# New obstructions to smooth nonpositively curved metrics in dimension 4

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München 2014

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Dissertation an der Fakultät für Mathematik,
Informatik und Statistik
der Ludwig–Maximilians–Universität
München

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München, den 15. April 2014

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Tag der mündlichen Prüfung: 16. Juli 2014

# Eidesstattliche Versicherung

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

*Ort, Datum* *Stephan Stadler*

# Zusammenfassung

Wir geben neue Beispiele geschlossener glatter 4-Mannigfaltigkeiten welche singuläre Metriken nichtpositiver Krümmung tragen aber keine glatten. Dies liefert eine positive Antwort auf eine Frage von Gromov. Die Obstruktion stammt von hinreichend komplizierten Mustern inkompressibler 2-Tori welche verzweigende Geodätische für nichtpositiv gekrümmte Metriken erzwingen.

# Abstract

We give new examples of closed smooth 4-manifolds which support singular metrics of nonpositive curvature, but no smooth ones, thereby answering affirmatively a question of Gromov. The obstruction comes from patterns of incompressible 2-tori sufficiently complicated to force branching of geodesics for nonpositively curved metrics.

# Acknowledgements

I am very grateful to my advisor, Bernhard Leeb, for calling my attention to the first exercise in [BGS85] and for his support and guidance during the last years. I would also like to thank Alexander Lytchak and Richard Bamler for many related and unrelated discussions as well as the geometry group at LMU for providing an inspiring working environment.

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# 1 Introduction

This thesis studies the rigidity of singular nonpositively curved metrics on certain 4-dimensional manifolds. It answers affirmatively the first problem, posed by Gromov, in the book *Manifolds of nonpositive curvature* by Ballmann, Gromov and Schroeder ([BGS85]). It has remained an open question since then and is asked on Bestvina's list of *Questions in geometric group theory*. (Question 2.5 in [Be04].)

The framework for the question is Gromov's expectation that closed smooth manifolds of dimension $\geq 4$, which admit singular metrics of nonpositive curvature, generically do *not* admit such smooth metrics (where "curvature" refers to sectional curvature). The problem concerns a natural candidate for this phenomenon in dimension 4, namely the singular pullback metrics on branched coverings of nonpositively curved 4-manifolds with totally-geodesic 2-dimensional branching locus. More precisely, it claims the following

**Theorem 1.1.** *Let $\Sigma$ be a closed hyperbolic surface, and let $\psi : V \to \Sigma \times \Sigma$ be a finite (non-trivially) branched covering with branching locus the diagonal $\Delta_\Sigma \subset \Sigma \times \Sigma$. Then $V$ does not admit a $\mathcal{C}^2$-smooth Riemannian metric of nonpositive curvature.*

We will prove Theorem 1.1 in Section 5.

The following results supporting Gromov's expectation were known before: Davis and Januszkiewicz ([DJ91]) constructed examples of closed manifolds with singular nonpositively curved metrics in dimension $\geq 5$, whose universal covers are not simply-connected at infinity. Recently (2012), Davis, Januszkiewicz and Lafont ([DJL12]) constructed such manifolds $M^4$ in dimension 4, whose universal covers $\tilde{M}$ are diffeomorphic to $\mathbb{R}^4$ (and hence simply-connected at infinity) but which contain immersed totally-geodesic flat 2-tori $T^2$ whose universal covering planes $\tilde{T} \subset \tilde{M}$ are knotted at infinity, i.e. their boundary circles $\partial_\infty \tilde{T} \cong S^1$ are wild knots in the boundary sphere $\partial_\infty \tilde{M} \cong S^3$. Both phenomena cannot occur for smooth nonpositively curved metrics (by the classical Cartan-Hadamard theorem) and therefore rule out their existence.

However, these methods do not apply to Gromov's problem because these obstructions vanish for the branched coverings. (See Section 4.4 and appendix.) In this thesis, a new obstruction is developed based on the pattern of $\pi_1$-injectively immersed flat 2-tori, equivalently, of copies of $\mathbb{Z}^2$ in the fundamental group. It yields sufficient information on the asymptotic (Tits) geometry of the universal covering to imply the branching of some geodesics, which again cannot happen for ($\mathcal{C}^2$-)smooth Riemannian metrics.

CAT(0) metrics are (possibly) singular geodesic metrics which have nonpositive sectional curvature in a triangle comparison sense. (That is, locally, geodesic triangles are not thicker than Euclidean ones, expressed by a secant comparison inequality.) Riemannian manifolds are locally CAT(0) if and only if they have nonpositive sectional curvature in the usual (differentiable) sense; important compact examples are locally symmetric spaces of noncompact type. Non-smooth CAT(0) structures arise naturally in geometric group theory, e.g. as piecewise flat polyhedral complexes; important examples here are Euclidean buildings and their quotients.

A celebrated theorem of Mostow ([Mo73]) asserts that irreducible locally symmetric spaces $\Gamma\backslash X = \Gamma\backslash G/K$ of higher rank are strongly rigid in the sense that the locally symmetric metric on the underlying manifold is unique up to scaling. Equivalently, the inclusion $\Gamma \to G$ of the fundamental group is unique up to conjugation as a discrete representation. (In particular, there are no nontrivial deformations.)

This result has been generalized by Gromov in the above mentioned book to the effect that the locally symmetric metric is unique up to scaling even among all smooth nonpositively curved Riemannian metrics (not only the locally symmetric ones). This in turn has been further generalized by Leeb ([L00]) to rigidity within all CAT(0) metrics, whether smooth or singular.

The reason for the rigidity of compact nonpositively curved metrics in higher rank is the presence of "much" extremal curvature zero: every geodesic lies in an immersed totally geodesic Euclidean plane. In contrast, rank one manifolds of nonpositive curvature, i.e. where not all geodesics have this property, tend to be non-rigid. For instance, manifolds with strictly negative curvature can be deformed, just because negative curvature is an open condition. So the question of deformability is most intriguing for rank one spaces which do not contain open subsets of negative curvature. A great source of such spaces is provided by branched covers of symmetric space or Euclidean buildings of higher rank. These are locally CAT(0) spaces which contain plenty of flat subspaces on the one hand, but where almost every geodesic is of rank one (i.e. does not bound a flat half-plane). Although these metrics have cone-type singularities along convex subsets of codimension 2, one can often arrange for their underlying spaces to be smooth. Hence, the question arises whether a deformation to a smooth metric of nonpositive curvature is possible.

Under very specific circumstances this is actually possible, e.g. if the branching locus is a fiber of a product of two surfaces. In this case one can deform, locally near the singular set, to a smooth product metric of nonpositive curvature. However, generically the global features of the metric are altered quite intricately by the branching process. As mentioned by Gromov ([G93]), local deformations supported near the singular set are out of the question. That there is no global deformation to a smooth metric of nonpositive curvature, in the particular case of a branched cover of a product of a surface with itself and branching locus the diagonal, is implied by Theorem 1.1. This space is especially critical because it comes with a large deformation space. Indeed, every locally CAT(0) metric on the base space, with the diagonal as a totally geodesic subspace, induces a locally CAT(0) metric on the total space.

The deformation space as a whole, and the deformability of branched covers of other symmetric spaces, in particular irreducible ones, will be discussed elsewhere.

This thesis is organized as follows: In Section 2 we recall some basic facts on spaces with curvature bounded above and branched coverings. In particular, we describe a specific construction for branched coverings of products of surfaces, due to Atiyah and Kodaira, and show that these carry natural locally CAT(0) metrics. We then reprove a theorem of Lang and Schroeder concerning quasi-flats in CAT(0) spaces, which plays an important role in the study of quasi-isometries. The section ends with a discussion of basic product splitting results where we give elementary proofs.

Section 3 begins by treating so called ideal books and their images under quasi-isometries in the general setting of CAT(0) spaces. We then introduce the concept of coarse intersection of flats, which is a quasi-isometry-invariant version of “nontrivial transversal intersection”. We conclude with a short discussion of intersecting product structures.

Section 4 is the technical heart of the thesis. We exploit an easy 2-dimensional observation to produce an obstruction to smooth nonpositive curvature in dimension 4. The second half reviews the few obstructions which have been known before.

The core of the thesis, Section 5, develops the proof of the main theorem.

We take the total space of the branched cover, with its natural locally CAT(0) metric, as a singular model space. We then discuss its geometry well enough to conclude that our obstruction applies.

The appendix is devoted to the study of the topology of CAT(0) branched coverings.

# 2 Preliminaries

## 2.1 Spaces with curvature bounded above

Our primary reference for this section is [KL97] but the reader is also pointed to [B95], [B04], [BGS85] and [BH99].

### 2.1.1 CBA spaces

For $\kappa \in \mathbb{R}$ denote $M_\kappa$ the simply connected model surface of constant curvature $\kappa$ and set $D_\kappa := \mathrm{diam}(M_\kappa)$. A complete metric space $(X, |\cdot|)$ is a CAT($\kappa$) space if

1. Every pair of points $x_1, x_2 \in X$ with $|x_1, x_2| < D_\kappa$ is joined by a unique geodesic segment denoted $x_1x_2$.
2. For every geodesic triangle $\triangle$ in $X$ with perimeter $< 2D_\kappa$ there is
   (i) A *comparison triangle* $\tilde{\triangle}$ in $M_\kappa$ with the same side lengths as $\triangle$;
   (ii) A 1-Lipschitz map from the convex hull of $\tilde{\triangle}$ to $X$ which maps the sides of $\tilde{\triangle}$ isometrically to the sides of $\triangle$.

We say that $X$ has curvature bounded above by $\kappa$, in symbols $K_X \leq \kappa$, if every point $x \in X$ has a neighborhood $B_x$ which is a CAT($\kappa$) space with respect to the restricted metric. $X$ will be called a *CBA space*, if there exists $\kappa \in \mathbb{R}$ such that $K_X \leq \kappa$.

**Example 1.** *A smooth Riemannian manifold $M$ has $K_M \leq \kappa$ if and only if its sectional curvature is $\leq \kappa$. Whereas $M$ is a CAT($\kappa$) space if and only if its sectional curvature is $\leq \kappa$ and its injectivity radius is $\geq D_\kappa$.*

A subset $C$ in a CAT($\kappa$) $X$ is called *convex* if for every $x_1, x_2 \in C$ with $|x_1, x_2| < D_\kappa$ the unique geodesic $x_1x_2$ is contained in $C$. If $C$ is a closed convex set and $r \leq \frac{D_\kappa}{2}$, then the tubular neighborhoods $N_r(C)$ are again convex and there are well defined continuous nearest point projections $\mathrm{pr} : N_r(C) \to C$. Every closed convex subset of a CAT($\kappa$) space is itself a CAT($\kappa$) space with respect to the induced metric. Besides passing to closed convex subsets there is a very useful tool to construct new CAT($\kappa$) spaces from old ones.

**Lemma 2.1** (Theorem 11.1 in [BH99])**.** *Let $X_1, X_2$ be CAT($\kappa$) spaces containing closed convex subsets $C_1, C_2$. If there is an isometry $\varphi : C_1 \to C_2$, then the space $Y = X_1 \cup_\varphi X_2$, obtained by gluing $C_1$ to $C_2$ via $\varphi$, is again CAT($\kappa$).*

In addition to this flexible construction, CAT($\kappa$) spaces are also stable under various limiting operations. We only mention Gromov-Hausdorff limits and ultralimits. For precise definitions and statements we refer the reader to [KL97].

Three points $x, y, z$ in a CAT($\kappa$) space which fulfill $|x, y| + |y, z| + |z, x| < 2D_\kappa$ define a geodesic triangle $\triangle(x, y, z)$. Moreover, we can associate to every vertex, say $x$, a *comparison angle* $\tilde{\angle}_x(y, z)$ which is the angle at the corresponding vertex $\tilde{x}$ of the comparison triangle $\tilde{\triangle}(x, y, z)$. If $y', z' \neq x$ are points on the geodesic segments $xy, xz$ then we have $\tilde{\angle}_x(y', z') \leq \tilde{\angle}_x(y, z)$ and we can define the angle between the geodesic segments $xy$ and $xz$ at $x$ by

$$\angle_x(y, z) := \lim_{y' \to x, z' \to x} \tilde{\angle}_x(y', z').$$

The set of *geodesics germs* $\Sigma_x^* X$ at a point $x \in X$ is obtained from the set of geodesic segments emanating from $x$ by identifying segments of zero angle at $x$. The angle descends to a metric on $\Sigma_x^* X$ and we define the *space of directions* or *link* $\Sigma_x X$ to be the metric completion of $\Sigma_x^* X$. There is a natural continuous *logarithm map* $\log_x : B_{D_\kappa} \setminus \{x\} \to \Sigma_x X$ which sends a point $y$ to the initial direction of the geodesic segment $xy$.

An important fact, due to Nikolaev [Ni95], is that the space of directions $\Sigma_x X$ of a CAT($\kappa$) space $X$ together with the angle metric is a CAT(1) space. This is why CAT(1) spaces play a distinguished role when studying spaces with curvature bounded above.

## 2.2 CAT(0) spaces

### 2.2.1 Globalization and topology

We saw in Example 1 above, that the CAT($\kappa$) condition is more than an upper curvature bound. For $\kappa$ equal to zero, one has the following relation.

**Theorem 2.2** (Cartan-Hadamard, Theorem 6.11 in [B04])**.** *Let $X$ be a complete and simply connected space with $K_X \leq 0$. Then $X$ is a CAT(0) space.*

Any two points in a CAT(0) space are joined by a unique geodesic and therefore CAT(0) spaces are contractible. Whereas every Hadamard manifold is diffeomorphic to a Euclidean space, a generic CAT(0) space is not homeomorphic to any Euclidean space. A simple example is obtained from gluing two Euclidean spaces of positive dimension along a point. Such a space is finite dimensional, locally compact and geodesically complete. Nevertheless it contains branching geodesics, as every finite dimensional example would[1]. Recall that a *branching geodesic* is a geodesic segment which extends in several ways to a longer geodesic.

### 2.2.2 Products and projections

The product of two CAT(0) spaces $X_1, X_2$ is a CAT(0) space with respect to the Pythagorean metric

$$|\cdot|_{X_1\times X_2} := \sqrt{|\cdot|^2_{X_1} + |\cdot|^2_{X_2}}.$$

Both spaces $X_i$ appear in $X_1 \times X_2$ as closed convex subsets and the natural projections $\mathrm{pr}_{X_i} : X_1 \times X_2 \to X_i$ are surjective 1-Lipschitz maps. More generally, the nearest point projection

$$\mathrm{pr}_C : X \to C$$

onto a closed convex subset $C$ in a CAT(0) space $X$ is always surjective and 1-Lipschitz. In the particular case of a projection between concentric distance balls we write

$$c_{t,s} : \overline{B_t(p)} \to \overline{B_s(p)}$$

where $0 < s < t$ and $p$ is a point in a CAT(0) space $X$.

[1]By Theorem B in [K99], for an n-dimensional CAT(0) space $X$ there is a point $x \in X$ such that the round (n-1)-sphere $S^{n-1}$ embeds isometrically into $\Sigma_x X$. If there is no branching in $X$, then $S^{n-1} \cong \Sigma_x X$.

### 2.2.3 Asymptotic data

Let $X$ be a CAT(0) space. The *ideal boundary* $\partial_\infty X$ is defined as the set of geodesic rays in $X$ modulo finite Hausdorff distance. Equivalence classes are called *ideal boundary points* or *points at infinity.*

For every point $x \in X$ and every ideal boundary point $\xi \in \partial_\infty X$ there is a unique ray $x\xi$ starting in $x$ and representing $\xi$. Therefore, pointed Hausdorff convergence of geodesic rays starting in $x$ defines a topology on $\partial_\infty X$. This topology is independent of the chosen point. It is called the *cone topology* $\mathcal{T}_{cone}$ and discribes the topology of larger and larger distance spheres [2]. The cone topology extends to $\bar{X} := X \cup \partial_\infty X$. If $X$ is locally compact, then both $\bar{X}$ and $\partial_\infty X$ are compact. The logarithmic maps $\log_x$ extend to continuous maps $\bar{X} \to \Sigma_x X$.

**Example 2.** *Let $X$ be a n-dimensional Hadamard manifold, i.e. a simply connected geodesically complete Riemannian manifold of nonpositive curvature. Then $\partial_\infty X \cong S^{n-1}$. If $Y$ is the universal cover of two circles glued together at a single point, then $\partial_\infty Y$ is a Cantor set.*

The monotonicity of comparison angles allows us to measure angles between ideal boundary points. More precisely, for $p \in X$ and $\xi, \eta \in \partial_\infty X$ we set

$$\angle_{Tits}(\xi, \eta) := \lim_{x \to \xi, y \to \eta} \tilde{\angle}_p(x, y)$$

where $x, y$ glide along $p\xi, p\eta$. We call $\partial_{Tits} X := (\partial_\infty X, \angle_{Tits})$ the *Tits boundary* of $X$. $\partial_{Tits} X$ is a CAT(1) space which encodes the geometry of larger and larger distance spheres[3]. The maps $\log_x : \partial_{Tits} X \to \Sigma_x X$ are surjective and 1-Lipschitz.

**Example 3.** *For $\mathbb{R}^n$, equipped with the Euclidean metric, $\partial_{Tits}\mathbb{R}^n$ is isometric to the (n-1)-dimensional round unit sphere. In contrast, the Tits boundary of hyperbolic space $\mathbb{H}^n$ is discrete.*

### 2.2.4 Quasi-isometries and group actions

A (possibly discontinuous) map $\Phi : X \to X'$ between metric spaces $(X, |\cdot|_X)$ and $(X', |\cdot|_{X'})$ is a quasi-isometric embedding, if there are constants $L \geq 1$

[2] If $X$ is smooth Riemannian away from a set of codimension 2 then $\partial_\infty X$ is homeomorphic to the inverse limit of distance spheres as the radius goes to infinity. See Theorem 6.5.

[3] See p.43 in [BGS85].

and $A \geq 0$ such that for every $x_1, x_2 \in X$

$$\frac{1}{L}|x_1, x_2|_X - A \leq |\Phi(x_1), \Phi(x_1)|_{X'} \leq L|x_1, x_2|_X + A.$$

If in addition there is a constant $C \geq 0$ such that $X' = N_C(\Phi(X))$, then $\Phi$ is a *quasi-isometry*.

An isometric action $\Gamma \curvearrowright X$ of a discrete group on a metric space is *properly discontinuous* if for every compact set $K \subset X$ the set

$$\{\gamma \in \Gamma |\ \gamma K \cap K \neq \emptyset\}$$

is finite; it is *cocompact* if there is a compact set $\bar{K} \subset X$ such that

$$X = \bigcup_{\gamma \in \Gamma} \gamma \bar{K}.$$

A discrete group acts *geometrically* on a metric space if it acts properly discontinuously, cocompactly and isometrically.

The fundamental lemma of geometric group theory relates geometric group actions $\Gamma \curvearrowright X$, $\Gamma \curvearrowright X'$ to quasi-isometries $X \to X'$.

**Lemma 2.3** (Lemma 5.35 in [DK14])**.** *If a discrete group $\Gamma$ acts geometrically on two proper geodesic spaces $X$ and $X'$, then there is a quasi-isometry $\Phi : X \to X'$ which is (quasi-)$\Gamma$-equivariant. I.e. there is a constant $D \geq 0$ such that*

$$|\Phi(\gamma x), \gamma\Phi(x)|_{X'} \leq D$$

*for all $\gamma \in \Gamma$, $x \in X$.*

### 2.2.5 Geometric dimension

We will rely on the notion of *geometric dimension* defined and investigated by Kleiner in [K99]. It reads as follows.

**Definition 2.4.** The *(geometric) dimension* $\dim(X)$ of a CBA space $X$ is the smallest function on the class of CBA spaces such that a) $\dim(X) = 0$ if $X$ is discrete, and b) $\dim(X) \geq 1 + \dim(\Sigma_p X)$ for every $p \in X$.

*Remark* 2.5. For Riemannian manifolds as for piecewise Riemannian polyhedral complexes the geometric dimension agrees with the usual dimension.

It was shown in [K99], as part of Theorem A, that the geometric dimension of a CAT(0) space $X$ equals the supremum of topological dimensions of compact subsets $K \subset X$.

By Theorem C in [K99], for a locally compact cocompact CAT(0) space $X$ is equivalent:

(i) There exists an isometric embedding of k-dimensional Euclidean space;

(ii) There exists a quasi-isometric embedding of k-dimensional Euclidean space;

(iii) The dimension of $\partial_{Tits} X$ is $\geq k-1$.

Therefore if a discrete group $\Gamma$ acts geometrically on a CAT(0) space $X$, then $\dim(\partial_{Tits} X)$ is an invariant of $\Gamma$.

## 2.3 Branched coverings

Our general reference for this section is [GS99].

**Definition 2.6.** A (non-singular) *k-fold branched covering* is a smooth (proper) map $f : M^n \to N^n$ with a critical subset $B \subset N$ called *branching locus* such that

$$f|_{M \setminus f^{-1}(B)} : M \setminus f^{-1}(B) \to N \setminus B$$

is a k-fold covering and such that for every point $p \in f^{-1}(B)$ there exists charts $U, V \to \mathbb{C} \times \mathbb{R}^{n-2}$ around $p$ respectively $f(p)$ such that $f$ is locally given by $(z, x) \to (z^l, x)$ for a natural number $l$, the *branching index at p*. Moreover, we define the *singular set* $M^{sing}$ to be the preimage of $B$ under $f$.

*Remark* 2.7.

- The singular set consits exactly of the points in $M$ where the pull-back of the metric on $N$ is not Riemannian.
- For critical points the branching index is greater than or equals 2. Moreover it is constant on components of $C$.
- If we write $\mathrm{ind}(p)$ for the branching index at $p$, we obtain the following formula for the degree: $\deg(f) = \sum_{q \in f^{-1}(p)} \mathrm{ind}(q)$.

- $f$ is completely determined by the subgroup $\pi_1(M \setminus f^{-1}(B)) \subset \pi_1(N \setminus B)$. The reason is that this group determines the covering on the complement of the critical sets. But the lift of the normal circle-bundle along $B$ extends uniquely to a disc-bundle.
- $f$ is called *cyclic*, if $M \setminus f^{-1}(B)$ is given by an epimorphism $\pi_1 : (N \setminus B) \twoheadrightarrow \mathbb{Z}_l$.

**Example 4.** *Probably the most common appearance of branched coverings are holomorphic maps between closed Riemannian surfaces.*

From now on we will restrict our attention to complex surfaces. Let $N$ be a smooth complex surface and $B$ a divisor in $N$. We denote by $L_B$ the line-bundle associated to $B$. If $B$ is topologically divisible, then there is a general construction which produces a cyclic branched covering of $N$ with branching locus $B$.

**Lemma 2.8** (Construction of branched covers)**.** *If $c_1(L_B)$ is divisible by $k$ in $H^2(N, \mathbb{Z})$, then there exists a $k$-fold cyclic branched covering $\Phi : M \to N$ with branching locus $B$.*

*Proof.* First we show that there is a line-bundle $L \in H^1(N, \mathcal{O}^*)$ such that $L^k \cong L_B$. Indeed, recall that the exponential sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp(2\pi i)} \mathcal{O}^* \to 0$$

induces the exact sequence

$$\ldots \to H^1(N, \mathcal{O}) \xrightarrow{i^*} H^1(N, \mathcal{O}^*) \xrightarrow{\delta} H^2(N, \mathbb{Z}) \xrightarrow{j^*} H^2(N, \mathcal{O}) \to \ldots$$

Now, we find an element $\alpha \in H^2(N, \mathbb{Z})$ such that $c_1(L_B) = k\alpha$. $c_1(L_B) = \delta(L_B)$ implies $0 = j^*(k\alpha) = kj^*(\alpha)$. Since $H^2(N, \mathcal{O})$ is a complex vector space, $\alpha$ lies in the kernel of $j^*$ and we find a line-bundle $L_\alpha$ with $\delta(L_\alpha) = \alpha$. Then $L_B - kL_\alpha$ lies in the kernel of $\delta$ and therefore in the image of $i^*$. Hence we find an element $\beta \in H^1(N, \mathcal{O})$ with $i^*(\beta) = L_B - kL_\alpha$. But $\beta$ is divisible by $k$ since $H^1(N, \mathcal{O})$ is a complex vector space and therefore $L_B$ is divisible by $k$ as well.
Let $s_B$ be a holomorphic section of $L_B$ vanishing exactly along $B$. We set $M := \{s_p \in L | \, s_p^k = s_B(p)\}$. Then $\Phi$ is provided by the restriction of $L \to N$ to $M$. □

We follow up with a short local description of the branched cover we have just constructed. In particular, we show that if $B$ is a smooth curve in $N$, then $M$ will be smooth.

**Lemma 2.9.** *If $B$ is a smooth curve in the smooth complex surface $N$, then $M$, as constructed above, is smooth as well.*

*Proof.* Let $\kappa : U \to \mathbb{C}^2$ be a chart for $N$ such that $\kappa(U \cap B)$ is given by the zero-locus of a holomorphic function $f$. Then, the inverse image $\Phi^{-1}(U)$ is given by the zero-locus of $g(x,y,z) = z^k - f(x,y)$. We need to show that the gradient of $g$ is nontrivial on the zero-locus of $g$. So let us assume that $g(x,y,z) = 0$. Then $\frac{\partial g}{\partial z}(x,y,z) = 0$ is equivalent to $z = 0$ which implies $f(x,y) = 0$. Hence we saw that on the zero-locus of $g$ the vanishing of the gradient $\nabla g(x,y,z)$ is equivalent to $z = 0$ and $\nabla f(x,y) = 0$. But $B$ is a smooth curve and therefore $\nabla f \neq 0$ on $\{f = 0\}$. $\square$

## 2.4 The Atiyah-Kodaira example

In this paragraph we will construct the object of main interest for this article: a branched cover $V \to \Sigma \times \Sigma$ with branching locus equal to the diagonal $\Delta \subset \Sigma \times \Sigma$ and where $\Sigma$ is a closed surface of higher genus. Such spaces first appeared, around the same time, in [A69] and [Ko67]. There, the viewpoint was topological and Atiyah and Kodaira, independently, constructed these spaces to provide examples of surface bundles over surfaces with non-zero signature. Our point of view is more geometric where the significance of such branched coverings lies in the fact that they often carry natural locally CAT(0)-metrics. (See Section 2.5 below.) We present their construction following [A69].

Let $\Sigma$ be a closed surface of genus $g \geq 2$. Choose a 2-fold covering $\Sigma' \xrightarrow{\varphi} \Sigma$ and denote by $g'$ the genus of $\Sigma'$. Let $\Sigma'' \xrightarrow{\psi} \Sigma'$ be the covering of $\Sigma'$ corresponding to

$$\pi_1(\Sigma') \to H_1(\Sigma', \mathbb{Z}) \to H_1(\Sigma', \mathbb{Z}_2).$$

We define a divisor $B \subset \Sigma'' \times \Sigma'$ by $B = \operatorname{graph}(\psi) \cup \operatorname{graph}(\tau \circ \psi)$ where $\tau$ is the nontrivial element in the deck group of $\varphi$. As before we will denote the associated line-bundle of $B$ by $L_B$. To apply Lemma 2.8 we have to show that $c_1(L_B)$ is divisible by 2 in $H^2(\Sigma'' \times \Sigma', \mathbb{Z})$. Note that under the identification

$$H^2(\Sigma'' \times \Sigma', \mathbb{Z}_2) \cong \bigoplus_{i=0}^{2} \mathrm{Hom}(H^i(\Sigma', \mathbb{Z}_2), H^i(\Sigma'', \mathbb{Z}_2)),$$

given by the Künneth formula and Poincaré duality, $c_1(L_B)$ maps to $\sum_{i=0}^{2}(\psi_i^* + (\tau \circ \psi)_i^*)$. Obviously we have $\psi_0^* = 1$ and $\psi_2^* = \deg(\psi) = 0$. By our choice of $\psi$ we also have $\psi_1^* = 0$ over $\mathbb{Z}_2$ as can be read off the following commutative diagram.

$$\begin{array}{ccccc} \pi_1(\Sigma'') & \xrightarrow{\Psi_*} & \pi_1(\Sigma') & \longrightarrow & \mathbb{Z}_2^{2g'} \\ \downarrow & & \downarrow & \nearrow & \\ H_1(\Sigma'') & \xrightarrow[(\Psi_1)_*]{} & H_1(\Sigma') & & \end{array}$$

Since we get the same results for the maps $(\tau \circ \psi)_i^*$, we conclude $c_1(L_B) = 0$ in $H^2(\Sigma'' \times \Sigma', \mathbb{Z}_2)$. Therefore Lemma 2.8 provides us with a 2-fold cyclic branched covering $\Phi : V \to \Sigma'' \times \Sigma'$ with branching locus $B$. The desired branched covering over $\Sigma \times \Sigma$ with branching locus $\Delta \subset \Sigma \times \Sigma$ is then given by the composition

$$V \xrightarrow{\Phi} \Sigma'' \times \Sigma' \xrightarrow{(\varphi,\varphi)\circ(\psi,\mathrm{id})} \Sigma \times \Sigma.$$

*Remark* 2.10. It is known that $V$ is algebraic. Also, the composition of $V \to \Sigma \times \Sigma$ with one of the projections $\Sigma \times \Sigma \to \Sigma$ is a smooth fiber bundle but *not* a complex analytic bundle. (See [A69].)

## 2.5 Branched coverings are locally CAT(0)

In Section 4.4 of [G87] Gromov observed that under some mild conditions branched coverings of nonpositively curved spaces are again nonpositively curved. Let $f : M^n \to N^n$ be a nontrivial finite branched covering between $n$-dimensional closed manifolds $M$ and $N$ and assume that $N$ is nonpositively curved and the branching locus $B \subset N$ is totally geodesic. The following argument is taken from [Al00].

**Lemma 2.11.** *$M$ is locally CAT(0) with respect to the induced path metric.*

*Proof.* Since the question is local, we may assume that $M$, $N$ and $B$ are simply connected. We are going to show that $M$ is a Gromov-Hausdorff limit of CAT(0) spaces. For $\epsilon > 0$ define $M_\epsilon := M \cup_f \overline{N_\epsilon(B)}$ and provide it with its path metric. More precisely, $M_\epsilon$ is obtained from $M$ by identifying preimages of points in $\overline{N_\epsilon(B)}$ while leaving the rest of $M$ untouched. We claim that $M_\epsilon$ is CAT(0). Since it is simply connected and complete it is enough, by Theorem 2.2, to show that it is locally CAT(0). This is clear for points $x \in M_\epsilon$ with $|x, B| \neq \epsilon$. If $x \in M_\epsilon$ has $|x, B| = \epsilon$, then we obtain a CAT(0) neighborhood of $x$ by gluing several copies of $\overline{B_{\frac{\epsilon}{2}}(x)} \subset N$ along the closed convex set $\overline{B_{\frac{\epsilon}{2}}(x)} \cap \overline{N_\epsilon(B)}$ (see Lemma 2.1). Hence $M_\epsilon$ is CAT(0). The natural projection $M \to M_\epsilon$ is a $(1, 2\epsilon)$-quasi-isometry and therefore $M_\epsilon$ Gromov-Hausdorff converges to $M$ for $\epsilon \to 0$. □

For now, let us equip $M$ with the induced locally CAT(0) metric.

**Lemma 2.12.** *At a singular point $x \in M^{sing}$ the map $f$ restricts to a radial isometry on small distance-balls. The link $\Sigma_x M$ decomposes as a spherical join of $\Sigma_x M^{sing}$ and a circle of length $2\pi \operatorname{ind}(x)$. Moreover, there is a natural differential $df_x : \Sigma_x M \to \Sigma_{f(x)} N$ which respects this splitting in the sense that it is given by a product of an isometry tangential to $M^{sing}$ and by $z \mapsto z^{\operatorname{ind}(x)}$ orthogonal to $M^{sing}$.*

*Proof.* Let $x$ be a singular point. Choose $\epsilon > 0$ small enough such that $B_\epsilon(x)$ intersects only one component of $M^{sing}$. For a point $y$ in $B_\epsilon(x)$ the geodesic segment $xy$ is either contained in $M^{sing}$ or intersects $M^{sing}$ only in $x$. Since $f$ is a local isometry on $B_\epsilon(x) \setminus M^{sing}$ and an isometry on $B_\epsilon(x) \cap M^{sing}$, it follows that $f$ restricts to a radial isometry on $B_\epsilon(x)$. Therefore there is a natural differential map $df_x : \Sigma_x M \to \Sigma_{f(x)} N$ sending the germ of a geodesic segment starting in $x$ to the geodesic germ of its image. By what we saw, $df_x$ restricts to an isometry on $\Sigma_x M^{sing}$ and preserves the distance to $\Sigma_x M^{sing}$. Hence $\Sigma_x M$ splits metrically as a spherical join. □

## 2.6 Periodic maximal flats

**Definition 2.13.** A *(k-)flat* $F$, for $k \geq 1$, in a CAT(0) space $X$ is a convex subset isometric to Euclidean space $\mathbb{R}^k$. $F$ will be called *maximal*, if it is not contained in another flat of strictly larger dimension. If $\Gamma$ is a group of isometries of $X$, then a flat $F$ is $\Gamma$-*periodic* if its stabilizer $\operatorname{Stab}_\Gamma(F)$ acts cocompactly on $F$.

*Remark* 2.14. By the classical theorem of Bieberbach, if $\Gamma$ acts discretely, then the stabilizer of a $\Gamma$-periodic flat is virtually abelian.

**Theorem 2.15** (Theorem B in [LS97])**.** *Let $(X, d)$ be a locally compact and cocompact CAT(0) space containing a $k$-flat but no $(k+1)$-flat, where $k \geq 1$. Then for all $L > 0$ and $C \geq 0$ there exists $D \geq 0$ such that the following holds. Let $F \subset X$ be a $k$-flat, $f : \mathbb{R}^k \to X$ an $(L, C)$-quasi-isometric map, and for $r > 0$ define $a(r) := \sup\{d(f(x), F) : \ |x| \leq r\}$. If $\limsup_{r\to\infty} a(r)/r < L^{-1}$, then $d_H(f(\mathbb{R}^k), F) \leq D$.*

We provide a proof of a qualitative version of this theorem, which is good enough for our applications. More precisely, we will prove:

**Proposition 2.16.** *Let $X$ be a locally compact and cocompact CAT(0) space with $\dim \partial_{Tits} X = k-1$. Then there is a universal constant $D = D(L, A, X)$ such that the following holds. Any $(L, A)$-quasi-flat $QF : \mathbb{R}^k \to X$ which has finite Hausdorff-distance from a $k$-flat $F$ is $D$-close to $F$.*

To obtain this result we will use support sets of locally finite homology classes in a similar vein as in [KL97] and [BKS07].

**Definition 2.17.** Let $X$ be a topological space, define the locally finite chain complex $C_k^{lf}(X)$ to be the set of all formal sums of singular $k$ simplices $\sum_{\sigma\in I} n_\sigma \sigma$ such that for every $x \in X$ there is an open neighborhood $U$ such that $\{\sigma \in I |\ n_\sigma \neq 0 \text{ and } \operatorname{im}\sigma \cap U \neq \emptyset\}$ is finite. $C_k^{lf}(X)$ with the usual boundary map is a chain complex. Its homology $H_*^{lf}(X)$ is called locally finite homology of $X$.

**Lemma 2.18.** *Let $f : X \to Y$ be a continuous $(L, A)$-quasi-isometric embedding between metric spaces $X$ and $Y$. Assume that $X$ is proper. Then $f$ induces a map on locally finite homology.*

*Proof.* Let $c = \sum_{\sigma\in I} n_\sigma \sigma$ be a locally finite cycle in $X$. Then $f_\# c = \sum_{\sigma\in I} n_\sigma (f \circ \sigma)$ is a locally finite cycle in $Y$. Indeed, for every $x \in X$ and $R \geq 0$ holds $\operatorname{im}(f \circ \sigma) \cap B_R(f(x)) \neq \emptyset \Rightarrow \operatorname{im}\sigma \cap B_{L(R+A)}(x) \neq \emptyset$ and $B_{L(R+A)}(x)$ is relatively compact. □

*Remark* 2.19. If $Y$ is locally compact, then any proper map from an arbitrary topological space $X$ induces a map on locally finite homology.

Recall that for an element $[Q]$ of the locally finite Homology group $H_k^{lf}(Z)$ its *support set* $\operatorname{supp}[Q]$ is given by the set of points $z \in Z$ such that the induced homomorphism $H_k^{lf}(Z) \to H_k(Z, Z \setminus \{z\})$ is nontrivial on $[Q]$.

**Lemma 2.20.** *Let $Z$ be a (not necessarily locally compact) CAT(0) space of geometric dimension $k$ and $E : \mathbb{R}^k \to Z$ a bilipschitz embedding. Then, $E$ induces a map $E_*$ in locally finite homology and it holds* $\operatorname{supp}(E_*[\mathbb{R}^k]) = \operatorname{im}(E)$*.*

*Proof.* $E$ induces a map on locally finite homology by Lemma 2.18. It is then enough to show that $E_* : H_k(\mathbb{R}^k, \mathbb{R}^k \setminus \{x\}) \to H_k(Z, Z \setminus \{E(x)\})$ is injective for every $x$ in $\mathbb{R}^k$. By Theorem A in [K99], geometric dimension equal to $k$ implies $H_r(U_1, U_2) = 0$ for every pair of open subsets $U_2 \subset U_1 \subset Z$, $r > k$. Therefore the lemma follows from Lemma 6.1.2 in [KL97]. □

*Proof of Proposition 2.16.* If the claim is false, then we find a sequence $(QF_i)$ of $(L, A)$-quasi-$k$-flats and a sequence $(F_i)$ of $k$-flats such that

$$d_H(QF_i, F_i) < \infty \text{ but } d_H(QF_i, F_i) \to \infty.$$

We choose points $x_i \in QF_i$ such that $d(x_i, F_i) \geq \sup_{x \in QF_i} d(x, F_i) - 1$. Let $y_i = \operatorname{pr}_{F_i} x_i$ and set $\lambda_i = d(x_i, y_i)$. By Theorem C in [K99], the asymptotic cone

$$(X_\omega, y_\omega, d_\omega) = \omega \lim(X, y_i, \lambda_i^{-1} d)$$

has dimension $k$. $QF_\omega = \omega \lim QF_i$ is a $L$-bilipschitz embedding of $\mathbb{R}^k$. By construction $\operatorname{im} QF_\omega \subset N_1(\operatorname{im} F_\omega)$ and $d_\omega(F_\omega, x_\omega) = 1$. Lemma 2.20 gives us $\operatorname{supp}(QF_\omega)_*[\mathbb{R}^k] = \operatorname{im} QF_\omega$. On the other hand, the negative gradient flow of $d(F_\omega, \cdot)$ provides a homotopy between $QF_\omega$ and $\operatorname{pr}_{F_\omega} QF_\omega$. This homotopy is a continuous $(L, 3)$-quasi-isometric embedding since $QF_\omega \subset N_1(F_\omega)$. Hence, $[QF_\omega] = [\operatorname{pr}_{F_\omega} QF_\omega] \in H_k^{lf}(X_\omega)$. Therefore,
$\operatorname{supp}[QF_\omega] = \operatorname{supp}[\operatorname{pr}_{F_\omega} QF_\omega] \subset \operatorname{im} \operatorname{pr}_{F_\omega} QF_\omega \subset \operatorname{im} F_\omega$. This shows that $\operatorname{im} QF_\omega \subset \operatorname{im} F_\omega$ and contradicts $d_\omega(F_\omega, x_\omega) = 1$. □

This result can be seen as a starting point for investigations concerning equivariant quasi-isometries between cocompact CAT(0) spaces. It tells us that periodic maximal flats map uniformly close to periodic maximal flats and, as a consequence, limits of such to limits of such. Hence there is a bijection between the families of parallel maximal flats which are limits of periodic flats.

## 2.7 Basic rigidity results

We will need special cases of some well known splitting results. We give a direct argument for the cases which we need using some ideas of [L00].

### 2.7.1 Flat strips

Throughout this section $\Gamma$ will be a group acting discretely by axial isometries on a locally compact CAT(0) space $X$.

**Lemma 2.21.** *Suppose that $X$ contains a non-empty $\Gamma$-invariant closed convex subset $C$ with discrete Tits boundary. Then $C$ contains a minimal non-empty $\Gamma$-invariant closed convex subset.*

*Proof.* We can assume that $\Gamma$ is nontrivial, otherwise we can take any point in $C$. Let $\gamma$ be a nontrivial element in $\Gamma$. Then any non-empty $\Gamma$-invariant closed convex subset contains an axis $c$ for $\gamma$. The parallel set $P(c) \cap C$ is the union of all $\gamma$-axes in $C$. Its cross section is compact because $C$ has discrete Tits boundary. Now we proceed as in the proof of Proposition 2.2 in [L00] to find a closed convex subset $C' \subset C$ such that the cross section of $P(c) \cap C'$ is minimal. Then, the closed convex hull of $\Gamma \cdot (P(c) \cap C')$ fulfills the requirements. □

**Lemma 2.22.** *Let $C$ be a minimal non-empty $\Gamma$-invariant closed convex subset such that $\partial_{Tits}C$ is no spherical suspension. Then every $\Gamma$-equivariant self-isometry $\Psi$ of $C$ is the identity.*

*Proof.* The displacement $\delta_\Psi$ is convex and $\Gamma$-periodic. By the minimality of $C$ $\delta_\Psi$ is constant. It cannot be positive, since this would imply that $C$ has to split off an $\mathbb{R}$-factor. (The axis-direction of $\Psi$.) Hence $\partial_{Tits}C$ would be a spherical suspension. □

Let $\mathcal{C}_\Gamma$ denote the family of minimal non-empty $\Gamma$-invariant closed convex subsets of $X$. In the rest of this section we assume that $\mathcal{C}_\Gamma$ is non-empty and contains a subset $C$ with discrete Tits boundary which consists of more than two points.

**Lemma 2.23** (Flat strip)**.** *For any two subsets $C, C' \in \mathcal{C}_\Gamma$ the closed convex hull of $C \cup C'$ is isometric to a product $C \times [0, d]$ with $C \times \{0\}$ corresponding to $C$ and $C \times \{d\}$ corresponding to $C'$.*

*Proof.* By minimality, the $\Gamma$-invariant convex function $d(\cdot, C')$ is constant on $C$. For any point $x$ in $C$ let $x'$ be the nearest point in $C'$. Since $d(\cdot, C')$ is constant on $C$, the angle at $x$ between the geodesic segment $xx'$ and $C$ is greater of equal to $\frac{\pi}{2}$. We conclude that quadrilaterals with vertices $x, y \in C$ and $x', y' \in C'$ are rigid, i.e. they can be filled with flat rectangles. Consequently, the nearest point projections $\mathrm{pr}_{C'C} : C \to C'$ and $\mathrm{pr}_{CC'} : C' \to C$ are isometries which are inverse to each other. Furthermore, the union over all elements $x$ in $C$ of the perpendiculars $xx'$ is a closed convex subset isometric to $C \times [0, d]$ with $d = d(C, C')$. □

Next, we observe that the pairwise identifications of elements in $\mathcal{C}_\Gamma$ by nearest point projections are consistent.

**Lemma 2.24.** *For minimal elements $C, C', C'' \in \mathcal{C}_\Gamma$ the self-isometry*

$$\Psi = \mathrm{pr}_{CC'} \circ \mathrm{pr}_{C'C''} \circ \mathrm{pr}_{C''C}$$

*is the identity.*

*Proof.* $\Psi$ is a $\Gamma$-equivariant isometry of $C$. Hence the claim is a consequence of Lemma 2.22. □

Let $Y$ denote the union of all subsets in $\mathcal{C}_\Gamma$. The existence of rigid flat strips (Lemma 2.23 and Lemma 2.24) implies that $Y$ is closed convex and splits metrically as

$$Y \cong C \times Q$$

where the layers $C \times \{q\}$ are precisely the subsets in $\mathcal{C}_\Gamma$. The group $\Gamma$ respects the product structure and acts trivially on $Q$.It follows that $Q$ is a CAT(0) space.

### 2.7.2 Product splittings

We will need the following product splitting result, compare Corollary 10 in [M06] and Theorem 1 in [S85].

**Lemma 2.25.** *Let $X$ be a locally compact CAT(0) space with $\dim \partial_{Tits} X = 1$. Furthermore, let $\Gamma \cong \Gamma_1 \times \Gamma_2$ be a product of non-abelian free groups $\Gamma_i$ and suppose that $\Gamma$ acts discretely by axial isometries on $X$. Then there exists a minimal non-empty $\Gamma$-invariant closed convex subspace $C \subset X$ which splits metrically as a product, $C \cong C_1 \times C_2$, such that $\Gamma$ preserves the product splitting and $\Gamma_i$ acts trivially on $C_{3-i}$.*

*Proof.* We start by showing that there exist minimal non-empty $\Gamma_1$-invariant closed convex subsets in $X$. To see this, we choose two non-commuting elements $\gamma_2$ and $\gamma_2'$ in $\Gamma_2$. Let $c_2$ respectively $c_2'$ be their axes. Observe that $\partial_\infty c_2 \cap \partial_\infty c_2' = \emptyset$, because $\Gamma$ acts discretely. The parallel sets $P(c_2)$ and $P(c_2')$ are non-empty and $\Gamma_1$-invariant. Therefore, the intersection $\partial_\infty P(c_2) \cap \partial_\infty P(c_2')$ contains the ideal end points of $\Gamma_1$-axes. Note that the Tits boundaries $\partial_{Tits} P(c_2)$ and $\partial_{Tits} P(c_2')$ are spherical suspensions of the discrete sets $\partial_{Tits} CS(c_2)$ respectively $\partial_{Tits} CS(c_2')$. Furthermore, the $\Gamma_1$-action preserves the suspension points. Because $\Gamma_1$ is non-abelian, there are no $\Gamma_1$ fixed points on the equators of the suspensions. Now if an ideal end point $c_1(+\infty)$ of a $\Gamma_1$-axis $c_1$ has distance smaller than $\frac{\pi}{2}$ from a suspension point, then so does the opposite ideal end point $c_1(-\infty)$. Since the convex hull of the $\Gamma_1$-orbit $\Gamma_1(\partial_\infty c)$ is contained in $\partial_\infty P(c_2) \cap \partial_\infty P(c_2')$, it follows that $c_2(+\infty)$ and $c_2(-\infty)$ are contained in $\partial_\infty P(c_2')$. Consequently, $\partial_\infty c_2 \cup \partial_\infty c_2'$ are contained in a common $2\pi$-circle contradicting the fact that $\gamma_2$ and $\gamma_2'$ are non-commuting. We conclude that all $\Gamma_1$-axes are horizontal. Hence the cross section $CS(c_2)$ is $\Gamma_1$-invariant. By Lemma 2.21, $CS(c_2)$ contains a minimal non-empty $\Gamma_1$-invariant closed convex subset.
The union $Y_1$ of all minimal non-empty $\Gamma_1$-invariant closed convex subsets splits metrically as $Y_1 \cong C_1 \times Q_1$. (See Section 2.7.1.) Since elements $\gamma_2 \in \Gamma_2$ are in the centralizer of $\Gamma_1$ in $\Gamma$, $\Gamma_2$ preserves $Y_1$ and the product splitting. The induced action of $\Gamma_2$ on $C_1$ commutes with the $\Gamma_1$-action and is therefore trivial in view of Lemma 2.22. Since $\partial_{Tits} Q_1$ is discrete by dimension reasons, Lemma 2.21 yields a minimal non-empty $\Gamma_2$-invariant closed convex subset $C_2 \subset Q_1$. $\square$

*Remark* 2.26. If $X$ is 4-dimensional and does not contain isometrically embedded tripods, then the set $C$ is unique. Indeed, the factors $C_i$ cannot be 1-dimensional, because this would yield tripods in $X$. Hence they are 2-dimensional and $C$ has dimension 4. Since two minimal non-empty $\Gamma$-invariant closed convex subsets are parallel by Lemma 2.23 and $X$ is 4-dimensional, $C$ is unique.

*Remark* 2.27. Any $\Gamma$-periodic 2-flat $F \subset C$ is the product of $\Gamma_i$-periodic geodesics $c_i \subset C_i$. Indeed, the Tits boundary $\partial_{Tits} F$ is a $2\pi$-circle in $\partial_{Tits} C \cong \partial_{Tits} C_1 \circ \partial_{Tits} C_2$. Hence, $\partial_{Tits} F$ decomposes as $\partial_{Tits} F \cong \{\xi_1^+, \xi_1^-\} \circ \{\xi_2^+, \xi_2^-\}$, with $\xi_i^\pm \in \partial_{Tits} C_i$. It follows that $F$ is a product of geodesics $F \cong c_1 \times c_2$, $c_i \in C_i$. Since $\mathrm{Stab}_\Gamma(F)$ acts cocompactly on $F$ and with discrete orbits on $C_i$, it has infinite intersection with $\Gamma_{3-i}$ which implies that the $c_i$ are

$\Gamma_i$-periodic.

# 3 Convex product subsets and their interaction

## 3.1 Ideal (quasi-)books

**Definition 3.1.** An *ideal book* in a CAT(0) space is a convex subset isometric to $\mathbb{R}^k \times Z$ where $Z$ is a CAT(0) space whose ideal boundary contains at least three points and is discrete with respect to the Tits metric.

**Lemma 3.2.** *Let $X$ be a CAT(0) space with $\dim(\partial_{Tits}X) = k \geq 1$. Let $\Psi$ be a quasi-isometric embedding of an ideal book $\mathbb{R}^k \times Z$ into $X$. If $\Psi$ maps every maximal flat in $\mathbb{R}^k \times Z$ at uniform Hausdorff distance from a maximal flat in $X$, then $\Psi$ induces an isometric embedding of $\partial_{Tits}(\mathbb{R}^k \times Z)$ into $\partial_{Tits}X$.*

*Proof.* For an ideal triangle in $Z$ let $F_i$, $i = 1, 2, 3$, be the associated maximal flats in $\mathbb{R}^k \times Z$. By assumption, there are maximal flats $F_i'$ in $X$ such that $d_H(\Psi(F_i), F_i') < D$ for some constant $D > 0$. For $F_i \neq F_j$ the intersection $\partial_\infty F_i' \cap \partial_\infty F_j'$ is contained in a hemisphere. On the other hand, since $\partial_\infty F_i \subset \partial_\infty F_{i-1} \cup \partial_\infty F_{i+1}$, we have $\partial_\infty F_i' = (\partial_\infty F_i' \cap \partial_\infty F_{i-1}') \cup (\partial_\infty F_i' \cap \partial_\infty F_{i+1}')$. Therefore, $\partial_\infty F_i' \cap \partial_\infty F_j'$ equals a hemisphere. The union of any two of these hemispheres is a sphere, because otherwise, by the lune lemma (Lemma 2.5 in [BB99] ), they would span a spherical lune of dimension $k+1$. The claim follows. □

**Lemma 3.3.** *Let $T$ be a discrete metric tree (i.e. branch points have uniform positive distance) and $\Gamma'$ a discrete group of isometries acting on $T$ with finite covolume. Then, $\Gamma'$-periodic geodesics are dense in the space of complete geodesics in $T$.*

*Proof.* The lengths of edges in the quotient $T/\Gamma$ are unifomly positive. In particular, since $\mathrm{vol}(T/\Gamma) < \infty$, there is only a finite number of edges in $T/\Gamma$. A complete geodesic $c$ in $T/\Gamma$ corresponds to an infinite sequence of such edges. There is a branch point $x$ in $T/\Gamma$ such that $c$ intersects $x$ infinitely often. Therefore $c$ can be approximated by geodesic loops. □

**Corollary 3.4.** *Suppose that a discrete group $\Gamma$ acts geometrically on a CAT(0) space $X$ which splits metrically as $X \cong T \times \mathbb{R}^{k-1}$, $k \geq 2$, where*

*$T$ is a discrete locally compact geodesically complete metric tree. Then every $k$-flat in $X$ is a limit of $\Gamma$-periodic $k$-flats in $X$.*

*Proof.* By Theorem 3.8 in [CM09], $\Gamma$ has a finite index subgroup $\Gamma_0$ which splits as a direct product $\Gamma_0 \cong \mathbb{Z}^{k-1} \times \Gamma'$. Moreover, $\mathbb{Z}^{k-1}$ acts trivially on $T$ and cocompactly on $\mathbb{R}^{k-1}$ and the projection of $\Gamma'$ to $\mathrm{Isom}(T)$ is discrete. Therefore the claim follows from Lemma 3.3. □

## 3.2 Coarse intersection of flats and quasi-isometry invariance

Let $F_1, F_2 \subset X$ be flats. We say that they *diverge* if $\partial_\infty F_1 \cap \partial_\infty F_2 = \emptyset$. Equivalently, the distance function $d(\cdot, F_2)|_{F_1}$ is proper and diverges linearly.

**Definition 3.5.** Let $F_1, F_2 \subset X$ be diverging flats. We say that $F_1$ *coarsely intersects* $F_2$ if there is $R > 0$ such that for every $r \geq R$ holds: If $B_1 \subset F_1$ is a round ball such that $F_1 \cap \overline{N_r(F_2)} \subset \mathrm{int}(B_1)$, then its boundary sphere $\partial B_1$ is not contractible inside $X \setminus \overline{N_r(F_2)}$.

*Remark* 3.6.

(i) This is independent of the choice of the ball $B_1 \subset F_1$.

(ii) The notion is asymptotic, in the sense that it only depends on the ideal boundaries of the flats involved, i.e. passing to parallel flats does not affect coarse intersection.

(iii) Coarse intersection is *not* a symmetric relation.

(iv) In general, disjoint flats can coarsely intersect. However, this cannot occur in geodesically complete smooth spaces, i.e. in Hadamard manifolds.

We need a criterion to recognize whether flats coarsely intersect. In the smooth case "coarse intersection" simply becomes "nontrivial transversal intersection", i.e. two flats in a Hadamard manifold intersect coarsely if and only if they intersect transversely in one point. This is clear, because for a flat $F$ in a Hadamard manifold $X$ there is a deformation retraction of $X \setminus F$ onto $X \setminus \overline{N_r(F)}$ using the gradient flow of $d(\cdot, F)$.

More generally, suppose that $X$ is a CAT(0) space and that $C \subset X$ is an open convex subset which is smooth Riemannian. Let $F_1, F_2 \subset X$ be flats such that $F_2 \subset C$ and $F_1, F_2$ intersect transversally in one point. Then $F_1$ coarsely intersects $F_2$. This is also clear, because otherwise spheres in $F_1 \setminus F_2$ around the intersection point $F_1 \cap F_2$ could be contracted in $X \setminus F_2$. But this would be absurd since $X \setminus F_2$ retracts to $\overline{C} \setminus F_2$ along normal geodesics. For future reference, we put this on record.

**Lemma 3.7.** *Let $F_1$ and $F_2$ be flats in a CAT(0) space $X$. Suppose that $F_2$ is contained in an open convex subset $C \subset X$ which is Riemannian, i.e. the metric on $C$ is induced by a Riemannian metric. If $F_1$ intersects $F_2$ transversely in one point, then $F_1$ coarsely intersects $F_2$.*

It will be crucial for us that coarse intersection is quasi-isometry invariant under suitable assumptions on the CAT(0) spaces involved.

**Lemma 3.8.** *Let $Y$ be a locally compact geodesically complete CAT(0) space which does not contain isometrically embedded tripods. Let $C$ be a closed convex subset in $Y$ and $\varphi : \bar{B}^n \to Y \setminus C$ a continuous map from the closed unit ball in $\mathbb{R}^n$. Suppose that $d(\mathrm{im}(\varphi|_{\partial \bar{B}^n}), C) > r > 0$. Then, $\varphi$ is homotopic relative $\partial \bar{B}^n$ to a map $\tilde{\varphi}$ with $d(\mathrm{im}(\tilde{\varphi}), C) > r$.*

*Proof.* To simplify notation we will just write $B$ for $\bar{B}^n$. We choose $\epsilon > 0$ such that $d(\mathrm{im}(\varphi|_{\partial \bar{B}^n}), C) > r + \epsilon$ and $d(\mathrm{im}(\varphi), C) > \epsilon$. Next, we fix $\rho < \frac{\epsilon}{4}$. For every $\delta < \rho$ we choose a triangulation of $B$ such that the image of every simplex under $\varphi$ has diameter less than $\delta$. Denote $B^{(0)} = \{v_i\}_{i \in I}$ the zero-skeleton and set $p_i = \varphi(v_i)$. For every point $p_i$ we choose a ray $r_i$ with starting point $\mathrm{pr}_C(p_i)$ in $Y$, extending the geodesic segment $\mathrm{pr}_C(p_i)p_i$. Now we will homotop $\varphi$ to a $PL$-map $\varphi_\delta$ as follows. We slide all points $p_i$ with $d(p_i, C) \leq r$ along the chosen rays $r_i$ to points $q_i$, thereby increasing the distance from $C$ by $\rho$. Using the usual orientation on simplices, we can extend this naturally to a homotopy relative $\partial B$, by deforming $\varphi$ on each simplex to a geodesic cone. Since the single deformations take place within the convex balls $B_{3\rho}(q_i)$, we see that the homotopy has its image in the complement of the closed $\frac{\epsilon}{2}$-neighborhood of $C$. We claim that we can choose $\delta$ small enough such that $d(\mathrm{im}(\varphi_\delta|_{B^{(l)}}), C) > \epsilon + \frac{\rho}{l+1}$ where $B^{(l)}$ denotes the $l$-skeleton of $B$ for $l = 1, \ldots, n$. We prove this by induction on $l$. If the claim fails for $l = 1$, then, for a sequence $\delta_k \to 0$, we find points $m_k$ in $\mathrm{im}(\varphi_\delta|_{B^{(1)}})$ with $d(m_k, C) \leq \epsilon + \frac{\rho}{2}$. The points $m_k$ lie on geodesic segments between points $q_k$

and $q_k'$ in $\mathrm{im}(\varphi_\delta|_{B(0)})$. By construction, the points $p_k$ and $p_k'$ on the geodesic segments $q_k \operatorname{pr}_C(q_k)$ and $q_k' \operatorname{pr}_C(q_k')$ at distance $\rho$ from $q_k$ and $q_k'$ respectively, fulfill $d(p_k, p_k') \leq \delta_k$. After choosing subsequences we can arrange that this configuration converges to a tripod. Contradiction. The argument for the induction step is similar. Repeating the whole process allows us to move $\varphi$ away from the closed $r$-neighborhood of $C$. $\square$

**Corollary 3.9.** *Let $Y$ be a locally compact geodesically complete CAT(0) space which does not contain isometrically embedded tripods. If $F_1$ and $F_2$ are coarsely intersecting flats, then their intersection $F_1 \cap F_2$ is nonempty.*

Consider a quasi-isometry $\Phi : X \to X'$ of CAT(0) spaces with a quasi-inverse $\Phi' : X' \to X$. Let $F_1, F_2 \subset X$ and $F_1', F_2' \subset X'$ be flats such that $\Phi(F_i)$ is Hausdorff close to $F_i'$. Then $F_1$ and $F_2$ diverge if and only if $F_1'$ and $F_2'$ do. If one of the CAT(0) spaces does not contain isometrically embedded tripods, we have:

**Lemma 3.10.** *Suppose that $X'$ does not contain isometrically embedded tripods. If $F_1$ coarsely intersects $F_2$, then $F_1'$ coarsely intersects $F_2'$.*

*Proof.* The quasi-flat $\Phi|_{F_1}$ may not be continuous, but since $X$ is CAT(0), it is uniformly (in terms of the quasi-isometry constants) Hausdorff close to a continuous quasi-flat $q : F_1 \to X'$. Suppose that $q(F_1)$ is $D$-Hausdorff close to $F_1'$. If $F_1'$ does not coarsely intersect $F_2'$, then $F_1'$ and $F_2'$ are disjoint since $X'$ is smooth. The $q$-image of a sphere in $F_1$ can be homotoped to $F_1'$ with a $D$-short homotopy and then contracted inside $F_1'$. Hence, the image of a large sphere in $F_1$ can be contracted far away from $F_2'$, using Lemma 3.8 to push the (contracting) homotopy away from $F_2'$. The $\Phi'$-image of the homotopy is again uniformly Hausdorff close to a continuous map. Since $\Phi' \circ \Phi$ is at finite distance from $\mathrm{id}_X$, it follows that we can for every radius $r > 0$ contract sufficiently large spheres in $F_1$ in the complement of the tubular $r$-neighborhood of $F_2$. Consequently, $F_1$ does not coarsely intersect $F_2$. $\square$

## 3.3 Interfering product structures

**Lemma 3.11.** *Let $X_1 \times X_2$ be a metric product of two CAT(0) spaces. Moreover, let $\triangle$ be a triangle in $X_1 \times X_2$. $\triangle_1$ and $\triangle_2$ shall denote the projections of $\triangle$ to the factors. Then, $\triangle$ is rigid if and only if both, $\triangle_1$ and $\triangle_2$, are rigid.*

*Proof.* Realize the comparison triangles $\tilde{\triangle}_1$ and $\tilde{\triangle}_2$ in $\mathbb{R}^2$. By the Pythagorean rule, $\tilde{\triangle}$ isometrically embeds into $\tilde{\triangle}_1 \times \tilde{\triangle}_2$ and the claim follows. □

**Lemma 3.12.** *Let $X$ be a CAT(0) space which contains two closed convex subsets, namely a product $Y_1 \times Y_2$ of smooth CAT(0) surfaces $Y_1$ and $Y_2$ such that* $\mathrm{int}(Y_1) \times \mathrm{int}(Y_2)$ *is open in $X$; and a product $Z \times \mathbb{R}$. Assume that there is a point $\eta \in \partial_\infty Z$ and points $\xi_i \in \partial_\infty Y_i$, $i = 1, 2$, such that $\eta$ is an* interior *point of the Tits arc $\xi_1\xi_2$ of length $\frac{\pi}{2}$ in $\partial_\infty X$. Then $Y_1 \times Y_2 \cap Z \times \mathbb{R}$ is either empty or flat.*

*Proof.* Let us assume that the intersection is nonempty. It is enough to show that both projections $\mathrm{pr}_{Y_i}(Y_1 \times Y_2 \cap Z \times \mathbb{R})$, $i = 1, 2$, are flat. Since $\eta$ lies in the interior of $\xi_1\xi_2$, we see that projections of geodesics parallel to the $\mathbb{R}$-factor in $Z \times \mathbb{R}$ foliate $\mathrm{pr}_{Y_i}(Y_1 \times Y_2 \cap Z \times \mathbb{R})$. But two parallel geodesics span a flat strip, hence the claim follows from Lemma 3.11. □

# 4 Configurations of convex product subsets in dimension 4

## 4.1 Flat half-strips in CAT(0) surfaces with symmetries

By a *flat strip*, respectively, *half-strip* of *width* $w \geq 0$ in a CAT(0) space we mean a convex subset isometric to $\mathbb{R} \times [0, w]$, respectively, to $[0, +\infty) \times [0, w]$.

The following observation restricts the possible positions of flat half-strips in a CAT(0) surface relative to the action of its isometry group.

**Lemma 4.1.** *Let $Y$ be a smooth CAT(0) surface, and let $h \subset Y$ be a flat half-strip. Suppose that $h$ is asymptotic to a periodic geodesic $c \subset Y$, i.e. to an axis $c$ of an axial isometry $\gamma$ of $Y$.*

*Then either $w = 0$, or $h$ extends to a (periodic) flat strip in $Y$ parallel to $c$.*

*Proof.* We may assume that $\gamma$ translates towards the ideal endpoint of $h$ and preserves the orientation transversal to $c$. If $w > 0$ and $r(t)$ is a ray in $\mathrm{int}(h)$, then the ray $\gamma^{-1}r$ is strongly asymptotic to $r$, i.e. $d(\gamma^{-1}r(t), r) \to 0$ as $t \to +\infty$. Therefore, $\gamma^{-1}r$ must enter $\mathrm{int}(h)$, because $\mathrm{int}(h)$ is open in $Y$. Consequently, $\gamma^{-1}r$ extends $r$, and $\gamma^{-1}h$ extends $h$. It follows by induction that $h$ is contained in a $\gamma$-invariant flat strip. □

## 4.2 Singular configurations

Let $X$ be a CAT(0) space.

We describe a configuration of convex product subsets which enforces branching of some geodesics.

We assume that $X$ contains two closed convex subsets, namely a product

$$Y_1 \times Y_2$$

of smooth CAT(0) surfaces $Y_1$ and $Y_2$ with boundary such that $\mathrm{int}(Y_1) \times \mathrm{int}(Y_2)$ is open in $X$; and a product

$$Z \times \mathbb{R}$$

whose (not necessarily smooth) cross section $Z$ contains an ideal triangle with three ideal vertices $\eta, \eta_+, \eta_-$. We denote the sides asymptotic to $\eta$ and $\eta_\pm$ by $l_\pm$ and the side asymptotic to $\eta_+$ and $\eta_-$ by $l_{+-}$.

We assume furthermore, that these product subsets interact as follows:

**(i)** The intersection of the flat $F_\pm = l_\pm \times \mathbb{R} \subset Z \times \mathbb{R}$ with $Y_1 \times Y_2$ contains a quadrant $r_1^\pm \times r_2^\pm$, where $r_i^\pm$ are asymptotic rays in $Y_i$. We denote their common ideal endpoint by $\xi_i \in \partial_\infty Y_i$.

**(ii)** $\eta$ is an *interior* point of the Tits arc $\xi_1\xi_2$ of length $\frac{\pi}{2}$ in $\partial_\infty X$.

Then the intersection $Y_1 \times Y_2 \cap Z \times \mathbb{R}$ is nonempty and, by condition (ii), the product structures (i.e. the directions of the factors) do not match on it. The latter implies, by Lemma 3.12, that the convex subset $Y_1 \times Y_2 \cap Z \times \mathbb{R}$ is *flat*. As a consequence, subrays of the rays $r_i^\pm$ bound a *flat half-strip* $h_i \subset Y_i$.

In addition, we impose a *periodicity* condition:

**(iii)** The rays $r_i^\pm$ are asymptotic to a periodic geodesic $c_i \subset Y_i$.

Using Lemma 4.1 above, we conclude: Either subrays of the rays $r_i^\pm$ coincide, or subrays extend to geodesics $c_i^\pm \subset Y_i$ parallel to $c_i$.

**Claim 1.** *If conditions (i)-(iii) hold, then $X$ contains branching geodesics.*

*Proof.* Suppose that geodesics in $X$ do not branch. Then our discussion implies that the flats $F_\pm$ either have a quadrant in common and therefore coincide, or contain parallel half-planes and their intersection of ideal boundaries $\partial_\infty F_+ \cap \partial_\infty F_-$ contains an arc of length $\pi$ of the form $\xi_1\xi_2\hat{\xi}_1$ or $\xi_2\xi_1\hat{\xi}_2$ with an antipode $\hat{\xi}_i \in \partial_\infty Y_i$ for $i = 1$ or 2. It follows that $\angle_{Tits}(\eta_\pm, \hat{\xi}_i) < \frac{\pi}{2}$ and hence $\angle_{Tits}(\eta_+, \eta_-) < \pi$, a contradiction. □

## 4.3 Not equivariantly smoothable configurations

Now, we restrict to *periodic* situations and consider *geometric* actions

$$\Gamma \curvearrowright X$$

by discrete groups on locally compact CAT(0) spaces, i.e. actions which are isometric, properly discontinuous and cocompact.

We will tie the configuration considered above sufficiently closely to the action so that it will carry over to other geometric actions $\Gamma \curvearrowright X'$ on CAT(0) spaces. This will then be used to rule out such actions on $\mathcal{C}^2$-*smooth* CAT(0) spaces, i.e. on Hadamard 4-manifolds with $\mathcal{C}^2$-smooth Riemannian metrics.

In addition to the conditions (i)-(iii) above, we assume:

**(iv)** $X$ contains no 3-flats.

**(v)** $Y_1 \times Y_2$ is preserved by a subgroup $\Gamma_1 \times \Gamma_2 \subset \Gamma$ with non-abelian free factors $\Gamma_i$, and the restricted action $\Gamma_1 \times \Gamma_2 \curvearrowright Y_1 \times Y_2$ is a product action (not necessarily cocompact).

**(vi)** The flats $F_\pm$ and the flat $F_{+-} = l_{+-} \times \mathbb{R}$ in $Z \times \mathbb{R}$ are $\Gamma$-*periodically approximable* (i.e. pointed Hausdorff limits of $\Gamma$-periodic flats).

**(vii)** The geodesics $c_i \subset Y_i$ are $\Gamma_i$-periodic. Moreover, there exist $\Gamma_i$-periodic geodesics $d_i \subset \text{int}(Y_i)$ which intersect the rays $r_i^\pm \subset Y_i$ transversally in points.

Under the assumptions (i)-(vii), we look for a corresponding configuration in $X'$. Let $\Phi : X \to X'$ denote a $\Gamma$-equivariant quasi-isometry.

By (v) and Lemma 2.25, there exists a $\Gamma_1 \times \Gamma_2$-invariant closed convex product subset (in general singular)

$$Y_1' \times Y_2' \subset X'$$

on which $\Gamma_1 \times \Gamma_2$ acts by a product action. The $\Gamma_i$-periodic image quasi-geodesics $\Phi(c_i)$ are Hausdorff close to $\Gamma_i$-periodic geodesics $c_i' \subset Y_i'$.

By (iv+vi) and Proposition 2.16, the quasi-flats $\Phi(F_\pm)$ and $\Phi(F_{+-})$ are Hausdorff close to flats $F'_\pm$ and $F'_{+-}$. By Lemma 3.2, $\Phi$ induces an isometric embedding of $\partial_{Tits}(Z \times \mathbb{R})$ into $\partial_{Tits} X'$. Therefore these flats are contained in a closed convex product subset

$$Z' \times \mathbb{R} \subset X'$$

whose cross section $Z'$ contains an ideal triangle with corresponding ideal vertices $\eta', \eta'_+, \eta'_-$ and sides $l'_+, l'_-, l'_{+-}$, such that $F'_\pm = l'_\pm \times \mathbb{R}$ and $F'_{+-} =$

$l'_{+-} \times \mathbb{R}$. Furthermore, if $\rho \subset Z$ is a ray asymptotic to one of the ideal vertices $\eta, \eta_+$ or $\eta_-$, then $\Phi$ carries the vertical half-plane $\rho \times \mathbb{R} \subset Z \times \mathbb{R}$ Hausdorff close to a vertical half-plane $\rho' \times \mathbb{R} \subset Z' \times \mathbb{R}$ where $\rho' \subset Z'$ is a ray with corresponding ideal endpoint $\eta', \eta'_+$ or $\eta'_-$.

Since the $c_i$ are periodic, $\Phi$ carries the quadrants $r_1^{\pm} \times r_2^{\pm}$ Hausdorff close to a quadrant $r'_1 \times r'_2$ for rays $r'_i \subset c'_i$. The quadrants $r_1^{\pm} \times r_2^{\pm}$ are contained in vertical half-planes with ideal boundary semicircle $\partial_\infty F_+ \cap \partial_\infty F_-$ and, by condition (ii), their ideal boundary arc $\xi_1\xi_2$ of length $\frac{\pi}{2}$ is contained in the interior of this semicircle. Denoting the ideal endpoints of the rays $r'_i$ by $\xi'_i = \partial_\infty r'_i$ it follows that the arc $\xi'_1\xi'_2$ of length $\frac{\pi}{2}$ is contained in the *interior* of the semicircle $\partial_\infty F'_+ \cap \partial_\infty F'_-$, and $\eta'$ is an *interior* point of the arc $\xi'_1\xi'_2$ of length $\frac{\pi}{2}$.

In summary, the interaction of the product subsets $Y'_1 \times Y'_2$ and $Z' \times \mathbb{R}$ at infinity is as for the configuration in $X$. However, without further assumptions, the intersection $Y'_1 \times Y'_2 \cap Z' \times \mathbb{R}$ could be empty.

**Claim 2.** *$X'$ contains branching geodesics.*

*Proof.* Suppose that geodesics in $X'$ do not branch. We show that then the intersection $Y'_1 \times Y'_2 \cap Z' \times \mathbb{R}$ must be nonempty.

Note that there exist $\Gamma_i$-periodic geodesics $d'_i \subset Y'_i$ with the same stabilizers as the geodesics $d_i$. By (vii), the periodic flat $d_1 \times d_2$ transversally intersects the flats $F_\pm$ in points inside the smooth region $\mathrm{int}(Y_1) \times \mathrm{int}(Y_2)$. Hence, by Lemma 3.7, $d_1 \times d_2$ *coarsely* intersects $F_\pm$. It follows from Lemma 3.10 that $d'_1 \times d'_2$ coarsely intersects $F'_\pm$. From Corollary 3.9 we deduce that $d'_1 \times d'_2$ has nontrivial intersection with $F'_\pm$. In particular, $Y'_1 \times Y'_2 \cap Z' \times \mathbb{R} \neq \emptyset$.

It follows that conditions (i)-(iii) are satisfied by the product subsets $Y'_1 \times Y'_2$ and $Z' \times \mathbb{R}$ of $X'$. By Claim 1, this is a contradiction. □

We have proved:

**Theorem 4.2** (Obstruction to $\mathcal{C}^2$-smooth action)**.** *If a discrete group $\Gamma$ admits a geometric action $\Gamma \curvearrowright X$ on a locally compact CAT(0) space satisfying conditions (i)-(vii), then $\Gamma$ does not act geometrically on any $\mathcal{C}^2$-smooth Hadamard 4-manifold.*

## 4.4 Related results

Before we give an appliction of Theorem 4.2, we review known obstructions to smooth metrics of nonpositive curvature.

**Definition 4.3.** A (smooth) manifold equipped with a locally CAT(0) metric will be called a *(smooth) locally CAT(0) manifold*.

Examples of smooth locally CAT(0) manifolds are Riemannian manifolds of nonpositive curvature. In dimensions 2 and 3 every closed locally CAT(0) manifold carries a smooth Riemannian metric of nonpositive curvature. In dimension 2 this is a consequence of the classification of surfaces. Whereas in dimension 3 this follows from Thurston's geometrization conjecture, proved by Perelman in [Pe02], [Pe03a], [Pe03b], in combination with [L00] and [BS04]. (See Proposition 1 in [DJL12].) By [DJL12], it is known that in every dimension $n \geq 4$ there exist closed smooth locally CAT(0) manifolds which do not carry smooth Riemannian metrics of nonpositive curvature.

**Question 4.4.** *When does a smooth closed locally CAT(0) manifold $V$ carry a Riemannian metric of nonpositive sectional curvature?*

*Remark* 4.5. Related questions for topological n-manifolds are reviewed in [DJL12].

We state known obstructions.

**Obstruction 1** (Cartan-Hadamard)**.** *Let $X$ be an n-dimensional simply connected (locally) CAT(0) manifold which is not diffeomorphic to $\mathbb{R}^n$. Then, $X$ does not carry a Riemannian metric of nonpositive sectional curvature.*

In [DJ91], Davis and Januszkiewicz constructed examples of locally CAT(0)-manifolds $V^n$ (for $n \geq 5$), with the property that their universal covers $\tilde{V}^n$ are not simply connected at infinity, and therefore not even homeomorphic to $\mathbb{R}^n$.

For the next obstruction we need to recall the following definition.

**Definition 4.6.** Let $X$ be a CAT(0) space. We say that $X$ has *isolated flats*, if each connected component of the Tits boundary $\partial_{Tits}X$ of $X$ is either a point or isometric to a Euclidean unit sphere.

*Remark* 4.7. By Theorem 5.2.4 in [HK09] this definition is equivalent to the usual one.

**Obstruction 2** (Davis-Januszkiewicz-Lafont)**.** *Let $V$ be a 4-dimensional closed locally CAT(0) manifold such that its universal cover $X$ is a CAT(0)* *space with isolated flats. Moreover, assume that the geometric boundary $\partial_\infty X$ is homeomorphic to $S^3$ and the Tits boundary $\partial_{Tits} X$ has dimension 1. If $X$ contains a maximal flat $F$ whose ideal boundary $\partial_\infty F$ is a nontrivial knot in $\partial_\infty X$, then $V$ does not admit a Riemannian metric of nonpositive sectional curvature.*

*Sketch.* Assume that there is a Hadamard manifold $X'$ such that $\Gamma := \pi_1(V)$ acts geometrically on $X'$. Since $X$ has isolated flats, Lemma 3.1.2 in [HK09] implies that $F$ is $\Gamma$-periodic. Hence, we find a $\Gamma$-periodic 2-flat $F'$ in $X'$ whose stabilizer in $\Gamma$ is commensurable to the stabilizer of $F$. By Corollary 4.1.8 in [HK09], $\partial_\infty X$ is $\Gamma$-equivariantly homeomorphic to $\partial_\infty X'$. Therefore $\partial_\infty F'$ is knotted in $\partial_\infty X'$. It follows that $\log_p(\partial_\infty F') = \Sigma_p F'$ is knotted in $\Sigma_p X'$ for every point $p$ in $F'$. Contradiction. □

In [DJL12] the authors construct an example of a 4-dimensional smooth closed locally CAT(0) manifold $V$ whose universal cover is diffeomorphic to $\mathbb{R}^4$ but where this obstruction applies. Their example is a Davis complex associated to a very special triangulation of the 3-sphere.
Finally, once one has found a closed locally CAT(0)-manifold which does not support a Riemannian metric of nonpositive sectional curvature one can produce new examples by taking products. (See Proposition 2 in [DJL12].)

**Obstruction 3** (Davis-Januszkiewicz-Lafont)**.** *Let $V^n$ be a locally CAT(0)-manifold which does not support a Riemannian metric of nonpositive sectional curvature, and assume that $n \geq 5$. Then for $W$ an arbitrary locally CAT(0)-manifold, the product $V \times W$ is a locally CAT(0)-manifold which does not support a Riemannian metric of nonpositive sectional curvature.*

*Remark* 4.8. This obstruction relies on the classical splitting theorems and the resolution of the Borel conjecture, therefore the restriction to dimensions greater or equal to 5.

# 5 An example

In this section, we consider the geometric actions on 4-dimensional singular CAT(0) spaces suggested by Gromov in the first exercise of [BGS85] and verify that they contain configurations satisfying conditions (i-vii) of Theorem 4.2.

Let $\Sigma$ be a closed surface of genus $\geq 2$, and let

$$\beta : V \to \Sigma \times \Sigma$$

be a non-trivial finite branched covering with branching locus the diagonal $\Delta_\Sigma \subset \Sigma \times \Sigma$. Then the group

$$\Gamma := \pi_1(V)$$

admits geometric actions on 4-dimensional singular CAT(0) spaces: Let $\pi_V : X \to V$ denote the universal covering, and $\pi := \beta \circ \pi_V : X \to \Sigma \times \Sigma$. We equip $\Sigma$ with a hyperbolic metric and pull back the corresponding product metric on $\Sigma \times \Sigma$ to singular metrics on $V$ and $X$. In this way the 4-manifold $X$ becomes a CAT(0) space, and the deck action

$$\Gamma \curvearrowright X$$

becomes a geometric action.

Regarding the geometry of $X$, note first that the *singular locus* $\pi^{-1}(\Delta_\Sigma) \subset X$ is a disjoint union of isometrically embedded hyperbolic planes. The restriction of $\pi$ to any of them is a universal covering of the *branching locus* $\Delta_\Sigma \subset \Sigma \times \Sigma$.

We look for patterns of flats in $X$ which obstruct the existence of geometric $\Gamma$-actions on Hadamard manifolds, as described in sections 4.2 and 4.3.

The space $X$ contains no 3-dimensional flats, but plenty of 2-dimensional ones. There are two kinds of them: flats disjoint from $\pi^{-1}(\Delta_\Sigma)$, and flats which intersect $\pi^{-1}(\Delta_\Sigma)$ orthogonally in one or several parallel geodesics.

Let $\mathcal{F}_0$ denote the set of flats disjoint from $\pi^{-1}(\Delta_\Sigma)$. There are obvious subfamilies of $\mathcal{F}_0$ which occur in convex product subsets of $X$. Namely, let

$$\Sigma = \Sigma^+ \cup \Sigma^- \tag{5.1}$$

be a decomposition of $\Sigma$ into two subsurfaces $\Sigma^\pm$ along a finite family of disjoint closed geodesics. Then the open product block $\operatorname{int}(\Sigma^+ \times \Sigma^-) \subset \Sigma \times \Sigma$ is disjoint from $\Delta_\Sigma$, and hence the connected components of its inverse image $\pi^{-1}(\operatorname{int}(\Sigma^+ \times \Sigma^-))$ in $X$ are convex subsets isometric to $\operatorname{int}(\widetilde{\Sigma}^+ \times \widetilde{\Sigma}^-)$ on which $\pi$ restricts to a universal covering of $\operatorname{int}(\Sigma^+ \times \Sigma^-)$.

The other flats in $X$ important for our argument are, somewhat unexpectedly, the flats which intersect $\pi^{-1}(\Delta_\Sigma)$ in precisely *one* geodesic; let us denote the set of these flats by $\mathcal{F}_1$. Understanding them leads us to considering flat half-planes.

We define $\mathcal{H}$ as the set of *injectively* immersed flat half-planes $H \subset \Sigma \times \Sigma$ which intersect the branching locus precisely along their boundary line, $H \cap \Delta_\Sigma = \partial H$, and are orthogonal to it, $H \perp \Delta_\Sigma$. Furthermore, we define $\widetilde{\mathcal{H}}$ as the set of isometrically embedded flat half-planes $\widetilde{H} \subset X$ such that $\widetilde{H} \cap \pi^{-1}(\Delta_\Sigma) = \partial\widetilde{H}$ and $\widetilde{H} \perp \pi^{-1}(\Delta_\Sigma)$. We say that a half-plane $\widetilde{H} \in \widetilde{\mathcal{H}}$ *covers* or is a *lift* of a half-plane $H \in \mathcal{H}$ if $\pi|_{\widetilde{H}}$ is a local isometry onto $H$. A flat in $\mathcal{F}_1$ is the union of two half-planes in $\widetilde{\mathcal{H}}$ with common boundary line.

We collect some facts about $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ needed for our argument.

If $H \in \mathcal{H}$, then $\partial H$ is an injectively immersed line in $\Delta_\Sigma$ and therefore of the form $\partial H = \Delta_c$ for a nonperiodic simple geodesic $c \subset \Sigma$. It follows that $H \subset c \times c$ because $H$ is flat. We also see that half-planes in $\mathcal{H}$ occur in pairs of opposite half-planes with common boundary line.

A half-plane $H \in \mathcal{H}$ lifts to a half-plane $\widetilde{H} \in \widetilde{\mathcal{H}}$ because it is simply-connected and the branched covering $\beta$ is a true covering over $\Sigma \times \Sigma \setminus \Delta_\Sigma$. More precisely, for a point $p \in H \setminus \partial H$ and a lift $\widetilde{p}$ of $p$ there exists a unique lift $\widetilde{H}$ of $H$ with $\widetilde{p} \in \widetilde{H}$. A lift $\widetilde{l} \subset \pi^{-1}(\Delta_\Sigma)$ of the boundary line $\partial H$ extends in several ways to a lift $\widetilde{H}$ of $H$, because points close to $\partial H$ can be lifted in several ways to points close to $\widetilde{l}$. The number of lifts is given by the local branching order of $\pi$ at $\widetilde{l}$.

If $\widetilde{H} \in \widetilde{\mathcal{H}}$, then its boundary line $\partial\widetilde{H}$ projects to an immersed line $\Delta_c$ in $\Delta_\Sigma$. The geodesic $c \subset \Sigma$ must be nonperiodic simple, because otherwise $(\widetilde{H} \setminus \partial\widetilde{H}) \cap \pi^{-1}(\Delta_\Sigma) \neq \emptyset$. Thus, all half-planes in $\widetilde{\mathcal{H}}$ are lifts of half-planes in $\mathcal{H}$.

If $\widetilde{H}_1, \widetilde{H}_2 \in \widetilde{\mathcal{H}}$ are distinct half-planes with the same boundary line, $\partial\widetilde{H}_1 = \partial\widetilde{H}_2$, then their projections $H_1, H_2 \in \mathcal{H}$ either coincide or are a pair of opposite half-planes. The local geometry of branched coverings implies, that $\widetilde{H}_1, \widetilde{H}_2$ have angle $\pi$ along their common boundary line and their union $\widetilde{H}_1 \cup \widetilde{H}_2$ is a flat in $\mathcal{F}_1$.

We will use the following consequence of this discussion: Let $c \times c \subset \Sigma \times \Sigma$ be an injectively immersed plane, and let $H_\pm$ be the half-planes into which it is divided by $\Delta_c$. Then for every lift $\widetilde{H}_+$ of $H_+$ there exist at least two distinct lifts $\widetilde{H}^1_-, \widetilde{H}^2_-$ of $H_-$ with the same boundary line $\partial\widetilde{H}^i_- = \partial\widetilde{H}_+$, and the union of any two of the three half-planes $\widetilde{H}_+, \widetilde{H}^1_-, \widetilde{H}^2_-$ is a flat in $\mathcal{F}_1$.

The flats in $\mathcal{F}_1$ are nonperiodic. Nevertheless, they are useful for investigating geometric $\Gamma$-actions on other CAT(0) spaces. This is due to the following fact:

**Lemma 5.1.** *Let $F \in \mathcal{F}_1$. Suppose that the nonperiodic simple geodesic $\pi(F \cap \pi^{-1}(\Delta_\Sigma))$ in $\Delta_\Sigma$ is the pointed Hausdorff limit of periodic simple geodesics in $\Delta_\Sigma$. Then $F$ is the pointed Hausdorff limit of $\Gamma$-periodic flats in $X$.*

*Proof.* We denote $\widetilde{l} = F \cap \pi^{-1}(\Delta_\Sigma)$. Let $(c_n, p_n) \to (c, p)$ be a sequence of pointed periodic simple geodesics in $\Sigma$ converging to the nonperiodic simple geodesic $c \subset \Sigma$ with $\pi(\widetilde{l}) = \Delta_c$. There exist geodesics $\widetilde{l}_n \subset \pi^{-1}(\Delta_\Sigma)$ lifting the $c_n$ and lifts $\widetilde{p}_n, \widetilde{p}$ of the base points $p_n, p$ such that $(\widetilde{l}_n, \widetilde{p}_n) \to (\widetilde{l}, \widetilde{p})$. We choose embedded subsegments $s_n \subset c_n$ of increasing lengths centered at the base points $p_n$ such that also $(s_n, p_n) \to (c, p)$ and lifted segments $\widetilde{s}_n \subset \widetilde{l}_n$ centered at the $\widetilde{p}_n$ such that $(\widetilde{s}_n, \widetilde{p}_n) \to (\widetilde{l}, \widetilde{p})$.

The main step of the argument is to approximate $F$ by isometrically embedded flat squares $\widetilde{Q}_n \subset \pi^{-1}(s_n \times s_n)$ with diagonals $\widetilde{s}_n$, $(\widetilde{Q}_n, \widetilde{p}_n) \to (F, \widetilde{p})$. This will imply the assertion because isometrically embedded flat squares in $\pi^{-1}(c_n \times c_n)$ are contained in $\Gamma$-periodic flats. Indeed, the subsets $\pi^{-1}(c_n \times c_n) \subset X$ have cocompact stabilizers in $\Gamma$, and their connected components are convex subsets which split as metric products of the line with discrete metric trees. All flats contained in them are limits of $\Gamma$-periodic ones.

To find the squares $\widetilde{Q}^n$, we proceed as follows. The flat $F$ is divided by $\widetilde{l}$ into two half-planes $\widetilde{H}_\pm \in \widetilde{\mathcal{H}}$. We will approximate these simultaneously by isometrically embedded right-angled isosceles triangles $\widetilde{T}^n_\pm \subset \pi^{-1}(s_n \times s_n)$ with sides $\widetilde{s}_n$.

Let $\widetilde{q}_\pm \in \widetilde{H}_\pm \backslash \partial \widetilde{H}_\pm$ be base points close to $\widetilde{p}$, and let $\bar{q}_\pm = \pi(\widetilde{q}_\pm) \in c \times c \backslash \Delta_c$ denote their projections. There exist sequences of points $\bar{q}^n_\pm \in s_n \times s_n \setminus \Delta_{s_n}$ approximating them, $\bar{q}^n_\pm \to \bar{q}_\pm$. More precisely, we choose them such that they are close to $\Delta_{p_n} \in \Delta_{s_n}$ intrinsically in $s_n \times s_n$, i.e. such that the segments $\Delta_{p_n} \bar{q}^n_\pm \subset s_n \times s_n$. Furthermore, there exists a sequence of lifts $\widetilde{q}^n_\pm \in \pi^{-1}(\bar{q}^n_\pm)$ close to $\widetilde{p}_n$ such that $\widetilde{q}^n_\pm \to \widetilde{q}_\pm$.

The injectively immersed square $s_n \times s_n \subset \Sigma \times \Sigma$ is divided by $\Delta_{s_n}$ into two triangles. Let $T^n_\pm$ be the subtriangle containing $\bar{q}^n_\pm$. (Possibly $T^n_+ = T^n_-$.) Since the injectively immersed flat triangles $T^n_\pm$ meet $\Delta_\Sigma$ only along their hypotenuses $\Delta_{s_n}$, we can lift them to isometrically embedded flat triangles $\widetilde{T}^n_\pm$ in $X$ with hypotenuses $\widetilde{s}_n$, as we could lift the half-planes in $\mathcal{H}$ to half-planes in $\widetilde{\mathcal{H}}$. The lifts are again uniquely determined by the lift of one off-hypotenuse point. Thus we can choose them such that $\widetilde{q}^n_\pm \in \widetilde{T}^n_\pm \subset \pi^{-1}(c_n \times c_n)$. Then the pointed triangles $(\widetilde{T}^n_\pm, \widetilde{q}^n_\pm)$ Hausdorff converge to a flat half-plane in $\widetilde{\mathcal{H}}$ with base point $\widetilde{q}_\pm$ and boundary line $\widetilde{l}$. The only such half-plane is $\widetilde{H}_\pm$, i.e. $(\widetilde{T}^n_\pm, \widetilde{q}^n_\pm) \to (\widetilde{H}_\pm, \widetilde{q}_\pm)$.

The two triangles $T^n_\pm$ either coincide or have angle $\pi$ along their common side $\Delta_{s_n}$. The local geometry of branched coverings implies that the lifted triangles $\widetilde{T}^n_\pm$ have angle $\pi$ along their common side $\widetilde{s}_n$. (They are distinct for large $n$, $\widetilde{T}^n_+ \cap \widetilde{T}^n_- = \widetilde{s}_n$.) Hence their union $\widetilde{Q}^n = \widetilde{T}^n_+ \cup \widetilde{T}^n_-$ is an embedded flat square in $X$. These are the squares we were looking for. As desired, they satisfy $(\widetilde{Q}^n, \widetilde{p}_n) \to (F, \widetilde{p})$. This finishes the proof. $\square$

Now we describe a configuration in $X$ which satisfies conditions (i-vii) formulated in sections 4.2 and 4.3.

We consider a decomposition (5.1) of $\Sigma$ and choose an injectively immersed geodesic line $c \subset \Sigma$ which intersects $\Sigma^+ \cap \Sigma^-$ transversally in precisely one point $p$. The geodesic $c$ is divided by $p$ into the injectively immersed rays $r^\pm = c \cap \Sigma^\pm$. We can arrange our choices (of $\Sigma$, $\Sigma^\pm$ and $c$) so that

(a) $r^\pm$ is asymptotic to a simple closed geodesic $c^\pm \subset \text{int}(\Sigma^\pm)$, and

(b) $c$ is a pointed Hausdorff limit of simple closed geodesics $c_n \subset \Sigma$.

Indeed, if $\Sigma^\pm$ and $c^\pm$ are chosen appropriately then there exists a simple closed curve $a$, which intersects $c^+$ and $c^-$ transversally in one point each and $\Sigma^+ \cap \Sigma^-$ transversally in two points. It is divided by its intersection points with $c^\pm$ into two arcs $a_{+-}$ and $a_{-+}$. The concatenations $a_{+-} * nc^- * a_{-+} * nc^+$ are freely homotopic to simple closed geodesics $c_n$ which, when equipped with suitable base points, Hausdorff converge to an injectively immersed line $c$ with the desired properties.

Let $H \in \mathcal{H}$ be the half-plane $H \subset c \times c$ with boundary line $\partial H = \Delta_c$ and containing the quadrant $r^+ \times r^-$. There exist two distinct flats $F_1, F_2 \in \mathcal{F}_1$ which contain the same lift $\widetilde{H} \in \widetilde{\mathcal{H}}$ of $H$ (and branch along its boundary line $\partial\widetilde{H}$). Their union $F_1 \cup F_2$ splits metrically as $Z \times \mathbb{R}$, and the cross section $Z$ is a degenerate ideal triangle (a tripod). By Lemma 5.1, the three flats contained in $Z \times \mathbb{R}$, i.e. $F_1, F_2$ and $(F_1 \cup F_2) \setminus \text{int}(\widetilde{H})$, are $\Gamma$-periodically approximable.

Let $\widetilde{r}^+ \times \widetilde{r}^- \subset \widetilde{H}$ be the quadrant lifting $r^+ \times r^-$. There exists a closed convex product subset $P = Y^+ \times Y^- \subset X$ such that $\pi|_P$ is a universal covering of $\Sigma^+ \times \Sigma^-$ and $F_j \cap P = \widetilde{r}^+ \times \widetilde{r}^-$ for $j = 1, 2$.

The product subsets $Y^+ \times Y^-$ and $Z \times \mathbb{R}$ satisfy conditions (i)-(vii).

Applying Theorem 4.2, we therefore obtain:

**Theorem 1.1** (Exercise 1 in [BGS85])**.** *Let $V$ be a closed 4-dimensional manifold which admits a non-trivial finite branched covering $\beta : V \to \Sigma \times \Sigma$ over the product of a hyperbolic surface $\Sigma$ with itself such that the branching locus equals the diagonal $\Delta_\Sigma \subset \Sigma \times \Sigma$. Then $V$ admits no $\mathcal{C}^2$-smooth Riemannian metric of nonpositive sectional curvature.*

*Remark* 5.2.

1. The theorem fails, if instead of the diagonal we take a fiber or a totally geodesic torus as branching locus. See [FS90].

# 6 Appendix

## 6.1 The topology of branched coverings

Throughout this section we let $f : M^n \to N^n$ be a finite branched covering between closed manifolds with branching locus $B \subset N$. Moreover, we assume that $N$ is nonpositively curved, $B$ is totally geodesic, and $M$ is equipped with its natural locally CAT(0) metric. (See 2.11.) We will study the topology of distance balls in $\hat{M}$, the universal cover of $M$. It will turn out that all distance balls are topological balls. This allows us to conclude that the ideal boundary of $\hat{M}$ is a topological sphere.

We begin with a corollary of Lemma 2.12 which establishes the topology of small distance balls.

**Corollary 6.1.** *Small distance-balls in $M$ are topological balls.*

*Proof.* The claim is clear for small enough distance-balls around regluar points. So let $x$ be a singular point and $B_\epsilon(x)$ a small distance ball intersecting only one component of $M^{sing}$. Then, by Lemma 2.12, radial geodesics in $B_\epsilon(x)$ do not branch. Hence $B_\epsilon(x)$ is homeomorphic to a cone over $\Sigma_x M$. By Lemma 2.12 $\Sigma_x M$ is homeomorphic to a sphere and therefore $B_\epsilon(x)$ is homeomorphic to a ball. □

Next, we want to understand the topology of the geometric boundary of the universal cover of $M$. Recall that the geometric boundary of a CAT(0) space is its ideal boundary equipped with the cone topology. In the following it will be useful to lift $f$ to a map $\hat{f} : \hat{M} \to \hat{N}$ between universal covers.

Denote $\pi_M : \hat{M} \to M$ respectively $\pi_N : \hat{N} \to N$ the covering maps. Then, $\hat{f}$ is a branched covering with branching locus $\hat{B} := \pi_N^{-1}(B)$ and singular set $\hat{M}^{sing} := \pi_M^{-1}(M^{sing})$. Note that the components of $\hat{M}^{sing}$ are isometric to the universal cover of $B$ which is a Hadamard manifold. The following proposition is crucial.

**Proposition 6.2.** *Distance-balls in $\hat{M}$ are topological manifolds with boundary.*

*Proof.* Let $p_1$ and $x$ be points in $\hat{M}$ at distance $R$ from each other. We will discribe the topology of the closed distance ball $\overline{B_R(p_1)}$, locally near $x$. More precisely, we will show that we can add appropriate functions to the distance function $d(p_1, \cdot)$ to obtain a chart around $x$. If $x$ is a regular point, then this is clear because we can find regular points $p_2, \dots, p_n$ such that the gradients $\{\nabla d(p_i, \cdot)\}_{1 \le i \le n}$ are uniformly close to an orthonormal basis near $x$. A similar argument applies if $p_1$ and $x$ lie in the same component of $\hat{M}^{sing}$.

So let us assume $x \in \hat{M}_0^{sing}$ where $\hat{M}_0^{sing}$ is a connected component of $\hat{M}^{sing}$ and $p_1 \notin \hat{M}^{sing}$. Then we choose points $p_2, \dots, p_{n-1} \in \hat{M}_0^{sing}$ such that the gradients $\{\nabla d(p_i, \cdot)\}_{2 \le i \le n-1}$ are uniformly close to an orthonormal system near $x$. Note that the angle between $\nabla d(p_1, \cdot)$ and the span of the family $\{\nabla d(p_i, \cdot)\}_{2 \le i \le n-1}$ is uniformly positive in a neighborhood $U$ of $x$. Hence, by the implicit function theorem, the fibers $\Pi_x$ of the map $F : U \to \mathbb{R}^{n-1}$ are smooth 1-manifolds away from $\hat{M}^{sing}$. Moreover, they have one-sided tangents at points in $\hat{M}^{sing}$ and since the family $\{d(p_i, \cdot)\}_{2 \le i \le n-1}$ forms a chart for $\hat{M}_0^{sing}$ near $x$, the fibers $\Pi_x$ are orthogonal to $\hat{M}_0^{sing}$. It follows that the fibers are locally rectifiable near $\hat{M}_0^{sing}$ and for the length $L(\Pi_x)$ holds $\lim_{\epsilon \to 0} L(\Pi_x \cap N_\epsilon(\hat{M}_0^{sing})) = 0$. As a consequence, the lengths of fibers $\Pi_x$ vary continuously. Let $W$ be a $(n-1)$-manifold near $x$ which is disjoint from $\hat{M}_0^{sing}$ and transversal to the fibers of $F$. Then we can measure the lengths of fibers $\Pi_x$ starting in $W$, thereby completing $F$ to a chart near $x$. □

To get our hands on the topology of distance-spheres we will compare larger distance-spheres with smaller ones, via radial projection. It turns out that these radial contraction maps are near-homeomorphisms (i.e. uniformly approximated by homeomorphisms). As a consequence all distance-spheres are topological spheres.

We need to recall some basics from geometric topology. For more information we refer the reader to [DJ91] and the references therein. A compact metric space $C$ is *cell-like* if there is an embedding of $C$ into the Hilbert cube $I^\infty$

such that for any neighborhood $U$ of $C$ in $I^\infty$, the embedding of $C$ is null-homotopic in $U$. A cell-like subspace $C$ of a n-manifold $M$ is *cellular*, if there is a sequence of n-cells $C_1, C_2, \dots$ in $M$ such that $C_{i+1} \subset \operatorname{int} C_i$ and $C = \bigcap_{i=1}^\infty C_i$. A *cellular map* is a proper continuous surjection such that each inverse image of a point is cellular. The composition of cell-like maps between ANR's is cell-like (see [Ed78] p. 116).

**Lemma 6.3.** *There is a constant $\epsilon > 0$ such that the radial contraction maps $c_{r,r+\epsilon} : \partial\overline{B_{r+\epsilon}(p)} \to \partial\overline{B_r(p)}$, $r < s$, between concentric distance-spheres in $\hat{M}$ are cellular.*

*Proof.* Since $\hat{M}$ is the universal cover of a the closed manifold $M$, there is a constant $\epsilon > 0$ such that the distance between two components of $\hat{M}^{sing}$ is bounded below by $2\epsilon$. It follows that radial rays, emanating from a point $p$ in $\hat{M}$, are either contained in $\hat{M}^{sing}$ or else intersect $\hat{M}^{sing}$ in an $\epsilon$-separated set. Let $x$ be a point in $\partial\overline{B_{r+\epsilon}(p)}$ and set $y = c_{r,r+\epsilon}(x)$. If the geodesic segment $xy$ is either contained in or disjoint from the singular set $\hat{M}^{sing}$, then $c^{-1}_{r,r+\epsilon}(y) = x$. Otherwise, $xy$ intersects $\hat{M}^{sing}$ in a unique point $z$. It follows that $c^{-1}_{r,r+\epsilon}(y)$ is homeomorphic to $\Sigma_z\hat{M}\setminus B_\pi(v)$ where $v$ is the direction pointing to $y$. From Lemma 2.12 we see that $\Sigma_z\hat{M} \setminus B_r(v)$ is homeomorphic to a closed (n-1)-ball for $0 < r < \pi$. Hence $\Sigma_z\hat{M} \setminus B_\pi(v)$ is cellular. □

The approximation theorem for cellular maps between n-manifolds (see [DJ91] p. 371) says that such a map is a near-homeomorphism, i.e. a uniform limit of homeomorphisms. Hence, from Corollary 6.1, we obtain

**Corollary 6.4.** *Distance-spheres in $\hat{M}$ are topological (n-1)-spheres.*

To show that the geometric boundary of $\hat{M}$ is homeomorphic to a sphere, we will need further results from geometric topology. The next theorem provides a connection between the topology of distance spheres and the topology of the geometric boundary.

**Theorem 6.5** (Theorem (2b.2) in [DJ91]). *Suppose that $P$ is a CAT(0) geodesic space and that $P$ is a Riemannian manifold on the complement of a set of codimension 2. Then for any $x \in P$, the natural map $\Psi : \varprojlim \partial\overline{B_r(x)} \to \partial_\infty P$ is a homeomorphism.*

*Remark* 6.6. The transition morphisms for the inverse limit in the above theorem are given by the radial contraction maps between concentric distance-spheres.

Our last ingredient is a theorem of Brown [Br60] which tells us that an inverse limit of near homeomorphisms is a near homeomorphism. Putting everything together we achieve our aim.

**Proposition 6.7.** *The geometric boundary of $\hat{M}$ is homeomorphic to a sphere.*

*Proof.* Since $\hat{M}$ is Riemannian away from $\hat{M}^{sing}$, $\partial_\infty \hat{M}$ is homeomorphic to $\varprojlim \partial \overline{B_r(x)}$ by Theorem 6.5. The inverse limit of radial contraction maps provides a homeomorphism between $\varprojlim \partial \overline{B_r(x)}$ and $\partial \overline{B_{r_0}(x)}$ by a combination of Lemma 6.3 with the approximation theorem and the above quoted theorem of Brown. Finally, $\partial \overline{B_{r_0}(x)}$ is homeomorphic to a sphere by Corollary 6.4. □

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