X \to *)^* F$, $[F_X, F_X]^\times$ denotes the invertible morphisms in $Ho(\mathcal{M}(X))$ and $F_X \wedge S^i$ denotes the external tensoring coming from the simplicial structure.

The space $B\underline{Aut}^h(F)$ has the same homotopy presheaves, and $LOC(\mathcal{M}, F)$ has the same homotopy presheaves except in degree zero (where it consists of the set of locally trivial fibrations with fibre F, up to homotopy).

Proof. The result for $B\underline{Aut}^h$ is clear by construction, since it is a sectionwise delooping of $\underline{Aut}^h$. The result for $GLOB$ follows from Proposition 2.13, and the result for LOC is again by construction. □

Corollary 2.16. *Let $(\mathcal{C}, \tau)$ be a Verdier site, $\mathcal{M}$ a simplicial τ-Quillen presheaf and $F_1, F_2 \in \mathcal{M}(*)$ both fibrant and cofibrant. Then if F_1 and F_2 are weakly equivalent, $B\underline{Aut}^h(F_1)$ and $B\underline{Aut}^h(F_2)$ define isomorphic objects of $Ho(sPre(\mathcal{P})_{gl})$.*

Proof. Clearly $GLOB(\mathcal{M}, F_1) = GLOB(\mathcal{M}, F_2)$, so this follows from 2.13. □

2.4.2 Comparing Stable and Unstable Fibrations

Suppose that $\mathcal{M}$ is a symmetric monoidal τ-Quillen presheaf and fix $P, Q \in \mathcal{M}(*)$. We would like to describe $LOC(Stab(\mathcal{M}, P), \Sigma^\infty Q)$ in terms of the various $LOC(\mathcal{M}, Q \otimes P^{\otimes n})$. This is much less formal than the results in the previous subsection. We begin with the following fairly trivial observation.

Lemma 2.17. *Let $\Theta : \mathcal{M} \to \mathcal{N}$ be a left morphism of τ-Quillen presheaves on the Verdier site $(\mathcal{C}, \tau)$ with final object $*$. Fix $F \in Ho(\mathcal{M})(*)$. There is a natural map*

$$LOC(\mathcal{M}, F) \to LOC(\mathcal{N}, L\Theta(F))$$

which is a weak equivalence provided that all of the $\Theta(X) : \mathcal{M}(X) \to \mathcal{N}(X)$ are Quillen equivalences.

Proof. We have a natural map $N(\Theta^r(X)) : CORE(\mathcal{M})(X) \to CORE(\mathcal{N}, L\Theta(F))$ and it is easy to check that this restricts to LOC. If $\Theta(X)$ is a Quillen equivalence then $N(\Theta(X)) : CORE(\mathcal{M})(X) \to CORE(\mathcal{N})(X)$ is a weak equivalence by Lemma A.15. The restriction to LOC is a weak equivalence because it is an isomorphism on π_0, as follows from the fact that $Ho(\mathcal{M})$ is equivalent to $Ho(\mathcal{N})$. □

This is useful because it means that we may prove results for $Stab(\mathcal{M}, P)$ instead of the more complicated $Stab^{\Sigma}(\mathcal{M}, P)$, at least under the reasonable conditions explained after Corollary A.112.

We now need to establish a technical result about spectra. This is mostly about formalism, the substance is well known. See section A.9.3 for details about the category of naive spectra and its model structure. We just recall that objects in $Stab(\mathcal{M}, P)$ are sequences $X_n \in \mathcal{M}, n = 0, 1, \dots,$ together with structure maps $X_n \otimes P \to X_{n+1}$. For $n \geq 0$ we define an endofunctor $[n] : Stab(\mathcal{M}, P) \to Stab(\mathcal{M}, P)$ by $(X[n])_k = X_{n+k}$, with the obvious structure maps.

Recall the notation $\Omega_P = \underline{\mathrm{Hom}}(\bullet, P)$.

Lemma 2.18. *Let $\mathcal{M}$ be a left proper, closed symmetric monoidal, combinatorial model category and $P \in \mathcal{M}$ be cofibrant.*

The functor $[n] : Stab(\mathcal{M}, P) \to Stab(\mathcal{M}, P)$ affords a left adjoint $[-n]^{\emptyset}$ and a right adjoint $[-n]^{\Omega}$. It is bi-Quillen, and the derived functors form adjoint pairs of equivalences.

Proof. We put $(X[-n]^{\emptyset})_k = \emptyset$ if $k < n$ and $(X[-n]^{\emptyset})_k = X_{k-n}$ if $k \geq n$. Similarly we put $(X[-n]^{\Omega})_k = \Omega_P^{n-k} X_n$ if $k < n$ and $(X[-n]^{\Omega})_k = X_{k-n}$ if $k \geq n$. In each case there are evident structure maps, and checking the adjunction is straightforward (see also Lemma A.94).

The functors $[n]$ and $[-n]^{\Omega}$ preserve fibrant objects (i.e. termwise fibrant homotopy cartesian objects), and also fibrations between fibrant objects and acyclic fibrations (since these are determined objectwise), so are right Quillen [43, Proposition 8.5.4].

To prove that the derived functors are adjoint equivalences, it is enough to show that for each fibrant X the natural maps $X[-n]^{\Omega}[n] \to X$ and $X \to X[n][-n]^{\Omega}$ are equivalences (recall that $[n]$ is bi-Quillen and so preserves fibrations, cofibrations and weak equivalences and is its own derived functor). Now $X[-n]^{\Omega}[n] \to X$ is actually an isomorphism, and $X_k \to (X[n][-n]^{\Omega})_k$ is either $X_k \to \Omega^{n-k} X_n$ if $n \geq k$, which is a weak equivalence by holim-fibrancy of X, or $X_k \to X_k$ if $k \geq n$, which is an isomorphism. This concludes the proof. □

Now let again $\mathcal{M}$ be a symmetric monoidal τ-Quillen presheaf on the Verdier site $(\mathcal{C}, \tau)$ satisfying the projection formula, fix $P \in \mathcal{M}(*)$ cofibrant and also $F \in \mathcal{M}(*)$ cofibrant. We would like to define a map $\mathrm{hocolim}_n \, LOC(\mathcal{M}, F \otimes P^{\otimes n}) \to LOC(Stab(\mathcal{M}, P), \Sigma^{\infty} F)$. There is an evident map

$$N(\otimes P) : LOC(\mathcal{M}, F \otimes P^{\otimes n}) \to LOC(\mathcal{M}, F \otimes P^{\otimes n+1}).$$

It is just the sectionwise nerve of the map which sends an object T which is locally equivalent to $F \otimes P^{\otimes n}$ to $T \otimes P$, which is locally equivalent to $F \otimes P^{\otimes n+1}$. There is also a natural map

$$N(\Sigma^{\infty - n}) : LOC(\mathcal{M}, F \otimes P^{\otimes n}) \to LOC(Stab(\mathcal{M}, P), \Sigma^{\infty} F).$$

This is just the nerve of the map which sends an object T locally equivalent to $F \otimes P^{\otimes n}$ to the spectrum

$$\Sigma^{\infty}(T)[-n]^{\emptyset} = (\emptyset, \emptyset, \dots, T, T \otimes P, T \otimes P \otimes P, \dots).$$

Note that Σ^{∞} and $[-n]^{\emptyset}$ are left Quillen functors, so this spectrum is cofibrant if T is, as necessary. Also note that $\Sigma^{\infty}(F \otimes P^{\otimes n})[-n]^{\emptyset} = \Sigma^{\infty}(F)[n][-n]^{\emptyset} \simeq \Sigma^{\infty}(F)$, so $\Sigma^{\infty}(T)[-n]^{\emptyset}$ is indeed locally equivalent to $\Sigma^{\infty} F$.

Lemma 2.19. *Let $\mathcal{M}$ be a left proper, closed symmetric monoidal τ-Quillen presheaf on the suitable Verdier site $(\mathcal{C}, \tau)$ with final object $*$. Fix $P, F \in \mathcal{M}(*)$ cofibrant. The above construction can be completed to a commutative diagram*

$$\begin{array}{ccc} LOC(\mathcal{M}, F) & \xrightarrow{N(\Sigma^\infty)} & LOC(Stab(\mathcal{M}, P), \Sigma^\infty F) \\ {\scriptstyle N(\otimes P)}\downarrow & & \downarrow{\scriptstyle N([1][-1])^\emptyset} \\ LOC(\mathcal{M}, F \otimes P) & \xrightarrow{N(\Sigma^{\infty-1})} & LOC(Stab(\mathcal{M}, P), \Sigma^\infty F) \\ {\scriptstyle N(\otimes P)}\downarrow & & \downarrow{\scriptstyle N([1][-1])^\emptyset} \\ \cdots & & \cdots \end{array}$$

The vertical maps on the right are global weak equivalences, and so there is for each $X \in \mathcal{C}$ an induced map

$$\operatorname{hocolim}_n LOC(\mathcal{M}, F \otimes P^{\otimes n})(X) \to LOC(Stab(\mathcal{M}, P), \Sigma^\infty F)(X)$$

in $Ho(sSet)$.

Proof. The diagram

$$\begin{array}{ccc} \mathcal{M}(X) & \xrightarrow{\Sigma^{\infty-n}} & Stab(\mathcal{M}(X), P_X) \\ {\scriptstyle \otimes P}\downarrow & & \downarrow{\scriptstyle [1][-1]^\emptyset} \\ \mathcal{M}(X) & \xrightarrow{\Sigma^{\infty-n-1}} & Stab(\mathcal{M}(X), P_X) \end{array}$$

commutes up to natural isomorphism. Hence the diagram obtained by extending infinitely in the vertical direction also commutes up to natural isomorphism. This yields a pseudo-morphism of pseudofunctors on $\mathbb{N}$ and, upon strictification, a strict morphism of strict functors. The spaces LOC are nerves of appropriate subcategories of the strictified functors to which the maps in the diagram restrict, hence we obtain the first claim.

The functor $[1][-1]^\emptyset$ is a left Quillen equivalence by Lemma 2.18, hence induces a weak equivalence on nerves by Lemma A.15. Since the indexing category $\mathbb{N}$ is contractible, Lemma A.19 implies that $\operatorname{hocolim}_n LOC(Stab(\mathcal{M}, P), \Sigma^\infty F)(X) \simeq LOC(Stab(\mathcal{M}, P), \Sigma^\infty F)(X)$. This concludes the proof. □

The goal now is to prove that the natural comparison map produced by this lemma is a weak equivalence. This seems to require some very non-trivial conditions on $(\mathcal{C}, \tau)$ and also on $\mathcal{M}$.

Lemma 2.20. *Homotopy colimits in $sPre(\mathcal{C})_\tau$ are computed sectionwise.*

Proof. Use the definitions of Section A.1.5. Given a diagram $X : I \to sPre(\mathcal{C})$, we may compute $\operatorname{hocolim}_I X$ as "$\operatorname{hocolim}_I$"$R_c(X)$, where R_c is a cofibrant replacement functor. Now "$\operatorname{hocolim}_I$" is a certain functor built out of the simplicial tensoring, which is defined sectionwise, whence ("$\operatorname{hocolim}_I$"$X)(T)$ = "$\operatorname{hocolim}_I$"$(X(T))$. Thus the result follows if $X(i)(T) \to R_c X(i)(T)$ is a weak equivalence for all $i \in I$ and all $T \in \mathcal{C}$. But the local model structure has the same cofibrations as the global one, so we may choose the same cofibrant replacement functor, i.e. one which induces a global weak equivalence. □

Corollary 2.21. *Sequential colimits in $sPre(\mathcal{C})$ are homotopy colimits (in both the local and the global model structure).*

Proof. Since homotopy colimits are computed sectionwise, and so are ordinary colimits, we need only know that sequential colimits of simplicial sets are homotopy colimits (this is well known; it follows for example from the fact that the standard presentation of the model category structure on simplicial sets is (almost) finitely generated in the sense of Section A.4, and Lemma A.43). □

Corollary 2.22. *If fibrant objects in $sPre(\mathcal{C})_{proj,\tau}$ are preserved under sequential colimits, then τ-local objects are preserved under sequential homotopy colimits.*

Proof. Since homotopy colimits are preserved under (local!) weak equivalences, we may assume given a sequence of τ-local fibrant objects. Now their (sequential) homotopy colimit is just given by their colimit by the previous corollary, and this sequential colimit is still τ-locally fibrant, by assumption. The result follows. □

We note that fibrant objects are preserved under sequential homotopy colimits in almost finitely generated model categories, so for example if the topology τ is generated by a complete, regular, bounded cd structure. See Section A.4.

Theorem 2.23. *Let $(\mathcal{C}, \tau)$ be a Verdier site with final object $*$ and $\mathcal{M}$ a simplicial, closed symmetric monoidal, almost finitely generated τ-Quillen presheaf on $\mathcal{C}$ satisfying the projection formula. Fix $P, F \in \mathcal{M}(*)$ cofibrant.*

Suppose that for each $X \in \mathcal{C}$, the functor $\Omega_{P_X} : \mathcal{M}(X) \to \mathcal{M}(X)$ commutes with sequential colimits and F_X defines a sequentially compact object of $Ho(\mathcal{M}(X))$.

Then the canonical map $\operatorname{hocolim}_n GLOB(\mathcal{M}, F \otimes P^{\otimes n}) \to GLOB(Stab(\mathcal{M}, P), \Sigma^\infty F)$ *is a global weak equivalence.*

Proof. Fix $X \in \mathcal{C}$. Since homotopy groups of simplicial sets commute with sequential colimits, by Corollary 2.15 it is thus enough to show that

$$\operatorname{colim}_n[(F \otimes P^{\otimes n})_X, (F \otimes P^{\otimes n})_X]^\times \to [\Sigma^\infty(F)_X, \Sigma^\infty(F)_X]^\times$$

is an isomorphism and also for all $i \geq 1$,

$$\operatorname{colim}_n[S^i \otimes (F \otimes P^{\otimes n})_X, (F \otimes P^{\otimes n})_X] \to [S^i \otimes \Sigma^\infty(F)_X, \Sigma^\infty(F)_X]$$

is an isomorphism. The former statement follows from the latter for $i = 0$, so we shall prove the latter for $i \geq 0$.

We note that $[S^i \otimes (F \otimes P^{\otimes n})_X, (F \otimes P^{\otimes n})_X] = [S^i \otimes F_X, R\Omega^n_{P_X}(F_X \otimes P_X^{\otimes n})]$. Now since F_X defines a compact object of $Ho(\mathcal{M}(X))$ so does $S^i \otimes F_X$. Since in an almost finitely generated model category sequential colimits are sequential homotopy colimits by Lemma A.43, we compute as follows:

$$\begin{aligned}\operatorname{colim}_n[S^i \otimes F_X, R\Omega^n_{P_X}(F_X \otimes P_X^{\otimes n})] &= [S^i \otimes F_X, \operatorname{hocolim}_n R\Omega^n_{P_X}(F_X \otimes P_X^{\otimes n})] \\ &= [S^i \otimes F_X, \operatorname{colim}_n R\Omega^n_{P_X}(F_X \otimes P_X^{\otimes n})].\end{aligned}$$

It follows from [46, Proposition 4.4] and our assumption that $\Omega^\infty_{P_X}$ commutes with sequential colimits that

$$\operatorname{colim}_n R\Omega^n_{P_X}(F_X \otimes P_X^{\otimes n}) \simeq R\Omega^\infty\Sigma^\infty F_X.$$

The result follows. □

In particular, under the assumptions of the theorem, the map $\operatorname{hocolim}_n LOC(\mathcal{M}, F \otimes P^{\otimes n}) \to LOC(Stab(\mathcal{M}, P), \Sigma^\infty F)$ is always a local weak equivalence. We can do better:

Corollary 2.24. *In the situation of the theorem, if the Verdier site $(\mathcal{C}, \tau)$ is suitable, $\mathcal{M}$ is a τ-Quillen sheaf and fibrant objects are preserved under sequential colimits in $sPre(\mathcal{C})_{proj,\tau}$, then* $\operatorname{hocolim}_n LOC(\mathcal{M}, F \otimes P^{\otimes n}) \to LOC(Stab(\mathcal{M}, P), \Sigma^\infty F)$ *is a global weak equivalence.*

Proof. By Proposition 2.13 (and the 2-out-of-3 property) and the theorem we conclude that the map is a local weak equivalence.

By Theorem 2.7 each $LOC(\mathcal{M}, F \otimes P^{\otimes n})$ is τ-local, and hence so is $\operatorname{hocolim}_n LOC(\mathcal{M}, F \otimes P^{\otimes n})$ by Corollary 2.22. Also $LOC(Stab(\mathcal{M}, P), \Sigma^\infty F)$ is τ-local, again by Theorem 2.7 (and Theorem A.102). But a local weak equivalence between τ-local objects is a global weak equivalence, so we are done. □

Remark 1. Both the requirement that fibrant objects in $sPre(\mathcal{C})_\tau$ be preserved under sequential colimits and the requirement that each $\mathcal{M}(X)$ be almost finitely generated seem very strong; they usually require the use of a topology which is "finitary" in some strong sense (e.g. the Nisnevich topology as opposed to the étale topology). We see that there are two distinct, but related, causes for this. The theorem only asks for almost finite generation, and this property is needed to control the stabilisation process sufficiently to be able to compute $Map(\Sigma^\infty F, \Sigma^\infty F)$ as a (homotopy) colimit. The corollary also uses finiteness of the topology τ. Essentially it is clear that the image of $\operatorname{colim}_n \pi_0 LOC(\mathcal{M}, F \otimes P^{\otimes n})(X)$ in $\pi_0 LOC(Stab(\mathcal{M}, P), \Sigma^\infty F)(X)$ can only consist of objects which are finite P-desuspensions of suspension spectra. But it is not a priori clear that every object on the right is of this form. This is true locally, by assumption, but if the topology is "infinitary" then it seems conceivable that the required "desuspension degree" may become unbounded globally.

Remark 2. Stabilisation is a homotopy *limit*, not colimit. This suggests that one may wish to describe $LOC(Stab(\mathcal{M}, P), F)$ (where F is a fibrant spectrum) via $\operatorname{holim}_n LOC(\mathcal{M}, F_n)$. There is a comparison map $LOC(Stab(\mathcal{M}, P), F) \to \operatorname{holim}_n LOC(\mathcal{M}, F_n)$ but in order for this to be a week equivalence one seems to need not only strong finiteness conditions, but also the vanishing of a $\lim^1$ obstruction. Additionally fibrations with fibres certain abstract infinite loop spaces F_n seem to carry much less geometric intuition than fibrations with fibres $F \otimes P^{\otimes n}$. Thus we do not pursue further here the holim description.

2.4.3 Comparing Pointed and Unpointed Fibrations

We begin with the following general observation. Recall that a *relative category* is a pair $(\mathcal{C}, W)$ of an ordinary category $\mathcal{C}$ and a subcategory W (sometimes required to satisfy certain properties), and a functor of relative categories $F : (\mathcal{C}, W) \to (\mathcal{D}, W')$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(W) \subset W'$. Dwyer and Kan associate functorially with any relative category $(\mathcal{C}, W)$ a simplicial category $L^H_W \mathcal{C}$ with the same objects as $\mathcal{C}$, called its *Hammock localization* [25]. For us the most important property of this construction is as follows: if $\mathcal{M}$ is a model category and W its subcategory of weak equivalences, then for $X, Y \in \mathcal{M}$ there is a weak equivalence $Map_{L^H_W \mathcal{M}}(X, Y) \simeq Map^d_{\mathcal{M}}(X, Y)$ [26]. Note that this does not require X, Y to be cofibrant or fibrant!

Our immediate application of this theory is as follows. Let $\mathcal{C}$ be an essentially small category and $\mathcal{M}$ a left Quillen (pseudo-) presheaf on $\mathcal{C}$. We define a pseudo-presheaf of simplicial categories $L^H_W \mathcal{M}^c$ by

$$(L^H_W \mathcal{M}^c)(X) = L^H_W \mathcal{M}(X)^c,$$

where $\mathcal{M}(X)^c$ denotes the category of cofibrant objects. By assumption for each $f : X \to Y \in \mathcal{C}$ the restriction f^* is left Quillen, so preserves cofibrant objects and weak equivalences between cofibrant objects. Thus this is indeed a pseudo-presheaf.

Now suppose that $\mathcal{C}$ has a final object $*$ and we are given $F, G \in \mathcal{M}(*)^c$. Recall that for $X \in \mathcal{C}$ we write $F_X := (X \to *)^* F$, and so on. We define a simplicial presheaf

$$R\underline{\operatorname{Hom}}(F, G) \in sPre(\mathcal{C}) \quad X \mapsto Map_{L^H_W \mathcal{M}^c(X)}(F_X, G_X),$$

where as usual we first strictify the pseudo-presheaf $\mathcal{M}$. The following result is clear by construction.

Lemma 2.25. *If $(\mathcal{C}, \tau)$ is a Verdier site and $\mathcal{M}$ a simplicial τ-Quillen presheaf, and $F, G \in \mathcal{M}(*)$ are both cofibrant and fibrant, then the simplicial presheaves $\underline{\operatorname{Hom}}(F, G)_{\mathcal{P}}$ (from Subsection 2.3.1) and $R\underline{\operatorname{Hom}}(F, G)|_{\mathcal{P}}$ are globally weakly equivalent.*

Now let us get to the actual pointing. For this, suppose given a cofibrant object $pt \in \mathcal{M}(*)$ and consider the τ-Quillen presheaf $pt/\mathcal{M} =: \mathcal{M}_*$ (see Subsection A.9.1). We have the underlying object functor $U : \mathcal{M}_* \to \mathcal{M}$ (with left adjoint adding a disjoint base point). To remind ourselves of the difference, we shall write Map_* etc. for mapping spaces computed in $\mathcal{M}_*$.

Lemma 2.26. *For $F_*, G_* \in \mathcal{M}_*(*)$ there is a natural sequence of presheaves of simplicial sets*

$$R\underline{\mathrm{Hom}}_*(F_*, G_*) \to R\underline{\mathrm{Hom}}(UF_*, UG_*) \to R\underline{\mathrm{Hom}}(pt, UG_*).$$

It is a sectionwise homotopy fibration sequence.

Proof. Since hammock localisation is strictly functorial and U preserves cofibrations and weak equivalences (and fibrations) and also cofibrant objects (since pt is cofibrant), we get the first map. The second is restriction along $pt \to UF_*$ and using functoriality of hammock localisation again.

To prove this is a homotopy fibration sequence, we fix $X \in \mathcal{C}$. Let $\tilde{G}_X$ be a fibrant replacement of $G_{*,X}$. Then $U\tilde{G}_X$ is also fibrant (U is right Quillen). Also $F_{*,X}$ is cofibrant, pt_X is cofibrant, $UF_{*,X}$ is cofibrant (because pt_X is and U preserves cofibrations) and $pt_X \to UF_{*,X}$ is a cofibration (this is the definition of $F_{*,X}$ being cofibrant). Consequently the sequence we are trying to prove is a homotopy fibration sequence is (weakly equivalent to)

$$Map_*(F_{*,X}, \tilde{G}_X) \to Map(UF_{*,X}, U\tilde{G}_X) \to Map(pt_X, U\tilde{G}_X).$$

But the map on the right is a fibration (because $pt_X \to UF_{*,X}$ is a cofibration) between fibrant simplicial sets with fibre the simplicial set on the left (by definition), so this *is* a fibration sequence. □

Next we need to review the theory of n-connected spaces and maps.

Definition 2.27. *Let $n > 0$.*

A map $f : X \to Y$ of simplicial sets is called n-connected *if $\pi_0(X) \to \pi_0(Y)$ is a bijection, for every base point $b \in X_0$ and every $i < n$ the map $\pi_i(X, b) \to \pi_i(Y, f(b))$ is an isomorphism, and also $\pi_n(X, b) \to \pi_n(Y, f(b))$ is surjective.*

A non-empty simplicial set X is called n-connected *if $\pi_0(X) = *$ and $\pi_i(X) = 0$ for every $i \leq n$.*

A map of simplicial presheaves $f : F \to G \in sPre(\mathcal{C})$ is called locally n-connected *if the map on homotopy* sheaves *$\underline{\pi}_0 F \to \underline{\pi}_0 G$ is an isomorphism, if for every $i < n$ and every* local *base point the map $\underline{\pi}_i F \to \underline{\pi}_i G$ is an isomorphism, and also $\underline{\pi}_n F \to \underline{\pi}_n G$ is a surjection (of sheaves).*

A simplicial presheaf F is called locally n-connected *if the homotopy sheaf $\underline{\pi}_0(F) = *$ and for every $i \leq n$ and every local base point we have $\underline{\pi}_i F = 0$.*

We emphasize that in the notion of local connectedness, homotopy *sheaves* are used and not homotopy presheaves, and one has to consider all *local* base points. See also the definition of local weak equivalences in Definition A.22; it is clear that local n-connectedness is preserved under local weak equivalence. Also note that a map of simplicial sets $X \to Y$ is n-connected if and only if every homotopy fibre F is n-connected, and a simplicial set X is n-connected if and only if every map $* \to X$ is n-connected. Neither of these statements is true about local n-connectedness (at least when interpreted naively).

Theorem 2.28. *Let $\mathcal{M}$ be a τ-Quillen presheaf on the Verdier site $(\mathcal{C}, \tau)$ with final object $*$. Suppose given a cofibrant object $pt \in \mathcal{M}(*)$ and write $\mathcal{M}_* := pt/\mathcal{M}$.*

Then given $F_* \in Ho(\mathcal{M}(*))$, there is a natural map $LOC(\mathcal{M}_*, F_*) \to LOC(\mathcal{M}, UF_*)$. If the simplicial presheaf $R\underline{\mathrm{Hom}}(pt, UF_*) \in sPre(\mathcal{C})$ is locally n-connected ($n > 1$) then $LOC(\mathcal{M}_*, F_*) \to LOC(\mathcal{M}, UF_*)$ is locally n-connected.

Proof. Since U preserves cofibrant objects and weak equivalences and commutes with base change, the existence of the natural map is clear. It remains to prove the n-equivalence statement. So assume that $R\underline{\mathrm{Hom}}(pt, UF_*)$ is locally n-connected. I claim that $R\underline{\mathrm{Hom}}_*(F_*, F_*) \to R\underline{\mathrm{Hom}}(UF_*, UF_*)$ is locally $(n-1)$-connected. Indeed Lemma 2.26 implies that for every local base point we can get a long exact sequence of sheaves

$$\cdots \to \underline{\pi}_i R\underline{\mathrm{Hom}}_*(F_*, F_*) \to \underline{\pi}_i R\underline{\mathrm{Hom}}(UF_*, UF_*) \to \underline{\pi}_i R\underline{\mathrm{Hom}}(pt, UF_*) \to \underline{\pi}_{i-1} R\underline{\mathrm{Hom}}_*(F_*, F_*) \to \dots$$

In particular $\underline{\pi}_0 R\underline{\mathrm{Hom}}_*(F_*, F_*) = \underline{\pi}_0 R\underline{\mathrm{Hom}}(UF_*, UF_*)$ and thus also $\underline{\pi}_0 R\underline{\mathrm{Hom}}_*(F_*, F_*)^\times = \underline{\pi}_0 R\underline{\mathrm{Hom}}(UF_*, UF_*)^\times$. But by Corollary 2.15 LOC has the same homotopy sheaves as $R\underline{\mathrm{Hom}}_*$ except shifted by one, and except in degree one, where we only get the invertible homotopy-endomorphisms. It follows from the long exact sequences that the natural map of LOC is locally n-connected (we gain one degree of connectedness by delooping). □

We have the following relationship between local and global n-connectedness.

Lemma 2.29. *Let $F \to G$ be a locally n-connected map of τ-locally fibrant simplicial presheaves and let $X \in \mathcal{C}$ have τ-cohomological dimension at most d. Then $F(X) \to G(X)$ is an $(n-d)$-connected map of simplicial sets.*

Proof. We have the induced map of strongly convergent unstable descent spectral sequences $E_2^{pq}(F) = H^p(X, \underline{\pi}_{-q} F) \to E_2^{pq}(G) = H^p(X, \underline{\pi}_{-q} G)$. Considering that the differentials have degree $(r, 1-r)$ on the E_r page, we see that the isomorphism $E_2^{pq}(F) = E_2^{pq}(G)$ for $p > d$ or $q > -n$ is preserved on the infinity page for $-p-q \geq -n+d$ and hence the result follows by standard spectral sequence comparison techniques. □

Corollary 2.30. *Under the assumptions of the theorem, if in addition $(\mathcal{C}, \tau)$ is suitable, $\mathcal{M}$ is a simplicial τ-Quillen sheaf and $X \in \mathcal{C}$ has τ-cohomological dimension at most d, then $LOC(\mathcal{M}_*, F_*)(X) \to LOC(\mathcal{M}, UF_*)(X)$ is $(n-d)$-connected.*

Proof. Combine the theorem and the lemma, using Theorem 2.7 to observe that a globally fibrant replacement of $LOC(\mathcal{M}, UF_*)$ is τ-locally fibrant, and Theorem A.93 for the case of $\mathcal{M}_*$. □

2.4.4 Examples

We are interested in studying the group $Pic^0(\mathcal{SH}, X) := \pi_0 PIC^0(\mathcal{SH})(X)$ and, by extension, the space $PIC^0(\mathcal{SH}) \in sPre(Ft)$ and the spectrum $\mathbb{P}IC^0(\mathcal{SH})$. We know that, up to local weak equivalence in $sPre(Sm)$, this space is the same as $B\underline{Aut}^h(\tilde{\mathbb{1}}_{\mathcal{SH}})$, where $\tilde{\mathbb{1}}$ is a cofibrant-fibrant replacement.

Note that by Corollary 2.15, its homotopy sheaves are

$$a_{Nis}\underline{\pi}_0 B\underline{Aut}^h(\tilde{\mathbb{1}}_{\mathcal{SH}}) = 0$$

$$a_{Nis}\underline{\pi}_1 B\underline{Aut}^h(\tilde{\mathbb{1}}_{\mathcal{SH}}) = \underline{GW}^\times$$

$$a_{Nis}\underline{\pi}_{i+1} B\underline{Aut}^h(\tilde{\mathbb{1}}_{\mathcal{SH}}) = \underline{\pi}_i^{\mathbb{A}^1} \mathbb{1}_{\mathbf{SH}}.$$

Here we have used the identification $\underline{\pi}_0^{\mathbb{A}^1}(\mathbb{1}_{\mathbf{SH}}) = \underline{GW}$ of [79, Section 6].

We start with the following result. It is a special (but easier) case of [115, Theorem 7.1].

Lemma 2.31. *The space $B\underline{Aut}^h(\tilde{\mathbb{1}}_{\mathcal{SH}})$ is $\mathbb{A}^1$-local (as an object of $sPre(Sm)$).*

Proof. Since the space is an infinite loop space, it suffices to show that the homotopy sheaves are all strictly invariant [79, Lemmas 4.3.6 and 4.3.7]. We have computed them above. Since the homotopy sheaves of an $\mathbb{A}^1$-local object are strictly invariant, it only remains to verify that $\underline{GW}^\times$ is strictly invariant. This is shown in [116, Theorem 5.7]. □

Corollary 2.32. *The space $B\underline{Aut}^h(\tilde{\mathbb{1}}_{\mathcal{SH}}) \simeq PIC^0(\mathcal{SH})|_{Sm}$ classifies locally trivial invertible spectra in the $\mathbb{A}^1$-homotopy category:*

$$[X, B\underline{Aut}^h(\tilde{\mathbb{1}}_{\mathcal{SH}})]_{\mathbb{A}^1} = Pic^0(\mathcal{SH})(X)$$

for any $X \in Sm(k)$.

Proof. Combine Lemma 2.31 and Corollary 2.14. □

Now in order to study $Pic^0(\mathcal{SH}, X)$ one might try to look at unstable fibrations. We have the following result.

Proposition 2.33. *The canonical map*

$$\mathrm{hocolim}_n\, LOC(\mathcal{H}_*, (\mathbb{P}^1)^{\wedge n}) \to PIC^0(\mathcal{SH})$$

is a global weak equivalence. In particular there is an isomorphism

$$Pic^0(\mathcal{SH}, X) \cong \mathrm{colim}_n\, \pi_0 LOC(\mathcal{H}_*, (\mathbb{P}^1)^{\wedge n})(X).$$

Proof. Combine Corollary 2.24, Theorem 2.3, Lemma 2.17, the paragraph after Corollary A.112 and the observation that $\Omega_{\mathbb{P}^1}$ commutes with sequential colimits (which is an immediate consequence of representable presheaves being sequentially compact).

For the in particular part, use that π_n commutes with sequential homotopy colimits of simplicial sets. □

Recall that by definition, $\pi_0 LOC(\mathcal{H}_*, (\mathbb{P}^1)^{\wedge n})(X)$ is the set of locally trivial pointed fibrations over X with fibre $(\mathbb{P}^1)^{\wedge n}$, up to homotopy. Here a *pointed fibration* just means a fibration with a section. This section is important, otherwise we cannot construct a stabilisation map

$$\pi_0 LOC(\mathcal{H}_*, (\mathbb{P}^1)^{\wedge n})(X) \to \pi_0 LOC(\mathcal{H}_*, (\mathbb{P}^1)^{\wedge n+1})(X)$$

(which is given by fibre-wise smash product with $\mathbb{P}^1$). It may nonetheless be geometrically more natural to look at unpointed fibrations. To this end we have the following result:

Proposition 2.34. *If X is smooth of dimension d, then the canonical map*

$$LOC(\mathcal{H}_*, (\mathbb{P}^1)^{\wedge n})(X) \to LOC(\mathcal{H}, (\mathbb{P}^1)^{\wedge n})(X)$$

is an $(n-d)$-connected map of simplicial sets. In particular it induces an isomorphism on π_0 for $n > d$.

Proof. This is just Corollary 2.30 in our situation: The Nisnevich cohomological dimension of X is bounded by its Krull dimension [83, Proposition 3.1.8] and the space $R\underline{\mathrm{Hom}}(pt, (\mathbb{P}^1)^{\wedge n})$ is just $\mathbb{P}^1 \in \mathcal{H}$. This is n-connected by the unstable connectivity theorem [82, Theorem 6.38] (this is where we need X smooth) together with the fact that $\mathbb{P}^1 \simeq \mathbb{G}_m \wedge S^1$. □

2.5 Comparison of cdh-locally Trivial Objects and Nisnevich-locally Trivial Objects

For the purpose of this section we need resolution of singularities, so we will work over a field k of characteristic zero. We need to compare the Nisnevich and cdh cohomology. This is a rather subtle issue in general, but the following result is enough for our purposes.

Theorem 2.35 (Voevodsky [108])**.** *If k is a field of characteristic zero, then the natural left Quillen functor*

$$\mathcal{SH}^{S^1}(k) \to \underline{\mathcal{SH}}^{S^1}(k)$$

is a Quillen equivalence.

Recall here that $\mathcal{SH}^{S^1}(k)$ is the S^1-stable $\mathbb{A}^1$-homotopy category built from smooth varieties over k, whereas $\underline{\mathcal{SH}}^{S^1}(k)$ is the S^1-stable $\mathbb{A}^1$-homotopy category built from all varieties over k. This result has many interesting consequences.

Corollary 2.36. *The following statements are true (over a field of characteristic zero).*

1. *The natural left Quillen functor*

$$\mathcal{SH}(k) \to \underline{\mathcal{SH}}(k)$$

is a Quillen equivalence.

2. *Consider the restriction functor $PSh(Ft(k)) \to PSh(Sm(k))$. It induces an equivalence between the categories of strictly homotopy invariant cdh sheaves on $Ft(k)$ and the category of strictly homotopy invariant Nisnevich sheaves on $Sm(k)$. Moreover if F is such an object, and X is a smooth variety, then for any $p \geq 0$ the natural map*

$$H^p_{Nis}(X, F) \to H^p_{cdh}(X, F)$$

is an isomorphism.

3. *In fact $\underline{\mathbf{SH}}^{S^1}(k)$ has a t-structure with heart the strictly homotopy invariant cdh sheaves, and $\mathbf{SH}^{S^1}(k) \to \underline{\mathbf{SH}}^{S^1}(k)$ is a t-exact equivalence.*

Proof. The first statement follows from invariance of stabilisation under Quillen equivalence, see e.g. Lemma A.101. For the second statement, recall that $\mathbf{SH}^{S^1}(k)$ affords a t-structure with heart the strictly homotopy invariant Nisnevich sheaves (on $Sm(k)$).

Similarly, the category $\underline{\mathbf{SH}}^{S^1}(k)$ affords a t-structure with heart the strictly homotopy invariant cdh sheaves on $Ft(k)$. Indeed the proof in [81] for $\mathbf{SH}^{S^1}(k)$ goes through: the category $Ho(Spt(sPre(Ft(k), cdh), S^1))$ affords a t-structure with heart the cdh sheaves by standard theory. Then an $\mathbb{A}^1$-localisation functor can be constructed as a countable direct limit as in [81, end of section 4.2] (the main reason this works is the generation of the topology by distinguished squares). The vanishing result [81, Lemma 4.3.1] (a weak form of stable connectivity) holds with the same proof (the main ingredient is the fact that the Krull dimension bounds the cohomological dimension). Now let $E \in Spt(sPre(Ft(k)), S^1)$ be non-negative in the simplicial t-structure. Write $e : Spt(sPre(Sm(k)), S^1) \leftrightarrows Spt(sPre(Ft(k)), S^1) : r$ for the adjunction. Then $rL_{\mathbb{A}^1}E$ is weakly connective (i.e. $\underline{\pi}_i(rL_{\mathbb{A}^1}E)(K) = 0$ for every field K and $i < 0$) and hence connective. Thus for every smooth Henselian local scheme X we get $0 = [\Sigma^\infty X_+[i], rL_{\mathbb{A}^1}E] = [e\Sigma^\infty X_+[i], L_{\mathbb{A}^1}E]$ and consequently the cdh sheaf $\underline{\pi}_i(L_{\mathbb{A}^1}E)$ vanishes, for $i < 0$, by resolution of singularities. Finally we get the t-structure as in [81, Lemma 6.2.6]. The identification of the heart is clear.

It remains to show that the equivalence $\mathbf{SH}^{S^1}(k) \simeq \underline{\mathbf{SH}}^{S^1}(k)$ is exact for the t-structures. For this it is enough to prove that $e : \mathbf{SH}^{S^1}(k) \to \underline{\mathbf{SH}}^{S^1}(k)$ identifies the subcategories $\mathbf{SH}^{S^1}(k)_{\geq 0}$ and $\underline{\mathbf{SH}}^{S^1}(k)_{\geq 0}$. The former is generated under homotopy colimits by $\Sigma^\infty X_+$ for $X \in Sm(k)$ and the latter by $\Sigma^\infty X_+$ for $X \in Ft(k)$. So e certainly preserves non-negative objects. Moreover distinguished cdh squares become homotopy pushouts in $\underline{\mathbf{SH}}^{S^1}(k)$ (essentially by the definition of the cdh topology), and so resolution of singularities implies that every $\Sigma^\infty X_+$ for $X \in Ft(k)$ reduced can be obtained as an iterated pushout of smooth schemes. Finally if X is non-reduced with reduced closed structure X^r, then $\Sigma^\infty X_+ \simeq \Sigma^\infty X^r_+$, as usual for the cdh topology.

Consequently if $F \in PSh(Ft(k))$ is a strictly homotopy invariant cdh sheaf, then F defines an object $HF \in \underline{\mathbf{SH}}^{S^1}$ such that $[\Sigma^\infty X_+, HF[n]] = H^n_{cdh}(X, F)$. Since r is t-exact (since e is) we have $rHF = HrF$ and thus $H^n_{Nis}(X, F) = [\Sigma^\infty X_+, rHF] = [\Sigma^\infty X_+, HF] = H^n_{cdh}(X, F)$ for all X smooth. □

We can then deduce our main result.

Theorem 2.37. *Let k be a field of characteristic zero and X smooth over k. The natural map*

$$PIC^0(\mathcal{SH})(X) \to PIC^0(\underline{\mathcal{SH}})(X)$$

is a weak equivalence of simplicial sets. In particular, an object of $\mathbf{SH}(X)$ is Nisnevich locally trivial if and only if it is cdh locally trivial.

Proof. We consider the classifying spectra $\mathbb{P}IC^0(\mathcal{SH}) \in \mathbf{SH}^{S^1}(k)$ and $\mathbb{P}IC^0(\underline{\mathcal{SH}}) \in \underline{\mathbf{SH}}^{S^1}(k)$, see Proposition 2.9. Clearly there is a comparison map $\mathbb{P}IC^0(\mathcal{SH}) \to r\mathbb{P}IC^0(\underline{\mathcal{SH}})$.

Now we note that $\underline{\pi}_{i+1}(\mathbb{P}IC^0)$ for $i > 0$ is the same as $\underline{\pi}_i \mathbb{1}$, for $i = 0$ we get the units of $\underline{\pi}_0 \mathbb{1}$, and finally $\underline{\pi}_0(\mathbb{P}IC^0) = *$. This holds for both of the Picard spectra, so in particular they are $\mathbb{A}^1$-local by [116, Theorem 5.7] (see also Lemma 2.31). Furthermore since $e : \mathbf{SH}^{S^1}(k) \leftrightarrows \underline{\mathbf{SH}}^{S^1}(k) : r$ is a t-exact adjoint equivalence, we conclude that $\mathbb{P}IC^0(\mathcal{SH}) \to r\mathbb{P}IC^0(\underline{\mathcal{SH}})$ is an equivalence.

The result follows since $PIC^0 = \Omega^\infty \mathbb{P}IC^0$ and the spaces satisfy descent (by Theorem 2.7), so the global sections have the correct homotopy types.

□

Chapter 3

Recollement and Reduction to Fields

In this chapter we study pointwise trivial objects. This is done by using quite extensively some of the properties of the six functors formalism, namely continuity and gluing. In particular we use very freely the language of *triangulated categories*. See [87] for one of many references.

In Section 3.1 we recall the notion of gluing (or recollement) of triangulated categories, and prove some potentially new results. Recall that a recollement consists of triangulated categories and triangulated functors between them:

$$\mathcal{D}' \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{D}''.$$

They are required to satisfy many compatibilities. In particular i_* has a left adjoint i^* and $j^* = j^!$ has a left adjoint $j_!$. Moreover for every $E \in \mathcal{D}$ the adjunction unit and co-unit fit into a (necessarily functorial) distinguished triangle

$$j_! j^! E \to E \to i_* i^* E \xrightarrow{\partial_E} j_! j^! E[1].$$

It is fairly clear from this, and well known, that the boundary $\partial_E \in [i_* i^* E, j_! j^! E[1]]$ classifies the possible ways of gluing i^*E and $j^!E = j^*E$. In fact fixing $E' \in \mathcal{D}', E'' \in \mathcal{D}''$, the isomorphism classes of objects $E \in \mathcal{D}$ with $i^*E \simeq E'$ and $j^*E \simeq E''$ are in bijection with

$$End(i_*E')^\times \backslash [i_*E', j_!E''[1]]/End(j_!E'')^\times.$$

(The quotienting by automorphisms accounts for the fact that we only require that there *is* an isomorphism $i^*E \simeq E'$, but do not fix one, and similarly for j^*E.) Here for an object X, we write $End(X)^\times$ for the invertible endomorphisms (i.e. automorphisms) of X.

If the categories $\mathcal{D}', \mathcal{D}, \mathcal{D}''$ are symmetric monoidal, and so are the functors i^* and j^*, we speak of a *symmetric monoidal recollement*. One may then prove that the set $[i_*\mathbb{1}, j_!\mathbb{1}[1]]$ acquires a product. (Essentially, if ∂_E and ∂_F are two elements classifying E and F respectively, then $\partial_E \otimes \partial_F$ is the element classifying $E \otimes F$. Note that if $i^*E \simeq \mathbb{1}$ and $i^*F \simeq \mathbb{1}$, also $i^*(E \otimes F) \simeq \mathbb{1}$ canonically, and similarly for j^*.) It is then fairly clear that there is an exact sequence as follows:

$$0 \to End(i_*\mathbb{1})^\times \backslash [i_*\mathbb{1}, j_!\mathbb{1}[1]]^\times / End(j_!\mathbb{1})^\times \to Pic(\mathcal{D}) \xrightarrow{(i^*, j^*)} Pic(\mathcal{D}') \times Pic(\mathcal{D}'').$$

The importance of this result is that we managed to identify, among those objects of $\mathcal{D}$ glued from $\mathbb{1} \in \mathcal{D}'$ and $\mathbb{1} \in \mathcal{D}''$, the ones which are actually invertible.

In order to use this result, we need to be able to compute the rings $[i_*\mathbb{1}, i_*\mathbb{1}], [j_!\mathbb{1}, j_!\mathbb{1}]$ and $[i_*\mathbb{1}, j_!\mathbb{1}[1]]$, or at least find relations among them. In order to think about this, we find it helpful to pretend that the categories $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ and the functors between them are enriched in (the homotopy category of) spectra in an appropriate sense. We write $\mathbb{R}\mathrm{Hom}(E, F)$ for the mapping spectra. Then contemplating the issue, one is naturally led to believe that there should be a distinguished triangle for $E, F \in \mathcal{D}$

$$\mathbb{R}\mathrm{Hom}(E, F) \to \mathbb{R}\mathrm{Hom}(i^*E, i^*F) \oplus \mathbb{R}\mathrm{Hom}(j^!E, j^!F) \to \mathbb{R}\mathrm{Hom}(i_*i^*E, j_!j^!F[1]).$$

(Basically this expresses how morphisms in $\mathcal{D}$ are glued from morphisms in $\mathcal{D}', \mathcal{D}''$.) It is possible to make this enrichment precise, but this would take us too far afield. Instead, in Section 3.2 we just prove that there is a long exact sequence relating the groups $[E, F[*]], [i^*E, i^*F[*]], [j^!E, j^!F[*]]$ and $[i_*i^*E, j_!j^!F[* + 1]]$ in the manner one would expect from the existence of the distinguished triangle.

The rest of this chapter consists of applications of the above two results, and a few more preparations needed for the applications. In section 3.3 we briefly review some aspects of the six functors formalism and prove the easy implication that the presheaf of Picard groups is continuous if the functor is.

In Section 3.4 we very rapidly recall Hironaka's theorem about resolution of singularities in the following strong form: if U is a smooth variety and Z is a reduced closed subscheme, then by strategic blowups one may find a proper birational morphism $U' \to U$ such that the preimage Z' of Z is a normal crossings scheme.

In Section 3.5 we prove the main theorem of this chapter: if k is a field of characteristic 0, X a variety over k (of finite 2-étale cohomological dimension) and $E \in Pic(\mathbf{SH}(X))$ is pointwise trivial (i.e. for every morphism $i : Spec(K) \to X$ with K a field, the pullback $i^*(E) \in \mathbf{SH}(K)$ is equivalent to $\mathbb{1}$), then E is cdh locally trivial. The strategy of the proof is to use continuity and generic triviality to find an everywhere dense open subset U of X such that $E|_U$ is trivial. Write Z for the reduced closed complement of U in X. Using resolution of singularities, we may assume that X is smooth and Z is normal crossings. By an inductive procedure we may assume that E is Nisnevich locally trivial on Z; using the long exact sequence and gluing for Picard groups results from above we conclude.

Finally in Section 3.6 we prove a similar but easier result for motives: if $E \in \mathbf{DM}(X)$ is pointwise trivial, then it is trivial.

3.1 Gluing and Picard Groups

Recall that a *recollement* is something like a short exact sequence of triangulated categories. There are various ways of presenting the data, ranging from one triangulated category with a subcategory to three triangulated categories with six functors between them; of course satisfying various conditions. The following definition is extracted from [34, Exercise IV.4.4].

Definition 3.1. *A* recollement *or* gluing *of triangulated categories consists of three triangulated categories and two triangulated functors*

$$\mathcal{D}' \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{D}''$$

satisfying the following conditions:

(a) $i_! := i_*$ *admits a left adjoint* i^* *and a right adjoint* $i^!$

(b) $j^! := j^*$ *admits a left adjoint* $j_!$ *and a right adjoint* j_*

(c) $j^*i_* = 0$

(d) given $d \in \mathcal{D}$ *there exist distinguished triangles*

$$i_!i^!d \to d \to j_*j^*d$$

$$j_!j^!d \to d \to i_*i^*d$$

(e) $i_*, j_*, j_!$ are full embeddings.

Standard examples of recollements of triangulated categories come from derived categories of sheaves on a topological space X with an open subset U and closed complement Z. This is also where the notation for the abstract case, with the slight idiosyncrasies such as $i_! := i_*$, comes from.

These axioms have many consequences. We note that the triangles of point (d) are necessarily unique and functorial, as explained in the reference. They are called *gluing triangles*. Also it follows from (c) by adjunction that $i^* j_! = 0$ and $i^! j_* = 0$. Together with the gluing triangles, this implies that $i^* i_* \cong \mathrm{id}, j^* j_* \cong \mathrm{id}$ and $j^* j_! \cong \mathrm{id}$.

It is tempting to believe that $\mathcal{D}$ can be recovered from $\mathcal{D}', \mathcal{D}''$. This is not quite true. However it is well known that the objects of $\mathcal{D}$ up to isomorphism can be recovered. This is done as follows.

Fix $X' \in \mathcal{D}', X'' \in \mathcal{D}''$. A *gluing datum* for (X', X'') consists of an object $X \in \mathcal{D}$ together with isomorphisms $\alpha' : i^* X \simeq X'$ and $\alpha'' : j^* X \simeq X''$. An isomorphism of gluing data (X, α', α'') and (Y, β', β'') consists is a morphism $f : X \to Y$ such that the following diagrams commute:

$$\begin{array}{ccccccc} i^*X & \xrightarrow{i^*f} & i^*Y & & j^*X & \xrightarrow{j^*f} & j^*Y \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \beta'} & & {\scriptstyle \alpha''}\downarrow & & \downarrow{\scriptstyle \beta''} \\ X' & = & X' & & X'' & = & X''. \end{array}$$

The second gluing triangle together with $j^! = j^*$ implies that (i^*, j^*) is conservative, hence f is necessarily an isomorphism.

Write $GD(X', X'')$ for the set of gluing data for (X', X''), up to isomorphism. Saying that objects of $\mathcal{D}$ can be recovered from gluing data means the following.

Proposition 3.2. *The set $GD(X', X'')$ is in natural bijection with $[i_* X', j_! X''[1]]$.*

Proof. If (X, α', α'') is a representative of a gluing datum, then the boundary map of the second gluing triangle yields an element $\partial_X \in [i_* X', j_! X''[1]]$. If (Y, β', β'') is an isomorphic representative and $f : X \to Y$ is the isomorphism, consider the commutative diagram

$$\begin{array}{ccccc} & & i_*X' & \xrightarrow{\partial_X} & j_!X''[1] \\ & & {\scriptstyle i_*\alpha'}\uparrow & & \uparrow{\scriptstyle j_!\alpha''[1]} \\ X & \longrightarrow & i_*i^*X & \longrightarrow & j_!j^!X[1] \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & i_*i^*Y & \longrightarrow & j_!j^!Y[1] \\ & & {\scriptstyle i_*\beta'}\downarrow & & \downarrow{\scriptstyle j_!\beta''[1]} \\ & & i_*X' & \xrightarrow{\partial_Y} & j_!X''[1] \end{array}$$

coming from functoriality of the gluing triangle and definition of ∂_X, ∂_Y. The maps $i_*\alpha'$ etc. are isomorphisms by assumption, and the induced maps $i_* X' \to i_* X', j_! X''[1] \to j_! X''[1]$ (obtained by going vertically from the very top to the very bottom of the diagram, which is possible because the "wrong way" arrow is an isomorphism) are the identity, also by assumption (i.e. the definition of isomorphism of gluing data). It follows that $\partial_X = \partial_Y$. We thus have a well-defined map $GD(X', X'') \to [i_* X', j_! X''[1]]$.

The map $[i_* X', j_! X''[1]] \to GD(X', X'')$ given by sending the map $\partial : i_* X' \to j_! X''[1]$ to the isomorphism class of a co-cone is easily checked to be inverse to the one constructed above. □

Let us write $End(X') := [X', X'] = [i_* X', i_* X']$ and similarly $End(X'') := [X'', X''] = [j_! X'', j_! X'']$. Also write $End(X')^\times$ for the invertible endomorphisms, and similarly for $End(X'')^\times$. Note that $GD(X', X'') \cong [i_* X', j_! X''[1]]$ carries a natural left $End(X')$-action and right $End(X'')$-action (coming from composition).

Corollary 3.3. *The set* $\{X \in \mathcal{D} | i^*X \simeq X', j^*X \simeq X''\}/iso$ *is in natural bijection with*

$$End(X')^{\times} \backslash GD(X', X'')/End(X'')^{\times}.$$

Proof. If $d_1 = (X, \alpha', \alpha'')$ and $d_2 = (Y, \beta', \beta'')$ are two gluing data and $f : X \to Y$ is an isomorphism in $\mathcal{D}$ (not necessarily compatible with the gluing data), then

$$(\beta'(i^*f)\alpha'^{-1})d_1(\beta''(j^*f)\alpha''^{-1}) \cong d_2$$

via f. Conversely, given $a' \in End(X')^{\times}$ and $a'' \in End(X'')^{\times}$ we have a commutative diagram of distinguished triangles

$$\begin{array}{ccccc} X & \longrightarrow & i_*X' & \xrightarrow{\partial} & j_!X''[1] \\ {\scriptstyle t}\downarrow & & {\scriptstyle a'^{-1}}\downarrow & & {\scriptstyle a''}\downarrow \\ Y & \longrightarrow & i_*X' & \xrightarrow{a'\partial a''} & j_!X''[1]. \end{array}$$

This is obtained by noting that the square on the right commutes and then applying one of the axioms of triangulated categories (often denoted $TR3$).

Here t is an isomorphism since a', a'' are and consequently the objects defined by the gluing data corresponding to ∂ and $a'\partial a''$ are isomorphic. This concludes the proof. □

The situation gets somewhat more interesting if we add a monoidal structure to our categories.

Definition 3.4. *A (symmetric) monoidal recollement consists of a recollement* $\mathcal{D}' \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{D}''$ *where* $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ *are (symmetric) monoidal categories and* i^*, j^* *are (symmetric) monoidal functors.*

In this situation there is a natural homomorphism $Pic(\mathcal{D}) \to Pic(\mathcal{D}') \times Pic(\mathcal{D}'')$. We intend to study its kernel.

Note that the monoidal structures induce a natural pairing $GD(X', X'') \times GD(Y', Y'') \to GD(X' \otimes Y', X'' \otimes Y'')$ (just given by $(X, \alpha', \alpha''), (Y, \beta', \beta'') \mapsto (X \otimes Y, \alpha' \otimes \beta', \alpha'' \otimes \beta'')$), and that both sides are $End(X') \times End(Y') - End(X'') \times End(Y'')$-bimodules.

Proposition 3.5. *Given a monoidal recollement, the pairing* $GD(X', X'') \times GD(Y', Y'') \to GD(X' \otimes Y', X'' \otimes Y'')$ *satisfies the following properties for* $d_1, d_2 \in GD(X', X'')$, $e_1, e_2 \in GD(Y', Y'')$, $a_1 \in End(X'), a_2 \in End(Y')$, $b_1 \in End(X''), b_2 \in End(Y''), \lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{Z}$*:*

1. *Bilinearity:* $(\lambda_1 d_1 + \lambda_2 d_2) \otimes (\mu_1 e_1 + \mu_2 e_2) = \lambda_1\mu_1(d_1 \otimes e_1) + \lambda_1\mu_2(d_1 \otimes e_2) + \lambda_2\mu_1(d_2 \otimes e_1) + \lambda_2\mu_2(d_2 \otimes e_2)$
2. $(a_1 \otimes a_2)(d_1 \otimes e_1) = (a_1 d_1) \otimes (a_2 e_1)$
3. $(d_1 \otimes e_1)(b_1 \otimes b_2) = (d_1 b_1) \otimes (e_1 b_2)$.

The proof will occupy the rest of this section. Before embarking on it, let us point out some related results. We write $GD(\mathbb{1}) := GD(\mathbb{1}', \mathbb{1}'') \cong [i_*\mathbb{1}', j_!\mathbb{1}''[1]]$. This is of course an abelian group. Moreover the pairing $GD(\mathbb{1}) \times GD(\mathbb{1}) \to GD(\mathbb{1} \otimes \mathbb{1})$ together with the isomorphism $\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$ gives this abelian group a multiplication.

Corollary 3.6. *The set* $GD(\mathbb{1})$ *is a ring with the above operations, which is commutative in the symmetric monoidal setting.*

The natural maps $\rho : End(i_*\mathbb{1}') \to GD(\mathbb{1}) \cong [i_*\mathbb{1}', j_!\mathbb{1}''[1]]$ *and* $\Theta : End(j_!\mathbb{1}'') \cong End(j_!\mathbb{1}''[1]) \to GD(\mathbb{1}) \cong [i_*\mathbb{1}', j_!\mathbb{1}''[1]]$ *coming from composition are ring homomorphisms and in the symmetric monoidal situation induce an isomorphism*

$$ker(Pic(\mathcal{D}) \to Pic(\mathcal{D}') \times Pic(\mathcal{D}'')) \cong End(i_*\mathbb{1}')^{\times} \backslash GD(\mathbb{1})^{\times}/End(j_!\mathbb{1}'')^{\times}.$$

Proof. Associativity of the multiplication in $GD(\mathbb{1})$ follows from the associativity in the monoidal structure. Distributivity of addition over multiplication is property (1) of the proposition.

That Θ and ρ respect addition is clear. To see that this holds for multiplication, observe that for $\alpha, \beta \in End(i_*\mathbb{1}')$ we have $\alpha \circ \beta = \alpha \otimes \beta$ under the isomorphism $\mathbb{1}' \cong \mathbb{1}' \otimes \mathbb{1}'$ coming from Lemma 3.7 below [8, sentence before Proposition 2.2], and then use part (2) of the proposition. Similarly on the other side.

Now we prove the statement about Picard groups. By Corollary 3.3, the set of isomorphism classes of objects $E \in \mathcal{D}$ with $i^*E \simeq \mathbb{1}'$ and $j^*E \simeq \mathbb{1}''$ is in bijection with the set $End(i_*\mathbb{1}')^\times \backslash GD(\mathbb{1})/End(j_!\mathbb{1}'')^\times$. By construction, tensor product of such objects corresponds to multiplication in $GD(\mathbb{1})$. Hence $x \in GD(\mathbb{1})$ represents an invertible object with inverse represented by $y \in GD(\mathbb{1})$ if and only if $xy \sim 1 \in End(i_*\mathbb{1}')^\times \backslash GD(\mathbb{1})/End(j_!\mathbb{1}'')^\times$. That is to say there are $t \in End(i_*\mathbb{1}')^\times$ and $u \in End(j_!\mathbb{1}'')^\times$ with $\rho(t)xy\Theta(u) = 1 \in GD(\mathbb{1})$. Since ρ, Θ are ring homomorphisms they preserve units and consequently x is a unit. The converse is clear. This concludes the proof. □

For the purpose of the proof of the proposition, let us call a functor F *weakly monoidal* if it satisfies $F(X \otimes Y) \cong F(X) \otimes F(Y)$, but not necessarily $F(\mathbb{1}) \cong \mathbb{1}$.

Lemma 3.7. *In the situation of a monoidal recollement, the functors $i_*, j_!$ are weakly monoidal. Moreover we have projection formulas $i_*(X) \otimes Y \simeq i_*(X \otimes i^*Y)$ and $j_!(Z) \otimes Y \cong j_!(Z \otimes j^!(Y))$ (for $X \in \mathcal{D}', Y \in \mathcal{D}, Z \in \mathcal{D}''$).*

Proof. We prove the statements about i_*, the proofs for $j_!$ are similar.

An object Y is in the essential image of i_* if and only if $j^*Y = 0$. Necessity follows from $j^*i_* = 0$ and sufficiency follows by considering the gluing triangle $j_!j^!Y \to Y \to i_*i^*Y$. Since $j^! = j^*$ is monoidal, it follows that $i_*\mathcal{D}'$ is a monoidal ideal: if X is in the essential image of i_* then so is $X \otimes Y$ for any Y.

Now let $X, X' \in \mathcal{D}'$. Then $i_*(X) \otimes i_*(X')$ is in the essential image of i_* by the previous remark, and since i^* is a section of the embedding i_* it suffices to prove that $i^*(i_*(X) \otimes i_*(X')) \simeq X \otimes X'$. This is clear because i^* is monoidal.

Finally we have $i_*(X) \otimes Y$ in the essential image of i_*, so to show it is isomorphic to $i_*(X \otimes i^*Y)$ we may again apply the monoidal section i^*. The result follows. □

Proof of Proposition 3.5. Let (X, α, β) represent an element $\partial_X \in GD(X', X'') \cong [i_*X', j_!X''[1]]$ and (Y, α', β') represent an element $\partial_Y \in GD(Y', Y'')$. We have the canonical commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & i_*i^*X & \longrightarrow & j_!j^!X[1] \\ & & {\scriptstyle i_*\alpha}\big\downarrow & & {\scriptstyle j_!\beta}\big\downarrow \\ & & i_*X' & \xrightarrow{\partial_X} & j_!X''[1]. \end{array}$$

We tensor the whole diagram with Y, use the projection formulas and thus build the following commutative diagram:

$$\begin{array}{ccccc}
 & & i_*i^*(X\otimes Y) & \longrightarrow & j_!j^!(X\otimes Y)[1] \\
 & & \uparrow & & \uparrow \\
X\otimes Y & \longrightarrow & (i_*i^*X)\otimes Y & \longrightarrow & (j_!j^!X[1])\otimes Y \\
 & & \downarrow{\scriptstyle i_*\alpha\otimes \mathrm{id}_Y} & & \downarrow{\scriptstyle j_!\beta\otimes\mathrm{id}_Y} \\
 & & (i_*X')\otimes Y & \xrightarrow{\partial_X\otimes\mathrm{id}_Y} & (j_!X''[1])\otimes Y \\
 & & \downarrow & & \downarrow \\
 & & i_*(X'\otimes i^*Y) & \longrightarrow & j_!(X''[1]\otimes j^!Y) \\
 & & \downarrow{\scriptstyle i_*(\mathrm{id}_{X'}\otimes\alpha')} & & \downarrow{\scriptstyle j_!(\mathrm{id}_{X''}\otimes\beta')} \\
 & & i_*(X'\otimes Y') & \xrightarrow{\omega} & j_!(X''\otimes Y''[1]).
\end{array}$$

Here all the vertical maps are isomorphisms, the unlabelled vertical maps are from the projection formulas, the unlabelled horizontal maps are the ones required to make the diagram commute, and similarly for the map labelled ω. I claim that $\omega = \partial_X \otimes \partial_Y$. To see this, it is enough to show that the vertical composites from the very top to the very bottom $i_*i^*(X\otimes Y) \to i_*(X'\otimes Y')$ and $j_!j^!(X\otimes Y)[1] \to j_!(X''\otimes Y'')[1]$ are respectively equal to $i_*(\alpha\otimes\alpha')$ and $j_!(\beta\otimes\beta')[1]$.

We prove the first equality, the second is similar. To do this, as in the lemma, we use that i^* is a section to i_*; hence it is enough to apply i^* to the left vertical composite and show that we get $\alpha\otimes\alpha'$. But under i^* the unlabelled vertical maps become identities (up to natural identifications coming from the monoidal structure isomorphisms) and so the claim follows from $\alpha\otimes\alpha' = (\alpha\otimes\mathrm{id})\circ(\mathrm{id}\otimes\alpha')$.

We conclude that under our particular isomorphisms $i_*X'\otimes Y \simeq i_*(X'\otimes Y')$ and $j_!X''[1]\otimes Y \simeq j_!(X''\otimes Y'')[1]$ (depending on ∂_Y but not ∂_X!), the element $\partial_X\otimes\partial_Y \in [i_*(X'\otimes Y'), j_!(X''\otimes Y'')[1]] \cong [i_*X'\otimes Y, j_!X''[1]\otimes Y]$ corresponds to $\partial_X\otimes\mathrm{id}_Y$. From this bilinearity on the left in statement (1) to be proved follows. We also find that $(a_1\otimes\mathrm{id})(d_1\otimes e_1) = (a_1d_1)\otimes e_1$ and that $(d_1\otimes e_1)(b_1\otimes\mathrm{id}) = (d_1b_1)\otimes e_1$.

Repeating the same argument but with the roles of X and Y reversed we get bilinearity on the right for statement (1) and also $(\mathrm{id}\otimes a_2)(d_1\otimes e_1) = d_1\otimes(a_2e_1)$ and similarly on the right. We then compute

$$(a_1\otimes a_2)(d_1\otimes e_1) = [(a_1\otimes\mathrm{id})\circ(\mathrm{id}\otimes a_2)](d_1\otimes e_1) = (a_1\otimes\mathrm{id})(d_1\otimes(a_2e_1)) = (a_1d_1)\otimes(a_2e_1).$$

Here we have used that $GD(X',X'')$ is a *module* over $End(X')$. This establishes (2); (3) is done similarly. □

3.2 A Long Exact Sequence

The following result will be useful for computations.

Proposition 3.8. *Let $\mathcal{D}' \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{D}''$ be a recollement and $X, Y \in \mathcal{D}$. There is a long exact sequence*

$$\cdots \to [X, Y[n]] \xrightarrow{(i_*i^*, j_!j^!)} [i_*i^*X, i_*i^*Y[n]] \oplus [j_!j^!X, j_!j^!Y[n]] \xrightarrow{p-q} [i_*i^*X, j_!j^!Y[n+1]] \xrightarrow{\partial} [X, Y[n+1]] \to \dots.$$

Here the map $p : [i_*i^*X, i_*i^*Y[n]] \to [i_*i^*X, j_!j^!Y[n+1]]$ is composition with the natural map $i_*i^*Y[n] \to j_!j^!Y[n+1]$ coming from the gluing triangle for Y, and $q : [j_!j^!X, j_!j^!Y[n]] \cong [j_!j^!X[1], j_!j^!Y[n+1]] \to [i_*i^*X, j_!j^!Y[n+1]]$ is composition with the natural map $i_*i^*X \to j_!j^!X[1]$ coming from the gluing triangle for X.

We will often use the following Corollary.

Corollary 3.9. *In the situation of a monoidal recollement, if* $[\mathbb{1},\mathbb{1}] \to [i_*i^*\mathbb{1}, i_*i^*\mathbb{1}]$ *is an isomorphism and* $[\mathbb{1},\mathbb{1}[1]] = 0$*, then*

$$Pic(\mathcal{D}) \to Pic(\mathcal{D}') \times Pic(\mathcal{D}'')$$

is injective.

Proof. It follows from the long exact sequence and our assumptions that $[j_!\mathbb{1}, j_!\mathbb{1}] \to [i_*\mathbb{1}, j_!\mathbb{1}[1]]$ is an isomorphism. Hence the result follows from Corollary 3.6. □

We now discuss the proposition. The status of this result is slightly peculiar. It clearly looks like a Mayer-Vietoris sequence. In fact if there is a sufficiently rich mapping spectrum functor $M : \mathcal{D} \to \mathbf{SH}$ then we can form the square

$$\begin{array}{ccc} M(X,Y) & \xrightarrow{i_*i^*} & M(i_*i^*X, i_*i^*Y) \\ {\scriptstyle j_!j^!}\downarrow & & {\scriptstyle p}\downarrow \\ M(j_!j^!X, j_!j^!Y) & \xrightarrow{q} & M(i_*i^*X, j_!j^*Y[1]). \end{array}$$

If it is homotopy cartesian, the exact sequence follows. The problem with working at the level of triangulated categories is that in order to establish homotopy cartesian squares one essentially has to use the octahedral axiom, but then one tends to lose control over the maps. One may indeed prove that there is a homotopy cartesian square with the correct vertices, but the author only manages to control two out of the four maps out of which it is made up.

Alternatively one may work in a better kind of category. If instead of working with triangulated categories we work with stable infinity categories (say), one may prove that the square is homotopy cartesian on the nose.[1]

We begin with the following purely algebraic lemma, which must surely be well known.

Lemma 3.10. *Given a morphism of long exact sequences*

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & A'^n & \longrightarrow & A^n & \xrightarrow{f} & B^n & \longrightarrow & A'^{n+1} & \longrightarrow & \dots \\ & & \cong\downarrow & & g\downarrow & & p\downarrow & & \cong\downarrow & & \\ \dots & \longrightarrow & C'^n & \longrightarrow & C^n & \xrightarrow{q} & D^n & \longrightarrow & C'^{n+1} & \longrightarrow & \dots \end{array}$$

where every third vertical map is an isomorphism as indicated, there is a long exact sequence

$$\dots \to A^n \xrightarrow{(f,g)} B^n \oplus C^n \xrightarrow{p-q} D^n \xrightarrow{\partial} A^{n+1} \to \dots.$$

Here ∂ *is the composite* $D^n \to C'^{n+1} \to A'^{n+1} \to A^{n+1}$*, where the arrow in the wrong direction can be inverted since it is an isomorphism by assumption.*

Proof. Verifying exactness at all places is a straightforward diagram chase. □

Proof of Proposition 3.8. Consider the gluing triangles

$$j_!j^!X \xrightarrow{\alpha_X} X \xrightarrow{\beta_X} i_*i^*X \xrightarrow{\partial_X} j_!j^!X[1]$$

$$j_!j^!Y \xrightarrow{\alpha_Y} Y \xrightarrow{\beta_Y} i_*i^*Y \xrightarrow{\partial_Y} j_!j^!Y[1].$$

Using the long exact sequence obtained by mapping $j_!j^!X$ into the second triangle and using $i^*j_! = 0$ we conclude that

$$\circ\alpha_Y : [j_!j^!X, j_!j^!Y[n]] \to [j_!j^!X, Y[n]]$$

[1] http://mathoverflow.net/a/236905/5181

is an isomorphism for all n. Consider now the following diagram:

$$\begin{array}{ccccccccc}
[i_*i^*X, Y[n]] & \xrightarrow{\beta_X\circ} & [X, Y[n]] & \xrightarrow{j_!j^!} & [j_!j^!X, j_!j^!Y[n]] \cong [j_!j^!X, Y[n]] & \xrightarrow{\partial_X\circ} & [i_*i^*X, Y[n+1]] \\
\| & & \downarrow{\scriptstyle i_*i^*} & & \downarrow{\scriptstyle \partial_X\circ} & & \| \\
[i_*i^*X, Y[n]] & \xrightarrow{\circ\beta_Y} & [i_*i^*X, i_*i^*Y[n]] & \xrightarrow{\circ\partial_Y} & [i_*i^*X, j_!j^!Y[n+1]] & \xrightarrow{\circ\alpha_Y} & [i_*i^*X, Y[n+1]].
\end{array}$$

For varying n, these diagrams splice together, and we get long exact sequences at the top and bottom. This is clear at the bottom, and at the top it follows from the fact that the composite $[X, Y[n]] \to [j_!j^!X, j_!j^!Y[n]] \to [j_!j^!X, Y[n]]$ is the same as $\alpha_X\circ$ (by standard facts about adjunctions), so the sequence at the top is just maps out of the gluing triangle for X into Y.

The result thus follows from our lemma, provided we prove that the diagram commutes. For the left hand square this follows from the same fact about adjunctions, for the middle square this follows from functoriality of the gluing triangles, and for the right hand square this is just true on the nose. □

3.3 Recollections about the Six Functors and Continuity

If $\mathcal{M}$ is a Quillen (pseudo-)presheaf on a site $(\mathcal{C}, \tau)$ then $Ho(\mathcal{M}) : X \mapsto Ho(\mathcal{M}(X))$ defines a pseudo-functor. If $\mathcal{M}$ is sufficiently nice then the pseudofunctor $Ho(\mathcal{M})$ has many very good properties. These are generally summarised under the heading "six functors formalism". A good reference at this level of generality is [18].

Basic Setup We fix a base category $\mathcal{S}$ of schemes. It is usually essentially small and consists of Noetherian schemes. Typically whenever $X \in \mathcal{S}$ then any subscheme of X is also in $\mathcal{S}$, as is any localisation of X and also any Henselisation. We then study pseudofunctors $T : \mathcal{S}^{op} \to SymMonTrCat$. This means that for every $X \in \mathcal{S}$ we are given a symmetric monoidal, triangulated category $T(X)$ and for every morphism $f : X \to Y$ in $\mathcal{S}$ we obtain a symmetric monoidal triangulated functor $f^* : T(Y) \to T(X)$. This functor always has a right adjoint $f_* : T(X) \to T(Y)$. If f is separated and of finite type then there is also the exceptional inverse image $f^! : T(Y) \to T(X)$ which affords a *left* adjoint $f_!$. All of $f_*, f_!, f^!$ form pseudo-functors. If f is proper then $f_! = f_*$, and if f is étale then $f^! = f^*$.

Gluing If $X \in \mathcal{S}$, $j : U \subset X$ is an open subscheme with closed complement $i : Z \to X$, then the functors

$$T(Z) \xrightarrow{i_*} T(X) \xrightarrow{j^*} T(U)$$

form a recollement (in the sense of Subsection 3.1). This is even a symmetric monoidal recollement.

Compact Objects We write $T(X)^c$ for the subcategory category of compact objects. We say that T has *stable compact objects* if $\mathbb{1}_X \in T(X)^c$, and this category is stable under the monoidal operation and restriction. In this case $X \mapsto T(X)^c$ defines a symmetric monoidal triangulated pseudo-presheaf on $\mathcal{S}$.

Continuity Let $\{S_\alpha\}_{\alpha\in A}$ be a projective system in $\mathcal{S}$ with dominant, affine transition morphisms, and assume that $S := \lim_\alpha S_\alpha$ exists in $\mathcal{S}$. Write $p_\alpha : S \to S_\alpha$ for the canonical morphism. Then T is said to satisfy continuity if the canonical functor

$$2 - \lim_\alpha T(S_\alpha)^c \to T(S)^c$$

is an equivalence [18, Proposition 4.3.4].

This means in particular that for $E \in T(S)^c$ there exists $\alpha \in A$ and $E_0 \in T(S_\alpha)^c$ such that $p_\alpha^*(E_0) \simeq E$ and that for $E_0, F_0 \in T(S_{\alpha_0})^c$ we have $[p_{\alpha_0}^*E_0, p_{\alpha_0}^*F_0] = \operatorname{colim}_\alpha[(S_\alpha \to S_{\alpha_0})^*E_0, (S_\alpha \to S_{\alpha_0})^*F_0]$.

Application to Picard Groups One upshot of the above discussion is the following.

Proposition 3.11. *Let T be a symmetric monoidal pseudo-presheaf on $\mathcal{S}$. Then the assignment $Pic(T) : \mathcal{S} \ni T \mapsto Pic(T(X))$ is a presheaf on $\mathcal{S}$. If T is continuous and has stable compact objects, then the presheaf $Pic(T)$ is also continuous: for any dominant affine system $\{S_\alpha\}_{\alpha \in A}$ with $\lim_\alpha S_\alpha =: S \in \mathcal{S}$, we have*

$$Pic(T(S)) = \operatorname{colim}_\alpha Pic(T(S_\alpha)).$$

Proof. Any symmetric monoidal functor preserves invertible objects, so given $f : X \to Y \in \mathcal{S}$ there is $f^* : Pic(T(Y)) \to Pic(T(X))$. Since any natural isomorphism of functors preserves isomorphisms of objects, and the Picard group consists of invertible objects up to isomorphism, the various morphisms f^* make the assignment $X \mapsto Pic(T(X))$ into a (strict!) presheaf (of abelian groups).

Now suppose that T is continuous with stable compact objects, and that $\{S_\alpha\}$ is an affine dominant system with $S := \lim_\alpha S_\alpha \in \mathcal{S}$. We wish to show that the natural map $\operatorname{colim}_\alpha Pic(T(S_\alpha)) \to Pic(T(S))$ is an isomorphism. Thus we need to show that (1) given $E \in Pic(T(S))$ there exist $\alpha \in A$ and $E_0 \in Pic(T(S_\alpha))$ such that $p_\alpha^*(E_0) = E$, and that (2) given $E_0 \in Pic(T(S_\alpha))$ with $p_\alpha^*(E_0) = \mathbb{1}$, there exists $\alpha' \in A$ such that $(\alpha' \to \alpha)^*(E_0) = \mathbb{1}$.

To prove (1), let $F \in Pic(T(S))$ be inverse to E. Note that both E, F are compact, since they are invertible and the tensor unit is assumed compact. Then by assumption of continuity of T there exist $\alpha, E_0 \in T(S_\alpha)^c, F_0 \in T(S_\alpha)^c$ such that $p_\alpha^*(E_0) \simeq E, p_\alpha^*(F_0) \simeq F$. It need not be true that $E_0 \otimes F_0 \simeq \mathbb{1}$, so neither E_0 nor F_0 need to be invertible. However $p_\alpha^*(E_0 \otimes F_0) \simeq \mathbb{1}$, so by the next lemma there is α' such that $(\alpha' \to \alpha)^*(E_0 \otimes F_0) \simeq \mathbb{1}$. This proves (1), and (2) also follows immediately from the lemma. □

Lemma 3.12. *Suppose $T(\alpha)^c = 2 - \lim_\alpha T(S_\alpha)^c$ and $\alpha \in A, E, F \in T(S_\alpha)^c$. If $p_\alpha^* E \simeq p_\alpha^* F$ then there is $\alpha' \in A$ such that $(\alpha' \to \alpha)^* E \simeq (\alpha' \to \alpha)^* F$.*

Proof. Let $i \in [p_\alpha^* E, p_\alpha^* F], j \in [p_\alpha^* F, p_\alpha^* E]$ be inverse isomorphisms. Since

$$[p_\alpha^* E, p_\alpha^* Y] = \operatorname{colim}_{\alpha'} [(\alpha' \to \alpha)^* E, (\alpha' \to \alpha)^* F]$$

and similarly the other way round, we find that there is α' such that i, j lift to $(\alpha' \to \alpha)^* E, (\alpha' \to \alpha)^* F$ in such a way that the compositions are both the identity. This proves the result. □

Examples. All our categories of interest satisfy the six-functors formalism.

Theorem 3.13 (Ayoub, Cisinski-Déglise)**.** *Let $\mathcal{S}$ be the category of Noetherian schemes of finite dimension. The assignment $\mathcal{S} \ni X \mapsto \mathbf{SH}(X)$ defines a symmetric monoidal, triangulated pseudo-presheaf satisfying the six functors formalism, in particular gluing and continuity, and it has stable compact objects.*

(This result is reviewed and extended in [50, Appendix C].)

Theorem 3.14 (Cisinski-Déglise, [20])**.** *Let k be a field of exponential characteristic e and $\mathcal{S}$ the category of Noetherian schemes of finite dimension over k. Then there is a symmetric monoidal, triangulated pseudo-presheaf $\mathcal{S} \ni X \mapsto \mathbf{DM}(X, \mathbb{Z}[1/p])$ extending Voevodsky's definition for fields. It satisfies the six functors formalism, in particular gluing, continuity, and it has stable compact objects.*

3.4 Recollections about Resolution of Singularities

We will be using resolution of singularities very much as a black box. The following suffices for our purposes.

Definition 3.15. *A field k is said to satisfy* resolution of singularities *if for every smooth variety X over k and every reduced, closed subvariety $Z \subset V$, there exists a proper birational morphism $p : X' \to X$ with X' smooth and $p^{-1}(Z)$ a normal crossings variety, and moreover there exists a smooth, closed subvariety $Z' \subset X'$ with $p(Z') \subset Z$ and $p : Z' \to Z$ proper and birational.*

Here a variety is said to be normal crossings *if it is smooth, or if it is a union of irreducible components $Z_1, \ldots, Z_n$ such that each Z_i is smooth and such that $Z_i \bigcap \cup_{j \neq i} Z_j$ is normal crossings for each i.*

We recall that a morphism $f : X \to Y$ is birational if there exist everywhere dense open subsets $U \subset X, V \subset Y$ with $f(U) = V$ and $f : U \to V$ an isomorphism.

The above definition is not really standard. It encapsulates however precisely the properties we are going to use.

Theorem 3.16 (Hironaka [42])**.** *Every field of characteristic zero admits resolution of singularities.*

Proof. A readable introduction to this material is [41]. We use the embedded strong resolution of singularities, as stated on page 329 of loc. cit. and elaborated upon on page 335 of loc. cit.

Alternatively, embedded resolution of singularities is also recalled in [38, Theorem 4.3]. The properties of normal crossing schemes which we use as a definition are recalled in the proof of Proposition 3.12 in loc. cit. □

We also have the following easy observation.

Lemma 3.17. *Let k be a perfect field with resolution of singularities and X a scheme of finite type. Then X admits a cdh cover by smooth schemes (of finite type) of dimension at most* $\dim X$.

Proof. We use Noetherian induction on X. Since $X_{red} \to X$ is a cdh cover [75, Example 12.24] we may assume that X is reduced. Since Zariski coverings are cdh coverings, we may assume that X is affine. Thus by resolution of singularities applied to $X \subset \mathbb{A}^n$ there is a proper birational morphism $p : X' \to X$ with X' smooth. Let $Z \subset X$ be the centre. By definition, $\{X', Z\}$ is a cdh cover of X, so we need only show that Z has a smooth cdh cover. But Z is a proper closed subset of X, so this finishes the induction. □

3.5 Application 1: Pointwise Trivial Motivic Spectra

Definition 3.18. *We call a presheaf F on $Sm(k)$* rigid *if for every essentially smooth, Henselian local scheme X with closed point x, the restriction homomorphism $F(X) \to F(x)$ is an isomorphism.*

This notion of rigidity is well established. We want to point out the following example.

Proposition 3.19. *Let k be an infinite perfect field of exponential characteristic $2 \neq e$ and finite 2-étale cohomological dimension.*

The sheaves $\underline{\pi}_i(S_e)_0$ are rigid for all i.

Proof. Let us note that rigid sheaves are stable under kernel, image, extension, limit and colimit. To simplify notation, we write S instead of S_e. We consider the arithmetic square for 2 on S [96, Lemma 3.9]. This furnishes a homotopy pullback square

$$\begin{array}{ccc} S & \longrightarrow & S[1/2] \\ \downarrow & & \downarrow \\ S_2^\wedge & \longrightarrow & S_2^\wedge[1/2]. \end{array}$$

Here $S_2^\wedge$ denotes the two-completion, which we will define below. The long exact sequence for this square together with the five lemma implies that it is enough to show that $\underline{\pi}_*(S[1/2])_0, \underline{\pi}_*(S_2^\wedge)_0$

and $\underline{\pi}_*(S_2^\wedge[1/2])$ are rigid. Since rigid sheaves are stable by colimit, the case of $S_2^\wedge[1/2]$ follows from $S_2^\wedge$.

We begin with $S[1/2]$. By motivic Serre finiteness [2, Theorem 6] (beware that their indexing convention for motivic homotopy groups differs from ours!), $\underline{\pi}_i(S[1/2])$ is torsion for $i > 0$. By design, it is of odd torsion prime to the exponential characteristic. Writing this sheaf as the colimit of its sheaves of l-torsion, where l ranges over odd integers prime to the exponential characteristic, and applying [117, Corollary 2.6] (note that finite 2-étale cohomological dimension implies that k is non-real), we find that it is rigid. (In fact one may prove that it is a (torsion) sheaf with transfers in the sense of Voevodsky.)

For $i = 0$ we have $\underline{\pi}_i(S[1/2])_0 = \underline{GW}[1/2]$ and the sheaf $\underline{GW}$ is known to be rigid [35, Theorem 2.4].

It remains to deal with $S_2^\wedge$. By [55, Theorem 1] and [95, proof of Theorem 8.1], we know that $S_2^\wedge$ is actually is actually the same as S^{Ad}, where S^{Ad} is the realisation of a semi-cosimplicial spectrum with $S_q^{Ad} = S \wedge H\mathbb{Z}/2^{\wedge q}$. (Here $H\mathbb{Z}/2$ is the motivic cohomology spectrum.) That is to say there is a convergent spectral sequence for $T \in \mathbf{SH}(k)$, $E_1^{**} \Rightarrow [T[*], S^{Ad}]$ [55, Corollary 3]. Here $E_1^{*q} = ker([T[*], S_q^{Ad}] \xrightarrow{s} \bigoplus_i [T[*], S_{q+1}^{Ad}])$. The map s comes from the $q+1$ cosimplicial structure maps. In particular it suffices to show that the homotopy sheaves of S_q^{Ad} are rigid, for any q. But these are torsion sheaves with transfers (in the sense of Voevodsky), so in particular oriented, whence rigidity holds for example by [45, Paragraph after Lemma 1.6]. □

Remark 1. In light of the fact that $\underline{\pi}_0(S)_0$ is always rigid, one may ask if the same is true about $\underline{\pi}_i(S)_0$ for all i (i.e. without assuming that k is of finite 2-étale cohomological dimension). The author knows a somewhat elaborate proof of this result which is not included here for reasons of space.

Remark 2. It is not true that $\underline{\pi}_i(S)_j$ is rigid for all i, j. For example $\underline{\pi}_0(S)_1 = \underline{K}_1^{MW}$ and $\underline{K}_1^{MW}(A)$ surjects onto $A^\times$, so this sheaf very far from rigid.

The reason why we wish to deal with rigid sheaves is the following.

Lemma 3.20. *Let F be a strictly homotopy invariant cdh sheaf on Sch/k, where k is a field of characteristic zero. Assume that F is rigid.*

If X is the Henselisation of a normal crossing scheme in a point x, then $H^p_{cdh}(X, F) = 0$ for $p > 0$ and $H^0_{cdh}(X, F) = F(x)$.

Proof. We use induction on the dimension and number of components of X. We know that X is the union of smooth closed subschemes $Z_1, \dots, Z_r$, all of which are Henselian local with closed point x, and smooth. The result holds for each of the Z_i by Corollary 2.36 part (2). We may assume by induction it holds for $X' = Z_1 \cup \dots \cup Z_{r-1}$ and also for $X'' := X' \cap Z_n$ (being normal crossings of smaller dimension). Consider the Mayer-Vietoris sequence for the closed cover $X = Z_n \cup X'$ (which is a rather degenerate abstract blowup)

$$0 \to H^0(X) \to H^0(X') \oplus H^0(Z_n) \to H^0(X'') \to H^1(X) \to H^1(X') \oplus H^1(Z_n) \to H^1(X'') \to \dots.$$

The result for X follows immediately. □

Corollary 3.21. *Let X be as in the lemma and k additionally of finite 2-étale cohomological dimension. Then*

$$\mathrm{Hom}_{\mathbf{SH}(X)}[\mathbb{1}, \mathbb{1}] = GW(x)$$

and for $i > 0$

$$\mathrm{Hom}_{\mathbf{SH}(X)}[\mathbb{1}, \mathbb{1}[i]] = 0.$$

In order to prove this, we use a slightly different kind of descent spectral sequence. First, recall that $\mathbf{SH}(X) \to \underline{\mathbf{SH}}(X)$ is fully faithful, by the same argument as for [18, Corollary 6.2.5]. Thus it is enough to compute $\mathrm{Hom}_{\underline{\mathbf{SH}}(X)}[\mathbb{1}, \mathbb{1}[i]]$. Let $f : X \to Spec(k)$ be the structural morphism.

Then the pullback $f^* : \underline{\mathbf{SH}}(k) \to \underline{\mathbf{SH}}(X)$ has a left adjoint $\underline{f}_\# : \underline{\mathbf{SH}}(X) \to \underline{\mathbf{SH}}(k)$. Consequently $[\mathbb{1}, \mathbb{1}[i]] = [\mathbb{1}, f^*\mathbb{1}[i]] = [\underline{f}_\#\mathbb{1}, \mathbb{1}[i]]$. Moreover, $\underline{f}_\#\mathbb{1} = \underline{\Sigma}^\infty X_+$. At this point we can use the usual descent spectral sequence in the category $\underline{\mathbf{SH}}(k)$ to compute $Map^d(\underline{\Sigma}^\infty X_+, \mathbb{1})$. This requires us to know the cdh homotopy sheaves of $\mathbb{1} \in \underline{\mathbf{SH}}(k)$ on $Ft(k)$. However by corollary 2.36 (and its proof) it is enough to know the Nisnevich homotopy sheaves of $\mathbb{1} \in \mathbf{SH}(k)$ on $Sm(k)$.

Proof. The homotopy sheaves $\underline{\pi}_*(S)_0$ are strictly homotopy invariant and rigid by Proposition 3.19. Hence the descent spectral sequence

$$H^p_{cdh}(X, \underline{\pi}_{-q}(S)_*) \Rightarrow [\mathbb{1}, \mathbb{1}[p+q]]_X$$

collapses (by the lemma) to yield $[\mathbb{1}, \mathbb{1}[i]] = H^0_{cdh}(X, \underline{\pi}_{-i}(S)_0)$. The result follows from the lemma. □

We are now ready to state our main theorem of this section.

Theorem 3.22. *Let k be a field of characteristic zero and finite 2-étale cohomological dimension, and $X \in Ft(k)$. If $E \in \mathbf{SH}(X)$ is pointwise trivial, i.e. for every $x \in X$ (not necessarily closed) we have $E_x \simeq \mathbb{1} \in \mathbf{SH}(x)$, then E is cdh-locally trivial. If X is a normal crossings scheme, then in fact E is Nisnevich-locally trivial.*

Proof. Let us write G_n for the assertion "the theorem holds in the weak form about cdh-local triviality for all X of dimension at most n", S_n for the assertion "the theorem holds in its strong form about Nisnevich-local triviality for all smooth X of dimension at most n" and N_n for the assertion "the theorem holds in its strong form for all X of dimension at most n which are normal crossings". Clearly when $n = 0$ all of these hold. We shall show that $G_n \Rightarrow S_n \Rightarrow N_n \Rightarrow G_{n+1}$. This proves the result by induction.

($G_n \Rightarrow S_n$). This is just Theorem 2.37.

($S_n \Rightarrow N_n$). Let X be the Henselisation of a normal crossings variety in a point p. We know that $X = Z_1 \cup \cdots \cup Z_n$ where each Z_i is smooth. We show by induction on n that E is trivial on X. By induction we know that E is trivial on $U := X \setminus Z_n \subset Z_1 \cup \cdots \cup Z_n$ and on $Z := Z_n$. We are in a recollement situation, and Corollary 3.21 applied to Z and Z_n says that we may apply Corollary 3.9, whence E is trivial.

($N_n \Rightarrow G_{n+1}$) Since X is cdh locally smooth by Lemma 3.17, we may assume that X is smooth. Considering the triviality of E over the generic point(s) and using continuity (i.e. Proposition 3.11), we find an (everywhere) dense open subset $U \subset X$ such that $E|_U$ is trivial. Let Z be the closed complement. Using resolution of singularities (see Definition 3.15 and Theorem 3.16), we find a proper cdh cover $X' \to X$ such that the preimage Z' of Z is normal crossings. Consequently we may assume that Z is normal crossings. Passing to the Henselisation of a point $x \in X$, we may assume further that X is Hensel local with closed point x. Now $E|_Z = \mathbb{1}$ by assumption N_n and $E|_U = \mathbb{1}$ by construction, so using Corollaries 3.21 and 3.9 as before, we conclude that E is trivial.

This concludes the proof. □

Remark 1. The result extends to arbitrary fields (of characteristic zero) containing $\sqrt{-1}$ by the usual continuity argument (a field finitely generated over $\mathbb{Q}(\sqrt{-1})$ is of finite 2-étale cohomological dimension, and every invertible object is defined over such a field by compactness).

Remark 2. The only reason why we need to assume finite 2-étale cohomological dimension is in the proof of Proposition 3.19. As mentioned after that proof, the assumption is not necessary (but we did not prove that claim).

We can also remove the reliance on that proposition altogether by the following technically more complicated argument. First, we prove the theorem not for the functor $X \mapsto \mathbf{SH}(X)$ but for $X \mapsto (S_{\le 0})$-**Mod**. This should be possible because only $\underline{\pi}_0(S)_0 = \underline{GW}$ matters here, and we know rigidity for this sheaf. Second, we check that for X smooth Henselian local, the homomorphism $Pic(\mathbf{SH}(X)) \to Pic((S_{\le 0})\text{-}\mathbf{Mod})$ is injective.

The main reason why we have chosen not to implement this strategy is that justifying the technical details would take up significantly more space.

3.6 Application 2: Pointwise Trivial Motives

A stronger result is true for motives, with an easier proof.

Theorem 3.23. *Let k be a field of characteristic zero and X Noetherian of finite dimension over k. If $E \in Pic(\mathbf{DM}(X))$ is pointwise trivial, i.e. $E_x \simeq \mathbb{1}$ for all $x \in X$, then $E \simeq \mathbb{1}$.*

Proof. By continuity, we may assume that X is of finite type. By [18, Proposition 2.3.6 (1)] we may assume that X is reduced.

Arguing as in Chapter 2, the space $PIC(\underline{\mathbf{DM}})$ satisfies cdh-descent. But we have $\underline{h}_i \mathbb{1} = 0$ for $i \neq 0$ and $\underline{h}_0 \mathbb{1} = \mathbb{Z}$. It follows from the descent spectral sequence that the presheaf $X \mapsto Pic(\underline{\mathbf{DM}}(X))$ is actually a sheaf in the cdh topology (use the lemma below). Consequently any cdh-locally trivial invertible motive is trivial. This holds for elements of $\mathbf{DM}(X)$ (no underline) as well, since $\mathbf{DM}(X) \to \underline{\mathbf{DM}}(X)$ is fully faithful (essentially by definition).

We now prove the result by induction on the dimension of X. By what we said above, it suffices to prove the result if X is reduced Henselian local with closed point x, open complement U. Then U is of smaller dimension than X, so $E|_U = \mathbb{1}$, and also $E|_x = \mathbb{1}$ by assumption. The lemma below implies (using a descent spectral sequence) that $[\mathbb{1}, \mathbb{1}[i]] = 0$ for $i \neq 0$ and $[\mathbb{1}, \mathbb{1}] = \mathbb{Z}$, over *any* (reduced and connected) X. We may thus apply Corollaries 3.21 and 3.9 as before to conclude that $E = \mathbb{1}$. □

Lemma 3.24. *Let A be an abelian group and $\underline{A}$ the associated constant sheaf. Then for any reduced, connected scheme essentially of finite type over k, we have $H^p_{cdh}(X, A) = 0$ for $p > 0$ and $H^0_{cdh}(X, A) = A$.*

Proof. The result holds for X smooth by Corollary 2.36 and the fact that constant sheaves are flasque for the Nisnevich topology (e.g. because they afford transfers and are homotopy invariant, so the Nisnevich and Zariski cohomology coincide, and the result for Zariski cohomology is well known). By resolution of singularities, we find a cdh cover $Z \coprod X' \to X$ where $Z \subset X$ is closed of smaller dimension and X' is smooth. The preimage Z' of Z in X also has smaller dimension. Considering the associated Mayer-Vietoris sequence concludes. □

Chapter 4

The Motivic Hurewicz Theorem

The results in this chapter are being published as [6].

We fix throughout a perfect ground field k of exponential characteristic e.

In this short chapter we establish the Motivic Hurewicz Theorem and its corollaries, the conservativity and Pic-injectivity theorems. In section 4.1 we prove a convenient result about abstract t-categories (triangulated categories with a t-structure) and adjunctions between them. It answers the following question. Suppose that $M : \mathcal{C} \to \mathcal{D}$ is a triangulated functor between t-categories, and assume that for $E \in \mathcal{C}$ with $\pi_i^{\mathcal{C}}(E) = 0$ for all $i < 0$ we also have $\pi_i^{\mathcal{D}}(ME) = 0$ for all $i < 0$ (we say that M is right-t-exact). How can we relate $\pi_0^{\mathcal{C}}(E) \in \mathcal{C}^\heartsuit$ and $\pi_0^{\mathcal{D}}(ME) \in \mathcal{D}^\heartsuit$? It turns out that if we assume that M has a right adjoint, then the answer is the best possible: there is a canonical isomorphism $\pi_0^{\mathcal{D}}(ME) \cong M^\heartsuit \pi_0^{\mathcal{C}}(E)$. We call this the *abstract Hurewicz Theorem* because of its similarity to the eponymous result in topology. (Here $M^\heartsuit : \mathcal{C}^\heartsuit \to \mathcal{D}^\heartsuit$ is the natural induced functor.)

In Section 4.2, we apply this result to the case of motivic stable homotopy theory, i.e. the functor $M : \mathbf{SH}(k) \to \mathbf{DM}(k)$. Recall that the algebraic Hopf map $\eta : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ induces a morphism of similar name in the stable motivic homotopy category $\eta : \mathbb{G}_m \to S$. The categories $\mathbf{SH}(k), \mathbf{DM}(k)$ have t-structures and M is left-t-exact with a t-exact right adjoint U. By a foundational result of Deglise, the functor $U : \mathbf{DM}(k)^\heartsuit \to \mathbf{SH}(k)^\heartsuit$ is an equivalence onto the full subcategory $\mathbf{SH}(k)^{\heartsuit,\eta=0}$ of objects F such that $\eta_F : F \wedge \mathbb{G}_m \to F$ is the zero map. As a consequence, we prove that if $E \in \mathbf{SH}(k)$ with $\underline{\pi}_i(E)_* = 0$ for all $i < 0$, then $\underline{h}_i(ME)_* = 0$ for all $i < 0$ and $\underline{h}_0(ME)_* = \underline{\pi}_0(ME)_*/\eta$, using the above identification of $\mathbf{DM}(k)^\heartsuit$ inside $\mathbf{SH}(k)^\heartsuit$. This is what we call the Motivic Hurewicz Theorem.

In Section 4.3 we recall Voevodsky's slice filtration and Levine's computation of the filtration it induces on $\mathbf{SH}(k)^\heartsuit$. Using Voevodsky's resolution of the Milnor conjectures we can prove: if k has finite 2-étale cohomological dimension, $E \in \mathbf{SH}(k)_e$ is slice-connective, $\underline{\pi}_i(E)_* = 0$ for $i < 0$ and additionally $\underline{h}_0(ME)_* = 0$, then we also have $\underline{\pi}_0(E)_* = 0$. The conservativity and Pic-injectivity theorems are easy corollaries of this.

In the final Section 4.4 we collect some straightforward applications: the invertibility of (suitable) spectra is detected by their motives, and so are (multiplicative) equations among invertible spectra. We will use these in the next chapter to prove invertibility of affine quadrics and establish the Hu-conjecture.

4.1 The Abstract Hurewicz Theorem

We begin by proving a natural result about t-categories. We assume that it must be well known in the right circles, so make no claim to originality.

We begin by recalling some definitions. The most important point to note is that we use *homological* indexing for t-structures. See for example [70, Definition 1.2.1.1].

Definition 4.1. *Let $\mathcal{C}$ be a triangulated category. By a t-structure on $\mathcal{D}$ we mean a pair of strictly full subcategories $\mathcal{C}_{\ge 0}, \mathcal{C}_{\le 0}$ with the following properties:*

1. *For $X \in \mathcal{C}_{\ge 0}, Y \in \mathcal{C}_{\le 0}$ we have $[X, Y[-1]] = 0$.*
2. *We have $\mathcal{C}_{\ge 0}[1] \subset \mathcal{C}_{\ge 0}$ and $\mathcal{C}_{\le 0}[1] \supset \mathcal{C}_{\le 0}$.*
3. *For any $X \in \mathcal{C}$ there is a distinguished triangle $X_{\ge 0} \to X \to X_{<0}$ with $X_{\ge 0} \in \mathcal{C}_{\ge 0}$ and $X_{<0} \in \mathcal{C}_{\le 0}[-1]$.*

By a t-category we mean a triangulated category with a fixed t-structure.

This definition has a plethora of consequences, see for example [11, Section 1.3] [34, Section IV §4]. Let us recall the following. We write $\mathcal{C}_{\le n} = \mathcal{C}_{\le 0}[n]$ and $\mathcal{C}_{\ge n} = \mathcal{C}_{\ge 0}[n]$. The inclusion $\mathcal{C}_{\le n} \hookrightarrow \mathcal{C}$ has a *left* adjoint $E \mapsto E_{\le n}$. Similarly the inclusion $\mathcal{C}_{\ge n} \hookrightarrow \mathcal{C}$ has a *right* adjoint $E \mapsto E_{\ge n}$. Both are called *truncation*. The distinguished triangle from point (3) of the definition is in fact unique, and functorially comprised of the truncations, as indicated in the notation.

The full subcategory $\mathcal{C}^{\heartsuit} = \mathcal{C}_{\le 0} \cap \mathcal{C}_{\ge 0}$ turns out to be abelian and is called the *heart* of the t-category. For $E \in \mathcal{C}$ the homotopy objects are $\pi_0^{\mathcal{C}}(E) := (E_{\le 0})_{\ge 0} \simeq (E_{\ge 0})_{\le 0} \in \mathcal{C}^{\heartsuit}$, and $\pi_i^{\mathcal{C}}(E) := \pi_0^{\mathcal{C}}(E[-i])$. The functor $\pi_*^{\mathcal{C}}$ is homological, i.e. turns exact triangles into long exact sequences.

Definition 4.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a triangulated functor between t-categories. We call F* right-t-exact *(respectively* left-t-exact*) if $F(\mathcal{C}_{\ge 0}) \subset \mathcal{D}_{\ge 0}$ (respectively if $F(\mathcal{C}_{\le 0}) \subset \mathcal{D}_{\le 0}$). We call it* t-exact *if it is both left and right t-exact.*

We note that any triangulated functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor $F^{\heartsuit} : \mathcal{C}^{\heartsuit} \to \mathcal{D}^{\heartsuit}, E \mapsto \pi_0^{\mathcal{D}} F(E)$.

The notions of left and right t-exactness are partly justified by the following.

Proposition 4.3. *Let $M : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be adjoint functors between t-categories, with M right-t-exact and U left-t-exact. Then*

$$M^{\heartsuit} : \mathcal{C}^{\heartsuit} \rightleftarrows \mathcal{D}^{\heartsuit} : U$$

is also an adjoint pair. In particular $M^{\heartsuit}$ is right exact and $U^{\heartsuit}$ is left exact.

Proof. Let $A \in \mathcal{C}^{\heartsuit}, B \in \mathcal{D}^{\heartsuit}$. We compute

$$\begin{aligned}[M^{\heartsuit} A, B] &\overset{(1)}{=} [\pi_0^{\mathcal{D}} MA, B] \overset{(2)}{=} [(MA)_{\le 0}, B] \overset{(3)}{=} [MA, B] \\ &\overset{(4)}{=} [A, UB] \overset{(3')}{=} [A, (UB)_{\ge 0}] \overset{(2')}{=} [A, \pi_0^{\mathcal{C}} UB] \overset{(1')}{=} [A, U^{\heartsuit} B].\end{aligned}$$

Here (1) is by definition, (2) is because $A \in \mathcal{C}_{\ge 0}$ and so $MA \in \mathcal{D}_{\ge 0}$ by right-t-exactness of M, and (3) because $B \in \mathcal{D}_{\le 0}$. Equality (4) is by adjunction, and finally (3'), (2'), (1') reverse (3), (2), (1) with $M \leftrightarrow U$, left $\leftrightarrow$ right, etc. □

The following lemma is a bit technical but naturally isolates a crucial step in the proofs of the two following results.

Lemma 4.4. *Let $M : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be adjoint functors between t-categories. If M is right-t-exact (respectively U left-t-exact) then there is a natural isomorphism $(UE)_{\ge n} \simeq (UE_{\ge n})_{\ge n}$ (respectively $(ME)_{\le n} \simeq (ME_{\le n})_{\le n}$).*

Proof. By duality, we need only prove one of the statements. Suppose that U is left-t-exact. We compute for $T \in \mathcal{D}_{\le n}$

$$\begin{aligned}[(ME)_{\le n}, T] &\overset{(1)}{=} [ME, T] \overset{(2)}{=} [E, UT] \\ &\overset{(3)}{=} [E_{\le n}, UT] \overset{(2)}{=} [ME_{\le n}, T] \overset{(1)}{=} [(ME_{\le n})_{\le n}, T].\end{aligned}$$

Since all equalities are natural, the result follows from the Yoneda lemma. Here (1) is because $T \in \mathcal{D}_{\le n}$, (2) is by adjunction, and (3) is because U is left-t-exact so $UT \in \mathcal{C}_{\le n}$. □

With this preparation, we can now formulate the main theorem of this section.

Theorem 4.5 (Abstract Hurewicz Theorem). *Let $M : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be adjoint functors between t-categories, such that M is right-t-exact and U is left-t-exact. Then for $E \in \mathcal{C}_{\geq 0}$ there is a natural isomorphism*

$$\pi_0^{\mathcal{D}} ME \simeq M^{\heartsuit} \pi_0^{\mathcal{C}} E.$$

Proof. We have

$$M^{\heartsuit}\pi_0^{\mathcal{C}} E = \pi_0^{\mathcal{D}} M \pi_0^{\mathcal{C}} E \overset{(1)}{=} \pi_0^{\mathcal{D}} M E_{\leq 0} \overset{(2)}{=} (ME_{\leq 0})_{\leq 0} \overset{(3)}{=} (ME)_{\leq 0} \overset{(2)}{=} \pi_0^{\mathcal{D}} ME.$$

This is the desired result. Here (1) is because $E \in \mathcal{C}_{\geq 0}$, (2) is because M is right-t-exact, and (3) is because of Lemma 4.4 applied to left-t-exactness of U. □

For convenience, we also include the following observation. What this says is that in order to apply the abstract Hurewicz theorem, only one of the two exactness properties needs to be checked.

Lemma 4.6. *Let $M : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be adjoint functors between t-categories. If M is right-t-exact (respectively U left-t-exact), then U is left-t-exact (respectively M right-t-exact).*

So in applying the theorem only one of the two exactness properties has to be checked.

Proof. By duality it suffices to prove only one of the statements. So assume that M is right-t-exact and let $E \in \mathcal{D}_{\leq 0}$. We need to show that $UE \in \mathcal{C}_{\leq 0}$. Because of the distinguished triangle $(UE)_{\geq 1} \to UE \to (UE)_{\leq 0}$ it suffices to show that $(UE)_{\geq 1} = 0$. But by Lemma 4.4, the right-t-exactness of M implies that $(UE)_{\geq 1} = (UE_{\geq 1})_{\geq 1} = 0$ (since $E \in \mathcal{D}_{\leq 0}$). □

Example Let us illustrate these rather abstract results with a well-known example. Write **SH** for the stable homotopy category $D(Ab)$ for the derived category of abelian groups, and $C_* : \mathbf{SH} \to D(Ab)$ for the singular chain complex functor. This is triangulated and preserves arbitrary sums, so has a right adjoint U, say by Neeman's version of Brown representability. (Of course U is just the Eilenberg-MacLane spectrum functor.) Both of these categories have well-known t-structures with heart the category of abelian groups. In fact one has $\pi_i^{\mathbf{SH}}(E) := \pi_i(E) = [S[i], E]_{\mathbf{SH}}$ the stable homotopy groups of the spectrum, and $\pi_i^{D(Ab)}(C) := H_i(C) = [\mathbb{Z}[i], C]_{D(Ab)}$ the homology groups of the chain complex.

Now it is well-known that $H_0(C_*S) = \mathbb{Z}$ and $H_i(C_*S) = 0$ for $i \neq 0$ and consequently $C_*S \simeq \mathbb{Z}$. Thus by adjunction we immediately find that $\pi_i UC = [S[i], UC] = [\mathbb{Z}[i], C] = H_i C$ and so U is t-exact. Thus by Lemma 4.6 we find that C_* is right-t-exact and we may apply the abstract Hurewicz Theorem 4.5. In our case this says: if $E \in \mathbf{SH}$ is such that $\pi_i(E) = 0$ for all $i < 0$, then also $H_i(E) = 0$ for all $i < 0$ and $H_0(E) = \pi_0(E)$. This is, of course, (a stable version of) the classical Hurewicz theorem!

4.2 The Motivic Hurewicz Theorem

We now apply the above results to the case of motivic homotopy theory. Recall from Section 2.2 that in this situation we are interested in the categories

$$\mathbf{SH}(k) = Ho(\mathcal{SH}(k)), \quad \mathcal{SH}(k) := Stab^{\Sigma}(L_{\mathbb{A}^1} sPre(Sm(k)_{Nis})_*, \mathbb{P}^1)$$

and

$$\mathbf{DM}(k) = Ho(\mathcal{DM}(k)), \quad \mathcal{DM}(k) := Stab^{\Sigma}(L_{\mathbb{A}^1} sPre(Cor(k)_{Nis}), \mathbb{P}^1).$$

Here $Cor(k)$ is the category whose objects are the smooth schemes and whose morphisms are the finite correspondences. By $sPre(Cor(k))$ we mean (simplicial) presheaves with transfers; such objects are automatically pointed (by 0). We did not discuss the construction of $\mathbf{DM}(k)$ in detail in section 2.2, mainly because the situation is much more complicated if the base is not just a perfect field. For more details see [98, Section 2] as well as [75].

There is a functor $M : Sm(k) \to Cor(k)$. It is the identity on objects and sends a morphism to its graph. By Kan extension this yields a functor $M : sPre(Sm(k))_* \to sPre(Cor(k))$. (That is to say M is the unique colimit preserving functor which maps the representable presheaf $\Delta^n \times X_+$ to $\Delta^n \times M(X)$, for $X \in Sm(k)$.) By general nonsense M has a right adjoint U, which just forgets the transfer structure. It is clear that U preserves fibrations and acyclic fibrations in the projective global model structure, thus there is a Quillen adjunction

$$M : sPre(Sm(k))_* \leftrightarrows sPre(Cor(k)) : U.$$

By the usual procedure this Quillen adjunction passes through Bousfield localisation and stabilisation, yielding a Quillen adjunction

$$M : \mathcal{SH}(k) \leftrightarrows \mathcal{DM}(k) : U.$$

We sometimes abuse notation and write, for $X \in Sm(k)$, $\Sigma^\infty X_+ \in \mathbf{SH}(k)$ and $MX \in \mathbf{DM}(k)$. This should not cause confusion.

Recall that $\mathbb{A}^1$-locally we have $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$. This implies that $\mathbf{SH}(k), \mathbf{DM}(k)$ are *triangulated* categories. They come with t-structures called the *homotopy t-structures* [79, Section 5.2]. Write $Shv(k)$ for the category of Nisnevich sheaves of abelian groups on $Sm(k)$, and $Shv^{tr}(k)$ for the category of sheaves with transfers, i.e. additive presheaves on $Cor(k)$ such that their restriction to $Sm(k)$ is a sheaf.

Definition 4.7. *For $E \in \mathbf{SH}(k)$ and $i, j \in \mathbb{Z}$ define $\underline{\pi}_i(E)_j \in Shv(k)$ as the sheaf associated to the presheaf $X \mapsto [\Sigma^\infty(X) \wedge S^i, E \wedge \mathbb{G}_m{}^{\wedge j}]$, and put*

$$\mathbf{SH}(k)_{\geq 0} = \{E \in \mathbf{SH}(k) : \underline{\pi}_i(E)_j = 0 \text{ for } i < 0 \text{ and } j \in \mathbb{Z}\}$$
$$\mathbf{SH}(k)_{\leq 0} = \{E \in \mathbf{SH}(k) : \underline{\pi}_i(E)_j = 0 \text{ for } i > 0 \text{ and } j \in \mathbb{Z}\}.$$

The same story can be repeated for $\mathbf{DM}$.

Definition 4.8. *For $E \in \mathbf{DM}(k)$ and $i, j \in \mathbb{Z}$ define $\underline{h}_i(E)_j \in Shv^{tr}(k)$ as the sheaf associated to the presheaf $X \mapsto [M(X)[i], E \wedge \mathbb{G}_m{}^{\wedge j}]$, and put*

$$\mathbf{DM}(k)_{\geq 0} = \{E \in \mathbf{DM}(k) : \underline{h}_i(E)_j = 0 \text{ for } i < 0 \text{ and } j \in \mathbb{Z}\}$$
$$\mathbf{DM}(k)_{\leq 0} = \{E \in \mathbf{DM}(k) : \underline{h}_i(E)_j = 0 \text{ for } i > 0 \text{ and } j \in \mathbb{Z}\}.$$

As the notation suggests, these subcategories define t-structures. In fact the hearts are known fairly explicitly. Recall that for a presheaf F of abelian groups (say), the contraction F_{-1} is defined as $F_{-1}(X) = ker(F(X \times (\mathbb{A}^1 \setminus \{0\})) \to F(X))$, where the pullback is along the inclusion $X \to X \times (\mathbb{A}^1 \setminus \{0\}), x \mapsto (x, 1)$. Recall also that a sheaf F is called *strictly homotopy invariant* if $H^i(X \times \mathbb{A}^1, F) = H^i(X, F)$ for all $i \geq 0$ and all $X \in Sm(k)$.

Definition 4.9. *A* homotopy module *consists of a family of strictly homotopy invariant sheaves $F_i \in Shv(k), i \in \mathbb{Z}$ together with isomorphisms $F_i \cong (F_{i+1})_{-1}$. A morphism of homotopy modules $f_* : F_* \to G_*$ consists of homomorphisms of sheaves $f_i : F_i \to G_i$ such that $(f_{i+1})_{-1} = f_i$ under the canonical identifications. The category of homotopy modules is denoted $\mathbf{HI}_*(k)$.*

Similarly, a homotopy module with transfers *is a family F_* of strictly homotopy invariant sheaves with transfers together with isomorphisms $(F_{i+1})_{-1} \cong F_i$. Morphisms are defined as before. The category is denoted $\mathbf{HI}_*^{tr}(k)$.*

The following result was first published by Morel, but known to Voevodsky.

Theorem 4.10. *Let k be a perfect field.*

(i) The subcategories $\mathbf{SH}(k)_{\leq 0}, \mathbf{SH}(k)_{\geq 0} \subset \mathbf{SH}(k)$ define a non-degenerate t-structure on $\mathbf{SH}(k)$. For $E \in \mathbf{SH}(k)$, the homotopy sheaves $\underline{\pi}_i(E)_*$ define a homotopy module in a natural way, and this yields an equivalence $\mathbf{SH}(k)^\heartsuit \simeq \mathbf{HI}_*(k)$.

(ii) The subcategories $\mathbf{DM}(k)_{\leq 0}, \mathbf{DM}(k)_{\geq 0} \subset \mathbf{DM}(k)$ *define a non-degenerate* t*-structure on* $\mathbf{DM}(k)$*. For* $E \in \mathbf{DM}(k)$*, the homology sheaves* $\underline{h}_i(E)_*$ *define a homotopy module with transfers in a natural way, and this yields an equivalence* $\mathbf{DM}(k)^\heartsuit \simeq \mathbf{HI}_*^{tr}(k)$*.*

Proof. The first claim is [79, Theorem 5.2.6]. The second obtained by a straightforward adaptation of that proof. ("Straightforward" because all the necessary prerequisites have been established by Voevodsky.) □

Now that we have t-structures, we wish to apply the abstract Hurewicz theorem. As a first step, we need to investigate the exactness properties of M, U.

Lemma 4.11. *The functor* $U : \mathbf{DM}(k) \to \mathbf{SH}(k)$ *is* t*-exact. In fact for* $E \in \mathbf{DM}(k)$ *we have* $\underline{\pi}_i(UE)_j = U(\underline{h}_i(E)_j)$ *and* $U : Shv^{tr}(k) \to Shv(k)$ *detects zero objects.*

Proof. It follows from the definitions of the t-structures that we need only prove the "in fact" part. Let $\underline{\pi}_i^{pre}(E)_j$ be the presheaf $V \mapsto [\Sigma^\infty(V_+) \wedge S^i, {\mathbb{G}_m}^{\wedge j} \wedge E]$, and similarly for $\underline{h}_i^{pre}(E)_j$. Then writing also $U : PreShv(Cor(k)) \to PreShv(Sm(k))$ we get immediately from the definitions that $U(\underline{h}_i^{pre}(E)_j) = \underline{\pi}_i^{pre}(UE)_j$. So we need to show that $U : PreShv(Cor(k)) \to PreShv(Sm(k))$ commutes with taking the associated sheaf. This is well known, see e.g. [75, Theorem 13.1]. □

Corollary 4.12 (Preliminary form of the Motivic Hurewicz Theorem)**.** *Let* $E \in \mathbf{SH}(k)_{\geq 0}$*. Then* $ME \in \mathbf{DM}(k)_{\geq 0}$ *and* $\underline{h}_0(ME)_* = M^\heartsuit(\underline{\pi}_0(E)_*)$*.*

Proof. We know that M is left adjoint to the t-exact functor U. Hence M is right-t-exact by Lemma 4.6. Thus $ME \in \mathbf{DM}(k)_{\geq 0}$, and the result about homotopy objects is just a concrete incarnation of Theorem 4.5. □

In order to arrive at a more useful form of the Hurewicz Theorem, we have to understand better the functor $M^\heartsuit$. Fortunately for us, the hard work has again been done already by someone else. We need a little more preparation.

Recall the algebraic Hopf map $\eta : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$. This defines by functoriality a map $\Sigma^\infty(\mathbb{A}^2 \setminus \{0\}) \to \Sigma^\infty(\mathbb{P}^1) \in \mathbf{SH}(k)$, where we point $\mathbb{A}^2 \setminus \{0\}$ by $(1, 1)$ and $\mathbb{P}^1$ by the corresponding point $(1 : 1)$. It is not hard to show that there is an $\mathbb{A}^1$-weak equivalence $\mathbb{A}^2 \setminus \{0\} \simeq S^1 \wedge {\mathbb{G}_m}^{\wedge 2}$ [83, Example 3.2.20] and we have already mentioned that $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$. Since $S^1, \mathbb{G}_m$ are invertible in $\mathbf{SH}(k)$ by design, we obtain a desuspended map still denoted by the same letter

$$\eta : \mathbb{G}_m \to S \in \mathbf{SH}(k).$$

Here we write $S = \mathbb{1}_{\mathbf{SH}}$ for the motivic sphere spectrum.

Given $H \in \mathbf{HI}_*(k) \simeq \mathbf{SH}(k)^\heartsuit$ we define $F(n) := \underline{\pi}_0(F \wedge {\mathbb{G}_m}^{\wedge n})_*$. There is then the map

$$\eta = \eta_F : F(1) \to F,$$

namely $\eta_F = \underline{\pi}_0(\eta \wedge \mathrm{id}_F)_*$. We call a homotopy module F such that $\eta_F = 0$ *orientable.* It is an important observation that if $F \in \mathbf{HI}_*^{tr}(k)$ then $(UF)_\eta = 0$, i.e. all homotopy modules with transfers are orientable. To see this, we note that there is a distinguished triangle [79, Lemma 6.2.1]

$$\Sigma^\infty(\mathbb{A}^2 \setminus \{0\}) \xrightarrow{\eta} \Sigma^\infty\mathbb{P}^1 \to \Sigma^\infty\mathbb{P}^2.$$

But in $\mathbf{DM}$ it is well known that the inclusion $M\mathbb{P}^1 \to M\mathbb{P}^2$ splits (this is a special case of the projective bundle formula, see e.g. [75, 14.5.2]), which implies that $M(\eta) = 0$, and the claim follows. We can now state and use the following foundational result.

Theorem 4.13 (Deglise [21])**.** *Let* k *be a perfect field. The functor* $U : \mathbf{DM}(k)^\heartsuit \to \mathbf{SH}(k)^{\heartsuit, \eta=0}$ *is an equivalence of categories.*

Proof. Modulo identifying $\mathbf{SH}(k)^\heartsuit = \Pi_*(k)$ and $\mathbf{DM}(k)^\heartsuit = \Pi_*^{tr}(k)$, this is Theorem 1.3.4 of Deglise. This identification is given by Theorem 4.10. □

Corollary 4.14. *For $F \in \mathbf{SH}(k)^{\heartsuit,\eta=0}$ we have $UM^{\heartsuit}F = F$.*

Proof. By the theorem we may write $F = UF'$. Using the fact that $M^{\heartsuit}$ is left adjoint to U $(= U^{\heartsuit})$ by Proposition 4.3, we compute $[M^{\heartsuit}F, T] = [M^{\heartsuit}UF', T] = [UF', UT] = [F', T]$, where the last equality is because U is fully faithful (by the theorem). Thus $M^{\heartsuit}F = F'$ by the Yoneda lemma, and finally $UM^{\heartsuit}F = UF' = F$. □

Corollary 4.15. *For $F \in \mathbf{SH}(k)^{\heartsuit}$ we have $UM^{\heartsuit}(F) = F/\eta$, where F/η denotes the cokernel of $\eta_F : F(1) \to F$ in the abelian category $\mathbf{SH}(k)^{\heartsuit}$.*

Proof. We have the right exact sequence

$$F(1) \xrightarrow{\eta} F \to F/\eta \to 0.$$

Since $M^{\heartsuit}$ is left adjoint it is right exact. Also U is exact, so we get the right exact sequence

$$UM^{\heartsuit}F(1) \to UM^{\heartsuit}F \to UM^{\heartsuit}(F/\eta) \to 0.$$

The first arrow is zero and $UM^{\heartsuit}(F/\eta) = F/\eta$ by the previous corollary (note that $F/\eta \in \mathbf{SH}(k)^{\heartsuit,\eta=0}$). The result follows. □

We thus obtain the Hurewicz theorem for $\mathbf{SH}(k) \to \mathbf{DM}(k)$.

Theorem 4.16 (Final Version of the Motivic Hurewicz Theorem)**.** *Let k be a perfect field and $E \in \mathbf{SH}(k)_{\ge 0}$.*

Then $ME \in \mathbf{DM}(k)_{\ge 0}$ and modulo the identification of $\mathbf{DM}(k)^{\heartsuit}$ as a full subcategory of $\mathbf{SH}(k)^{\heartsuit}$ (via Theorem 4.13) we have

$$\underline{h}_0(ME)_* = \underline{\pi}_0(E)_*/\eta.$$

Proof. Combine Corollary 4.12 with Corollary 4.15. □

4.3 The Slice Filtration and the Conservativity Theorem

In order to make use of the motivic Hurewicz theorem, we need to understand better the map η_F on a homotopy module F. Let us recall some more facts about the structure of the category $\mathbf{HI}_*(k)$. It is a symmetric monoidal abelian category. We denote the monoidal operation by $\wedge$. The monoidal unit is denoted $\underline{\pi}_0(S)_* =: \underline{K}_*^{MW}$ and called *unramified Milnor-Witt K-theory.* It has been explicitly described by Morel [82, Chapter 3 and Section 6.3]. In particular η defines an element (of the same name) in $\underline{K}_{-1}^{MW}(k)$.

There is for $E, F \in \mathbf{SH}(k)$ a natural map $\underline{\pi}_0(E)_i \otimes \underline{\pi}_0(F)_j \to \underline{\pi}_0(E \wedge F)_{i+j}$. This induces for $E_*, F_* \in \mathbf{HI}_*(k)$ homomorphisms of sheaves $E_i \otimes F_j \to (E \wedge F)_{i+j}$. In particular putting $E_* = \underline{K}_*^{MW}$ we get $\underline{K}_i^{MW} \otimes F_j \to F_{i+j}$. This makes every homotopy module a module over the unramified Milnor-Witt K-theory.

If $F_* \in \mathbf{HI}_*(k)$ is a homotopy module, then each of the sheaves F_i is strictly homotopy invariant. Since strictly invariant sheaves are unramified [81, Lemma 6.4.4], $F_i = 0$ if and only if $F_i(L) = 0$ for every finitely generated field extension L/k.

It is also known that such sheaves have transfers. More specifically, if L/K is a finite field extension with K/k finitely generated, then there is the so-called *cohomological transfer* $tr_{L/K} : F_i(L) \to F_i(K)$ [82, Chapter 4]. These transfers have many intricate properties which we do not state in detail here.

Given $F_* \in \mathbf{HI}_*(k)$ and K/k a finitely generated extension, we define

$$(\underline{K}_i^{MW} F_j)^{tr}(K) = \langle tr_{L/K}(K_i(L)F_j(L)) \rangle_{L/K \text{ finite}} \subset F_{i+j}(K).$$

We shall use this notation momentarily. First we need to recall the *slice filtration* [106, Section 2]. Write $\mathbf{SH}(k)^{eff}(i)$ for the localising subcategory of $\mathbf{SH}(k)$ generated by $(\Sigma^\infty X_+) \wedge {\mathbb{G}_m}^{\wedge i}$ for all

$X \in Sm(k)$. The inclusion $\mathbf{SH}(k)^{eff}(i) \hookrightarrow \mathbf{SH}(k)$ commutes with arbitrary sums by construction and so affords a right adjoint f_i by Neeman's version of Brown representability. The object $f_i E$ is called the i-th *slice cover* of E. It is easy to see that there is a commutative diagram of natural transformations

$$\begin{array}{ccc} f_i & \longrightarrow & f_{i-1} \\ \downarrow & & \downarrow \\ \mathrm{id} & = & \mathrm{id}\,. \end{array}$$

We call E such that $E \in \mathbf{SH}(k)^{eff}(n)$ for some n (equivalently $E = f_n E$) *slice-connective.*

Now suppose that $E \in \mathbf{SH}(k)_{\geq 0}$. We define a filtration on $\underline{\pi}_0(E)_*$ by putting

$$F_N \underline{\pi}_0(E)_* := im(\underline{\pi}_0(f_{-N}E)_* \to \underline{\pi}_0(E)_*) \subset \underline{\pi}_0(E)_*.$$

The above commutative diagram implies that $F_N \underline{\pi}_0(E)_* \subset F_{N+1}\underline{\pi}_0(E)_*$.

There is now the following highly interesting result. (We caution that Levine uses somewhat different indexing conventions than we do.)

Theorem 4.17 (Levine [66], slightly adapted Theorem 2)**.** *Let k be a perfect field of characteristic different from 2 and $E \in \mathbf{SH}(k)_{\geq 0}$. Then for $m \geq i$ and any perfect field extension F/k we have*

$$(F_i \underline{\pi}_0(E)_*)_m(F) = (\underline{K}^{MW}_{m-i}\underline{\pi}_0(E)_i)^{tr}(F).$$

We can fruitfully combine this result with the motivic Hurewicz theorem and the work of Voevodsky and others on the Milnor conjectures. The upshot is the following. Write $\mathbf{SH}(k)_e$ for the (co-localising) subcategory of $\mathbf{SH}(k)$ consisting of those $E \in \mathbf{SH}(k)$ such that $E \xrightarrow{e} E$ is an isomorphism. Equivalently, all the homotopy sheaves $\underline{\pi}_i(E)_j$ are modules over $\mathbb{Z}[1/e]$.

Theorem 4.18. *Let k be a perfect field of finite 2-étale cohomological dimension and exponential characteristic e. Let $E \in \mathbf{SH}(k)_e$ be slice-connective and 0-connective (i.e. $\underline{\pi}_i(E)_* = 0$ for $i < 0$). Then if $\underline{h}_0(ME)_* = 0$ also $\underline{\pi}_0(E)_* = 0$.*

Write $\mathbf{SH}(k)_e^{conn,slconn}$ for the subcategory of e-local, connective and slice-connective spectra. Then in particular the functor

$$M : \mathbf{SH}(k)_e^{conn,slconn} \to \mathbf{DM}(k)$$

is conservative.

We note that the theorem can definitely fail if k has infinite 2-étale cohomological dimension. See [15, Example 2.1.2(4)] for an example.

Before proving the theorem, let us mention two auxiliary results.

Lemma 4.19. *Let $E \in \mathbf{SH}(k)_e$ be compact. Then E is connective and slice-connective.*

Proof. The category $\mathbf{SH}(k)_e$ is generated as a localising category by the e-localisations of $\Sigma^\infty X_+ \wedge {\mathbb{G}_m}^{\wedge i}$ for $i \in \mathbb{Z}$ and $X \in Sm(k)$. Each of these objects is slice-connective by definition and connective by Morel's stable connectivity theorem [81]. Consequently if $\mathcal{T}$ is the thick triangulated subcategory of $\mathbf{SH}(k)_e$ generated by these objects, all $E \in \mathcal{T}$ are connective and slice-connective.

It remains to show that all compact objects $E \in \mathbf{SH}(k)_e$ are contained in $\mathcal{T}$. But each of the generators $\Sigma^\infty X_+ \wedge {\mathbb{G}_m}^{\wedge i}$ is compact (see e.g. [98], Lemma 2.27 and paragraph thereafter); it follows from general results [86, Lemma 2.2] that $\mathcal{T}$ is precisely the subcategory of compact objects. □

Theorem 4.20. *The homomorphism*

$$M : Pic(\mathbf{SH}(k)_e) \to Pic(\mathbf{DM}(k, \mathbb{Z}[1/e]))$$

is injective (if k is a perfect field of finite 2-étale cohomological dimension).

Proof. Invertible objects are compact (since S is) and hence connective and slice-connective by the lemma. Let $E \in Pic(\mathbf{SH}(k)_e)$ such that $ME = \mathbb{1}$. Then $ME \in \mathbf{DM}(k)_{\geq 0}$ and hence by the theorem we have $E \in \mathbf{SH}(k)_{\geq 0}$. Applying the motivic Hurewicz theorem we find that $\underline{\pi}_0(E)_*/\eta = \underline{h}_0(ME)_* = \underline{h}_0(\mathbb{1})_* = K_*^{MW}[1/e]/\eta$. Consequently there exists an element $a \in [S, E] = \underline{\pi}_0(E)_0(k)$ with Ma an isomorphism. By the conservativity we find that a is an isomorphism and so $E = \mathbb{1}$. This concludes the proof. □

Proof of Theorem 4.18. The "in particular" part follows from the first part by induction and non-degeneracy of the t-structure.

Now let $E \in \mathbf{SH}(k)_e$ be slice-connective and zero-connective, and $\underline{h}_0(ME)_* = 0$. We need to show that $\underline{\pi}_0(E)_* = 0$. By the motivic Hurewicz theorem, we know that $\underline{\pi}_0(E)_*/\eta = 0$, i.e. η is surjective on $\underline{\pi}_0(E)_*$.

Let us assume first that k has characteristic zero. Since E is slice-connective, we know that $f_{-N}E \simeq E$ for some N sufficiently large. Thus $F_N\underline{\pi}_0(E)_* = \underline{\pi}_0(E)_*$. By Levine's Theorem 4.17 we find that for any finitely generated field extension K/k and any $n > 0$

$$\underline{\pi}_0(E)_{n+N}(K) = (F_N\underline{\pi}_0(E)_{n+N})(K) = (\underline{K}_n^{MW}\underline{\pi}_0(E)_N)^{tr}(K).$$

It is thus enough to show that there exists $R > 0$ such that for $n > R$ and L/K finite, we have

$$\underline{K}_n^{MW}(L)\underline{\pi}_0(E)_N(L) = 0.$$

Using surjectivity of η, this is the same as $\underline{K}_n^{MW}(L)\eta^n\underline{\pi}_0(E)_{N+n}(L)$. But $\underline{K}_n^{MW}(L)\eta^n = I(L)^n$, where $I(L)$ is the fundamental ideal in the Witt ring (see again [82, Chapter 3]). By the resolution of the Milnor conjectures, $I(L)^n = 0$ as soon as $n > cd_2(L)$, where $cd_2(L)$ is the 2-étale cohomological dimension. See [78] for Voevodsky's resolution of the Milnor conjectures. Let $R = cd_2(K)$. Then $R < \infty$ since k has finite 2-étale cohomological dimension by assumption and K/k is finitely generated [102, Theorem 28 of Chapter 4]. But then since L/K is a finite extension, $cd_2(L) \leq R = cd_2(K)$ by loc. cit. Hence this R works. This concludes the proof in characteristic zero.

If $2 \neq e > 1$ we may use exactly the same argument, but eventually we cannot conclude that $\underline{\pi}_0(E)_*(K) = 0$ for all K/k finitely generated, but only for the perfect closures K^p of such fields. (Note that $cd_2(K^p) \leq cd_2(K) < \infty$ by loc. cit. Actually equality holds but we do not need this.) The result then follows from the Lemma below.

If $2 = e$, we use that $\mathbf{SH}(k)_2$ decomposes as $\mathbf{SH}(k)_2^+ \times \mathbf{SH}(k)_2^-$. One has $\eta_2^+ = 0$ and consequently $(\mathbf{SH}(k)_2^+)^\heartsuit \subset \mathbf{SH}(k)^{\heartsuit,\eta=0}$ and in particular $\underline{\pi}_0(E)_*^+ = \underline{\pi}_0(E)_*^+/\eta = 0$. On the other hand $End_{\mathbf{SH}(k)_2^-}(\mathbb{1}) = W(k)[1/2] = 0$ [77, Theorem III.3.6]. Consequently $\mathbf{SH}(k)_2^- = 0$ in our case and thus $\underline{\pi}_0(E)_*^- = 0$. Thus $\underline{\pi}_0(E)_* = \underline{\pi}_0(E)_*^+ \oplus \underline{\pi}_0(E)_*^- = 0$, concluding the proof in the final case. □

Lemma 4.21. *Let k be a perfect field of characteristic $p > 0$ and H_* a homotopy module on which p is invertible. Let K/k be a finitely generated field and K'/K a purely inseparable extension. Then $H_*(K) \to H_*(K')$ is injective.*

Proof. Let L/k be a finitely generated field extension and $x \in L$ not a p-th root. Write $L' = L(x^{1/p})$. I claim that $H_*(L) \to H_*(L')$ is injective. Once this is done we conclude that $H_*(K) \to H_*(K')$ is injective for any purely inseparable finitely generated extension (being a composite of finitely many extensions of the form L'/L) and hence for any purely inseparable extension by continuity.

In order to prove the claim, we shall use the transfer $tr_{L'/L} : H_*(L') \to H_*(L)$. This satisfies the projection formula: if $\alpha \in H_*(L)$ then $tr_{L'/L}\alpha|_{L'} = tr_{L'/L}(1)\alpha$, where $1 \in \underline{G}W(L')$ is the unit. (This is because transfer comes from an actual map of pro-spectra $\Sigma^\infty Spec(L)_+ \to \Sigma^\infty Spec(L')_+$.) Hence it is enough to show that $t := tr_{L'/L}(1)$ is a unit in $\underline{G}W(L)[1/p]$. But $t = \sum_{i=1}^p \langle(-1)^{i-1}\rangle$. Indeed this may be checked by direct computation, using the fact that (cohomological) transfers on $\underline{K}_*^{MW}$ coincide with Scharlau transfers, as follows from their definition [82, Section 4.2] and Scharlau's reciprocity law [99, Theorem 4.1]; this is explained in more detail in [16, Lemma

2.2]. To show that $t \in GW(L)[1/p]$ is invertible, it is enough to consider the canonical images $dim(t) \in \mathbb{Z}[1/p]$ and $cl(t) \in W(L)[1/p]$. We know that $dim(t) = p$ is invertible by design. If $p = 2$ then $W(L)[1/p] = 0$ [77, Theorem III.3.6], so $cl(t)$ is clearly a unit. Otherwise we have that $t = \frac{p-1}{2}(\langle 1\rangle + \langle -1\rangle) + 1$ and so $cl(t) = 1$ is also invertible. This concludes the proof. □

4.4 Applications

We list some rather abstract consequences. More concrete ones will follow in the next chapter.

Recall that an object E in a symmetric monoidal category $\mathcal{C}$ is called *rigid* with dual DE if the functors $\otimes E, \otimes DE$ are both left and right adjoint to one another. Rigid objects are preserved by monoidal functors since they are detected by the zig-zag equations [74, Theorem 2.6]. If $\mathcal{C}$ is a tensor triangulated category then the subcategory $\mathcal{C}^{rig}$ of rigid objects is a thick tensor triangulated subcategory [48, Theorem A.2.5].

Lemma 4.22. *Let $\mathcal{C}, \mathcal{D}$ be symmetric monoidal categories and $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal, conservative functor. Suppose given $E \in \mathcal{C}$ rigid. Then E is invertible if and only if $F(E) \in \mathcal{D}$ is invertible.*

Proof. The object E is invertible if and only if the natural map $\alpha : DE \otimes E \to \mathbb{1}_{\mathcal{C}}$ is an isomorphism. By assumption, this happens if and only if $F(\alpha)$ is an isomorphism. But now F is symmetric monoidal so preserves duals, whence $F(\alpha) : DF(E) \otimes F(E) \to \mathbb{1}_{\mathcal{D}}$ is the canonical map, which is an isomorphism if and only if $F(E)$ is invertible. □

Of course, if $\mathcal{C}, \mathcal{D}, F$ are triangulated then F is conservative if and only if it detects zero objects, which may be easier to verify.

Proposition 4.23. *Let k be a perfect field of exponential characteristic e and finite 2-étale cohomological dimension. Let $E \in \mathbf{SH}(k)_e$ be compact. Then E is invertible if and only if $ME \in \mathbf{DM}(k, \mathbb{Z}[1/e])$ is invertible.*

Proof. The triangulated category $\mathbf{SH}(k)_e$ is compact-rigidly generated [67, Corollary B.2]. The functor $M : \mathbf{SH}(k)_e^c \to \mathbf{DM}(k, \mathbb{Z}[1/e])$ is symmetric monoidal triangulated, and conservative by Lemma 4.19 and the Conservativity Theorem 4.18. Thus the result follows from Lemma 4.22. □

We wish to globalise this result, i.e. extend it to more general bases.

Proposition 4.24. *Let X be a scheme of finite type over a field k and $E \in \mathbf{SH}(k)$ be rigid. Then E is invertible if and only if for every point $i_x : \{x\} \to X$ of X, the pullback $i_x^*(E) \in \mathbf{SH}(x)$ is invertible.*

Proof. Necessity is clear, we show sufficiency. Since $X_{red} \to X$ is a cdh cover, pullback along $X_{red} \to X$ is conservative and hence detects invertibility of rigid objects, by Lemma 4.22. We may thus assume that X is reduced.

Let C be a cone on the canonical morphism $DE \otimes E \to \mathbb{1}$. Since the tensor unit in $\mathbf{SH}(X)$ is compact so is any rigid object such as C. We may thus use continuity (see Theorem 3.13 and the preceding discussion). Since for every generic point $\eta \in X$ we know that $i_\eta E \in \mathbf{SH}(E)$ is invertible, we know that $i_\eta C \simeq 0$ and hence we find an everywhere dense open subset $U \subset X$ such that $E|_U$ is invertible. Let Z be the closed complement. The pair $\mathbf{SH}(X) \to \mathbf{SH}(U), \mathbf{SH}(X) \to \mathbf{SH}(Z)$ is conservative (e.g. by considering gluing triangles), and hence it follows from Lemma 4.22 that it suffices to show that $E|_Z$ is invertible. This follows by induction on dimension. □

Of course, this result is most useful in characteristic zero, where all the residue fields are automatically perfect, and the previous result applies.

Corollary 4.25. *Let k be a field of characteristic zero and finite 2-étale cohomological dimension, and X a scheme of finite type over k. Let $E \in \mathbf{SH}(X)$ be rigid. Then E is invertible if and only if $Mi_x^*E \in \mathbf{DM}(x)$ is invertible for all $x \in X$.*

The following result helps identifying rigid objects.

Lemma 4.26. *Let X be a scheme of finite type over a field k.*

(i) For $f : Y \to X$ smooth and proper and $E \in \mathbf{SH}(Y)$ rigid, the object $f_\#(E) \in \mathbf{SH}(X)$ is rigid.

(ii) If $\{U_\alpha \to X\}$ is a Nisnevich cover and $E \in \mathbf{SH}(X)$, then E is rigid if and only if $E|_{U_\alpha}$ is rigid for every α.

(iii) If $V \to X$ is a vector bundle, then the Thom spectrum $Th(N) \in \mathbf{SH}(X)$ is rigid (in fact invertible).

(iv) If $Y \to X$ is smooth and proper, $Z \to Y$ a closed immersion such that $Z \to X$ is also smooth, and $U = Y \setminus Z$, then $\Sigma^\infty(U) \in \mathbf{SH}(X)$ is rigid.

Proof. Part (i) is a direct generalization of [18, Proposition 2.4.31]: the functor $\mathbf{SH}(X) \ni A \mapsto A \otimes f_\#(E)$ is isomorphic to $A \mapsto f_\#(E \otimes f^*A)$. This has a right adjoint $A \mapsto f_*(E' \otimes f^*A)$, where E' is the dual of E. By purity, this functor is isomorphic to $A \mapsto f_\#\Omega_f(E' \otimes f^*A)$, where Ω_f is tensoring with an invertible object T. Using the projection formula, this functor is isomorphic to $A \mapsto f_\#(T \otimes E') \otimes A$. Finally, the functor $A \mapsto f_\#\Omega_f(E' \otimes f^*)$ has as right adjoint $A \mapsto f_*(E \otimes \Sigma_f f^*A)$, which can be identified as $A \mapsto f_\#(E) \otimes A$ by the same method as before. Thus $f_\#(E)$ is rigid with dual $f_\#(T \otimes E')$.

For part (ii), necessity is clear. We show sufficiency. Note that E is rigid if and only if $T \otimes \underline{\mathrm{Hom}}(E, \mathbb{1}) \to \underline{\mathrm{Hom}}(E, T)$ is an isomorphism for all T [48, Definition A.2.4]. But for $f : U \to X$ étale we know that by the six functors formalism, $\underline{\mathrm{Hom}}(f^*E, f^*T) = f^*\underline{\mathrm{Hom}}(E, T)$ [18, A.5.1 (6), (4)]. Since pullback along a Nisnevich cover is conservative since $\mathbf{SH}$ has Nisnevich descent, and the morphisms in a Nisnevich cover are étale, we are done.

Part (iii) now reduces, by local triviality of vector bundles, to rigidity of Thom spectra of affine spaces, which holds by construction.

Part (iv) follows from homotopy purity [18, Theorem 2.4.35], i.e. the existence of a distinguished triangle

$$\Sigma^\infty(U_+) \to \Sigma^\infty(Y_+) \to Th(N_Z Y).$$

Indeed both $\Sigma^\infty(Y_+)$ and $Th(N_Z Y)$ are rigid by (i) and (iii), and hence so is $\Sigma^\infty(U_+)$. □

4.5 Extensions

The results in this chapter are not optimal. We have decided to prove only the version of the conservativity theorem which has the simplest proof. For stronger statements, see [6]. We can summarize the results of that article as follows:

1. Let k be a perfect field of exponential characteristic e, $E \in \mathbf{SH}(k)$ and assume that $0 \simeq ME \in \mathbf{DM}(k)$. We can conclude that $E \simeq 0$ in the following situations.

   (a) $E \in \mathbf{SH}(k)_e$ and E is connective and slice-connective, and k is of finite 2-étale cohomological dimension. (This is the version proved here.)

   (b) $E \in \mathbf{SH}(k)$ is compact and k is non-orderable (i.e. -1 is a sum of squares in k).

   There are two ingredients to going from (a) to (b). Firstly, one may prove the result for compact $E \in \mathbf{SH}(k)$ with k of finite 2-étale cohomological dimension. This is done by replacing the slice filtration on the homotopy modules by an algebraically constructed one. Note that while compact implies connective and slice-connective, the proof really needs compactness in its current form, so this is not a strict generalisation. Secondly, any non-orderable perfect field is a colimit of perfect fields of finite 2-étale cohomological dimension. The result over any perfect non-orderable field for compact objects thus follows by continuity.

2. The above deals with all perfect fields of positive characteristic. Suppose now that k has characteristic zero. If k can be embedded into $\mathbb{R}$, the set of orderings $Sper(k)$ coincides with the set of embeddings into $\mathbb{R}$. For $\alpha \in Sper(k)$ such an embedding, there is a real realisation functor
$$\mathbf{SH}(k) \to \mathbf{SH}(\mathbb{R}) \to \mathbf{SH}$$
coming from looking at the topological space of real points. We denote the composite
$$\mathbf{SH}(k) \to \mathbf{SH}(\mathbb{R}) \to \mathbf{SH} \xrightarrow{C_*} D(Ab) \to D(\mathbb{Z}[1/2])$$
by $M_\alpha[1/2] : \mathbf{SH}(k) \to D(\mathbb{Z}[1/2])$. Here C_* denotes the ordinary singular complex functor. We call $M_\alpha[1/2]$ the real motive functor associated with $\alpha \in Sper(k)$. It turns out that a similar functor can be defined for any field k of characteristic zero, whether it can be embedded into $\mathbb{R}$ or not.

   One may then prove: if k is of finite virtual 2-étale cohomological dimension (and characteristic zero), $E \in \mathbf{SH}(k)$ is connective and slice-connective, $0 \simeq ME \in \mathbf{DM}(k)$ and also $0 \simeq M_\alpha[1/2]E \in D(\mathbb{Z}[1/2])$ for every $\alpha \in Sper(k)$, then in fact $E \simeq 0$.

3. By the same argument as in (1), the previous result holds for arbitrary fields k of characteristic zero when E is assumed compact.

These results allow us to strengthen Corollary 4.25. Recall that for a scheme X, the set $R(X)$ consists of pairs (p, α) with $p \in X$ and $\alpha \in Sper(k(x))$ [100, (1.2)]. For $x \in X$ we write $M_x : \mathbf{SH}(X) \to \mathbf{DM}(x)$ for the composite $\mathbf{SH}(X) \xrightarrow{i_x^*} \mathbf{SH}(x) \xrightarrow{M} \mathbf{DM}(x)$ and for $r = (x, \alpha) \in R(X)$ we write $M_r[1/2]$ for the composite $\mathbf{SH}(X) \to \mathbf{SH}(x) \xrightarrow{M_\alpha[1/2]} D(\mathbb{Z}[1/2])$.

Corollary 4.27. *Let k be a field of characteristic zero, X/k of finite type, and $E \in \mathbf{SH}(X)$ rigid. Then E is invertible if and only if*

(i) for every $x \in X$, the motive $M_x E \in \mathbf{DM}(x)$ is invertible, and

(ii) for every $r \in R(X)$, the real motive $M_r[1/2]E \in D(\mathbb{Z}[1/2])$ is invertible.

Proof. Necessity is clear, we show sufficiency. By Proposition 4.24, we may assume that X is the spectrum of a field (of characteristic zero). Rigid objects are compact since the unit is, so we can use the strongest form of the conservativity theorem as explained in point (3) of the above summary of extensions. The result follows. □

Chapter 5

Computations

Some of the results in this chapter are being published as [7].

In this chapter we concentrate on more concrete results. In section 5.1 we begin by recalling Voevodsky's construction of $\mathbf{DM}(k)$ and its basic properties. We prove some basic results about change of coefficients and base. In particular, if l/k is a finite separable field extension and A is a ring of coefficients such that $[l:k] \in A^\times$, then the functor $\mathbf{DM}(k,A) \to \mathbf{DM}(l,A)$ is conservative, and there is a bijection $ker(Pic(\mathbf{DM}(k,A)) \to Pic(\mathbf{DM}(l,A))) \cong \mathrm{Hom}(Gal(l/k), A^\times)$. We also show that if A is a PID, then the study of $Pic(\mathbf{DM}(k,A))$ can roughly be broken up into studying $Pic(\mathbf{DM}(k,k(P)))$, where $k(P)$ runs through the various residue fields of A, including $Frac(A)$.

In Section 5.2 we recall the basics about weight structures and prove our abstract fixed point functors theorem. This essentially says the following. Suppose that F is a finite (coefficient) field, k is a base field, and for every field l/k we are given a set S_l of smooth projective varieties. We write $\mathbf{D}\langle S\rangle\mathbf{TM}(l,F)$ for the thick, triangulated, symmetric monoidal subcategory of $\mathbf{DM}(l,F)$ generated by the motives of varieties in S_l and the Tate motives. Recall also that the category $Chow(k,F)$ of Chow motives embeds into $\mathbf{DM}(k,F)$, and that we write $Tate(F)$ for the subcategory of $Chow(k,F)$ consisting of the Tate motives. Then for good choices of S, there is a functor $\Phi^k : \mathbf{D}\langle S\rangle\mathbf{TM}(k,F) \to K^b(Tate(F))$ which is essentially uniquely determined by the following property: if $X \in S_k$ write $M(X) \cong M \oplus T \in Chow(k,F)$, where T is a Tate motive and M is a *Tate-free* motive, i.e. affords no (non-zero) summands which are Tate motives. Then $\Phi^k(MX) \cong T[0]$.

If S is suitably stable under base change, then for a field extension l/k the base change $\mathbf{DM}(k,F) \to \mathbf{DM}(l,F)$ restricts to the subcategories $\mathbf{D}\langle S\rangle\mathbf{TM}(k,F) \to \mathbf{D}\langle S\rangle\mathbf{TM}(l,F)$, and we may construct a composite functor $\Phi^l : \mathbf{D}\langle S\rangle\mathbf{TM}(k,F) \to \mathbf{D}\langle S\rangle\mathbf{TM}(l,F) \to K^b(Tate(F))$.

The abstract fixed point functors theorem given conditions on S under which the constructions outlined above are possible and yield a (weight) conservative and Pic-injective collection of functors. Apart from trivial compatibility requirements (ensuring that we get base change functors $\mathbf{D}\langle S\rangle\mathbf{TM}(k,F) \to \mathbf{D}\langle S\rangle\mathbf{TM}(l,F)$) there are basically two such conditions. Firstly, we have to ensure that the functors Φ^l are well-defined and symmetric monoidal. It turns out that a good way to achieve this is to require that for every $X \in S_l$, every zero-cycle of X has degree divisible by the characteristic of F. Secondly we need to ensure that the functors Φ^l "see all of MX", for $X \in S$. This certainly requires that each MX be geometrically Tate. In order for our argument to work we need a somewhat stronger property to be true. Basically, whenever X has an l-rational point, we need the base change $(MX)_l$ to break up into a Tate motive and some other motives built from various $Y \in S_l$, and we need these Y to be "smaller" in some sense (in order to allow inductive arguments).

With this result established, we perform the computations outlined in the introduction. In Section 5.3 we study the motives of quadrics. Using the results about change of coefficients, base change, and the abstract fixed point functors theorem, we cook up a reasonably computable collection functors on $\mathbf{DQM}^{gm}(k,\mathbb{Z})$ which is jointly conservative and Pic-injective. (These functors are basically reduction modulo two followed by the abstract fixed point functors, which exist essentially by Springer's theorem, and the geometric base change functor.) We use these to prove that all reduced motives of smooth affine quadrics are invertible, and to prove the Hu-conjecture for **DM**. We also use the Motivic Hurewicz Theorem from the previous section to lift these results to some extent to **SH**.

In Section 5.4 we study Artin and Artin-Tate motives, following essentially the same strategy: using the general results, we cook up a conservative and Pic-injective family of functors. There are some minor differences compared with quadrics: we need to consider reductions modulo all primes, not just two, and the fixed point functors exist only after "p-special base change", i.e. they exist with coefficients of characteristic p over a field k such that every finite separable extension of k is of degree a power of p. Fortunately if k is any field then there is an extension k_p/k with this property and which moreover has the property that every finite subextension $k_p/l/k$ has degree $[l : k]$ coprime to p (essentially, k_p is a maximal extension with this property). Then the base change $\mathbf{DAM}(k, \mathbb{Z}/p) \to \mathbf{DAM}(k_p, \mathbb{Z}/p)$ is conservative and (so) can be controlled by our general results. With these constructions and results out of the way, we can perform the computations outlined in the introduction.

In Section 5.5 we look at a somewhat different example: we study the Picard group of the subcategory of $\mathbf{SH}(k)$ generated by the spectra ${\mathbb{G}_m}^{\wedge n}$, which might be called the category of Tate spectra. Here we cannot apply the abstract fixed point functors theorem, but there is still a weight structure. We use it, together with some non-commutative algebra, to prove that every invertible Tate spectrum is (uniquely) of the form $S_{\mathcal{L}} \wedge S^n \wedge {\mathbb{G}_m}^{\wedge m}$. Here $m, n \in \mathbb{Z}$ and $S_{\mathcal{L}}$ is the invertible Tate spectrum corresponding to an element $\mathcal{L} \in Pic(GW(k)) = Pic(End(\mathbb{1}))$ under the embedding $Pic(End(\mathbb{1})) \to \mathbf{SH}(k)$ explained in the introduction.

Finally in Section 5.6 we come to the first computation mentioned in the introduction: assuming suitable standard conjectures, we compute $Pic(\mathbf{DM}(k, \mathbb{Q}))$ completely. We also explain how to extend this to integral coefficients, if one looks at *étale motives* instead of the usual Nisnevich ones.

5.1 Change of Coefficients and Base for DM

In this section we recall the construction of Voevodsky's category $\mathbf{DM}(k, A)$ and prove some basic results about change of base and/or coefficients.

5.1.1 Recollections about DM

Let k be a field of exponential characteristic e. In [20] there was constructed a *cdh*-Quillen presheaf $Ft(k) \ni X \mapsto \mathcal{DM}(X, \mathbb{Z}[1/e])$ such that $\mathbf{DM} := Ho(\mathcal{DM})$ satisfies the six functors formalism. From this it follows (more or less) that $\mathcal{DM}$ is a cdh-Quillen sheaf. If l/k is a purely inseparable field extension, then the base change functor $\mathbf{DM}(k, \mathbb{Z}[1/e]) \to \mathbf{DM}(l, \mathbb{Z}[1/e])$ is an equivalence of categories [20, Proposition 8.1(d)]. If k is a perfect field there is a full subcategory $\mathbf{DM}^-(k, \mathbb{Z}[1/e]) \subset \mathbf{DM}(k, \mathbb{Z}[1/e])$ which can be described very explicitly, thanks to the work of Voevodsky [107].

For this, recall the category $Cor(k)$ from Section 4.2: the objects are the smooth schemes over k and the morphisms are the finite correspondences. Let A be a commutative ring. We can form the category $Shv^{tr}(k, A)$ consisting of presheaves of A-modules on $Cor(k)$ such that the restriction to $Sm(k)$ is a sheaf in the Nisnevich topology. This affords an unbounded derived category $D(Shv^{tr}(k, A))$ which is the homotopy category of a model category $C(Shv^{tr}(k, A))$. We can perform $\mathbb{A}^1$-localisation on this model category and obtain $\mathbf{DM}^{\text{eff}}(k, A) := Ho(L_{\mathbb{A}^1} C(Shv^{tr}(k, A))) \subset D(Shv^{tr}(k, A))$.

Write $\mathbf{HI}^{tr}(k, A)$ for the subcategory of $Shv^{tr}(k, A)$ consisting of homotopy invariant sheaves with transfers. Voevodsky has proved that if k is perfect and F is a homotopy invariant *presheaf*, F is actually strictly homotopy invariant: $H^p(X \otimes \mathbb{A}^1, F) = H^p(X, F)$ for any $X \in Sm(k)$ and any $p \geq 0$. Consequently an object $E \in D(Shv^{tr}(k, A))$ is $\mathbb{A}^1$-local (i.e. in $\mathbf{DM}^{\text{eff}}(k, A)$) if and only if all the homology sheaves $\underline{h}_i(E) \in \mathbf{HI}^{tr}(k)$. This also implies that the $\mathbb{A}^1$-localisation functor has a very nice and explicit description: let C_* denote the $\mathbb{A}^1$-chain complex functor, i.e. for $F \in Shv^{tr}(k,A)$, $C_n(F)(X) = F(X \times \Delta^n)$, where Δ^n is the algebraic n-simplex. Then one easily checks that the homology *presheaves* of C_*F are homotopy invariant and that $C_*F \to F$ is an $\mathbb{A}^1$-weak equivalence. It follows that C_* is an explicit model of the $\mathbb{A}^1$-localisation functor.

The next step in constructing $\mathbf{DM}(k,A)$ is to invert the object $\mathbb{G}_m$ under tensor product, i.e. to consider $\mathcal{DM}(k,A) = Stab^{\Sigma}(L_{\mathbb{A}^1}C(Shv^{tr}(k,A)), \mathbb{G}_m)$. Again a result of Voevodsky makes this construction much more accessible. Write $\mathbf{DM}^{\text{eff},-}(k,A)$ for the full subcategory of $\mathbf{DM}^{\text{eff}}(k,A)$ consisting of objects $E \in \mathbf{DM}^{\text{eff}}(k,A)$ such that $\underline{h}_i(E) = 0$ for i sufficiently small. Then we have Voevodsky's *Cancellation Theorem*: if k is perfect then for $E, F \in \mathbf{DM}^{\text{eff},-}(k,A)$ we have $[E,F] = [E \otimes \mathbb{G}_m, F \otimes \mathbb{G}_m]$ [109, Corollary 4.10]. Consequently the stabilisation functor Σ^∞ : $\mathbf{DM}^{\text{eff},-}(k,A) \to \mathbf{DM}(k,A)$ is a full embedding. In fact from this one deduces using standard techniques that $\mathbf{DM}^{\text{eff}}(k,A) \to \mathbf{DM}(k,A)$ is a full embedding, but we do not need this.

Some further notation. If $X \in Sm(k)$ we denote by $MX = M_AX$ its image in $\mathbf{DM}(k,A)$. The thick triangulated subcategory of $\mathbf{DM}(k,A)$ generated by $M(X) \otimes {\mathbb{G}_m}^{\wedge n}$ for $X \in Sm(k)$ and $n \in \mathbb{Z}$ is denoted $\mathbf{DM}^{gm}(k,A)$. If one restricts to $n \geq 0$ (equivalently $n = 0$) then one obtains a subcategory of $\mathbf{DM}^{\text{eff},-}(k,A)$ denoted $\mathbf{DM}^{\text{eff},gm}(k,A)$. Each of the generators of $\mathbf{DM}^{gm}(k,A)$ or $\mathbf{DM}^{\text{eff},gm}(k,A)$ is compact and so it follows that $\mathbf{DM}^{gm}(k,A)$ and $\mathbf{DM}^{\text{eff},gm}(k,A)$ are the subcategories of compact objects [86, Lemma 2.2].

A further important theorem of Voevodsky is that the Karoubi-closed, additive subcategory of $\mathbf{DM}(k,A)$ spanned by MX for X smooth and projective is equivalent to the category $Chow(k,A)$ of Chow motives with A-coefficients. (See Subsection 5.2.2 for our conventions regarding Chow motives.) It is well-known that the category $Chow(k,A)$ is rigid, i.e. every object is strongly dualisable (i.e. rigid). Since rigid objects are preserved under symmetric monoidal functors, every $MX \otimes {\mathbb{G}_m}^{\otimes n} \in \mathbf{DM}^{gm}(k,A)$ is rigid (here X is smooth and *projective*). If e is invertible on A then by [14], the category $\mathbf{DM}(k,A)$ is generated (as a localising subcategory) by $MX \otimes {\mathbb{G}_m}^{\otimes n}$ for X smooth and *projective* (and $n \in \mathbb{Z}$). Consequently $\mathbf{DM}^{gm}(k,A)$ is generated by $MX \otimes {\mathbb{G}_m}^{\otimes n}$ for X smooth and projective and coincides with the subcategory of rigid objects, and $\mathbf{DM}(k,A)$ is compact-rigidly generated.

5.1.2 Base Change

Theorem 5.1. *Let A be a (commutative) ring and $d \in \mathbb{Z}$ be invertible on A. If $f : Spec(l) \to Spec(k)$ is a separable extension of perfect fields such that every finite subextension $l/l'/k$ has degree $[l' : k]$ dividing a power of d,then $f^* : \mathbf{DM}(k,A) \to \mathbf{DM}(l,A)$ is conservative. Moreover there is an exact sequence*

$$0 \to \mathrm{Hom}_{cts}(Gal(l/k), A^\times) \to Pic(\mathbf{DM}(k,A)) \to Pic(\mathbf{DM}(l,A)).$$

Proof. Suppose first that f itself is finite. In this case f defines a transfer map denoted $f^\dagger \in \mathrm{Hom}_{Cor(k)}(Spec(k), Spec(l))$. The composite $ff^\dagger$ equals $d\,\mathrm{id}_X$. Consequently for $F \in Shv^{tr}(k,A)$, the homomorphism $F(k) \to F(l)$ is injective. By the same argument, for any $X \in Sm(k)$, the homomorphism $F(X) \to F(X_l)$ is injective.

The functor $f^* : \mathbf{DM}(k,A) \to \mathbf{DM}(l,A)$ is exact, i.e. commutes with homology sheaves. The conservativity result follows immediately. For general, possibly infinite, f it follows from continuity.

We now prove the result about Picard groups, in a slightly roundabout way.

Write $d - Nis$ for the topology on $Sm(k)$ where the coverings are jointly surjective families of étale maps $Y_\alpha \to X$ such that for every $x \in X$ there is an α and a $y \in Y_\alpha$ over x such that the residue field extension $k(y)/k(x)$ is of degree dividing a power of d. One may check that this defines a topology; write a_d for the associated sheaf functor. I claim that a $d - Nis$-local weak equivalence in $\mathbf{DM}^{\text{eff}}(k,A)$ is an ordinary weak equivalence. Indeed it suffices to show that if $E \in \mathbf{DM}^{\text{eff}}(k,A)$ is $d - Nis$-locally weakly equivalent to 0 then $E \simeq 0$. But this can be checked on homology sheaves. Let l/k be a finitely generated field extension and l^d/l be obtained by taking the union of all finite separable extensions of l (in a separable closure, say) of degree dividing a power of d. By what we have said before (and continuity), $F(l) \to F(l^d)$ is injective. But it is easy to see that $(a_d F)(l^d) = F(l^d)$, hence if $a_d F = 0$ then $F(l) = 0$ for every finitely generated field extension l/k. It follows that $F = 0$, because strictly homotopy invariant sheaves are unramified [81, Lemma 6.4.4].

It follows in the usual way that $\mathcal{DM}^{\text{eff}}(k, A)$ has descent in the $d - Nis$ topology, and hence so does $\mathbf{DM}^{gm}(k, A)$ (in an appropriate sense). We apply the descent spectral sequence for the space $PIC(\mathbf{DM}(k, A)) = PIC(\mathbf{DM}^{gm}(k, A))$ to the d-Nisnevich cover $f : Spec(l) \to Spec(k)$. Now we have $\pi_0(PIC(\mathbf{DM}(?, A))) = Pic(\mathbf{DM}(?, A))$, $\pi_1(PIC(\mathbf{DM}(?, A))) = A^\times$, and $\pi_i(PIC(\mathbf{DM}(?, A))) = [\mathbb{1}[i-1], \mathbb{1}] = 0$ [75, Corollary 4.2]. It follows from the form of the spectral sequence that there is an exact sequence

$$0 \to \check{H}^1(l/k, A^\times) \to Pic(\mathbf{DM}(k, A)) \to \check{H}^0(l/k, Pic(\mathbf{DM}(?, A))).$$

Now $\check{H}^0(l/k, F) \hookrightarrow F(l)$ for any presheaf F, and $\check{H}^1(l/k, F) = \mathrm{Hom}_{cts}(Gal(l/k), F)$ is well-known. The result follows. □

Remark. It is possible to prove the exact sequence result about Picard groups much more directly, if somewhat less transparently [7, Proposition 13]. The injection displayed in loc. cit. is in fact an isomorphism, as we shall explain more directly in section 5.4.

Proposition 5.2. *Let k be an algebraically closed field of exponential characteristic e, A a coefficient ring on which e is invertible, and l/k an arbitrary field extension.*

Then the homomorphism $Pic(\mathbf{DM}(k, A)) \to Pic(\mathbf{DM}(l, A))$ is injective.

Proof. Let $E \in Pic(\mathbf{DM}(k))$ be such that $E|_l = \mathbb{1}$. Note that E is compact. Write l as a colimit of smooth, finitely generated subalgebras (this is possible since k is algebraically closed so in particular perfect). By continuity, we find that there exists a smooth affine finite type scheme $f : X \to Spec(k)$ such that $f^*E = \mathbb{1}$. Let $i : x \hookrightarrow X$ be any closed point. Then $i^*f^*E = \mathbb{1}$ as well. But $fi : x \to Spec(k)$ is an isomorphism since k is algebraically closed. This concludes the proof. □

Remark. The same result and proof apply to **SH**, or more generally any pseudofunctor satisfying continuity in which the units are compact.

We will later need the following observation. For $M \in \mathbf{DM}(k, A)$ we write $M\{n\} = M \otimes {\mathbb{G}_m}^{\otimes n}[n]$.

Lemma 5.3. *Let k be a perfect field, X/k a smooth variety, A a ring in which the exponential characteristic of k is invertible, and $M \in \mathbf{DM}(k, A)$.*

If for all $n \in \mathbb{Z}$ and all $x \in X$ (not necessarily closed) we have that $\mathrm{Hom}_{\mathbf{DM}(x,A)}(\mathbb{1}\{n\}, M_x) = 0$, then also for all $n \in \mathbb{Z}$ we have $\mathrm{Hom}_{\mathbf{DM}(k,A)}(MX\{n\}, M) = 0$.

Proof. We will prove the result by induction on $\dim X$. Thus in order to prove it for X we may assume it proved for every smooth, locally closed $X' \subset X$ with $\dim X' < \dim X$ (because the residue fields of X' form a subset of those of X). If $\dim X = 0$ then X is a disjoint union of spectra of fields, and the result is clear.

To prove the general case, we may assume that X is connected. Let $n \in \mathbb{Z}$ and $\alpha \in \mathrm{Hom}(MX\{n\}, M)$. It suffices to show that $\alpha = 0$. By considering the generic point and using continuity [20, Example 2.6(2)] we conclude that there exists a non-empty open subvariety $U \subset X$ such that $\alpha|_U = 0$. Let $Z = X \setminus U$.

If Z is empty there is nothing to do. Otherwise there exists a non-empty, smooth, connected open subvariety $U_1 \subset Z$, since k is perfect.

Let $Z' = Z \setminus U_1$, $U' = U \cup U_1 = X \setminus Z'$. Then U' is smooth open in X and we have $X \setminus U' = Z'$, which is strictly smaller than Z. We shall prove that $\alpha|_{U'} = 0$. By repeating this argument with U replaced by U' (i.e. Noetherian induction on Z) it will follow that $\alpha = 0$.

Note that $U_1 = U' \setminus U$ is closed in U', say of codimension c. Thus we get the exact Gysin triangle

$$MU\{n\} \to MU'\{n\} \to MU_1\{n-c\}.$$

Now $\mathrm{Hom}(MU_1\{n-c\}, M) = 0$ by the induction on dimension. Thus $\mathrm{Hom}(MU'\{n\}, M) \to \mathrm{Hom}(MU\{n\}, M)$ is injective. But $(\alpha|_{U'})|_U = \alpha|_U = 0$ by assumption, so $\alpha|_{U'} = 0$.

This concludes the proof. □

5.1.3 Change of Coefficients

One of the advantages of working with an algebraically constructed category like **DM** is that we can get very good control over the change of coefficients functors, and use them to simplify problems.

First recall the construction. Let $\alpha : A \to B$ be a homomorphism of (commutative) rings. There is an adjunction $\alpha_\# : A\textbf{-Mod} \leftrightarrows B\textbf{-Mod} : \alpha^*$, where $\alpha_\#(M) = M \otimes_A B$ and α^* is the forgetful functor. This extends to an adjunction $\alpha_\# : Shv^{tr}(k, A) \leftrightarrows Shv^{tr}(k, B) : \alpha^*$. Here $\alpha_\#(F)$ is the sheaf associated with $X \mapsto F(X) \otimes_A B$. The forgetful functor α^* is exact and so immediately descends to $\alpha^* : D(Shv^{tr}(k, B)) \to D(Shv^{tr}(k, A))$. We also have $\alpha^*(\mathbf{HI}^{tr}(k, B)) \subset \mathbf{HI}^{tr}(k, A)$ and so α^* defines $R\alpha^* : \mathbf{DM}^{\mathrm{eff}}(k, B) \to \mathbf{DM}^{\mathrm{eff}}(k, A)$.

The situation with $\alpha_\#$ is more complicated. There is $L\alpha_\# : D(Shv^{tr})(k, A) \to D(Shv^{tr})(k, B)$. This is essentially just derived tensor product. In particular $L\alpha_\#$ is a symmetric monoidal functor. One also sees easily that $L\alpha_\# R_{X,A} = R_{X,B}$, where $R_{X,A}$ is the representable sheaf with transfers. It follows that $L\alpha_\#$ passes through $\mathbb{A}^1$-localisation and defines $L\alpha_\# : \mathbf{DM}^{\mathrm{eff}}(k, A) \to \mathbf{DM}^{\mathrm{eff}}(k, B)$, which is still a symmetric monoidal functor. Consequently $L\alpha_\#$ extends to the stabilisation $L\alpha_\# : \mathbf{DM}(k, A) \to \mathbf{DM}(k, B)$.

Resolving B projectively as an A-module, one sees that for $E \in \mathbf{DM}^{\mathrm{eff},-}(k, B)$ we have $R\alpha^*(E \otimes \mathbb{G}_m) \in \mathbf{DM}^{\mathrm{eff},-}(k, A) \otimes \mathbb{G}_m$. Then an easy calculation using adjunction of $L\alpha_\#, R\alpha^*$ and cancellation shows that $R\alpha^*(E \otimes \mathbb{G}_m) \simeq R\alpha^*(E) \otimes \mathbb{G}_m$.

One may check that $R\alpha^*$ commutes with filtered homotopy colimits. From this it follows that $R\alpha^*(E \otimes \mathbb{G}_m) \simeq R\alpha^*(E) \otimes \mathbb{G}_m$ for all $E \in \mathbf{DM}^{\mathrm{eff}}(k, B)$, not just the connective objects. Thus $R\alpha^*$ also extends to $R\alpha^* : \mathbf{DM}(k, B) \to \mathbf{DM}(k, A)$.

We point out that as usual, all parallel versions of $L\alpha_\#, R\alpha^*$ are adjoint. Also any f^* commutes with $L\alpha_\#, R\alpha^*$. There are two basic properties of the base change functors we shall need.

Proposition 5.4. *Let k be perfect, $\alpha : A \to B$ be flat, $E \in \mathbf{DM}^{gm}(k, A)$ and $F \in \mathbf{DM}(k, A)$. Then*

$$\mathrm{Hom}(E, F) \otimes_A B \cong \mathrm{Hom}(L\alpha_\# E, L\alpha_\# F).$$

Proof. There is a natural homomorphism $\gamma_{E,F} : \mathrm{Hom}(E, F) \otimes_A B \to \mathrm{Hom}(L\alpha_\# E, L\alpha_\# F)$. For fixed E let $\mathcal{C}_X$ be the class of objects in $\mathbf{DM}(k, A)$ such that $\gamma_{E,F}$ is an isomorphism. We want to show that $F \in \mathcal{C}_E$. The class $\mathcal{C}_E$ is stable under cones, isomorphisms, and arbitrary sums (E being compact). Hence it suffices to show that $\mathbf{DM}^{gm}(k, A) \subset \mathcal{C}_E$. Thus we may assume that F is compact.

By the cancellation theorem, we may assume that E, F are effective. Using the 5-lemma and the fact that $\mathbf{DM}^{\mathrm{eff},gm}(k, A)$ is generated by MX for $X \in Sm(k)$, we may reduce to $E = MX[i]$. In this case $\mathrm{Hom}(MX[i], F)$ is given by the hypercohomology $H^{-i}(X, F^\bullet)$. Since $\otimes_A B$ is exact it commutes with hypercohomology and preserves sheaves, so we have $H^{-i}(X, F^\bullet) \otimes_A B = H^{-i}(X, F^\bullet \otimes_A B) = H^{-i}(X, (L\alpha_\# F)^\bullet)$. □

Proposition 5.5. *Let k be perfect, A a ring, $a \in A$ not a zero-divisor and $\alpha : A \to A/(a)$ the natural map. Then for $E \in \mathbf{DM}(k, A)$ there is a natural distinguished triangle*

$$E \xrightarrow{\cdot a} E \to R\alpha^* L\alpha_\# E.$$

This triangle yields the typical *Bockstein sequences* one expects for reduction of coefficients.

Proof. Let $C(E)$ denote a cone on $E \xrightarrow{a} E$. There is a *canonical* map $\gamma_E : C(E) \to R\alpha^* L\alpha_\# E$ coming from adjunction and the fact that there is a *canonical* isomorphism $L\alpha_\# C(E) = L\alpha_\# E \oplus L\alpha_\# E[1]$. We wish to show that γ_E is an isomorphism. Let $\mathcal{C}$ denote the class of objects in $E \in \mathbf{DM}(k, A)$ such that γ_E is an isomorphism. Then $\mathcal{C}$ is closed under isomorphism, arbitrary sum (because triangles are stable by sums) and cones (because of the canonicity of γ_E). Hence we may assume that $E = MX\{i\}$ for some $X \in Sm(k)$ and $i \in \mathbb{Z}$ (since these objects generate $\mathbf{DM}(k, A)$ as a localising subcategory).

Since $R\alpha^*$ and $L\alpha_\#$ commute with $\otimes M\mathbb{G}_m$, we may assume that $i = 0$. Then $R\alpha^* L\alpha_\# MX = C_\bullet R_X/(c)$. (Note that since $C_\bullet R_X$ has homotopy invariant cohomology, so does $\alpha_\# C_\bullet R_X = C_\bullet R_X/(a)$, by considering the (ordinary) Bockstein sequence. Hence we may apply α^* immediately to $\alpha_\# C_\bullet R_X$ instead of having to $\mathbb{A}^1$-localise first.) Since a is not a zero divisor the sequence $0 \to C_\bullet R_X \to C_\bullet R_X \to C_\bullet R_X/(a) \to 0$ is exact and yields the desired triangle. □

Theorem 5.6. *Let k be a perfect field, A a PID and $f : Spec(k^s) \to Spec(k)$ a separable closure. Write $Max(A)$ for the set of maximal ideals of A. For $P \in Spec(A)$ write $\alpha^P : A \to k(P)$ for the residue homomorphism.*

(i) The collection $\{L\alpha^P_\#\}_{P \in Spec(A)}$ is conservative and Pic-injective.

(ii) If A is of characteristic zero, the collection $\{L\alpha^P_\#\}_{P \in Max(A)} \cup \{f^*\}$ is conservative.

(iii) If A has residue fields of arbitrarily large characteristic, then the collection $\{L\alpha^P_\#\}_{P \in Max(A)} \cup \{f^*\}$ is Pic-injective.

Proof. Write $K = Frac(A)$ for the field of fractions. We first prove the conservativity results. Fix $T \in \mathbf{DM}^{gm}(k, A)$. Let $E \in \mathbf{DM}(k, A)$ be such that $L\alpha^P_\# E \simeq 0$ for some $P \in Max(A)$. We have $P = (\pi)$ for some prime π, and hence a Bockstein triangle

$$E \xrightarrow{\pi} E \to R\alpha^{P,*} L\alpha^P_\# E \simeq 0.$$

Consequently multiplication by π on $[T, E]$ is an isomorphism. Thus if $L\alpha^P_\#(E) = 0$ for all $P \in Max(A)$, then $[T, E]$ is a K-vector space. Since $K \otimes_A K \neq 0$ we conclude that $[T, E] = 0$ as soon as $[T, E] \otimes_A K = 0$. Write $\alpha^0 : A \to K$ for the (flat) localisation. By Proposition 5.4, $[L\alpha^0_\# T, L\alpha^0_\# E] = [T, E] \otimes_A K$. Thus we have shown: if $L\alpha^P_\# E \simeq 0$ for all $P \in Spec(A)$, then $[T, E] = 0$. Since $T \in \mathbf{DM}^{gm}(k, A)$ was arbitrary and $\mathbf{DM}^{gm}(k, A)$ generates $\mathbf{DM}(k, A)$, we conclude that $E \simeq 0$. This proves the conservativity part of (i).

If A is of characteristic zero then K is of characteristic zero and hence $f^* : \mathbf{DM}(k, K) \to \mathbf{DM}(k^s, K)$ is conservative by the first part of Theorem 5.1. Since $L\alpha^0_\# f^* \simeq f^* L\alpha^0_\#$, (ii) follows from (i).

Next we prove the Pic-injectivity part of (i). Thus let $E \in Pic(\mathbf{DM}(k, A))$ be such that $\mathbb{1} = L\alpha^P_\# E \in Pic(\mathbf{DM}(k, k(P)))$ for all $P \in Spec(A)$. We need to find $a \in [\mathbb{1}, E]$ which is an isomorphism. By the conservativity result, this happens if and only if $L\alpha^P_\#(a)$ is an isomorphism for all $P \in Spec(A)$.

We know that $[\mathbb{1}[n], \mathbb{1}]_{\mathbf{DM}(k,B)} = 0$ for any $n \neq 0$ and any B, and also that $[\mathbb{1}, \mathbb{1}]_{\mathbf{DM}(k,B)} = B$. Consider for a prime π the Bockstein sequence

$$0 = [\mathbb{1}, L\alpha^\pi_\# E[-1]] \to [\mathbb{1}, E] \xrightarrow{\pi} [\mathbb{1}, E] \to [\mathbb{1}, L\alpha^\pi_\# E] = A/\pi.$$

We conclude that multiplication by π on $[\mathbb{1}, E]$ is injective, and that if we consider some $a \in [\mathbb{1}, E]$ then either $L\alpha^\pi_\#(a)$ is an isomorphism or else $a = \pi a'$ for some (unique) $a' \in [\mathbb{1}, E]$.

I claim there exists $a \in [\mathbb{1}, E]$ which is not divisible (in this way) by any prime π. If this is so then a must be an isomorphism and we are done (with (i)). Indeed $0 \neq a$, so $0 \neq L\alpha^0_\#(a)$ because $[\mathbb{1}, E] \to [\mathbb{1}, L\alpha^0_\# E] \cong [\mathbb{1}, E] \otimes_A K$ is injective, as multiplication by any π on $[\mathbb{1}, E]$ is injective. But then $[\mathbb{1}, L\alpha^0_\# E] \cong [\mathbb{1}, \mathbb{1}]_{\mathbf{DM}(k,K)} = K$ (since $L\alpha^0_\# E \simeq \mathbb{1}$ by assumption) and so $L\alpha^0_\#(a)$ is an isomorphism as well.

Write DE for the dual (monoidal inverse) of E. There is a pairing

$$p : [\mathbb{1}, E] \times [\mathbb{1}, DE] \to [\mathbb{1}, E \otimes DE] \cong [\mathbb{1}, \mathbb{1}] = A.$$

This pairing is A-bilinear. Note that since $[\mathbb{1}, E] \otimes_A K \cong [\mathbb{1}, L\alpha^0_{\#} E] \cong [\mathbb{1}, \mathbb{1}]_{\mathbf{DM}(k,K)} \neq 0$ there exists $0 \neq a \in [\mathbb{1}, E]$. Similarly there exists $0 \neq a' \in [\mathbb{1}, DE]$. But then $0 \neq p(a, a') \in A$, as follows from the following commutative diagram

$$\begin{array}{ccc}
[\mathbb{1}, E] \times [\mathbb{1}, DE] & \xrightarrow{p} & A \\
{\scriptstyle L\alpha^0_{\#}}\downarrow & & \downarrow \\
[\mathbb{1}, L\alpha^0_{\#} E] \times [\mathbb{1}, L\alpha^0_{\#} DE] & \xrightarrow{p} & K \\
{\scriptstyle \cong}\downarrow & & \| \\
K \times K & \xrightarrow{m} & K,
\end{array}$$

where m is the ordinary multiplication in K.

Since $0 \neq p(a, a')$ is divisible by only finitely many primes finitely many times, and the pairing p is A-bilinear, it follows that a also can only be divisible by finitely many primes finitely many times. Doing as many divisions as possible, we arrive at an a which cannot further be divided. This proves the claim.

It remains to establish (iii). Let $E \in \mathbf{DM}(k, A)$ be such that $f^*E \cong \mathbb{1}$ and $L\alpha^{\pi}_{\#} E \cong \mathbb{1}$. As a first step, I claim that there exists a finite extension $k \subset l \subset k^s$ such that $g^*E \cong \mathbb{1}$, where $g : Spec(l) \to Spec(k)$. Indeed it follows from continuity that $[\mathbb{1}, f^*E] = \operatorname{colim}_{k \subset l \subset k^s} [\mathbb{1}, (l/k)^*E]$, where the colimit is over finite subextensions. Hence there exist l and an element $t \in [\mathbb{1}, g^*E]$ such that $(k^s/l)^*(t)$ is an isomorphism. The commutative diagram

$$\begin{array}{ccc}
[\mathbb{1}, g^*E] & \longrightarrow & [\mathbb{1}, f^*E] \cong A \\
\downarrow & & \downarrow \\
[\mathbb{1}, L\alpha_{\pi\#} g^*E] & \xrightarrow{\cong} & [\mathbb{1}, L\alpha^{\pi}_{\#} f^*E] \cong A/(\pi)
\end{array}$$

shows that $L\alpha^{\pi}_{\#}(t)$ is an isomorphism. Thus by (ii), t is an isomorphism.

Now we consider $[\mathbb{1}, E]$. From the Bockstein triangles and the assumption $L\alpha^{\pi}_{\#} E \cong \mathbb{1}$ we get the exact sequences

$$\begin{aligned}
[\mathbb{1}, L\alpha^{\pi}_{\#} E[-1]] = 0 \to [\mathbb{1}, E] \xrightarrow{\pi} [\mathbb{1}, E] \\
\to [\mathbb{1}, L\alpha^{\pi} \# E] \cong A/(\pi) \to [\mathbb{1}, E[1]]
\end{aligned}$$

It follows that $[\mathbb{1}, E]$ is a torsion-free A-module (hence abelian group). Thus by transfer it follows that $[\mathbb{1}, E] \to [\mathbb{1}, g^*E] \cong A$ is injective. Let us denote the image by $I \subset A$. This is a free A-module (of rank zero or one).

Since $[\mathbb{1}, g^*(E)[1]] = 0$ it follows by transfer that $[\mathbb{1}, E[1]]$ is $[l : k]$-torsion. Choosing π of sufficiently large characteristic, we find that $A/(\pi) \to [\mathbb{1}, E[1]]$ is the zero map. Thus $I = [\mathbb{1}, E] \neq 0$, i.e. $I \cong A$. It follows that $[\mathbb{1}, E] \to [\mathbb{1}, L\alpha^{\pi}_{\#} E] \cong A/(\pi)$ is surjective for each π.

Consider the commutative diagram

$$\begin{array}{ccc}
[\mathbb{1}, E] & \longrightarrow & [\mathbb{1}, g^*E] \cong A \\
{\scriptstyle (*)}\downarrow & & {\scriptstyle (**)}\downarrow \\
[\mathbb{1}, L\alpha_{\pi\#} E] & \xrightarrow{\cong} & [\mathbb{1}, L\alpha^{\pi}_{\#} g^*E] \cong A/(\pi)
\end{array}$$

The map (**) is the natural surjection and (*) is surjective as we just proved. It follows that $I + (\pi) = A$ for each π and so $I = A$. Thus there exists $t' \in [\mathbb{1}, E]$ with $g^*(t') = t$ an isomorphism. Considering the diagram again one finds that $L\alpha^{\pi}_{\#}(t')$ is also an isomorphism. Thus t' is an isomorphism (by (ii), again) and we are done. □

Remark. With only slight adaptations, the theorem holds for any *Dedekind domain* A, not just principal ideal domains. Indeed the conservativity results hold as stated. The Pic-injectivity need no longer be true. There is a natural injection $Pic(A) \to Pic(\mathbf{DM}(k, A))$ (coming from the fact that the Karoubi-closed additive subcategory of $\mathbf{DM}(k, A)$ generated by $\mathbb{1}$ is equivalent to the category of finitely generated, projective A-modules). One may prove that elements in this subgroup are the only obstruction to Pic-injectivity.

5.2 Weight Structures and Fixed Point Functors

In this section we prove the abstract fixed point functors theorem. Before doing so, we need to recall Bondarko's weight structures and prove some simple lemmas about them, and review some basic properties of Chow motives.

5.2.1 Generalities about Weight Structures

We shall work extensively in this section with weight structures [13], which we now review rapidly. We follow the cohomological notation of Bondarko's earlier papers, in contrast to the homological notation we use for t-structures (and also in contrast to the notation in some of Bondarko's newer work).

Definition 5.7. *Let $\mathcal{C}$ be a triangulated category and $\mathcal{C}^{w\geq 0}, \mathcal{C}^{w\leq 0} \subset \mathcal{C}$ two classes of objects. We call this a weight structure if the following hold:*

(i) $\mathcal{C}^{w\geq 0}, \mathcal{C}^{w\leq 0}$ are additive and Karoubi-closed in $\mathcal{C}$.

(ii) $\mathcal{C}^{w\geq 0} \subset \mathcal{C}^{w\geq 0}[1]$, $\mathcal{C}^{w\leq 0}[1] \subset \mathcal{C}^{w\leq 0}$

(iii) For $X \in \mathcal{C}^{w\geq 0}, Y \in \mathcal{C}^{w\leq 0}$ we have $\mathrm{Hom}(X, Y[1]) = 0$.

(iv) For each $X \in \mathcal{C}$ there is a distinguished triangle

$$B[-1] \to X \to A$$

with $B \in \mathcal{C}^{w\geq 0}$ and $A \in \mathcal{C}^{w\leq 0}$.

These axioms look quite similar to those of a t-structure, but in practice weight structures behave rather differently. We call a decomposition as in (iv) a *weight decomposition*. It is usually far from unique. We put $\mathcal{C}^{w\geq n} = \mathcal{C}^{w\geq 0}[-n]$ and $\mathcal{C}^{w\leq n} = \mathcal{C}^{w\leq 0}[-n]$. We also write $\mathcal{C}^{w>n} = \mathcal{C}^{w\geq n+1}$ etc. The intersection $\mathcal{C}^{w=0} := \mathcal{C}^{w\geq 0} \cap \mathcal{C}^{w\leq 0}$ is called the *heart* of the weight structure.

A weight structure is called *non-degenerate* if $\cap_n \mathcal{C}^{w\geq n} = 0 = \cap_n \mathcal{C}^{w\leq n}$. It is called *bounded* if $\cup_n \mathcal{C}^{w\geq n} = \mathcal{C} = \cup_n \mathcal{C}^{w\leq n}$

A functor $F : \mathcal{C} \to \mathcal{D}$ between categories with weight structures is called *right-w-exact* if $F(\mathcal{C}^{w\leq 0}) \subset \mathcal{D}^{w\leq 0}$ and called *left-w-exact* $F(\mathcal{C}^{w\geq 0}) \subset \mathcal{D}^{w\geq 0}$. It is called *w-exact* if it is both left and right w-exact. It is called *right-w-conservative* if given $X \in \mathcal{C}$ with $F(X) \in \mathcal{D}^{w\leq 0}$ we have $X \in \mathcal{C}^{w\leq 0}$, and similarly it is called *left-w-conservative* if the same holds for $w \geq 0$. The functor is called *w-conservative* if it is both left- and right-w-conservative. Note that a (left or right) w-conservative functor on a non-degenerate weight structure is conservative.

In the following proposition we summarise properties of weight structures we use.

Proposition 5.8 (Bondarko). *(1) $\mathcal{C}^{w\leq 0}$ and $\mathcal{C}^{w\geq 0}$ are extension-stable: if $A \to B \to C$ is a distinguished triangle and $A, C \in \mathcal{C}^{w\leq 0}$ (respectively $A, C \in \mathcal{C}^{w\geq 0}$) then $B \in \mathcal{C}^{w\leq 0}$ (respectively $B \in \mathcal{C}^{w\geq 0}$).*

Moreover $X \in \mathcal{C}^{w\geq 0}$ if and only if $\mathrm{Hom}(X, Y) = 0$ for all $Y \in \mathcal{C}^{w<0}$, and similarly $X \in \mathcal{C}^{w\leq 0}$ if and only if $\mathrm{Hom}(Y, X) = 0$ for all $Y \in \mathcal{C}^{w>0}$.

(2) Bounded weight structures are non-degenerate.

(3) *If $\mathcal{C}$ admits a DG-enhancement and the weight structure is bounded, then there exists a w-exact, w-conservative triangulated functor*

$$t : \mathcal{C} \to K^b(\mathcal{C}^{w=0})$$

called the weight complex*. Its restriction to $\mathcal{C}^{w=0}$ is the natural inclusion.*

(4) *If the weight structure is bounded and $\mathcal{C}^{w=0}$ is Karoubi-closed then so is $\mathcal{C}$.*

(5) *If $H \subset \mathcal{C}$ is a* negative *subcategory of a triangulated category (i.e. for $X, Y \in H$ we have $\mathrm{Hom}(X, Y[n]) = 0$ for $n > 0$) generating it as a thick subcategory, then there exists a unique weight structure on $\mathcal{C}$ with $H \subset \mathcal{C}^{w=0}$. Moreover $\mathcal{C}^{w\leq 0}$ is the smallest extension-stable Karoubi-closed subcategory of $\mathcal{C}$ containing $\cup_{n\geq 0} H[n]$, and similarly for $\mathcal{C}^{w\geq 0}$. The weight structure is bounded and $\mathcal{C}^{w=0}$ is the Karoubi-closure of H in $\mathcal{C}$.*

(6) *If $\mathcal{D} \subset \mathcal{C}$ is a triangulated subcategory such that $\mathcal{D}^{w\leq 0} := \mathcal{D} \cap \mathcal{C}^{w\leq 0}$ and $\mathcal{D}^{w\geq 0} := \mathcal{D} \cap \mathcal{C}^{w\geq 0}$ define a weight structure on $\mathcal{D}$ (we say the weight structure* restricts *to $\mathcal{D}$) then the Verdier quotient $\mathcal{C}/\mathcal{D}$ affords a weight structure with $(\mathcal{C}/\mathcal{D})^{w\leq 0}$ the Karoubi-closure of the image of $\mathcal{C}^{w\leq 0}$ in $\mathcal{C}/\mathcal{D}$, and similarly for $(\mathcal{C}/\mathcal{D})^{w\geq 0}$, $(\mathcal{C}/\mathcal{D})^{w=0}$.*

The natural "quotient" functor $Q : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ is w-exact. If $X, Y \in \mathcal{C}^{w=0}$ then

$$\mathrm{Hom}(QX, QY) = \mathrm{Hom}(X, Y)/\Sigma_{Z\in\mathcal{D}^{w=0}} \mathrm{Hom}(Z, Y) \circ \mathrm{Hom}(X, Z).$$

The weight structure on $\mathcal{C}/\mathcal{D}$ is bounded if the one on $\mathcal{C}$ is.

(7) *Let $(\mathcal{C}, w)$ be a bounded w-category with heart H, and $E \subset H$ a Karoubi-closed subcategory. Assume that $\mathcal{C}$ affords a DG-enhancement. Write $\langle E\rangle$ for the thick triangulated subcategory of $\mathcal{C}$ generated by E. Then for $X \in \mathcal{C}$ we have $X \in \langle E\rangle$ if and only if $t(X) \in K^b(E) \subset K^b(H)$.*

Proof. (1) [13, Proposition 1.3.3 (1-3)]. (2) [13, Proposition 1.3.6 (3) and comment after proof]. (3) [13, Proposition 3.3.1 (I), (IV) and Section 6.3]. (4) [13, Lemma 5.2.1]. (5) [13, Theorem 4.3.2 (II) and its proof]. (6) [13, Proposition 8.1.1]. Weight exactness holds by definition of the weight structure on $\mathcal{C}/\mathcal{D}$. (7) [13, Corollary 8.1.2]. □

We shall call a triangulated category with a fixed weight structure a w-category.

Weight structures mostly come from "stupid truncation" of (generalised) complexes, and this intuition allows us to formulate many true results about weight structures. Here are some examples of that intuition.

Lemma 5.9. *Let $\mathcal{C}$ be a w-category with heart H, and $H' \subset H$ an additive subcategory. Let $\mathcal{C}'$ be the triangulated category generated by H' inside $\mathcal{C}$.*

Then the weight structure of $\mathcal{C}$ restricts to $\mathcal{C}'$. In particular, if $X \in \mathcal{C}'$ then we may choose a weight decomposition $A \to X \to X'$ (i.e. $A \in \mathcal{C}^{w\geq 0}$ and $X' \in \mathcal{C}^{w<0}$) with $A, X' \in \mathcal{C}'$.

Proof. This is just Proposition 5.8 (5) which says that $\mathcal{C}'$, being negatively generated by H', carries a natural unique weight structure. By the description provided we find $\mathcal{C}'^{w\leq 0} \subset \mathcal{C}^{w\leq 0}, \mathcal{C}'^{w\geq 0} \subset \mathcal{C}^{w\geq 0}$. Hence a weight decomposition in $\mathcal{C}'$ is also a weight decomposition in $\mathcal{C}$. The rest follows from the definitions. (It follows from the orthogonality characterisation that $\mathcal{C}'^{w\leq 0} = \mathcal{C}^{w\leq 0} \cap \mathcal{C}'$, but we do not need this.) □

Lemma 5.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a triangulated functor of w-categories, and assume that the weight structure on $\mathcal{C}$ is bounded. Then F is w-exact if and only if $F(\mathcal{C}^{w=0}) \subset \mathcal{D}^{w=0}$.*

Proof. Necessity is clear, we show sufficiency. We find by induction that the subcategory of $\mathcal{C}$ generated by $\mathcal{C}^{w=0}$ contains $\mathcal{C}^{w\leq n} \cap \mathcal{C}^{w\geq -n}$ for all n, and hence all of $\mathcal{C}$ by boundedness. It follows that the weight structure on $\mathcal{C}$ is the one described in Proposition 5.8 (5), i.e. $\mathcal{C}^{w\geq 0}, \mathcal{C}^{w\leq 0}$ are obtained as extension closures of $\bigcup_{n\geq 0} \mathcal{C}^{w=n}, \bigcup_{n\leq 0} \mathcal{C}^{w=n}$. The result follows since $\mathcal{D}^{w\geq 0}, \mathcal{D}^{w\leq 0}$ are extension-stable. □

Lemma 5.11. *Let $\mathcal{C}$ be a w-category which is also a symmetric monoidal category. Assume that $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}^{w=0}$ and that tensoring is weight-bi-exact, i.e. that $\mathcal{C}^{w\leq 0} \otimes \mathcal{C}^{w\leq 0} \subset \mathcal{C}^{w\leq 0}$ and similarly for $\mathcal{C}^{w\geq 0}$.*

Then the weight complex functor is symmetric monoidal whenever $\mathcal{C}$ affords a symmetric monoidal DG-enhancement and Pic-injective whenever additionally the weight structure is bounded.

Moreover the dualisation $D : \mathcal{C}^{op} \to \mathcal{C}$ is w-exact to the extent that it is defined (i.e. if $X \in \mathcal{C}^{w\geq 0}$ is dualisable, then $DX \in \mathcal{C}^{w\leq 0}$, and if $X \in \mathcal{C}^{w\leq 0}$ is dualisable, then $DX \in \mathcal{C}^{w\geq 0}$).

Proof. If $\mathcal{D}$ is a negative DG symmetric monoidal category, then $H^0(\mathcal{D})$ is symmetric monoidal in a natural way and the weight complex functor t manifestly respects the symmetric monoidal structure. If $\mathcal{C}$ is a symmetric monoidal DG category with the property that $H^n(\mathrm{Hom}(X, Y)) = 0$ for all $X, Y \in \mathcal{D}$ and $n > 0$ then the good truncation $\tau_{\leq 0}\mathcal{D}$ is symmetric monoidal in a natural way, and the quasi-equivalence $\tau_{\leq 0}\mathcal{D} \to \mathcal{D}$ is a symmetric monoidal equivalence.

Hence the weight complex functor is symmetric monoidal as soon as there is any symmetric monoidal DG enhancement of $\mathcal{C}^{w=0}$. Moreover by Proposition 5.8 (3) if the weight structure is bounded t is w-conservative. Since it induces an isomorphism on hearts it is a fortiori Pic-injective. This proves the first part.

For the second part, let $X \in \mathcal{C}$. X being dualisable means that there exists an object DX such that $\otimes DX$ is both right and left adjoint to $\otimes X$.

If $X \in \mathcal{C}^{w\geq 0}$ and $Y \in \mathcal{C}^{w>0}$ then $\mathrm{Hom}(Y, DX) = \mathrm{Hom}(Y \otimes X, \mathbb{1}) = 0$ because $Y \otimes X \in \mathcal{C}^{w>0}$ whereas $\mathbb{1} \in \mathcal{C}^{w=0}$. It follows that $DX \in \mathcal{C}^{w\leq 0}$ by Proposition 5.8 (1). The case of $X \in \mathcal{C}^{w\leq 0}$ is similar. □

Lemma 5.12. *Let $\mathcal{C}, \mathcal{D}$ be w-categories with bi-w-exact symmetric monoidal structures. Suppose that $\mathcal{C}$ is rigid and let $\Phi : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal triangulated functor.*

Then Φ is right-w-exact if and only if it is left-w-exact if and only if it is w-exact. Moreover Φ is right-w-conservative if and only if it is left-w-conservative if and only if it is w-conservative.

Proof. Since our axioms are self-dual, we need only show that right-something implies left-something.

Suppose that Φ is right-w-exact and let $X \in \mathcal{C}^{w\geq 0}$. We need to show that $\Phi(X) \in \mathcal{D}^{w\geq 0}$. But $\mathcal{C}$ is rigid so X is dualisable and $DX \in \mathcal{C}^{w\leq 0}$ by the second part of Lemma 5.11. Then $\Phi(DX) \in \mathcal{D}^{w\leq 0}$ by assumption, and $\Phi(X)$ is dualisable with dual $\Phi(DX)$. Consequently $\Phi(X) = D(\Phi(DX)) \in \mathcal{D}^{w\geq 0}$ by the same Lemma.

Suppose now the weight structure is bounded and Φ is right-w-conservative. Let $X \in \mathcal{C}$ be such that $\Phi(X) \in \mathcal{D}^{w\geq 0}$. We need to show that $X \in \mathcal{C}^{w\geq 0}$. Now since $\mathcal{C}$ is rigid X is dualisable, and as before $\Phi(X)$ is dualisable with dual $\Phi(DX)$. Thus $\Phi(DX) = D\Phi(X) \in \mathcal{D}^{w\leq 0}$, by the Lemma again. Hence $DX \in \mathcal{C}^{w\leq 0}$ by assumption, and finally $X \in \mathcal{C}^{w\geq 0}$ by a final application of the Lemma. □

Lemma 5.13. *Let $\mathcal{C}$ be a w-category, $X \in \mathcal{C}^{w\leq 0}$. Suppose given weight decompositions $A \to X \to X'$ and $B[1] \to X' \to X''$ (i.e. $A, B \in \mathcal{C}^{w\geq 0}$, $X' \in \mathcal{C}^{w<0}$ and $X'' \in \mathcal{C}^{w<-1}$).*

Then $A, B \in \mathcal{C}^{w=0}$ and for $T \in \mathcal{C}^{w=0}$ there is an exact sequence

$$\mathrm{Hom}(T, B) \to \mathrm{Hom}(T, A) \to \mathrm{Hom}(T, X) \to 0,$$

where the morphisms are composition with the canonical maps $B \to X'[-1] \to A$ and $A \to X$ coming from the chosen weight decompositions.

Proof. We have $A, B \in \mathcal{C}^{w=0}$ by (the dual of) [13, Proposition 1.3.3 (6)]. There is an exact sequence

$$\mathrm{Hom}(T, X'[-1]) \to \mathrm{Hom}(T, A) \to \mathrm{Hom}(T, X) \to \mathrm{Hom}(T, X') = 0$$

where the last term is zero because $T \in \mathcal{C}^{w\geq 0}, X' \in \mathcal{C}^{w<0}$. In particular $\mathrm{Hom}(T, A) \to \mathrm{Hom}(T, X)$ is surjective. Applying the same reasoning to $\mathrm{Hom}(T, X'[-1])$ we find that the homomorphism $\mathrm{Hom}(T, B) \to \mathrm{Hom}(T, X'[-1])$ is surjective and hence

$$\mathrm{Hom}(T, B) \to \mathrm{Hom}(T, A) \to \mathrm{Hom}(T, X) \to 0$$

is exact. This concludes the proof. □

We say that a morphism $f : X \to Y$ in a category $\mathcal{C}$ *admits (or has) a section* if there exists $s : Y \to X$ such that $fs = \mathrm{id}_X$. We say that f *admits a retraction* if there exists $r : Y \to X$ such that $rf = \mathrm{id}_X$.

Corollary 5.14. *In the situation of the Lemma, we have $X \in \mathcal{C}^{w<0}$ if and only if the composite $B \to X'[-1] \to A$ has a section.*

Proof. If $X \in \mathcal{C}^{w<0}$ then $\mathrm{Hom}(T, X) = 0$ and so $\mathrm{Hom}(T, B) \to \mathrm{Hom}(T, A)$ must be surjective. Putting $T = A$ this precisely says that $B \to A$ has a section. Conversely, if $B \to A$ has a section then $\mathrm{Hom}(T, B) \to \mathrm{Hom}(T, A)$ is always surjective and so $\mathrm{Hom}(T, X) = 0$. Consequently the map $A \to X$ in the weight decomposition $A \to X \to X'$ must be zero and so $X' \simeq X \oplus A[1]$. Hence $X \in \mathcal{C}^{w<0}$ since $\mathcal{C}^{w<0}$ is Karoubi-closed by definition. □

We say that a functor $F : \mathcal{C} \to \mathcal{D}$ *detects sections* if a morphism $f : X \to Y \in \mathcal{C}$ admits a section whenever $F(f)$ does, and we say that F *detects retractions* if f admits a retraction whenever $F(f)$ does. The following corollary is one method of proving that a functor is w-conservative.

Corollary 5.15. *Let $F : \mathcal{C} \to \mathcal{D}$ be a w-exact triangulated functor of w-categories. Assume that the weight structure on $\mathcal{C}$ is bounded.*

Then F is right-w-conservative if and only if the induced functor $F^{w=0} : \mathcal{C}^{w=0} \to \mathcal{D}^{w=0}$ detects sections, and F is left-w-conservative if and only if $F^{w=0}$ detects retractions.

Proof. Since the axioms are self-dual, and right/left, section/retraction are interchanged upon passing to opposite categories, it suffices to prove the first statement.

If F is right-w-conservative, let $f : B \to A \in \mathcal{C}^{w=0}$ be any morphism. Let X be a cone on f, so we get a distinguished triangle $B \to A \to X$. We can view $A \to X \to B[1]$ as a weight decomposition of X with $X' = B[1]$, and $B[1] \to X' \to 0$ as a further weight decomposition with $X'' = 0$. Since $\mathcal{C}^{w\le 0}$ is extension stable, $X \in \mathcal{C}^{w\le 0}$. By the previous corollary, we have $X \in \mathcal{C}^{w<0}$ if and only if f admits a section. Similarly $F(X) \in \mathcal{D}^{w<0}$ if and only if $F(f)$ admits a section. Hence if $F(f)$ admits a section then $F(X) \in \mathcal{D}^{w<0}$, so $X \in \mathcal{C}^{w<0}$ by assumption, and finally f admits a section.

Conversely, if $F^{w=0}$ detects sections, the weight structure is bounded, and $X \in \mathcal{C}$ is such that $F(X) \in \mathcal{D}^{w\le 0}$, we need to prove that $X \in \mathcal{C}^{w\le 0}$. We know that $X \in \mathcal{C}^{w\le n}$ for some n sufficiently large by boundedness. Consequently it suffices to prove: if $X \in \mathcal{C}^{w\le 0}$ and $FX \in \mathcal{D}^{w<0}$ then $X \in \mathcal{D}^{w<0}$. This follows from the previous corollary and the assumption that $F^{w=0}$ detects sections. □

5.2.2 More About Chow Motives

We need to recall the category of Chow motives in somewhat more detail. By $SmProj(k)$ we denote the category of smooth projective varieties over the field k. It is a symmetric monoidal category using cartesian product as monoidal product. We shall assume understood the existence and functoriality properties of the *Chow ring* $A^*(X)$, see e.g. [32]. Grading is by codimension and the equivalence relation we use is rational equivalence. Lower index means grading by dimension. For convenience if $\mathbb{F}$ is any coefficient ring, we put $A^*(X, \mathbb{F}) = A^*(X) \otimes_{\mathbb{Z}} \mathbb{F}$. It is then possible to construct a Karoubi-closed, symmetric monoidal, rigid, additive category $Chow(k, \mathbb{F})$ together with a covariant symmetric monoidal functor $M = M_{\mathbb{F}} : SmProj(k) \to Chow(k, \mathbb{F})$ which has the following properties (see e.g. [107, p. 6]). The unit object is $\mathbb{1}_{Chow(k,\mathbb{F})} = \mathbb{1} = M(Spec(k))$. There exists an object $\mathbb{1}\{1\}$ such that $M(\mathbb{P}^1) \cong \mathbb{1} \oplus \mathbb{1}\{1\}$. We call $\mathbb{1}\{1\}$ the Lefschetz motive. It is invertible. For any $n \in \mathbb{Z}$ and $M \in Chow(k, \mathbb{F})$ we write $M\{n\} := M \otimes \mathbb{1}\{1\}^{\otimes n}$. For any $X, Y \in SmProj(k)$ with X equidimensional, and $i, j \in \mathbb{Z}$ we have

$$\mathrm{Hom}_{Chow(k,\mathbb{F})}(M(X)\{i\}, M(Y)\{j\}) = A_{\dim X + i - j}(X \times Y).$$

In particular we have $\mathrm{Hom}(MX, \mathbb{1}\{i\}) = A^i(X, \mathbb{F})$ and $\mathrm{Hom}(\mathbb{1}\{i\}, MX) = A_i(X, \mathbb{F})$. Composition is by the usual push-pull convolution.

We shall need the following results. None of them are hard, so most of them are probably well known.

Recall first that if l/k is a field extension then $SmProj(k) \to SmProj(l), X \mapsto X_l$ induces a functor $Chow(k, \mathbb{F}) \to Chow(l, \mathbb{F})$ called *base change* and denoted $M \mapsto M_l$. We need to know something about this in the inseparable case.

Lemma 5.16. *Let l/k be a purely inseparable extension of fields of characteristic p and $\mathbb{F}$ a coefficient ring in which (the image of) p is invertible. Then the base change $Chow(k, \mathbb{F}) \to Chow(l, \mathbb{F})$ is fully faithful.*

Proof. It suffices to prove that for $X \in SmProj(k)$ we have $A_*(X, \mathbb{F}) = A_*(X_l, \mathbb{F})$. By the definition of rational equivalence as in [32, Section 1.6] it is enough to show that $Z_*(X, \mathbb{F}) \to Z_*(X_l, \mathbb{F})$ is an isomorphism for all X.

Let $Z \subset X$ be a reduced closed subscheme and $|Z_l|$ the reduced closed subscheme underlying Z_l. Then the image of $[Z]$ under $Z_*(X, \mathbb{F}) \to Z_*(X_l, \mathbb{F})$ is $n[|Z_l|]$, where n is the multiplicity of Z_l. This is easily seen to be a power of p, whence $Z_*(X, \mathbb{F}) \to Z_*(X_l, \mathbb{F})$ is injective. It is also surjective since $X_l \to X$ is a homeomorphism on underlying spaces. This concludes the proof. □

We now investigate "Tate summands". Denote by $Tate(k, \mathbb{F}) \subset Chow(k, \mathbb{F})$ the smallest (strictly) full Karoubi-closed additive subcategory containing $\mathbb{1}\{i\}$ for all i. This is independent up to equivalence of k and we will just write $Tate(\mathbb{F})$ if no confusion can arise. (It is a symmetric monoidal category.)

We call $M \in Chow(k, \mathbb{F})$ *Tate-free* if whenever $M \cong T \oplus M'$ with $T \in Tate(k, \mathbb{F})$, then $T \cong 0$. The next proposition holds in much greater generality, but this version is all we need.

Proposition 5.17. *Let $\mathbb{F}$ be a finite ring and $M \in Chow(k, \mathbb{F})$. Then there exist $T \in Tate(\mathbb{F})$ and $M' \in Chow(k, \mathbb{F})$ with M' Tate-free and $M \cong T \oplus M'$.*

Proof. Splitting off Tate summands inductively, the only problem which could occur is that M might afford arbitrarily large Tate summands. The impossibility of this follows (for example) from the finiteness of étale cohomology of complete varieties [76, Corollary VI.2.8]. □

Lemma 5.18. *Let $\mathbb{F}$ be a field. Then if $M, N \in Chow(k, \mathbb{F})$ are Tate-free so is $M \oplus N$.*

Proof. A motive with $\mathbb{F}$-coefficients is Tate-free if and only if it affords no summand of the form $\mathbb{1}\{n\}$ for any n.

Let $i : \mathbb{1}\{n\} \to M \oplus N$ and $p : M \oplus N \to \mathbb{1}\{n\}$ be inclusion of and projection to a summand, for M, N arbitrary. Write $i = (i_M, i_N)^T$ and $p = (p_M, p_N)$. Then $\mathrm{id} = pi = p_M i_M + p_N i_N$. Since $\mathrm{Hom}(\mathbb{1}\{n\}, \mathbb{1}\{n\}) = \mathbb{F} \neq 0$ we must have $p_M i_M \neq 0$ or $p_N i_N \neq 0$. Suppose the former holds. Then since $\mathbb{F}$ is a field we may replace i_M by a multiple ci_M such that $p_M(ci_M) = 1$. Thus $\mathbb{1}\{n\}$ is a summand of M. Similarly in the other case. This establishes the contrapositive of the lemma. □

Lemma 5.19. *Let $\mathbb{F}$ be a field. Then any morphism in $Tate(k, \mathbb{F})$ factoring through a Tate-free object is zero.*

Proof. Since $\mathbb{F}$ is a field any Tate motive is a sum of $\mathbb{1}\{n\}$ for various n, so it suffices to consider a morphism $\mathbb{1}\{n\} \to \mathbb{1}\{m\}$ factoring through a Tate-free object. Since $\mathrm{Hom}(\mathbb{1}\{n\}, \mathbb{1}\{m\}) = 0$ for $n \neq m$ we may assume $n = m$. Consider $a \in \mathrm{Hom}(\mathbb{1}\{n\}, M)$ and $b \in \mathrm{Hom}(M, \mathbb{1}\{n\})$. If $ba \neq 0$ then there exists $c \in \mathbb{F}$ such that $(cb)a = \mathrm{id}$. It follows that $(cb), a$ present $\mathbb{1}\{n\}$ as a summand of M. This establishes the contrapositive. □

We need tools to recognise Tate-free motives. To do so, we introduce some more notation. For $X \in SmProj(k)$ there exists the degree map $deg : A_0(X, \mathbb{F}) \to \mathbb{F}$ (corresponding to pushforward along the structure map $\mathrm{Hom}(\mathbb{1}, MX) \to \mathrm{Hom}(\mathbb{1}, \mathbb{1})$). Write $I_{\mathbb{F}}(X) = deg(A_0(X, \mathbb{F}))$ for the image of the degree map. This is the ideal inside $\mathbb{F}$ generated by the degrees of closed points. The utility of this notion is as follows.

Lemma 5.20. *Let $\mathbb{F}$ be a field and suppose $I_{\mathbb{F}}(X) \neq \mathbb{F}$. Then MX is Tate-free.*

Proof. As before MX is Tate-free if and only if it affords no summand $\mathbb{1}\{N\}$ for any N. Given $i \in \mathrm{Hom}(\mathbb{1}\{N\}, MX) = A_N(X, \mathbb{F})$ and $p \in \mathrm{Hom}(MX, \mathbb{1}\{N\}) = A^N(X, \mathbb{F})$, the composite $pi \in \mathrm{Hom}(\mathbb{1}\{N\}, \mathbb{1}\{N\}) = \mathbb{F}$ is obtained by push-pull convolution. In this case it is just $deg(p \cap i)$ and so is contained in $I_{\mathbb{F}}(X)$. Thus $pi \neq 1$ and (p, i) is *not* a presentation of $\mathbb{1}\{N\}$ as a summand of X. □

Lemma 5.21. *Let $X, Y \in SmProj(k)$. Then $I_{\mathbb{F}}(X \times Y) \subset I_{\mathbb{F}}(X) \cap I_{\mathbb{F}}(Y)$.*

Proof. We recall that $I_{\mathbb{F}}(X \times Y)$ is just the ideal generated by degrees of closed points. So let $z \in X \times Y$ be a closed point. Then $z \to X \times Y$ corresponds to morphisms $z \to X$ and $z \to Y$. It follows that $deg(z) \in I_{\mathbb{F}}(X)$ and similarly $deg(z) \in I_{\mathbb{F}}(Y)$. This implies the result. □

Suppose $S \subset SmProj(k)$ is a set of smooth projective varieties. We write $\langle S \rangle^{\otimes, T}_{Chow(k,\mathbb{F})}$ for the smallest strictly full, additive, Karoubi-closed, tensor subcategory of $Chow(k, \mathbb{F})$ containing all Tate motives and also MX for each $X \in S$. Assuming $\mathbb{F}$ is a field, this means that a general object of $\langle S \rangle^{\otimes, T}_{Chow(k,\mathbb{F})}$ is (isomorphic to) a summand of

$$T \oplus M(X_1^{(1)} \times \cdots \times X_{n_1}^{(1)})\{i_1\} \oplus \cdots \oplus M(X_1^{(m)} \times \cdots \times X_{n_m}^{(m)})\{i_m\},$$

with $T \in Tate(\mathbb{F})$ and $X_i^{(j)} \in S, i_r \in \mathbb{Z}$.

The following proposition (or rather its failure to generalise) is the basic reason why in the construction of fixed point functors we will need to restrict to subcategories.

Proposition 5.22. *Let $\mathbb{F}$ be a finite field and $S \subset SmProj(k)$ be such that $I_{\mathbb{F}}(X) = 0$ for all $X \in S$ (i.e. such that all closed points of X have degree divisible by the characteristic of $\mathbb{F}$). Then any object $M \in \langle S \rangle^{\otimes, T}_{Chow(k,\mathbb{F})}$ can be written as $T \oplus M'$, where $T \in Tate(\mathbb{F})$ and M' is (isomorphic to) a summand of*

$$M(X_1^{(1)} \times \cdots \times X_{n_1}^{(1)})\{i_1\} \oplus \cdots \oplus M(X_1^{(m)} \times \cdots \times X_{n_m}^{(m)})\{i_m\},$$

for some $X_i^{(j)} \in S, i_r \in \mathbb{Z}$. Moreover any such M' is Tate-free.

Proof. By Lemma 5.21 we know that $I_{\mathbb{F}}(X_1^{(j)} \times \dots X_{n_j}^{(j)}) = 0$ and thus by Lemmas 5.20 and 5.18 we conclude that any M' as displayed is indeed Tate-free. So it suffices to establish the first part.

By definition we may write

$$M \oplus M'' \cong T \oplus M(X_1^{(1)} \otimes \cdots \otimes X_{n_1}^{(1)})\{i_1\} \oplus \cdots \oplus M(X_1^{(m)} \otimes \cdots \otimes X_{n_m}^{(m)})\{i_m\},$$

with $T \in Tate(\mathbb{F})$ and $X_i^{(j)} \in S$. Using Proposition 5.17 we write $M \oplus M'' \cong M' \oplus M''' \oplus T'$, where M', M'' are maximal Tate-free summands in M, M'' respectively and T' is Tate. Writing out the inverse isomorphisms $M' \oplus M''' \oplus T' \leftrightarrows T \oplus M(X_1^{(1)} \dots) \oplus \dots$ in matrix form and using Lemma 5.19 we conclude that $T' \cong T$ via the induced map. The Lemma below yields that $M' \oplus M'' \cong M(X_1^{(1)} \dots) \oplus \dots$. This finishes the proof. □

Lemma 5.23. *Let $\mathcal{C}$ be an additive category and let $U, T, X, T' \in \mathcal{C}$ be four objects. Suppose we are given an isomorphism $\phi : U \oplus T \to X \oplus T'$ such that the component $T \to T'$ is also an isomorphism. Then there is an isomorphism $\tilde{\phi} : U \to X$.*

Proof. Let us write

$$\phi = \begin{pmatrix} \alpha & a \\ b & f \end{pmatrix} \quad \psi = \begin{pmatrix} \beta & a' \\ b' & g \end{pmatrix},$$

where ψ is the inverse of ϕ. By assumption f is an isomorphism. Writing out $\phi\psi = \mathrm{id}_{X \oplus T}$ and $\psi\phi = \mathrm{id}_{U \oplus T'}$ one obtains

$$b\beta = -fb' \quad \beta a = -a'f \quad \alpha\beta + ab' = \mathrm{id}_U \quad \beta\alpha + a'b = \mathrm{id}_X\,.$$

Put $\tilde{\alpha} = \alpha - af^{-1}b : U \to X$. Then the above relations imply that $\tilde{\alpha}$ is an isomorphism with inverse β. □

5.2.3 The Abstract Fixed Point Functors Theorem

We can now establish our general abstract fixed point functors theorem. Recall that the category $Chow(k, \mathbb{F}) \subset \mathbf{DM}^{gm}(k, \mathbb{F})$ is negative for any $\mathbb{F}$ provided that k is perfect [75, Theorem 19.1, Property (14.5.6)] and generates $\mathbf{DM}^{gm}(k, \mathbb{F})$ if in addition the exponential characteristic e of k is invertible on $\mathbb{F}$ [14]. Thus there is a canonical weight structure on $\mathbf{DM}^{gm}(k, \mathbb{F})$. For any field k with perfect closure k^p and $\mathbb{F}$ on which e is invertible the base change functor $\mathbf{DM}(k, \mathbb{F}) \to \mathbf{DM}(k^p, \mathbb{F})$ is an equivalence [20, Proposition 8.1 (d)]. Consequently for any k (and such $\mathbb{F}$) the category $\mathbf{DM}^{gm}(k, \mathbb{F})$ has a canonical weight structure. Its heart is $Chow(k^p, \mathbb{F})$ and contains $Chow(k, \mathbb{F})$ as a full subcategory by Lemma 5.16. All of the weight structures are bounded. All of the base change functors are w-exact, by Lemma 5.10.

In the remainder of this section we will be dealing with the following situation. The coefficient ring $\mathbb{F}$ is a finite field of characteristic p (necessarily $p \neq e$, where e is the exponential characteristic of the ground field k). For every extension l/k we are given a set $S_l \subset SmProj(l)$ such that for all closed points $x \in X \in S_l$ we have $p | deg(x)$. Recall the categories $\langle S_l \rangle^{\otimes, T}_{Chow(l,\mathbb{F})}$ of Section 5.2.2. We will assume that they are stable by base change, i.e. that for $X \in S_l$ and l'/l another extension we have $MX_{l'} \in \langle S_{l'} \rangle^{\otimes, T}_{Chow(l',\mathbb{F})}$.

We write $\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})$ for the thick triangulated subcategory of $\mathbf{DM}(l, \mathbb{F})$ generated by $\langle S_l \rangle^{\otimes, T}_{Chow(l,\mathbb{F})} \subset Chow(l, \mathbb{F}) \subset \mathbf{DM}(l, \mathbb{F})^{w=0}$. It is symmetric monoidal. The triangulated categories $\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})$ are also stable by base change in the sense that if $f : Spec(l') \to Spec(l)$ is a field extension then $f^*(\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})) \subset \mathbf{D}\langle S \rangle\mathbf{TM}(l', \mathbb{F})$. By Proposition 5.8 (5) the weight structure on $\mathbf{DM}^{gm}(l, \mathbb{F})$ restricts to $\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})$, and the heart is $\langle S_l \rangle^{\otimes, T}_{Chow(l,\mathbb{F})}$.

We write $\langle S_l \rangle^{\otimes}_{Chow(l,\mathbb{F})}$ for the Karoubi-closed symmetric monoidal subcategory of $Chow(l, \mathbb{F})$ generated by Tate twists of motives of varieties in S_l (i.e. this is $\langle S_l \rangle^{\otimes, T}_{Chow(l,\mathbb{F})}$ "without the Tate motives"). By Proposition 5.22 this subcategory consists of Tate-free objects. Let $\langle S_l \rangle^{tri} \subset \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})$ be the triangulated subcategory generated by $\langle S_l \rangle^{\otimes}_{Chow(l,\mathbb{F})}$. As before, the weight structure restricts to $\langle S_l \rangle^{tri}$. We write $\varphi^l_0 : \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F}) \to \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})/\langle S \rangle^{tri}$ for the Verdier quotient.

Proposition 5.24. *The category $\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})/\langle S \rangle^{tri}$ carries natural weight and symmetric monoidal structures, and φ^l_0 is a w-exact symmetric monoidal functor. The composite*

$$Tate(\mathbb{F}) \to \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F}) \to \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})/\langle S \rangle^{tri}$$

is a full embedding with essential image $\left(\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})/\langle S \rangle^{tri}\right)^{w=0}$.

Proof. The existence of the weight structure and weight exactness is Proposition 5.8 (6). This also says that $\left(\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})/\langle S \rangle^{tri}\right)^{w=0}$ is generated as a Karoubi-closed category by $\varphi^l_0\left(\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})^{w=0}\right)$. If $M \in \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})^{w=0} = \langle S_l \rangle^{\otimes, T}_{Chow(l,\mathbb{F})}$ then we may write $M \cong M' \oplus T$ with T a Tate and $M' \in \langle S_l \rangle^{\otimes}_{Chow(l,\mathbb{F})}$, by Proposition 5.22. Thus $\varphi^l_0(M) \cong \varphi^l_0(T)$ and so $\varphi^l_0 : Tate(\mathbb{F}) \to \left(\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})/\langle S \rangle^{tri}\right)^{w=0}$ is essentially surjective up to Karoubi-completing. We shall show it is fully faithful whence its essential image is Karoubi-closed and so $\varphi^l_0 : Tate(\mathbb{F}) \to \left(\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})/\langle S \rangle^{tri}\right)^{w=0}$ will be an equivalence. But by the description in Proposition 5.8 (6) it suffices to prove that any morphism between Tate objects factoring through $\langle S_l \rangle^{\otimes}_{Chow(l,\mathbb{F})}$ is zero. This follows from Lemma 5.19.

For the existence of the symmetric monoidal structure we need $\langle S \rangle^{tri} \otimes \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F}) \subset \langle S \rangle^{tri}$; then φ^l_0 is automatically symmetric monoidal. Considering generators, it suffices to show that $\langle S_l \rangle^{\otimes}_{Chow(l,\mathbb{F})} \otimes \langle S_l \rangle^{\otimes, T}_{Chow(l,\mathbb{F})} \subset \langle S_l \rangle^{\otimes}_{Chow(l,\mathbb{F})}$. This follows from Proposition 5.22. □

Let l/k be any extension. We write $\Phi^l : \mathbf{D}\langle S \rangle\mathbf{TM}(k, \mathbb{F}) \to K^b(Tate(\mathbb{F}))$ for the composite

$$\Phi^l : \mathbf{D}\langle S \rangle\mathbf{TM}(k, \mathbb{F}) \to \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F}) \to \mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})/\langle S_l \rangle^{tri}$$
$$\xrightarrow{t} K^b\left(\left(\mathbf{D}\langle S \rangle\mathbf{TM}(l, \mathbb{F})/\langle S \rangle^{tri}\right)^{w=0}\right) \cong K^b(Tate(\mathbb{F}))$$

of base change, the Verdier quotient functor φ_0^l, and the weight complex t. It is a w-exact triangulated symmetric monoidal functor. We can now state the main theorem of this section.

Theorem 5.25 (Abstract Fixed Point Functors)**.** *Let k be a ground field of exponential characteristic e, $\mathbb{F}$ a finite field of characteristic $p \neq e$. Suppose given for each field extension l/k a set $S_l \subset SmProj(l)$ and a function $ex = ex_l : S_l \to \mathbb{N}$. Assume that the following hold (for all fields l/k):*

(1) For $x \in X \in S_l$ closed, $p | deg(x)$.

(2) If l'/l is a field extension and $X \in S_l$ has no rational point over l', then $X_{l'}$ is isomorphic to an object of $S_{l'}$ and $ex(X_{l'}) \leq ex(X)$.

(3) If l'/l is a field extension and $X \in S_l$ has a rational point over l', then $MX_{l'}$ is a summand of a motive of the form

$$T \oplus M(X_1^{(1)} \otimes \cdots \otimes X_{n_1}^{(1)})\{i_1\} \oplus \cdots \oplus M(X_1^{(m)} \otimes \cdots \otimes X_{n_m}^{(m)})\{i_m\},$$

with $T \in Tate(\mathbb{F})$, $X_i^{(j)} \in S_{l'}$ and $ex(X_i^{(j)}) < ex(X)$ for all i, j.

Then the family $\{\Phi^l\}_l$, as l runs through finitely generated extensions of k is w-conservative (so in particular conservative) and Pic-injective.

We note that (2) and (3) imply that $\langle S_l \rangle^{\otimes,T}_{Chow(l,\mathbb{F})}$ are stable by base change, i.e. we are in the situation we have been discussing. Also (1) implies that none of the $X \in S_l$ have rational points over l. The somewhat obscure functions ex_l are necessary to make an induction step in the proof work. We will mostly use $ex = \dim$ in applications.

Before proving the result we explain how to compute Φ^l in the case that k is perfect (but l need not be).

Proposition 5.26. *Assume in addition that k is perfect. Let l/k be a field extension.*

There exists an essentially unique additive functor $\Phi_0^l : \langle S_l \rangle^{\otimes,T}_{Chow(l,\mathbb{F})} \to Tate(\mathbb{F})$ such that $\Phi_0^l|_{Tate(l,\mathbb{F})} = \mathrm{id}$ and $\Phi_0^l(M) = 0$ if M is Tate-free. It is symmetric monoidal and the following diagram commutes (up to natural isomorphism; the lower horizontal arrow is base change of Chow motives):

$$\begin{array}{ccc} \mathbf{D}\langle S \rangle \mathbf{TM}(k,\mathbb{F}) & \xrightarrow{\Phi^l} & K^b(Tate(\mathbb{F})) \\ {\scriptstyle t}\big\downarrow & & \big\uparrow{\scriptstyle \Phi_0^l} \\ K^b\left(\langle S_k \rangle^{\otimes,T}_{Chow(k,\mathbb{F})}\right) & \longrightarrow & K^b\left(\langle S_l \rangle^{\otimes,T}_{Chow(l,\mathbb{F})}\right) \end{array}$$

Proof. Certainly Φ_0^l is essentially unique, using e.g. Proposition 5.22. The functor $t \circ \varphi_0^l$ satisfies the required properties, so Φ_0^l exists. It is symmetric monoidal by construction.

To establish the commutativity claim, consider the diagram

$$\begin{array}{ccc} \mathbf{D}\langle S \rangle \mathbf{TM}(k,\mathbb{F}) & \xrightarrow{t} & K^b\left(\langle S_k \rangle^{\otimes,T}_{Chow(k,\mathbb{F})}\right) \\ \big\downarrow & & \big\downarrow \\ \mathbf{D}\langle S \rangle \mathbf{TM}(l,\mathbb{F}) & \xrightarrow{t} & K^b\left(\langle S_l \rangle^{\otimes,T}_{Chow(k,\mathbb{F})}\right) \\ {\scriptstyle \varphi_0^l}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_0^l} \\ \mathbf{D}\langle S \rangle \mathbf{TM}(l,\mathbb{F})/\langle S_l \rangle^{tri} & \xrightarrow{t} & K^b(Tate(\mathbb{F})). \end{array}$$

It suffices to prove that the two squares commute (up to natural isomorphism). This is most readily seen using DG-enhancements: let $\mathcal{D}(r)$ be a functorial negative DG-enhancement of

$\langle S_r\rangle^{\otimes,T}_{Chow(r,\mathbb{F})} \subset \mathbf{D}\langle S\rangle\mathbf{TM}(r,\mathbb{F})$, for fields r/k. Then it suffices to establish strict commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{D}(k) & \longrightarrow & \mathcal{D}(k)_0 \\
\downarrow & & \downarrow \\
\mathcal{D}(l) & \longrightarrow & \mathcal{D}(l)_0 \cong \langle S_l\rangle^{\otimes,T}_{Chow(l,\mathbb{F})} \\
\downarrow & & \Phi^l_0 \downarrow \\
\mathcal{D}(l)/\langle S_l\rangle^{tri} & \longrightarrow & \left(\mathcal{D}(l)/\langle S_l\rangle tri\right)_0 \cong Tate(\mathbb{F}),
\end{array}$$

where $\mathcal{D}_0$ for a negative DG-category means zero-truncation. (Indeed the previous diagram is obtained by passing to $Ho(\text{Pre-Tr}(\bullet))$.) The upper square commutes by functoriality, and the lower square commutes if and only if it commutes on degree zero morphisms, which is true essentially by definition of Φ^l_0. □

We establish the abstract fixed points functors Theorem 5.25 through the following two lemmas.

Lemma 5.27. *Let $X \in \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F})^{w\le 0}$ have a weight decomposition $T \to X \to X'$ with $T \in Tate(\mathbb{F})$ (and $X' \in \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F})^{w<0}$). Suppose that $\varphi^k(X) \in \left(\mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F})/\langle S_k\rangle^{tri}\right)^{w<0}$.*

Then for $T' \in Tate(\mathbb{F})$ we have $\mathrm{Hom}(T', X) = 0$.

Proof. Let $B[1] \to X' \to X''$ be a further weight decomposition. Using Lemma 5.13 yields the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{Hom}(T',B) & \xrightarrow{\gamma} & \mathrm{Hom}(T',T) & \longrightarrow & \mathrm{Hom}(T',X) & \longrightarrow & 0 \\
\alpha\downarrow & & \beta\downarrow & & \downarrow & & \\
\mathrm{Hom}(\varphi^k(T'),\varphi^k(B)) & \xrightarrow{\delta} & \mathrm{Hom}(\varphi^k(T'),\varphi^k(T)) & \longrightarrow & \mathrm{Hom}(\varphi^k(T'),\varphi^k(X)) & \longrightarrow & 0.
\end{array}$$

Since φ^k is weight exact we have $\mathrm{Hom}(\varphi^k(T'),\varphi^k(X)) = 0$ and so δ is surjective. The construction of φ^k (in particular Proposition 5.24) implies that α is surjective and β is an isomorphism. It follows that γ is surjective, whence $\mathrm{Hom}(T', X) = 0$. This concludes the proof. □

We let $\varphi^l : \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F}) \to \mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F})/\langle S_l\rangle^{tri}$ be the composite of φ^l_0 and base change.

Lemma 5.28. *Let $X \in \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F})^{w\le 0}$ and suppose that for all l/k finitely generated, $\varphi^l(X) \in \left(\mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F})/\langle S_l\rangle^{tri}\right)^{w<0}$. Then $X \in \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F})^{w<0}$.*

Proof. We begin by pointing out that Lemma 5.3 also applies if k is not perfect. Indeed if k^p/k is the perfect closure then X_{k^p} is homeomorphic to X, so has the same set of points, and the residue field extensions of $X_{k^p} \to X$ are purely inseparable, so induce equivalences on $\mathbf{DM}(?,\mathbb{F})$. Thus the Lemma holds over k if and only if it holds over k^p.

Let $\mathcal{R}$ be the set of finite multi-subsets of $\mathbb{N}$ (i.e. the set of finite non-increasing sequences in $\mathbb{N}$). It is well-ordered lexicographically and so can be used for induction. We extend ex to a function $ex_l : \mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F}) \to \mathcal{R}$. First, for $X_1,\ldots,X_n \in S_l$ put $ex(X_1,\ldots,X_n) = \{\{ex(X_1),\ldots,ex(X_n)\}\}$. Next, if $Y \in \mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F})$ then there exist $X_1,\ldots,X_n \in S_l$ such that $Y \in \langle Tate(\mathbb{F}), X_1,\ldots,X_n\rangle^{tri}$, i.e. Y is in the thick tensor triangulated subcategory generated by the MX_i and the Tate motives. We let $ex(Y)$ be the minimum of $ex(X_1,\ldots,X_n)$ such that this holds. We shall abuse notation and write $ex(Y) = ex(X_1,\ldots,X_n)$ to additionally mean that $Y \in \langle Tate(\mathbb{F}), X_1,\ldots,X_n\rangle^{tri}$.

Let us observe that if $ex(Y) = ex(X_1,\ldots,X_n)$ and l'/l is an extension in which one of the X_i acquires a rational point, then $ex(Y_{l'}) < ex(Y_l)$, using assumptions (2) and (3).

We shall prove the result by induction on $ex(X)$. Note that is suffices to prove that there is a weight decomposition $A \xrightarrow{\alpha} X \to X'$ (i.e. $A \in \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F})^{w=0}$ and $X' \in \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F})^{w<0}$) with $\alpha = 0$ (because then $X' \cong X \oplus A[1]$ and so $X \in \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F})^{w<0}$, the latter being Karoubi-closed by definition).

If $ex(X)=\emptyset$ then X must must be Tate. By Lemma 5.9 we may choose a weight decomposition $T \xrightarrow{\alpha} X \to X'$ with $T \in Tate(\mathbb{F})$. By the corollary above (applied to $T'=T$) we find that $\alpha=0$. This finishes the base case of our induction.

Suppose now $ex(X)=ex(X_1,\dots,X_n)>\emptyset$. If l/k is any extension such that one of the $X_1,\dots,X_n$ acquires a rational point over l, then we may assume the lemma proved over l by induction, so $X_l \in \mathbf{D}\langle S\rangle\mathbf{TM}(l,\mathbb{F})^{w<0}$. Let $A \xrightarrow{\alpha} X \to X'$ be a weight decomposition; as before way may choose $A \in \langle\{X_1,\dots,X_n\}\rangle^{\otimes,T}_{Chow(k,\mathbb{F})}$. Write $A \cong T \oplus A'$ as in Proposition 5.22. I claim that $\alpha|_{A'}=0$. It is enough to show that if Y is a product of the X_i then $\mathrm{Hom}(MY\{n\},X)=0$ for all n. By Lemma 5.3, it is enough to show that for all $n \in \mathbb{Z}$ and $p \in Y$ we have that $\mathrm{Hom}_{\mathbf{DM}(p,\mathbb{F})}(\mathbb{1}\{n\},X_p)=0$. But every variety has a rational point after base change to any one of its points, so $X_p \in \mathbf{D}\langle S\rangle\mathbf{TM}(p,\mathbb{F})^{w<0}$ by induction. This proves the claim.

We thus have a weight decomposition $T\oplus A' \xrightarrow{(\alpha,0)^T} X \to X'$. Let Y be a cone on $\alpha : T \to X$. We find that $X' \cong Y \oplus A'[1]$ and hence $Y \in \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F})^{w<0}$. Thus $T \xrightarrow{\alpha} X \to Y$ is a weight decomposition. Using the corollary again we get $\mathrm{Hom}(T,X)=0$ and so $\alpha=0$. This finishes the induction step. □

Proof of the abstract fixed point functors Theorem 5.25. By Lemma 5.12, in order to show w-conservativity, it suffices to show right-w-conservativity. The above Lemma establishes right-w-conservativity of the family $\{\phi^l\}_l$, and since our weight structures are bounded, the weight complex functors are w-conservative. Thus the family $\{\Phi^l\}_l$ is w-conservative.

Finally for Pic-injectivity, let $X \in \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F})$ be invertible with $\Phi^l(X)\cong\mathbb{1}$ for all l. Since $\mathbb{1} \in K^b(Tate(\mathbb{F}))^{w=0}$, w-conservativity implies that $X \in \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F})^{w=0} = \langle S_k\rangle^{\otimes,T}_{Chow(k,\mathbb{F})}$. Write $X \cong T \oplus X'$, with T Tate and X' Tate-free. Then $\mathbb{1}\cong\Phi^k(X)=T$ and so $T\cong\mathbb{1}$. It follows that $\Phi^l(X)=\mathbb{1}\oplus\Phi^l(X')\in Tate(\mathbb{F})$. For this to be invertible we need $\Phi^l(X')=0$. Since this is true for all l, conservativity implies that $X'=0$. □

5.3 Quadrics

5.3.1 The Geometric Fixed Point Functors Theorem for Motives of Quadrics

If k is a field and ϕ is a non-degenerate quadratic form over k, write $Y_\phi = Proj(\phi=0)$ for the associated projective quadric. If $\dim\phi=1$ we put $Y_\phi=\emptyset$. Given $a\in k^\times$ we denote $Y_\phi^a = Proj(\phi=aZ^2)=Y_{\phi\perp\langle -a\rangle}$ and write $X_\phi^a = Spec(\phi=a)$ for the affine quadric.

Fix now a perfect field k of exponential characteristic $e\neq 2$. We write $\mathbf{QM}(k,A)$ for the Karoubi-closed, additive, monoidal subcategory of $Chow(k,A)$ generated by the $M(Y_\phi)$, as ϕ ranges over non-degenerate quadrics, and $\mathbb{1}\{-1\}$. Note that $\mathbb{1}\{1\}\in\mathbf{QM}(k,A)$ and so $\mathbf{QM}(k,A)$ contains all Tate motives. We write $\mathbf{DQM}^{gm}(k,A)$ for the thick triangulated subcategory of $\mathbf{DM}^{gm}(k,A)$ generated by $\mathbf{QM}(k,A)\subset Chow(k,A)\subset\mathbf{DM}^{gm}(k,A)$. This is a symmetric monoidal, Karoubi-closed, triangulated category. By Lemma 5.9 $\mathbf{DQM}^{gm}(k,A)$ carries a weight structure inherited from $\mathbf{DM}^{gm}(k,A)$.

We write $\mathbf{QM}(k) := \mathbf{QM}(k,\mathbb{Z}[1/e])$ and $\mathbf{DQM}^{gm}(k) := \mathbf{DQM}^{gm}(k,\mathbb{Z}[1/e])$.

Lemma 5.29 (Rost). *Let ϕ be an isotropic non-degenerate quadratic form. Then there exists a non-degenerate form ψ such that*

$$M(Y_\phi)\cong\mathbb{1}\oplus M(Y_\psi)\{1\}\oplus\mathbb{1}\{\dim Y_\phi\}.$$

Moreover for $a\in k^\times$ the natural inclusion $M(Y_\phi)\to M(Y_\phi^a)$ is given by

$$\begin{pmatrix}\mathrm{id} & 0 & 0\\ 0 & 0 & 0\\ 0 & s\{1\} & i\{1\}\end{pmatrix} : \mathbb{1}\oplus\mathbb{1}\{\dim Y_\phi\}\oplus M(Y_\psi)\{1\}\to\mathbb{1}\oplus\mathbb{1}\{\dim Y_\phi+1\}\oplus M(Y_\psi^a)\{1\},$$

where $i : M(Y_\psi) \to M(Y_\psi^a)$ *is the natural inclusion and* $s : \mathbb{1}\{\dim Y_\psi^a\} \to M(Y_\psi^a)$ *is the fundamental class (dual of the structure map).*

Proof. This is basically [97, Proposition 2]. Rost starts with $\phi = \mathbb{H} \perp \psi$, but this is equivalent to ϕ having a rational point.

For the explicit form of the inclusion, note first that all matrix entries shown as zero have to be so for dimensional reasons. The entries "id" and "$i\{1\}$" follow from naturality of Rost's construction. For the final entry, we can argue as follows. Note that $\mathbb{Z} = CH^0(Y_\psi^a) = \mathrm{Hom}(\mathbb{1}\{\dim Y_\psi^a + 1\}, MY_\psi^a\{1\}) \cong \mathrm{Hom}(\mathbb{1}\{\dim Y_\psi^a + 1\}, MY_\phi) = CH^1(Y_\phi)$. The induced map we are interested in corresponds under this identification to the cycle class of the closed subvariety $Y_\phi \subset Y_\phi^a$. So up to verifying a sign (which is irrelevant for all our applications), it is enough to show that this class is a generator, which one sees for example by considering the embedding into ambient projective space. □

Lemma 5.30. *For a field extension l/k let S_l be the set of anisotropic projective smooth quadrics over l, and let $ex_l : S_l \to \mathbb{N}$ be the dimension function $ex(X) = \dim X$. Then Theorem 5.25 applies, with $\mathbb{F} = \mathbb{F}_2$.*

We note that $\mathbf{D}\langle S\rangle\mathbf{TM}(k, \mathbb{F}_2) = \mathbf{DQM}^{gm}(k, \mathbb{F}_2)$, in the notation of the Theorem.

Proof. Points on an anisotropic quadric have degree divisible by two by Springer's theorem [64, Chapter 7, Theorem 2.3], hence condition (1) holds. Condition (2) is satisfied essentially by definition. Finally condition (3) follows from Lemma 5.29. □

It is well known, and also follows from Lemma 5.29, that motives of quadrics are geometrically Tate. Let $f : Spec(k^s) \to Spec(k)$ be a separable closure. It follows that $\mathbf{QM}(k^s, A) \simeq Tate(A)$ and so the weight complex functor on $\mathbf{DQM}^{gm}(k^s, A)$ takes values in $K^b(Tate(A))$. Write Ψ for the composite

$$\Psi : \mathbf{DQM}^{gm}(k, A) \xrightarrow{f^*} \mathbf{DQM}^{gm}(k^s) \xrightarrow{t} K^b(Tate(\mathbb{Z}[1/e])).$$

If $g : Spec(l) \to Spec(k)$ is any field extension, then via Lemma 5.30 (i.e. Theorem 5.25) we obtain a geometric fixed point functor $\Phi^l : \mathbf{DQM}^{gm}(k, \mathbb{F}_2) \to K^b(Tate(\mathbb{F}_2))$. By abuse of notation we also write Φ^l for the composite $\mathbf{DQM}^{gm}(k) \xrightarrow{\alpha_\#^2} \mathbf{DQM}^{gm}(k, \mathbb{F}_2) \to K^b(Tate(\mathbb{F}_2))$.

Theorem 5.31. *The functors Ψ, Φ^l are tensor triangulated. Together (as l ranges over all finitely generated extension of k) they are conservative and Pic-injective.*

Proof. The functors are composites of tensor triangulated functors, so are tensor triangulated.

By Theorem 5.6 parts (ii, iii) the collection $f^*, \{L\alpha_{p\#}\}_p$ (where p ranges over all primes) is conservative and Pic-injective. Since all weight complex functors are conservative and Pic-injective by Lemma 5.11, the collection $tf^*, \{tL\alpha_{p\#}\}_p$ is conservative and Pic-injective. We have $tf^* = \Psi$. By Theorem 5.25 we may replace $tL\alpha_{2\#}$ in our collection by $\{\Phi^l\}_l$.

It remains to deal with $L\alpha_{p\#}$ at odd p. Let $M \in \mathbf{DQM}^{gm}(k, \mathbb{Z}[1/e])$. By repeated application of Lemma 5.29 we can find an extension L/k (which we may assume Galois) of degree a power of 2, such that M_L is in the triangulated subcategory generated by the Tate motives. In particular $t(L\alpha_{p\#}M_L) \cong L\alpha_{p\#}\Psi(M)$ (as complexes of Tate motives). Since $[L : k]$ is a power of two, base change along L/k is conservative in odd characteristic by Theorem 5.1. Thus if $\Psi(M) \simeq 0$ then also $L\alpha_{p\#}M \simeq 0$ and our collection is conservative.

We need to work a bit harder for Pic-injectivity. Let $M \in \mathbf{DQM}^{gm}(k, \mathbb{Z}[1/e])$ be invertible with $\Phi^l(M) \simeq \mathbb{1}[0]$ for all l/k and $\Psi(M) \simeq \mathbb{1}[0]$. Then we know that $L\alpha_{2\#}(M) \simeq \mathbb{1}$ by Theorem 5.25. We also have $t(M_L) = \Psi(M) \simeq \mathbb{1}$, so $M_L \simeq \mathbb{1}$ by Lemma 5.11. Consider the mod 2 Bockstein sequence

$$\mathrm{Hom}(\mathbb{1}, L\alpha_{2\#}M[-1]) = 0 \to \mathrm{Hom}(\mathbb{1}, M) \xrightarrow{2} \mathrm{Hom}(\mathbb{1}, M) \to$$
$$\mathrm{Hom}(\mathbb{1}, L\alpha_{2\#}M) \to \mathrm{Hom}(\mathbb{1}, M[1]) \xrightarrow{2} \mathrm{Hom}(\mathbb{1}, M[1]) \to \mathrm{Hom}(\mathbb{1}, L\alpha_{2\#}M[1]) = 0.$$

The extremal terms are zero because $L\alpha_{2\#}M \simeq \mathbb{1}$, and for the same reason we have that $\mathrm{Hom}(\mathbb{1}, L\alpha_{2\#}M) \cong \mathbb{F}_2$. Thus $\mathrm{Hom}(\mathbb{1}, M)$ has no 2-torsion, whereas $\mathrm{Hom}(\mathbb{1}, M[1])$ has no 2-cotorsion. The composite $M \to M_L \to M$ of base change and transfer is multiplication by $[L : k] = 2^N$. We conclude that $\mathrm{Hom}(\mathbb{1}, M)$ injects into $\mathrm{Hom}_L(\mathbb{1}_L, M_L) \cong \mathbb{Z}[1/e]$ and that the kernel of $\mathrm{Hom}(\mathbb{1}, M[1]) \to \mathrm{Hom}_L(\mathbb{1}_L, M_L[1]) = 0$ (i.e. the whole group) is contained in the 2^N-torsion. But multiplication by 2 is surjective on $\mathrm{Hom}(\mathbb{1}, M[1])$, whence so is multiplication by 2^N, and we conclude that $\mathrm{Hom}(\mathbb{1}, M[1]) = 0$. Consequently we have $\mathrm{Hom}(\mathbb{1}, M) \cong \mathbb{Z}[1/e]$ (since it is an ideal of $\mathbb{Z}[1/e]$ with a non-vanishing quotient, i.e. $\mathbb{F}_2$).

We shall now apply the second part of Theorem 5.1. As we have seen $M_L \simeq \mathbb{1}$, so we obtain a $G = Gal(L/k)$-action on $\mathrm{Hom}(\mathbb{1}, M_L) \cong \mathbb{Z}[1/e]$, i.e. a group homomorphism $\kappa_M : G \to \mathbb{Z}[1/e]^\times$. Since e is prime we have $\mathbb{Z}[1/e]^\times = \{\pm 1\} \times \{e^k | k \in \mathbb{Z}\}$ and since G is finite the image of κ_M must be contained in $\{\pm 1\}$. Note that if $\kappa_M = 1$ then $M \simeq \mathbb{1}$ and we are done. Indeed it suffices by Theorem 5.6 to show that $L\alpha_{p\#}M \simeq \mathbb{1}$ for odd p. Since $(L\alpha_{p\#}M)_L \simeq \mathbb{1}$, by the second part of theorem 5.1 this happens if and only if an appropriate Galois action is trivial, but this action is just the reduction $G \xrightarrow{\kappa_M} \mathbb{Z}[1/e]^\times \to (\mathbb{Z}/p)^\times$. So assume now that κ_M is non-trivial.

Let $\beta : \mathbb{Z}[1/e] \to \mathbb{Z}[1/(2e)]$ be the natural map. Note that $\kappa_M : G \to \{\pm 1\}$ has a kernel index 2, i.e. corresponds to a quadratic subextension $k \subset k_2 \subset L$. I claim that $L\beta_\# M \simeq L\beta_\# \tilde{M}Spec(k_2)$. Indeed this follows from Theorem 5.1 applied to $A = \mathbb{Z}[1/(2e)]$, where base change to L becomes conservative, and the observation that $\kappa_{\tilde{M}Spec(k_2)} = \kappa_M$.

In particular we must have $\mathrm{Hom}(\mathbb{1}, L\beta_\# \tilde{M}Spec(k_2)) \cong \mathrm{Hom}(\mathbb{1}, M) \otimes_{\mathbb{Z}[1/e]} \mathbb{Z}[1/(2e)] = \mathbb{Z}[1/(2e)]$, by Proposition 5.4 and our previous computation. But one may compute easily that $\mathrm{Hom}(\mathbb{1}, L\beta_\# \tilde{M}Spec(k_2)) = 0$. This contradiction concludes the proof. □

Note that if A is a PID, then $Pic(K^b(Tate(A))) = \mathbb{Z} \oplus \mathbb{Z}$. Consequently we have the following corollary.

Corollary 5.32. *The abelian group $Pic(\mathbf{DM}^{gm}(k, \mathbb{Z}[1/e]))$ is torsion-free (where k is a perfect field of exponential characteristic $e \neq 2$).*

Remark. As can be seen from the proof, this result is completely false in $\mathbf{DQM}^{gm}(k, A)$ as soon as $1/2 \in A$. See also Example 1 in Subsection 5.4.2.

5.3.2 The Invertibility of Affine Quadrics

We will use the conservative, Pic-injective collection $\{\Phi^l\}_l \cup \{\Psi\}$ on $\mathbf{DQM}^{gm}(k)$ to study affine quadrics. First we need to verify that the affine quadrics even live in this category. Fortunately that is not hard.

Lemma 5.33. *If ϕ is a non-degenerate quadratic form over the perfect field k of characteristic not two, and $a \in k^\times$, then the affine quadric X_ϕ^a satisfies $M(X_\phi^a) \in \mathbf{DQM}^{gm}(k, A)$.*

Proof. We have $X_\phi^a = Y_\phi^a \setminus Y_\phi$ and $M(Y_\phi^a), M(Y_\phi), \mathbb{1}\{1\} \in \mathbf{DQM}^{gm}(k, A)$, so the result follows from the Gysin triangle. □

We can now prove that affine quadrics are invertible. Recall the *reduced motive* $\tilde{M}(X) = cone(M(X) \to M(Spec(k)))[-1]$.

Theorem 5.34. *Let k be a perfect field of characteristic not two, ϕ a non-degenerate quadratic form over k and $a \in k^\times$. Then $\tilde{M}(X_\phi^a)$ is invertible in $\mathbf{DM}^{gm}(k)$.*

Proof. We have $\tilde{M}X_\phi^a \in \mathbf{DQM}^{gm}(k)$ by Lemma 5.33 and so we can use Theorem 5.31. Since the category $\mathbf{DQM}^{gm}(k)$ is generated by rigid objects (Chow motives) it is rigid and so conservative tensor functors detect invertibility, by standard arguments. We thus need to show that $\Psi(\tilde{M}X_\phi^a)$ is invertible and that for each l/k, $\Phi^l(\tilde{M}X_\phi^a)$ is invertible.

Let $d+2 = \dim \phi$. Let us put $V_\phi^a = D(MX_\phi^a)\{d+1\}$ and $\tilde{V}_\phi^a = D(\tilde{M}X_\phi^a)\{d+1\}$. Then $\tilde{M}X_\phi^a$ is invertible if and only if $\tilde{V}_\phi^a$ is. From the closed inclusion $i : Y_\phi \to Y_\phi^a$ with complement X_ϕ^a we get the dual Gysin triangle

$$MY_\phi \xrightarrow{i} MY_\phi^a \to V_\phi^a.$$

It follows that $t(V_\phi^a) = [MY_\phi \xrightarrow{i} \dot{M}Y_\phi^a]$. Here the dot is used to indicate the term of degree zero in the chain complex. Dualising the defining triangle of $\tilde{M}X_\phi^a$ we obtain

$$\mathbb{1}\{d+1\} \xrightarrow{s} V_\phi^a \to \tilde{V}_\phi^a,$$

where s is the fundamental class (dual of the structure map). Hence we finally obtain

$$t(\tilde{V}_\phi^a) = [MY_\phi \oplus \mathbb{1}\{d+1\} \xrightarrow{(i,s)} \dot{M}Y_\phi^a] =: C(\phi).$$

The functor Ψ is computed by first applying geometric base change, so ϕ becomes completely split. In particular it has to be isotropic. An induction on dimension using Lemma 5.35 below show that we may reduce to $\dim \phi = 1$ or 2, i.e. $\{x^2 = 1\}$ or $\{xy = 1\}$ (recall that completely split quadrics are characterised by their dimension, so we can choose any non-degenerate model quadric of the correct dimension). But $\tilde{M}(\{x^2 = 1\}) = \mathbb{1}$ and $\tilde{M}(\{xy = 1\}) = \tilde{M}(\mathbb{G}_m)$ are both invertible.

Dealing with Φ^l is a bit harder.

The expression $C(\phi) \in K^b(\mathbf{QM}(k))$ makes sense even if k is not perfect. Using Proposition 5.26 it suffices to prove: if l/k is any field extension, then $\Phi_0^l C(\phi_l)$ is invertible. We drop the subscript zero from now on. We may as well prove: if k is any field and ϕ is any non-degenerate quadratic form over k, then $\Phi^k(C(\phi))$ is invertible. By Lemma 5.35 below, if $\phi \cong \psi \perp \mathbb{H}$ then $C(\phi) \simeq C(\psi)\{1\}$. We may thus assume that either ϕ is anisotropic, or $\phi = \mathbb{H}$, or ϕ is of dimension one.

If $\phi = \mathbb{H}$ then $Y_\phi \cong \mathit{Spec}(k \times k)$, $Y_\phi^a \cong \mathbb{P}^1$ and the result follows easily. If ϕ is of dimension one then $MY_\phi = 0$ and either $MY_\phi^a = \mathbb{1} \oplus \mathbb{1}$ or $MY_\phi^a = M(k')$, where k'/k is a quadratic extension. Again the result follows easily.

So we may assume that ϕ is anisotropic. There are three cases. If $\phi \perp \langle -a \rangle$ is also anisotropic, then none of MY_ϕ, MY_ϕ^a afford Tate summands, by Proposition 5.22. Thus $\Phi^k(C(\phi)) = \mathbb{1}\{d+1\}[1]$ is invertible.

If $\phi \perp \langle -a \rangle$ is isotropic, then $\phi \perp \langle -a \rangle = \psi \perp \mathbb{H}$. Suppose that ψ has dimension greater than one. Then by (the contrapositive of) Lemma 5.36 below, ψ is anisotropic. It follows that $MY_\phi^a \cong \mathbb{1} \oplus \mathbb{1}\{d+1\} \oplus MY_\psi^a$ and $\Phi^k(C(\phi)) = [\mathbb{1}\{d+1\} \to \dot{\mathbb{1}} \oplus \mathbb{1}\{d+1\}]$. The component $\mathbb{1}\{d+1\} \to \mathbb{1}\{d+1\}]$ comes from the fundamental class of M_ψ^a and so is an isomorphism. Thus $\Phi^k(C(\phi)) \simeq \mathbb{1}$ is invertible.

Finally it might be that ψ has dimension one. Then $Y_\phi^a \cong \mathbb{P}^1$ whereas MY_ϕ affords no Tate summands, and the result follows as in the case of dimension greater than one. This concludes the proof. □

Lemma 5.35. *Notation as in the theorem. If $\phi = \psi \perp \mathbb{H}$ then $C(\phi) \simeq C(\psi)\{1\}$.*

Proof. Using the explicit form for the inclusion $MY_\phi \to MY_\phi^a$ from Lemma 5.29 we find that

$$C(\phi) = [(\mathbb{1} \oplus \mathbb{1}\{d\} \oplus MY_\psi\{1\}) \oplus \mathbb{1}\{d+1\} \xrightarrow{\alpha} \dot{\mathbb{1}} \oplus \mathbb{1}\{d+1\} \oplus MY_\psi^a\{1\}],$$

where α is given by the matrix

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 \\ 0 & 0 & 0 & f \\ 0 & s\{1\} & i\{1\} & 0 \end{pmatrix}.$$

Here f comes from the fundamental class and so is an isomorphism. It follows that $C(\phi) \cong C(\psi)\{1\} \oplus \mathit{cone}(\mathrm{id}_{\mathbb{1}})[-1] \oplus \mathit{cone}(\mathrm{id}_{\mathbb{1}\{d+1\}})[-1] \simeq C(\psi)\{1\}$. This is the desired result. □

Lemma 5.36. *If $\phi \perp \langle a \rangle \cong \psi \perp \mathbb{H} \perp \mathbb{H}$, then ϕ is isotropic.*

Proof. Let $X = Y_{\langle a \rangle \perp \phi}$. Then $Y_\phi = X \cap \{X_0 = 0\}$. Since $\langle a \rangle \perp \phi \cong \psi \perp \mathbb{H} \perp \mathbb{H}$, we find that $Y_{\mathbb{H} \perp \mathbb{H}} \subset X$. Then $Y_\phi \cap Y_{\mathbb{H} \perp \mathbb{H}} = Y_{\mathbb{H} \perp \mathbb{H}} \cap \{X_0 = 0\}$ (intersecting inside X). Now we know that after a *linear* change of coordinates $(X_0 : \cdots : X_r) \mapsto (T_0 : \cdots : T_r)$ the subvariety $Y_{\mathbb{H} \perp \mathbb{H}}$ of X is given by the equations $T_0T_1 + T_2T_3 = 0$, $T_i = 0$ for $i > 3$. Thus $Y_\phi \cap Y_{\mathbb{H} \perp \mathbb{H}}$ is obtained by adding a further *linear* constraint in the T_0, T_1, T_2, T_3. It is easy to see that there must be a rational, non-zero solution, so Y_ϕ has a rational point. This was to be shown. □

From this we also deduce invertibility in **SH**.

Corollary 5.37. *Let S be either the spectrum of a perfect field of exponential characteristic $e > 2$ and finite 2-étale cohomological dimension, or a scheme of finite type over a field of characteristic zero and finite 2-étale cohomological dimension.*

Let (E, ϕ) be a vector bundle on S with a non-degenerate quadratic form ϕ, and $a \in \mathcal{O}(X)^\times$. The affine quadric bundle

$$Q := \{\phi = a\}$$

has invertible reduced suspension spectrum: $\Sigma^\infty \tilde{Q} \in Pic(\mathbf{SH}(S)_e)$.

Proof. Let $\mathbb{P}(E \oplus \mathcal{O}) \to S$ be the associated projective bundle. This contains the smooth (over S) closed subvariety $Y = \{\phi = aZ^2\}$, which itself has a smooth closed subvariety $Z = \{Z = 0\}$. Then $Q = Y \setminus Z$. By Lemma 4.26 (iv), $\Sigma^\infty \tilde{Q}$ is rigid.

If S is the spectrum of a perfect field Proposition 4.23 implies that it is enough to prove that $\tilde{M}Q$ is invertible, which is the above theorem.

If S is of finite type over a field of characteristic zero (of finite 2-étale cohomological dimension), then Corollary 4.25 shows that we may reduce to showing that the fibres of Q have invertible motives, which is again the above theorem. This concludes the proof. □

Remark 1. Using the extensions described in Corollary 4.27, one may prove invertibility of quadric bundles for any field of characteristic zero, not just those of finite 2-étale cohomological dimension.

Remark 2. The author does not know how to globalise the invertibility result in positive characteristic. The problem is that in positive characteristic, even very well-behaved varieties have residue fields that are not perfect.

5.3.3 Pfister Quadrics and the Conjectures of Po Hu

For $\underline{a} = (a_1, \ldots, a_n) \in (k^\times)^n$, $b \in k^\times$ let us put

$$U^b_{\underline{a}} = X^b_{\langle\langle a_1, \ldots, a_n \rangle\rangle},$$

where $\langle\langle a_1, \ldots, a_n \rangle\rangle$ is the n-fold Pfister quadric associated with the symbol $\underline{a}$. We use notation such as $\underline{a}, a' = (a_1, \ldots, a_n, a') \in (k^\times)^{n+1}$ for concatenation of tuples.

Theorem 5.38. *Let k be a perfect field of characteristic not two, and $\underline{a} \in (k^\times)^n, b \in k^\times$.*

In $\mathbf{DM}^{gm}(k)$ there is an isomorphism

$$\tilde{M}(U^1_{\underline{a},b}) \otimes \tilde{M}(U^b_{\underline{a}})[1] \cong \tilde{M}(U^1_{\underline{a}})\{2^n\}. \tag{5.1}$$

To prove this, we have to recall some facts about *Rost motives.* If $\underline{a} \in (k^\times)^n$, then there is the associated Rost motive $R_{\underline{a}} \in \mathbf{QM}(k)$. Recall that one has $H^1_{et}(k, \mathbb{F}_2) = k^\times/2$, and hence cup product yields a natural map $\partial = \partial^k : (k^\times)^n \to H^n_{et}(k, \mathbb{Z}/2)$. The Rost motives have the remarkable property that $R_{\underline{a}}$ is irreducible if and only if $\partial(\underline{a}) \neq 0$. In fact there are canonical maps

$$\mathbb{1}\{2^{n-1} - 1\} \to R_{\underline{a}} \to \mathbb{1} \tag{5.2}$$

(which we call structure maps) and if $\partial(\underline{a}) = 0$ then this is a splitting distinguished triangle. The same statements hold true with $\mathbb{F}_2$ coefficients. These results follow from the work of a number of people, see [78] for an overview.

The relationship between Rost motives and $U^b_{\underline{a}}$ is encapsulated in the following proposition.

Proposition 5.39. *For $\underline{a} \in (k^\times)^n, b \in k^\times$ there is a distinguished triangle*

$$\tilde{M}(U^b_{\underline{a}}) \to R_{\underline{a},b} \to R_{\underline{a}}\{2^{n-1}\} \oplus \mathbb{1}.$$

Here $R_{\underline{a},b} \to \mathbb{1}$ is the structure map, and the composite

$$\mathbb{1}\{2^n - 1\} \to R_{\underline{a},b} \to R_{\underline{a}}\{2^{n-1}\}$$

is the $\{2^{n-1}\}$ twist of the structure map $\mathbb{1}\{2^{n-1} - 1\} \to R_{\underline{a}}$.

Proof. This is essentially [54, proof of Proposition 5.5].

We know that $U := U^b_{\underline{a}}$ is the complement of $X := Y_{\langle\langle \underline{a} \rangle\rangle}$ in $Y := Y^b_{\langle\langle \underline{a} \rangle\rangle}$. By the work of Rost [97, Theorem 17 and Proposition 19], if we put $R_n := R_{\underline{a},b}$ and $R_{n-1} = R_{\underline{a}}$, then

$$M(Y) = R_n \oplus \bigoplus_{k=1}^{2^{n-1}-1} R_{n-1}\{k\} := R_n \oplus R', \quad M(X) = \bigoplus_{k=0}^{2^{n-1}-1} R_{n-1}\{k\} := R_{n-1} \oplus R'$$

and the natural map $M(X) \to M(Y)$ is the identity on R'.

The localisation triangle $M^c(X) = M(X) \to M^c(Y) = M(Y) \to M^c(U)$ fits into the following commutative diagram of (distinguished) triangles:

$$\begin{array}{ccccc} R' & = & R' & & \\ \downarrow & & \downarrow & & \\ M(X) & \longrightarrow & M(Y) & \longrightarrow & M^c(U) \\ \downarrow & & \downarrow & & \\ R_{n-1} & & R_n & & \end{array}$$

An application of the octahedral axiom yields a distinguished triangle $R_{n-1} \to R_n \to M^c(U)$. Noting that $DM^c(U) = M(U)\{-(2^n-1)\}, DR_n = R_n\{-(2^n-1)\}$ and $DR_{n-1} = R_{n-1}\{-(2^{n-1} - 1)\}$, by dualising and twisting the triangle, we find a distinguished triangle $M(U) \to R_n \to R_{n-1}\{2^{n-1}\}$. Adding in the copy of $\mathbb{1}$ implied in $\tilde{M}(U)$, we get the claimed triangle with the correct map $R_n \to \mathbb{1}$.

To see the second claim about the differential, the important point is that in the triangle $R_{n-1} \to R_n \to M^c(U)$ the map $R_{n-1} \to R_n$ is induced from the inclusion $M(X) \to M(Y)$ by passing to the appropriate summands. It follows that $R_{n-1} \to R_n \to \mathbb{1}$ is the structure map of $R_{n-1} \to \mathbb{1}$. The desired result now follows by dualising. ☐

Proof of Theorem 5.38. By Lemma 5.33, we have $\tilde{M}(U^b_{\underline{a}}) \in \mathbf{DQM}^{gm}(k)$, etc. We also know by Theorem 5.34 that both sides of equation (5.3) are invertible. Hence if $F : \mathbf{DQM}^{gm}(k) \to \mathcal{C}$ is a Pic-injective functor, it suffices to prove that $F(LHS) \cong F(RHS)$.

Of course we use the Pic-injective collection from Theorem 5.31.

From Proposition 5.39 we know that

$$t(\tilde{M}(U^b_{\underline{a}})) = [\dot{R}_{\underline{a},b} \to R_{\underline{a}}\{2^n\} \oplus \mathbb{1}],$$

and we also know certain things about the differential. To compute Ψ, we have to consider geometric base change, where the triangle (5.2) is splitting distinguished. One obtains

$$\Psi(\tilde{M}(U^b_{\underline{a}})) = [\dot{\mathbb{1}} \oplus \mathbb{1}\{2^n - 1\} \to \mathbb{1}\{2^{n-1}\} \oplus \mathbb{1}\{2^n - 1\} \oplus \mathbb{1}]$$

Table 5.1: Terms needed to compute Φ^l.

	$\partial^l(\underline{a}, b) \neq 0$	$\partial^l(\underline{a}, b) = 0$ but $\partial^l(\underline{a}) \neq 0$
$\Phi^l(U^1_{\underline{a},b})$	$[\dot{\mathbb{1}} \oplus \mathbb{1}\{2^{n+1}-1\} \to \mathbb{1}]$	$[\dot{\mathbb{1}} \oplus \mathbb{1}\{2^{n+1}-1\} \to \mathbb{1}\{2^n\} \oplus \mathbb{1}\{2^{n+1}-1\} \oplus \mathbb{1}]$
$\Phi^l(U^b_{\underline{a}})$	$[\dot{0} \to \mathbb{1}]$	$[\dot{\mathbb{1}} \oplus \mathbb{1}\{2^n-1\} \to \mathbb{1}]$
$\Phi^l(U^1_{\underline{a}})$	$[\dot{\mathbb{1}} \oplus \mathbb{1}\{2^n-1\} \to \mathbb{1}]$	$[\dot{\mathbb{1}} \oplus \mathbb{1}\{2^n-1\} \to \mathbb{1}]$

Table 5.2: Terms needed to compute Φ^l, simplified form.

	$\partial^l(\underline{a}, b) \neq 0$	$\partial^l(\underline{a}, b) = 0$ but $\partial^l(\underline{a}) \neq 0$
$\Phi^l(U^1_{\underline{a},b})$	$\mathbb{1}\{2^{n+1}-1\}$	$\mathbb{1}\{2^n\}[-1]$
$\Phi^l(U^b_{\underline{a}})$	$\mathbb{1}[-1]$	$\mathbb{1}\{2^n-1\}$
$\Phi^l(U^1_{\underline{a}})$	$\mathbb{1}\{2^n-1\}$	$\mathbb{1}\{2^n-1\}$

and from the information about the differential given in proposition 5.39 we deduce that $\Psi(\tilde{M}(U^b_{\underline{a}})) \simeq \mathbb{1}\{2^{n-1}\}[-1]$. Thus $\Psi(LHS) \cong \Psi(RHS)$ reads

$$\mathbb{1}\{2^n\}[-1] \otimes \mathbb{1}\{2^{n-1}\}[-1][1] \cong \mathbb{1}\{2^{n-1}\}[-1]\{2^n\},$$

which is certainly true.

Now let l/k be an arbitrary field extension. We need to prove $\Phi^l(LHS) \cong \Phi^l(RHS)$. This involves $R_{\underline{a}}, R_{\underline{a},b}, R_{\underline{a},1}$ and $R_{\underline{a},b,1}$. Depending on l these may or may not split into Tate motives, so may or may not survive Φ. We see that $R_{\underline{a},1}$ and $R_{\underline{a},b,1}$ always split (because $\partial^l(1) = 0$), and that $R_{\underline{a},b}$ splits whenever $R_{\underline{a}}$ splits (because $\partial(\underline{a}, b) = \partial(\underline{a}) \cup \partial(b)$).

If $R_{\underline{a}}$ splits then everything is split and Φ^l is just mod two reduction of Ψ, so we know the equation is satisfied. Thus there are just two cases and three things in each to compute, which we gather in Table 5.1.

The differentials can again be figured out using Proposition 5.39. Using these one can simplify the expressions. We have gathered the results in Table 5.2.

To complete the proof, we check that $\Phi^l(LHS) \cong \Phi^l(RHS)$ in both cases. This is easy. □

Corollary 5.40. *Let k be a perfect field of characteristic not two, and $\underline{a} \in (k^\times)^n, b \in k^\times$.*

In $\mathbf{SH}(k)_e$ there is an isomorphism

$$\Sigma^\infty \widetilde{U^1_{\underline{a},b}} \otimes \Sigma^\infty \widetilde{U^b_{\underline{a}}}[1] \cong \Sigma^\infty \widetilde{U^1_{\underline{a}}}\{2^n\}. \tag{5.3}$$

Proof. By Corollary 5.37 both sides of the equation are invertible. The result thus follows from Theorem 5.38 and Pic-injectivity of $M : \mathbf{SH}(k)_e \to \mathbf{DM}(k, \mathbb{Z}[1/e])$, i.e. Theorem 4.20. □

5.4 Artin Motives

In this section we study Artin- and Artin-Tate motives. In subsection 5.4.1 we introduce the category $\mathbf{DAM}^{gm}(k, A)$ of (geometric, i.e. compact) derived Artin motives and explain some basics. In Subsection 5.4.2 we construct geometric fixed point functors for $\mathbf{DAM}^{gm}(k, A)$ and prove that in reasonable cases, they yield a conservative and Pic-injective collection.

One obtains from the fixed point functors a homomorphism $\phi : Pic(\mathbf{DAM}^{gm}(k, A)) \to C(k, A)$ where $C(k, A)$ is a product of copies of $\mathbb{Z}$. That is to say ϕ associates with every invertible Artin motive a collection of numerical invariants. In Subsection 5.4.3 we show that these invariants cannot take on arbitrary values, but instead satisfy certain congruence conditions called Borel-Smith conditions. That subsection is probably the messiest part of the entire thesis. This can be explained as follows: when the author took up this project, Artin motives were a very accessible first target of study, and he did so using all methods available, which were often messy. During the course of the project the methods were generalised and abstracted considerably, and are now hidden behind weight structures calculations and descent spectral sequences. However, the proof of the Borel-Smith conditions does not seem to fit into our more general framework, and so still employs the original messy computations.

In Subsection 5.4.4 we explain the relationship between derived Artin motives and stable homotopy theory equivariant with respect to the Galois group. This makes good use of the Borel-Smith conditions we so painfully established before. Finally in subsection 5.4.5 we study Artin-Tate motives.

5.4.1 The Category of Derived Artin Motives

For a field k and a coefficient ring A, write $\mathbf{AM}(k, A)$ for the Karoubi-closed, additive subcategory of $Chow(k, A)$ spanned by the motives of spectra of finite separable extensions of k, and $\mathbf{DAM}^{gm}(k, A)$ for the thick triangulated subcategory of $\mathbf{DM}(k, A)$ generated by $\mathbf{AM}(k, A)$. The Chow weight structure on $\mathbf{DM}^{gm}(k, A)$ restricts to $\mathbf{DAM}^{gm}(k, A)$ by Lemma 5.9. Thus we obtain the conservative and Pic-injective weight complex functor $t : \mathbf{DAM}^{gm}(k, A) \to K^b(\mathbf{AM}(k, A))$. This functor is in fact an equivalence [107, Proposition 3.4.1]. In the case that k is imperfect of characteristic p not invertible on A, we use this equivalence to *define* $\mathbf{DAM}^{gm}(k, A)$.

The category $\mathbf{AM}(k, A)$ is symmetric monoidal, and hence so is $\mathbf{DAM}^{gm}(k, A)$.

If l/k is a Galois extension write $\mathbf{AM}(l/k, A)$ for the full subcategory of $\mathbf{AM}(k, A)$ spanned by (spectra of) subextensions of l/k, and $\mathbf{DAM}^{gm}(l/k, A)$ for the thick triangulated category of $\mathbf{DM}^{gm}(k, A)$. Again $\mathbf{AM}(l/k, A)$ is symmetric monoidal, so is $\mathbf{DAM}^{gm}(l/k, A)$, and the weight complex functor is an equivalence.

The category $\mathbf{AM}(l/k, A)$ has a further convenient description. Namely, it is equivalent to the Karoubi-closed subcategory $Perm(l/k, A)$ of the category of $A[Gal(l/k)]$-**Mod** spanned by modules $A[Gal(l/l')]$ for $l/l'/k$ a subextension. This is also known as the category of permutation representations. The identification comes from Galois theory and is explained in [107, paragraphs before Proposition 3.4.1].

We can illustrate now very directly Theorem 5.1. This also proves surjectivity of the map alluded to in the paragraph after the proof of that theorem.

Proposition 5.41. *Let l/k be a finite Galois extension and A a coefficient ring on which $[l : k]$ is invertible. If $\rho : Gal(l/k) \to A^\times$ is a character, then the corresponding representation $A_\rho \in A[Gal(l/k)]$-**Mod** is in $Perm(l/k, A)$ and hence defines an invertible element of* $\mathbf{AM}(l/k, A)$.

The proof will show that the invertible element is a summand of $M(l)$; this is not surprising since for every subextension $l/l'/k$ the motive $M(l')$ is a summand of $M(l)$ (the composite $M(l') \to M(l) \to M(l')$ of base change and transfer is multiplication by $[l : l']|d$ and so is invertible).

Proof. The trivial Galois module A corresponds to $M(k)$ and hence is a summand of the free Galois module $A[Gal(l/k)$, by what we just said. Consequently A is a projective $A[Gal(l/k)]$-module and hence so is A_ρ, being invertible (in $A[Gal(l/k)]$-**Mod**). Since there is an evident surjection $A[Gal(l/k)] \to A_\rho$ and A_ρ is projective, we find that A_ρ is a summand of $A[Gal(l/k)]$ and thus an element of $Perm(l/k, A)$. □

We also have the following straightforward observation.

Lemma 5.42. *Let A a Noetherian ring and $E, F \in \mathbf{DAM}^{gm}(k, A)$. Then $[E, F]$ is a finitely generated A-module.*

Proof. Since $\mathbf{DAM}^{gm}(k, A)$ is rigid it suffices to prove the claim for $E = \mathbb{1}$. The result is clear for $F = Spec(l)$ and hence for $F \in \mathbf{AM}(k, A)$. Since also $\mathbf{DAM}^{gm}(k, A) \cong K^b(\mathbf{AM}(k, A))$ the claim holds for $F \in \mathbf{AM}(k, A)[i]$, $i \in \mathbb{Z}$. Now let $\mathcal{C} \subset \mathbf{DAM}^{gm}(k, A)$ be the subclass of objects $X \in \mathbf{DAM}^{gm}(k, A)$ such that $[\mathbb{1}, X[i]]$ is finitely generated for all i. Then $\mathbf{AM}(k, A) \subset \mathcal{C}$ as we have just seen. Also $\mathcal{C}$ is closed under isomorphisms, finite sums, summands, and taking cones (the latter because A is Noetherian), so $\mathcal{C} = \mathbf{DAM}^{gm}(k, A)$. □

5.4.2 Geometric Fixed Point Functors

A field k is called p-special if every finite separable extension l/k is of degree a power of p. For every field k and prime p there exists an extension k_p/k such that k_p is p-special and every finite subextension $k_p/l/k$ has degree $[l:k]$ coprime to p. This follows from infinite Galois theory and is proved for example in [28, Proposition 101.16].

Lemma 5.43. *For any field k, let S_k be the set of finite disjoint unions of spectra of finite separable* proper *extensions of k.*

If k is a p-special field, then Theorem 5.25 applies to S_k, $\mathbb{F}$ any finite field of characteristic p, and $ex(l) = [l:k]$.

Proof. Condition (i) holds because k is p-special, (ii) and (iii) are clear. (The theorem only applies of $char(k) \neq p$, but it is easy to see from the proof that this does not matter in our case.) □

Note that the category $\mathbf{D}\langle S\rangle\mathbf{TM}^{gm}(k,\mathbb{F})$ is *not* the same as $\mathbf{DAM}^{gm}(k,\mathbb{F})$; it also contains the Tate motives $\mathbb{1}\{i\}$ for $i \neq 0$. This does not matter much. We get fixed points functors $\mathbf{DAM}^{gm}(k,\mathbb{F}) \to \mathbf{D}\langle S\rangle\mathbf{TM}(k,\mathbb{F}) \xrightarrow{\Phi^l} K^b(Tate(\mathbb{F}))$, and the image of the composite is essentially the subcategory $\mathbf{DAM}(k/k,\mathbb{F}) = K^b(\mathbb{F}\text{-}\mathbf{Mod}^f) \subset K^b(Tate(\mathbb{F}))$. We then write $\Phi^l : \mathbf{DAM}(k,\mathbb{F}) \to K^b(\mathbb{F}\text{-}\mathbf{Mod}^f)$ for the functor with domain and codomain restricted. (Here $\mathbb{F}\text{-}\mathbf{Mod}^f$ denotes the category of finite-dimensional $\mathbb{F}$-vector spaces.)

Now let k be any field and A a PID with finite residue fields. For every prime $\pi \in A$, choose a $char(A/\pi)$-special extension k_π/k. Then for every finite extension l/k_π we obtain from the lemma (i.e. Theorem 5.25) and the discussion of the previous paragraph a "geometric fixed points" functor $\Phi^l_\pi : \mathbf{DAM}(k_p, A/\pi) \to K^b(A/\pi)$. We also write $\Phi^l_\pi : \mathbf{DAM}(k,A) \to K^b(A/\pi)$ for the evident composite.

Proposition 5.44. *Let A be a PID with finite residue fields of unbounded characteristics. The collection $\{\Phi^l_\pi : \mathbf{DAM}(k,A) \to K^b(A/\pi)\}_{l,\pi}$ is conservative.*

Proof. Let $f : Spec(k^s) \to Spec(k)$ be a separable closure. By Theorem 5.6 part (iii), the collection $\{\alpha^\pi_\#\}_\pi \cup \{f^*\}$ is conservative. For each π, the base change $\mathbf{DAM}(k, A/\pi) \to \mathbf{DAM}(k_\pi, A/\pi)$ is conservative by Theorem 5.1, and the $\Phi^l_\pi : \mathbf{DAM}(k_\pi, A/\pi) \to K^b(A/\pi)$ form a conservative collection by the Lemma. Since change of coefficients and base change commute, it follows that it suffices to prove that for $k = k^s$, the change of coefficient functors $\alpha^\pi_\# : \mathbf{DAM}(k,A) \to \mathbf{DAM}(k, A/\pi)$ are conservative.

But $\mathbf{DAM}(k,A) \simeq D^b(A)$ for k separably closed, and so by using Corollary 5.15 it is enough to show that if M is a finitely generated A-module such that $M/\pi = 0$ for all primes π then $M = 0$. This is well known. □

Write $\phi^l_\pi : Pic(\mathbf{DAM}(k,A)) \to Pic(K^b(A/\pi)) = \mathbb{Z}$ for the evident homomorphism (built from change of coefficients, base change, and geometric fixed points). Let $C(k,A)$ denote the abelian group

$$C(k,A) = \prod_{\pi, l/k_\pi} \mathbb{Z},$$

where the product is over all primes $\pi \in A$ (up to associates) and all finite separable extensions l/k_π. We obtain a combined homomorphism $\phi : Pic(\mathbf{DAM}(k,A)) \to C(k,A)$.

Proposition 5.45. *If $E \in Pic(\mathbf{DAM}(k,A))$ and $\phi(E) = 0$ then there is a finite separable extension l/k such that $E|_l = \mathbb{1}$.*

There is an injection

$$ker\,[Pic(\mathbf{DAM}(k,A)) \to C(k,A)] \to \mathrm{Hom}_{cts}(Gal(k), A^\times).$$

Proof. Note that $Pic(\mathbf{DAM}(k^s, A)) = \mathbb{Z} = Pic(\mathbf{DAM}(k^s, A/\pi))$. From this we conclude that $E|_{k^s} = \mathbb{1}$ and hence the same is true after a finite extension by compactness of E and continuity of **DM**.

We conclude from Theorem 5.1 that there is an injection

$$ker(Pic(\mathbf{DAM}(k, A/\pi)) \to Pic(\mathbf{DAM}(k_\pi, A/\pi)) \to \mathrm{Hom}_{cts}(Gal(k), (A/\pi)^\times),$$

and hence by combining it with the first statement, part (iii) of Theorem 5.6 and the Pic-injectivity of fixed point functors (i.e. Theorem 5.25) we find an injection

$$ker(Pic(\mathbf{DAM}(k, A)) \to C(k)) \to \prod_\pi \mathrm{Hom}_{cts}(Gal(k), A^\times).$$

The proof of Theorem 5.1 shows that there is a homomorphism $ker(Pic(\mathbf{DAM}(k, A)) \to Pic(\mathbf{DAM}(k^s, A))) \to \mathrm{Hom}_{cts}(Gal(k), A^\times)$ (basically $E \in ker(\dots)$ is a complex of Galois modules which has homology A concentrated in degree zero, and the corresponding element of the set $\mathrm{Hom}_{cts}(Gal(k), A^\times)$ specifies the action on A) such that for every π, the following diagram commutes:

$$\begin{array}{ccc} ker(Pic(\mathbf{DAM}(k, A)) \to C(k, A)) & \longrightarrow & \mathrm{Hom}_{cts}(Gal(k), A^\times)) \\ \downarrow & & \downarrow \\ ker(Pic(\mathbf{DAM}(k, A/\pi)) \to Pic(\mathbf{DAM}(k_\pi, A/\pi))) & \longrightarrow & \mathrm{Hom}_{cts}(Gal(k), (A/\pi)^\times). \end{array}$$

Here the right vertical homomorphism is reduction modulo π. The result follows. □

Corollary 5.46. *Let A be a localisation of $\mathbb{Z}$ on which* 2 *is* not *invertible. The homomorphism*

$$Pic(\mathbf{DAM}^{gm}(k, A)) \to C(k, A)$$

is injective.

In particular, in this situation $Pic(\mathbf{DAM}^{gm}(k, A))$ is torsion-free.

Proof. The "in particular" part follows because $C(k, A)$ is torsion-free. So we prove the first part.

Let $E \in ker(Pic(\mathbf{DAM}^{gm}(k, A)) \to C(k, A))$ be classified by $\rho : Gal(k) \to A^\times$. If ρ is trivial we are done. Otherwise, since the only elements of $A^\times$ of finite order are $\{\pm 1\}$, ρ determines a surjection $Gal(k) \to \{\pm 1\}$ and so an index two subgroup of $Gal(k)$, i.e. a quadratic extension q/k. Write $\tilde{q}$ for the invertible motive in $\mathbf{AM}(k, A[1/2])$ corresponding to ρ via Proposition 5.41.

Write $\beta : A \to A[1/2]$ for the flat localisation. It follows from Proposition 5.45 that $\beta_\#(E) \simeq \tilde{q}$. We have $0 = [\mathbb{1}, \tilde{q}] = [\mathbb{1}, \tilde{q}[1]]$ and hence $[\mathbb{1}, E]$ and $[\mathbb{1}, E[1]]$ are two-primary torsion groups, by Proposition 5.4. They are also finitely generated A-modules by Lemma 5.42 and thus in fact finite, by the classification of finitely generated modules over a PID.

I claim that $\alpha^2_\#(E) \simeq \mathbb{1}$. Indeed we know that this is true after base change to k_2/k by Lemma 5.43. But this is an infinite Galois extension of degree coprime to 2, so $\alpha^2_\#(E)$ is classified by an element of $\mathrm{Hom}_{cts}(Gal(k_2/k), \mathbb{F}_2^\times) = 0$.

Now we use the Bockstein sequence.

$$0 = [\mathbb{1}, \alpha^2_\#(E)[-1]] \to [\mathbb{1}, E] \xrightarrow{2} [\mathbb{1}, E] \to [\mathbb{1}, \alpha^2_\#(E)] \to [\mathbb{1}, E[1]] \xrightarrow{2} [\mathbb{1}, E[1]] \to [\mathbb{1}, \alpha^2_\# E[1]] = 0.$$

We conclude that multiplication by 2 is injective on the (finite) 2-primary torsion group $[\mathbb{1}, E]$ which is thus zero, and multiplication by 2 is surjective on the finite 2-primary torsion group $[\mathbb{1}, E[1]]$, which is thus also zero! Thus $A/2 \cong [\mathbb{1}, \alpha^2_\#(E)] = 0$ and 2 is invertible on A. This establishes the contrapositive. □

Remark. One may show that the image of $\phi : Pic(\mathbf{DAM}^{gm}(k, A)) \to C(k, A)$ is free abelian for reasonable A, but this takes some further effort. In particular $Pic(\mathbf{DAM}^{gm}(k, \mathbb{Z}))$ is a free abelian group.

Example 1: quadratic extensions. Let l/k be a quadratic (separable) extension and $\tilde{l}$ fit in the distinguished triangle $\tilde{l} \to M(l) \to M(k)$. If $1/2 \in A$ then $M(l) \to M(k)$ has a section yielding a splitting $M(l) \simeq M(k) \oplus \tilde{l}$. This proves that $\tilde{l}$ then coincides with $\tilde{q}$ from the proof of the corollary, i.e. is an invertible object.

Now suppose that k is 2-special and let l'/k be a finite extension. Either l embeds into l' in which case $\tilde{l}|_{l'} \simeq \mathbb{1}$ and so $\Phi^{l'}(\tilde{l}) = \mathbb{1}$, or l does not embed into l' in which case $\Phi^{l'}(\tilde{l})$ is the complex $[\Phi^{l'}(M(l)) \to \Phi^{l'}(M(k))]$. The Chow motive $M(l)$ is Tate-free since it is indecomposable, hence $\Phi^{l'}(M(l)) = 0$ and so $\Phi^{l'}(\tilde{l}) = \mathbb{1}[-1]$. This is invertible for any l' and so we find that $\tilde{l}$ is invertible by Proposition 5.44.

Combining the two observations we find that $\tilde{l}$ is always invertible. If $1/2 \in A$ then $\tilde{l}^{\otimes 2} \simeq \mathbb{1}$ (because $\phi(\tilde{l}) = 0$). However if 2 is not invertible on A then no power of $\tilde{l}$ is trivial, even though $\tilde{l}$ certainly becomes trivial after geometric base change.

Example 2: cyclic extensions. Let l/k be a cyclic Galois extension of degree p and let $\sigma : l \to l$ generate the Galois group. Consider the complex

$$C(l) = [M(l) \xrightarrow{1-\sigma} M(l) \to M(k)] \in K^b(\mathbf{AM}(k, A)) \simeq \mathbf{DAM}^{gm}(k, A).$$

If l'/k is an extension splitting l then one checks easily that $C(l)_{l'} = \mathbb{1}$. It follows that $\Phi_q^{l'}(C(l)) = \mathbb{1}$ for any $q \neq p$ and that $\Phi_p^{l'}(C(l))$ is either $\mathbb{1}$ or $\mathbb{1}[-2]$, depending on whether or not l' splits l. (See also Theorem 5.50, part (i).) In particular, $C(l)$ is always an invertible object. If $1/p \in A$ then actually $C(l) \simeq \mathbb{1}$ because one easily checks that $C(l)$ is classified by the trivial homomorphism $Gal(l/k) \to A^{\times}$. On the other hand if $1/p \notin A$ then $C(l)$ has infinite order in $Pic(\mathbf{DAM}^{gm}(k, A))$.

5.4.3 Borel-Smith Conditions

We now wish to establish some numerical conditions on invertible Artin motives. Unfortunately doing so requires us to get down to the level of complexes, and leave the world cushioned by weight structures and triangulated categories. We begin with the following observation.

Lemma 5.47. *Let k be a field and l/k an extension.*

(i) Let l_1/k be a separable extension which does not embed into l, and let l_2/k be any separable extension. Then in the decomposition

$$l_1 \otimes_k l_2 = F_1 \times \cdots \times F_n$$

where the F_i are fields, and none of the F_i embed into l.

(ii) Let L/l be a p-extension (p some prime), and assume that $L/k, l/k$ are Galois extensions. Let F be a ring of characteristic p. If $l_1, l_2 \subset l$ and $l_3 \subset L$ does not embed into l, then any composite in $\mathbf{AM}(k, F)$ of the form

$$M(l_1) \to M(l_3) \to M(l_2)$$

is zero.

Proof. Using continuity, we may assume that all extensions are finite.

(i). Since l_1/k is separable it is of the form $l_1 \cong k[T]/P$ for some irreducible polynomial P. Since l_1 does not embed into l, the polynomial P has no roots in l. Now $l_2 \otimes l_1 \cong l_2[T]/P$. The image of T in F_i is a root of P, so F_i cannot embed into l.

(ii). Since $\mathbf{AM}(L/k, F) \subset F[Gal(L/k)]$**-Mod** and base change corresponds to restriction of the Galois action to a subgroup, base change of Artin motives is faithful. We base change to l. We find that $l_1 \otimes l, l_2 \otimes l$ split into products of l (since l/k is Galois), whereas by (i) $l_3 \otimes l$ splits into extensions not embedding into l. It follows that we may assume that $l_1 = l_2 = l = k$. In this case $\mathrm{Hom}(M(l), M(l_3))$ is generated as an F-module by the transfer, and $\mathrm{Hom}(M(l_3), M(l))$

is generated by the canonical (structure) map. But the composite of transfer and structure map is multiplication by $[l_3 : l]$ which is a power of p, since F/l is a p-extension. This concludes the proof. □

Corollary 5.48. *Let k be a field, p a prime, F a coefficient field of characteristic p. Let $L/l/k$ extensions with $L/k, l/k$ Galois and L/l a p-extension.*

There is an essentially unique additive functor

$$\Phi^{l/k} : \mathbf{AM}(L/k, F) \to \mathbf{AM}(l/k, F)$$

with the property that $\Phi^{l/k}(M(l')) = M(l')$ if l' embeds into l (more precisely, $\Phi^{l/k}|_{\mathbf{AM}(l/k,F)} \simeq \mathrm{id}$), and $\Phi^{l/k}(M(l')) = 0$ else. It is symmetric monoidal.

Proof. The functor $\Phi^{l/k}$ is essentially unique if it exists, because the motives $M(l')$ generate $\mathbf{AM}(L/k, F)$ as a Karoubi-closed category.

We shall prove existence. Recall the basic results about weight structures, i.e. Proposition 5.8.

We can proceed essentially as in the proof of Proposition 5.24. Namely, the triangulated category $\mathbf{DAM}^{gm}(L/k, F)$ affords a weight structure with heart $\mathbf{AM}(L/k, F)$. Let $\mathcal{C} \subset \mathbf{AM}(L/k, F)$ be spanned by $M(l')$ for l' not embedding into l. Write $\langle\mathcal{C}\rangle \subset \mathbf{DAM}^{gm}(L/k, F)$ for the thick triangulated subcategory generated by $\mathcal{C}$. Then the weight structure restricts to $\langle\mathcal{C}\rangle$ and we may form the Verdier quotient $\mathcal{D} := \mathbf{DAM}^{gm}(L/k, F)/\langle\mathcal{C}\rangle$. This inherits a weight structure with heart $\mathcal{D}^{w=0}$. It follows from part (i) of the above Lemma and Proposition 5.8 part (5) that $\mathcal{D}^{w=0}$ is equivalent to $\mathbf{AM}(l/k, F)$, via the natural functor

$$\mathbf{AM}(l/k, F) \to \mathbf{DAM}^{gm}(L/k, F)^{w=0} \to \mathcal{D}^{w=0}.$$

Denote the functor $\mathbf{AM}(L/k, F) \to \mathbf{DAM}^{gm}(L/k, F)^{w=0} \to \mathcal{D}^{w=0} \simeq \mathbf{AM}(l/k, F)$ by $\Phi^{l/k}$. Then $\Phi^{l/k}$ has the stated properties (note that it is symmetric monoidal by part (ii) of the Lemma). □

We can thus define $\Phi^{l/k} : C^b(\mathbf{AM}(L/k, F)) \to C^b(\mathbf{AM}(l/k), F)$. This induces

$$\Phi^{l/k} : K^b(\mathbf{AM}(L/k, F)) \simeq \mathbf{DAM}^{gm}(L/k, F) \to K^b(\mathbf{AM}(l/k, F)) \simeq \mathbf{DAM}^{gm}(l/k, F)$$

coinciding with the construction in the proof. In particular if k is p-special then $\Phi^{k/k}$ coincides with Φ^k from the previous section, and so the collection of the $\Phi^{k/k}$ is conservative and Pic-injective.

Corollary 5.49. *Let L/k be a Galois p-extension and F of characteristic p. Given $L/l_0/k$ and $L/l/k$ we have $\Phi^{l_0}(M(l)) = 0$ unless l embeds into l_0.*

Proof. The functor Φ^{l_0} is computed by base change to l_0 and then applying Φ^{l_0/l_0}. So we need only show that the base change of $M(l)$ to l_0 has no summand $M(l_0)$. Equivalently, $l \otimes l_0$ has no factor l_0. But since l does not embed into l_0 so does no factor of $l \otimes l_0$, by Lemma 5.47 part (i). This concludes the proof. □

Suppose now that $E \in \mathbf{DAM}^{gm}(L/k, F)$ is invertible. Then $\Phi^{l/l}(E) \in \mathbf{DAM}^{gm}(l/l, F)$ is invertible, so of the form $\mathbb{1}[\phi(E)(l)]$ for some $\phi(E)(l) \in \mathbb{Z}$. Of course, if $L/l'/k$ is a conjugate intermediate extension then $\phi(E)(l') = \phi(E)(l)$.

More is true. The object $\Phi^{l/k}(E) \in \mathbf{DAM}^{gm}(l/k, F)$ is also invertible. Using the identification $\mathbf{DAM}^{gm}(l/k, F) \simeq K^b(Perm(l/k, F))$ we can view $\Phi^{l/k}(E)$ as a complex of $F[Gal(l/k)]$-modules. In order for this to be invertible, its homology (as $Gal(l/k)$-modules) must be one-dimensional. This way $\Phi^{l/k}(E)$ defines a homomorphism $Gal(l/k) \to F^\times$. (This can only be non-trivial if l/k is not a p-extension!) We denote the composite

$$Gal(L/k) \to Gal(l/k) \to F^\times$$

by $\rho^{l/k}(E)$. The aim of this subsection is to prove the following result. What this does is constrain the image of $\phi(Pic(\mathbf{DAM}^{gm}(k, F)))$ inside $C(k, F)$.

Theorem 5.50. *Let F be a field of characteristic p, L/k a Galois extension, and let $X \in \mathbf{DAM}^{gm}(L/k,F)$ be invertible. Let p be a rational prime. Then the following conditions hold:*

(i) If given $L/l/k'/k$ with l/k' Galois of degree $p \neq 2$, and L/k' a p-extension, then $\phi(X)(l) \equiv \phi(X)(k') \mod 2$.

(ii) Suppose given $L/l/k'/k''/k$ with l/k'' Galois, k'/k'' Galois, l/k' of degree 2 and L/k'' a 2-extension. (a) If l/k'' is cyclic of degree four, then $\phi(X)(l) \equiv \phi(X)(k') \mod 2$. (b) If l/k'' is a quaternion extension (of degree eight), then $\phi(X)(l) \equiv \phi(X)(k') \mod 4$.

(iii) Suppose given $L/l/k'/k$ with l/k' Galois with group $\mathbb{Z}/p \times \mathbb{Z}/p$ and L/k' a p-extension. Let $l_0, \ldots, l_p$ be the proper non-trivial subfields of l/k'. Then $\phi(X)(l) - \phi(X)(k') = \sum_i (\phi(X)(l_i) - \phi(X)(k'))$.

(iv) Let $L/l/k'/k''/k$ be extensions, such that L/l is a p-extension, l/k'' and k'/k'' are Galois extension, and with $Gal(l/k') = \mathbb{Z}/p$, $Gal(k'/k'') = \mathbb{Z}/q^r$. Then $\mathbb{Z}/q^r$ acts naturally on $\mathbb{Z}/p$ with kernel of some size q^l.

Assume that $\rho^{l/k''}(X) = \rho^{k'/k''}(X)$. Then $\phi(X)(l) \equiv \phi(X)(k') \mod 2q^{r-l}$, provided $2 \neq p \neq q$.

We will prove these results using explicit manipulations in the chain homotopy category of bounded complexes. Inspiration for the proofs comes from [39]. The basis for all our manipulations is the following result which the author learned from Jake Rasmussen.

Lemma 5.51 (Cancellation)**.** *Let $\mathcal{C}$ be an additive category and $X^\bullet$ a cochain complex. Suppose we are given $N \in \mathbb{Z}$ and decompositions $X^N \cong X'^N \oplus A$, $X^{N+1} \cong X'^{N+1} \oplus B$ such that the component of $d^N : X^N \to X^{N+1}$ from A to B is an isomorphism. Then $X^\bullet$ is chain homotopy equivalent to a new complex $X'^\bullet$ with $X'^n = X^n$ for $n \neq N, N+1$ and $d'^n = d^n$ for $n \neq N-1, N, N+1$.*

The proof is an easy but slightly elaborate computation. We defer it to the end of this subsection.

In order to use this effectively, we need to better understand Φ^l and $\mathbf{AM}(L/k,F)$. Since $\mathbf{AM}(L/k,F)$ is equivalent to a Karoubi-closed subcategory of $F[Gal(L/k)]$**-Mod** and $F[Gal(L/k)]$ is a finite-dimensional algebra over the field F, every object in $\mathbf{AM}(L/k,F)$ can be written as a sum of indecomposable objects, in an essentially unique way. This is known as the Krull-Schmidt theorem [4, Theorem I.4.10]

We now concentrate on proving points (i) to (iii) of the theorem. We may thus assume that F has characteristic p and that L/k is a p-extension. Then each of the objects $M(l)$ for $L/l/k$ is indecomposable. Indeed it corresponds to the $F[Gal(L/k)]$-module $F[Gal(L/k)/H]$ for some (not necessarily normal) subgroup H. But being a quotient of $F[Gal(L/k)]$ it has a simple head (this is well known; see e.g. [68, Theorem 2.2] for a proof), so this module is indecomposable. In particular, every object in $\mathbf{AM}(L/k,F)$ is a sum of objects of the form $M(l)$ for $L/l/k$ (still provided that L/k is a p-extension and F has characteristic p).

Lemma 5.52 (Jeremy Rickard)**.** *Let L/k be a Galois p-extension and $L/l/k$ a subextension, F of characteristic p.*

(i) If $\alpha : M \to M(l)^n$ is such that $\Phi^l(\alpha)$ is surjective, then α is surjective (as a map of G-modules).

(ii) If $\alpha : M(l)^m \to M(l)^n$ corresponds to a surjective homomorphism of G-modules, then α splits.

Proof. Let $G = Gal(L/k)$, H the subgroup corresponding to l. Then $M(l)$ corresponds to the G-module $F[G/H]$. The functor Φ^l corresponds to restricting to the subgroup H and then looking at fixed points of the G-set, i.e. $\Phi^l(M(l))$ corresponds to $F[(G/H)^H]$ (with the trivial action). The object M corresponds to $F[X]$ for some finite G-set X, by Krull-Schmidt and indecomposability as earlier.

We show first that as a map of G-modules, α is surjective. Indeed the element $eH \in G/H$ is stabilised by H. We know that $\Phi^l(\alpha) : F[X^H] \to F[(G/H)^H]^n$ is surjective. There are thus elements $x_1, \dots, x_n \in F[X^H] \subset F[X]$ such that $\alpha(x_i) = (0, \dots, 0, eH, 0, \dots) =: y_i$, with eH in place i. But the element eH generates $F[G/H]$ as a G-module, so the y_i generate $F[G/H]^n$ as a G-module, and α is surjective. This proves the claim.

The next part of the proof was explained to the author by Jeremy Rickard [91]. We consider $M = M(l)^m$, i.e. $X = (G/H)^m$. We need to show that α splits. Let $A = End(M(l))$. This is a local ring since $M(l)$ is indecomposable [4, Corollary I.4.8(b)]. Put $P_1 = \mathrm{Hom}(M(l), M(l)^m), P_2 = \mathrm{Hom}(M(l), M(l)^n)$. These are free A-modules. The induced homomorphism $\alpha' : P_1 \to P_2$ is surjective: If not there is a non-zero map $\beta' : P_2 \to A$ such that $\beta'\alpha' = 0$ (the cokernel of α' is non-zero and thus admits a non-zero map to the socle of A) and then this corresponds to a non-zero map $\beta : M(l)^m \to M(l)$ such that $\beta\alpha = 0$. Thus α was not surjective.

But now $\alpha' : P_1 \to P_2$ must split and consequently so does α. □

Lemma 5.53. *Let L/k be a Galois p-extension, F a field of characteristic p. Suppose given $X \in \mathbf{DAM}^{gm}(L/k, F) \simeq K^b(\mathbf{AM}(L/k, F))$ and a subextension $L/l_0/k$, and assume that for all $l_0/l/k$ we have $\Phi^l(X) \simeq F[0]$.*

Then X can be represented by a finite complex of sums of objects of the form $M(l)$, where $L/l/k$ and l does not embed into l_0, with the exception of a single term $M(k)$ in X^0.

If we are given a particular complex representing X, then no new fields are introduced into the complex by this process.

Proof. Fix any representation of X as a finite complex $X^\bullet$. By the Krull-Schmidt theorem and indecomposability of the $M(l)$, this is (isomorphic to) a finite complex of sums of objects of the form $M(l)$ for various l.

We first show how to eliminate all but one copy of $M(k)$. To do so, let n be maximal such that $M(k)$ occurs in X^n. Suppose $n > 0$. Since $\Phi^k(X) \simeq F[0]$ we find that $\Phi^k(X)^{n-1} \to \Phi^k(X)^n$ must split. Consequently if $X^{n-1} = M(k)^m \oplus X'^{n-1}$ and $X^n = M(k)^n \oplus X'^n$ (with no $M(k)$ occurring in X'^{n-1} or X'^n), then $M(k)^m \to M(k)^n$ splits, and so by the Cancellation Lemma 5.51 we may pass to a new chain homotopy equivalent complex X' with $X'^k = X^k$ for $k \neq n, n-1$. Thus we find a representation as a finite complex with no $M(k)$ in degrees > 0. Applying the same argument to DX we find a representation with no $M(k)$ in degree < 0, and hence $\Phi^k(X) = \Phi^k(X^0)$. This implies there must be a unique copy of $M(k)$ in degree zero.

Next we show how to eliminate subextensions embedding into l_0. Let $L/l_0/l/k$ be a minimal (over k) subextension not yet eliminated. As before we find n maximal such that X^n contains $M(l)$. Suppose that $n > 1$. Write $X^n = M(l)^n \oplus X'^n$, $X^{n-1} = M(l)^m \oplus X'^{n-1}$, where X'^n, X'^{n-1} contain no copies of l. Then by minimality and since $n - 1 > 0$ we have that X' contains no copies of $M(l')$ for l' embedding into l, and so $\Phi^l(X'^n) = 0 = \Phi^l(X'^{n-1})$ by Corollary 5.49. Since $\Phi^l(X) \simeq F[0]$ by assumption, $\Phi^l(M(l)^n) \to \Phi^l(M(l)^m)$ must split and hence $M(l)^m \to M(l)^n$ splits by Lemma 5.52. Thus appealing to the Cancellation Lemma we may eliminate $M(l)^m$.

Suppose now that we have eliminated $M(l)$ from X^n for $n > 1$. There is a problem with applying the same strategy to $n = 1$. We can write $X^0 = M(k) \oplus M(l)^m \oplus X'^0$ and $X^1 = M(l)^n \oplus X'^1$ with X'^0, X'^1 free of fields embedding into l. But now we only know that $\Phi^l(M(k) \oplus M(l)^m) \to \Phi^l(M(l)^n)$ splits. From this we can conclude that $M(k) \oplus M(l)^m \to M(l)^n$ is surjective by Lemma 5.52 part (i). I claim that $M(l)^m \to M(l)^n$ is also surjective. If not write C for the cokernel, considered as a G-module. This is a non-zero image of the trivial G-module F, so just F. But any composite $M(k) \to M(l)^m \to M(k) = C$ is zero, which is a contradiction.

Consequently we may appeal to Lemma 5.52 part (ii) again to split $M(l)^m \to M(l)^n$ and eliminate $M(l)$ from X^1. Dualising and repeating the process as before, $M(l)$ is eliminated from all but X^0, and considering $F[0] \simeq \Phi^l(X) = \Phi^l(X^0)$ shows that there are no copies of $M(l)$ in X^0 either.

This concludes the proof. □

Proof of Theorem 5.50, parts (i) to (iii). To unify notation, let us put $k'' = k'$ in cases (i), (iii), and $p = 2$ in case (ii). We may base change to k'' and so assume $k'' = k$. Since L/l is a p-extension and l/k is Galois we may apply $\Phi^{l/k}$ and assume $L = l$. Let $G = Gal(l/k)$.

In all cases the strategy will be to use X to build a periodic resolution of the trivial G-module F and then apply Lemma 5.54 below.

(i). We can view $\Phi^k(X) \in \mathbf{DAM}^{gm}(k, F) \subset \mathbf{DAM}^{gm}(L/k, F)$ as an element trivially satisfying the stated condition. Since Φ^k is a monoidal functor it preserves invertible objects and thus $\Phi^k(X)$ is invertible. We may replace X by $X \otimes \Phi^k(X)^{-1}$. Then $\Phi^k(X) \simeq F[0]$. By Lemma 5.53 we may represent X as a finite complex in $K^b(\mathbf{AM}(L/k, F))$, consisting only of $M(L)$, except for one copy of $M(k)$ in degree zero. We have $\Phi^L(X) \simeq F[n]$ for some $n \in \mathbb{Z}$, in fact $n = \phi(X)(l)$. If $n = 0$ there is nothing to prove. Otherwise, dualising if necessary, we may assume that $n > 0$. Since l corresponds to a projective G-module we find that can split off copies of $M(l)$ from the right, so arrive at a new representation where $X^i = 0$ for $i > n$. Dualising and repeating the process, then dualising again, we may assume that $X^i = 0$ for $i < 0$ or $i > n$. As a complex of G-modules, the homology of $X^\bullet$ is F concentrated in degree n (with trivial G-action, since any p-group acts trivially on a field of characteristic p[1]). Consequently we may append a single copy of $M(k)$ in X^{n+1} to obtain an acyclic complex of G-modules looking like

$$F \to M(l)^{k_1} \to M(l)^{k_2} \cdots \to M(l)^{k_n} \to F.$$

This yields a periodic resolution of period n, which must thus be divisible by 2 (by the lemma below).

The proof of (ii) is essentially the same. Replace X by $X \otimes \Phi^{k'/k}(X)^{-1}$ and apply Lemma 5.53 to obtain a representation with a unique copy of $M(k)$ in degree zero and all other terms $M(l)$. (Note that any subextension $l/l_0/k$ with $l \neq l_0$ is contained in k', by the structure of the Galois group.) Now the same argument as before applies.

(iii). Shifting X, we may assume that $\Phi^k(X) \simeq F[0]$. Any two distinct non-trivial proper subfields of l/k generate l, so no such field embeds into the other. Hence $\Phi^{l_i/k}(X)$ consists precisely of the copies of $M(l_i)$ and of $M(k)$, by Corollary 5.49. Thus replace X by $X \otimes \Phi^{l_0/k}(X)^{-1} \otimes \cdots \otimes \Phi^{l_p/k}(X)^{-1}$. This means that we may assume that $\Phi^k(X) \simeq F[0]$ and $\Phi^{l_i}(X) \simeq F[0]$. Appealing to Lemma 5.49 p+1 times we find a representation of X with a unique copy of $M(k)$ in degree zero, no copies of $M(l_i)$ for any i, and so everything else copies of $M(l)$. We need to show now that $\Phi^l(X) \simeq F[0]$. As before if this is not true we can find a periodic resolution of F as a $F[G]$-module, which is impossible by Lemma 5.54 part (iii). □

Lemma 5.54. *Let F be a field of characteristic p and G a finite group. We say that G is periodic of period n if any periodic resolution of F as an $F[G]$-algebra must have period divisible by n.*

(i) If p odd: The group $G = \mathbb{Z}/p$ has period 2.

(ii) If $p = 2$: The group $G = \mathbb{Z}/4$ has period 2, and the quaternion group (of order eight) has period 4.

(iii) The group $\mathbb{Z}/p \times \mathbb{Z}/p$ does not admit a periodic resolution.

(iv) The group $0 \to \mathbb{Z}/p \to G \to \mathbb{Z}/q^r \to 0$, where $2 \neq p \neq q$ and q is a prime, and $\mathbb{Z}/q$ acts on $\mathbb{Z}/p$ with kernel of size q^l, has period $2q^{r-l}$.

Proof. These results are well-known, except for perhaps (iv), which Jeremy Rickard has established on Math.StackExchange [92]. □

We now have to prove part (iv) of the theorem. In this situation G is no longer a p-group, so most of our lemmas no longer apply directly.

[1] If $G \hookrightarrow F^\times$ where $|G| = p^n$ and F is a field of characteristic p, then the image of G consists of elements of finite order and so is contained in a finite field $\mathbb{F}_{p^m}$. Thus $p^n | p^m - 1$, which can only happen if $n = 0$.

Proof of Theorem 5.50, parts (i) to (iii). We may again assume that $L = l$ and $k'' = k$.

The category $\mathbf{AM}(L/k, F)$ is equivalent to the category of permutation representations of G in F. Let us write N for the normal subgroup $\mathbb{Z}/p$ of G. Since $|G/N| = q^r$ is not divisible by p, the category of $F[G/N]$-modules is semi-simple. This means that $F[G/N]$ splits as a G/N-module (and hence as a G-module) into simple $F[G/N]$-modules, and every $F[G/N]$-module is a sum of these simples (uniquely up to order). So every $F[G/N]$-module is both projective and injective (as a $F[G/N]$-module).

If T is a one-dimensional $F[G/N]$-module then there is a surjection $F[G/N] \to T$ which must split, so T defines an invertible element of $\mathbf{AM}(L/k, F)$. We may replace X by $X \otimes T^{-1}$ to assume that $\rho^{k'/k}(X)$ is the trivial action.

Now $\Phi^{k'/k}(X)$ is a complex of $F[G/N]$-modules, and so any surjections or injections between these split. Since this complex is invertible its homology must be one-dimensional, with action $\rho^{k'/k}(X)$ which is trivial by our reduction. Applying the cancellation lemma sufficiently many times, we find a representation of X as a complex with a unique copy of $M(k)$ in degree zero, and otherwise only summands of $M(l')$, where l' does not embed into k'. Let H' be the corresponding subgroup. Then H' does not contain any conjugate of N and thus is of order prime to p (by Sylow theory). It follows that $[l : l']$ is prime to p and $M(l')$ is a summand of $M(l)$, by transfer.

Thus X is now a complex of projective modules, except for a single term $M(k)$ in degree zero. It has a single non-zero homology group, which is one-dimensional (since X is invertible) and carries the trivial G-action (by assumption). Suppose this homology is in degree n. If $n = 0$ there is nothing to do. If $n > 0$ dualise for definiteness. We can split off the projective terms from the right, using that any surjection with projective target splits. Dualising and splitting off all terms in degree $> -n$ we build a periodic resolution, as before. Then Lemma 5.54 part (iv) finishes to proof. $\square$

Remark. It is natural to ask if the condition $\rho^{l/k''}(X) = \rho^{k'/k''}(X)$ is really necessary. For this we would (at least) need examples of invertible complexes X such that $\rho^{l/k''}(X) \neq \rho^{k'/k''}(X)$. The author does not know any such examples. However (weaker) examples of Jeremy Rickard seem to indicate that the existence of such complexes is plausible.[2]

Proof of the Cancellation Lemma 5.51. We need to write down the complex $X'^{\bullet}$, chain maps $f : X^{\bullet} \to X'^{\bullet}$ and $g : X'^{\bullet} \to X^{\bullet}$, and chain homotopies proving that $gf \simeq \mathrm{id}_{X^{\bullet}}$ and that $fg \simeq \mathrm{id}_{X'^{\bullet}}$. We choose f and g to be the identity whenever possible. This means we only have to specify g^N, g^{N+1} and similarly for f. The essentially unique choice for f^N is the projection $p : X'^N \oplus A \to X'^N$ and the essentially unique choice for g^{N+1} is the inclusion $i : X'^{N+1} \to X'^{N+1} \oplus B$. For the differentials $d'^{\bullet}$ in $X'^{\bullet}$ we use $d'^n = d^n$ for $n \neq N-1, N, N+1$ and the obvious component of the old differential for $n = N-1, N+1$. Thus we have reduced to figuring out $d := d'^N, g^N$ and $f^{(N+1)}$ (and various checks).

We write the differential $d^N : X'^N \oplus A \to X'^{N+1} \oplus B$ as the matrix $\begin{pmatrix} d' & \alpha \\ \beta & f \end{pmatrix}$. Here f is an isomorphism by assumption. We choose the ansatz $g^N = (\mathrm{id}, j)^T : X'^N \to X'^N \oplus A$ and $f^{N+1} = (\mathrm{id}, q) : X'^{N+1} \oplus B \to X'^{N+1}$.

In order for f, g to be chain maps there are two squares which must commute. Writing out the conditions one obtains

$$0 = \beta + fj$$
$$d = d' + \alpha j$$
$$dp = d' + \alpha pr_A + q(\beta + f pr_A).$$

The first condition allows us to solve for $j = -f^{-1}\beta$ and then the second specifies $d = d' - \alpha f^{-1}\beta$. The third condition is then satisfied if we put $q = -\alpha f^{-1}$.

Next one needs to check that $X'^{\bullet}$ is indeed a chain complex and that f, g are indeed chain maps (some more commutativity conditions). All of this is trivial and left to the reader. We

[2] `http://math.stackexchange.com/a/855302/14762`

compute that $fg = \mathrm{id}_{X'^\bullet}$ and so we need only show that $gf \simeq \mathrm{id}_{X^\bullet}$. The required chain homotopy $K^\bullet : X^\bullet \to X^{\bullet-1}$ has $K^n = 0$ for $n \neq N+1$ and $K^{N+1} = \begin{pmatrix} 0 & 0 \\ 0 & -f^{-1} \end{pmatrix} : X'^{N+1} \oplus B \to X'^N \oplus A$. Again we leave it to the reader to check that this works. □

5.4.4 Relationship to Equivariant Stable Homotopy Theory

Let L/k be a finite Galois extension of the perfect field k. In [52, Theorem 3.5] it is shown (more or less) that there is a symmetric monoidal triangulated functor $\mathbf{SH}(G) \to \mathbf{SH}(k)$, where $\mathbf{SH}(G)$ denotes the G-equivariant stable homotopy category. We wish to study the induced homomorphism on Picard groups.

Let use recall the construction. We begin with the G-equivariant stable homotopy category [31]. Let G be a finite group. Write $Spc(G)_*$ for the category of pointed simplicial sets with a G-action (equivalently simplicial objects in the category of pointed G-sets). For $H \subset G$ a subgroup and $X \in Spc(G)_*$, write X^H for the subspace of points fixed by H. For $X, Y \in Spc(G)_*$ the space $X \wedge Y$ carries a natural G-action.

The category $Spc(G)_*$ affords a model structure in which the weak equivalences are the maps $X \to Y$ such that for all subgroups H of G, the induced map $X^H \to Y^H$ is a weak equivalence of simplicial sets. We denote this model category by $\mathcal{H}(G)_*$. This is a closed symmetric monoidal model category. There are natural functors $\bullet^H : \mathcal{H}(G)_* \to \mathcal{H}_*, X \mapsto X^H$. (Here $\mathcal{H}_*$ is the ordinary model category of pointed simplicial sets.) They preserve weak equivalences by construction and so descend to the homotopy category.

Write $S^{\wedge G}$ for the simplicial set $(S^1)^{\wedge n}$, where $n = |G|$ and G acts by permuting the factors. This is an element of $Spc(G)_*$. The category $\mathcal{SH}(G)$ is obtained as $Stab^\Sigma(\mathcal{H}_*, S^{\wedge G})$. One may prove that in $\mathbf{SH}(G) := Ho(\mathcal{SH}(G))$ in fact all the spheres $S^{\wedge T}$ are invertible, where T is any finite G-set. In particular S^1 with the trivial action is invertible and so this is a triangulated, closed symmetric monoidal category. It is generated by $\Sigma^\infty (G/H)_+$ for subgroups $H \subset G$.

For $f : H \to G$ a group homomorphism, there is a natural functor $f^* : Spc_*(G) \to Spc_*(H)$ by letting H act through f. That is to say $f^*(X)$ is the same space as X, but the H-action is $hx = f(h)x$. This extends to a symmetric monoidal functor $f^* : \mathbf{SH}(G) \to \mathbf{SH}(H)$.

The functor $\bullet^G : Spc(G)_* \to Spc_*$ extends to $\Phi^G : \mathbf{SH}(G) \to \mathbf{SH}$. It is the essentially unique symmetric monoidal triangulated functor with the property that $\Phi^G(\Sigma^\infty(G/H)_+) = 0$ for $H \neq G$, and $\Phi^G \Sigma^\infty (G/G)_+ = S$. Note that $(S^{\wedge G})^G = S^1$ is given by the diagonal subset, so Φ^G is essentially just obtained by taking fixed points levelwise.

For $H \subset G$ one defines a functor $\Phi^H : \mathbf{SH}(G) \to \mathbf{SH}$ as $\Phi^H \circ i_H^*$, where $i_H : H \to G$ is the inclusion and $\Phi^H : \mathbf{SH}(H) \to \mathbf{SH}$ is the functor constructed above.

More generally, for $X \in Spc(G)_*$ and subgroups $H \subset K \subset G$ such that H is normal in K, we can view X^H as an element of $Spc(K/H)_*$. We denote the induced functor by $\Phi^{K/H} : \mathbf{SH}(G) \to \mathbf{SH}(K/H)$. It is symmetric monoidal and triangulated.

Now suppose L/k is a Galois extension with group G. By Grothendieck's Galois theory, there is an equivalence of categories between the category of G-sets and the category of étale algebras over k with all residue fields embedding into L. The latter category embeds into $PSh(Sm(k))$ and hence there is a well-defined functor $F : Spc(G)_* \to Spc(k)_*$, where $Spc(k)_*$ denotes the presheaves of pointed simplicial sets on $Sm(k)$. This functor is symmetric monoidal and one may check that $F : \mathcal{H}(G)_* \to \mathcal{H}(k)_*$ is left Quillen.

Po Hu proves that $\Sigma^\infty F(S^{\wedge G}) \in \mathbf{SH}(k)$ is invertible [52, Theorem 3.5]. It follows that we obtain an induced functor $F : \mathbf{SH}(G) \to \mathbf{SH}(k)$.

Proposition 5.55. *For $X \in \mathbf{SH}(G)$ compact, the motive $MF(G) \in \mathbf{DM}(k)$ lies in the subcategory $\mathbf{DAM}^{gm}(L/k)$. Write $MF : \mathbf{SH}(G)^{cpt} \to \mathbf{DAM}^{gm}(L/k)$ for the composite functor.*

The following diagrams commute up to natural isomorphism.

(i)

$$\begin{array}{ccc} \mathbf{SH}(H) & \xrightarrow{F} & \mathbf{SH}(l) \\ {\scriptstyle i_H^*}\uparrow & & {\scriptstyle f^*}\uparrow \\ \mathbf{SH}(G) & \xrightarrow{F} & \mathbf{SH}(k), \end{array}$$

where $H \subset G$ is a subgroup and $f : Spec(l) \to Spec(k)$ is the corresponding extension.

(ii)

$$\begin{array}{ccccc} \mathbf{SH}(K/H)^{cpt} & \xrightarrow{MF} & \mathbf{DAM}^{gm}(l/k') & \xrightarrow{\alpha^p_\#} & \mathbf{DAM}^{gm}(l/k', \mathbb{F}_p) \\ {\scriptstyle \Phi^{K/H}}\uparrow & & & & {\scriptstyle \Phi^{l/k'}}\uparrow \\ \mathbf{SH}(G)^{cpt} & \xrightarrow{MF} & \mathbf{DAM}^{gm}(L/k) & \xrightarrow{\alpha^p_\#} & \mathbf{DAM}^{gm}(L/k, \mathbb{F}_p), \end{array}$$

where $H \subset K \subset G$ is a subgroup such that H is normal in K, H is a p-group, and $L/l/k'/k$ are the field extensions corresponding to $\{1\} \subset H \subset K \subset G$ (so in particular l/k' is Galois and L/l is a p-extension).

Proof. It is enough to prove the space level equivalents. That is we may replace $\mathbf{SH}(G)$ by $Spc(G)_*$ and $\mathbf{SH}(k)$ by $Spc(k)_*$. Then (i) just says that Grothendieck's Galois theory is natural in the base, which is clear.

For (ii), we note that we may write every G-set X as a disjoint union of transitive G-sets. It is then easy to see using the definition of $\Phi^{l/k'}$ as being induced from a functor at the level of $\mathbf{AM}(L/k, \mathbb{F}_p)$ that it is enough to show that for a transitive G-set X corresponding to the extension $L/l'/k$, the (not necessarily transitive) K/H-set X^H corresponds to $\Phi^{l/k'}(\alpha^p_\# M(l))$. But by definition this latter term is computed by base change to k', i.e. restriction to K, and then retaining only those fields embedding into l, i.e. those G-orbits fixed by H.

This concludes the proof. □

Now we need to recall two more groups associated with G and $\mathbf{SH}(G)$. The first is $RO(G)$, namely the ring of virtual real representations of G. The other is $JO(G)$, the quotient of this group under "stable J-equivalence" [65]. If V is an actual real representation of G, then the one-point compactification S^V is a pointed *topological* G-space. By the usual comparison theorems, this defines an element $S^V \in Ho(\mathcal{H}(G)_*)$. It turns out (essentially by design) that $\Sigma^\infty S^V$ is invertible. Moreover $S^{V \oplus W} \simeq S^V \wedge S^W$ and so one obtains a group homomorphism $RO(G) \to Pic(\mathbf{SH}(G))$. See also [31].

Let us write $C(L/k)$ for the group

$$C(L/k) = \prod_{p, H \subset G_p} \mathbb{Z},$$

where the product is over primes p and subgroups H of a (fixed) Sylow p-subgroup. Put differently, for every p we fix a maximal prime-to-p subextension $L/k_p/k$ (then L/k_p is a p-extension) and consider the subextensions $L/l/k_p$. We then get the fixed points homomorphism

$$\phi : Pic(\mathbf{DAM}^{gm}(L/k)) \to C(L/k)$$

defined by $\Phi^l(E) \simeq \mathbb{F}_p[\phi(E)(l)]$.

We also define a group

$$c(L/k) = \prod_{p, l/k'} \mathrm{Hom}(Gal(l/k'), \mathbb{F}_p^\times).$$

Here the product is over extensions $L/l/k'/k$ such that L/l is a p-extension and l/k' is Galois. Then we get the homomorphism $\rho : Pic(\mathbf{DAM}^{gm}(k)) \to c(k)$ defined by saying that $\rho(E)(l/k')$ is the Galois action $\rho^{l/k'}(E)$ on $\Phi^{l/k'}(E)$.

Theorem 5.56. *The homomorphism* $Pic(\mathbf{SH}(G)) \to Pic(\mathbf{DAM}^{gm}(k))$ *has the same image as the homomorphism* $RO(G) \to Pic(\mathbf{SH}(G)) \to Pic(\mathbf{DAM}^{gm}(k))$. *The latter homomorphism factors through* $JO(G)$, *and the following sequence is exact.*

$$0 \to JO(G) \to Pic(\mathbf{DAM}^{gm}(k)) \xrightarrow{\rho^2} c(k).$$

Proof. To prove that the image of $Pic(SH(G))$ inside $Pic(\mathbf{DAM}(L/k, \mathbb{Z}))$ coincides with the image of $RO(G)$, we appeal to [10, Proposition 1.2 and Theorem 1.3] and [31, Proposition 2.1]. These say that $Pic(SH(G))$ is generated by the suspension spectra of suitable G-spaces, and that the set of dimension functions of such G-spaces, when restricted to p-subgroups of G, is already exhausted by $RO(G)$. But by Proposition 5.55 on compatibility of MF and Φ, our map $\phi \circ MF$ precisely assigns to every space its dimension function restricted to p-subgroups.

Now we prove that the map $RO(G) \to Pic(\mathbf{DAM}(k, \mathbb{Z}))$ factors through $JO(G)$. Here we appeal to [65]. Their main Theorem 3.20 says that $V, W \in RO(G)$ are identified in $JO(G)$ if and only if their dimension functions coincide on all *cyclic* p-subgroups (here and everywhere else in this proof, we mean varying p). Write $\mathcal{P}$ for the set of p-subgroups of G and $\mathcal{C}$ for the set of cyclic P-subgroups of G. Let $D_{\mathcal{P}}(G)$ be the set of functions $\mathcal{P} \to \mathbb{Z}$ arising as dimension functions of elements in $RO(G)$. Then by the opening remark in the proof of theorem 1.3 in [10] the map $D_{\mathcal{P}} \to \mathbb{Z}^{\mathcal{C}(G)}$ is injective (this is essentially because of the Borel condition, see Theorem 5.50). Thus V and W are identified in $JO(G)$ if and only if $\phi(MF(V)) = \phi(MF(W))$, proving the claim.

It remains to identify the image of $JO(G)$ in $Pic(\mathbf{DAM}(L/k, \mathbb{Z}))$. We now use the full strength of [10, Theorem 1.3]. It says that an element of $f \in D_{\mathcal{P}}(G)$ arises as dimension function of an element of $RO(G)$ if and only if f satisfies the Borel-Smith conditions (i) to (iv) of [10, Definition 1.1].

Hence let $X \in Pic(\mathbf{DAM}(L/k, \mathbb{Z}))$. Then $\phi(X)$ satisfies conditions (i) to (iii), since these coincide with our Theorem 5.50 (i) to (iii). The trouble is with condition (iv). Bauer's condition (iv) differs from condition (iv) in our result in three aspects: it always applies, the congruence is modulo q^{r-l} and not modulo $2q^{r-l}$, and it is stated for all p and q, allowing $p = 2$ and $p = q$. The last of these restrictions is least serious: if $p = 2$ or $p = q$ the homomorphism $\mathbb{Z}/q^r \to Aut(\mathbb{Z}/p) = \mathbb{Z}/p - 1$ is zero, so the condition is vacuous.

Suppose now $\rho^2(X) = 1$. Then $X \otimes X$ has trivial Galois actions, and so Theorem 5.50 part (iv) applies to $X \otimes X$. We conclude that X satisfies Bauer's condition (iv) (the point being that considering $X \otimes X$ instead of X we lose a factor of 2 in the congruence modulus, but this is not a problem, since our result is too strong by a factor of two).

Finally let $X \in RO(G)$. We need to show that $\rho(MFX)^2 = 1$. This follows from part (ii) of Proposition 5.55. Indeed this says that the Galois actions of MFX on $\mathbb{F}_p$ are reductions of Galois actions on $\mathbb{Z}$, and thus have trivial squares. □

Remark 1. We have worked in this section with finite Galois extensions L/k. The results extend to arbitrary extensions without much difficulty. (The major obstacle being the definition of $\mathbf{SH}(Gal(L/k))$ if L/k is not finite.) The point is that any element of $Pic(\mathbf{DAM}(k))$ is compact and so comes from $\mathbf{DAM}(L/k)$ for some finite Galois extension L/k.

Remark 2. By Theorem 4.20, the homomorphism $Pic(\mathbf{SH}(k)) \to Pic(\mathbf{DM}(k))$ is often injective. Thus in general the functor $\mathbf{SH}(G) \to \mathbf{SH}(k)$ is very far from being an embedding.

Example. Let L/k be a cyclic extension of degree p, an odd prime. Let $G = Gal(L/k)$ be generated by σ. Then there is an obvious ("rotation") representation V of G on $\mathbb{R}^2$. The unit sphere $S(V)$ in V can be "equivariantly triangulated" as one 0-cell G and one 1-cell $G \times I$ (I the interval). From this it follows easily that $MF(S^V)$ is the invertible Artin motive $C(L)$ from Example 2 of subsection 5.4.2.

5.4.5 Artin-Tate Motives

We can treat (derived) Artin-Tate motives without much further difficulty. Write $\mathbf{ATM}(k, A)$ for the Karoubi-closed, additive subcategory of $Chow(k, A)$ spanned by the motives of the form $M(l)\{i\}$ with l/k a finite separable extension and $i \in \mathbb{Z}$. This is a symmetric monoidal subcategory containing $\mathbf{AM}(k, A)$. Write $\mathbf{DATM}^{gm}(k, A) \subset \mathbf{DM}^{gm}(k, A)$ for the thick triangulated subcategory generated by $\mathbf{ATM}(k, A)$. As usual this affords a weight structure with heart $\mathbf{ATM}(k, A)$. In contrast to Artin motives, the weight complex functor $t : \mathbf{DATM}^{gm}(k, A) \to K^b(\mathbf{ATM}(k, A))$ is *not* an equivalence.

Theorem 5.57. *Let A be an indecomposable ring (i.e. $Spec(A)$ is connected). Then every $E \in Pic(\mathbf{DATM}^{gm}(k, A))$ is uniquely of the form $F\{i\}$, with $F \in Pic(\mathbf{DAM}^{gm}(k, A))$. In other words,*

$$Pic(\mathbf{DATM}^{gm}(k, A)) = Pic(\mathbf{DAM}^{gm}(k, A)) \oplus \mathbb{Z}.$$

Proof. If $\mathcal{C}$ is an additive category, write $Gr(\mathcal{C})$ for the category whose objects are families $(X_i)_{i \in \mathbb{Z}}$ with each $X_i \in \mathcal{C}$ and $X_i = 0$ for all but finitely many i. We put

$$\mathrm{Hom}_{Gr(\mathcal{C})}((X_i)_i, (Y_i)_i) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_i).$$

It is easy to check that this is also an additive category. We have the inclusions $\bullet\{i\} : \mathcal{C} \to Gr(\mathcal{C}), X \mapsto X\{i\}$, where $(X\{i\})_j = X$ if $i = j$ and $X\{i\}_j = 0$ if $i \neq j$. Then $\mathrm{Hom}(X\{i\}, Y\{j\}) = 0$ if $i \neq j$ and $\mathrm{Hom}(X\{i\}, Y\{i\}) = \mathrm{Hom}(X, Y)$. Every object of $Gr(\mathcal{C})$ is canonically a sum of objects of the form $X\{i\}$. Indeed

$$(X_i)_{i \in \mathbb{Z}} = \bigoplus_i X_i\{i\}.$$

If $\mathcal{C}$ is symmetric monoidal, then $Gr(\mathcal{C})$ is symmetric monoidal in a natural way, with $X\{i\} \otimes Y\{j\} = (X \otimes Y)\{i + j\}$ and unit $\mathbb{1}\{0\}$. Also for any additive category $\mathcal{C}$, we have $K^b(Gr(\mathcal{C})) \cong Gr(K^b(\mathcal{C}))$.

Now note that for $l_1, l_2/k$ finite separable and $i \neq j \in \mathbb{Z}$ we have

$$\mathrm{Hom}(M(l_1)\{i_1\}, M(l_2)\{i_2\}) \cong \mathrm{Hom}(\mathbb{1}, M(l_1) \otimes M(l_2)\{i_2 - i_1\}) = CH^{i_2 - i_1}(Spec(l_1) \times Spec(l_2)) = 0,$$

whereas

$$\mathrm{Hom}(M(l_1)\{i\}, M(l_2)\{i\}) = \mathrm{Hom}(M(l_1), M(l_2)).$$

From this it easily follows that $\mathbf{ATM}(k, A) \cong Gr(\mathbf{AM}(k, A))$ and so we have the conservative, Pic-injective weight complex functor $\mathbf{DATM}^{gm}(k, A) \to K^b(Gr(\mathbf{AM}(k, A))) \cong Gr(K^b(k, A))$. Since there is an obvious homomorphism $Pic(\mathbf{DAM}^{gm}(k, A)) \oplus \mathbb{Z} \to Pic(\mathbf{DATM}^{gm}(k, A))$, the theorem follows from the Lemma below (applied to $\mathcal{C} = K^b(\mathbf{AM}(k, A))$). □

Lemma 5.58. *Let $\mathcal{C}$ be a symmetric monoidal additive category, and assume that* $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ *is an indecomposable ring. Then in the notation of the proof of the theorem*

$$Pic(Gr(\mathcal{C})) = Pic(\mathcal{C}) \oplus \mathbb{Z}.$$

Proof. There is clearly an injection $Pic(\mathcal{C}) \oplus \mathbb{Z} \to Pic(Gr(\mathcal{C})), (X, i) \mapsto X\{i\}$. We show it is also a surjection.

Let $E \in Gr(\mathcal{C})$ be invertible, with inverse F. We may write $E = \bigoplus E_i\{i\}, F = \bigoplus F_i\{i\}$, with $E_i, F_i \in \mathcal{C}$. Then $E \otimes F = \bigoplus_{i,j}(E_i \otimes F_j)\{i + j\}$. This must be isomorphic to $\mathbb{1}_{Gr(\mathcal{C})} = \mathbb{1}_{\mathcal{C}}\{0\}$. Consequently $\bigoplus_i E_i \otimes F_{-i} \cong \mathbb{1}$ whereas $\bigoplus_{i+j=n} E_i \otimes F_j = 0$ for $n \neq 0$. Also $\mathbb{1}_{\mathcal{C}}$ is assumed to be indecomposable, so in fact there exists i_0 such that $E_{i_0} \otimes F_{-i_0} \cong \mathbb{1}$.

But now for $i \neq i_0$ we find that $E_i \otimes F_{-i_0}$ is a summand of $(E \otimes F)_{i - i_0} = 0$, so $E_i \otimes F_{-i_0} = 0$, whence $0 = E_i \otimes F_{-i_0} \otimes E_{i_0} \cong E_i$. We thus find that $E = E_{i_0}\{i_0\}$. Since $E \in Pic(Gr(\mathcal{C}))$ was arbitrary, the result follows. □

5.5 Tate Spectra

In some situations we can use weight structures to study (subgroups of) $Pic(\mathbf{SH}(k))$ directly. Of course, by the Pic-injectivity Theorem 4.20, this is only interesting if k is of infinite 2-étale cohomological dimension.

We cannot hope to find a weight structure on $\mathbf{SH}(k)$ corresponding to the Chow weight structure on $\mathbf{DM}(k)$. This is because we would need the negativity condition $[\Sigma^\infty X, \Sigma^\infty Y[i]] = 0$ for all X, Y smooth and projective. But for example $[\Sigma^\infty \mathbb{P}^1, \Sigma^\infty S[1]] = [\mathbb{G}_m \wedge S[1], S[1]] = W(k)$. However, if X is essentially smooth, Henselian local, and Y is (essentially) smooth, then $[\Sigma^\infty X, \Sigma^\infty Y \wedge {\mathbb{G}_m}^{\wedge n}[i]] = \underline{\pi}_{-i}(Y)_n(X)$, and this is zero provided $i > 0$ [79, Example 5.2.2]. Hence write $\mathbf{SH}(k)^{loc}$ for the thick triangulated subcategory of $\mathbf{SH}(k)$ generated by $\Sigma^\infty X_+$, for X essentially smooth Henselian local. By Proposition 5.8 part (5), we obtain on $\mathbf{SH}(k)^{loc}$ a (unique) weight structure with heart the Karoubi-closure of the additive category spanned by the $\Sigma^\infty X_+$ for X essentially smooth and Henselian local.

We would like to say that there is a weight complex functor $\mathbf{SH}(k)^{loc} \to K^b(\mathbf{SH}(k)^{loc,w=0})$. However, to know this we need DG enhancements, and $\mathbf{SH}(k)$ probably does not afford such an enhancement.[3] However, recall the triangulated category $D_{\mathbb{A}^1}(k)$ of "not so naive motives" [80, Section 5.2] [18, Section 5.3]. This is a DG-enhanced category built just like $\mathbf{SH}(k)$, but starting with the category of abelian sheaves, not sheaves of sets.

Lemma 5.59. *The natural functor $M' : \mathbf{SH}(k) \to D_{\mathbb{A}^1}(k)$ is Pic-injective, and conservative on connective objects.*

Proof. The category $D_{\mathbb{A}^1}(k)$ affords a homotopy t-structure and an adjunction $M' : \mathbf{SH}(k) \leftrightarrows D_{\mathbb{A}^1}(k) : U'$ with U' right-t-exact. It follows from Morel's Hurewicz Theorem [82, Theorem 6.37] that $M'^{\heartsuit} : \mathbf{SH}(k)^{\heartsuit} \to D_{\mathbb{A}^1}(k)^{\heartsuit}$ is an equivalence of categories, and hence by the abstract Hurewicz Theorem 4.5 we conclude: if $E \in \mathbf{SH}(k)_{\ge 0}$ then $\underline{\pi}_0(E)_* = \underline{h}_0(M'E)_*$, and M' is conservative on connective objects.

Now suppose that $E \in \mathbf{SH}(k)$ is invertible and $M'E \simeq \mathbb{1}$. Since E is compact it is connective, and hence we may conclude in the usual way from $M'E = \mathbb{1} \in D_{\mathbb{A}^1}(k)_{\ge 0}$ that $E \in \mathbf{SH}(k)_{\ge 0}$ and that $\underline{\pi}_0(E)_* = \underline{h}_i(M'E)_* = \underline{h}_i(\mathbb{1})_* = \underline{K}^{MW}_*$. In particular there exists $a \in [\mathbb{1}, E] = \underline{\pi}_0(E)_0(k)$ such that $M'a$ is an isomorphism. By the conservativity of M' for connective objects, a is an isomorphism. □

We can repeat the construction of the homotopy weight structure on $D_{\mathbb{A}^1}(k)^{loc}$, here there now *is* a DG enhancement and hence a strong weight complex functor.

Proposition 5.60. *The functor M' is w-exact and induces an isomorphism on the hearts. The composite $\mathbf{SH}(k)^{loc} \to D_{\mathbb{A}^1}(k)^{loc} \to K^b(\mathbf{SH}(k)^{loc,w=0})$ is w-conservative and Pic-injective.*

Proof. It follows from Morel's Hurewicz theorem, together with the abelian stable connectivity theorem [82, Theorem 6.22] that for X, Y smooth Henselian local and $i > 0$ we have

$$[M'\Sigma^\infty X_+, M'\Sigma^\infty Y_+ \wedge {\mathbb{G}_m}^{\wedge n}[i]] = [\Sigma^\infty X_+, \Sigma^\infty Y_+ \wedge {\mathbb{G}_m}^{\wedge n}[i]].$$

Thus there is a weight structure on $D_{\mathbb{A}^1}(k)^{loc}$ with heart $M'\mathbf{SH}(k)^{loc,w=0}$. It follows from Lemma 5.10 that M' is w-exact, and from Corollary 5.15 (and the fact that $M'^{w=0}$ is an equivalence) that M' is w-conservative.

We know that M' is Pic-injective by the previous lemma. Since the weight complex functor is always w-conservative and Pic-injective by Lemma 5.11, we are done. □

Note that if l/k is a finite separable extension and $n \in \mathbb{Z}$, then $\Sigma^\infty Spec(l)_+ \wedge {\mathbb{G}_m}^{\wedge n} \in \mathbf{SH}(k)^{loc}$. Thus the above proposition should allow us to study "Artin-Tate spectra", and one might hope to prove results analogous to those about Artin-Tate motives. Unfortunately the algebra involved

[3] Actually we know that a strong weight complex functor exists for triangulated categories with "f-enhancements" [13, Section 8.4], and $\mathbf{SH}(k)$ probably does have such an enhancement.

is much more complicated. Let us write $Tate'(k)$ for the additive subcategory of $\mathbf{SH}(k)$ spanned by ${\mathbb{G}_m}^{\wedge m}$ with $m \in \mathbb{Z}$, and $\mathbf{SH}(k)^{Tate}$ for the thick triangulated subcategory of $\mathbf{SH}(k)$ generated by $Tate'(k)$. We think of these as "Tate spectra" and propose to study $Pic(\mathbf{SH}(k)^{Tate})$. In fact we shall prove the following result.

Theorem 5.61. *Let k be a perfect field. There is a canonical isomorphism*

$$Pic(\mathbf{SH}(k)^{Tate}) \cong Pic(GW(k)) \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

We explain first how the isomorphism is supposed to work. If $\mathcal{L}$ is an invertible $GW(k)$-module, then $\mathcal{L}$ is projective and so a summand of a free $GW(k)$-module. However the category of free $GW(k)$-modules embeds into $\mathbf{SH}(k)$, simply because $\mathbb{1} \in \mathbf{SH}(k)$ is compact and with endomorphism ring $GW(k)$. Consequently we find a canonically defined object $S_\mathcal{L} \in \mathbf{SH}(k)$ corresponding to $\mathcal{L}$, by taking the relevant summand of a wedge of spheres. The functor thus defined $P(GW(k)) \to \mathbf{SH}(k)$, where P is the category of projective $GW(k)$-modules, respects the tensor products. It follows that $S_\mathcal{L}$ is invertible and hence compact, so actually $S_\mathcal{L} \in \mathbf{SH}(k)^{Tate}$. The claim is now that every invertible element of $\mathbf{SH}(k)^{Tate}$ is of the form $S_\mathcal{L} \wedge S^n \wedge {\mathbb{G}_m}^{\wedge m}$, for unique $\mathcal{L} \in Pic(GW(k)), m, n \in \mathbb{Z}$.

The "weight complex functor" $\mathbf{SH}(k)^{Tate} \hookrightarrow \mathbf{SH}(k)^{loc} \to K^b(\mathbf{SH}(k)^{loc,w=0})$ has image contained in $K^b(Tate'(k))$. Since it is also Pic-injective, it is enough to prove the claim for $K^b(Tate'(k))$.

We now observe that $Tate'(k)$ has another description: it is the category of finitely generated, projective $K^{MW}(k)$-modules, with their graded homomorphisms.

If A is any graded ring (not necessarily commutative), write $K^b(A)$ for the homotopy category of the bounded chain complexes of finitely generated, projective graded (left) A-modules. We note that $K^b(A)^{op}$ is equivalent, via dualisation, to $K^b(A^{op})$, and if A is sufficiently commutative (e.g. ϵ-commutative like K^{MW}_*) then $K^b(A^{op})$ is equivalent to $K^b(A)$. In such cases, the category $K^b(A)$ is symmetric monoidal. We note that $K^b(A)$ affords an obvious weight structure.

Suppose now $\phi : A \to B$ is a morphism of ϵ-commutative graded rings. Then there is an obvious symmetric monoidal functor $\phi : K^b(A) \to K^b(B)$. We consider $\phi_1 : K^{MW}_*(k) \to K^{MW}_*(k)/\eta = K^M_*(k)$ and $\phi_2 : K^{MW}_*(k) \to K^{MW}_*(k)[\eta^{-1}] = W(k)[\eta, \eta^{-1}]$.

Lemma 5.62. *The functor $(\phi_1, \phi_2) : K^b(K^{MW}(k)) \to K^b(K^M(k)) \times K^b(W(k)[\eta, \eta^{-1}])$ is weight conservative.*

Proof. This has very little to do with K^{MW} and holds for many more rings. By Lemma 5.12, it is enough to show that (ϕ_1, ϕ_2) is right-w-conservative. By Corollary 5.15, it suffices to show that the induced functor on hearts detects sections.

So Let $\alpha : M_1 \to M_2$ be a morphism of finitely generated projective graded $K^{MW}_*(k)$-modules. We need to show that α has a section if α/η and $\alpha[\eta^{-1}]$ do. Let C be the (finitely generated) cokernel. If $\alpha[\eta^{-1}]$ has a section then from the exact sequence $M_1 \otimes K^{MW}[\eta^{-1}] \to M_2 \otimes K^{MW}[\eta^{-1}] \to C \otimes K^{MW}[\eta^{-1}] \to 0$ we conclude that $\eta^n C = 0$ for some $n > 0$. But from α/η having a section we similarly conclude that $C = \eta C$. Thus $C = 0$ and we are done. □

We now investigate the two "pieces".

Lemma 5.63. *The group $Pic(K^b(K^M(k)))$ is free abelian of rank two, generated by $K^M(k)[1]$ and $K^M(k)(1)$.*

Here we write $M(1)$ for the shift of the graded module structure, corresponding to multiplication by $\mathbb{G}_m$. Note that this is inconsistent with the use of $M(1)$ in the normal motivic situation, where it corresponds to multiplication by $\mathbb{G}_m[-1]$.

Proof. Let $q : K^M(k) \to K^M(k)/K^M(k)_+ \cong \mathbb{Z}$ be the natural quotient map, where $K^M(k)_+$ is the ideal of positive elements. Then q detects sections on finitely generated, graded projective modules: if $f : M \to N$ is a morphism of finitely generated, graded projective $K^M(k)_*$-modules, then f admits a section if and only if the cokernel C vanishes. This is true if and only if $C = K^M_+ C$

(by the graded Nakayama lemma), which is true if and only if $q(f) : M/K^M_+ M \to N/K^M_+ N$ admits a section.

It follows that $K^b(K^M(k)) \to K^b(Gr(\mathbb{Z}))$ (where we write $K^b(Gr(\mathbb{Z}))$ to emphasize that we view $\mathbb{Z}$ here as a trivially graded ring) is weight-conservative, whence all objects of $Pic(K^b(K^M(k)))$ are of the form $E[n]$, where $E \in Pic(K^M(k))$. Moreover we know that $E/K^M(k)_+ E \cong \mathbb{Z}(i)$ for some i. It is thus enough to show: if $E \in Pic(K^M(k))$ and $E/K^M(k)_+ E \cong \mathbb{Z}(0)$ then $E \cong K^M(k)(0)$.

But $\mathbb{Z}(0)$ is generated by a single degree zero element, and hence so is E, by the graded Nakayama lemma again. It follows that E is a summand of $K^M(k)(0)$ and then we are done because q induces an equivalence of categories of degree zero modules. □

Let us for completeness recall the graded Nakayama Lemma.

Lemma 5.64 (Graded Nakayama Lemma)**.** *Let A be a (not necessarily commutative) non-negatively graded ring and M a finitely generated, graded A-module. Write $A_+ \subset A$ for the ideal of positively graded elements. If $A_+ M = M$ then $M = 0$. If $N \subset M$ is a submodule and $M = N + A_+ M$, then $N = M$.*

Proof. The version about $N = M$ follows as usual by considering M/N. So we prove the first statement.

If $M \neq 0$ then, since M is finitely generated, there exists a minimal i with $M_i \neq 0$. But then $M_i = (A_+ M)_i = 0$, a contradiction. □

Lemma 5.65. *The natural morphism $Pic(W(k)) \oplus \mathbb{Z} \to Pic(K^b(W(k)[\eta, \eta^{-1}]))$ is an isomorphism.*

Proof. First note that $K^b(W(k)[\eta, \eta^{-1}]) \cong K^b(W(k))$ where the right hand side uses *ungraded* modules. The result then follows in the usual way (see e.g. [30, Theorem 3.5]) from the fact that $Spec(W(k))$ is connected [62, Prop. 2.22] □

We thus have a homomorphism $Pic(K^b(K^{MW}(k))) \to \mathbb{Z}^2 \oplus \mathbb{Z}$, corresponding to shift and twist in $K^b(K^M(k))$ and (only) shift in $K^b(W(k)[\eta, \eta^{-1}])$. The next lemma will be proved at the end.

Lemma 5.66. *If $E \in Pic(K^b(K^{MW}(k)))$ is such that $E/\eta \cong \mathbb{Z}[i](j)$, then there exists $F \in Pic(W(k))$ with $E \otimes K^{MW}(k)[\eta^{-1}] \cong F[i]$.*

Proof of Theorem 5.61, assuming the lemma. As we have said we may replace $Pic(\mathbf{SH}^{Tate}(k))$ by $Pic(K^b(Tate'(k))) = Pic(K^b(K^{MW}_*(k)))$.

Let $E \in Pic(K^b(K^{MW}(k)))$ and assume that $E/\eta \cong \mathbb{Z}(0)[0]$. It is enough to show that $E \in Pic(GW(k))$. By the above lemma and the weight conservativity result in Lemma 5.62 we conclude that E is an invertible graded $K^{MW}(k)$-module. We need to show that it is generated in degree zero.

Certainly E/η is generated in degree zero by assumption. Let $E_0 \subset E$ be a submodule obtained by lifting a generator of E/η. Then $E = \eta E + E_0$. By induction $E = \eta^n E + E_0$. Since E is finitely generated $\eta^n E$ is negatively generated for n sufficiently large, and thus E is non-positively generated. Being projective we conclude that E is a summand of $\bigoplus_{i=1}^N K^{MW}(k)(n_i)$ where $n_i \geq 0$ for all i. Then DE is a summand (so quotient) of $\bigoplus_{i=1}^N K^{MW}(k)(-n_i)$ and so is non-negatively generated. But we can use the same reasoning as before to conclude that $DE = (DE)_0 + \eta(DE)$ and so on, and hence push all generators in positive degrees into degree zero. We conclude that DE is generated in degree zero and thus so is E. □

To prove the lemma, we need a little (slightly) non-commutative ring theory. In fact all our rings are ϵ-commutative so probably all results from commutative theory go through without a problem, but it is easier to just prove what we need. So let A be an associative unital ($\mathbb{Z}$-graded) ring. We call A local if the set of non-units forms a two-sided (graded) ideal.

Lemma 5.67 (Nakayama)**.** *Let A be a local (not necessarily commutative) ring with maximal ideal m. If M is a finitely-generated A-module such that $M/mM = 0$ then $M = 0$*

Proof. The usual proof works. If M is generated by $x_1, \ldots, x_n$ then $x_1 = m_1 x_1 + \ldots m_n x_n$ with $m_i \in m$, and $1 - m_1$ is a unit, so M is generated by $x_2, \ldots, x_n$. Conclude by induction. □

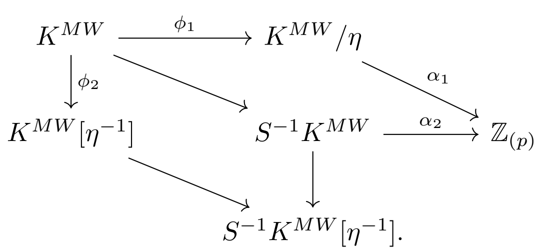
Corollary 5.68. *Let A be as above. A finitely generated projective A-module is free.*

Proof. Let P be finitely generated projective. Since A/m is a (skew-) field, $P/mP \cong (A/mA)^n$ for some (unique!) n and, lifting generators, we get $A^n \to P$ inducing an isomorphism after modding out by m. By Nakayama this is surjective as usual. We have an exact sequence $0 \to K \to A^n \to P \to 0$. Since P is projective $A^n \cong P \oplus K$ whence K is finitely generated and $K/mK = 0$. Thus $K = 0$. This concludes the proof. □

Corollary 5.69. *Let A be (not necessarily commutative) graded local, affording duality (e.g. ϵ-commutative). Then every element of $Pic(K^b(A))$ is of the form $A(i)[j]$.*

Proof. The functor $M \mapsto M/mM$ detects sections (of finitely generated projective modules) by Nakayama and hence is weight conservative (as a functor $K^b(A) \to K^b(A/m)$). It follows that every element of $Pic(K^b(A))$ is of the form $E[j]$ for some graded invertible A-module E. By the previous corollary $E \cong A$ as *ungraded* modules. Any element in $E \setminus mE$ is a free generator by the reasoning of the corollary (since E/mE is a skew-field), so we may pick a homogeneous free generator. This concludes the proof. □

Proof of Lemma 5.66. Let $I \subset K^{MW}(k)$ be the ideal generated by some prime (say p), all elements of positive degree, and all elements of negative degree. Then $K^{MW}(k)/I = K^M(k)/(K^M(k)_+ + pK^M(k)) = \mathbb{Z}/p$ is a field. Moreover $K^{MW}(k) \setminus I = GW(k) \setminus \langle I_0, p\rangle =: S$, where I_0 is the fundamental ideal. This is a multiplicative subset of central degree zero elements and so we can form the localisation $A = S^{-1}K^{MW}(k)$ which is a non-commutative, graded local ring. Now consider the commutative diagram



(Here α_1 is obtained by killing K^M_+ and localising at (p), whereas α_2 is obtained by killing all elements of positive or negative degree. This explains the upper/right trapezoid. The lower/left trapezoid is obtained by observing that we only invert central elements, so it does not matter in what order we do it.)

For $E \in Pic(K^b(K^{MW}(k)))$ we may write $\phi_1 E \simeq K^M(r)[s]$, $\phi_2 E \simeq F[t]$ and $S^{-1}E \simeq A(u)[v]$, where $F \in Pic(W(k))$. The upper/right trapezoid implies that $s = v$ (and $u = r$), whereas the lower/left trapezoid implies that $t = v$. Hence $t = s$ as was to be shown. □

5.6 Rational Coefficients, the Standard Conjectures, and the Étale Topology

In this short section we shall explain that some of our questions can be answered conditional on the "standard conjectures" of Grothendieck, Hodge, Beilinson etc.

First, recall that a *mixed Hodge structure* with $\mathbb{Q}$-coefficients consists of a rational, finite-dimensional vector space V together with a *weight filtration* $W_\bullet V \subset V$ and two filtrations (called the *Hodge filtration* and its opposite) $F^\bullet V, \bar{F}^\bullet V \subset V_\mathbb{C} := V \otimes_\mathbb{Q} \mathbb{C}$, satisfying certain compatibility conditions. See e.g. [111, Chapter 7] for an overview. The upshot is that the category $MHS_\mathbb{Q}$ of mixed hodge structures with $\mathbb{Q}$-coefficients is abelian symmetric monoidal.

If $V = W_n V$ and $W_{n-1}V = 0$, then we say that V is *pure of weight* n.

If $V \cong \mathbb{Q}$ is one-dimensional then any Hodge structure on V is necessarily pure. Given $n \in \mathbb{Z}$, there is a unique mixed Hodge structure on V of weight n denoted $\mathbb{Q}(n)$.

There is a *Hodge realisation* functor $R_H : \mathbf{DM}^{gm}(\mathbb{C}, \mathbb{Q}) \to D^b(MHS_{\mathbb{Q}})$ which is symmetric monoidal, triangulated, and satisfies $R_H \mathbb{1}(i) = \mathbb{Q}(i)$ [56, 57].

We may now summarise our take on the standard conjectures as follows.

Definition 5.70. *We will say that the* usual conjectures hold *if the following is true. The category* $\mathbf{DM}^{gm}(\mathbb{C}, \mathbb{Q})$ *affords a t-structure (called the motivic t-structure) with heart a symmetric monoidal rigid abelian category* $\mathcal{MM}(\mathbb{C})$. *The hodge realisation functor is t-exact for this t-structure, and the induced functor* $\mathcal{MM}(\mathbb{C}) \to MHS_{\mathbb{Q}}$ *is fully faithful.*

We should explain where these conjectures come from. The existence of a motivic t-structure with heart $\mathcal{MM}(\mathbb{C})$ and the exactness of the realisation is generally referred to as Beilinson's conjecture. See [59] for an overview. This conjecture also stipulates that the category has an abelian subcategory $\mathcal{M}(\mathbb{C})$ of simple objects, and that every object in $\mathcal{MM}(\mathbb{C})$ can be obtained by iterated extension from objects in $\mathcal{M}(\mathbb{C})$. The objects in $\mathcal{M}$ are called *pure motives*, and the objects in $\mathcal{MM}$ are called *mixed motives*. The category of pure motives is supposed to be obtained from the category $Chow(\mathbb{C})$ by using a stronger equivalence relation than rational equivalence on algebraic cycles. By celebrated work of Jannsen [58], this is abelian if and only if the equivalence relation is numerical equivalence. One of Grothendieck's standard conjectures asserts that numerical equivalence coincides with homological equivalence. This is equivalent to the functor $\mathcal{M}(\mathbb{C}) \to MHS_{\mathbb{Q}}$ being faithful. The Hodge conjecture asserts fullness of this functor. Since $\mathcal{MM}(\mathbb{C})$ is rigid abelian the functor Hom is exact, and similarly for $MHS_{\mathbb{Q}}$ [22, Proposition 1.16]. Thus fully faithfulness extends to all objects of $\mathcal{MM}(\mathbb{C})$ which are obtained by iterated extension from $\mathcal{M}(\mathbb{C})$, which is to say all of them.

Proposition 5.71. *Assume that the usual conjectures hold.*

Then $Pic(\mathbf{DM}(\mathbb{C}, \mathbb{Q})) = \mathbb{Z} \oplus \mathbb{Z}$, *i.e. if* $E \in Pic(\mathbf{DM}(\mathbb{C}, \mathbb{Q}))$ *is invertible there exist (unique) integers* $m, n \in \mathbb{Z}$ *such that* $E \simeq \mathbb{1}(m)[n]$.

Proof. Invertible objects are compact, so $E \in \mathbf{DM}^{gm}(\mathbb{C}, \mathbb{Q})$. Consider the Hodge realisation $R_H(E) \in D^b(MHS_{\mathbb{Q}})$. The objects in $D^b(MHS_{\mathbb{Q}})$ are represented by finite complexes of $\mathbb{Q}$-vector spaces, with certain additional data. Since $\otimes$ is exact on $MHS_{\mathbb{Q}}$, the tensor product on such complexes is the derived tensor product. Thus in order for $R_H(E)$ to be invertible it must consist of just a single vector space with some extra structure, that is to say $R_H(E) \simeq \mathbb{Q}(m)[n]$ for some (unique) m, n.

We may twist and shift conveniently to assume that $R_H(E) \simeq \mathbb{Q}[0]$ and shall prove that $E \simeq \mathbb{1}$. Indeed since R_H is t-exact, we find that E is in the heart $\mathcal{MM}(\mathbb{C})$ of the motivic t-structure. But then fully faithfulness of $\mathcal{MM}(\mathbb{C}) \to MHS_{\mathbb{Q}}$ implies the result. □

Corollary 5.72. *Assume that the usual conjectures hold.*

Let k be a field of characteristic zero. Then $Pic(\mathbf{DM}(k, \mathbb{Q})) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Hom}_{cts}(Gal(k), \mathbb{Z}/2)$. *More precisely, if* $E \in Pic(\mathbf{DM}(k, \mathbb{Q}))$ *then there exists a unique quadratic extension l/k (possibly the "trivial extension" $k \to k \times k$) and unique $m, n \in \mathbb{Z}$ such that* $E \simeq \tilde{M}(Spec(l))(m)[n]$.

Proof. The objects $\tilde{M}(Spec(l))$ are invertible by Example 1 in Section 5.4.2. Moreover if $f : Spec(\bar{k}) \to Spec(k)$ is an algebraic closure, then the kernel of $f^* : Pic(\mathbf{DM}(k, \mathbb{Q})) \to Pic(\mathbf{DM}(\bar{k}, \mathbb{Q}))$ is isomorphic to $\mathrm{Hom}_{cts}(Gal(k), \mathbb{Q}^\times)$ by Theorem 5.1. As usual the image of any continuous homomorphism from $Gal(k)$ to $\mathbb{Q}^\times$ must be contained in a torsion subgroup, and the torsion subgroup of $\mathbb{Q}^\times$ is precisely $\{\pm 1\} \cong \mathbb{Z}/2$. That the elements $\tilde{M}(Spec(l))$ for l/k quadratic exhaust this latter group follows easily from the proof of Proposition 5.41.

It thus remains to show that for any algebraically closed field $\bar{k}$ we have $Pic(\mathbf{DM}(\bar{k}, \mathbb{Q})) = \mathbb{Z} \oplus \mathbb{Z}$ generated by $\mathbb{1}[1]$ and $\mathbb{1}(1)$. We can write $\bar{k}$ as the colimit of its finitely generated subfields, and hence by Proposition 3.11 there exist a finitely generated field k_0 and $E \in Pic(\mathbf{DM}(k_0, \mathbb{Q}))$ with $E_{\bar{k}} = E$. Of course the algebraic closure $\bar{k_0}$ of k_0 embeds into $\bar{k}$, so in order to prove the claim we may assume that $\bar{k}$ is the algebraic closure of a finitely generated field (of characteristic zero). Thus $\bar{k}$ embeds into $\mathbb{C}$. The result follows from Proposition 5.71 and Proposition 5.2. □

It would be desirable to compute $Pic(\mathbf{DM}(k,\mathbb{Z}))$, at least conditionally. The author does not know how to do this. The problem is that with our current methods, this would require understanding $Pic(\mathbf{DM}(k,\mathbb{Z}/p))$. Using Theorem 5.1 one may assume here that k is p-special. However it is not clear how to reduce to k algebraically closed, and even for an algebraically closed field the author does not know the answer, even conditionally.

There is one way to make both of these problems disappear, and that is using the étale topology. Étale motives are very well developed [19]. We do not really need this heavy machinery, the ideas in Voevodsky's original work [107] are sufficient. We write $\mathbf{DM}_{et}(k,\mathbb{Z})$ for the category of integral étale motives.

Passing to étale motives is not completely arbitrary; we have $\mathbf{DM}_{et}(k,\mathbb{Q}) \simeq \mathbf{DM}(k,\mathbb{Q})$ [107, Proposition 3.3.2].

Proposition 5.73. *Assume the usual conjectures.*

Let k be a field of characteristic zero and finite étale cohomological dimension. Then

$$Pic(\mathbf{DM}_{et}(k,\mathbb{Z})) = \mathrm{Hom}_{cts}(Gal(k),\mathbb{Z}/2) \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

More precisely, if $E \in Pic(\mathbf{DM}_{et}(k,\mathbb{Q}))$ then there exists a unique quadratic extension l/k (possibly the "trivial extension" $k \to k \times k$) and unique $m,n \in \mathbb{Z}$ such that $E \simeq \tilde{M}(Spec(l))(m)[n]$.

Proof. We need to use formal properties of $\mathbf{DM}_{et}$ analogous to those of $\mathbf{DM}$. We will comment on when we do.

The analogue of Theorem 5.1 is true. Namely, without restriction on the coefficients, the base change along any (separable) field extension is conservative, and the kernel on the Picard group is always $\mathrm{Hom}_{cts}(Gal(k),A^\times)$. The first claim is true by design, since any finite separable extension is an étale cover. The second claim follows if we know that the homology sheaves $\underline{h}_i\mathbb{1}$ satisfy $\underline{h}_0\mathbb{1} = A$ (the constant sheaf associated with A) and $\underline{h}_i\mathbb{1} = 0$ for $i \neq 0$. Now we know that $[\mathbb{1},\mathbb{1}[i]] = H^i_{et}(k)$. The claim follows since in order to check an isomorphism of sheaves in the étale topology, we may pass to an algebraic closure of k, which has no higher étale cohomology.

Thus, as in the proof of the result with rational coefficients, we may assume that k is algebraically closed. In this case Theorem 5.6 part (i) remains valid, with the same proof. Indeed the only problematic part in that proof is that it uses $[\mathbb{1},\mathbb{1}[i]] = 0$ for $i \neq 0$, and this holds in the étale topology if k is algebraically closed.

Next we use that $\mathbf{DM}_{et}(k,\mathbb{Z}/p) \simeq D(Spec(k)_{et},\mathbb{Z}/p)$ is the ordinary derived category of abelian sheaves [75, Theorem 9.35]. Consequently $Pic(\mathbf{DM}_{et}(k,\mathbb{Z}/p)) = \mathbb{Z}$ generated by $\mathbb{1}[1]$, if k is algebraically closed.

Now let $E \in Pic(\mathbf{DM}_{et}(k,\mathbb{Z}))$. Then $\alpha^p_\#(E) \simeq \mathbb{1}[n_p] \in \mathbf{DM}_{et}(k,\mathbb{Z}/p)$ and $\alpha^{(0)}_\#(E) \simeq \mathbb{1}(m)[n] \in \mathbf{DM}_{et}(k,\mathbb{Q}) \simeq \mathbf{DM}(k,\mathbb{Q})$. Here we have used the identification of étale and Nisnevich motivic cohomology with rational coefficients, as well as the conditional computation of $Pic(\mathbf{DM}(k,\mathbb{Q}))$ of Proposition 5.71. Thus by the analogue of Theorem 5.6 part (i) explained above, it is enough to show that $n_p = n$ for all p.

As before we may assume that $k = \mathbb{C}$ for this (the analogue of Proposition 5.2 remains valid, and $\mathbf{DM}^{gm}_{et}$ also satisfies continuity).

Consider the ordinary Betti realisation $R_{A,b} : \mathbf{DM}_{et}(\mathbb{C},A) \to D(A\text{-}\mathbf{Mod})$, which sends $M(X)$ to the singular chain complex of the complex points $C_*(X(\mathbb{C}),A)$. Now we have for any homomorphism $\mathbb{Z} \to A$ a commutative diagram

$$\begin{array}{ccc} Pic(\mathbf{DM}_{et}(\mathbb{C},\mathbb{Z})) & \longrightarrow & Pic(D(\mathbb{Z}\text{-}\mathbf{Mod})) \\ \downarrow & & \downarrow \\ Pic(\mathbf{DM}_{et}(\mathbb{C},A)) & \longrightarrow & Pic(D(A\text{-}\mathbf{Mod})). \end{array}$$

The result follows applying this to $A = \mathbb{Q}$ and $A = \mathbb{Z}/p$, since the morphisms $Pic(D(\mathbb{Z}\text{-}\mathbf{Mod})) \to Pic(D(\mathbb{Q}\text{-}\mathbf{Mod}))$ and $Pic(D(\mathbb{Z}\text{-}\mathbf{Mod})) \to Pic(D(\mathbb{Z}/p\text{-}\mathbf{Mod}))$ are isomorphisms (all these groups are isomorphic to $\mathbb{Z}$). □

Remark 1. The assumption of finite cohomological dimension can probably eliminated, using the sophisticated techniques from [19].

Remark 2. The above results are highly dependent on the ring of coefficients. Of course $Pic(\mathbf{DM}(k, A))$ always contains $Pic(A)$, but far more interesting things can happen, as explained to the author by Joseph Ayoub. Roughly, if E is an appropriate elliptic curve and $K/\mathbb{Q}$ a sufficiently big field, then $M(E) \in \mathbf{DM}(k, K)$ has a summand, some power of which is the Tate motive!

Appendix A

Model Categories of Simplicial Presheaves

The purpose of this rather long appendix is twofold. Firstly, we review some standard and not-so-standard language and results about model categories. Secondly, we employ Barwick's notion of homotopy limits of diagrams of model categories to define a notion of a sheaf of model categories, and prove basic results about such sheaves. As explained in subsection 2.1.2, this is the main technical ingredient to the proofs of our representability results.

We now provide an overview of the sections of this appendix.

In Section A.1 we review the basics of model categories. All of this is standard.

In Section A.2 we start with model categories of simplicial presheaves. If $(\mathcal{C}, \tau)$ is a site, then there are (at least) for model category structures on the category $sPre(\mathcal{C})$: the projective/injective, global/local model structures. We establish their standard properties. We make one potentially original observation. Suppose that $X \in \mathcal{C}$. Then the overcategory $\mathcal{C}/X$ is a site in a natural way, and there is a well known equivalence of categories $sPre(\mathcal{C}/X) \simeq sPre(\mathcal{C})/X$. We show that under this equivalence, the above mentioned four model structures are taken into one another. That is to say, a morphism in $sPre(\mathcal{C}/X)$ is a cofibration/fibration/weak equivalence (in one of the model structures) if and only if the corresponding morphism in $sPre(\mathcal{C})/X$ is a cofibration/fibration/weak equivalence (in the model category structure on the overcategory $sPre(\mathcal{C})/X$ induced from the corresponding model structure on $sPre(\mathcal{C})$).

In Section A.3 we review the notion of a hypercover. Basically, a hypercover $U_\bullet \to X$ is a simplicial object in $\mathcal{C}/X$ satisfying certain conditions (depending on the choice of topology). One example is the Cech complex of a cover, but in general there are others. The main purpose for us is a theorem of Dugger et al. which says that the local model structures on simplicial presheaves can be obtained from the global model structures by localisation at the hypercovers.

Section A.4 treats almost finitely generated model categories. This is a particularly convenient class of model categories which are small in a precise sense.

In Section A.5 we review descent spectral sequences.

The rest of the appendix deals with (pre)sheaves of model categories. In section A.6 we review pseudofunctors and strictification. Essentially, the usual construction of a "functor" valued in categories such as $X \mapsto D(X)$ or $X \mapsto \mathbf{SH}(X)$ is not actually a functor at all, because given $X \xrightarrow{f} Y \xrightarrow{g} Z$, the two functors $(f \circ g)^*$ and $g^* f^*$ are only naturally isomorphic, not *equal*. This is of course a standard problem. The resolution is reasonably easy: the "functor" is a weaker structure known as *pseudo-functor*. Moreover every pseudofunctor is equivalent to an ordinary (or *strict*) functor, in a strong sense. This means that with an eye on applications we can formulate results about pseudofunctors, but deduce them by strictifying as the first step and then working entirely with strict functors.

In Section A.7, we begin in earnest the study of presheaves of model categories. If $\mathcal{C}$ is an essentially small category, by a Quillen pseudo-presheaf we mean a pseudofunctor on $\mathcal{C}^{op}$ with values in the category of model categories. In fact we usually talk about left or right pseudo-presheaves, meaning that the restriction functors are (left or right) Quillen functors. Next we want to define what it means for a Quillen pseudo-presheaf $\mathcal{M}$ to be a sheaf. Recall that a presheaf (of sets) F on a site $(\mathcal{C}, \tau)$ is a sheaf if for every cover $U_\bullet \to X$, the natural map $F(X) \to \lim F(U_\bullet)$ is an isomorphism. For a simplicial presheaf F, being a sheaf should mean being globally weakly equivalent to its fibrant replacement in one of the local model structures. The theorem of Dugger et al. we reviewed earlier says that this happens precisely if for every hypercover $U_\bullet \to X$, the natural map $F(X) \to \operatorname{holim} F(U_\bullet)$ is a weak equivalence.

This suggests that we should call the pseudo-presheaf $\mathcal{M}$ a sheaf if for every hypercover $U_\bullet \to X$ the (hypothetical) functor $\mathcal{M}(X) \to \operatorname{holim} \mathcal{M}(U_\bullet)$ is a Quillen equivalence. The problem here is that the category of model categories is not a model category in any obvious sense, so making sense of the homotopy limit is not trivial. Fortunately Barwick has defined such a notion, so we can just re-use it.

Of course any definition is only as good as its consequences. Barwick's definition is justified essentially by the fact that it coincides with the homotopy limit of the underlying infinity categories. Since we do not use the language of infinity categories, this is not very useful to us. The main result about homotopy limits we prove is the following. Let $\mathcal{M}$ be a suitable left Quillen pseudo-presheaf on some indexing category I. We can assume that $\mathcal{M}$ is strict. Then for $X \in I$ write $N(\mathcal{M}(I)^c_{we})$ for the nerve of the category of cofibrant objects, with weak equivalences between them. Since $\mathcal{M}$ is left Quillen, the restriction functors preserve cofibrant objects and weak equivalences between such. Consequently we obtain a diagram $N(\mathcal{M}^c_{we})$ on I. Then our result is that taking homotopy limits commutes with taking nerves in this way: there is a canonical weak equivalence

$$N((\operatorname{holim}_I \mathcal{M})^c_{we}) \simeq \operatorname{holim}_I N(\mathcal{M}^c_{we}).$$

This result is apparently folklore, but without written up proof. We present a proof outlined to the author by Bill Dwyer. We should remark that the ∞-categorical version of this result has a much nicer proof: the functor N sending a model category to its nerve of weak equivalences models the functor $\iota : \infty Cat \to Spc$ which sends an ∞-category to its core. It is the left adjoint of the inclusion $Spc \to \infty Cat$ regarding a space (∞-groupoid) as an ∞-category [93, Section 17.2], and consequently preserves (homotopy) limits.

In Section A.8 we finally define the notion of a sheaf of model categories and show how to use Rezk's theory of model toposes to construct examples. The main application is that $X \mapsto sPre(Sm(k))_\tau$ is a sheaf of model categories in the τ-topology, for τ one of Zariski, Nisnevich or étale.

In the final Section A.9 we study constructions with sheaves of model categories. Essentially, any sufficiently functorial construction with model categories can be performed sectionwise on a presheaf of model categories. If the presheaf we started with was a actually a sheaf, one might hope that the resulting presheaf is also a sheaf. This is the kind of result we prove in that section. We have three examples: pointing, localisation, and stabilisation. Pointing refers to passing from a model category $\mathcal{M}$ to the pointed version $\mathcal{M}_*$, localisation refers to turning chosen maps into weak equivalence, and stabilisation means passing to spectra. (Not surprisingly, these are the operations needed to get from the model category $sPre(Sm(X))_{Nis}$ to the model category $\mathcal{SH}(X)$.) In each of the three cases, we prove a theorem saying that under certain reasonable conditions, the sheaf property is preserved.

A.1 Review of Model Categories

A.1.1 Basic Definitions; Properness

There are various slightly different definitions of model categories. Good references are [43] [36] [47]. We take the definition from the last reference, where also the terms right and left lifting property and retract are explained.

Definition A.1. *A model category is a category $\mathcal{C}$ together with three subcategories called fibrations, cofibrations and weak equivalences, such that:*

(1) The category $\mathcal{C}$ has all (small) limits and colimits.

(2) (2-out-of-3) If f, g are composable morphisms in $\mathcal{C}$ and any two of f, g, fg are weak equivalences, then so is the third.

(3) Weak equivalences, fibrations and cofibrations are stable under retracts.

(4) Cofibrations have the left lifting property (LLP) with respect to maps which are both fibrations and weak equivalences, and maps which are both cofibrations and weak equivalences have the LLP with respect to all fibrations.

(5) Every morphism in $\mathcal{C}$ can be factored functorially *into a cofibration followed by a fibration which is also a weak equivalence, or a cofibration which is also a weak equivalence followed by a fibration.*

We call maps which are both fibrations and weak equivalences acyclic fibrations, and maps which are both cofibrations and weak equivalences acyclic cofibrations. These axioms have a long list of standard consequences, as illustrated for example in [36, Section II.1].

One of these standard results is that fibrations are stable under pullback, and cofibrations are stable under pushout [36, Corollary II.1.3]. This is not in general true for weak equivalences.

Definition A.2. *A model category $\mathcal{C}$ is called* left proper *if weak equivalences are stable under pushout along cofibrations. It is called* right proper *if weak equivalences are stable under pullback along fibrations. We call $\mathcal{C}$* proper *if it is both left and right proper.*

General results about properness can be found in [43, Chapter 13].

Here is a curious characterisation of right properness.[1] Recall that if $\mathcal{M}$ is a model category and $X \in \mathcal{M}$, then the overcategory $\mathcal{M}/X$ with its forgetful functor $U : \mathcal{M}/X \to \mathcal{M}$ affords a model structure in which U detects weak equivalences, fibrations and cofibrations [43, Theorem 7.6.5]. Now if $f : X \to Y \in \mathcal{M}$ is a morphism, then there is an obvious adjunction $f_\# : \mathcal{M}/X \rightleftarrows \mathcal{M}/Y : f^*$, which is a Quillen adjunction because $f_\#$ preserves cofibrations, fibrations and weak equivalences.

Proposition A.3 (Charles Rezk)**.** *1. If $f_\#$ is a Quillen equivalence, then f is a weak equivalence.*

2. If X, Y are fibrant and f is a weak equivalence, then $f_\#$ is a Quillen equivalence.

3. The model category $\mathcal{M}$ is right proper if and only if $f_\#$ is a Quillen equivalence for any weak equivalence $f : X \to Y$.

Proof. Since $f_\#$ preserves cofibrations and weak equivalences, $Lf_\# = f_\#$, and so $Lf_\#(\mathrm{id} : X \to X) = (f : X \to Y)$. Next since $U(\mathrm{id} : Y \to Y) = \mathrm{id}_Y$ is a fibration, the object $Y' := (\mathrm{id} : Y \to Y) \in \mathcal{M}/Y$ is fibrant and thus $Rf^*Y' = f^*Y' = (\mathrm{id} : X \to X)$. If $f_\#$ is a Quillen equivalence then $Lf_\#, Rf^*$ are ordinary equivalences of homotopy categories. We then have $Y' \simeq Lf_\# Rf^* Y' = (f : X \to Y)$. Thus f is a weak equivalence, proving (1).

To prove (2), we use the fact that in any model category weak equivalences between *fibrant* objects are stable under pullback along fibrations [43, Proposition 13.1.2(2)]. Now let $f : X \to Y$ be a weak equivalence between fibrant objects. If $T \to Y$ is a fibration, then $Lf_\# Rf^*(T \to Y) = f_\# f^*(T \to Y) = (T \times_Y X \to Y)$. But $T \times_Y X \to T$ is a weak equivalence by the quoted result, so the natural transformation $Lf_\# Rf^* \to \mathrm{id}$ on $Ho(\mathcal{M}/Y)$ is a natural isomorphism. Similarly let $(T \to X) \in \mathcal{M}/X$. Then $Lf_\#(T \to X) = (T \to X \to Y)$. Factor $T \to Y$ as $T \to T' \to Y$ with $T \to T'$ a weak equivalence and $T' \to Y$ a fibration. Then $Rf^* Lf_\#(T \to X) = (T' \times_Y X \to X)$.

[1] `https://golem.ph.utexas.educategory/2012/05/the_mysterious_nature_of_right.html#c041294`

Now $T' \times_Y X \simeq T'$ by the quoted result, and $T' \simeq T$ by construction. Hence $\mathrm{id} \cong Rf^* Lf_\#$ and we are done.

Finally if $\mathcal{M}$ is right proper then the proof of (2) works for any X, Y and so any $f_\#$ is a Quillen equivalence, provided f is a weak equivalence. Conversely, assume that $f_\#$ is a Quillen equivalence whenever f is a weak equivalence. We need to show that $\mathcal{M}$ is right proper. So let $f : X \to Y$ be a weak equivalence and $T \to Y$ a fibration. We need to show that $T' := T \times_Y X \to T$ is a weak equivalence. But $Rf^*(T \to Y) = (T' \to X)$ because $T \to Y \in \mathcal{M}/Y$ is fibrant, and so $Lf_\# Rf^*(T \to Y) = (T' \to Y)$. Since $Lf_\# Rf^* \simeq \mathrm{id}$ we conclude that $T' \to T$ is a weak equivalence, as required. □

Note that as a consequence, if $\mathcal{M}$ is a category and $(W, C_1, F_1), (W, C_2, F_2)$ are two model structures on $\mathcal{M}$ with the same weak equivalences, then $(\mathcal{M}, W, C_1, F_1)$ is right proper if and only if $(\mathcal{M}, W, C_2, F_2)$ is right proper. The same result holds for left properness by duality.

A.1.2 Adjunctions in Two Variables; Simplicial and Monoidal Model Categories

Next we come to the notions of simplicial and monoidal model categories. All model categories we use in this work are simplicial and most are monoidal. In fact both notions are subsumed under the common concept of a Quillen adjunction of two variables. First recall the concept of an ordinary adjunction in two variables. Such an object consists of three categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and three bifunctors $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}, \hom_l : \mathcal{C}^{op} \times \mathcal{E} \to \mathcal{D}, \hom_r : \mathcal{D}^{op} \times E \to \mathcal{C}$, together with natural isomorphisms $\mathrm{Hom}_{\mathcal{E}}(c \otimes d, e) \cong \mathrm{Hom}_{\mathcal{C}}(c, \hom_l(d, e)) \cong \mathrm{Hom}_{\mathcal{D}}(d, \hom_r(c, e))$.

There are two common examples. The first is a closed monoidal category. In this case $\mathcal{C} = \mathcal{D} = \mathcal{E}$ and $\hom_l = \hom_r = \underline{\mathrm{Hom}}$, the internal hom object. We denote the unit by $\mathbb{1}$.

The second case is a simplicially enriched category with tensors and cotensors. Then $\mathcal{C} = sSet$ is the category of simplicial sets, $\mathcal{D} = \mathcal{E} =: \mathcal{M}$ is the category we are enriching. The functor $\otimes$ is external tensor, $\hom_l(S, M) = M^S$ is external cotensor, and $\hom_r(M, N) = Map(M, N)$ is the simplicial mapping space.

The model category analogues of these definitions are obtained by adding the pushout-product axiom to the requirements of an adjunction in two variables [9, Section 3].

Definition A.4. *Suppose $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are model categories and $(\otimes, \hom_l, \hom_r)$ is an adjunction in two variables. We call this data a* Quillen adjunction in two variables *if the* pushout-product *axiom holds, i.e. if for any pair of cofibrations $c : Q \to R \in \mathcal{C}$ and $d : S \to T \in \mathcal{D}$, the pushout-product*

$$c \square d : (Q \otimes T) \coprod_{Q \otimes S} (R \otimes S) \to R \otimes T$$

is a cofibration in $\mathcal{E}$, which is acyclic if either c or d is.

Suppose that $\mathcal{M}$ is a model category and also a closed symmetric monoidal category. We call $\mathcal{M}$ a symmetric monoidal model category if (1) the data $(\otimes_{\mathcal{M}}, \underline{\mathrm{Hom}}_{\mathcal{M}}, \underline{\mathrm{Hom}}_{\mathcal{M}})$ is a Quillen adjunction in two variables, and if (2) for some cofibrant replacement $\tilde{\mathbb{1}} \to \mathbb{1}$ and for any object A, the composite $\tilde{\mathbb{1}} \otimes A \to \mathbb{1} \otimes A \cong A$ is a weak equivalence.

Suppose that $\mathcal{M}$ is a model category and also a simplicially enriched category with tensors and cotensors. We call $\mathcal{M}$ a simplicial model category if the data $(\otimes, \bullet^\bullet, Map)$ is a Quillen adjunction in two variables.

Some of the consequences of the simplicial model category axioms are expounded in [36, Section II.3]. The pushout-product axiom has many equivalent forms, see e.g. [9, Lemma 3.4].

We use the symbol $\otimes$ for both internal and external tensor, this should never cause confusion (whenever $\mathcal{C}$ is a simplicial closed monoidal category with tensors, there will usually be a natural embedding $sSet \to \mathcal{C}$ under which internal and external tensors coincide).

Note the following trivial consequence of the pushout-product axiom which we use extensively.

Lemma A.5. *Let $\mathcal{M}$ be a symmetric monoidal model category. Then tensor product with cofibrant objects preserves cofibrations and acyclic cofibrations.*

More generally if $(\mathcal{C}, \mathcal{D}, \mathcal{E}, \otimes, \hom_l, \hom_r)$ is a Quillen adjunction in two variables, then each of the functors $\otimes, \hom_l, \hom_r$ is (left) bi-Quillen.

Proof. Let $Q \to R$ be an (acyclic) cofibration and $\emptyset = S \to T$ a cofibration. Then $R \otimes S \cong \emptyset \cong Q \otimes S$, and so the pushout-product axiom implies that $Q \otimes T \to R \otimes T$ is an (acyclic) cofibration.

The more general assertion follows from the same reasoning applied to the equivalent (adjoint) formulations of the pushout-product axiom [9, Lemma 3.4]. □

We call a functor $R : \mathcal{N} \to \mathcal{M}$ between simplicial model categories simplicial right Quillen if it is right Quillen in the ordinary sense, with adjoint L, and the adjunction enriches: $Map(LX, Y) \cong Map(X, RY)$.

Lemma A.6. *If $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ is a simplicial adjunction, then L preserves tensors and R preserves cotensors.*

Proof. For $T \in \mathcal{M}$, $Y \in \mathcal{N}$ and $K \in sSet$, we obtain by adjunctions

$$\begin{aligned}\mathrm{Hom}(T, R(Y^K)) &\cong \mathrm{Hom}(LT, Y^K) \cong \mathrm{Hom}(K \otimes LT, Y) \\ &\cong \mathrm{Hom}(K, Map(LT, Y)) \cong \mathrm{Hom}(K, Map(T, RY)) \cong \mathrm{Hom}(K \otimes T, RY).\end{aligned}$$

The last term is isomorphic to both $\mathrm{Hom}(T, (RY)^K)$ and $\mathrm{Hom}(L(K \otimes T), Y)$. We have thus obtained canonical isomorphisms

$$\mathrm{Hom}(T, R(Y^K)) \cong \mathrm{Hom}(T, (RY)^K)$$

and

$$\mathrm{Hom}(K \otimes LT, Y) \cong \mathrm{Hom}(L(K \otimes T), Y),$$

natural in T and Y (and K). The result follows from the Yoneda lemma. □

A.1.3 Bousfield Localization

We now need some technical conditions on model categories which are used to control localizations. The slickest definitions and proofs are found in [9, Section 1] and are repeated below. It is worth pointing out right away that there are equivalent conditions which are easier to check.

Definition A.7. *Let $\mathcal{C}$ be a category with small colimits. An object $X \in \mathcal{C}$ is called* small *if there exists a regular cardinal λ such that $\mathrm{Hom}(X, \bullet)$ commutes with λ-filtered colimits.*

A category $\mathcal{C}$ is called locally presentable *if it has small colimits and is generated under colimits by a (small) set of small objects.*

A model category $\mathcal{M}$ is called combinatorial *if it is locally presentable and there exist (small) sets I and J of morphisms such that a morphism satisfies RLP with respect to I if and only if it is an acyclic fibration, and a morphism satisfies RLP with respect to J if and only if it is a fibration.*

A model category $\mathcal{M}$ is called tractable *if it is combinatorial and if the domains and codomains of the maps in I, J can be chosen to be cofibrant objects.*

We note that every object in a locally presentable category is small.

We finally come to Bousfield localization. Since we only deal with simplicial model categories, we shall only review the enriched version [9, Section 3]. So let $\mathcal{M}$ be a simplicial model category and H a set of homotopy classes of maps in $\mathcal{M}$. Recall that for $X, Y \in \mathcal{M}$ there is the *derived mapping space* $Map^d(X, Y)$ which is obtained by replacing X cofibrantly and Y fibrantly.

Definition A.8. *A left Bousfield localization of $\mathcal{M}$ at H is a simplicial model category $L_H\mathcal{M}$ together with a simplicial left Quillen functor $\mathcal{M} \to L_H\mathcal{M}$ which is initial among simplicial left Quillen functors $F : \mathcal{M} \to \mathcal{M}'$ such that for any f representing a class in H, the image $F(f)$ is a weak equivalence of $\mathcal{M}'$.*

Clearly $L_H\mathcal{M}$ is essentially unique if it exists. There is a standard strategy for constructing $L_H\mathcal{M}$ by "adding weak equivalences to $\mathcal{M}$".

Definition A.9. *An object $Z \in \mathcal{M}$ is H-local if for any $A \to B$ representing an element of H, the morphism*

$$Map^d(B,Z) \to Map^d(A,Z)$$

is a weak equivalence of simplicial sets.

A morphism $A \to B \in \mathcal{M}$ is an H-local weak equivalence if for any fibrant H-local object Z, the morphism

$$Map^d(B,Z) \to Map^d(A,Z)$$

is a weak equivalence of simplicial sets.

A morphism $A \to B$ is an H-fibration if it satisfies RLP with respect to all H-local weak equivalences that are cofibrations.

Theorem A.10 ([9], Theorem 3.18)**.** *If H is a (small) set and $\mathcal{M}$ is a left proper tractable simplicial model category, then there exists a left proper tractable simplicial model structure $L_H\mathcal{M}$ on $\mathcal{M}$ which has the same cofibrations as $\mathcal{M}$, the H-local weak equivalences as weak equivalences, and the H-fibrations as fibrations. This is a Bousfield localization of $\mathcal{M}$ at H. The H-fibrant objects are the H-local fibrant objects.*

We would like to say something about symmetric monoidal structures. For this, we need the notion of homotopy generators: a class $G \subset Ob(\mathcal{M})$ is called homotopy generating if every object in $\mathcal{M}$ is weakly equivalent to a homotopy colimit of objects in G.

Theorem A.11 ([9], Theorem 3.19)**.** *Let H be a (small) set of homotopy classes of maps in the left proper, tractable, simplicial, symmetric monoidal model category $\mathcal{M}$. Suppose further there exists a class G of cofibrant homotopy generators such that for every $A \in G$ and every fibrant H-local object $B \in \mathcal{M}$, the internal mapping object $\underline{\mathrm{Hom}}(A,B)$ is H-local.*

Then the symmetric monoidal structure on $\mathcal{M}$ makes the model category structure $L_H\mathcal{M}$ from the previous theorem into a symmetric monoidal model category.

A.1.4 Mapping Spaces and Nerves

Recall the cosimplicial small category $[\bullet]$, where $[n]$ is the category corresponding to the ordered set $\{1,2,\ldots,n\}$.

Definition A.12. *The functor $N : Cat \to sSet$ from (small) categories to (small) simplicial sets represented by the cosimplicial object $[\bullet]$ is called the* nerve *functor.*

By abstract nonsense the nerve functor has a left adjoint, and so preserves limits. Recall also that the functor $N : Cat \to \Delta^{op}Set$ is fully faithful, and natural transformations of categories correspond precisely to simplicial homotopies [113, IV.3.2]. In particularly any two adjoint categories have homotopy equivalent nerves.

Definition A.13. *Let $\mathcal{M}$ be a model category. We write $\mathcal{M}^c$ for the full subcategory of cofibrant objects, and similarly $\mathcal{M}^f$ for the category of fibrant objects, $\mathcal{M}^{cf}$ for the cofibrant-fibrant objects.*

We write $\mathcal{M}_w, \mathcal{M}_w^c, \mathcal{M}_w^f, \mathcal{M}_w^{cf}$ for the non-full subcategory with the same objects, but only weak equivalences as maps.

We want to study nerves of model categories. Unfortunately model categories are not usually essentially small, and so their nerves are *large* simplicial sets in the sense of Section 1.4. This is not really a problem, as explained there. Moreover let $\mathcal{M}$ be a model category and $S \subset Ob(Ho(\mathcal{M}))/\simeq$ a (small) set of weak equivalence classes of objects. Then there is a diagram $N(\mathcal{M}_w) \to Ob(Ho(\mathcal{M}))/\simeq \supset S$; write $N(\mathcal{M}_w)_S$ for the pullback. It follows from results of Dwyer-Kan about mapping spaces [27] [25] that $N(\mathcal{M}_w)_S$ is weakly equivalent to a small simplicial set.

Lemma A.14. *Let $\mathcal{M}$ be a model category. In the diagram*

$$\begin{array}{ccc} N(\mathcal{M}_w^{cf}) & \longrightarrow & N(\mathcal{M}_w^c) \\ \downarrow & & \downarrow \\ N(\mathcal{M}_w^f) & \longrightarrow & N(\mathcal{M}_w) \end{array}$$

all maps are simplicial homotopy equivalences.

Proof. Let R be a cofibrant replacement functor. Then there is a natural transformation $R \Rightarrow \mathrm{id}$. It follows that NR is a homotopy inverse to $N(\mathcal{M}_w^c) \to N(\mathcal{M}_w)$. The other maps are treated similarly. □

Lemma A.15. *Let*

$$L : \mathcal{M} \leftrightarrows \mathcal{N} : R$$

be a Quillen adjunction. Then there are induced maps of simplicial sets

$$N(L) : N(\mathcal{M}_w^c) \to N(\mathcal{N}_w^c)$$

$$N(R) : N(\mathcal{N}_w^f) \to N(\mathcal{M}_w^f).$$

If L, R form a Quillen equivalence then $N(L), N(R)$ are inverses in $Ho(sSet)$ under the natural identifications.

Proof. Since left Quillen functors preserve cofibrant objects and weak equivalences between cofibrant objects, we have a restricted functor $L : \mathcal{M}_w^c \to \mathcal{N}_w^c$, whence there is an induced map on nerves as claimed. Similarly for the right adjoint.

Write M_c for cofibrant replacement in $\mathcal{M}$, N_f for fibrant replacement in $\mathcal{N}$. We have a string of natural transformations $M_c \Rightarrow RLM_c \Rightarrow RN_fLM_c$ of endofunctors of $\mathcal{M}$. Both the very left and the very right term restrict to functors $\mathcal{M}_w \to \mathcal{M}_w$, and hence $N(M_c), N(RN_fLM_c) : N(\mathcal{M}_w) \to N(\mathcal{M}_w)$ are homotopic. We have seen in the previous proof that the former is a homotopy equivalence, hence so is the latter. We may similarly argue the other way round and hence conclude that the maps

$$N(LM_c) : N(\mathcal{M}_w) \leftrightarrows N(\mathcal{N}_w) : N(RN_f)$$

are inverse homotopy equivalences. The result follows. □

A.1.5 Homotopy Limits and Colimits

We shall have to deal with homotopy limits and colimits in general (simplicial) model categories. These have been treated amply elsewhere, so we just recall some salient points from [43, Chapter 18] and point out some non-standard notations we use. We first recall some categorical notions.

Definition A.16 ([43], Definition 18.3.2)**.** *Let $\mathcal{C}$ be a small category and $\mathcal{M}$ a category which is tensored and cotensored over simplicial sets. Assume that $\mathcal{M}$ is complete and cocomplete.*

If $X : \mathcal{C}^{op} \to \mathcal{M}$ is a diagram in $\mathcal{M}$ and $K : \mathcal{C} \to sSet$ is a diagram of simplicial sets, then we define

$$X \otimes_{\mathcal{C}} K = coeq \left[\coprod_{d \to c \in \mathcal{C}} X_c \otimes K_d \rightrightarrows \coprod_{c \in \mathcal{C}} X_c \otimes K_c \right],$$

where the two maps on the right are the obvious ones. This construction is called a coend.

Dually, if $X : \mathcal{C}^{op} \to \mathcal{M}$ is a diagram in $\mathcal{M}$ and $K : \mathcal{C}^{op} \to sSet$ is a diagram of simplicial sets, we define

$$\hom^{\mathcal{C}}(K, X) = eq \left[\prod_{c \in \mathcal{C}} X_c^{K_c} \rightrightarrows \prod_{c \to d \in \mathcal{C}} X_c^{K_d} \right],$$

where the two maps on the right are the obvious ones. This construction is called an end.

Recall that if $\mathcal{C}$ is a small category then we get canonical diagrams $\mathcal{C}/ : \mathcal{C} \to Cat, c \mapsto \mathcal{C}/c$ and $/\mathcal{C} : \mathcal{C}^{op} \to Cat, c \mapsto c/\mathcal{C}$. Combined with the nerve functor from the last subsection, this yields canonical diagrams of simplicial sets $N(\mathcal{C}/)$ and $N(/\mathcal{C})$.

Definition A.17. *Let $\mathcal{C}$ be a small category and $\mathcal{M}$ a simplicial model category. Given a diagram $X : \mathcal{C}^{op} \to \mathcal{M}$ we define the* "homotopy colimit" *as*

$$\text{“hocolim}_{\mathcal{C}}\text{”} X = X \otimes_{\mathcal{C}} N(\mathcal{C}/).$$

Similarly we define the "homotopy limit" *as*

$$\text{“holim}_{\mathcal{C}}\text{”} X = \hom^{\mathcal{C}}(N(/\mathcal{C}), X).$$

Beware of the quotation marks, which are usually omitted! We have the following result:

Theorem A.18 ([43], Theorem 18.5.3)**.** *Let $\mathcal{C}$ be a small category, $\mathcal{M}$ a simplicial model category, $X, Y : \mathcal{C}^{op} \to \mathcal{M}$ be diagrams, and $X \to Y$ a morphism of diagrams which is an objectwise weak equivalence.*

1. *If X, Y are diagrams of cofibrant objects, then the induced map* $\text{“hocolim}_{\mathcal{C}}\text{”}X \to \text{“hocolim}_{\mathcal{C}}\text{”}Y$ *is a weak equivalence.*
2. *If X, Y are diagrams of fibrant objects, then the induced map* $\text{“holim}_{\mathcal{C}}\text{”}X \to \text{“holim}_{\mathcal{C}}\text{”}Y$ *is a weak equivalence.*

What this theorem tells us, and what the quotation marks are supposed to remind us of, is that the "hocolim" and "holim" constructions are only guaranteed to yield the correct homotopy type when they are applied to objectwise cofibrant or fibrant diagrams. We shall omit the quotation marks when referring to this "correct" homotopy time. So if R_c and R_f are cofibrant and fibrant replacement functors for $\mathcal{M}$, we put

$$\operatorname{hocolim}_{\mathcal{C}} X := \text{“hocolim}_{\mathcal{C}}\text{”} R_c X$$

and

$$\operatorname{holim}_{\mathcal{C}} X := \text{“holim}_{\mathcal{C}}\text{”} R_c X.$$

Of course other constructions are possible (for example using model structures on diagram categories) and could be used interchangeably.

The following result does not seem to be as well known as it should be.

Theorem A.19. *Let $\mathcal{C}$ be a small category, $\mathcal{M}$ a simplicial model category and $X : \mathcal{C}^{op} \to \mathcal{M}$ a diagram.*

Suppose that $N(\mathcal{C})$ is contractible. Then for $c \in \mathcal{C}$, the natural map

$$\operatorname{holim}_{\mathcal{C}} X \to X(c)$$

is a weak equivalence, and similarly for homotopy colimits.

Proof. The result about homotopy colimits is proved in [17, Corollary 29.2]. The result about homotopy limits follows because $\operatorname{holim}_{\mathcal{C}} X = \operatorname{hocolim}_{\mathcal{C}^{op}} X^{op}$.

Here is an alternative proof. By the above argument it is enough to establish the case of homotopy limits. If $\mathcal{M} = sSet$ the result is stated in [61, 9.10]. As pointed out to the author by Zhen Lin[2], the case of general simplicial model categories $\mathcal{M}$ follows by considering the jointly conservative collection of functors $Map^d(X, \bullet)$, which preserves homotopy limits. □

Certain homotopy limits and colimits have special names, just like ordinary limits and colimits. For example the limit of a diagram indexed on the category $\bullet \to \bullet \leftarrow \bullet$ is called a pullback and so the homotopy limit is called a homotopy pullback. Similarly for pushouts.

The next result (or rather some strengthening of it) is usually referred to as "pasting law".

[2] `http://mathoverflow.net/q/227961`

Lemma A.20. *Let $\mathcal{M}$ be a right proper model category and*

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

a commutative diagram, such that both the left and the right inner squares are homotopy pullbacks. Then the outer rectangle is also a homotopy pullback.

Proof. Since homotopy pullback squares are detected and preserved by termwise weak equivalences, we may freely replace the diagram by a weakly equivalent one in order to prove the Lemma. We shall do so repeatedly.

We can factor $Z \to Z'$ as $Z \to \hat{Z} \to Z'$, with $Z \to \hat{Z}$ a weak equivalence and $\hat{Z} \to Z'$ a fibration. We may thus pass to a weakly equivalent diagram where $Z \to Z'$ is a fibration. Similarly we may assume that $X' \to Y'$ is a fibration. We then recall that in a right proper model category, homotopy pullbacks can be computed as ordinary pullbacks, provided that one of the maps is a fibration [43, Proposition 19.5.3, Corollary 13.3.8]. We can map our diagram to a new one, with the top row replaced by $X \to Y' \times_{Z'} Z \to Z$. The map $Y \to Y' \times_{Z'} Z$ is a weak equivalence because the right square is homotopy cartesian by assumption, and $Z \to Z'$ is a fibration. So we may assume that the right hand square is cartesian. By a similar construction (using that $X' \to Y'$ is fibrant) we may assume that the left hand square is also cartesian. It is then well known that the outer rectangle is cartesian, and it is thus homotopy cartesian (because $Z \to Z'$ is a fibration). This concludes the proof. □

A.2 The Four Model Category Structures

In this section we exhibit the basic model categories we will be working with. Throughout $\mathcal{C}$ is a small category, possibly with a Grothendieck topology τ. All result extend in obvious ways if $\mathcal{C}$ is only essentially small. The best general reference for this material is probably [60].

For any small category $\mathcal{C}$ we can form the category $Fun(\mathcal{C}^{op}, sSet)$ of contravariant functors from $\mathcal{C}$ to the category of simplicial sets. We call such functors simplicial presheaves and denote the category by $sPre(\mathcal{C}) := Fun(\mathcal{C}^{op}, sSet)$. There is a functor $R : \mathcal{C} \to sPre(\mathcal{C})$, associating with an object $c \in \mathcal{C}$ the representable (simplicially constant) presheaf $R_c(d) = \mathrm{Hom}_{\mathcal{C}}(d, c)$. Since we think of $sSet$ as a combinatorial avatar for the category of spaces, we think of $sPre(\mathcal{C})$ as generalized spaces. We shall put various model structures on this category.

Definition A.21. *A map $F \to G \in sPre(\mathcal{C})$ is called* global weak equivalence *or* objectwise weak equivalence *if for every $X \in \mathcal{C}$ the section $F(X) \to G(X)$ is a weak equivalence.*

A map $F \to G \in sPre(\mathcal{C})$ is called projective fibration *if it is an objectwise fibration, and* projective cofibration *if it satisfies the LLP with respect to all projective fibrations which are also global weak equivalences.*

A map $F \to G \in sPre(\mathcal{C})$ is called injective cofibration *if it is an objectwise cofibration (i.e. monomorphism), and* injective fibration *if it satisfies the RLP with respect to all injective cofibrations which are also global weak equivalences.*

If $F \in sPre(\mathcal{C})$ write $\underline{\pi}_0(F)^p$ for the presheaf $X \mapsto \pi_0(F(X))$. For $n > 0$ define a presheaf $\underline{\pi}_n(F)^p \to F_0$ *over* F_0 by $\underline{\pi}_n(F)^p(X) = \coprod_{b \in F(X)_0} \pi_n(F(X), b)$, with the evident map to $F(X)_0$. Clearly π_0, π_n are functors.

Definition A.22. *Let τ be a Grothendieck topology on $\mathcal{C}$. Write a_τ for the associated sheaf functor. A map $\theta : F \to G \in sPre(\mathcal{C})$ is called a* τ-local weak equivalence *if (1) $a_\tau(\underline{\pi}_0(\theta)^p) : a_\tau \underline{\pi}_0(F)^p \to a_\tau \underline{\pi}_0(G)^p$ is an isomorphism, and (2) for each $n > 0$ the following diagram is a* *pullback*

$$\begin{array}{ccc} a_\tau \underline{\pi}_n(X)^p & \longrightarrow & a_\tau \underline{\pi}_n(Y)^p \\ \downarrow & & \downarrow \\ a_\tau \underline{\pi}_0(X)^p & \longrightarrow & a_\tau \underline{\pi}_0(Y)^p. \end{array}$$

A map $F \to G$ is called a τ-local injective (respectively projective) fibration *if it satisfies RLP with respect to all injective (respectively projective) cofibrations which are also τ-local weak equivalences.*

If no confusion can arise, we write $\underline{\pi}_n(F) := a_\tau \underline{\pi}_n(F)^p$, and so on.
We note that $sPre(\mathcal{C})$ is closed symmetric monoidal and simplicial in an evident way.

Theorem A.23. *Let $\mathcal{C}$ be a small category and τ a Grothendieck topology. The category $sPre(\mathcal{C})$ admits four model structures:*

1. *The* global injective model structure*, with the global weak equivalences, injective cofibrations and injective fibrations.*
2. *The* global projective model structure*, with the global weak equivalences, projective cofibrations and projective fibrations.*
3. *The* τ-local injective model structure*, with the τ-local weak equivalences, injective cofibrations and injective τ-local fibrations.*
4. *The* τ-local projective model structure*, with the τ-local weak equivalences, projective cofibrations and projective τ-local fibrations.*

The local model structures are Bousfield localizations of the global ones. All model structures are proper simplicial tractable, and the representable presheaves are always cofibrant. Both injective model structures are symmetric monoidal. The projective model structures are symmetric monoidal if $\mathcal{C}$ admits binary products.

Proof. The existence of the global model structures is [9, Theorems 1.17, 1.19]; these theorems also establish tractability. Clearly representable presheaves are cofibrant in the injective global model structure (all presheaves are), and for the projective global model structure this follows from the LLP characterisation and the fact that projective acyclic fibrations are objectwise acyclic fibrations. Properness is [9, Theorem 1.21], simpliciality is [9, Theorem 3.30]. Symmetric monoidality is established in [9, Theorems 3.31 and Corollary 3.33] (note that in that Corollary, only binary products are needed).

Now we deal with the local model structures. By the main results of [23] (in particular Theorem 6.2, Corollary 6.3, Proposition 6.7) the local model structures can be obtained as Bousfield localizations at the class of all τ-hypercovers (we will review hypercovers in the next section), or equivalently at a certain subset (i.e. actual set) of the hypercovers. We may thus apply Theorem A.10 to conclude that the injective and projective local model structures exist and are simplicial, left proper, and tractable. Since Bousfield localization preserves cofibrations, the representables are still cofibrant.

The injective local model structure is also right proper, by [60, Theorem 5.8]. As explained after the proof of Proposition A.3, a model category being right proper depends only on the weak equivalences. Hence the projective local model structure is also right proper.

Finally we apply Theorem A.11 to prove that the local model structures are monoidal. The representable presheaves are cofibrant homotopy generators. The assumption in the theorem boils down to hypercovers being stable by tensor product with a representable, which is clear. □

We denote the model structures by $sPre(\mathcal{C})_{proj,gl}$, $sPre(\mathcal{C})_{inj,gl}$, $sPre(\mathcal{C})_{proj,\tau}$ and $sPre(\mathcal{C})_{inj,\tau}$.

Suppose now that $\alpha : \mathcal{C} \to \mathcal{D}$ is any functor. There is an induced functor $\alpha^* : sPre(\mathcal{D}) \to sPre(\mathcal{C})$ via $\alpha^*(F)(c) = F(\alpha(c))$. Since limits and colimits of simplicial presheaves are computed objectwise, this functor commutes with all limits and colimits, so has a left and a right adjoint, by general nonsense. The left adjoint is denoted $\alpha_\# : sPre(\mathcal{C}) \to sPre(\mathcal{D})$ and is the essentially unique colimit-preserving functor which satisfies $\alpha_\#(R_c) = R_{\alpha(c)}$. The right adjoint is denoted $\alpha_* : sPre(\mathcal{C}) \to sPre(\mathcal{D})$ and satisfies $(\alpha_* F)(d) = \mathrm{Hom}(\alpha^* R_d, F)$.

Definition A.24. *A functor $\alpha : \mathcal{C} \to \mathcal{D}$ between sites is called* continuous *if the functor $\alpha^* : PSh(\mathcal{D}) \to PSh(\mathcal{C})$ preserves sheaves. It is called* strongly continuous *if $\alpha^* : sPre(\mathcal{D}) \to sPre(\mathcal{C})$ preserves local objects. It is called* morphism *of sites if it is continuous and $\alpha_\# : PSh(\mathcal{C}) \to PSh(\mathcal{D})$ preserves finite limits.*

The notions of continuity and morphisms are standard; the notion of strong continuity is not usually given a separate name.

Proposition A.25. *Let $\alpha : \mathcal{C} \to \mathcal{D}$ be a functor of small categories. Then there are induced Quillen adjunctions*

$$\alpha_\# : sPre(\mathcal{C})_{proj} \leftrightarrows sPre(\mathcal{D})_{proj} : \alpha^* \quad \alpha^* : sPre(\mathcal{C})_{inj} \leftrightarrows sPre(\mathcal{D})_{inj} : \alpha_*.$$

If α is a functor between sites which is strongly continuous, then

$$\alpha_\# : sPre(\mathcal{C})_{proj,\tau} \leftrightarrows sPre(\mathcal{D})_{proj,\tau} : \alpha^*$$

is a Quillen adjunction.

If α is a morphisms of sites, then

$$\alpha_\# : sPre(\mathcal{C})_{proj,\tau} \leftrightarrows sPre(\mathcal{D})_{proj,\tau} : \alpha^* \quad \alpha_\# : sPre(\mathcal{C})_{inj,\tau} \leftrightarrows sPre(\mathcal{D})_{inj,\tau} : \alpha^*.$$

are both Quillen adjunctions.

Proof. Clearly α^* preserves objectwise weak equivalences, fibrations and cofibrations, so the first part is trivial.

If α^* preserves local objects then it preserves projective fibrant objects, so the second part follows from [43, Proposition 8.5.4].

If α is a morphism of sites then we get a Quillen adjunction in the injective local model structures by [60, Corollary 5.24]. In particular it follows that α^* preserves injective fibrant objects. Now let $E \in sPre(\mathcal{D})$ be local. Then E is globally weakly equivalent to an injective fibrant replacement E^f. Now $\alpha^* E$ is globally weakly equivalent to $\alpha^* E^f$, since α^* preserves global weak equivalences, and $\alpha^* E^f$ is injective fibrant since α^* is right Quillen in the injective local model structure. Consequently $\alpha^* E$ is local and α is strongly continuous, so by the second part α^* is right Quillen in the projective local model structure as well. ☐

Let us record the following fact which was demonstrated as a part of the above proof.

Corollary A.26. *A morphism of sites is strongly continuous.*

Remark 1. Suppose that $\alpha : \mathcal{C} \to \mathcal{D}$ is a functor of sites. Then α is continuous if (but not only if) it preserves covering families. We will see in the next section that α is strongly continuous if it preserves *hypercovers* (see Corollary A.38).

Remark 2. Suppose that $\alpha : \mathcal{C} \to \mathcal{D}$ is a continuous functor of sites. If $\mathcal{C}$ has fibre products and α preserves fibre products, then α is a morphism of sites [83, Remark 2.1.45]. We will see in the next section that even if $\mathcal{C}$ does not have all fibre products, α is still often strongly continuous (see Corollary A.38).

Lemma A.27. *Let $\mathcal{C}$ be a small category and $X \in \mathcal{C}$. The functor $e : sPre(\mathcal{C}/X) \to sPre(\mathcal{C})/X$ mapping a simplicial presheaf F to $eF : U \mapsto \coprod_{f:U \to X} F(f)$ is an equivalence of categories.*

Proof. Write $R_X(U) = \mathrm{Hom}_{\mathcal{C}}(U, X)$. Then there is a natural map $eF(U) \to R_X(U)$, and together these induce $eF \to R_X$. Thus e indeed takes values in $sPre(\mathcal{C})/X$.

Define $\bar{e} : sPre(\mathcal{C})/X \to sPre(\mathcal{C}/X)$ as follows. Given $\eta : F' \to R_X$, and $f : U \to X$ put $\bar{e}(\eta)(f) = \{t \in F'(U) | \eta(t) = f \in R_X(U)\}$.

One checks easily that $\bar{e}$ is essentially inverse to e. □

Note that if $\mathcal{C}$ is provided with a topology τ, then we can put a topology on $\mathcal{C}/X$ (which we denote by τ or τ/X) by declaring that $U : \mathcal{C}/X \to \mathcal{C}$ detects covering families. The following is the main result of this section.

Lemma A.28. *If we give $sPre(\mathcal{C})$ one of the four canonical model structures (injective global, projective global, injective τ-local, projective τ-local), then the induced model structure on $sPre(\mathcal{C})/X \cong sPre(\mathcal{C}/X)$ is the corresponding canonical model structure.*

Proof. We begin with the following observation: if $e : \mathcal{M} \leftrightarrows \mathcal{M}' : \bar{e}$ is an adjoint equivalence of categories and $\mathcal{M}, \mathcal{M}'$ are provided with model structures, then this is an equivalence of model categories (not to be confused with a Quillen equivalence) if and only if one of the following equivalent conditions holds: (i) $e, \bar{e}$ preserve cofibrations and weak equivalences, (ii) $e, \bar{e}$ preserve fibrations and weak equivalences. This follows because the third class is determined by lifting properties.

To deal with the global model structures, it is thus enough to show that $e : sPre(\mathcal{C}/X) \leftrightarrows sPre(\mathcal{C})/X : \bar{e}$ preserve objectwise weak equivalences, fibrations and cofibrations. Looking back at the explicit formulas for $e, \bar{e}$ this means we have to show that in the model category of simplicial sets, the weak equivalences, fibrations and cofibrations are stable by disjoint union and restriction to compatible families of connected components. This is straightforward. (Cofibrations being the monomorphisms are clearly stable. Fibrations are stable because the generating acyclic cofibrations have connected domains. Weak equivalences are stable by considering the definition using homotopy groups.)

The local model structures are obtained as localisations of the global model structures, hence have the same cofibrations. It thus suffices to show that $e, \bar{e}$ preserve local weak equivalences.

By [60, Lemma 5.25] the composite $Ue : sPre(\mathcal{C}/X) \to sPre(\mathcal{C})$ preserves local weak equivalences. Since U detects local weak equivalences (by definition), it follows that e preserves local weak equivalences.

We show that $\bar{e}$ preserves local weak equivalences as well. For this first note that if $F \in PSh(\mathcal{C})/X$ is such that UF is a sheaf, then $\bar{e}F$ is also a sheaf. Indeed if $\{V_\alpha\} \to V$ is a cover over X then we need

$$\bar{e}F(V) \to \prod_\alpha \bar{e}F(V_\alpha) \rightrightarrows \prod_{\alpha,\beta} \bar{e}F(V_\alpha \times_V V_\beta)$$

to be an equalizer diagram. But it is a pullback of

$$(UF)(V) \to \prod_\alpha (UF)(V_\alpha) \rightrightarrows \prod_{\alpha,\beta} (UF)(V_\alpha \times_V V_\beta),$$

which is an equalizer because UF is a sheaf by assumption. Similarly one finds that $\bar{e}$ commutes with the "associated sheaf" functor.

Now let $\alpha : F \to G \in sPre(\mathcal{C})/X$ be a weak equivalence. Then we have the presheaves $\underline{\pi}_0(F), \underline{\pi}_0(G) \to \underline{\pi}_0(X) = X$. We have $\bar{e}(\underline{\pi}_0(F)) = \underline{\pi}_0(\bar{e}F)$, and similarly for G. Hence we conclude that $a\underline{\pi}_0(\bar{e}\alpha) : a\underline{\pi}_0(\bar{e}F) \to a\underline{\pi}_0(\bar{e}G)$ is an isomorphism. To show that $\bar{e}(\alpha)$ is a weak equivalence it suffices, replacing X by some $X' \to X$ if necessary, to show that for $x \in \bar{e}F_0(X)$, the induced map on higher homotopy sheaves is an isomorphism. This follows by essentially the same argument. □

A.3 Review of Hypercovers

We rapidly review hypercovers and their utility in local homotopy theory. The primary reference is [23]. Actually we shall only ever deal with internal hypercovers, so we restrict attention to those.

Definition A.29. *A* Verdier site *is an essentially small category $\mathcal{C}$ together with a given collection of covering families $\{U_\alpha \to X\}_\alpha$ satisfying the properties below. A map $U \to X$ is called* basal *if it belongs to one of these covering families. The properties are as follows:*

(i) Isomorphisms are covering "families".

(ii) Pullbacks of covering families along arbitrary maps exist and are covering families.

(iii) Covering families are stable by composition.

(iv) If $U \to X$ is a basal map then so is the diagonal $U \to U \times_X U$.

A Verdier site is called suitable *(λ-suitable) if there exists a regular cardinal λ such that (λ is a regular cardinal such that) all the covering families are of size less than λ, coproducts of size less than λ exist in $\mathcal{C}$, and for any family of objects $(X_i)_{i\in I} \in \mathcal{C}$ of size less than λ, the natural map $\coprod_i R_{X_i} \to R_{\coprod_i X_i}$ becomes an isomorphism after sheafification.*

We point out right away that pullbacks along basal maps exist (ii) and basal maps are stable by composition (iii). The notion of a suitable Verdier site builds in the assumptions of [23, Section 9]. The first three axioms of a Verdier site are those of the covering family for a Grothendieck site, so we obtain a category of sheaves $Sh(\mathcal{C})$. Of course distinct sets of covering families may yield the same category of sheaves. This non-uniqueness can be avoided by formulating everything in terms of sieves, but for our purposes it is important to work with covering families directly.

There are plenty of suitable Verdier sites:

Lemma A.30. *Let X be a scheme of finite type over a field k. The categories $Sm(X), Ft(X)$ with the usual finite Zariski, Nisnevich or Étale covering families form $\aleph_0$-suitable Verdier sites.*

Since schemes of finite type are quasi-compact and étale morphisms are open [103, Point (7) of paragraph after Tag 039N / Lemma 40.11.5], the assumption on X implies that these covering families determine the same topology as the usual ones.

Proof. For the Zariski topology, the only non-trivial observation is that $R_A \coprod R_B \to R_{A\coprod B}$ becomes an isomorphism after sheafification. We have $(R_A \coprod R_B)(T) = R_{A\coprod B}(T)$ for T connected, and schemes of finite type are (Zariski-) locally connected (e.g. because a Noetherian ring can only have finitely many orthogonal idempotents). The result follows.

For the étale and Nisnevich topology, the basal maps are the étale maps, so property (iv) of Definition A.29 holds. The suitability condition follows from suitability of the Zariski site. □

Recall that the cdh topology on the category $Ft(X)$ of finite type X-schemes is the minimal topology generated by the Nisnevich covers and the proper cdh covers [18, Example 2.1.11]. In fact a proper cdh cover cover is the same as a cover by proper maps satisfying the Nisnevich lifting property [75, Lemma 12.26]. We shall call a (finite) family $\{U_i \to X\}$ a cdh covering family if $\coprod U_i \to X$ is a cdh cover, i.e. an epimorphism locally in the cdh topology. We note that this is a bit of a cop-out; for example any map is basal in this sense.

Lemma A.31. *If X is a scheme of finite type over a field k, then the topology on $Ft(X)$ generated by the finite cdh covering families is the cdh topology, and this specifies an $\aleph_0$-suitable Verdier site.*

Proof. Finite cdh covering families generate the topology by [75, Lemma 12.28]. Conditions (i), (ii) and (iii) of definition A.29 are automatic. Condition (iv) is clear because all maps are basal, and the topology is suitable by the same argument as before. □

Next we need to define what hypercovers actually are. We need some machinery, slightly adapting [23, Section 4] to work with internal hypercovers from the start.

If $\mathcal{D}$ is any category we write $s\mathcal{D}$ for the category of simplicial objects in $\mathcal{D}$. We write $s_+\mathcal{D}$ for the category of *augmented* simplicial objects. Let λ be a regular cardinal and write λSet for the category of sets of size less than λ. Suppose $\mathcal{C}$ is any category with coproducts of size less than λ.

For $Z \in \mathcal{C}$ and $K \in s_+\lambda Set$, write $Z \otimes K \in s_+\mathcal{C}$ for the augmented simplicial object in $\mathcal{C}$ which in degree n consists of the coproduct of copies of Z, indexed by K_n. Let $W \in s_+\mathcal{C}$. By the Yoneda lemma, there exists at most one object (up to unique isomorphism) $\hom_+(K, W) \in \mathcal{C}$ such that for $Z \in \mathcal{C}$ there is a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(Z, \hom_+(K, W)) = \mathrm{Hom}_{s_+\mathcal{C}}(Z \otimes K, W)$.

Definition A.32. *If $\mathcal{C}$ is a category with coproducts of size less than λ, $K \in s_+\lambda Set$, $W \in s_+\mathcal{C}$, and $\hom_+(K, W)$ exists, then we call it the* augmented matching space*.*

Lemma A.33 ([23], Lemma 4.7)**.** *For an augmented simplicial object $W \to X$ in the category $\mathcal{C}$ with coproducts of size less than λ, the following hold:*

(i) $\hom_+(\emptyset, W)$ *exists and is naturally isomorphic to X*

(ii) $\hom_+(\Delta^n, W)$ *exists and is naturally isomorphic to W_n*

(iii) The "functor" $\hom_+(\bullet, W)$ *converts colimits of size less than λ into limits in the following sense: If I is a category of size less than λ, $F : I \to \lambda Set$ a diagram such that* $\hom_+(Fi, W)$ *exists for every $i \in I$, and additionally* $\lim_I \hom_+(F, W)$ *exists in $\mathcal{C}$, then* $\hom_+(\mathrm{colim}_I F, W)$ *exists and is naturally isomorphic to* $\lim_I \hom_+(F, W)$*.*

Proof. Only the third statement requires proof. By adjunction it is enough to prove that for $Z \in \mathcal{C}$, the functor $\bullet \otimes Z : s_+\lambda Set \to s_+\mathcal{C}$ preserves colimits. Since colimits in diagram categories are computed termwise, it suffices to consider $\bullet \otimes Z : \lambda Set \to \mathcal{C}$. This follows from $\mathrm{Hom}_{\mathcal{C}}(S \otimes Z, W) = \mathrm{Hom}_{Set}(S, \mathrm{Hom}_{\mathcal{C}}(Z, W))$. $\square$

We can finally define hypercovers.

Definition A.34. *Let $(\mathcal{C}, \tau)$ be a suitable Verdier site.*

A map $X \to Y$ in $\mathcal{C}$ is called an internal cover if it is isomorphic (over X) to a map of the form $\coprod_i X_i \to Y$, where each $X_i \to Y$ is basal, and $\{X_i \to Y\}_i$ generate a covering sieve.

By an (internal) hypercover *of $X \in \mathcal{C}$ we mean an augmented simplicial object $W \to X$ such that for each n, the matching space $\tilde{M}_n W :=$* $\hom_+(\partial\Delta^n, W)$ *exists and such that the natural maps $W_n =$* $\hom_+(\Delta^n, W) \to \tilde{M}_n W$ *are internal covers.*

We need the slightly awkward definition of an internal cover because in general, not every internal cover is a covering family (although this is true in the Zariski, Nisnevich, étale and cdh sites).

Now we can state the main result of Dugger et al. we are going to use.

Theorem A.35 ([23], Theorem 9.3)**.** *Let $(\mathcal{C}, \tau)$ be a λ-suitable Verdier site. An object $X \in sPre(\mathcal{C})$ is τ-local (i.e. globally weakly equivalent to a fibrant replacement in the τ-local model structure) if and only if the following two conditions hold:*

1. *For a family $\{U_i\}_{i \in I} \in \mathcal{C}$ of size less than λ, the natural map $X(\coprod_i U_i) \to \prod_i X(U_i)$ is a weak equivalence.*
2. *For every hypercover $W \to U$ the natural map $X(U) \to$* $\mathrm{holim}\, X(W)$ *is a weak equivalence.*

Here are a few more auxiliary results about hypercovers we shall use.

Corollary A.36. *Let $\mathcal{C}$ be a suitable Verdier site. Then for any hypercover $U_\bullet \to X$, the natural map* $\mathrm{hocolim}\, U_\bullet \to X$ *is a weak equivalence (in either of the local model structures).*

Proof. Of course we really mean the natural map $\mathrm{hocolim}\, R_{U_\bullet} \to R_X$. It comes from $\mathrm{hocolim}\, U_\bullet \to$ "hocolim"$U_\bullet \to \mathrm{colim}\, U_\bullet \to X$.

Since representable presheaves are cofibrant, we have $\mathrm{hocolim}\, U_\bullet \simeq$ "hocolim"$U_\bullet$, and this latter object is cofibrant [43, Theorem 18.5.2 (1)]. It thus suffices to prove that for $T \in sPre(\mathcal{C})_\tau$ fibrant, we have that $Map(X, T) \to Map($"hocolim"$U_\bullet, T)$ is a weak equivalence (by the Yoneda lemma applied on the level of homotopy categories). Now we know that $Map($"hocolim"$U_\bullet, T) \cong$

"holim"$Map(U_\bullet, T)$ [43, Theorem 18.1.10]. Since each $U_\bullet$ is cofibrant and T is fibrant, the simplicial model category axioms imply that $Map(U_\bullet, T)$ is fibrant. We also have $Map(U_\bullet, T) = T(U_\bullet)$, essentially by definition. Thus "holim"$Map(U_\bullet, T) =$ "holim"$T(U_\bullet) \simeq \operatorname{holim} T(U_\bullet)$, and this latter object is weakly equivalent to $T(X) = Map(X, T)$, by the theorem. This concludes the proof. □

Proposition A.37. *If $W \to X$ is an internal hypercover and $A \to B$ is a map of λ-simplicial sets, then $\hom_+(A, W)$ and $\hom_+(B, W)$ exist, and $\hom_+(B, W) \to \hom_+(B, W)$ is a basal map.*

In particular all of the simplicial structure maps in W are basal maps.

Proof. A slight adaptation of [23, Proposition 8.5]. □

We can now prove a criterion for a functor of sites to be strongly continuous.

Corollary A.38. *Let $\alpha : (\mathcal{C}, \tau_{\mathcal{C}}) \to (\mathcal{D}, \tau_{\mathcal{D}})$ be a functor of Verdier sites.*

If α preserves hypercovers, then α is strongly continuous. This happens for example if α preserves covering families and pullbacks along basal maps.

Proof. If α preserves hypercovers then α^* preserves local objects, by Theorem A.35, so α is strongly continuous (by definition).

In order to prove the second part, it is enough to show that if $W \to X$ is an internal hypercover in $\mathcal{C}$ and K is a finite simplicial set, then $\hom_+(K, \alpha W)$ exists and is isomorphic to $\alpha \hom_+(K, W)$.

We use induction on the dimension of K in the usual way. If K has dimension zero this is clear. Now suppose K is of dimension n. There is the usual pushout

$$\begin{array}{ccc} \coprod_{\Delta^n \to K} \partial\Delta^n & \longrightarrow & K_{(n-1)} \\ \downarrow & & \downarrow \\ \coprod_{\Delta^n \to K} \Delta^n & \longrightarrow & K, \end{array}$$

where $K_{(n-1)}$ is the $(n-1)$-skeleton. In this way $\hom_+(K, W)$ is obtained as a pullback along basal maps

$$\begin{array}{ccc} \hom_+(\coprod_{\Delta^n \to K} \partial\Delta^n, W) & \longleftarrow & \hom_+(K^{(n-1)}, W) \\ \uparrow & & \uparrow \\ \hom_+(\coprod_{\Delta^n \to K} \Delta^n, W) & \longleftarrow & \hom_+(K, W), \end{array}$$

by Proposition A.37 and Lemma A.33 part (iii). But then α preserves this pullback along basal maps (by assumption) and the matching spaces except possibly for $\hom_+(K, W)$ (by induction). Thus $\hom_+(K, \alpha W)$ exists and equals $\alpha \hom_+(K, W)$ by Lemma A.33 part (iii) again. This concludes the induction step. □

A.4 Almost Finitely Generated Model Categories

The notion of almost finitely generated model categories was defined in [46]. It is related to the notion of ($\aleph_0$-)combinatoriality. It is fairly convenient for us.

Definition A.39. *Call a model category* almost finitely generated *if it is cofibrantly generated, the domains and codomains of generating cofibrations are sequentially compact, and if there is a set of trivial cofibrations J' with sequentially compact domains and codomains such that a map f with fibrant codomain is a fibration if and only if f has the right lifting property with respect to J'.*

(An object A is called sequentially compact if the functor $\operatorname{Hom}(A, \bullet)$ *preserves limits of sequences $X_1 \to X_2 \to \dots$.)*

This notion has a number of good properties. They are stated in [46].

Proposition A.40. *Let $\mathcal{M}$ be an almost finitely generated model category. Then sequential colimits in $\mathcal{M}$ preserve trivial fibrations, fibrant objects, and fibrations between fibrant objects.*

Proposition A.41. *If $\mathcal{M}$ is an almost finitely generated, simplicial, left proper, cellular model category and S is a set of cofibrations such that $X \otimes K$ is sequentially compact for every finite simplicial set K and every domain or codomain X of a map in S, then $L_S\mathcal{M}$ is almost finitely generated.*

Next we want to give conditions under which the local model category of simplicial presheaves is almost finitely generated. This uses the notion of a cd structure, see [110] for details.

Corollary A.42. *Let $\mathcal{C}$ be an essentially small category and τ the topology defined by a complete, regular, bounded cd structure. The model category $sPre(\mathcal{C})_{proj,\tau}$ is almost finitely generated.*

Proof. The category $sPre(\mathcal{C})_{proj,gl}$ is almost finitely generated, basically because $sSet$ is. By [3, Remark 3.2.6], the local model structure is obtained by localising at the distinguished squares defining the cd structure. (In fact, we localise at appropriate mapping cylinders, so that the maps are cofibrations.) This preserves almost finite generation by the proposition. □

To apply this result, recall that the Zariski, Nisnevich and cdh topology can be defined by cd structures, which satisfy the requirements of the corollary as long as the base scheme is reasonable (e.g. Noetherian of finite dimension).

We will also use the following easy observation.

Lemma A.43. *In an almost finitely generated simplicial model category $\mathcal{M}$, sequential colimits are sequential homotopy colimits.*

Proof. We have the colimit functor $\operatorname{colim} : \mathcal{M}^{\mathbb{N}} \to \mathcal{M}$. The homotopy colimit functor is its left derived functor. It may be computed as $\operatorname{hocolim}_n X_\bullet = \operatorname{colim}_n R_c X_\bullet$, where R_c is a cofibrant replacement functor in the projective model structure on $\mathcal{M}^{\mathbb{N}}$. In particular $R_c(X) \to X$ is an acyclic fibration in this model structure, i.e. a sectionwise acyclic fibration. But in an almost finitely generated model category acyclic fibrations are preserved under sequential colimits, so the natural map

$$\operatorname{hocolim}_n X_\bullet = \operatorname{colim}_n R_c(X_\bullet) \to \operatorname{colim}_n X_\bullet$$

is an acyclic fibration and in particular a weak equivalence. □

A.5 Descent Spectral Sequences and t-structures

In this section we will explain the proof of the following result.

Theorem A.44. *Let $(\mathcal{C}, \tau)$ be a site and F a presheaf of spectra on $\mathcal{C}$. Fix $X \in \mathcal{C}$. There exists a natural (in F and X) spectral sequence with E_2 page*

$$E_2^{pq}(F) = H_\tau^p(X, \underline{\pi}_{-q}F).$$

If F satisfies τ-descent and there exists N such that for all $p > N$ and all q we have $H_\tau^p(X, \underline{\pi}_{-q}F) = 0$, then the spectral sequence converges strongly to $\pi_{-p-q}F(X)$.

This result is of course well known, but it seems delicate to locate in this form. Descent spectral sequences are treated in many places, but convergence is usually only treated in special cases.

The most important feature of this spectral sequence is that $E_2^{pq}(F)$ (and hence all of the spectral sequence $E(F)$) is invariant under local weak equivalences. Consequently, as long as the vanishing condition is satisfied, the spectral sequence can be used to determine the homotopy groups of $F^f(X)$, the sections of the τ-local replacement.

A similar theorem also holds on the space level, but then one has to deal with the additional complication that π_1 need not be abelian and π_0 need not even be a group.

Before embarking on the proof, let us now explain the terms in the theorem. Write *Spt* for a model category of spectra; see subsection A.9.3 for some choices. Essentially an object $X \in Spt$ consists of simplicial sets X_i $(i = 0, 1, \dots)$ together with maps $X_i \wedge S^1 \to X_{i+1}$. The weak equivalences are the stable homotopy equivalences. A *presheaf of spectra* is then a functor $F : \mathcal{C}^{op} \to Spt$. This is the same thing as a spectrum object in the category of simplicial presheaves on $\mathcal{C}$. The category of presheaves of spectra can be given global and local model structures, just as in the case of presheaves of simplicial sets. If $U_\bullet \to X$ is a τ-hypercover, there is a natural map $F(X) \to \operatorname{holim} F(U_\bullet)$; F is said to *satisfy τ-descent* if this map is a weak equivalence (of spectra) for all such hypercovers. It is easy to see from the results in the previous section that F satisfies τ-descent if and only if it is globally weakly equivalent to a τ-local fibrant replacement.

Next, by a *spectral sequence* we mean a tri-graded abelian group E_r^{pq} with $p, q, r \in \mathbb{Z}$, though usually we will only consider $r > 0$. This is required to come with differentials $d_r^{pq} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ and isomorphisms $E_{r+1}^{pq} = ker(d_r^{pq})/im(d_r)$. This somewhat peculiar setup is explained in many places, see e.g. [112, Definition 5.2.3] (but beware that indexing conventions differ wildly!).

Given such a spectral sequence, note that E_r^{pq} is a subquotient Z_r^{pq}/B_r^{pq} of E_1^{pq} (say). One puts $Z_\infty^{pq} = \cap_i Z_i^{pq} \subset E_1^{pq}$ and $B_\infty^{pq} = \cup_i B_i^{pq} \subset E_1^{pq}$. Then one puts $E_\infty^{pq} = Z_\infty^{pq}/B_\infty^{pq}$. We say that E_r^{pq} *converges strongly to A_n* if each A_n is a filtered abelian group, we are given isomorphisms between the subquotients of the filtrations and appropriate terms of the E_∞ page, and if all the filtrations are exhaustive ($A_n = \cup_i F_i A_n$), separated ($\cap_i F_i A_n = 0$) and complete ($A_n = \lim_i A_n/F_i A_n$). This is the most desirable mode of convergence, enabling us to reconstruct A_n from E_∞, at least up to extension.

The theorem is a special case of the following more general result. Our notations regarding t-categories are outlined in section 4.1. For homotopy limits in triangulated categories, see for example [87, Section 1.6].

Theorem A.45. *Let $\mathcal{C}$ be a t-category, and $X, F \in \mathcal{C}$. For $G \in \mathcal{C}^\heartsuit$, write $H^p(X, G) := [X, G[p]]$.*

There is a spectral sequence with

$$E_2^{pq} = H^p(X, \pi_{-q}^{\mathcal{C}}(F)),$$

natural in X and F.

Suppose that X is connective (i.e. $X \in \mathcal{C}_{\geq n}$ for some n) and that the natural map

$$F \to \operatorname{holim}_n F_{\leq n}$$

is an isomorphism (in particular, the homotopy limit exists). Suppose furthermore there exists N such that $p > N$ implies that $H^p(X, \pi_{-q}^{\mathcal{C}}(F)) = 0$ for all q. Then the spectral sequence converges strongly to $[X, F[p+q]]$.

Proof. We build an exact couple in the sense of [12, Section 0]. We will freely use notation from that article.

To do this, consider the tower

$$\cdots \to F_{\leq n+1} \to F_{\leq n} \to F_{\leq n-1} \to \dots .$$

We put

$$A^{s+1,t} = [X[t], F_{\leq s}]$$

and

$$E^{s,t} = [X[t], hofib(F_{\leq s} \to F_{\leq s-1})].$$

Here t is the internal grading of the graded group A^s, in Boardman's notation. Again in his notation, the maps $i : A^{s+1} \to A^s$ and $k : E^s \to A^{s+1}$ have degree 0 (in t). We have exact triangles

$$F_{\leq s} \to F_{\leq s-1} \to hofib[1]$$

which induce $k : A^{s,t} \to E^{s,t-1}$, i.e. maps of degree -1. Consequently the differentials in Boardman's spectral sequence are maps

$$d_r^{st} : E_r^{s,t} \to E_r^{s+r,t-1}.$$

Note that we have $hofib(F_{\leq s} \to F_{\leq s-1}) = \pi_s^{\mathcal{C}}(F)[s]$ and so

$$E_1^{s,t} = E^{s,t} = [X[t], \pi_s^{\mathcal{C}}(F)[s]] = H^{s-t}(X, \pi_s^{\mathcal{C}}(F)).$$

We define a new spectral sequence by re-indexing

$$\tilde{E}_r^{pq} := E_{r-1}^{-p-q,-q}.$$

Then one checks easily that $\tilde{E}_2^{pq} = H^p(X, \pi_{-q}^{\mathcal{C}}(F))$ and that the differentials are maps $\tilde{d}_r^{pq} : \tilde{E}_r^{pq} \to \tilde{E}_r^{p+r,q+1-r}$. Thus we have a spectral sequence as claimed.

In order to establish convergence, we wish to use [12, Theorem 8.13]. (Note that in this section of the cited article, the author has done re-indexing akin to $E \to \tilde{E}$.) We have

$$A^{-\infty} = \operatorname{colim}_{n \to -\infty} A^n = \operatorname{colim}_n [X[*], F_{\leq n}].$$

This group is zero for connective X, by orthogonality. Next we need to show that $RE_\infty = 0$ and $W = 0$. Both are immediate consequences of our assumption on the vanishing of $H^p(X, \pi_{-q}^{\mathcal{C}}(F))$ outside a strip (use [12, Lemma 8.1] for W).

Boardman's result now says that $RA^\infty = 0$ and the spectral sequence converges strongly to A^∞. By definition, $A^\infty = \lim_n A^n$ and $RA^\infty = \lim_n^1 A^n$. Now by assumption we have an isomorphism

$$F \to \operatorname{holim}_n F_{\leq n} = hofib\left(\prod_n F_{\leq n} \to \prod_n F_{\leq n}\right).$$

Consider the map

$$\left[X[*], \prod_n F_{\leq n}\right] \to \left[X[*], \prod_n F_{\leq n}\right].$$

By definition its kernel is A^∞ and its cokernel is $RA^\infty = 0$. It follows from the exact triangle defining the holim that $[X[*], F] = A^\infty$. This concludes the proof. □

Theorem A.44 thus follows from the following well-known result, together with the observation that for F a presheaf of spectra with τ-descent, we have $\pi_i F(X) = [\Sigma^\infty X_+[i], F]$ (which holds true because satisfying τ-descent is the same as being τ-local, as explained in the previous sections).

Theorem A.46. *Let $(\mathcal{C}, \tau)$ be a site. The τ-local homotopy category of presheaves of spectra on $\mathcal{C}$, denoted $\mathbf{SH}(\mathcal{C}, \tau)$, affords a t-structure with the following properties:*

1. *$\mathcal{C}^\heartsuit \cong Shv(\mathcal{C})_\tau$*

2. *For $X \in \mathcal{C}$ and $F \in \mathbf{SH}(\mathcal{C}, \tau)^\heartsuit$ we have $[\Sigma^\infty X_+, F[n]] = H_\tau^n(X, F)$.*

3. *For $X \in \mathcal{C}$ we have $\Sigma^\infty X_+ \in \mathbf{SH}(\mathcal{C}, \tau)_{\geq 0}$.*

4. *The category $\mathbf{SH}(\mathcal{C}, \tau)$ has small products and coproducts, so all (sequential) homotopy limits and colimits.*

5. *For any $F \in \mathbf{SH}(\mathcal{C}, \tau)$ the natural map $F \to \operatorname{holim}_n F_{\leq n}$ is an isomorphism.*

Proof. This can be assembled from results in [60]. □

A.6 Pseudofunctors and Fibred Categories

In this section we elaborate on the notion of a presheaf of categories. This material is well-understood at least since [37, Expose VI]. We review the notions, relying mainly on [105] for definitions.

A.6.1 Pseudofunctors and Strictification

Let $\mathcal{C}$ be a category. We wish to define the notion of a presheaf on $\mathcal{C}$ with values in categories. Now the category Cat of categories is a perfectly fine (large) category, so we can build the (large) functor category $Fun(\mathcal{C}^{op}, Cat)$. An element $F \in Fun(\mathcal{C}^{op}, Cat)$ consists of the following data: for each $c \in \mathcal{C}$ a category $F(c)$, and for every morphism $f : c \to c' \in \mathcal{C}$ a functor $f^* : F(c') \to F(c)$. These have to satisfy $\mathrm{id}_c^* = \mathrm{id}_{F(c)}$ for all $c \in \mathcal{C}$ and for $c \xrightarrow{f} c' \xrightarrow{g} c''$ we must have $f^*g^* = (g \circ f)^*$. Both of these are equalities of functors, and therein lies the problem. Indeed it would be more natural to assume that these are just natural isomorphisms (perhaps with certain additional conditions) rather than equalities.

In fact this problem has two equivalent solutions, pseudofunctors and fibred categories. The former is closer to the intuition of a "weak functor with values in categories", whereas the latter is technically simpler. Since we shall mainly use, and not manipulate, presheaves of categories, we shall focus on the pseudofunctor approach, explaining the notion of fibred categories only in passing in the next subsection. The reader is warned however that the latter is found more often in the literature.

Definition A.47 ([105], Definition 3.10)**.** *A* pseudo-functor F *on* $\mathcal{C}$ *consists of the following data.*

(i) For each $c \in \mathcal{C}$ *a category* $F(c)$*.*

(ii) For each morphism $f : c \to c'$ *a functor* $f^* : F(c') \to F(c)$*.*

(iii) For each $c \in \mathcal{C}$ *an isomorphism of functors* $\epsilon_c : \mathrm{id}_c^* \cong \mathrm{id}_{F(c)}$*.*

(iv) For each $c \xrightarrow{f} c' \xrightarrow{g} c''$ *an isomorphism of functors* $\alpha_{f,g} : f^*g^* \cong (gf)^*$*.*

These data are required to satisfy the following conditions:

(a) For $f : c \to c'$ *and* $\eta \in F(c')$*, we have*

$$\alpha_{\mathrm{id}_c,f}(\eta) = \epsilon_c(f^*\eta) : \mathrm{id}_c^* f^*\eta \to f^*\eta$$

and

$$\alpha_{f,\mathrm{id}_{c'}}(\eta) = f^*\epsilon_{c'}(\eta) : f^*\,\mathrm{id}_{c'}^*\,\eta \to f^*\eta.$$

(b) For $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$ *and* $\theta \in F(d)$*, the following diagram commutes:*

$$\begin{array}{ccc} f^*g^*h^*\theta & \xrightarrow{\alpha_{f,g}(h^*\theta)} & (gf)^*h^*\theta \\ {\scriptstyle f^*\alpha_{g,h}(\theta)}\downarrow & & \downarrow{\scriptstyle \alpha_{gf,h}(\theta)} \\ f^*(hg)^*\theta & \xrightarrow{\alpha_{f,hg}(\theta)} & (hgf)^*\theta \end{array}$$

The conditions (a) and (b) are also known as coherence conditions. As usual there is a coherence theorem to the following effect: any natural isomorphism of functors built out of the α and ϵ between two universally equal strings of compositions, obtained by re-bracketing and inserting/deleting identities, are equal [89].

We denote the class of pseudofunctors on $\mathcal{C}$ by $PsFun(\mathcal{C}^{op}, Cat)$. We make it into a category as follows.

Definition A.48 ([69], Definition 2.14). *Let F, G be two pseudofunctors on $\mathcal{C}$. By a* pseudonatural transformation Θ *from F to G we mean the following data.*

(i) *For each $c \in \mathcal{C}$, a functor $\Theta_c : F(c) \to G(c)$.*

(ii) *For each $f : c \to d \in \mathcal{C}$ an isomorphism of functors*

$$\Theta_f : \Theta_c f^*_F \cong f^*_G \Theta_d.$$

These data are required to satisfy the following constraints:

(a) *For $c \in \mathcal{C}, \theta \in F(c)$, the natural morphism*

$$\Theta_c(\theta) \xrightarrow{\Theta_c(\epsilon_{c,F}(\theta))} \Theta_c(\mathrm{id}^*_{c,F}\,\theta) \xrightarrow{\Theta_{\mathrm{id}_c}} \mathrm{id}^*_{c,G}\,\Theta_c(\theta)$$

is equal to $\epsilon_{c,G}(\theta)$.

(b) *For each $c \xrightarrow{f} d \xrightarrow{g} e \in \mathcal{C}$ and $\theta \in F(e)$, the following diagram commutes:*

$$\begin{array}{ccc} \Theta_c((gf)^*\theta) & \xrightarrow{\Theta_c(\alpha_{f,g})} & \Theta_c(f^*g^*\theta) \\ {\scriptstyle \Theta_{gf}}\downarrow & & \downarrow{\scriptstyle \Theta_f\Theta_g} \\ (gf)^*\Theta_e(\theta) & \xrightarrow{\alpha_{f,g}} & f^*g^*\Theta_e(\theta) \end{array}$$

We call a pseudonatural transformation $\Theta : F \to G$ a pseudonatural equivalence *if for all $c \in \mathcal{C}$, the component functor $\Theta_c : F(c) \to G(c)$ is an equivalence.*

One may check that $PsFun(\mathcal{C}^{op}, Cat)$ forms a category, with morphisms the pseudonatural transformations. This is our category of weak presheaves on $\mathcal{C}$ with values in categories.

In some cases we will wish to deal with covariant pseudofunctors. We shall always treat these as pseudo-presheaves on $\mathcal{C}^{op}$.

It would sometimes be useful to replace a pseudofunctor F by a pseudonaturally equivalent *strict* functor. It turns out that this is always possible, even canonically so.

Theorem A.49. *The (non-full) embedding $i : Fun(\mathcal{C}^{op}, Cat) \to PsFun(\mathcal{C}^{op}, Cat)$ affords a left adjoint $L : PsFun(I^{op}, Cat) \to Fun(I^{op}, Cat)$. Moreover the unit and co-unit of adjunction are (pseudo-)natural equivalences.*

The last statement can be souped up to say that i and L are essentially-inverse natural equivalences of bicategories. We shall denote the composite $iL : PsFun(I^{op}, Cat) \to PsFun(I^{op}, Cat)$ by $F \mapsto F^r$. It has the properties of turning pseudofunctors into strict functors, pseudonatural transformations into strict transformations, and if G is a strict functor, then the set of strict transformations from F^r to G is in bijection with the set of pseudonatural transformations from F to G, via composition with the adjunction pseudo-transformation $F \to F^r$.

Proof. The proof is actually reasonably straightforward. It can be found abstractly in [89] [63], and the concrete argument is sketched at

`http://ncatlab.org/nlab/show/pseudofunctor`. □

A.6.2 Grothendieck Construction, Fibred Categories and Homotopy Colimits

We begin with the Grothendieck construction for a strict functor. Thus let $F : \mathcal{C}^{op} \to Cat$ be an ordinary (strict) functor. Recall that the category of categories is tensored over small categories in an obvious way. Consider the coend (see also Definition A.16)

$$\mathcal{C} \int F := F \otimes_{\mathcal{C}} \mathcal{C}/ = coeq \left[\coprod_{d \to c} F(c) \times (\mathcal{C}/d) \rightrightarrows \coprod_{c \in \mathcal{C}} F(c) \times (\mathcal{C}/c) \right].$$

It is amusing to note that this agrees with the following more conventional description.

Lemma A.50. *The category $\mathcal{C} \int F$ is isomorphic to the category whose objects are pairs (c, X) with $c \in \mathcal{C}$ and $X \in F(\mathcal{C})$, and where morphisms $(c, X) \to (d, Y)$ are pairs (f, ϕ) with $f : c \to d$ and $\phi : X \to f^*Y$.*

Proof. Write $\mathcal{D}$ for the category defined in the statement. There is a functor $\kappa : \mathcal{C} \int F \to \mathcal{D}$ coming from $(X, f : c \to d) \mapsto (c, f^*X)$. To show it factors through the coequaliser one has to use that F is a strict functor. It is not difficult to check that κ induces bijections on objects and morphisms. □

This latter description actually extends to the case where F is just a pseudofunctor.

Definition A.51 ([105], Section 3.1.3)**.** *Let $\mathcal{C}$ be a small category and F a pseudofunctor on $\mathcal{C}$. Define the category $\mathcal{C} \int F$ to have objects the pairs (c, X) with $c \in \mathcal{C}$ and $X \in F(c)$, and morphisms* $\mathrm{Hom}((c, X), (d, Y))$ *the pairs (f, α) with $f : c \to d$ and $\alpha : X \to f^*Y$. Composition is defined as before, but with judiciously inserted structural transformations $\alpha_{\bullet,\bullet}$ and $\epsilon_\bullet$.*

It is proved in the reference that this does, in fact, define a category. Note that there is an obvious functor $\mathcal{C} \int F \to \mathcal{C}$. The equivalence between pseudofunctors and fibred categories alluded to before can now be expressed as follows.

Theorem A.52. *The construction $F \mapsto \mathcal{C} \int F$ extends to a faithful functor*

$$\mathcal{C} \int : PsFun(\mathcal{C}^{op}, Cat) \to Cat/\mathcal{C}.$$

The essential image consists of the fibred categories and their morphisms.

Proof. This is well known. Parts of it are explained in [37, Expose VI, Chapter 8] and [105, Section 3.1.3]. Neither references details functoriality, i.e. the relevance of pseudonatural transformations. We sketch this part.

Let F, G be pseudofunctors on $\mathcal{C}$. Then a pseudonatural transformation $\Theta : F \to G$ consists of the following data: (i) for every $c \in \mathcal{C}$ and every $X \in F(c)$ an object $\Theta_c(X) \in G(c)$, (ii) for $c \in \mathcal{C}$, $\alpha : X \to Y \in F(c)$ a morphism $\Theta_c(\alpha) : \Theta_c(X) \to \Theta_c(Y)$, (iii) for every $f : c \to d \in \mathcal{C}$ and every $X \in F(d)$, an isomorphism $f^*\Theta_d(X) \cong \Theta_c(f^*X)$. Of course these data have to satisfy certain constraints.

In contrast, a functor $\Phi : \mathcal{C} \int F \to \mathcal{C} \int G$ over $\mathcal{C}$ consists of the following data: (a) for every $c \in \mathcal{C}$ and every $X \in F(c)$ an object $\Phi((c, X)) \in G(c)$, (b) for every $f : c \to d \in \mathcal{C}, X \in F(c), Y \in F(d), \alpha : X \to f^*Y$ a morphism $\Phi((f, \alpha)) : \Phi((c, X)) \to f^*\Phi((d, Y))$; again these have to satisfy some constraints. The data looks a bit different, but note that the morphism $(f, \alpha) : (c, X) \to (d, Y)$ can be factored as $(c, X) \to (c, f^*Y) \to (d, Y)$, so that the data in (b) can be split as (b') for every $c \in \mathcal{C}, \alpha : X \to Y \in F(c)$ a morphism $\Phi((\mathrm{id}, \alpha)) : \Phi((c, X)) \to \Phi((d, Y))$ and (b") for every $f : c \to d \in \mathcal{C}$ and $Y \in F(d)$ a morphism $\Phi((c, f^*Y)) \to f^*\Phi((d, Y))$.

It is then easy to check that the data (i), (ii), (iii) define in a natural way data (a), (b'), (b") and that the constraints (i) to (iii) imply the constraints on (a), (b'), (b"). We thus have a faithful functor $PsFun(\mathcal{C}^{op}, Cat) \to Cat/\mathcal{C}$. One checks that the image consists of fibred categories. A morphism of fibred categories is just a functor over $\mathcal{C}$ preserving cartesian arrows [105, Definition 3.6]; one checks that this precisely means that the morphism in (b") has to be an isomorphism, which then implies that every morphism of fibred categories $\mathcal{C} \int F \to \mathcal{D} \int F$ comes from a pseudonatural transformation. □

The Grothendieck construction has another use, namely in computing homotopy colimits. For this, suppose first that F is a strict functor on $\mathcal{C}$. We can compose with the nerve construction to get a diagram $NF : \mathcal{C}^{op} \to sSet$. This has a homotopy colimit $\mathrm{hocolim}_{\mathcal{C}} NF \in sSet$. If F is only a pseudofunctor, then NF is not a diagram (but some sort of homotopy coherent diagram), and there is no immediate way to define its homotopy colimit. However all is not lost, because we can use the rectification functor.

Definition A.53. *Let $\mathcal{C}$ be a small category and F a pseudofunctor on $\mathcal{C}$. We define the homotopy colimit of its nerves as*

$$\operatorname{hocolim}^*_{\mathcal{C}} NF := \operatorname{hocolim}_{\mathcal{C}} N(F^r).$$

This definition has many desirable properties:

Proposition A.54. *(i) If $\Theta : F \to G$ is a pseudonatural transformation, then there is a functorial morphism $\operatorname{hocolim}^*_{\mathcal{C}} N(F) \to \operatorname{hocolim}^*_{\mathcal{C}} N(G)$.*

(ii) If Θ is a pseudonatural transformation such that for every $c \in \mathcal{C}$, the induced map of simplicial sets $N(F(c)) \to N(G(c))$ is a weak equivalence (for example if Θ is a pseudonatural equivalence), then the induced morphism $\operatorname{hocolim}^*_{\mathcal{C}} N(F) \to \operatorname{hocolim}^*_{\mathcal{C}} N(G)$ is a weak equivalence.

(iii) If F is a strict functor then $\operatorname{hocolim}^*_{\mathcal{C}} N(F)$ is naturally weakly equivalent to the ordinary $\operatorname{hocolim}_{\mathcal{C}} N(F)$.

Because of (iii), we will usually just write $\operatorname{hocolim}_I N(F)$ for either construction, dropping the "*". Note that (i) is useful even in the case of strict functors, since the usual construction is not functorial for pseudonatural transformations.

Proof. (i) holds because rectification $F \mapsto F^r$ is a functor. (ii) is just homotopy invariance of the ordinary homotopy colimit (together with the fact that equivalences of categories induce homotopy equivalences of simplicial sets). For (iii), note that for a strict functor F there is via adjunction a (natural) *strict* transformation $(iF)^r = iL(iF) \to iF$ which is a component-wise equivalence. Now ordinary hocolim of the nerve of the left hand side is $\operatorname{hocolim}^*$, which is thus equivalent to ordinary hocolim of the nerve of the right hand side, by the same argument as for (ii). □

We can now explain the relationship between the Grothendieck construction an homotopy colimits:

Theorem A.55. *Let F be a pseudofunctor on $\mathcal{C}$. There is a natural zig-zag of weak equivalences*

$$N\left(\mathcal{C} \int F\right) \to N\left(\mathcal{C} \int F^r\right) \leftarrow \operatorname{hocolim}^*_{\mathcal{C}} NF.$$

Proof. If $\Theta : F \to G$ is a pseudonatural equivalence then $\mathcal{C} \int \Theta : \mathcal{C} \int F \to \mathcal{C} \int G$ is an equivalence of categories [105, 3.36]. Hence in the zig-zag the first map, which is obtained from the adjunction morphism $F \to F^r$, is a weak equivalence. It is thus enough to exhibit for an *ordinary* functor F a natural weak equivalence $\operatorname{hocolim}_{\mathcal{C}} NF \to N(\mathcal{C} \int F)$. Using the fact that both $\operatorname{hocolim}_{\mathcal{C}} NF$ and $N(\mathcal{C} \int F)$ are defined by similar ends, and that the nerve functor N is a right adjoint, such a natural map is easily found. The main content of the theorem is that this map is always a weak equivalence, which is proved in [104]. □

A.6.3 Homotopy Limits and Further Comments

We wish to repeat the above treatment for homotopy limits.

Definition A.56. *Let $\mathcal{C}$ be a small category and F a pseudofunctor on $\mathcal{C}$. We define the homotopy limit of its nerves as*

$$\operatorname{holim}^*_{\mathcal{C}} NF := \operatorname{holim}_{\mathcal{C}} N(F^r).$$

This definition has the same desirable properties as before:

Proposition A.57. *(i) If $\Theta : F \to G$ is a pseudonatural transformation, then there is a functorial morphism $\operatorname{holim}^*_{\mathcal{C}} N(F) \to \operatorname{holim}^*_{\mathcal{C}} N(G)$.*

(ii) If Θ is a pseudonatural transformation such that for every $c \in \mathcal{C}$, the induced map of simplicial sets $N(F(c)) \to N(G(c))$ is a weak equivalence (for example if Θ is a pseudonatural equivalence), then the induced morphism $\operatorname{holim}^*_{\mathcal{C}} N(F) \to \operatorname{holim}^*_{\mathcal{C}} N(G)$ is a weak equivalence.

(iii) If F is a strict functor then $\operatorname{holim}^*_{\mathcal{C}} N(F)$ *is naturally weakly equivalent to the ordinary* $\operatorname{holim}_{\mathcal{C}} N(F)$.

The proof is basically the same as for homotopy colimits, so we omit it. As before we just write $\operatorname{holim}_{\mathcal{C}} NF$, omitting the "*", if no confusion can arise. We would like to find a category $\mathcal{L}$ such that $N(\mathcal{L}) \simeq \operatorname{holim}_{\mathcal{C}} NF$, just like $N(\mathcal{C} \int F) \simeq \operatorname{hocolim}_{\mathcal{C}} NF$. Unfortunately this does not seem to work in general. Here is a natural candidate for $\mathcal{L}$:

Definition A.58. *Let F be a pseudofunctor on $\mathcal{C}$. Define the category of* (right) sections *of F, $Sect(\mathcal{C}, F)$ (sometimes denoted $Sect^R(\mathcal{C}, F)$) to be the full subcategory of $Fun(\mathcal{C}, \mathcal{C} \int F)$ consisting of functors which are sections of the natural projection $\mathcal{C} \int F \to \mathcal{C}$.*

We record the following more conventional description:

Lemma A.59. *The category $Sect(\mathcal{C}, F)$ is isomorphic to the category whose objects are families $(X_c)_{c \in \mathcal{C}}$, where $X_c \in F(c)$, together with structural maps $X_f : X_c \to f^* X_d$ for any $f : c \to d \in \mathcal{C}$, satisfying evident cocycle and identity conditions (which involve the composition and identity transformations $\alpha_{\bullet,\bullet}$ and $\epsilon_\bullet$ if F is not a strict functor). The morphisms from $(X_c)_c$ to $(Y_c)_c$ are the families of morphisms $(\phi_c)_c, \phi_c : X_c \to Y_c$, such that for any $f : c \to d$ the following diagram commutes:*

$$\begin{array}{ccc} X_c & \xrightarrow{X_f} & f^* X_d \\ {\scriptstyle \phi_c}\downarrow & & \downarrow{\scriptstyle f^*\phi_d} \\ Y_c & \xrightarrow{Y_f} & f^* Y_d. \end{array}$$

The following is a weaker version of Theorem A.55 for the case of homotopy limits.

Proposition A.60. *Let $\mathcal{C}$ be a small category.*

(i) If $\Theta : F \to G$ is a pseudonatural equivalence of pseudofunctors, then the induced functor $Sect(\mathcal{C}, F) \to Sect(\mathcal{C}, G)$ is an equivalence of categories.

(ii) If F is a strict functor, there is a natural isomorphism

$$N(Sect(\mathcal{C}, F)) \cong \text{"}\operatorname{holim}_{\mathcal{C}}\text{"} NF.$$

(iii) Consequently, for any pseudofunctor F, there is a natural string of morphisms

$$N(Sect(\mathcal{C}, F)) \to N(Sect(\mathcal{C}, F^r)) \to \operatorname{holim}_{\mathcal{C}} N(F^r),$$

where the first map is always a weak equivalence.

The problem is that the second map does not seem to be a weak equivalence in general. One case in which this works is explained in the next section.

Proof. If $\Theta : F \to G$ is a pseudonatural equivalence then $\mathcal{C} \int \Theta : \mathcal{C} \int F \to \mathcal{C} \int G$ is an equivalence of categories [105, 3.36], as before. It follows that $Sect(\mathcal{C}, F) \to Sect(\mathcal{C}, G)$ is an equivalence of categories, as desired.

Suppose now F is a strict functor. Then the description of $\mathcal{C} \int F$ as a coend via Lemma A.50 implies that

$$Sect(\mathcal{C}, F) \cong \hom^{\mathcal{C}}(/\mathcal{C}, F) = eq \left[\prod_{c \in \mathcal{C}} Fun(c/\mathcal{C}, F(c)) \rightrightarrows \prod_{c \to d \in \mathcal{C}} Fun(d/\mathcal{C}, F(c)) \right].$$

The nerve functor being right adjoint commutes with limits, and since it is fully faithful and commutes with products (being limits) it preserves cotensors. Consequently the nerve functor preserves ends, and we conclude that

$$N(Sect(\mathcal{C}, F)) \cong N(\hom^{\mathcal{C}}(/\mathcal{C}, F)) \cong \hom^{\mathcal{C}}(N(/\mathcal{C}), NF) = \text{"}\operatorname{holim}_{\mathcal{C}}\text{"} NF.$$

Here the last identification is just by (our) definition of holim, i.e. Definition A.17. This proves (ii).

To obtain the second map in (iii) we compose the isomorphism from (ii) with the natural map "$\mathrm{holim}_{\mathcal{C}} NF$" $\to \mathrm{holim}_{\mathcal{C}} NF$. The first map is an equivalence by (i). □

We also have the following observation, which is surely well known. We say that a pseudofunctor F on $\mathcal{C}^{op}$ is a right (left) pseudofunctor if for each $f : X \to Y \in \mathcal{C}$ the restriction f^* is a right (left) adjoint.

Lemma A.61. *Let F, G be right (left) pseudofunctors on $\mathcal{C}$ and $\Theta : F \to G$ a pseudonatural transformation. Suppose that each component Θ_c affords a left (right) adjoint Ω_c. Then the induced functor $Sect(\Theta) : Sect(\mathcal{C}, F) \to Sect(\mathcal{C}, G)$ affords a left (right) adjoint Ω which on the level of objects satisfies $\Omega(Y)_i = \Omega_i Y_i$.*

Proof. The problem is invariant under pseudonatural equivalence, so we may assume that F, G are strict. The result then follows from the proof of [9, Lemma 1.25]. □

A.7 Homotopy Limits of Model Categories and their Nerves

In order to call the (pseudo-) presheaves of the above section (homotopical) sheaves, we have to make sense of homotopy limits of model categories. Since the category of model categories is not (known to be) a model category, there is no obvious way of doing this. We shall use a definition of Barwick, and prove that it has all the properties we need.

A.7.1 Quillen (Pseudo-) Presheaves and Homotopy Limits of Model Categories

We begin by recalling Barwick's definition of homotopy limits of model categories. That takes some preparation.

Definition A.62. *Let $\mathcal{C}$ be a small category. A* right (left) Quillen (pseudo-) presheaf *$\mathcal{M}$ on $\mathcal{C}$ consists of a (pseudo-) functor $\mathcal{M}$ on $\mathcal{C}$, together with a model category structure on $\mathcal{M}(c)$ for every $c \in \mathcal{C}$, such that for each $f : c \to d \in \mathcal{C}$, the pullback $f^* : \mathcal{M}(d) \to \mathcal{M}(c)$ is a right (left) Quillen functor.*

We call $\mathcal{M}$ (left/right) proper, tractable, combinatorial etc. *if $\mathcal{M}(c)$ is, for every $c \in \mathcal{C}$.*

We call $\mathcal{M}$ simplicial *if $\mathcal{M}(c)$ is provided with the structure of a simplicial model category for every $c \in \mathcal{C}$, and all the pullbacks f^* are simplicial (right/left) Quillen functors.*

If $\Theta : \mathcal{M} \to \mathcal{N}$ is a pseudonatural transformation of Quillen pseudo-presheaves, we call Θ a right (left) morphism if each $\Theta_c : \mathcal{M}(c) \to \mathcal{N}(c)$ is a right (left) Quillen functor. If $\mathcal{M}$ and $\mathcal{N}$ are simplicial we call Θ a simplicial morphism if each Θ_c is a simplicial functor.

Note that if $\mathcal{M} \to \mathcal{N}$ is a pseudonatural equivalence of pseudofunctors, and $\mathcal{N}$ (or $\mathcal{M}$) is a Quillen pseudo-presheaf, then $\mathcal{M}$ (or $\mathcal{N}$) is a Quillen pseudo-presheaf in a unique way such that each $\mathcal{M}(c) \to \mathcal{N}(c)$ is an equivalence of model categories. Thus Quillen pseudo-presheaves can be rectified to (strict) Quillen presheaves, etc.

Now let $\mathcal{M}$ be a right Quillen pseudo-presheaf on a small category I. We wish to define a model category which is the homotopy limit of $\mathcal{M}$ over I. As hinted in the last section, a good candidate would be $Sect(I, \mathcal{M})$. We need an appropriate model structure. A first candidate is as follows.

Lemma A.63. *Let I be a small category and $\mathcal{M}$ a combinatorial right Quillen pseudo-presheaf on I. The category $Sect(I, \mathcal{M})$ affords a combinatorial model structure (called the* projective model structure*) in which fibrations and weak equivalences are determined objectwise.*

If $\mathcal{M}$ is (left/right) proper, tractable or simplicial and tractable so is $Sect(I, \mathcal{M})$.

The objectwise fibrations and weak equivalences are also known as *projective* fibrations and weak equivalences.

Proof. By the remarks after the definition of Quillen pseudo-presheaves, we may rectify $\mathcal{M}$, i.e. assume that $\mathcal{M}$ is an ordinary Quillen presheaf (use also part one of Proposition A.60). In this case the combinatorial model structure is established in [9, Theorem 1.30], and properness is established in [9, Proposition 1.33].

For the simplicial structure, we put $((X_i)_i \otimes K)_i = X_i \otimes K$, $((X_i)_i^K)_i = X_i^K$ and define $Map((X_i)_i, (Y_i)_i) \subset \prod_i Map(X_i, Y_i)$ to be the subset of elements commuting with the structure maps, just as in the definition of $\mathrm{Hom}((X_i)_i, (Y_i)_i)$. Here we use that the structure maps are of simplicial degree zero, so promote unambiguously via degeneracies to higher degree. □

The projective cofibrations (i.e. cofibrations in the projective model structure on $Sect(I, \mathcal{M})$) are in general hard to describe, but we have the following result.

Lemma A.64. *If $X \to Y$ is a (projective) cofibration in $Sect(I, \mathcal{M})$ then each entry $X_i \to Y_i$ is a cofibration in $\mathcal{M}(i)$.*

Proof. Fix $i \in I$. The natural functor $F^* : Sect(I, \mathcal{M}) \to \mathcal{M}(i), X \mapsto X_i$ has a right adjoint $F_* : \mathcal{M}(i) \to Sect(I, \mathcal{M})$ given by

$$(F_* X)_{i'} = \prod_{\alpha : i' \to i} \alpha^* X.$$

Since fibrations and acyclic fibrations are stable under products (being definable by RLP), the functor F_* is right Quillen. Thus F^* is left Quillen and so preserves cofibrations. □

This model structure does not describe the homotopy limit. This is fairly clear: the limit of a set-valued functor $X : I^{op} \to Set$ consists of families $L \subset \prod_i X(i)$ which are *compatible*. The homotopy limit of a Quillen presheaf $\mathcal{M}$ should certainly have something to do with the product $\prod_i \mathcal{M}(i)$. Since asking for two objects in a category to be equal is not in general very sensible (as opposed to asking for two objects in a set to be equal) we have to introduce comparison morphisms, this is what the category $Sect(I, \mathcal{M})$ does. But it is clear that not all elements of $Sect(I, \mathcal{M})$ should model elements of the homotopy limit, because the comparison morphisms need not be equivalences in any sense (i.e. the families of objects are only very loosely compatible). The following definition introduces the right kind of compatible family.

Definition A.65. *A section $(X_i)_i \in Sect(I, \mathcal{M})$ is called* homotopy cartesian *if for each $f : i \to j \in I$ the comparison morphism $X_i \to Rf^* X_j$ is a weak equivalence.*

Theorem A.66 (Barwick)**.** *Let I be a small category and $\mathcal{M}$ a left proper, combinatorial right Quillen pseudo-presheaf on I. The model structure on $Sect(I, \mathcal{M})$ affords a unique Bousfield localization such that the fibrant objects are the projective fibrant homotopy cartesian sections. The new model structure is left proper and combinatorial.*

If $\mathcal{M}$ is tractable or simplicial and tractable, so is the new model structure on $Sect(I, \mathcal{M})$.

We write $\mathrm{holim}_I \mathcal{M}$ for the category $Sect(I, \mathcal{M})$ with this model structure. The weak equivalences are known as holim-local weak equivalences, and the fibrations as holim-local fibrations.

Proof. As before we may assume that $\mathcal{M}$ is strict. The existence of the model structure is then [9, Theorem 2.42]. It is clear that any Bousfield localization is determined by its local objects, so uniqueness follows. Left properness, combinatoriality and tractability are always preserved by left Bousfield localization [9, Theorem 2.11, Proposition 2.15]. The Bousfield localization of a simplicial model category is a simplicial model category, simply because it coincides with the simplicial Bousfield localization. □

We now intend to establish certain properties of the $\mathrm{holim}_I \mathcal{M}$ construction parallel to ordinary (homotopy) limits. We begin with two simple ones.

Proposition A.67. *Let $\Theta : \mathcal{M} \to \mathcal{N}$ be a right morphism of combinatorial right Quillen pseudo-presheaves. Then the induced functor* $\operatorname{holim}_I \Theta : \operatorname{holim}_I \mathcal{M} \to \operatorname{holim}_I \mathcal{N}$ *is right Quillen.*

If each component Θ_c is a Quillen equivalence, then so is $\operatorname{holim}_I \Theta$.

If Θ is simplicial so is $\operatorname{holim}_I \Theta$

Proof. By Lemma A.61 the functor $\operatorname{holim}_I \Theta$ affords a left adjoint which we shall denote $\Theta_\#$. Let us also write Θ for $\operatorname{holim}_I \Theta$. It is clear that Θ is right Quillen in the projective (instead of holim) model structure. Indeed each Θ_i preserves fibrations and acyclic fibrations (being right Quillen), and thus Θ preserves fibrations and acyclic fibrations (these being defined objectwise).

For the holim model structure, it suffices to prove that Θ preserves fibrant objects, fibrations between fibrant objects, and acyclic fibrations [43, Proposition 8.5.4]. The acyclic fibrations and fibrations between fibrant objects in the holim model structure are the same as in the projective model structure [43, Propositions 3.3.3(1)(b) and 3.3.16] (fibrant objects in the Bousfield localisation are $\mathcal{C}$-local by [43, Propositions 3.3.11 and 3.3.14 with $Y = W = *$]). It hence suffices to show that fibrant objects are preserved. But a section is (holim-) fibrant if and only if it is projectively fibrant and all the structure maps are weak equivalences. Since weak equivalences between fibrant objects are preserved by all the Θ_i (being right Quillen), we are done.

Next we show that if each Θ_i is a Quillen equivalence, then so is $\operatorname{holim}_i \Theta$. Again for the projective model structure this is clear: if $X \in Sect(I, \mathcal{M})$ is projective cofibrant and $Y \in Sect(I, \mathcal{N})$ is projective fibrant, we need that a morphism $X \to \Theta Y$ is a weak equivalence if and only if the adjoint $\Theta_\# X \to Y$ is a weak equivalence [43, Definition 8.5.20]. Note that each Y_i is fibrant (by definition of projective fibrancy) and each X_i is cofibrant, by Lemma A.64. Now $X \to \Theta Y$ is a weak equivalence if and only if each $X_i \to (\Theta Y)_i = \Theta_i Y_i$ is a weak equivalence for all i (by definition of projective weak equivalences and $\Theta = \operatorname{holim}_i \Theta$), which happens if and only if $\Theta_{i\#} X_i \to Y_i$ is a weak equivalence (because $\Theta_{i\#}, \Theta_i$ form a Quillen equivalence and X_i is cofibrant, Y_i is fibrant), which happens if and only if $\Theta_\# X \to Y$ is a weak equivalence (by definition of projective weak equivalence and Lemma A.61).

To extend the result to the holim model structure, it is enough to show that both $R\Theta$ and $L\Theta_\#$ preserve holim-local objects. This is clear for $R\Theta$, this functor being right Quillen. For $L\Theta_\#$ this is harder. Let $Y \in Sect(I, \mathcal{N})$ be a cofibrant holim-local section and fix $f : i \to j \in I$. We need to show that the composite $\Theta_{i\#} Y_i \to f^* \Theta_{j\#} Y_j \to f^* R_f \Theta_{j\#} Y_j$ is a weak equivalence. Since Θ_i is a Quillen equivalence and f^* is right Quillen (so preserves fibrant objects), this is the same as requiring that the adjoint $Y_i \to \Theta_i f^* R_f \Theta_{j\#} Y_j$ is a weak equivalence. To do this consider the following commutative diagram.

$$\begin{array}{ccccc}
Y_i & \longrightarrow & \Theta_i f^* \Theta_{j\#} Y_j & \longrightarrow & \Theta_i f^* R_f \Theta_{j\#} Y_j \\
{\scriptstyle Y_f}\downarrow & & \cong\downarrow & & \downarrow{\scriptstyle w} \\
f^* Y_j & \longrightarrow & f^* \Theta_j \Theta_{j\#} Y_j & \longrightarrow & f^* \Theta_j R_f \Theta_{j\#} R_f Y_j \\
\downarrow & & & & \| \\
f^* R_f Y_j & \longrightarrow & f^* \Theta_j \Theta_{j\#} R_f Y_j & \longrightarrow & f^* \Theta_j R_f \Theta_{j\#} R_f Y_j
\end{array}$$

Here the upper left square commutes by definition of the functor $\Theta_\#$. All the other maps come from adjunction units and fibrant replacement, making the other two squares easy to check. The map from top left to bottom left is a weak equivalence because Y is holim-local. The map from bottom left to bottom right is a weak equivalence because Θ_j is a Quillen equivalence, f^* preserves weak equivalences between fibrant objects, R_f preserves cofibrant objects, and Y_j is cofibrant by Lemma A.64. The map labelled w is a weak equivalence by a similar argument.

It follows that the map from top left to top right is a weak equivalence, as was to be shown.

If Θ is simplicial then clearly so is $\operatorname{holim}_I \Theta$. □

Proposition A.68. *Let I, J be small categories and $\mathcal{M}$ be a left proper, combinatorial right Quillen pseudo-presheaf on $I \times J$. Then the model categories* $\operatorname{holim}_{I \times J} \mathcal{M}$ *and* $\operatorname{holim}_I \operatorname{holim}_J \mathcal{M}$ *are canonically isomorphic.*

Before the proof we record the following immediate corollary.

Corollary A.69. *Let $\mathcal{M}$ be a combinatorial right Quillen pseudo-presheaf on $I \times J$. Then* $\operatorname{holim}_I \operatorname{holim}_J \mathcal{M}$ *and* $\operatorname{holim}_J \operatorname{holim}_I \mathcal{M}$ *are canonically isomorphic as model categories (in particular Quillen equivalent).*

Proof of Proposition. Recall that $\operatorname{holim}_{I\times J} \mathcal{M}$ means $Sect(I\times J, \mathcal{M})$ with a certain localised model structure. The same holds for the double holim, of course.

As ordinary categories, there is an evident isomorphism

$$\Theta : Sect(I \times J, \mathcal{M}) \to Sect(I, Sect(J, \mathcal{M})).$$

We have to prove that it identifies the classes of cofibrations, fibrations and weak equivalences. I claim that Θ identifies the fibrant objects, fibrations between fibrant objects, and acyclic fibrations.

Suppose the claim holds and let θ be the inverse of Θ. Then θ, Θ are both left and right adjoint to each other. We shall use a result of Dugger [43, Proposition 8.5.4] which states that a pair of adjoint functors between model categories is Quillen if and only if the right adjoint preserves acyclic fibrations and fibrations between fibrant objects. It follows from the claim that this result applies to both Θ and θ, so both are right Quillen. Consequently both are also left Quillen. It follows that both Θ and θ preserve fibrations, cofibrations and weak equivalences (every weak equivalence can be factored into an acyclic cofibration followed by an acyclic fibration), so are inverse isomorphisms of model categories.

Let us prove the claim. By construction, the fibrant objects of $\operatorname{holim}_{I\times J} \mathcal{M}$ are the objectwise fibrant (homotopy) cartesian sections. A fibration between fibrant objects in the holim model structure is the same as a fibration in the projective model structure [43, Proposition 3.3.16], i.e. an objectwise fibration. Finally an acyclic fibration in the holim model structure is the same thing as an acyclic fibration in the projective model structure, i.e. an objectwise acyclic fibration. The analysis of $\operatorname{holim}_I \operatorname{holim}_J \mathcal{M}$ is similar (using that a weak equivalence or fibration between fibrant objects in a left Bousfield localisation is the same as a weak equivalence or fibration in the original model structure). Thus Θ identifies the classes of acyclic fibrations, fibrant objects, and fibrations between fibrant objects, as claimed. □

In the next three subsections we establish more complicated properties of homotopy limits of model categories, concluding with the crucial Theorem A.80. The method of proof was suggested to the author by Bill Dwyer.

A.7.2 Changing the Index Category

Definition A.70. *Let $F : I \to J$ be a functor of small categories.*

1. *By precomposition we obtain obvious functors $F^* : PsFun(J^{op}, Cat) \to PsFun(I^{op}, Cat)$ and for any category $\mathcal{C}$, $F^* : Fun(J^{op}, \mathcal{C}) \to Fun(I^{op}, \mathcal{C})$.*
2. *For $j \in J$ we define the overcategory j/F to have as objects the pairs (i, f) with $i \in I$ and $f : j \to Fi$, and as morphisms from (i_1, f_1) to (i_2, f_2) the maps $\alpha : i_1 \to i_2$ such that*

$$\begin{array}{ccc} j & = & j \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ Fi_1 & \xrightarrow{F\alpha} & Fi_2 \end{array}$$

commutes.

3. *The functor F is called* homotopy cofinal *if for each $j \in J$ the nerve $N(j/F)$ is contractible.*

We then have the following well-known result.

Theorem A.71. *Let $F : I \to J$ be a homotopy cofinal functor of small categories and $\mathcal{M}$ a simplicial model category. If $X : J^{op} \to \mathcal{M}$ is a diagram, then there is a natural weak equivalence*

$$\operatorname{holim}_J X \to \operatorname{holim}_I F^* X.$$

Proof. See e.g. [43, Theorem 19.6.7 (2)]. Note that we do not need the objectwise fibrancy assumption because of our "holim"/holim convention. Our definition of homotopy cofinal is the same as Hirschhorn's homotopy left cofinal for F^{op}. Simplicial model categories are canonically framed by [43, Proposition 16.6.4]. □

We intend to prove a similar result for homotopy limits of model categories. We begin with the following.

Proposition A.72. *Let $F : I \to J$ be a functor of small categories and $\mathcal{M}$ a pseudo-presheaf on J.*

1. *There is a canonical functor $F^* : Sect(J, \mathcal{M}) \to Sect(I, F^*\mathcal{M})$ defined on objects by $(F^*X)_i = X_{Fi}$.*
2. *The functor F^* has a left adjoint $F_\#$ and a right adjoint F_*.*
3. *If $\mathcal{M}$ is simplicial then so are $F^*, F_\#, F_*$.*
4. *If $\mathcal{M}$ is a left proper, combinatorial right Quillen pseudo-presheaf, then $F^* : \operatorname{holim}_I \mathcal{M} \to \operatorname{holim}_J F^*\mathcal{M}$ is right Quillen.*

Proof. (1) and (3) are clear. Limits and colimits in $Sect(I, \mathcal{M})$ are computed objectwise and similarly for J, so F^* commutes with all limits and colimits, and so affords a right and a left adjoint by general nonsense. Hence (2).

For (4), first note that F^* preserves projective fibrations and weak equivalences, hence is right Quillen in the projective Model structure. To prove it remains right Quillen in the holim model structure, by [43, Proposition 8.5.4] it suffices to prove that F^* preserves holim-fibrant objects, fibrations between such objects, and general acyclic holim-fibrations. The latter are the same as projective acyclic fibrations and so are preserved. Holim-fibrant objects are (homotopy) cartesian projective fibrant sections, which are preserved. Fibrations between holim-fibrant objects are projective fibrations [43, Proposition 3.3.16] and so are also preserved. □

Let us point out that usually F^* does not preserve cofibrations, and so F_* is usually not right Quillen. This *is* the case if I is discrete, as we implicitly exploited in the proof of Lemma A.64.

The functor $F_\#$ is some kind of relative homotopy colimit (left Kan extension). We can make this explicit in the case that $J = *$ is the final category, and thus $\mathcal{M}$ is a constant presheaf.

Proposition A.73. *Let $\mathcal{M}$ be a combinatorial model category and I a small category. Let $F : I \to *$ be the unique functor and view $\mathcal{M}$ as a right Quillen presheaf on $*$.*

1. *The model category $Sect(I, F^*\mathcal{M})$ (with its projective model structure) is canonically isomorphic to the model category $Fun(I, \mathcal{M})$ of covariant I-diagrams in $\mathcal{M}$, with its projective model structure.*
2. *Suppose $\mathcal{M}$ is left proper. The following diagram commutes up to natural isomorphism.*

$$\begin{array}{ccc} Ho(\operatorname{holim}_I F^*\mathcal{M}) & \xrightarrow{LF_\#} & Ho(\mathcal{M}) \\ \downarrow & & \| \\ Ho(Fun(I, \mathcal{M})) & \xrightarrow{\text{hocolim}} & Ho(\mathcal{M}). \end{array}$$

Here the functor $Ho(\operatorname{holim}_I F^*\mathcal{M}) \to Ho(Fun(I, \mathcal{M}))$ is the canonical embedding coming from the fact that $\operatorname{holim}_I F^*\mathcal{M}$ is a Bousfield localization of $Sect(I, F^*\mathcal{M}) \cong Fun(I, \mathcal{M})$. The functor $\text{hocolim} : Ho(Fun(I, \mathcal{M})) \to Ho(\mathcal{M})$ is obtained by either applying "hocolim_I" to an objectwise cofibrant diagram, or by applying colim_I to a projective cofibrant diagram.

Proof. Statement (1) is immediate. For statement (2), note first that as explained in [33, Section 4], the two descriptions of hocolim : $Ho(Fun(I,\mathcal{M})) \to Ho(\mathcal{M})$ we have given agree. Thus the functor is right adjoint to the constant diagram functor, just as is $LF_\#$. The result follows by essential uniqueness of adjoints. □

The main result of this section is the following.

Theorem A.74. *Let $F : I \to J$ be a homotopy cofinal functor of small categories and $\mathcal{M}$ a left proper, combinatorial, simplicial right Quillen pseudo-presheaf on J.*

Then the Quillen adjunction

$$F_\# : \operatorname{holim}_I F^*\mathcal{M} \leftrightarrows \operatorname{holim}_J \mathcal{M} : F^*$$

is a Quillen equivalence.

The proof is somewhat complicated and occupies the rest of this subsection. By (a weak version of) Proposition A.67, the theorem is invariant under pseudonatural (Quillen) equivalences in $\mathcal{M}$. In particular we may and will from now on assume that $\mathcal{M}$ is a *strict* functor.

Recall that a Quillen adjunction is a Quillen equivalence if and only if the derived adjunction $LF_\# : Ho(\operatorname{holim}_I F^*\mathcal{M}) \rightleftarrows Ho(\operatorname{holim}_J \mathcal{M}) : RF^*$ consists of equivalences.

Unfortunately, cofibrations in the projective model structure are hard to understand, and hence so is the derived functor $LF_\#$. We shall employ a trick similar to [61, Section 6.4]. Namely, we will prove that RF^* actually has a *right* adjoint RF_* (even though F^* is not left Quillen!) which is easier to understand. To prepare, we note that the right adjoint F_* is given by

$$(F_*X)_j = \lim_{\alpha:j\to Fi} \alpha^* X_i.$$

Here the limit is taken over the category j/F. (Note that this is the limit of a *covariant* diagram, not a contravariant one as we usually see.)

We consider a "souped-up" version of F_* as follows:

$$(\text{“}RF_*\text{”}X)_j = \text{“}\operatorname*{holim}_{\alpha:j\to Fi}\text{”}\alpha^* X_i.$$

The "homotopy limit" is still over the category j/F (recall our conventions on "holim"/holim from Subsection A.1.5).

Proposition A.75. *The above assignment yields a well-defined functor "RF_*" : $Sect(I,F^*\mathcal{M}) \to Sect(J,\mathcal{M})$. The functor "$RF_*$" is right Quillen (with respect to the projective model structures) and so descends to a functor $RF_* : Ho(Sect(I,F^*\mathcal{M})) \to Ho(Sect(J,\mathcal{M}))$. The functor RF_* is right adjoint to $F^* = RF^* : Ho(Sect(J,\mathcal{M})) \to Ho(Sect(I,F^*\mathcal{M}))$.*

Proof. First we need to provide, for each $\gamma : j \to j' \in J$ a structure map $(\text{“}RF_*\text{”}X)_j \to \gamma^*(\text{“}RF_*\text{”}X)_{j'}$. Since γ^* commutes with limits (being a right adjoint) and cotensors (by Lemma A.6), it commutes with "holim". We thus need to provide a map

$$\text{“}\operatorname*{holim}_{\alpha:j\to Fi}\text{”}\alpha^* X_i \to \text{“}\operatorname*{holim}_{\beta:j'\to Fi}\text{”}\gamma^*\beta^* X_i.$$

Now $\gamma^*\beta^* = (\beta\circ\gamma)^*$ and so the holim on the right is naturally over a subcategory of the holim on the left, whence there is a natural comparison map (projecting to the components). One may verify that these structure maps satisfy our cocycle condition and are natural in morphisms $X \to Y$. This establishes that "RF_*" is a functor. (It is easy to see that "RF_*" preserves weak equivalences between termwise fibrant objects, but this also follows from the rest of what we do.)

The second half of the proposition is more difficult. We begin by defining a functor "LF^*" : $Sect(J,\mathcal{M}) \to Sect(I,F^*\mathcal{M})$ as follows. For $j \in J, i \in I$ write $j/F/i$ for the category whose objects are pairs $(i' \to i \in I, j \to Fi' \in J)$. Given $\alpha : j \to Fi \in J$ write $(j/F/i)_\alpha$ for the full subcategory of those pairs $(i' \to i, j \to Fi')$ such that the composite $j \to Fi' \to Fi$ is α. (Observe that $j/F/i = \coprod_\alpha (j/F/i)_\alpha$.) Denote by $(J/F/i)$ the J/Fi-diagram $(\alpha : j \to Fi) \mapsto (j/F/i)_\alpha$ and put

$$(\text{“}LF^{*}\text{”}Y)_i = N(J/F/i) \otimes_{J/Fi} \alpha_{\#} Y.$$

Here we denote by $\alpha_{\#}Y$ the J/Fi-diagram which associates with $\alpha : j \to Fi$ the object $\alpha_{\#}Y_j$, and we use the notation for coends from Section A.1.5. One shows as before that these coends fit together via structure maps and so this defines a functor “LF^*”.

Define the functor $\tilde{F}^* : Sect(J, \mathcal{M}) \to Sect(I, F^*\mathcal{M})$ in the same way as “LF^*”, but where instead of using the J/Fi-diagram $N(J/F/i)$ we use the constant diagram $*$. Then there is an evident natural transformation “LF^*” $\Rightarrow \tilde{F}^*$. Then since J/Fi has a terminal object one just finds that $\tilde{F}^* \cong F^*$, and so we have a natural transformation “LF”$^* \Rightarrow F^*$.

I claim that “LF^*” and “RF_*” are adjoint. This is a messy but straightforward computation, which we defer to the end of the proof. The functor “RF_*” preserves projective fibrations and acyclic fibrations by [43, Corollary 18.4.2(2)]. It follows that “RF_*” is right Quillen and “LF^*” is left Quillen. We denote the derived functors by LF^* and RF_*. The natural transformation “LF^*” $\Rightarrow F^*$ is simplicial and hence induces a natural transformation $LF^* \Rightarrow F^*$ (restrict to cofibrant objects and use that F^* preserves all weak equivalences). I claim this is a natural isomorphism. To see this, let $Y \in Sect(J, \mathcal{M})$ be (projective) cofibrant. We need only show that the natural map “LF^*”$Y \to F^*Y$ is a weak equivalence. But observe that $(\text{“}LF^{*}\text{”}Y)_i \simeq \operatorname{hocolim}_{\alpha : j \to Fi} \alpha_{\#} Y_j \simeq Y_{Fi}$. The second weak equivalence follows from the homotopy cofinality theorem for homotopy colimits [43, Corollary 19.6.8(1)], the fact that the category J/Fi has a terminal object and Lemma A.64. The first weak equivalence follows from the fact that $N(J/F/i)$ is a *cofibrant* J/Fi-diagram [61, 6.4(i)] consisting of (weakly) contractible spaces (the category $(j/F/i)_\alpha$ has terminal object $(\mathrm{id} : i \to i, \alpha : j \to Fi)$) and [43, Corollary 18.4.5].

Thus to finish the proof, we need to show that “LF^*” and “RF_*” are adjoints. For this, let $X \in Sect(I, F^*\mathcal{M})$ and $Y \in Sect(J, \mathcal{M})$.

Then one checks directly from the definitions that both $Map(\text{“}LF^*\text{”}Y, X)$ and $Map(Y, \text{“}RF_*\text{”}X)$ are isomorphic to the following equaliser

$$eq\left(\prod_{(\alpha : j \to Fi) \in J/F} Map(\alpha_{\#}Y_j, X_i)^{N(j/J/Fi)_\alpha} \rightrightarrows \prod_{\beta : \alpha_1 \to \alpha_2 \in J/F} Map(\alpha_{\#}\beta_{\#}Y_{j_1}, X_{i_2})^{N(j_2/J/Fi_1)} \right).$$

Here the first product is over all objects in J/F, i.e. all triples $(j \in J, i \in I, j \to Fi)$ and the second product is over morphisms in J/F, i.e. commutative squares of the form

$$\begin{array}{ccc} j_1 & \xrightarrow{\alpha_1} & Fi_1 \\ {\scriptstyle \beta'}\downarrow & & \downarrow{\scriptstyle F\beta} \\ j_2 & \xrightarrow{\alpha_2} & Fi_2. \end{array}$$

□

The rest of the proof of Theorem A.74 is relatively straightforward. Suppose that $F : I \to J$ is homotopy cofinal. We first observe that “RF_*” preserves holim-fibrant objects. To see this, let $X \in Sect(I, F^*M)$ be holim-fibrant, i.e. projective fibrant and (homotopy) cartesian. Let $\gamma : j \to j' \in J$. Since “RF_*” is right Quillen in the projective model structure we need only show that the comparison map

$$(\text{“}RF_*\text{”}X)_j = \text{“}\operatorname*{holim}_{\alpha : j \to Fi}\text{”} \alpha^* X_i \to R\gamma^*(\text{“}RF_*\text{”}X)_{j'} = \gamma^*(\text{“}RF_*\text{”}X)_{j'} = \gamma^* \text{“}\operatorname*{holim}_{\alpha' : j' \to Fi}\text{”} \alpha'^* X_i$$

is a weak equivalence. As before (when defining the comparison map) we can commute γ^* and “holim”. The result now follows from the assumption that j/I and j'/I have contractible nerves, the assumption that X is fibrant (so all its structure maps are weak equivalences), and Theorem A.19.

We know already that F^* also preserves holim-local objects (being right Quillen) and thus we obtain an adjunction (by restriction)

$$F^* : Ho(\mathrm{holim}_I\, F^*\mathcal{M}) \rightleftarrows Ho(\mathrm{holim}_J\, \mathcal{M}) : RF_*.$$

I claim this is a pair of equivalences. The theorem follows from this. In order to prove the claim we need only show that the unit and co-unit $\mathrm{id} \Rightarrow RF_*F^*, F^*RF_* \Rightarrow \mathrm{id}$ are isomorphisms. So let $X \in Sect(I, F^*\mathcal{M})$ be holim-fibrant. Then

$$(F^*RF_*X)_i = (F^*\text{“}RF_*\text{”}X)_i = \text{“}\underset{\alpha:Fi\to Fi'}{\mathrm{holim}}\text{”}\alpha^* X_{i'}$$

and this is weakly equivalent to X_i by homotopy cofinality, holim-fibrancy and Theorem A.19 again. Similarly for $Y \in Sect(J, \mathcal{M})$ holim-fibrant F^*Y is also holim-fibrant and so

$$(RF_*F^*Y)_j = (\text{“}RF_*\text{”}F^*Y)_j = \text{“}\underset{\alpha:j\to Fi}{\mathrm{holim}}\text{”}\alpha^* X_{Fi}.$$

This is a weak equivalence by the same argument.

A.7.3 Decomposition along Pushouts

Definition A.76. *Let I be a small category and $I' \subset I$ a full subcategory. We call I'* initial *in I if whenever there is a morphism $i \to i' \in I$ with $i' \in I'$ we have $i \in I'$.*

As in the previous subsection we begin with a well-known result [61, Lemma 7.2].

Theorem A.77. *Let*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{f} & \mathcal{C} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ \mathcal{D}' & \xrightarrow{g} & \mathcal{D} \end{array}$$

be a pushout of small categories, where i and f (and hence j and g) map their source isomorphically into an initial subcategory of the target.

Suppose that $\mathcal{M}$ is a simplicial model category and let $X : \mathcal{D}^{op} \to \mathcal{M}$ be a diagram.

Then the natural square

$$\begin{array}{ccc} \mathrm{holim}_{\mathcal{C}'} X & \xleftarrow{f^*} & \mathrm{holim}_{\mathcal{C}} X \\ {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle j^*} \\ \mathrm{holim}_{\mathcal{D}'} X & \xleftarrow{g^*} & \mathrm{holim}_{\mathcal{D}} X \end{array}$$

is a homotopy pullback.

Proof. There is a natural map $\mathrm{holim}_{\mathcal{D}} X \to \mathrm{holim}_{\mathcal{D}'} X \times^h_{\mathrm{holim}_{\mathcal{C}'} X} \mathrm{holim}_{\mathcal{C}} X$. By the Yoneda lemma it is enough to show that for each $T \in \mathcal{M}$, the natural map

$$Map^d(T, \mathrm{holim}_{\mathcal{D}}) \to Map^d(T, \mathrm{holim}_{\mathcal{D}'} X \times^h_{\mathrm{holim}_{\mathcal{C}'} X} \mathrm{holim}_{\mathcal{C}} X)$$

is a weak equivalence of simplicial sets. Now

$$\begin{aligned} &Map^d(T, \mathrm{holim}_{\mathcal{D}'} X \times^h_{\mathrm{holim}_{\mathcal{C}'} X} \mathrm{holim}_{\mathcal{C}} X) \\ \simeq\ &Map^d(T, \mathrm{holim}_{\mathcal{D}'} X) \times^h_{Map^d(T, \mathrm{holim}_{\mathcal{C}'} X)} Map^d(T, \mathrm{holim}_{\mathcal{C}} X) \end{aligned}$$

by [43, Proposition 18.3.10 (2)] and so it enough to prove the lemma in the special case when $\mathcal{M} = sSet$.

We now use the fact that the functor $\mathrm{holim} : Ho(Fun(I^{op}, sSet)) \to Ho(sSet)$ is the right derived functor of the ordinary limit, and exploit the fact that there are several useful model structures on diagram categories. This is explained in detail in [33]. The upshot is that we have $\operatorname{holim}_{\mathcal{D}} X = Map^d_{sSet}(*, \operatorname{holim}_{\mathcal{D}} X) \simeq Map^d_{Fun(I^{op},sSet)}(*, X)$.

We shall use the *injective* model structures on our diagram categories from now on, i.e. where cofibrations and weak equivalences are determined objectwise. Since we are dealing with diagrams of simplicial sets this is entirely classical. Consider the functor $j : \mathcal{C} \to \mathcal{D}$. The induced functor $j^* : Fun(\mathcal{D}^{op}, sSet) \to Fun(\mathcal{C}^{op}, sSet)$ is left Quillen. It has a left adjoint $j_\#$ satisfying

$$(j_\# X)(d) = \operatorname{colim}_{d/\mathcal{C}} X,$$

this is explained for example in [103, Tag 00VC]. Since $\mathcal{C}$ is initial in $\mathcal{D}$ the categories $d/\mathcal{C}$ are either empty or have an initial object, so $(j_\# X)(d) = X(d)$ if $d \in \mathcal{C}$ or $(j_\# X)(d) = \emptyset$ else. It follows that $j_\#$ preserves objectwise cofibrations and weak equivalences, so is left Quillen. Thus j^* is bi-Quillen. Similar comments apply to i, f, g.

We have $(f_\# *)(d) = *$ if $d \in \mathcal{C}$ and $(f_\# *)(d) = \emptyset$ else. From this it is easy to see that there is a natural pushout square in $Fun(\mathcal{D}^{op}, sSet)$

$$\begin{array}{ccc} f_\# j_\# * & \longrightarrow & j_\# * \\ \downarrow & & \downarrow \\ i_\# * & \longrightarrow & *. \end{array}$$

Each of the maps displayed is an objectwise monomorphism, i.e. a cofibration, so the diagram is a homotopy pushout, $sSet$ being (left) proper. We conclude that there is a homotopy pullback

$$\begin{array}{ccc} Map^d(*, X) & \longrightarrow & Map^d(j_\# *, X) \\ \downarrow & & \downarrow \\ Map^d(i_\# *, X) & \longrightarrow & Map^d(f_\# j_\# *, X). \end{array}$$

But $Map^d(i_\# *, X) \simeq \operatorname{holim}_{\mathcal{D}'} X$, and so on. With these identifications the above homotopy pullback square is weakly equivalent to the square in the statement of the theorem. This concludes the proof. □

We remark that if we are willing to assume that $\mathcal{M}$ is combinatorial and right proper, then in the proof one may avoid reducing to the case of simplicial sets. This does not really simplify the argument, though.

The real point of this subsection is that a similar result holds for the nerves of homotopy limits of Quillen pseudo-presheaves.

Theorem A.78. *Let*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{g} & \mathcal{C} \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathcal{D}' & \longrightarrow & \mathcal{D} \end{array}$$

be a pushout of small categories, where f and g map their source isomorphically into an initial subcategory of the target. Let $\mathcal{M}$ be a left proper, simplicial, combinatorial right Quillen pseudo-presheaf on $\mathcal{D}$.

Then the natural diagram

$$\begin{array}{ccc} N(\operatorname{holim}_{\mathcal{C}'} \mathcal{M})^f_w & \longleftarrow & N(\operatorname{holim}_{\mathcal{C}} \mathcal{M})^f_w \\ \uparrow & & \uparrow \\ N(\operatorname{holim}_{\mathcal{D}'} \mathcal{M})^f_w & \longleftarrow & N(\operatorname{holim}_{\mathcal{D}} \mathcal{M})^f_w \end{array}$$

is a homotopy pullback.

Proof. Since homotopy pullback squares are detected and preserved under weak equivalence, using Lemma A.15 and Proposition A.67 we find that the statement is invariant under pseudonatural (Quillen) equivalences in $\mathcal{M}$. It follows that we may assume that $\mathcal{M}$ is a strict functor.

We follow closely the proof of [61, Lemma 7.2]. To make the proof more readable we separate out certain claims which are proved separately at the end. We number them for easy reference.

Let W be the category with five objects $\{u, v, w, x, y\}$ and four non-identity maps $u \to v \leftarrow w \to x \leftarrow y$. Define categories $\mathcal{E}'$ and $\mathcal{E}$ by the pushouts

$$\begin{array}{ccc} \mathcal{C}' = \mathcal{C}' \times u & \longrightarrow & \mathcal{D}' \\ \downarrow & & \downarrow \\ \mathcal{C}' \times W & \longrightarrow & \mathcal{E}' \end{array} \quad \begin{array}{ccc} \mathcal{C}' = \mathcal{C}' \times y & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{E}' & \longrightarrow & \mathcal{E}. \end{array}$$

There are canonical functors $\mathcal{E}' \to \mathcal{D}'$ and $\mathcal{E} \to \mathcal{D}$ which we claim are homotopy cofinal (C1). We also obtain a morphism from the left diagram to the right in the following

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{E}' & \longrightarrow & \mathcal{E} \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{D}' & \longrightarrow & \mathcal{D}. \end{array}$$

Since homotopy pullback squares are detected and preserved under weak equivalence, the change of index category Theorem A.74 allows us to reduce to proving the proposition for the left square. We factor it further as

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{C}' \coprod \mathcal{D}' & \longrightarrow & \mathcal{C} \coprod \mathcal{D}' \\ \downarrow & & \downarrow \\ \mathcal{E}' & \longrightarrow & \mathcal{E}. \end{array}$$

The pasting law for homotopy pullbacks, i.e. Lemma A.20, allows us to prove the proposition for the top and bottom square separately.

Now the nerve functor commutes with products and so $N(\mathrm{holim}_{\mathcal{C} \coprod \mathcal{D}'} \mathcal{M})^f_w = N(\mathrm{holim}_{\mathcal{C}} \mathcal{M})^f_w \times N(\mathrm{holim}_{\mathcal{D}'} \mathcal{M})^f_w$. Hence for the top square it suffices to observe that (C2) if $A \to B$ is any map of simplicial sets and C is any simplicial set, then the obvious diagram

$$\begin{array}{ccc} B \times C & \longrightarrow & A \times C \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \end{array}$$

is homotopy cartesian.

For the bottom square, the induced diagram of nerves is

$$\begin{array}{ccc} N(\mathrm{holim}_{\mathcal{E}} \mathcal{M})^f_w & \longrightarrow & N(\mathrm{holim}_{\mathcal{E}'} \mathcal{M})^f_w \\ \downarrow & & \downarrow \\ N(\mathrm{holim}_{\mathcal{C} \coprod \mathcal{D}'} \mathcal{M})^f_w & \longrightarrow & N(\mathrm{holim}_{\mathcal{C}' \coprod \mathcal{D}'} \mathcal{M})^f_w. \end{array}$$

The vertical maps are "restriction to the end points". Unravelling the category $\mathit{Sect}(\mathcal{E}, \mathcal{M})$ one sees easily that the objects are families $(X^u, X^v, X^w, X^x, X^y)$ with $X^u \in \mathit{Sect}(\mathcal{D}', \mathcal{M}), X^v, X^w, X^x \in \mathit{Sect}(\mathcal{C}', \mathcal{M})$ and $X^y \in \mathit{Sect}(\mathcal{C}, \mathcal{M})$ together with maps $f^* X^u \to X^v \leftarrow X^w \to X^x \leftarrow g^* X^y$. Morphisms are just the compatible families of morphisms. A similar observation applies to $\mathit{Sect}(\mathcal{E}', \mathcal{M})$.

It follows that the square of section categories is cartesian, and hence so is the square of nerves (since the nerve functor commutes with limits). A square is homotopy cartesian if and only if it induces an equivalence of all homotopy fibres [84, Proposition 3.3.18], so a cartesian square is homotopy cartesian if and only if the inclusion of the fibres into the homotopy fibres is a weak equivalence. We shall show that this applies to the nerves of the functors $B : (\operatorname{holim}_{\mathcal{E}} \mathcal{M})^f_w \to (\operatorname{holim}_{\mathcal{C} \coprod \mathcal{D}'} \mathcal{M})^f_w$ and similarly for $\mathcal{E}'$. In fact the proofs are essentially the same so we treat only B.

A point in $N(\operatorname{holim}_{\mathcal{C} \coprod \mathcal{D}'} \mathcal{M})^f_w$ corresponds to a pair $(Y^y, Y^u) \in Sect(\mathcal{C}, \mathcal{M}) \times Sect(\mathcal{D}', \mathcal{M})$ such that each object is termwise fibrant and homotopy cartesian. The fibre $N(B)^{-1}(Y^y, Y^u)$ is the nerve of the category $B^{-1}(Y^y, Y^u) \subset Sect(\mathcal{E}, \mathcal{M})$ consisting of objects $f^*X^u \to X^v \leftarrow X^w \to X^x \leftarrow g^*X^y$ such that each of the $X^\bullet$ is termwise fibrant and homotopy cartesian, and each of the structure maps is a termwise weak equivalence, and such that $X^u = Y^u, X^y = Y^y$. The morphisms in $B^{-1}(Y^y, Y^u)$ are the compatible families which are the identities on the endpoints. By construction, this category is related to *Hammock localization*. In fact it follows from [25, 6.2] that $N(B^{-1}(Y^y, Y^u))$ is the space of homotopy automorphisms of $f^*Y^u \simeq g^*Y^y$. Next, I claim (C3) that the natural map $B^{-1}(Y^y, Y^u) \to (Y^y, Y^u)/B$ induces a weak equivalence on nerves. Since the space of homotopy automorphisms of f^*Y^u is independent of f^*Y^u up to weak equivalence, it follows that $N(Y^y, Y^u)/B$ is independent of (Y^y, Y^u) up to weak equivalence, and hence a homotopy fiber by Quillen's Theorem B [113, Theorem 3.8]. This concludes the proof, modulo the claims.

Proof of (C1). We treat only $F : \mathcal{E} \to \mathcal{D}$, the case of $\mathcal{E}'$ being similar. We need to prove that for each $d \in \mathcal{D}$ the category d/F has (weakly) contractible nerve. Let $d \to Fe \in d/F$. If $d \notin \mathcal{C}$ then $Fe \notin \mathcal{C}$ because $\mathcal{C} \subset \mathcal{D}$ is initial. It follows that $e \in \mathcal{D}' \times u$ and thus we have that $d/F \cong d/\mathcal{D}'$ (recall that $d \notin \mathcal{C}$, so $d \in \mathcal{D}' \setminus \mathcal{C}'$). This has an initial object and thus contractible nerve. Similarly if $d \notin \mathcal{C}$. Hence assume that $d \in \mathcal{D}' \cap \mathcal{C} = \mathcal{C}'$. The full subcategory $i : d/F|_{d \times W} \subset d/F$ is equivalent to W and so contractible. There is a functor $R : d/F \to d/F|_{d \times W}$. Given $t \in W$, $a \in \mathcal{C}, \mathcal{C}', \mathcal{D}'$ as appropriate, so $(a, t) \in \mathcal{E}$ and $d \to a$, we put $R(d \to (a, t)) = (d, t)$. Clearly $Ri = \mathrm{id}$ and there is a natural transformation $iR \Rightarrow \mathrm{id}$, whence $N(d/F)$ is a deformation retract of $N(d/F|_{d \times W})$ [113, 3.2] and so is contractible.

Proof of (C2). Let $C \to \tilde{C}$ be a fibrant replacement. Since fibrations are stable under base change, the map $A \times \tilde{C} \to A$ is a fibration. Since the category of simplicial sets satisfies the pushout-product axiom and all objects are cofibrant, the maps $A \times C \to A \times \tilde{C}$ and $B \times C \to B \times \tilde{C}$ are weak equivalences. Since homotopy pullbacks are stable under termwise weak equivalence, we have that $B \times^h_A (A \times C) \simeq B \times^h_A (A \times \tilde{C})$. But $A \times \tilde{C} \to A$ is a fibration and the model category of simplicial sets is proper, so $B \times^h_A (A \times \tilde{C}) \simeq B \times_A (A \times \tilde{C}) \cong B \times \tilde{C} \simeq B \times C$. This proves the claim.

Proof of (C3). Since adjoint categories have homotopy equivalent nerves [113, 3.2], it suffices to show that the natural functor $i : B^{-1}(Y^y, Y^u) \to (Y^y, Y^u)/B$ affords a right adjoint R. An object of $(Y^y, Y^u)/B$ consists of objects $f^*X^u \to X^v \leftarrow X^w \to X^x \leftarrow g^*X^y$ and additional morphisms $Y^u \to X^u, Y^y \to X^y$. We denote this object by $(Y^{u,y} \to X^\bullet)$. Let $R(Y^{u,y} \to X^\bullet) = f^*(Y^u \to X^u) \to X^v \leftarrow X^w \to X^x \leftarrow g^*(X^y \leftarrow Y^y)$. It is not hard to prove adjunction, but we actually only need a natural transformation $iR \Rightarrow \mathrm{id}$, the existence of which is basically obvious. □

A.7.4 Commutation of Homotopy Limits and Nerves

We now come to the most crucial result of this section. Let I be a small category and $\mathcal{M}$ a left proper, combinatorial right Quillen pseudo-presheaf on I.

Lemma A.79. *There is a canonical isomorphism of categories* $(\operatorname{holim}_I \mathcal{M})^f_w \cong Sect(I, \mathcal{M}^f_w)$.

(Here $\mathcal{M}^f_w$ denotes the pseudo-presheaf $i \mapsto \mathcal{M}(i)^f_w$.)

Proof. This just says that the objects of $(\operatorname{holim}_I \mathcal{M})^f$ are those sections consisting of fibrant objects and weak equivalences between them, and that the weak equivalences between such objects are the entry-wise weak equivalences. □

We thus obtain, using Lemma A.60 part (3), a string of morphisms

$$N(\operatorname{holim}_I \mathcal{M})_w^f \cong N(Sect(I, \mathcal{M}_w^f)) \to \operatorname{holim}_I N((\mathcal{M}_w^f)^r). \tag{A.1}$$

Theorem A.80. *Let I be a small category and $\mathcal{M}$ a left proper, combinatorial, simplicial right Quillen pseudo-presheaf on $\mathcal{I}$. Then the natural map*

$$N(\operatorname{holim}_I \mathcal{M})_w^f \to \operatorname{holim}_I N((\mathcal{M}_w^f)^r)$$

is a weak equivalence of simplicial sets.

The proof will occupy the rest of this section. Using homotopy invariance of (ordinary) homotopy limits, Lemma A.15 and Proposition A.67, we find that the statement of the theorem is invariant under pseudonatural equivalences in $\mathcal{M}$. Thus we may and shall for the rest of the section assume that $\mathcal{M}$ is a strict functor.

The idea of the proof is to use the results from the previous subsections to simplify the indexing category I, eventually reducing to cases which can be checked by hand. We follow more or less [61, Section 8].

A category I is called *direct* if for each $i \in I$ the undercategory I/i has finite-dimensional nerve. That is to say for every commutative diagram

$$\begin{array}{ccccccc}
i_1 & \xrightarrow{f_1} & i_2 & \xrightarrow{f_2} & \dots & \xrightarrow{f_{n-1}} & i_n \\
\downarrow & & \downarrow & & & & \downarrow \\
i & = & i & = & \dots & = & i
\end{array}$$

and n sufficiently large, one of the f_k is an identity morphism. We write $dim(i) = dim N(I/i)$. The category I is called *finite length* (at most n) if it is direct and the dimensions of objects are bounded (by n).

Lemma A.81. *Let I be a direct category and $\alpha : i \to j$ a morphism. If $dim(j) \leq dim(i)$, then α is an identity morphism (i.e. $i = j$).*

Proof. Let

$$\begin{array}{ccccccc}
i_1 & \longrightarrow & i_2 & \longrightarrow & \dots & \longrightarrow & i \\
\downarrow & & \downarrow & & & & \downarrow \\
i & = & i & = & \dots & = & i
\end{array}$$

be a non-degenerate simplex of maximal dimension (we may always assume that the right-most map to i is the identity, by maximality). Then

$$\begin{array}{ccccccccc}
i_1 & \longrightarrow & i_2 & \xrightarrow{f_2} & \dots & \longrightarrow & i & \xrightarrow{\alpha} & j \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
j & = & j & = & \dots & = & j & = & j
\end{array}$$

is a simplex of strictly larger dimension (where the maps are $i_k \to i \xrightarrow{\alpha} j$), which must be degenerate by assumption. Since the original simplex was non-degenerate it must be that $\alpha =$ id. □

One may show that an appropriate converse of this lemma is also true.

We will be using the following construction as a black box. There is a functor $\overline{sd} : Cat \to Cat$ together with a natural transformation $\overline{sd} \to \mathrm{id}$ which has the following properties. For any small category I, the category $\overline{sd}I$ is direct [24, 5.3] [61, 8.I]. The natural functor $\overline{sd}I \to I$ is homotopy cofinal [61, 6.10(ii), Proposition 6.7]. If I is direct of length at most n then so is $\overline{sd}I$. (This is because objects in $\overline{sd}I$ are non-degenerate functors $[k] \to I$ for varying k, and one proves that $dim([k] \to I) = k$ provides a dimension function. However in a category I of length n there are no non-degenerate functors $[n+1] \to I$.)

Lemma A.82. *The map* (A.1) *is a weak equivalence if I is of finite length.*

Proof. This is essentially a verbatim copy of [61, 8.III].

The map is a weak equivalence if I is discrete, since then both sides are just products. Whenever $I \to J$ is homotopy cofinal, the map is an equivalence for I if and only if it is an equivalence for J, by Theorems A.71 and A.74. For example the category $[n]$ has an initial object, hence $* \to [n]$ is homotopy cofinal, and so the map is a weak equivalence for $I = [n]$.

Let I be of dimension $n > 0$; we shall prove the result by induction on n. Write I^k for the subcategory of objects of dimension at most k. There is a pushout [61, 8.6]

$$\begin{array}{ccc} \mathcal{A}' = \coprod_{i \in (sdI)_n} \overline{sd}(I^{n-1})/i & \longrightarrow & \overline{sd}(I^{n-1}) = \mathcal{A} \\ \downarrow & & \downarrow \\ \mathcal{B}' = \coprod_{i \in (sdI)_n} \overline{sd}I/i & \longrightarrow & \overline{sd}I = \mathcal{B}. \end{array}$$

One may further factor this diagram as

$$\begin{array}{ccccc} \mathcal{A}' & \xrightarrow{f} & \mathcal{A}'' & \xrightarrow{p} & \mathcal{A} \\ {\scriptstyle g}\downarrow & & \downarrow & & \downarrow \\ \mathcal{B}' & \longrightarrow & \mathcal{B}'' & \xrightarrow{q} & \mathcal{B}, \end{array}$$

where f, g are inclusions of initial subcategories, the left hand square is a pushout, and p, q are homotopy cofinal [61, 8.8, 6.15, 6.6]. The map is an equivalence for $\mathcal{A}'$ and $\mathcal{A}$ by induction, for $\mathcal{B}'$ because $\overline{sd}I/i \cong \overline{sd}[n]$ (and both kinds of hocolim turn coproducts of index categories into products). It is an equivalence for $\mathcal{A}''$ because p is homotopy cofinal, hence it is an equivalence for $\mathcal{B}''$ by Theorems A.77 and A.78 (and the fact that homotopy pullbacks are preserved under weak equivalence). It is thus an equivalence for $\mathcal{B}$ since q is homotopy cofinal, and finally an equivalence for I because $\mathcal{B} = \overline{sd}I \to I$ is homotopy cofinal. □

Lemma A.83. *Let I be a small category which is an increasing union of initial subcategories $I^0 \subset I^1 \subset \dots$, such that the map* (A.1) *is a weak equivalence for each I^k. Then the map* (A.1) *is a weak equivalence for I.*

Proof. Again we follow [61, Proof of Proposition 8.2] very closely. Put $\mathcal{C} = \coprod_k I^k$ and recall that both kinds of homotopy limits turn disjoint unions into products. Thus the map (A.1) is an equivalence for C. There is a map of pushout diagrams

$$\begin{array}{ccccccc} \mathcal{C}\coprod\mathcal{C} & \xrightarrow{a} & \mathcal{C}\times V & \Rightarrow & \mathcal{C}\coprod\mathcal{C} & \longrightarrow & \mathcal{C} \\ {\scriptstyle b}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}\times V & \longrightarrow & I^{\#} & & \mathcal{C} & \longrightarrow & I, \end{array}$$

where $I^{\#} \to I$ and $\mathcal{C} \times V \to \mathcal{C}$ are homotopy cofinal functors. Thus, Since the map (A.1) is an equivalence for $\mathcal{C}$ it is one for $\mathcal{C} \times V$, and it is an equivalence for I if and only if it is one for $I^{\#}$. But it is an equivalence for $\mathcal{C}\coprod\mathcal{C}$, and the functors a, b are inclusions of initial subcategories. Thus we conclude as before using Theorems A.77 and A.78 (and the fact that homotopy pullbacks are preserved under weak equivalence). □

The proof of the theorem is now easy. If I is any small category, then $\overline{sd}I$ is a direct category and $\overline{sd}I \to I$ is homotopy cofinal, so the theorem holds for I if and only if it holds for $\overline{sd}I$, whence we may assume that I is direct. But then $I = \bigcup_k I^k$, where I^k is the full subcategory on objects of dimension at most k. The theorem holds for each I^k by the lemma before last, and hence holds for I by the last lemma.

A.8 Descent in τ-Quillen Presheaves

Suppose that $\mathcal{C}$ is a small category and $\mathcal{M}$ is a left proper, combinatorial right Quillen pseudo-presheaf on $\mathcal{C}$. For any augmented simplicial object $\phi : U \to X \in s_+\mathcal{C}$, we can consider the restricted pseudo-presheaf $\mathcal{M}|_U : \Delta \to MCat^R$, and its homotopy limit $\operatorname{holim}_{\Delta^{op}} \mathcal{M}_U =: \operatorname{holim}_U \mathcal{M}$.

For any map of simplicial objects $\psi : U \to V \in s_+\mathcal{C}$ over X, we get a right morphism $\psi^* : \mathcal{M}|_V \to \mathcal{M}|_U$, and correspondingly by proposition A.67 a morphism $\psi^* : \operatorname{holim}_V \mathcal{M} \to \operatorname{holim}_U \mathcal{M}$.

This in particular applies to the map $\phi_0 : U \to sX$, where sX is the constant simplicial object with value X. There is a unique functor $s : \Delta^{op} \to *$ (where $*$ is the category with one object and one morphism). Write $\mathcal{M}_X$ for the Quillen presheaf on $*$ with unique object $\mathcal{M}_X(*) = \mathcal{M}(X)$. Then $\mathcal{M}|_{sX} = s^*\mathcal{M}_X$. In particular the change of index category functor from section A.7.2 furnishes a right Quillen functor $s^* : \mathcal{M}(X) = \operatorname{holim}_* \mathcal{M}_X \to \operatorname{holim}_{\Delta^{op}} s^*\mathcal{M}_X = \operatorname{holim}_{\Delta^{op}} \mathcal{M}|_X$. We denote the composite $\phi_0^* s^*$ by

$$\phi^* : \mathcal{M}(X) \to \operatorname{holim}_U \mathcal{M}$$

and call it the descent functor. It is right Quillen, and simplicial if $\mathcal{M}$ is.

Note that to define the descent functor, we did not really need a right Quillen presheaf, we only needed left adjoints to the maps appearing in the hypercovering $U \to X$. This is what the next definition takes care of.

Definition A.84. *Let $(\mathcal{C}, \tau)$ be a λ-suitable Verdier site. By a τ-Quillen presheaf on $\mathcal{C}$ we mean a left Quillen presheaf such that for every basal map $f : X \to Y$, the restriction $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$ is also right Quillen. We call $\mathcal{M}$ τ-Quillen simplicial if $\mathcal{M}$ is left Quillen simplicial and τ-Quillen, and the extra left adjoints enrich to simplicial adjoints.*

If $\mathcal{M}$ is left proper and combinatorial and $\phi : U \to X$ is a hypercover (by definition internal), then the above discussion together with Proposition A.37 (guaranteeing that the structure maps of U are basal) furnishes a descent functor

$$\phi^* : \mathcal{M}(X) \to \operatorname{holim}_U \mathcal{M}.$$

We say that $\mathcal{M}$ has descent *or* is a sheaf *if $\mathcal{M}$ is left proper and combinatorial, for every hypercover the descent functor ϕ^* is a Quillen equivalence, and if for every family $\{X_i\}_{i \in I} \in \mathcal{C}$ of size less than λ, the natural right Quillen functor*

$$\mathcal{M}\left(\coprod_i X_i\right) \to \prod_i \mathcal{M}(X_i)$$

is a Quillen equivalence.

We compare this definition to the notions of Cisinski-Deglise [18]. There, the central notion is that of a (pre-) $\mathcal{P}$-fibred (model) category. Our definition of a τ-Quillen pseudo-presheaf coincides with their definition of a $\mathcal{P}$-fibred model category, where $\mathcal{P}$ is the class of basal maps. If $\mathcal{M}$ is such a $\mathcal{P}$-fibred model category / τ-Quillen pseudo-presheaf (with some cofibrant generation assumptions), then for any diagram $\mathcal{X}$ in $\mathcal{C}$ they define a model category $\mathcal{M}(\mathcal{X})$. It coincides with our category $Sect(\mathcal{X}, \mathcal{M})$, where the projective model structure is used as usual.

Cisinski-Deglise also define a notion of descent, called τ-descent. In fact $\mathcal{M}$ satisfies τ-descent if and only if for every hypercover $\phi : U \to X$, the (derived) descent functor

$$R\phi^* : Ho(\mathcal{M}(X)) \to Ho(Sect(U, \mathcal{M}))$$

is fully faithful [18, Corollary 3.2.7].

It is clear that the image of $R\phi^*$ consists of homotopy cartesian sections, so there is a commutative diagram

$$\begin{array}{ccc} Ho(\mathcal{M}(X)) & = & Ho(\mathcal{M}(X)) \\ {\scriptstyle R\phi^*}\downarrow & & \downarrow{\scriptstyle R\phi^*} \\ Ho(\operatorname{holim}_U \mathcal{M}) & \xrightarrow{i} & Ho(Sect(U, \mathcal{M})). \end{array}$$

Here the functor i is the fully faithful embedding of the holim-local (i.e. homotopy cartesian) sections into the homotopy category of all sections. Our definition of descent requires the left vertical arrow to be an equivalence, whereas $\mathcal{M}$ satisfying τ-descent only requires the right vertical arrow to be fully faithful, a condition which is clearly weaker.

Our next task is to construct τ-Quillen sheaves. It turns out that the easiest way of doing so is to go through an auxiliary notion, developed by Charles Rezk [90].

Definition A.85 (Rezk)**.** *Let I be a small category and $\mathcal{M}$ a model category.*

A natural transformation $f : X \to Y$ of functors $X, Y : I \to \mathcal{M}$ is called equifibred *if for each map $i \to j \in I$ the induced square*

$$\begin{array}{ccc} X(i) & \longrightarrow & X(j) \\ \downarrow & & \downarrow \\ Y(i) & \longrightarrow & Y(j) \end{array}$$

is homotopy cartesian. We say that $\mathcal{M}$ has homotopical patching *if two conditions hold:*

P1 *Let I be a small category, $X : I \to \mathcal{M}$ a functor, and $\bar{X} = \operatorname{hocolim}_I X$. Let $f : \bar{Y} \to \bar{X}$ be a map. Define the functor $Y : I \to \mathcal{M}$ by $Y(i) := X(i) \times^h_{\bar{X}} \bar{Y}$. (So in particular f is equifibred.) Then $\operatorname{hocolim}_I Y \to \bar{Y}$ is a weak equivalence.*

P2 *Let I be a small category, $f : Y \to X$ an equifibred transformation. Let $\bar{f} : \bar{Y} \to \bar{X}$ be the map on homotopy colimits. Then for each i, $Y(i) \to X(i) \times^h_{\bar{X}} \bar{Y}$ is a weak equivalence.*

Suppose now that $D : I \to \mathcal{M}$ is a diagram in a model category. We can form a right Quillen pseudo-presheaf $\mathcal{M}/D$ on I given by $(\mathcal{M}/D)(i) = \mathcal{M}/D(i)$, with its canonical model structure. If there is an augmentation $\phi : D \to X$ with $X \in \mathcal{M}$ then as usual we get a Quillen adjunction

$$\phi_\# : \operatorname{holim}_I \mathcal{M}/D \leftrightarrows \mathcal{M}/X : \phi^*.$$

Lemma A.86. *Let $\mathcal{M}$ be a proper, combinatorial model category which satisfies homotopical patching. Suppose that I is a small category, $D : I \to \mathcal{M}$ a diagram and $\phi : D \to X$ an augmentation, such that the composite*

$$\operatorname{hocolim}_I D \to \text{“}\operatorname{hocolim}_I\text{”} D \to \operatorname{colim}_I D \to X$$

is a weak equivalence. Then the Quillen adjunction $\phi_\# : \operatorname{holim}_I \mathcal{M}/D \leftrightarrows \mathcal{M}/X : \phi^*$ is a Quillen equivalence.

Proof. Let us make the category $Sect(I, \mathcal{M}/D)$ more explicit. Objects are families $\{T_i \in \mathcal{M}/D(i)\}_i$ together with structure maps $T_i \to f^*T_j$ for each $f : i \to j$. Now $f^*T_j = T_j \times_{D(j)} D(i)$, so we really just have $Sect(I, \mathcal{M}/D) \cong Fun(I, \mathcal{M})/D$.

A section $T_\bullet$ is homotopy cartesian if for every $f : i \to j$ we have that $T_i \to Rf^*T_j$ is a weak equivalence. Now Rf^*T_j is just f^* of a fibrant replacement T'_j of T_j in $\mathcal{M}/D(j)$, i.e. we form $T_j \to T'_j \to D(j)$ such that the first map is a weak equivalence and the second map is a fibration. It follows from right properness of $\mathcal{M}$ that $Rf^*T_j = T_j \times^h_{D(j)} D(i)$. Consequently the homotopy cartesian sections in $Sect(I, \mathcal{M}/D)$ correspond precisely to the equifibred transformations to D in $Fun(I, \mathcal{M})/D$.

As explained at the beginning of this section, the functor $\phi_\#$ is obtained as a composite $\operatorname{holim}_I \mathcal{M}/D \xrightarrow{\phi_{0\#}} \operatorname{holim}_I \mathcal{M}/X \xrightarrow{s_\#} \mathcal{M}/X$ and similarly for the right adjoints. Here $\operatorname{holim}_I \mathcal{M}/X$ denotes the homotopy limit of the constant right Quillen presheaf on I with value $\mathcal{M}/X$. As explained in Proposition A.73, the functor $Ls_\#$ (on homotopy categories) is naturally isomorphic to the functor $T_\bullet \mapsto \operatorname{hocolim}_I T$.

Now to prove that $\phi_\# \vdash \phi^*$ is a Quillen equivalence it is necessary and sufficient to prove that (1) for every $T \in \mathcal{M}/X$ the natural map $L\phi_\# R\phi^* T \to T$ is a weak equivalence, and that (2) for every homotopy cartesian section $S \in Sect(\mathcal{M}/D)$ the natural map $S \to R\phi^* L\phi_\# S$ is a weak equivalence. These correspond precisely to the conditions (P1) and (P2).

We first prove (1). We know that $(R\phi^* T)_i = T \times^h_X D(i) \in \mathcal{M}/D(i)$, by right properness again. The functor $\phi_{0\#}$ preserves weak equivalences and so $(L\phi_{0\#} R\phi^* T)_i = (\phi_{0\#} R\phi^* T)_i = T \times^h_X D(i) \in \mathcal{M}/X$. Finally $L\phi_\# R\phi^* T = Ls_\# \phi_{0\#} R\phi^* T = \operatorname{hocolim}_I T \times^h_X D(i)$, by the identification of $Ls_\#$ recalled above. This is weakly equivalent to T by the assumption that $X \simeq \operatorname{hocolim}_I D$ and (P1).

Now we prove (2). Let $Y \in Sect(I, \mathcal{M}/D)$ be homotopy cartesian. By the remarks from the beginning this corresponds to an equifibred natural transformation $Y \to D$ of functors $I \to \mathcal{M}$. Thus by the assumption that $\operatorname{hocolim}_I D \simeq X$ and (P2) we know that Y_i is weakly equivalent to $D(i) \times^h_X \operatorname{hocolim}_I Y$. By the same arguments as for (1), the right hand side is (naturally) weakly equivalent to $(R\phi^* L\phi_\# Y)_i$. This concludes the proof. □

Definition A.87. *A Quillen pseudo-presheaf $\mathcal{M}$ on a small category $\mathcal{C}$ is called a (symmetric) monoidal pseudo-presheaf if each category $\mathcal{M}(c)$ is a (symmetric) monoidal model category and each of the restriction functors $f^* : \mathcal{M}(d) \to \mathcal{M}(c)$ (for $f : c \to d \in \mathcal{C}$) is monoidal.*

We say that a (symmetric) monoidal τ-Quillen pseudo-presheaf on a Verdier site $\mathcal{C}$ satisfies the projection formula *if for every basal map $f : c \to d \in \mathcal{C}$ and every $T \in \mathcal{M}(d)$ and every $S \in \mathcal{M}(c)$, the natural map*

$$f_\#(S \otimes f^* T) \to f_\#(S) \otimes T$$

is an isomorphism.[3]

The natural map is constructed as follows: by adjunction, it corresponds to a natural map $S \otimes f^* T \to f^* f_\#(S) \otimes f^* T$. It is thus enough to find a canonical map $S \to f^* f_\# S$. There is such a map, the one corresponding by adjunction to $\mathrm{id} : f_\# S \to f_\# S$.

We now state a technical result which allows us to construct many Quillen sheaves. To do so, we introduce the notion of a λ-suitable τ-fibred Verdier site. This just means a λ-suitable Verdier site $(\mathcal{C}, \tau)$, a pseudofunctor $\mathcal{D}$ on $\mathcal{C}$, and the structure of a λ-suitable Verdier site $(\mathcal{D}(c), \tau_{\mathcal{D}(c)})$ for every $c \in \mathcal{C}$, such that each restriction $f^* : \mathcal{D}(d) \to \mathcal{D}(c)$ preserves covering families, and such that if f is basal, then f^* has a left adjoint $f_\#$ which also preserves covering families.

Theorem A.88. *Let $\mathcal{C}, \mathcal{D}$ be a λ-suitable τ-fibred Verdier site as above. Assume that*

(i) Each $\mathcal{D}(c)$ has finite products and a final object,

(ii) For each basal map $f : c \to d$, the functor $\mathcal{D}(c) \to \mathcal{D}(d)/f_\#(*)$ is an equivalence of sites,

(iii) The functors f^* preserve pullbacks along basal maps,

(iv) If $\phi : U_\bullet \to X$ is a hypercover in $\mathcal{C}$, then $\phi_\#(*_{\mathcal{D}(U_\bullet)}) \to *_{\mathcal{D}(X)}$ is a hypercover of $*$ in $\mathcal{D}(X)$,

(v) For every family $\{X_i\}_{i \in I}$ of size less than λ, we have $\mathcal{D}(\coprod_i X) \cong \prod_i \mathcal{D}(X_i)$ (as sites).

Then the assignment

$$\mathcal{C} \ni X \mapsto sPre(\mathcal{D}(X))_{proj, \tau_{\mathcal{D}(X)}}$$

forms a simplicial, symmetric monoidal, proper, tractable τ-Quillen pseudo-sheaf on $\mathcal{C}$ which satisfies the projection formula.

Proof. Let us put $\mathcal{M}(c) = sPre(\mathcal{D}(c))_{proj, \tau_{\mathcal{D}(c)}}$.

By Theorem A.23, each $\mathcal{M}(c)$ is a symmetric monoidal, proper, simplicial, tractable model category (using (i) for monoidality). By Proposition A.25 and Corollary A.38, each of the restriction functors $f : \mathcal{D}(d) \to \mathcal{D}(c)$ induces a left Quillen functor $f^* : \mathcal{M}(d) \to \mathcal{M}(c)$. (In that proposition the left functor is denoted $\alpha_\#$ and the right functor α^*, we choose to call them f^* and f_* instead.)

Suppose $f : c \to d$ is a basal map. Then we have $\mathcal{D}(c) \cong \mathcal{D}(d)/f_\#(*)$ by (ii), and under this identification the functor $f_\# : \mathcal{D}(d)/f_\#(*) \to \mathcal{D}(d)$ is just the forgetful one. Its right adjoint is $T \mapsto T \times f_\#(*)$, so this identifies f^*. Since the monoidal structures we use come just from ordinary products, we conclude that the projection formula holds for all representable objects, and hence

[3] It may be tempting to only require the map to be a weak equivalence. This is probably possible, but would seriously complicate our treatment of spectra later.

it holds for all objects because both $f_\#$ and f^* (on categories of simplicial presheaves) commute with colimits (since both have right adjoints).

Additionally the functor $f_\# : \mathcal{M}(c) \to \mathcal{M}(d)$ is left Quillen, by Proposition A.25 and Corollary A.38 again (use that $f_\#$ preserves pullbacks along basal maps, by its identification as a forgetful functor we produced above).

We have thus constructed a simplicial, symmetric monoidal, proper, tractable τ-Quillen pseudo-presheaf on $\mathcal{C}$, satisfying the projection formula, as claimed. It remains to show that this is a sheaf.

The condition about coproducts of families of size less than λ follows immediately from assumption (v).

Hence let $\phi : U_\bullet \to X$ be a hypercover in $\mathcal{C}$. We need to show that the natural right Quillen functor $\mathcal{M}(X) \to \operatorname{holim}_U \mathcal{M}$ is a Quillen equivalence. Let $R_n = \phi_{n\#}(*_{\mathcal{D}(U_n)})$. Then by assumption (iv) the R_n form a hypercover of $* \in \mathcal{D}(X)$. We have $\mathcal{M}(U_n) \cong \mathcal{M}(X)/R_n$, by assumption (ii) and Lemma A.28. The result thus follows from Lemma A.86, provided that $\mathcal{M}(X)$ satisfies homotopical patching, and the natural map $\operatorname{hocolim} R_n \to *$ is a weak equivalence in $\mathcal{M}(X)$.

For any site $(\mathcal{E}, \nu)$, the model category $sPre(\mathcal{E})_{proj,\nu}$ satisfies patching [90, Example 6.3 and Proposition 6.6]. If $\mathcal{E}$ is a suitable Verdier site, then for any hypercover $V_\bullet \to Y$ in $\mathcal{E}$ the map $\operatorname{hocolim} V_\bullet \to Y$ is a weak equivalence, by Corollary A.36. This concludes the proof. □

Corollary A.89. *Let $(\mathcal{C}, \tau)$ be a suitable Verdier site with fibre products. Then the assignment*

$$\mathcal{C} \ni X \mapsto sPre(\mathcal{C}/X)_{proj \tau_X}$$

defines a simplicial, symmetric monoidal, tractable, proper τ-Quillen pseudo-sheaf which satisfies the projection formula.

Proof. Apply the theorem with $\mathcal{D}(X) = (\mathcal{C}/X, \tau_X)$. □

Recall the suitable cdh Verdier site on a scheme X of finite type over a field k from Lemma A.31.

Corollary A.90. *Let S be a scheme of finite type over a field k. The assignment*

$$Ft(S) \ni X \mapsto sPre(Ft(X))_{cdh}$$

defines a simplicial, symmetric monoidal, tractable, proper bi-Quillen pseudo-sheaf which satisfies the projection formula.

Proof. Apply the previous corollary, using that for $X \in Ft(S)$ we have that $Ft(X) \cong Ft(S)/X$. (Note that all maps are basal in this Verdier site, so we get a bi-Quillen pseudo-presheaf.) □

Recall the suitable Zariski/Nisnevich/étale Verdier sites on a scheme X of finite type over a field k from Lemma A.30. The following corollary is a reformulation of an argument of Marc Hoyois [51, Proposition 4.8(1)] in the language of model categories.

Corollary A.91. *Let S be a scheme of finite type over a field k and $\tau \in \{Zar, Nis, et\}$. The assignment*

$$Ft(S) \ni X \mapsto sPre(Sm(X))_\tau$$

defines a simplicial, symmetric monoidal, tractable, proper τ-Quillen pseudo-sheaf which satisfies the projection formula.

Proof. Apply the theorem with $\mathcal{D}(X) = Sm(X)_\tau$, using that for $X \to Y$ étale we have $Sm(X) \cong Sm(Y)/X$, and that the basal maps are étale in all three cases. □

A.9 Constructions with τ-Quillen Sheaves

In this section we prove a few theorems along the following lines: if $\mathcal{M}$ is a pseudo-presheaf and we perform a familiar operation in a compatible way on each of the $\mathcal{M}(X)$, then we obtain another pseudo-presheaf, and any good properties of the old presheaf (being simplicial, a sheaf, monoidal, satisfying the projection formula, etc.) also hold for the new one.

Definition A.92. *Let $\mathcal{C}$ be a Verdier site and $\mathcal{M}, \mathcal{N}$ Quillen presheaves on $\mathcal{C}$. We call a pseudonatural transformation $F : \mathcal{M} \to \mathcal{N}$ a left (right) morphism if each of the sections $F(X) : \mathcal{M}(X) \to \mathcal{N}(X)$ is a left (right) Quillen functor.*

A.9.1 Pointing

Theorem A.93. *Let $\mathcal{M}$ be a τ-Quillen pseudo-presheaf on the suitable Verdier site $(\mathcal{C}, \tau)$. Assume that $\mathcal{C}$ has a final object $*$, pick $pt \in \mathcal{M}(*)$, and for $X \in \mathcal{C}$ put $pt(X) = (X \to *)^* pt$. We consider the assignment $pt/\mathcal{M} : X \mapsto pt(X)/\mathcal{M}(X)$.*

1. *The assignment $pt/\mathcal{M}$ defines a τ-Quillen pseudo-presheaf. There is a canonical right morphism $U : pt/\mathcal{M} \to \mathcal{M}$, the underlying object morphism.*
2. *If $\mathcal{M}$ has any of the following properties, so does $pt/\mathcal{M}$: left/right proper, combinatorial, tractable, simplicial.*
3. *Suppose that $\mathcal{M}$ is a sheaf. If $pt \in \mathcal{M}(*)$ is cofibrant, then $pt/\mathcal{M}$ is a τ-Quillen sheaf.*
4. *If $\mathcal{M}$ is monoidal, for each $X \in \mathcal{C}$ we have $pt(X) \cong * \cong \mathbb{1}$, and this object is cofibrant, then $pt/\mathcal{M}$ is monoidal, with the smash product as monoidal operation. Also $pt/\mathcal{M}$ satisfies the projection formula if $\mathcal{M}$ does.*

In the situation of statement (4), we also denote $pt/\mathcal{M}$ by $\mathcal{M}_*$. We remark that instead of assuming that $\mathcal{C}$ has a final object, we could just assume given a cartesian section $X \mapsto pt(X)$. We do not need this additional generality.

Proof. Each $pt(X)/\mathcal{M}(X)$ is a model category where cofibrations, fibrations and weak equivalences are the underlying ones [43, Theorem 7.6.5].

If $f : X \to Y \in \mathcal{C}$ and $T \in pt(Y)/\mathcal{M}(Y)$ then $f^*(T)$ is naturally pointed by $f^*(pt(X)) \cong pt(Y)$. This defines $f^* : (pt/\mathcal{M})(Y) \to (pt/\mathcal{M})(X)$. Similarly for $S \in pt(X)/\mathcal{M}(X)$, $f_*(S)$ is pointed by $pt(Y) \to f_* f^* pt(Y) \cong f_* pt(X) \to f_* S$. This defines $f_* : (pt/\mathcal{M})(X) \to (pt/\mathcal{M})(Y)$. It is then easy to check using Lemma A.94 below that there is an adjunction

$$f^* : (pt/\mathcal{M})(Y) \leftrightarrows (pt/\mathcal{M})(X) : f_*.$$

If $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$ affords a left adjoint $f_\#$, we define a functor $\tilde{f}_\# : (pt/\mathcal{M})(X) \to (pt/\mathcal{M})(Y)$ by the pushout

$$\begin{array}{ccc} f_\# pt(X) & \longrightarrow & f_\# S \\ \downarrow & & \downarrow \\ pt(Y) & \longrightarrow & \tilde{f}_\# S. \end{array}$$

(Note that $\tilde{f}_\# S$ *is* naturally pointed by $pt(Y)$.) Again $\tilde{f}_\#$ is left adjoint to f^* by an easy application of Lemma A.94 below.

Since the cofibrations/fibrations/weak equivalences in $pt/\mathcal{M}$ are essentially the same as those in $\mathcal{M}$, the functor $f^* : pt(Y)/\mathcal{M}(Y) \to pt(X)/\mathcal{M}(X)$ is left/right Quillen if $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$ is. Consequently we have a τ-Quillen pseudo-presheaf. The (right Quillen) underlying object functors assemble to a right morphism as claimed, proving (1).

The non-trivial parts of (2) are proved in [44, Theorem 2.8].

Statement (4), without the claim that $\mathcal{M}$ satisfies the projection formula, is essentially [47, Proposition 4.2.9].

Let us show that in the situation of (4), $pt/\mathcal{M}$ satisfies the projection formula if $\mathcal{M}$ does. So let $f : X \to Y$ be a basal morphism, $S \in pt(X)/\mathcal{M}(X)$, $T \in pt(Y)/\mathcal{M}(Y)$, and suppose that $\mathcal{M}$ satisfies the projection formula. We obtain a commutative diagram

$$\begin{array}{ccccc}
f_\# S \coprod T \otimes f_\# * & & & & \\
\| & & & & \\
f_\#(S \coprod f^*T) & \longrightarrow & f_\# * & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
f_\#(S \otimes f^*T) & \longrightarrow & f_\#(S \wedge f^*T) & \longrightarrow & U\tilde{f}_\#(S \wedge f^*T) \\
\| & & & & \\
f_\#(S) \otimes T. & & & &
\end{array}$$

Here the top and bottom identifications come from the projection formula for $\mathcal{M}$, the middle left square is a pushout by definition of the smash product and because $f_\#$ preserves colimits, and the right middle square is a pushout by definition of $\tilde{f}_\#$. In contrast we have pushout squares

$$\begin{array}{ccccc}
f_\#(*) \otimes T & \longrightarrow & T & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
f_\#(S) \otimes T & \longrightarrow & \tilde{f}_\#(S) \otimes T & \longrightarrow & \tilde{f}_\#(S) \wedge' T.
\end{array}$$

(Use that $\otimes$ preserves colimits, having a right adjoint.) Here $\tilde{f}_\#(S) \wedge' T$ means that in order to obtain $\tilde{f}_\#(S) \wedge T$ one still has to "quotient out by the image of $\tilde{f}_\# S$". This is the same as quotienting out by the image of $f_\# S$, of which $\tilde{f}_\# S$ is itself a quotient. Hence $U(\tilde{f}_\#(S) \wedge T)$ satisfies the same universal property as $U\tilde{f}_\#(S \wedge f^*T)$, whence they are isomorphic. Thus $pt/\mathcal{M}$ satisfies the projection formula.

We finally prove (3). So suppose that $\mathcal{M}$ is a sheaf. We need to prove that $pt/\mathcal{M}$ is also a sheaf. First let $X = \coprod_i X_i$ be a suitably small coproduct. We need to show that $(pt/\mathcal{M})(X) \to \prod_i (pt/\mathcal{M})(X_i)$ is a Quillen equivalence. This follows from Lemma A.96 below.

Now let $\phi : V_\bullet \to X$ be a hypercover. We need to prove that $\tilde{\phi}_\# : \operatorname{holim}(pt/\mathcal{M})|_V \to pt/\mathcal{M}(X)$ is a Quillen equivalence. We know that ϕ^* preserves weak equivalences, so $\phi^* T \simeq R\phi^* T$, and consequently $UR\phi^* T \simeq R\phi^* UT$. I claim that similarly $UL\tilde{\phi}_\# S \simeq L\phi_\# US$. The result will follow since U detects weak equivalences.

To prove the claim, consider the right Quillen presheaf $\mathcal{M}'$ on Δ^{op} given by $\mathcal{M}'([n]) = \phi_\#(pt(V_n))/\mathcal{M}(X)$. Then the pullback $\phi^* : pt(X)/\mathcal{M}(X) \to \operatorname{holim}(pt/\mathcal{M})|_V$ can be factored as $pt(X)/\mathcal{M}(X) \xrightarrow{\alpha^*} \operatorname{holim}\mathcal{M}' \xrightarrow{\beta^*} \operatorname{holim}(pt/\mathcal{M})|_V$. Here $\alpha^*([n]) : pt(X)/\mathcal{M}(X) \to \phi_\#(pt(V_n))/\mathcal{M}(X)$ is just precomposition, and $\beta^*([n]) : \phi_\# pt(V_n)/\mathcal{M}(X) \to pt(V_n)/\mathcal{M}(V_n)$ is $\beta([n])^*(\phi_\#(pt)(V_n) \to T) = (pt(V_n) \to \phi^*(T))$, where the structure morphism is given by adjunction.

Note that β^* has a left adjoint $\beta_\#$, with $\beta_\#([n]) : pt(V_n)/\mathcal{M}(V_n) \to \phi_\#(pt(V_n))/\mathcal{M}(X)$ given by $\phi_\#$. The functor α^* has a left adjoint given by the usual pushout. Moreover α^* preserves cofibrations, fibrations and weak equivalences, and β^* preserves fibrations and acyclic fibrations (because ϕ^* is right Quillen). Consequently α^* and β^* are both right Quillen.

Write $\mathcal{M}''$ for the constant presheaf on Δ^{op} given by $\mathcal{M}''([n]) = \mathcal{M}(X)$, $\alpha'_\# : \operatorname{holim}\mathcal{M}'' \leftrightarrows \mathcal{M}(X) : \alpha'^*$ for the evident adjunction and U for the underlying object functors $\mathcal{M}' \to \mathcal{M}''$ etc. Let $S \in \operatorname{holim}_V \mathcal{M}$. Then $L\tilde{\phi}_\# S = (L\alpha_\#)(L\beta_\#)S$. By Lemma A.95 below, there is a homotopy pushout

$$\begin{array}{ccc} L\alpha'_{\#}U\beta_{\#}pt(V_\bullet) & \longrightarrow & L\alpha'_{\#}UL\beta_{\#}S \\ \big\downarrow & & \big\downarrow \\ pt(X) & \longrightarrow & UL\alpha_{\#}L\beta_{\#}S. \end{array}$$

It is easy to see that the top left hand corner is weakly equivalent to $L\phi_{\#}R\phi^* pt(X)$ and hence the left hand map is a weak equivalence, by the sheaf condition for $\mathcal{M}$. Consequently so is the right hand map. The bottom right corner is $UL\tilde{\phi}_{\#}S$. It is easy to see that $L\beta_{\#}$ commutes with U, and hence the top right hand corner is $L\phi_{\#}US$. This proves the claim. □

We contend the following lemma must be well known, but could not locate a source.

Lemma A.94. *Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjunction between categories. Suppose given $f : X \to Y$ in $\mathcal{C}$ and a diagram*

$$\begin{array}{ccc} LY & \xrightarrow{\alpha} & Y' \\ {\scriptstyle Lf}\big\uparrow & & {\scriptstyle g}\big\uparrow \\ LX & \xrightarrow{\beta} & X'. \end{array}$$

Then the diagram commutes if and only if the adjoint diagram

$$\begin{array}{ccc} Y & \xrightarrow{\alpha^T} & RY' \\ {\scriptstyle f}\big\uparrow & & {\scriptstyle Rg}\big\uparrow \\ X & \xrightarrow{\beta^T} & RX'. \end{array}$$

commutes.

Proof. Since passing to opposite categories interchanges left and right adjoints but preserves and detects commutativity, it suffices to show that the adjoint diagram commutes whenever the original one does. So suppose that the top diagram commutes.

Let $\eta_X = \mathrm{id}^T \in \mathrm{Hom}(X, RLX) \cong \mathrm{Hom}(LX, LX)$ be the unit of adjunction, and similarly for Y. Then using the unit-co-unit formulas for determining the adjoint of α, β, the equality we have to prove is $R\alpha \circ \eta_Y \circ f = Rg \circ R\beta \circ \eta_X$. Consider the following diagram.

$$\begin{array}{ccccc} Y & \xrightarrow{\eta_Y} & RLY & \xrightarrow{R\alpha} & RY' \\ {\scriptstyle f}\big\uparrow & & {\scriptstyle RLf}\big\uparrow & & {\scriptstyle Rg}\big\uparrow \\ X & \xrightarrow{\eta_X} & RLX & \xrightarrow{R\beta} & RX'. \end{array}$$

We need to prove commutativity of the big rectangle. The right hand square certainly commutes (it is obtained by applying the functor R to the square we assumed commutative), so we need only prove commutativity of the left hand square.

Thus we have reduced to $g = Lf, \alpha = \mathrm{id}, \beta = \mathrm{id}$. We have to show that $\eta_Y \circ f = RL(f) \circ \eta_X$. Let ϵ_{LY} be the co-unit. Then $\eta_Y f$ corresponds by adjunction to $\epsilon_{LY} L(\eta_Y) L(f)$, which equals $L(f)$ by the unit-co-unit formulas. Finally the adjoint of $L(f)$ is by definition $RL(f)\eta_X$. In other words $\eta_Y f$ and $RL(f)\eta_X$ are both adjoint to $L(f)$, and so equal. □

Lemma A.95. *Let I be a small category, $\mathcal{M}$ a combinatorial model category, $D : I \to \mathcal{M}$ a diagram, $X \in \mathcal{M}$ and $\alpha : D \to X$ an augmentation. Consider the right Quillen pseudo-presheaves $\mathcal{M}'(i) = D(i)/\mathcal{M}$ and $\mathcal{M}''(i) = \mathcal{M}$ (a constant presheaf). There is an essentially commutative diagram of right Quillen functors*

$$\begin{array}{ccc} \mathrm{holim}_I\, \mathcal{M}' & \xrightarrow{U} & \mathrm{holim}_I\, \mathcal{M}'' \\ {\scriptstyle \alpha^*}\big\uparrow & & {\scriptstyle \alpha'^*}\big\uparrow \\ X/\mathcal{M} & \xrightarrow{U} & \mathcal{M}. \end{array}$$

Suppose now that each object $D(i)$ is cofibrant. Write $D_\bullet \in \operatorname{holim}_i \mathcal{M}'' \cong Fun(I, \mathcal{M})$ for the section corresponding to D via Proposition A.73, part 1. Then for $S \in \operatorname{holim}_I \mathcal{M}'$ there is a homotopy pushout

$$\begin{array}{ccc} L\alpha'_{\#} D_\bullet & \longrightarrow & L\alpha'_{\#} US \\ \downarrow & & \downarrow \\ X & \longrightarrow & UL\alpha_{\#} S. \end{array}$$

Proof. The existence of the first commutative diagram is clear.

It remains to establish the second statement. I claim that it is invariant under projective (objectwise) weak equivalences in D. Indeed let $D' : I \to \mathcal{M}$ be another objectwise cofibrant diagram and $f : D' \to D$ an objectwise weak equivalence. Write $\mathcal{N}$ for the right Quillen pseudo-presheaf $\mathcal{N}(i) = D'(i)/\mathcal{M}$. Then we get a right morphism $f^* : \mathcal{M}' \to \mathcal{N}$ given by precomposition and hence a Quillen adjunction

$$f_\# : \operatorname{holim}_I \mathcal{N} \leftrightarrows \operatorname{holim}_I \mathcal{M}' : f^*,$$

by Proposition A.67. By that same proposition this is a Quillen equivalence provided that each $f^* : \mathcal{M}'(i) \to \mathcal{N}(i)$ is a Quillen equivalence. This is true by the dual of Proposition A.3, part 2, since each $D(i), D'(i)$ is assumed cofibrant. This proves the claim.

Next note that there is a pushout

$$\begin{array}{ccc} \alpha'_{\#} D_\bullet & \longrightarrow & \alpha'_{\#} US \\ \downarrow & & \downarrow \\ X & \longrightarrow & U\alpha_{\#} S. \end{array}$$

Indeed using Lemma A.94 one shows that this pushout defines a left adjoint to α^* (the bottom right hand corner is naturally pointed by X).

By the claim, we may assume that $D_\bullet$ is cofibrant as a diagram (i.e. in the holim model structure, equivalently in the projective model structure). In order to prove something about derived functors, we may also assume that S is cofibrant, i.e. that $D_\bullet \to US$ is a cofibration. Consequently US is also cofibrant. Now $L\alpha_\# S = \alpha_\# S$ (since $\alpha_\#$ is left Quillen), and similarly $L\alpha'_\# D_\bullet = \alpha'_\# D_\bullet, L\alpha'_\# US = \alpha'_\# US$. We thus have a pushout diagram of derived functors in the claimed form. This pushout is a homotopy pushout because the top map is a cofibration between cofibrant objects ($\alpha'_\#$ being left Quillen). This concludes the proof. □

Lemma A.96. *Let $F : \mathcal{M} \to \mathcal{N}$ be a bi-Quillen functor which is a Quillen equivalence. If $X \in \mathcal{M}$ is cofibrant, then $F : X/\mathcal{M} \to F(X)/\mathcal{N}$ is a Quillen equivalence.*

Proof. Let G be the left adjoint to F. As usual we get an induced Quillen adjunction $\tilde{G} : F(X)/\mathcal{N} \leftrightarrows X/\mathcal{M} : F$, where for $FX \to T \in F(X)/\mathcal{N}$ the object $\tilde{G}(FX \to T)$ is computed as the pushout

$$\begin{array}{ccc} GFX & \xrightarrow{a} & X \\ {\scriptstyle c}\downarrow & & \downarrow \\ GT & \xrightarrow{b} & \tilde{G}T. \end{array}$$

Now $GFX \simeq (LG)(RF)X$ because F is bi-Quillen and so FX is cofibrant. But $(LG)(RF)X \simeq X$ because F is a Quillen equivalence, hence $a : GFX \to X$ is a weak equivalence. If $(FX \to T) \in F(X)/\mathcal{M}$ is cofibrant then b is the pushout of a weak equivalence between cofibrant objects along a cofibration, so is a weak equivalence [43, Proposition 13.1.2(1)]. We conclude that $L(\tilde{G})$ commutes with the underlying object functors (up to weak equivalence). Since F always does, and the underlying object functors detect weak equivalences (i.e. isomorphisms in the homotopy category), it follows that $L(\tilde{G}) \vdash RF$ are adjoint equivalences of homotopy categories. □

We remark that for point (3) of the theorem, it is not necessary to assume that pt is cofibrant. Indeed in order to talk about sheaves we need to assume that $\mathcal{M}$ is left proper, and then the undercategories $pt(X)/\mathcal{M}(X)$ are invariant up to Quillen equivalence under change of $pt(X)$ (such as cofibrantly replacing), by the dual of Proposition A.3. However cofibrancy of pt is essential for point (4), and we shall only ever use that result.

A.9.2 Localisation

The result in this section is essentially a reformulation of [51, Proposition 4.8(2)].

Theorem A.97. *Let $\mathcal{M}$ be a τ-Quillen pseudo-presheaf on the suitable Verdier site $(\mathcal{C}, \tau)$. Suppose given for each $X \in \mathcal{C}$ a set $H(X) \subset Mor(Ho(\mathcal{M}(X)))$ such that for each $f : Y \to X \in \mathcal{C}$ we have $Lf^*(H(X)) \subset H(Y)$, and such that if f is basal, then additionally $Lf_\#(H(Y)) \subset H(X)$. Assume further that for each X the left Bousfield localisation $L_{H(X)}\mathcal{M}(X)$ exists. We consider the assignment $L_H\mathcal{M} : X \mapsto L_{H(X)}\mathcal{M}(X)$.*

1. *The assignment $L_H\mathcal{M}$ defines a τ-Quillen pseudo-presheaf. There is a canonical left morphism $L : \mathcal{M} \to L_H\mathcal{M}$, the localisation morphism.*
2. *If $\mathcal{M}$ has any of the following properties, so does $L_H\mathcal{M}$: left proper, combinatorial, tractable, simplicial and tractable, satisfies the projection formula.*
3. *Suppose that $\mathcal{M}$ is a sheaf (so in particular combinatorial and left proper). Assume that if $X = \coprod_i X_i$ is a suitable coproduct, then $L_H\mathcal{M}(X) \to \prod_i L_H\mathcal{M}(X_i)$ is a Quillen equivalence. Then $L_H\mathcal{M}$ is a τ-Quillen sheaf.*

We point out that it is *not* automatic that $L_H\mathcal{M}$ is monoidal if $\mathcal{M}$ is. This can be established in practice by using Theorem A.11.

Proof. If $F : \mathcal{M}_1 \to \mathcal{M}_2$ is a left Quillen functor of model categories, $H_i \subset Mor(Ho(\mathcal{M}_i))$, and $LF(H_1) \subset H_2$, then the universal property of left Bousfield localization [9, Definition 2.6] implies that there is a unique left Quillen functor $F' : L_{H_1}\mathcal{M}_1 \to L_{H_2}\mathcal{M}_2$ factoring $\mathcal{M}_1 \to \mathcal{M}_2 \to L_{H_2}\mathcal{M}_2$ through $\mathcal{M}_1 \to L_{H_1}\mathcal{M}_1$. In fact the localisation functors $L_i : \mathcal{M}_i \to L_{H_i}\mathcal{M}_i$ are always the identity on underlying categories, so $F' = F$ as ordinary functors, and this really only says that F remains left Quillen in the new model structures.

As a consequence of this discussion, $L_H\mathcal{M}$ is a Quillen pseudo-presheaf in the canonical way, simply because $\mathcal{M}$ is, and all the functors $f^*, f_\#$ remain left Quillen in the localised model structures by our assumptions on H. The morphism $L : \mathcal{M} \to L_H\mathcal{M}$ is just the identity. We have thus established (1).

All the claims in (2) except for the projection formula just say that certain properties of model categories are preserved under localization and are standard. The projection formula does not mention the model structures at all and so is satisfied for $L_H\mathcal{M}$ if and only if it is satisfied for $\mathcal{M}$.

Now to prove (3), let $U_\bullet \to X$ be a hypercover. We have a commutative diagram

$$\begin{array}{ccc} Ho(\operatorname{holim}_U L_H\mathcal{M}) & \xrightarrow{d} & Ho(\operatorname{holim}_U \mathcal{M}) \\ {\scriptstyle a}\uparrow & & {\scriptstyle c}\uparrow \\ Ho((L_H\mathcal{M})(X)) & \xrightarrow{b} & Ho(\mathcal{M}(X)). \end{array}$$

We wish to show that the functor a is an equivalence. The functor b is fully faithful by construction and c as an equivalence because $\mathcal{M}$ is a sheaf, so a must be fully faithful. It remains to show that a is essentially surjective.

To see this, I first claim that d is fully faithful. Indeed $i : L_H\mathcal{M} \to \mathcal{M}$ is a right morphism of combinatorial Quillen presheaves, so by Proposition A.67 there is an induced adjunction $L : \operatorname{holim}_U \mathcal{M} \leftrightarrows \operatorname{holim}_U L_H\mathcal{M} : i$ of homotopy limits. But it is easy to see that L is a Bousfield localisation, and also $d = Ho(i)$, so d has a section (namely $Ho(L)$), and in particular is fully faithful.

Now let $T \in Ho(\operatorname{holim}_U L_H\mathcal{M})$. Since c is an equivalence there exists an essentially unique $S \in Ho(\mathcal{M}(X))$ such that $cS \cong dT$. If $\phi_n : U_n \to X$ is one of the maps in the cover, then we have the commutative diagram

$$\begin{array}{ccc} Ho((L_H\mathcal{M})(U_n)) & \longrightarrow & Ho(\mathcal{M}(U_n)) \\ {\scriptstyle q}\uparrow & & {\scriptstyle p}\uparrow \\ Ho(\operatorname{holim}_U L_H\mathcal{M}) & \longrightarrow & Ho(\operatorname{holim}_U \mathcal{M}) \\ \uparrow & & {\scriptstyle c}\uparrow \\ Ho((L_H\mathcal{M})(X)) & \longrightarrow & Ho(\mathcal{M}(X)), \end{array}$$

where the two upper vertical arrows come from the change of diagram functor $[n] : * \to \Delta$ via Proposition A.72, and the vertical composites are the pullbacks ϕ_n^*. It follows that $\phi_n^* S \cong pcS \cong qT \in Ho((L_H\mathcal{M})(U_n)) \subset Ho(\mathcal{M}(U_n))$. Since n was arbitrary the Lemma below applies and we conclude that $S \in Ho((L_H\mathcal{M})(X))$, as needed. □

Lemma A.98. *In the situation of the theorem part (3), let $\phi : U_\bullet \to X$ be a hypercover and $T \in Ho(\mathcal{M})(X)$ be such that if $\phi_0 : U_0 \to X$ is the cover in level zero, then the object $\phi_0^* T \in Ho(\mathcal{M}(U_0))$ is $H(U_0)$-local. Then T is $H(X)$-local.*

Proof. We need to prove that if $p : A \to B$ is any one of the maps in $H(X)$, then $Map^d(p, T) : Map^d(B, T) \to Map^d(A, T)$ is a weak equivalence of simplicial sets. Since $\mathcal{M}$ is a sheaf we have $p \simeq L\phi_\# \phi^* p$. Recall that $\phi_\# : \operatorname{holim}_U \mathcal{M} \to \mathcal{M}(X)$ can be factored as $\operatorname{holim}_U \mathcal{M} \xrightarrow{\phi^0_\#} \operatorname{holim}_U \mathcal{M}_X \xrightarrow{s_\#} \mathcal{M}(X)$. Here $\phi^0_\#$ is just objectwise $\phi_{n\#}$, whereas $s_\#$ is $\operatorname{hocolim}_\Delta$, by Proposition A.73. Consequently we have

$$Map^d(p, T) \simeq \operatorname{holim}_n Map^d(L\phi_{n\#}\phi_n^* p, T).$$

By homotopy invariance of homotopy limits, it is thus enough to show that $Map^d(L\phi_{n\#}\phi_n^* p, T)$ is a weak equivalence for every n. By adjunction this is the same as showing that $Map^d(\phi_n^* p, \phi_n^* T)$ is a weak equivalence. For this it is enough to know that $\phi_n^* T$ is $H(U_n)$-local, which follows from $\phi_0^* T$ being $H(U_0)$-local (because the pullbacks to higher levels of the cover can be factored through pullback to level zero). □

A.9.3 Stabilisation

Definition A.99. *Let $\mathcal{M}$ be a closed symmetric monoidal model category and $P \in \mathcal{M}$. We write Ω_P for the functor $\underline{\mathrm{Hom}}(P, \bullet) : \mathcal{M} \to \mathcal{M}$. Its left adjoint is denoted $\Sigma_P : X \mapsto X \otimes P$.*

Let $\mathbb{N}$ be the small category with objects the natural numbers $\{0, 1, \dots\}$ and a unique morphism from $a \to b$ if and only if $a > b$. We obtain a right pseudo-presheaf $\mathcal{M}^{\Omega_P}$ on $\mathbb{N}$ in which the transition morphism $(b \to a)^*$ is given by Ω_P^{b-a}. This is a right Quillen pseudo-presheaf if P is cofibrant.

If $\mathcal{M}$ is left proper and combinatorial and P is cofibrant, then we denote by $Stab(\mathcal{M}, P)$ the homotopy limit $\operatorname{holim}_{\mathbb{N}} \mathcal{M}^{\Omega_P}$. Write $i_0 : * \to \mathbb{N}$ for the inclusion of the object 0. In the Quillen adjunction

$$i_{0\#} : \mathcal{M} = \operatorname{holim}_* \mathcal{M}^{\Omega_P} \leftrightarrows \operatorname{holim}_{\mathbb{N}}^{\Omega_P} : i_0^*$$

from Proposition A.67, we write Σ^∞ for $i_{0\#}$ and Ω^∞ for i_0^*.

Note that an object $X \in Stab(\mathcal{M}, P)$ consists of a objects $X_n \in \mathcal{M}$ $(n = 0, 1, \dots)$ together with maps $X_n \to \Omega_P X_{n+1}$ (or equivalently $X_n \otimes P \to X_{n+1}$). In these terms the functor Ω^∞ is given by $\Omega^\infty(X) = X_0$, whereas $\Sigma^\infty(S)_n = S \otimes P^{\otimes n}$. The holim-fibrant objects are those $X \in Stab(\mathcal{M}, P)$ such that each X_n is fibrant and such that $X_n \to \Omega_P X_{n+1}$ is a weak equivalence.

For example, if $\mathcal{M}$ is the model category of pointed simplicial sets and $P = \partial\Delta^2$, then $Ho(Stab(\mathcal{M}, P))$ is the classical stable homotopy category [9, Example 2.43]. We will say more about how $Stab(\mathcal{M}, P)$ models spectra after establishing some properties.

Lemma A.100. *Let $\mathcal{M}$ be a left proper, combinatorial model category and $P \in \mathcal{M}$ be fibrant. Then $Stab(\mathcal{M}, P)$ is left proper and combinatorial. If $\mathcal{M}$ is tractable or simplicial and tractable then so is $Stab(\mathcal{M}, P)$.*

Proof. This is just a special case of Theorem A.66. □

Lemma A.101. *Let $F : \mathcal{M} \leftrightarrows \mathcal{N} : R$ be an adjunction of closed symmetric monoidal categories, $P \in \mathcal{M}$ and $Q \in \mathcal{N}$. Assume given for $T \in \mathcal{M}$ a natural isomorphism $F(T \otimes P) \cong FT \otimes Q$. Then there exists for $S \in \mathcal{N}$ a natural isomorphism $R(\Omega_Q S) \cong \Omega_P RS$.*

Consequently if $\mathcal{M}, \mathcal{N}$ are left proper, combinatorial model categories, P is cofibrant, Q is cofibrant and F is a left Quillen functor, then there is an induced Quillen adjunction

$$Stab(F) : Stab(\mathcal{M}, P) \leftrightarrows Stab(\mathcal{N}, Q),$$

which is a Quillen equivalence if F is. The induced adjunction is simplicial if the original one is.

Proof. The first part follows immediately from the Yoneda lemma and using the adjunctions $F \vdash R, \Sigma_P \vdash \Omega_P$. We thus get a right morphism $R : \mathcal{N}^{\Omega_P} \to \mathcal{M}^{\Omega_P}$. Under the extra assumptions this is a right morphism of left proper, combinatorial right Quillen pseudo-presheaves, so the remaining claims are a special case of Proposition A.67. □

Theorem A.102. *Let $(\mathcal{C}, \tau)$ be a suitable Verdier site with final object $*$, $\mathcal{M}$ a symmetric monoidal, combinatorial, left proper τ-Quillen presheaf on $\mathcal{C}$, and $P \in \mathcal{M}$ cofibrant. Assume that $\mathcal{M}$ satisfies the projection formula. Put $P(X) = (X \to *)^* P$ and consider the assignment $Stab(\mathcal{M}, P) : X \mapsto Stab(\mathcal{M}(X), P(X))$.*

1. *The assignment $Stab(\mathcal{M}, P)$ defines a left proper, combinatorial τ-Quillen pseudo-presheaf.*
2. *If $\mathcal{M}$ has any of the following properties, so does $Stab(\mathcal{M}, P)$: tractable, simplicial and tractable, is a sheaf.*
3. *The various functors $\Sigma^\infty \vdash \Omega^\infty$ assemble into a pair of left/right morphisms*

$$\Sigma^\infty : \mathcal{M} \leftrightarrows Stab(\mathcal{M}, P) : \Omega^\infty.$$

We remark that as explained after Theorem A.93, instead of assuming that $\mathcal{C}$ has a final object, we could assume given a cartesian section $X \mapsto P(X)$.

Proof. If $f : X \to Y \in \mathcal{C}$, then the adjoint pair $f^* \vdash f_*$ satisfies Lemma A.101 with $Q = f^*P$. Hence there is a canonical induced left Quillen functor $f^* : Stab(\mathcal{M}, P)(Y) \to Stab(\mathcal{M}, P)(X)$. This defines a left Quillen presheaf.

If f is basal then the adjoint pair $f_\# \vdash f^*$ satisfies Lemma A.101, because $\mathcal{M}$ satisfies the projection formula: $f_\#(T \otimes f^*P) \cong f_\#(T) \otimes P$. Hence we have a τ-Quillen presheaf.

The statements about being left proper, combinatorial, tractable, simplicial follow from Lemma A.100.

It remains to show that $Stab(\mathcal{M}, P)$ is a sheaf. So let $\phi : U_\bullet \to X$ be a hypercover. Then we have $\operatorname{holim}_U Stab(\mathcal{M}, P) \cong \operatorname{holim}_{\Delta \times \mathbb{N}} \mathcal{M}_U^{\Omega_P} \cong \operatorname{holim}_{\mathbb{N}} \operatorname{holim}_U \mathcal{M}^{\Omega_P}$, by Corollary A.69.

There is a canonical right Quillen pseudofunctor

$$(\phi^*)^{\Omega_P} : \mathcal{M}(X)^{\Omega_P} \to (\operatorname{holim}_U \mathcal{M})^{\Omega_P}.$$

Here the transition morphisms in the right Quillen presheaf $(\operatorname{holim}_U \mathcal{M})^{\Omega_P}$ are induced by the right morphism $\Omega_P(U_n) := \Omega_{P(U_n)} : \mathcal{M}(U_n) \to \mathcal{M}(U_n)$ via Proposition A.67. This works because restrictions f^* commute with loops Ω_P, as we have seen. The morphism $(\phi^*)^{\Omega_P}$ is an objectwise Quillen equivalence because $\mathcal{M}$ is a sheaf, hence $\operatorname{holim}_{\mathbb{N}}(\phi^*)^{\Omega_P}$ is a Quillen equivalence, by Proposition A.67 again. But

$$\begin{aligned}\operatorname{holim}_{\mathbb{N}}(\phi^*)^{\Omega_P} : \operatorname{holim}_{\mathbb{N}} \mathcal{M}(X)^{\Omega_P} &= Stab(\mathcal{M}(X), P(X)) \to \\ &\operatorname{holim}_{\mathbb{N}}(\operatorname{holim}_U \mathcal{M})^{\Omega_P} \cong \operatorname{holim}_U Stab(\mathcal{M}, P)\end{aligned}$$

is the descent morphism. This proves that $Stab(\mathcal{M}, P)$ is a sheaf.

To prove statement (3), we need only show that $\Sigma^\infty, \Omega^\infty$ commute with restrictions. This follows from the explicit descriptions of the functors we have given (together with the fact that restrictions commute with tensor products by assumption). □

The *Stab* construction is somewhat inconvenient. Indeed even though $\mathcal{M}$ is a symmetric monoidal model category, and $Ho(Stab(\mathcal{M}, P))$ is known to be a symmetric monoidal category, the category $Stab(\mathcal{M}, P)$ does not afford a monoidal structure. This can be remedied by passing to *symmetric spectra*. Our main reference is [46].

Definition A.103. *Let Σ be the category whose objects are the finite sets and whose morphisms are the isomorphisms. For a category $\mathcal{C}$, the category of* symmetric sequences *in $\mathcal{C}$ is $Fun(\Sigma, \mathcal{C})$.*

If $\mathcal{C}$ is symmetric monoidal, define a symmetric monoidal structure on $Fun(\Sigma, \mathcal{C})$ by

$$(X \otimes Y)(C) = \coprod_{A \cup B = C, A \cap B = \emptyset} X(A) \otimes Y(B).$$

For $P \in \mathcal{C}$, denote by $Sym(P)$ the free commutative monoid in $Fun(\Sigma, \mathcal{C})$ on the object $\tilde{P} \in Fun(\Sigma, \mathcal{C})$ with $\tilde{P}(C) = \emptyset$ if C has cardinality different from one, and $\tilde{P}(C) = P$ else (with all structure maps the identity).

Write $Stab^\Sigma(\mathcal{C}, P)$ for the category of modules in $Fun(\Sigma, \mathcal{C})$ over $Sym(P)$.

This definition needs some elaborations. A skeleton of the category Σ is provided by the category $\tilde{\Sigma}$ whose objects are the natural numbers $0, 1, 2, \ldots$, where there a no maps between m and n if $m \neq n$, and where $\operatorname{Hom}(m, m) = \Sigma_m$, the symmetric group on m letters. Then an object of $Fun(\Sigma, \mathcal{C}) \cong Fun(\tilde{\Sigma}, \mathcal{C})$ is just a sequence of objects $X_i \in \mathcal{C}$, $i = 0, 1, \ldots$, with Σ_n acting on X_n. In this notation, $Sym(P) = (\mathbb{1}, P, P \otimes P, P^{\otimes 3}, \ldots)$. Here Σ_n acts on $P^{\otimes n}$ via the symmetry isomorphisms. We shall assume understood the notion of monoids and modules over them.

Lemma A.104. *If $\mathcal{C}$ has one of these properties, so does $Fun(\Sigma, \mathcal{C})$: complete, cocomplete, locally λ-presentable, simplicial, closed (symmetric monoidal).*

Proof. The first four properties hold in any functor category. For closed symmetric monoidality, see [46, Section 6]. □

Lemma A.105. *Let $\mathcal{C}$ be a bicomplete, closed symmetric monoidal category, and $R \in \mathcal{C}$ a commutative monoid.*

1. *The category R-**Mod** is a bicomplete, closed symmetric monoidal. The functor $U : R\text{-}\mathbf{Mod} \to \mathcal{C}$ affords a left adjoint $F : X \mapsto R \otimes X$ and a right adjoint $C : X \mapsto \underline{\operatorname{Hom}}(R, X)$.*

2. *The category R-**Mod** is locally presentable if $\mathcal{C}$ is.*

3. *Suppose now $\mathcal{D}$ is also bicomplete closed symmetric monoidal, $\alpha : \mathcal{C} \leftrightarrows \mathcal{D} : \beta$ is an adjunction and we are given a commutative monoid S in $\mathcal{D}$, together with natural isomorphisms $\alpha(X \otimes R) \cong \alpha(X) \otimes S$, such that the following diagrams commute:*

$$\begin{array}{ccc} S \otimes S \otimes \alpha X & \longrightarrow & \alpha(R \otimes R \otimes X) \\ \Big\downarrow{\scriptstyle m_S \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle \alpha(m_R \otimes \mathrm{id})} \\ S \otimes \alpha X & \longrightarrow & \alpha(R \otimes X) \end{array}$$

$$\begin{array}{ccccc} \mathbb{1}\otimes\alpha X & \xrightarrow{e_S\otimes\mathrm{id}} & S\otimes\alpha X & \longrightarrow & \alpha(R\otimes X) \\ \| & & & & \| \\ \alpha X & = & \alpha(\mathbb{1}\otimes X) & \xrightarrow{\alpha(e_R\otimes\mathrm{id})} & \alpha(R\otimes X) \end{array}$$

Then there is an induced adjunction $\alpha : R\text{-}\boldsymbol{Mod} \leftrightarrows S\text{-}\boldsymbol{Mod} : \beta$ commuting with the underlying object functors.

4. *If in the above adjunction α is in fact a tensor functor, $\alpha(R) \cong S$ and the compatibility isomorphisms are the natural ones coming from the tensor structure isomorphisms of α, then the induced functor $\alpha : R\text{-}\boldsymbol{Mod} \to S\text{-}\boldsymbol{Mod}$ is also tensor.*

Of course, here we mean the symmetric monoidal structure on R-**Mod** with

$$M \otimes_R N = coeq(M \otimes N \otimes R \rightrightarrows M \otimes N),$$

where the two right arrows correspond to multiplication by R in either M or N.

Proof. (1) This is fairly well known. A reference is [71]. In particular their proof of their Proposition 1.2.14 shows that R-**Mod** is bicomplete and that the forgetful functor U preserves limits and colimits. Symmetric monoidality is their Proposition 1.2.15, closedness is (part of) Proposition 1.2.17. The left adjoint F is their Proposition 1.3.3.

Let us establish the right adjoint. First, for $X \in \mathcal{C}$ we need to make $CX = \underline{\mathrm{Hom}}(R, X)$ into an R-module. This means providing a structure map $R \otimes CX \to CX$. By adjunction this is the same as a map $R \otimes R \otimes \underline{\mathrm{Hom}}(R, X) \to X$. Using the multiplication $R \otimes R \to R$ it suffices to find a map $R \otimes \underline{\mathrm{Hom}}(R, X) \to X$. By adjunction this corresponds to $\underline{\mathrm{Hom}}(R, X) \to \underline{\mathrm{Hom}}(R, X)$, and we choose the identity map. One checks easily that this defines a functor $C : \mathcal{C} \to R$-**Mod**.

This functor is left adjoint to U. Indeed let $T \in R$-**Mod**. We compute

$$\begin{aligned} \mathrm{Hom}_R(T, CX) &= \mathrm{Hom}_R(T, \underline{\mathrm{Hom}}(R, X)) \\ &= eq(\mathrm{Hom}(UT, \underline{\mathrm{Hom}}(R, X)) \rightrightarrows \mathrm{Hom}(UT \otimes R, \underline{\mathrm{Hom}}(R, X))) \\ &= eq(\mathrm{Hom}(UT \otimes R, X) \rightrightarrows \mathrm{Hom}(UT \otimes R \otimes R, X)) \\ &= \mathrm{Hom}(U(T \otimes_R R), X) = \mathrm{Hom}(UT, X). \end{aligned}$$

(2) Suppose that $X \in \mathcal{C}$ is λ-small. Then FX is λ-small because U commutes with filtered colimits (being a left adjoint). If $S \subset \mathcal{C}$ is a set of small objects generating $\mathcal{C}$, then I claim that FS generates R-**Mod**. Indeed let $X \in R$-**Mod**. Then there is a diagram D on a small category I and an isomorphism $UX \cong \mathrm{colim}_I D$. Consequently FUX is in the subcategory of R-**Mod** generated by FS. Since there is a coequaliser diagram $FUFUX \rightrightarrows FUX \to X$ (which is isomorphic to the diagram $R \otimes R \otimes X \rightrightarrows R \otimes X \to R \otimes_R X \cong X$), the claim is proved.

(3) If $M \in R$-**Mod**, then αM naturally defines an element of S-**Mod**. Indeed we have a structure map $S \otimes \alpha M \cong \alpha(R \otimes M) \to \alpha M$, and the unit/multiplication axioms follow immediately from our two assumptions.

Given $Y \in S$-**Mod**, there is a natural structure map $R \otimes \beta Y \to \beta Y$ corresponding to $\alpha(R \otimes \beta Y) \cong S \otimes \alpha\beta Y \to S \otimes Y \to Y$ via adjunction. The unit/multiplication axioms from our two assumptions, by using the adjunction $\alpha \vdash \beta$. The induced functors $\alpha : R$-**Mod** $\leftrightarrows S$-**Mod** $: \beta$ are adjoint as one sees by using that $\mathrm{Hom}_R(X, \beta Y) = eq(\mathrm{Hom}(X, \beta Y) \rightrightarrows \mathrm{Hom}(X \otimes R, \beta Y))$ and similarly for Hom_S. The induced adjunction commutes with U by construction.

(4) Let $X, Y \in R$-**Mod**. By our assumptions on the structure isomorphisms, the coequalizer diagram defining $X \otimes_R Y$ is mapped by α to the diagram defining $\alpha(X) \otimes_S \alpha(Y)$. Since α commutes with colimits (being a left adjoint), it preserves these coequalizers. This proves the result. □

There is a slight inconvenience in reference [46], in that they use the language of *cellular* model categories, whereas we use the *combinatorial* model categories. These two notions are closely related but not equivalent. We elect to reprove the existence of the model structure on symmetric spectra, but shall add the cellularity assumption when we want to use the deeper results of that reference. Of course our proofs are strongly inspired by the proofs in [46].

Recall that if $\mathcal{M}$ is a symmetric monoidal category and $P \in \mathcal{M}$, then an object of $Stab^{\Sigma}(\mathcal{M}, P)$ consists of objects $X_0, X_1, \cdots \in \mathcal{M}$ together with maps $X_i \otimes P \to X_{i+1}$, satisfying certain properties.

Definition A.106. *If $\mathcal{M}$ is a symmetric monoidal model category and P is cofibrant, then we call $(X_i) \in Stab^{\Sigma}(\mathcal{M}, P)$ a* homotopy Ω_P-spectrum *if the adjoint maps $X_i \to \Omega_P X_{i+1}$ induce weak equivalences $X_i \to R\Omega_P X_{i+1}$.*

A map $X \to Y \in Stab^{\Sigma}(\mathcal{M}, P)$ is called a level fibration, level cofibration, level acyclic fibration, level acyclic cofibration, or level equivalence, if each of the maps $X_i \to Y_i$ is a fibration, cofibration, acyclic fibration, acyclic cofibration, or weak equivalence.

A map is called projective cofibration if it has LLP with respect to all level acyclic fibrations, and similarly for projective acyclic cofibrations.

Theorem A.107. *Let $\mathcal{M}$ be a symmetric monoidal combinatorial model category and $P \in \mathcal{M}$ cofibrant.*

1. *The category $Stab^{\Sigma}(\mathcal{M}, P)$ affords the* projective model structure *in which cofibrations, fibrations and weak equivalences are the projective cofibrations, level fibrations and level equivalences. Every projective cofibration is a level cofibration.*

2. *If $\mathcal{M}$ is left proper, this model structure affords a (unique) Bousfield localization where the local objects are the homotopy Ω_P-spectra, known as the stable model structure.*

3. *The stable model structure is left proper, combinatorial, and symmetric monoidal. If $\mathcal{M}$ is tractable or tractable and simplicial, so is $Stab^{\Sigma}(\mathcal{M}, P)$.*

4. *There is a Quillen adjunction*
$$\Sigma^{\infty} : \mathcal{M} \leftrightarrows Stab^{\Sigma}(\mathcal{M}, P) : \Omega^{\infty}.$$
The functor Σ^{∞} is tensor.

5. *Let*
$$G : \mathcal{M} \leftrightarrows \mathcal{N} : R$$
be a Quillen adjunction. Assume given $Q \in \mathcal{N}$ cofibrant, and for $T \in \mathcal{M}$ a natural isomorphism $G(T \otimes P) \cong GT \otimes Q$. Then there exists a natural induced Quillen adjunction
$$Stab^{\Sigma}(G) : Stab^{\Sigma}(\mathcal{M}, P) \leftrightarrows Stab^{\Sigma}(\mathcal{N}, Q) : Stab^{\Sigma}(R),$$
which is a Quillen equivalence if $G \vdash R$ is.

6. *If in the above adjunction G is a tensor functor, $Q \cong GT$, and the structure isomorphisms $(G(T \otimes P) \cong GT \otimes Q)$ are the canonical ones, then $Stab(G)$ is a tensor functor.*

Unless stated otherwise, we will always consider the stable model structure on the category $Stab^{\Sigma}(\mathcal{M}, P)$.

Proof. (1) The category $Fun(\Sigma, \mathcal{M})$ affords the projective model structure, which is combinatorial, by [9, Theorem 1.17].

The category $Stab(\Sigma, \mathcal{M})$ is locally presentable by the previous Lemma.

Let I, J be sets of generating cofibrations and acyclic cofibrations, with presentable domains and codomains, as exist by the definition of combinatoriality. We have the adjunction $F : Fun(\Sigma, \mathcal{M}) \leftrightarrows Stab^{\Sigma}(\mathcal{M}, P) = Sym(P)$**-Mod** $: U$. We shall use [101, Lemma 2.3] to transfer the model structure. The functor U commutes with filtered colimits since it affords a right adjoint. The domains of I_T, J_T are (absolutely) small. We need to show that every "regular

J_T-cofibration" is a weak equivalence. For this it is enough to show that if $f : X \to Y$ is an acyclic cofibration in $Fun(\Sigma, \mathcal{M})$ and $x \in \Sigma$, then $(UFf)(x)$ is an acyclic cofibration in $\mathcal{M}$. Indeed the functor U commutes with colimits, and weak equivalences in $Fun(\Sigma, \mathcal{M})$ are determined objectwise, as are colimits.

By an argument similar to Lemma A.64, the map $f(y) \in \mathcal{M}$ is an acyclic cofibration for every $y \in \Sigma$. Now $UFf = Sym(P) \otimes f$, and so $(UFf)(x)$ is a coproduct of terms of the form $Sym(P)(y) \otimes f(z)$. Each of these is an acyclic cofibration because either $Sym(P)(y) = \mathbb{1}$ or $Sym(P)(y) = P^{\otimes k}$ for some k, and the latter object is cofibrant, so tensoring with it preserves acyclic cofibrations by the pushout-product axiom.

We thus have established the projective model structure on $Stab^{\Sigma}(\mathcal{M}, P)$. The proof shows that it is combinatorial. It is tractable if $Fun(\Sigma, \mathcal{M})$ is, which happens as soon as $\mathcal{M}$ is tractable by [9, Theorem 1.17].

Every "regular I_T-cofibration" is a level cofibration, by the same argument as for acyclic cofibrations. By [101, Lemma 2.1] it follows that every projective cofibration is a retract of a regular I_T-cofibration, so is a retract of a level cofibration, which is a level cofibration.

(2) We follow the standard techniques. Let G be a small set of cofibrant homotopy generators for $\mathcal{M}$, which exists by [9, Theorem 2.34]. Write $i_k : * \to \Sigma$ for the inclusion of the k-element set. For $X \in \mathcal{M}$ let $s_k(X) : Fi_{k+1\#}(X \otimes P) \to Fi_{k\#}X$ be the map adjoint to $X \otimes P \to i_k^* UFi_{k\#}X = \Sigma_k \times (P \otimes X)$ corresponding to the inclusion of the identity in Σ_k. Let $H = \{s_k(X) | k = 0, 1, \dots; X \in G\}$. I claim that the Bousfield localisation at H, which exists by [9, Theorem 2.11], is the model structure we are looking for. Indeed a projective fibrant object $E \in Stab^{\Sigma}(\mathcal{M}, P)$ is H-local if and only if for each $X \in G$ and $k = 0, 1, \dots$, the canonical map $s_k^*(X) : Map^d(Fi_{k\#}X, E) = Map^d(X, E_k) \to Map^d(Fi_{k+1\#}(X \otimes P), E) = Map^d(X, \Omega_P E_{k+1})$ is a weak equivalence. This happens if and only if $E_k \to \Omega_P E_{k+1}$ is a weak equivalence, since G is a homotopy generating set. But this is exactly the definition of an Ω-spectrum.

(3) All the properties except for monoidality are standard. Let us show as a preliminary step that the projective model structure is monoidal. We first verify the pushout-product axiom. By [9, Lemma 3.5] it suffices to do this for generating (acyclic) cofibrations. These are of the form Ff, for $f \in Fun(\Sigma, \mathcal{M})$. For $X, Y \in Fun(\Sigma, \mathcal{M})$ one has $FX \otimes_{Sym(P)} FY \cong F(X \otimes Y)$. One then finds that $Ff \square Fg \cong F(f \square g)$. Since F is left Quillen the pushout product axiom for $Stab^{\Sigma}$ reduces to that of $Fun(\Sigma, \mathcal{M})$. But for $x \in \Sigma$ one finds that $(f \square g)(x) = \coprod_{x = c \coprod d} (f(c) \square g(d))$, and so the pushout-product axiom for $Fun(\Sigma, \mathcal{M})$ follows from that for $\mathcal{M}$.

Now let $f : S \to \mathbb{1}$ be a cofibrant replacement in $\mathcal{M}$. Consider the functor $i_0 : * \to \Sigma$ mapping the unique object $*$ to the empty set. One checks that $i_{0\#}S = (S, \emptyset, \emptyset, \dots)$ and $i_{0\#}\mathbb{1} = \mathbb{1}$. Thus $i_{0\#}f$ is a cofibrant replacement in $Fun(\Sigma, \mathcal{M})$. Next $UFi_{0\#}S = (S, S \otimes P, S \otimes P^{\otimes 2}, \dots)$ whereas $UF\mathbb{1} = Sym(P)$. Consequently $Fi_{0\#}(f)$ is a cofibrant replacement in $Stab^{\Sigma}$. It is easy to check that for $X \in Fun(\Sigma, \mathcal{M})$ and $Y \in Stab^{\Sigma}(\mathcal{M}, P)$ we have $U(F(X) \otimes Y) \cong X \otimes UY$. We thus need to show that $U(Fi_{0\#}(f) \otimes Y) : (i_{0\#}S) \otimes UY \to \mathbb{1} \otimes UY$ is a weak equivalence. But $((i_{0\#}S) \otimes UY)(x) = S \otimes Y(x)$, so this follows because $\mathcal{M}$ is a monoidal model category.

To prove that $Stab^{\Sigma}(\mathcal{M}, P)$ remains a monoidal model category in the stable model structure, we appeal to Theorem A.11. Recall that G is a set of cofibrant homotopy generators for $\mathcal{M}$. It follows that $\{Fi_{k\#}X | k = 0, 1, \dots; X \in G\}$ is a set of cofibrant homotopy generators for $Stab^{\Sigma}(\mathcal{M}, P)$. We thus need to show that for every fibrant Ω-spectrum E, $k = 0, 1, \dots$ and $X \in G$, the internal hom object $\underline{\mathrm{Hom}}(Fi_{k\#}X, E)$ is an Ω-spectrum. It is thus enough to show that for $Y \in G$, $l = 0, 1, \dots$, the canonical map

$$Map^d(Fi_{l\#}Y, \underline{\mathrm{Hom}}(Fi_{k\#}X, E)) \to Map^d(Fi_{l+1\#}(Y \otimes P), \underline{\mathrm{Hom}}(Fi_{k\#}X, E))$$

is a weak equivalence. Since the domains are cofibrant and the codomains are fibrant, the derived mapping spaces are just ordinary mapping spaces. Thus by adjunction, this is the same as

$$Map((Fi_{l\#}Y) \otimes (Fi_{k\#}X), E) \to Map(Fi_{l+1\#}(Y \otimes P) \otimes Fi_{k\#}X, E).$$

Now observe that F is a tensor functor and $(i_{k\#}X) \otimes (i_{l\#}Y) \cong i_{k+l\#}X \otimes Y$. Hence the map is $Map(X \otimes Y, E_{k+l}) \to Map(X \otimes Y, \Omega_P E_{k+l+1})$, which is a weak equivalence because E is an Ω-spectrum.

(4) Let us first describe the adjunction $\Sigma^\infty \vdash \Omega^\infty$. We have the functor $i : * \to \Sigma$ sending the unique object of $*$ to the empty set in Σ. We get an adjunction $i_\# : \mathcal{M} = Fun(*, \mathcal{M}) \leftrightarrows Fun(\Sigma, \mathcal{M}) : i^*$. We put $\Sigma^\infty = Fi_\#$ and $\Omega^\infty = i^*U$. The adjunction $i_\# \vdash i^*$ is a Quillen adjunction by [9, Proposition 1.22], and the adjunction $F \vdash U$ is a Quillen adjunction in the projective model structure by construction. This remains a Quillen adjunction when passing to the stable model structure (to put it differently, we compose with the Quillen adjunction $\mathrm{id} : Stab^\Sigma(\mathcal{M}, P)_{proj} \leftrightarrows Stab^\Sigma(\mathcal{M}, P)_{stable} : \mathrm{id}$).

One checks easily that $i_\#$ is tensor, and clearly F is tensor as well. Thus Σ^∞ is tensor, as required.

(5) We first describe the adjunction. To construct it we shall use the second to last part of Lemma A.105. The adjunction $G : \mathcal{M} \leftrightarrows \mathcal{N} : R$ extends objectwise to an adjunction $G : Fun(\Sigma, \mathcal{M}) \leftrightarrows Fun(\Sigma, \mathcal{N})$. Note that we have for $X \in \mathcal{M}$ and $n = 0, 1, 2, \dots$ that $G(P^{\otimes n} \otimes X) \cong Q^{\otimes n} \otimes G(X)$, naturally in X. Consequently we have for $X \in Fun(\Sigma, \mathcal{M})$ a natural isomorphism $G(Sym(P) \otimes X) \cong Sym(Q) \otimes GX$ (note that G commutes with coproducts, being a left adjoint). The two required diagrams then commute essentially by definition.

We thus get an induced adjunction $Stab(G) : Stab^\Sigma(\mathcal{M}, P) \leftrightarrows Stab^\Sigma(\mathcal{N}, Q) : Stab(R)$ (called $\alpha \vdash \beta$ in that lemma). We first show this is a Quillen adjunction in the projective model structure. For that, we need to show that $Stab(R)$ preserves fibrations and acyclic fibrations. These are detected by U, and the adjunction commutes with U, so it suffices to show that $R : Fun(\Sigma, \mathcal{N}) \to Fun(\Sigma, \mathcal{M})$ preserves fibrations and acyclic fibrations. These are determined objectwise, and the functor R is defined as an objectwise extension of the right Quillen functor $R : \mathcal{N} \to \mathcal{M}$, so this is true.

To pass to the stable model structures, we use [43, Propositions 8.5.4 and 3.3.16], as usual. The cited results imply that $Stab(R)$ remains right Quillen as long as it preserves (projective) fibrant homotopy-Ω-spectra. This is true because $Stab(R)$ is just the levelwise extension of R, and we have $\Omega_P RY \cong R\Omega_Q Y$ by the first part of Lemma A.101.

It remains to show that $Stab(G)$ is a Quillen equivalence if G is. Again we first deal with the projective model structures. Let $X \in Stab^\Sigma(\mathcal{M}, P)$. Since the induced adjunction commutes with U it is just obtained by applying G, R termwise. Since the weak equivalences are the level equivalences, and all cofibrations (fibrations) are level cofibrations (fibrations), cofibrant/fibrant replacement of spectra induces levelwise cofibrant/fibrant replacement. In particular $(Stab(R)(Stab(G)(X^c)^f))_i \simeq R(G(X_i^c)^f)$. It follows that $(LG)(RR)X \simeq X$. Similarly the other way round.

In order to prove that the Quillen adjunction remains an equivalence when passing to the stable model structure, it suffices to show that LG preserves local objects (since RR always does). Hence let $E \in Ho(Stab^\Sigma(\mathcal{M}, P))$ be local and let $T \in Ho(\mathcal{M})$. Note that $(LG)i_k^* UE \simeq i_k^* ULStab(G)E$, because $Stab(G)$ is computed levelwise, and projective cofibrations are level cofibrations. We thus have

$$\begin{aligned} Map^d((LF)(Li_{k\#})T, E) &\simeq Map^d(T, i_k^* UE) \\ &\simeq Map^d(LGT, (LG)i_k^* UE) \\ &\simeq Map^d((LF)(Li_{k\#})(LG)T, LStab(G)E), \end{aligned}$$

with the first and last equivalences by adjunction, and the middle one because $LG : Ho(\mathcal{M}) \to Ho(\mathcal{N})$ is an equivalence. Since $LF(T \otimes P) \simeq LF(T) \otimes Q$ we conclude that $LStab(G)E$ is local with respect to the maps $s_k(GT) : Fi_{k+1\#}GT \otimes Q \to Fi_{k\#}GT$, for all cofibrant T. Since G is a Quillen equivalence such GT form a class of homotopy generators for $\mathcal{N}$, whence $LStab(G)E$ is indeed local in the stable model structure.

(6) It is easy to check by hand that $G : Fun(\Sigma, \mathcal{M}) \to Fun(\Sigma, \mathcal{N})$ is a tensor functor. The result now follows from Lemma A.105, part (4). □

Theorem A.108. *Let $(\mathcal{C}, \tau)$ be a suitable Verdier site with final object $*$, $\mathcal{M}$ a symmetric monoidal, combinatorial, left proper τ-Quillen presheaf on $\mathcal{C}$, and $P \in \mathcal{M}$ cofibrant. Assume* *that $\mathcal{M}$ satisfies the projection formula. Put $P(X) = (X \to *)^*P$ and consider the assignment $Stab^\Sigma(\mathcal{M}, P) : X \mapsto Stab^\Sigma(\mathcal{M}(X), P(X))$.*

1. *The assignment $Stab^\Sigma(\mathcal{M}, P)$ defines a left proper, combinatorial, symmetric monoidal τ-Quillen pseudo-presheaf.*
2. *If $\mathcal{M}$ has any of the following properties, so does $Stab^\Sigma(\mathcal{M}, P)$: tractable, simplicial and tractable.*
3. *The various functors $\Sigma^\infty \vdash \Omega^\infty$ assemble into a pair of left/right morphisms*

$$\Sigma^\infty : \mathcal{M} \leftrightarrows Stab^\Sigma(\mathcal{M}, P) : \Omega^\infty.$$

4. *The τ-presheaf $Stab^\Sigma(\mathcal{M}, P)$ satisfies the projection formula.*

We remark that as explained after Theorem A.93, instead of assuming that $\mathcal{C}$ has a final object, we could assume given a cartesian section $X \mapsto P(X)$.

Note that the theorem does not claim that $Stab^\Sigma$ is a sheaf if $\mathcal{M}$ is. This is usually true, and we will give an indirect proof later. Apart from that, it exactly mirrors Theorem A.102, and so does the proof.

Proof. (1) If $f : X \to Y \in \mathcal{C}$ is a morphism then $f^* : \mathcal{M}(Y) \leftrightarrows \mathcal{M}(X) : f_*$ is an adjunction satisfying the assumptions of Theorem A.107 part (5), since f^* is tensor. Consequently there is an induced Quillen adjunction $Stab(f^*) : Stab^\Sigma(\mathcal{M}, P)(Y) \leftrightarrows Stab^\Sigma(\mathcal{M}, P)(X) : Stab(f_*)$. If additionally f is basal then $f_\# : \mathcal{M}(X) \leftrightarrows \mathcal{M}(Y) : f^*$ also satisfies the assumptions of Theorem A.107 part (5), since $\mathcal{M}$ satisfies the projection formula, and so there is an induced Quillen adjunction $Stab(f_\#) : Stab^\Sigma(\mathcal{M}, P)(X) \leftrightarrows Stab^\Sigma(\mathcal{M}, P)(Y) : Stab(f^*)$. We thus have a τ-Quillen presheaf, as claimed.

The presheaf is left proper, combinatorial and symmetric monoidal by Theorem A.107 parts (3) and (6).

(2) This is again Theorem A.107 part (3).

(3) We just need to check that $\Sigma^\infty, \Omega^\infty$ commute with restrictions f^*. This is essentially the same proof as in the non-symmetric case.

(4) Let $f : X \to Y \in \mathcal{C}$ be basal, $E \in Stab^\Sigma(\mathcal{M}, P)(X)$ and $F \in Stab^\Sigma(\mathcal{M}, P)(Y)$. We need to show that the natural map $f_\#(X \otimes f^*Y) \to f_\#(X) \otimes Y$ is an isomorphism. Let us write $\otimes$ for the tensor product of symmetric spectra and $\hat{\otimes}$ for the tensor product of symmetric sequences. Then there is a coequalizer

$$E \otimes f^*F = coeq(E \hat{\otimes} f^*F \hat{\otimes} Sym(P(X)) \rightrightarrows E \hat{\otimes} f^*F).$$

Since $Sym(P(X)) \cong f^*Sym(P(Y))$, the functor f^* is tensor and the functor $f_\#$ is left adjoint, so preserves colimits, the projection formula for symmetric spectra follows from the projection formula for symmetric sequences.

Thus let $E \in Fun(\Sigma, \mathcal{M}(X)), F \in Fun(\Sigma, \mathcal{M}(Y))$ and C be a finite set. We compute

$$\begin{aligned} f_\#(E \hat{\otimes} f^*F)(C) &= \coprod_{C = A \coprod B} f_\#(E(A) \otimes f^*F(B)) \\ &\cong \coprod_{C = A \coprod B} f_\#(E(A)) \otimes F(B) \\ &= (f_\#(E) \otimes Y)(C), \end{aligned}$$

where in the middle we have used that $\mathcal{M}$ satisfies the projection formula. □

Now suppose given a left proper, combinatorial, symmetric monoidal model category $\mathcal{M}$ and a cofibrant object P. We would like to compare $Stab(\mathcal{M}, P)$ and $Stab^\Sigma(\mathcal{M}, P)$. Note that $Stab^\Sigma(\mathcal{M}, P)$ is again a left proper, combinatorial, symmetric monoidal model category.

We can thus form $Stab(Stab^\Sigma(\mathcal{M}, P), \Sigma^\infty(P))$. There is the usual right Quillen functor $\Omega^\infty : Stab(Stab^\Sigma(\mathcal{M}, P), \Sigma^\infty(P)) \to Stab^\Sigma(\mathcal{M}, P)$.

Note that an object X of $Stab(Stab^\Sigma(\mathcal{M}, P), \Sigma^\infty(P))$ consists of symmetric spectra X_i, together with structure maps $X_i \otimes \Sigma^\infty(P) \to X_{i+1}$. Now $X_i \otimes \Sigma^\infty(P)$ is a symmetric spectrum with zeroth term $(X_i)_0 \otimes (\Sigma^\infty(P))_0 = (X_i)_0 \otimes P$. Hence we can extract a non-symmetric spectrum $\Omega^\dagger(X)$ with $\Omega^\dagger(X)_i = (X_i)_0$.

Lemma A.109. *The above assignment extends to a right Quillen functor*

$$\Omega^\dagger : Stab(Stab^\Sigma(\mathcal{M}, P), \Sigma^\infty(P)) \to Stab(\mathcal{M}, P).$$

It is natural in $\mathcal{M}$ (under left Quillen tensor functors).

Proof. It is essentially clear that we have a natural tensor functor. We prove it is right Quillen. If we were to use the projective instead of stable/holim model structures throughout this would be clear. To pass to the localised model structures, we use [43, Propositions 8.5.4 and 3.3.16] once more. The proof is essentially the same as the proof of Proposition A.68. The main point to observe is that for $X \in Stab^\Sigma(\mathcal{M}, P)$ we have that $\Omega^\infty \Omega_{\Sigma^\infty P} X \cong \Omega_P \Omega^\infty X$, which by adjunction follows from the fact that Σ^∞ is tensor. □

We can sheafify this comparison functor.

Lemma A.110. *Let $\mathcal{M}$ be a left proper, combinatorial, symmetric monoidal τ-Quillen presheaf satisfying the projection formula, on the suitable Verdier site $(\mathcal{C}, \tau)$ with final object $*$. Let $P \in \mathcal{M}(*)$ be cofibrant. Consider the assignment*

$$Stab(Stab^\Sigma(\mathcal{M}, P), \Sigma^\infty P) : X \mapsto Stab(Stab^\Sigma(\mathcal{M}(X), P(X)), \Sigma^\infty P(X)).$$

1. *The assignment $Stab(Stab^\Sigma(\mathcal{M}, P), \Sigma^\infty P)$ defines a τ-Quillen presheaf.*
2. *There is a zig-zag of right morphisms*

$$Stab^\Sigma(\mathcal{M}, P) \xleftarrow{\Omega^\infty} Stab(Stab^\Sigma(\mathcal{M}, P), \Sigma^\infty P) \xrightarrow{\Omega^\dagger} Stab(\mathcal{M}, P).$$

Proof. By Theorem A.108, $Stab^\Sigma(\mathcal{M}, P)$ is a left proper, combinatorial, symmetric monoidal τ-Quillen presheaf which satisfies the projection formula. Since Σ^∞ is a left Quillen functor, $\Sigma^\infty P$ is cofibrant. Consequently by Theorem A.102, the assignment $Stab(Stab^\Sigma(\mathcal{M}, P), \Sigma^\infty P)$ is a τ-Quillen presheaf.

Since both Ω^∞ and $\Omega^\dagger$ are right Quillen morphisms, natural under restrictions f^*, we obtain the zig-zag of right morphisms as claimed. □

In good cases, the right Quillen functors $\Omega^\infty, \Omega^\dagger$ above are Quillen equivalences. These are the "deep results" which we are not going to prove, alluded to earlier. Let us first record a consequence.

Lemma A.111. *Let $(\mathcal{C}, \tau)$ be a suitable Verdier site and $\mathcal{M}_1, \mathcal{M}_2$ be two left proper, combinatorial τ-Quillen presheaves on $\mathcal{C}$.*

Write $\mathcal{C}_b$ for the category with the same objects as $\mathcal{C}$, but only the basal morphisms. Suppose given a right morphism $\Theta : \mathcal{M}_1|_{\mathcal{C}_b} \to \mathcal{M}_2|_{\mathcal{C}_b}$, such that each $\Theta(X) : \mathcal{M}_1(X) \to \mathcal{M}_2(X)$ is a Quillen equivalence.

Then $\mathcal{M}_1$ is a sheaf if and only if $\mathcal{M}_2$ is a sheaf.

Proof. Let $\phi : U_\bullet \to X$ be a hypercover. By Proposition A.37, each of the maps involved is basal, which is to say that $\phi : U_\bullet \to X$ is an augmented simplicial object in $\mathcal{C}_b$. We thus get by Proposition A.67 a Quillen equivalence $\operatorname{holim}_U \Theta : \operatorname{holim}_U \mathcal{M}_1 \to \operatorname{holim}_U \mathcal{M}_2$. Consider the commutative diagram

$$\begin{array}{ccc} Ho(\operatorname{holim}_U \mathcal{M}_1) & \xrightarrow{\operatorname{holim}\Theta} & Ho(\operatorname{holim}_U \mathcal{M}_2) \\ {\scriptstyle \phi^*}\uparrow & & {\scriptstyle \phi^*}\uparrow \\ Ho(\mathcal{M}_1(X)) & \xrightarrow{\Theta(X)} & Ho(\mathcal{M}_2(X)). \end{array}$$

The top horizontal arrow is an equivalence by what we just said, the bottom horizontal arrow is an equivalence by assumption. Hence the left vertical arrow is an equivalence if and only if the right one is. The sheaf condition means that the vertical arrows are equivalences. This concludes the proof. □

Corollary A.112. *Let $\mathcal{M}$ be as in Lemma A.110. If each of the functors*

$$\Omega^\infty(X) : Stab(Stab^\Sigma(\mathcal{M}(X), P(X)), \Sigma^\infty P(X)) \to Stab^\Sigma(\mathcal{M}(X), P(X))$$

$$\Omega^\dagger(X) : Stab(Stab^\Sigma(\mathcal{M}(X), P(X)), \Sigma^\infty P(X)) \to Stab(\mathcal{M}(X), P(X))$$

is a Quillen equivalence, then $Stab^\Sigma(\mathcal{M}, P)$ is a sheaf (and not just a presheaf).

It is proved in [46, Section 9] that the assumptions in the Corollary are satisfied if each $\mathcal{M}(X)$ is cellular and tractable, and if additionally the unit in $\mathcal{M}(*)$ is cofibrant and $P \in \mathcal{M}(*)$ is *symmetric*, which means that the cyclic permutation on $P \otimes P \otimes P$ is homotopic to the identity, in an appropriate sense. It seems to the author that Hovey's argument goes through without cellularity, relying on combinatoriality instead. But the proof is long and delicate, so we choose not to re-do it in the new language.

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