Non-commutative recurrence relations for scattering amplitudes

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit Differenzengleichungen mit nichtkommutativen Koeffizienten und deren Auftreten in Berechnungen von Streuamplituden. Wir betrachten lineare Differenzengleichungen beliebiger Ordnungen mit konstanten Koeffizienten, welche nicht vertauschbar bezüglich der Multiplikation sind. Sowie für die homogenen als auch für die inhomogenen Relationen diesen Typs präsentieren wir die allgemeine Lösung. Den Beweis legen wir mittels vollständiger Induktion dar.

Zur Handhabung der nichtkommutativen Koeffizienten und für eine kompaktere Darstellung unserer Ergebnisse führen wir ein Operatorprodukt ein. Neben einer einfachen kombinatorischen Interpretation geben wir auch eine formale Definition dieses Produkts. Hiermit leiten wir für weitere Anwendungen erforderliche grundlegende Eigenschaften des Operatorprodukts konsistent her.

Eine Anwendung legen wir für verallgemeinerte hypergeometrische Funktionen dar. Deren Laurent-Reihenentwicklung ist sowohl in der Quantenfeldtheorie als auch in der Stringtheorie von Interesse. Zur Berechnung einer endlichen Anzahl von Gliedern einer Reihe können rekursive Differentialgleichungen verwendet werden. Stellt man die Integrationen in den formalen Lösungen dieser Gleichungen durch Integraloperatoren dar, dann erhält man die oben beschriebenen Differenzengleichungen mit den (nichtkommutativen) Integraloperatoren als Koeffizienten. Unsere allgemeine Lösung diesen Typs von Differenzengleichungen liefert einen Ausdruck für die Reihenglieder, welcher für alle Ordnungen der Entwicklung gilt. Im Gegensatz zu bisherigen Betrachtungen ist es damit möglich eine beliebige Ordnung ohne die Berechnung der vorhergehenden zu ermitteln. Darüber hinaus erlauben diese Ergebnisse verallgemeinerte hypergeometrische Funktionen als unendliche Reihen und damit exakt darzustellen. Das demonstrieren wir für verallgemeinerte hypergeometrische Funktionen mit ganzzahligen Parametern als auch für eine hypergeometrische Funktion mit einem halbzahligen Parameter.

Auch bei der Berechnung von Feynman Integralen lässt sich diese Methode anwenden. Zur Handhabung von Divergenzen ist man hier an der Laurent-Reihe um $\epsilon = 0$ (mit dem dimensionellen Regulator ϵ) interessiert. Diese lässt sich ebenso wie die Entwicklung verallgemeinerter hypergeometrischer Funktionen mittels Differentialgleichungen Ordnung für Ordnung berechnen. Wir demonstrieren anhand eines massiven Ein-Schleifen-Integrals, wie man die zugehörige nichtkommutative Differenzengleichung herleitet, um schließlich einen allgemeinen Ausdruck für alle Reihenglieder der ϵ-Entwicklung zu erhalten. Desweiteren

verwenden wir die daraus resultierende exakte Reihendarstellung des Feynman Integrals, um es durch eine hypergeometrische Funktion auszudrücken.

In der Stringtheorie findet man verallgemeinerte hypergeometrische Funktionen in den Baum-Niveau-Streuamplituden offener Strings. Für die 4- und 5-Punkt Amplituden präsentieren wir die entsprechenden Reihenentwicklungen um $\alpha' = 0$ für alle Ordnungen, wobei α' die inverse Stringspannung darstellt. Neben der phänomenologischen Bedeutung der Entwicklung ist deren Struktur vor allem von mathematischem Interesse. Wir nutzen die zyklische Symmetrie der Amplitude bezüglich der kinematischen Invarianten, um bisher unbekannte Familien von Identitäten für multiple Zetawerte zu gewinnen. Das wird vor allem durch eine Reihe nicht-trivialer Relationen des von uns eingeführte Operatorprodukts ermöglicht, welche wir durch kombinatorische Überlegungen konstruieren.

Abstract

In this thesis recurrence relations with non-commutative coefficients and their applications for computing scattering amplitudes are discussed. We consider n-th order linear recurrence relations with constant coefficients, which do not commute with respect to multiplication. The general solution is presented both for homogeneous and inhomogeneous relations of this kind. The prove is given through mathematical induction.

We introduce a generalized operator product, which is useful to handle non-commutative coefficients and which allows to represent our results in a compact form. Besides a simple combinatorial interpretation of this product, we also give a formal definition. The latter is used to consistently derive basic properties of the generalized operator product, which are required for further applications.

One application is explained for generalized hypergeometric functions. Their Laurent series expansion is of interest both in quantum field theory and in string theory. Recursive differential equations can be used to compute a finite number of orders of an expansion. Replacing the integrations in the formal solution of the differential equations by integral operators, yields recurrence relations of the type described above, where the coefficients include the (non-commutative) integral operators. Our general solution for this kind of recurrence relations results in all-order expansions for generalized hypergeometric functions. In contrast to previous calculations, this makes it possible to determine any order of a series without having to compute lower orders in advance. Moreover, these results allow the representation of generalized hypergeometric functions as infinite series. This is demonstrated for generalized hypergeometric functions with integer parameters and for a hypergeometric function with a half-integer parameter.

This method is also applicable for the calculation of Feynman integrals. The latter are expanded in a Laurent series around $\epsilon = 0$ (with the dimensional regulator ϵ) to handle divergences. Just like expansions of generalized hypergeometric functions, this can be computed by means of recursive differential equations. We use a massive one-loop integral to demonstrate, how to derive the corresponding non-commutative recurrence relations in order to obtain a general expression for all orders of the ϵ-expansion. Furthermore, we use the resulting exact representation of the complete Feynman integral as an infinite series to write it in terms of a hypergeometric function.

In string theory generalized hypergeometric functions appear in disk-level open superstring amplitudes. The all-order expansions around $\alpha' = 0$ (with the inverse string tension

α') are given for the four- and five-point amplitudes. These expansion are of both phenomenological and mathematical importance. We use the symmetry of string amplitudes with respect to cyclic permutations of kinematic invariants to extract previously unknown families of identities for multiple zeta values from the all-order expansions. This is achieved, using a variety of non-trivial relations satisfied by the generalized operator product, for which we use combinatorial approaches.

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Chapter 1

Introduction

The physics at the Large Hadron Collider (LHC) at high luminosities [3, 4] and at future colliders, like the Linear Collider ILC [5] or the Super Proton-Proton Collider (SPPC) [6], concentrates on measuring the properties of the Higgs boson, the top quark and vector bosons. Besides more accurate tests of established theories, the focus is on searches for signals beyond the Standard Model to check for example models of large extra dimensions [7] or low string scale models [8]. All measurements will be performed at high precision. In order to compare the experimental data with theoretical predictions of these models, higher precision for the latter is required as well. In perturbative treatment of QFT this boils down to the calculation of higher loop orders for scattering amplitudes. The computation of these objects, according to the application of Feynman rules, essentially consists of two steps. The first is to find a compact expression in terms of a minimal set of Feynman integrals. These scalar functions capture the integrations over momenta of virtual particles. The second step is to evaluate all Feynman integrals.

While there are various methods leading to different representations, many Feynman integrals can be given in terms of hypergeometric functions and their generalizations [9]. The study of (generalized) hypergeometric functions is more than 200 years old and includes contributions from Euler, Gauss and Riemann. Many classes of functions, e.g. trigonometric functions, logarithms and Bessel functions, arise as special cases of hypergeometric functions. The latter can be characterized in three ways: (i) as power series whose coefficients satisfy certain recursion properties, (ii) as integrals of Euler or Mellin-Barnes type, and (iii) as solutions to a system of differential equations [10]. The first version (i) can be understood as a generalization of the geometric series, which justifies the term *hyper*geometric. A representation of a Feynman integral in terms of hypergeometric functions provides a compact expression for the complete integral. Furthermore, this framework is suitable for integrals in general dimensions with arbitrary powers of propagators. However, not all relevant information on Feynman integrals can directly be extracted from their hypergeometric representation.

In the computation of Feynman integrals one often has to deal with divergences. To handle those, a regularization is necessary. One technique is dimensional regularization. The idea is to introduce a small parameter ϵ and add it to the dimension, e.g. in four dimensions $D = 4 - 2\epsilon$. Integrals are then expanded in a Laurent series around $\epsilon = 0$. Therefore, with a representation of a Feynman integral in terms of generalized hypergeometric functions at hand, the ϵ-expansion of the latter is of great interest. The third representation (iii) provides one method to obtain the ϵ-expansion of hypergeometric functions [11, 12]. This can schematically be described as follows:

- The goal is to determine the coefficient functions f_k of the Laurent expansion

$$f = \sum_k \epsilon^k f_k \ . \tag{1.1}$$

- The function f represent hypergeometric functions, which satisfy differential equations of the following form:

$$\partial_x f = A(x, \epsilon) f \ . \tag{1.2}$$

  The quantity x stands for some variable the function f depends on. The coefficient A depends on x and ϵ.

- Inserting the expansion (1.1) in eq. (1.2) yields recursive differential equations for the coefficient functions f_k:

$$\partial_x f_k = B(x) f_{k-1} \ . \tag{1.3}$$

  The coefficient B follows directly from A. The differential equation (1.3) can be used to determine the coefficient functions f_k iteratively, starting with the lowest order and proceeding up to any order in ϵ. This is achieved by integrating over x and using appropriate sets of boundary conditions.

These steps apply not only to hypergeometric function but to a larger class of functions. In fact, the above discussion is not relevant only for those Feynman integrals, which are known in terms of hypergeometric functions, but for Feynman integrals in general. The form of a l-loop Feynman integral in D space-time dimensions is:

$$\int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \cdots \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \frac{1}{D_1^{a_1} \dots D_j^{a_j}} \ . \tag{1.4}$$

For every loop there is an integration over a loop momentum k_α, $\alpha = 1, \dots, l$. The integrand is directly related to the Feynman diagrams it stems from. The (inverse) propagators D_β, $\beta = 1, \dots, j$, appear with integer powers a_β, also called indices. The general form of propagators is

$$D_\beta = q_\beta^2 - m_\beta^2 + i0 \ ,$$

with some momentum q_β and mass m_β. Besides the loop momenta k_α, the momentum q_β can depend on external momenta, for which there might be additional conditions like conservation of momentum or on-shell conditions. The latter can include kinematic invariants, such as the Mandelstam variables in the four-point case.

Feynman integrals can be classified into families or topologies. A topology is determined by a set of propagators. Thus, it consists of all integrals appearing with arbitrary indices of these propagators. For each topology there is a set of integration-by-parts (IBP) identities [13] that relate integrals with different indices. These relations are linear in the integrals, with coefficients being rational functions of the space-time dimension D and kinematic invariant. IBP identities allow to reduce all integrals of a topology to a finite set of so-called master integrals (MI) [14]. The are several public codes to solve IBP relations [15]–[20], some of which are based on the Laporta method [21, 22].

The method of differential equations is one of the most powerful techniques for the computation of the MIs. It was suggested in [23, 24] and first applications can be found in [25]–[28]. Detailed reviews of the method of differential equations are given in [29, 30]. Instead of solving the loop integrations directly, the idea in this formulation is to set up differential equations in the masses or kinematic invariants. The key to this method is that derivatives of propagators $D = q^2 - m^2 + i0$, for instance w.r.t. the squared mass m^2, can again be written in terms of propagators:

$$\frac{\partial}{\partial m^2}\frac{1}{D} = \frac{1}{D^2} \ . \tag{1.5}$$

This allows to write derivatives of MIs as linear combinations of integrals with different powers of propagators, i.e. in terms of integrals of the same topology. The application of IBP identities allows to reduce this linear combination to MIs. Performing these steps for every MI yields a system of linear differential equations of the form (1.2). As explained in eqs. (1.1)–(1.3) for hypergeometric functions, this system of differential equations allows to construct the ϵ-expansions of MIs.

Not only in the computation of Feynman integrals, but also in the evaluation of disk-level string amplitudes, the module of generalized hypergeometric functions is ubiquitous. It is an important discovery, that the complete tree-level amplitude of N massless open strings has a striking simple and compact form in terms minimal building blocks [31, 32]:

$$A_{OS} = F(\alpha')A_{YM} \ . \tag{1.6}$$

The $(N-3)!$-dimensional vector of N-point open superstring amplitudes A_{OS} is expressed in terms of N-point Yang-Mills subamplitudes contained in the $(N-3)!$-dimensional vector A_{YM}. The full α'-dependence of disk-level string amplitudes is encoded in the matrix $F(\alpha')$, with the elements being generalized Euler integrals, which integrate to multiple Gaussian hypergeometric functions [33]. The quantity α' describes the string length scale. For example with α' going to zero, strings become point particles. In other words, $F(\alpha')$ describes string theory corrections to field amplitudes. The Yang-Mills amplitude is reproduced in the limit $\alpha' \to 0$, i.e. $F(0) = 1$, while its modifications can be derived by studying higher orders in α' of the functions contained in the matrix $F(\alpha')$. Therefore, extracting from the latter the Laurent expansion around $\alpha' = 0$ is also of phenomenological interest [8]. The computation of α'-expansions of generalized Euler integrals has been initiated in [33, 34], while a more systematic way by making profit of the underlying algebra of multiple polylogarithms has been presented in [35]. Further attempts can be found in [36]–[38]. For four- and five-point amplitudes, the functions in $F(\alpha')$ can be written as products of generalized hypergeometric functions.

Although ϵ and α' are of completely different physical origin, the dependence of the hypergeometric functions on these quantities in the corresponding amplitudes is the same. As a consequence, the coefficient functions f_k appearing in the ϵ-expansion (1.1) are of the same class as those appearing in the Laurent series in α', namely Chen's iterated integrals [39], elliptic integrals and maybe generalizations thereof. The type of iterated integrals entering string amplitudes are multiple polylogarithms at unity, i.e. multiple zeta values (MZV) [40, 41, 33]. As a generalization of the famous Riemann zeta function $\zeta(n)$, MZVs are defined as infinite series of the form

$$\zeta(n_1, n_2, \ldots, n_d) = \sum_{0<k_d<\ldots<k_2<k_1} k_1^{-n_1} \cdot k_2^{-n_2} \cdot \ldots \cdot k_d^{-n_d} \, , \tag{1.7}$$

with positive integers $n_1, \ldots, n_d$ and $n_1 > 1$. Despite their simple definition as convergent series, MZVs are subject to many interesting identities [42], with the simplest one, $\zeta(3) = \zeta(2,1)$, already known to Euler. For string amplitudes (1.6) the matrix $F(\alpha')$ decomposes into factors accounting for different classes of multiple zeta values [43]. Furthermore, in that paper it was found that in terms of elements of a specific Hopf algebra the α'-expansion of the open superstring amplitudes assumes a very simple and symmetric form, which carries all the relevant information. This Hopf algebra is related to the algebra spanned by all MZVs [44, 45].

Although a variety of methods for obtaining ϵ- and α'-expansions of hypergeometric type functions have been established in the context of both QFT and string theory, computing in a fully systematic way a closed, compact and analytic expression for a given order in ϵ or α', respectively, which is given explicitly in terms of iterated integrals, is desirable and still lacking. In [1, 2] we presented two methods, which exactly meet these requirements and which are straightforwardly applicable. One way matches a given order in the power series expansion with the corresponding coefficient of some fundamental and universal solution of the Knizhnik–Zamolodchikov equation. This reduces the calculation of ϵ- or α'-expansions to simple matrix multiplications. The other method solves recurrence relations satisfied by the corresponding coefficient functions and shall be addressed in this thesis.

Our general idea can be understood as the next steps after setting up eqs. (1.1)–(1.3). The solution to the recursive differential equation (1.3) in combination with boundary conditions can be written as a recurrence relation for the functions f_k. This is accomplished by replacing the integrations by integral operators. As consequence, the coefficients of the recurrence relations are non-commutative. One of the main achievement described in this thesis, is the construction of the general solution of this type of recurrence relations. This allows to present f_k explicitly in terms of iterated integrals, without having to compute lower orders of the expansion in advance. Furthermore, since the resulting expression for f_k is valid at all orders in ϵ, the expansion (1.1) can be given as an infinite series, thereby providing an exact representation for the complete function f. In other words, while previous calculations solved eq. (1.3) to obtain a finite number of orders of the expansion (1.1), we can give all orders and therefore a solution to eq. (1.2). As applications, we demonstrate how to obtain all-order expansions for generalized hypergeometric functions, Feynman integrals and disk-level open superstring amplitudes. Another application arises for the products of generalized hypergeometric functions, which appear in string amplitudes. Their symmetry w.r.t. cyclic permutations of kinematic invariants is not automatically fulfilled in the corresponding all-order expansions. Instead, identities for MZVs are generated.

This thesis is organized as follows:

- In chapter 2 we discuss linear homogeneous recurrence relations with constant non-commutative coefficients. Before presenting the general solution to this mathematical problem, we introduce the generalized operator product and its basic properties. This product is useful to handle non-commutative objects and allows compact representations of our results. We also prove our solution via mathematical induction and give the general solution for the inhomogeneous version.

- Chapter 3 concentrates on various types of generalized hypergeometric functions. After the basics and some notations for the integral operators are introduced, we review the order-by-order computation of Laurent series around integer and half-integer values of parameters of generalized hypergeometric functions via differential equations. Then, we set up the associated recurrence relations and apply our general solution from chapter 2 to eventually obtain all-order expansions, which solve the underlying hypergeometric differential equations. We also give the mathematical framework for the general class of differential equations, to which our method can be applied. Finally, we discuss some detailed restrictions on the form of the differential equations, which is of particular interest for applications to Feynman integrals.

- In chapter 4 we use a simple example to demonstrate, how the method of differential equations can be extended to obtain all-order ϵ-expansions for Feynman integrals. After we point out the advantages of this expression, we use it to construct a representation of the Feynman integral in terms of a hypergeometric function.

- In chapter 5 we apply our results for generalized hypergeometric functions to α'-expansions of open superstring amplitudes. For demonstration purposes this is performed to obtain the already known all-order expansion of the four-point amplitude. The five-point functions are presented in two different ways. While one version is more compact, the other is more suitable for further applications to be discussed in chapter 6.

- In chapter 6 we use combinatorial approaches to derive a variety of relations for the generalized operator product. These identities are used to provide alternative representation of our results from chapters 3 and 5. Furthermore, we discuss identities for MZVs, which are generated through cyclic symmetry of string amplitudes. This includes previously unknown families of identities of MZVs. Both the alternative representations of the all-order expansions and the general identities of MZVs involve functions, for which we present relations involving hypergeometric functions, binomial coefficients and (generalized) Fibonacci numbers.

- As a conclusion we summarize our achievements, before discussing possible directions for future research. The appendix contains some intermediate results from chaper 6, which serve as a consistency check. One of the more interesting new identities for MZVs is analysed in detail in the appendix as well.

Chapter 2

Recurrence relations

One essential step of this work involves n-th order linear homogenous recurrence relations

$$w_k = \sum_{\alpha=1}^{n} c_\alpha \, w_{k-\alpha} \ , \tag{2.1}$$

with constant coefficients c_i and initial values $w_k = \bar{w}_k$ for $k = 0, 1, \ldots, n-1$. In particular we have to deal with coefficients, which do not commute: $c_i c_j \neq c_j c_i$. In section 2.1 useful definitions and notations for non-commutative coefficients are introduced. The solution, i.e. a formula that expresses all w_k $(k \geq n)$ in terms of initial values $\bar{w}_k$ only, is presented in section 2.2.

2.1 The generalized operator product

A simple example is the following second order recurrence relation,

$$w_k = c_1 w_{k-1} + c_2 w_{k-2} \ , \tag{2.2}$$

with initial values $w_0 = 1$ and $w_1 = c_1$. For $k = 5$ this gives:

$$w_5 = c_1^5 + c_1^3 c_2 + c_1^2 c_2 c_1 + c_1 c_2 c_1^2 + c_2 c_1^3 + c_1 c_2^2 + c_2 c_1 c_2 + c_2^2 c_1 \ . \tag{2.3}$$

It can be related to the integer partitions of 5, which use only 2 and 1:

$$5 = 1 + 1 + 1 + 1 + 1 = 2 + 1 + 1 + 1 = 2 + 2 + 1 \ . \tag{2.4}$$

Denoting how often 1 appears in a partition by j_1 and the number of 2's by j_2, then each of these three partitions can be identified by a product $c_1^{j_1} c_2^{j_2}$, which for the case of interest are c_1^5, $c_1^3 c_2$ and $c_1 c_2^2$. All these terms appear on the r.h.s. of (2.3). The remaining terms in (2.3) are permutations of these three products. Let us introduce the bracket $\{c_1^{j_1}, c_2^{j_2}\}$

as the sum of all possible distinct permutations of factors c_i, each one appearing j_i times ($i = 1, 2$). For example $j_1 = 1$, $j_2 = 2$ yields the following sum of three products:

$$\{c_1, c_2^2\} = c_1c_2^2 + c_1c_2c_1 + c_2^2c_1 \ . \tag{2.5}$$

With this bracket we can now write w_5 more compact as:

$$w_5 = \sum_{j_1+2j_2=5} \{c_1^{j_1}, c_2^{j_2}\} \ . \tag{2.6}$$

The sum over non-negative integers j_1 and j_2 represents all the integer partitions (2.4). It turns out, that the generalization of (2.6) solves the recurrence relation (2.2):

$$w_k = \sum_{j_1+2j_2=k} \{c_1^{j_1}, c_2^{j_2}\} \ . \tag{2.7}$$

Before the solution of the more general recurrence relation (2.1) and its proof are discussed, in the following subsection a proper definition and some basic properties of a generalized version of the brackets $\{c_1^{j_1}, c_2^{j_2}\}$ is given.

2.1.1 Definition

The object

$$\left\{c_1^{j_1}, c_2^{j_2}, \ldots, c_n^{j_n}\right\} \tag{2.8}$$

is defined as the sum of all the

$$\binom{\sum\limits_{\alpha=1}^{n} j_\alpha}{j_1, j_2, \ldots, j_n} \tag{2.9}$$

possible distinct permutations of non-commutative factors c_i, each one appearing j_i times (with $j_i \in \mathbb{N}$, $i = 1, 2, \ldots, n$). The non-negative integers j_i are referred to as indices and the factors c_i as arguments of the generalized operator product (2.8). For example:

$$\{c_1, c_2\} = c_1c_2 + c_2c_1 \ , \tag{2.10}$$

$$\begin{aligned}\{c_1^2, c_2, c_3\} &= c_1^2c_2c_3 + c_1c_2c_1c_3 + c_1c_2c_3c_1 + c_2c_1^2c_3 + c_2c_1c_3c_1 + c_2c_3c_1^2 \\ &\quad + c_1^2c_3c_2 + c_1c_3c_1c_2 + c_1c_3c_2c_1 + c_3c_1^2c_2 + c_3c_1c_2c_1 + c_3c_2c_1^2 \ .\end{aligned} \tag{2.11}$$

For the case of two arguments the object (2.8) was used in [46] to solve a second order recurrence relation with non-commutative coefficients. There is a useful recursive definition[1] for (2.8) as:

$$\left\{c_1^{j_1}, c_2^{j_2}, \ldots, c_n^{j_n}\right\} = \sum_{\substack{\alpha=1 \\ j_\alpha \neq 0}}^{n} c_\alpha \left\{c_1^{j_1}, c_2^{j_2}, \ldots, c_\alpha^{j_\alpha - 1}, \ldots, c_n^{j_n}\right\} + \prod_{\beta=1}^{n} \delta_{0 j_\beta} \ . \tag{2.12}$$

[1] The same formula with c_α to the right of the generalized operator product holds as well.

Furthermore we have[2]:

$$\begin{aligned} \{c_1^0, c_2^{j_2}, \ldots, c_n^{j_n}\} &= \{c_2^{j_2}, \ldots, c_n^{j_n}\} , \\ \{c^j\} &= c^j \ , \\ \text{and especially } \{c^0\} &= 1 \ . \end{aligned} \tag{2.13}$$

The product of Kronecker deltas in (2.12) gives a non-vanishing contribution in case all indices $j_1, \ldots, j_n$ are zero. Without this product an inconsistency would occur: for $j_1 = \ldots = j_n = 0$ the sum on the r.h.s. of (2.12) becomes zero, while the l.h.s. should equal one according to (2.13).

The definition (2.12) together with eqs. (2.13) allow to decrease step by step the indices and the number of arguments. This way the object (2.8) can be written in terms of non-commutative products. For instance applying (2.12) twice to all generalized operator products on the l.h.s. of (2.11) yields:

$$\begin{aligned} \{c_1^2, c_2, c_3\} &= c_1\{c_1, c_2, c_3\} + c_2\{c_1^2, c_3\} + c_3\{c_1^2, c_2\} \\ &= c_1^2\{c_2, c_3\} + c_1c_2\{c_1, c_3\} + c_1c_3\{c_1, c_2\} + c_2c_1\{c_1, c_3\} + c_2c_3c_1^2 \\ &+ c_3c_1\{c_1, c_2\} + c_3c_2c_1^2 \ . \end{aligned} \tag{2.14}$$

Applying (2.12) once again or using (2.10) gives the r.h.s. of (2.11).

The definition of (2.8) can be extended to negative integer indices as:

$$\{c_1^{j_1}, c_2^{j_2}, \ldots, c_n^{j_n}\} = 0 \ , \quad j_1 < 0 \ . \tag{2.15}$$

This extension turns out to be useful, when the indices of generalized operator products include summation indices. It allows to reduce the conditions for the summation regions. E.g. the condition $j_\alpha \neq 0$ in the sum on the r.h.s. of (2.12) can be dropped with this extension.

To prove that (2.12) gives indeed all distinct permutations, it is sufficient to show that:

1. the number of terms equals (2.9),
2. there are no identical terms
3. and all terms contain each non-commutative factor c_i exactly j_i times.

The third point is obviously fulfilled. The second one is also quite obvious, since every summand of the sum in (2.12) starts with a different factor. Using the definition again, yields that all terms, which come from the same summand and therefore have the same

[2]Since in (2.8) the order of the arguments is irrelevant, we write identities, such as the first line of (2.13), without loss of generality for the first arguments only.

first factor, have a different second factor and so on. The first point is also true, since the number of terms on the r.h.s. of (2.12) is:

$$\sum_{\alpha=1}^{n} \binom{-1+\sum_{\beta=1}^{n} j_\beta}{j_1, j_2, \ldots, j_\alpha - 1, \ldots, j_n} . \tag{2.16}$$

The above expression can easily be transformed into (2.9) using the definition of the multinomial coefficient in terms of factorials.

The generalized operator product (2.8) is closely related to the shuffle product:

$$\left\{c_1^{j_1}, c_2^{j_2}, \ldots, c_n^{j_n}\right\} = \underbrace{c_1 \ldots c_1}_{j_1} \sqcup\!\sqcup \underbrace{c_2 \ldots c_2}_{j_2} \sqcup\!\sqcup \ldots \sqcup\!\sqcup \underbrace{c_n \ldots c_n}_{j_n} . \tag{2.17}$$

However, the notation on the l.h.s. is more compact, in particular for the applications in the following chapters.

2.1.2 Basic properties

With the definition (2.12) and eqs. (2.13) the following basic properties can easily be proven. Factors a, which commute with all arguments, i.e. $c_i a = a c_i$, can be factorized:

$$\left\{(ac_1)^{j_1}, c_2^{j_2}, \ldots, c_n^{j_n}\right\} = a^{j_1} \left\{c_1^{j_1}, c_2^{j_2}, \ldots, c_n^{j_n}\right\} . \tag{2.18}$$

Identical arguments can be combined:

$$\left\{c_1^{j_1}, c_1^{j_2}, c_3^{j_3}, \ldots, c_n^{j_n}\right\} = \left\{c_1^{j_1+j_2}, c_3^{j_3}, \ldots, c_n^{j_n}\right\} \binom{j_1+j_2}{j_1} . \tag{2.19}$$

The binomial coefficient[3] ensures that the number of terms is the same on both sides. While sums can be treated according to

$$\{c_1 + c_2, c_3^{j_3}, \ldots, c_n^{j_n}\} = \{c_1, c_3^{j_3}, \ldots, c_n^{j_n}\} + \{c_2, c_3^{j_3}, \ldots, c_n^{j_n}\} , \tag{2.20}$$

one has to be careful, when such arguments appear with exponents greater than one. Before these cases are discussed, note that the generalized operator product can be used for a generalized version of the binomial theorem, which is also valid for non-commutative quantities c_1 and c_2:

$$(c_1 + c_2)^j = \sum_{j_1+j_2=j} \left\{c_1^{j_1}, c_2^{j_2}\right\} , \tag{2.21}$$

[3] In order not to conflict with (2.15), we use $\binom{j_1+j_2}{j_1} = 0$ for $(j_1 < 0) \vee (j_2 < 0)$.

with non-negative integers j, j_1 and j_2. Applying naively this relation to arguments of (2.8) leads to inconsistencies. E.g.:

$$\begin{aligned}\left\{(c_1+c_2)^2, c_3\right\} &\stackrel{?}{=} \left\{\sum_{\alpha=0}^{2}\left\{c_1^{2-\alpha}, c_2^{\alpha}\right\}, c_3\right\} \\ &= \left\{c_1^2, c_3\right\} + \left\{c_2^2, c_3\right\} + \left\{c_1 c_2, c_3\right\} + \left\{c_2 c_1, c_3\right\} \ . \end{aligned} \tag{2.22}$$

Eq. (2.20) is used in the last step. Using instead the definition (2.12) gives:

$$\begin{aligned}\left\{(c_1+c_2)^2, c_3\right\} &= (c_1+c_2)(c_1+c_2)c_3 + (c_1+c_2)c_3(c_1+c_2) \\ &\quad + c_3(c_1+c_2)(c_1+c_2) \ . \end{aligned} \tag{2.23}$$

Comparing (2.22) and (2.23) shows that $c_1c_3c_2 + c_2c_3c_1$ is missing in (2.22). To avoid this problem, one simply has to ignore the inner curly brackets, when applying (2.21) to arguments of (2.8). Hence, the following relation is consistent:

$$\{(c_1+c_2)^j, c_3^{j_3}, \ldots, c_n^{j_n}\} = \sum_{j_1+j_2=j} \{c_1^{j_1}, c_2^{j_2}, c_3^{j_3}, \ldots, c_n^{j_n}\} \ . \tag{2.24}$$

This can be easily generalized to multinomials:

$$\{(c_1+c_2+\ldots+c_n)^j, c_{n+1}^{j_{n+1}}, \ldots\} = \sum_{j_1+j_2+\ldots+j_n=j} \{c_1^{j_1}, c_2^{j_2}, \ldots, c_n^{j_n}, c_{n+1}^{j_{n+1}}, \ldots\} \ . \tag{2.25}$$

Besides these basic properties, there are more intricate identities satisfied by generalized operator products. They are discussed in chapter 6.

2.2 Solution

The n-th order linear homogeneous recurrence relation (2.1) is solved by:

$$w_k = \sum_{\alpha=0}^{n-1} \sum_{j_1+2j_2+\ldots+nj_n=k-n-\alpha} \{c_1^{j_1}, c_2^{j_2}, \ldots, c_n^{j_n}\} \sum_{\beta=\alpha+1}^{n} c_\beta \, \bar{w}_{n-\beta+\alpha} \ , \quad k \geq n \ . \tag{2.26}$$

Note that the r.h.s. of (2.26) contains only initial values $\bar{w}_l$. The second sum is over n-tuples of non-negative integers $j_1, \ldots, j_n$, which solve the equation:

$$\sum_{\gamma=1}^{n} \gamma \, j_\gamma = k - n - \alpha \ . \tag{2.27}$$

In the following we shall prove by induction that (2.26) solves (2.1). The regions $2n > k \geq n$ and $k \geq 2n$ are discussed separately. The first region is required to prove the base case

$k = 2n$ of the induction for $k \geq 2n$. The induction for $2n > k \geq n$ has the two base cases $k = n$ and $k = n+1$. In the first case ($k = n$) the only non-zero contribution comes from $\alpha = j_1 = j_2 = \ldots = j_n = 0$:

$$w_n = \sum_{\beta=1}^{n} c_\beta \; \bar{w}_{n-\beta} \; . \tag{2.28}$$

The second case ($k = n+1$) has two parts, one with $\alpha = 1$, $j_1 = j_2 = \ldots = j_n = 0$ and the other with $j_1 = 1$, $\alpha = j_2 = \ldots = j_n = 0$:

$$\begin{aligned} w_{n+1} &= c_1 \sum_{\gamma=1}^{n} c_\gamma \; \bar{w}_{n-\gamma} + \sum_{\beta=2}^{n} c_\beta \; \bar{w}_{n+1-\beta} \\ &= c_1 w_n + \sum_{\beta=2}^{n} c_\beta \; \bar{w}_{n+1-\beta} \\ &= \sum_{\beta=1}^{n} c_\beta \; w_{n+1-\beta} \; . \end{aligned} \tag{2.29}$$

Eq. (2.28) is used in the second line of (2.29). Both cases, (2.28) and (2.29), are in agreement with the eq. (2.1) and the initial conditions. The recursive definition (2.12) of the generalized operator product is particularly useful for the inductive step:

$$\begin{aligned} w_k = &\sum_{\alpha=0}^{n-1} \sum_{j_1+2j_2+\ldots+nj_n=k-n-\alpha} \sum_{\gamma=1}^{n} c_\gamma \left\{ c_1^{j_1}, c_2^{j_2}, \ldots, c_\gamma^{j_\gamma - 1}, \ldots, c_n^{j_n} \right\} \sum_{\beta=\alpha+1}^{n} c_\beta \; \bar{w}_{n-\beta+\alpha} \\ &+ \sum_{\alpha=0}^{n-1} \sum_{j_1+2j_2+\ldots+nj_n=k-n-\alpha} \prod_{\gamma=1}^{n} \delta_{0 j_\gamma} \sum_{\beta=\alpha+1}^{n} c_\beta \; \bar{w}_{n-\beta+\alpha} \; . \end{aligned} \tag{2.30}$$

Shifting $j_\gamma \rightarrow j_\gamma + 1$ on the r.h.s. of the first line gives:

$$\begin{aligned} &\sum_{\gamma=1}^{n} c_\gamma \sum_{\alpha=0}^{n-1} \sum_{j_1+2j_2+\ldots+nj_n=k-n-\alpha-\gamma} \left\{ c_1^{j_1}, c_2^{j_2}, \ldots, c_n^{j_n} \right\} \sum_{\beta=\alpha+1}^{n} c_\beta \; \bar{w}_{n-\beta+\alpha} \\ &= \sum_{\gamma=1}^{n} c_\gamma \cdot \begin{cases} w_{k-\gamma} & \text{for } k-\gamma \geq n \\ 0 & \text{else} \end{cases} \\ &= \sum_{\gamma=1}^{\min\{n,k-n\}} c_\gamma \; w_{k-\gamma} \; . \end{aligned} \tag{2.31}$$

The second line of (2.30) is non-zero only if there is a solution for $k - n - \alpha = 0$. Inserting the upper bound $\alpha \leq n-1$ of the first sum gives the condition $k < 2n$. Combining both lines of (2.30) for this region yields

$$w_k = \sum_{\gamma=1}^{k-n} c_\gamma \, w_{k-\gamma} + \sum_{\beta=k-n+1}^{n} c_\beta \, \bar{w}_{k-\beta} \ , \tag{2.32}$$

which is identical to (2.1). In the region $k \geq 2n$ the second line of (2.30) becomes zero and the upper bound in (2.31) is n, since $k - n \geq n$. This also results in (2.1).

Finally, it is easy to prove that the inhomogeneous recurrence relation ($k \geq n$)

$$w_k^{(\text{inh})} = \sum_{\alpha=1}^{n} c_\alpha \, w_{k-\alpha}^{(\text{inh})} + d_k \tag{2.33}$$

is solved by

$$w_k^{(\text{inh})} = w_k + \sum_{\alpha=n}^{k} \sum_{j_1+2j_2+\ldots+nj_n=k-\alpha} \{c_1^{j_1}, c_2^{j_2}, \ldots, c_n^{j_n}\} \ d_\alpha \ , \tag{2.34}$$

where w_k is the solution of the corresponding homogeneous recurrence relation and d_k is a inhomogeneity, which depends on k.

Chapter 3

Generalized hypergeometric functions

The generalized Gauss function or generalized hypergeometric function ${}_pF_q$ is given by the power series [10]

$$ {}_pF_q(\vec{a};\vec{b};z) \equiv {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix};z\right] = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)^m}{\prod_{j=1}^{q}(b_j)^m}\,\frac{z^m}{m!}\ ,\quad p,q\in\mathbb{N}\ , \tag{3.1} $$

with parameters $a_i, b_j \in \mathbb{R}$ and the Pochhammer (rising factorial) symbol:

$$ (a)^n = \frac{\Gamma(a+n)}{\Gamma(a)} = a\ (a+1)\ldots(a+n-1)\ . $$

Generalized hypergeometric functions, which appear in calculations of Feynman integrals and string amplitudes, have $q = p-1$. In that case the series (3.1) converges absolutely at the unit circle $|z| = 1$ if the parameters meet the following condition:

$$ \sum_{\beta=1}^{p-1} b_\beta - \sum_{\alpha=1}^{p} a_\alpha > 0\ . \tag{3.2} $$

Dimensional regularization is a common method to handle divergences in field theory calculations. The idea is to add a small parameter ϵ to the dimension D, e.g. in four-dimensional spacetime: $D = 4-2\epsilon$. The parameters a_i, b_j of the functions (3.1) we are interested in are linear functions of ϵ. Likewise the parameters of generalized hypergeometric functions, which appear in our string theory calculations, are linear functions of the string tension α'. In this chapter we shall present and apply our new technique to solve for recurrences to compute the corresponding ϵ- or α'-expansion:[1]

$$ {}_pF_{p-1}\left[\begin{matrix} m_1+\alpha' a_1,\ldots,m_p+\alpha' a_p \\ n_1+\alpha' b_1,\ldots,n_{p-1}+\alpha' b_{p-1} \end{matrix};z\right] = \sum_k (\alpha')^k u_{p,k}(z)\ ,\quad m_i,n_j\in\mathbb{Z}\ . \tag{3.3} $$

[1]In this chapter we use α' only. Of course all equation hold with α' replaced by ϵ as well.

The coefficient functions $u_{p,k}(z)$ of this expansion are expressible in terms of multiple polylogarithms (MPLs) with coefficients, that are ratios of polynomials [11, 12]. In section 3.1 we introduce our notation for MPLs and related functions. The starting point of our calculations are differential equations satisfied by generalized hypergeometric functions. We discuss these equations and how to use them to obtain expansions of the form (3.3) in section 3.2. In section 3.3 we demonstrate how to transform the differential equations to recurrence relations. This allows the application of our general solution (2.26) for recurrences to obtain all-order expansions. When the parameters m_i, n_j are not restricted to integers, MPLs are not sufficient to represent the expansion (3.3). One such case, a $p = 2$ hypergeometric function with one half-integer parameter, is discussed in sections 3.2 and 3.3 as well. Harmonic polylogarithms (HPLs), which we also introduce in section 3.1, can be used to represent the coefficient functions of that expansion. The differential equations for the generalized hypergeometric functions under consideration are special cases of generic first-order Fuchsian equations. In section 3.4 we give the mathematical description of the latter and show how to perform our method using reccurences to obtain all-order expressions for this general case. Finally, in section 3.5 we discuss the form of the differential equations of this chapter, which is of particular interest when comparing with the ones satisfied by Feynman integrals subject to chapter 4.

3.1 Integral operators for MPLs, MZVs and HPLs

We introduce the differential operator

$$\theta = z\,\frac{d}{dz} \tag{3.4}$$

and the integral operators

$$\begin{aligned} I(1)\ f(z) &= \int_0^z \frac{dt}{1-t}\ f(t)\ , \\ I(0)\ f(z) &= \int_0^z \frac{dt}{t}\ f(t)\ . \end{aligned} \tag{3.5}$$

Up to boundary values the differential operator θ is the inverse of $I(0)$:

$$\theta\ I(0) f(z) = I(0)\ \theta f(z) = f(z)\ . \tag{3.6}$$

It is useful to combine products of these operators into the shorter form:

$$I(m_1, m_2, \ldots, m_w) \equiv I(m_1)I(m_2)\ldots I(m_w)\ ,\quad m_i \in \{0, 1, \theta\}\ ,\quad I(\theta) \equiv \theta\ . \tag{3.7}$$

Acting with the operator

$$I(\underbrace{0,\ldots,0}_{n_1-1}, 1, \underbrace{0,\ldots,0}_{n_2-1}, 1, \ldots, \underbrace{0,\ldots,0}_{n_d-1}, 1) \equiv I(0)^{n_1-1}I(1)I(0)^{n_2-1}I(1)\ldots I(0)^{n_d-1}I(1) \tag{3.8}$$

on the constant function[2] 1 yields MPLs:

$$I(\underbrace{0,\ldots,0}_{n_1-1},1,\ldots,\underbrace{0,\ldots,0}_{n_d-1},1)\;1=\mathcal{L}i_{\vec{n}}(z)\equiv\mathcal{L}i_{n_1,\ldots,n_d}(z,\underbrace{1,\ldots,1}_{d-1})=\sum_{0<k_d<\ldots<k_1}\frac{z^{k_1}}{k_1^{n_1}\cdot\ldots\cdot k_d^{n_d}}\;, \tag{3.9}$$

with the multiple index $\vec{n}=(n_1,n_2,\ldots,n_d)$. For $z=1$ the MPLs become multiple zeta values (MZVs)

$$\zeta(\vec{n})=\mathcal{L}i_{\vec{n}}(1)=I(\underbrace{0,\ldots,0}_{n_1-1},1,\underbrace{0,\ldots,0}_{n_2-1},1,\ldots,\underbrace{0,\ldots,0}_{n_d-1},1)\;1\Big|_{z=1}\;, \tag{3.10}$$

with the following definition of MZVs:

$$\zeta(\vec{n})\equiv\zeta(n_1,n_2,\ldots,n_d)=\sum_{0<k_d<\ldots<k_2<k_1}k_1^{-n_1}\cdot k_2^{-n_2}\cdot\ldots\cdot k_d^{-n_d}\;. \tag{3.11}$$

Both for MPLs and for MZVs the weight w is defined as the sum of all indices:

$$w=n_1+n_2+\ldots+n_d\;. \tag{3.12}$$

Using the representations in terms of integral operators the weight is equivalent to the total number of integral operators. The depth d is defined as the number of indices, i.e.:

$$d=\dim(\vec{n})\;. \tag{3.13}$$

In terms of integral operators, this is the number of operators $I(1)$. For example

$$I(0)I(1)I(1)=\mathcal{L}i_{2,1}(z)\overset{z=1}{\to}\zeta(2,1) \tag{3.14}$$

are weight $w=3$ and depth $d=2$ MPLs and MZVs, respectively.

For hypergeometric functions with half-integer parameters we introduce the integral operators

$$\begin{aligned}J(0)\;f(y)&=\int_1^y\frac{dt}{t}f(t)\;,\\ J(1)\;f(y)&=\int_1^y\frac{dt}{1-t}f(t)\;,\\ J(-1)\;f(y)&=\int_1^y\frac{dt}{1+t}f(t)\;,\end{aligned} \tag{3.15}$$

[2]In the sequel we do not write this 1.

and similar to eq. (3.7) we use a shorter notation for products of these operators:

$$J(m_1, m_2, \ldots, m_w) \equiv J(m_1)J(m_2)\ldots J(m_w) \ , \quad m_i \in \{0, 1, -1, \theta\} \ , \quad J(\theta) \equiv y\frac{d}{dy} \ . \tag{3.16}$$

Acting with the operators (3.15) on 1 gives HPLs [47]. HPLs of weight $w \geq 2$ are defined recursively as

$$\begin{aligned} H(m_0, \vec{m}; y) &= \int_0^y dt\, g(m_0; t) H(\vec{m}; t) \ , \quad m_i \in \{0, 1, -1\} \ , \quad (m_0, \vec{m}) \neq (\underbrace{0, \ldots, 0}_{w}) \ , \\ H(\underbrace{0, \ldots, 0}_{w}; y) &= \frac{1}{w!} \ln^w y \ , \end{aligned} \tag{3.17}$$

with the multiple index $\vec{m} = (m_1, m_2, \ldots, m_{w-1})$ and

$$g(0; y) = \frac{1}{y} \ , \quad g(1; y) = \frac{1}{1-y} \ , \quad g(-1; y) = \frac{1}{1+y} \ . \tag{3.18}$$

Weight $w = 1$ functions are simple logarithms:

$$H(0; y) = \ln(y) \ , \quad H(1; y) = -\ln(1-y) \ , \quad H(-1; y) = \ln(1+y) \ . \tag{3.19}$$

MPLs (3.9) can be written as HPLs, which include the indices 0 and 1 only:

$$\mathcal{L}i_{\vec{n}}(y) = H(\underbrace{0, \ldots, 0}_{n_1 - 1}, 1, \ldots, \underbrace{0, \ldots, 0}_{n_d - 1}, 1; y) \ . \tag{3.20}$$

However, we use HPLs only for hypergeometric functions with half-integer parameters and the corresponding integral operators (3.15). Products of the latter acting on functions independent of y, like the constant function 1, can recursively be written as HPLs using

$$\begin{aligned} J(m_1) &= H(m_1; y) - H(m_1; 1) \ , \\ J(m_1)H(m_2, \ldots, m_w; y) &= H(m_1, m_2, \ldots, m_w; y) - H(m_1, m_2, \ldots, m_w; 1) \ , \end{aligned} \tag{3.21}$$

e.g.:

$$\begin{aligned} J(\underbrace{0, \ldots, 0}_{w}) &= H(\underbrace{0, \ldots, 0}_{w}; y) \ , \\ J(1) &= H(1; y) - H(1; 1) \ , \\ J(1, 0) &= H(1, 0; y) - H(1, 0; 1) \ , \\ J(0, 1, 0) &= H(0, 1, 0; y) - H(0, 1, 0; 1) - H(0; y)H(1, 0; 1) \ , \\ J(0, -1, 0) &= H(0, -1, 0; y) - H(0, -1, 0; 1) - H(0; y)H(-1, 0; 1) \ . \end{aligned} \tag{3.22}$$

The essential difference between the integral operators (3.15) and the integrations in the definition (3.17) of HPLs is the lower bound. In later sections we encounter differential equations satisfied by functions, for which we know boundary conditions at $y = 1$. Since the integrations in (3.15) start at this point, it is advantageous to use integral operators over HPLs.

The results of the following chapters often contain sums of the form:

$$\sum_{\dots} \zeta(\vec{n}) \; . \tag{3.23}$$

Above, the dots may represent conditions for the weight w, the depth d, specific indices n_i or other quantities referring to the MZVs $\zeta(\vec{n})$ in the sum. The sum runs over all sets of positive integers $\vec{n} = (n_1, \dots, n_d)$, that satisfy these requirements. It is understood that $n_1 > 1$. For example the sum of all MZVs of weight $w = 5$ and depth $d = 2$ is represented as:

$$\sum_{\substack{w=5\\d=2}} \zeta(\vec{n}) = \zeta(4,1) + \zeta(3,2) + \zeta(2,3). \tag{3.24}$$

Further conditions could include the first index n_1 or the number of indices d_1, which equal one:

$$\sum_{\substack{w=5;\; d=2\\n_1\geq 3;\; d_1=0}} \zeta(\vec{n}) = \zeta(3,2) \tag{3.25}$$

Obviously $d_1 = 0$ is equivalent to $n_i \geq 2$ $(i = 1, \dots, d)$. In general we use d_i as the number of indices, which equal i, so that $d = \sum_i d_i$. In some cases a weighting ω is included, which can depend on the indices $\vec{n}$ or other quantities. For example the following sum has $\omega = d_1$:

$$\sum_{\substack{w=6\\d=3}} \zeta(\vec{n})\; d_1 = 2\zeta(4,1,1) + \zeta(3,2,1) + \zeta(3,1,2) + \zeta(2,3,1) + \zeta(2,1,3) \; . \tag{3.26}$$

In our notation the well known sum theorem [48] reads

$$\sum_{\substack{w=a\\d=b}} \zeta(\vec{n}) = \zeta(a) \; , \tag{3.27}$$

which means that for given weight and depth the sum of all MZVs equals the single zeta value (depth one MZV) of that weight (independent of the given depth). The same notation is used for MPLs.

Some sums use multiple indices $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d)$:

$$\sum_{\vec{\alpha}\in L(\vec{\alpha})} f(\alpha_1, \alpha_2, \dots, \alpha_d) = \sum_{\alpha_1\in L_1(\alpha_1)} \sum_{\alpha_2\in L_2(\alpha_2)} \cdots \sum_{\alpha_d\in L_d(\alpha_d)} f(\alpha_1, \alpha_2, \dots, \alpha_d) \; . \tag{3.28}$$

In this context it is necessary to distinguish between functions $f(\alpha_1, \alpha_2, \ldots, \alpha_d)$, which depend on elements of multi-indices, and functions $g(\vec{\alpha})$, which have multi-indices as arguments:

$$\sum_{\vec{\alpha}\in L(\vec{\alpha})} g(\vec{\alpha}) = \sum_{\alpha_1\in L_1(\alpha_1)} g(\alpha_1) \sum_{\alpha_2\in L_2(\alpha_2)} g(\alpha_2) \ldots \sum_{\alpha_d\in L_d(\alpha_d)} g(\alpha_d) \ . \tag{3.29}$$

The latter only occur in combination with multi-index sums and they represent functions $g(\alpha_i)$, which have only one element as argument. The summation regions L_i follow from L in a natural way. All indices of these sums are non-negative integers ($\alpha_i \geq 0$). The sum of all elements of a multi-index $\vec{\alpha}$ is written as $|\vec{\alpha}| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$. This notation is especially used for weightings in sums of MZVs (3.23). It is understood, that the number of elements of the multi-indices equals the depth of the corresponding MZVs.

3.2 Differential equations for generalized hypergeometric functions

In this section we describe some of the achievements originally developed in [11, 12] for the calculation of expansions of the form (3.3). Some of the formulas derived here are the foundation of the method we presented in [1] to obtain all-order expansions, which is described in the next section.

Applying the differential operator (3.4) to the series (3.1), it is easy to show that generalized hypergeometric functions satisfy:

$$\begin{aligned}
(\theta + a_i)\, {}_pF_{p-1}(\vec{a};\vec{b};z) &= a_i\ {}_pF_{p-1}\left[\begin{matrix} a_1,\ldots,a_i+1,\ldots,a_p \\ b_1,\ldots,b_{p-1}\end{matrix};z\right], \\
(\theta + b_j - 1)\, {}_pF_{p-1}(\vec{a};\vec{b};z) &= (b_j-1)\ {}_pF_{p-1}\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_j-1,\ldots,b_{p-1}\end{matrix};z\right], \\
\frac{d}{dz}\ {}_pF_{p-1}(\vec{a};\vec{b};z) &= \frac{\prod_{i=1}^p a_i}{\prod_{j=1}^{p-1} b_j}\ {}_pF_{p-1}\left[\begin{matrix} a_1+1,\ldots,a_p+1 \\ b_1+1,\ldots,b_{p-1}+1\end{matrix};z\right] .
\end{aligned} \tag{3.30}$$

Combining these relations yields the differential equation:

$$z \prod_{i=1}^{p}(\theta + a_i)\ {}_pF_{p-1}(\vec{a};\vec{b};z) = \theta \prod_{j=1}^{p-1}(\theta + b_j - 1)\ {}_pF_{p-1}(\vec{a};\vec{b};z) \ . \tag{3.31}$$

This equation is the starting point for the calculation of expansions (3.3). For each p it is sufficient to derive the expansion (3.3) for one set of integers $\vec{m} = (m_1, \ldots, m_p)$ and $\vec{n} = (n_1, \ldots, n_{p-1})$ only. By using eqs. (3.30) any function ${}_pF_{p-1}(\vec{a};\vec{b};z)$ can be expressed as a linear combination of other functions with parameters that differ from the original ones by an integer shift, i.e. ${}_pF_{p-1}(\vec{m}+\vec{a};\vec{n}+\vec{b};z)$ and the first $p-1$ derivatives thereof [49]. It is clear from eq. (3.31) that the differential equation for the function (3.3) takes the simplest form with $\vec{m}=(0,\ldots,0)$ and $\vec{n}=(1,\ldots,1)$. Thus we consider the α'-expansion:

$$ {}_pF_{p-1}\left[\begin{matrix}\alpha' a_1,\ldots,\alpha' a_p\\ 1+\alpha' b_1,\ldots,1+\alpha' b_{p-1}\end{matrix};z\right]=\sum_k(\alpha')^k v_{p,k}(z)\ . \tag{3.32} $$

For more detailed explanations we start with the $p=2$ version of (3.32) in section 3.2.1, before discussing the case for general p in section 3.2.2. Finally, the expansion of a $p=2$ hypergeometric function with a half-integer parameter is presented in section 3.2.3.

3.2.1 Integer parameters and $p = 2$

According to eq. (3.31) the function (3.32) with $p=2$

$$ v(z):={}_2F_1\left[\begin{matrix}\alpha' a_1,\alpha' a_2\\ 1+\alpha' b\end{matrix};z\right]=\sum_k(\alpha')^k v_{2,k}(z)\ , \tag{3.33} $$

satisfies the second order differential equation:

$$ z(\theta+a_1\alpha')(\theta+a_2\alpha')\ v(z)=\theta(\theta+b\alpha')\ v(z)\ . \tag{3.34} $$

Introducing $\rho(z):=\theta v(z)$ allows to write this as a system of two first order differential equations:

$$ \begin{aligned} \partial_z v(z)&=\frac{1}{z}\rho(z)\ ,\\ \partial_z\rho(z)&=\frac{a_1+a_2-b}{1-z}\alpha'\rho(z)-\frac{b}{z}\alpha'\rho(z)+\frac{a_1a_2}{1-z}(\alpha')^2v(z)\ , \end{aligned} \tag{3.35} $$

with $\partial_z:=\frac{d}{dz}$. Inserting the expansion (3.33) and using that the resulting differential equation is valid at any order in α', yields a recursive differential equation for the coefficient functions $v_{2,k}(z)$ and $\rho_k(z):=\theta v_{2,k}(z)$:

$$ \begin{aligned} \partial_z v_{2,k}(z)&=\frac{1}{z}\rho_k(z)\ ,\\ \partial_z\rho_k(z)&=\frac{a_1+a_2-b}{1-z}\rho_{k-1}(z)-\frac{b}{z}\rho_{k-1}(z)+\frac{a_1a_2}{1-z}v_{2,k-2}(z)\ , \end{aligned} \tag{3.36} $$

Boundary conditions follow from the series representation (3.1):

$$ \begin{aligned} v_{2,k}(0)&=0\ ,\quad k\geq 1\ ,\\ \theta v_{2,k}(z)\big|_{z=0}&=0\ ,\quad k\geq 0\ , \end{aligned} \tag{3.37} $$

Using those, it is possible to solve eqs. (3.36) iteratively for $k \geq 1$:

$$\begin{aligned} v_{2,k}(z) &= \int_0^z \frac{dt}{t} \rho_k(t) \ , \\ \rho_k(z) &= (a_1 + a_2 - b) \int_0^z \frac{dt}{1-t} \rho_{k-1}(t) - b \int_0^z \frac{dt}{t} \rho_{k-1}(t) + a_1 a_2 \int_0^z \frac{dt}{1-t} v_{2,k-2}(t) \ . \end{aligned} \tag{3.38}$$

Furthermore we have:

$$\begin{aligned} v_{2,0}(z) &= 1 \ , \\ v_{2,k}(z) &= 0 \ , \quad k < 0 \ . \end{aligned} \tag{3.39}$$

The first line follows from eq. (3.1) and the second line from the convergence condition (3.2). Eqs. (3.38) and (3.39) are sufficient to calculate the expansion (3.33) order by order starting with $k = 1$, e.g.:

$$\begin{aligned} v_{2,1}(z) &= 0 \ , \\ v_{2,2}(z) &= a_1 a_2 \ \mathcal{L}i_2(z) \ , \\ v_{2,3}(z) &= a_1 a_2 (a_1 + a_2 - b) \ \mathcal{L}i_{2,1}(z) - a_1 a_2 b \ \mathcal{L}i_3(z) \ . \end{aligned} \tag{3.40}$$

Eqs. (3.5) and (3.9) allow to write the integrations directly in terms of MPLs.

3.2.2 Integer parameters and general p

Similar results can be obtained for (3.32) with general p. The same steps as in the $p = 2$ case lead to a system of p first order differential equations for the functions:

$$\rho_k^{(0)}(z) := v_{p,k}(z) \ , \quad \rho_k^{(j)}(z) := \theta^j v_{p,k}(z) \ , \quad j = 1, \ldots, p-1 \ . \tag{3.41}$$

The iterative solution of their differential equations reads

$$\begin{aligned} \rho_k^{(p-1)}(z) &= \sum_{\alpha=1}^{p} \left(\Delta_{p,\alpha} \int_0^z \frac{dt}{1-t} \ \rho_{k-\alpha}^{(p-\alpha)}(t) - Q_{p,\alpha} \ \rho_{k-\alpha}^{(p-\alpha-1)}(z) \right) + Q_{p,p-1} \delta_{0,k-p+1} \ , \\ \rho_k^{(j-1)}(z) &= \int_0^z \frac{dt}{t} \ \rho_k^{(j)}(t) \ , \quad k \geq 1 \ , \quad j = 1, \ldots, p-1 \ , \end{aligned} \tag{3.42}$$

with

$$\Delta_{p,\alpha} = P_{p,\alpha} - Q_{p,\alpha} \ , \quad \alpha = 1, \ldots, p \ , \tag{3.43}$$

and $P_{p,\alpha}$ the α-th symmetric product (elementary symmetric function) of the parameters $a_1,\ldots,a_p$ and $Q_{p,\beta}$ the β-th symmetric product of the parameters $b_1,\ldots,b_{p-1}$, i.e.:

$$\begin{aligned} P_{p,\alpha} &= \sum_{\substack{i_1,\ldots,i_\alpha=1 \\ i_1<i_2<\ldots<i_\alpha}}^{p} a_{i_1}\cdot\ldots\cdot a_{i_\alpha} \ , \quad \alpha=1,\ldots,p \ , \\ Q_{p,\beta} &= \sum_{\substack{i_1,\ldots,i_\beta=1 \\ i_1<i_2<\ldots<i_\beta}}^{p-1} b_{i_1}\cdot\ldots\cdot b_{i_\beta} \ , \quad \beta=1,\ldots,p-1 \ , \quad Q_{p,p}=0 \ . \end{aligned} \tag{3.44}$$

The lowest orders of the expansion (3.32) are

$$\begin{aligned} v_{p,k}(z) &= 0 \ , \quad k<0 \ , \\ v_{p,0}(z) &= 1 \ , \\ v_{p,k}(z) &= 0 \ , \quad k=1,\ldots,p-1 \ , \\ v_{p,p}(z) &= \Delta_{p,p}\ \mathcal{L}i_p(z) \ , \\ v_{p,p+1}(z) &= \Delta_{p,p}\Delta_{p,1}\ \mathcal{L}i_{p,1}(z) - \Delta_{p,p}Q_{p,1}\ \mathcal{L}i_{p+1}(z) \ , \end{aligned} \tag{3.45}$$

with the first two lines following from eqs. (3.2) and (3.1), respectively. As in the $p=2$ case higher orders can be calculated straightforwardly with the iterative solution (3.42).

3.2.3 Half-integer parameters and $p = 2$

As an example for generalized hypergeometric functions with half-integer parameters we analyse the expansion:

$${}_2F_1\left[\begin{matrix}\alpha' a_1, \alpha' a_2 \\ \frac{1}{2}+\alpha' b\end{matrix}; z\right] = \sum_k (\alpha')^k w_k(z) \ . \tag{3.46}$$

Introducing the new variable y, with

$$y = \frac{1-\sqrt{\frac{z}{z-1}}}{1+\sqrt{\frac{z}{z-1}}} \ , \quad z = -\frac{(1-y)^2}{4y} \ , \quad \theta = -\frac{1-y}{1+y}\ y\partial_y \ , \tag{3.47}$$

and new functions $\rho_k(y)$, which are related to $w_k(z)$ and $w_k(y)$ via

$$\theta w_k(z) = -\frac{1-y}{1+y}\ y\partial_y w_k(y) = -\frac{1-y}{1+y}\ \rho_k(y) \ , \tag{3.48}$$

the following system of first order differential equations can be obtained from eq. (3.31):

$$\begin{aligned} \partial_y w_k(y) &= \frac{1}{y}\rho_k(y) \ , \\ \partial_y \rho_k(y) &= \frac{2b}{1-y}\rho_{k-1}(y) + \frac{a_1+a_2}{y}\rho_{k-1}(y) - 2\frac{a_1+a_2-b}{1+y}\rho_{k-1}(y) - \frac{a_1a_2}{y}w_{k-2}(y) \ . \end{aligned} \tag{3.49}$$

The iterative solution of this system has the form ($k \geq 1$):

$$\begin{aligned} w_k(y) &= \int_1^y \frac{dt}{t} \rho_k(t) \ , \\ \rho_k(y) &= \int_1^y dt \left[\frac{2b}{1-t} + \frac{a_1+a_2}{t} - 2\frac{a_1+a_2-b}{1+t} \right] \rho_{k-1}(t) - a_1 a_2 \int_1^y \frac{dt}{t} w_{k-2}(t) \ . \end{aligned} \tag{3.50}$$

The point $z = 0$ transforms to the point $y = 1$ under (3.47), so that the boundary conditions are $\rho_k(1) = 0$ for $k \geq 0$ and $w_k(y = 1) = 0$ for $k \geq 1$. Using the lowest order $w_0(y) = 1$ and eqs. (3.50), the α'-expansion (3.46) can be calculated straightforwardly order by order, e.g.:

$$\begin{aligned} w_1(y) &= 0 \ , \\ w_2(y) &= -a_1 a_2 \ J(0,0) = -a_1 a_2 \ H(0,0;y) \ , \\ w_3(y) &= -2a_1 a_2 b \ J(0,1,0) + 2a_1 a_2 (a_1+a_2-b) J(0,-1,0) - a_1 a_2 (a_1+a_2) J(0,0,0) \\ &= -2a_1 a_2 b \left[H(0,1,0;y) - H(0,1,0;1) - H(0;y) H(1,0;1) \right] + 2a_1 a_2 (a_1+a_2-b) \\ &\quad \times \left[H(0,-1,0;y) - H(0,-1,0;1) - H(0;y) H(-1,0;1) \right] - a_1 a_2 (a_1+a_2) H(0,0,0;y) \ , \\ w_4(y) &= -4b^2 a_1 a_2 J(0,1,1,0) - 4(a_1+a_2-b)^2 a_1 a_2 J(0,-1,-1,0) \\ &\quad - (a_1+a_2)^2 a_1 a_2 J(0,0,0,0) + a_1^2 a_2^2 J(0,0,0,0) + 4a_1 a_2 (a_1+a_2-b) b \\ &\quad \times \left[J(0,1,-1,0) + J(0,-1,1,0) \right] - 2a_1 a_2 b (a_1+a_2) \left[J(0,1,0,0) + J(0,0,1,0) \right] \\ &\quad + 2a_1 a_2 (a_1+a_2)(a_1+a_2-b) \left[J(0,-1,0,0) + J(0,0,-1,0) \right] \ . \end{aligned} \tag{3.51}$$

We used eqs. (3.21) to give $w_2(y)$ and $w_3(y)$ in terms of HPLs. This can be achieved for $w_4(y)$ and higher order coefficient functions as well.

The iterative computation of Laurent expansions of hypergeometric functions described in this section has first been presented in [11] for the $p = 2$ cases (3.33) and (3.46). It has been applied to similar functions, e.g. to generalized hypergeometric functions with integer parameters (3.32) in [12]. Eqs. (3.38), (3.42) and (3.50) are the main results of this section. They allow to straightforwardly calculate the expansion (3.33), (3.32) and (3.46), respectively, up to any order in α'. It is, however, not possible to obtain an order without knowing the previous ones. This issue is solved in the next section.

3.3 Recurrence relations for generalized hypergeometric functions

In this section we present all-order expansions for generalized hypergeometric functions. The idea is to write the differential equations for the coefficient functions as recurrence relations. This is achieved by replacing the derivatives and integrations in the iterative solutions of these differential equations with differential and integral operators, respectively. In [37] such recurrence relations have been used to calculate expansions for $p = 2$ and $p = 3$ hypergeometric functions order by order. The recurrence relations have the form (2.1), where the non-commutative coefficients c_i represent differential and integral operators. The iterative solutions of the differential equations are actually equivalent to the recurrence relations. With the latter it becomes, however, more obvious how to calculate α'-expansions iteratively.

More importantly, with the general solution (2.26) for recurrence relations of this type, the all-order expansions can now systematically be constructed and straightforwardly be given in closed form. By *all-order* we mean, representations for example for the coefficient functions $u_{p,k}$ of the expansion (3.32), which include k as a variable and therefore hold for all orders. In contrast to that, the method of the previous section allows to compute α'-expansions order by order starting with $u_{p,0}$, $u_{p,1}$, $u_{p,2}$ and so on. In other words, the formula for $u_{p,k}$, no matter if in the form of a iterative solution to differential equations as discussed in the previous section or as a recurrence relation as presented in the following, is not given in terms of MPLs or similar functions. Instead coefficient functions of lower orders are included. On the other hand our all-order results, which follow from (2.26), give coefficient functions for all orders *explicitly* in terms of MPLs, HPLs or the related integral operators.

To warm up we begin with a hypergeometric function, whose all-order expansion is already known: the $p = 2$ hypergeometric function (3.33) with parameters $(a_1, a_2) = (-a, b)$. Section 3.3.2 deals with the $p = 3$ version of the expansion (3.32). This function is of particular interest for the open string amplitudes to be discussed in chapter 5. The case (3.32) with general p is elaborated in section 3.3.3. Finally, the all-order expansion (3.46) of the $p = 2$ hypergeometric function with a half-integer parameter is constructed in section 3.3.4. This function plays an important role in the calculation of a Feynman integral in chapter 4.

3.3.1 Integer parameters and $p = 2$

For demonstration purposes in this subsection the solution of the recurrence relation for the coefficients $u_k(z)$ of the expansion

$$
{}_2F_1\left[\begin{matrix}-\alpha' a, \alpha' b\\ 1+\alpha' b\end{matrix}; z\right] = \sum_{k=0}^{\infty} (\alpha')^k u_k(z) \tag{3.52}
$$

is calculated. This result is already known [50]. Combining eqs. (3.38) and applying the replacements

$$
(a_1, a_2) \to (-a, b)\ , \quad v_{2,k}(z) \to u_k(z)\ , \tag{3.53}
$$

yields the recurrence relation

$$
u_k(z) = c_1 u_{k-1}(z) + c_2 u_{k-2}(z)\ , \quad k \geq 2\ , \tag{3.54}
$$

with the non-commutative coefficients

$$\begin{aligned} c_1 &= -a\, I(0,1,\theta) - b\, I(0) \ , \\ c_2 &= -ab\, I(0,1) \ , \end{aligned} \tag{3.55}$$

for which we used the integral operators (3.5). Furthermore we use the initial values $u_0(z) = 1$ and $u_1(z) = 0$. It is easy to check that this reproduces (3.40) with (3.53). According to (2.26) the solution of (3.54) is:

$$u_k(z) = -ab \sum_{j_1+2j_2=k-2} \left\{(-aI(0,1,\theta) - bI(0))^{j_1}, (-abI(0,1))^{j_2}\right\} I(0,1) \ . \tag{3.56}$$

Eq. (3.9) implies that the final expression only contains the integral operators $I(0)$ and $I(1)$. Therefore, the first step in simplifying the solution (3.56) should be to eliminate the differential operator $I(\theta)$. This is achieved by the relation (3.6) and the following identity:

$$\left\{I(0,\vec{p}_1,\theta)^{j_1}, I(0,\vec{p}_2,\theta)^{j_2}, \ldots, I(0,\vec{p}_n,\theta)^{j_n}\right\} = I(0) \left\{I(\vec{p}_1)^{j_1}, I(\vec{p}_2)^{j_2}, \ldots, I(\vec{p}_n)^{j_n}\right\} I(\theta) \ . \tag{3.57}$$

The vectors $\vec{p}_i$ represent arbitrary sequences of the elements $\{0, 1, \theta\}$. The removal of $I(\theta)$ works, because every argument starts with an $I(0)$ and ends with an $I(\theta)$. With the relations (2.18), (2.25) and (3.57) the result (3.56) can be transformed to:

$$u_k(z) = \sum_{\alpha=1}^{k-1} (-1)^{k+1} a^{k-\alpha} b^{\alpha} \sum_{\beta} (-1)^{\beta} I(0) \left\{I(1)^{k-\alpha-1-\beta}, I(0)^{\alpha-1-\beta}, I(1,0)^{\beta}\right\} I(1) \ . \tag{3.58}$$

An identity, which we discuss in chapter 6, allows to simplify the generalized operator product and the sum over β to arrive at:

$$u_k(z) = \sum_{\alpha=1}^{k-1} (-1)^{k+1} a^{k-\alpha} b^{\alpha} I(0)^{\alpha} I(1)^{k-\alpha} = \sum_{\alpha=1}^{k-1} (-1)^{k+1} a^{k-\alpha} b^{\alpha}\ \mathcal{L}i_{(\alpha+1,\{1\}^{k-\alpha-1})}(z) \ . \tag{3.59}$$

In the final step eq. (3.9) has been used to express the result in terms of MPLs. Therefore, the hypergeometric function (3.52) can be written as:

$${}_2F_1 \begin{bmatrix} -\alpha' a, \alpha' b \\ 1 + \alpha' b \end{bmatrix} ; z \Big] = 1 - \sum_{k=2}^{\infty} (-\alpha')^k \sum_{\alpha=1}^{k-1} a^{k-\alpha} b^{\alpha}\ \mathcal{L}i_{(\alpha+1,\{1\}^{k-\alpha-1})}(z) \ . \tag{3.60}$$

Of particular interest is the case $z = 1$, since the resulting object arises in the four-point open string amplitude. This is discussed in section 5.1.

3.3.2 Integer parameters and $p = 3$

The recurrence relation for the coefficients $v_{3,k}(z)$ of the series

$$ {}_3F_2\left[\begin{matrix}\alpha' a_1, \alpha' a_2, \alpha' a_3 \\ 1+\alpha' b_1, 1+\alpha' b_2\end{matrix}; z\right] = \sum_{k=0}^{\infty} (\alpha')^k v_{3,k}(z) \tag{3.61} $$

follows from eqs. (3.42) with $p = 3$. It reads

$$ v_{3,k}(z) = c_1 v_{3,k-1}(z) + c_2 v_{3,k-2}(z) + c_3 v_{3,k-3}(z) \ , \quad k \geq 3 \ , \tag{3.62} $$

with

$$ \begin{aligned} c_1 &= \Delta_{3,1}\ I(0,0,1,\theta,\theta) - Q_{3,1}\ I(0) \ , \\ c_2 &= \Delta_{3,2}\ I(0,0,1,\theta) - Q_{3,2}\ I(0,0) \ , \\ c_3 &= \Delta_{3,3}\ I(0,0,1) \end{aligned} \tag{3.63} $$

and $v_{3,0}(z) = 1$, $v_{3,1}(z) = v_{3,2}(z) = 0$ as initial values. According to the definitions (3.44) we have:

$$ \begin{aligned} \Delta_{3,1} &= a_1 + a_2 + a_3 - b_1 - b_2 \ , \\ \Delta_{3,2} &= a_1 a_2 + a_2 a_3 + a_3 a_1 - b_1 b_2 \ , \\ \Delta_{3,3} &= a_1 a_2 a_3 \ , \\ Q_{3,1} &= b_1 + b_2 \ , \\ Q_{3,2} &= b_1 b_2 \ . \end{aligned} \tag{3.64} $$

The solution of the recurrence relation (3.62) follows straightforwardly from (2.26):

$$ v_{3,k}(z) = \sum_{j_1+2j_2+3j_3=k-3} \left\{ c_1^{j_1}, c_2^{j_2}, c_3^{j_3} \right\} c_3 \ . \tag{3.65} $$

Inserting eqs. (3.63) and applying the identities (2.18), (2.25), (3.6) as well as (3.57) gives:

$$ \begin{aligned} v_{3,k}(z) = &\sum_{m_1+l_1+2(l_2+m_2)+3m_3=k-3} (-1)^{l_1+l_2} \Delta_{3,1}^{m_1} \Delta_{3,2}^{m_2} \Delta_{3,3}^{m_3+1} Q_{3,1}^{l_1} Q_{3,2}^{l_2} \\ &\times I(0,0) \left\{ I(0)^{l_1}, I(0,0)^{l_2}, I(1)^{m_1}, I(1,0)^{m_2}, I(1,0,0)^{m_3} \right\} I(1) \ . \end{aligned} \tag{3.66} $$

As already mentioned, expansions of other hypergeometric functions ${}_3F_2(\vec{m}+\vec{a};\vec{n}+\vec{b};z)$ can be obtained from the result (3.66) with the relations (3.30). Two such functions, which enter the five-point open superstring amplitude, are a topic of chapter 5.

As in the second order case there is an identity, which allows to remove the generalized operator product yielding a representation in terms of MPLs. This alternative representation for $v_{3,k}(z)$ can be found in (6.60).

3.3.3 Integer parameters and general p

From eqs. (3.42) we obtain the following recurrence relation for the coefficients $v_{p,k}(z)$ of the expansion (3.32):

$$v_{p,k}(z) = \sum_{\alpha=1}^{p} c_{p,\alpha}\ v_{p,k-\alpha}(z) \quad , \quad k \geq p \ , \tag{3.67}$$

with the coefficients

$$c_{p,\alpha} = \Delta_{p,\alpha}\ I(0)^{p-1} I(1)\ \theta^{p-\alpha} - Q_{p,\alpha}\ I(0)^{\alpha} \ , \quad \alpha = 1, \ldots, p \ . \tag{3.68}$$

The initial values are $v_{p,0}(z) = 1$ and $v_{p,k}(z) = 0$ for $0 < k < p$. According to (2.26) the solution is:

$$v_{p,k}(z) = \sum_{j_1+2j_2+\ldots+pj_p=k-p} \{c_{p,1}^{j_1}, c_{p,2}^{j_2}, \ldots, c_{p,p}^{j_p}\}\ c_{p,p} \ . \tag{3.69}$$

Performing the same steps as for $p = 2$ and $p = 3$ leads to the following result:

$$\begin{aligned} v_{p,k}(z) = \sum_{\vec{l},\vec{m}} & (-1)^{|\vec{l}|} \Delta_{p,1}^{m_1} \Delta_{p,2}^{m_2} \ldots \Delta_{p,p-1}^{m_{p-1}} \Delta_{p,p}^{m_p+1} Q_{p,1}^{l_1} Q_{p,2}^{l_2} \ldots Q_{p,p-1}^{l_{p-1}} \\ & \times I(0)^{p-1} \{I(0)^{l_1}, \ldots, I(\underbrace{0,\ldots,0}_{p-1})^{l_{p-1}}, I(1)^{m_1}, \ldots, I(1,\underbrace{0,\ldots,0}_{p-1})^{m_p}\} I(1) \ . \end{aligned} \tag{3.70}$$

The sum is over the multi-indices $\vec{l} = (l_1, l_2, \ldots, l_{p-1})$ and $\vec{m} = (m_1, m_2, \ldots, m_p)$, which solve the equation:

$$\sum_{\alpha=1}^{p-1} \alpha(l_\alpha + m_\alpha) + p\ m_p = k - p \ . \tag{3.71}$$

A representation for $v_{p,k}(z)$ explicitly in terms of MPLs is given in eq. (6.61).

Note that the operators $I(0)$ to the left and $I(1)$ to the right of the generalized operator products in the expansions (3.58), (3.66) and (3.70) ensure that all MPLs are finite. These operators arise automatically with the application of our method. As a consequence no regularization has to be performed.

3.3.4 Half-integer parameters and $p = 2$

Combining eqs. (3.50) for the coefficients $w_k(y)$ of the expansion (3.46) yields the second order recurrence relation

$$w_k(y) = c_1 w_{k-1}(y) + c_2 w_{k-2}(y) \ , \quad k \geq 2 \ , \tag{3.72}$$

with the non-commutative coefficients:

$$\begin{aligned} c_1 &= 2b\ J(0,1,\theta) - 2(a_1+a_2-b)\ J(0,-1,\theta) + (a_1+a_2)\ J(0)\ , \\ c_2 &= -a_1 a_2\ J(0,0)\ . \end{aligned} \tag{3.73}$$

With the initial values $w_0(y) = 1$ and $w_1(y) = 0$ the solution of eq. (3.72) reads:

$$w_k(y) = \sum_{\substack{l_1+l_2+l_3+2m \\ =k-2}} (-1)^{l_2+m+1} 2^{l_1+l_2} b^{l_1} (a_1+a_2-b)^{l_2} (a_1+a_2)^{l_3} (a_1 a_2)^{m+1} \\ \times J(0)\{J(1)^{l_1}, J(-1)^{l_2}, J(0)^{l_3}, J(0,0)^m\} J(0)\ . \tag{3.74}$$

Let us use this all-order expression as an example to demonstrate that, in contrast to the findings in [11, 12], the results (3.60), (3.66), (3.70) and (3.74) allow to express any order of the corresponding expansions directly without using lower orders. The Mathematica routine 'DistinctPermutations' is useful to evaluate the generalized operator products. For instance $k = 3$ in (3.74) yields the condition $l_1 + l_2 + l_3 + 2m = 1$ for the sum over non-negative integers l_1, l_2, l_3 and m. This equation has three solutions: $(l_1, l_2, l_3, m) = (1,0,0,0)$, $(0,1,0,0)$ and $(0,0,1,0)$. It is easy to check that they give

$$-2ba_1a_2\ J(0,1,0)\ , \quad 2a_1a_2(a_1+a_2-b)\ J(0,-1,0) \ \text{ and } \ -a_1a_2(a_1+a_2)\ J(0,0,0)\ ,$$

respectively. This is in accordance with the expression for $w_3(y)$ given in (3.51). Higher orders can be evaluated the same way, e.g. for $k = 5$ there are 13 terms with the following summation indices:

$$(l_1, l_2, l_3, m) = (3,0,0,0),\ (0,3,0,0),\ (0,0,3,0),\ (1,0,0,1),\ (0,1,0,1),\ (0,0,1,1), \\ (1,1,1,0),\ (2,1,0,0),\ (2,0,1,0),\ (1,2,0,0),\ (0,2,1,0),\ (1,0,2,0) \text{ and } (0,1,2,0)\ .$$

Every summand can be calculated straightforwardly, e.g. for $(l_1, l_2, l_3, m) = (0,0,1,1)$ we get $-2a_1^2a_2^2(a_1+a_2)J(0,0,0,0,0)$.

It has been shown in [51] that a variety of other hypergeometric functions ${}_2F_1$ with parameters, that differ from those in (3.46) by integer or half-integer shifts, can be written as linear combinations of the function (3.46) and derivatives thereof. Thus our result (3.74) allows to construct the all-order expansions for these functions as well. One of these functions appears in chapter 4 in a representation of a Feynman integral.

3.4 Generic first-order Fuchsian equations and recurrences

A generic system $\partial_z \boldsymbol{g}(z) = A(z)\boldsymbol{g}(z)$ of n equations of (first-order) Fuchsian class has the form

$$\partial_z \boldsymbol{g}(z) = \sum_{\alpha=0}^{\alpha_0} \frac{A_\alpha}{z - z_\alpha}\ \boldsymbol{g}(z)\ , \tag{3.75}$$

with the $\alpha_0 + 1$ distinct points $z_0, \ldots, z_{\alpha_0}$ and constant non-commutative $n \times n$ matrices A_α. If $\sum_{\alpha=0}^{\alpha_0} A_\alpha \neq 0$ the system of equations (3.75) has $\alpha_0 + 2$ regular singular points at $z = z_\alpha$ and $z = \infty$ and is known as Schlesinger system.

At a regular singular point any solution can be expressed explicitly by the combination of elementary functions and power series convergent within a circle around the singular point. A solution to (3.75) taking values in $\mathbf{C}\langle A\rangle$ with the alphabet $A = \{A_0, \ldots, A_{\alpha_0}\}$ can be given as formal weighted sum over iterated integrals

$$\boldsymbol{g}(z) = \sum_{w \in A^*} L_w(z)\ w \ , \tag{3.76}$$

leading to hyperlogarithms [52]. The latter are defined recursively from words w built from an alphabet $\{w_0, \ldots, w_{\alpha_0}\}$ (with $w_\alpha \simeq A_\alpha$) with $\alpha_0 + 1$ letters:

$$\begin{aligned} L_{w_0^m}(z) &:= \frac{1}{m!}\ \ln^m(z - z_0)\ ,\ m \in \mathbb{N}\ , \\ L_{w_\alpha^m}(z) &:= \frac{1}{m!}\ \ln^m\left(\frac{z - z_\alpha}{z_0 - z_\alpha}\right)\ ,\ 1 \leq \alpha \leq \alpha_0\ , \\ L_{w_\alpha w}(z) &:= \int_0^z \frac{dt}{t - z_\alpha}\ L_w(t)\ ,\quad L_1(z) = 1\ . \end{aligned} \tag{3.77}$$

The functions (3.77) may also be written as Goncharov polylogarithms [53, 54]

$$L_{w_{\sigma_1} \ldots w_{\sigma_m}}(z) = G(z_{\sigma_1}, \ldots, z_{\sigma_m}; z) = \int_0^z \frac{dt}{t - z_{\sigma_1}}\ G(z_{\sigma_2}, \ldots, z_{\sigma_m}; t)\ , \tag{3.78}$$

with $G(;z) = 1$ except $G(;0) = 0$. Typically, for a given class of amplitudes one only needs a certain special subset of allowed indices z_α referring to a specific alphabet. E.g. for the evaluation of loop integrals arising in massless quantum field theories one has $z_\alpha \in \{0, 1\}$. However, the inclusion of particle masses in loop integrals may give rise to $z_\alpha \in \{0, 1, -1\}$. The objects (3.78) are related to the MPLs (cf. also eq. (3.9))

$$\mathcal{L}i_{n_1,\ldots,n_d}(z_1, \ldots, z_d) = \sum_{0<k_d<\ldots<k_1} \frac{z_1^{k_1} \cdot \ldots \cdot z_d^{k_d}}{k_1^{n_1} \cdot \ldots \cdot k_d^{n_d}} \tag{3.79}$$

as follows:

$$\mathcal{L}i_{n_1,\ldots,n_d}(z_1, \ldots, z_d) = (-1)^d\ G\left(\underbrace{0, \ldots, 0}_{n_1-1}, \frac{1}{z_1}, \ldots, \underbrace{0, \ldots, 0}_{n_d-1}, \frac{1}{z_1 z_2 \cdot \ldots \cdot z_d}; 1\right)\ . \tag{3.80}$$

Furthermore we have:

$$L_{w_0^{n_1-1} w_{\sigma_1} \ldots w_0^{n_d-1} w_{\sigma_d}}(z) = (-1)^d\ \mathcal{L}i_{n_1,\ldots,n_d}\left(\frac{z - z_0}{z_{\sigma_1} - z_0}, \frac{z_{\sigma_1} - z_0}{z_{\sigma_2} - z_0}, \ldots, \frac{z_{\sigma_{d-2}} - z_0}{z_{\sigma_{d-1}} - z_0}, \frac{z_{\sigma_{d-1}} - z_0}{z_{\sigma_d} - z_0}\right)\ . \tag{3.81}$$

In the following let us assume that in (3.75) the matrices A_α have some polynomial dependence on α' with integer powers as:

$$A_\alpha = \sum_{\beta=1}^{\beta_0} a_{\alpha\beta}\ (\alpha')^\beta\ . \tag{3.82}$$

We are looking for a power series solution in α':

$$\boldsymbol{g}(z) = \sum_{k\geq 0} \boldsymbol{u}_k(z)\ (\alpha')^k\ . \tag{3.83}$$

Eventually, each order $(\alpha')^k$ of the power series is supplemented by a $\mathbb{Q}$-linear combination of iterated integrals of weight k. After inserting the Ansatz (3.83) into (3.75) we obtain a recursive differential equation for the functions $\boldsymbol{u}_k(x)$, which can be integrated to the iterative solution:

$$\boldsymbol{u}_k(z) = \boldsymbol{u}_k(0) + \sum_{\alpha=0}^{\alpha_0} \sum_{\beta=1}^{\min\{\beta_0,k\}} a_{\alpha\beta} \int_0^z \frac{\boldsymbol{u}_{k-\beta}(t)}{t-z_\alpha}\ dt\ . \tag{3.84}$$

This translates into the following operator equation:

$$\boldsymbol{u}_k(z) = \boldsymbol{u}_k(0) + \sum_{\alpha=0}^{\alpha_0} \sum_{\beta=1}^{\min\{\beta_0,k\}} a_{\alpha\beta}\ I(z_\alpha)\ \boldsymbol{u}_{k-\beta}(z)\ . \tag{3.85}$$

Above, $\boldsymbol{u}_k(0)$ represents a possible inhomogeneity accounting for an integration constant, which is determined by boundary conditions. Evidently, we have[3] $\boldsymbol{u}_0(z) = \boldsymbol{u}_0(0) = const.$ We may find a general solution to (3.85) by considering the recurrence relation

$$\boldsymbol{u}_k(z) = \boldsymbol{u}_k(0) + \sum_{\beta=1}^{\min\{\beta_0,k\}} c_\beta\ \boldsymbol{u}_{k-\beta}(z)\ , \tag{3.86}$$

with the coefficients:

$$c_\beta = \sum_{\alpha=0}^{\alpha_0} a_{\alpha\beta}\ I(z_\alpha)\ . \tag{3.87}$$

For (3.86) we can directly apply our general solution for inhomogeneous recurrence relations (2.34) to obtain

$$\boldsymbol{u}_k(z) = \sum_{\gamma=0}^{\beta_0-1} \sum_{\substack{|\vec{j}_1|+2|\vec{j}_2|+\ldots+\beta_0|\vec{j}_{\beta_0}| \\ =k-\beta_0-\gamma}} \{\ldots\} \sum_{\beta=\gamma+1}^{\beta_0} \sum_{\alpha=0}^{\alpha_0} a_{\alpha\beta}\ I(z_\alpha)\ \bar{\boldsymbol{u}}_{\beta_0-\beta-\gamma} \tag{3.88}$$

[3] Already at $k=1$ the equation (3.84) translates into the non-trivial recursion $\boldsymbol{u}_1(z) = \boldsymbol{u}_1(0) + \sum_{\alpha=0}^{\alpha_0} a_{\alpha 1} \boldsymbol{u}_0 \int_0^z \frac{dt}{t-z_\alpha}$.

$$+\sum_{\alpha=\beta_0}^{k}\ \sum_{\substack{|\vec{j}_1|+2|\vec{j}_2|+\ldots+\beta_0|\vec{j}_{\beta_0}|\\=k-\alpha}} \{\ldots\}\ \boldsymbol{u}_\alpha(0)\ ,$$

with the generalized operator product

$$\{\ldots\} = \{(\hat{I}_{1,0})^{j_{1,0}},\ldots,(\hat{I}_{1,\alpha_0})^{j_{1,\alpha_0}},\ldots,(\hat{I}_{\beta_0,0})^{j_{\beta_0,0}},\ldots,(\hat{I}_{\beta_0,\alpha_0})^{j_{\beta_0,\alpha_0}}\}\ , \tag{3.89}$$

and

$$\hat{I}_{\beta,\alpha} := a_{\alpha\beta}\ I(z_\alpha)\ . \tag{3.90}$$

Eq. (3.88) is valid for $k \geq \beta_0$ and it uses the initial values $\boldsymbol{u}_k(z) = \bar{\boldsymbol{u}}_k$, $k = 0, 1, \ldots, \beta_0 - 1$. The multiple indices $\vec{j}_\beta$, $\beta = 1, \ldots, \beta_0$ consist of $\alpha_0 + 1$ non-negative integers: $\vec{j}_\beta = (j_{\beta,0}, \ldots, j_{\beta,\alpha_0})$. Each of these elements $j_{\beta,\alpha}$ appears as an index of a corresponding argument $\hat{I}_{\beta,\alpha}$ in the generalized operator product (3.89).

If the matrices $a_{\alpha\beta}$ commute, the generalized operator product (3.89) simplifies to

$$\{\ldots\} = \left(\prod_{\substack{0\leq\alpha\leq\alpha_0\\1\leq\beta\leq\beta_0}} (a_{\alpha\beta})^{j_{\beta,\alpha}}\right)\binom{j_{1,0}+\ldots+j_{\beta_0,0}}{j_{1,0},\ldots,j_{\beta_0,0}}\cdots\binom{j_{1,\alpha_0}+\ldots+j_{\beta_0,\alpha_0}}{j_{1,\alpha_0},\ldots,j_{\beta_0,\alpha_0}} \tag{3.91}$$

$$\times\ \{I(z_0)^{j_{1,0}+\ldots+j_{\beta_0,0}},\ldots,I(z_{\alpha_0})^{j_{1,\alpha_0}+\ldots+j_{\beta_0,\alpha_0}}\}\ , \tag{3.92}$$

where the integral operators are now separated from the matrices. Furthermore, every integral operator $I(z_\alpha)$ appears only once. Their products in the second line can be written in terms of iterated integrals given above, e.g. in terms of hyperlogarithms (3.77). However, in general the matrices $a_{\alpha\beta}$ are non-commutative, i.e. they cannot be factorized from the generalized operator products. Therefore, if possible it is better to avoid matrix notations. This is what we have done for the generalized hypergeometric functions. For instance for the $p = 2$ case with integer parameters, instead of writing the system (3.38) as a first order recurrence relation for the vector $(v_{2,k}, \rho_k)$, we combined both eqs. to a second order relation for v_k. The system (3.75) also appears after expressing Feynman integrals as first-order coupled systems of differential equations. In this context alternatively we could consider higher order differential equations instead of coupled first order equations and treat equations, that can be decoupled, as inhomogeneities.

3.5 Canonical form of differential equations

Let us conclude this chapter by discussing a detail we skipped in the previous sections. Note that the series in (3.82) starts with a term linear in α', i.e. has no term of order $(\alpha')^0$. This property is crucial to present the iterative solution (3.84) to eventually obtain an all-order expansion. The differential equations of the generalized hypergeometric functions we study throughout this chapter have a similar characteristic. Consider for example eqs. (3.35), which can be written in the form (3.75) as

$$\partial_z \boldsymbol{g}(z) = \left(\frac{A_0}{z} + \frac{A_1}{1-z} \right) \boldsymbol{g}(z) \ , \tag{3.93}$$

with $g_1(z) = v(z)$, $g_2(z) = \rho(z) \equiv \theta v(z)$ and the matrices

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & -b\alpha' \end{pmatrix} \quad , \quad A_1 = \begin{pmatrix} 0 & 0 \\ a_1 a_2 (\alpha')^2 & (a_1 + a_2 - b)\alpha' \end{pmatrix} \ , \tag{3.94}$$

There is a term of order $(\alpha')^0$ in A_0. However, as mentioned at the end of the previous section, we presented the iterative solution not for $\boldsymbol{g}(z)$ but for each of its two elements $v(z)$ and $\rho(z)$ individually to eventually combine both to a second order recurrence relation. For this purpose it is necessary that the diagonal elements of the matrices (3.94) have no term of order $(\alpha')^0$ and that the same holds for at least one of the two other elements $(A_1 + A_2)|_{1,2}$ and $(A_1 + A_2)|_{2,1}$. This is obviously fulfilled. Alternatively we could obtain a coefficient matrix linear in α' by including α' in the vector $\boldsymbol{g}$ through the transformation $v(z) \to \alpha' v(z)$.

The suitable form of the differential equation (3.93) comes with the introduction of $\theta v(z)$ as an independent function. This is the first step in section 3.2.1 to write the second order differential equation (3.34) as system of two first order differential eqs. (3.35) or equivalently (3.93). This choice for $\rho(z)$ follows naturally from the form of eq. (3.34) due to the use of the differential operator θ therein.

For differential equations satisfied by Feynman integrals similar restrictions for the ϵ-dependence of the coefficient matrices are useful (cf. chapter 4). In general there is, however, no natural choice for the integrals represented by the elements of $\boldsymbol{g}$, which yields this canonical form. Instead, transformations of $\boldsymbol{g}$ are necessary to find this form. It is therefore interesting to see how to obtain the system (3.93) for hypergeometric functions ${}_2F_1$ starting with a different vector $\boldsymbol{f}(z)$. For this, we assume that the differential equation (3.34) would be given in a different form without the operator θ, e.g.:

$$z(z-1)\, \partial_z^2 v(z) + \left[(a_1 + a_2)\alpha' z - b\alpha' + z - 1 \right]\, \partial_z v(z) + a_1 a_2 (\alpha')^2\, v(z) = 0 \ . \tag{3.95}$$

A naive choice for $\boldsymbol{f}(z)$ would be $f_1(z) = v(z)$, $f_2 = \partial_z v(z)$, which yields the system

$$\partial_z \boldsymbol{f}(z) = B(z) \boldsymbol{f}(z) \ , \tag{3.96}$$

with the coefficient matrix

$$B(z) = \begin{pmatrix} 0 & 1 \\ \frac{(a_1 a_2)(\alpha')^2}{z} + \frac{(a_1 a_2)(\alpha')^2}{1-z} & \frac{(a_1 + a_2 - b)\alpha'}{1-z} - \frac{1 + b\alpha'}{z} \end{pmatrix} \ . \tag{3.97}$$

Before worrying about the α'-dependence, we note that this system is not of the form (3.75) with regular poles in $z_0 = 0$ and $z_1 = 1$. Instead, it implies non-simple or spurious poles, which can be transformed away by a suitable transformation $\boldsymbol{f}(z) = T(z)\ \boldsymbol{f}'(z)$. This leads to the system $\partial_z \boldsymbol{f}'(z) = B'(z)\boldsymbol{f}'(z)$ with:

$$B'(z) = T^{-1}(z)\ B(z)\ T(z) - T^{-1}(z)\ \partial_z T(z)\ . \tag{3.98}$$

Using the transformation matrix

$$T(z) = \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix}\ , \tag{3.99}$$

we arrive at the system (3.93), i.e.:

$$\boldsymbol{f}'(z) = \boldsymbol{g}(z)\ ,\quad B'(z) = \frac{A_0}{z} + \frac{A_1}{1-z}\ . \tag{3.100}$$

The transformation yields a coefficient matrix with a proper form concerning the poles in z as well as to a suitable α'-dependence[4].

The above discussion can be generalized for the function (3.32) with generic p. For the elements

$$\begin{aligned} f_1(z) &= {}_pF_{p-1}(\vec{a}\alpha'; \vec{1} + \vec{b}\alpha'; z)\ , \\ f_2(z) &= \partial_z\ f_1(z)\ , \\ f_3(z) &= \partial_z^2\ f_1(z)\ , \\ &\ \ \vdots \\ f_p(z) &= \partial_z^{p-1}\ f_1(z)\ , \end{aligned} \tag{3.101}$$

of the vector $\boldsymbol{f}(z)$ a system of first order linear differential equations of the form $\partial_z \boldsymbol{f}(z) = B_p(z)\boldsymbol{f}(z)$ can be obtained. The quadratic matrix $B_p(z)$ is given by the parameters a_i, b_j. The system implies non-simple or spurious poles, which can be transformed away by a suitable transformation $\boldsymbol{f}(z) = T_p(z)\boldsymbol{g}(z)$. One can recursively define the transformation matrix

$$T_p = \left(\begin{array}{c|c} T_{p-1} & \mathbf{0}_{p-1} \\ \hline z^{-1}\ \omega_1\ T_{p-1}\ \omega_2 & z^{-p} \end{array}\right)\ , \tag{3.102}$$

[4]It is worth mentioning, that this happens also due to the particular set of parameters we choose for the hypergeometric function under consideration. This followed from the underlying differential equation as well (cf. the discussion between eqs. (3.31) and (3.32)).

with

$$\omega_1 = (0^{p-2}, 1) \ , \quad \omega_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -(p-2)! & 1 & 0 & \dots & 0 & 0 \\ \vdots & & & \ddots & \ddots & & \vdots \\ 0 & \dots & & 0 & & 1 & 0 \\ 0 & \dots & & 0 & & -(p-2)! & 1 \\ 0 & \dots & & 0 & & 0 & -(p-2)! \end{pmatrix} ,$$

e.g.:

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{z} & 0 \\ 0 & -\frac{1}{z^2} & \frac{1}{z^2} \end{pmatrix} .$$

This transformation yields the system

$$\partial_z \boldsymbol{g}(z) = \left(\frac{A_{p,0}}{z} + \frac{A_{p,1}}{1-z} \right) \boldsymbol{g}(z) \ , \tag{3.103}$$

with

$$\boldsymbol{g}(z) = \begin{pmatrix} f_1 \\ \theta f_1 \\ \vdots \\ \theta^{(p-1)} f_1 \end{pmatrix} , \tag{3.104}$$

and the matrices

$$A_{p,0} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & -Q_{p,p-1}(\alpha')^{p-1} & -Q_{p,p-2}(\alpha')^{p-2} & \dots & -Q_{p,2}(\alpha')^2 & -Q_{p,1}\alpha' \end{pmatrix} , \tag{3.105}$$

$$A_{p,1} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ \Delta_{p,p}(\alpha')^p & \Delta_{p,p-1}(\alpha')^{p-1} & \dots & \Delta_{p,2}(\alpha')^2 & \Delta_{p,1}\alpha' \end{pmatrix} . \tag{3.106}$$

This is equivalent to the system of differential equations (3.42).

There is a whole family of transformations yielding the form (3.103). E.g. for $p = 2$ the two transformations

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\lambda}{1-z} \end{pmatrix} \ , \quad T = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\lambda}{z} \end{pmatrix} \ , \quad \lambda \in \mathbb{R} - \{0\} \ , \tag{3.107}$$

yield

$$\begin{aligned} A_0 &= \begin{pmatrix} 0 & 0 \\ \frac{a_1 a_2 (\alpha')^2}{\lambda} & -b\alpha' \end{pmatrix} \ , \quad A_1 = \begin{pmatrix} 0 & \lambda \\ 0 & (a_1 + a_2 - b)\alpha' \end{pmatrix} \ , \\ A_0 &= \begin{pmatrix} 0 & \lambda \\ 0 & -b\alpha' \end{pmatrix} \ , \quad A_1 = \begin{pmatrix} 0 & 0 \\ \frac{a_1 a_2 (\alpha')^2}{c} & (a_1 + a_2 - b)\alpha' \end{pmatrix} \ , \end{aligned} \tag{3.108}$$

respectively.

For systems (3.75) appearing for Feynman integrals it was observed in [55] that the suitable α'- (or in that context rather the ϵ-) dependence of the coefficients A_α is related to the transcendental properties of the functions $\boldsymbol{g}$. The transcendental degree $T(f)$ of a function f is given by the number of iterated integrals needed to define the function f, which in our notation corresponds to the number of integral operators, e.g.: $T(\mathcal{L}i_{\vec{n}}(z)) = |\vec{n}|$, $T(H(0,0;z)) = 2$, $T(\zeta(2,1)) = 3$. Algebraic factors have degree zero. The degree of transcendentality of a product is defined to be the sum of the degrees of each factor: $T(f_1 f_2) = T(f_1) + T(f_2)$. A function f has a uniform degree of transcendentality if f is a sum of terms and all summands have the same degree. Moreover, such functions are called pure if their degree of transcendentality is lowered by taking a derivative: $T(\partial_z f(z)) = T(f(z)) - 1$. In other words, the coefficients of the transcendental functions in f do not depend on z.

The coefficient functions (3.59), (3.66) and (3.70) of the expansions we calculated in this chapter are obviously all pure functions. At each order in α' the MPLs are of uniform degree of transcendentality. Furthermore the degree is identical to the order in α', which simply follows from the underlying recursion relations. With f_1 in (3.104) being a pure function, the other vector elements are pure as well. Due to (3.6), the operators θ reduce the degree of transcendentality by one without producing rational functions of z. This is not the case for the naive choice of functions (3.101). The choice of parameters $\vec{m} = (0, \ldots, 0)$ and $\vec{n} = (1, \ldots, 1)$ in (3.3), which leads to (3.32), is essential for the transcendental properties. Eqs. (3.30) can be used to show that the expansions of generalized hypergeometric functions with different integer shifts in the parameters are not pure or not even uniform, unless they appear with an appropriate prefactor. Identical discussions apply to the half-integer case (3.74). General discussions for transcendental properties of multiple Gaussian hypergeometric functions have been given in [32, 56].

Chapter 4

Feynman integrals

In this chapter we the family of one-loop bubble integrals

$$I(a_1, a_2) = \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{D_1^{a_1} D_2^{a_2}} \ , \quad a_1, a_2 \in \mathbb{Z} \ , \tag{4.1}$$

with massive propagators

$$D_1 = k^2 - m^2 + i0 \ ,$$
$$D_2 = (k+p)^2 - m^2 + i0 \ ,$$

and $p^2 = s$ as a simple example to demonstrate, how to obtain all-order expansions in ϵ (with $D = 4 - 2\epsilon$) for Feynman integrals. This is achieved for a MI of the family (4.1) in section 4.1. In section 4.2 we discuss how to obtain the representation for that MI in terms of a hypergeometric function from the all-order expansion.

Various representations of a more general version of $I(1,1)$ (with propagators of different masses) are discussed in [57]. This includes all-order expansions, for example in terms of Nielsen polylogarithms, and representations in terms of hypergeometric functions ${}_2F_1$.

4.1 All-order Laurent series in ϵ

IBP identities [13] allow to express any integral $I(a_1, a_2)$ in terms of two MIs (cf. Fig. 4.1).

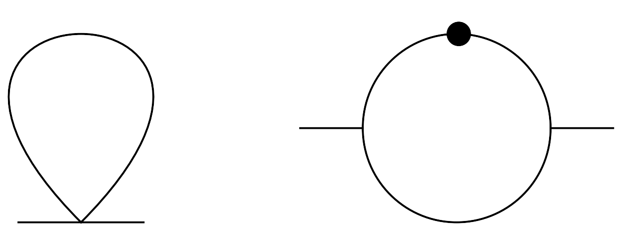

Figure 4.1: Master integrals of topology (4.1): The massive tadpole integral (left) and the massive bubble integral (right) with the dot representing a squared propagator.

A natural choice for one of the MIs is the massive tadpole integral $I(1,0)$, which follows from the well-known result

$$I(a,0) = (-1)^a \frac{\Gamma(a+\epsilon-2)}{\Gamma(a)} (m^2)^{2-\epsilon-a} \tag{4.2}$$

for $a = 1$. As second MI we consider the finite integral $I(2,1)$. We use the ansatz

$$I(2,1) = \Gamma(\epsilon+1) \frac{(m^2)^{-\epsilon}}{\sqrt{s(s-4m^2)}} \, g(x) \ , \tag{4.3}$$

where the function $g(x)$ depends on the ratio of the squared mass m^2 and the kinematic invariant s through the variable:

$$x = \frac{\sqrt{1-\frac{4m^2}{s}}-1}{\sqrt{1-\frac{4m^2}{s}}+1} \ . \tag{4.4}$$

In the region $s < 0$, which transforms to $0 < x < 1$ under eq. (4.4), the function $g(x)$ satisfies the following first order differential equation in x [23]:

$$\frac{d}{dx} g(x) = \left(\frac{1}{x} - \frac{2}{1+x}\right) \epsilon \, g(x) + \frac{1}{x} \ . \tag{4.5}$$

Instead of solving this equation directly, we consider the Laurent expansion around $\epsilon = 0$:

$$g(x) = \sum_{k\geq 0} g_k(x) \, \epsilon^k \ . \tag{4.6}$$

Inserting (4.6) into eq. (4.5) and using the fact, that the resulting differential equation is valid at any order in ϵ, yields first order differential equations for the coefficient functions $g_k(x)$:

$$\begin{aligned} \frac{d}{dx} g_0(x) &= \frac{1}{x} \ , \\ \frac{d}{dx} g_k(x) &= \left(\frac{1}{x} - \frac{2}{1+x}\right) g_{k-1}(x) \ , \quad k \geq 1 \ . \end{aligned} \tag{4.7}$$

We use the limit of vanishing external momenta, i.e. $s = 0$ (which corresponds to $x = 1$), in $I(2,1) \overset{s=0}{\to} I(3,0)$ and eq. (4.2) to determine the boundary conditions $g_k(1) = 0$. This allows to straightforwardly solve the differential equations (4.7). We can directly calculate $g_0(x)$ and give an iterative solution for all other coefficient functions:

$$g_0(x) = H(0;x) \ , \tag{4.8}$$

$$g_k(x) = \int_1^x \frac{dx'}{x'} \, g_{k-1}(x') - 2 \int_1^x \frac{dx'}{1+x'} \, g_{k-1}(x') \ , \quad k \geq 1 \ . \tag{4.9}$$

For example we obtain the next orders in terms of HPLs:

$$g_1(x) = H(0,0;x) - 2H(-1,0;x) - \zeta(2) \; , \tag{4.10}$$

$$\begin{aligned} g_2(x) = & H(0,0,0;x) - 2H(0,-1,0;x) - 2H(-1,0,0;x) + 4H(-1,-1,0;x) \\ & - \zeta(2)H(0;x) + 2\zeta(2)H(-1;x) - 2\zeta(3) \; . \end{aligned} \tag{4.11}$$

Representing the Feynman integral by the lowest orders of the ϵ-expansion is sufficient to describe the behaviour around $\epsilon = 0$. But not all information on the integral are contained in a finite number of orders. Therefore, we are interested in the all-order expression for $g_k(x)$: an equation valid for all k, which in contrast to eq. (4.9) is given explicitly in terms of iterated integrals. An expression for $g_k(x)$ of this type would allow to give the ϵ-expansion (4.6) as an infinite series thereby providing a closed representation of the complete Feynman integral.

Applying the definitions of the integral operators (3.15) in the iterative solution (4.9) yields the first order recurrence relation:

$$g_k(x) = [J(0) - 2J(-1)]\, g_{k-1}(x) \; , \quad k \geq 1 \; . \tag{4.12}$$

With the general solution for this type of relations given in eq. (2.26) and the initial value (4.8) the all-order expression

$$g_k(x) = \sum_{j_1+j_2=k} (-2)^{j_2} \{J(0)^{j_1}, J(-1)^{j_2}\} J(0) \tag{4.13}$$

follows directly. The sum is over non-negative integers j_1, j_2 that fulfil the condition $j_1 + j_2 = k$. While eq. (4.9) depends on lower orders of the expansion, eq. (4.13) allows to extract the coefficient function of any order in ϵ directly. For example with $k = 2$ the condition in the sum over j_1, j_2 has three solutions, which give the following terms:

$$\begin{aligned} (j_1, j_2) = & (2,0): \quad J(0,0,0) \; , \\ & (1,1): \quad -2J(0,-1,0) - 2J(-1,0,0) \; , \\ & (0,2): \quad 4J(-1,-1,0) \; . \end{aligned} \tag{4.14}$$

In agreement with eq. (4.11) the sum of all three terms gives $g_2(x)$.

We can now give a compact expression for $g(x)$ in terms of integral operators:

$$g(x) = \sum_{j_1, j_2 \geq 0} \epsilon^{j_1+j_2} (-2)^{j_2} \{J(0)^{j_1}, J(-1)^{j_2}\} J(0) \; . \tag{4.15}$$

Using the recursive definition (2.12) of the generalized operator product, it easy to prove that (4.15) solves the differential equation (4.5). On the other hand, (4.9) or the equivalent recurrence relation (4.12) only solve the differential eqs. (4.7) iteratively.

Alternatively to study a Laurent expansion in ϵ in terms of iterated integrals, one can look for a solution exact in ϵ in terms of hypergeometric functions and their generalizations.

However, for the latter the behaviour of the integral at $\epsilon = 0$ is not obvious. The all-order representation combines the advantages of both approaches. It is given explicitly in terms of iterated integrals, which allows to straightforwardly determine any order of the ϵ-expansion and therefore the behaviour at $\epsilon = 0$. Moreover, the all-order result contains all information on the Feynman integral in a compact form like a representation in terms of hypergeometric functions. In the next section we give an application for the all-order expression, which confirms this statement.

4.2 Hypergeometric series

Starting with eq. (4.15), it is possible to construct the representation of $I(2,1)$ in terms of a hypergeometric function, because the all-order expansion of the latter is known as well. The type of integral operators, which are part of (4.15), also appear in the expansion of the hypergeometric function ${}_2F_1$ with a half-integer parameter, which we considered in section 3.3.4. Combining eqs. (3.46) and (3.74), this expansion can be written as:

$$ {}_2F_1\left[\begin{matrix} a\epsilon, b\epsilon \\ \frac{1}{2}+c\epsilon \end{matrix}; z\right] = \sum_{l_1,l_2,l_3,m\geq 0} \epsilon^{l_1+l_2+l_3+2m+2}(-1)^{l_2+m+1}2^{l_1+l_2}c^{l_1}(a+b-c)^{l_2}(a+b)^{l_3}(ab)^{m+1} \\ \times J(0)\{J(1)^{l_1}, J(-1)^{l_2}, J(0)^{l_3}, J(0,0)^m\}J(0)\ . \quad (4.16) $$

Let us start by comparing the products of integral operators in the all-order expressions (4.15) and (4.16). One difference is the operator $J(0)$ to the left of the generalized operator product in (4.16), which is missing in (4.15). Therefore we shall deal with a hypergeometric function, which is the product of the differential operator $y\frac{d}{dy}$ and (4.16). According to eqs. (3.30) and (3.47), this function can be written in the form:

$$ y\frac{d}{dy}\,{}_2F_1\left[\begin{matrix} a\epsilon, b\epsilon \\ \frac{1}{2}+c\epsilon \end{matrix}; z\right] = -\frac{1+y}{1-y}\,\frac{zab\epsilon^2}{\frac{1}{2}+c\epsilon}\,{}_2F_1\left[\begin{matrix} 1+a\epsilon, 1+b\epsilon \\ \frac{3}{2}+c\epsilon \end{matrix}; z\right]\ . \quad (4.17) $$

Inserting the expansion (4.16) on the l.h.s., we arrive at

$$ {}_2F_1\left[\begin{matrix} 1+a\epsilon, 1+b\epsilon \\ \frac{3}{2}+c\epsilon \end{matrix}; z\right] = \frac{4y}{1-y^2}\left(\frac{1}{2}+c\epsilon\right)\sum_{l_1,l_2,l_3,m\geq 0}\epsilon^{l_1+l_2+l_3+2m}(-1)^{l_2+m+1}2^{l_1+l_2} \\ \times c^{l_1}(a+b-c)^{l_2}(a+b)^{l_3}(ab)^m\{J(1)^{l_1}, J(-1)^{l_2}, J(0)^{l_3}, J(0,0)^m\}J(0)\ , \quad (4.18) $$

where identical to eq. (4.15) the operator $J(0)$ to the left of the generalized operator product is now missing. In the next step we want to remove the operators $J(1)$ and $J(0,0)$ in (4.18), which do not appear in (4.15). To achieve this, we set $a = c = 0$, so that only the part of the sum with $l_1 = m = 0$ remains:

$$ {}_2F_1\left[\begin{matrix} 1, 1+b\epsilon \\ \frac{3}{2} \end{matrix}; z\right] = -\frac{2y}{1-y^2}\sum_{l_2,l_3\geq 0}\epsilon^{l_2+l_3}(-2)^{l_2}b^{l_2+l_3}\{J(-1)^{l_2}, J(0)^{l_3}\}J(0)\ . \quad (4.19) $$

Now the integral operators are identical to the ones in (4.15). To adjust the other factors, we set $b = 1$ and replace y and z via

$$y = x \ , \quad z = \frac{s}{4m^2} \ , \tag{4.20}$$

which yields the final result:

$$g\left(\frac{s}{m^2}\right) = \frac{\sqrt{s(s-4m^2)}}{2m^2} \, {}_2F_1\left[\begin{matrix} 1, 1+\epsilon \\ \frac{3}{2} \end{matrix} ; \frac{s}{4m^2}\right] \ . \tag{4.21}$$

For completeness we give the expression for the complete integral (4.3) as well:

$$I(2,1) = \frac{1}{2}\,\Gamma(\epsilon+1)(m^2)^{-\epsilon-1}{}_2F_1\left[\begin{matrix} 1, 1+\epsilon \\ \frac{3}{2} \end{matrix} ; \frac{s}{4m^2}\right] \ . \tag{4.22}$$

Let us conclude this chapter by pointing out similarities in the transformations we used to obtain proper differential equations both for the Feynman integral and hypergeometric function in the previous chapter. We have not yet explained the factor in the ansatz (4.3) for the Feynman integral. Since Feynman integrals are scalar functions, they depend only on scalar quantities, which in this case are the mass m and the kinematic invariant s. In general one has to consider one differential equation in each of these scalars in order to receive all information on a MI. One can, however, read off the mass dimension of an integral directly from its definition of the form (4.1). This allows to write the integral as a product of a factor carrying the mass dimension and a dimensionless function. The latter depends only on dimensionless quantities. With the two variables m and s there is only one independent dimensionless quantity, e.g. the ratio $\frac{m^2}{s}$ or the variable x. So in general one chooses an ansatz with a factor, which carries the mass dimension of the integral, in order to reduce the number of differential equations to be considered. Hence, a reasonable ansatz for $I(2,1)$ would be:

$$I(2,1) = (m^2)^{-\epsilon-1} f(x) \ . \tag{4.23}$$

However, this does not explain the use of the square root in (4.3). Actually, the factor in (4.3) seems to be unnecessarily complicated, since the square root cancels once the expression (4.21) for $g(x)$ is inserted. But this square root is in fact crucial to perform the calculations presented above. Let us demonstrate how far we can go with the alternative approach (4.23). The differential equation for the function $f(x)$ in x takes the form

$$\frac{d}{dx} f(x) = \left[A_0(x) + \epsilon\, A_1(x)\right] f(x) + \Gamma(\epsilon+1) B(x) \ , \tag{4.24}$$

with some functions $A_1(x)$, $B(x)$ and:

$$A_0(x) = \frac{1}{1-x} + \frac{1}{x} - \frac{1}{1+x} \ . \tag{4.25}$$

The crucial difference to the differential equation (4.5) for $g(x)$ is the term $A_0(x)$ of order ϵ^0. It is necessary to remove the latter in order to obtain a straightforward iterative solution of the form (4.9) for the coefficient functions of the ϵ-expansion[1]. This property is generic to Feynman integrals [55]. Furthermore, without this iterative solution we are not able to write a recurrence relation to eventually obtain an all-order expression. In order to remove this term in eq. (4.24), we use the transformation

$$f(x) = T(x)\, g(x) \tag{4.26}$$

to obtain the new differential equation:

$$\begin{aligned}\frac{d}{dx}g(x) = T^{-1}(x)\left[A_0(x)\, T(x) - \frac{d}{dx}T(x)\right] g(x) \\ + \epsilon\, T^{-1}(x)\, A_1(x)\, T(x)\, g(x) + \Gamma(\epsilon+1)\, T^{-1}(x)\, B(x)\ .\end{aligned} \tag{4.27}$$

We want the leading term in ϵ on the r.h.s. to vanish, i.e.:

$$A_0(x)\, T(x) = \frac{d}{dx}T(x)\ . \tag{4.28}$$

Then, the general solution of this differential equation is:

$$T(x) = \frac{x\, C}{1-x^2} = \frac{m^2 C}{\sqrt{s(s-4m^2)}}\ . \tag{4.29}$$

Choosing the constant $C = \Gamma(1+\epsilon)$ to remove the gamma function from the inhomogeneous term in eq. (4.27) gives the ansatz (4.3) and the corresponding differential equation (4.5).

The form of the differential equation we aimed at and the transformation we used to obtain it, are identical to those in the discussion about hypergeometric differential equations in section 3.5. We elaborated in detail how the convenient form of the differential equations follows naturally for the generalized hypergeometric functions we considered. While it has been observed for Feynman integrals in [55] that a canonical form of differential equations allows a straightforward iterative compution of ϵ-expansions, for generalized hypergeometric functions this form has already been used in [11, 12].

[1] Note also, that $A_0(x)$ has a pole at $x = 1$, which is not the case for $g(x)$.

Chapter 5

String amplitudes

In this chapter we want to apply our new method to the four-point [58]–[60] and five-point [33, 34, 61, 62] superstring disk amplitudes. An elegant and unifying picture of both amplitudes in terms of super Yang-Mills building blocks and generic string form factors has been elaborated in [31, 32, 56]. The latter are given by generalized hypergeometric functions (3.1) with $p = 2$ for the four-point case and $p = 3$ for the five-point case. The all-order expressions (3.60) and (3.66) of these functions are now used to obtain all order expansions for these superstring amplitudes.

5.1 Four-point superstring amplitude

The four-point amplitude is written in terms of the hypergeometric function

$$F(s,u) = {}_2F_1\left[\begin{matrix}-s, u\\ 1+s\end{matrix};1\right] = \sum_{k\geq 0} u_k(s,u)\ , \tag{5.1}$$

with the Mandelstam variables[1] $s = \alpha'(k_1+k_2)^2$ and $u = \alpha'(k_1+k_4)^2$. According to (3.10) and (3.60) the coefficient functions $u_k(s,u)$ of the α'-expansion can be written as:

$$u_k(s,u) = \begin{cases} 1 & \text{for } k=0,\\ \displaystyle\sum_{\alpha=1}^{k-1}(-1)^{k+1}s^{k-\alpha}u^{\alpha}\zeta(\alpha+1,\{1\}^{k-\alpha-1}) & \text{otherwise.}\end{cases} \tag{5.2}$$

For later convenience we combined the lowest order $u_0(s,u) = 1$ and all higher orders into one expression and we added the kinematic variables s and u as arguments. The all-order

[1] Since α' enters the definition of kinematic invariants, it does not appear explicitely in α'-expansions of string amplitudes.

representation of the four-point amplitude (5.1) reads:

$$F(s,u) = 1 - \sum_{k=2}^{\infty} (-1)^k \sum_{\alpha=1}^{k-1} s^{k-\alpha} u^{\alpha} \zeta(\alpha+1, \{1\}^{k-\alpha-1}) \ . \tag{5.3}$$

By this result the string corrections to the four-point Yang-Mills amplitudes can easily be calculated for any order in α'. The well known duality-symmetry of $F(s,u)$ w.r.t. the exchange $s \leftrightarrow u$ is not automatically fulfilled in (5.3). Instead, this leads to the MZV identity:

$$\zeta(\alpha_1+1, \{1\}^{\alpha_2-1}) = \zeta(\alpha_2+1, \{1\}^{\alpha_1-1}) \ , \quad \alpha_1, \alpha_2 \geq 1 \ . \tag{5.4}$$

Both this identity, which is a special case of the well-known duality formula for MZVs, and the representation for $F(s,u)$ are already known. However, it is interesting to compare these results with those for the five-point amplitude.

5.2 Five-point superstring amplitude

For five-point amplitudes the basis of generalized Euler integrals is two-dimensional. A possible choice are the two functions [31, 32]

$$F_1 = F(s_1,s_2)\ F(s_3,s_4)\ {}_3F_2 \left[\begin{matrix} s_1, 1+s_4, -s_{24} \\ 1+s_1+s_2, 1+s_3+s_4 \end{matrix} ; 1 \right] \ , \tag{5.5}$$

$$F_2 = s_{13}s_{24} \frac{F(s_1,s_2)\ F(s_3,s_4)}{(1+s_1+s_2)(1+s_3+s_4)}\ {}_3F_2 \left[\begin{matrix} 1+s_1, 1+s_4, 1-s_{24} \\ 2+s_1+s_2, 2+s_3+s_4 \end{matrix} ; 1 \right] \ , \tag{5.6}$$

which depend on the kinematic invariants $s_{ij} = \alpha'(k_i+k_j)^2$ and $s_i = s_{i\,i+1}$. Both hypergeometric functions ${}_3F_2$ can be related to (3.61) and derivatives thereof by using eqs. (3.30):

$${}_3F_2 \left[\begin{matrix} \alpha' a_1, 1+\alpha' a_2, \alpha' a_3 \\ 1+\alpha' b_1, 1+\alpha' b_2 \end{matrix} ; 1 \right] = \left(\frac{\theta}{\alpha' a_2} + 1 \right) {}_3F_2 \left[\begin{matrix} \alpha' a_1, \alpha' a_2, \alpha' a_3 \\ 1+\alpha' b_1, 1+\alpha' b_2 \end{matrix} ; z \right] \Bigg|_{z=1} \ , \tag{5.7}$$

$${}_3F_2 \left[\begin{matrix} 1+\alpha' a_1, 1+\alpha' a_2, 1+\alpha' a_3 \\ 2+\alpha' b_1, 2+\alpha' b_2 \end{matrix} ; 1 \right] = \frac{(1+\alpha' b_1)(1+\alpha' b_2)}{a_1 a_2 a_3 (\alpha')^3 z}\ \theta\ {}_3F_2 \left[\begin{matrix} \alpha' a_1, \alpha' a_2, \alpha' a_3 \\ 1+\alpha' b_1, 1+\alpha' b_2 \end{matrix} ; z \right] \Bigg|_{z=1} \ , \tag{5.8}$$

with

$$\begin{aligned} &\alpha' a_1 = s_1 \ , \quad \alpha' a_2 = s_4 \ , \quad \alpha' a_3 = s_2 + s_3 - s_5 \ , \\ &\alpha' b_1 = s_1 + s_2 \ , \quad \alpha' b_2 = s_3 + s_4 \ . \end{aligned} \tag{5.9}$$

Eqs. (5.7) and (5.8) lead to:

$$F_1 = F(s_1,s_2)\ F(s_3,s_4)\ {}_3F_2 \left[\begin{matrix} s_1, s_4, -s_{24} \\ 1+s_1+s_2, 1+s_3+s_4 \end{matrix} ; 1 \right] \tag{5.10}$$

$$- s_1 s_{24}\ F(s_1,s_2)\ F(s_3,s_4)\ \frac{\theta}{(\alpha')^3\Delta_{3,3}}\ {}_3F_2\left[\begin{matrix} s_1, s_4, -s_{24} \\ 1+s_1+s_2, 1+s_3+s_4 \end{matrix}; z\right]\Bigg|_{z=1},$$

$$F_2 = s_{13}s_{24}\ F(s_1,s_2)\ F(s_3,s_4)\ \frac{\theta}{(\alpha')^3\Delta_{3,3}}\ {}_3F_2\left[\begin{matrix} s_1, s_4, -s_{24} \\ 1+s_1+s_2, 1+s_3+s_4 \end{matrix}; z\right]\Bigg|_{z=1}. \quad (5.11)$$

In these two expressions the α'-expansions of all factors are known. As a consequence we are able to derive the expansions of F_1 and F_2. In the following this is accomplished in two different ways. According to eqs. (3.64) and (5.9) the kinematic invariants, appearing in the parameters of the hypergeometric function ${}_3F_2$, enter the corresponding expansion via the elementary symmetric functions

$$\begin{aligned} \Delta_1 &= -s_5\ , \\ \Delta_2 &= s_1s_2 - s_2s_3 + s_3s_4 - s_4s_5 - s_5s_1\ , \\ \Delta_3 &= s_1s_2s_4 + s_1s_3s_4 - s_1s_4s_5\ , \\ Q_1 &= s_1+s_2+s_3+s_4\ , \\ Q_2 &= s_1s_3+s_2s_3+s_1s_4+s_2s_4\ , \end{aligned} \quad (5.12)$$

with $\Delta_i = (\alpha')^i\Delta_{3,i}$, $i=1,2,3$ and $Q_j = (\alpha')^j Q_{3,j}$, $j=1,2$. While in subsection 5.2.1 these elementary symmetric functions are left as symbols, in subsection 5.2.2 we take the explicit representation (5.12) in terms of kinematic invariants to derive an alternative expression for F_2. Although this representation is not as compact as the one in subsection 5.2.1, it is more suitable for further applications to be discussed in chapter 6.

5.2.1 Representation in terms of elementary symmetric functions

Applying the results of chapter 3 to (5.10) and (5.11) yields

$$\begin{aligned} F_1 = &\sum_{k=0}^{\infty} \sum_{\substack{k_1,k_2\geq 0\\ k_1+k_2\leq k}} u_{k_1}(s_1,s_2)\ u_{k_2}(s_3,s_4)\ v_{k-k_1-k_2}(1) \\ &- \sum_{k=2}^{\infty} s_1s_{24} \sum_{\substack{k_1,k_2\geq 0\\ k_1+k_2\leq k-2}} u_{k_1}(s_1,s_2)\ u_{k_2}(s_3,s_4)\ \frac{\theta}{\Delta_3} v_{k-k_1-k_2+1}(z)\Bigg|_{z=1}, \end{aligned} \quad (5.13)$$

$$F_2 = \sum_{k=2}^{\infty} s_{13}s_{24} \sum_{\substack{k_1,k_2\geq 0\\ k_1+k_2\leq k-2}} u_{k_1}(s_1,s_2)\ u_{k_2}(s_3,s_4)\ \frac{\theta}{\Delta_3} v_{k-k_1-k_2+1}(z)\Bigg|_{z=1}, \quad (5.14)$$

with $u_k(a,b)$ defined in (5.2) and with the coefficient functions $v_k(z)$ of the expansion:

$${}_3F_2\left[\begin{matrix} s_1, s_4, -s_{24} \\ 1+s_1+s_2, 1+s_3+s_4 \end{matrix}; z\right] = \sum_{k\geq 0} v_k(z)\ . \quad (5.15)$$

From the results of section 3.3.2 follows $v_0(z) = 1$ and:

$$v_k(z) = \sum_{l_1+m_1+2(l_2+m_2)+3m_3=k-3} (-1)^{l_1+l_2} \Delta_1^{m_1} \Delta_2^{m_2} \Delta_3^{m_3+1} Q_1^{l_1} Q_2^{l_2} \\ \times I(0,0) \left\{ I(0)^{l_1}, I(0,0)^{l_2}, I(1)^{m_1}, I(1,0)^{m_2}, I(1,0,0)^{m_3} \right\} I(1) \ , \quad k \geq 1 \ . \tag{5.16}$$

The derivative of this coefficient at $z = 1$ with the factor $\Delta_3^{-1} z$ enters eqs. (5.13) and (5.14) as well. The all-order expression follows straightforwardly from (5.16):

$$\left. \frac{\theta}{\Delta_3} v_k(z) \right|_{z=1} = \sum_{l_1+m_1+2(l_2+m_2)+3m_3=k-3} (-1)^{l_1+l_2} \Delta_1^{m_1} \Delta_2^{m_2} \Delta_3^{m_3} Q_1^{l_1} Q_2^{l_2} \\ \times I(0) \left\{ I(0)^{l_1}, I(0,0)^{l_2}, I(1)^{m_1}, I(1,0)^{m_2}, I(1,0,0)^{m_3} \right\} I(1) \Big|_{z=1} \ . \tag{5.17}$$

Eqs. (5.13) and (5.14) give all orders of the α'-expansions of the five-point open superstring amplitude. Besides the kinematic variables at each order only products of at most three MZVs are produced. This is to be contrasted with the procedure in [35] where also higher products of MZVs appear.

As already discussed at the end of section 3.3.2, there is a way to remove the generalized operator product to obtain a representation in terms of MZVs (see eqs. (6.62) and (6.63)).

5.2.2 Representation in terms of kinematic invariants

The kinematic parts of $u_{k_1}(s_1, s_2)$, $u_{k_2}(s_3, s_4)$ and $v_k(z)$ can be combined in eq. (5.14) to obtain

$$F_2 = s_{13} s_{24} \sum_{k=2}^{\infty} \sum_{j_1+j_2+j_3+j_4+j_5=k-2} s_1^{j_1} s_2^{j_2} s_3^{j_3} s_4^{j_4} s_5^{j_5} f(j_1, j_2, j_3, j_4, j_5) \ , \tag{5.18}$$

where all MZVs and integral operators are contained in:

$$f(j_1, j_2, j_3, j_4, j_5) = \sum_{\vec{l}=(l_1,l_2,l_3,l_4)} (-1)^{|\vec{l}|} \ \zeta'_{l_1,l_2} \ \zeta'_{l_3,l_4} v(j_1 - l_1, j_2 - l_2, j_3 - l_3, j_4 - l_4, j_5) \ . \tag{5.19}$$

This function involves the MZVs of (5.2),

$$\zeta'_{i_1,i_2} = \begin{cases} \zeta(i_1+1, \{1\}^{i_2-1}) & \text{for } i_1, i_2 \geq 1, \\ -1 & \text{for } i_1, i_2 = 0, \\ 0 & \text{else,} \end{cases} \tag{5.20}$$

and the integral operators of (5.17):

$$v(j_1, j_2, j_3, j_4, j_5)$$

$$= \sum_{\vec{\alpha},\vec{\beta},\gamma,\vec{\delta},\vec{\epsilon}\in L} I(0)\left\{I(0)^{|\vec{\alpha}|}, I(0,0)^{|\vec{\beta}|}, I(1)^{\gamma}, I(1,0)^{|\vec{\delta}|}, I(1,0,0)^{|\vec{\epsilon}|}\right\} I(1)\Big|_{z=1} \tag{5.21}$$
$$\times (-1)^{|\vec{\alpha}|+|\vec{\beta}|+\delta_3+j_5} \binom{|\vec{\alpha}|}{\alpha_1,\alpha_2,\alpha_3,\alpha_4}\binom{|\vec{\beta}|}{\beta_1,\beta_2,\beta_3,\beta_4}\binom{|\vec{\delta}|}{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5}\binom{|\vec{\epsilon}|}{\epsilon_1,\epsilon_2,\epsilon_3} .$$

The summation is over non-negative integers γ and the multiple indices:

$$\begin{aligned} \vec{\alpha} &= (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \ , \\ \vec{\beta} &= (\beta_1,\beta_2,\beta_3,\beta_4) \ , \\ \vec{\delta} &= (\delta_1,\delta_2,\delta_3,\delta_4,\delta_5) \ , \\ \vec{\epsilon} &= (\epsilon_1,\epsilon_2,\epsilon_3) \ . \end{aligned} \tag{5.22}$$

The summation region of $\vec{\alpha}$, $\vec{\beta}$, γ, $\vec{\delta}$ and $\vec{\epsilon}$ is the solution set L of the five equations:

$$\begin{aligned} j_1 &= \alpha_1+\beta_1+\beta_2+\delta_1+\delta_2+|\vec{\epsilon}| \ , \\ j_2 &= \alpha_2+\beta_3+\beta_4+\delta_1+\delta_3+\epsilon_1 \ , \\ j_3 &= \alpha_3+\beta_1+\beta_3+\delta_3+\delta_4+\epsilon_2 \ , \\ j_4 &= \alpha_4+\beta_2+\beta_4+\delta_4+\delta_5+|\vec{\epsilon}| \ . \\ j_5 &= \gamma+\delta_2+\delta_5+\epsilon_3 \ . \end{aligned} \tag{5.23}$$

The function $v(j_1,j_2,j_3,j_4,j_5)$ is related to $v_k(z)$ through:

$$\frac{\theta}{\Delta_3} v_k(z)\Big|_{z=1} = \sum_{j_1+j_2+j_3+j_4+j_5=k-3} s_1^{j_1} s_2^{j_2} s_3^{j_3} s_4^{j_4} s_5^{j_5} \ v(j_1,j_2,j_3,j_4,j_5) \ . \tag{5.24}$$

Obviously, the r.h.s. of (5.24) together with (5.21) is less compact than the r.h.s. of (5.17), which uses the quantities Δ_1, Δ_2, Δ_3, Q_1 and Q_2. But the advantage of the former is, that the symmetry of $F_2\ (s_{13}s_{24})^{-1}$ can directly be analyzed in (5.19). This function is invariant w.r.t. cyclic permutation of the kinematic invariants (s_1,s_2,s_3,s_4,s_5). Therefore the function $f(j_1,j_2,j_3,j_4,j_5)$ is invariant w.r.t. cyclic permutations of its arguments (j_1,j_2,j_3,j_4,j_5). Just like for the four-point amplitude (cf. section 5.1) this symmetry is non-trivially fulfilled and various MZV identities are generated. To obtain them, the generalized operator product in (5.21) has to be written in terms of MZVs. This is of course the same operator product as the one in (3.66) and (5.17). In addition, there are summations over a total of 17 indices in (5.21), of which five can be evaluated with eqs. (5.23). Thus, besides the issue of converting the generalized product of integral operators into MZVs, it is interesting to see, whether some of the remaining twelve sums can be evaluated along the way. For the α'-expansion of the four-point amplitude (5.1) this is performed in section 3.3.1. The identity, which leads from eq. (3.58) to (3.59) eliminates both the generalized operator product and the inner sum. This identity and similar ones, which apply to the five-point case, are discussed in chapter 6.

Chapter 6

From generalized operator products to MZVs

The results of chapters 3 and 5 left some open questions: how to obtain the all-order expression (3.59) for the hypergeometric function ${}_2F_1$ from its representation (3.58) with the latter involving generalized operator products. Likewise, how to achieve similar transformations on the operator products arising for the hypergeometric functions ${}_3F_2$ and ${}_pF_{p-1}$ in general. These questions are discussed in section 6.1. The relations, to be derived there, are applied to the results from chapters 3 and 5 in section 6.2. The MZV identities, which follow from cyclic symmetry of the function $f(j_1, j_2, j_3, j_4, j_5)$, are the topic of section 6.3.

6.1 Identities for generalized operator products

In this section three types of operator products are discussed. Starting from simple examples, involving independent arguments, the complexity increases step by step to finally obtain identities, which can be applied to expressions (3.66), (3.70) and (5.21). Therefore, not all identities are needed, at least not for the results of this thesis. However, simpler identities provide a consistency check for the most complicated ones.

All identities for generalized operator products, presented in this section, contain MZVs. Identical relations hold for MPLs as well. Before and after every generalized operator product there is an $I(0)$ and an $I(1)$ operator, respectively, to ensure finiteness of the corresponding MZV. The following equations make extensive use of two notations, which we introduced at the end of section 3.2 in eqs. (3.23)–(3.29): sums over all sets of indices $\vec{n} = (n_1, n_2, \ldots, n_d)$ of MZVs $\zeta(\vec{n})$ and multiple index sums.

6.1.1 Independent arguments

We start with generalized operator products, which include only independent arguments.

Example 1.1: The simplest example is:

$$I(0)\{I(0)^{j_1}, I(1)^{j_2}\}I(1)\Big|_{z=1} = \sum_{\substack{w=j_1+j_2+2\\ d=j_2+1}} \zeta(\vec{n}) \ . \tag{6.1}$$

It is clear that the sum is over the given weight and depth, since these quantities correlate directly to the number of integral operators. It needs to be proven that *all* MZVs of given weight and depth are generated by the generalized operator product on the l.h.s. To do this, it is sufficient to show that

1. both sides contain the same number of terms
2. and that there are no identical terms on the l.h.s.

The second point is clear due to the definition of the generalized operator product as the sum of *distinct* permutations and the fact that both arguments $I(0)$ and $I(1)$ are independent. The first property is also true. The number of different MZVs of weight w and depth d is $\binom{w-2}{d-1}$, which equals $\binom{j_1+j_2}{j_2}$ in this case. According to eq. (2.9), this quantity is identical to the number of terms on the l.h.s. of (6.1).

Example 1.2: With $I(1,0)$ instead of $I(1)$ the identity (6.1) becomes:

$$I(0)\{I(0)^{j_1}, I(1,0)^{j_2}\}I(1)\Big|_{z=1} = \sum_{\substack{w=j_1+2j_2+2\\ d=j_2+1;\ n_i\geq 2}} \zeta(\vec{n}) \ . \tag{6.2}$$

The additional condition $n_i \geq 2$ is self-explanatory. For the first integer n_1 it is obvious that $n_1 \geq 2$, because the l.h.s. starts with the operator $I(0)$. For any other integers n_i to be one, any sequence of integral operators would have to include the product $I(1,1)$. With the arguments $I(0)$ and $I(1,0)$ this is obviously not possible. To prove that the l.h.s. of (6.2) produces *all* MZVs of given weight and depth, which do not include an index $n_i = 1$, similar arguments as for (6.1) hold here. The generalization to cases with $I(1,0)$ replaced by other arguments of the type $I(1,0,\ldots,0)$ is straightforward.

Examples 1.3 and 1.4: Other generalized operator products with independent arguments are

$$\begin{aligned} I(0)\{I(1)^{j_1}, I(1,0)^{j_2}\}I(1)\Big|_{z=1} &= I(0,1)\{I(1)^{j_1}, I(0,1)^{j_2}\}\Big|_{z=1} \\ &= \sum_{\substack{w=j_1+2j_2+2;\ d=j_1+j_2+1\\ n_1=2;\ d_1=j_1;\ d_2=j_2+1}} \zeta(\vec{n}) \ , \end{aligned} \tag{6.3}$$

and

$$I(0)\{I(1)^{j_1}, I(1,0)^{j_2}, I(1,0,0)^{j_3}\}I(1)\Big|_{z=1} = I(0,1)\{I(1)^{j_1}, I(0,1)^{j_2}, I(0,0,1)^{j_3}\}\Big|_{z=1}$$

$$= \sum_{\substack{w=j_1+2j_2+3j_3+2;\ d=j_1+j_2+j_3+1 \\ n_1=2;\ d_1=j_1;\ d_2=j_2+1;\ d_3=j_3}} \zeta(\vec{n}) \ , \tag{6.4}$$

with two and three arguments, respectively. In both first equations we used

$$\{I(1)^{j_1}, I(1,0)^{j_2}, \ldots, I(1, \underbrace{0,\ldots,0}_{j_n-1})^{j_n}\}I(1) = I(1)\{I(1)^{j_1}, I(0,1)^{j_2}, \ldots, I(\underbrace{0,\ldots,0}_{j_n-1}, 1)^{j_n}\} \ , \tag{6.5}$$

to make evident the conditions on d_i. To check consistency, note that $d = \sum_i d_i$ and that relation (6.4) becomes (6.3) for $j_3 = 0$. For a strict proof, the same strategy as for (6.1) should work. Even though more combinatorics is needed here, to determine for example the number of MZVs of given w, d, d_1, d_2 and d_3. The generalization to more arguments of the kind $I(1,0,\ldots,0)$ is straightforward.

6.1.2 Dependent arguments

The following identities involve generalized products of dependent operators. Distinct permutations of dependent factors can be identical. Consider for instance the generalized operator product:

$$\{I(0), I(1), I(1,0)\} = I(0,1,1,0) + I(0,1,0,1) + I(1,0,0,1) + I(1,1,0,0) + 2I(1,0,1,0) \ . \tag{6.6}$$

Clearly, the third arguments $I(1,0)$ can be written as a product of the first two, $I(1)$ and $I(0)$. As a consequence the two distinct permutations

$$I(1)I(0)I(1,0) \ \ \text{and} \ \ I(1,0)I(1)I(0) \tag{6.7}$$

are identical and the corresponding product $I(1,0,1,0)$ appears twice in (6.6). No other products appears more than once, since $I(1,0,1,0)$ is the only one to contain twice the sequence $I(1)I(0)$. For the more complicated identities in the following, the task is to count identical permutations such as (6.7). Since these identities translate generalized operator products into sums of MZVs, identical permutations correspond to MZVs, which appear more than once. Therefore, a weighting is required in the sum of MZVs, i.e. a function, which can be evaluated for every single MZV in that sum and thereby describes how often every single MZV appears. On one hand, weightings can depend on quantities, which are identical for all MZVs of the corresponding sum. These quantities are determined by the relevant generalized operator product. Examples are the indices of the generalized operator product or the weight w and depth d, which are determined by the number of integral operators. On the other hand, in order to give different factors for different MZVs, the weighting has to depend on the indices $\vec{n}$ of MZVs $\zeta(\vec{n})$. This dependence can be explicit or in terms of related quantities such as the number d_j of indices, which equal j.

We start with the discussion on the generalized operator product of the most general case (3.70). There are two types of arguments in the generalized operator products we encountered in our results of chapters 3 and 5. Those, that consist only of integral operators $I(0)$ and those, which have an operator $I(1)$ to the left of all $I(0)$ operators. In the following relation the former have an index j_μ, $\mu = 1, \dots, a$ and the latter have j'_ν, $\nu = 1, \dots, b$:

$$\begin{aligned} &I(0)\{I(0)^{j_1}, I(0,0)^{j_2}, \dots, I(\underbrace{0,\dots,0}_{a})^{j_a}, I(1)^{j'_1}, I(1,0)^{j'_2}, \dots, I(1,\underbrace{0,\dots,0}_{b-1})^{j'_b}\}I(1)\big|_{z=1} \\ &\qquad = \sum_{\substack{w=j_1+2j_2+\ldots+aj_a \\ +j'_1+2j'_2+\ldots+bj'_b+2 \\ d=j'_1+j'_2+\ldots+j'_b+1}} \zeta(\vec{n})\, \omega_{a,b}(\vec{n}'-\vec{1}; j_1, j_2, \dots, j_a; j'_1, j'_2, \dots, j'_b)\ , \end{aligned} \tag{6.8}$$

with the constant vector $\vec{1} = (\underbrace{1,\dots,1}_{d})$ and the elements

$$n'_i = \begin{cases} n_1 - 1 & \text{for } i = 1\ , \\ n_i & \text{for } i = 2, \dots, d\ , \end{cases} \tag{6.9}$$

of the vector $\vec{n}'$. The conditions for the weight w and the depth d in the sum of MZVs are self-explanatory. They are related to the number of integral operators, which can easily be read off from the first line. Of greater interest is the weighting

$$\begin{aligned} &\omega_{a,b}(\vec{n}; j_1, j_2, \dots, j_a; j'_1, j'_2, \dots, j'_b) \\ &\qquad = \sum_{\substack{\vec{\beta}_1+\vec{\beta}_2+\ldots+\vec{\beta}_b=\vec{1} \\ |\vec{\beta}_\nu|=j'_\nu;\ \beta_{\nu,1}=0}} \omega_a(\vec{n}-\vec{\beta}_2-2\vec{\beta}_3-\ldots-(b-1)\vec{\beta}_b; j_1, j_2, \dots, j_a)\ , \end{aligned} \tag{6.10}$$

with:

$$\omega_a(\vec{n}; j_1, j_2, \dots, j_a) = \sum_{\substack{\vec{\alpha}_1+2\vec{\alpha}_2+\ldots+a\vec{\alpha}_a=\vec{n} \\ |\vec{\alpha}_\mu|=j_\mu}} \binom{\vec{\alpha}_1+\vec{\alpha}_2+\ldots+\vec{\alpha}_a}{\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_a}\ . \tag{6.11}$$

The weighting $\omega_{a,b}$ depends on the indices $\vec{n}$ of the MZVs and the indices j_μ and j'_ν of the generalized operator product. Besides the multinomial coefficient, essentially there are two summations in the definitions (6.10) and (6.11). The one in (6.10) over the indices

$$\vec{\beta}_\nu = (\beta_{\nu,1}, \dots, \beta_{\nu,d})\ ,\quad \nu = 1, \dots, b\ , \tag{6.12}$$

take permutations into account, which involve the operators $I(1,0,\dots,0)$, while the sums in (6.11) over the indices

$$\vec{\alpha}_\mu = (\alpha_{\mu,1}, \dots, \alpha_{\mu,d})\ ,\quad \mu = 1, \dots, a\ , \tag{6.13}$$

refer to the operators of the type $I(0,\ldots,0)$. To explain the expressions (6.10) and (6.11) in detail, we consider permutations of the arguments of the type $I(0,\ldots,0)$ before discussing those of the arguments of the type $I(1,0,\ldots,0)$. Finally, we explain how both types are related.

Both sums in (6.10) and (6.11) use several multi-indices (6.12) and (6.13) with d elements each. This is motivated by the following idea. First we count the number of identical permutations of operators, which are part of the integral operator representation

$$I(\underbrace{0,\ldots,0}_{n_i-1},1) \tag{6.14}$$

of a single MZV index n_i. Then we use the multi-index notation to combine the results of all indices $\vec{n}$, which yields the total number of identical permutations, i.e. the weighting.

Let us start with identical permutations of arguments of the type $I(0,\ldots,0)$, which are part of one index (6.14). Assuming there are a different arguments of the type $I(\underbrace{0,\ldots,0}_{\mu}) \equiv I(0)^\mu$, $\mu = 1,\ldots,a$, which appear α_μ times, then for fixed α_μ there are

$$\binom{\alpha_1+\alpha_2+\ldots+\alpha_a}{\alpha_1,\alpha_2,\ldots,\alpha_a} \tag{6.15}$$

distinct permutations of these arguments. Every permutation is a sequence of $\alpha_1+2\alpha_2+\ldots+a\alpha_a$ operators $I(0)$. Now we assume the α_μ are not fixed, but the total number of operators $I(0)$ has to add up to n_i-1. To count the permutations, we have to sum over all sets $(\alpha_1,\alpha_2,\ldots,\alpha_a)$ and take the fixed total number of n_i-1 operators $I(0)$ into account:

$$\sum_{\alpha_1+2\alpha_2+\ldots+a\alpha_a=n_i-1} \binom{\alpha_1+\alpha_2+\ldots+\alpha_a}{\alpha_1,\alpha_2,\ldots,\alpha_a}\,. \tag{6.16}$$

These are the number the possibilities to permute the arguments of the type $I(0,\ldots,0)$, which contribute to one index (6.14). Multiplying the possibilities of all indices $\vec{n}$ of a MZV yields the multi-index notation presented in (6.11) and (6.13), with $\alpha_{\mu,i}$ being the number of operators $I(0)^\mu$, which contribute to the index n_i. The elements of a multi-index are not independent, since their sum $\alpha_{\mu,1}+\ldots+\alpha_{\mu,d} = |\vec{\alpha}_\mu|$, i.e. the total number of operators $I(0)^\mu$, is fixed by the index j_μ of the generalized operator product. As a consequence, we have the additional conditions $|\vec{\alpha}_\mu| = j_\mu$, $\mu = 1,\ldots,a$ in (6.11). The first index n_1 is an exceptional case. The first $I(0)$ in the integral operator representation is the one to the left of the generalized operator product in (6.8). Thus, only n_1-2 operators are relevant for the permutations. This explains the use of (6.9) for the arguments of the weighting $\omega_{a,b}$ in the identity (6.8).

Now that we have discussed the arrangements of the arguments $I(0,\ldots,0)$, described by ω_a, let us consider the arguments $I(1,0,\ldots,0)$. All identical permutations of the b

arguments $I(1,\underbrace{0,\dots,0}_{\nu-1}) \equiv I(1)I(0)^{\nu-1}$, $\nu = 1,\dots,b$ are taken into account by the multi-index sums in (6.10). The summation indices (6.12) take only two different values:

$$\beta_{\nu,i} \in \{0,1\}\ , \quad \nu = 1,\dots,b\ , \quad i = 2,\dots,d\ . \tag{6.17}$$

The case $\beta_{\nu,i} = 1$ corresponds to $I(1)I(0)^{\nu-1}$ contributing[1] to the index n_{i-1}. On the other hand, $\beta_{\nu,i} = 0$ means that $I(1)I(0)^{\nu-1}$ does not contribute to n_{i-1}. Of course, no more (and no less) than one argument of the type $I(1,0,\dots,0)$ can contribute to a single index n_i, therefore (6.10) uses the conditions $\beta_{1,i} + \ldots + \beta_{b,i} = 1$ for all $i = 1,\dots,d$ or in the multi-index notation: $\vec{\beta}_1 + \ldots + \vec{\beta}_b = \vec{1}$. Similar to the conditions for $|\vec{\alpha}_\mu|$ in (6.11), the conditions for $|\vec{\beta}_\nu|$ are necessary due to the fixed total number j'_ν of arguments $I(1)I(0)^{\nu-1}$ in the generalized operator product. The first index n_1 provides again an exceptional case. Since there is no previous index to which any argument could contribute, we set $\beta_{\nu,1} = 0$, $\nu = 1,\dots,b$.

Finally, we can analyse how permutations between arguments of type $I(0,\dots,0)$ and those between arguments of type $I(1,0,\dots,0)$ affect each other. For every configuration of the arguments of the type $I(1,0,\dots,0)$, i.e. for every set of indices $\beta_{\nu,i}$, we need to count the identical permutations of the arguments of the type $I(0,\dots,0)$ via ω_a. That is why ω_a appears in (6.10). However, this combination of the two types of permutations is not just a product, since they are not independent. In other words, it is not possible to obtain identical products out of permutations of two operators $I(1)I(0)^{\nu_1}$ and $I(1)I(0)^{\nu_2}$ with $\nu_1 \neq \nu_2$, without changing the positions of operators of the type $I(0,\dots,0)$ as well. This effect of the sums over (6.12) on the sums over (6.13) is described by the first argument of ω_a in (6.10). It contains the contributions $-\vec{\beta}_2 - 2\vec{\beta}_3 - \ldots - (b-1)\vec{\beta}_b$. As a result, after inserting (6.10) and (6.11) in (6.8), we get the conditions

$$\alpha_{1,i} + 2\alpha_{2,i} + \ldots + a\alpha_{a,i} = n'_i - 1 - \beta_{2,i} - 2\beta_{3,i} - \ldots - (b-1)\beta_{b,i}\ , \quad i = 1,\dots,d \tag{6.18}$$

for the sums over the indices (6.13). This can be explained as follows. Recall that the l.h.s. represents the number of operators $I(0)$, which are relevant for permutations of arguments $I(0,\dots,0)$. In general this does not equal $n'_i - 1$ as suggested by (6.16) and the discussions in that paragraph. It depends on which of the arguments $I(1,0,\dots,0)$ contributes to the previous index n_{i-1}. E.g. there are only $n_i - 2$ relevant operators in case $I(1,0)$ contributes to n_{i-1} ($i > 1$), since the first $I(0)$ in (6.14) comes from the argument $I(1,0)$ and is therefore fixed. This case is represented by $\beta_{2,i} = 1$, which indeed gives $n_i - 2$ on the r.h.s. of (6.18). In general the number of relevant operators $I(0)$ is $n_i - \nu$ with $I(1)I(0)^{\nu-1}$ contributing to n_{i-1}. In accordance with eq. (6.18), this is represented by $\beta_{\nu,i} = 1$ for $i > 1$.

The function (6.11) has a remarkable property. Dropping the conditions for $|\vec{\alpha}_\mu|$, which is equivalent to summing over all sets $\vec{j} = (j_1, j_2, \dots, j_a)$, gives

$$\sum_{\vec{j}} \omega_a(\vec{n}; j_1, j_2, \dots, j_a) = F^{(a)}_{n_1+1} F^{(a)}_{n_2+1} \ldots F^{(a)}_{n_d+1}\ , \tag{6.19}$$

[1] By saying "n_i originates from c_j" or "c_j contributes to n_i" it is meant, that $I(1)$ in the integral operator representation (6.14) of the index n_i is part of the operator c_j.

with the generalized Fibonacci numbers:

$$F_n^{(k)} = \sum_{\alpha=1}^{k} F_{n-\alpha}^{(k)} \ , \quad F_1^{(k)} = F_2^{(k)} = 1 \ , \quad F_{n\leq 0}^{(k)} = 0 \ . \tag{6.20}$$

The weighting (6.10) depends on the indices of the MZVs, but not on their order except for n_1. Furthermore, $\omega_{a,1}(\vec{n}; j_1, j_2, \ldots, j_a; j_1')$ is equivalent to $\omega_a(\vec{n}; j_1, j_2, \ldots, j_a)$ as long as both functions are used as weightings in identical sums of MZVs. A few special cases of the generalized operator product in identity (6.8) are discussed in the following.

Example 2.1: Identity (6.8) with $(a, b) = (2, 3)$ is relevant for the generalized hypergeometric function ${}_3F_2$:

$$I(0)\{I(0)^{j_1}, I(0,0)^{j_2}, I(1)^{j_3}, I(1,0)^{j_4}, I(1,0,0)^{j_5}\}I(1)\big|_{z=1} = \sum_{\substack{w=j_1+2j_2+j_3+2j_4+3j_5+2 \\ d=j_3+j_4+j_5+1}} \zeta(\vec{n})\ \omega_{2,3}(\vec{n}' - \vec{1}; j_2, j_4, j_5) \ , \tag{6.21}$$

with

$$\omega_{2,3}(\vec{n}; j_x, j_y, j_z) = \sum_{\substack{\vec{\beta}+\vec{\gamma}\leq\vec{1};\ \beta_1=\gamma_1=0 \\ |\vec{\beta}|=j_y;\ |\vec{\gamma}|=j_z}} \omega_2(\vec{n} - \vec{\beta} - 2\vec{\gamma}; j_x) \ , \tag{6.22}$$

and

$$\omega_2(\vec{n}; j) = \sum_{\substack{\vec{\alpha}\leq\lfloor\vec{n}/2\rfloor \\ |\vec{\alpha}|=j}} \binom{\vec{n} - \vec{\alpha}}{\vec{\alpha}} \ . \tag{6.23}$$

Eq. (6.19) gives the relation to the Fibonacci numbers $F_n \equiv F_n^{(2)}$:

$$\sum_j \omega_2(\vec{n}; j) = F_{n_1+1} F_{n_2+1} \ldots F_{n_d+1} \ . \tag{6.24}$$

Alternatives to the multi-index sum representation (6.8) are possible for generalized operator products with $a = 1$, i.e. cases where the weighting needs only to consider permutations of operators $I(1, 0, \ldots, 0)$. As a result the weighting depends on d_j rather than on n_i.

Example 2.2: The simplest nontrivial case is:

$$I(0)\{I(0)^{j_1}, I(1)^{j_2}, I(1,0)^{j_3}\}I(1)\big|_{z=1} = \sum_{\substack{w=j_1+j_2+2j_3+2 \\ d=j_2+j_3+1}} \zeta(\vec{n}) \binom{d - 1 - d_1}{j_3} \ . \tag{6.25}$$

In contrast to $\omega_{1,2}$, the weighting is written as a binomial coefficient without any sums. It can be understood as the number of ways to distribute the operators $I(1,0)$ among the indices $\vec{n}$. This explains the lower line of the binomial coefficient, since the number of operators $I(1,0)$ is j_3. The upper line of the binomial coefficient represents the number of indices n_i, to which the third argument $I(1,0)$ can contribute. This is the depth d minus one, because the $I(1)$ of n_d is fixed. Furthermore, one has to subtract d_1, since the third argument $I(1,0)$ cannot contribute to n_i, when $n_{i+1} = 1$.

For example $(j_1, j_2, j_3) = (1,1,1)$ gives $w = 6$ and $d = 3$. One MZV with these properties is $\zeta(3,2,1)$. Since the $I(1)$ of n_3 is fixed, the third argument $I(1,0)$ can only come with n_1 and n_2. The latter is not possible, since otherwise the sequence for n_3 would start with an $I(0)$ and therefore $n_3 \geq 2$, which contradicts $n_3 = 1$. So there is only one way to obtain this MZV with the given arguments: $I(0,0,(1,0),1,1)$. Parentheses are included to indicate the position of the third argument. This is in accordance with the weighting in (6.25): $\binom{3-1-1}{1} = 1$. The same holds for all other MZVs of weight $w = 6$ and depth $d = 3$, which have $d_1 = 1$: $\zeta(3,1,2)$, $\zeta(2,3,1)$ and $\zeta(2,1,3)$. There is one MZV with $d_1 = 2$, namely $\zeta(4,1,1)$. The corresponding weighting is zero, since the third argument cannot contribute to any n_i. The last MZV to consider for the given weight and depth is $\zeta(2,2,2)$ with $d_1 = 0$. This one appears twice, since $I(1,0)$ can contribute both to n_1 and n_2: $I(0,1,0,(1,0),1) + I(0,(1,0),1,0,1)$. It is easy to check that this example gives indeed

$$I(0)\{I(0), I(1), I(1,0)\}I(1)\big|_{z=1} \\ = \zeta(3,2,1) + \zeta(3,1,2) + \zeta(2,3,1) + \zeta(2,1,3) + 2\zeta(2,2,2) \;, \quad (6.26)$$

in agreement with (6.25).

Example 2.3: A similar relation applies to the case with $I(1,0,0)$ instead of $I(1,0)$:

$$I(0)\{I(0)^{j_1}, I(1)^{j_2}, I(1,0,0)^{j_3}\}I(1)\big|_{z=1} = \sum_{\substack{w=j_1+j_2+3j_3+2 \\ d=j_2+j_3+1}} \zeta(\vec{n}) \binom{d-1-d_1-\bar{d}_2}{j_3} \,. \quad (6.27)$$

The third argument $I(1,0,0)$ contributing to n_i implies $n_{i+1} \geq 3$. Therefore, the number of indices n_i, which can originate from $I(1,0,0)$ (corresponding to the upper line of the binomial coefficient) is the total number d minus one due to n_d. In addition, the term $(d_1 + \bar{d}_2)$, representing the number of integers with $n_{i+1} < 3$, has to be subtracted. The first integer n_1 has to be excluded from these considerations, simply because there is no preceding integer to which $I(1,0,0)$ could contribute. So $\bar{d}_2$, which is the number of indices n_i which equal 2, does not count $n_1 = 2$: $\bar{d}_2 = d_2 - \delta_{2,n_1}$. This distinction is not necessary for d_1, as $n_1 \neq 1$.

Example 2.4: The following relation includes three arguments of the kind $I(1,0,\ldots,0)$:

$$I(0)\{I(0)^{j_1}, I(1)^{j_2}, I(1,0)^{j_3}, I(1,0,0)^{j_4}\}I(1)\big|_{z=1}$$
$$= \sum_{\substack{w=j_1+j_2+2j_3+3j_4+2 \\ d=j_2+j_3+j_4+1}} \zeta(\vec{n}) \binom{d-1-d_1-j_4}{j_3} \binom{d-1-d_1-\bar{d}_2}{j_4} \ . \tag{6.28}$$

The weighting has a similar explanation as the ones in (6.25) and (6.27). The first binomial coefficient counts all identical terms, which follow from the distribution of the third argument. The second binomial coefficient represents the same for the fourth argument. It is easy to check that the cases $j_3 = 0$ and $j_4 = 0$ reproduce (6.27) and (6.25), respectively.

Identities (6.25), (6.27) and (6.28) give more compact weightings than the ones following from (6.8). It is possible to derive these binomial coefficients from the multi-index sums, but it is not obvious how to achieve this. Also, note that $j_1 = 0$ in (6.25) and (6.28) is in accordance with eqs. (6.3) and (6.4), respectively.

6.1.3 Identities with sums

All identities discussed so far are sufficient to write all generalized operator products, which appear in chapters 3 and 5, in terms of MPLs or MZVs.

Example 3.1: For instance, by using eq. (6.25) it is possible to close the gap in the calculation of the hypergeometric function ${}_2F_1$ in section 3.3.1:

$$\begin{aligned}
\sum_\alpha (-1)^\alpha \, I(0)\{I(0)^{j_1-\alpha}, I(1)^{j_2-\alpha}, I(1,0)^\alpha\}I(1)\big|_{z=1} &= \sum_\alpha (-1)^\alpha \sum_{\substack{w=j_1+j_2+2 \\ d=j_2+1}} \zeta(\vec{n}) \binom{j_2-d_1}{\alpha} \\
&= \sum_{\substack{w=j_1+j_2+2 \\ d=j_2+1}} \zeta(\vec{n}) \sum_\alpha (-1)^\alpha \binom{j_2-d_1}{\alpha} \\
&= \sum_{\substack{w=j_1+j_2+2 \\ d=j_2+1}} \zeta(\vec{n}) \, \delta_{j_2,d_1} \\
&= \zeta(j_1+2, \{1\}^{j_2}) \ .
\end{aligned} \tag{6.29}$$

The sum in the last step disappears, since there is only one set of indices $\vec{n}$ with weight $w = j_1 + j_2 + 2$, depth $d = j_2 + 1$ and j_2 times the index 1. In this simple case it is possible to combine the outer sum over α with the sum of MZVs to obtain a simple expression. However, in (5.21) there are summations over 17 indices (5.22) and performing the evaluation in the same way gives a rather complicated weighting.

Example 3.2: Summations over two indices are already problematic:

$$\sum_{\alpha_1,\alpha_2} (-1)^{\alpha_1+\alpha_2} I(0)\{I(0)^{j_1-\alpha_1-\alpha_2}, I(0)^{j_2-\alpha_1}, I(0,0)^{\alpha_1}, \, I(1)^{j_3-\alpha_2}, I(1,0)^{\alpha_2}\}I(1)\big|_{z=1}$$

$$
\begin{aligned}
&= \sum_{\alpha_1,\alpha_2} (-1)^{\alpha_1+\alpha_2} \binom{j_1+j_2-2\alpha_1-\alpha_2}{j_2-\alpha_1} \\
&\times I(0)\{I(0)^{j_1+j_2-2\alpha_1-\alpha_2}, I(0,0)^{\alpha_1}, I(1)^{j_3-\alpha_2}, I(1,0)^{\alpha_2}\}I(1)\big|_{z=1} \\
&= \sum_{\substack{w=j_1+j_2+j_3+2\\ d=j_3+1}} \zeta(\vec{n}) \sum_{\alpha_1,\alpha_2} (-1)^{\alpha_1+\alpha_2} \binom{j_1+j_2-2\alpha_1-\alpha_2}{j_2-\alpha_1} \\
&\times \omega_{2,2}(\vec{n}'-\vec{1}; j_1+j_2-2\alpha_1-\alpha_2, \alpha_1; j_3-\alpha_2, \alpha_2) \ . \qquad (6.30)
\end{aligned}
$$

In the first step (2.19) is used to combine the identical arguments. Applying identity (6.8) in the next step leads to the given weighting. There is no obvious way to simplify this expression. The summations in the first line have a form similar to the one on the l.h.s. of the first line of (6.29). Hence, the question arises, whether they can be evaluated as well. The formalism, discussed in the following, yields indeed a much simpler expression for (6.30).

The general form of the expressions, which are discussed in this subsection, is

$$
\sum_{\alpha} (-1)^{\alpha} \{c_1^{j_1-\alpha}, c_2^{j_2-\alpha}, (c_2c_1)^{\alpha}, \ldots\} \ , \qquad (6.31)
$$

i.e. one argument is a product of two others and their indices share the same summation index α. The dots represent additional arguments. Their indices may depend on other summation indices but not on α. Obviously, all operator products in (6.31) contain the same number of c_1's and c_2's independent of α. Thus, identical products arise, not only from single generalized operator products due to dependent arguments, but also from products with different α. The goal is to combine all the identical terms. Due to the factor $(-1)^{\alpha}$, many of these terms cancel, which leads to simplifications in both the generalized operator product and the summation regions. Eventually, it is possible to obtain for (6.31) a compact representation in terms of MZVs, not only for the generalized operator product, but for the complete expression including the sum.

Obviously, products arising from the generalized product of (6.31) can only be identical if they contain the same number m of sequences c_2c_1. Denoting the sum of all terms, which include m times the sequence c_2c_1, by s_m, the $\alpha = 0$ operator product of (6.31) can be written as:

$$
\{c_1^{j_1}, c_2^{j_2}, \ldots\} = \sum_{m\geq 0} s_m \ . \qquad (6.32)
$$

The sum is over all possible numbers of sequences c_2c_1. The next term ($\alpha = 1$) gives:

$$
\{c_1^{j_1-1}, c_2^{j_2-1}, c_2c_1, \ldots\} = \sum_{m\geq 1} m \ s_m \ . \qquad (6.33)
$$

Here the sum starts with $m = 1$, since there is at least one sequence c_2c_1 coming from the third argument. In addition, the summands are weighted by m. The reason is, that some terms appear more than once: the sequences c_2c_1, which come from the first two arguments, can be exchanged with the ones coming from the third argument without changing the product. E.g. for $m = 2$ there are terms of the form $(\ldots c_2c_1 \ldots (c_2c_1)\ldots)$, where the inner brackets indicate, that the second sequence comes from the third argument. To all of these products, there is one identical term: $(\ldots(c_2c_1)\ldots c_2c_1\ldots)$. This explains the weight 2 for the case $m = 2$ in (6.33). For general α and m there are α sequences c_2c_1 coming from the third argument, while the remaining $m-\alpha$ sequences c_2c_1 originate from the first two arguments. This explains the binomial coefficient in the general relation:

$$\{c_1^{j_1-\alpha}, c_2^{j_2-\alpha}, (c_2c_1)^\alpha, \ldots\} = \sum_{m\geq\alpha} s_m \binom{m}{\alpha} . \tag{6.34}$$

Inserting this expression in (6.31) yields:

$$\begin{aligned}\sum_\alpha (-1)^\alpha \{c_1^{j_1-\alpha}, c_2^{j_2-\alpha}, (c_2c_1)^\alpha, \ldots\} &= \sum_{m\geq 0} s_m \sum_{\alpha=0}^{m} (-1)^\alpha \binom{m}{\alpha} \\ &= \sum_{m\geq 0} s_m \delta_{m,0} = s_0 .\end{aligned} \tag{6.35}$$

Only s_0 is left. This is the sum of all products, which do not include the sequence c_2c_1. Thus, summations of the form (6.31) can be interpreted as restrictions for the sequences of operators, which appear in the non-commutative products.

For expressions with more sums of the form (6.31) eq. (6.35) has to be applied to each of these individually. It appears that the indices of some arguments include more than one summation index. A compact representation is possible, when all summation indices appear in the first entry of the generalized operator product. In other words, all composed arguments contain c_1:

$$\begin{aligned}&\sum_{\vec\alpha} (-1)^{|\vec\alpha|} \{c_1^{j_1-|\vec\alpha|}, c_2^{j_2-\alpha_1}, (c_2c_1)^{\alpha_1}, c_3^{j_3-\alpha_2}, (c_3c_1)^{\alpha_2}, \ldots, c_n^{j_n-\alpha_{n-1}}, (c_nc_1)^{\alpha_{n-1}}\} \\ &\qquad = c_1^{j_1}\{c_2^{j_2}, c_3^{j_3}, \ldots, c_n^{j_n}\} .\end{aligned} \tag{6.36}$$

Applying (6.35) to each summation allows to identify the forbidden sequences c_2c_1, c_3c_1, ..., c_nc_1, i.e. only those products remain, in which all c_1's appear to the left of all other operators c_2, c_3, ..., c_n. These are the terms on the r.h.s. of (6.36). All composed arguments and the sums over the corresponding indices are removed. The number of arguments is reduced from $2n-1$ to $n-1$.

The identity (6.36) provides an alternative to determine the example 3.1. Setting $n = 2$, $c_1 = I(0)$ and $c_2 = I(1)$ gives the relation:

$$\sum_\alpha (-1)^\alpha\, I(0)\{I(0)^{j_1-\alpha}, I(1)^{j_2-\alpha}, I(1,0)^\alpha\}I(1)\big|_{z=1} = I(0)^{j_1+1}I(1)^{j_2+1} . \tag{6.37}$$

This matches eq. (6.29). The $n = 3$ version of (6.36), with $c_1 = I(0)$, $c_2 = I(0)$ and $c_3 = I(1)$, can be used to evaluate the sums over α_1 and α_2 in example 3.2. What remains is:

$$I(0)^{j_1+1}\{I(0)^{j_2}, I(1)^{j_3}\}I(1)\big|_{z=1} = \sum_{\substack{w=j_1+j_2+j_3+2 \\ d=j_3+1;\; n_1\geq j_1+2}} \zeta(\vec{n}) \ . \tag{6.38}$$

After the sums are removed, the generalized operator product can be written easily in terms of MZVs by using (6.1). Instead of the complicated weighting in (6.30), there is only the additional condition for n_1 in the sum of MZVs in (6.38).

The formalism, used to obtain the important relations (6.35) and (6.36), can be generalized to cases involving general functions $f(\alpha)$ instead of $(-1)^\alpha$:

$$\sum_{\alpha\geq\alpha_0} f(\alpha)\{c_1^{j_1-\alpha}, c_2^{j_2-\alpha}, (c_2c_1)^\alpha, \ldots\} = \sum_{m\geq\alpha_0} s_m \sum_{\alpha=\alpha_0}^{m} f(\alpha)\binom{m}{\alpha} \ . \tag{6.39}$$

Furthermore, the lower bound $\vec{\alpha}_0$ of the summation is kept general. This allows to handle expressions, where the indices of composed arguments are not simply summation indices, but depend on other quantities as well. Eq. (5.21) has indeed the more general form (6.39), since it contains multinomial coefficients. However, all of them can be removed by using eq. (2.19). On the other hand, the resulting expression involves an increased number of arguments. Hence, in general one has to decide, whether to handle more complicated functions $f(\alpha)$ or to deal with a larger number of arguments in the generalized operator product. The latter turned out to be more appropriate for the relations considered in this thesis, because in this case only the factors $(-1)^\alpha$ remain, which allows to apply the advantageous relation (6.35).

The strategy to simplify (5.21) after all multinomial coefficients are removed, is to use shifts in the summation indices to bring as many of the twelve sums (after application of eqs. (5.23)) as possible into the form (6.35). This allows us to identify all forbidden sequences. However, the configuration for the generalized operator product in (5.21) is not as convenient as the one in identity (6.36). In contrast to c_1 in (6.36), there is no argument of the generalized operator product in (5.21) with an index, that includes all indices of summations of the form (6.35). Furthermore, there are arguments in (5.21), whose indices do not depend on indices of summations of the form (6.35) at all. As a consequence of these two issue, we are not able to present (5.21) in terms of a simplified generalized operator product, as it happens in identity (6.36). Instead, the corresponding forbidden sequences are used to write generalized operator products directly in terms of MZVs. This is achieved in a similar manner as for the relations in section 6.1.2. First all permutations of integral operators, which consist only of $I(0)$, are counted. Then it is analysed how the contribution of operators, which include $I(1)$, affects the weighting. Thus, inner multiple index sums are related to the first step and outer ones to the second step. This is demonstrated on some examples in the following. We start with generalized operator products with only a few arguments and minor deviations from the form in (6.36), to ultimately present an identity, which can be applied to (5.21).

Example 3.3: A simple case to start with is

$$\sum_{\alpha}(-1)^{\alpha}\, I(0)\{I(0)^{j_1-\alpha}, I(0)^{j_2-\alpha}, I(0,0)^{\alpha}, I(1,0)^{j_3}\}I(1)\Big|_{z=1}$$
$$= \sum_{\substack{w=2+j_1+j_2+2j_3\\ d=j_3+1}} \zeta(\vec{n})\;\omega_2'(\vec{n}-\vec{2};j_1)\;, \quad (6.40)$$

with the weighting:

$$\omega_2'(\vec{n};j) = \sum_{\substack{\vec{\alpha}\le\vec{n}\\ |\vec{\alpha}|=j}} 1\;. \quad (6.41)$$

The arguments[2], which can contribute to a sequence of operators $I(0)$, are c_1 and c_2. The sum over α removes all products with the sequence c_2c_1. Therefore, there is only one way to arrange them: all c_1 to the left of all c_2, resulting in the factor of 1 in the multiple index sum (6.41). The sequence of n_i-1 operators $I(0)$, related to n_i, starts with one $I(0)$ stemming either from c_3 for $i>1$ or from the $I(0)$ to the left of the generalized operator product for $i=1$. As a consequence, there can be up to n_i-2 arguments c_1 contributing to n_i. This explains the range of the sum in (6.41), when the arguments of ω_2' given in identity (6.40) are inserted. The additional condition for $|\vec{\alpha}|$ takes the fixed number of factors c_1 into account. Since the argument c_3 is independent of all others, it is irrelevant for the weighting.

Example 3.4: The following relation includes two arguments, which are not related to summations:

$$\sum_{\alpha}(-1)^{\alpha}\, I(0)\{I(0)^{j_1-\alpha}, I(0)^{j_2-\alpha}, I(0,0)^{\alpha}, I(1)^{j_3}, I(1,0)^{j_4}\}I(1)\Big|_{z=1}$$
$$= \sum_{\substack{w=2+j_1+j_2+j_3+2j_4\\ d=j_3+j_4+1}} \zeta(\vec{n})\;\omega_{2a}'(\vec{n}-\vec{2};j_1,j_3)\;, \quad (6.42)$$

with

$$\omega_{2a}'(\vec{n};j_x,j_y) = \sum_{\substack{\vec{\beta}\le\vec{1}\\ |\vec{\beta}|=j_y;\ \beta_1=0}} \omega_2'(\vec{n}+\vec{\beta};j_x)\;. \quad (6.43)$$

The generalized operator product contains the same operators of the type $I(0,\ldots,0)$ as example 3.3. This is why the inner sum uses ω_2'. The only difference is the range. It can be either n_i-1 or n_i-2. This depends on wether c_3 or c_4 contribute to n_{i-1} $(i>1)$. Similar

[2]Some of the arguments in the generalized operator products of this and the following examples are identical. In order to avoid confusion, the argument with the index j_i is referred to as c_i.

to the identity (6.8), this is taken into account by the outer sum over the multi-index $\vec{\beta} = (\beta_1, \ldots, \beta_d)$. Identical terms, which follow from the exchange of arguments c_3 and c_4 are counted this way. For $\beta_i = 1$ the argument c_3 contributes to n_{i-1}, while for $\beta_i = 0$ c_4 does. The condition for $|\vec{\beta}|$ is due to the fixed number j_3 of arguments c_3. Again, n_1 is not affected by these discussions, therefore $\beta_1 = 0$.

Example 3.5: Next, there are two sums of the form (6.35):

$$\sum_{\alpha_1,\alpha_2} (-1)^{\alpha_1+\alpha_2} I(0)\{I(0)^{j_1-\alpha_1-\alpha_2}, I(0)^{j_2-\alpha_1}, I(0,0)^{\alpha_1}, I(0)^{j_3-\alpha_2}, I(0,0)^{\alpha_2},$$
$$I(1,0)^{j_4}\}I(1)\big|_{z=1} = \sum_{\substack{w=2+j_1+j_2+j_3+2j_4\\ d=j_4+1;\ n_i\geq 2}} \zeta(\vec{n})\, \omega_3'(\vec{n}-\vec{2}; j_2, j_3)\ , \quad (6.44)$$

with

$$\omega_3'(\vec{n}; j_x, j_y) = \sum_{\substack{\vec{\alpha}+\vec{\beta}\leq\vec{n}\\ |\vec{\alpha}|=j_x;\ |\vec{\beta}|=j_y}} \binom{\vec{\alpha}+\vec{\beta}}{\vec{\beta}}\ . \quad (6.45)$$

The relevant operators of the type $I(0,\ldots,0)$ are c_1, c_2 and c_3. The forbidden sequences are c_2c_1 and c_3c_1. So all c_1's have to appear to the left of all the c_2's and c_3's, which are contributing to the same n_i. The positions of c_2 and c_3 are not completely fixed, since they may be permuted. With α_i and β_i being the number of factors c_2 and c_3, respectively, the number of permutations are of course $\binom{\alpha_i+\beta_i}{\alpha_i}$. The range of the sum is identical to the one in example 3.3, since the operators of the type $I(1,0,\ldots,0)$ are the same. The additional conditions in the sum in (6.45) are self-explanatory.

Example 3.6: This example includes three sums:

$$\sum_{\alpha_1,\alpha_2,\alpha_3} (-1)^{\alpha_1+\alpha_2+\alpha_3} I(0)\{I(0)^{j_1-\alpha_1-\alpha_2}, I(0)^{j_2-\alpha_1-\alpha_3}, I(0,0)^{\alpha_1}, I(0)^{j_3-\alpha_2}, I(0,0)^{\alpha_2},$$
$$I(1)^{j_4-\alpha_3}, I(1,0)^{\alpha_3}, I(1,0)^{j_5}\}I(1)\big|_{z=1} = \sum_{\substack{w=j_1+j_2+j_3+j_4+2j_5+2\\ d=j_4+j_5+1}} \zeta(\vec{n})\, \omega_4'\left(\vec{n}-\vec{2};\ \begin{matrix} j_4 \\ j_2, j_3\end{matrix}\right)\ , \quad (6.46)$$

with

$$\omega_4'\left(\vec{n};\ \begin{matrix} j_x \\ j_a, j_b\end{matrix}\right) = \sum_{\substack{\vec{\mu}\leq\vec{1}\\ |\vec{\mu}|=j_x;\ \mu_1=0}} \sum_{\substack{\vec{\alpha}+\vec{\beta}\leq\vec{n}+\vec{\mu}\\ |\vec{\alpha}|=j_a;\ |\vec{\beta}|=j_b}} \binom{\vec{\alpha}+\vec{\beta}+\vec{\mu}(\delta_{\vec{\alpha},0}-1)}{\vec{\alpha}}\ . \quad (6.47)$$

Forbidden products are c_1c_2, c_1c_3 and c_4c_2. Therefore, from all arguments contributing to the same sequence of integral operators $I(0)$, the c_1's appear to the right of all c_2's and c_3's. With α_i and β_i being the numbers of c_2's and c_3's, respectively, there are

$$\binom{\alpha_i+\beta_i}{\alpha_i} \tag{6.48}$$

possibilities to arrange given numbers of c_1's, c_2's and c_3's without the sequences c_1c_2 or c_1c_3. In case c_5 appears to the left of these operators, the coefficient (6.48) stays the same. But for c_4 the third forbidden product c_4c_2 has to be respected. For $\alpha_i > 0$ there are

$$\binom{\alpha_i+\beta_i-1}{\alpha_i} \tag{6.49}$$

possibilities, while for $\alpha_1 = 0$ the coefficient remains as in (6.48). Thus, for all α_i there are

$$\binom{\alpha_i+\beta_i+\delta_{\alpha_i,0}-1}{\alpha_i} \tag{6.50}$$

permutations of given numbers of c_1's, c_2's and c_3's with c_4 appearing to the left and without the sequences c_1c_2, c_1c_3 and c_4c_2. Through the sums over the multi-index $\vec{\mu} = (\mu_1, \ldots, \mu_d)$ both cases are taken into account: $\mu_i = 1$ represents c_4 contributing to n_{i-1} $(i > 1)$, which yields the coefficient (6.50) in (6.47). On the other hand, $\mu_i = 0$ represents c_5 contributing to n_{i-1} $(i > 1)$, which gives the coefficient (6.48). Also the range of the inner sums depends on wether c_4 or c_5 contribute to n_{i-1}. It is $n_i - 1$ for the former and $n_i - 2$ for the latter $(i > 1)$. The additional conditions for the sums in (6.47) are there for the same reasons as the ones in the previous examples.

Example 3.7: The following expression involves sums over six indices $\vec{\alpha} = (\alpha_1, \ldots, \alpha_6)$:

$$\begin{aligned}
&\sum_{\vec{\alpha}}(-1)^{|\vec{\alpha}|}I(0)\{I(0)^{j_1-\alpha_1-\alpha_2-\alpha_5}, I(0)^{j_2-\alpha_3-\alpha_4}, I(0)^{j_3-\alpha_1-\alpha_3}, I(0)^{j_4-\alpha_2-\alpha_4-\alpha_6},\\
&I(0,0)^{\alpha_1}, I(0,0)^{\alpha_2}, I(0,0)^{\alpha_3}, I(0,0)^{\alpha_4}, I(1,0)^{j_5-\alpha_5}, I(1,0,0)^{\alpha_5}, I(1,0)^{j_6-\alpha_6},\\
&I(1,0,0)^{\alpha_6}, I(1,0)^{j_7}\}I(1)\big|_{z=1}\\
=&\sum_{\substack{w=2+j_1+j_2+j_3+j_4+2j_5+2j_6+2j_7\\ d=1+j_5+j_6+j_7}} \zeta(\vec{n})\, \omega_5'\left(\vec{n}-\vec{2};\, \begin{matrix} j_5, j_6 \\ j_1, j_2, j_3, j_4 \end{matrix}\right)\ ,
\end{aligned} \tag{6.51}$$

with the weighting

$$\begin{aligned}
&\omega_5'\left(\vec{n};\, \begin{matrix} j_x, j_y \\ j_a, j_b, j_c, j_d \end{matrix}\right)\\
&\quad = \sum_{\substack{\vec{\mu}+\vec{\nu}\leq\vec{1}\\ |\vec{\mu}|=j_x;\ |\vec{\nu}|=j_y\\ \alpha_1,\beta_1=0}} \sum_{\substack{\vec{\alpha}+\vec{\beta}+\vec{\gamma}+\vec{\delta}=\vec{n}\\ |\vec{\alpha}|=j_a;\ |\vec{\beta}|=j_b\\ |\vec{\gamma}|=j_c;\ |\vec{\delta}|=j_d}} \binom{\vec{\alpha}+\vec{\beta}+\vec{\mu}(\delta_{\vec{\alpha},0}-1)}{\vec{\alpha}}\binom{\vec{\gamma}+\vec{\delta}+\vec{\nu}(\delta_{\vec{\delta},0}-1)}{\vec{\delta}}\ .
\end{aligned} \tag{6.52}$$

The forbidden sequences related to the six summations are c_3c_1, c_4c_1, c_3c_2, c_4c_2, c_5c_4 and c_6c_1. The first four sequences exclusively affect the operators $I(0)$ and are therefore relevant for the inner sums. A sequence of operators $I(0)$, consisting of the arguments c_1, c_2, c_3 and c_4 without the forbidden sequences, has to start with all c_1 and all c_2. All permutations of these two factors are allowed. Also all permutations of c_3 and c_4 are allowed. Hence, there are

$$\binom{\alpha_i + \beta_i}{\alpha_i} \binom{\gamma_i + \delta_i}{\delta_i} \tag{6.53}$$

possibilities to build this sequence with α_i, β_i, γ_i and δ_i being the numbers of factors c_1, c_2, c_3 and c_4, respectively. Some of the forbidden sequences interfere with each other: c_5c_4 with c_4c_1 and c_4c_2. This means, that products, in which these sequences are combined ($c_5c_4c_1$ and $c_5c_4c_2$), do not just vanish but they appear with a negative sign. In other words, an expression, which simply ignores the sequences c_5c_4, c_4c_1 and c_4c_2, involves too many products. An elegant way to solve this problem, is to introduce the additional forbidden sequences c_1c_4 and c_2c_4, which have to be considered if and only if c_5 contributes to n_i. This is possible, because $c_4 = c_1 = c_2$.

The freedom of using either c_2c_1 or c_1c_2 as forbidden sequence, when $c_2 = c_1$, is used in the previous examples to avoid interfering forbidden sequences. However, this manipulation is not possible for the example under consideration.

The coefficient within the inner sums of (6.52) depends on which argument of the type $I(1, 0, \ldots, 0)$ contributes to n_{i-1}: c_5, c_6 or c_7. The upper bound of the inner sums is $n_i - 2$ in all three cases. For c_7 the coefficient is identical to (6.53). For c_6 those contributions have to be subtracted, which start with c_1. For $\alpha_i > 0$ the first binomial coefficient in (6.53) changes to (6.49). Using Kronecker deltas, this can be written for all α_i as (6.50). For c_5 the second binomial coefficient in (6.53) has to be modified. In case $\delta_i > 0$ there are

$$\binom{\gamma_i - 1 + \delta_i}{\delta_i} \tag{6.54}$$

possible permutations of c_3 and c_4. Therefore, for all δ_i the second binomial coefficient becomes:

$$\binom{\gamma_i + \delta_{\delta_i,0} - 1 + \delta_i}{\delta_i} . \tag{6.55}$$

All these cases are respected by the summations over the d-dimensional multi-indices $\vec{\mu}$ and $\vec{\nu}$. For $\mu_i = \nu_i = 0$ the binomial coefficients remain as in (6.53), so this represents the contribution of c_7 to n_{i-1}. The first binomial coefficient changes to (6.50) for $\nu_i = 1$, while the second becomes (6.55) for $\mu_i = 1$, thus representing the contribution of c_6 and c_5, respectively.

Note that the introduction of the additional forbidden sequences leads to the symmetric form of the binomial coefficients in (6.52), which is in agreement with the symmetry of the corresponding generalized operator product. Of course, one of the multiple indices $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ and $\vec{\delta}$ can be removed, e.g. to obtain the summation region $\vec{\alpha}+\vec{\beta}+\vec{\gamma}\leq\vec{n}$ for the inner sum. This would, however, destroy the symmetric form of the binomial coefficients.

Example 3.8: This is the most general relation, which includes the previous examples 3.3–3.7 as special cases:

$$\begin{aligned}
&\sum_{\vec{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_9)} (-1)^{|\vec{\alpha}|} I(0)\{I(0)^{j_1-\alpha_1-\alpha_2-\alpha_5-\alpha_7-\alpha_8}, I(0)^{j_2-\alpha_3-\alpha_4},\\
&\qquad I(0)^{j_3-\alpha_1-\alpha_3}, I(0)^{j_4-\alpha_2-\alpha_4-\alpha_6-\alpha_9}, I(0,0)^{\alpha_1}, I(0,0)^{\alpha_2}, I(0,0)^{\alpha_3}, I(0,0)^{\alpha_4},\\
&\qquad I(1,0)^{j_5-\alpha_5}, I(1,0,0)^{\alpha_5}, I(1,0)^{j_6-\alpha_6}, I(1,0,0)^{\alpha_6}, I(1)^{j_7-\alpha_7-\alpha_9}, I(1,0)^{\alpha_7},\\
&\qquad I(1,0)^{\alpha_9-\alpha_8}, I(1,0,0)^{\alpha_8}, I(1,0)^{j_8}\} I(1)\Big|_{z=1}\\
&\quad = \sum_{\substack{w=j_1+\ldots+j_5+2j_6+2j_7+2j_8+2\\ d=j_5+j_6+j_7+j_8+1}} \omega_6'\left(\vec{n}-\vec{2};\, \begin{matrix} j_5, j_6, j_7\\ j_1, j_2, j_3, j_4\end{matrix}\right)\ ,
\end{aligned} \tag{6.56}$$

with the weighting

$$\begin{aligned}
\omega_6'\left(\vec{n};\, \begin{matrix} j_x, j_y, j_z\\ j_a, j_b, j_c, j_d\end{matrix}\right) = &\sum_{\substack{\vec{\mu}+\vec{\nu}+\vec{\sigma}\leq\vec{1}\\ |\vec{\mu}|=j_x;\ |\vec{\nu}|=j_y;\ |\vec{\sigma}|=j_z\\ \mu_1,\nu_1,\sigma_1=0}} \ \sum_{\substack{\vec{\alpha}+\vec{\beta}+\vec{\gamma}+\vec{\delta}=\vec{n}+\vec{\sigma}\\ |\vec{\alpha}|=j_a;\ |\vec{\beta}|=j_b\\ |\vec{\gamma}|=j_c;\ |\vec{\delta}|=j_d}}\\
&\times \binom{\vec{\alpha}+\vec{\beta}+(\vec{\mu}+\vec{\sigma})(\delta_{\vec{\alpha},0}-1)}{\vec{\alpha}} \binom{\vec{\gamma}+\vec{\delta}+(\vec{\nu}+\vec{\sigma})(\delta_{\vec{\delta},0}-1)}{\vec{\delta}}\ .
\end{aligned} \tag{6.57}$$

Forbidden sequences are c_3c_1, c_4c_1, c_3c_2, c_4c_2, c_5c_1, c_6c_4, c_7c_1 and c_7c_4. The arguments of the type $I(0,\ldots,0)$ are the same as in example 3.7. Hence, the number of possibilities (6.53) for a sequence of operators $I(0)$ applies here as well. There are four arguments of the type $I(1,0,\ldots,0)$: c_5, c_6, c_7 and c_8. The range of the inner sums is n_i-1 for c_7 and n_i-2 for all others. For c_8 the coefficient (6.53) is unaffected. For c_5 the first binomial coefficient changes again to (6.50) and for c_6 the second one changes again to (6.55). For c_7 both forbidden sequences including c_5 and c_6 are combined, so both binomial coefficients change to (6.50) and (6.55), respectively. All these modifications are taken into account in (6.57).

The following relations hold, when the functions on both sides are used as weightings within identical sums of MZVs. These relations provide a consistency check, because they result both from the definitions of the weightings and the corresponding generalized operator products:

$$\begin{aligned}
\omega'_6\left(\vec{n};\begin{matrix} j_x, j_y, 0 \\ j_a, j_b, j_c, j_d\end{matrix}\right) &= \omega'_5\left(\vec{n};\begin{matrix} j_x, j_y \\ j_a, j_b, j_c, j_d\end{matrix}\right) \ , \\
\omega'_6\left(\vec{n};\begin{matrix} 0, 0, j_z \\ j_a, j_b, j_c, 0\end{matrix}\right) &= \omega'_4\left(\vec{n};\begin{matrix} j_z \\ j_a, j_b\end{matrix}\right) \ , \\
\omega'_6\left(\vec{n};\begin{matrix} 0, 0, 0 \\ j_a, j_b, j_c, 0\end{matrix}\right) &= \omega'_3(\vec{n}; j_a, j_b) \ , \\
\omega'_6\left(\vec{n};\begin{matrix} 0, 0, j_z \\ 0, j_b, j_c, 0\end{matrix}\right) &= \omega'_{2a}(\vec{n}; j_b, j_z) \ , \\
\omega'_6\left(\vec{n};\begin{matrix} 0, 0, 0 \\ 0, j_b, j_c, 0\end{matrix}\right) &= \omega'_2(\vec{n}; j_b) \ .
\end{aligned} \tag{6.58}$$

Furthermore the following symmetry holds:

$$\omega'_6\left(\vec{n};\begin{matrix} j_x, j_y, j_z \\ j_a, j_b, j_c, j_d\end{matrix}\right) = \omega'_6\left(\vec{n};\begin{matrix} j_y, j_x, j_z \\ j_d, j_c, j_b, j_a\end{matrix}\right) \ . \tag{6.59}$$

6.2 Applications

Two methods to get from the expression (3.58) to (3.59) for the hypergeometric function ${}_2F_1$ have already been demonstrated in subsection 6.1.3 by using either identity (6.25) or (6.36).

Identity (6.21) allows to express the all-order expansion (3.66) of the hypergeometric function ${}_3F_2$ in terms of MPLs:

$$\begin{aligned}
v_{3,k}(z) = &\sum_{l_1+m_1+2(l_2+m_2)+3m_3=k-3} (-1)^{l_1+l_2} \Delta_{3,1}^{m_1} \Delta_{3,2}^{m_2} \Delta_{3,3}^{m_3+1} Q_{3,1}^{l_1} Q_{3,2}^{l_2} \\
&\times \sum_{\substack{w=k;\ n_1\geq 3 \\ d=m_1+m_2+m_3+1}} \mathcal{L}i_{\vec{n}}(z)\ \omega_{2,3}(\vec{n}''-\vec{1}; l_2, m_2, m_3) \ ,
\end{aligned} \tag{6.60}$$

with $\omega_{2,3}$ defined in (6.22), $n''_1 = n_1 - 2$ and $n''_i = n_i$ for $i = 2, \ldots, d$.

Eq. (6.8) leads to the representation of the coefficient function (3.70) of ${}_pF_{p-1}$ in terms of MPLs:

$$\begin{aligned}
v_{p,k}(z) = &\sum_{\vec{l},\vec{m}} (-1)^{|\vec{l}|} \Delta_{p,1}^{m_1} \Delta_{p,2}^{m_2} \ldots \Delta_{p,p-1}^{m_{p-1}} \Delta_{p,p}^{m_p+1} Q_{p,1}^{l_1} Q_{p,2}^{l_2} \ldots Q_{p,p-1}^{l_{p-1}} \\
&\times \sum_{\substack{w=k;\ n_1\geq p \\ d=m_1+m_2+\ldots+m_p+1}} \mathcal{L}i_{\vec{n}}(z)\ \omega_{p-1,p}(\vec{n}^*-\vec{1}; l_1, l_2, \ldots, l_{p-1}; m_1, m_2, m_3, \ldots, m_p) \ ,
\end{aligned} \tag{6.61}$$

with the weighting $\omega_{p-1,p}$ defined in (6.10), $n_1^* = n_1 - (p-1)$ and $n_i^* = n_i$ for $i = 2, \ldots, d$.

The coefficient function (5.16) at $z = 1$, which enters the all-order expansions (5.13) and (5.14) of the five-point open string amplitude, can be written in terms of MZVs using identity (6.21):

$$v_k(1) = \sum_{l_1+m_1+2(l_2+m_2)+3m_3=k-3} (-1)^{l_1+l_2} \Delta_1^{m_1} \Delta_2^{m_2} \Delta_3^{m_3+1} Q_1^{l_1} Q_2^{l_2} \times \sum_{\substack{w=k-1;\ n_1\geq 3 \\ d=m_1+m_2+m_3+1}} \zeta(\vec{n})\ \omega_{2,3}(\vec{n}' - \vec{1}; l_2, m_2, m_3)\ . \quad (6.62)$$

For completeness we give the MZV representation for (5.17) as well:

$$\frac{\theta}{\Delta_3} v_k(z)\Big|_{z=1} = \sum_{l_1+m_1+2(l_2+m_2)+3m_3=k-3} (-1)^{l_1+l_2} \Delta_1^{m_1} \Delta_2^{m_2} \Delta_3^{m_3} Q_1^{l_1} Q_2^{l_2} \times \sum_{\substack{w=k-1 \\ d=m_1+m_2+m_3+1}} \zeta(\vec{n})\ \omega_{2,3}(\vec{n}' - \vec{1}; l_2, m_2, m_3)\ . \quad (6.63)$$

For the alternative representation (5.18) of the five-point string amplitude we use eqs. (5.23) in (5.21) to evaluate five of the 17 sums. Some shifts in the remaining 12 summation indices allow to bring nine summations into the form (6.35). Then it is possible to apply the identity (6.56) to arrive at:

$$\begin{aligned} & v(j_1, j_2, j_3, j_4, j_5) \\ &= \sum_{\vec{\beta},\vec{\delta},\vec{\epsilon}} (-1)^{|\vec{j}|+\delta_3+|\vec{\beta}|+\delta_2+\delta_5+|\vec{\epsilon}|} I(0)\{I(0)^{j_1-\delta_1-\beta_1-\beta_2-\delta_2-\epsilon_2-\epsilon_3}, I(0)^{j_2-\delta_1-\delta_3-\beta_3-\beta_4}, \qquad (6.64) \\ & I(0)^{j_3-\delta_3-\delta_4-\beta_1-\beta_3}, I(0)^{j_4-\delta_4-\delta_5-\beta_2-\beta_4-\epsilon_1}, I(0,0)^{\beta_1}, I(0,0)^{\beta_2}, I(0,0)^{\beta_3}, I(0,0)^{\beta_4}, \\ & I(1,0)^{\delta_4-\epsilon_2}, I(1,0,0)^{\epsilon_2}, I(1,0)^{\delta_1-\epsilon_1}, I(1,0,0)^{\epsilon_1}, I(1)^{j_5-\delta_5-\delta_2}, I(1,0)^{\delta_2}, I(1,0)^{\delta_5-\epsilon_3}, \\ & I(1,0,0)^{\epsilon_3}, I(1,0)^{\delta_3}\} I(1)\Big|_{z=1} \\ &= (-1)^{|\vec{j}|} \sum_{\delta_1,\delta_3,\delta_4} (-1)^{\delta_3} \sum_{\substack{w=j_1+j_2+j_3+j_4+j_5+2 \\ d=j_5+\delta_1+\delta_3+\delta_4+1}} \zeta(\vec{n}) \\ & \times \omega_6' \left(\vec{n} - \vec{2}; \begin{matrix} \delta_4, \delta_1, j_5 \\ j_1-\delta_1, j_2-\delta_1-\delta_3, j_3-\delta_3-\delta_4, j_4-\delta_4, j_5 \end{matrix} \right) \\ &= (-1)^{|\vec{j}|} \sum_{w=j_1+j_2+j_3+j_4+j_5+2} \zeta(\vec{n}) \sum_{\delta_1,\delta_4} (-1)^{d-1-j_5-\delta_1-\delta_4} \\ & \times \omega_6' \left(\vec{n} - \vec{2}; \begin{matrix} \delta_4, \delta_1, j_5 \\ j_1-\delta_1, j_2+j_5+\delta_4-d+1, j_3+j_5+\delta_1-d+1, j_4-\delta_4 \end{matrix} \right)\ . \qquad (6.65) \end{aligned}$$

The weighting ω_6' is defined in (6.57). In the second step the three sums over δ_1, δ_3 and δ_4 are combined with the sum of MZVs. Only two of the sums over the indices (5.22) remain, even though more summations appear in ω_6'.

The transformations of the generalized operator products to sums of MPLs or MZVs in the all-order expressions for $v_{3,k}$, $v_{p,k}$, v_k and $v(j_1, j_2, j_3, j_4, j_5)$, as presented in eqs. (6.60)–(6.63) and (6.65), involve weightings, which are rather complicated, due to the variety of multiple index sums and the large number of conditions therein. On one hand, the expressions presented in this section have the advantage that they allow to pick a specific MPL or MZV, respectively, and directly determine the factor they appear with via the weighting. On the other hand, the corresponding representations (3.66), (3.70), (5.16), (5.17) and (5.21) in terms of generalized operator products provide more compact alternatives.

We showed in the previous section that there are identities for generalized operator products, which yield sums of MZV with less complicated weightings, e.g. binomial coefficients. Our objective in the following is to use such identities for limits of the function $v(j_1, j_2, j_3, j_4, j_5)$ with some of its arguments j_1, j_2, j_3, j_4, j_5 set to zero. Not only is it interesting to see how certain generalized operator products simplify significantly this way, but we are also using these results in the next section to identify particular MZV identities. These limits do not follow directly from (6.65). Going instead one step backwards to the generalized operator product in (6.64) allows to use different identities than (6.56) (in many cases eq. (6.36)). For completeness and as a consistency check we give the MZV representation for all limits. To reduce the number of limits to be calculated we use:

$$v(j_1, j_2, j_3, j_4, j_5) = v(j_4, j_3, j_2, j_1, j_5) \ . \tag{6.66}$$

This symmetry can easily be proven, using for instance (5.21) or the symmetry (6.59) of the weighting ω_6' in (6.65). Those cases, that involve weightings of similar complexity as ω_6', i.e. weighting with multiple index sums, are summarized in the appendix.

Four j_i set to zero: We start with the simplest limits, where four of the arguments of $v(j_1, j_2, j_3, j_4, j_5)$ are zero. These calculations are trivial and do not require any identities of section 6.1. No generalized operator products remain and therefore the integral operators can easily be written in terms of MZVs using (3.10). Setting $j_2 = j_3 = j_4 = j_5 = 0$ in (6.64) yields:

$$v(j_1, 0, 0, 0, 0) = (-1)^{j_1} \ I(0)^{j_1+1} I(1)\big|_{z=1} = (-1)^{j_1} \zeta(j_1 + 2) \ . \tag{6.67}$$

The case $j_1 = j_3 = j_4 = j_5 = 0$ results in the same MZV and we can then use the symmetry (6.66) to determine $v(0, 0, j_3, 0, 0)$ and $v(0, 0, 0, j_4, 0)$:

$$v(j_1, 0, 0, 0, 0) = v(0, j_1, 0, 0, 0) = v(0, 0, j_1, 0, 0) = v(0, 0, 0, j_1, 0) \ . \tag{6.68}$$

The limit $j_1 = j_2 = j_3 = j_4 = 0$ in (6.64) gives:

$$v(0, 0, 0, 0, j_5) = (-1)^{j_5} \ I(0) I(1)^{j_5+1}\big|_{z=1} = (-1)^{j_5} \zeta(2, \{1\}^{j_5}) \ . \tag{6.69}$$

Three j_i set to zero: There are ten different cases with three arguments set to zero. For $j_3 = j_4 = j_5 = 0$ we get:

$$\begin{aligned} v(j_1, j_2, 0, 0, 0) &= (-1)^{j_1+j_2} \sum_{\delta_1} \binom{j_1 + j_2 - 2\delta_1}{j_1 - \delta_1} I(0)\{ I(0)^{j_1+j_2-2\delta_1}, I(1,0)^{\delta_1} \} I(1)\big|_{z=1} \\ &= (-1)^{j_1+j_2} \sum_{\delta_1} \binom{j_1 + j_2 - 2\delta_1}{j_1 - \delta_1} \sum_{\substack{w=j_1+j_2+2 \\ d=\delta_1+1;\ n_i \geq 2}} \zeta(\vec{n}) \\ &= (-1)^{j_1+j_2} \sum_{\substack{w=j_1+j_2+2 \\ n_i \geq 2}} \zeta(\vec{n}) \binom{w - 2d}{j_1 + 1 - d} . \end{aligned} \tag{6.70}$$

We used identity (6.2) in the first step and combined the sum over δ_1 with the sum of MZVs in the last line. For $j_2 = j_3 = j_4 = 0$ the $n = 2$ version of identity (6.36) can be used to obtain:

$$\begin{aligned} v(j_1, 0, 0, 0, j_5) &= (-1)^{j_1+j_5} \sum_{\delta_2} (-1)^{\delta_2} I(0)\{I(0)^{j_1-\delta_2}, I(1)^{j_5-\delta_2}, I(1,0)^{\delta_2}\} I(1)\big|_{z=1} \\ &= (-1)^{j_1+j_5}\ I(0)^{j_1+1} I(1)^{j_5+1}\big|_{z=1} = (-1)^{j_1+j_5} \zeta(j_1 + 2, \{1\}^{j_5}) . \end{aligned} \tag{6.71}$$

The same identity applies to the limit $j_2 = j_4 = j_5 = 0$, thus we get:

$$\begin{aligned} v(j_1, 0, j_3, 0, 0) &= (-1)^{j_1+j_3} \sum_{\beta_1} (-1)^{\beta_1} I(0)\{I(0)^{j_1-\beta_1}, I(0)^{j_3-\beta_1}, I(0,0)^{\beta_1}\} I(1)\big|_{z=1} \\ &= (-1)^{j_1+j_3}\ I(0)^{j_1+j_3+1} I(1)\big|_{z=1} = (-1)^{j_1+j_3} \zeta(j_1 + j_3 + 2) . \end{aligned} \tag{6.72}$$

The calculations for the case $j_2 = j_3 = j_5 = 0$ take the same steps as for $v(j_1, 0, j_3, 0, 0)$, so that:

$$v(j_1, 0, 0, j_4, 0) = v(j_1, 0, j_4, 0, 0) . \tag{6.73}$$

Identity (6.1) is useful for the limit $j_1 = j_3 = j_4 = 0$:

$$\begin{aligned} v(0, j_2, 0, 0, j_5) &= (-1)^{j_2+j_5} I(0)\{I(0)^{j_2}\ , I(1)^{j_5}\} I(1)\big|_{z=1} \\ &= (-1)^{j_2+j_5} \sum_{\substack{w=j_2+j_5+2 \\ d=j_5+1}} \zeta(\vec{n}) . \end{aligned} \tag{6.74}$$

Obtaining the MZV representation for $v(0, j_2, j_3, 0, 0)$ requires the identity (6.40) and therefore involves the multi-index sum ω_2'. The result for $v(0, j_2, j_3, 0, 0)$ is given in (A.1). The limits $v(0, 0, j_3, j_4, 0)$, $v(0, 0, 0, j_4, j_5)$, $v(0, j_2, 0, j_4, 0)$ and $v(0, 0, j_3, 0, j_5)$ follow from eqs. (6.70), (6.71), (6.72) and (6.74), respectively, through the symmetry (6.66).

Two j_i set to zero: There are ten different cases with two j_i set to zero. Setting $j_2 = j_3 = 0$ in (6.64) yields:

$$\begin{aligned} v(j_1,0,0,j_4,j_5) &= (-1)^{j_1+j_4+j_5} \sum_{\delta_5,\beta_2,\delta_2,\epsilon_3} (-1)^{\delta_5+\beta_2+\delta_2+\epsilon_3} I(0)\{I(0)^{j_1-\beta_2-\delta_2-\epsilon_2}, I(0)^{j_4-\delta_5-\beta_2}, \\ &\quad I(0,0)^{\beta_2}, I(1)^{j_5-\delta_5-\delta_2}, I(1,0)^{\delta_2}, I(1,0)^{\delta_5-\epsilon_3}, I(1,0,0)^{\epsilon_3}\} I(1)\big|_{z=1} \\ &= (-1)^{j_1+j_4+j_5} \sum_{\delta_5} (-1)^{\delta_5} I(0)^{j_1+1}\{I(0)^{j_4-\delta_5}, I(1)^{j_5-\delta_5}, I(1,0)^{\delta_5}\} I(1)\big|_{z=1} \\ &= (-1)^{j_1+j_4+j_5}\, I(0)^{j_1+j_4+1} I(1)^{j_5+1}\big|_{z=1} \\ &= (-1)^{j_1+j_4+j_5} \zeta(j_1+j_4+2,\{1\}^{j_5}) \ . \end{aligned} \tag{6.75}$$

The identity (6.36) has to be applied two times. In the first step with $n = 4$ and in the second with $n = 2$. This limit demonstrates how valuable the identities, derived in section 6.1, can be. Starting with four sums of generalized operator products, involving seven arguments, only a single MZV remains in (6.75). For the limit $j_3 = j_4 = 0$ identity (6.25) leads to

$$\begin{aligned} v(j_1,j_2,0,0,j_5) &= (-1)^{j_1+j_2+j_5} \sum_{\delta_1,\delta_2} (-1)^{\delta_1+\delta_2} \binom{j_1+j_2-\delta_1-\delta_2}{j_2-\delta_1} \binom{\delta_2}{\delta_1} I(0) \\ &\quad \times \{I(0)^{j_1+j_2-\delta_1-\delta_2}, I(1)^{j_5+\delta_1-\delta_2}, I(1,0)^{\delta_2}\} I(1)\big|_{z=1} \\ &= (-1)^{j_1+j_2+j_5} \sum_{\delta_1=0}^{\min\{j_1,j_2\}} \sum_{\substack{w=j_1+j_2+j_5+2 \\ d=j_5+\delta_1+1}} \zeta(\vec{n})\, \omega(j_1{+}1, j_2{+}1, d{-}d_1, \delta_1{+}1) \\ &= (-1)^{j_1+j_2+j_5} \sum_{\substack{w=j_1+j_2+j_5+2 \\ j_5<d\le j_5+1+\min\{j_1,j_2\}}} \zeta(\vec{n})\, \omega(j_1{+}1, j_2{+}1, d{-}d_1, d{-}j_5) \ , \end{aligned} \tag{6.76}$$

with the weighting[3]:

$$\omega(j_x,j_y,\delta_x,\delta_y) = \binom{\delta_x-1}{\delta_y-1} \binom{j_x+j_y-2\delta_y}{j_x-\delta_y} {}_2F_1\left[\begin{matrix}\delta_y-j_x,\delta_y-\delta_x \\ 2\delta_y-j_x-j_y\end{matrix};1\right]. \tag{6.77}$$

For $j_3 = j_5 = 0$ we obtain:

$$\begin{aligned} & v(j_1,j_2,0,j_4,0) \\ &= (-1)^{j_1+j_2+j_4} \sum_{\delta_1,\beta_2,\beta_4,\epsilon_1} (-1)^{\beta_2+\beta_4+\epsilon_1} I(0)\{I(0)^{j_4-\beta_2-\beta_4-\epsilon_1}, I(0)^{j_1-\delta_1-\beta_2}, I(0,0)^{\beta_2}, \end{aligned}$$

[3]To write summation regions for sums of MZVs more compact, we set binomial coefficients to zero for negative arguments (cf. fn. 3 on p. 10). This is, however, not sufficient in (6.76), since the hypergeometric function in the weighting (6.77) has singularities in the region, where the binomial coefficients are set to zero. To ensure convergence, we introduce explicit bounds of summation for the sum over δ_1 in the third line of (6.76). This leads to the condition $j_5{+}1 \le d \le j_5{+}1{+}\min\{j_1,j_2\}$ for the sum of MZVs in the last line of (6.76).

$$I(0)^{j_2-\delta_1-\beta_4}, I(0,0)^{\beta_4}, I(1,0)^{\delta_1-\epsilon_1}, I(1,0,0)^{\epsilon_1}\}I(1)\big|_{z=1}$$
$$= (-1)^{j_1+j_2+j_4} \sum_{\delta_1} \binom{j_1+j_2-2\delta_1}{j_1-\delta_1} I(0)^{j_4+1}\{I(0)^{j_1+j_2-2\delta_1}, I(1,0)^{\delta_1}\}I(1)\big|_{z=1}$$
$$= (-1)^{j_1+j_2+j_4} \sum_{\delta_1} \binom{j_1+j_2-2\delta_1}{j_1-\delta_1} \sum_{\substack{w=j_1+j_2+j_4+2 \\ d=\delta_1+1;\ n_1\geq j_4+2;\ n_i\geq 2}} \zeta(\vec{n})$$
$$= (-1)^{j_1+j_2+j_4} \sum_{\substack{w=j_1+j_2+j_4+2 \\ n_1\geq j_4+2;\ n_i\geq 2}} \zeta(\vec{n}) \binom{w-j_4-2d}{j_1-d+1} . \tag{6.78}$$

The first step requires the $n=4$ version of (6.36) and the second step eq. (6.2). Eventually, the sum over δ_1 is combined with the sum of MZVs. With $j_2=j_4=0$ eq. (6.64) becomes:

$$v(j_1,0,j_3,0,j_5)$$
$$= (-1)^{j_1+j_3+j_5} \sum_{\beta_1,\delta_2} (-1)^{\beta_1+\delta_2} I(0)\{I(0)^{j_1-\beta_1-\delta_2}, I(0)^{j_3-\beta_1}, I(0,0)^{\beta_1},\ I(1)^{j_5-\delta_2}, I(1,0)^{\delta_2}\}I(1)\big|_{z=1}$$
$$= (-1)^{j_1+j_3+j_5} I(0)^{j_1+1}\{I(0)^{j_3},\ I(1)^{j_5}\}I(1)\big|_{z=1}$$
$$= (-1)^{j_1+j_3+j_5} \sum_{\substack{w=j_1+j_3+j_5+2 \\ d=j_5+1;\ n_1\geq j_1+2}} \zeta(\vec{n}) . \tag{6.79}$$

Here the $n=3$ version of identity (6.36) is used in the first and (6.1) in the second step. The MZV representations for $v(j_1,j_2,j_3,0,0)$ and $v(0,j_2,j_3,0,j_5)$ include the weightings ω_3' and ω_{2a}', respectively, which involve multi-index sums. These cases can be found in eqs. (A.2) and (A.3). The symmetry (6.66) allows to straightforwardly determine $v(0,j_2,j_3,j_4,0)$, $v(0,0,j_3,j_4,j_5)$, $v(j_1,0,j_3,j_4,0)$ and $v(0,j_2,0,j_4,j_5)$ using the results in (A.2), (6.76), (6.78) and (6.79), respectively.

One j_i set to zero: Setting $j_3=0$ in (6.64) yields:

$$v(j_1,j_2,0,j_4,j_5)$$
$$= (-1)^{j_1+j_2+j_4+j_5} \sum_{\delta_1,\delta_2,\beta_2,\beta_4,\delta_5,\epsilon_1,\epsilon_3} (-1)^{\delta_2+\beta_2+\beta_4+\delta_5+\epsilon_1+\epsilon_3} I(0)\{I(0)^{j_4-\delta_5-\beta_2-\beta_4-\epsilon_1-\epsilon_3},$$
$$I(0)^{j_1-\delta_1-\delta_2-\beta_2}, I(0,0)^{\beta_2}, I(0)^{j_2-\delta_1-\beta_4}, I(0,0)^{\beta_4}, I(1)^{j_5-\delta_2-\delta_5}, I(1,0)^{\delta_5}, I(1,0)^{\delta_1-\epsilon_1},$$
$$I(1,0,0)^{\epsilon_1},\ I(1,0)^{\delta_2-\epsilon_3}, I(1,0,0)^{\epsilon_3}\}I(1)\big|_{z=1}$$
$$= (-1)^{j_1+j_2+j_4+j_5} \sum_{\delta_1,\delta_2} (-1)^{\delta_1+\delta_2} \binom{j_1+j_2-\delta_1-\delta_2}{j_1-\delta_2} \binom{\delta_2}{\delta_1}$$
$$\times\ I(0)^{j_4+1}\{I(0)^{j_1+j_2-\delta_1-\delta_2}, I(1)^{j_5+\delta_1-\delta_2},\ I(1,0)^{\delta_2}\}I(1)\big|_{z=1}$$
$$= (-1)^{j_1+j_2+j_4+j_5} \sum_{\delta_1=0}^{\min\{j_1,j_2\}} \sum_{\substack{w=j_1+j_2+j_4+j_5+2 \\ d=j_5+\delta_1+1;\ n_1\geq j_4+2}} \zeta(\vec{n})\ \omega(j_1{+}1, j_2{+}1, d{-}d_1, \delta_1{+}1)$$

$$= (-1)^{j_1+j_2+j_4+j_5} \sum_{\substack{w=j_1+j_2+j_4+j_5+2;\ n_1 \geq j_4+2 \\ j_5<d\leq j_5+1+\min\{j_1,j_2\}}} \zeta(\vec{n})\, \omega(j_1{+}1, j_2{+}1, d{-}d_1, d{-}j_5)\ . \tag{6.80}$$

The $n = 5$ version of (6.36) is applied in the first step and identity (6.25) in step two. The weighting ω is defined in (6.77). The weightings in the MZV representations of the limits $j_5 = 0$ and $j_4 = 0$ involve multiple index sums. They are given in eqs. (A.4) and (A.5). The limits $j_1 = 0$ and $j_2 = 0$ follow from eqs. (A.5) and (6.80), respectively, through the symmetry (6.66).

6.3 Identities for MZVs

In section 5.2.2 we presented the α'-expansion (5.18) of the five-point integral F_2, in which the kinematic part is separated from the MZVs. The latter are summarized in the function $f(j_1, j_2, j_3, j_4, j_5)$. Combining eqs. (5.6) and (5.18) allows to write $f(j_1, j_2, j_3, j_4, j_5)$ as the coefficient function of the series:

$$\frac{{}_2F_1\left[\substack{-s_1,s_2\\1+s_1};1\right]\,{}_2F_1\left[\substack{-s_3,s_4\\1+s_3};1\right]}{(1+s_1+s_2)(1+s_3+s_4)}\ {}_3F_2\left[\begin{matrix}1+s_1, 1+s_4, 1+s_2+s_3-s_5\\ 2+s_1+s_2, 2+s_3+s_4\end{matrix};1\right] \\ = \sum_{j_1,j_2,j_3,j_4,j_5\geq 0} s_1^{j_1} s_2^{j_2} s_3^{j_3} s_4^{j_4} s_5^{j_5} f(j_1, j_2, j_3, j_4, j_5)\ . \tag{6.81}$$

The formula for $f(j_1, j_2, j_3, j_4, j_5)$ can be found in eq. (5.19), which essentially is a sum of products of three types of MZVs, representing the three generalized hypergeometric functions on the l.h.s. of (6.81). Two of the factors in (5.19) are given as the object ζ'_{i_1,i_2}, subject to the definition (5.20), which represent MZVs of the kind $\zeta(i_1 + 1, \{1\}^{i_2-1})$ stemming from the hypergeometric functions ${}_2F_1$. These MZVs can be written in terms of single zeta values [42, 50]. The MZVs, which originate from ${}_3F_2$, are contained in the third factor $v(j_1, j_2, j_3, j_4, j_5)$. This function is given in (5.21) in terms of generalized operator products and in (6.65) in the MZV representation. Combining eqs. (5.19) and (6.65) we can write $f(j_1, j_2, j_3, j_4, j_5)$ in terms of MZVs as well:

$$f(j_1, j_2, j_3, j_4, j_5) = (-1)^k \sum_{\vec{l}=(l_1,l_2,l_3,l_4)} \zeta'_{l_1,l_2}\, \zeta'_{l_3,l_4} \sum_{w=k-|\vec{l}|} \zeta(\vec{n}) \sum_{\delta_1,\delta_4} (-1)^{d-1-j_5-\delta_1-\delta_4} \\ \times \omega'_6\left(\vec{n}-\vec{2};\ \substack{\delta_4,\ \delta_1,\ j_5\\ j_1-l_1-\delta_1,\ j_2+j_5-l_2+\delta_4-d+1,\ j_3+j_5-l_3+\delta_1-d+1,\ j_4-l_4-\delta_4}\right)\ , \tag{6.82}$$

with $k = j_1 + j_2 + j_3 + j_4 + j_5 + 2$.

Our motive in presenting the α'-expansion of F_2 in the form (5.18) is to directly extract identities for MZVs. The object $F_2(s_{13}s_{24})^{-1}$, which equals the product of generalized hypergeometric functions on the l.h.s. of eq. (6.81), is invariant w.r.t. cyclic permutations of the kinematic invariants s_1, s_2, s_3, s_4, s_5. For this symmetry to hold on the r.h.s. of eq. (6.81), the function $f(j_1, j_2, j_3, j_4, j_5)$ has to be invariant w.r.t. cyclic permutations of its arguments:

$$\begin{aligned} f(j_1, j_2, j_3, j_4, j_5) = f(j_5, j_1, j_2, j_3, j_4) &= f(j_4, j_5, j_1, j_2, j_3) \\ &= f(j_3, j_4, j_5, j_1, j_2) = f(j_2, j_3, j_4, j_5, j_1) \ . \end{aligned} \tag{6.83}$$

This is not trivially fulfilled. Instead, identities for MZVs are generated. Since we know the representation (6.82) for $f(j_1, j_2, j_3, j_4, j_5)$ in terms of MZV, these identities can now be analysed. We mentioned a similar symmetry for the four-point function, which generates identities for MZVs as well (cf. section 5.1).

The general identities, which follow from (6.82) through eqs. (6.83), are rather complicated due to the multi-index sums appearing in ω_6'. However, some more interesting families of MZV identities are included. Instead of multi-index sums, they involve known functions such as binomial coefficients and hypergeometric functions ${}_2F_1$. These identities appear for specific limits, where some of the arguments j_1, j_2, j_3, j_4, j_5 are set to zero. Below we present the MZV identities related to the limits we considered for $v(j_1, j_2, j_3, j_4, j_5)$ in section 6.2. Hence, we ignore the cases, which yield multi-index sums, i.e. the ones, which satisfy $(j_2 \neq 0) \wedge (j_3 \neq 0)$ (cf. appendix A). Similar to the previous section, we start with limits, where up to four arguments of $f(j_1, j_2, j_3, j_4, j_5)$ equal zero and eventually discuss cases with only one j_i set to zero. This way we see, which families of MZV identities are included in more general ones. Of course, all identities follow from (6.82). However, we do not use eq. (6.82) to compute the limits for $f(j_1, j_2, j_3, j_4, j_5)$, since it is not obvious how the weighting ω_6' simplifies in many cases. Instead, we work with eq. (5.19) and insert the corresponding limits for $v(j_1, j_2, j_3, j_4, j_5)$, which are computed in the previous section. With those expressions for the MZVs, that originate from ${}_3F_2$, already given, it is straightforward to combine them in eq. (5.19) with the contributions of the hypergeometric functions ${}_2F_1$. This step is particular trivial in case the condition $[(j_1 = 0) \vee (j_2 = 0)] \wedge [(j_3 = 0) \vee (j_4 = 0)]$ holds for $f(j_1, j_2, j_3, j_4, j_5)$. Eq. (5.19) then becomes $f(j_1, j_2, j_3, j_4, j_5) = v(j_1, j_2, j_3, j_4, j_5)$. Thus, instead of giving explicit expressions for limits of $f(j_1, j_2, j_3, j_4, j_5)$, we directly present the corresponding MZV identities.

Setting $s_i = 0$ on the r.h.s. of eq. (6.81) gives non-zero contributions for $j_i = 0$ only. Therefore, the generating function of MZV identities, appearing for $j_i = 0$, can be obtained by setting $s_i = 0$ on the l.h.s. of (6.81).

Similar to the symmetry (6.66) we have:

$$f(j_1, j_2, j_3, j_4, j_5) = f(j_4, j_3, j_2, j_1, j_5) \ . \tag{6.84}$$

It can be seen on the l.h.s. of eq. (6.81) that the corresponding replacements

$$(s_1, s_2, s_3, s_4, s_5) \to (s_4, s_3, s_2, s_1, s_5)$$

vary only the hypergeometric functions ${}_2F_1$. As a consequence, the symmetry (6.84) generates solely the MZV identity (5.4). In contrast to that, the cyclic symmetry (6.83) generates more interesting identities, as can be seen in the following.

Four j_i set to zero: There are five different limits for the simplest case:

$$f(j_1,0,0,0,0)=f(0,j_1,0,0,0)=f(0,0,j_1,0,0)=f(0,0,0,j_1,0)=f(0,0,0,0,j_1)\ . \tag{6.85}$$

According to eqs. (6.67)–(6.69) the first 4 terms in (6.85) are identical, while $f(0,0,0,0,j_1)$ contains a different MZV. Hence, we obtain the relation:

$$\zeta(\alpha_1+1)=\zeta(2,\{1\}^{\alpha_1-1})\ ,\quad \alpha_1\geq 1\ . \tag{6.86}$$

This is an instance of the more general relation (5.4).

Three j_i set to zero: The limits with three arguments j_i set to zero can be divided into two types:

$$f(j_1,j_2,0,0,0)=f(0,0,j_1,j_2,0)=f(0,0,0,j_1,j_2)=f(j_2,0,0,0,j_1) \tag{6.87}$$

and

$$f(j_1,0,j_3,0,0)=f(0,j_1,0,j_3,0)=f(0,0,j_1,0,j_3)$$
$$=f(j_3,0,0,j_1,0)=f(0,j_3,0,0,j_1)\ . \tag{6.88}$$

We ignore $f(0,j_1,j_2,0,0)$, since it involves multi-index sums (cf. (A.1)). While (6.87) includes three independent relations, only one independent family of MZV identities is generated. Comparing eqs. (6.70) and (6.71) allows to write:

$$\sum_{l_1,l_2\geq 1}\zeta(l_1+1,\{1\}^{l_2-1})\sum_{\substack{w=\alpha_1+\alpha_2-l_1-l_2\\ n_i\geq 2}}\zeta(\vec{n})\binom{w-2d}{\alpha_1-l_1-d}$$
$$=\sum_{\substack{w=\alpha_1+\alpha_2\\ n_i\geq 2}}\zeta(\vec{n})\binom{w-2d}{\alpha_1-d}-\zeta(\alpha_1+1,\{1\}^{\alpha_2-1})\ . \tag{6.89}$$

with $\alpha_1,\alpha_2\geq 1$. Up to the relation (5.4) this identity is invariant w.r.t. $\alpha_1\leftrightarrow\alpha_2$, which allows us to give the additional restriction $\alpha_1\geq\alpha_2$ in order to generate less linear dependent identities. For examples at weight $w=5$ with $\alpha_1=3$, $\alpha_2=2$ identity (6.89) yields

$$2\zeta(2)\zeta(3)=3\zeta(5)+\zeta(2,3)+\zeta(3,2)-\zeta(4,1)\ , \tag{6.90}$$

and at weight $w=6$ with $\alpha_1=4$, $\alpha_2=2$ we get:

$$\zeta(3)^2+2\zeta(2)\zeta(4)=4\zeta(6)+\zeta(2,4)+\zeta(3,3)+\zeta(4,2)-\zeta(5,1)\ . \tag{6.91}$$

The second type of limits (6.88) yields another class of identities. From eqs. (6.72) and (6.74) follows the well-known sum theorem [48] :

$$\zeta(\alpha_1)=\sum_{\substack{w=\alpha_1\\ d=\alpha_2}}\zeta(\vec{n})\ ,\qquad \alpha_1>\alpha_2\geq 1\ . \tag{6.92}$$

At weights $w = 5$ and $w = 6$ it includes the following identities:

$$\begin{aligned}
\zeta(5) &= \zeta(4,1) + \zeta(3,2) + \zeta(2,3) \\
&= \zeta(3,1,1) + \zeta(2,2,1) + \zeta(2,1,2) \\
&= \zeta(2,1,1,1)\ , \\
\zeta(6) &= \zeta(5,1) + \zeta(4,2) + \zeta(3,3) + \zeta(2,4) \\
&= \zeta(4,1,1) + \zeta(3,2,1) + \zeta(3,1,2) + \zeta(2,2,2) + \zeta(2,3,1) + \zeta(2,1,3) \\
&= \zeta(3,1,1,1) + \zeta(2,2,1,1) + \zeta(2,1,2,1) + \zeta(2,1,1,2) \\
&= \zeta(2,1,1,1,1)\ .
\end{aligned} \tag{6.93}$$

The sum theorem is a particular beautiful identity, because it can be described in one sentence: The sum of all MZVs of given weight w and depth d is independent of d.

Two j_i set to zero: There are two classes of limits:

$$f(0,0,j_3,j_4,j_5) = f(j_5,0,0,j_3,j_4) = f(j_4,j_5,0,0,j_3) \tag{6.94}$$

and

$$f(j_1,j_2,0,j_4,0) = f(j_4,0,j_1,j_2,0) = f(0,j_4,0,j_1,j_2) = f(j_2,0,j_4,0,j_1)\ . \tag{6.95}$$

We obtain one interesting relation via (6.94) with eqs. (6.75) and (6.76):

$$\begin{aligned}
&\sum_{l_1,l_2\geq 1} \zeta(l_1+1,\{1\}^{l_2-1}) \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3-l_1-l_2 \\ \alpha_3<d\leq\alpha_3+\min\{\alpha_1-l_1,\alpha_2-l_2\}}} \zeta(\vec{n})\, \omega(\alpha_1-l_1,\alpha_2-l_2,d-d_1,d-\alpha_3) \\
&\quad = \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3 \\ \alpha_3<d\leq\alpha_3+\min\{\alpha_1,\alpha_2\}}} \zeta(\vec{n})\, \omega(\alpha_1,\alpha_2,d-d_1,d-\alpha_3) - \zeta(\alpha_2+\alpha_3+1,\{1\}^{\alpha_1-1})\ ,
\end{aligned} \tag{6.96}$$

with $\alpha_1, \alpha_2 \geq 1$, $\alpha_3 \geq 0$. This identity contains a hypergeometric function ${}_2F_1$ through the function ω, given in (6.77). Examples are

$$\begin{aligned}
\zeta(2)\zeta(2,1) &= \zeta(2,3) + \zeta(3,2) + \zeta(4,1) + \zeta(2,1,2) + \zeta(2,2,1)\ , \\
\zeta(3)\zeta(2,1) + \zeta(2)\zeta(3,1) &= \zeta(2,4) + \zeta(3,3) + \zeta(4,2) + 3\zeta(5,1) + \zeta(2,1,3) \\
&\quad + \zeta(2,3,1) + \zeta(3,1,2) + \zeta(3,2,1) - \zeta(4,1,1)\ ,
\end{aligned} \tag{6.97}$$

for $(\alpha_1,\alpha_2,\alpha_3) = (2,2,1)$ and $(\alpha_1,\alpha_2,\alpha_3) = (3,2,1)$, respectively. From the results (6.78), (6.79) and eqs. (6.95) follows the family of MVZ identities:

$$\sum_{l_1,l_2\geq 1} \zeta(l_1+1,\{1\}^{l_2-1}) \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3-l_1-l_2 \\ n_1\geq\alpha_3+2;\ n_i\geq 2}} \zeta(\vec{n}) \binom{w-\alpha_3-2d}{\alpha_1-l_1-d}$$

$$= \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3\\ n_1\geq\alpha_3+2;\ n_i\geq 2}} \zeta(\vec{n}) \binom{w-\alpha_3-2d}{\alpha_1-d} - \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3\\ n_1>\alpha_2;\ d=\alpha_1}} \zeta(\vec{n}) \ , \quad (6.98)$$

with $\alpha_1, \alpha_2 \geq 1$, $\alpha_3 \geq 0$. Two examples with $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, 1)$ and $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, 1)$, respectively, are

$$\begin{aligned} \zeta(2)\zeta(3) &= 2\zeta(5) - \zeta(4,1) \ , \\ \zeta(2)\zeta(4) + \zeta(3)\zeta(2,1) &= 3\zeta(6) + \zeta(3,3) - \zeta(5,1) \ . \end{aligned} \quad (6.99)$$

The transformation $\alpha_1 \leftrightarrow \alpha_2$ in (6.98) changes only the second sum on the r.h.s. Thus, we can write:

$$\sum_{\substack{w=\alpha_3\\ d=\alpha_1;\ n_1>\alpha_2}} \zeta(\vec{n}) = \sum_{\substack{w=\alpha_3\\ d=\alpha_2;\ n_1>\alpha_1}} \zeta(\vec{n}) \ , \quad \alpha_1 > \alpha_2 \geq 1 \ , \quad \alpha_3 \geq \alpha_1 + \alpha_2 \ . \quad (6.100)$$

This is an interesting generalization of the sum theorem (6.92). The latter arises for $\alpha_2 = 1$. For MZVs of weights $w = 5$ and $w = 6$ additionally to eqs. (6.93) the independent relations

$$\begin{aligned} \zeta(4,1) &= \zeta(3,1,1) \ , \\ \zeta(5,1) &= \zeta(3,1,1,1) \ , \\ \zeta(5,1) + \zeta(4,2) &= \zeta(4,1,1) + \zeta(3,2,1) + \zeta(3,1,2) \ , \end{aligned} \quad (6.101)$$

are generated. The identity (5.4) arises from (6.100) for $\alpha_3 = \alpha_1 + \alpha_2$. All relations following from (6.100) are contained in (6.98), when using the regions for the parameters $\alpha_1, \alpha_2, \alpha_3$ given below that identity. Alternatively, we could use the additional condition $\alpha_1 \geq \alpha_2$ in (6.98) and generate the remaining identities with (6.100).

One j_i set to zero: Finally, we discuss the identities generated through:

$$f(j_1, j_2, 0, j_4, j_5) = f(j_2, 0, j_4, j_5, j_1) \ . \quad (6.102)$$

Using the symmetry (6.84) and eq. (6.80) we obtain

$$\begin{aligned} \sum_{l_1,l_2\geq 1} \zeta(l_1+1, \{1\}^{l_2-1}) & \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3+\alpha_4-l_1-l_2;\ n_1\geq\alpha_3+2\\ \alpha_4<d\leq\alpha_4+\min\{\alpha_1-l_1,\alpha_2-l_2\}}} \zeta(\vec{n})\, \omega(\alpha_1-l_1, \alpha_2-l_2, d-d_1, d-\alpha_4) \\ & - \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3+\alpha_4;\ n_1\geq\alpha_3+2\\ \alpha_4<d\leq\alpha_4+\min\{\alpha_1,\alpha_2\}}} \zeta(\vec{n})\, \omega(\alpha_1, \alpha_2, d-d_1, d-\alpha_4) \ , \end{aligned} \quad (6.103)$$

with $\alpha_1, \alpha_2 \geq 1$ and $\alpha_3, \alpha_4 \geq 0$. This combination of MZVs appears on one side of the general identity and the other one can be obtained through the transformation:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \rightarrow (\alpha_4 + 1, \alpha_3 + 1, \alpha_2 - 1, \alpha_1 - 1) \ .$$

For example the parameters $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (3, 1, 1, 1)$ lead to:

$$\begin{aligned}-\zeta(5,1) = \zeta(2)\zeta(2,1,1) - \zeta(2,1,3) - \zeta(2,3,1) - \zeta(3,1,2) - \zeta(3,2,1) - 2\zeta(4,1,1) \\ - \zeta(2,1,1,2) - \zeta(2,1,2,1) - \zeta(2,2,1,1) \ . \quad (6.104)\end{aligned}$$

Let us emphasize that identities related to limits, where less arguments j_i equal zero, include those with more j_i set to zero as special cases. All identities presented in this section can be generated using the single expression (6.82). Moreover, additional identities arise, which are not explicitly given here.

The most interesting identities listed above are certainly those, which include hypergeometric functions ${}_2F_1$, i.e. eq. (6.96) and the more general relation involving (6.103). Similar families of MZV identities, that contain hypergeometric functions, are not known. Some explicit relations following from eq. (6.96) can be found in appendix B.

Chapter 7

Conclusion

Let us briefly summarize our achievements described in this thesis. We also discuss some unsolved problems and ideas for future studies.

We give the general solution to linear recurrence relations with constant non-commutative coefficients. With this solution at hand, we present a new method, which allows to construct all-order Laurent series of functions satisfying generic first-order Fuchsian differential equations. We also provide the tools required to apply our method. This includes the generalized operator product, its consistent definition and basic properties as well as a variety of non-trivial identities.

We apply our method to obtain all-order expansions of generalized hypergeometric functions around integer and half-integer parameters. It is worth pointing out the compactness of our results, for example the expansion of the general case ${}_pF_{p-1}(\alpha'\vec{a};\vec{1}+\alpha'\vec{b};z)$, in which the dependence on the variable z through MPLs is separated from the parameters $\vec{a}$ and $\vec{b}$. For the latter the simple form is due to the use of elementary symmetric functions, while for the MPLs the generalized operator product of the corresponding integral operators is the key to the compact expression. The next obvious step would be to consider multi-variable hypergeometric series, such as Horn and Appell functions, or expansions around rational values of parameters. These functions also appear in representations of Feynman integrals. Work in these directions has already been initiated in [63]–[66],[2]. Since the outcome of our method is an *exact* representation, it offers a complete new way to characterize generalized hypergeometric functions: as infinite series with iterated integrals as coefficient functions. This opens up a multitude of possibilities for the study of both types of functions. Iterated integrals as well as hypergeometric functions and their generalizations are subject to various identities, for instance the Kummer relation of ${}_2F_1$ and shuffle relations of MZVs. Since our all-order expansions connect both types of functions, they can be used to translate relations of iterated integrals to those of generalized hypergeometric functions and vice versa. This is exactly what we describe in chapter 6, by using a symmetry of a product of hypergeometric functions ${}_2F_1$ and ${}_3F_2$ to generate families of identities for MZVs.

We propose the application of our new method to the calculation of Feynman integrals and provide a simple example. In this context, our approach can be understood as an extension of the method of differential equations. Assuming a canonical form of the corresponding differential equation is available, the current version of this method allows to straightforwardly determine an ϵ-expansion up to a finite number of orders. Our upgrade of this method can be used to obtain a representation for *all* orders and thus for an expression of the complete Feynman integral in dimensional regularization. Furthermore, we showed that it is possible to construct a representation in terms of hypergeometric functions from the all-order result. This step is a nice additional application, however, we prefer the all-order ϵ-expansion over the hypergeometric representation, since the former directly gives the behaviour of the Feynman integral around $\epsilon = 0$. With this advantage in mind, the all-order expansion could be established as a standard representation for Feynman integrals. Analysing how the structure of products of integral operators evolves with increasing complexity of Feynman integrals, could provide a new access to the computation of the latter. So far we only considered a simple example, which is already known. But we did not use this to our advantage at any point of the calculation. The next step would be to consider a more complicated (multi-loop) Feynman integral. In this regard we can benefit from the fact that our method builds on the method of differential equations, which is one of the most powerful modern techniques to obtain analytic results for Feynman integrals. Furthermore, one of the requirements — the canonical form of the differential equations — has already been found for several classes of Feynman integrals (see [67]–[69] for some recent examples).

We present the all-order α'-expansions of four-point and five-point disk-level open string amplitudes. As already mentioned above, we use these expressions in combination with their symmetries to extract families of relations among MZVs, some of which were previously unknown. We only studied the MZV identities following from the cyclic invariance of $F_2(s_{13}s_{24})^{-1}$. Another independent symmetric functions can be found as a combination of F_1 and F_2. Additionally, it is interesting that string theory provides through N-point disk-level open string amplitudes an infinite set of functions, which can be used in the same way to generate relations among MZVs. Deriving their all-order α'-expansions as well, would therefore be of particular interest from the number theoretical point of view.

We give our results in two ways. The all-order expansions in chapters 3, 4 and 5 are written in terms of integral operators, while in chapter 6 the same expressions are given explicitly as sums of MPLs and MZVs, respectively. The latter involve weightings with a rather cumbersome representation in terms of multiple index sums. Due to these weightings, the results in chapter 6, which also affects some of the identities of MZVs, are to complicated to be of special interest and expressions in terms of integral operators provide more compact alternatives. We discovered, however, some limits leading to known functions, such as binomial coefficients and Fibonacci numbers, which indicates that there might be more interesting representations for these weightings.

It would be desirable to find recurrence relations with non-commutative coefficients, which are not necessarily related to scattering amplitudes, and see what we can achieve with our general solution. Our general solution holds not only for integral operators, but for the most general case with any type of non-commutative coefficients, which is after all a fundamental mathematical problem.

Appendix A

Limits of $v(j_1, j_2, j_3, j_4, j_5)$

The results for limits of the function $v(j_1, j_2, j_3, j_4, j_5)$, which involve weightings with multi–index sums, are listed below. Note that $(j_2 \neq 0) \wedge (j_3 \neq 0)$ holds for all of them. They follow directly from (6.65) and eqs. (6.58). However, we calculate these limits starting from (6.64) and using proper identities of section 6.1 to check their consistency.

The case $j_1 = j_4 = j_5 = 0$ uses identity (6.40):

$$\begin{aligned}
&v(0, j_2, j_3, 0, 0)\\
&= (-1)^{j_2+j_3} \sum_{\delta_3,\beta_3} (-1)^{\delta_3+\beta_3} I(0)\{I(0)^{j_2-\delta_3-\beta_3}, I(0)^{j_3-\delta_3-\beta_3}, I(0,0)^{\beta_3}, I(1,0)^{\delta_3}\} I(1)\big|_{z=1}\\
&= (-1)^{j_2+j_3} \sum_{\delta_3} (-1)^{\delta_3} \sum_{\substack{w=j_2+j_3+2\\ d=\delta_3+1}} \zeta(\vec{n})\ \omega_2'(\vec{n}-\vec{2}; j_2-\delta_3)\\
&= (-1)^{j_2+j_3} \sum_{w=j_2+j_3+2} \zeta(\vec{n})(-1)^{d-1} \omega_2'(\vec{n}-\vec{2}; j_2-d+1)\ . \qquad \text{(A.1)}
\end{aligned}$$

Setting $j_4 = j_5 = 0$ and applying identity (6.44) yields:

$$\begin{aligned}
&v(j_1, j_2, j_3, 0, 0)\\
&= (-1)^{j_1+j_2+j_3} \sum_{\delta_1,\delta_3} (-1)^{\delta_3} \binom{\delta_1+\delta_3}{\delta_3} \sum_{\beta_1,\beta_3} (-1)^{\beta_1+\beta_3} I(0)\{I(0)^{j_3-\delta_3-\beta_1-\beta_3}, I(0)^{j_1-\delta_1-\beta_1},\\
&\quad I(0,0)^{\beta_1}, I(0)^{j_2-\delta_1-\delta_3-\beta_3}, I(0,0)^{\beta_3}, I(1,0)^{\delta_1+\delta_3}\} I(1)\big|_{z=1}\\
&= (-1)^{j_1+j_2+j_3} \sum_{\delta_1,\delta_3} (-1)^{\delta_3} \binom{\delta_1+\delta_3}{\delta_3} \sum_{\substack{w=j_1+j_2+j_3+2\\ d=\delta_1+\delta_3+1}} \zeta(\vec{n})\ \omega_3'(\vec{n}-\vec{2}; j_1-\delta_1, j_2-\delta_1-\delta_3)\\
&= (-1)^{j_1+j_2+j_3} \sum_{w=j_1+j_2+j_3+2} \zeta(\vec{n}) \sum_{\delta_3} (-1)^{\delta_3} \binom{d-1}{\delta_3} \omega_3'(\vec{n}-\vec{2}; j_1+\delta_3-d+1, j_2-d+1)\ . \qquad \text{(A.2)}
\end{aligned}$$

Identity (6.42) is useful for the limit $j_1 = j_4 = 0$:

$$
\begin{aligned}
&v(0, j_2, j_3, 0, j_5) \\
&= (-1)^{j_2+j_3+j_5} \sum_{\delta_3,\beta_3} (-1)^{\delta_3+\beta_3} I(0)\{I(0)^{j_2-\delta_3-\beta_3},\, I(0)^{j_3-\delta_3-\beta_3}, I(0,0)^{\beta_3}, I(1)^{j_5}, I(1,0)^{\delta_3}\} I(1)\Big|_{z=1} \\
&= (-1)^{j_2+j_3+j_5} \sum_{\delta_3} (-1)^{\delta_3} \sum_{\substack{w=j_2+j_3+j_5+2 \\ d=1+j_5+\delta_3}} \zeta(\vec{n})\, \omega'_{2a}(\vec{n}-\vec{2}; j_2-\delta_3, j_5) \\
&= (-1)^{j_2+j_3+j_5} \sum_{\substack{w=j_2+j_3+j_5+2 \\ d\geq 1+j_5}} \zeta(\vec{n}) (-1)^{d-1-j_5} \omega'_{2a}(\vec{n}-\vec{2}; j_2+j_5+1-d, j_5)\ .
\end{aligned} \tag{A.3}
$$

For $j_5 = 0$ we use identity (6.51) to obtain:

$$
\begin{aligned}
&v(j_1, j_2, j_3, j_4, 0) \\
&= (-1)^{j_1+j_2+j_3+j_4} \sum_{\delta_1,\delta_3,\delta_4,\vec{\beta},\epsilon_1,\epsilon_2} (-1)^{\delta_3+|\vec{\beta}|+\epsilon_1+\epsilon_2} I(0)\{I(0)^{j_1-\delta_1-\beta_1-\beta_2-\epsilon_2}, I(0)^{j_2-\delta_1-\delta_3-\beta_3-\beta_4}, \\
&\quad I(0)^{j_3-\delta_3-\delta_4-\beta_1-\beta_3}, I(0)^{j_4-\delta_4-\beta_2-\beta_4-\epsilon_1}, I(0,0)^{\beta_1}, I(0,0)^{\beta_2}, I(0,0)^{\beta_3}, I(0,0)^{\beta_4}, \\
&\quad I(1,0)^{\delta_4-\epsilon_2}, I(1,0,0)^{\epsilon_2}, I(1,0)^{\delta_1-\epsilon_1},\, I(1,0,0)^{\epsilon_1}, I(1,0)^{\delta_3}\} I(1)\Big|_{z=1} \\
&= (-1)^{j_1+j_2+j_3+j_4} \sum_{\delta_1,\delta_3,\delta_4} (-1)^{\delta_3} \sum_{\substack{w=j_1+j_2+j_3+j_4+2 \\ d=\delta_1+\delta_3+\delta_4+1}} \zeta(\vec{n}) \\
&\quad \times \omega'_5 \left(\vec{n}-\vec{2}; \begin{matrix} & \delta_4,\ \delta_1 & \\ j_1-\delta_1,\ j_2-\delta_1-\delta_3,\ j_3-\delta_3-\delta_4 & , j_4-\delta_4 \end{matrix} \right) \\
&= (-1)^{j_1+j_2+j_3+j_4} \sum_{w=j_1+j_2+j_3+j_4+2} \zeta(\vec{n}) \sum_{\delta_1,\delta_4} (-1)^{d-1+\delta_1+\delta_4} \\
&\quad \times \omega'_5 \left(\vec{n}-\vec{2}; \begin{matrix} & \delta_4,\ \delta_1 & \\ j_1-\delta_1,\ j_2+\delta_4-d+1,\ j_3+\delta_1-d+1,\ j_4-\delta_4 \end{matrix} \right)\ .
\end{aligned} \tag{A.4}
$$

Identity (6.46) allows to determine the MZV representation for the limit $j_4 = 0$:

$$
\begin{aligned}
&v(j_1, j_2, j_3, 0, j_5) \\
&= (-1)^{j_1+j_2+j_3+j_5} \sum_{\delta_1,\delta_3} (-1)^{\delta_3} \binom{\delta_1+\delta_3}{\delta_1} \sum_{\beta_1,\beta_3,\delta_2} (-1)^{\beta_1+\beta_3+\delta_2} I(0)\{I(0)^{j_3-\delta_3-\beta_1-\beta_3}, \\
&\quad I(0)^{j_1-\delta_1-\delta_2-\beta_1}, I(0,0)^{\beta_1}, I(0)^{j_2-\delta_1-\delta_3-\beta_3}, I(0,0)^{\beta_3}, I(1)^{j_5-\delta_2}, I(1,0)^{\delta_2}, \\
&\quad I(1,0)^{\delta_1+\delta_3}\} I(1)\Big|_{z=1} \\
&= (-1)^{j_1+j_2+j_3+j_5} \sum_{\delta_1,\delta_3} (-1)^{\delta_3} \binom{\delta_1+\delta_3}{\delta_1} \sum_{\substack{w=j_1+j_2+j_3+j_5+2 \\ d=j_5+\delta_1+\delta_3+1}} \zeta(\vec{n}) \\
&\quad \times \omega'_4 \left(\vec{n}-\vec{2}; \begin{matrix} & j_5 \\ j_1-\delta_1,\ j_2-\delta_1-\delta_3 \end{matrix} \right)
\end{aligned}
$$

$$= (-1)^{j_1+j_2+j_3+j_5} \sum_{w=j_1+j_2+j_3+j_5+2} \zeta(\vec{n}) \sum_{\delta_1} (-1)^{\delta_1+d-1-j_5} \binom{d-1-j_5}{\delta_1}$$

$$\times\, \omega_4' \left(\vec{n} - \vec{2};\, \begin{matrix} j_5 \\ j_1{-}\delta_1,\ j_2{+}j_5{-}d{+}1 \end{matrix} \right) \ . \tag{A.5}$$

Appendix B

Some identities for MZVs

In the following we list all independent identities included in the family (6.96) up to weight $w = 6$:

weight 3

$$\zeta(2,1) = \zeta(3) \tag{B.1}$$

weight 4

$$\zeta(4) = \zeta(3,1) + \zeta(2,2) \tag{B.2}$$

$$\zeta(4) = \zeta(2,1,1) \tag{B.3}$$

$$\zeta(3,1) = 2\zeta(4) + \zeta(2,2) - \zeta(2)^2 \tag{B.4}$$

weight 5

$$\zeta(5) = \zeta(4,1) + \zeta(3,2) + \zeta(2,3) \tag{B.5}$$

$$\zeta(5) = \zeta(3,1,1) + \zeta(2,2,1) + \zeta(2,1,2) \tag{B.6}$$

$$\zeta(5) = \zeta(2,1,1,1) \tag{B.7}$$

$$\zeta(3,1,1) = \zeta(4,1) \tag{B.8}$$

$$\zeta(3,1,1) = 3\zeta(5) + \zeta(3,2) + \zeta(2,3) - 2\zeta(3)\zeta(2) \tag{B.9}$$

$$\zeta(3,1,1) = 3\zeta(5) + \zeta(3,2) + \zeta(2,3) - \zeta(3)\zeta(2) - \zeta(2,1)\zeta(2) \tag{B.10}$$

weight 6

$$\zeta(6) = \zeta(5,1) + \zeta(4,2) + \zeta(3,3) + \zeta(2,4) \tag{B.11}$$

$$\zeta(6) = \zeta(4,1,1) + \zeta(3,2,1) + \zeta(3,1,2) + \zeta(2,3,1) + \zeta(2,1,3) + \zeta(2,2,2) \tag{B.12}$$

$$\zeta(6) = \zeta(3,1,1,1) + \zeta(2,2,1,1) + \zeta(2,1,2,1) + \zeta(2,1,1,2) \tag{B.13}$$

$$\zeta(6) = \zeta(2,1,1,1,1) \tag{B.14}$$

$$\zeta(3,1,1,1) = \zeta(5,1) \tag{B.15}$$

$$\zeta(3,1,1,1) = 4\zeta(6) + \zeta(4,2) + \zeta(3,3) + \zeta(2,4) - 2\zeta(4)\zeta(2) - \zeta(3)\zeta(3) \tag{B.16}$$

$$\zeta(5,1) = 4\zeta(6) + \zeta(4,2) + \zeta(3,3) + \zeta(2,4) - \zeta(2,1,1)\zeta(2) - \zeta(4)\zeta(2) - \zeta(2,1)\zeta(3) \tag{B.17}$$

$$\begin{aligned}\zeta(4,1,1) = {} & 6\zeta(6) + 2\zeta(4,2) + 2\zeta(3,3) + 2\zeta(2,4) + \zeta(2,2,2) - 2\zeta(4)\zeta(2) - \zeta(2,2)\zeta(2) \\ & - \zeta(3,1)\zeta(2) - \zeta(3)\zeta(3) - \zeta(2,1)\zeta(3)\end{aligned} \tag{B.18}$$

$$\begin{aligned}\zeta(4,1,1) = {} & 3\zeta(5,1) + \zeta(4,2) + \zeta(3,3) + \zeta(2,4) + \zeta(3,2,1) + \zeta(3,1,2) + \zeta(2,3,1) \\ & + \zeta(2,1,3) - \zeta(3)\zeta(2,1) - \zeta(2)\zeta(3,1)\end{aligned} \tag{B.19}$$

$$\begin{aligned}\zeta(5,1) = {} & 3\zeta(5,1) + 2\zeta(4,2) + 2\zeta(3,3) + 2\zeta(2,4) + \zeta(3,2,1) + \zeta(3,1,2) + \zeta(2,3,1) \\ & + \zeta(2,1,3) + 2\zeta(2,2,2) - \zeta(2,1)\zeta(2,1) - \zeta(2)\zeta(2,2) - \zeta(2)\zeta(3,1)\end{aligned} \tag{B.20}$$

$$\begin{aligned}\zeta(5,1) = {} & 2\zeta(4,1,1) + \zeta(3,2,1) + \zeta(3,1,2) + \zeta(2,3,1) + \zeta(2,1,3) \\ & + \zeta(2,2,1,1) + \zeta(2,1,2,1) + \zeta(2,1,1,2) - \zeta(2)\zeta(2,1,1)\end{aligned} \tag{B.21}$$

Applying lower weight identities, e.g. eq. (B.1) to the last term $-\zeta(2,1)\zeta(2)$ of eq. (B.10), reduces the number of independent identities to 5 and 8 for the weights 5 and 6, respectively. Another example at weigth 8 with $(\alpha_1, \alpha_2, \alpha_3) = (2,3,3)$ is:

$$\begin{aligned}\zeta(7,1) = {} & -\zeta(2,1)\zeta(2,1,1,1) - \zeta(2)\zeta(3,1,1,1) - \zeta(2)\zeta(2,1,1,2) - \zeta(2)\zeta(2,1,2,1) \\ & - \zeta(2)\zeta(2,2,1,1) + 2\zeta(2,1,1,4) + \zeta(2,1,2,3) + \zeta(2,1,3,2) + 2\zeta(2,1,4,1) \\ & + \zeta(2,2,1,3) + \zeta(2,2,3,1) + \zeta(2,3,1,2) + \zeta(2,3,2,1) + 2\zeta(2,4,1,1) + 2\zeta(3,1,1,3) \\ & + \zeta(3,1,2,2) + 2\zeta(3,1,3,1) + \zeta(3,2,1,2) + \zeta(3,2,2,1) + 2\zeta(3,3,1,1) \\ & + 2\zeta(4,1,1,2) + 2\zeta(4,1,2,1) + 2\zeta(4,2,1,1) + 3\zeta(5,1,1,1) + \zeta(2,1,1,1,3) \\ & + 2\zeta(2,1,1,2,2) + \zeta(2,1,1,3,1) + 2\zeta(2,1,2,1,2) + 2\zeta(2,1,2,2,1) + \zeta(2,1,3,1,1) \\ & + 2\zeta(2,2,1,1,2) + 2\zeta(2,2,1,2,1) + 2\zeta(2,2,2,1,1) + \zeta(2,3,1,1,1) + \zeta(3,1,1,1,2) \\ & + \zeta(3,1,1,2,1) + \zeta(3,1,2,1,1) + \zeta(3,2,1,1,1)\end{aligned} \tag{B.22}$$

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