Quantum many-body effects in gravity and Bosonic theories

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ABSTRACT

Many-body quantum effects play a crucial role in many domains of physics, from condensed matter to black-hole evaporation. The fundamental interest and difficulty in studying this class of systems is the fact that their effective coupling constant become rescaled by the number of particles involved $g = \alpha N$, and thus we observe a breakdown of perturbation theory even for small values of the $2 \to 2$ coupling constant. We will study three very different systems which share the property that their behaviour is dominated by non-perturbative effects.
The strong CP problem - the problem of why the θ angle of QCD is so small - can be solved by the Peccei-Quinn mechanism, which promotes the θ angle to a physical particle, the axion. The essence of the PQ mechanism is that the coupling will generate a mass gap, and thus the expectation value of the axion will vanish at the vacuum. It has been suggested that topological effects in gravity can spoil the axion solution. By using the dual formulation of the Peccei-Quinn mechanism, we are able to show that even in the presence of such dangerous contributions from gravity, the presence of light neutrinos can stabilize the axion potential. This effect also puts an upper bound on the lightest neutrino mass.
We know that at high energies, gravitational scattering is dominated by black-hole formation. The typical size of black-holes is a growing function of the total center-of-mass energy involved in the scattering process. In the asymptotic future, these black-holes will decay into Hawking radiation, which has a typical wave-length of the size of the black-hole. Thus high energy gravitational scattering is dominated by low energy out states. It has been suggested that gravity is self-complete due to this effect, and that furthermore, there is a class of bosonic theories which can also be self-complete due to the formation of large classical field configurations: UV completion by Classicalization.
We explore the idea of Classicalization versus Wilsonian UV completion in derivatively coupled scalars. We seek to answer the following question: how does the theory decide which road to take at high energies? We find out that the information about the high energy be-

haviour of the theory is encoded in the sign of the quartic derivative coupling. There is one sign that allows for a consistent Wilsonian UV-completion, and another sign that contains continuous classical field configurations for localized sources.

In the third part of the thesis we explore non-perturbative properties of black holes. We consider the model proposed by Dvali and Gomez where black holes are described as Bose-Einstein condensates of N gravitons. These gravitons are weakly interacting, however their collective coupling constant puts them exactly at the critical point of a quantum phase transition $\alpha N = 1$. We focus on a toy model which captures some of the features of information storage and processing of black holes. The carriers of information and entropy are the Bogoliubov modes, which we are able to map to pseuo-Goldstone bosons of a broken SU(2) symmetry. At the quantum phase transition this gap becomes $1/N$, which implies that the cost of information storage disappears in the $N \to \infty$ limit. Furthermore, the storage capacity and lifetime of the modes increases with N, becoming infinite in the $N \to \infty$ limit.

The attractive Bose gas which we considered is integrable in 1+1d. All the eigenstates of the system can be constructed using the Bethe ansatz, which transforms the Hamiltonian eigenvalue problem into a set of algebraic equations - the Bethe equations - for N parameters which play the role of generalize momenta. While the ground state and excitation spectrum are known in the repulsive regime, in the attractive case the system becomes more complicated due to the appearance of bound states. In order to solve the Bethe equations, we restrict ourselves to the $N \to \infty$ limit and transform the algebraic equations into a constrained integral equation. By solving this integral equation, we are able to study the phase transition from the point of view of the Bethe ansatz. We observe that the phase transition happens precisely when the constraint is saturated, and manifests itself as a change in the functional form of the density of momenta. Furthermore, we are able to show that the ground state of this system can be mapped to the saddle-point equation of 2-dimensional Yang–Mills on a sphere, with a gauge group U(N).

ZUSAMMENFASSUNG

Kollektive Quanteneffekte spielen eine entscheidende Rolle in vielen Bereichen der Physik, von Festkrperphysik bis hin zur Verdampfung von schwarzen Lchern. Das grundlegende Interesse und auch die Schwierigkeit beim Untersuchen von solchen Systemen liegt darin begrndet, dass ihre effektive Kopplungskonstante mit der Anzahl der beteiligten Teilchen reskaliert wird $g = \alpha N$. Daher beobachten wir den Zusammenbruch von Strungstheorie, selbst fr kleine Werte der $2 \to 2$ Kopplungskonstante.

In dieser Doktorarbeit erforschen wir drei verschiedene Systeme, die von kollektiven, nicht-strungstheoretischen Effekten dominiert sind. Wir untersuchen die Axion-Lsung des starken CP-Problems und zeigen, dass mgliche gravitative Korrekturen zum Axion-Potential dank nicht-perturbativer Effekte aus dem Neutrinosektor unterdrckt werden knnen. Wir erkunden die Idee der Klassikalisierung als Gegenstck zu Wilsonscher UV-Vervollstndigung in Ableitungs-gekoppelten skalaren Theorien und zeigen, dass das Hochenergieverhalten dieser Theorien im Vorzeichen der quartischen Kopplung verschlsselt ist. Wir zeigen, dass viele Eigenschaften von Informationsverarbeitung und -speicherung in schwarzen Lchern mit einem Bose-Einstein Kondensat in einem Quantenphasenbergang erklrt werden knnen. Des Weiteren untersuchen wir die Eigenschaften desselben Systems am kritischen Punkt mithilfe seiner Integrabilittseigenschaften und zeigen, dass sein Grundzustand auf Yang-Mills Theorie auf der Sphre abgebildet werden kann.

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CONTENTS

1

INTRODUCTION

It has been over one hundred years since Einstein discovered the theory of General Relativity. As I write this, we witness the announcement of one of the most awaited experimental discoveries since the discovery of GR: gravitational waves [1]. The importance of this discovery cannot be overstated, since it confirms one of the last remaining untested predictions of GR. Besides being the first detection of gravitational waves, it was also the first observation of a binary black-hole merger! The source of the observed gravitational waves was a system of two merging black-holes, initially with 36 and 29 solar masses, and the total energy radiated in this process was 3 solar masses.

Hopefully this discovery will be just the first step in a century where observation of gravitational waves will become standard practice. Despite the fact that it took an extraordinary event to allow for this observation, there are many unanswered questions which can be solved by the direct detection of gravitational waves but require a higher degree of sensitivity. As a matter of fact, there is a whole background of gravitational waves from the early universe that are just waiting to be probed, and the consequences of this observation will be just as revolutionary as the discovery of the CMB.

Besides the cosmological implications, the discovery of primordial gravitational waves will have also a rather direct impact on high energy physics. As of now, the reasons why it is believed that GR should be treated as a Quantum Field Theory all rely on consistency arguments. While it is true that there is very little chance that Quantum Mechanics is ignored by GR, a direct observation of primordial gravitational waves would mean an indirect detection of gravitons - the fundamental degrees of freedom in a quantum theory of gravity.

While many opponents of this idea cite issues such as breakdown of (perturbative) unitary and non-renormalizability as objections to a consistent theory of quantum gravity, we will take these as features of the theory, rather than problems. In a modern perspective, renormalizability is no longer a sacred requirement in order to construct consistent theories. Theories can be well-defined up to some energy scale, which corresponds to the length scale at which we expect new degrees of freedom to become important. Until that scale is reached, we can encapsulate our ignorance about microscopic degrees of freedom using a few parameters: this is the Effective Field Theory approach.

When an Effective Field Theory reaches the limit of its domain of validity, we need to incorporate these new degrees of freedom in the theory. This is called a UV completion: completing the theory by introducing new physics that will be responsible for making it predictive again. According to the Wilsonian paradigm of physics, whenever we observe the breakdown of predictability in a theory, we will uncover new degrees of freedom which will appear before the problematic scale and will be responsible for making the theory sensible again. In the Wilsonian view, less fundamental theories are embedded in more fundamental ones, and we can always uncover new microscopic degrees of freedom by increasing the energy at which we measure physical process.

Gravity indeed behaves differently from the known Quantum Field Theories, but its difference precisely due to the enhancement of quantum mechanical effects, rather than the lack of it. We will also see that the breakdown of unitarity is indeed just a perturbative problem, and in the center of all this discussion we will encounter the very objects which dominate gravity at high energies: black-holes.

It is impossible to talk about the interplay between General Relativity and Quantum Mechanics without discussing black-holes. For the past 40 years, black-holes became the theoretical laboratory where thought experiments about quantum gravity are tested. Today, before a theory of Quantum Gravity is even required to actually reproduce GR at low energies, it is supposed to predict the correct Bekenstein–Hawking entropy of a black-hole. Understanding what are the correct micro-states of a black-hole is indeed a very interesting and rather complex puzzle. Solving this puzzle requires going beyond the view that GR is a theory of a purely geometric background over which quantum fields propagate. A quantum description of space–time itself is necessary.

It is embedded in the folklore that a quantum description of space–time requires a fundamental theory of gravity - one that is valid for every energy. As we stated previously, GR works perfectly as an Effective Field Theory, so it is natural to ask whether it is possible to explain the puzzling features of black-holes with only low-energy degrees of freedom. As we will see, recent research seems to indicate that this is in fact the case, and large black-holes are insensitive to the short-distance features of the theory. Whichever UV-completes gravity does leave an imprint on large black-holes, they are purely dominated by infrared physics.

The underlying theme throughout this thesis is the following: how can we understand many-body states in gravity and other Bosonic theories, and what are their phenomenological effects? With the aid of both perturbative and non-perturbative methods, we will study black-holes, topological objects, bound states and solitons.

1.1 HOW NEUTRINO PROTECTS THE AXION

Non-perturbative collective effects play a major role in gauge theories. The most obvious example is QCD, where the vacuum is dominated by a condensate of quarks. Perhaps the most interesting aspect of the QCD vacuum is the existence of instantons: these are time-dependendent field configurations that interpolate between different vacua in different times. The existence of these boundary effects allow for QCD to break CP symmetry, and the magnitude of this process can be measured experimentally and can be parametrized by one number: the θ angle. Explaining why this angle is so small is a puzzle known as the strong CP problem.

The most embraced solution to the strong CP problem is known as the Peccei–Quinn mechanism, which predicts the existence of a new particle - the axion - whose expectation value plays the role of the θ angle. By minimizing its potential, it is possible to predict that θ should be very small. This solution, however, is not without its issues. Nothing stops extra couplings to destabilize its potential and thus ruin the solution to the strong CP problem. More specifically, it has been suggested that gravity might be the source of these contributions, and that non-perturbative gravitational effects - akin to instantons - will ressurrect the strong CP problem.

It is nevertheless possible to parametrize the possible dangerous contributions from gravity if one uses a dual formulation of the theory, in which the axion is in fact a 2-form, coupling to the QCD 3-form. In this formulation, we have an additional gauge redundancy which must be protected. This limits the possible ways in which gravity can interact with the axion, and thus allows us to parametrize its effects on the axion potential.

In the Part I of this thesis, we will present the results we obtained in [6]. We show that even in the presence of these dangerous gravitational contributions to the axion potential, the axion can be saved due to the fact that there are very light neutrinos that are gravitational anomalous. Should there be non-perturbative gravitational effects, there will be a bound state of neutrinos, similar to the η' in QCD, which will play the role of the axion and thus restore the Peccei–Quinn solution to the strong CP problem.

1.2 ROAD SIGNS FOR UV-COMPLETION

Resolving the quantum properties of black-holes and understanding the correct way to UV-complete gravity are intimately connected issues. If black-holes do exist in nature, it's unavoidable that they will be formed in high energy scattering experiments. If large black-holes are really insensitive the high-energy details of the theory, gravity breaks the link between high energy and low distances, and breaking this link has dramatic consequences, since we can no longer rely on the Wilsonian paradigm when it comes to gravity - we need a different approach to UV-completion.

In this approach, the UV-completion of the system is accomplished by the creation of large objects. These objects will dominate the spectrum of the theory at large energies, thus providing a bridge between the infrared and the ultraviolet degrees of freedom. This mapping allows the theory to be self-complete without the necessity of introducing degrees of freedom. In the Part II of this thesis, we will focus on the results we presented in [24]. We argue that this paradigm of UV-completion can be applied to theories other than gravities, and we will derive some of the theoretical and phenomenological consequences of this method of UV-completion.

1.3 BLACK-HOLE INFORMATION AND QUANTUM CRITICALITY

It is generally true that large occupation number leads to classical world: when we look at the sun, we observe classical electromagnetic radiation, even though at a fundamental level we have many single quanta being emitted in elementary processes. The same can be said for most of our interactions with nature: we observe a mostly classical world. In order to discover quantum mechanics, it took observations of very small length scales, where the granularity associated with seemingly continuous phenomena becomes apparent.

In reality, large does not necessarily mean quantum. Under certain conditions, a macroscopic number of bosons can occupy the lowest quantum state of the system, forming a Bose-Einstein condensate. With the advent of condensed matter, other examples of macroscopic quantum phenomena became known, such as superfluidity, superconductivity and the quantum Hall effect.

Once we turn on interactions, quantum systems, no matter how weakly coupled they are, seem to have a turning point whenever there are so many particles that the collective coupling exceeds unity. At this point, collective quantum effects become dominant and the systems become intrinsically non-perturbative. In the case we will study, the system undergoes a phase transition and the ground state becomes dominated by the formation of a soliton - an intrinsically non-perturbative object. By studying the system near the critical point, we find striking similarity with some of the mysterious properties of black-holes.

In the Part III of this thesis, we will focus on the results we obtained in [43]. We will argue that in fact black-holes should be viewed as a large quantum system which is stuck in a quantum phase transition, and we will show how to understand information-theoretical properties of black-holes through quantum criticality.

1.4 ATTRACTIVE LIEB-LINIGER MODEL AND YANG-MILLS

The physics of 1+1D systems is very peculiar. While in higher dimensions propagating particles can avoid interacting by just going around each other, in one spatial dimension, if they are converging to the same point, they have no choice but to scatter. For certain theories, any multiparticle scattering can be factorized into multiple elementry $2 \to 2$ processes: theories that have this property satisfy the so-called Yang-Baxter equation.

Theories that satisfy this property are said to be integrable. For our application, the one property of integrable systems we need to retain is that it allows us to convert the N-particle Hamiltonian eigenvalue problem into the problem of N algebraic equations for N parameters.

One intuitive way to understand why this is possible is the following: in two dimensions, the S-Matrix of the theory is just a phase shift that depends on the incoming momenta of the particles. If we define the theory on a ring, we can perform an experiment in which a particle is dragged around the circle until it returns to the same position, scattering with all other $N-1$ particles along the way. Due to the fact that the scattering is factorizable, this can be written as a product of $N-1$ elementary $2 \to 2$ scattering processes. Since we have periodic boundary conditions, going around the circle restricts the total phase shift of the wave-function to be a multiple of 2π. Equating the phase shifts, we have a constraint on the possible momenta of the particles.

This system of constraints is known as the Bethe equations, and it completely determines the spectrum of the system.

In Part III of this thesis, we will present the results obtained in [68]. We will develop a method to solve the Bethe equations for the attractive Bose gas. By doing so, we are able to characterize the phase transition using the Bethe ansatz, and also show that the ground state of the system can be mapped to the saddle-point of 2D Yang-Mills on a sphere.

Part I

HOW NEUTRINOS PROTECT THE AXION

2

FROM WEINBERG TO PECCEI – QUINN

Shortly after the discovery of strong interactions and the advent of QCD, a problem, which became known as the $U(1)_A$ problem, was uncovered. Solving the $U(1)_A$ was possible due to the discovery that global axial transformations are in fact not symmetries of QCD at quantum level - they are anomalous. The effect of this anomaly, as we will see, is intimately connected to the vacuum of QCD, which is not unique and contains non-perturbative topological objects - instantons - which contribute to observable quantities in the theory. We observe, however, that the parameter which controls these effects - the θ angle, has a very very small upper bound. In order to explain the smallness of this parameter, Peccei and Quinn introduced another particle in the spectrum - the Axion - whose expectation value corresponds to the θ angle, successfully being able to naturally have a theory in which θ is small.

In this chapter we will briefly review each one of these steps, starting from the $U(1)_A$ problem, and finishing with the introduction of the Axion.

2.1 THE $u(1)_a$ NON-SYMMETRY

In this section we will review the developments that led to the discovery of the $U(1)_A$ problem. For a comprehensive treatment, please refer to [2].

Take the QCD Lagrangian with N flavours of quarks, and suppose that all of the quark masses are 0. In this theory, we can identify two possible ways of independently rotating the quark flavours, one through a global vector $U(N)_V$ transformation, and another with an axial $U(N)_A$ transformation. We expect then that in this theory, we have a global symmetry of $U(N)_V \times U(N)_A$.

In reality, quarks are not massless, thus we don't expect to observe any exact global flavour symmetry in nature. We do, however, have a big hierarchy of quark masses. While the *up* and *down* quarks are very light, around the MeV scale, all the other quark flavours are much heavier. The next lightest quark is the strange, with a mass of 95 MeV, while all the other quarks are already at the GeV scale. In fact, the *up* and *down* quarks have masses way smaller than the QCD scale ($\approx$ 200 MeV). It is then reasonable to expect that there is a global *approximate* $U(2)_V \times U(2)_A$ symmetry in strong interactions.

Below the QCD scale quarks are not asymptotic states of strong interactions. The QCD vacuum is characterized by a quark condensate

$$\langle \bar{q}q \rangle \neq 0 \, , \tag{1}$$

so these global symmetries will be manifest in the spectrum of the theory, which will be composed of bound state of quarks. It was observed that indeed the global vector $U(2)_V = SU(2)_I \times U(1)_B$ symmetry is present in strong interactions, due to the existence of the pion and the nucleons in the hadronic spectrum.

In the case of the axial current, we would also expect the appearence of a $U(1)_A$ from the conserved $U(2)_A$ current. However it is easy to check that the condensate (1) dynamically breaks global axial symmetries. That means that if the QCD lagrangian was originally (approximately) invariant under some $U(1)_A$, we would observe its corresponding (pseudo-) Goldstone boson in the low energy spectrum of the theory, after symmetry breaking.

We arrive at a puzzle then, which is the following: the effect of the explicit symmetry breaking terms in the QCD lagrangian predicts that the mass of this pseudo-Goldstone boson of $U(1)_A$ should be very close to the mass of the pion, the lightest hadron in the spectrum. We observe, however, the second lightest particle in the spectrum, the η meson, has a much higher mass than the pi meson. This puzzle, discovered by Weinberg [2], is known as the $U(1)_A$ problem. Weinberg then suggested that the only natural way to escape this problem is to find a way to get of the initial axial symmetry of light quarks in QCD.

2.2 THE ANOMALY

It had been known, however, that the axial symmetry was not a true symmetry of the theory [3]. Associated with an axial transformation, there is a current

$$J^5_\mu = \bar{q}\gamma_\mu\gamma_5 q \;, \tag{2}$$

the axial current, which is conserved whenever the theory is invariant under an axial transformation. Classically, it is easy to check that indeed this current is divergenceless in the theory. Quantum mechanically, there is a one-loop diagram which contributes to the divergence of the current. The divergence of the axial current, in the presence of the gluon fields $F_{\mathrm{a}\mu\nu}$, will be

$$\partial_\mu J^\mu_5 = \frac{g^2 N}{32\pi^2} F^{\mu\nu}_a \tilde{F}_{\mathrm{a}\mu\nu} \;\;, \tag{3}$$

where we introduce the dual gauge field $\tilde{F}$, defined as

$$\tilde{F}_{\mathrm{a}\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}_{\mathrm{a}} \tag{4}$$

The $F^{\mu\nu}_{\mathrm{a}}\tilde{F}_{\mathrm{a}\mu\nu}$ term on the right hand side of (3) is a very special entity, since it can be written in the following form

$$F^{\mu\nu}_{\mathrm{a}}\tilde{F}_{\mathrm{a}\mu\nu} = \partial_\mu K^\mu \;. \tag{5}$$

The vector K_μ, called the Bardeen current, is the dual of the Chern-Simons three-form, which will play a major role in our discussion later. This three-form can be written as

$$K_\mu = \epsilon_{\mu\alpha\beta\gamma}C^{\alpha\beta\gamma} \tag{6}$$

$$C^{\alpha\beta\gamma} = A^\alpha_{\mathrm{a}}\left(F^{\mathrm{a}\beta\gamma} - \frac{g}{3}f_{\mathrm{abc}}A^\beta_{\mathrm{b}}A^\gamma_{\mathrm{c}}\right) \tag{7}$$

Despite the fact that the classical theory is invariant under $U(1)_A$, what we find from (3) is that under the transformation

$$q \to e^{i\epsilon\gamma_5}q \;, \tag{8}$$

we will the following contribution to the action

$$\delta S = \epsilon\int d^4x\,\partial_\mu J^\mu_5 = \epsilon\frac{g^2N}{32\pi^2}\int d^4x\, F^{\mu\nu}_a\tilde{F}_{\mathrm{a}\mu\nu}$$
$$= \epsilon\frac{g^2N}{32\pi^2}\int d^4x\,\partial_\mu K^\mu \;\;. \tag{9}$$

While the divergence of the J_μ^5 current is non-zero, what we find is that it corresponds to a boundary term, since the $F_\mathrm{a}^{\mu\nu}\tilde{F}_{\mathrm{a}\mu\nu}$ contribution is nothing but a total derivative. One could be tempted to assign $A_\mathrm{a}^\mu(\infty) = 0$ and quickly disregard this term, however as we will see, the situation is more tricky, since the possible gauge choices of gauge for the asymptotic behaviour of A can lead to a non-vanishing boundary contribution.

2.3 TOPOLOGY AND THE θ VACUUM

In the past sections, we have seen that the existence of a conserved $U(1)_A$ current in QCD would lead to the appearance of a light Goldstone boson, since this symmetry would be spontaneously broken by the quark condensate in the QCD vacuum. In reality, this symmetry is anomalous, and thus only conserved classically. What we find is that axial transformations induce a new term in the action which happens to be a total derivative. Usually total derivatives can be disregarded, since we choose vanishing boundary conditions for the fields at infinity, but in this case we have the extra hassle of dealing with gauge dependent fields, so any gauge transformation of 0 is a physically equivalent choice of boundary condition.

Our goal is then to enumerate the possible choices of field configurations that are asymptotically pure gauge, since these will be the ones which will minimize the hamiltonian of the theory. In order to do so, let us notice that a pure gauge (pg) field has the form (let us suppress color indices for simplicity)

$$A_\mu^{\mathrm{pg}} = -iU(x)\partial_\mu U^\dagger(x) \tag{10}$$

where $U(x) \in SU(3)$ is a gauge transformation.

It seems like this is an ungrateful task, since these choices are neither unique nor gauge invariant. However, it can be seen that the choices of A_μ^{pg} fall into a countable infinity of equivalence classes. In order to see this, let us make two restrictions on the gauge transformations: first, we restrict ourselves to those that will approach 1 at infinity

$$\lim_{x\to\infty} U(\vec{x}) = 1 \tag{11}$$

and we stick to the temporal gauge in order to eliminate the time-dependence

$$A_0^{\mathrm{pg}} = 0 \ . \tag{12}$$

The choice of asymptotics for $U(x)$ has the effect of identifying every point at the spacial infinity, since they can't effectively be distinguished from the point of view of the gauge field. This identification thus transforms R^3 into S^3 through a transformation akin to a stereographic projection.

The gauge transformation $U(x)$ will then define a map

$$S^3 \to SU(3) \tag{13}$$

In order to classify this map in terms of homotopy, only the $SU(2)$ subgroup matters [5]. Furthermore, $SU(2)$ can be mapped into S^3 using the Pauli matrices representation of the group. This map is then topologically equivalent to

$$S^3 \to S^3 \ , \tag{14}$$

which maps that the gauge transformations of 0 at spatial infinity essentially map spheres into spheres. We know that in general it is not possible to continuously deform one mapping into another, which means that they fall into distinct homotopy classes. The homotopy classes are elements of a ground - the homotopy ground. In the case of mappings between spheres of the same dimension, this group is

$$\pi_d(S^d) = Z \ ,$$

meaning that every class of $U(\vec{x})$ can be characterized by an integer n, which is the d-dimensional generalization of a winding number.

For a given gauge field configuration $A^\mu_\mathrm{a}(\vec{x})$, this integer n is given by

$$n = \frac{ig^3}{24\pi^2} \int d^3\vec{x} \ \mathrm{Tr} \ \left[\epsilon_{ijk} A^i A^j A^k\right] \ . \tag{15}$$

This quantity is related to the Bardeen current we introduced previously. With the choice of normalization we used, and sticking to the $A^0 = 0$ gauge, we have

$$K^0 = \frac{4}{3} ig\epsilon_{ijk} \ \mathrm{Tr} \ \left[A^i A^j A^k\right] \ . \tag{16}$$

so the integral over the charge density K^0 precisely measures the winding number of the field.

As we see, there is actually a countable infinite number of distinct vacua we can choose. Although these vacua are physically equivalent, nothing forbids us from having field configurations which transition between different vacua at some point in time. Since all the vacua belong to different homotopic classes, any field configuration that interpolates between different vacua will have to cross regions where the gauge field is no longer pure gauge. This corresponds to field configurations which will have a non-zero action, similar to the problem of a particle tunneling between different minima of a potential, separated by a wall. These field configurations of finite action that interpolate between different vacua at different times are called instantons.

As a matter of fact, the existence of instantons lift the degeneracy between the different vacua and forces us to consider a different ground state for the theory. Two vacua characterized by different winding numbers are not orthogonal, since there are finite action configurations which will interpolate between these vacua.

It is instructive to consider a quantum mechanical example: consider a theory with infinitely many degenerate vacua characterized by an integer, and take the partition function of the system, with fixed boundary conditions n and m for asymptotic past and future. Let us consider only the simplest instanton, which interpolates between n and $n+1$, and its corresponding anti-instanton. We have have infinitely many tunneling process in between, as long as the number of instantons and anti-instantons obey $N_I - N_{\bar{I}} = m - n$. The partition function of the system can be then written as as sum over all the possible field configurations that respect this constraint. For some large time T, we have

$$Z(n,m) = Z_0 \sum_{N_I, N_{\bar{I}}} \frac{(Z_I T)^{N_I + N_{\bar{I}}}}{N_I! N_{\bar{I}}!} \delta(N_I - N_{\bar{I}}, m - n) \tag{17}$$

where Z_I is the action for an single instanton. Using the representation

$$\delta(a,b) = \int \frac{d\theta}{2\pi} e^{i\theta(a-b)}$$

we can rewrite the partition function as

$$Z(n,m) = Z_0 \int \frac{d\theta}{2\pi} e^{i\,\theta(m-n) + 2\,T\,Z_I \cos(\theta)} \tag{18}$$

In order to interpret this partition function, let us use the energy representation of the matrix element $Z(n,m)$

$$Z(n,m) = \sum_k e^{-E_k T} \langle n|k\rangle \langle k|m\rangle \ . \tag{19}$$

For $T \to \infty$, only the ground state will contribute, so we can write

$$E_0 = \lim_{T \to \infty} \frac{1}{T} \ln Z(n, m) \ . \tag{20}$$

And thus we have

$$E_0 \propto 2 Z_I \cos(\theta) \tag{21}$$

This parameter θ labels the true vacuum of the system. It is easy to see that choosing

$$|\theta\rangle = \sum_n e^{in\theta} |n\rangle \tag{22}$$

will lead to orthogonal, and most importantly, gauge invariant vacua.

In order to see how the choice of this vacuum affects the partition function of our theory, consider the vaccum-to-vacuum transition amplitude for a given angle θ. We have

$$\langle\theta|\theta\rangle = \sum_{m,n} e^{i\theta(m-n)} \langle m|n\rangle = \sum_v e^{i\theta v} \sum_n \langle m+v|n\rangle \tag{23}$$

and the integer v can be expressed as an integral over the divergence of the Badeen current, which is precisely

$$v = \frac{g^2}{32\pi^2} \int d^4x \, F_{\mathrm{a}}^{\mu\nu} \tilde{F}_{\mathrm{a}\mu\nu} \tag{24}$$

Using the path integral representation of $\langle\theta|\theta\rangle$, we hae

$$< \theta|\theta >= \sum_v \int_v \mathcal{D}A \, e^{iS[A] + i\theta \frac{g^2}{32\pi^2} \int d^4x \, F_{\mathrm{a}}^{\mu\nu} \tilde{F}_{\mathrm{a}\mu\nu}} \tag{25}$$

where the functional integration is taken, for each term in the sum, over field configurations that satisfy the constraint (24).

We started by attempting to construct the vacuum of the theory. This task was not so simple, since what we find is that due to the gauge redundancy of the theory, there is an infinite number of classes of field configurations that minimize the action. These classes are parametrized by an integer, which measures how many times the maps from the gauge group manifold winds around the spatial manifold. In reality, these are not the appropriate vacua of the theory, since there are instantons which transition between the vacua with finite action. The orthogonal and gauge invariant way to construct the QCD vacuum is to consider an infinite superposition of the n-vacua. The QCD vacuum, or the θ vacuum, is parametrized by a parameter θ, and these vacua are all orthogonal and gauge invariant. Choosing this as the vacuum of the theory forces us, however, to introduce a term in the lagrangian, which is precisely the CP violating $F\tilde{F}$ term which we encountered before.

2.4 THE STRONG CP PROBLEM AND THE AXION SOLUTION

In this section we review the works of Peccei and Quinn [7], which proposed the axion as a solution to the strong CP problem.

Let us first remark that the term $F\tilde{F}$ is not invariant under CP transformations. Secondly, the vacuum expectation value $\langle F\tilde{F}\rangle$ is proportional to the choice of θ, which completely parametrizes the vacuum of the theory.

The value of θ, however, can be related to observable quantities. There is a bound on the neutron dipole moment, which in turns implies that $\theta < 10^{-9}$. There is no specific reason why θ should be so small, so explaining why θ is so close to 0 is known as the strong CP problem.

In order to solve this problem, Peccei and Quinn [7] introduced a new symmetry $U(1)_{PQ}$ which can be realized due to the addition of a new pseudo-scalar field, the axion a. Under a $U(1)_{PQ}$ transformation, the axion receives a shift

$$a(x) \to a(x) + \alpha f_a \tag{26}$$

f_a is an order parameter which is associated with the breaking of $U(1)_{PQ}$. For $f_a \to 0$, the axion is no longer dynamical and the symmetry ceases to be realized. This field is then coupled to the $F\tilde{F}$ term in a way that ensures that its current obeys

$$\partial_\mu J^\mu_{PQ} = \xi \frac{g^2}{32\pi^2} F^{\mu\nu}_a \tilde{F}_{a\mu\nu}. \tag{27}$$

We have then the following lagrangian, neglecting the possible couplings of axion to matter

$$L_{PQ} = \left(\theta + \frac{a}{f_a}\right) \frac{g^2}{32\pi^2} F^{\mu\nu}_a \tilde{F}_{a\mu\nu} + \frac{1}{2}\partial_\mu a \partial^\mu a \ . \tag{28}$$

The essence of the PQ solution is that the effective potential for the axion field, after integrating out the QCD dynamics, has a minimum at

$$\langle a \rangle = -\theta f_a \ , \tag{29}$$

which precisely cancels the θ contribution, thus getting rid of the $\langle F\tilde{F}\rangle$ term.

In the next chapter we will reformulate the strong CP problem in terms of the Chern-Simons 3-form. By doing so we are able to exploit a more powerful formulation of the PQ solution, in which the axion becomes a 2-form field. The shift-symmetry is promoted into a gauge redundancy, and the goal of the axion will be to provie a mass gap for the QCD Chern-Simons 3-form.

3

THREE-FORM GAUGING OF AXION SYMMETRY

In this chapter we review the formulation of the Axion symmetry in terms of Three-Form gauging. For a comprehensive treatment of the subject, we refer to the original work of Dvali [12].

The strong CP problem is a problem of vacuum superselection. There is an arbitrary parameter θ which sets the strength of the the CP violating term in QCD. Whenever θ is nonzero, we observe that

$$\langle F_{\mathrm{a}}^{\mu\nu} \tilde{F}_{\mathrm{a}\mu\nu} \rangle \neq 0 \tag{30}$$

and thus the theory would predict processes that violate CP symmetry. The observed smallness of the neutron dipole moment can be related to the parameter θ, which is bounded by $\theta < 10^{-9}$. Peccei and Quinn observed that this problem can be avoided by promoting the parameter θ into a dynamical field. This has the effect of restoring the original anomalous chiral symmetry as a new shift symmetry under which the this dynamical field will transform. The strong CP problem is then solved because the minimum of the axion potential is for $\theta = 0$, which implies a vanishing of the $\langle F_{\mathrm{a}}^{\mu\nu} \tilde{F}_{\mathrm{a}\mu\nu} \rangle$ correlator.

In this chapter, we seek to review the formulation of the PQ theory in terms of a 3-form field [12]. In this formulation, the strong CP problem can be viewed as the problem of a massless 3-form gauge field which creates a constant (4-form) electric field in the vacuum. This phenomenon is similar to that of the Schwinger model in 1+1d, which a 1-form gauge field creates a constant 2-form electric field, which in the absence of charges remains constant - and thus the vacuum is determined by the value of this electric field. In QCD, this 4-form electric field has the peculiarity that it violates CP, and thus we know, by phenomenology, that it must be screened somehow.

Screening an electric field is possible by giving the gauge field a mass. In order to give a 3-form gauge field a mass, we need to introduce a 2-form gauge field that will be responsible for "Higgsing" this three-form. A 2-form field in 3+1 dimensions is precisely dual to a pseudo-scalar, which corresponds to the axion. It is clear, then, that the shift symmetry of the axion field will be promoted to a gauge redundancy of the 3-form field, whose gauge transformation have to be offset by this shift of the axion 2-form.

It is clear that this formulation gives us a lot more power when it comes to constructing the axion potential, since now we are bound by a gauge redundancy instead of a global symmetry, and thus the couplings of the axion are extremely constrained. Precisely for that reason, we can use this machinery to speculate on how, and if, the coupling of gravity to the axion might destabilize its potential and thus ruin the PQ solution to the strong CP problem. In the next chapter, however, we will extend this idea to include ways to protect the axion, in case gravity does spoil the solution.

3.1 2-FORM AXION AND 3-FORM GAUGE FIELDS

The essence of the axion solution is the following: we have a shift-symmetric (pseudo-)scalar field which plays the role of the θ angle, and this scalar is coupled to the $F_{\text{a}}^{\mu\nu}\tilde{F}_{\text{a}\mu\nu}$ field, which as we'll see, is nothing more than the field strength of a 3-form gauge field. In the vacuum, the electric field will be screened by the dynamical production of a mass gap, but there are no new degrees of freedom and the shift symmetry of the scalar field cannot be explicitly broken! In order to see how this is possible, we need to introduce the 2-form field $B_{\mu\nu}$ which will play the role of the axion, and we need to see how this field will couple to the QCD 3-form.

Let us start by introducing the Kalb-Ramond 2-form antisymmetric field $B_{\mu\nu}$. The kinetic term for this field is

$$P_{\alpha\beta\gamma}P^{\alpha\beta\gamma} \ , \tag{31}$$

where the field strength P is defined as

$$P_{\alpha\beta\gamma} = \partial_{[\alpha}B_{\beta\gamma]} = \mathrm{d}B \ , \tag{32}$$

and $\mathrm{d}B$ is the exterior derivative of the 2-form B, in a coordinate independent notation.

The kinetic term is invariant under the global shift

$$B_{\mu\nu} \to B_{\mu\nu} + \Omega_{\mu\nu} \ , \tag{33}$$

and it also contains a redundancy

$$B \to B + \mathrm{d}\eta \,, \tag{34}$$

where η is a 1-form. This redundancy, together with the antisymmetry of B and the fact that one of the components is not dynamical, restricts the number of degrees of freedom to 1. We also need to introduce the 3-form gauge field $C_{\alpha\beta\gamma}$, whose kinetic term is

$$F_{\mu\alpha\beta\gamma}F^{\mu\alpha\beta\gamma} \tag{35}$$

and the field strength tensor F is naturally

$$F_{\mu\alpha\beta\gamma} = \partial_{[\mu}C_{\alpha\beta\gamma]} = \mathrm{d}C \,. \tag{36}$$

The theory of invariant under the gauge transformation

$$C \to C + \mathrm{d}\Omega \,, \tag{37}$$

where Ω is a three-form.

In 3+1d the gauge redundancy is enough to strip the 3-form field $C_{\alpha\beta\gamma}$ of all its propagating degrees of freedom. The situation is similar to that of the vector field in 1+1 dimension, in which despite the absence of degrees of freedom, there is still a long range interaction through means of a constant electric field. While in 1+1d this field is sourced and screened by particle charges, in 3+1d the 3-form field couples instead to branes or domain walls. We can visualize the effect of C as that of a global capacitor plate - it creates a constant electric field which can only be screened by a co-dimensional source.

In the presence of this constant electric field, the strength tensor can be parametrized as

$$F_{\mu\alpha\beta\gamma} = F_0\epsilon_{\mu\alpha\beta\gamma} \,. \tag{38}$$

In the absence of sources the equations of motion will be simply

$$\partial_\mu F^{\mu\alpha\beta\gamma} = 0 \,, \tag{39}$$

which imply that the value F_0 is arbitrary.

3.2 THE MASS GAP

In the case of a vector field A_μ which transforms as $\delta A_\mu = \partial_\mu\omega$ under gauge transformations, the way to introduce a mass gap in the theory is by coupling the vector field to the derivative of a shift-symmetric scalar in a way that the scalar field will absorb the gauge transformations of A_μ - or in other words, a 1-form gauge field can acquire a mass gap by eating a 0-form.

One should be however careful, since we always have the choice to add a mass gap by hand without the necessity to worry about who the gauge field is eating - which is just Proca theory. While doing that, however, we are promoting A from a massless to a massive representation of the Lorentz group, and thus gaining an extra degree of freedom - which is just the longitudinal polarization of the gauge field which becomes physical. If the mass gap is dynamically generated, however, we need to keep track of where the extra degree of freedom comes from, and that is the essence of this whole discussion: the axion is a necessary ingredient in screening the QCD 3-form precisely because it provides the degree of freedom necessary to create a mass gap.

In the case of the 3-form, it can become massive in the vacuum by eating the 2-form field. Let us construct the simplest lagrangian which can do this

$$L = -\frac{1}{2} F_{\mu\alpha\beta\gamma} F^{\mu\alpha\beta\gamma} - 12m^2 \left(P_{\alpha\beta\gamma} - C_{\alpha\beta\gamma}\right)^2 \tag{40}$$

It is clear that this theory contains a mass gap and it only propagates 1 degree of freedom. Furthermore, the equations of motion now contain a term linear in $P_{\alpha\beta\gamma}$, which makes the field strength decay exponentially. Instead of propagating a long range force, the correlators have now a typical length scale of m^{-1}.

We can attempt to dualize the theory in order to check how it looks in terms of the pseudo-scalar degree of freedom. In order to dualize the theory, we treat P is a fundamental 3-form and enforce the Bianchi identity as a constraint

$$\epsilon^{\mu\alpha\beta\gamma} \partial_{[\mu} P_{\alpha\beta\gamma]} = 0, \tag{41}$$

Introducing the lagrange multiplier a for the constraint, we will end up with the following lagrangian

$$L = -\frac{1}{2} F_{\mu\alpha\beta\gamma} F^{\mu\alpha\beta\gamma} - 12m^2 (P_{\alpha\beta\gamma} - C_{\alpha\beta\gamma})^2 - \frac{\Lambda^2}{24} a\, \epsilon^{\mu\alpha\beta\gamma} \partial_{[\mu} P_{\alpha\beta\gamma]} \tag{42}$$

Integrating out P we arrive at the following lagrangian, where the lagrange multiplier a is now the pseudo-scalar degree of freedom

$$L = \frac{\Lambda^4}{2m^2} (\partial_\mu a)^2 - \Lambda^2 \partial_{[\alpha} a C_{\beta\gamma\delta]} \epsilon^{\alpha\beta\gamma\delta} - \frac{1}{2} F_{\mu\alpha\beta\gamma} F^{\mu\alpha\beta\gamma} \,. \tag{43}$$

While the lagrangian defined in terms of P had an explicit mass gap, it is not completely clear that the same is true for this theory. In order to verify that indeed we have a mass gap, we can integrate out C through its equations of motion and write the effective equations for a

$$\Box a + \frac{m^2}{\Lambda^2}(a - \kappa) = 0. \tag{44}$$

where κ is an integration constant. At the level of the equations of motion, we see that a mass gap is generated, although the field a remains shift-symmetric in the lagrangian!

If we identify $C_{\alpha\beta\gamma}$ as being the Chern-Simons three-form of QCD, the second term in the lagrangian (43) is nothing more than the coupling between the axion a and $F_{\mathrm{a}}^{\mu\nu}\tilde{F}_{\mathrm{a}\mu\nu}$. It is clear that both formulations are equivalent, and that indeed the role of the axion is to generate a mass gap - and thus the ground state of the theory is for $a = 0$, which implies $\langle F\tilde{F}\rangle = 0$.

3.3 QCD CHERN SIMONS 3-FORM

As we have seen, we can formulate the strong CP problem in the language of a 3-form, the Chern-Simons 3-form $C_{\alpha\beta\gamma}$ which mediates a long range correlation. Another way to see that this is indeed the case is to consider the fact that the topological susceptibility of the QCD vacuum is non-zero

$$\langle F\tilde{F}\,,\, F\tilde{F}\rangle_{q\to 0} \neq 0\,. \tag{45}$$

Since $F = dC$, we have

$$\langle C\,,\, C\rangle_{q\to 0} \propto \frac{1}{q^2}\,. \tag{46}$$

Since we have a pole in the $q \to 0$ limit, C is a massless field in the effective theory. We can understand the physics of the θ angle in the following way: whenever this correlator has a pole at zero momenta, the low energy theory is dependend on the θ angle. Removing the θ dependence means introducing a mass gap for C, which will eliminate the $q = 0$ pole.

It is instructive to ask the following question: what happens when the quark masses are exactly 0 and chiral symmetry is classically exact? As we have seen, in the presence of a chiral theory, we can simply rotate the θ angle away. But in order to interpret this in terms of mass gap generation, we need an extra degree of freedom which will Higgs the Chern-Simons 3-form. The only candidate is the η' meson, which is the would-be Goldstone of spontaneous chiral symmetry breaking! So it is no surprise that the η' ends up being heavier than the π meson, since its mass gap comes from two sources: the explicit breaking of chiral symmetry by the light quark masses, and also the mixing with the QCD 3-form which will dynamically generate a mass gap.

It is important to notice the following, though: although the η' meson is able to play the role of the axion in the 0 mass limit, the explicit chiral symmetry breaking in real QCD is not small enough to comply with the bounds on the θ angle, as it was shown in [18].

It is important to notice that we can generate this mass gap in purely topological terms, following [13]. In this language, it doesn't matter whether the mass gap generation comes from a fundamental boson, like the axion, or some composite goldstone boson. All that matters is that there is a current J_μ which is anomalous, thus satisfying

$$\partial_\mu J^\mu = F\tilde{F} = E \, , \tag{47}$$

where we have $E = dC$, the field strength tensor of the Chern-Simons 3-form.

Let us show that this current automatically introduces a mass gap in the theory. This anomalous current will introduce a unique coupling in the action [13]

$$E\frac{1}{\Box}\partial_\mu J^\mu \, , \tag{48}$$

While we don't know the precise form of the low energy action of the theory, we can use the power of effective field theory to make a $1/\lambda$ expansion. Since we are interested in the zero momentum limit, higher order terms will be irrelevant. To lowest order, we have

$$L = \frac{1}{\Lambda^4}E^2 + \frac{E}{\Lambda^2}\frac{\partial_\mu}{\Box}J^\mu \, . \tag{49}$$

Upon variation with respect to C, we have the following equations of motion

$$\partial_\nu E = -\Lambda^2\partial_\nu\frac{\partial_\mu}{\Box}J^\mu \, , \tag{50}$$

which yields, after taking into account the anomalous divergence of J_μ

$$\Box E + \Lambda^2 E = 0 \, , \tag{51}$$

where we have also ignored an integration constant which does not affect the physics of the result. At the end, it is clear that whenever we have an anomalous current, a mass gap is generated and the $q^2 = 0$ pole is removed. This is true regardless of the form of the current J_μ.

In the presence of an axion, the current J_μ is simply

$$J_\mu = f_a \partial_\mu a \tag{52}$$

whereas in a chiral theory, we have

$$J_\mu = \bar{Q}\gamma_\mu\gamma_5 Q \, , \tag{53}$$

3.4 THE ROLE OF GRAVITY

In the PQ formulation of the axion, the reason why the strong CP problem is eliminated lies on the fact that the minimum of the axion potential happens for $a = 0$. This fact is heavily dependent on the form of axion potential, though. In the presence of extra fields which couple to the axion, the potential can receive corrections which spoil the solution. In other words, the global shift-symmetry of the axion is too weak to constraint its interaction, and the corrections are uncontrollable.

By gauging this symmetry, we are sure that the possible interactions of the axion, now described by a 2-form $B_{\mu\nu}$ are within control. This is because interactions which would spoil the gauge redundancy of the theory will inevitable add new - likely unhealthy - degrees of freedom. By enforcing the sanity of the theory, we are able to say exactly how extra corrections might couple to the axion.

As we have seen (42), the lagrangian

$$L = (\partial_\mu a)^2 + \frac{a}{f_a} E + \frac{1}{\Lambda^4} E^2 \tag{54}$$

can be written in the following form after dualization, in a coordinate independent notation

$$L = \frac{1}{\Lambda^4} E^2 + \frac{1}{f_a^2} (C - \mathrm{d}B)^2 \, , \tag{55}$$

This theory contains a gauge redundancy, under which both C and B transform

$$\begin{aligned} C &\to C + \mathrm{d}\Omega \, , \\ B &\to B + \Omega \, , \end{aligned} \tag{56}$$

where Ω is an arbitrary two-form.

We would like to stress that the possibility of rewriting the QCD axion in a dual language of $B_{\mu\nu}$ and coupling it to a QCD Chern-Simons was already considered in [11], but unfortunately was abandoned, because the authors assumed that the duality between a and $B_{\mu\nu}$ does not hold for a massive axion. This is not the case, since it is exclusively the coupling to C that generates a mass for the axion in both formulations.[1]

The power of the 3-form formulation is manifest here: since we have a strong gauge redundancy that has to be respected, the gravitational coupling to the axion must come in a similar way. The only possible field that can absorb a 2-form in a gauge transformation is another 3-form, C_G. The only possible choice of C_G in gravity is the gravitational Chern-Simons field

$$C_G \equiv \Gamma \mathrm{d}\Gamma - \frac{3}{2}\Gamma\Gamma\Gamma \,, \tag{57}$$

with

$$\mathrm{d}C_G = R\tilde{R} \equiv E_G \,, \tag{58}$$

where R is the Riemann tensor and $\tilde{R}$ is its dual. For gravity thus to ruin the axion solution the following two conditions must be satisfied:

- We must have a non-zero topological vacuum susceptibility

  $$\langle R\tilde{R}, R\tilde{R}\rangle_{q\to 0} = \text{const} \neq 0 \tag{59}$$

  in the absence of the axion.

- The divergence of the axion current must contain a contribution from $aR\tilde{R}$.

These two conditions can be set by two parameters: the scale Λ_G which sets how big the vacuum susceptibility is; and the coupling constant α_G which determines the coupling of the axion to the gravitational Chern-Simons 3-form. Whenever either one of these terms in 0, the axion is safe since gravity can have no effect on the axion potential.

The Lagrangian which renders the QCD θ-term physical has the unique form

$$L = \frac{1}{\Lambda^4}E^2 + \frac{1}{\Lambda_G^4}E_G^2 + \frac{1}{f_a^2}(\alpha C + \alpha_G C_G - \mathrm{d}B)^2 \,, \tag{60}$$

1 The dual formulation of a massive $B_{\mu\nu}$ in terms of massive three-forms was studied in [14] up to quadratic order in fields. The duality between the massive three-form and the massive axion for an arbitrary form of the axion potential was proven in [13].

where as just explained α and α_G are two dimensionless parameters which determine the respective coupling strengths. Normalizing the three-forms canonically, rescaling $B \to B\sqrt{(\alpha^2\Lambda^4 + \alpha_G^2\Lambda_G^4)}$, and introducing the mixing angle,

$$\cos\beta \equiv \frac{\alpha\Lambda^2}{\sqrt{\alpha^2\Lambda^4 + \alpha_G^2\Lambda_G^4}} ,$$

we can rewrite,

$$L = E^2 + E_G^2 + m^2(C\cos\beta + C_G\sin\beta - \mathrm{d}B)^2 , \quad (61)$$

where

$$m^2 \equiv (\alpha^2\Lambda^4 + \alpha_G^2\Lambda_G^4)/f_a^2 .$$

We see that, due to mixing, only one combination of three-forms is becoming massive, whereas the orthogonal one, $C\sin\beta - C_G\cos\beta$, is massless. The natural value of the QCD θ-term is measured by the relative weight of the QCD three-form in this combination,

$$\theta_{QCD} = \sin\beta \equiv \frac{\alpha_G\Lambda_G^2}{\sqrt{\alpha^2\Lambda^4 + \alpha_G^2\Lambda_G^4}} . \quad (62)$$

We don't know if either one of the gravitational parameters can be non-zero, but let us assume that there exists a gravitational coupling and that the scale Λ_G is large enough that this contribution would indeed spoil the axion solution.

If that were to be the case, we would need to introuce another axionic-type particle in order to Higgs the massless combination. From our previous discussion, all we need is another current J'_μ whose divergence will be sourced by the gravitational E_G. The exact source of this current is irrelevant, as long as it is anomalous under gravity.

In the next chapter, we will show that the standard model already contains such a protection mechanism in the form of light neutrinos.

4

HOW NEUTRINO PROTECTS THE AXION

As we have seen, the axion is able to solve the strong CP problem by providing a mass gap to the QCD 3-form. This can be easily seen in the language where an axion is an antisymmetric 2-form $B_{\mu\nu}$, since now the shift-symmetry of the axion becomes a gauge redundancy, and the axion plays the role of the extra degree of freedom which gets eaten up by the 3-form in order to create a gauge invariant mass term. The way the axion solution can be ruined by gravity is if gravity provides an additional 3-form field which also couples to the axion 2-form.

In order to protect the axion, we need to provide the theory an extra current which is anomalous under gravity. As a matter of fact, the standard model already contains such a current: the neutrino lepton number current. If the neutrinos were exactly massless, this current would be enough to rotate the gravitational θ angle away, but in fact neutrinos have a tiny mass. What we will show in this chapter is that if the mass of the lightest neutrino is small enough, we are able to use this anomalous current to save the axion. Conversely, if we do find that there is a non-vanishing Λ_G and nonzero coupling of C_G to the axion, determining these parameters puts an upper bound on the lightest neutrino mass. We will closely follow the derivation and expand on the results we obtained in [6].

4.1 MASSLESS NEUTRINOS

For definiteness, we consider a single massless neutrino species of left chirality ν_L. The chiral symmetry

$$\nu_L \to e^{i\phi}\nu_L \tag{63}$$

is anomalous with respect to gravity and the corresponding current

$$J^L_\mu = \bar{\nu}_L\gamma_\mu\nu_L \ , \tag{64}$$

has an anomalous divergence [15]

$$\partial_\mu J^{L\mu} = R\tilde{R} = E_G \,, \tag{65}$$

where we have set the known anomaly coefficient to one. It is now obvious that the massless pole is no longer there for E and E_G. Indeed from the effective Lagrangian

$$L = \frac{1}{\Lambda^4}E^2 + \frac{1}{\Lambda_G^4}E_G^2 + \frac{\alpha}{f_a^2}E\frac{\partial_\mu}{\Box}J^\mu + \frac{\alpha_G}{f_a^2}E_G\frac{\partial_\mu}{\Box}J^\mu + \frac{1}{f_\nu^2}E_G\frac{\partial_\mu}{\Box}J^{L\mu} \tag{66}$$

where $f_\nu \sim \Lambda_G$ is a scale that sets the strength of anomaly-induced coupling to neutrino current.

One can obtain the equations of motion for C and C_G which together with (65) show that there are no massless modes in E and E_G. Thus, massless neutrino protects the axion solution to the strong-CP problem. In the next section we shall take into the account a possible effect of the small neutrino mass.

4.2 SMALL NEUTRINO MASSES

Neutrino masses are, unfortunately, nonzero. Due to this reason, the neutrinos cannot exactly rotate the gravitational θ angle away, or conversely, the axion being minimized will not necessarily imply that that $\theta_Q CD$ will sit exactly at 0. The effects of small netrino masses - small compared to the other relevant scales in the problem, $\Lambda_Q CD$ and Λ_G should not change the story too much in a qualitative way.

In this section we will calculate exactly the effect of a small neutrino mass on the QCD θ angle. Observationally [16], there is an upper bound on neutrino masses $\sum m_\nu \lesssim 0.3$eV [17], where the sum is over all neutrino flavors. We therefore parametrize the mass of the lightest neutrino by m_ν. In case m_ν is non-zero, the neutrino lepton number is explicitly broken. This introduces an additional factor to the divergence of the current (64)

$$\partial_\mu J^\mu_{\nu_L} = R\tilde{R} + m_\nu \bar{\nu}\gamma^5\nu \,. \tag{67}$$

As we have shown before, the generation of the mass gap for C_G automatically implies that this current must be identified with a pseudoscalar degree of freedom. We shall denote it by η_ν in analogy with the η' meson of QCD.

Notice, that we are not making any extra assumption. The necessity of a physical η_ν degree of freedom follows from the matching of high-energy and low-energy theories [12, 13]. From the high energy point of view, the axion is protected because the would-be gravitational θ-term is rendered unphysical by a chiral neutrino rotation. In order to match this effect in low energy description, the correlator (59) must be screened. By gauge symmetry, this is only possible if there is a corresponding Goldstone-type degree of freedom that plays the role of the Stückelberg field for the gravitational three-form (57). In other words, the same physics that provides the correlator (59), by consistency, must also provide the physical degree of freedom, η_ν, necessary for generation of the mass-gap in the presence of anomaly.

We can think of η_ν as of a pseudo-Goldstone boson of the spontaneously-broken lepton-number symmetry (63) by non-perturbative gravity. In a sense, η_ν can be thought of as a low-energy limit of the neutrino bilinear operator,

$$\eta_\nu \to \frac{1}{\Lambda_G^2}\bar{\nu}\gamma^5\nu \quad \text{and} \quad J^\mu_{\nu_L} \to \Lambda_G \partial^\mu \eta_\nu \, , \tag{68}$$

in a way similar to relation of η' of QCD in terms of a quark bilinear operator. With this connection, the effect of a small neutrino mass on η_ν is similar to the effect of a small quark mass on η'. Namely, to the leading order in $\frac{m_\nu}{\Lambda_G}$ such a deformation of the theory should result into a small explicit mass of η_ν in effective low energy Lagrangian. Because physics must be periodic in the Goldstone field, this mass term should be thought of as the leading order term in an expansion of the periodic function. The higher order terms in this expansion cannot affect the mechanism of mass-generation and are unimportant for the present discussion. Thus, the dynamics of the theory is now governed by the Lagrangian (66) with an additional mass term $m_\nu \Lambda_G \eta_\nu^2$ appearing from the explicit symmetry breaking by the neutrino mass. Replacing the currents with their corresponding pseudo-Goldstone bosons yields

$$\begin{aligned} L &= \frac{1}{\Lambda^4}E^2 + \frac{1}{\Lambda_G^4}E_G^2 - \frac{a}{f_a}E - \alpha_G \frac{a}{f_a}E_G - \frac{\eta_\nu}{\Lambda_G}E_G \\ &\quad + \partial_\mu a \partial^\mu a + \partial_\mu \eta_\nu \partial^\mu \eta_\nu - m_\nu \Lambda_G \eta_\nu^2 \, . \end{aligned} \tag{69}$$

Here we have absorbed α into the definition of f_a and have set the decay constant of η_ν to be equal to Λ_G [1]. Ignoring numerical factors, the equations of motion for C and G are

$$\begin{aligned} \mathrm{d}\left(E - \Lambda^4 \frac{a}{f_a}\right) &= 0 \\ \mathrm{d}\left(E_G - \alpha_G \Lambda_G^4 \frac{a}{f_a} - \Lambda_G^3 \eta_\nu\right) &= 0\,, \end{aligned} \tag{70}$$

and the ones for a and η_ν read

$$\begin{aligned} f_a \Box a &= -\alpha_G E_G - E \\ (\Box + m_\nu \Lambda_G)\eta_\nu &= -\frac{E_G}{\Lambda_G}\,. \end{aligned} \tag{71}$$

It is already clear from the last two equations that small enough neutrino mass will continue to keep θ-term of QCD under control. Indeed, these two equations imply that in the vacuum (that is, for $\eta_\nu = a =$ constant) the value of the QCD electric four-form is $E = m_\nu \alpha_G \Lambda_G^2 \eta_\nu$. Since, the vacuum expectation value of η_ν cannot exceed its decay constant, Λ_G, the corresponding maximal possible value of the θ_{QCD} is

$$\theta_{max} = \frac{E_{max}}{\Lambda^4} = \frac{m_\nu \alpha_G \Lambda_G^3}{\Lambda^4}\,, \tag{72}$$

which vanishes for $m_\nu \to 0$.

Indeed, the vacuum solutions of (70) and (71) are given by the following expressions

$$\begin{aligned} E &= \alpha_G \Lambda^4 \frac{m_\nu \Lambda_G^4 (\alpha_G \beta_2 - \beta_1)}{\Lambda_G^4 m_\nu \alpha_G^2 + \Lambda^4(\Lambda_G + m_\nu)} \\ E_G &= \Lambda_G^4 \frac{m_\nu \Lambda^4 (\beta_1 - \alpha_G \beta_2)}{\Lambda_G^4 m_\nu \alpha_G^2 + \Lambda^4(\Lambda_G + m_\nu)}\,, \end{aligned} \tag{73}$$

where β_1 and β_2 are dimensionless integration constants of (70), i.e. $E = \Lambda^4 \frac{a}{f_a} + \Lambda^4 \beta_2$ and $E_G = \Lambda_G^4 \alpha_G \frac{a}{f_a} + \Lambda_G^3 \eta_\nu + \beta_1 \Lambda_G^4$. Since the maximal values of the two electric fields are bounded by the scales Λ^4 and Λ_G^4, the maximal values of the crresponding integration constants can be order one.

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1 In the absence of other scales in the problem, this is the only natural possibility, in full analogy to the decay constant of η' meson being set by the QCD scale. The possible difference between f_ν and Λ_G can easily be taken into the account and changes nothing in our analysis.

Let us parametrize this result by the ratio of the neutrino mass to the gravitational scale by defining $\epsilon \equiv \frac{m_\nu}{m_\nu + \Lambda_G}$. The maximal value for the quantity $\alpha_G \beta_2 - \beta_1$ is either order one or α_G depending whether α_G is less or larger than one. For definiteness, we shall assume $\alpha_G > 1$. The maximal value for the QCD electric four-form field E_G then depends on ϵ as follows[2]

$$E = \Lambda^4 \frac{\epsilon \alpha_G^2}{\epsilon \alpha_G^2 + \frac{\Lambda^4}{\Lambda_G^4}} \, . \tag{74}$$

The limit of massless neutrinos, $\epsilon \to 0$ leads to $E = E_G = 0$. In other words, massless neutrino fully protects the axion from gravity.

Considering the value of the neutrino mass m_ν much smaller than the gravitational scale Λ_G, $m_\nu \ll \Lambda_G$, we get $\epsilon \simeq \frac{m_\nu}{\Lambda_G}$. Then, the value of E in the QCD vacuum is instead

$$E = \Lambda^4 \frac{\alpha_G^2 m_\nu \Lambda_G^3}{\Lambda^4 + \alpha_G^2 m_\nu \Lambda_G^3} \, . \tag{75}$$

In terms of θ this gives

$$\theta = \frac{\alpha_G^2 m_\nu \Lambda_G^3}{\Lambda^4 + \alpha_G^2 m_\nu \Lambda_G^3} \, . \tag{76}$$

In order to be compatible with observations (cf. the electric dipole moment [18, 19]), θ must satisfy the bound $\theta < 10^{-9}$.

It is instructive to look at the values of E for different choices of parameters. If the denominator in (76) is dominated by $\alpha_G^2 m_\nu \Lambda_G^3$, there is essentially no screening of the four-form electric field, $E \sim \Lambda^4$, and correspondingly the un-fine-tuned value is $\theta \sim 1$. In this case, the axion solution of strong-CP problem is ruined. On the other hand, if $\Lambda^4 \gg \alpha_G^2 m_\nu \Lambda_G^3$, then requirement of the protection of the successful axion mechanism, translates into the following bound on the lightest neutrino mass,

$$m_\nu \lesssim 10^{-9} \frac{\Lambda^4}{\alpha_G^2 \Lambda_G^3} \, . \tag{77}$$

In turn, the experimental measurement of the neutrino mass would introduce an upper bound on the non-perturbative gravitational scale of the anomaly Λ_G.

2 Note that we are not concerned with the actual value the gravitational Chern-Pontryagin density takes in the vacuum as it is not constrained by measurements.

The experimental searches [21, 20] currently focus on the mass range $0.2\,\text{eV} < m_\nu < 2\,\text{eV}$. A detection of the lightest neutrino mass in this window would give the bound $\sqrt[3]{\alpha_G^2}\Lambda_G \lesssim 0.2\,\text{GeV}$.

4.3 SUMMARY

In this chapter, we unveiled a very interesting connection between particle physics an collective quantum phenomena in gravity. We have seen how the existence of non-perturbative gravitational objects might interfere with the axion solution of the strong CP problem, and we showed how light neutrinos may cancel this effect in a very unexpected way.

By relating the mass of the neutrinos with the scale that sets the gravitational anomaly, Λ_G, we have been able to create an upper bound on the lightest neutrino mass, in case the gravitational Chern Simons 3-form does indeed couple to the axion. In case we are able to measure accurately the lightest neutrino mass, this relation gives us a bound on the gravitational topological vacuum susceptibility.

One interesting consequence of this idea is the fact that it predicts a new effective low energy pseudo-scalar degree of freedom, η_ν, that plays the role analogous to η' meson of QCD. The phenomenological consequences of the existence of this new degree of freedom has been recently studied in [23], where the authors argue that the gravitational θ term itself may be responsible for generating small neutrino masses.

Part II

ROAD SIGNS FOR UV-COMPLETION

5

A QUESTION OF SCALES

5.1 The Wilsonian paradigm

It has been the dominant assumption in physics that nature obeys the Wilsonian paradigm. According to this paradigm, at each energy scale we have the presence of weakly coupled degrees of freedom, and these degrees of freedom might change as we raise or lower the energy, depending on the dynamics of the theory. While at high energies, we have quarks and gluons as fundamental degrees of freedom of QCD, at low energies hadrons, baryons and glueballs appear.

The fact that we can identify effective degrees of freedom at any given energy scale means that it is always possible to parametrize our knowledge of short distance physics by fixing observed symmetries and measured parameters. Although the dynamics of low-energy QCD in terms of quarks and gluons remains unsolved, we have a reasonably good understanding of the measurable low-energy processes through the pion lagrangian. The internal dynamics of quarks and gluons become frozen and manifest themselves through the pion decay constant and the QCD scale.

One very important feature of Wilsonian theories is the fact whenever our degrees of freedom become strongly coupled, there should be another weakly coupled degree of freedom that restores perturbative unitarity in our theory. Using the example of QCD, when the pion lagrangian violates unitarity, pions cease to be good degrees of freedom and we need the introduction of quarks and gluons.

This assumption is surely justified by past observations: from the 4-Fermi interactions to QCD, it seems like there is always some new particle(s) which are hidden when we reach a unitarity crisis. As it stands, it seems as if the standard model is following the same path: the scattering of W bosons becomes strong around the electroweak scale, and a weakly coupled boson was discovered. Whether it was really the Higgs, some composite particle or the first resonance in a larger tower, it is impossible to say just yet.

The Wilsonian paradigm has a deep implication for our understanding of physics: a theory can never be deemed as fundamental in a phenomenological sense, since it is only as good as our resolution power. In Wilsonian theories, it is always possible to introduce degrees of freedom that are limited to length scales smaller than those we can probe, and those modifications will have no effect on the outcome of experiments within our reach.

One natural question we can ask is: can Gravity also be viewed as an effective field theory, in the Wilsonian sense? The answer is most likely yes, and no. While it makes perfect sense to treat GR as an effective field theory, there are very good reasons as to why it might not be possible to UV-complete it simply by introducing new degrees of freedom - gravity seems to break the Wilsonian paradigm.

In the next section we will try to convince you that due to the existence of Black-Holes, GR behaves very differently at large energies.

5.2 WILSON AND GRAVITY

We define GR through the Einstein-Hilbert action

$$S = \frac{1}{16\pi G} \int \sqrt{-g}\, R\, \mathrm{d}^4 x \tag{78}$$

where R is the Ricci scalar and G is Newton's constant. In natural units, G has dimension of $[M]^-2$ or $[L]^2$. This allows us to define a mass scale, the Planck scale

$$M_p = \frac{1}{\sqrt{8\pi G}} \tag{79}$$

The Planck mass takes the value of 10^19 GeV, which is orders of magnitude higher than any mass scale we have access to, through laboratory experiments. The electroweak scale, defined analogously through the Fermi coupling constant G_F, sits at 246 GeV, orders of magnitude below the Planck scale.

In order to define the theory quantum mechanically, we take the theory of perturbations on top of the Minkowski vacuum and perform a polynomial expansion of the action: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. This turns out to be the unique theory of a spin 2 representation of the Lorentz group invariant under the following gauge transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)} \tag{80}$$

After canonical normalization, the action will contain a quadratic part and a series of terms which are polynomials of $h_{\mu}\nu$, all with two derivatives. Each one of this terms will be suppressed by an appropriate power of M_p. Schematically, we can write the action as

$$S = \int \partial h\, \partial h + \frac{1}{M_p} h\, \partial h\, \partial h + \frac{1}{M_p^2} h\, h\, \partial h\, \partial h \tag{81}$$

where we disregard the contraction of the indices.

If we introduce an external source on the theory, by consistency, we have the following coupling

$$S_T = \frac{1}{M_p} \int h_{\mu\nu} T^{\mu\nu} \tag{82}$$

Suppose that this source describes a fundamental particle localized within a region of typical size R. The typical energy of this particle will be $E \approx 1/R$. In this situation, for distances much larger than R, the h_{00} component will simply give us the Newtonian potential

$$h_{00} = \frac{M}{M_p} \frac{1}{r} \tag{83}$$

Suppose we are external observers attempting to probe the gravitational field of this source, how close to we need to go in order to see deviations from the Newtonian potential? We can simply compare at which scale the quadratic and cubic terms of the action become comparable,

$$\partial h\, \partial h \approx \frac{1}{M_p} h\, \partial h\, \partial h \tag{84}$$

which gives us

$$r = r_S(E) = E/M_P^2 \tag{85}$$

assuming that $R < r_S(E)$. As a matter of fact, it is easy to check that this scale is the same for any of the interaction terms, which tells us that at this point every single term in the action has the same order of magnitude.

Indeed something dramatic happens at this scale, since it is precisely the Schwarschild radius: the radius of the Horizon of a black-hole of mass M. So the whole exercise was senseless from the beginning, since the moment we prepared a particle confined to a small region $R < r_S(M)$, we stopped probing the particle and started probing a much larger object of radius r_S.

The effect of Black-holes create a minimal scale $L_P = 1/M_P$. Any particle localized within a region $R < L_P$ ceases to be an effective degree of freedom of the theory and starts being a black-hole instead.

UV-Completing GR by adding new degrees of freedom would require the introduction of particles which have a mass of the order of the M_P. If we hold on to the previous assumptions, then these degrees of freedom are inherently strongly coupled, since they always come following by a very strong gravitational field. One way to avoid the inherent strong coupling of heavier degrees of freedom would be to make gravity weaker at high energies / high field values, which is inconsistent due to the appearance of negative normed states [25].

The Wilsonian paradigm seems to be fundamentally in conflict with the existence of black-holes. And due to the very nature of black-holes, high energy and short distances can no longer be used interchangeably - modifying the theory at high energies can have consequences which trickle up to large distance phenomena. Trans-planckian physics is not the physics of very small objects, it is the physics of very large distances!

Perturbatively, it seems as if high-energy gravitational scattering still encounters a crisis of strong coupling, and we still need a way to address this issue.

5.3 BLACK-HOLES AS UNITARIZERS

Suppose we prepare a scattering experiment in gravity. Our initial state is that of two gravitons colliding with a center of mass energy $\sqrt{s} \gg M_P^2$. Perturbatively, the tree level $2 \to 2$ matrix element for this process can be easily estimated, and we have

$$A_{2\to 2}^{tree} \approx \frac{s}{M_P^2} \ . \tag{86}$$

In this process, perturbative unitarity is violated for trans-planckian energies. In this situation, we can no longer trust our expansion, since the higher order diagrams contributing to the process are no longer suppressed, but rather enhanced. We know that this is in fact an artifact of perturbation theory, since as we discussed previously, this scattering experiment will inevitably result in the formation of a black-hole.

It is easy to understand why this happens: the scattering experiment will inevitably localize of a pair of particles with center of mass energy $\sqrt{s} \gg M_P$ on a region of typical size $1/\sqrt{s} \ll r_S(\sqrt{s})$. Instead of a scattering experiment whose end-product are perturbative states of the theory, we are producing a black-hole, which is a purely non-perturbative object.

Due to the intrinsic non-perturbative nature of the black-hole, we cannot probe it in any finite order of perturbation theory. While the black-hole itself might be composed of soft objects for which there is a natural perturbative expansion, the collective effects become order 1 and thus the field is strong.

It is extremely difficult to avoid the fact that (quantum) high-energy scattering leads to black-hole formation, since this reasoning only relies on the fact that the gravitational field is sourced by the expected value of the energy momentum tensor - which is a requirement for a consistent theory of gravity. In order to state that black-holes are formed, all we need is universality and Gauss law. Avoiding black-hole formation implies that we are no longer in a theory in which the fundamental degrees of freedom are massless spin-2. Modifications of gravity that get rid of black-holes must be severe!

Furthermore, should we attempt to modify gravity at high energies in order to prevent black-hole formation, we need to either abandon universality or in fact modify the theory at large distances, since black-hole formation is insensitive to the short-distance details of the theory. Abandoning universality will inevitably add extra degrees of freedom to the theory, whereas modifying the theory at large distances is phenomenologically unwanted.

In order to probe the short-distance behavior of the theory, we need to design an experiment where we scatter degrees of freedom in that length scale, thus the inevitability of black-hole formation tells us that no length scale $L \ll L_P$ can be probed, in principle. The attempt to probe the short-distance behavior of the theory will inevitably lead to the formation of a much larger object of length $r = L_P^2/L \gg L_P$.

While perturbative reasoning tells us that $2 \to 2$ trans-planckian scattering violates unitarity, we know for a fact that in this regime we are bound by black-hole formation. Furthermore, black-holes are not good asymptotic states of the theory, since they will eventually decay by hawking radiation. In order to paint a consistent picture of what is the end-product of the scattering experiment, we need to introduce black-holes in the spectrum of the theory as intermediate states.

5.4 BLACK-HOLES AS S-MATRICES

From the previous reasoning, we know for a fact that the full non-perturbative amplitude for a transplanckian $2 \to 2$ process cannot be a growing function of s, as predicted by perturbation theory. As a matter of fact, the higher is the center of mass energy, the more certain we are that there will be black-hole formation, and the larger the black-hole will be. The $s \to \infty$ limit of the $2 \to 2$ amplitude should be in fact 0.

In reality, black-holes are not stable: they eventually decay by hawking radiation. For a black-hole of mass $M \gg M_P$, the typical wavelength of the hawking quanta will be given by its Schwarschild radius $\lambda \approx M/M_P^2 \gg L_P$. If this black-hole was created in a trans-planckian scattering process with $\sqrt{s} \approx M$, what we observe is that the true end-products of the scattering are not high-energy modes, they are rather in the deep infrared.

The dominant process in a trans-planckian scattering experiment should be in fact

$$2 \to BH \to \text{Many IR quanta} \quad . \tag{87}$$

where black-holes act as intermediary states in the process of energy flow from the UV down to the IR of the theory.

As long as we have $r_S \ll L_P$, we know that the decay of the black-hole into a 2-particle state should be exponentially suppressed, so we can estimate that the true $2 \to 2$ amplitude in a trans-planckian scattering should be

$$A_{2\to 2} \approx e^{-(r_S/L_P)^2} \tag{88}$$

The exponential factor can be understood in the following way: $(r_S/L_P)^2$ is nothing more than the entropy S of the black-hole, and since a two particle state has very little degeneracy in the phase space, creating a two particle state means choosing 1 out of e^S microstates, thus creating the exponential suppression.

Furthermore, since the radiation is mostly thermal with a temperature at the order of r_S^-1, we know that the probability that we are going to find trans-planckian modes in Hawking radiation is exponentially suppressed. This UV-screening device is what allows us to conjecture that gravity is in fact self-complete, regardless of any Wilsonian UV-completion. Whichever are the high energy behavior of the theory, it cannot be probed since processes with high momentum transfer are blocked.

It seems that gravity has all in itself the power to both unitarize scattering amplitudes and to flow energy down from UV modes into the IR. It is natural that we ask if this is unique to gravity or whether we can construct theories that share the same features. In the next section we will discuss about what are the fundamental ingredients that make self-unitarization possible.

5.5 THE r_* PHENOMENA

As we discussed, gravity has a built in mechanism that allows high-energy scattering to be dominated by infrared processes. This happens due to the fact that the length scale at which nonlinearities become important actually grows with the total energy we source the gravitational field with. Because large energies are always followed by large field values, we have both a breakdown of perturbation theory and a flow of energy from the UV to the IR.

The important feature here is that the effective gravitational charge which sources the growth of the classical field is in fact energy. In a theory where the charge is fixed as an external parameter, the growth of the classical field remains the same regardless of the energy we put into a scattering process.

Any theory that has built in this mechanism should in fact be sourced by the energy. Nonlinearities in the theory then should be derivatively coupled. Intuitively, it is easy to see why: asymptotically, fields that obey Gauss law and are weakly coupled obey the $1/r$ law. The effect of non-linearities on these solutions is to increase the power of r, which will increase the length scale at which they will also become dominant. This is consistent with the fact that the first nonlinear term in the perturbative expansion of GR is in fact

$$S_1 = \frac{1}{M_P}\int h_{\mu\nu}T_0^{\mu\nu} \tag{89}$$

where T_0 is the energy momentum tensor of the quadratic term in the expansion, schematically $\partial h\,\partial h$.

Let us take a similar but much simpler theory, in which we couple a real scalar field to its own energy momentum tensor and an external source T.

$$L = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{M_*}\phi\left(\partial_\mu \phi\right)^2 + \frac{1}{M_*}\phi T \tag{90}$$

A static spherically symmetric external source is going to produce a potential which for $r \to \infty$ behaves as

$$\phi_0(r) \approx \frac{E}{M_*}\frac{1}{r} \tag{91}$$

where E is the energy of the source.

Analogously to what happens in gravity, in this theory the nonlinearities will become dominant at a scale given by

$$r_* = \frac{E}{M_*^2} \ , \tag{92}$$

so the stronger the source, the larger is the distance at which the nonlinearities become dominant. Analogously, should we prepare a scattering experiment in this theory, by the time the two particles reach a distance r_* the classical field will be dominant and we will leave the perturbative regime.

This phenomenon, the r_* effect, plays a major role in infrared modifications of gravity (see, e.g., [32]). As a matter of fact, it is a fundamental property of theories which are not Wilsonian, since the attempt to UV-Complete this theory by weakly coupled physics will inevitably spoil the r_* effect.

Whenever we enter the r_* region, we should abandon perturbation theory altogether [34]. At this point, the background over which we define the fock states - the ϕ quanta - has changed altogether, and we are in the presence of a large non-perturbative object that needs to be taken into account when calculating physical observables.

Although black-holes are unique to gravity, the presence of nontrivial scales which increase with energy is universal. Any scalar theory coupled in the appropriate way can exhibit this phenomenon. It's instructive to ask what that implies from the point of view of UV-completion, given the special role that black-holes play in the UV sector of gravity.

5.6 UV-COMPLETION BY CLASSICALIZATION

As we have seen, in derivatively coupled theories trans-cutoff scattering is followed by the creation of large classical field configurations. In the case of gravity, we know that in high energy scattering we have no other choice but to form a black-hole. The important feature of the black-hole is that it absorbs all the possible momentum transfer which would in principle probe the trans-planckian regime of gravity.

We can in fact find analogues of black-holes in other bosonic theories: large field configurations whose size grows with their total mass. We call these field configurations *classicalons*. In the next chapter we will discuss in detail a few examples of classicalon field configurations.

We can formulate the idea of classicalization in the following way: a theory is classcalizing if it UV-completes itself without the necessity to add new weakly coupled degrees of freedom beyond the cutoff scale. In classicalizing theories, the role of UV-completion is taken up by collective excitations of the original quanta. This many-body states is composed of soft constituents, in the sense that their wave-length will increase with the total energy of the configuration.

Suppose we are interested in resolving the structure of these classical field configurations in terms of the original quanta, and let us take a theory in which we have the following $2 \to 2$ coupling constant

$$\alpha = \left(\frac{L_*}{\lambda}\right)^n \tag{93}$$

where λ is the wave-length of the particles involed in the process.

It is easy to see that we will have the following scaling for radius of a classicalon of mass M.

$$r_* = L_* \left(L_* M\right)^{\frac{1}{n-1}} \tag{94}$$

For gravity we have $\alpha = (L_P/\lambda)^2$, and thus we have $n = 2$ and we recover the Schwarschild radius.

If this is a bound state of many quanta, their typical wave-length will scale as r_*, thus the total number of quanta should obey

$$M = \frac{N}{r_*} \tag{95}$$

and thus from (93) and (94), we have

$$\alpha = \frac{1}{N} \tag{96}$$

As we will see further, the fact that we have this scaling is no accident, since it is precisely the condition that needs to be satisfied for collective effects to become dominant. At this point, systems tend to undergo quantum phase transitions, and the ground state changes dramatically.

We are still left with the task of verifying whether explicit classicalon solutions exist in these theories and constructing them explicitly. It turns out that not every theory admits classicalons, and the criteria reveals something very fundamental about which road the theory chooses for UV-completion: Wilsonian or classicalization.

5.7 SUMMARY

In this chapter we reviewed important scales, first in gravity and then in general derivatively coupled theories. We have seen that in Wilsonian theories, we always have the chance of probing smaller distances through high-energy scattering; in gravity, the existence of black-holes introduces another scale in the problem - the Schwarschild radius - which transforms high energies into large distances.

We reviewed the idea that gravity has the possibility to unitarize itself through black-hole creation, and that black-holes act as intermediate states, turning high-energy initial states into low energy hawking radiation.

Extending this concept to general bosonic theories, we have revisted the r_* phenomena, in which derivatively coupled theories also display this peculiar UV-IR mixing of gravity. We have also reviewed the idea of classicalization, the hypothesis that not only gravity, but also bosonic derivatively coupled theories are able to unitarize themselves through creation of classical field configurations.

6

A QUESTION OF SIGNS

In this chapter, we attempt to answer the following question: what can the sign of an interaction tell us about the high energy behaviour of the theory?

As we discussed previously, coupling constants of irrelevant operators in effective field theories are simply parametrizing our ignorance about the degrees of freedom that mediate these interactions at high energies. When we perform experiments at high enough energies to seemingly violate unitarity, we will, according to the Wilsonian paradigm, probe instead the hidden structure of the interaction.

Although every derivatively coupled theory exhibit the r_* effect, it is often the case that there is a weakly coupled UV-Completion sets in before it is possible to observe the collective phenomena that give rise to classicalization.

We would like to argue that for a class of theories, there is no possible weakly coupled UV-completion and the theory has no choice but to classicalize. Furthermore, precisely for these theories we are able to find continuous classicalon solutions. Conversely, whenever there is the possibility of a weakly coupled UV-completion, we are not able to find these static solutions in the theory. We will closely follow the derivations and present the results we obtained in [24].

6.1 STATIC SOLUTIONS

We start with the theory which we will refer from now on as the Goldstone Lagrangian. It describes a single self-coupled shift-symmetric scalar field

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \epsilon\,\frac{L_*^4}{4}\left(\partial_\mu \phi\right)^4 , \qquad (97)$$

where the parameter ϵ can take the value ± 1. Our goal is to study how static spherically-symmetric field configurations in the above system depend on the choice of the sign. The solutions of this system satisfy the equation of motion

$$\partial^\mu \left\{ \partial_\mu \phi + \epsilon L_*^4 \partial_\mu \phi \left(\partial_\nu \phi\right)^2 \right\} = 0\,. \tag{98}$$

Since this is a total divergence, we can reduce the problem in the static spherically symmetric case to an algebraic equation on $\partial_r \phi$

$$\partial_r \phi \left(1 - \epsilon L_*^4 \left(\partial_r \phi\right)^2\right) = \frac{M L_*}{r^2} \tag{99}$$

where we have introduced ML_* as an integration constant which measures the total energy in the field configuration.

The solution will have two distinct regimes. For $r \to \infty$, the contribution from the nonlinearities are irrelevant and we have the linear solution

$$\phi(r \to \infty) \sim \frac{r_*^2}{L_*^2} \frac{1}{r} \tag{100}$$

while for $r \to 0$, we have

$$\phi(r \to 0) \sim \frac{r_*}{L_*^2} \left(\frac{r}{r_*}\right)^{1/3} \tag{101}$$

and r_* is defined as the scale at which the two solutions become comparable,

$$r_* = \left(M L_*\right)^{1/2} L_*\,. \tag{102}$$

Although we have the same asymptotics for both signs, a real continuous solution that interpolates the linear and non-linear regime only exists for $\epsilon = -1$. With positive ϵ we have an extra real solution in which $\partial_r \phi \to L_*^{-2}$ as $r \to \infty$. This solution has diverging energy and is thus unphysical.

This can be easily seen by focusing on the sign on $\partial_r \phi$: when $r \to \infty$, $\partial_r \phi$ is positive, whereas for $r \to 0$, $\partial_r \phi$ takes the opposite sign of ϵ. Since $\partial_r \phi = 0$ is never a solution of the equation, it can never switch sign, thus ϵ must be negative.

The physical solution for negative ϵ has the following expression

$$\partial_r\phi = \frac{-\frac{2}{3} + \left(\sqrt{2}\, G\left(\frac{r_*}{r}\right)\right)^{2/3}}{\left(4\, G\left(\frac{r_*}{r}\right)\right)^{1/3}} \tag{103}$$

$$G(x) = x^2 + \sqrt{x^4 + \frac{4}{27}} \tag{104}$$

and it is clear that it interpolates between the two regimes (100) and (101).

6.2 THE ROLE OF HIGHER ORDER OPERATORS

It may be seen as artificial to stop at quartic order. In the spirit of effective field theory, one could be interested in investigating the behaviour of the theory once we supplement it with higher order operators, since inside the r_* scale, higher order operators are dominant.

Suppose that we take $\epsilon = -1$ and add a $(\partial\phi)^{2n}$ vertex suppressed by some scale Λ that may or may not coincide with L_*^{-1},

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{L_*^4}{4}\left(\partial_\mu\phi\right)^4 + \frac{\epsilon_n}{\Lambda^{4n-4}}\left(\partial_\mu\phi\right)^{2n} . \tag{105}$$

Even if it is still possible, for $\epsilon_n > 0$, to unitarize the $(\partial\phi)^{2n}$ vertex above the scale Λ by integrating-in some weakly coupled physics, this will only tame the growth of the scattering amplitude for processes with $2n$ or more external legs. Its contribution to the $2 \to 2$ scattering amplitude will be suppressed, since it will be on high loop order.

This operator – and whatever physics it is embedded into – cannot help with the unitarization of the $2 \to 2$ amplitude coming from the quartic term. Thus, regardless of higher order operators, for $\epsilon = -1$, the theory is not expected to have any weakly-coupled Wilsonian UV-completion and the only chance is to self-complete by classicalization. [1]

1 Generalizing the previous analysis, in case $\epsilon_n = (-1)^n$, then we can be sure that a classicalon solution exists regardless of whether or not this higher order vertex is unitarized by weakly coupled physics. In case the theory choses to embed this term into Wilsonian physics, then as long as the scale m of this new weakly coupled physics is much lower than the unitarity-violating scale, $m \ll \Lambda$, the contribution from this vertex will always be subdominant with respect to the classicalon background, and we still can expect the theory to classicalize.

6.3 THE CASE OF THE LINEAR SIGMA MODEL

In order to understand the connection between the sign of ϵ and the nature of UV-completion, it is instructive to complete the theory by embedding it into a weakly-coupled linear sigma model,

$$\mathcal{L} = |\partial_\mu \Phi|^2 - \frac{\lambda}{8}(2|\Phi|^2 - v^2)^2 , \tag{106}$$

where Φ is a complex scalar field carrying two real degrees of freedom, and the Lagrangian is invariant under the global $U(1)$ transformation

$$\Phi \to e^{i\theta}\Phi . \tag{107}$$

In the ground state of the theory, the symmetry (107) is spontaneously broken by the vacuum expectation value of Φ, $\langle \Phi^* \Phi \rangle = \frac{1}{2}v^2$, so we can parametrize the degrees of freedom as

$$\Phi = \frac{1}{\sqrt{2}}(v + \rho)e^{i\phi/v} ,$$

in terms of a radial (Higgs) mode ρ and a Goldstone boson ϕ, which under spontaneously broken $U(1)$ transforms as,

$$\frac{\phi}{v} \to \frac{\phi}{v} + \theta . \tag{108}$$

In terms of the fields ρ and ϕ the Lagrangian becomes,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}\left(1 + \frac{\rho}{v}\right)^2 (\partial_\mu \phi)^2 - \frac{\lambda^2}{8}(\rho^2 + 2\rho v)^2 , \tag{109}$$

from where ρ acquires a mass of $m = \lambda v$.

For energies $E \ll \lambda v$, we can integrate out ρ and write down an effective low energy theory for ϕ. Integrating out ρ through its linear order equation,

$$\left\{ \Box + m^2 \right\} \rho = \frac{1}{v}(\partial_\mu \phi)^2 , \tag{110}$$

we get the following leading order low energy effective equation for ϕ,

$$\partial^\mu \left\{ \partial_\mu \phi \left(1 + L_*^4 \frac{m^2}{\Box + m^2} (\partial_\nu \phi)^2 \right) \right\} = 0 \tag{111}$$

where we have defined a cutoff length $L_*^2 = \sqrt{2}(mv)^{-1}$. In the low energy limit, $m^2 \gg \Box$, this recovers (98) with $\epsilon = 1$.

The origin of the positive sign is now clear. Notice that the only way to obtain a theory with $\epsilon = 1$ would be to flip the sign of m^2, making it tachyonic and thus unstable. It is clear that the only sensible weakly coupled sigma model whose effective low energy theory admits classicalon solutions.

Furthermore, it can be checked explicitly [26] that the r_* radius of static classical sources collapses when the Goldstone theory is embedded in the linear sigma model. This effect is a particular manifestation of a very general *de-classicalization* phenomenon by weakly-coupled UV-completing physics [28] that we shall discuss in more details below. The above discussion shows why classicalizing theory cannot be obtained as a low energy limit of a weakly-coupled UV-completion.

6.4 DBI

Another example that illustrates incompatibility between the classicalons and UV-completion by a weakly-coupled theory, is provided by the embedding of the Goldstone model (97) into a Dirac-Born-Infeld (DBI) theory. The Lagrangian (97) can be thought as an expansion of DBI type Langangian

$$\mathcal{L}_{DBI} = \epsilon_1 L_*^{-4} \sqrt{1 + \epsilon_2 \, L_*^4 \, \left(\partial_\mu \phi\right)^2}, \tag{112}$$

where parameters $\epsilon_{1,2}$ can take values ± 1. For the values $\epsilon_1 = \epsilon_2 = 1$ this theory admits a classicalon solution [26],

$$\partial_r \phi = \frac{r_*^2}{L_*^2} \frac{1}{\sqrt{r^4 + r_*^4}}, \tag{113}$$

where r_* is an integration constant. On the other hand, the sensible embedding into a weakly-coupled theory is only possible for $\epsilon_1 = \epsilon_2 = -1$. To see this, note that the action (112) can be viewed as an effective low energy action (Nambu-type action) describing the embedding of a 3-brane (domain wall) in a five-dimensional space-time, with ϕ being a Nambu-Goldstone mode of spontaneously broken translational invariance. Expanding this action in powers of ϕ we get the action very similar to (97),

$$\mathcal{L} = \epsilon_1 L_*^{-4} + \epsilon_1 \epsilon_2 \frac{1}{2} \left(\partial_\mu \phi\right)^2 - \epsilon_1 \epsilon_2^2 \frac{L_*^4}{4} \left(\partial_\mu \phi\right)^4 + \dots . \tag{114}$$

The first term represents the brane tension with negative sign and this fixes $\epsilon_1 = -1$. The positivity of the kinetic term fixes $\epsilon_2 = -1$. As a result the sign of the next term is fixed to be the one that does not admit classicalon solutions. Having a classicalizing theory requires $\epsilon_1 = +1$, which would give a wrong sign brane tension. We thus see that, just like in the Higgs case, the weakly-coupled UV-completion is possible for the sign that does not admit the classicalon solutions and vice-versa.

6.5 EVIDENCE FROM SPECTRAL REPRESENTATION

The impossibility of sensible weakly-coupled UV-completion for $\epsilon = -1$ can be seen from the following general argument. Suppose that there is such a UV-completion, that would imply that the 4-derivative vertex is an effective is obtained by integrating out some heavier weakly coupled degrees of freedom that couples to ϕ in the following form

$$\partial_\mu \phi \partial_\nu \phi \, J^{\mu\nu} \,, \tag{115}$$

where $J_{\mu\nu}$ is some effective (in general composite) operator that encodes information about the given UV-completing physics. From the symmetry properties it is clear that the current $J_{\mu\nu}$ can transform either as spin-2 or spin-0 under the Poincare group. The effective four-derivative vertex of ϕ is then result of a non-trivial $\langle J_{\mu\nu} J_{\alpha\beta} \rangle$ correlator. The positivity of ϵ follows from the positivity of the spectral function in the Källen-Lehmann spectral representation of this correlator. The most general ghost and tachyon-free spectral representation of this current-current correlator is (see, e.g., [39]),

$$\begin{aligned} \langle J_{\mu\nu} \, J_{\alpha\beta} \rangle \;\; &= \;\; \int_0^\infty dm^2 \, \rho_2(m^2) \frac{\frac{1}{2}(\tilde{\eta}_{\mu\alpha}\tilde{\eta}_{\nu\beta} + \tilde{\eta}_{\mu\beta}\tilde{\eta}_{\nu\alpha}) - \frac{1}{3}\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\alpha\beta}}{\Box + m^2} + \\ &+ \;\; \int_0^\infty dm^2 \, \rho_0(m^2) \frac{\eta_{\mu\nu}\eta_{\alpha\beta}}{\Box + m^2} \,, \end{aligned} \tag{116}$$

where $\tilde{\eta}_{\mu\nu} = \eta_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m^2}$ and $\rho_2(m)$ and $\rho_0(m)$ are the spectral functions corresponding to massive spin-2 and spin-0 poles respectively. The crucial point is that the absence of ghost and tachyonic poles demands that these spectral functions are strictly positive-definite and vanish for $m^2 < 0$ (the latter condition fixes the lower bound of integration). The entire weakly-coupled UV-dynamics is encoded in the detailed form of these spectral functions, which is completely unimportant for us except for the signs.

Convoluting this expression with $\partial_\mu\phi\partial_\nu\phi$, and ignoring high derivatives, it is clear that the coefficient of an effective low energy vertex is strictly positive,

$$\partial_\mu\phi\partial_\nu\phi\langle J^{\mu\nu}J^{\alpha\beta}\rangle\partial_\alpha\phi\partial_\beta\phi \rightarrow (\partial_\mu\phi\partial^\mu\phi)^2\int_0^\infty dm^2\left(\frac{2}{3}\frac{\rho_2(m)}{m^2}+\frac{\rho_0(m)}{m^2}\right). \tag{117}$$

Having a negative sign for the coefficient requires either a ghost or a tachyonic pole. The linear sigma model example considered in the previous section corresponds to a particular choice $\rho_2(m^2) = 0$ and $\rho_0(m^2) \propto \delta(m^2 - (\lambda v)^2)$.

Thus, the negative sign cannot be obtained by integrating out any weakly-coupled Wilsonian physics. However, instead of dismissing such a possibility, we should take this as a message that the theory tells us that we have to abandon the Wilsonian view, and treat the quartic vertex as *fundamental*. The road that the theory chooses in such a case is UV-completion through classicalization.

6.6 classicalization and superluminality

In the Goldstone example, the static classicalon solutions appear precisely for the sign of the interaction for which we have backgrounds with superluminal propagation [30]. In the presence of a weakly coupled UV-completion, that would allow for closed timelike curves, thus ruining the causality of the theory. Since no weakly coupled UV-completion exists in the first place, the theory is still safe. If the UV-completion happens by classicalization, superluminality need not imply violation of causality.

To explain the reason, let us first reproduce the argument why superluminality appears and why this may lead to a problem. Consider the goldstone Lagrangian (97). In this theory one can consider an extended field configuration that locally has a form $\phi_{cl} = c_\alpha x^\alpha$, where c_α are constants that are chosen to be sufficiently small, so that the invariant is well-below the cutoff scale, $\left(\partial_\mu\phi\right)^2 = c_\alpha c^\alpha \ll L_*^{-4}$. On such a background the linearized perturbations $\phi = \phi_{cl} + \delta\phi$ sees an effective Lorentz-violating metric,

$$\eta_{\mu\nu} + \epsilon L_*^4 c_\mu c_\nu + \dots, \tag{118}$$

which gives a superluminal dispersion relation for $\epsilon < 0$.

In order for this superluminal dispersion relation to become an inconsistency, one should be able to create closed time-like curves and to send signals into the past. Such a situation can be arranged by a set of highly boosted observers. However, here we reach a subtle point. In order to send a signal into the past at least some of the observers must be boosted relative to the background with trans-cutoff center of mass energies.

Such a boost relative to a background is *not* a symmetry transformation and is physical. So to rely on such a thought experiment we have to be sure that the interaction between an observer and the background allows for such a boost. Here comes the issue of the UV-completion. Since the center of mass energies are trans-cutoff, the legitimacy of the boost depends on the UV-completion.

If the UV-completion is by weakly-coupled physics (which is an implicit assumption of ref[30]), then boosts are allowed, since for such UV-completions the cross-sections diminish at high energies, and a background is not an obstacle for the boost. However, as we have seen, for the superluminal sign the Wilsonian UV-completion is absent anyway. Instead, the theory allows classicalons, which is an indication that the theory chooses the classicalization path for UV-completion.

In this case, the trans-cutoff boosts are a problem, since the cross-section increases with energy and any attempt to boost an observer relative to the background with trans-cutoff energy per particle should result into creation of many soft quanta that will cutoff the boost. In this way, the system is expected to self-protect against creation of closed time-like curves and violation of causality.

6.7 CLASSICALIZATION AND THE a-THEOREM

The a-theorem is the four dimensional generalization of Zamolodchikov's c-theorem in two dimensions [36]. What the theorem establishes is that for a RG flow between two CFT fixed points, at the UV and the IR respectively, Cardy's a-function [35] fulfills the strong inequality $a(UV) > a(IR)$.

Since along the flow the theory is not conformal, the t'Hooft's anomaly matching conditions should be appropriately improved. This has been recently done in [33] based on previous results in [37]. The key idea is to interpret the theory along the flow as a spontaneously broken CFT with the dilaton as the corresponding Nambu-Goldstone boson. More precisely, given the UV CFT we add a relevant deformation to induce the flow as well as the coupling to the dilaton field. This is done in order to restore the conformal invariance leading to a total $T^\mu_\mu = 0$.

The vacuum expectation value (VEV) f setting the breaking of the conformal symmetry defines the decay constant of the dilaton field. Along the flow some massless UV degrees of freedom will become massive. The IR fixed point is obtained after integrating these out. Thus, the final theory in the IR contains in addition to the IR CFT the low energy effective theory for the dilaton. This effective theory is

$$\mathcal{L} = (\partial\phi)^2 + 2a\,\frac{1}{f^4}\,(\partial\phi)^4\,, \tag{119}$$

with a satisfying the anomaly matching condition: $a(UV) - a(IR) = a$. Thus, the a-theorem follows from the sign of the derivative coupling of the effective low energy theory for the dilaton.

The connection with classicalization is now pretty clear. In fact the effective low energy theory for the dilaton is, as we have discussed in the previous sections, of the type of theories that, *depending on the sign of the derivative coupling*, can be self-completed in the UV by classicalization.

In order to understand the meaning of the a-theorem let us focus on the effective Lagrangian (119). By itself this theory has a unitarity bound at energies of order f. This is obvious from the scattering amplitude that scales like

$$A(s) \sim \frac{2as^2}{f^4}\,. \tag{120}$$

In order to make sense of this theory we need to complete it at energies $E > f$. In the previous setup it is obvious how the effective theory of the dilaton is UV completed. Namely, the completion takes place by the UV degrees of freedom of the UV CFT fixed point we have started with. In other words, the effective theory (119) is, by construction, completed in the UV $E > f$ in a Wilsonian sense. The interesting thing is that this Wilsonian completion determines the sign of a to be positive and therefore the proof of the a-theorem. This result directly follows from our previous discussion in the sense that for a negative sign of a the theory cannot be completed in Wilsonian sense.

In other words, the sign of the derivative coupling – and therefore the a-theorem – depends on how the theory tames the growth of the amplitude (120), i.e., on how the low energy effective theory of the dilaton unitarizes. A Wilsonian unitarization forces this sign to be positive. Therefore, once we embed the dilaton dynamics in a flow with a well-defined UV CFT fixed point, the sign is forced to be positive, leading to the a-theorem. We cannot reach this conclusion directly from the Lagrangian (119). In fact, general arguments based on dispersion relations for the dilaton scattering amplitude necessarily hide the key assumption on how the growth of the amplitude at high energies has been tamed.

Classicalization tells us however what is the physics when the sign is negative. In this case the theory unitarizes at high energies by classicalization. This means that the scale f setting the unitarity bound becomes the limit on length-resolution in the sense that the theory at higher UV energies turns into a theory that probes IR scales. In particular, we can suggest the following conjecture. Let us start with a CFT in the IR and let us add an *irrelevant* operator and a coupling to a dilaton in order to keep the conformal invariance. If the effective theory of this dilaton has negative sign for the derivative self-coupling, the theory will classicalize in the UV. It's important to notice that there is a subtlety in the derivation given in [33], which relies on the analytical structure of the $2 \to 2$ scattering amplitude in the forward limit. In massless theories, one has branch cuts all over the real line, which makes it impossible to write a dispersion relation by closing the integration contour, as done in [30], in the case where the theory is already supplemented with a mass gap. In order to attempt to make this work in a massless theory, one needs to regulate the theory in the infrared by introducing a small mass, which is not a continuous change in the theory, as noted in [34] in the context of gravity.

6.8 CLASSICALIZATION AND THE STANDARD MODEL

Our observations have important implication for determining the UV-completion of the Standard Model. If the scattering of the longitudinal W-s is unitarized by Higgs or any other weakly coupled Wilsonian physics, the sign of the four-derivative self-coupling must be positive. In the opposite case no weakly-coupled Wilsonian UV-completion is possible, but the theory makes up due to the existence of classicalons, indicating that unitarization happens through classicalization. This is remarkable, since measurement of the sign of the longitudinal WW-

scattering amplitude can give a decisive information about the UV-completion of the theory.

For completeness, let us repeat our arguments for the non-abelian case. Since we are concerned with the scattering of the longitudinal W-s, which are equivalent to Goldstone bosons, we shall work in the gaugeless limit.

The Standard Model Lagrangian (with Higgs) then reduces to a Nambu-Goldstone model with spontaneously broken $SU(2)$ global symmetry,

$$\partial_\mu H^{a*}\partial^\mu H_a - \frac{\lambda^2}{2}\left(H^{a*}H_a - \frac{v^2}{2}\right)^2, \tag{121}$$

where H_a $(a = 1, 2)$ is an $SU(2)$-doublet scalar field.

Following [26], we shall now represent the doublet field in terms of the radial (Higgs) and Goldstone degrees of freedom,

$$H_a = U_a(x)\,\rho(x)/\sqrt{2} = (cos\theta e^{i\alpha},\ -sin\theta e^{-i\beta})\rho/\sqrt{2} \tag{122}$$

where θ, α, and β are the three Goldstone fields of the spontaneously broken global $SU(2)$ group.

In this parameterization the Higgs Lagrangian becomes,

$$\frac{1}{2}(\partial_\mu\rho)^2 + \frac{\rho^2}{2}(\partial_\mu U^\dagger\partial_\mu U) - \frac{\lambda^2}{8}(\rho^2 - v^2)^2. \tag{123}$$

where

$$\partial^\mu U(x)^\dagger\partial_\mu U(x) = \left[(\partial_\mu\theta)^2 + \cos^2\theta(\partial_\mu\alpha)^2 + \sin^2\theta(\partial_\mu\beta)^2\right]. \tag{124}$$

Integrating out the Higgs through its equation of motion, which at low energies becomes an algebraic constraint

$$\rho^2 = v^2 + \frac{2}{\lambda^2}(\partial_\mu U^\dagger\partial_\mu U), \tag{125}$$

and rescaling, $U \to vU$, we obtain the following effective theory

$$\frac{1}{2}(\partial_\mu U^\dagger\partial^\mu U) + \frac{1}{2\lambda^2 v^4}(\partial_\mu U^\dagger\partial^\mu U)^2. \tag{126}$$

This is similar to (97) with $\epsilon = 1$ and $L_*^4 = 2/(\lambda^2 v^4)$. The gauge case can be trivially restored by replacing ∂_μ with the covariant derivatives of the $SU(2) \times U(1)$ group. The positive sign of the four-derivative term indicates that the theory can be UV-completed by the Higgs particle. On the other hand, for the negative sign no such completion is possible, and the theory chooses the classicalization route. Thus, by detecting the sign of this operator at low energies we obtain the information about which route the theory chooses for its UV-completion. This sign can in principle be read-off from measuring the sign of the amplitude of longitudinal WW-scattering.

6.9 SUMMARY

In this chapter, we have addressed the issue of how a derivatively coupled theory chooses the way to UV-completion, and how the low energy effective theory signals which road it will follow.

Using the e Källen-Lehmann spectral representation, we have shown that the sign of the quartic coupling tells us whether it is possible or not, in principle, to embed this theory in a weakly coupled theory with extra degrees of freedom. While for the positive sign there are sensible Wilsonian UV-completions, for the negative sign that is forbidden. Furthermore, classicalon solutions only exist for theories with the negative sign for the coupling constant.

Our message is that for theories with the negatie sign, this vertex should be treated as fundamental, rather than an effective low-energy description of a more fundamental interaction. Since we have classicalon solutions, the theory will unitarize itself by classicalization and thus can be deemed as fundamental.

We can apply this reasoning to the standard model and make the following prediction: should the sign of the quartic longitudinal WW scattering amplitude be negative, this theory does not admit a weakly coupled UV-completion. Even though a Higgs-like resonance has been already discovered, measuring the sign will provide a powerful cross-check that we are in fact dealing with the Higgs boson.

Part III

BLACK-HOLE INFORMATION AND QUANTUM CRITICALITY

7

THE PROBLEM WITH BLACK-HOLES

This chapter covers the history of Black-Hole thermodynamics. For a comprehensive treatment of the subject we refer to the original works of Hawking [47] and Bekenstein [49].

As we have seen in the previous chapter, black-holes play a crucial role in gravity. They provide a connection between high energy and long distance physics, being the intermediate states between transplanckian scattering and soft Hawking radiation.

What is often referred to as black-holes are in fact a set of classical solutions of Einstein's equations. These solutions are universal and only depend on the choice of 3 parameters, mass M, angular momentum J and electric charge Q. This universality of black-holes is referred to as the no-hair *theorem*: these are the only measurable charges of black-holes, and they have no more hair attached. We emphasize that the word theorem is a stretch, since this is a property of classical black-holes, and it is well known that black-holes can indeed carry quantum mechanical hair [62].

In this chapter we will briefly recap the problems concerning the classical description of black-holes. The essence of these problems is the incompatibility of the *hairlessness* of black-hole with the unitary evolution of quantum mechanics. It is not our goal to formally rederive 40 years of results in black-hole physics, but rather to provide an intuitive path so that readers can understand the motivation for our work in the next chapter.

We will then argue that the only way to address these issues is by abandoning the semi-classical treatment of black-holes and quantum mechanically resolve the metric itself. By doing so, we need to create an ansatz about how to describe the black-hole state in terms of the degrees of freedom we have available. We will see that this is a very powerful ansatz, which opens the door to many possibilities to explore the quantum properties of black-holes.

7.1 A BRIEF HISTORY OF BH THERMODYNAMICS

Bekenstein and Hawking discovered in the early 1970s that black-holes have thermodynamic properties. Their discoveries can be summarized in two formulas,

$$kT = \frac{\hbar\kappa}{2\pi}, \qquad S = \frac{A}{4\hbar G} \tag{127}$$

which relate classical properties of black-holes: area an surface gravity, with intrinsically quantum mechanic ones: temperature and entropy. The presence of $\hbar$ in the equations make manifest the fact that this is indeed a quantum relation, and cannot be explained in a purely classical setting.

The historical derivation of these quantities traces back to Hawking's [47] proof that the area of a black-hole can never decrease, which is a very similar to the second law of thermodynamic. This observation went hand in hand with Bekenstein's gedanken experiment:

Suppose that indeed black-holes carry no hair, and let us throw inside the black-hole an object which carries entropy. Once this object crosses the boundary of the black-hole, we have effectively decreased entropy from the black-hole exterior. If the black-holes do not increase its entropy by eating this object, we have violated the second law of thermodynamics [49].

The correspondence between area and entropy was strengthened by the discovery of other analog laws of quantum mechanics. This culminated in [48] publishing the "4 laws of Black Hole mechanics" for a black-hole characterized by mass M, charge Q and angular momentum J:

0. The surface gravity κ is constant over the horizon.

1. Small variations of M, J, and Q obey

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega\delta J + \Phi\delta Q \tag{128}$$

   where Ω is the angular velocity and Φ is the electric potential at the horizon.

2. The area of the black-hole cannot decrase

$$\delta A \geq 0\ . \tag{129}$$

3. The surface gravity κ cannot be reach 0 through any physical process.

These laws are analogous to the laws of thermodynamics, with κ playing the role of temperature and A playing the role of entropy. Note that for Schwarschild black-holes, we have $\kappa = 1/2r_s$, meaning that large black-holes have small surface gravity.

We should take this with a word of warning, however: if we shine a laser into a room full of gas, the temperature will increase; also, if the gas in the room radiates, its temperature will decrease. This is a very simple observation that can be summarized as: the heat capacity is positive. Temperature reacts positively to heat flow.

In a black-hole, the surface gravity decreases as the mass increase: the black-hole temperature reacts negatively to the absorption of heat. Analogously, a radiating black-hole will increase its temperature. If the black-hole is to be viewed as an object in thermal equilibrium, a negative heat capacity would render it thermodynamically unstable.

The more physical interpretation of this temperature κ was given by Hawking [46], who discovered that black-holes radiate. This radiation, for large black-holes, has a thermal spectrum with a temperature given by (127), and its density matrix is that of a mixed state.

This derivation is based on the fact that the vacuum of a past observer watching the formation of a black-hole differs from the vacuum of the future observer who already sees the black-hole. Relating these vacua through a Bogoliubov transformation, it can be seen that the future vacuum is actually populated with radiation. It is important to notice that the spectrum of Hawking radiation is obtained by assuming that the radiation is a vacuum phenomenon, in the sense that it only relies in the Schwarschild geometry.

Regardless of the thermodynamical interpretation, it is clear that black-holes should carry some form of entropy from a statistical point of view: the fact that black-holes only carry 3 conserved charges means that for an outside observer, there is a panoply of micro-states which can conspire to form the same black-hole. In other words, a classical observation cannot distinguish between a black-hole that was formed by a particle collision or another formed by the collapse of a star, as long as they carry the same conserved charges.

The fact that black-holes carry a statistical entropy means that there are in fact e^S micro-states which remain unresolved in the classical description. This is a huge contrast with the no-hair theorem, since it automatically tells us that a resolution of the black-hole entropy can only be attained quantum mechanically.

7.2 THE INFORMATION PROBLEM

Let us now assume that black-holes carry indeed no hair and that Hawking radiation is truly thermal. If these assumptions hold, we can create a black-hole from a pure initial state with a sufficiently large energy, and its resulting black-hole will be described by one of the classical geometries. Once the black-hole evaporates, all that is left will be Hawking radiation, which is in a mixed state. So the black-holes have effectively converted a pure state into a mixed state, thus violating the unitary evolution of quantum mechanics.

Since the evaporation only depends on the physics at the horizon, we also cannot rely on Planck scale physics to modify the spectrum of the radiation. If the background remains classical, thermality is exact up to $1/M$ corrections. By the time the black-hole is small enough such that Planck scale effects may become dominant, all the radiation has already left the black-hole.

Another way to phrase this problem is the following observation, made by Page [60]: Suppose that the radiation that the black-hole emits is maximally entangled with the black-hole. The entanglement entropy between the black-hole and its exterior will initially grow as a function of time, however it is bounded by the size of the Hilbert space of the smallest system. When the area of the black-hole is halved, the Hilbert spaces of the black-hole and its exterior have the same size, and that is when the entanglement entropy should be maximum. After this time - known as the Page time, entanglement entropy should start decreasing.

This result is naturally contradicting to Hawking's calculation, since nothing lead us to believe that something dramatic happens at that time, if we accept that radiation is a pure vacuum process. The implicit assumptions for Hawking's results are still valid, however something must be dramatically changed.

Many physicists, starting from Hawking [45], have used this result to show that quantum mechanics must break down in the presence of gravity. It is of course obvious that if we ignore the quantum effects of the background itself throughout its whole evaporation, the unitary evolution will be lost. In order to make a statement about the validity of quantum mechanics, it's just natural that one should really take quantum mechanics into account.

8

A QUANTUM RESOLUTION

In order to resolve what are the true quantum microstates of the theory, we need to first redefine what we mean when we talk about a black-hole. Classically, as we said, a black-hole is a specific solution of Einstein's equations in the presence of a localized source. Quantum mechanically, however, we need to define a black-hole state

$$|BH; \{Q_i\}\rangle$$

which lives in the Fock space of our gravitational theory and might depend on a set of charges $\{q_i\}$.

The classical limit is that in which the black-hole is infinitely massive, so that backreaction effects are negligible; and the coupling constant is 0, so that there are no fluctuations around the background. We then keep the Schwarzschild radius constant, so that the black-hole has a finite size

$$\begin{aligned} M &\to \infty \\ L_P^2 &\to 0 \\ ML_P^2 &\to r_s \end{aligned} \tag{130}$$

In this limit, we should recover the classical notion of geometry

$$\langle BH; \{Q_i\}|\hat{g}_{\mu\nu}|BH; \{Q_i\}\rangle \to g_{\mu\nu} \tag{131}$$

where $g_{\mu\nu}$ is the classical metric corresponding to the charges M, Q and J. There are infinitely many states that satisfy this requirement, all of which have differing expectation values of $\hat{g}_{\mu\nu}\hat{g}_{\alpha\beta}$ and higher order correlation functions.

Determining what is the correct state of the theory in the presence of source that will induce black-hole creation is equivalent to solving the infinite hierarchy of Dyson-Schwinger equations. The classical equations of motion are nothing more than the first order truncation of this hierarchy of equations, on which it is implicitly assumed that

$$\langle \hat{g}_{\mu\nu}\hat{g}_{\alpha\beta}\rangle = \langle \hat{g}_{\mu\nu}\rangle\langle \hat{g}_{\alpha\beta}\rangle. \tag{132}$$

This assumption is justified if the state is an exact coherent state, however the buildup of correlations due to particle creation will inevitably be responsible for the growth of the the variance $\langle \hat{g}_{\mu\nu}\hat{g}_{\alpha\beta}\rangle - \langle \hat{g}_{\mu\nu}\rangle\langle \hat{g}_{\alpha\beta}\rangle$, which is related to particle creation.

Let us go back to Page's formulation of the information problem, and let us try to resolve the following puzzle: by the time a large black-hole has evaporated half of its Hilbert space, it is still large, in the sense that an expansion in $1/M$ is still valid; however, since entanglement entropy must start decreasing, something dramatic must happen in order to change the spectrum of Hawking radiation.

As a matter of fact, it is not entirely true that something dramatic must happen. Let us attempt to define the previous problem in a more formal language. We have a state specified at some time t_0. This may be described by a density matrix $\rho_D(t_0)$ in either a pure or mixed state. In the case of a large black-hole, this can be taken to be a coherent state, which is the most classical state we can construct.

Equivalent to the specification of the density matrix is the specification of the time ordered equal time correlators for $t = t_0$: the one-point function

$$\mathrm{Tr}\ \{\rho_D(t_0)\Phi(t_0,x)\}$$

, the two-point function

$$\mathrm{Tr}\ \{\rho_D(t_0)\Phi(t_0,x)\,\Phi(t_0,y)\}\ ,$$

etc. Φ denotes the Heisenberg field operator. For the case of a coherent state, we know that every higher order correlation function can be written in terms of the one-point function, since it is a Gaussian state with 0 variance.

Whenever we have Hawking quanta, we are populating the state with modes that have zero expectation value. In the semi-classical language, we write

$$\begin{aligned}\Phi(t,x) =& \Phi_0(t,x) + \delta\Phi(t,x)\\ \langle\Phi(t,x)\rangle =& \langle\Phi_0(t,x)\rangle\end{aligned} \tag{133}$$

where the expectation value denotes the trace over the density matrix ρ_D. Nevertheless, these quanta have a non-zero contribution to the two-point function of the field operator

$$\langle \Phi(t,x)\Phi(t,y)\rangle - \langle \Phi_0(t,x)\Phi_0(t,y)\rangle = \langle \delta\Phi(t,x)\delta\Phi(t,y)\rangle \qquad (134)$$

The semi-classical approximation not only relies on the fact that the mass of the black-hole is large, but it also requires that the deviations away from coherence are small. In this basis of operators, the deviations away from coherence are precisely given by the magnitude of the Hawking quanta, which is a growing function of time.

By the time the Hilbert spaces are comparable, the interaction of the mean-field $\langle\Phi\rangle$ with the fluctuations $\langle\delta\Phi\delta\Phi\rangle$ become comparable to the self interactions of the mean-field itself. In other words, in each emission process there are corrections which scale as $1/N$, where N is the number of degrees of freedom that make up the black-hole. For these N degrees of freedom, the size of the Hilbert space will scale exponentially with N. When the cumulative effect of these corrections become of order 1, $t \propto N$, we reach Page's time.

In condensed matter, this behaviour is known as a problem of secularity (see e.g., [67]). When performing perturbative expansions using approximation schemes that are not self-consistent, there are terms that arise which will spoil the validity of the approximation at a finite time which typically scale as an inverse function of the coupling constant. In this example, that is exactly what we observe, since we expect N to be the timescale at which the semi-classical method fails completely. In order to make the approximation self-consistent, it is necessary to include the back-reaction effect on the background.

We will not attempt to solve the full problem of time-evolution, since it is challenging both numerically and analytically. One way of incorporating the quantum effects of the background itself is to adopt a reasonable ansatz for the black-hole state and verify that it indeed reproduces all the properties of black-holes we are familiar with. This was the philosophy was adopted by Dvali and Gomez [51], and we will review their results, and see that it is in fact possible to derive all the feature of black-holes from a purely quantum mechanical ansatz.

8.1 THE QUANTUM n PICTURE OF BLACK HOLES

As we have seen, most paradoxes of quantum mechanics arise from treating the black-hole as a purely classical objected on top of which there are quantum fluctuations. In order to depart ourselves from the semi-classical treatment, it is necessary to promote the background to a quantum object, and effectively resolve it in terms of the quantum degrees of freedom of the theory.

We will follow the prescription suggested by Dvali and Gomez [51] and review the essential points in their model.

If we take a bound state of size r_S, we expect that the typical degree of freedom localized in that state should have a wavelength of order r_S as well. For the case of the black-hole, we have that

$$\lambda = r_S = L_P(L_P M) \ ,$$

where M is the total mass of the black-hole. In order to determine the number of degrees of freedom N in the bound state, we observe that $N\lambda^-1 = M$, by which we have

$$N = (ML_P)^2 = \left(\frac{r_S}{L_P}\right)^2 = \left(\frac{\lambda}{L_P}\right)^2 \ . \tag{135}$$

We find that the number of constituents in the black-hole follows the area law, just like the Bekenstein-Hawking entropy. For macroscopic black-holes, N is a very big number, so it is natural to assume that collective quantum effects will be of major importance and we will have the formation of a Bose-Einstein condensate. In order to check how important collective effects are, we can calculate the collective coupling $\alpha_G N$, where α_G is the $2 \to 2$ gravitational coupling, given by

$$\alpha_G = \lambda^{-2} L_P^2 \ . \tag{136}$$

We find that for N given by (135), we have

$$\alpha_G N = 1 \ , \tag{137}$$

which precisely tells us that we sit on a very special point in the coupling space.

In this prescription, Hawking radiation arises naturally as the depletion out of the condensate. In this condensate, the collective binding energy is $\alpha_G N / r_S$, which in this special point becomes r_S^{-1}, which is exactly the kinetic energy of a single degree of freedom. The condensate is then leaky: any excitation is enough to scatter particles out of it. The time-scale of evaporation can be easily computed using combinatoric arguments: there are N^2 possible scattering events, and the scattering rate is α^2. The typical energy of the process is of order r_S^{-1}, so we have

$$\Gamma = N^2 \alpha_G^2 \frac{1}{r_S} = \frac{1}{r_S} \ . \tag{138}$$

which precisely matches the semi-classical calculations.

8.2 THE PHASE TRANSITION IN THE ATTRACTIVE BOSE GAS

Before we continue our discussion about black-holes, it is instructive to introduce the toy model we will consider and derive some of the features that will be fundamental in the future discussions. The model we will now consider is that of a Bose gas with attractive δ interactions, and for simplicity we will consider the system on a $1D$-circle of radius R. The Hamiltonian is given by

$$\hat{H} = \frac{1}{R}\frac{\hbar^2}{2m}\int_0^{2\pi} d\theta \left[-\hat{\psi}^\dagger(\theta)\partial_\theta^2\hat{\psi}(\theta) - \frac{\pi\alpha R}{2}\hat{\psi}^\dagger(\theta)\hat{\psi}^\dagger(\theta)\hat{\psi}(\theta)\hat{\psi}(\theta)\right] \tag{139}$$

where α is a dimensionless coupling constant. We chose the sign of the interaction in such a way that for positive α, we are in the attractive regime. For the sake of simplicity, we will use units in which $m = R = \hbar = 1$.

8.3 MEAN FIELD THEORY

The Gross-Pitaevskii energy functional can be derived using a mean-field approach, we have

$$E_{GP} = \int_0^{2\pi} d\theta \left[|\partial_\theta \Psi(\theta)|^2 - \frac{\alpha}{2}|\Psi(\theta)|^4 \right] \tag{140}$$

We focus on the N-particle sector of the theory, enforcing the constraint $\int d\theta |\Psi(\theta)|^2 = N$ through a Lagrange multiplier. Minimizing the constrained energy functional leads to the time independent Gross-Pitaevskii equation

$$\left[\partial_\theta^2 + \pi\alpha|\Psi_0(\theta)|^2\right]\Psi_0(\theta) = \mu\Psi_0(\theta)\,, \tag{141}$$

where $\mu = dE/dN$ is the chemical potential.

This solutions of the GP equations depend on the value of α. For small values of α the system sits in a homogeneous phase where the wave-function is constant

$$\Psi_0(\theta) = \sqrt{\frac{N}{2\pi}} \qquad \alpha N < 1 \tag{142}$$

whereas for $\alpha N > 1$ the system undergoes a phase transition and we have [59]

$$\Psi_0(\theta) = \sqrt{\frac{NK(m)}{2\pi E(m)}} \mathrm{dn}\left(\frac{E(m)}{\pi}(\theta - \theta_0)|m\right) \qquad \alpha N > 1 \tag{143}$$

where $\mathrm{dn}(u|m)$ is the Jacobi elliptic function and $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind and m is determined by the equation

$$K(m)E(m) = \frac{\pi^2}{4}\alpha N\,. \tag{144}$$

8.4 BOGOLIUBOV TRANSFORMATION

The spectrum of this theory can be studied by performing a mode expansion of the field operators. We have

$$\hat{\psi}(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{a}_k e^{ik\theta}\,, \tag{145}$$

where $\hat{a}_k$ are the annihilation operators of for the k-th mode. This decomposition leads to

$$\hat{H} = \sum_{k=-\infty}^{\infty} k^2 \hat{a}_k^\dagger \hat{a}_k - \frac{\alpha}{4} \sum_{k,l,m=-\infty}^{\infty} \hat{a}_k^\dagger \hat{a}_l^\dagger \hat{a}_{m+k} \hat{a}_{l-m} \tag{146}$$

Since the ground state in the weak coupling phase is homogeneous, we can make the assumption that the 0-mode of the system will be macroscopically occupied, in such a way that

$$\begin{aligned} N_0 &= \langle \hat{a}_0^\dagger \hat{a}_0 \rangle \gg 1 \\ \frac{N - N_0}{N} &= \frac{1}{N} \sum_{k\neq 0} \langle \hat{a}_k^\dagger \hat{a}_k \rangle \ll 1 \ \ \forall_{k\neq 0} \qquad . \end{aligned} \tag{147}$$

With these conditions met, we can replace $\hat{a}$ and $\hat{a}^\dagger$ by $\sqrt{N_0}$ and disregard the interaction terms that are not at least linear in N_0. We arrive at the following Hamiltonian

$$\mathcal{H} = \sum_{k\neq 0} \left(k^2 - \alpha N/2\right) a_k^\dagger a_k - \frac{1}{4}\alpha N \sum_{k\neq 0} \left(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}\right) . \tag{148}$$

The Hamiltonian can be diagonalized

$$\mathcal{H} = \sum_{k\neq 0} \epsilon_k b_k^\dagger b_k \qquad \epsilon_k = \sqrt{k^2(k^2 - \alpha N)} \tag{149}$$

though the Bogoliubov transformation

$$a_k = u_k b_k + v_k^\star b_{-k}^\dagger, \tag{150}$$

where we have the following Bogoliubov coefficients

$$u_k^2 = \frac{1}{2}\left[1 + \frac{k^2 - \frac{\alpha N}{2}}{\epsilon_k}\right] \tag{151}$$

$$v_k^2 = \frac{1}{2}\left[-1 + \frac{k^2 - \frac{\alpha N}{2}}{\epsilon_k}\right] . \tag{152}$$

Whenever we approach the phase transition, the mode with $k = 1$ becomes gapless. It should be noted, however, that right at the phase transition the Bogoliubov approximation breaks down. This is due to the fact that it becomes free to populate the first excited mode, thus the assumption that only the 0 mode is macroscopically is no longer valid. At any finite distance from the phase transition, we can still choose N large enough that the Bogoliubov approximation is still valid.

Furthermore, looking back at (149), we see that the Bogoliubov approximation is equivalent to taking $N \to \infty$ while keeping αN constant.

8.5 BH QUANTUM CRITICALITY AND INFORMATION STORAGE

Based on the observations from the attractive Bose gas, in [50] the following idea was proposed: black-holes are in fact condensates that sit exactly on the critical point of a quantum phase transition.

A critical point is characterized by mode crossing: at this value of the coupling, at least one excited mode will have the same energy as the ground state. This makes it such that a certain number of states will be degenerate and thus the entropy of the system will grow dramatically at the quantum phase transition, since the ground state is not uniquely defined.

One way to study the ground state and excitation spectrum of the system is to perform the Bogoliubov transformation (149), in which we take one macroscopically occupied mode - the ground state - and find the modes that diagonalize the excitations around this state. This procedure is very similar to the semi-classical treatment of black-holes, in which we take the background to be infinitely heavy and calculate the fluctuations around the black-hole, thus finding Hawking radiation.

In both cases, the backreaction effects are taken to be exactly 0, since there is no dynamical evolution of the background mode. In the condensate case, this is enforced by technically sending N to ∞ while keeping αN constant, whereas in the second case, we take M to infinity while keeping ML_P^2 constant.

Technically, once a mode becomes gapless in the $N \to \infty$ limit, there are infinitely many states which will be degenerate, since any n-particle occupation of that mode will have the same energy. For finite N, however, we have backreaction effects which provide a natural regularization of the gap of the excited modes. We have $1/N$-type corrections which allow us to distinguish between the different micro-states of the macroscopic ground state.

The entropy of a condensate with infinitely many particles at the critical point is technically infinite, since we need infinite resolution power to distinguish between the infinitely many micro-states of the now infinitely degenerate ground state. In the same way, an infinitely massive black-hole with finite radius also has infinite entropy, since fluctuations cost nothing due to the fact that the coupling is technically 0.

The essence of the black-hole information problem can be very simply explained in terms of Bose-Einstein condensates: $1/N$ effects provide a natural labeling of the many would-be degenerate states which overlap with the ground state. These extra charges that are associated with the occupation of the almost-degenerate modes disappear completely in the strict $N \to \infty$ limit.

Let us reformulate this point by reintroducing $\hbar$ in the formulas. The entropy of a black-hole scales as

$$\frac{A}{4\hbar G} .$$

This expression diverges in the classical limit, which would lead us to conclude that the entropy, and thus the capacity to store information, is infinite. This is consistent with the statement that a Bose-Einstein condensate at the critical point also has technically infinite entropy in the $N \to \infty$ limit, but at the same time the ground state is essentially featureless.

It is clear that if we want to understand how information is processed and retrieved from a black-hole, we need to study the effective theory of the quasi-degenerate excited modes which make-up the black-hole state. While we do not have the technology to tackle this issue on a purely gravitational theory, we will continue with the bosons, since it provides remarkable insights without the technical issues associated with gravitational theories.

9

GOLDSTONE THEORY AND QUANTUM CRITICALITY

As we have seen, there is an inherent puzzle regarding the ability of a classical black-hole to hold and process information. In one hand, the entropy of a black-hole

$$S = \frac{A}{4\hbar G}$$

diverges in the classical limit, while on the other hand, a classical black-hole is only described by 3 parameters. The lack of extra possible ways to parametrize a black-hole state would lead a classical observer to believe that the entropy of a black-hole is essentially 0, since for this observer it would be impossible to create such a large degeneracy of microstates.

Obviously, the only possible resolution is to identify what are the quantum micro-states that act as information carriers. By identifying the underlying degrees of freedom, we will be able to prescribe an extra family of states that will be responsible for carrying the exponentially large information that the black-hole is able to store.

In order to explain the origin of these microstates, we will focus on the microscopic theory offered in [51], in which the black hole is described as a composite multi-particle state of soft gravitons at a quantum critical point [50]. This system can approximately be described as a Bose-Einstein condensate of gravitons of characteristic wavelength $\sim R_S$ and occupation number $N \sim R_S^2/L_P^2$.

In this picture, the microscopic carriers of black hole entropy and information are collective Bogoliubov-type excitations. As we have seen from studying the model (139), around the critical point they become nearly gapless and decouple from the rest of the modes in the $N \to \infty$ limit. When we deviate from the Bogoliubov approximation and consider the extra interactions that are $1/N$ suppressed, we find that these states have a typical energy separation of $\sim 1/N$. Hence, if a system has N gapless Bogoliubov modes, there are expected to exist $\sim e^N$ nearly-degenerate quantum states that can be labeled by the occupation numbers of their respective modes, and the entropy should scale as $\sim N$.

In the strict limit where the Bogoliubov approximation is exact, which we'll refer as the decoupling limit, we have $N \to \infty$ while αN remains constant. In this limit, the system is supposed to be mostly classical, since the 0 mode is overwhelmingly more occupied than all the others. This of course breaks down at the phase transition, where the ground state of the system becomes degenerate and thus Bogoliubov theory fails. The system is mostly quantum at this point [53, 54]. At any finite distance from the critical point, we can still choose N sufficiently large to offset the these effects and make the Bogoliubov theory valid.

Naturally the decoupling limit is artificial, since physical systems do have a finite occupation number. The $1/N$ corrections, or deviations from Bogoliubov theory, introduce a natural regularization of the gap and the occupation number of the nearly gapless modes.

Without this regularization, we are left on a situation in which the entropy of the system is formally infinite, since there is not limit to how much we can populate any mode that becomes gapless. The entropy puzzle of the black-hole is also manifest here: Since while the ground state is infinitely-degenerate, there are no features that describe it: it carries no information.

Despite the fact that we have an infinite number of gapless states, the required time for resolving their differences also becomes infinite. This time-delay reconciles the views of the two observers. The classical ground state of the system appears as featureless for any finite interval of the observation time.

One important question we may ask is the following: How important is the nature of the constituent bosons? One might argue that it is impossible to expect that these results hold in a theory where our degrees of freedom have non-0 spin. However such a quick dismissive reasoning would miss the striking similarity between the information processing features we observe in the Bose gas and the mysterious properties of black-holes which still lack a quantum resolution. While we do not claim that in fact quantum constituents of black-hole should display the same properties as the bose gas, we suspect that the information-processing features that originate from the fact of criticality may not be so sensitive to the precise nature of the constituents.

If this is true, then a wide class of critical systems must share some of the black hole information-processing abilities.

This would be a very important conclusion, since it would allow us to study black hole properties on much simpler systems, both theoretically and experimentally. Furthermore, it would open up the possibility to implement table-top systems that mimic some of the quantum properties of black-holes, beyond the classical approximation.

In this chapter, we will follow the derivations and present the results we obtained in [43].

9.1 CRITICALITY AND PSEUDO-GOLDSTONE PHENOMENA

As it was noticed in [50], the simplest prototype models of attractive Bose-Einstein condensates already exhibit similarities with the black hole quantum portrait. The cost of the information-storage per qubit is by a factor of $1/N$ cheaper relative to the energy-cost exhibited by non-critical quantum systems. In the latter systems, for the same amount of information storage, one typically pays the energy price of the inverse size of the system, $\sim \frac{\hbar}{R}$, since this is the scale that is typically associated with the excitation with single modes that act as information carriers. Also, the degeneracy of states within the $1/N$ energy gap increases with N. Moreover, a sharp increase of one-particle entanglement takes place near the critical point [53], and the scrambling of information becomes maximally efficient [54].

The time-dependent evolution of the critical condensate uncovers a scaling solution in which the condensate is stuck at the critical point throughout the collapse [55]. This is the behavior that one would expect if the microscopic foundation of Hawking radiation were through the collapse and quantum depletion of the condensate. [1]

In order to study the effective theory of information processing near the phase transition, we will start with the system that was used in [50] and introduced in the previous chapter.

$$\mathcal{H} = \int d^d x\, \psi^+ \frac{-\hbar^2 \Delta}{2m} \psi - g\hbar \int d^d x\, \psi^+ \psi^+ \psi\psi\,, \qquad (153)$$

1 Let us remark here that there is interesting evidence from the S-matrix that the above multi-particle picture represents the correct microscopic description of black hole physics. Indeed, the $2 \to N$ scattering S-matrix element of two trans-Planckian gravitons into N soft ones reproduces an entropy suppression factor e^{-N} when the number N is given by the quantum critical value [56]. Similar conclusions were reached in [57] in a different approach. For other aspects and related work, see [61].

where $\psi = \sum_{\vec{k}} \frac{1}{\sqrt{V}} e^{i\frac{\vec{k}}{R}\vec{x}} a_k$, $V = R^d$ is the d-dimensional volume and $\vec{k}$ is the d-dimensional wave-number vector. $a^\dagger_{\vec{k}}, a_{\vec{k}}$ are creation and annihilation operators of bosons of momentum $\vec{k}$. They satisfy the usual commutation relation $[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta_{\vec{k}\vec{k}'}$. g is the coupling constant, which is positive in the attractive regime. Rescaling the Hamiltonian, we can write

$$\mathcal{H} \equiv \frac{\hbar^2}{2R^2 m} H$$

Introducing a notation $\alpha \equiv \left(\frac{g}{VR}\right)\frac{2Rm}{\hbar}$ and taking $d = 1$ we arrive at the following Hamiltonian,

$$H = \sum_k k^2 a^\dagger_k a_k - \frac{\alpha}{4} \sum_{k_1+k_2-k_3-k_4=0} a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} \,. \tag{154}$$

As we have seen in the previous chapter, at the phase transition only the modes with $|k| = 1$ become gapless. One approximation we can perform is to disregard the contribution of higher order modes, since their qualitative effect on the behaviour of the system near the phase transition will be subdominant. We will later justify this approximation in a more quantitative way.

Let us then define the triplet operator

$$a_i \equiv (a_{-1}, a_0, a_1) \tag{155}$$

and the corresponding number operators

$$n_i \equiv a^\dagger_i a_i \tag{156}$$

We treat the operator a_i as a triplet under a global symmetry group $SU(3)$,

$$a_i \to a'_i = U_i^j a_j \,, \tag{157}$$

where U_i^j is a unitary transformation matrix which keeps invariant the total number operator

$$n \equiv \sum_i n_i \,. \tag{158}$$

Thus, in any state with non-zero particle occupation number,

$$\langle n \rangle = N \,, \tag{159}$$

the $SU(3)$-symmetry is spontaneously broken and the order parameter of breaking is N. This fact allows us to map the dynamics of the quantum phase transition in a Bose-Einstein condensate described by the Hamiltonian (154) onto the pseudo-Goldstone phenomenon in an $SU(3)$ sigma model. The Hamiltonian in terms of the triplet a_i can be written as

$$H = \sum_{i=-1,1} n_i - \frac{\alpha}{2}\left(a_1^\dagger a_{-1}^\dagger a_0 a_0 + a_0^\dagger a_0^\dagger a_1 a_{-1}\right) + \frac{\alpha}{4}\sum_{i=-1}^{1} n_i^2 + H_{SU(3)}\,. \tag{160}$$

where, $H_{SU(3)}$ is the $SU(3)$-invariant part of the Hamiltonian and has the following form,

$$H_{SU(3)} = -\frac{\alpha}{2}n^2 + \frac{\alpha}{4}n + \mu\,(n - N)\,. \tag{161}$$

Here, μ is the Lagrange multiplier that fixes the total particle number. Let us first investigate the ground-state of the $SU(3)$-symmetric Hamiltonian. Minimization with respect to n_i gives the following equations,

$$\frac{\partial H_{SU(3)}}{\partial n_i} = -\alpha\, n + \frac{\alpha}{4} + \mu = 0 \tag{162}$$

and

$$\frac{\partial H_{SU(3)}}{\partial \mu} = n - N = 0\,. \tag{163}$$

The vacuum is achieved at $n = N$ and $\mu = \alpha(N - 1/4)$. The $SU(3)$-symmetry is spontaneously broken down to $SU(2)$ and there is a doublet of Nambu-Goldstone bosons. For example, choosing the vacuum at $n_1 = n_{-1} = 0$, $n_3 = N$ and ignoring $\mathcal{O}(1)$-corrections to the zero mode occupation, the Goldstone doublet is (a_1, a_{-1}). The existence of the Goldstone boson is the manifestation of the indifference of the system with respect to redistributing occupation numbers among the different levels.

The addition of the first three terms in (160) breaks the $SU(3)$ symmetry explicitly and lifts the vacuum degeneracy. Let us investigate the effect of these terms on the state $n_i = (0, N, 0)$. Notice, this VEV continues to be the extremum of the Hamiltonian even after the addition of the explicit-breaking terms, but the value of μ is now shifted to $\mu = \frac{\alpha}{2}(N - 1/2)$. The would-be Goldstone modes now acquire a non-trivial mass-matrix,

$$\begin{pmatrix} 1 - \frac{\alpha}{2}N\,, & -\frac{\alpha}{2}N \\ -\frac{\alpha}{2}N\,, & 1 - \frac{\alpha}{2}N \end{pmatrix} \tag{164}$$

This matrix is diagonalized by states $a_{\text{light}} \equiv \frac{1}{\sqrt{2}}(a_1 + a^\dagger_{-1})$ and $a_{\text{heavy}} \equiv \frac{1}{\sqrt{2}}(a_1 - a^\dagger_{-1})$, with eigenvalues 1 and $1 - \alpha N$ respectively. In this basis the Hamiltonian becomes,

$$\begin{pmatrix} 1, & 0 \\ 0, & 1 - \alpha N\,. \end{pmatrix} \tag{165}$$

Thus, a would be Goldstone doublet is now split into a light mode and its heavy mode partner. Since the explicit breaking terms leave unbroken the $U(1)$ subgroup of $SU(3)$, corresponding to the generator $Q \equiv diag(1,0,-1)$, the mass eigenstates a_{light} and a_{heavy} are also $U(1)$-charge eigenstates. They carry charges equal to $+1$ and -1 respectively. In the usual language, the charge Q is a momentum operator.

Let us at this point highlight the relation between the Goldstone modes a_{light} and a_{heavy} and the usual Bogoliubov eigenmodes of the mass matrix (164). The latter are obtained by diagonalizing the mass matrix through a canonical transformation; in our setup, they are given by

$$b_{\pm 1} = u_1 a_{\pm 1} - v_1 a^\dagger_{\mp 1}\,, \tag{166}$$

with

$$u_1 = \frac{1 + \sqrt{1-\alpha N}}{2(1-\alpha N)^{1/4}}\,, \qquad v_1 = \frac{1 - \sqrt{1-\alpha N}}{2(1-\alpha N)^{1/4}}\,. \tag{167}$$

In this description, the Hamiltonian has two degenerate eigenvalues

$$\epsilon_{1,-1} = \sqrt{1-\alpha N}\,. \tag{168}$$

The modes a_{light} and a_{heavy} relate to the Bogoliubov modes through

$$a_{\text{light}} = \frac{1}{\sqrt{2}}(u_1 + v_1)\left(b_1 + b^\dagger_{-1}\right) = \frac{1}{\sqrt{2}(1-\alpha N)^{1/4}}\left(b_1 + b^\dagger_{-1}\right)\,, \tag{169}$$

$$a_{\text{heavy}} = \frac{1}{\sqrt{2}}(u_1 - v_1)\left(b_1 - b^\dagger_{-1}\right) = \frac{(1-\alpha N)^{1/4}}{\sqrt{2}}\left(b_1 - b^\dagger_{-1}\right)\,. \tag{170}$$

The Goldstone mode thus encodes a light direction in configuration space, reachable through occupying both Bogolyubov modes simultaneously. Note that a state with Bogolyubov occupation number of $\mathcal{O}(1)$ corresponds to a Goldstone configuration with $a^\dagger_{\text{light}} a_{\text{light}} \sim 1/\sqrt{1-\alpha N}$, which is responsible for the $\sqrt{1-\alpha N}$-difference in the eigenvalues.

The heavy partner, on the other hand, corresponds to a heavy multi-particle direction in configuration space, reachable through populating either one of the Bogolyubov modes with $\mathcal{O}\left(1/\sqrt{1-\alpha N}\right)$ particles. In turn, this explains the αN-independence of the second eigenvalue of (164).

For $1-\alpha N < 0$ one of the pseudo-Goldstone bosons becomes tachyonic and the vacuum is destabilized. The physics of this instability is that for $\alpha N > 1$ the $SU(2)$ preserving vacuum is no longer energetically favorable and the system flows towards the new ground-state. This new ground-state can be found by minimizing the full Hamiltonian, (160). In order to minimize this Hamiltonian, let us set $a_i = \sqrt{n_i}e^{i\theta_i}$. The only phase-dependent term in the Hamiltonian is the second term in (160), which takes the form

$$-\frac{\alpha}{2}\left(a_1^\dagger a_{-1}^\dagger a_0 a_0 + a_0^\dagger a_0^\dagger a_1 a_{-1}\right) = -\alpha\left(\sqrt{n_1 n_{-1}}n_0\right)\cos(\theta_1+\theta_{-1}-2\theta_0)\,. \tag{171}$$

Since there is no other conflicting phase-dependent term in the energy, in the minimum we will have $\cos(\theta_1+\theta_{-1}-2\theta_0) = 1$. We can thus set without any loss of generality,

$$\theta_1+\theta_{-1}-2\theta_0 = 0\,. \tag{172}$$

Another simplifying observation is that in the extremum n_1 and n_{-1} must be equal. This can be seen by extremizing the Hamiltonian with respect to n_1 and n_{-1},

$$\frac{\partial H}{\partial n_1} = 1 - \frac{\alpha}{2}\sqrt{\frac{n_{-1}}{n_1}}n_0 + \frac{\alpha}{2}n_1 - \alpha\, n + \frac{\alpha}{4} + \mu = 0 \tag{173}$$

$$\frac{\partial H}{\partial n_{-1}} = 1 - \frac{\alpha}{2}\sqrt{\frac{n_1}{n_{-1}}}n_0 + \frac{\alpha}{2}n_{-1} - \alpha\, n + \frac{\alpha}{4} + \mu = 0 \tag{174}$$

which show that n_1 and n_{-1} can be non-zero only together. Moreover they must be equal, since they satisfy the same equation, with only one positive root. This can be seen by multiplying (173) and (174) by n_1 and n_-1 respectively. The resulting quadratic equations are identical,

$$n_1\frac{\partial H}{\partial n_1} = \frac{\alpha}{2}n_1^2 + n_1(1-\alpha N + \frac{\alpha}{4} + \mu) - \frac{\alpha}{2}\sqrt{n_{-1}n_1}n_0 = 0 \tag{175}$$

$$n_-1\frac{\partial H}{\partial n_{-1}} = \frac{\alpha}{2}n_{-1}^2 + n_{-1}(1-\alpha N + \frac{\alpha}{4} + \mu) - \frac{\alpha}{2}\sqrt{n_{-1}n_1}n_0 = 0 \tag{176}$$

and have only one positive root. Thus, without any loss of generality we can minimize the Hamiltonian for the configuration $n_i = (x, N - 2x, x)$, which gives

$$H = \frac{7}{2}\alpha x^2 + 2x(1 - \alpha N) - \frac{\alpha}{4} N(N-1) \,. \tag{177}$$

Since x is positive definite, for $\alpha N < 1$, the minimum is achieved at $x = 0$. For $\alpha N > 1$, the minimum is at

$$x = \frac{2}{7\alpha}(\alpha N - 1) \,. \tag{178}$$

Not surprisingly, this corresponds to the first Fourier modes of the bright soliton solution to the Gross-Pitaevskii equation in the overcritical regime [58]. The energy of the ground-state is given by

$$H_{min} = -\frac{2}{7\alpha}(\alpha N - 1)^2 - \frac{\alpha}{4} N(N-1) \,, \tag{179}$$

where the first term only exists for $\alpha N > 1$.

We can now understand the critical phenomenon in terms of this Goldstone-mode. The large occupation number serves as an order parameter for a spontaneous breaking of a global $SU(3)$-symmetry. This symmetry corresponds to redistribution of the particle occupation numbers between the different momentum states, without changing their total number. This symmetry is only approximate, and normally the would-be Goldstone mode has the mass of the order of the first momentum level (that is order one in our units). However, the phase transition corresponds to the point where the particle number distribution changes. This necessarily implies that the corresponding pseudo-Goldstone mode must become massless at this point.

Notice that since x parameterizes the occupation number of (pseudo-)Goldstones, $x = n_{gold} = a^{\dagger}_{gold} a_{gold}$, the Hamiltonian (177) essentially represents the effective action for this mode,

$$H_{Gold} = (n_{gold})^2 \alpha_{gold} + n_{gold} m^2_{gold} - \frac{\alpha}{4} N(N-1) \,. \tag{180}$$

Thus, the mass of the pseudo-Goldstone is given by $m^2_{gold} \equiv 2(1 - \alpha N)$ and the self-coupling is given by $\alpha_{gold} \equiv \alpha \frac{7}{2}$. At the critical point, the Goldstone mass term vanishes and the theory is described by a gapless mode with a self-interaction strength given by $\alpha_{gold} = 7/2N$,

$$H_{Gold} = (n_{gold})^2 \frac{1}{N}\frac{7}{2} - \frac{1}{4} N \,. \tag{181}$$

Of course, the phenomena uncovered in the Goldstone formulation are in one-to-one correspondence to those that are seen in a mean-field and Bogoliubov description of the model. In fact, as hinted before, the minima of the Hamiltonian Eq.(177) correspond to the $k = -1, 0, 1$ modes of the solutions to the mean-field Gross-Pitaevskii equation: for $\alpha N < 1$ we found the homogenous condensate, while the solution for $\alpha N > 1$ corresponds to a localized bright soliton. In this context, one should also keep in mind that the solution $(x, N-2x, x)$ only agrees with the exact quantum mechanical ground state in the limit $N \to \infty$. At any finite N, the ground state is a smeared distribution centered around $(x, N-2x, x)$ with x given by (178), as can be seen in Fig. (1). In particular, the ground state at the critical point is characterized by strong entanglement and is therefore not well described by a mean field. This, however, will not alter our conclusion on state-counting and information processing to be performed in the following sections.

We reemphasize that in the Goldstone language, the critical point amounts to destabilization of the $SU(2)$-invariant vacuum. With the pseudo-Goldstone method we have traded the diagonalization of the Hamiltonian with the minimization procedure. Substituting the small deformations of the order parameter by the Goldstone mode allowed us to derive the effective action for the latter. This is similar to deriving an effective Hamiltonian of a phonon field in a background external magnetic field, which breaks the rotational symmetry of a ferromagnet. The analogous role in our treatment is assumed by the terms in the Hamiltonian that explicitly break $SU(3)$-symmetry. [2]

We conclude this section by pointing out that one can continue an analytic treatment of the system into the solitonic regime by performing an x-dependent canonical rotation on the creation and annihilation operators a_i. We define the condensate mode

$$c_{\text{cond}} = \sqrt{\frac{x}{N}} a_{-1} + \sqrt{1 - \frac{2x}{N}} a_0 + \sqrt{\frac{x}{N}} a_1 \,, \tag{182}$$

2 Notice also a curious analogy with the sigma-model of large-N-color QCD. In this case our N would be mapped on the number of colors, whereas the levels $k = \pm, 0$ to quark flavors. The pseudo-Goldstone boson then is analogous to pion, which also has $1/N$ suppressed self-coupling.

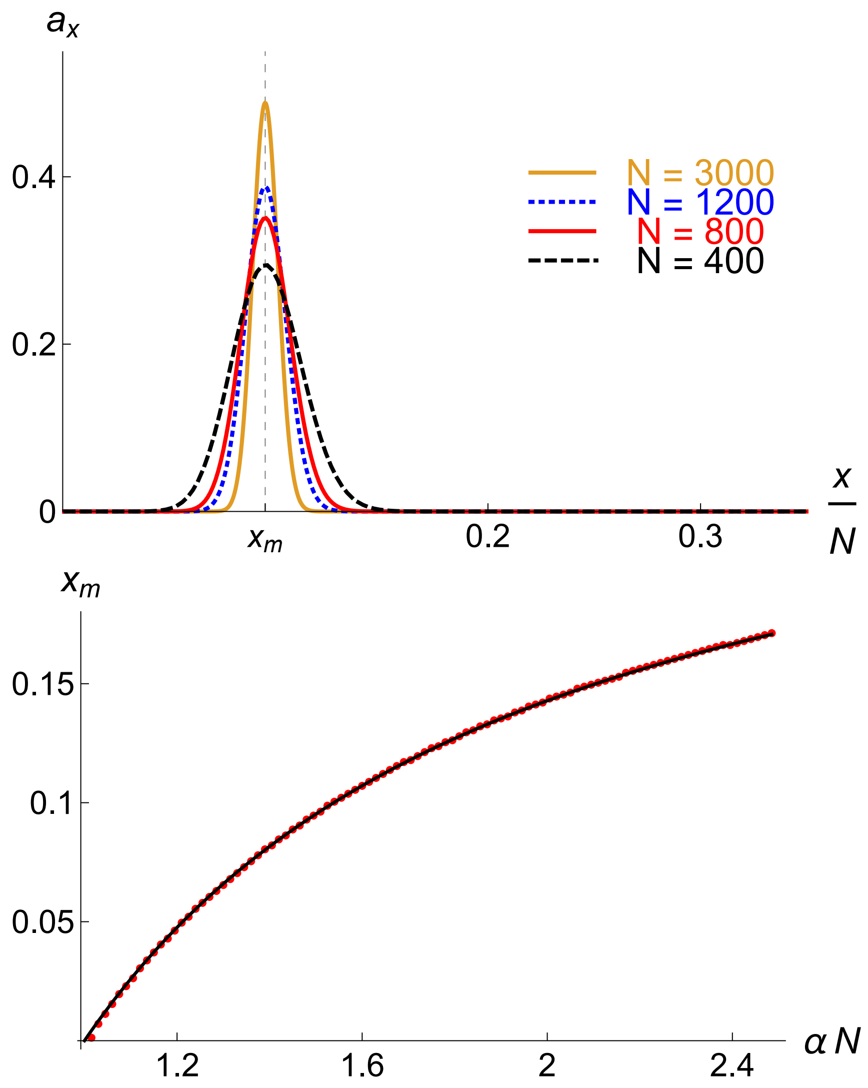

Figure 1: **(a)** Ground state for $\alpha N = 1.5$ at various values of N, using the parametrization $|GS\rangle = \sum_{x=0}^{N/2} a_x |x, N-2x, x\rangle$. a_x are normalized according to $\sum_{x=0}^{N/2} a_x^2 = N$, since as $N \to \infty$, $\int_0^{1/2} a(Ny)^2\, dy = 1$ and thus it's possible to compare them at different N. As N increases, the distribution becomes increasingly more peaked at x_m. **(b)** x_m for $N = 800$ particles compared with the analytic prediction $x_m = \frac{2}{7\alpha N}(\alpha N - 1)$.

and the corresponding orthogonal modes

$$c_{\pm 1} = \frac{1}{2}\left[\left(\pm 1 + \sqrt{1 - \frac{2x}{n}}\right) a_{-1} - 2\sqrt{\frac{x}{N}} a_0 + \frac{1}{2}\left(\mp 1 + \sqrt{1 - \frac{2x}{n}}\right) a_{-1}\right] . \tag{183}$$

The resultant Hamiltonian is now always minimized by the configuration $(c_{-1}, c_{\text{cond}}, c_1) = (0, N, 0)$. Expanding around this vacuum leads to a quadratic Hamiltonian which can be written in matrix form as

$$H_2 = \left(c_1^\dagger, c_{-1}^\dagger, c_1, c_{-1}\right) \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ m_2 & m_1 & m_4 & m_3 \\ m_3 & m_4 & m_1 & m_2 \\ m_4 & m_3 & m_2 & m_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_{-1} \\ c_1^\dagger \\ c_{-1} \end{pmatrix} \tag{184}$$

with

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\alpha N}{2} + x\left(8\alpha - \frac{3+14\alpha x}{N}\right) \\ \frac{x}{N}\left(-1 + 3\alpha N - 7\alpha x\right) \\ \frac{\alpha x}{2}\left(3 - 7\frac{x}{N}\right) \\ \frac{\alpha}{2}\left(4x - N - \frac{7x^2}{N}\right) \end{pmatrix} \tag{185}$$

Canonical diagonalization again leads to two distinct eigenmodes; for $\alpha N < 1$, their energy is $\epsilon = \sqrt{1 - \alpha N}$. For $\alpha N > 1$, one of them remains exactly gapless, while the other has a frequency

$$\epsilon = \sqrt{1 - \alpha N}\sqrt{\frac{2}{7}(4 + 3\alpha N)} \, . \tag{186}$$

The former is the mode that corresponds to translations of the soliton, whereas the other is the Bogolyubov mode that is only light in the vicinity of the critical point. In the Goldstone language, the Goldstone mode a_{light} splits into two parts, essentially given by its real and imaginary parts. One remains gapless and generates translations of the localized ground state; the other is gapless only at the critical point and obtains a frequency $\epsilon = (1 - \alpha N)\frac{2(4+3\alpha N)}{7\alpha N}$ for $\alpha N > 1$.

The appearance of the light modes can of course again be related to the breaking of $SU(3)$ generators. Keeping in mind that in the rotated description the symmetry transformations are given by $c' = SUS^{-1}c$, where S is the symplectic matrix that generates the canonical transformation, we immediately see that the translation generator $Q = diag(1, 0, -1)$, which commutes exactly with the Hamiltonian for all x, does no longer annihilate the ground state. The generator that redistributes particle numbers, on the other hand, only commutes with H at the critical point.

9.2 INFORMATION PROCESSING

The above form (180) of the effective Hamiltonian displays the role of quantum criticality for information storage and processing. The quantum information in the above system is encoded in the state of the Goldstone mode. The remarkable thing about it is the low energy cost of information-qubit-storage, which is suppressed by powers of $1/N$ relative to the inverse size of the system. This phenomenon is a manifestation of quantum criticality.

What is the role of quantum criticality for information storage and processing? From the effective Hamiltonian (180) we can read that the quantum information in the system is encoded in the state of the Goldstone mode.

In the absence of a Goldstone mode, the price of information storage would be related to the gap of the theory, which is typically related to the inverse size of the system. Since in this system information is stored in a Goldstone mode, we have a $1/N$ suppression, which is a manifestation of quantum criticality.

Since encoding information is so cheap, it is naturally that we ask whether information is actually encoded or the time-evolution of the system will eventually spoil the storage. In the time evolution of the Goldstone state we can distinguish two sources:
one is the interaction governed by a quartic self-interaction Hamiltonian (181). The rate of this process is suppressed by powers of $1/N$, and correspondingly the time-scale of evolution is very long.
The second source of time evolution is the Goldstone mass term that parametrizes the departure from quantum criticality. In the overcritical regime, this mass is imaginary and results in an exponential growth of the Goldstone occupation number. This instability is described by a Lyapunov exponent that, as shown in [54], leads to the generation of one-particle entanglement and potentially to the scrambling of information.

9.3 NUMERICAL RESULTS OF STATE EVOLUTION

An exact numerical diagonalization of the Hamiltonian (154) provides a complementary analysis which is valid also at the critical point. Using the same technique as in [53], we can verify the above results by comparing the the first Bogoliubov state as well as a lowly occupied

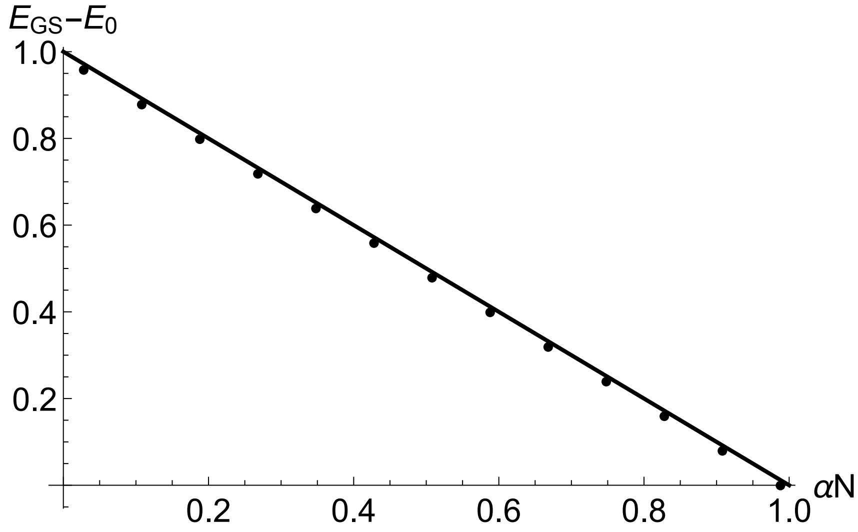

Figure 2: Expectation value of energy in state with Goldstone occupation $\left\langle a^\dagger_{\text{light}} a_{\text{light}} \right\rangle = 1$ as a function of αN for $N = 500$.

Goldstone state to the exact eigenstates of the Hamiltonian (154). Several quantities are instructive.

In Fig.2 we plot the expectation value of the energy in a state with Goldstone occupation $\left\langle a^\dagger_{\text{light}} a_{\text{light}} \right\rangle = 1$ as a function of αN. The solid line corresponds to the analytic result (180). The increase of energy around the phase transition is suppressed by $1/N$ and thus not visible.

In Fig.3a we plot the exact time evolution of the Bogolyubov state

$$|1_B\rangle = b_1^\dagger b_{-1}^\dagger |0_B\rangle$$

where $|0_B\rangle$ is the Bogolyubov ground state. The results are obtained for fixed particle number $N = 500$, while the effective coupling αN is varied. We observe an decrease of the frequency as αN approaches the critical point. To illustrate this point better, we plot the frequency of oscillations versus αN for $N = 100$, 300 and 500 in Fig 3b. We observe the square root behavior (168), with the frequency scaling with αN as $\sqrt{1 - \alpha N} + \mathcal{O}(1/N)$

9.4 OCCUPATION AND STABILITY OF HIGHER MODES

In the previous sections we have made the assumption that it's justified to ignore higher momentum modes, since their contribution to

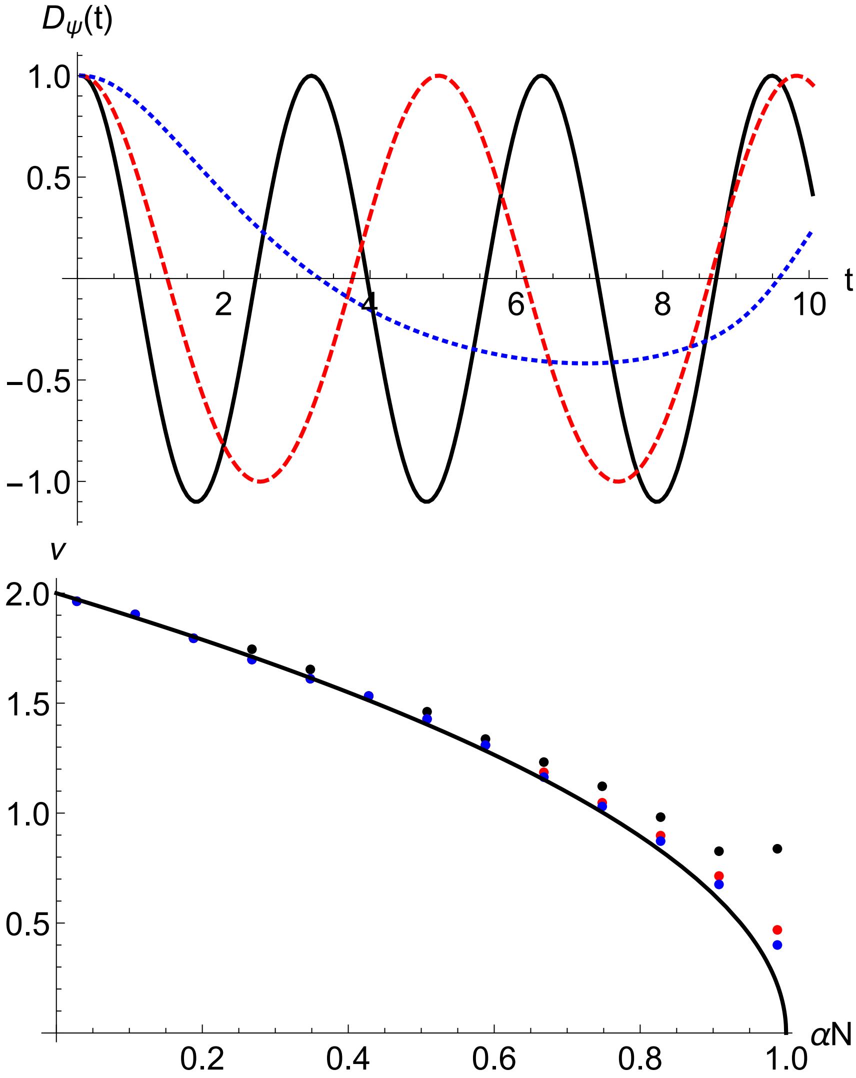

Figure 3: **(a)** Normalized exact time evolution of the time dependent part of the first Bogolyubov state for $N = 500$ and $\alpha N = 0.004$ (black, solid), 0.596 (red, dashed) and 0.976 (blue, dotted). **(b)** Frequency of oscillation as a function of αN at various values of N. The solid line corresponds to the analytic result $2\sqrt{1-\alpha N}$

the dynamics of the lowest lying excitations of the system near the phase transition was negligible due to their big gap, compared to the first momentum mode. In this section we will verify this assumption.

As we shall see, the occupation numbers of modes with momentum number k are suppressed by a factor of $(x/N)^{|k|-1}$.

Looking again at (154), we notice the following: we can decompose this Hamiltonian into a part that preserves and other that breaks a global $SU(2|k|+1)$ symmetry, similarly to what we did for $SU(3)$. This symmetry, which corresponds to the redistribution of occupation numbers among the fist k levels, is explicitly broken by the momentum term.

We can think of this breaking as the decoupling limit of a *spontaneous* breaking by a spurion order parameter, Σ, in the adjoint representation of the $SU(2|k|+1)$ group, with the expectation value [3]

$$\Sigma = \text{diag}(-|k|, \ldots, -1, 0, 1, \ldots, |k|)$$

Obviously, the kinetic term in the Hamiltonian (154), can then simply be written as an $SU(2|k|+1)$-invariant product, $a^\dagger \Sigma^\dagger \Sigma \, a$.

The spurion $\Sigma^\dagger\Sigma$ can be thought of as an expectation value of the number operator of a "hidden" Bose-gas of some σ_i-particles, interacting with a_i by an infinitely weak coupling κ, s.t. $\Sigma^+\Sigma = \kappa \sum \langle \sigma_i^\dagger \sigma_i \rangle$. In that case, the momentum operator is replaced by a peculiar $SU(2|k|+1)$ invariant interaction term and both occupation numbers spontaneously break one and the same $SU(2|k|+1)$-symmetry. The Goldstones bosons are superpositions of a-s and σ-s. The orthogonal combinations are pseudo-Goldstones.

Now, let us take the limit $\kappa \to 0$, while simultaneously taking the occupation numbers of σ-particles to infinity in such a way that the products $\kappa\langle \sigma_i^\dagger \sigma_i \rangle = i^2$ are kept fixed for all $i = -1, 1, ...$ We arrive at the situation in which in the a-particle sector, the entire information about the breaking is summed up in the expectation value of the spurion Σ. Correspondingly, all the Goldstones in the a-sector become pseudo-Goldstones. With increasing momentum level $|k|$, the masses of pseudo-Goldstones increase whereas their occupation numbers rapidly diminish.

In order to see this, let us compute the occupation numbers of the $k = \pm 2$ modes. The corresponding annihilation operators for $k = +2$ and $k = -2$ are a_2 and a_{-2}, respectively. The bilinear mass matrix of

3 This expectation value leaves invariant the $U(1)$-subgroup with the generator Q which is proportional to Σ. As in the case of $SU(3)$, this charge represents a momentum operator.

these modes expanded about the state $n_i = (0, x, N-2x, x, 0)$, has the form

$$\begin{pmatrix} 4 - \alpha N, & -\frac{\alpha}{2}N \\ -\frac{\alpha}{2}N, & 4 - \alpha N \end{pmatrix} \tag{187}$$

This has two eigenvalues, $4 - \frac{1}{2}\alpha N$ and $4 - \frac{3}{2}\alpha N$. Both eigenstates are stable as long as $\alpha N < 8/3$.

However, the modes $n_{-2,2}$ nevertheless can get populated due to the mixing with the lower levels. The occupation number of the lower modes acts as a source for the higher ones. The occupation numbers of the higher modes can be easily obtained by minimizing the bilinear Hamltonian including the effective source terms for $a_{-2,2}$. This source is generated from the interaction term after plugging the expectation values of $n_{-1,0,1}$. The phases of the expectation values can again safely be set to zero, because all the source terms are negative. The corresponding effective Hamiltonian for a_2 and a_{-2} has the form,

$$\begin{aligned} H_{2,-2} = & \frac{1}{4}(8 - 3\alpha N)(a_2^\dagger + a_{-2})(a_2 + a_{-2}^\dagger) \\ & + \frac{1}{4}(8 - \alpha N)(a_2^\dagger - a_{-2})(a_2 - a_{-2}^\dagger) \\ & - \frac{3}{2}\alpha x \sqrt{(N-2x)}(a_2 + a_{-2}^\dagger) + \text{h.c.} \end{aligned} \tag{188}$$

Minimizing this with respect to $a_{-2,2}$ we see that

$$n_{-2,2} \sim x^2/N\,. \tag{189}$$

Similarly, we can derive the occupation numbers of higher momentum modes. In general, modes with momentum $|k|$ have occupation numbers $\sim x(x/N)^{|k|-1}$, which rapidly approaches zero for $x/N \ll 1$. Thus in this regime, our assumption is well justified. In particular, this regime covers the neighborhood of the phase transition, around which x is small.

This behavior can also be confirmed by a numerical study of the system. To this end, we consider the Hamiltonian (154) with a momentum cut-off at $|k| = 2$. An instructive quantity is the occupation number $n_{-2,2}$ of the $|k| = 2$ modes for low lying states around the phase transition. As can be seen in Fig. 4, there is no appreciable contribution of the $|k| = 2$ modes to the low lying states at least until $\alpha N \sim 2$. We can moreover confirm the quadratic behavior (189).

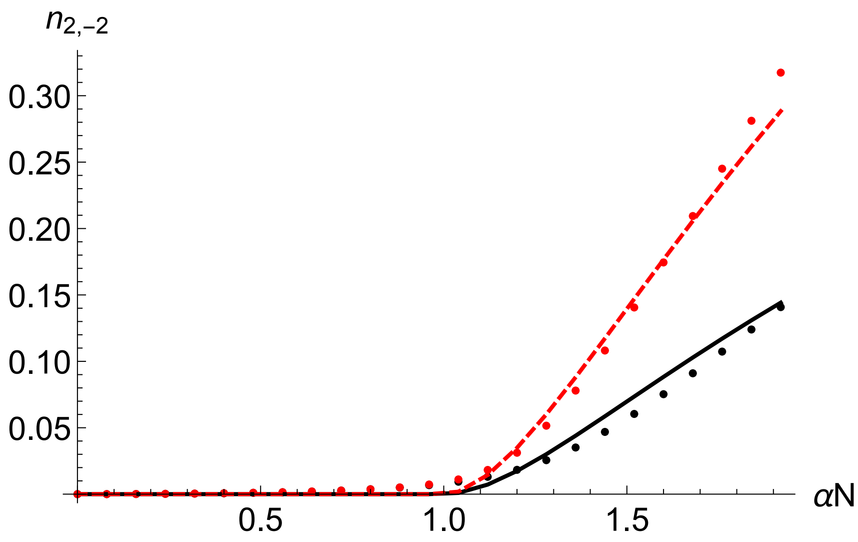

Figure 4: Contribution of the $|k| = 2$ modes to the lowest lying eigenstates for $N = 10$ (black) and $N = 20$ (red). The dots correspond to the exact numerical results, while the solid line depicts the analytical behavior (189).

9.5 DERIVATIVELY-COUPLED CASE

The model considered in the previous sections admits two solutions that become degenerate for $N \to \infty$ at the critical point. In d dimensions, this number can be increased by a factor of d; this corresponds to the number of gapless Bogolyubov modes. For finite N, we may define an entropy for a given N-particle system at the critical point by counting all states with $\frac{E-E_0}{E} < 1/N$. Given the gap of the light modes, $\Delta E \sim 1/N$, we obtain $\sum_k^N \binom{d+k-1}{k} \sim N^d$ states within the accountable range of energies. This implies an entropy $S \sim d \log N$.

We may attempt to reproduce an entropy that scales with N like the Bekenstein-Hawking entropy of black holes, $S \sim N$, by increasing the number of pseudo-Goldstone modes at the critical point. This can be achieved, for example, by taking the coupling to be momentum-dependent. In order to see this, consider a Hamiltonian,

$$\mathcal{H} = \int d^d x \, \psi^+ \frac{-\hbar^2 \Delta}{2m} \psi - L_*^{d+1} \hbar \int d^d x \left(\psi^+ \vec{\nabla} \psi^+ \right) \left(\psi \vec{\nabla} \psi \right) , \quad (190)$$

where $\psi = \sum_{\vec{k}} \frac{1}{\sqrt{V}} e^{i\frac{\vec{k}}{R}\vec{x}} a_k$, $V = R^d$ is the d-dimensional volume and $\vec{k}$ is the d-dimensional wave-number vector. L_* is a fundamental length and sets the cutoff of the effective theory. Rescaling the Hamiltonian, we can write $\mathcal{H} \equiv \frac{\hbar^2}{2R^2 m} H$, where

$$H = \sum_{\vec{k}} \vec{k}^2 a_k^\dagger a_k - \alpha_0 \sum_{k_1+k_2-k_3-k_4=0} \left(\vec{k}_2 \vec{k}_4\right) a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} . \tag{191}$$

and $\alpha_0 \equiv \left(\frac{L_*^{d+1}}{VR}\right) \frac{2Rm}{\hbar}$. Let us again find the effective bilinear Hamiltonian for $k \neq 0$ modes, about the point $n_0 = N$ and $n_{k\neq 0} = 0$. We obtain

$$H = \sum_{\vec{k}\neq 0} \vec{k}^2 \left(1 - \alpha_0 N\right) a_{\vec{k}}^\dagger a_{\vec{k}} \tag{192}$$

Thus, at $\alpha_0 = 1/N$, all modes are critical and there is the same number of massless pseudo-Goldstones as the number of momentum modes.

This peculiar behavior can be equivalently understood in a mean field analysis. Minimizing Eq. (190) under the constraint of fixed total particle number yields the Gross-Pitaevskii equation

$$-\frac{\Delta}{2m}\psi + \frac{L_*^{d+1}}{2}\psi^\dagger \Delta \psi^2 = \mu\psi , \tag{193}$$

where μ is again the Lagrange multiplier fixing the total particle number $\int dV \psi^\dagger \psi = N$. Using the ansatz $\psi_0^{\vec{k}} = \sqrt{\frac{N}{V}} e^{i\vec{k}\vec{x}}$, Eq.(193) becomes

$$\frac{k^2}{m}\left(\frac{1}{2} - \alpha_0 N\right) = \mu . \tag{194}$$

Hence plane waves of arbitrary wavenumber solve the GP equation (193). By inserting $\psi_0^{\vec{k}}$ into the Hamiltonian (190), we read off the energy of the plane wave solutions

$$E_k = \frac{\vec{k}^2}{2m}\left(1 - \alpha_0 N\right) . \tag{195}$$

At the critical point, all plane wave solutions become degenerate; the corresponding modes $a_{\vec{k}}$ are the Bogolyubov modes.

In order to give a lower bound on the entropy of this system for a given N, we have to keep in mind that for the validity of the non-relativistic treatment we should only include modes with $|\vec{k}| < k_{max} \equiv mR/\hbar$. Their number is $N_{max} \approx (k_{max})^d$. This number sets the number of legitimate pseudo-Goldstone modes. However, not all the massless-pseudo-Goldstone modes contribute the same weight into the entropy. This is because their self-couplings have different strengths. Hence exciting different pseudo-Goldstone species will contribute into the Hamiltonian differently. In order to identify which species give the maximal contribution into the entropy of the system, let us consider the effective Hamiltonian for the pseudo-Goldstone modes. It has the following form (we ignore numerical factors OF order one),

$$H_{Gold} = \sum_{\vec{k}\neq 0} |\vec{k}|^2 \left(\left(n_{gold}(\vec{k}) \right)^2 \alpha_0 + n_{gold}(\vec{k}) \left(1 - \alpha_0 N\right) \right) + \text{cross-couplings}\,, \tag{196}$$

where $n_{gold}(\vec{k}) \equiv a_{gold}(\vec{k})^\dagger a_{gold}(\vec{k})$ is the occupation number of pseudo-Goldstone of a given momentum number $\vec{k}$. At the critical point, the mass terms vanish and the Hamiltonian is given by the quartic couplings,

$$H_{Gold} = \sum_{\vec{k}\neq 0} |\vec{k}|^2 \alpha_0 \left(n_{gold}(\vec{k}) \right)^2 + \text{cross-couplings}\,. \tag{197}$$

We can obtain a lower bound for the number of states in the allowed range by looking for the $|\vec{k}|$ that provides the maximal contribution to the partition sum. We need to take into account two competing effects: The larger $|\vec{k}|$, the more modes, and thus states, are supplied. On the other hand, the energy cost of a given mode grows with $|\vec{k}|$, thereby limiting the maximum allowed occupation. Moreover, we pay a penalty for occupying modes more than once as seen from the quadratic dependence of (197) on $n_{gold}(\vec{k})$. A conservative lower bound is thus obtained by summing over all states in a shell around the limiting momentum k_l which allows for all Goldstone modes of given wavenumber to be occupied exactly once. Lower momenta will yield subdominant contributions due to the reduced number of modes and the related penalty for higher occupation; higher momentum modes, on the other hand, are too costly to excite. We obtain the contribution,

$$\delta_k H_{Gold} \sim |\vec{k}_l|^d \left(|\vec{k}_l|^2 \alpha_0 \right)\,. \tag{198}$$

where we ignored a possible cancellation from the cross-couplings. The factor $|\vec{k}_l|^d$ comes from the total number of modes. The maximal contribution to the entropy will come from the species with largest $|\vec{k}|$, subject to the condition $\delta_k H_{Gold} < 1$. Taking into the account that $\alpha_0 = 1/N$, and assuming $L_*/\hbar = m^{-1}$, we obtain

$$|\vec{k}_l| = N^{\frac{1}{d+2}}\ ,\ \ n_{\text{modes}} = N^{\frac{d}{d+2}}\,. \tag{199}$$

for the limiting momentum and the corresponding number of modes. We get the following contribution to the entropy from the pseudo-Goldstone species,

$$N_* \sim \log \sum_k^{N^{d/d+2}} \binom{N^{d/d+2}}{k} \sim N^{\frac{d}{d+2}}\,. \tag{200}$$

Note that we have also evaluated the full partition sum numerically and thereby verified that the wavenumbers with $|\vec{k}| \approx |\vec{k}_l|$ provide the dominant contributions.

If we take into the account the effect of cross-couplings, ignoring possible "flat directions" on which the different contributions cancel, we get a more conservative lower bound on the entropy. The number of cross-couplings for a given momentum level $|k|$ scales as the number of corresponding pseudo-Goldstones squared, $\sim (kd)^{2d}$. Then, equation (198) changes to

$$\delta_k H_{Gold} \ \sim\ |\vec{k}|^{2d}(|\vec{k}|^2\alpha_0)\,. \tag{201}$$

This lowers the allowed number of simultaneously-excitable Goldstones to

$$N_* \ =\ N^{\frac{d}{2d+2}}\,. \tag{202}$$

Independently, we observe the general message that the derivative self-coupling of bosons leads to a dramatic increase of pseudo-Goldstone modes.

9.6 THE CARRIERS OF BLACK HOLE ENTROPY

We finally wish to address the following question. Which of the large number of degenerate states are good candidates for carrying the Bekenstein-type entropy of a black hole? In order to answer this question, we shall distinguish two categories of states.

1) The first category represents states that *cannot* be resolved semi-classically in any macroscopic measurement. That is, the quantum information stored in such states becomes unreadable in the infinite-N limit. We shall refer to this category of states as type-A.

2) The second category are states that can be resolved in some macroscopic interference experiments. That is, the quantum information stored in such states can be read out even for $N = \infty$. We shall correspondingly refer to these states as type-B.

What distinguishes these two categories of states microscopically? In our picture the states of both categories can be labeled by the occupation numbers of some nearly-gapless quantum degrees of freedom. As discussed above, these information-carriers can be described as Bogolyubov and/or Goldstone degrees of freedom. What distinguishes the two category of states is the scaling behavior of information-carrier occupation numbers in large N limit.

The type-A category of states refers to those in which the relative occupation number of gapless degrees of freedom vanishes in the large-N limit. That is, none of these information-carrier degrees of freedom are macroscopically occupied. Correspondingly, the type-A states cannot be resolved in any macroscopic measurements.

For type-B states, some of the Goldstone modes can be macroscopically occupied with an occupation number that scales as a non-vanishing fraction of N in the $N \to \infty$ limit. Such states can be resolved in macroscopic interference experiments.

This discussion is more relevant for the derivatively-coupled model (190), since it exhibits a large diversity of Goldstone modes. However, to keep the discussion as simple as possible, we shall here consider only the model (153). Therein, let us now exemplify which states can be attributed to the type-A and type-B categories.

As discussed previously, at the critical point, two gapless modes emerge. The appearance of these states is a direct consequence of quantum criticality. They come from those components of the $SU(2)$-doublet pseudo-Goldstone that are not affected by the explicit breaking. One of them is only an approximate Goldstone, corresponding to one linear superposition of the off-diagonal generators from the quotient $SU(3)/SU(2) \times U(1)$. This Goldstone is massless only near the critical point. In contrast, the other degree of freedom is an exact Goldstone corresponding to the spontaneously broken $U(1)$-generator $Q = diag(1, 0, -1)$ of $SU(3)$. This Goldstone is massless everywhere at and beyond the critical point.

A counting of states in the Goldstone as well as in the Bogolyubov language suggests that the number of independent states below the $1/N$ energy gap near the critical point scales at least as $N^{1/4}$. However, the total number of almost orthogonal states with expectation value of energy below the same gap becomes larger if we also include coherent states in the counting (see Appendix). This is because the coherent states are neither energy nor number eigenstates and involve superpositions of arbitrary energetic modes with arbitrarily large occupation numbers. Correspondingly, they explore a much bigger fraction of the Hilbert space than the states constructed out of a finite number of energy or number eigenstates.

Only the states with vanishing fractions of Goldstone (or Bogolyubov) occupation numbers belong to the type-A category. On the other hand, states with macroscopic occupation numbers of Goldstones, such as coherent states, belong to the type-B category. The latter states can be resolved in macroscopic measurements, even in $N = \infty$ limit. This is a manifestation of the fact that the coherent states are "classical".

Given the fact that interference experiments with black holes have never been performed, both of these categories are very interesting, since they both reveal the internal microscopic structure of black holes. However, the question is whether both of the types of states should be counted as carriers of black hole Bekenstein entropy.

Should Bekenstein entropy originate exclusively from the quantum states with small-occupation numbers of many species of gapless Bogolyubov - Goldstone degrees of freedom, or should the states with large occupation numbers of few species also count?

For now, we shall keep both options open.

9.7 SUMMARY

We started this chapter with a question: how does one reconcile the fact that the entropy of a classical black-hole is infinite with the fact that there are only three parameters that describe every possible classical black-hole? It is clear that any possible model that can address this question should have two defining features: an exponential set of almost generate quantum states; and a decoupling limit in which these states are indistinguishable given a finite timescale.

In this chapter, we have shown that this question can in fact be answered if we consider a many-body system near a quantum phase transition. Furthermore, we have shown that this system is a viable candidate for the underlying mechanism for black hole information-processing, along the lines of [50, 53, 54]. We showed that the simplest multi-particle systems with critical behavior capture qualitative features of information-processing that is expected for black holes, such as the low energy cost of information storage, large degeneracy of states and scrambling of information.

In order to visualize the nature of the critical phase transition in the language of spontaneous symmetry breaking, we developed a new description in which we mapped the quantum phase transition in attractive Bose-gas on a Goldstone phenomenon in a sigma model. The two systems represent two realizations of one and same unitary symmetry that rotates different momentum modes into each other. This symmetry is broken both spontaneously as well as explicitly, but the explicit breaking vanishes, up to $1/N$ effects, at the critical point, resulting onto the gapless pseudo-Goldstone modes.

Our findings, within the validity-domain of the description, confirm the results of previous studies [50, 53, 54] and shed light on the criticality phenomenon from a novel angle. Our studies indicate that the key information-processing properties of black holes are shared by a larger class of the critical systems, including the ones that can be designed in table-top labs. This opens up a possibility of "borrowing" black hole information processing abilities for implementing them in the laboratory systems.

Since our findings suggest that critical instability is the key for efficient information processing, it would be interesting to find out, by generalizing our approach, whether other unstable systems, for example, the ones with parametric resonance instabilities [65] and large classical statistical fluctuations, studied in [66], also exhibit some analogous properties of information scrambling and processing.

Part IV

1D ATTRACTIVE BOSE GAS AND YANG–MILLS

10

1D BOSE GAS AND INTEGRABILITY

In the previous chapter we have introduced the 1-D Bose gas with attractive coupling as a prototype in order to understand the properties of quantum systems near a quantum phase transition, with the goal or relating its information-processing properties to those of Black-Holes. It is clear from the previous discussion that this system is an interesting theoretical laboratory to study properties of strongly interacting quantum many-body systems. Experimentally, it has also been featured quite extensively in recent years, since the Feshbach resonance makes it possible to simulate effective one-dimensional systems at arbitrary couplings using cold atoms [95, 81, 91, 78].

As we discussed, the theory undergoes a large particle number phase transition [80]. This phase transition interpolates between a homogeneous phase in the weak coupling limit to a phase dominated by a solitonic bound state in the strong coupling limit, known as a bright soliton. The dynamics of the phase transition has been extensively studied, both using the mean-field analysis [80] and also by a truncation and numerical diagonalization of the Hamiltonian [96, 94, 76, 75].

Another interesting feature of this model is that it is exactly integrable [84]. As we'll see, this implies that the Schrödinger equation of the system can be mapped to a set of algebraic equations - the Bethe equations - which fully determine the complete spectrum of the theory. Despite the fact that the system can be in principle solved using this technique, in practice the equations are transcendental and cannot be analytically solved without any approximations. The only regime where it is possible to obtain exact solutions is in the $c \to \infty$ limit, where we are in the deep solitonic regime. In this regime, it is possible to explicitly construct exact solutions of the Bethe equations, due to the string hypothesis [87], which we'll revisit later.

One way to simplify the problem is to consider the large particle limit of the theory, using some scaling limit. There is an extensive literature covering the thermodynamic limit, in which we take $N \to \infty$ while keeping N/L constant - where L is the size of the system. In this limit, it is possible to study the properties of the system through means of integrability techniques (e.g. [70, 72, 71, 83, 90]). This is possible due to the fact that the thermodynamic limit of the theory is intrinsically a strong coupling limit, since the interaction term is irrelevant, becoming stronger in the infrared.

The scaling limit which allows us to probe the weak coupling limit of the theory is the one in which we take $N \to \infty$ while keeping $NcL = g$ constant. In this limit, the phase transition happens at finite g, and we are able to probe exactly what happens with the Bethe states.

Finally we observe that the ground state can be mapped exactly to the large-N saddle point of U(N) Yang–Mills theory on a two-sphere, where the phase transition manifests itself as the confinement / deconfinement phase transition of Douglas and Kazakov [74], which is deeply connected to random matrix theory [85] and has diverse manifestations [77, 73].

We describe the one-dimensional Bose gas in terms of the canonical fields Ψ and $\Psi^\dagger$, obeying the equal time commutation relations

$$[\Psi(x,t), \Psi^\dagger(y,t)] = \delta(x-y) \ . \tag{203}$$

The Hamiltonian for the model, defined on an interval of length L, is

$$H = \int_0^L dx \ \partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x) \ . \tag{204}$$

We can define the following integrals of motion, which can be interpreted as the momentum P and the particle number N, both commuting with the Hamiltonian

$$N = \int dx \ \psi^\dagger(x) \psi(x) \tag{205}$$

$$P = -\frac{i}{2} \int dx \ \psi^\dagger(x) \partial_x \psi(x) - (\partial_x \psi^\dagger(x)) \psi(x) \tag{206}$$

Since the Hamiltonian commutes with the particle number operator, we can construct its eigenstates by simply writing down the most general for of a N-particle state

$$|\psi_N; E, p\rangle = \frac{1}{\sqrt{N!}} \int d^N z \ \chi(z_1, ..., z_N; E, p) \Psi^\dagger(z_1) .... \Psi^\dagger(z_N) |0\rangle \ . \tag{207}$$

Each eigenstate is characterized by a symmetric function χ, which besides the spatial dependence, will in general depend on a finite set of parameters which will determine the momentum and energy of the corresponding eigenstate. In order to determine this function, we act on $|\psi_N; E, p\rangle$ with the operators E and P and find the solution of the eigenvalue equations.

The eigenvalue problem

$$H|\psi_N; E, p\rangle = E|\psi_N; E, p\rangle$$

becomes a Schrödinger equation for the wavefunction χ, with the quantum mechanical Hamiltonian:

$$H_N = \sum_{i=1}^{N} -\frac{\partial^2}{\partial x_i^2} + c\sum_{i\neq j}\delta(x_i - x_j) \tag{208}$$

and the solutions χ of the eigenvalue equation will have a momentum given by

$$P_N\chi(x_1, ..., x_N) = -i\sum_{i=1}^{N}\left(\frac{\partial}{\partial x_i}\right)\chi(x_1, ..., x_N) \tag{209}$$

10.1 THE COORDINATE BETHE ANSATZ

The model defined by the Hamiltonian (208) is known as the Lieb-Liniger model, it was first studied in detail by Lieb and Liniger (XX) who used the coordinate Bethe Ansatz (XX) in order to construct the ground state and excitation spectrum of the theory in the repulsive regime ($c > 0$).

As it turns out, the solutions obtained in the attractive and repulsive regime and qualitatively very different, which reflect on the form of the eigenstates obtained by the Bethe Ansatz. While the repulsive case has been extensively studied, both at 0 and finite temperature, the appearance of bound states make the treatment of the attractive case significantly more difficult.

In this section, we will review the coordinate Bethe Ansatz and how it can be applied to construct the eigenstates of (208).

10.1.1 *2-body problem*

To the idea behind the Bethe Ansatz, let us consider first an explicit solution of the 2-body problem defined by the Hamiltonian (208). This model has the following property: whenever the particles lie on different positions, the potential is 0, so the eigenstates can be constructed by separating the wave-function in the domains in which $x_1 > x_2$ and vice-versa.

The most general Ansatz we can write is then the following:

$$\chi(x_1, x_2) = f(x_1, x_2)\theta(x_2 - x_1) + f(x_2, x_1)\theta(x_1 - x_2) \tag{210}$$

where θ is the Heaviside step function. Let us assume a superposition of plane waves for $f(x_1, x_2)$

$$f(x_1, x_2) = A_{12}e^{ix_1k_1+x_2k_2} + A_{21}e^{ix_1k_2+x_2k_1} \tag{211}$$

The parameters k_j play the role as a generalization of the momenta for the interacting problem, so they are often referred to as quasi-momenta, rapidities or roots. The action of the free Hamiltonian on the χ will yield

$$\begin{aligned}\left(\partial^2_{x_1} + \partial^2_{x_2}\right)\chi(x_1, x_2) =& \partial^2_{x_1} f(x_1, x_2)\theta(x_2 - x_1) + \partial^2_{x_1} f(x_2, x_1)\theta(x_1 - x_2)\\ &+\partial^2_{x_2} f(x_1, x_2)\theta(x_2 - x_1) + \partial^2_{x_2} f(x_2, x_1)\theta(x_1 - x_2)\\ &-\partial_{x_1} f(x_1, x_2)\delta(x_2 - x_1) - \partial_{x_1} f(x_2, x_1)\delta(x_1 - x_2)\\ &-\partial_{x_2} f(x_1, x_2)\delta(x_2 - x_1) - \partial_{x_2} f(x_2, x_1)\delta(x_1 - x_2)\end{aligned}$$

where we have used the fact that $u(x)\delta'(x) = -u'(x)\delta(x)$, and we have

$$\begin{aligned}\partial_{x_1} f(x_1, x_2) = \partial_{x_2} f(x_2, x_1) &= ik^1_1 A_{12}e^{ix_1k_1+x_2k_2} + ik^1_2 A_{21}e^{ix_1k_2+x_2k_1}\\ \partial_{x_2} f(x_1, x_2) = \partial_{x_1} f(x_2, x_1) &= ik^1_2 A_{12}e^{ix_1k_1+x_2k_2} + ik^1_1 A_{21}e^{ix_1k_2+x_2k_1}\\ \partial^2_{x_1} f(x_1, x_2) = \partial^2_{x_2} f(x_2, x_1) &= -k^2_1 A_{12}e^{ix_1k_1+x_2k_2} - k^2_2 A_{21}e^{ix_1k_2+x_2k_1}\\ \partial^2_{x_2} f(x_1, x_2) = \partial^2_{x_1} f(x_2, x_1) &= -k^2_2 A_{12}e^{ix_1k_1+x_2k_2} - k^2_1 A_{21}e^{ix_1k_2+x_2k_1}\end{aligned}$$

Summing all the contribution, the eigenvalue equation becomes

$$\begin{aligned}H\chi =& \left(k_1^2 + k_2^2\right)\chi +\\ & 2\delta(x_1 - x_2)\left(c(A_{12} + A_{21}) - i(A_{12} - A_{21})(k_1 - k_2)\right)e^{i(k_1+k_2)x_1}\end{aligned} \tag{212}$$

which is satisfied as long as

$$\frac{A_{12}}{A_{21}} = \frac{(k_1 - k_2) - ic}{(k_1 - k_2) + ic} \tag{213}$$

Whenever the k_i are real, this ratio has unit norm, so it is simply a phase shift. The ratio of amplitudes for the different permutation of particles is physically the phase shift that the wave-function acquires whenever the particles cross each other:

$$\frac{A_{12}}{A_{21}} = e^{i\alpha(k_1 - k_2)} \tag{214}$$

with

$$\alpha(z) = 2\arctan(z/c) + \pi \tag{215}$$

As it stands, the amplitudes are still not completely determined, since we have not yet imposed the periodic boundary conditions on the wave-function. Imposing boundary conditions will also fixed the possible values that the rapidities can have

10.1.2 *Bethe Equations*

We will now generalize the previous result for the N-particle problem, and subsequently impose boundary conditions in order to derive the Bethe equations for the system.

Suppose that we assemble all particles on distinct positions, and let us define a domain D_i as one specific ordering of the particle positions, with $D_1 : x_i > x_j$ for $i > j$. Because of the symmetry of the wave-function, any other ordering ordering domain can be obtained from D_1 by a permutation of the particles. Since the potential is 0, in this domain the wave-function is an eigenstate of the free Hamiltonian, but subject to non-trivial boundary conditions.

Anagolously to (210), we write the wave-function as a superposition of plane waves on the various domains

$$\chi(x_1, ..., x_N) = \sum_{i=1}^{N} \chi(x_1, ..., x_N; D_i)\, \mathbb{1}\,(D_i) \tag{216}$$

$$\chi(x_1, ..., x_N; D_i) = \sum_{P} A(P, D_i) e^{ik_{P_1}x_1 + ... + ik_{P_N}x_N} \quad , \tag{217}$$

P is a permutation of the set $\{1, ..., N\}$.

Since the wave-function is bosonic, a permutation of the particle positions should leave it unchanged. By permuting the particle positions we are going to a different domain, so if a permutation takes from a domain D_i to a domain D_j, we should have by consistency that $A(P, D_i) = A(P', D_j)$, where P' is the permutation that will differ from P in the same particle indices as D_i differs from D_j.

To construct the boundary condition, we consider the variable $\Delta_j = x_{j+1} - x_j$. Let us integrate (208) from $\Delta_j = -\epsilon$ to $\Delta_j = +\epsilon$. Since the wave-function is analytic, the left-hand-side will be 0 in the $\epsilon \to 0$ limit. On the right-hand-side, we have a contribution of c from the δ-function, and the boundary terms from the second-derivative. On the domain D_1, the problem is equivalent to

$$H_N^0 \chi(x_1, ..., x_N; D_1) = E\,\chi(x_1, ..., x_N; D_1) \tag{218}$$

$$\lim_{x_j \to x_{j+1}} \left(\frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j} - c \right) \chi(x_1, ..., x_N; D_1) = 0 \tag{219}$$

We can now substitute the original problem with the interacting Hamiltonian by a system of N free particles constrained to the domain D_1 and subject to the boundary conditions (219). In order to construct the wave-function, take the following ansatz for the amplitudes, inspired by the 2-body problem

$$A(P) = (-1)^P \prod_{j>l} \left(k_{P_j} - k_{P_l} - ic \right) \tag{220}$$

The wave-function can be written in a compact form using the fact that

$$\sum_P (-1)^P \exp\{i \sum_j x_j k_{P_j}\} = \det[\exp\{i x_j k_i\}] \tag{221}$$

We have then, up to normalization

$$\chi = \prod_{N \geq j > l \geq 1} \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_l} + c \right) \det[\exp\{i x_j k_i\}] \tag{222}$$

It is easy to see that this ansatz also satisfies the boundary conditions (219). We can start by verifying the following equality

$$\lim_{x_1 \to x_2} \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} - c \right) \chi = 0 \tag{223}$$

We write $\chi = \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} + c\right)\tilde{\chi}$, with

$$\begin{aligned}\tilde{\chi} = &\prod_{j\geq 3}\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_1} + c\right)\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_2} + c\right) \times \\ &\prod_{j>l\geq 3}\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_l} + c\right)\det[\exp\{ix_jk_i\}]\end{aligned} \tag{224}$$

and under this decomposition (223) can be written as

$$\lim_{x_1\to x_2}\left(\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right)^2 - c^2\right)\tilde{\chi} = 0 \tag{225}$$

It is clear that $\tilde{\chi}$ is antisymmetric under the exchange $x_1 \leftrightarrow x_2$, while the prefactor is symmetric under the same exchange. Since the left-hand-side whenever $x_1 \to x_2$, the equality is proven. It is easy to see by the same construction that the the the condition is also satisfied for any pair $\{i, i+1\}$.

We are still left with the task of imposing periodic boundary conditions on the wave-function and thus constraint the set of possible quasi-momenta. A simple way to derive the equations is the following: suppose that we start on domain D_1, pick particle j and move it all the way around the circle. On one hand, the wave-function will acquire a phase given by e^{ik_jL}. On the other hand, it will scatter with the $N-1$ particles it will find on the way, and each scattering event will contribute with a phase shift, given by (213). We have then the following set of identities

$$e^{ik_jL} = \prod_{i\neq j}\frac{(k_j - k_i) + ic}{(k_j - k_i) - ic} \tag{226}$$

These identities constraints the possible quasi-momenta for a given eigenstate of the system, and are known Bethe Equations.

The energy and momentum eigenvalues of the Bethe eigenstates are given, as a function of the quasi-momenta, simply as

$$P = \sum_{j=1}^{N} k_j \tag{227}$$

$$E = \sum_{j=1}^{N} k_j^2 \tag{228}$$

10.1.3 *The sign of c*

In the repulsive regime, it was shown [98] that the Bethe states with real momenta form a complete set of the N-particle Hilbert space, which implies that all roots of the Bethe equations for $c > 0$ will be real. This is consistent with the intuition that the ground state of a repulsive gas will be characterized by scattering states occupying a Fermi sea.

The Bethe Equations (226) do not have unique solutions for k_i. More specifically, when taking the logarithm on both sides, we must specify in which branch we are, which can be done by choosing a set of integers n_i, one for each k_i. The physical role of the k_i can be understood as setting the excitation level of each of the particles. We have, with $\alpha(z) = 2\arctan(z/c) + \pi$,

$$k_j L = 2\pi n_j - \sum_{i=1}^{N} \alpha(k_j - k_i) \tag{229}$$

It is clear by the form of the wave-function ansatz that whenever two k_j coincide, the wave-function vanishes. This creates an exclusion principle for the n_j: as long as the quasi-momenta are all real, all n_j must differ. It is easy to check by going to the non-interacting limit that the lowest energy state we can build is one in which the n_j are symmetrically distributed around 0:

$$n_j = -\frac{N+1}{2} + j \qquad j = 1, 2 \ldots N \quad . \tag{230}$$

When c is negative, however, we are in the attractive regime of the Bose gas, and bound states appear in the spectrum. The appearence of bound states is reflected in the fact that the quasi-momenta are now allowed to become complex. Whenever the quasi-momenta are complex, the wave-function develops exponentially decreasing components, which corresponds to localized states.

In the attractive gas, we are then allowed to have multiple particle sitting on the same excitation level, as long as the imaginary part of their quasi-momentum differs. Particles that share the same real part of the quasi-momenta but have different imaginary part are in a bound state, and the real part of the quasi-momenta tells us the total momentum of the bound state.

Although the quasi-momenta are allowed to become complex, we need to make sure that the total energy and momentum of the system are still real. Take $k_j = r_j + im_j$, the the moment and energy of the state can be written as

$$P = \sum_{j=1}^{N} r_j + i \sum_{j=1}^{N} m_j \tag{231}$$

$$E = \sum_{j=1}^{N} \left(r_j^2 - m_j^2 \right) + 2i \sum_{j=1}^{N} r_j m_j \tag{232}$$

so to ensure that both imaginary parts are real, we need that all quasi-momenta must come in complex conjugate pairs.

10.2 SUMMARY

In this chapter, we reviewed the Bethe ansatz technique to solve the 1D bose gas. We have seen how to construct the Bethe wave-function as a function of the quasi-momenta k_i and how the boundary conditions impose a set of algebraic equations for the quasi-momenta. Although solving these equations completely determine the spectrum of the system, in reality there is no known way to solve them for arbitrary couplings.

The system for $c > 0$ differs dramatically from the attractive system. While in the repulsive case the ground state is dominated by a fermi-liquid type distribution for the quasi-momenta, in the attractive case the formation of bound states reflects in the appearence of complex momenta. These complex momenta imply exponentially decaying wave-functions, which signals localized particle configurations.

In the next chapter we will introduce a method to solve the ground state of the system at arbitrary coupling in the $N \to \infty$ limit. This method allows us to show that the ground state can be mapped to the saddle point of 2D Yang-Mills on a sphere.

11

THE GROUND STATE OF THE ATTRACTIVE BOSE GAS AND 2D YANG-MILLS

In the previous chapter we have seen how the attractive Bose gas can be solved using the coordinate Bethe ansatz. In practice, building the eigenstates of the model implies solving a system of N algebraic equations for the quasi-momenta k_i. While there is no general method to construct these Eigenstates at arbitrary N, in this chapter we will introduce the formalism that will allow us to obtain both the ground state and the lowest lying excitations in the decoupling limit, $N \to \infty$ with cN fixed.

Just by looking at (232), it is clear that in order to minimize the energy, we must choose all quasi-momenta to have 0 real part. The ground state of the attractive gas will be then simply a static bound state of N particles. In order to find how the quasi-momenta are distributed in the imaginary line, however, we still need to solve the Bethe equations.

Doing the following replacement $k_j \to -ik_j$ and $c \to -c$, we arrive at the Bethe equations that we will solve:

$$k_i L = \sum_{j \neq i} \log \frac{k_i - k_j + c}{k_i - k_j - c} \tag{233}$$

Constructing real solutions of these equations is equivalent to finding the ground state of the system.

In this chapter we will show that it is possible to solve (233) at leading order in a $1/N$ expansion, while keeping the 't Hooft coupling of the system, cN, fixed. This will be done by going to the continuum limit of the k_i, and transforming the set of algebraic equations into one constrained integral equation. This integral equation can be solved using known methods and will give us the distribution of k_i for any coupling.

One interesting curiosity that arises is that the system can be explicitly mapped to the saddle point of 2D Yang–Mills on a sphere. We will show here the derivations and results we obtained in [68].

11.1 THE DEEP SOLITONIC REGIME

Suppose first that that we take $L \to \infty$. In this limit, the left-hand-side is either $-\infty$ or $+\infty$. One the right-hand-side we must have then that at least one of the arguments of the logarithms must be 0. We can consistently create a solution that satisfies the equation by putting all the k_j on a "string" in which the distance between adjacent quasi-momenta is approximately c, with exponentially small deviations [87]:

$$k_j \approx cL(j - (N+1)/2) \tag{234}$$

Using (232), we find that the energy of this configuration is

$$E = -\frac{1}{12}c^2L^2N\left(N^2 - 1\right) \tag{235}$$

Although this expression diverges in the $N \to \infty$, the energy per particle is finite in the scaling limit:

$$E/N \approx -\frac{1}{12}c^2L^2N^2 = -\frac{1}{12}g^2 \tag{236}$$

To convince ourselves that the $L \to \infty$ limit corresponds to the strong coupling limit, we can write the Bethe equation in dimensionless units by taking $k \to ck$. In this choice of units, the left-hand-side of (233) becomes k_iLc, and thus the dimensionless $2 \to 2$ coupling constant of the theory becomes cL.

11.2 CONTINUUM LIMIT

Before we attempt to construct solutions for arbitrary coupling, we need to introduce the variables that will be used in the $N \to \infty$ limit treatment of the theory. In order to proceed, we need to assume that the N roots k_i of (233) converge to a continuous function, defined as

$$k_i \equiv g\,k(i/N) \quad . \tag{237}$$

This assumption is justified by the following observation: take the root distribution in the strong coupling limit (234). In the scaling limit we have

$$k_i/g - k_{i-1}/g \approx cL/g + \mathcal{O}(e^{-cLN}) = 1/N + \mathcal{O}(e^{-gN}) \tag{238}$$

thus making $k(x)$ a continuous function of x in the strong coupling limit.

Since (233) is a continuous function of c and k_i, we find that the $k(x)$ must be a continuous function of x for all c, as long as the roots k_i do not fall on the poles of the logarithm. This is equivalent to the constraint that

$$|k_i - k_j| > c \tag{239}$$

for all pairs $\{i, j\}$ which is a fundamental property of this system and is the reason why we encounter a phase transition.

In order to understand the origin of this constraint, imagine changing the coupling adiabatically. Suppose as well that for some value of c the inequality is satisfied for all pairs $\{i, j\}$. If at some point only a pair of roots $\{i, i+1\}$ violate the inequality, then there will be only one diverging contribution in the sum for the root i and j. In this situation, by iterating the argument, it must happen that all of the pairs $\{i, i+1\}$ violate the inequality for the same value of c, since the divergent contributions can be pairwise cancelled. Nevertheless, we are still left with root 1 and root N, which have only one near neighbour and thus only one divergent contribution, so they must diverge.

Since for $g \to \infty$ we have that the inequality is satisfied, it is clear then that there are no finite solutions of the Bethe equations for any g. It is important to notice, however, that that in the strict $N \to \infty$ limit, the inequality can be saturated.

In terms of continuum variables, we can write the inequality (239) for consecutive roots as

$$\lim_{N\to\infty} g|k(x + 1/N) - k(x)| \geq g/N \tag{240}$$

which is nothing more than a constraint on the first derivative of $k(x)$

$$|k'(x)| \geq 1 \tag{241}$$

We can also define the density of roots

$$\rho(k) = \frac{\partial x}{\partial k} \tag{242}$$

which satisfies $\rho(k) \leq 1$. This quantity appears when transforming discrete sums into integrals, since we have

$$\frac{1}{N}\sum_{i=1}^{N} F(k_i) \to \int_0^1 dx F(k(x)) = \int_{k_{min}}^{k_{max}} dk\rho(k)F(k) \tag{243}$$

with the endpoints of integration determined by the identity

$$\int_{k_{min}}^{k_{max}} \rho(k)\, dk = 1 \tag{244}$$

Given that the roots always appear in complex conjugates, $\rho(k) = \rho(-k)$, and thus $-k_{min} = k_{max}$.

We need now to rewrite the Bethe equations (233) in terms of continuum variables. The sum on the right-hand-side of can be split in a near contribution from $|j-i| < \epsilon N$ and the rest, for some $\epsilon > 0$. As long as $k' > 1$, the near contribution vanishes when taking the double limit $\lim_{\epsilon\to 0}\lim_{N\to\infty}$ (see appendix). For the rest of the sum, the following Taylor expansion is valid

$$\lim_{N\to\infty} \log \frac{k_i - k_j + \frac{g}{N}}{k_i - k_j - \frac{g}{N}} = \frac{2g}{N}\frac{1}{k_i - k_j}. \tag{245}$$

Using (243), we find that in the continuum limit, the Bethe equations become the integral equation

$$gk = 2\,\mathcal{P}\int_{-k_{\max}}^{k_{\max}} \frac{\rho(u)}{k-u}\, du \tag{246}$$

together with the constraint

$$\rho(k) \leq 1 \tag{247}$$

11.3 WEAK AND STRONG COUPLING SOLUTIONS

We have now replaced the problem of finding the N solutions of the Bethe equations (233) to determining the constrained solution $\rho(k)$ of (246).

As long as the constraint is satisfied, the solution of the integral equation (246) is a semi-circle [92]

$$\rho(k) = \frac{1}{\pi}\sqrt{g - \frac{g^2k^2}{4}} \tag{248}$$

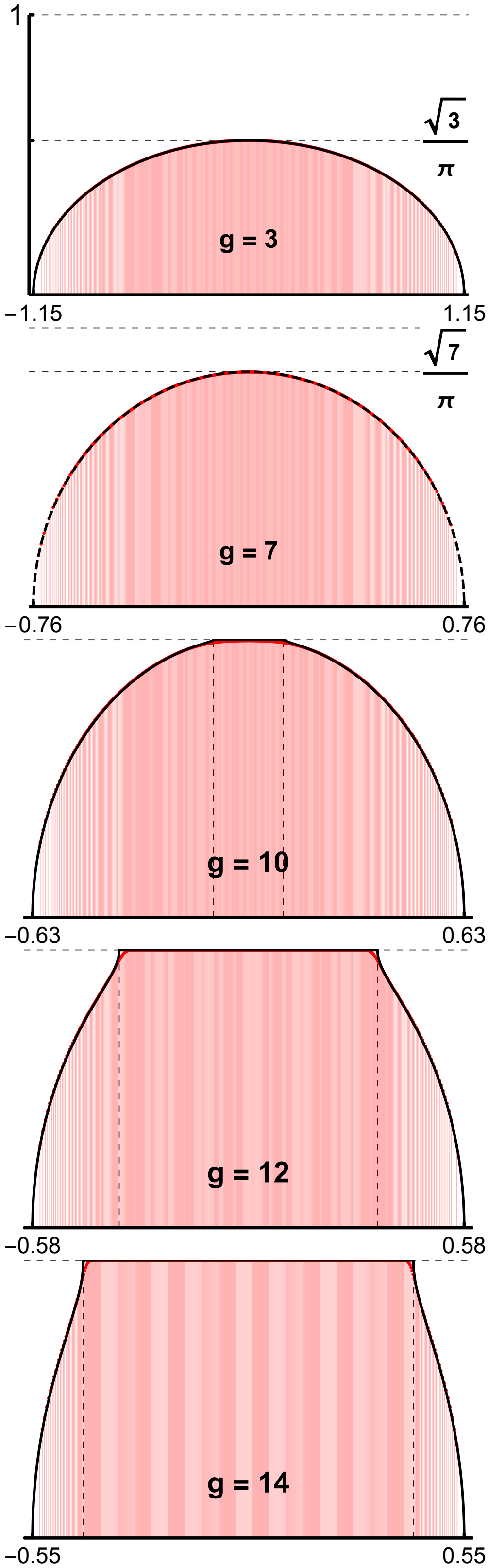

Figure 5: Continuum root distribution $\rho(k)$ (black) and numerical data points at $N = 400$ (red fill)

This distribution satisfies $\rho(k) \leq 1$ for $g < \pi^2$. When $g = \pi^2$, $\rho(0) = 1$ and the system undergoes a phase transition.

When $g > \pi^2$, we use the strong coupling asymptotics (234) make the following ansatz for ρ

$$\rho(k) = \begin{cases} 1 & k \in [-b, b] \\ \tilde{\rho}(k) & k \in [-a, -b) \cup (b, a] \end{cases} \tag{249}$$

which means that for some interval $[-b, b]$, ρ will saturate the constraint. Outside this interval, ρ has a different functional for, which is to be determined from

$$gk - 2\log\left(\frac{k-b}{k+b}\right) = 2\,\mathcal{P}\!\int_b^a \tilde{\rho}(u)\left(\frac{1}{k-u} + \frac{1}{k+u}\right) du \tag{250}$$

The solution of this integral equation is known [92, 74]

$$\tilde{\rho}(k) = \frac{2}{\pi a|k|}\sqrt{(a^2-k^2)(k^2-b^2)}\ \Pi_1\left(\frac{b^2}{k^2}, \frac{b^2}{a^2}\right) \tag{251}$$

and the parameters a and b are determined from the following conditions:

$$\begin{aligned} 4\,\mathrm{K}(x)(2\,\mathrm{E}(x) - (1-x)\,\mathrm{K}(x)) &= g \\ ag = 4\,\mathrm{K}(x) \quad \text{and} \quad x &= b^2/a^2 \end{aligned} \tag{252}$$

The functions $\mathrm{E}(x)$ and $\mathrm{K}(x)$ are the elliptic functions of the first and second kind, which in our conventions can be written as

$$\mathrm{E}(x) = \int_0^1 \frac{\sqrt{1-xu^2}}{\sqrt{1-u^2}}\, du \tag{253}$$

$$\mathrm{K}(x) = \int_0^1 \frac{1}{\sqrt{1-xu^2}\sqrt{1-u^2}}\, du \tag{254}$$

$\Pi_1(x, y)$ is the elliptic function of the third kind, defined as

$$\Pi_1(x,y) = \int_0^1 \frac{1}{(1-xu^2)\sqrt{1-yu^2}} \frac{du}{\sqrt{1-u^2}}. \tag{255}$$

Note that for $g \to \pi^2$, we have $b \to 0$ and $\tilde{\rho}(\pi^2)$ becomes a semicircle and thus the root distribution changes continuously at the phase transition.

In figure 5 we show the continuum limit root distribution for several values of the effective coupling. The numerical results for $N = 400$, obtained directly from (233), are superimposed on the graphs and match very well.

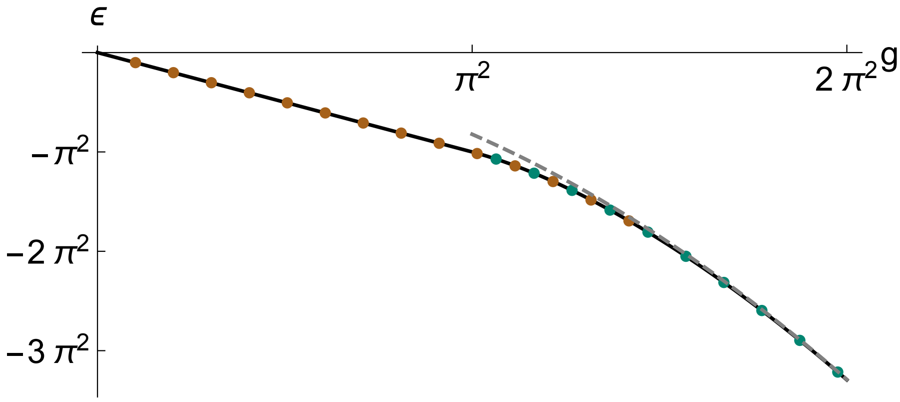

Figure 6: Ground state energy per particle. Numerical results for 400 particles are shown in brown. In green is the mean-field result in the strong coupling phase. The dashed line shows the thermodynamic limit [87].

11.4 GROUND STATE ENERGY

Although the density changes continuously at the point of phase transition, we will show that the second derivative of the ground state energy per particle is discontinuous, thus making the phase transition third order.

We will also compare the expression for the ground state energy at arbitrary coupling with the result obtained from mean-field theory, and verify that both expressions indeed coincide, as it should be expected since the mean-field theory should yield the correct large-N ground state.

Using (232) and (243), the ground state energy per particle is given by the following expression

$$\epsilon = -\frac{1}{N}\sum_i k_i^2 = -g^2 \int k^2 \rho(k)\, dk \qquad (256)$$

In the weak coupling phase (248) this expression has a very simple form. In the solitonic regime (251), we can use the integral representation of Π_1 and contour integration to calculate the energy. We have

$$-\epsilon = \begin{cases} g & \text{for} \quad g \leq \pi^2 \\ \frac{1}{48} g^2 \Big(8(a^2 + b^2) + g(a^2 - b^2)^2 \Big) & \text{for} \quad g > \pi^2 \end{cases} \qquad (257)$$

In order to check the order of the phase transition, we need to calculate the derivatives of (257) with respect to g for $g = \pi^2$. This can be done by performing a series expansion of a and b, using (252), in the vicinity of $g = \pi^2$. We find

$$-\epsilon = g + \frac{2}{\pi^2}(g - \pi^2)^2 + \mathcal{O}\big((g - \pi^2)^3\big), \text{ for } \quad g > \pi^2$$

We observe that $\epsilon(g)$ and $\epsilon'(g)$ are continuous at π^2, whereas the second derivative is discontinuous, confirming that it is indeed a second order phase transition.

We can relate (257) with the mean-field theory expression for the ground state energy. In the large-N limit, mean-field theory is expected to produce the correct ground state energy, so this serves as a nontrivial check for our method. We have, from [80], for $g > \pi^2$

$$-\epsilon_{\text{mf}} = \frac{4}{3}\frac{\mathrm{K}(m)^2}{\mathrm{E}(m)}\big((2-m)\,\mathrm{E}(m) + (1-m)\,\mathrm{K}(m)\big) \tag{258}$$

with m determined from

$$4\,\mathrm{E}(m)\,\mathrm{K}(m) = g \tag{259}$$

The parameter m has the physical interpretation of describing the physical width of the soliton in the mean-field analysis, measured in units of L. In order to reproduce our results, we need to find a transformation that relates it to x from equation (252), which is the parameter that measures how many roots are condensed in the constraint surface. This can be done through the substitution

$$\begin{aligned} m_1 &= 1 - m \\ x &= \frac{(1-\sqrt{m_1})^2}{(1+\sqrt{m_1})^2} \end{aligned} \tag{260}$$

And using the following relations [1]

$$\begin{aligned} \mathrm{K}(m) =& \frac{2}{1+\sqrt{m_1}}\,\mathrm{K}\left[\left(\frac{1-\sqrt{m_1}}{1+\sqrt{m_1}}\right)^2\right] \\ \mathrm{E}(m) =& (1+\sqrt{m_1})\,\mathrm{E}\left[\left(\frac{1-\sqrt{m_1}}{1+\sqrt{m_1}}\right)^2\right] - \frac{2\sqrt{m_1}}{1+\sqrt{m_1}}\,\mathrm{K}\left[\left(\frac{1-\sqrt{m_1}}{1+\sqrt{m_1}}\right)^2\right] \end{aligned} \tag{261}$$

1 (17.3.29) and (17.3.30) from [69].

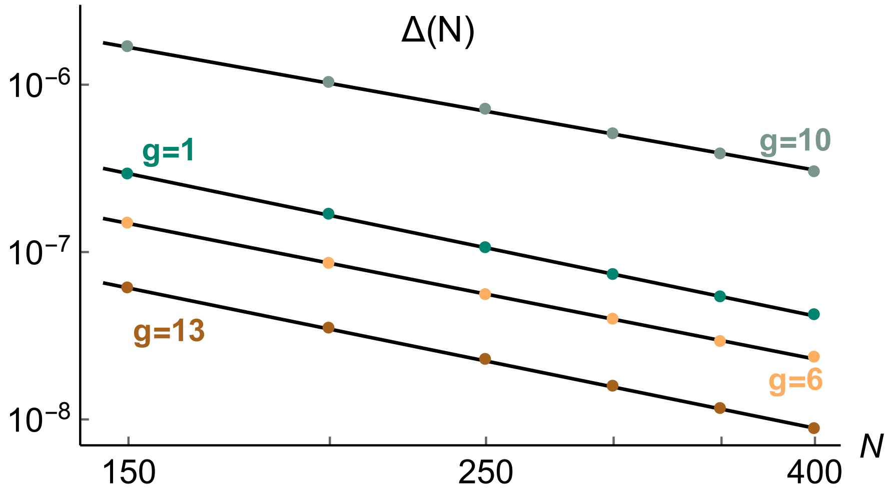

Figure 7: Asymptotic behavior of $\Delta(N)$.

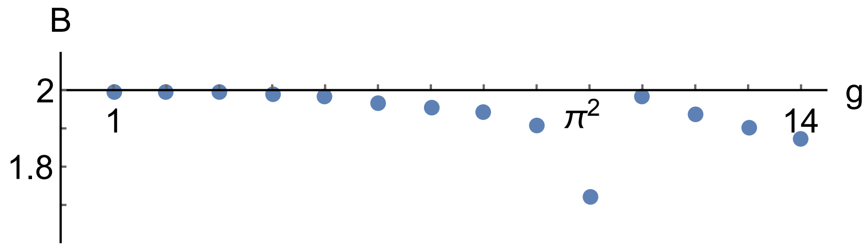

Figure 8: Best fit parameters B for different couplings.

it is possible to show the equivalence between the mean-field ground state energy and (257).

11.5 NUMERICAL CHECKS

Although there is no analytic solution of the discrete version of the Bethe equations (233), we can solve them numerically for chosen values of g. The algorithm uses the Levenberg-Marquadt solver provided by Mathematica, and in order to optimize the process, we use tracking: some initial guess is chosen for small g and then the solution for g is used as the starting searching point for $g + \delta g$.

As a measure of how close the numerical solution is from the large-N expression, we compute the mean square deviation

$$\Delta\left(N,g\right)=\frac{1}{N}\sum_{i=1}^{N}\left(k_i\left(g\right)-g\bar{k}\left(\frac{i}{N},g\right)\right)^2 \tag{262}$$

and $\bar{k}(x,g)$ is defined by numerical integration of (251, 248).

The results are shown in Fig. 7, where we display $\Delta\left(N,g\right)$ as a function of N for different values of g.

We observe that $\Delta\left(N,g\right)$ behaves like $A(g)N^{-B(g)}$. We show the best fit parameters $B(g)$ in figure 8. Based on these numbers, we conjecture that $B=2$ at large N, and we notice that the convergence seems to be slower at the point of phase transition, probably due to the increased numerical accuracy needed at that point.

11.6 EQUIVALENCE TO 2D YANG-MILLS ON A SPHERE

As we have seen, the phase transition in the 1D Bose gas can be understood using integrability techniques. In this language, the phase transition is characterized by a change in the functional form of the density of Bethe roots for the critical coupling $g=\pi^2$. This change is due to the saturation of a constraint which is lost in the continuum formulation and is then superimposed to the integral equation which defines the ground state of the system.

In a sense, this phase transition can be understood as a confinement to deconfinement phase transition. The effective coupling g grows linearly with the size of the system L, which signals that the interactions are stronger in the infrared. Furthermore, while for low couplings the mean-field ground state is homogeneous, after the phase transition the ground state is dominated by the formation of a non-perturbative bound state.

Interestingly, we can make this connection formal by mapping the ground state of this system to the saddle-point of 2D Yang-Mills on a sphere. We will briefly recap how 2D Yang-Mills can be solved, and then we will make the connection explicit.

The partition function of two-dimensional Yang-Mills can be written as a sum over representations R of the gauge group, given a manifold G of area A. Determining the saddle-point of this partition function is equivalent to determining the representation that has the highest contribution to the partition function [93].

$$Z_G(A) = \sum_R (\dim R)^{2-2G} e^{-A\lambda^2 C_2(R)/2N} \tag{263}$$

where λ is the 't Hooft coupling.

For the case of the group $U(N)$, the sum goes over all Young tableaux, which are characterized by the components of the highest weight. These are integer numbers obeying the inequality

$$\infty \geq n_1 \geq n_2 \geq ... \geq n_N \geq -\infty \tag{264}$$

and in terms of this set, we have

$$C_2(R) = \sum_{i=1}^{N} n_i(n_i - 2i + N + 1) \tag{265}$$

$$\dim R = \prod_{i>j} \left(1 - \frac{n_i - n_j}{i - j}\right) \tag{266}$$

In the large-N limit the representations can be characterized by a continuous function h, defined as

$$N\, h(i/N) \equiv -n_i + i - N/2 \tag{267}$$

and the partition function becomes

$$Z_{G=0}(A) = \int Dh(x) \exp(-N^2 S_{\text{eff}}[h])$$
$$S_{\text{eff}}[h] = -\int_0^1 \int_0^1 \log|h(x) - h(y)|\, dx\, dy + \frac{A\lambda^2}{2} \int_0^1 h(x)^2\, dx - \frac{A\lambda^2}{24} \tag{268}$$

Since the n_i are monotonic, $h(x)$ obeys the inequality $h(x) - h(y) \geq x - y$, which is equivalent to

$$\frac{\partial x}{\partial h} = \rho(h) \leq 1. \tag{269}$$

In terms of the density $\rho(h)$, the saddle point of the partition function (268) yields the following integral equation

$$A\lambda^2 h = 2\, \mathcal{P}\!\int \frac{\rho(s)}{h - s}\, ds \tag{270}$$

Comparing (246) with (270), it becomes clear that both systems obey the same integral equation. Most importantly, both systems obey the same constraint (269), which is responsible for the existence of the phase transition.

This correspondence maps the density of Young tablaux boxes h to the density of Bethe roots k and the coupling $A\lambda^2$ to the effective LL coupling g, while the number of particles N is mapped to the degree of the gauge group $U(N)$. Furthermore, the phase transition at $g = \pi^2$ in the Lieb–Liniger model appears as the confinement/deconfinement phase transition in the gauge theory.

11.7 SUMMARY

In this chapter we have studied the ground state of the attractive Bose gas using integrability techniques. We derived the integral equations that provide the density of Bethe roots in the continuum limit. We have shown that in the continuum limit, a constraint on the density arises, and by solving the constrained integral equations we were able to derive the root distribution in the ground state in the decoupling limit: $N \to \infty$ with cN fixed.

We have also shown a remarkable property of this system: in the decoupling limit, we can map the ground state of the Bose gas to the saddle point of 2D Yang-Mills on a sphere. We don't know, however, whether this correspondence has a deeper physical origin or whether it is purely a mathematical coincidence. Nevertheless, it seems like a promising avenue for future investigations, especially considering that various relations between (supersymmetric) YM theory and integrable systems have already been uncovered [88].

12

CONCLUSION

In this thesis, we have dealt with the problem of how quantum collective phenomena arises in both gravity and other bosonic theories. The nature of collective phenomena is intrinsically non-perturbative: these are regimes where the coupling constant α is not necessarily the expansion parameter of the theory: the expansion parameter becomes the collective coupling αN. When this expansion parameter becomes 1, we expect the many-body quantum phenomena to dominate over the the perturbative effects of the theory. This is often associated to a large-Ncriticality phenomena: the point in the coupling space where the system undergoes a quantum phase transition.

A clear cut definition of an occupation number of quanta does not arise immediately from classical solutions. In a standard perturbative treatment, these solutions are treated as backgrounds over which quantum fields propagate, and classical backgrounds are not treated as quantum objects themselves. In the simplest attempt to define an occupation number, we saw how both black-holes and classicalons share the criticality property: in both cases, the number of constituents scale exactly as $1/\alpha$. These systems can be taken to by dynamically stuck in a quantum phase transition. As a model for black-holes, we took the attractive Bose gas in order to exploit the properties of the quantum phase transition.

We exploited two non-perturbative methods to probe the quantum phase transition of the attractive Bose gas.

The first method was based on a truncation of the Hamiltonian, which reduces the system to a quantum mechanical problem of 3 interacting levels. We observe that we can identify the lowest lying excitations of the system as (pseuo-)goldstone bosons of a broken SU(2) symmetry. The breaking of the SU(2) happens due to the fact that the 0-momentum mode acquires an expectation value in the ground state. The gap of the excitations is given by the fact that there are explicit breaking terms, and right at the phase transition the explicit terms vanish, up to $1/N$ corrections which correspond to the frequency of these modes.

The fact that the modes have a tiny frequency has interesting consequences for information processing and storage. To start, it is extremely cheap to store information on the system: typically, one pays a price which is of the order of the inverse size of the system. Since we have a tiny gap, exciting this mode becomes easy, even in a finite sized system, and thus there is very little cost to store information. In the strict decoupling limit, we observe that the cost of information storage is 0 and also the number of states which can be occupied for these given modes diverges. Furthermore, we see that this information storage is not destabilize by the time-evolution of the system, which means that these are long lived modes, having a lifetime which is proportional to N - and become infinitely lived in the strict N inf limit.

These properties are very similar to what one observes in Black-Hole physics. Also for black-holes, we observe that the entropy of a classical system is infinite, and the system is long lived. By taking the mass to be finite, we observe that for $t \propto N$, which corresponds to Page time, information release for the black-hole should be maximal. This leads us to believe that in fact the information processing properties of black-holes can be shared by many other systems, which may even be accessible in table-top experiments.

The second method we used was integrability. It is well-known that this system can be solved through means of the Bethe ansatz. Using the Bethe ansatz, we are able to transform the Hamiltonian eigenvalue problem into a system of algebraic equations, which can be solved either analytically or numerically. Although we know that the system undergoes a large-N phase transition at $\alpha N = 1$, there is no known way to analytically solve the Bethe equations in this regime in order to study this phase transition using integrability techniques.

By going to the continuum limit of the equations, we were able to transform the system of algebraic equations into one constrained integral equation. This integral equation, together with the constraint, is very familiar from the study of Yang-Mills in 2D. It corresponds to the saddle-point equation of this theory defined on a 2-sphere. It is not a surprise that this system also undergoes a phase transition: the Douglas-Kazakov (Gross-Witten) phase transition.

In Yang-Mills, this phase transition interpolates between the confinement and deconfinement phases of the theory. Although there are no propagating degrees of freedom, this statement can be made since one can still check the scaling of Wilson loops. Although we do not know yet whether this correspondence goes beyond the mathematical statement, we are optimistic, since recent discoveries seem to indicate that there is an intimate general relationship between critical systems and gauge theories defined on boundaries [97].

It would be instructive to use this method to calculate the correlation functions of the system, as well as the excitation spectrum. We have made progress in both of these problems, and will include the results in a future publication.

Many-body states also manifest themselves in a topological nature. We know that the vacuum of Yang-Mills is infinitely degenerate, and there are classical field configurations that interpolate between these vacua at different times. These are called instantons, and play a very important role in QCD.

In a quantum field theory, symmetries of the classical action need not be symmetries of the full theory. In the case of massless fermions coupled to Yang-Mills, the classically conserved chiral symmetry is anomalous, and the divergence of the chiral anomaly is given by $F\tilde{F}$. In the effective action, this corresponds to a shift of the θ angle of the QCD vacuum, thus making this angle unobservable.

When we have a mass gap, we no longer have the ability to rotate the θ angle away by a chiral rotation, thus promoting the θ angle to an observable which can be related to the topological susceptibility of the vacuum. Whenever this quantity, given by the correlator $\langle F\tilde{F}, F\tilde{F}\rangle$, is non-zero, physical quantities, such as the neutron dipole moment, are dependent on the θ angle. It is important to notice that $F\tilde{F}$ is a total derivative, and the only reason why one should expect physical consequences from $F\tilde{F}$ is due to the fact that the theory contains instantons in the spectrum. Using current algebra methods, it is possible to relate this quantity to the physical parameters of the theory, and that gives us an upper bound $\theta < 10^{-9}$.

The problem of explaining the smallness of this parameter is known as the strong CP problem. The most embraced solution, the Peccei-Quinn mechanism [7], relies on introducing a new particle in the spectrum, whose expectation value corresponds to the value of θ. The coupling to the $F\tilde{F}$ dual term ensures that a mass gap for the axion will be generated, thus forcing its vacuum expectation value to be 0.

This problem can be reformulated in the language of 3-forms [12]. In this language, the axion is a 2-form which is eaten up by the QCD

Chern Simons 3-form. The strong CP problem exists whenever this 3-form is massless, since it then can propagate a long range correlation. In this formulation of the PQ solution, the axion 2-form is eaten up in order to generate a mass gap. This mass gap the "de-electrifies" the vacuum, thus solving the strong CP problem.

In the presence of gravity, we can ask the following question: how does GR affect the solution of the strong CP problem? If this is true, then there should be non-perturbative gravitational effects that will render the gravitational equivalent of $F\tilde{F}$ observable. This in fact exists, and it is the $R\tilde{R}$ correlator, which corresponds to the divergence of the gravitational Chern-Simons three-form.

In the first part of the paper, we provided the necessary conditions that gravity must fulfil in order to spoil the axion solution to the strong CP problem. More specifically, we have shown, using the language of 3-forms, that there is only one possible way in which gravity can couple to the axion in order to make θ_{QCD} non-zero again: the gravitational Chern-Simons 3-form must also mix with the axion.

In this situation, we have two 3-forms and only one extra 2-form to provide a mass gap, so one combination of the QCD and gravitational 3-form will be massive, while the other remains massless. In order to solve the problem, we need to make both combinations massive, thus we need another particle that plays the role of an axion. In a chiral symmetry, as we said, this issue is automatically solved, since we can just rotate the θ term away. Unfortunately there are no chiral quarks that we know of. There are, however, very light particles which are insensitive to QCD, only caring about gravity: these are the neutrinos.

We have shown that despite the fact that neutrinos are massive, as long as the lightest neutrino mass is small enough compared to the scale at which the gravitational non-perturbative effects take place, it is able to screen the gravitational Chern-Simons 3-form.

Quantum mechanically, GR is the theory of a self-interacting spin-2 particle. The self-interactions of this degree of freedom - the graviton - are a necessary condition in order to enforce diffeomorphism invariance, which becomes a gauge redundancy of the quantum theory. It is impossible to change the structure of the self-interactions without introducing extra degree of freedom, and for that reason, the theory is bound to be nonrenormalizable, with a perturbative unitarity breaking scale of $M_P \approx 10^{19}$GeV.

The breakdown of perturbative unitarity is related to the process of black-hole creation. Whenever we try to scatter particles with a center of mass momentum higher than the Planck scale, we necessarily end up creating a black-hole. Based on this observation, it was conjectured [25] that there are no microscopic degrees of freedom beyond the Planck scale, and in reality black-holes are the true degrees of freedom of the theory in the deep ultraviolet.

This statement has very puzzling consequences, since it implies a departure from the Wilsonian paradigm of quantum field theory. In the Wilsonian paradigm, whenever we reach the unitarity breaking scale of the theory, there is the appearance of new microscopic, weakly coupled degrees of freedom that restore the unitarity of the theory. These degrees of freedom need not be elementary degrees of freedom of the theory, but they should be able to be treated as elementary in the scale at which they set it. The UV-completion of a Wilsonian theory is then equivalent of "opening up" the interaction vertices. This corresponds to replacing coupling constants by propagators of heavier degrees of freedom with some massive pole.

In gravity, this procedure fails for two reasons: the attempt to change the interaction vertices inevitably introduce new, possibly dangerous, degrees of freedom which would be observed a low energies. The second source of problems is the fact that in gravity, high energy degrees of freedom are no longer confined to short distances: black-holes provide a UV - IR correspondence, since heavy black-holes are large!

If black-holes are indeed the UV degrees of freedom of the theory, it is impossible to probe distances shorter than the Planck length. Any attempt to probe these scales will inevitably create a mean field configuration which is much larger than the Planck scale. This motivated the proposal that in fact, gravity is not UV-completed by new weakly coupled degrees of freedom, it is self-complete thanks to the fact that black-holes make hide the microscopic structure of the theory. The theory is UV-completed by Classicalization: high energy scattering leads to the creation of large mean-field configurations.

One can extend this reasoning and ask whether it is possible that this behaviour is reproduced in other bosonic theories. In the second part of this thesis, we exploited the properties that these theories must fulfil in order to have the possibility to be UV-completed by classicalization. More specifically, the sign of the quartic derivative coupling encodes information about the path which the theory will following in the UV.

While for the positive sign we can treat this theory as an Effective Field Theory of, say, a linear sigma model, for the negative sign there are obstructions to UV-completing this theory in the Wilsonian way. It is possible to show that any theory that contains a heavy massive pole that is integrated out will lead to a positive sign: we are show this using spectral representation, thus avoiding the regularization issues in dealing with the analyticity analysis of the S-Matrix [33].

Furthermore, the theory with the negative sign allows for the propagation of superluminal modes. If the theory is UV-completed by weakly coupled physics, then it would be possible to use these modes to create closed timelike curves and thus violate causality. In a classicalizing theory, however, the large background remains at high energies, and thus the Lorentz group is no longer a symmetry of the theory, and we are not able to have boosted observers in the UV: boosting will create large mean-fields.

The connection between the negative sign and classicalization can be made explicit by noticing that there are only continuous classical field configurations for localized sources in the theory in which the sign is negative. That means that localizing a source will be followed by the creation of this large mean-field, whereas in the positive sign there is no possible field configuration that can be formed around this source: the theory begs for extra degrees of freedom.

BIBLIOGRAPHY

[1] Abbott, B. P., et al, "Observation of gravitational waves from a binary black hole merger." Physical Review Letters **116.6** (2016) 061102.

[2] S. Weinberg, Phys. Rev. D **11**, 3583 (1975). doi:10.1103/PhysRevD.11.3583

[3] S. L. Adler, Phys. Rev. **177**, 2426 (1969): J. S Bell and R. Jackiw, Nuovo Cimento **60**, 47 (1969); W. A. Bardeen, Phys. Rev. **184**, 1848 (1969)

[4] W. A. Bardeen, Nucl. Phys. **B75**, 246 (1974)

[5] R. Bott, Bulletin de la Societe Mathematique de France **84**, (1956): 251-281.

[6] G. Dvali, S. Folkerts and A. Franca, Phys. Rev. D **89**, no. 10, 105025 (2014) doi:10.1103/PhysRevD.89.105025 [arXiv:1312.7273 [hep-th]].

[7] R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. **38**, 1440 (1977).

[8] F. Wilczek, Phys. Rev. Lett. **40**, 279 (1978).

[9] S. Weinberg, Phys. Rev. Lett. **40** (1978) 223.

[10] C. Vafa and E. Witten, Phys. Rev. Lett. 53 (1984) 535.

[11] R. Kallosh, A. D. Linde, D. A. Linde and L. Susskind, "Gravity and global symmetries," Phys. Rev. D **52** (1995) 912 [hep-th/9502069].

[12] G. Dvali, "Three-form gauging of axion symmetries and gravity," hep-th/0507215.

[13] G. Dvali, R. Jackiw and S. -Y. Pi, "Topological mass generation in four dimensions," Phys. Rev. Lett. **96** (2006) 081602 [hep-th/0511175].

[14] F. Quevedo and C. A. Trugenberger, Nucl.Phys. B501 (1997) 143-172, hep-th/9604196.

[15] R. Delbourgo and A. Salam, Phys. Lett. B40 (1972) 381;

T. Eguchi and P. Freund, Phys. Rev. Lett. 37 (1976) 1251;

L. Alvarez-Gaume and E. Witten, Nucl. Phys. B234 (1983) 269.

[16] Q. R. Ahmad *et al.* [SNO Collaboration], Phys. Rev. Lett. **89** (2002) 011301 [nucl-ex/0204008].

[17] Particle Data Group, Amsler, C., Doser, M., et al. 2008, Physics Letters B, 667, 1

[18] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **166** (1980) 493.

[19] C. A. Baker, D. D. Doyle, P. Geltenbort, K. Green, M. G. D. van der Grinten, P. G. Harris, P. Iaydjiev and S. N. Ivanov *et al.*, Phys. Rev. Lett. **97**, 131801 (2006) [hep-ex/0602020].

[20] M. Agostini *et al.* (GERDA), Phys.Rev.Lett. **111**, 122503 (2013), arXiv:1307.4720 [nucl-ex].

[21] M. Sturm [KATRIN Collaboration], PoS DSU **2012**, 037 (2012).

[22] M. Kamionkowski and J. March-Russell, "Planck scale physics and the Peccei-Quinn mechanism," Phys. Lett. B **282** (1992) 137 [hep-th/9202003].

[23] G. Dvali and L. Funcke, arXiv:1602.03191 [hep-ph].

[24] G. Dvali, A. Franca and C. Gomez, arXiv:1204.6388 [hep-th].

[25] G. Dvali and , C. Gomez, Self-Completeness of Einstein Gravity, arXiv:1005.3497 [hep-th];

G. Dvali, S. Folkerts, C. Germani, Physics of Trans-Planckian Gravity, [arXiv:1006.0984 [hep-th]] Phys.Rev.D84:024039,2011.

[26] G. Dvali, G. F. Giudice, C. Gomez, A. Kehagias, UV-Completion by Classicalization, arXiv:1010.1415 [hep-ph]. JHEP 2011 (2011) 108;

For recent discussions, see

E. Spallucci, S. Ansoldi, Regular black holes in UV self-complete quantum gravity, Phys.Lett. B701 (2011) 471-474, arXiv:1101.2760 [hep-th]

Cosmological Classicalization: Maintaining Unitarity under Relevant Deformations of the Einstein-Hilbert Action, Phys.Rev.Lett. 106 (2011) 191102, arXiv:1102.0313 [hep-th]; F. Berkhahn, D. D. Dietrich, S. Hofmann Consistency of Relevant Cosmological Deformations on all Scales. JCAP 1109 (2011) 024 e-Print: arXiv:1104.2534 [hep-th]; F. Berkhahn, S. Hofmann, F. Kuhnel, P. Moyassari, Dennis Dietrich, Island of Stability for Consistent Deformations of Einstein's Gravity. arXiv:1106.3566 [hep-th]

B. Bajc, A. Momen, G. Senjanovic, Classicalization via Path Integral, arXiv:1102.3679 [hep-ph]

G. Dvali, C. Gomez, Alex Kehagias.Classicalization of Gravitons and Goldstones. arXiv:1103.5963 [hep-th], JHEP 1111 (2011) 070

C. Grojean, R. S. Gupta, Theory and LHC Phenomenology of Classicalon Decays, arXiv:1110.5317 [hep-ph].

[27] G. Dvali, D. Pirtskhalava, Dynamics of Unitarization by Classicalization, Phys.Lett. B699 (2011) 78-86, arXiv:1011.0114 [hep-ph]

[28] G. Dvali, Classicalize or not to Classicalize?, arXiv:1101.2661 [hep-th]

[29] G. Dvali, C. Gomez, "Black Hole's Quantum N-Portrait," arXiv:1112.3359 [hep-th];

"Landau-Ginzburg Limit of Black Hole's Quantum Portrait: Self Similarity and Critical Exponent," arXiv:1203.3372 [hep-th];

"Black Hole's 1/N Hair," arXiv:1203.6575 [hep-th].

[30] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi, "Causality, analyticity and an IR obstruction to UV completion," JHEP **0610**, 014 (2006) [hep-th/0602178].

[31] J. Rizos, N. Tetradis, Dynamical classicalization, arXiv:1112.5546 [hep-th]

N. Brouzakis, J. Rizos, N. Tetradis, On the dynamics of classicalization, Phys.Lett. B708 (2012) 170-173, arXiv:1109.6174 [hep-th]

[32] G. Dvali, S. Hofmann, J. Khoury, "Degravitation of the cosmological constant and graviton width", Phys.Rev. D76 (2007) 084006, hep-th/0703027 [HEP-TH]

[33] Z. Komargodski and A. Schwimmer, "On Renormalization Group Flows in Four Dimensions," JHEP **1112** (2011) 099 [arXiv:1107.3987 [hep-th]].

[34] C. Deffayet, G. R. Dvali, G. Gabadadze and A. I. Vainshtein, Phys. Rev. D **65** (2002) 044026 [hep-th/0106001].

[35] J. L. Cardy, "Is There a c Theorem in Four-Dimensions?," Phys. Lett. B **215**, 749 (1988).

[36] A. B. Zamolodchikov, "Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory," JETP Lett. **43**, 730 (1986) [Pisma Zh. Eksp. Teor. Fiz. **43**, 565 (1986)].

[37] A. Schwimmer and S. Theisen, Nucl. Phys. B **847**, 590 (2011) [arXiv:1011.0696 [hep-th]].

[38] C. Burrage, C. de Rham, L. Heisenberg, A.J. Tolley, "Chronology Protection in Galileon Models and Massive Gravity", arXiv:1111.5549 [hep-th]

[39] G. Dvali, " Predictive power of strong coupling in theories with large distance modified gravity", New J. Phys. 8, 326 (2006); [arXiv:hep-th/0610013]

[40] R. Percacci, L. Rachwal, On classicalization in nonlinear sigma models, arXiv:1202.1101 [hep-th]

[41] Gerard 't Hooft, A Planar Diagram Theory for Strong Interactions. Nucl. Phys. B72 (1974) 461

[42] M. A. Luty, J. Polchinski and R. Rattazzi, "The a-theorem and the Asymptotics of 4D Quantum Field Theory", arXiv:1204.5221 [hep-th]

[43] G. Dvali, A. Franca, C. Gomez and N. Wintergerst, "Nambu-Goldstone Effective Theory of Information at Quantum Criticality," Phys. Rev. D **92**, no. 12, 125002 (2015) doi:10.1103/PhysRevD.92.125002 [arXiv:1507.02948 [hep-th]].

[44] J. D. Bekenstein, Black holes and the second law, *Lett. Nuovo Cim.* **4**, (1972), 737–740.

[45] S. W. Hawking, Black hole explosions, *Nature* **248**, (1974), 30–31

[46] S. W. Hawking, Particle creation by black holes, *Commun. Math. Phys.* **43**, (1975), 199–220

[47] S. W. Hawking, Black holes in general relativity, *Commun. Math. Phys.* **25**, (1972), 152–166.

[48] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. 31, 161 (1973).

[49] J. D. Bekenstein, "Black holes and entropy," Phys. Rev. D **7**, 2333 (1973).

[50] G. Dvali and C. Gomez, "Black Holes as Critical Point of Quantum Phase Transition," Eur. Phys. J. C **74**, 2752 (2014) [arXiv:1207.4059 [hep-th]].

[51] G. Dvali and C. Gomez, "Black Hole's Quantum N-Portrait," Fortsch. Phys. **61** (2013) 742 [arXiv:1112.3359 [hep-th]];

[52] G. Dvali and C. Gomez, "Quantum Compositeness of Gravity: Black Holes, AdS and Inflation", JCAP01(2014)023, [arXiv:1312.4795].

[53] D. Flassig, A. Pritzel and N. Wintergerst, "Black Holes and Quantumness on Macroscopic Scales," Phys. Rev. D **87**, 084007 (2013) [arXiv:1212.3344].

[54] G. Dvali, D. Flassig, C. Gomez, A. Pritzel and N. Wintergerst, "Scrambling in the Black Hole Portrait," Phys. Rev. D **88**, no. 12, 124041 (2013) [arXiv:1307.3458 [hep-th]].

[55] V.F. Foit, N. Wintergerst, "Self-similar Evaporation and Collapse in the Quantum Portrait of Black Holes", arXiv:1504.04384 [hep-th]

[56] G. Dvali, C. Gomez, R.S. Isermann, D. Lüst, S. Stieberger, "Black hole formation and classicalization in ultra-Planckian $2 \to N$ scattering", Nucl. Phys. B893 (2015) 187-235, arXiv:1409.7405 [hep-th]

[57] F. Kuhnel and B. Sundborg, "High-Energy Gravitational Scattering and Bose-Einstein Condensates of Gravitons," arXiv:1406.4147 [hep-th].

[58] Kanamoto, R., Saito, H., & Ueda, M. 2003, Phys. Rev. A, 67, 013608

[59] Carr, L. D., Charles W. Clark, and W. P. Reinhardt, "Stationary Solutions of the One-dimensional Nonlinear Schrdinger Equation. II. Case of Attractive Nonlinearity." Phys. Rev. A **62**, 063611 (2000)

[60] D. N. Page, Phys. Rev. Lett. **71**, 3743 (1993) doi:10.1103/PhysRevLett.71.3743 [hep-th/9306083].

[61] G. Dvali and C. Gomez, "Black Hole's 1/N Hair," Phys. Lett. B **719** (2013) 419 [arXiv:1203.6575 [hep-th]].

G. Dvali, C. Gomez and D. Lüst, "Black Hole Quantum Mechanics in the Presence of Species," Fortsch. Phys. **61** (2013) 768 [arXiv:1206.2365 [hep-th]].

G. Veneziano, "Quantum hair and the string-black hole correspondence," Class. Quant. Grav. **30** (2013) 092001 [arXiv:1212.2606 [hep-th]].

P. Binetruy, "Vacuum energy, holography and a quantum portrait of the visible Universe," arXiv:1208.4645 [gr-qc].

R. Casadio, "Charged shells and elementary particles," Int. J. Mod. Phys. A **28**, 1350088 (2013) [arXiv:1303.1274 [gr-qc]].

R. Casadio and A. Orlandi, "Quantum Harmonic Black Holes," JHEP **1308**, 025 (2013) [arXiv:1302.7138 [hep-th]].

R. Casadio, A. Giugno, O. Micu and A. Orlandi, "Black holes as self-sustained quantum states, and Hawking radiation," arXiv:1405.4192 [hep-th].

W. Mück, "Counting Photons in Static Electric and Magnetic Fields," Eur. Phys. J. C **73**, 2679 (2013) [arXiv:1310.6909 [hep-th]].

R. Casadio, O. Micu and F. Scardigli, "Quantum hoop conjecture: Black hole formation by particle collisions," Phys. Lett. B **732**, 105 (2014) [arXiv:1311.5698 [hep-th]].

F. Kuhnel, "Bose-Einstein Condensates with Derivative and Long-Range Interactions as Set-Ups for Analog Black Holes," arXiv:1312.2977 [gr-qc].

F. Kuhnel and B. Sundborg, "Modified Bose-Einstein Condensate Black Holes in d Dimensions," arXiv:1401.6067 [hep-th].

M. Chen and Y.C. Huang, "Gas Model of Gravitons with Light Speed," arXiv:1402.5767 [gr-qc].

S. Hofmann and T. Rug, "A Quantum Bound State Description of Black Holes," arXiv:1403.3224 [hep-th].

F. Kuhnel and B. Sundborg, "Decay of Graviton Condensates and their Generalizations in Arbitrary Dimensions," arXiv:1405.2083 [hep-th].

C.J. Hogan, "Directional Entanglement of Quantum Fields with Quantum Geometry," arXiv:1312.7798 [gr-qc].

R. Casadio, A. Giugno, O. Micu and A. Orlandi, "Black holes as self-sustained quantum states, and Hawking radiation," arXiv:1405.4192 [hep-th].

R. Casadio, O. Micu and P. Nicolini, "Minimum length effects in black hole physics," arXiv:1405.1692 [hep-th].

L. Gruending, S. Hofmann, S. Müller and T. Rug, "Probing the Constituent Structure of Black Holes," arXiv:1407.1051 [hep-th].

W. Mück, "On the number of soft quanta in classical field configurations, Canadian Journal of Physics, 2014, 92(9): 973-975, arXiv:1306.6245 [hep-th].

W. Mück, G. Pozzo, "Quantum Portrait of a Black Hole with Pöschl-Teller Potential", JHEP05(2014)128, arXiv:1403.1422 [hep-th].

R. Casadio, A. Giugno, O. Micu and A. Orlandi, "Black holes as self-sustained quantum states, and Hawking radiation," arXiv:1405.4192 [hep-th].

R. Casadio, A. Giugno, A. Orlandi, "Thermal corpuscular black holes", Phys.Rev. D91 (2015) 124069, arXiv:1504.05356 [gr-qc].

F. Kühnel, M. Sandstad, "Baryon number conservation in Bose-Einstein condensate black holes", arXiv:1506.08823 [gr-qc].

[62] G. Dvali, "Black holes with quantum massive spin-2 hair," Phys. Rev. D **74**, 044013 (2006) doi:10.1103/PhysRevD.74.044013 [hep-th/0605295].

[63] F. Berkhahn, S. Müller, F. Niedermann and R. Schneider, "Microscopic Picture of Non-Relativistic Classicalons," JCAP **1308**, 028 (2013) [arXiv:1302.6581 [hep-th]].

[64] P. Hayden and J. Preskill, "Black holes as mirrors: Quantum information in random subsystems," JHEP 0709 (2007) 120, arXiv:0708.4025 [hep-th]

[65] J.H. Traschen and R.H. Brandenberger, Phys. Rev. D 42 (1990) 2491; L. Kofman, A. D. Linde and A. A. Starobinsky, Phys. Rev. Lett. 73 (1994) 3195.

[66] J. Berges, A. Rothkopf, J. Schmidt, "Non-thermal fixed points: effective weak-coupling for strongly correlated systems far from equilibrium", Phys.Rev.Lett.101 (2008) 041603, arXiv:0803.0131 [hep-ph].

[67] J. Berges, "Introduction to nonequilibrium quantum field theory," AIP Conf. Proc. **739** (2005) 3 doi:10.1063/1.1843591 [hep-ph/0409233].

[68] D. Flassig, A. Franca and A. Pritzel, "Large-N ground state of the Lieb-Liniger model and Yang-Mills theory on a two-sphere," Phys. Rev. A **93** (2016) 1, 013627 doi:10.1103/PhysRevA.93.013627 [arXiv:1508.01515 [cond-mat.quant-gas]].

[69] Milton Abramowitz and Irene A Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*. Number 55. Courier Corporation, 1964.

[70] P. Calabrese and J.-S. Caux. Dynamics of the attractive 1D Bose gas: analytical treatment from integrability. *J. Stat. Mech. Theor. Exp.*, 8:32, August 2007.

[71] P. Calabrese and P. Le Doussal. Interaction quench in a Lieb-Liniger model and the KPZ equation with flat initial conditions. *Journal of Statistical Mechanics: Theory and Experiment*, 5:4, May 2014.

[72] Pasquale Calabrese and Jean-Sébastien Caux. Correlation functions of the one-dimensional attractive bose gas. *Phys. Rev. Lett.*, 98:150403, Apr 2007.

[73] F. Colomo and A. G. Pronko. Third-order phase transition in random tilings. *Phys. Rev. E*, 88(4):042125, October 2013.

[74] Michael R. Douglas and Vladimir A. Kazakov. Large N phase transition in continuum QCD in two-dimensions. *Phys. Lett.*, B319:219–230, 1993.

[75] Gia Dvali, Andre Franca, Cesar Gomez, and Nico Wintergerst. Nambu-Goldstone Effective Theory of Information at Quantum Criticality. 2015.

[76] Daniel Flassig, Alexander Pritzel, and Nico Wintergerst. Black holes and quantumness on macroscopic scales. *Phys. Rev.*, D87(8):084007, 2013.

[77] P. J. Forrester, S. N. Majumdar, and G. Schehr. Non-intersecting Brownian walkers and Yang-Mills theory on the sphere. *Nuclear Physics B*, 844:500–526, March 2011.

[78] Elmar Haller, Mattias Gustavsson, Manfred J Mark, Johann G Danzl, Russell Hart, Guido Pupillo, and Hanns-Christoph Nägerl. Realization of an excited, strongly correlated quantum gas phase. *Science*, 325(5945):1224–1227, 2009.

[79] C. Herzog, M. Olshanii, and Y. Castin. Une transition liquide-gaz pour des bosons en interaction attractive à une dimension. *Comptes Rendus Physique*, 15:285–296, February 2014.

[80] Rina Kanamoto, Hiroki Saito, and Masahito Ueda. Quantum phase transition in one-dimensional bose-einstein condensates with attractive interactions. *Phys. Rev. A*, 67:013608, Jan 2003.

[81] L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon. Formation of a Matter-Wave Bright Soliton. *Science*, 296:1290–1293, May 2002.

[82] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin. *Quantum Inverse Scattering Method and Correlation Functions*. Cambridge University Press, 1993.

[83] M. Kormos, A. Shashi, Y.-Z. Chou, J.-S. Caux, and A. Imambekov. Interaction quenches in the one-dimensional Bose gas. *Phys. Rev. B*, 88(20):205131, November 2013.

[84] Elliott H. Lieb and Werner Liniger. Exact analysis of an interacting bose gas. i. the general solution and the ground state. *Phys. Rev.*, 130:1605–1616, May 1963.

[85] S. N. Majumdar and G. Schehr. Top eigenvalue of a random matrix: large deviations and third order phase transition. *Journal of Statistical Mechanics: Theory and Experiment*, 1:12, January 2014.

[86] O Marichev and M Trott. functions.wolfram.com, 2015.

[87] J. B. McGuire. Study of exactly soluble one-dimensional n-body problems. *J. Math. Phys.*, 5(5):622–636, 1964.

[88] Nikita A. Nekrasov and Samson L. Shatashvili. Supersymmetric vacua and Bethe ansatz. *Nucl. Phys. Proc. Suppl.*, 192-193:91–112, 2009.

[89] N. Oelkers and J. Links. Ground-state properties of the attractive one-dimensional Bose-Hubbard model. *Phys. Rev. B*, 75(11):115119, March 2007.

[90] M. Panfil, J. De Nardis, and J.-S. Caux. Metastable Criticality and the Super Tonks-Girardeau Gas. *Physical Review Letters*, 110(12):125302, March 2013.

[91] Belén Paredes, Artur Widera, Valentin Murg, Olaf Mandel, Simon Fölling, Ignacio Cirac, Gora V Shlyapnikov, Theodor W Hänsch, and Immanuel Bloch. Tonks–girardeau gas of ultracold atoms in an optical lattice. *Nature*, 429(6989):277–281, 2004.

[92] Allen C Pipkin. *A course on integral equations*. Number 9. Springer-Verlag New York, 1991.

[93] B.E. Rusakov. Loop averages and partition functions in U(N) gauge theory on two-dimensional manifolds. *Mod. Phys. Lett.*, A5:693–703, 1990.

[94] Kaspar Sakmann, Alexej I. Streltsov, Ofir E. Alon, and Lorenz S. Cederbaum. Exact ground state of finite bose-einstein condensates on a ring. *Phys. Rev. A*, 72:033613, Sep 2005.

[95] Kevin E Strecker, Guthrie B Partridge, Andrew G Truscott, and Randall G Hulet. Formation and propagation of matter-wave soliton trains. *Nature*, 417(6885):150–153, 2002.

[96] A. G. Sykes, P. D. Drummond, and M. J. Davis. Excitation spectrum of bosons in a finite one-dimensional circular waveguide via the Bethe ansatz. *Phys. Rev. A*, 76(6):063620, December 2007.

[97] G. Dvali, C. Gomez and N. Wintergerst, arXiv:1511.03525 [hep-th].

[98] Chen-Ning Yang and CP Yang. Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction. *J. Math. Phys.*, 10(7):1115–1122, 1969.