The free particle on q-Minkowski space

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Zusammenfassung

Die Annahme, daß die Raumzeit-Struktur durch kontinuierliche Koordinaten beschrieben werden kann, ist ein sehr erfolgreiches Konzept in der Physik. Bei sehr kleinen Abständen jedoch ist auch diese Struktur einer Quantisierung unterworfen, und man muß nach neuen physikalischen Modellen zu ihrer Beschreibung suchen. Eine Möglichkeit ist es, den Raum durch eine nichtkommutative Algebra darzustellen und auf diese Weise die entstehende Diskontinuität abzubilden. In dieser Arbeit wird der q-Minkowski Raum als ein konkretes Modell solch eines "Quantenraumes" betrachtet. Das besondere dieser q-deformierten Räume ist, daß sie eine so genannte Quantengruppe als Hintergrundsymmetrie besitzen. Dies macht es möglich sich die in der Physik äußerst wichtigen darstellungstheoretischen Aspekte auch für die q-deformierte Quantenräume zunutze zu manchen.

In den zwei Teilen dieser Arbeit werden irreduzible Darstellungen der q-deformierten Poincaré-Algebra berechnet. Im ersten Abschnitt werden wir sie als unitäre Darstellungen in einem abstrakten Hilbertraum realisieren, während wir sie im zweiten Teil als Lösungen der q-deformierten Klein-Gordon und Dirac-Gleichung erhalten werden.

Wir beginnen die Konstruktion der irreduziblen Hilbertraum Darstellungen mit der Wahl eines maximalen Satzes von miteinander kommutierenden Operatoren. Deren Eigenwerte repräsentieren die gleichzeitig beobachtbaren Meßgrößen und die gemeinsamen Eigenvektoren spannen eine Basis des Hilbertraumes auf. Die Bestimmung der Matrixelemente der Generatoren der q-Poincaré-Algebra erfolgt durch sukzessives Auswerten der zwischen ihnen bestehenden Vertauschungsrelationen. Dazu wird zuerst eine Darstellung für die Koordinaten des q-Minkowski Raumes konstruiert, dann werden die Generatoren der Drehungen dargestellt, um schließlich mit Hilfe dieser Ergebnisse auch die Darstellungen der Boost Operatoren zu erhalten. Indem wir die Algebra der Ableitungen in die q-Poincaré-Algebra einbetten, ist es am Ende auch möglich für diese die Matrixelemente zu finden, und somit den kompletten q-Minkowski Phasenraum darzustellen.

Um die Klein-Gordon Gleichung auf dem q-Minkowski Raum lösen zu können, ist es erst einmal nötig beliebige Funktionen ableiten zu können. Dies ist aufgrund der komplizierten Algebra Relationen zwischen den Koordinaten und Ableitungen ein schwieriges kombinatorisches Problem. Wie wir zeigen werden kann man es mit Hilfe von erzeugenden Funktionen lösen. Dies erlaubt es uns dann den Ruhezustand zu bestimmen, welcher die korrekte q-deformierte Verallgemeinerung der zeitabhängigen Exponentialfunktion auf dem q-Minkowski Raum darstellt. Durch Boosten dieses Zustandes wird anschließend eine Basis für die gesamte irreduzible Darstellung gefunden, die den Lösungsraum der Klein-Gordon Gleichung umfasst. Dieselben Methoden können nun auch dazu benutzt werden die Dirac-Gleichung zu lösen und Zustände mit einem Spin-$\frac{1}{2}$ Freiheitsgrad zu beschreiben.

Contents

Chapter 1

Introduction

Two of the most fundamental principles to describe physical phenomena are the strongly interrelated concepts of space and symmetry. Usually we model these entities mathematically by differential manifolds and Lie groups, a formulation which is confirmed to be very successful by experiment. Nevertheless we meet profound difficulties, originating from the short distance behaviour of field theory, if we try to quantise gravitation. Quantum gravity has an uncertainty principle which prevents one from measuring positions to better accuracies than the Planck length: the momentum and energy required to make such a measurement will itself modify the geometry at these scales. This gives rise to the question, whether a differential manifold, which imposes strong constraints on the local structure of space, is really an adequate model at small scales or equivalently, at high energies.

One way to generalise the description of space is to postulate that the coordinates form a non-commutative algebra, its intrinsic structure being encoded in the two operations of the algebra: addition and multiplication. This is by no means a new idea. In quantum mechanics one replaces the phase space, which is in classical mechanics represented by a symplectic manifold, by the Heisenberg algebra. Here a coordinate and its conjugate momentum do not commute any more, leading to the position-momentum uncertainty. If we similarly assume that the coordinates themselves no longer commute with each other, we naturally introduce also a position uncertainty. H. Snyder was the first who gave a concrete model for such a quantised space-time by postulating a Lie algebra structure for the coordinates [1].

The starting point for this algebraic setup is the free associative algebra $\mathbb{C}[[\hat{x}^1, .., \hat{x}^n]]$ generated by the space coordinates $\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^n$. Because we are only interested in algebraic properties we will admit formal power series. The space algebra $\mathcal{M}_x$ is then constructed by factoring out a two sided ideal $\mathcal{I}$ generated by the commutation relations:

$$\mathcal{M}_x = \mathbb{C}[[\hat{x}^1, .., \hat{x}^n]] \big/ \mathcal{I}$$

In general the commutator for two coordinates is given by

$$[\hat{x}^i, \hat{x}^j] = \theta^{ij}(\hat{x})$$

As additional condition we impose the so called Poincaré-Birkhoff-Witt property. It says that the vector spaces generated by homogeneous polynomials with fixed degree, should all have the same dimension as in the commutative case with $\theta^{ij}(\hat{x}) = 0$. This ensures that the relations defining the algebra allow us to normal order the coordinates[1].

We get the main examples of non-commutative spaces if we choose θ to be constant, linear or quadratic in the coordinates. The constant case with $\theta^{ij} \in \mathbb{C}$ is called the canonical case, because it is well known from quantum mechanics. The Lie algebras are described with a θ that is a linear function of the coordinates $\theta^{ij}(\hat{x}) = \Theta^{ij}_k \hat{x}^k$, $\Theta^{ij}_k \in \mathbb{C}$, and quadratic relations with $\theta^{ij}(\hat{x}) = \Theta^{ij}_{kl}\hat{x}^k\hat{x}^l$ comprise spaces that are representations of quantum groups.

In this thesis the model for the space will be the q-Minkowski space. This space is a quadratic algebra which originates form the usual Minkowski space by a continuous one parameter deformation. One characteristic feature of this kind of non-commutativity is that we deform the space together with its symmetry structure. This does not work in the category of Lie groups itself, but can be achieved, if we change the mathematical description of symmetry and use Hopf algebras instead of groups [3–5]. Since the invention of quantum groups by Drinfeld [6], a systematic procedure to deform Lie algebras [7, 8] and matrix groups [9–11] within the category of Hopf algebras has been developed. This led to the construction of the q-deformed plane [12, 13], the quantum Euclidean space [10] and the q-Minkowski space [14–18], all spaces being representations of $U_q(su_2)$, resp. $U_q(sl_2(\mathbb{C}))$, the q-deformed analogues of the classical symmetry algebras [16, 19–22]. It is also possible to define a covariant differential calculus on these spaces [17, 23–25].

It is the aim of this dissertation to pave the way for the construction of a quantum field theory based on the q-Minkowski space. The propagation of free particles in space is the most elementary process in field theory and therefore this is the first thing that has to be generalised to the non-commutative world. In classical physics the description of free particles on the Minkowski space is completely controlled by group theory. Free elementary particles are modelled as irreducible, unitary representations of the Poincaré algebra [26, 27]. Free wave equations represent projectors which single out irreducible representations from the space spanned by the wave functions. But this rigorous mathematical framework is also present for the q-deformed spaces. The background symmetry of the q-Minkowski space is the q-Poincaré algebra, consequently its irreducible, unitary representations will model free q-particles.

[1] More precisely, we ask for algebra relations which form a "convergent reduction relation" [2]. That is, every polynomial has a unique normal form, which can be found by performing transformations with a finite set of rules.

The two parts of this thesis are devoted to the construction of such representations. In the first part we will calculate irreducible Hilbert space representations and in the second part we will find the solutions of the q-Klein-Gordon and q-Dirac equation, realising irreducible representations on the space of q-Minkowski wave functions.

Let us give a more detailed outline:

In chapter 2, we start calculating the Hilbert space representations of the q-Minkowski phase space. Like in ordinary quantum mechanics, we first have to choose a maximal set of commuting operators, which will determine the quantum numbers of a state in the representation. In our case the observables will be the 4-dimensional length $(X)^2$, the time X^0, the coordinate X^3, the third component of angular momentum T^3, the helicity H and the spin Casimir $\mathfrak{C}$. This set of operators differs from the one used in previous papers [28–32], which also deal with the computation of Hilbert space representations. There, instead of X^3 the square of the angular momentum $\vec{T}^2$ is used. But here we follow [33], where the 3-dimensional q-Euclidean space $\mathbb{R}_q^3$ was scrutinised and diagonalise an additional space coordinate. Furthermore our representations are not limited to spin zero as in [28–31] and different from [32], where only the case $(X)^2 < 0$ was treated, we also consider the regions $(X)^2 > 0$ and $(X)^2 = 0$. The generic procedure to compute the matrix elements of the various operators is to transform the relations defining the algebra into equations for the matrix elements by multiplying them from both sides with state vectors. Then we plug in all the matrix elements we already have determined and see what we find for the unknown expressions. Usually we encounter a system of recursion relations which we try to solve by successively eliminating the dependencies on the quantum numbers. In 2.3 we begin with the space coordinates. Because by construction the coordinates X^0 and X^3 are already diagonal, we can easily solve for X^+ and X^-. Using these results we proceed in 2.4 to the generators of rotations. The discrete spectrum of the space observables is calculated in 2.5 and the evaluation of the matrix element of H in 2.6 finally allows us to fix the representation of the q-Euclidean subalgebra.

In chapter 3, we deal with the matrix elements of the boost generators. At first we evaluate in section 3.1 and 3.2 the commutation relations of the boosts with the coordinates and rotations. As a result, we find what transitions the boosts induce on the state vectors and we can partially determine the dependency of the matrix elements on the quantum numbers. To completely fix the matrix elements we also have to consider the relations of the boosts among each other. This is done in chapter 3.3. These relations allow us to deduce recursion relations for the remaining unknown dependencies, which can be solved successively. In the end, there is only one free constant left. As it is shown in chapter 3.4, where we calculate the action of the spin Casimir, this constant is directly related to the spin of the representation.

In chapter 4, we calculate the representations of the derivatives. We show, that one can express the derivatives by the coordinates and Lorentz generators. The representations can then easily be obtained by inserting the previous results.

In the second part of this thesis we start in chapter 5 with the necessary preparations for our next task. We want to solve differential equations, so we have in any case to be able to differentiate functions. Because of the complicated commutation relations between the derivatives and the coordinates the generalisation to arbitrary functions amounts to a combinatorial problem. In section 5.1, we show how one can overcome this difficulty by using generating functions. Applying the same methods we calculate in 5.2 also the action of the Lorentz generators on functions.

In chapter 6, we solve the free q-Klein-Gordon equation. Different from the previously studied Hilbert space representations, we choose this time an angular momentum basis and simultaneously diagonalise the operators $(\partial)^2, \partial_0, T^3$ and $\vec{T}^2$. These eigenvectors will constitute the irreducible spin-0 representations of the Poincaré algebra in the space of q-Minkowski space functions $\mathcal{M}_q$. In the classical case they would correspond to solutions of the Klein-Gordon equation calculated in spherical coordinates. To construct these states we first determine in section 6.1 the rest state, which shall be deemed as the q-deformed generalisation of the exponential function. In section 6.2 this state is boosted to give us basis vectors spanning the whole irreducible representation. In contrast to the formal solutions given in [34, 35] we will obtain here concrete expressions for the spherical waves.

Chapter 7 is devoted to the solution of the free q-Dirac equation. In section 7.1, we construct the q-gamma matrices and examine their commutation relations with the coordinates, derivatives and spin degrees. In section 7.2, we generalise the methods used in the Klein-Gordon case to find irreducible representations on the tensor product spaces $\mathcal{M}_q \otimes \mathrm{D}^{(\frac{1}{2},0)}$ and $\mathcal{M}_q \otimes \mathrm{D}^{(0,\frac{1}{2})}$, giving us the Weyl spinors. Finally, we combine them in section 7.3 to the solutions of the q-Dirac equation.

Notation Throughout this work, we assume that the deformation parameter q is real, with $q > 1$. We frequently use the abbreviations

$$\lambda = q - q^{-1}, \qquad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad \{n\}_q = q^n + q^{-n} = \frac{[2n]_q}{[n]_q}$$

with $n \in \mathbb{Z}$.

Part I

Representations

Chapter 2

The matrix elements of the coordinates and rotations

2.1 The algebraic setup

In this section we start to calculate the irreducible Hilbert space representations of the q-Minkowski phase space. This algebra comprises the q-deformed Minkowski space, the q-Lorentz algebra and the algebra of the q-derivatives[1]. With the help of the $\mathcal{R}$-matrices we can easily write down the covariant commutation relations for the space, the derivatives and the q-deformed Leibniz rule:

$$X^i X^j = \mathcal{R}_{I\,kl}^{ij} X^k X^j, \quad \partial_i \partial_j = \mathcal{R}_{I\,ji}^{lk} \partial_k \partial_l\,, \quad \partial_a X^b = \delta_a^b + \mathcal{R}_{II\;ad}^{\;bc} X^d \partial_c$$

The $\mathcal{R}$-matrices can be found in section B.1 of the appendix and the explicit list of resulting relations is given in paragraph B.4.1. The form of the q-Lorentz algebra we will use here has been introduced in [16], [17]. It consists of seven elements: T^+, T^-, τ^3 generate an $\mathcal{U}_q(su_2)$ subalgebra interpreted as the rotations and $T^2, \tau^1, S^1, \sigma^2$ are four additional generators for the Lorentz boosts. There is an extra relation in the algebra which allows the elimination of one generator. If we consider the quantity

$$Z = \tau^1 \sigma^2 - q^2 \lambda^2 T^2 S^1$$

we find, that Z is central in the algebra and commutes with all of the coordinates. Therefore we set $Z = 1$ and one could make for example the substitution $\sigma^2 = (\tau^1)^{-1}(1 + q^2 \lambda^2 T^2 S^1)$. Instead of introducing inverse powers of τ^1, it will be convenient in the following to keep all seven generators, having in mind that they are not independent. The explicit commutation relations of these generators among themselves and with the coordinates are again listed in the appendix, paragraph B.5 and B.6. We prefer to work with this seven generator version

[1]We can define hermitian momenta if we combine the derivatives with their conjugates.

of the q-Lorentz algebra, because compared to other forms of the algebra, e.g. the RS-form [16, 19, 20] defined in paragraph B.2, the commutation relations are simpler.

2.2 The set of observables

The independent observables chosen to fix the basis of the representation are the operators $(X)^2$, X^0, X^3, τ^3 together with the helicity[2] $H = g_{ij}L^i X^j$ and the spin Casimir[3] $\mathfrak{C}$. These hermitian elements of the q-deformed Poincaré algebra constitute a maximal commuting set of operators. Therefore, our representation will live on the Hilbert space spanned by their common eigenvectors:

$$\begin{aligned}
(X)^2|l,t,z,m,h\rangle &= l\,|l,t,z,m,h\rangle \\
X^0|l,t,z,m,h\rangle &= t\,|l,t,z,m,h\rangle \\
X^3|l,t,z,m,h\rangle &= z\,|l,t,z,m,h\rangle \\
\tau^3|l,t,z,m,h\rangle &= \varphi(m)\,|l,t,z,m,h\rangle \\
H|l,t,z,m,h\rangle &= h\,|l,t,z,m,h\rangle \\
\mathfrak{C}|l,t,z,m,h\rangle &= \mathfrak{c}\,|l,t,z,m,h\rangle
\end{aligned}$$

Note that the length $(X)^2$ is a Casimir of the whole Poincaré algebra, whereas X^0 and H generate the centre of the Euclidean subalgebra, which is the semidirect product of the space with the rotations, see [32]. In the following we will use the light-cone coordinates defined in (B.21) for the q-Minkowski space, replacing X^0 and X^3 by the diagonal operators C and D:

$$\begin{aligned}
C|l,t,z,m,h\rangle &= \frac{(q^2t+z)}{\sqrt{q[2]_q}}|l,t,z,m,h\rangle \\
D|l,t,z,m,h\rangle &= \frac{(t-z)}{\sqrt{q[2]_q}}|l,t,z,m,h\rangle
\end{aligned} \tag{2.1}$$

Being the results for possible measurements, the eigenvalues of these operators are all real. Anticipating later results we have already introduced an integer label for the possible quantum numbers of τ^3 with the eigenvalue $\varphi(m) \in \mathbb{R}$. The scalar product for states with the same spin is defined as

$$\langle l',t',z',m',h'|l,t,z,m,h\rangle = \delta_{l',l}\delta_{t',t}\delta_{z',z}\delta_{m',m}\delta_{h',h}$$

[2]The components of the 4-vector $L = (L^+, L^3, L^-, W)$ are closely related to the operators T^+, T^3, T^- and also generate the subalgebra of rotations, see (B.3). In (B.4) the helicity operator is given in terms of our usual set of generators.

[3]see section 3.4

Starting from the diagonal matrix elements of the observables, we will successively determine the matrix elements of all the generators. For that we will evaluate the algebra relations by multiplying them from both sides with arbitrary state vectors and then try to solve for the unknown matrix elements.

2.3 The representation of the space

Let us first consider the algebra of the q-Minkowski space, defined in (B.20) and look at the relations:

$$AD = \frac{1}{q^2}DA, \qquad AC = CA + q\lambda\, AD$$

We multiply from the left with the bra vector $\langle l, t', z', m', h|$ and from the right with ket vector $|l, t, z, m, h\rangle$. The action of D is given in (2.1), so we find for the operator A:

$$\begin{aligned}
\langle l, t', z', m', h|A|l, t, z, m, h\rangle\,(q^2(t-z) - (t'-z')) &= 0 \\
\langle l, t', z', m', h|A|l, t, z, m, h\rangle\,((t-z') - q^2(t'-z)) &= 0
\end{aligned}$$

From these equations we can see, that the matrix element of A can only be non-zero if $t = t'$ and $z' = q^2 z - q\lambda t$. We have already used that $l' = l$, because the Casimir $(X)^2$ as well as X^0 commute with A and therefore the operator A neither changes the length l nor the time t. As we will see later on, the transition rule for z will lead to a discrete spectrum of X^3. To account for these facts we write:

$$\begin{aligned}
\langle l', t', z', m', h'|A|l, t, z, m, h\rangle &= \\
&\delta_{l',l}\delta_{t',t}\delta_{h',h}\delta_{z',q^2z-q\lambda t}\langle l, t, q^2z - q\lambda t, m', h|A|l, t, z, m, h\rangle
\end{aligned}$$

In exactly the same way we proceed with the coordinate B evaluating the relations:

$$BD = q^2 DB, \qquad BC = CB - \frac{\lambda}{q}\, BD$$

They give us

$$\begin{aligned}
\langle l', t', z', m', h'|B|l, t, z, m, h\rangle &= \\
&\delta_{l',l}\delta_{t',t}\delta_{h',h}\delta_{z',q^{-2}z+q^{-1}\lambda t}\langle l, t, \frac{z}{q^2} + \frac{\lambda}{q}t, m', h|B|l, t, z, m, h\rangle
\end{aligned}$$

Next we consider the relation:

$$(X)^2 = AB - \frac{1}{q^2}CD$$

We insert the above results and find

$$\delta_{m',m}(l+\frac{(t-z)(q^2t+z)}{q^3[2]_q}) = \sum_k \langle l,t,z,m',h|A|l,t,\frac{z}{q^2}+\frac{\lambda}{q}t,k,h\rangle\langle l,t,\frac{z}{q^2}+\frac{\lambda}{q}t,k,h|B|l,t,z,m,h\rangle \tag{2.2}$$

To eliminate B we use $B=\overline{A}$, which means for the matrix elements

$$\langle l,t,\frac{z}{q^2}+\frac{\lambda}{q}t,m',h|B|l,t,z,m,h\rangle = \overline{\langle l,t,z,m,h|A|l,t,\frac{z}{q^2}+\frac{\lambda}{q}t,m',h\rangle}$$

If we take this into account and additionally perform a shift $z \to q^2z - q\lambda t$ in the quantum number z, (2.2) becomes

$$\mathcal{A}\mathcal{A}^\dagger = (l+\frac{(t-z)(t+q^2z)}{q[2]_q})$$

where $\mathcal{A}$ is a matrix with matrix elements

$$\mathcal{A}_k^m = \langle l,t,q^2z-q\lambda t,m,h|A|l,t,z,k,h\rangle$$

We can solve this equation, and the corresponding one for B, by setting:

$$\begin{aligned} A|l,t,z,m,h\rangle) &= \sqrt{l+\frac{(t-z)(t+q^2z)}{q[2]_q}}|l,t,q^2z-q\lambda t,m+1,h\rangle \\ B|l,t,z,m,h\rangle) &= \sqrt{l+\frac{(t-z)(q^2t+z)}{q^3[2]_q}}|l,t,\frac{z}{q^2}+\frac{\lambda}{q}t,m-1,h\rangle \end{aligned} \tag{2.3}$$

2.4 The representation of the rotations

Let us continue with the generators of the rotations and first evaluate the relations with the space coordinates, listed in (B.24). We start with the equations

$$T^+D - DT^+ + \frac{1}{q}A = 0$$

and

$$\tau^3T^+ - \frac{1}{q^4}T^+\tau^3 = 0 \tag{2.4}$$

We insert the result of (2.3) and get[4]

$$\frac{1}{q}\sqrt{l+\frac{(t-z)(t+q^2z)}{q[2]_q}}\delta_{m',m+1}\delta_{z',q^2z-q\lambda t}+\frac{(z'-z)}{\sqrt{q[2]_q}}\langle l,t,z',m',h|T^+|l,t,z,m,h\rangle = 0 \tag{2.5}$$

[4] $[(X)^2,T^+]=[X^0,T^+]=0 \quad \Rightarrow \quad l'=l, t'=t$

and

$$\langle l,t,z',m',h|T^+|l,t,z,m,h\rangle(\varphi(m') - \frac{1}{q^4}\varphi(m)) = 0 \tag{2.6}$$

(2.4) shows that $T^+|l,t,z,m,h\rangle$ is again an eigenvector of τ^3. Therefore we assume that the label m is chosen in such a way that this state is indexed by $m+1$:

$$T^+|l,t,z,m,h\rangle \propto |l,t,z',m+1,h\rangle$$

This means that the matrix element of T^+ in (2.6) can only be non-zero, if

$$\varphi(m+1) \;=\; \frac{1}{q^4}\varphi(m) \qquad \Longrightarrow \qquad \varphi(m) = d\,q^{-4m}$$

with a constant $d \in \mathbb{R}$. As was shown in [33], we have to set $d = 1$ to describe a representation of $su_q(2)$.

From relation (2.5) we read off the possible transitions for the quantum number z. The equation is only true if either $z' = z$ or $z' = q^2 z - q\lambda t$. This fact allows us to make the following ansatz for the matrix element of T^+:

$$\begin{aligned}\langle l',t',z',m',h'|T^+|l,t,z,m,h\rangle \;&=\\ \delta_{l',l}\delta_{t',t}\delta_{h',h}\delta_{m',m+1}&\left(\delta_{z',z}\Gamma_{m+1,m}(z) + \delta_{z',q^2z-q\lambda t}\frac{\sqrt{q[2]_q l + (t-z)(t+q^2z)}}{q^2\lambda(t-z)}\right)\end{aligned}$$

Because the conjugate of T^+ is proportional to T^-, $T^- = q^2\,\overline{T^+}$, we immediately get for the matrix element of T^-

$$\begin{aligned}\langle l',t',z',m',h'|T^-|l,t,z,m,h\rangle \;&=\\ \delta_{l',l}\delta_{t',t}\delta_{h',h}\delta_{m',m-1}&\left(q^2\delta_{z',z}\Gamma_{m,m-1}(z) + q\,\delta_{z',\frac{z}{q^2}+\frac{\lambda}{q}t}\frac{\sqrt{q^3[2]_q l + (t-z)(q^2t+z)}}{\lambda(t-z)}\right)\end{aligned}$$

assuming that all matrix elements are real.

To obtain information about the matrix $\Gamma(z)$, we evaluate the commutation relation of $T^+$with T^-:

$$T^+T^- - q^2T^-T^+ + \frac{q}{\lambda}(\tau^3 - 1) = 0$$

Abbreviating $\gamma_m(z) := \Gamma_{m+1,m}(z)$ we find two independent equations:

$$\gamma_{m+1}(z) \;=\; \frac{1}{q^2}\gamma_m(\frac{z}{q^2} + \frac{\lambda}{q}t) \tag{2.7}$$

$$\gamma^2_{m+1}(z) \;=\; \frac{1}{q^2}\left(\gamma^2_m(z) + \frac{q^{-4m-5} + \frac{[2]_q l}{(t-z)^2}}{\lambda}\right) \tag{2.8}$$

The recurrence relations (2.8) for $\gamma_m^2(z)$ can easily be solved:

$$\gamma_m^2(z) = C(l,t,z,h)\, q^{-2m} + \frac{1}{q^4\lambda^2}\left(\frac{q^3[2]_q l}{(t-z)^2} - q^{-4m}\right) \tag{2.9}$$

Because it is a first order recursion, the set of solutions is parametrised by a free parameter $C(l,t,z,h)$, which may depend on the remaining quantum numbers. In the following it will we expedient to change the description of this parametrisation. We introduce the variable $x := q^{-2m}$ and consider the right side of equation (2.9) as a polynomial in x. Now we use a zero of the quadratic polynomial as a free variable replacing the constant $C(l,t,z,h)$. Defining $x_0(l,h,t,z)$ by the equation

$$C(l,t,z,h) = \frac{x_0(l,t,z,h)^2 - \frac{lq^3[2]_q}{(t-z)^2}}{q^4\lambda^2 x_0(l,h,t,z)}, \qquad x_0(l,t,z,h) > 0$$

we obtain

$$\gamma_m(z) = \sqrt{\frac{\left(q^{2m}x_0(l,t,z,h) - 1\right)\left(lq^{3+2m}[2]_q + (t-z)^2 x_0(l,t,z,h)\right)}{q^{4(1+m)}(t-z)^2\lambda^2 x_0(l,t,z,h)}} \tag{2.10}$$

where we have factorised the polynomial to identify the newly introduced parameter $x_0(l,t,z,h)$ explicitly with a zero.

To evaluate equation (2.7) we insert (2.10) and find the following transformation property of x_0 with respect to a shift $z \to q^2 z - q\lambda t$ and its inverse $z \to q^{-2}z - q^{-1}\lambda t$:

$$\begin{aligned} x_0(l,t,\frac{z}{q^2} + \frac{\lambda}{q}t, h) &= q^2 x_0(l,t,z,h) \\ x_0(l,t,q^2 z - q\lambda t, h) &= \frac{1}{q^2} x_0(l,t,z,h) \end{aligned} \tag{2.11}$$

2.5 The space-time lattice

In this section we will reveal the discrete structure of the space. Because X^0 is a singlet under rotations, it commutes with $T^{\pm}, \tau^3$ and therefore we also have to consider commutation relations containing boost generators to get transitions changing the time eigenvalue. Here we start to examine the matrix element of the boost τ^1. First note that

$$[\tau^1, \tau^3] = 0$$

Hence τ^1 does not change the quantum number m. The first relation we evaluate is

$$\tau^1 D - qD\tau^1 = 0$$

We get

$$\frac{1}{\sqrt{q[2]_q}}(t - z - q(t' - z'))\langle l, t', z', m', h'|\tau^1|l, t, z, m, h\rangle = 0$$

which means that

$$z' = \frac{1}{q}(qt' - t + z) \tag{2.12}$$

if the matrix-element of τ^1 is non zero. We proceed with the commutation relation with the coordinate A:

$$\tau^1 A - qA\tau^1 - q\lambda^2 DT^2 = 0$$

Because we can invert the coordinate D this relation allows us to express the matrix element of T^2 in terms of τ^1:

$$\begin{aligned}\langle l, t', z', m', h'|T^2|l, t, z, m, h\rangle \; &= \\ \frac{1}{q\lambda^2 (t' - z')}\Bigg[&\sqrt{t^2 + qtz\lambda + q\left([2]_q l - qz^2\right)}\langle l, t', z', m', h'|\tau^1|l, q^2 z - q\lambda t, z, m + 1, h\rangle \\ &-\langle l, t', \frac{z'}{q^2} + \frac{\lambda}{q}t', m' - 1, h'|\tau^1|l, t, z, m, h\rangle\sqrt{(t' - z')\left(q^2 t' + z'\right) + q^3[2]_q l'}\Bigg]\end{aligned} \tag{2.13}$$

Let us insert this expression into the relations

$$T^2 C - qA\tau^1 - qCT^2 = 0 \qquad \tau^1 C - \frac{1}{q}C\tau^1 - q\lambda^2 BT^2 - q\lambda^2 D\tau^1$$

In addition to (2.12) we find, that the transitions induced by τ^1 have to satisfy

$$q^3\lambda^2 l + q[2]_q\,(qt - t')\,(qt' - t) = 0 \tag{2.14}$$

Having two equation for z' and t' we can express them in terms of z and t:

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{2z}{q} + \lambda t \\ [2]_q t \end{pmatrix} \pm \frac{\lambda}{2\sqrt{q[2]_q}}\sqrt{t^2 + q^2\left(4l + t^2\right)}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.15}$$

Now we can determine the lattice structure of the spectrum.

2.5.1 The discretisation of z

From (2.3) we see that the action of A on a state vector shifts the quantum number z by the rule $z \to q^2 z - q\lambda t$. The inverse transformation is performed by the action of B: $z \to \frac{z}{q^2} + \frac{\lambda}{q}t$. Iterating these actions several times, z passes through the sequence

$$z(\nu) = t + q^{2\nu}(z_0 - t), \qquad \nu \in \mathbb{Z} \tag{2.16}$$

where the operator A, resp. B, increases, resp. decreases, the quantum number ν by one unit. z_0 is not yet fixed and will be determined later.

2.5.2 The discretisation of t

The case $l = 0$ The two possible transitions allowed by (2.15) simplify to:

$$t' = q\,t \qquad \text{or} \qquad t' = \frac{1}{q}t$$

Therefore the spectrum of X^0 is described by the sequence

$$t(n) = q^n \tau_0$$

with some constant τ_0. We insert this in expression (2.16) and find for $z(\nu)$:

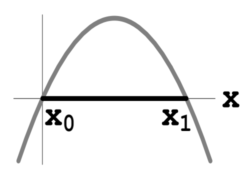
$$z(\nu) = q^\nu \left(q^\nu z_0 - q^n \lambda[\nu]_q \tau_0\right)$$

To ensure that the argument of the square root in the matrix element of the coordinate A in (2.3) never becomes negative, we have to fix the constant z_0 in the right way. The inequality we have to fulfil is:

$$l + \frac{(t-z)(t+q^2 z)}{q[2]_q} \geq 0 \tag{2.17}$$

Because for fixed time $|z|$ can not become arbitrarily large, we have to restrict the domain of the sequence $z(\nu)$. To clarify this we plug the expression for $z(\nu)$ and $t(n)$ into (2.17) and introduce the new variable $x := q^{2\nu}$. This gives the following quadratic polynomial in x:

$$-\frac{q\left(q^n\tau_0 - z_0\right)^2}{[2]_q} x \left(x - \frac{q^{-1+n}[2]_q \tau_0}{q^n \tau_0 - z_0}\right) \geq 0$$



To terminate the series for $z(\nu)$ we require that the zero x_1 coincides with $q^{2(n-1)}$:

$$q^{2(n-1)} = \frac{q^{-1+n}[2]_q \tau_0}{q^n \tau_0 - z_0}$$

This fixes the constant z_0 to

$$z_0 = q^{-n}\left(q^{2n} - q[2]_q\right)\tau_0$$

and our sequence $z(\nu)$ is

$$z(\nu) = q^\nu \left(\lambda[n-\nu]_q - q^{2-n+\nu}\right)\tau_0$$

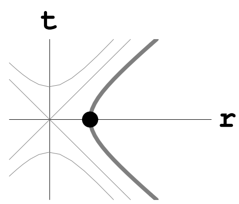
(2.17) simplifies to:

$$q^{2-n+3\nu}\lambda[2]_q\tau_0^2\,[n-1-\nu]_q \geq 0$$

Therefore, the square root is well defined for $\nu < n$, $n \in \mathbb{Z}$. For $\nu = n-1$, the operator A will annihilate the state vector and the series $z(\nu)$ will terminate. We can also check, that the matrix element of B is well defined.

Of course, you may choose z_0 in such a way that the series $z(\nu)$ stops at any given point $n+u \in \mathbb{Z}$. But this differs from the fixing above only by a renaming of the label ν: $\nu_{new} = \nu + (u+1)$. Here we have used $u = -1$ to be in accordance with the choice made in [31].

The case $l = \frac{l_0^2}{q[2]_q} > 0$ We assume that there is a state with $t = 0$. Graphically this state is depicted as the following point on the space like hyperboloid of constant positive length[5].



Iterating the time transformation in (2.15) for this initial value, we will produce the sequence

$$t(n) = \frac{\lambda[n]_q}{[2]_q} l_0$$

Using the plus sign in (2.15) will increase n by one: $t(n)' = t(n+1)$, whereas $t(n)' = t(n-1)$ for the minus sign.

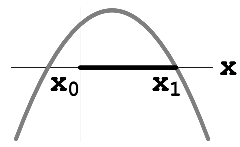
Plugging this into (2.16) we get for $z(\nu)$:

$$z(\nu) = q^\nu \left(q^\nu z_0 - \frac{\lambda^2[n]_q[\nu]_q}{[2]_q} l_0 \right)$$

Again we have to restrict the domain of the sequence $z(\nu)$ to ensure that the matrix element of A is well defined. This time the left side of (2.17) gives the following polynomial in $x = q^{2\nu}$:

$$-\frac{(q[2]_q z_0 - q\lambda[n]_q l_0)^2}{q{[2]_q}^3} \left(x + \frac{q^{-n}[2]_q l_0}{q\lambda[n]_q l_0 - q[2]_q z_0} \right) \left(x - \frac{q^n[2]_q l_0}{q\lambda[n]_q l_0 - q[2]_q z_0} \right) \geq 0$$

[5] In this picture r denotes the 3-dimensional radius and t the time.



In the above figure we have marked the allowed region for x by a thick line. Similarly to the the case $l = 0$ we fix the termination point of the series by demanding the right zero x_1 to be $q^{2(n-1)}$:

$$q^{2(n-1)} = \frac{q^n [2]_q l_0}{q\lambda [n]_q l_0 - q[2]_q z_0}$$

We solve for z_0

$$z_0 = \frac{l_0}{[2]_q} \left(q\lambda [n-1]_q - 2q^{-n} \right)$$

giving us

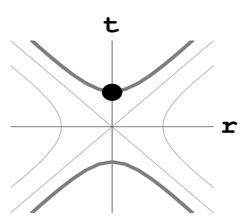
$$z(\nu) = \frac{q^{-n+\nu} l_0}{[2]_q} \left(q^n \lambda [n-\nu]_q - q\{1+\nu\}_q \right)$$

Furthermore, (2.17) reduces to

$$\frac{q^{1-n+2\nu} \lambda l_0^2}{[2]_q} \{\nu+1\}_q [n-1-\nu]_q \geq 0$$

Therefore the quantum number ν is bounded by n: $\nu < n$ with $n \in \mathbb{Z}$. This ensures also that the matrix element of B is well defined.

The case $l = -\frac{t_0^2}{q[2]_q} < 0$ Now we assume that there is a rest state with $t = t_0$.



If we successively apply the transformation (2.15), we will produce the sequence

$$t(n) = \frac{\{n+1\}_q}{[2]_q} t_0,$$

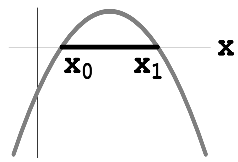
which gives for $z(\nu)$:

$$z(\nu) = q^{2\nu} z_0 - \frac{q^\nu \lambda \{\nu+1\}_q [\nu]_q}{[2]_q} t_0$$

The inequality (2.17) is now given by

$$-\frac{q([2]_q z_0 - \{\nu+1\}_q t_0)^2}{[2]_q{}^3}\left(x - \frac{q^{-2-n}[2]_q t_0}{\{n+1\}_q t_0 - [2]_q z_0}\right)\left(x - \frac{q^{n}[2]_q t_0}{\{n+1\}_q t_0 - [2]_q z_0}\right) \;\geq\; 0$$

This inequality selects a finite range for ν:



We choose

$$q^{2n} = \frac{q^n[2]_q t_0}{\{n+1\}_q t_0 - [2]_q z_0}$$

leading to

$$z_0 = \frac{q\lambda[n]_q}{[2]_q} t_0$$

and

$$z(\nu) = \frac{q^{-n+\nu}\lambda t_0}{[2]_q}\left(q^n[1+n-\nu]_q - [1+\nu]_q\right)$$

Now (2.17) is

$$\frac{q^{-n+2\nu}\lambda^2[n-\nu]_q[1+\nu]_q t_0^2}{[2]_q} \geq 0$$

and therefore the allowed region for ν is: $0 \leq \nu \leq n$, if we also take care that the matrix element of B is well defined.

2.5.3 The discretisation of l

To obtain a discrete spectrum for the Casimir $(X)^2$, we have to introduce the scaling operator Λ. From its commutation relations with the coordinates it follows

$$\Lambda^{\frac{1}{2}}|l,t,z,m,h\rangle \sim |q^2\, l, q\, t, z, m, h\rangle$$

As was shown in [30], we can represent Λ for $l \neq 0$ just by replacing $t_0 \to \widetilde{t}_0 q^M$ and $l_0 \to \widetilde{l}_0 q^M$ and setting

$$\Lambda^{\frac{1}{2}}|\mathrm{M},n,\nu,m,h\rangle = q^2|\mathrm{M}+1,n,\nu,m,h\rangle$$

For the case $l = 0$ the representation of Λ is realised by

$$\Lambda^{\frac{1}{2}}|0,n,\nu,m,h\rangle = q^2|0,n+1,\nu+1,m,h+1\rangle$$

The lattice

Let us summarise the results for the spectrum of the operators $(X)^2$, X^0, X^3 and the 3-dimensional length $(\vec{X})^2$:

For $l = 0$: $n \in \mathbb{Z}$ and $\nu < n$

$$\begin{aligned}
(X)^2|\mathrm{M}, n, \nu, m, h\rangle &= 0 \\
X^0|\mathrm{M}, n, \nu, m, h\rangle &= q^n \tau_0 \,|\mathrm{M}, n, \nu, m, h\rangle \\
X^3|\mathrm{M}, n, \nu, m, h\rangle &= q^\nu \left(\lambda[n-\nu]_q - q^{2-n+\nu}\right) \tau_0|\mathrm{M}, n, \nu, m, h\rangle \\
(\vec{X})^2|\mathrm{M}, n, \nu, m, h\rangle &= q^{2n}\tau_0^2 \,|\mathrm{M}, n, \nu, m, h\rangle
\end{aligned}$$

For $l = \frac{q^{2M}\tilde{l}_0^2}{q[2]_q} > 0$: $n \in \mathbb{Z}$ and $\nu < n$

$$\begin{aligned}
(X)^2|\mathrm{M}, n, \nu, m, h\rangle &= \frac{q^{2M}}{q[2]_q}\tilde{l}_0^2 \,|\mathrm{M}, n, \nu, m, h\rangle \\
X^0|\mathrm{M}, n, \nu, m, h\rangle &= \frac{\lambda[n]_q}{[2]_q} q^M \,\tilde{l}_0 \,|\mathrm{M}, n, \nu, m, h\rangle \\
X^3|\mathrm{M}, n, \nu, m, h\rangle &= \frac{q^{-n+\nu}}{[2]_q} \left(q^n\lambda[n-\nu]_q - q\{1+\nu\}_q\right) q^M \,\tilde{l}_0 \,|\mathrm{M}, n, \nu, m, h\rangle \\
(\vec{X})^2|\mathrm{M}, n, \nu, m, h\rangle &= \left(1 + \frac{\lambda^2 [n]_q{}^2}{[2]_q{}^2}\right) q^{2M}\,\tilde{l}_0^2 \,|\mathrm{M}, n, \nu, m, h\rangle
\end{aligned}$$

For $l = -\frac{t_0^2}{q[2]_q} < 0$: $n \in \mathbb{N}_0$ and $0 \le \nu \le n$

$$\begin{aligned}
(X)^2|\mathrm{M}, n, \nu, m, h\rangle &= -\frac{q^{2M}}{q[2]_q}\tilde{t}_0^2 \,|\mathrm{M}, n, \nu, m, h\rangle \\
X^0|\mathrm{M}, n, \nu, m, h\rangle &= \frac{\{n+1\}_q}{[2]_q} \, q^M \,\tilde{t}_0 \,|\mathrm{M}, n, \nu, m, h\rangle \\
X^3|\mathrm{M}, n, \nu, m, h\rangle &= \frac{q^{-n+\nu}\lambda}{[2]_q} \left(q^n[1+n-\nu]_q - [1+\nu]_q\right) q^M \,\tilde{t}_0 \,|\mathrm{M}, n, \nu, m, h\rangle \\
(\vec{X})^2|\mathrm{M}, n, \nu, m, h\rangle &= \frac{\lambda^2}{[2]_q{}^2}[n]_q[n+2]_q \, q^{2M}\,\tilde{t}_0^2 \,|\mathrm{M}, n, \nu, m, h\rangle
\end{aligned}$$

To get an idea how these spectra look like, we draw the eigenvalues of X^0 versus the values of $\sqrt{(\vec{X})^2}$. The resulting space-time lattice, where all three cases are combined, is shown in figure (2.1). You can still recognise the hyperbolas of constant length (dotted lines), which are now fixed by the quantum number M. But this time they are set up by a discrete series of allowed spectral points. A more detailed discussion of the spectrum is given in [31].

The matrix elements of the coordinates with discrete quantum numbers are given in the appendix (A.2).

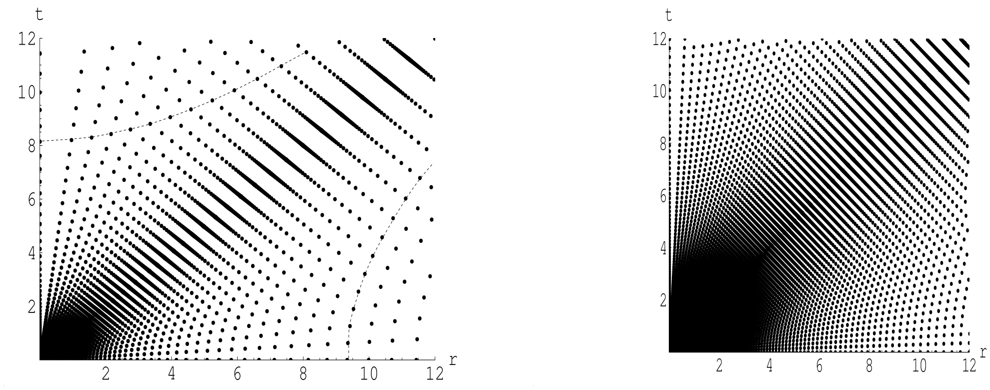


Figure 2.1: space-time lattice for $q = 1.07$ and $q = 1.03$

2.6 The matrix element of H

If we insert the discrete values for the quantum numbers in equation (2.11) we get the ν dependence of $x_0(l, t, z, h)$:

$$x_0(l, n, \nu, h) = q^{-2\nu} x_0(l, n, h)$$

Because H is a Casimir operator[6] of $\mathcal{U}_q(su_2)$ like X^0, the possible eigenvalues of H are the same as for X^0. Therefore we know:

$$H|l, n, \nu, m, h\rangle = \begin{cases} q^h \tau_0 |0, n, \nu, m, h\rangle & \text{for } l = 0 \\ \frac{1}{[2]_q}\{h+1\}_q t_0| - \frac{t0^2}{q[2]_q}, n, \nu, m, h\rangle & \text{for } l < 0 \\ \frac{\lambda[h]_q}{[2]_q}{}_0 l_0|\frac{l0^2}{q[2]_q}, n, \nu, m, h\rangle & \text{for } l > 0 \end{cases}$$

But H is given by

$$H = \frac{1}{\sqrt{q[2]_q}}\left[D\big(\tau^3\big)^{\frac{1}{2}} + \big(C - \lambda A T^- + q^2\lambda\left(\lambda D T^- T^+ - B T^+\right)\big)\left(\tau^3\right)^{-\frac{1}{2}}\right]$$

So we can calculate its action also in terms of the matrix elements of the coordinates and the rotations. Doing this we find:

$$H|l, n, \nu, m, h\rangle = \begin{cases} q^n \tau_0\, x_0(0, n, h)\, |0, n, \nu, m, h\rangle & \text{for } l = 0 \\ \frac{q^{-1-n}\left(q^{2(1+n)} + x_0(-\frac{t0^2}{q[2]_q}, n, h)^2\right)}{[2]_q x_0(-\frac{t0^2}{q[2]_q}, n, h)} t_0| - \frac{t0^2}{q[2]_q}, n, \nu, m, h\rangle & \text{for } l < 0 \\ \frac{\left(x_0(\frac{l0^2}{q[2]_q}, n, h)^2 - q^{2n}\right)}{q^n [2]_q x_0(\frac{l0^2}{q[2]_q}, n, h)} l_0|\frac{l0^2}{q[2]_q}, n, \nu, m, h\rangle & \text{for } l > 0 \end{cases}$$

[6]See [32, 36]

These two actions have to be equal, which allows us to solve for $x_0(l, n.h)$:

$$x_0(l,n,h) = \begin{cases} \qquad q^{h+n} & \text{for } l = 0 \\ q^{n-h} \quad \text{or} \quad q^{h+n+2} & \text{for } l < 0 \\ q^{h+n} \quad \text{or} \quad -q^{n-h} & \text{for } l > 0 \end{cases}$$

We find for the case $l \neq 0$ two possible solutions for x_0. But inserting them in (2.10), they both give the same expression for the function γ_m and therefore give also the same final result for the rotations, which can be found in the appendix A.3.

Chapter 3

The representation of the boosts

3.1 The relations with the coordinates

Equation (2.15) shows the two possible transitions induced by the generator τ^1. For the discrete quantum numbers these transitions reduce to the shifts: $n' = n+1$ and $\nu' = \nu$ or $n' = n-1$ and $\nu' = \nu - 1$. This enables us to make the following ansatz for the action of τ^1:

$$\begin{aligned}\tau^1|l,n,\nu,m,h\rangle &= \\ \sum_{h'} \tau^1_1(l,n,\nu,m,h,h')|l,n-1,\nu-1,m,h'\rangle &+ \tau^1_2(l,n,\nu,m,h,h')|l,n+1,\nu,m,h'\rangle\end{aligned} \tag{3.1}$$

Using this ansatz we reevaluate the commutation relations of τ^1 and T^2 with the coordinates. What we find are rules, that allow us to shift the arguments ν and m of the functions $\tau^1_{1/2}$ simultaneously about one unit. In the case $l = 0$, we get

$$\begin{aligned}\tau^1_1(0,n,1+\nu,1+m,h,h') &= q^2\tau^1_1(0,n,\nu,m,h,h') \\ \tau^1_2(0,n,1+\nu,1+m,h,h') &= q^{\frac{3}{2}}\sqrt{\frac{[-1+n-\nu]_q}{[n-\nu]_q}}\tau^1_2(0,n,\nu,m,h,h')\end{aligned} \tag{3.2}$$

and for $l \neq 0$ the relations are listed in (A.2). This is all what we can deduce from the commutation relations with the coordinates.

3.2 The relations with the rotations

Let us proceed with the relation

$$\tau^1T^+ - T^+\tau^1 - \lambda T^2 = 0$$

We rewrite T^2 with the help of (2.13) in terms of τ^1 and insert the matrix elements of the rotations and the ansatz for τ^1. This time we can derive rules which accomplish a shift in the quantum number m alone. Again, due to the length of the expressions, we only show here the case $l = 0$

$$\tau^1_{1/2}(0,n,\nu,1+m,h,h') \;=\; q^{\frac{1-h+h'}{4}}\sqrt{\frac{[\frac{1+2m+n-2\nu+h'}{2}]_q}{[\frac{h+2m+n-2\nu}{2}]_q}}\,\tau^1_{1/2}(0,n,\nu,m,h,h') \quad (3.3)$$

and list the other cases in the appendix, see equation (A.3). Together with (3.2) it is now possible to shift ν and m separately.

The next relation we consider is

$$\tau^1 T^- - \frac{1}{q^2}T^-\tau^1 + \lambda S^1 = 0$$

To evaluate it we plug in the ansatz for τ^1 and the known action for T^-. Because $S^1 = -(\tau^3)^{\frac{1}{2}}\overline{T^2}$ and (2.13), also the matrix element of S^1 can be expressed by the ansatz of τ^1. In the end we get for the case $l = 0$ the following four independent relations[1]:

$$\begin{aligned}
0 &= q^2\tau^1_1(0,n,-1+\nu,-1+m,h,h') - \tau^1_1(0,n,\nu,m,h,h') \\
0 &= q^{\frac{3}{2}}\sqrt{[n-\nu]_q}\,\tau^1_2(0,n,-1+\nu,-1+m,h,h') - \sqrt{[1+n-\nu]_q}\,\tau^1_2(0,n,\nu,m,h,h') \\
0 &= q^{\frac{7+h+2m}{4}}\sqrt{[\frac{-2+h+2m+n-2\nu}{2}]_q}\,\tau^1_1(0,n,\nu,-1+m,h,h') \\
&\quad -q^{\frac{h'+2m}{4}}\sqrt{[\frac{-1+h'+2m+n-2\nu}{2}]_q}\,\tau^1_1(0,n,\nu,m,h,h') \\
&\quad +q^{\frac{5+n}{4}}\sqrt{[n-\nu]_q}\,\tau^1_2(0,-1+n,-1+\nu,-1+m,h',h) \\
&\quad -q^{\frac{3+n}{4}}\sqrt{[n-1-\nu]_q}\,\tau^1_2(0,-1+n,\nu,m,h',h) \\
0 &= q^{\frac{5+n}{4}}\sqrt{[n-\nu]_q}\left(\tau^1_1(0,1+n,\nu,-1+m,h',h) - \tau^1_1(0,1+n,1+\nu,m,h',h)\right) \\
&\quad +q^{\frac{7+h+2m}{4}}\sqrt{[\frac{-2+h+2m+n-2\nu}{2}]_q}\,\tau^1_2(0,n,\nu,-1+m,h,h') \\
&\quad -q^{\frac{h'+2m}{4}}\sqrt{[\frac{-1+h'+2m+n-2\nu}{2}]_q}\,\tau^1_2(0,n,\nu,m,h,h')
\end{aligned}$$

These relations can now be further simplified, if we apply the rules (3.2) and (3.3). We find that the left side of the first two equation give 0, therefore these

[1] Due to the length of the relations for the cases $l \neq 0$ and because the procedure for their evaluation is same as for the case $l = 0$, we do not show them here.

relations contain no new information. But the third relation allows us to express τ_2^1 in terms of τ_1^1:

$$\begin{aligned}\tau_2^1(0,n,\nu,m,h,h') &= q^{\frac{2-h-2m+n-8\nu}{4}}\sqrt{\frac{[n-\nu]_q}{[\frac{h+2m+n-2\nu}{2}]_q}} \\ &\left(q^{\frac{1}{2}+2\nu}-q^{\frac{h+4m+2n+h'}{2}}[\frac{h'+1-h}{2}]_q\right)\tau_1^1(0,1+n,\nu,m,h',h)\end{aligned}$$

Again, we list the results for the cases $l>0$ and $l<0$ only in the appendix, see (A.4). Inserting this into the last equation, the left side turns out to be proportional to $[\frac{1}{2}(h'-h-1)]_q[\frac{1}{2}(h'-h+1)]_q * \tau_1^1(l,n+1,\nu,m,h,h')$. In fact, this is also true for the cases $l\neq 0$. Therefore the function $\tau_1^1(l,n,\nu,m,h,h')$ can only be non-zero for $h'=h+1$ or $h'=h-1$ and our ansatz (3.1) simplifies to

$$\begin{aligned}\tau^1|l,n,\nu,m,h\rangle &= \\ &\tau_{1,-1}^1(l,n,\nu,m,h)|l,n-1,\nu-1,m,h-1\rangle+\tau_{2,-1}^1(l,n,\nu,m,h)|l,n+1,\nu,m,h-1\rangle \\ &\tau_{1,1}^1(l,n,\nu,m,h)|l,n-1,\nu-1,m,h+1\rangle+\tau_{2,1}^1(l,n,\nu,m,h)|l,n+1,\nu,m,h+1\rangle\end{aligned} \tag{3.4}$$

where we have used the abbreviation $\tau_{1,\pm1}^1(l,n,\nu,m,h):=\tau_1^1(l,n,\nu,m,h,h\pm1)$, likewise for τ_2^1.

3.3 The relations with the boost generators

To get information about the function $\tau_{1,\pm1}^1(l,n,\nu,m,h)$, we now consider the relations

$$\begin{aligned}\tau^1T^2-q^2\,T^2\tau^1 &= 0\\ \tau^1S^1-S^1\tau^1 &= 0\\ \tau^1\sigma^2-\sigma^2\tau^1-q\lambda^3\,T^2S^1 &= 0\end{aligned}$$

The evaluation of these relations is a straight forward but lengthy calculation. The procedure is clear. We plug in the ansatz (3.4) for the action of τ^1 and also express T^2, $S^1=-(\tau^3)^{\frac{1}{2}}\overline{T^2}$ and $\sigma^2=(\tau^3)^{\frac{1}{2}}\overline{\tau^1}$ through the matrix element of τ^1. Then we use the rules in (A.4) to express the function τ_2^1 by τ_1^1 and the rules (A.2) and (A.3) to eliminate any shift in the quantum numbers ν or m in the arguments of the functions $\tau_{1,\pm1}^1(l,n,\nu,m,h)$.

We will find two sorts of relations. The first set of equations does contain the function $\tau_{1,\pm1}^1(l,n,\nu,m,h)$ only linearly. They will give us rules, which translate a shift in the quantum number h to a shift in the quantum number n. Explicitly, we get for the case $l = 0$

$$\tau^1_{1,1}(0,n,\nu,m,1+h) = q^{-\frac{5}{2}}\sqrt{\frac{[\frac{h+1+2m+n-2\nu}{2}]_q}{[\frac{h-1+2m+n-2\nu}{2}]_q}}\tau^1_{1,1}(0,n-1,\nu,m,h) \quad (3.5)$$

$$\tau_{1,-1}(0,n,\nu,m,1+h) = q\,\tau_{1,-1}(0,1+n,\nu,m,h) \quad (3.6)$$

and for the case $l \neq 0$ we refer to the appendix (A.5) and (A.6).

The second set of relations contains the functions $\tau^1_{1,\pm1}(l,n,\nu,m,h)$ only quadratically. Here we treat the cases $l = 0$ and $l \neq 0$ separately:

The case $l = 0$ If we simplify the expressions we result in the following two independent relations:

$$\tau^1_{1,1}(0,2+n,\nu,m,h)^2 = \frac{[\frac{2+h+2m+n-2\nu}{2}]_q}{q^7[\frac{h+2m+n-2\nu}{2}]_q}\,\tau^1_{1,1}(0,n,\nu,m,h)^2 \quad (3.7)$$

$$\begin{aligned}\tau^1_{1,1}(0,n,\nu,m,h)^2 &= q^{5+\frac{h}{2}+m+\frac{n}{2}-\nu}[\frac{h+2m+n-2\nu}{2}]_q \\ &\quad \left(\tau^1_{1,-1}(0,n+2,\nu,m,h)^2 - q^{-6}\tau^1_{1,-1}(0,n,\nu,m,h)^2\right)\end{aligned}$$

You see that the second equation expresses $\tau^1_{1,1}$ in terms of $\tau^1_{1,-1}$. If we plug this into the first relation we deduce a recursion relation for the n dependence of $\tau^1_{1,-1}$:

$$\tau^1_{1,-1}(0,n,\nu,m,h)^2 - q^7[2]_q\tau^1_{1,-1}(0,n+2,\nu,m,h)^2 + q^{14}\tau^1_{1,-1}(0,n+4,\nu,m,h)^2 = 0$$

The cases $l > 0$ and $l < 0$ Similar to the case $l = 0$ we get relations which allow us to eliminate the function $\tau^1_{1,-1}$

$$\begin{aligned}
\tau^1_{1,-1}(\frac{{l_0}^2}{q[2]_q}, n, \nu, m, h)^2 &= \frac{\{1+h\}_q\{\frac{h-2m-n}{2}+\nu\}_q}{q^h\lambda^3[h-n]_q[\frac{h+2m+n-2\nu}{2}]_q[\frac{2+h+2m+n-2\nu}{2}]_q\{-1+h\}_q\{n\}_q} \\
&\quad \Bigg[\{-1+h\}_q{\{n\}_q}^2[\frac{2+h+2m+n-2\nu}{2}]_q\tau^1_{1,1}(\frac{{l_0}^2}{q[2]_q}, n, \nu, m, h)^2 \\
&\quad -q^3\{h\}_q\{1+n\}_q\{2+n\}_q[\frac{h+2m+n-2\nu}{2}]_q\tau^1_{1,1}(\frac{{l_0}^2}{q[2]_q}, n, \nu, m, h)^2\Bigg] \\
\tau^1_{1,-1}(-\frac{{t_0}^2}{q[2]_q}, n, \nu, m, h)^2 &= \frac{q^{-1-h}[2+h]_q[\frac{h}{2}-m-\frac{n}{2}+\nu]_q}{[h]_q[h-n]_q[1+n]_q} \\
&\quad \Bigg[\frac{q^3[1+h]_q[2+n]_q[3+n]_q}{[\frac{4+h+2m+n-2\nu}{2}]_q}\tau^1_{1,1}(-\frac{{t_0}^2}{q[2]_q}, 2+n, \nu, m, h)^2 \\
&\quad -\frac{[h]_q{[1+n]_q}^2}{[\frac{2+h+2m+n-2\nu}{2}]_q}\tau^1_{1,1}(-\frac{{t_0}^2}{q[2]_q}, n, \nu, m, h)^2\Bigg]
\end{aligned} \tag{3.8}$$

In addition we find recursion relations for the n dependence of $\tau^1_{1,1}$:

$$\begin{aligned}
0 &= \frac{\{n-1\}_q}{[\frac{h+2m+n-2\nu}{2}]_q}\,\tau^1_{1,1}(\frac{{l_0}^2}{q[2]_q}, n, \nu, m, h)^2 \\
&\quad -\frac{q^3[2]_q[-1+h-n]_q\{n+2\}_q\{n+1\}_q}{[-2+h-n]_q\{n\}_q[\frac{2+h+2m+n-2\nu}{2}]_q}\,\tau^1_{1,1}(\frac{{l_0}^2}{q[2]_q}, 2+n, \nu, m, h)^2 \\
&\quad +\frac{q^6[h-n]_q\{n+3\}_q\{n+4\}_q}{[-2+h-n]_q\{n\}_q[\frac{4+h+2m+n-2\nu}{2}]_q}\,\tau^1_{1,1}(\frac{{l_0}^2}{q[2]_q}, 4+n, \nu, m, h)^2
\end{aligned}$$

$$\begin{aligned}
0 &= \frac{[n]_q}{[\frac{2+h+2m+n-2\nu}{2}]_q}\,\tau^1_{1,1}(-\frac{{t_0}^2}{q[2]_q}, n, \nu, m, h, 1, 1)^2 \\
&\quad -\frac{q^3[2]_q[-1+h-n]_q[2+n]_q[3+n]_q}{[-2+h-n]_q[1+n]_q[\frac{4+h+2m+n-2\nu}{2}]_q}\,\tau^1_{1,1}(-\frac{{t_0}^2}{q[2]_q}, 2+n, \nu, m, h, 1, 1)^2 \\
&\quad +\frac{q^6[h-n]_q[4+n]_q[5+n]_q}{[-2+h-n]_q[1+n]_q[\frac{6+h+2m+n-2\nu}{2}]_q}\,\tau^1_{1,1}(-\frac{{t_0}^2}{q[2]_q}, 4+n, \nu, m, h, 1, 1)^2
\end{aligned}$$

These equations can be simplified a good deal more, if we perform the following substitutions, defining the functions $R(l_0, n)$ and $R(t_0, n)$:

$$\begin{aligned}
\tau^1_{1,1}(\frac{{l_0}^2}{q[2]_q}, n, \nu, m, h)^2 &= \frac{q^{1+\frac{h}{2}+m-\frac{3n}{2}+\nu}\lambda[n-1]_q[n]_q[\frac{h+2m+n-2\nu}{2}]_q}{[2n]_q[2n-2]_q}R(l_0,n) \\
\tau^1_{1,1}(-\frac{{t_0}^2}{q[2]_q}, n, \nu, m, h, 1, 1)^2 &= -\frac{q^{\frac{h}{2}+m-\frac{3n}{2}+\nu}[\frac{2+h+2m+n-2\nu}{2}]_q}{\lambda[n]_q[1+n]_q}R(t_0, n)
\end{aligned} \tag{3.9}$$

Indeed, using these transformations we will end up with only one recursion relation, valid in both cases:

$$[h-n-2]_q R(l_0/t_0, n) - [2]_q[h-n-1]_q R(l_0/t_0, 2+n) + [h-n]_q R(l_0/t_0, 4+n) = 0 \tag{3.10}$$

Remember that the generators τ^1, T^2, σ^2 and S^1 are not independent. They are related by the relation

$$1 = \sigma^2\tau^1 - \lambda^2 S^1 T^2 \tag{3.11}$$

which can be used to find a second recursion relation for the n-dependence of $\tau_{1,1}(0, n, \nu, m, h)$ and $R(l_0/t_0, n)$.

For the case $l = 0$ equation (3.11) reduces to

$$\begin{aligned}
1 &= q^{-2+h+3n-4\nu}\tau^1_{1,-1}(0, n, \nu, m, h)^2 + q^{-2(m-n+\nu)}\tau_{1,1}(0, n, \nu, m, h)^2 \\
&+ \frac{q^{7-\frac{h}{2}-3m+\frac{3n}{2}-\nu}}{\lambda[\frac{2+h+2m+n-2\nu}{2}]_q}\tau^1_{1,1}(0, 2+n, \nu, m, h)^2
\end{aligned} \tag{3.12}$$

and for $l \neq 0$ we again refer to the appendix (A.7, A.8).

Now we are prepared to solve the various recurrence relations. We start with

the case $l = 0$ With the help of (3.12) we simplify (3.7) and obtain:

$$\tau^1_{1,1}(0, n, \nu, m, h)^2 = q^{-2-\frac{h}{2}+m-\frac{5n}{2}-\nu}\lambda[\frac{h+2m+n-2\nu}{2}]_q\left(q^{2+4\nu} - q^{h+3n}\tau^1_{1,-1}(0, n, \nu, m, h)^2\right)$$

$$0 = q^{3+4\nu}\lambda + q^{h+3n}\tau^1_{1,-1}(0, n, \nu, m, h)^2 - q^{8+h+3n}\tau^1_{1,-1}(0, 2+n, \nu, m, h)^2$$

It is easy to solve these equations:

$$\begin{aligned}
\tau^1_{1,-1}(0,n,\nu,m,h)^2 &= q^{2-h-3n+4\nu} \\
&\quad + \frac{1}{2q^{4n}}\left(\left(-1+(-1)^n\right)C_1(0,\nu,m,h) + \left(1+(-1)^n\right)C_2(0,\nu,m,h)\right) \\
\tau^1_{1,1}(0,n,\nu,m,h)^2 &= -\frac{1}{2}q^{\frac{-4+h+2m-7n-2\nu}{2}}\lambda[\frac{h+2m+n-2\nu}{2}]_q \\
&\quad \Big[\left(-1+(-1)^n\right)C_1(0,\nu,m,h) + \left(1+(-1)^n\right)C_2(0,\nu,m,h)\Big]
\end{aligned} \tag{3.13}$$

The n independent functions $C_1(0,\nu,m,h)$ and $C_2(0,\nu,m,h)$ parametrise the possible solutions and are not fixed by the recursion relations. Even though we have a first order recurrence, we get here two constants to determine the initial value, because n is only coupled with $n+2$, giving us for the even and odd integers independent sequences.

To determine the dependency of the functions C_1 and C_2 on the other quantum numbers, we refer to the relations (A.2) and (A.3). Inserting the expressions (3.13) we are able to find the ν and m dependence:

$$C_{1/2}(0,\nu,m,h) = q^{4\nu}C_{1/2}(h)$$

and from (3.6) we deduce a recursion relation for the h dependence

$$\begin{aligned}
C_1(h+1) &= q^{-2}C_2(h) \\
C_2(h+1) &= q^{-2}C_1(h)
\end{aligned}$$

The solution is

$$\begin{aligned}
C_1(h) &= q^{-2h}\left(\frac{1}{2}(1+(-1)^h)C_1(0) + \frac{1}{2}(-1+(-1)^h)C_2(0)\right) \\
C_2(h) &= q^{-2h}\left(\frac{1}{2}(-1+(-1)^h)C_1(0) + \frac{1}{2}(1+(-1)^h)C_2(0)\right)
\end{aligned}$$

and gives us finally the following expressions for $\tau^1_{1,\pm 1}$:

$$\begin{aligned}
\tau^1_{1,-1}(0,n,\nu,m,h) &= q^{-h-2n+2\nu}\sqrt{q^{2+h+n} - C(0,h+n)} \\
\tau^1_{1,1}(0,n,\nu,m,h) &= q^{-\frac{1}{4}(3h-2m+7n-6\nu+4)}\sqrt{\lambda[\frac{h+2m+n-2\nu}{2}]_q C(0,h+n)}
\end{aligned} \tag{3.14}$$

with the abbreviation $C(0,n+h) := -\frac{1}{2}\Big[(1+(-1)^{n+h})C_2(0) + (-1+(-1)^{n+h})C_1(0)\Big]$.

In a similar way we proceed with the other cases.

The cases $l \neq 0$ If we combine (3.10) with (A.7) and (A.8) we can simplify the second order recurrences to a first order one:

$$\begin{aligned} R(l_0, n+2) &= R(l_0, n) - \frac{q^\nu \lambda^2 [h-n]_q \{\nu\}_q}{\{h\}_q \{h+1\}_q} \\ R(t_0, n+2) &= R(t_0, n) - \frac{q^\nu \lambda [h-n]_q [\nu]_q}{[1+h]_q [2+h]_q} \end{aligned}$$

Again, we get for the even and odd integers two independent sets of solutions, parametrised by the n independent functions $C_{1/2}(l_0/t_0, \nu, m, h)$:

$$\begin{aligned} R(l_0, n) &= \frac{(1+(-1)^n)}{2} C_2(l_0, \nu, m, h) - \frac{(-1+(-1)^n)}{2} C_1(l_0, \nu, m, h) \\ &+ \frac{q^\nu \{\nu\}_q}{\{h\}_q \{1+h\}_q} \left(\frac{(-1+(-1)^n)}{2} \{h\}_q - \frac{(1+(-1)^n)}{2} \{1+h\}_q + \{1+h-n\}_q \right) \\ R(t_0, n) &= \frac{(1+(-1)^n)}{2} C_2(t_0, \nu, m, h) - \frac{(-1+(-1)^n)}{2} C_1(t_0, \nu, m, h) + \\ &+ \frac{q^\nu [\nu]_q}{\lambda [1+h]_q [2+h]_q} \left(\frac{(-1+(-1)^n)}{2} \{h\}_q - \frac{(1+(-1)^n)}{2} \{1+h\}_q + \{1+h-n\}_q \right) \end{aligned}$$

Then we fix the ν and m dependence of the functions $C_{1/2}$ with the help of the relations (A.2) and (A.3):

$$\begin{aligned} C_{1/2}(l_0, \nu, m, h) &= q^\nu \{\nu\}_q C_{1/2}(l_0, h) \\ C_{1/2}(t_0, \nu, m, h) &= \lambda q^\nu [\nu]_q C_{1/2}(t_0, h) \end{aligned}$$

(A.5,A.6) is used to find a recursion relation for the h-dependence:

$$\begin{aligned} C_2(l_0, h) &= \frac{\{2+h\}_q}{\{h\}_q} C_1(l_0, h+1) \\ C_1(l_0, h) &= \frac{1}{\{h\}_q} \left(\{h+2\}_q C_2(l_0, h+1) - \frac{\lambda^2 [1+h]_q}{\{h+1\}_q} \right) \end{aligned}$$

$$\begin{aligned} C_2(t_0, h) &= \frac{[3+h]_q}{[1+h]_q} C_1(t_0, h+1) \\ C_1(t_0, h) &= \frac{[3+h]_q}{[1+h]_q} C_2(t_0, h+1) - \frac{1}{[2+h]_q} \end{aligned}$$

The solution is

$$\begin{aligned} C_1(l_0, h) &= \frac{\lambda^2 [\frac{h}{2}]_q^{\,2} + [2]_q \left((1+(-1)^h) C_1(l_0) - (-1+(-1)^h) C_2(l_0) \right)}{\{h\}_q \{h+1\}_q} \\ C_2(l_0, h) &= \frac{\lambda^2 [\frac{1+h}{2}]_q^{\,2} + [2]_q \left((1+(-1)^h) C_2(l_0) - (-1+(-1)^h) C_1(l_0) \right)}{\{h\}_q \{h+1\}_q} \end{aligned}$$

$$C_1(t_0,h) = \frac{2[\frac{h}{2}]_q{}^2 + [2]_q\left((1+(-1)^h)C_1(t_0) - (-1+(-1)^h)C_2(t_0)\right)}{2[1+h]_q[2+h]_q}$$

$$C_2(t_0,h) = \frac{2[\frac{1+h}{2}]_q{}^2 + [2]_q\left((1+(-1)^h)C_2(t_0) - (-1+(-1)^h)C_1(t_0)\right)}{2[1+h]_q[2+h]_q}$$

Altogether we have for $R(l_0,n)$ and $R(t_0,n)$

$$\begin{aligned} R(l_0,n) &= \frac{q^\nu\{\nu\}_q}{\{h\}_q\{h+1\}_q} \\ &\left[\lambda^2[\frac{1+h-n}{2}]_q{}^2 + (1+(-1)^{h+n})[2]_qC_2(l_0) - (-1+(-1)^{h+n})[2]_qC_1(l_0)\right] \\ R(t_0,n) &= \frac{q^\nu\lambda[\nu]_q}{2[1+h]_q[2+h]_q} \\ &\left[2[\frac{1+h-n}{2}]_q{}^2 + (1+(-1)^{h+n})[2]_qC_2(t_0) - (-1+(-1)^{h+n})[2]_qC_1(t_0)\right] \end{aligned}$$

Finally we can insert this in (3.9) and together with (3.8) we obtain:

$$\begin{aligned} \tau^1_{1,1}(\frac{l_0^2}{q[2]_q},n,\nu,m,h) &= \\ &q^{\frac{1}{4}(2+h+2m-3n+4\nu)}\sqrt{\frac{\lambda[\frac{h+2m+n}{2}-\nu]_q\{\nu\}_q}{\{h\}_q\{h+1\}_q\{n\}_q\{n-1\}_q}\left(2C(l_0,h+n)+\lambda^2[\frac{1+h-n}{2}]_q{}^2\right)} \\ \tau^1_{1,-1}(\frac{l_0^2}{q[2]_q},n,\nu,m,h) &= \\ &q^{\frac{1}{2}(1-\frac{h}{2}+m-\frac{3n}{2}+2\nu)}\sqrt{\frac{\{\frac{h-2m-2}{2}+\nu\}_q\{\nu\}_q}{\{h_q\}\{h-1\}_q\{n\}_q\{n-1\}_q}\left(\{\frac{h+n-1}{2}\}_q^2-2C(l_0,h+n)\right)} \\ \tau^1_{1,1}(-\frac{t_0^2}{q[2]_q},n,\nu,m,h) &= \\ &q^{\frac{h+2m-3n+4\nu}{4}}\sqrt{\frac{[\frac{2+h+2m+n}{2}-\nu]_q[\nu]_q}{[1+h]_q[2+h]_q[n]_q[1+n]_q}\left(C(t_0,h+n)-[\frac{1+h-n}{2}]_q{}^2\right)} \\ \tau^1_{1,-1}(-\frac{t_0^2}{q[2]_q},n,\nu,m,h) &= \\ &q^{\frac{1}{2}(-1-\frac{h}{2}+m-\frac{3n}{2}+2\nu)}\sqrt{\frac{[\nu]_q[\frac{h-2m-n}{2}+\nu]_q}{[h]_q[1+h]_q[n]_q[1+n]_q}\left([\frac{1+h+n}{2}]_q{}^2-C(t_0,h+n)\right)} \end{aligned} \tag{3.15}$$

with

$$\begin{aligned} C(l_0, h+n) &= \frac{[2]_q}{2}\Big((1+(-1)^{n+h})C_2(l_0) - (-1+(-1)^{n+h})C_1(l_0)\Big) \\ C(t_0, h+n) &= -\frac{[2]_q}{2}\Big((1+(-1)^{n+h})C_2(t_0) - (-1+(-1)^{n+h})C_1(t_0)\Big) \end{aligned}$$

3.4 The Casimir

Up to now, we have determined the matrix element of τ^1 except for the constants $C_{1/2}(0/l_0/t_0)$. Here we will see that these constants will be fixed if we specify the spin of the representation. For that we calculate the action of the spin Casimir $\mathfrak{C}$. As it was shown in [32, 36], this spin Casimir is given by the square of the Pauli-Lubanski-vector $\mathfrak{P}$, whose components, using our set of generators, are listed in the appendix (A.9):

$$\mathfrak{C} = (\mathfrak{P})^2 = \mathfrak{P}_A\mathfrak{P}_B - \frac{1}{q^2}\mathfrak{P}_C\mathfrak{P}_D \tag{3.16}$$

We will proceed in the same way and first calculate the representation of the Pauli-Lubanski-vector. It can be found in the appendix A.5. Then we evaluate the action of the Casimir, now expressed via equation (3.16) by the components of the Pauli-Lubanski-vector, on an arbitrary state vector. Despite the complicated looking expressions, we find in the end a very simple result:

$$\mathfrak{C}|l,n,\nu,m,h\rangle = \begin{cases} \frac{2\tau_0^2[2]_q}{q^3\lambda^2}C(0,h+n)|l,n,\nu,m,h\rangle & \text{for } l=0 \\ \frac{2t_0^2\left((1+q)^2C(t_0,h+n)-q\right)}{(1+q)^2[2]_q}|l,n,\nu,m,h\rangle & \text{for } l<0 \\ \frac{2l_0^2\left((-1+q)^2+2qC(l_0,h+n)\right)}{q\lambda^2[2]_q}|l,n,\nu,m,h\rangle & \text{for } l>0 \end{cases}$$

This allows us to write the constants in terms of the eigenvalue $\mathfrak{c}$ of the Casimir:

$$\begin{aligned} C(0,h+n) &= \frac{q^3\lambda^2}{2\tau_0^2[2]_q}\mathfrak{c} \\ C(t_0,h+n) &= \frac{q}{(1+q)^2} + \frac{[2]_q}{2t_0^2}\mathfrak{c} \\ C(l_0,h+n) &= -\frac{(-1+q)^2}{2q} + \frac{\lambda^2[2]_q}{4l_0^2}\mathfrak{c} \end{aligned}$$

What remains is to specify the possible eigenvalues of the Casimir. We can proceed as in section 2.5, where we have determined the spectra of the space observables. Again, we have to take care that the square roots appearing in the expression for the representation of τ^1 are well defined. That is, the arguments of these roots have to be bigger than zero.

This means for the $l = 0$, if we look at (3.14), that we have two possibilities:

$$C(0, h+n) = \begin{cases} 0 \\ q^{2+s} \end{cases} \qquad \Rightarrow \qquad \mathfrak{c} = \begin{cases} 0 \\ \frac{2\tau_0^2[2]_q}{q^3\lambda^2} q^{2+s} \end{cases}$$

where $s \in \mathbb{Z}$ with $h + n \geq s$ and $h + n \geq -2(m - \nu)$. Remember that in the classical case $\mathfrak{c} = 0$ [37] and therefore we also have to set $C(0, h + n) = \mathfrak{c} = 0$ to correctly generalise to the q-deformed case.

For the case $l < 0$ it follows from (3.15) that we have to choose

$$C(t_0, h+n) = [s + \frac{1}{2}]_q^2$$

which gives us for the Casimir

$$\mathfrak{c} = \frac{2t_0^2}{[2]_q}[s]_q[s+1]_q \tag{3.17}$$

with $s \in \frac{1}{2}\mathbb{N}_0$. The allowed region for the quantum numbers n and h is described by the inequalities

$$s \geq \frac{|h-n|}{2}\,,\ \frac{|h+n|}{2} \geq s\,,\ h-n \geq 2(m-\nu)\,,\ h+n \geq -2(m-\nu+1) \tag{3.18}$$

Note that (3.17) coincides with the expression given in [32] for the eigenvalues of the spin-Casimir.

For the case $l > 0$ we get

$$C(l_0, h+n) = -\frac{1}{2}\lambda^2[s + \frac{1}{2}]_q^2$$

and

$$\mathfrak{c} = -\frac{2l_0^2}{[2]_q}[s]_q[s+1]_q$$

This time the region for the quantum numbers n and h is:

$$\frac{|h-n|}{2} \geq s\,,\ h-n \geq 2(m-\nu)\,,\ h+n \geq -2(m-\nu)$$

To complete the calculation of the matrix elements of the Lorentz boosts, we listen the final results for the representations of the generators τ^1, T^2, σ^2 and S^1 in the appendix A.6.

Chapter 4

The representation of the derivatives

In this section we will derive the representation of the derivatives. Different from the method used previously, where we successively evaluated commutation relations by acting on state vectors, we will here solve the problem algebraically by expressing the derivatives in terms of the coordinates and q-Lorentz generators. In [20] it was shown, how to realise the q-Lorentz algebra via the coordinates and derivatives:

$$V^{ij} = \Lambda^{1/2} P_{A\,kl}^{ij} X^k \hat{\partial}^l$$

where $P_{A\,kl}^{ij}$ is the antisymmetric projector of the $\mathcal{R}$-matrix, see (B.2). The decomposition of V^{ij} into its selfdual and anti-selfdual components gives the vectorial generators of the q-Lorentz algebra:

$$R^A = P_{+\,cd}^{A0} V^{cd}, \; S^A = \frac{1}{q^2} P_{-\,cd}^{A0} V^{cd}$$

The explicit form of these relations can be found in the appendix (B.5) and converted in our set of generators in (B.6). Looking at the first relation of (B.6) we can immediately read off

$$\hat{\partial}^A = D^{-1}(q^2 A \hat{\partial}^D - T^2 \Lambda^{-\frac{1}{2}})$$

In the same way we can also express $\hat{\partial}^B$ through $\hat{\partial}^D$:

$$\hat{\partial}^B = \frac{1}{q^2} D^{-1}(B \hat{\partial}^D - \frac{1}{q^2} S^1 (\tau^3)^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}})$$

Plugging this into (B.7) we find for $\hat{\partial}^C$:

$$\hat{\partial}^C = \frac{1}{\lambda} D^{-2} \left(qD \left(\tau^1 - \sigma^2\right) + \lambda B T^2 + \lambda D C \hat{\partial}^D \Lambda^{\frac{1}{2}} - \lambda \left(\lambda D T^2 T^- + A S^1 \left(\tau^3\right)^{-\frac{1}{2}} \right) \right) \Lambda^{-\frac{1}{2}}$$

We can further simplify the expression for $\hat{\partial}^C$, if we make use of the equation $U^1 = U^2$, which is additionally satisfied by our realisation of the q-Lorentz generators. As was shown in [19, 20], this relation is related to the intuitive picture that V^{ij} represents orbital angular momentum. The above expression for $\hat{\partial}^C$ reduces to:

$$\hat{\partial}^C = \frac{1}{q\lambda} D^{-1} \left(\tau^1 \Lambda^{-\frac{1}{2}} + q\lambda C \hat{\partial}^D - \sigma^2 \left(\tau^3\right)^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} \right)$$

What is left is to determine $\hat{\partial}^D$. To this end we consider the relation

$$\hat{\partial}^B A = \frac{1}{q^2} + q^2 A \hat{\partial}^B - q\lambda D \hat{\partial}^D$$

which gives the action of $\hat{\partial}^B$ on the coordinate A. If we multiply this relation from the right with B, replace AB with $(X)^2 + q^{-2}CD$ and insert for $\hat{\partial}^B$ what we have found above, we can solve it for $\hat{\partial}^D$:

$$\hat{\partial}^D = -\frac{1}{q^5\lambda} \left(D\Lambda^{\frac{1}{2}} + q^2 \left(D\sigma^2 - q\lambda A S^1 \right) \left(\tau^3\right)^{-\frac{1}{2}} \right) (X)^{-2} \Lambda^{-\frac{1}{2}}$$

Now we can insert our results for the action of the coordinates and q-Lorentz generators. Because V^{ij} represents the orbital angular momentum, we have to choose the spin zero representation for the generators. Notice that we have divided by the square of the length $(X)^2$, so the above derivation is only valid for $l \neq 0$. Indeed, it was shown [31], that it is not possible to construct a representation for derivatives on the light-cone. The explicit expressions for the action of the derivatives on a state vector are listed in the appendix, paragraph A.7.

Above we have calculated the representation for the hatted derivatives, which we have introduced in equation (B.18) to be proportional to the conjugated derivatives. Therefore we can also easily get a representation for the ∂_i, by just transposing the corresponding matrix elements of the $\hat{\partial}_i$. As an consistency check we can finally evaluate the nonlinear relation (B.19), which relates the conjugate derivatives with the ordinary ones. Indeed if we multiply the equation from both sides with arbitrary state vectors and evaluate the actions, we can verify the representation listed in A.7.

If we combine the derivatives with their conjugates, it is possible to define a hermitian momentum [20]

$$P_i = -\frac{i}{2}(\partial_i + q^4 \hat{\partial}_i)$$

with the same conjugation properties as the coordinates

$$\overline{P}_A = P_B,\ \overline{P}_B = P_A,\ \overline{P}_C = P_C \text{ and } \overline{P}_D = P_D$$

The representation for these momenta can be found in appendix A.8.

Part II

Solution of wave equations

Chapter 5

Acting on functions

Before we can start to find solutions of wave equations on the q-Minkowski space, we have to be able to differentiate functions. In this section we will show how to cope with the complicated differential calculus on the q-Minkowski space. We will present closed expressions for the action of the derivatives and q-Lorentz generators on a function in one variable. Such representations for the derivatives and symmetry operators on functions were already given in [38]. But there the action on functions $f(X^0, X^+, X^-, X^3)$ depending on all the space coordinates were calculated, and therefore, due to the complexity of this problem, the results are given in power series expansions. Here we restrict ourselves to functions depending on only one variable. This simplification and the introduction of a matrix notation, which neatly encodes the braiding, allows us to use "generating functions" to solve the combinatorial problem inherent in the commutation relations. With the help of these results it is then also possible to compute closed forms for the derivatives of functions depending on more than one variable.

5.1 The derivatives of q-Minkowski space functions

To have an efficient calculus for the derivatives of the q-Minkowski space, it is expedient to assemble them in a vector and to perform the calculations using a matrix notation. In tensor notation the Leibniz rule is given by

$$\partial_a X^b = \delta_a^b + \mathcal{R}_{II\;ad}^{\;bc} X^d \partial_c$$

Let us define the matrix $(L_{X^b})^a_c := \mathcal{R}_{II\;ad}^{\;bc} X^d$ and from now on write the commutation relation as:

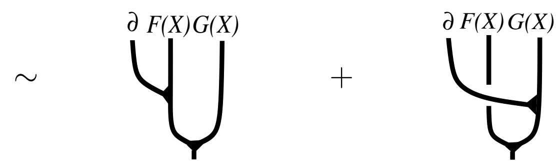
$$\partial X = \partial \triangleright X + L_X \partial \tag{5.1}$$

with the 4-vector $\partial = (\partial_0, \partial_3, \partial_+, \partial_-)^T$ and $X \in \{X^0, X^{0/3}, X^+, X^-\}$. In this equation $L_X\partial$ is the usual matrix multiplication of the vector ∂ with the matrix L_X, whereas the other operations are performed componentwise. The explicit form of the matrices L_X can be found in the appendix, see (B.22).

The advantage of this notation is that it neatly encodes the braiding between space functions living on $\mathcal{M}_q$ and the derivatives. We just have to replace the coordinates X with L_X if we commute it with a derivative. For example, we can generalise equation (5.1) and write down the Leibniz rule for any two function F and G as

$$\partial \triangleright (F(X^i)G(X^j)) = (\partial \triangleright F(X^i))\, G(X^j) + (F(L_{X^i})\partial) \triangleright G(X^j) \tag{5.2}$$

which we may also depict graphically as:



Next let us use this new notation and try to calculate the derivative of an arbitrary function in one variable. The first step towards this goal is to consider only the derivative of powers $(X)^n$ of this coordinate. $\frac{1}{1-zX}$ is the generating function of the powers, so let us see what its derivative is like:

$$\begin{aligned} \partial \frac{1}{1-zX} &= \partial\,(1 + zX\,\frac{1}{1-zX}) \\ &= \partial + z\,[\partial \triangleright X + L_X\partial\,]\,\frac{1}{1-zX} \\ \Longrightarrow\ (1 - zL_X)\ \partial\frac{1}{1-zX} &= \partial + z\,(\partial \triangleright X)\frac{1}{1-zX}, \qquad \text{so} \end{aligned}$$

$$\partial\frac{1}{1-zX} = \frac{1}{1-zL_X}\left(\partial + \frac{z}{1-zX}(\partial \triangleright X)\right) \tag{5.3}$$

If we now expand in powers of z and compare the resulting coefficients, we will find the result for $\partial(X^i)^n$. Of course, before we can make this expansion, we have to find a explicit formula for $\frac{1}{1-zL_X}$.

5.1.1 Commuting the derivatives with space functions

To make use of the Leibniz-rule, we have to be able to calculate functions of the L_X matrices. In fact, this is possible, because the matrices L_X satisfy the following finiteness conditions, which resemble characteristic equations found in classical linear algebra:

$$\begin{aligned}
L_0^2 &= \frac{\lambda^2}{q[2]_q}(X)^2 + \frac{[2]_q}{q}X^0 L_0 - \frac{1}{q^2}(X^0)^2 \\
(L_{0/3})^3 &= [3]_q(X^{0/3})(L_{0/3})^2 - [3]_q(X^{0/3})^2(L_{0/3}) + (X^{0/3})^3 \\
(L_+)^4 &= q[4]_q X^+(L_+)^3 - q^2\frac{[3]_q[4]_q}{[2]_q}(X^+)^2(L_+)^2 + q^3[4]_q(X^+)^3(L_+) - q^4(X^+)^4 \\
(L_-)^3 &= \frac{[3]_q}{q^2}X^-(L_-)^2 - \frac{[3]_q}{q^4}(X^-)^2 L_- + \frac{1}{q^6}(X^-)^3 \\
L_{(X)^2} &= \frac{1}{q^2}(X^2)
\end{aligned} \tag{5.4}$$

Using these relations, it is a simple task to calculate arbitrary powers of the L_X matrices. To demonstrate the procedure, consider for example $\frac{1}{1-zL_0}$:

$$\frac{1}{1-zL_0} = 1 + zL_0 + z^2 L_0^2 \frac{1}{1-zL_0}$$

We insert (5.4) to eliminate L_0^2 and use $L_0\,\frac{1}{1-zL_0} = \frac{1}{z}(\frac{1}{1-zL_0} - 1)$. We find

$$\frac{1}{1-zL_0} = 1 + z(L_0 - \frac{[2]_q}{q}X^0) + \left[\frac{z^2}{q^2}\left(\frac{q\lambda^2}{[2]_q}(X)^2 - (X^0)^2\right) + \frac{z[2]_q}{q}X^0\right]\frac{1}{1-zL_0}$$

and then solve for $\frac{1}{1-zL_0}$:

$$\frac{1}{1-zL_0} = \frac{q[2]_q\,(q + qzL_0 - [2]_q X^0 z)}{[2]_q\,(q^2 + (X^0)^2 z^2 - q[2]_q X^0 z) - z^2 q\lambda^2 (X)^2} \tag{5.5}$$

Since the matrix L_0 doesn't appear any more in the denominator, we can expand without problems in powers of z by making a partial fraction decomposition.

$$\begin{aligned}
\frac{1}{1-zL_0} = \sum_{n\geq 0} \frac{z^n}{2W\lambda\,(2q^3[2]_q)^n}\Bigg[&\left(\lambda W + \frac{\sqrt{q[2]_q}}{q}\left(X^0 - q^2\left(2L_0 - X^0\right)\right)\right)\alpha_-^n \\
&+ \left(\lambda W + \sqrt{q[2]_q}\left(2qL_0 - [2]_q X^0\right)\right)\alpha_+^n\Bigg]
\end{aligned} \tag{5.6}$$

with

$$\alpha_\pm = q^2[2]_q^2 X^0 \pm q\lambda\sqrt{q[2]_q}W \quad\text{and}\quad W = \sqrt{4(X)^2q^2 + q(X^0)^2[2]_q}$$

The coefficients of this power series are either constant or itself powers of n, therefore we can also immediately read off the result for an arbitrary function

$F(L_0)$:

$$
\begin{aligned}
F(L_0) &= \frac{1}{2qW\lambda}\Bigg[F(\frac{\alpha_-}{2q^3[2]_q})\left(q\lambda W + \sqrt{q[2]_q}\left(X^0 + q^2\left(X^0 - 2L_0\right)\right)\right) + \\
&\quad qF(\frac{\alpha_+}{2q^3[2]_q})\left(\lambda W + \sqrt{q[2]_q}\left(2qL_0 - [2]_q X^0\right)\right)\Bigg]
\end{aligned}
$$

Functions of the other L_X matrices can be calculated in the same way. You start with the generating function $\frac{1}{1-zL_X}$, separate from it the first terms of the power series and insert the characteristic equation. After solving this equation for $\frac{1}{1-zL_X}$, the matrix L_X will appear only in the numerator and it is possible to expand in powers of z. Again the results for general functions can easily be read off:

$$
\begin{aligned}
F(L_+) &= \frac{1}{q^6\lambda^3[2]_q}\Big[\frac{(q^2-\Lambda_{q^2})(q^4-\Lambda_{q^2})(q^6-\Lambda_{q^2})}{[3]_q\Lambda_{q^2}}F(X^+) \qquad (5.7)\\
&\quad -\frac{(q^4-\Lambda_{q^2})(q^6-\Lambda_{q^2})(1-\Lambda_{q^2})}{X^+\Lambda_{q^2}}F(X^+)\,L_+\\
&\quad -\frac{(q^2-\Lambda_{q^2})(q^6-\Lambda_{q^2})(1-\Lambda_{q^2})}{(X^+)^2\Lambda_{q^2}}F(X^+)\,(L_+)^2\\
&\quad -\frac{(q^2-\Lambda_{q^2})(q^4-\Lambda_{q^2})(1-\Lambda_{q^2})}{[3]_q(X^+)^3\Lambda_{q^2}}F(X^+)\,(L_+)^2\Big]\\
F(L_-) &= \frac{1}{\lambda^2}\Big[\frac{(1-q^2\Lambda_{q^{-2}})(1-q^4\Lambda_{q^{-2}})}{q^3[2]_q}F(X^-)\\
&\quad -\frac{(1-\Lambda_{q^{-2}})(1-q^4\Lambda_{q^{-2}})}{X^-}F(X^-)L_-\\
&\quad +\frac{q^3(1-\Lambda_{q^{-2}})(1-q^2\Lambda_{q^{-2}})}{[2]_q(X^-)^2}F(X^-)(L_-)^2\Big]\\
F(L_{0/3}) &= \frac{1}{q^3\lambda^2[2]_q}\Big[\frac{(q^2-\Lambda_{q^2})(q^4-\Lambda_{q^2})}{\Lambda_{q^2}}F(X^{0/3})\\
&\quad -q[2]_q\frac{(q^4-\Lambda_{q^2})(1-\Lambda_{q^2})}{X^{0/3}\Lambda_{q^2}}F(X^{0/3})L_{0/3}\\
&\quad +q^2\frac{(q^2-\Lambda_{q^2})(1-\Lambda_{q^2})}{(X^{0/3})^2\Lambda_{q^2}}F(X^{0/3})(L_{0/3})^2\Big]\\
F(L_{(X)^2}) &= F(\frac{(X)^2}{q^2})
\end{aligned}
$$

Here the scaling operator Λ_a ($\Lambda_a F(X) = F(aX)$, $\frac{1}{\Lambda_a} := \Lambda_{1/a}$) acts only on the function F, but not any more on the coordinates appearing in the L matrices.

5.1.2 Calculation of the derivatives

After we know how to calculate functions of L matrices we can go back to equation (5.3) and compute the derivatives of the powers $(X^i)^n$. For that we only have to expand $\frac{1}{1-zL_X}\frac{z}{1-zX}$ in a power series. $\frac{1}{1-zL_X}$ was calculated in (5.5), resp. (5.7), and a partial fraction decomposition again will yield the results:

$$\begin{aligned} \partial \triangleright F(X^0) &= \frac{1}{2(X)^2 qW\lambda^2}\Big(\Big[2[2]_q W F(X^0)\left(X^0 - q^2 L_0\right) + \\ &\quad F(\frac{\alpha_+}{2q^3[2]_q})A_+ - F(\frac{\alpha_-}{2q^3[2]_q})A_-\Big]\partial\Big) \triangleright X^0 \\ \partial \triangleright F(X^+) &= D_{\frac{1}{q^2}X^+} F(X^+)(\partial \triangleright X^+) \\ \partial \triangleright F(X^-) &= D_{\frac{1}{q^2}X^-} F(X^-)(\partial \triangleright X^-) \\ \partial \triangleright F(X^{0/3}) &= D_{\frac{1}{q^2}X^{0/3}} F(X^{0/3})(\partial \triangleright X^{0/3}) \\ \partial \triangleright F((X)^2) &= D_{\frac{1}{q^2}(X)^2} F((X)^2)(\partial \triangleright (X)^2) \end{aligned} \tag{5.8}$$

with

$$A_\pm = \left([2]_q W(q^2 L_0 - X^0) \pm \sqrt{q[2]_q}\left(2q^2\lambda(X)^2 + [2]_q X^0\left(q^2 L_0 - X^0\right)\right)\right).$$

The derivatives with respect to the coordinates X^+, X^- and $X^{0/3}$ are the well known Jackson derivatives[1]. Due to their simplicity they can also be deduced directly from the commutation relations, without using generating function at all. Only the time derivative can not be found so easily and it is necessary to go through the procedure described in 5.1.1.

[1] $D_{aX} f(X) = \frac{f(aX)-f(X)}{X(a-1)}$

5.2 The action of the symmetry operators on q-Minkowski space functions

5.2.1 The rotations

Again we first calculate the action on the generating function for the powers and then generalise to functions. To demonstrate the method we only show the calculations for T^- acting on a function $F(X^+)$:

$$\begin{aligned} T^-\frac{1}{1-zX^+} &= T^- + z\,T^-X^+\,\frac{1}{1-zX^+} \\ &\overset{(B.23)}{=} \frac{z}{q^2}X^+T^-\frac{1}{1-zX^+} + z\sqrt{q[2]_q}\,X^3\frac{1}{1-zX^+} + T^- \\ \Longrightarrow T^-\frac{1}{1-zX^+} &= \frac{1}{1-\frac{z}{q^2}X^+}T^- + z\sqrt{q[2]_q}\left(X^0\frac{1}{1-\frac{z}{q^2}X^+} - X^{0/3}\frac{1}{1-\frac{z}{q^4}X^+}\right)\frac{1}{1-zX^+} \end{aligned}$$

The expansion in z gives the result for a general function:

$$T^-F(X^+) = \sqrt{q[2]_q}\left(X^0 D_{q^2X^+} - X^{0/3}D_{q^4X^+}\right)F(X^+) + F(\frac{X^+}{q^2})T^-$$

The remaining relations can be found in the same way:

$$\begin{aligned} T^-F(X^{0/3}) &= F(X^{0/3})T^- - q\sqrt{q[2]_q}\,D_{q^2X^{0/3}}F(X^{0/3})X^- \\ T^-F(X^-) &= F(q^2X^-)T^- \\ T^-F(X^0) &= F(X^0)T^- \end{aligned}$$

$$\begin{aligned} T^+F(X^-) &= F(q^2X^-)T^+ + \frac{[2]_q}{\sqrt{q[2]_q}}\left(X^0D_{q^2X^-} - X^{0/3}D_{q^4X^-}\right)F(X^-) \\ T^+F(X^+) &= F(\frac{X^+}{q^2})T^+ \\ T^+F(X^0) &= F(X^0)T^+ \\ T^+F(X^{0/3}) &= F(X^{0/3})T^+ - \frac{[2]_q}{q\sqrt{q[2]_q}}D_{\frac{X^{0/3}}{q^2}}F(X^{0/3})X^+ \end{aligned}$$

$$\begin{aligned} \tau^3F(X^0) &= F(X^0)\tau^3 \\ \tau^3F(X^{0/3}) &= F(X^{0/3})\tau^3 \\ \tau^3F(X^-) &= F(q^4X^-)\tau^3 \\ \tau^3F(X^+) &= F(\frac{X^+}{q^4})\tau^3 \end{aligned}$$

5.2.2 The action of the boosts

We will proceed in an analogous manner as above, but now the calculations will become a bit more involved, because we have to solve a coupled system of equations. To illustrate the computation, we consider as an example the action of τ^1 and T^2 on the coordinate X^0 :

$$T^2 X^0 = q\, X^0 T^2 - \frac{\lambda}{q[2]_q} X^{0/3} T^2 + \frac{1}{\sqrt{q[2]_q}} X^+ \tau^1 \tag{5.9}$$

$$\tau^1 X^0 = \frac{1}{q}\, X^0 \tau^1 + \frac{q\lambda}{[2]_q} X^{0/3} \tau^1 - \frac{q\lambda^2}{\sqrt{q[2]_q}} X^- T^2. \tag{5.10}$$

Plugging these relations into the expansion of the generating function $\frac{1}{1-zX^0}$ we get:

$$\begin{aligned} T^2 \frac{1}{1-zX^0} &= T^2 + z\, T^2 X^0 \frac{1}{1-zX^0} \\ &\overset{(5.9)}{=} T^2 + \left(qzX^0 - \frac{z\lambda}{q[2]_q} X^{0/3} \right) T^2 \frac{1}{1-zX^0} + \frac{z}{\sqrt{q[2]_q}}\, X^+ \tau^1 \frac{1}{1-zX^0} \end{aligned}$$

$$\Longrightarrow T^2 \frac{1}{1-zX^0} = \frac{1}{1 - qzX^0 + \frac{z\lambda}{q[2]_q} X^{0/3}} \left(T^2 + \frac{z}{\sqrt{q[2]_q}}\, X^+\ \tau^1 \frac{1}{1-zX^0} \right) \tag{5.11}$$

$$\begin{aligned} \tau^1 \frac{1}{1-zX^0} &= \tau^1 + z\, \tau^1 X^0 \frac{1}{1-zX^0} \\ &\overset{(5.10)}{=} \tau^1 + \left(\frac{z}{q} X^0 + \frac{qz\lambda}{[2]_q} X^{0/3} \right) \tau^1 \frac{1}{1-zX^0} - \frac{qz\lambda^2}{\sqrt{q[2]_q}}\, X^- T^2 \frac{1}{1-zX^0} \end{aligned}$$

$$\Longrightarrow \tau^1 \frac{1}{1-zX^0} = \frac{1}{1 - \frac{z}{q} X^0 - \frac{qz\lambda}{[2]_q} X^{0/3}} \left(\tau^1 - \frac{qz\lambda^2}{\sqrt{q[2]_q}}\, X^-\ T^2 \frac{1}{1-zX^0} \right) \tag{5.12}$$

Now we can solve (5.11) and (5.12) for $T^2 \frac{1}{1-zX^0}$ and $\tau^1 \frac{1}{1-zX^0}$:

$$\tau^1 \frac{1}{1-zX^0} = \frac{\left(q[2]_q X^0 z - z\lambda q X^{0/3} - [2]_q \right) \tau^1 + z\lambda^2 \sqrt{q[2]_q} X^- T^2}{qz^2\lambda^2 (X)^2 - [2]_q \left(1 - zX^0 \left([2]_q - zX^0\right)\right)}$$

$$T^2 \frac{1}{1-zX^0} = \frac{\frac{1}{q} \left(\lambda X^{0/3} z - q[2]_q + [2]_q X^0 z \right) T^2 - \frac{z[2]_q}{\sqrt{q[2]_q}} X^+ \tau^1}{qz^2\lambda^2 (X)^2 - [2]_q \left(1 - zX^0 \left([2]_q - zX^0\right)\right)}$$

This can be expanded in a power series and then easily generalised to yield the commutation relations with a general function:

$$
\begin{aligned}
T^2 F(X^0) &= \frac{1}{\lambda W}\left(F(\frac{\alpha_-}{2q^2[2]_q}) - F(\frac{\alpha_+}{2q^2[2]_q})\right)\left(\frac{\lambda}{\sqrt{q[2]_q}}X^{0/3}T^2 - X^+\tau^1\right) \\
&\quad + \frac{1}{2qW[2]_q}\left(F(\frac{\alpha_+}{2q^2[2]_q})A_+ + F(\frac{\alpha_-}{2q^2[2]_q})A_-\right)T^2 \\
\tau^1 F(X^0) &= \frac{q\lambda}{W}\left(F(\frac{\alpha_-}{2q^2[2]_q}) - F(\frac{\alpha_+}{2q^2[2]_q})\right)\left(X^-T^2 - \frac{q}{\lambda\sqrt{q[2]_q}}X^{0/3}\tau^1\right) \\
&\quad + \frac{1}{2qW[2]_q}\left(F(\frac{\alpha_+}{2q^2[2]_q})A_- + F(\frac{\alpha_-}{2q^2[2]_q})A_+\right)\tau^1
\end{aligned}
$$

Here we have defined $A_\pm := \left(qW[2]_q \pm qX^0[2]_q\sqrt{q[2]_q}\right)$. $\alpha_\pm$ and W are taken from equation (5.6). Similarly we deduce for S^1 and σ^2:

$$
\begin{aligned}
S^1 F(X^0) &= \frac{q}{\lambda W}\left(F(\frac{\alpha_+}{2q^2[2]_q}) - F(\frac{\alpha_-}{2q^2[2]_q})\right)\left(\frac{q\lambda}{\sqrt{q[2]_q}}X^{0/3}S^1 - X^-\sigma^2\right) \\
&\quad + \frac{1}{2qW[2]_q}\left(F(\frac{\alpha_+}{2q^2[2]_q})A_- + F(\frac{\alpha_-}{2q^2[2]_q})A_+\right)S^1 \\
\sigma^2 F(X^0) &= \frac{1}{\lambda W}\left(F(\frac{\alpha_+}{2q^2[2]_q}) - F(\frac{\alpha_-}{2q^2[2]_q})\right)\left(X^+S^1 - \frac{1}{\lambda\sqrt{q[2]_q}}X^{0/3}\sigma^2\right) \\
&\quad + \frac{1}{2qW[2]_q}\left(F(\frac{\alpha_+}{2q^2[2]_q})A_+ + F(\frac{\alpha_-}{2q^2[2]_q})A_-\right)\sigma^2
\end{aligned}
$$

The formulas for the remaining operators follow again directly from the corresponding relations with the coordinate:

$$
\begin{aligned}
T^2 F(X^-) &= F(\frac{1}{q}X^-)T^2 - \frac{1}{q^{\frac{3}{2}}\sqrt{[2]_q}}X^{0/3}D_{qX^-}F(X^-)\tau^1 \\
\tau^1 F(X^+) &= F(qX^+)\tau^1 + \frac{q^2\lambda^2}{\sqrt{q[2]_q}}X^{0/3}D_{qX^+}F(X^+)T^2 \\
S^1 F(X^+) &= F(\frac{1}{q}X^+)S^1 + \frac{1}{\sqrt{q[2]_q}}X^{0/3}D_{qX^+}F(\frac{1}{q^2}X^+)\sigma^2 \\
\sigma^2 F(X^-) &= F(qX^-)\sigma^2 - \frac{q\lambda^2}{\sqrt{q[2]_q}}X^{0/3}D_{qX^-}F(q^2X^-)S^1
\end{aligned}
$$

$$
\begin{aligned}
T^2 F(X^{0/3}) &= F(\frac{1}{q}X^{0/3})T^2 \\
T^2 F(X^+) &= F(qX^+)T \\
\tau^1 F(X^{0/3}) &= F(qX^{0/3})\tau^1 \\
\tau^1 F(X^-) &= F(\frac{1}{q}X^-)\tau^1
\end{aligned}
\qquad
\begin{aligned}
S^1 F(X^{0/3}) &= F(qX^{0/3})S^1 \\
S^1 F(X^-) &= F(qX^-)S^1 \\
\sigma^2 F(X^{0/3}) &= F(\frac{1}{q}X^{0/3})\sigma^2 \\
\sigma^2 F(X^+) &= F(\frac{1}{q}X^+)\sigma^2
\end{aligned}
$$

Chapter 6

The solution of the free q-Klein-Gordon equation

In this chapter, we want to find irreducible spin-0 representations of the q-Poincaré algebra in the space of q-Minkowski space functions $\mathcal{M}_q$. The basis vectors we are going to construct for the irreducible subspace are the common eigenvectors of the operators $(\partial)^2, \partial_0, T^3, \vec{T}^2$, which constitute in the spin-0 case, a complete set of commuting observables. We will start by solving the q-Klein-Gordon equation in the rest frame, the solution being the q-deformed generalisation of the exponential function. The remaining states of the irreducible representation can then be calculated by successively acting on this state with the symmetry generators.

6.1 The q-exponential function

Let us try to find a function being a common eigenvector of the derivatives with only the time eigenvalue not being zero:

$$(\partial_0, \vec{\partial}) \triangleright F(X^0, \vec{X}) = (\alpha F(X^0, \vec{X}), \vec{0}) \ , \ \alpha \in \mathbb{C} \tag{6.1}$$

with $\vec{X} = (X^3, X^+, X^-)$ and $\vec{\partial} = (\partial_3, \partial_+, \partial_-)$. If we can find such a function it would constitute the rest state, having the eigenvalues $(-\frac{\alpha^2}{q[2]_q}, \alpha, 0, 0)$ with respect to our set of observables $(\partial)^2 = \frac{1}{q[2]_q} g^{ij} \partial_i \partial_j,\ \partial_0, T^3, \vec{T}^2$ [32, 36].

(6.1) is a first order differential equation, so we must specify an initial value to get a unique solution. We choose $F(0, \vec{0}) = 1$, because then we get the usual time dependent exponential function in the classical limit $q = 1$ and the above state is a reasonable candidate for a q-deformed generalisation of the exponential function in the q-Minkowski space.

First notice, that the function F can only depend on X^0 and $(X)^2$:

To see this, we expand F with respect to a Poincaré-Birkhoff-Witt basis into a power series. Choosing the polynomials $(X^0)^n(X^+)^k(X^3)^i(X^-)^j$ as the vectorspace basis of $\mathcal{M}_q$ we can write:

$$F(X^0, \vec{X}) = \sum_{n,k,i,j\geq 0} C_{n,m,k,i,j}(X^0)^n(X^+)^k(X^3)^i(X^-)^j$$

The initial condition sets $C_{0,0,0,0,0} = 1$ and the first order[1] term is X^0, exactly as in the classical case. But what happens with the second resp. higher order terms ?

To handle these terms, we will make use of the covariance of the differential calculus with respect to the rotations. Consider for example the action of ∂_0 on an $\mathcal{U}_q(\mathrm{su}_2)$ irreducible subspace V_j of $\mathcal{M}_q$ with quantum number j. The highest weight vector $|j,j\rangle$ of this representation will be proportional to $(X^0)^n(X^2)^l(X^+)^j$, for some $n, l \in \mathbb{N}$, and by successively applying T^- we will get a basis of V_j. Because the actions of the symmetry operators do not change the grading, all the vectors $|j,m\rangle$ will have the same one, given by $\mathrm{gr}(|j,m\rangle) = n+2l+j$. ∂_0 is itself a singlet under the rotations, so if we act with it on V_j we again get an irreducible space V_j, but of course with the grading diminished by one. Let us apply this observation now to the second order term v_2 of $F(X^0, \vec{X})$. We decompose it into $\mathcal{U}_q(\mathrm{su}_2)$ irreducible components and write $v_2 = s_2 + r_2$, where s_2 contains all the singlets and r_2 the rest. The action of ∂_0 must give the singlet X^0 , hence we have $\partial_0 \triangleright r_2 = 0$ and together with (6.1) $(\partial_0, \vec{\partial}) \triangleright r_2 = 0$, which means that the function r_2 is a constant. To fulfil the initial condition this constant has to be set to zero, thus v_2 only depends on the singlets X^0 and (X^2). In fact, because the above consideration is also valid for higher order terms, induction on the grading will give us the desired result.

Now we can apply the formulas from (5.8) in order to differentiate $F(X^0, (X^2))$,

[1] with respect to the natural grading: $\mathrm{gr}((X^0)^n(X^+)^k(X^3)^i(X^-)^j) = n+k+i+j$

with the result:

$$\partial \triangleright F(X^0,(X)^2) \quad = \tag{6.2}$$

$$\frac{1}{2\lambda^2\sqrt{q[2]_q}(X)^2W}\Bigg[2q^3[2]_q\left(F(\frac{\alpha_+}{2q^3[2]_q},\frac{(X)^2}{q^2})-F(\frac{\alpha_-}{2q^3[2]_q},\frac{(X)^2}{q^2})\right)\begin{pmatrix}(X^0)^2+\frac{\lambda}{q}(X)^2\\ -X^0X^3\\ X^0qX^-\\ \frac{1}{q}X^0X^+\end{pmatrix}$$

$$-\Bigg(2q^{\frac{3}{2}}\sqrt{[2]_q}\lambda WF(X^0,(X)^2)$$

$$+\left(2X^0q^2{[2]_q}^2-\alpha_+\right)F(\frac{\alpha_+}{2q^3[2]_q},\frac{(X)^2}{q^2})-\left(2X^0q^2{[2]_q}^2-\alpha_-\right)F(\frac{\alpha_-}{2q^3[2]_q},\frac{(X)^2}{q^2})\Bigg)\begin{pmatrix}X^0\\ -X^3\\ qX^-\\ \frac{1}{q}X^+\end{pmatrix}\Bigg]$$

Plugging this into the differential equation (6.1) we will find only two independent equations,

$$\partial_{X^0}\triangleright F(X^0,(X)^2) \quad = \quad \alpha F(X^0,(X)^2) \tag{6.3}$$

$$\partial_{X^3}\triangleright F(X^0,(X)^2) \quad = \quad 0 \tag{6.4}$$

because the relations for the space derivatives turn out to be equivalent. Explicitly we get from (6.4):

$$2\,q\,\sqrt{q\,[2]_q}\,\lambda\,WF(X^0,(X)^2)\,+ \tag{6.5}$$

$$F(\frac{\alpha_+}{2\,q^3\,[2]_q},\frac{(X)^2}{q^2})\,\left(2\,q\,X^0\,[2]_q-\alpha_+\right)-F(\frac{\alpha_-}{2\,q^3\,[2]_q},\frac{(X)^2}{q^2})\,\left(2\,q\,X^0\,[2]_q-\alpha_-\right)=0$$

and from (6.3) together with (6.4):

$$\alpha W\,\lambda\,F(X^0,(X)^2)-q\sqrt{q\,[2]_q}\left(F(\frac{\alpha_+}{2\,q^3\,[2]_q},\frac{(X)^2}{q^2})-F(\frac{\alpha_-}{2\,q^3\,[2]_q},\frac{(X)^2}{q^2})\right)=0 \tag{6.6}$$

At first sight, if one remembers the square roots in the definition of $\alpha_\pm$ (see (5.6)), it seems, that these functional equations are very difficult to solve. But it turns out, that we can get rid of the square roots by performing the following coordinate transformation for the two central elements X^0 and $(X)^2$:

$$\begin{pmatrix}X^0\\ (X)^2\end{pmatrix}=\begin{pmatrix}\frac{1}{2}(B-A)\\ \frac{[2]_q}{4q}AB\end{pmatrix} \tag{6.7}$$

with the inverse:

$$\begin{pmatrix}A\\ B\end{pmatrix}=\begin{pmatrix}-X^0+\sqrt{(X^0)^2+\frac{4q}{[2]_q}(X^2)}\\ X^0+\sqrt{(X^0)^2+\frac{4q}{[2]_q}(X^2)}\end{pmatrix}$$

Notice, we have $\sqrt{(X^0)^2 + \frac{4q}{[2]_q}(X^2)} = \frac{1}{[2]_q}\sqrt{([2]_q^2 - 4)(X^0)^2 + 4\vec{X}^2}$, and therefore the argument of the square root is bigger than zero. The classical limit $q \to 1$ of the transformation is simply

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \sqrt{\vec{X}^2} - X^0 \\ \sqrt{\vec{X}^2} + X^0 \end{pmatrix}.$$

In the new coordinates A and B the equations (6.5) and (6.6) are:

$$\begin{aligned} (A+B)\; F(\frac{1}{2}(B-A), c\,AB) \;&=\; \\ & B\,F(\frac{1}{2}(B - \frac{A}{q^2}), \frac{c}{q^2}\,A\,B) + A\,F(\frac{1}{2}(\frac{B}{q^2} - A), \frac{c}{q^2}\,A\,B) \\ \alpha\,\frac{\lambda}{2q}\,(A+B)\; F(\frac{1}{2}(B-A), c\,AB) \;&=\; \\ & F(\frac{1}{2}(B - \frac{A}{q^2}), \frac{c}{q^2}\,A\,B) - F(\frac{1}{2}(\frac{B}{q^2} - A), \frac{c}{q^2}\,A\,B) \end{aligned}$$

with $c := \frac{[2]_q}{4\,q}$. These equations can be further simplified if we introduce the scaling operator $\Lambda_{ax} \triangleright f(x) = f(ax)$ and observe that the functions on the right hand side of the relations can be rewritten in terms of the function on the left side:

$$\begin{aligned} F(\frac{1}{2}(B - \frac{A}{q^2}), \frac{c}{q^2}\,A\,B) \;&=\; \Lambda_{\frac{A}{q^2}} \triangleright F(\frac{1}{2}(B-A), c\,AB) \\ F(\frac{1}{2}(\frac{B}{q^2} - A), \frac{c}{q^2}\,A\,B) \;&=\; \Lambda_{\frac{B}{q^2}} \triangleright F(\frac{1}{2}(B-A), c\,AB) \end{aligned}$$

This allows us to write:

$$\begin{aligned} \Big(A(1 - \Lambda_{\frac{B}{q^2}}) + B(1 - \Lambda_{\frac{A}{q^2}})\Big) \triangleright F'(A,B) \;&=\; 0 \\ \Big(\alpha\,\lambda\,(A+B) - 2\,q\,(\Lambda_{\frac{A}{q^2}} - \Lambda_{\frac{B}{q^2}})\Big) \triangleright F'(A,B) \;&=\; 0 \end{aligned} \tag{6.8}$$

abbreviating $F'(A,B) := F(\frac{1}{2}(B-A), c\,AB) = F(X^0, (X)^2)$. To bring this in a more familiar form, let us replace the scaling operator Λ_{ax} with the Jackson derivative. We have $D_{ax} \triangleright f(x) = \frac{f(x) - f(ax)}{x - ax} = \frac{1}{(1-a)x}(1 - \Lambda_{ax}) \triangleright f(x)$ and therefore $\Lambda_{ax} = (a-1)xD_{ax} + 1$. Plugging this into (6.8) we get

$$\begin{aligned} (D_{\frac{A}{q^2}} + D_{\frac{B}{q^2}}) \triangleright F'(A,B) \;&=\; 0 \\ A(\alpha + 2D_{\frac{A}{q^2}}) + B(\alpha - 2D_{\frac{B}{q^2}}) \triangleright F'(A,B) \;&=\; 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} D_{\frac{B}{q^2}} \triangleright F'(A,B) &= -D_{\frac{A}{q^2}} \triangleright F'(A,B) \\ D_{\frac{A}{q^2}} \triangleright F'(A,B) &= -\frac{\alpha}{2} F'(A,B) \\ D_{\frac{B}{q^2}} \triangleright F'(A,B) &= \frac{\alpha}{2} F'(A,B) \end{aligned} \tag{6.9}$$

These are well known differential equations for the q-exponential function in one variable. The solution of (6.9) is:

$$F'(A,B) = e_q^{-\frac{\alpha}{2}A} e_q^{\frac{\alpha}{2}B}$$

with $e_q^x := \sum_{k\geq 0} \frac{q^{\frac{1}{2}k(k-1)}}{[k]_q!} x^k$ the q-exponential function satisfying $D_{\frac{1}{q^2}x} e_q^{\alpha x} = \alpha\, e_q^{\alpha x}$. Transforming back to our original coordinates X^0 and $(X)^2$ we finally obtain:

$$\boxed{F(X^0,(X)^2) = e_q^{\frac{\alpha}{2}\left(X^0 - \sqrt{(X^0)^2 + \frac{4q}{[2]_q}(X)^2}\right)} e_q^{\frac{\alpha}{2}\left(X^0 + \sqrt{(X^0)^2 + \frac{4q}{[2]_q}(X)^2}\right)}} \tag{6.10}$$

Even though square roots appear in the arguments of our q-exponential function, they will all disappear, when we make an explicit expansion in powers of α:

$$\begin{aligned} & e_q^{\frac{\alpha}{2}\left(X^0 - \sqrt{(X^0)^2 + \frac{4q}{[2]_q}(X)^2}\right)} e_q^{\frac{\alpha}{2}\left(X^0 + \sqrt{(X^0)^2 + \frac{4q}{[2]_q}(X)^2}\right)} = \\ & \quad 1 + \alpha X^0 + \alpha^2 \left(\frac{q\lambda}{{[2]_q}^2}(X^2) + \frac{q}{[2]_q}(X^0)^2 \right) \\ & \quad +\alpha^3 \left(\frac{q^6}{1+2q^2+2q^4+q^6}(X^0)^3 - \frac{q^6\,(1+q^2-2q^4)\,\lambda^2[3]_q}{{[2]_q}^2} X^0(X^2) \right) \\ & \quad +\alpha^4 \left(\frac{q^4\lambda^2}{{[2]_q}^3[4]_q}(X^2)^2 + \frac{q^8\,(1+2q^2+3q^4)\,\lambda^3[3]_q}{{[2]_q}^2[4]_q}(X^0)^2(X^2) + \frac{q^{12}\lambda^2[3]_q}{[2]_q[4]_q}(X^0)^4 \right) \\ & \quad +\alpha^{k\geq 5} \ldots \end{aligned}$$

In the classical limit $q \to 1$ the spatial scalar $\sqrt{\vec{X}^2}$ cancels and we indeed end up with the usual exponential function:

$$F(X^0,(X)^2) = e^{-\frac{\alpha}{2}(\sqrt{\vec{X}^2} - X^0)} e^{\frac{\alpha}{2}(\sqrt{\vec{X}^2} + X^0)} = e^{\alpha X^0}$$

6.2 Boosting of the rest state

6.2.1 The highest weight vectors

The q-exponential function we have calculated in the previous section is the rest state. To get a moving particle we have to boost it, or more precisely, we want to construct the basis of the irreducible representation by using the rest state as a cyclic vector.

In fact, it is possible to find a closed formula for the action of powers $(T^2)^j$ and $(S^1)^j$ on our q-exponential. As an example, we apply the operator T^2 j-times[2]:

$$V_j^j := (T^2)^j \triangleright e_q^{-\frac{\alpha}{2}A} e_q^{\frac{\alpha}{2}B} = q^{j(j-\frac{3}{2})} \frac{\alpha^j}{[2]_q^{\frac{j}{2}}} e_q^{-q^j \frac{\alpha}{2}A} e_q^{q^j \frac{\alpha}{2}B} (X^+)^j$$

Here we used the relation in (B.26) for the action of T^2 and in addition simplified the expression with the rule $e_q^{\alpha x} = (1 - q\lambda\alpha\, x)\, e_q^{q^2\alpha x}$, which is a consequence of $D_{\frac{1}{q^2}x} e_q^{\alpha x} = \alpha\, e_q^{\alpha x}$, to have a common argument for all the q-exponential functions appearing in the calculation.

What eigenvalues have these states with respect to our maximal commuting set of observables $(\partial)^2, \partial_0, T^3$ and $\vec{T}^2$?

A and B are functions of the $su_q(2)$ scalars X^0 and $(X)^2$ and therefore it is only $(X^+)^j$ which determines the quantum numbers of T^3 and $\vec{T}^2$. Of course $(X^+)^j$ is a highest weight vector and we find in accordance with (A.1):

$$T^3 \triangleright V_j^j = q^{-2j}[2j]_q\, V_j^j \qquad \text{and} \qquad (\vec{T})^2 \triangleright V_j^j = q[j]_q[j+1]_q\, V_j^j$$

Because $(\partial)^2$ is by definition a Lorentz-scalar, T^2 commutes with it, and therefore $(\partial)^2 V_j^j = (T^2)^j \triangleright ((\partial)^2 \triangleright e_q^{-\frac{\alpha}{2}A} e_q^{\frac{\alpha}{2}B}) = -\frac{\alpha^2}{q[2]_q} V_j^j \quad \forall j \geq 0$. Remember that our definition for the Klein-Gordon operator was $(\partial)^2 = \frac{1}{q[2]_q} g^{ij}\partial_i\partial_j$, which is the reason for the factor $\frac{1}{q[2]_q}$.

To calculate the action of ∂_0, we use (B.27) giving us:

$$\partial_0 \triangleright V_j^j = \frac{\alpha}{[2]_q}\{j+1\}_q\, V_j^j$$

These eigenvalues already appeared in section 2.5.3 on page 26 where we considered the space-time lattice of the Hilbert-space representation.

[2]For S^1 we get a similar formula with X^- instead of X^+.
The action of $(\tau^1)^n$ or $(\sigma^2)^n$ is much more complicated.

Altogether we see that when we successively apply the boost T^2 on our rest state we can generate states which pass through all possible time eigenvalues. They are all of the same length, determined by the parameter α and all have, for a given time, the greatest possible quantum numbers of $\vec{T}^2$ and T^3. To state this result in a nice way, let us introduce the Dirac notation and write

$$V_j^j = | -\frac{\alpha^2}{q[2]_q}, j, j, j\rangle$$

where the quantum numbers correspond to $(\partial^2, \partial_0, \vec{T}^2, T^3)$ as the set of our observables.

6.2.2 Irreducible representations of $\mathcal{U}_q(su_2)$ in $\mathcal{M}_q$

Next we calculate basis vectors spanning an irreducible representation of the rotations. These are easy to construct, because we just have to act repetitively with T^- on our highest weight vector. Each action will decrease the magnetic quantum number by one and therefore we pass through all possible eigenvectors $|-\frac{\alpha^2}{q[2]_q}, j, j, m\rangle$, $m \in \{j, \ldots, -j\}$. To do this explicitly we have to use the relations in (B.23) and commute arbitrary powers $(T^-)^n(X^+)^m$ of the generators. As always, this is done with the help of generating functions. We start with

$$\frac{1}{1-yT^-}\frac{1}{1-zX^+} = \frac{1}{1-yT^-} + z\frac{1}{1-yT^-}X^+\frac{1}{1-zX^+} \tag{6.11}$$

To continue we need an expression for $\frac{1}{1-yT^-}X^+$:

$$\begin{aligned} \frac{1}{1-yT^-}X^+ &= X^+ + y\,\frac{1}{1-yT^-}T^-X^+ \\ &\overset{(B.23)}{=} X^+ + \frac{y}{q^2}\frac{1}{1-yT^-}X^+T^- + \sqrt{q[2]_q}\,yX^0\frac{1}{1-yT^-} \\ &\quad -\sqrt{q[2]_q}\,y\,\frac{1}{1-yT^-}X^{0/3} \end{aligned} \tag{6.12}$$

But what is $\frac{1}{1-yT^-}\,X^{0/3}$? Again using the same method, we get:

$$\begin{aligned} \frac{1}{1-yT^-}X^{0/3} &= X^{0/3} + y\frac{1}{1-yT^-}T^-X^{0/3} \\ &\overset{(B.23)}{=} X^{0/3} + y\frac{1}{1-yT^-}X^{0/3}T^- - q\sqrt{q[2]_q}\,y\,\frac{1}{1-yT^-}X^- \\ \Rightarrow \quad \frac{1}{1-yT^-}X^{0/3} &= X^{0/3}\frac{1}{1-yT^-} - q\sqrt{q[2]_q}\,yX^-\frac{1}{1-q^2yT^-}\frac{1}{1-yT^-} \end{aligned}$$

plugging this into relation (6.12) we can solve it for $\frac{1}{1-yT^-}X^+$:

$$\begin{aligned}\frac{1}{1-yT^-}X^+ &= X^+\frac{1}{1-\frac{y}{q^2}T^-} - \sqrt{q[2]_q}\,y\,X^3\frac{1}{1-yT^-}\frac{1}{1-\frac{y}{q^2}T^-}\\ &\quad +q^2[2]_q y^2 X^-\frac{1}{1-q^2yT^-}\frac{1}{1-yT^-}\frac{1}{1-\frac{y}{q^2}T^-}\end{aligned}$$

Now we can proceed with equation (6.11) and find:

$$\begin{aligned}\frac{1}{1-yT^-}\frac{1}{1-zX^+} &= \frac{1}{1-yT^-} + zX^+\frac{1}{1-\frac{y}{q^2}T^-}\frac{1}{1-zX^+} \\ &\quad -\sqrt{q[2]_q}yzX^3\frac{1}{1-yT^-}\frac{1}{1-\frac{y}{q^2}T^-}\frac{1}{1-zX^+}\\ &\quad +q^2[2]_q\,y^2zX^-\frac{1}{1-q^2yT^-}\frac{1}{1-yT^-}\frac{1}{1-\frac{y}{q^2}T^-}\frac{1}{1-zX^+}\end{aligned} \tag{6.13}$$

We have to solve this equation for $\frac{1}{1-yT^-}\frac{1}{1-zX^+}$. In fact this is possible if we apply a partial fraction decomposition to the products on the left side of the relation. We have:

$$\frac{1}{1-yT^-}\frac{1}{1-\frac{y}{q^2}T^-} = -\frac{1}{q\lambda}\frac{1}{1-\frac{y}{q^2}T^-} + \frac{q}{\lambda}\frac{1}{1-yT^-}$$

and

$$\frac{1}{1-q^2yT^-}\frac{1}{1-yT^-}\frac{1}{1-\frac{y}{q^2}T^-} = -\frac{1}{\lambda^2}\frac{1}{1-yT^-}+\frac{1}{q^3\lambda^2[2]_q}\frac{1}{1-\frac{y}{q^2}T^-}+\frac{q^3}{\lambda^2[2]_q}\frac{1}{1-q^2yT^-}.$$

Inserting this into (6.13) and rewriting $\frac{1}{1-ayT^-}$ with the help of the scaling operator as $\Lambda_{ay}\frac{1}{1-yT^-}$ we finally get the result:

$$\frac{1}{1-yT^-}\frac{1}{1-zX^+} = \frac{1}{1-\frac{yz\sqrt{q[2]_q}}{q\lambda}(q^2-\Lambda_{\frac{y}{q^2}})X^3 - \frac{y^2z}{q\lambda^2}\frac{(q^2-\Lambda_{\frac{y}{q^2}})(q^4-\Lambda_{\frac{y}{q^2}})}{\Lambda_{\frac{y}{q^2}}}X^- - z\Lambda_{\frac{y}{q^2}}X^+}\frac{1}{1-yT^-}$$

This relation holds in the semidirect product algebra $\mathcal{M}_q \rtimes U_q(su_2)$. To read off the action we just have to apply the counit ε to the second factor on the right side. In our case $\varepsilon(T^-) = 0$ and therefore we get the following polynomials with magnetic quantum number m, setting up an basis of an irreducible $U_q(su_2)$ representation in $\mathcal{M}_q$:

$$\begin{aligned}\mathcal{Y}^j_m &:= (T^-)^{j-m} \triangleright (X^+)^j \\ &= [y^{j-m}]\left(\Lambda_{\frac{y}{q^2}}X^+ + \frac{y\sqrt{q[2]_q}}{q\lambda}(q^2-\Lambda_{\frac{y}{q^2}})X^3 + \frac{y^2}{q\lambda^2}\frac{(q^2-\Lambda_{\frac{y}{q^2}})(q^4-\Lambda_{\frac{y}{q^2}})}{\Lambda_{\frac{y}{q^2}}}X^-\right)^j\end{aligned} \tag{6.14}$$

The symbol $[y^n]f(y)$ denotes the coefficient of y^n in the series expansion of $f(y)$. Notice that the time coordinate X^0 does not appear in this formula. Since the 3-dimensional space algebra is obtained by setting $X^0 = 0$ in $\mathcal{M}_q$, we see that (6.14) holds for the q-deformed 3-dimensional space[3] as well as for the q-deformed Minkowski space $\mathcal{M}_q$. Not until we start to normal order the polynomial in (6.14) the coordinate X^0 will reappear by using the commutation relations of $\mathcal{M}_q$ [40].

Above we have constructed the basis by using $(X^+)^j$ as the cyclic vector of the irreducible representation. Of course we can also take $(X^-)^j$ and act with $(T^+)^{j+m}$ to generate the basis vectors of the representation. To find these polynomials we can repeat the previous calculation in the same way, giving us:

$$\begin{aligned}\widetilde{\mathcal{Y}}^j_m &:= (T^+)^{j+m} \triangleright (X^-)^j \\ &= [y^{j+m}]\left(\Lambda_{q^2y}X^- - \frac{y\sqrt{q[2]_q}}{\lambda}(\frac{1}{q^2} - \Lambda_{q^2y})X^3 + \frac{qy^2}{\lambda^2}\frac{(\frac{1}{q^2}-\Lambda_{q^2y})(\frac{1}{q^4}-\Lambda_{q^2y})}{\Lambda_{q^2y}}X^-\right)^j\end{aligned}$$

The two bases are connected by the following factors:

$$\mathcal{Y}^j_m = q^{m^2-j(j-2)}\frac{[j-m]!_q}{[j+m]!_q}\,\widetilde{\mathcal{Y}}^j_m \tag{6.15}$$

In the end let us see, what happens in the classical limit $q \to 1$. Certainly one expects, that the above polynomials are related to the ordinary spherical harmonics. For example, if we compute the polynomials for $j = 2$ equation (6.14) gives us

$$\begin{aligned}\mathcal{Y}^2_2 &= X^+X^+ \\ \mathcal{Y}^2_1 &= \frac{\sqrt{q[2]_q}}{q^2}\left(q^2X^3X^+ + X^+X^3\right) \\ \mathcal{Y}^2_0 &= \frac{[2]_q}{q^2}\left(q^4X^-X^+ + X^+X^- + q^2[2]_qX^3X^3\right) \\ \mathcal{Y}^2_{-1} &= \sqrt{q[2]_q}[2]_q[3]_q\left(X^3X^- + q^2X^-X^3\right) \\ \mathcal{Y}^2_{-2} &= q^4[2]_q[3]_q[4]_qX^-X^-\end{aligned} \tag{6.16}$$

We don't need to normal order the coordinates, because in the classical limit they commute in any case. But be careful with the scaling operator $\Lambda_{\frac{y}{q^2}}$. It has to be evaluated before[4] we set $q \to 1$. The classical values of the space generators, transformed into spherical coordinates, are proportional to the ordinary spherical harmonics [41]:

[3]For the general quantum Euclidean space E^N_q the harmonic polynomials are calculated in [39].

[4]For example: $\frac{1}{\lambda}(q^4 - \Lambda_{\frac{y}{q^2}})y^2 = y^2\frac{1}{\lambda}(q^4 - \frac{1}{q^4}) \stackrel{q\to1}{\longrightarrow} 4y^2$ but $\frac{1}{\lambda}(q^4 - 1) \stackrel{q\to1}{\longrightarrow} 2$

$$
\begin{aligned}
X^+ &\stackrel{q\to 1}{=} -\frac{1}{\sqrt{2}} r\sin(\theta)e^{i\varphi} = 2\sqrt{\frac{\pi}{3}}\, rY_1^1(\varphi,\theta)\\
X^3 &\stackrel{q\to 1}{=} r\cos(\theta) = 2\sqrt{\frac{\pi}{3}}\, rY_0^1(\varphi,\theta)\\
X^- &\stackrel{q\to 1}{=} \frac{1}{\sqrt{2}} r\sin(\theta)e^{-i\varphi} = 2\sqrt{\frac{\pi}{3}}\, rY_{-1}^1(\varphi,\theta)
\end{aligned}
$$

If we plug them into (6.16) we find that the $\mathcal{Y}_m^2$ are for $q = 1$ also proportional to the normal spherical harmonics :

$$
\begin{aligned}
\mathcal{Y}_2^2 &\stackrel{q\to 1}{=} \frac{1}{2} r^2 e^{2i\varphi}\sin(\theta)^2 = 2\sqrt{\frac{2\pi}{15}}\, r^2 Y_2^2(\varphi,\theta)\\
\mathcal{Y}_1^2 &\stackrel{q\to 1}{=} -2r^2 e^{i\varphi}\cos(\theta)\sin(\theta) = 4\sqrt{\frac{2\pi}{15}}\, r^2 Y_1^2(\varphi,\theta)\\
\mathcal{Y}_0^2 &\stackrel{q\to 1}{=} r^2\left(1+3\cos(2\theta)\right) = 8\sqrt{\frac{\pi}{5}}\, r^2 Y_0^2(\varphi,\theta)\\
\mathcal{Y}_{-1}^2 &\stackrel{q\to 1}{=} 12 r^2 e^{-i\varphi}\cos(\theta)\sin(\theta) = 8\sqrt{\frac{6\pi}{5}}\, r^2 Y_{-1}^2(\varphi,\theta)\\
\mathcal{Y}_{-2}^2 &\stackrel{q\to 1}{=} 12 r^2 e^{-2i\varphi}\sin^2(\theta) = 16\sqrt{\frac{6\pi}{5}}\, r^2 Y_{-2}^2(\varphi,\theta)
\end{aligned}
$$

Of course, this is also valid for higher quantum numbers j: $\mathcal{Y}_m^j \stackrel{q\to 1}{\propto} Y_m^j$.

6.2.3 The solution of the q-Klein-Gordon equation

By using the Dirac notation for the states, we can summarise what we have found so far by writing:

$$
|-\frac{\alpha^2}{q[2]_q}, j, j, m\rangle = q^{j(j-\frac{3}{2})}\frac{\alpha^j}{[2]_q^{\frac{j}{2}}} e_q^{-q^j\frac{\alpha}{2}A} e_q^{q^j\frac{\alpha}{2}B}\, \mathcal{Y}_m^j
$$

with $-j \leq m \leq j$ and $j \in \mathbb{N}_0^+$.

Our next task is to construct the states $|-\frac{\alpha}{q[2]_q}, n, j, m\rangle$ with $0 \leq j < n$. Again we restrict ourselves to the highest weight vectors of the $U_q(su_2)$ representations, that is we are looking for the states $|-\frac{\alpha}{q[2]_q}, n, j, j\rangle$ with $0 \leq j < n$. We assert that we can make the following ansatz for these vectors:

$$
|-\frac{\alpha^2}{q[2]_q}, n, j, j\rangle = q^{n(n-\frac{3}{2})}\frac{\alpha^n}{[2]_q^{\frac{n}{2}}} e_q^{-q^n\frac{\alpha}{2}A} e_q^{q^n\frac{\alpha}{2}B}\; P(X^0,(X)^2)\;(X^+)^j, \qquad (6.17)
$$

where $P(X^0,(X)^2) = \sum_{0\leq k+2l\leq n-j} C_{n,m}(X^0)^k(X)^{2l}$ is a polynomial whose monomials have a grading smaller or equal than $n-j$. It is clear that this ansatz has already the correct eigenvalues with respect to $\vec{T}^2$ and T^3. What we have to take care of is, that we also get an eigenvector of ∂_0 and ∂^2. Consider first the action of the time derivative. For (6.17) to be an eigenvector of ∂_0 we have to fulfil:

$$\partial_0 | -\frac{\alpha^2}{q[2]_q}, n, j, j\rangle = \frac{\alpha}{[2]_q}\{n+1\}_q | -\frac{\alpha^2}{q[2]_q}, n, j, j\rangle$$

If we plug in our ansatz and differentiate it with the help of (B.27), we end up with a partial q-differential equation for the polynomial $G(A,B) := P(X^0,(X)^2)$, which we now write in the coordinates A, B introduced in (6.7). Defining the operator

$$\begin{aligned} O(n,j) \;:=\; & (A+B)\, q^{1+j}\alpha\lambda[n-j]_q \\ & - \left(A + q^{2(1+j)}B\right)(\alpha\lambda q^n A + 2q)\, D_{\frac{A}{q^2}} - \left(B + q^{2(1+j)}A\right)(\alpha\lambda q^n B - 2q)\, D_{\frac{B}{q^2}} \end{aligned} \tag{6.18}$$

which we will also need later on, the partial q-differential equation reads

$$O(n,j) \triangleright G(A,B) = 0$$

In fact, it is possible to find the solution of this equation. Up to a constant factor we get for fixed n and j a unique polynomial:

$$G_{n,j}(A,B) \;=\; \frac{[2j+1]_q!}{([j]_q!)^2} \sum_{a,b\geq 0}^{n-j} (-1)^b \left(\frac{\alpha\lambda}{2q}\right)^{a+b} \begin{bmatrix} n-j \\ b \end{bmatrix}_q \begin{bmatrix} n-j-b \\ a \end{bmatrix}_q \frac{[a+j]_q![b+j]_q!}{[a+b+2j+1]_q!}\, A^a B^b$$

You see, that this also fixes the state vector up to an overall normalisation constant. We even didn't need the eigenvalue equation for the Klein-Gordon operator ∂^2. Nevertheless, we have to check that our state possesses the correct eigenvalue with respect to ∂^2. For arbitrary functions $g(A,B)$ and $f(X^+)$ we deduce with the help of (B.27):

$$\begin{aligned} (\partial)^2 \triangleright [g(A,B)f(X^+)] \;=\; & \frac{4q}{[2]_q\,(A+B)}\Big[D_{\frac{A}{q^2}} D_{\frac{B}{q^2}} (A+B) g(A,B) f(X^+) \\ & + \left(D_{\frac{A}{q^2}} + D_{\frac{B}{q^2}} - \frac{\lambda}{q}(A+B) D_{\frac{A}{q^2}} D_{\frac{B}{q^2}}\right) g(A,B) X^+ D_{\frac{X^+}{q^2}} f(X^+)\Big] \end{aligned} \tag{6.19}$$

Here we insert our result for the vector $| -\frac{\alpha^2}{q[2]_q}, n, j, j\rangle$ and indeed we can validate $-\frac{\alpha^2}{q[2]_q}$ as the desired eigenvalue.

The vectors $|-\frac{\alpha^2}{q[2]_q}, n, j, m\rangle$ with $m < j$ are constructed by acting with T^- on $|-\frac{\alpha^2}{q[2]_q}, n, j, j\rangle$. Because T^- commutes with $(\partial)^2$, these vectors are also eigenvector of $(\partial)^2$ with the same eigenvalue $-\frac{\alpha^2}{q[2]_q}$.

Finally we can summarise our results:

The common eigenvectors of the operators $(\partial)^2, \partial_0, T^3, \vec{T}^2$ in the space of q-Minkowski space functions $\mathcal{M}_q$ are given by

$$\boxed{|-\frac{\alpha^2}{q[2]_q}, n, j, m\rangle = q^{n(n-\frac{3}{2})}\frac{\alpha^n}{[2]_q^{\frac{n}{2}}} e_q^{-q^n\frac{\alpha}{2}A} e_q^{q^n\frac{\alpha}{2}B}\; G_{n,j}(A,B)\; \mathcal{Y}_m^j}$$

They form a basis for the space of solutions of the Klein-Gordon equation $(\partial_i\partial^i - m^2)\,\psi = 0$, where $\alpha = \pm im$.

Chapter 7

The solution of the free q-Dirac equation

In this section, we want to realise an irreducible spin-$\frac{1}{2}$-representation of the q-Poincaré algebra by means of spinor wave functions. As in the classical case, we consider a spinor field $\psi = e_i \otimes \psi^i$ living in the tensor product space $(\mathrm{D}^{(\frac{1}{2},0)} \oplus \mathrm{D}^{(0,\frac{1}{2})}) \otimes \mathcal{M}_q$ and use a Lorentz invariant projector which is only linear in the derivatives to select an irreducible subspace. Indeed, we can formulate in complete analogy to the undeformed situation the q-Dirac equation as [32, 42]:

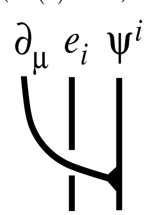
$$(m - \gamma^\mu \partial_\mu)\psi = 0 \tag{7.1}$$

But to give this equation also in the q-deformed case a precise meaning, we carefully have to examine the different terms which appear in this equation. First of all, we need to determine the gamma matrices. These matrices, defined on $(\mathrm{D}^{(\frac{1}{2},0)} \oplus \mathrm{D}^{(0,\frac{1}{2})})$, must constitute a 4-vector operator in order to combine with the derivatives to the scalar $\gamma^\mu \partial_\mu$ and as we will see in the next section, this property will fix them almost uniquely. Secondly we have to explain how the action of the derivatives on a spinor field is defined.

$$\partial_\mu \triangleright \psi = \partial_\mu \triangleright (e_i \otimes \psi^i)$$

In the classical case, we would simply commute ∂_μ with the basis vectors e_i to let it act on the functions ψ^i. However in the q-deformed setting the swapping of two tensor factors is no longer a covariant operation. Instead we have to use the braiding to assure the right transformation property:

$$\partial_\mu \triangleright (e_i \otimes \psi^i) \;=\; (\mathcal{R}_{(2)} \triangleright e_i) \otimes ((\mathcal{R}_{(1)} \triangleright \partial_\mu) \triangleright \psi^i) \tag{7.2}$$

$\sim$ 

$$\Longrightarrow \quad h \triangleright (\partial_\mu \triangleright \psi) \quad = \quad (h_{(1)} \triangleright \partial_\mu) \triangleright (h_{(2)} \triangleright \psi))$$

You might think, that this is nothing else than a slight generalisation of our usual Leibniz rule (5.2), if we set $\partial_\mu \triangleright e_i = 0$, because the spin degrees are not space dependent. Indeed this would be the case if we choose the second $\mathcal{R}$-Matrix $\mathcal{R}_{II}$ to accomplish the braiding. But as we will see later on, only if we use here the first $\mathcal{R}$-matrix $\mathcal{R}_I$ it is possible to solve the Dirac equation.

Keeping this modification in mind, the usual Dirac equation (7.1) has also in the q-deformed setting a well defined meaning.

7.1 The q-Gamma matrices

To construct the q-gamma matrices, we will proceed in this section in a similar way as in [32, 42]. The task is to find a 4-vector operator γ^μ in the space of matrices defined on the representation space $(\mathrm{D}^{(\frac{1}{2},0)} \oplus \mathrm{D}^{(0,\frac{1}{2})})$. On these matrices the q-Lorentz algebra act via the adjoint action

$$h \triangleright \gamma^\mu = \rho^{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(h_{(1)})\gamma^\mu \rho^{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(S(h_{(2)}))$$

where the matrix representations for the q-Lorentz generators can be found in the appendix (B.9). On the other hand, we can read off the action of the generators on a 4-vector operator from the relations with the space coordinates in (B.23,B.25). Each of these relations provide us with an equation for the matrices γ^μ. For example, if we look at the zero component of the 4-vector, we see that it has to fulfil the following four independent equations:

$$T^+ \triangleright \gamma^0 = T^- \triangleright \gamma^0 = T^3 \triangleright \gamma^0 = 0$$

and

$$\frac{[2]_q}{[4]_q}(\frac{1}{q}\,\tau^1 + q\,\sigma^2) \triangleright \gamma^0 = \gamma^0$$

Plugging in an arbitrary 4×4 matrix as an ansatz, we find that there are only two variables not yet fixed

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -\frac{b}{q} \\ 0 & 0 & b & 0 \\ 0 & -\frac{a}{q} & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}$$

If we require in addition that we are consistent with our choice of the metric, we have to set $(\gamma^0)^2 = -1$ and therefore $b = \frac{q}{a}$. Of course, our Dirac equation should also be covariant under the parity transformation, that means γ^0 has to commute with the parity operator $\mathcal{P}$: $[\gamma^0, \mathcal{P}] = 0$. In our basis, the parity operator, which exchanges the left and right chiral part of the $(\mathrm{D}^{(\frac{1}{2},0)} \oplus \mathrm{D}^{(0,\frac{1}{2})})$ representation, is given by the matrix

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

This matrix commutes with γ^0, if we choose $a = \sqrt{q}$.

The zeros component of the 4-vector at hand, the other components can be found in a straight forward way. In fact, by inspecting the relations for the action of the q-Lorentz generators, we can immediately read off:

$$\begin{aligned} \frac{1}{\lambda}(\sigma^2 - \tau^1) \triangleright \gamma^0 &= \gamma^3 \\ \sqrt{q[2]_q}\, T^2 \triangleright \gamma^0 &= \gamma^+ \\ -\sqrt{\frac{[2]_q}{q}}\, S^1 \triangleright \gamma^0 &= \gamma^- \end{aligned}$$

and after the evaluating of these actions we end up with

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{q}} \\ 0 & 0 & \sqrt{q} & 0 \\ 0 & -\frac{1}{\sqrt{q}} & 0 & 0 \\ \sqrt{q} & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{q}} \\ 0 & 0 & -q^{-\frac{3}{2}} & 0 \\ 0 & -\frac{1}{\sqrt{q}} & 0 & 0 \\ -q^{\frac{5}{2}} & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{[2]_q} \\ q^2\sqrt{[2]_q} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^- = \begin{pmatrix} 0 & 0 & -\frac{\sqrt{[2]_q}}{q^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{[2]_q} & 0 & 0 \end{pmatrix}$$

With this matrix representation of the q-deformed gamma matrices it is now also possible to specify the commutation relations they obey. We find[1]

$$\gamma^i \gamma^j = g^{ij} + P_{A\,kl}^{ij} \gamma^k \gamma^l \tag{7.3}$$

which also can be rewritten in the equivalent forms

$$P_{S\,kl}^{ij} \gamma^k \gamma^l = g^{ij} \qquad \text{or} \qquad \gamma^i \gamma^j + \mathcal{R}_{II\,kl}^{\;ij}\, \gamma^k \gamma^l = q[2]_q g^{ij}$$

where the antisymmetric projector $P_{A\,kl}^{ij}$, the symmetriser $P_{S\,kl}^{ij}$ and the $\mathcal{R}_{II}$-matrix are defined in (B.2, B.1). These relations generate the q-deformed Clifford algebra.

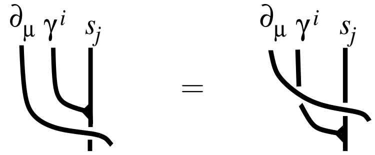
The commutation relations with the coordinates, derivatives and spinors

Next we want to calculate the square of the q-Dirac operator $(\gamma^\mu \partial_\mu)^2 = \gamma^\mu \partial_\mu \gamma^\nu \partial_\nu$. In order to simplify this expression by using the Clifford algebra relations (7.3), we first have to commute the gamma matrices γ^ν with the derivatives ∂_μ to have the

[1]The explicit form of these relations are shown in the appendix, equation (B.10)

gammas side by side. Of course, you might expect that this swapping has to be done via the braiding: $\partial^i \gamma^j = c\,\mathcal{R}^{ij}_{kl}\,\gamma^k \partial^l$. But which of the $\mathcal{R}$ matrices can we use and what value has the prefactor c ? To answer these questions, we can proceed in a straightforward manner. First we make an general ansatz for the commutation relations: $\partial^i \gamma^j = A^{ij}_{kl}\,\gamma^k \partial^l$, with an arbitrary 16×16 matrix $A^{(ij)}_{(kl)}$. We want covariant relations, so we require $h \triangleright (\partial^i \gamma^j) - A^{ij}_{kl}\, h \triangleright (\gamma^k \partial^l) = 0,\ h \in \mathcal{U}_q(sl_2(\mathbb{C}))$. Because the tensor product of two four vectors splits into four different irreducible components: $D^{(\frac{1}{2},\frac{1}{2})} \otimes D^{(\frac{1}{2},\frac{1}{2})} \cong D^{(0,0)} \oplus D^{(1,0)} \oplus D^{(0,1)} \oplus D^{(1,1)}$ every morphism on this tensor product is described by four variables. Therefore if we insert in the above covariance equation for h our symmetry generators and use the formulas in (B.23) and (B.25) to evaluate the relation, we can fix all but four coefficients of the matrix $A^{(ij)}_{(kl)}$. The remaining entries are determined by demanding compatibility with the various other algebraic structures.

Consider for example the relations in (B.11) showing the action of the gamma matrices on the spinors, which at once follow from their matrix representation. We want that the swapping of the derivatives with the gammas and spinors commutes with the action of the gammas on the spinor. Graphically, we may depict this as:

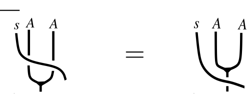


So far, we are still missing the commutation relations of the derivatives with the spinors. Therefore let us fix them now together with the relations for the gammas. In equation (7.2), we have already anticipated that these relations are also implemented by the braiding, but nevertheless let us make an general ansatz $\partial^i s^j = B^{ij}_{kl} s^k \partial^l$ and require covariance. Here the composition into the irreducible components again shows that we will end up with four coefficients not yet fixed: $D^{(\frac{1}{2},\frac{1}{2})} \otimes \left(D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}\right) \cong D^{(0,\frac{1}{2})} \oplus D^{(1,\frac{1}{2})} \oplus D^{(\frac{1}{2},0)} \oplus D^{(\frac{1}{2},1)}$. So we in addition require compatibility with the commutation relations between the derivatives and the coordinates, which can be illustrated as

(7.4)

Indeed, if we evaluate this condition we are left with only 8 different possibilities for the remaining unknowns, listed in (B.12, B.13). Compatibility of these solutions with the space and spinor algebra can then also be checked.[2]

[2] That is for $A = X$ or $A = s$ we want :



After this we can go back to the consistency condition concerning the action of the gammas on the spinors. This will give us in the end 16 different solutions, representing all the possibilities to combine the $\partial\gamma$-relations and ∂s-relations in a consistent way:

$$A^{(ij)}_{(kl)} = \begin{cases} c\,\mathcal{R}_I & \text{with} \quad c = \begin{cases} 1 \\ -1 \end{cases} \quad \text{if} \quad \mathbf{c} = \begin{cases} \pm(\mathbf{v}_1, \mathbf{v}_1) \\ \pm(\mathbf{v}_1, -\mathbf{v}_1) \end{cases} \\ c\,\mathcal{R}_I^{-1} & \text{with} \quad c = \begin{cases} 1 \\ -1 \end{cases} \quad \text{if} \quad \mathbf{c} = \begin{cases} \pm(\mathbf{v}_2, \mathbf{v}_3) \\ \pm(\mathbf{v}_2, -\mathbf{v}_3) \end{cases} \end{cases}$$
$$A^{(ij)}_{(kl)} = \begin{cases} c\,\mathcal{R}_{II} & \text{with} \quad c = \begin{cases} q \\ -q \end{cases} \quad \text{if} \quad \mathbf{c} = \begin{cases} \pm(\mathbf{v}_2, \mathbf{v}_1) \\ \pm(\mathbf{v}_2, -\mathbf{v}_1) \end{cases} \\ c\,\mathcal{R}_{II}^{-1} & \text{with} \quad c = \begin{cases} \frac{1}{q} \\ -\frac{1}{q} \end{cases} \quad \text{if} \quad \mathbf{c} = \begin{cases} \pm(\mathbf{v}_1, \mathbf{v}_3) \\ \pm(\mathbf{v}_2, -\mathbf{v}_3) \end{cases} \end{cases} \tag{7.5}$$

Here the vector $\mathbf{c} = (a, b, c, d)$ names the coefficients from (B.12, B.13) and the vectors $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (\frac{[4]_q}{[2]_q^2}, -\frac{q\lambda}{[2]_q})$ and $\mathbf{v}_3 = (\frac{[4]_q}{[2]_q^2}, \frac{\lambda}{q[2]_q})$, the various values for these variables.

Now we have everything needed to calculate the square of the q-Dirac operator. With the help of the Clifford algebra relations (7.3) we find:

$$(\gamma^\mu \partial_\mu)^2 = \begin{cases} (\partial)^2 & \text{for} \quad A^{(ij)}_{(kl)} = \mathcal{R}_I \quad \text{and} \quad \mathcal{R}_I^{-1} \\ q^{+3}(\partial)^2 & \text{for} \quad A^{(ij)}_{(kl)} = q\,\mathcal{R}_{II} \\ q^{-3}(\partial)^2 & \text{for} \quad A^{(ij)}_{(kl)} = \frac{1}{q}\,\mathcal{R}_{II}^{-1} \end{cases}$$

7.2 Spinor fields

In this chapter, we shall generalise the considerations which lead us to the solution of the q-Klein-Gordon equation and also incorporate spin degrees of freedom. The fields are now elements of the tensor product space $\mathcal{M}_q \otimes \mathrm{D}^{(\frac{1}{2},0)}$, resp. $\mathcal{M}_q \otimes \mathrm{D}^{(0,\frac{1}{2})}$, and again we try to find the common eigenvectors for a maximal set of observables. This set of operators is $(\partial)^2, \partial_0, T^3, \vec{T}^2$, as in the case of the q-Klein-Gordon equation, but now we also have to add the helicity-operator H to correctly describe the spin degrees of freedom. Note, that we have placed the space algebra $\mathcal{M}_q$ in the first tensor factor, because then the action of the derivatives is not affected by the spin part: $(\partial \otimes 1) \triangleright (f(X) \otimes v) = (\partial \triangleright f(X)) \otimes v$. This can easily be changed to the original order, as will be described in section 7.3.

7.2.1 The highest weight vectors on $\mathcal{M}_q \otimes \mathbf{D}^{(\frac{1}{2},0)}$

Let us first consider the space $\mathcal{M}_q \otimes \mathrm{D}^{(\frac{1}{2},0)}$. Using what we have learned in the Klein-Gordon case, we make the following ansatz for a highest weight vector of the rotations:

$$
\begin{aligned}
&|-\frac{\alpha^2}{q[2]_q}, h, n, j, j\rangle \;=\; \qquad\qquad (7.6)\\
&\frac{q^{\frac{-3n}{2}+n^2}\alpha^n}{[2]_q^{\frac{n}{2}}} e_q^{-q^n \frac{\alpha}{2} A} e_q^{q^n \frac{\alpha}{2} B} \Big(G_1(A,B,X^{0/3})(X^+)^{j-\frac{1}{2}} \otimes y + G_2(A,B,X^{0/3})(X^+)^{j+\frac{1}{2}} \otimes x \Big)
\end{aligned}
$$

Remember, that the magnetic quantum number of y and x is $\frac{1}{2}$ and $-\frac{1}{2}$, respectively, so $(X^+)^{j-\frac{1}{2}} \otimes y$ and $(X^+)^{j+\frac{1}{2}} \otimes x$ indeed have j as their total magnetic quantum number. From relation (3.18) on page 39 we know already the possible values of h and n, and the allowed region for j follows from the one in the Klein-Gordon case if we take into account the coupling with the spin-$\frac{1}{2}$-representation. Therefore the domain for the quantum numbers is

$$
\begin{aligned}
h = n+1: &\quad n \geq 0 \quad \text{and} \quad j = \frac{1}{2}, \dots, n + \frac{1}{2} \quad \text{with} \quad m = -j, \dots, j. \qquad \text{or}\\
h = n-1: &\quad n \geq 1 \quad \text{and} \quad j = \frac{1}{2}, \dots, n - \frac{1}{2} \quad \text{with} \quad m = -j, \dots, j.
\end{aligned}
$$

7.2.1.1 The action of T^+

First we have to make sure, that (7.6) is really a highest weight vector. For that to be true we must have $T^+ \triangleright |-\frac{\alpha^2}{q[2]_q}, n, j, j\rangle = 0$ and therefore by using (B.26) we get

$$
\begin{aligned}
T^{+} \triangleright |-\frac{\alpha^2}{q[2]_q}, h, n, j, j\rangle &= \frac{\alpha^n}{[2]_q^{\frac{n}{2}}} e_q^{-q^n \frac{\alpha}{2} A} e_q^{q^n \frac{\alpha}{2} B} \Bigg[\Big\{ q^{1-2j-\frac{3n}{2}+n^2} G_2(A, B, X^{0/3}) \\
&\quad -q^{-\frac{3}{2}-\frac{3n}{2}+n^2} \sqrt{[2]_q} D_{\frac{X^{0/3}}{q^2}} G_1(A, B, X^{0/3}) \Big\} (X^{+})^{j-\frac{1}{2}} \otimes y \\
&\quad -q^{-\frac{3}{2}-\frac{3n}{2}+n^2} \sqrt{[2]_q} D_{\frac{X^{0/3}}{q^2}} G_2(A, B, X^{0/3}) (X^{+})^{j+\frac{1}{2}} \otimes x \Bigg] \\
&\stackrel{!}{=} 0
\end{aligned}
$$

which means

$$
\begin{aligned}
G_2(A, B, X^{0/3}) &= q^{2j-\frac{5}{2}} \sqrt{[2]_q} D_{\frac{X^{0/3}}{q^2}} G_1(A, B, X^{0/3}) \\
D_{\frac{X^{0/3}}{q^2}} G_2(A, B, X^{0/3}) &= 0
\end{aligned}
$$

These differential equations can easily be solved, yielding

$$
\begin{aligned}
G_1(A, B, X^{0/3}) &= X^{0/3} F(A, B) + H(A, B) \\
G_2(A, B, X^{0/3}) &= q^{2j-1} \sqrt{q[2]_q}\, F(A, B)
\end{aligned}
\tag{7.7}
$$

with some arbitrary functions F and H.

7.2.1.2 The eigenvector equation of ∂_0

The eigenvalue equation for ∂_0 is:

$$
\partial_0 | -\frac{\alpha^2}{q[2]_q}, h, n, j, j\rangle = \frac{\alpha}{[2]_q} \{n+1\}_q | -\frac{\alpha^2}{q[2]_q}, h, n, j, j\rangle \tag{7.8}
$$

To evaluate the left side, we need to differentiate a function of the form $f(A, B, X^{3/0})(X^{+})^k$, which can be done with the help of (B.27) and (B.28):

$$
\begin{aligned}
\partial_0 \triangleright [f(A, B, X^{0/3})(X^{+})^k] &= \frac{1}{\lambda[2]_q} \Bigg(\frac{2q^2 (A-B) f(A, B, X^{0/3})}{AB} \\
&+ \frac{qAf(A, \frac{B}{q^2}, X^{0/3}) \left([2]_q B - 2qX^{0/3}\right)}{B(A+B) X^{0/3}} + \frac{f(\frac{A}{q^2}, B, X^{0/3}) \left(2q^2 B X^{0/3} + q[2]_q AB\right)}{A(A+B) X^{0/3}} \\
&+ \frac{f(\frac{A}{q^2}, B, \frac{X^{0/3}}{q^2}) \left(2q^{-2k} X^{0/3} - q[2]_q B\right)}{(A+B) X^{0/3}} - \frac{f(A, \frac{B}{q^2}, \frac{X^{0/3}}{q^2}) \left(2q^{-2k} X^{0/3} + q[2]_q A\right)}{(A+B) X^{0/3}} \Bigg) (X^{+})^k
\end{aligned}
$$

Applying this to the state vector and inserting the result in (7.8), we get two independent equations. The first relation, containing the spinor x, provides an equation for the function $F(A,B)$. It is the same differential equation we have also found in the case of the Klein-Gordon equation (see (6.18)):

$$O(n, j+\frac{1}{2}) \triangleright F(A,B) = 0$$

The only difference is that we have $j+\frac{1}{2}$ instead of j, which is due to the factor $(X^+)^{j+\frac{1}{2}}$ accompanying the spinor x. Therefore $F(A,B)$ is given by

$$\begin{aligned} F^N_{n,j}(A,B) &= N\,\frac{[2j+2]_q!}{([j+\frac{1}{2}]_q!)^2} \sum_{a,b\geq 0}^{n-j-\frac{1}{2}} (-1)^b \left(\frac{\alpha\lambda}{2q}\right)^{a+b} \begin{bmatrix} n-j-\frac{1}{2} \\ b \end{bmatrix}_q \begin{bmatrix} n-j-\frac{1}{2}-b \\ a \end{bmatrix}_q \\ &\quad \frac{[a+j+\frac{1}{2}]_q![b+j+\frac{1}{2}]_q!}{[a+b+2(j+1)]_q!}\,A^a B^b \end{aligned}$$

where N is a normalisation constant that will be fixed later.

The part of the ansatz containing the y spinor gives us the following equation:

$$\begin{aligned} O(n, j-\frac{1}{2}) \triangleright H(A,B) &= \\ \left[q^{2(j+\frac{1}{2})}[2]_q\,(A+B) - \frac{\lambda}{2}AB\left((2q+\alpha\lambda q^n A)\,D_{\frac{A}{q^2}} + (2q-\alpha\lambda q^n B)\,D_{\frac{B}{q^2}}\right)\right] F(A,B) \end{aligned} \tag{7.9}$$

7.2.1.3 The eigenvector equation of H

Additional relations for the function $H(A,B)$ can be derived from the eigenvalue equations of the helicity operator[3] H:

$$\begin{aligned} H|-\frac{\alpha^2}{q[2]_q}, h=n+1, n, j, j\rangle &= \frac{\alpha}{[2]_q}\{n+2\}_q\,|-\frac{\alpha^2}{q[2]_q}, n+1, n, j, j\rangle \\ H|-\frac{\alpha^2}{q[2]_q}, h=n-1, n, j, j\rangle &= \frac{\alpha}{[2]_q}\{n\}_q\,|-\frac{\alpha^2}{q[2]_q}, n-1, n, j, j\rangle \end{aligned}$$

To be able to calculate the action of H on our states, we first compute $H \triangleright [f(A,B,X^{3/0})(X^+)^k \otimes x]$ and $H \triangleright [f(A,B,X^{3/0})(X^+)^k \otimes y]$ for some arbitrary function f. This is done via the relations in (B.26) and (B.27), and yields the formulas shown in the appendix (B.30) and (B.31). Utilising these formulas for the state vectors, we finally end up with the following q-differential equations for the two possible values of h:

[3] H is given explicitly in (B.29)

$\mathbf{h = n + 1:}$

$$\left[\alpha\lambda q^{3+2j+n}\,(A+B) - O(n,j-\frac{1}{2})\right] F(A,B) \quad = \quad \frac{2q^2\lambda}{[2]_q}\left[(2q + Aq^n\alpha\lambda)\,D_{\frac{A}{q^2}} + (2q - Bq^n\alpha\lambda)\,D_{\frac{B}{q^2}}\right] H(A,B)$$

$$[2]_q q^{4(j+1)}\,(A+B)\,F(A,B) + \left[\alpha\lambda q^{1+2j-n}\,(A+B) + O(n,\frac{1}{2}+j)\right] H(A,B) = 0 \tag{7.10}$$

$\mathbf{h = n - 1:}$

$$\left[\alpha\lambda q^{1+2j-n}\,(A+B) - O(n,j-\frac{1}{2})\right] F(A,B) = \frac{2q^2\lambda}{[2]_q}\left[(2q + Aq^n\alpha\lambda)\,D_{\frac{A}{q^2}} + (2q - Bq^n\alpha\lambda)\,D_{\frac{B}{q^2}}\right] H(A,B)$$

$$[2]_q q^{4(j+1)}\,(A+B)\,F(A,B) + \left[\alpha\lambda q^{3+2j+n}\,(A+B) + O(n,\frac{1}{2}+j)\right] H(A,B) = 0 \tag{7.11}$$

Together with relation (7.9), we now have three equations containing the function $H(A,B)$ at our disposal. They allow us to express $H(A,B)$ in terms of $F(A,B)$. The easiest way to do this, is to rewrite all the Jackson derivatives in their explicit form again, $D_a f(x) = \frac{f(ax)-f(x)}{x(a-1)}$, because then we get three equations for the three functions $H(A,B)$, $H(\frac{A}{q^2},B)$ and $H(A,\frac{B}{q^2})$. By eliminating $H(\frac{A}{q^2},B)$ and $H(A,\frac{B}{q^2})$, we find for the case $h = n+1$:

$$\begin{aligned} H(A,B) \;&=\; O_H \triangleright F(A,B) := \\ \frac{1}{4\alpha\lambda[n+\frac{1}{2}-j]_q}&\left[\frac{[2]_q}{q^{4+j+n}}\left(\alpha\,(A-B)\left(q^{\frac{3}{2}+2n}\left(2q^{3+2j}-1\right) - q^{\frac{5}{2}+2j}\right) + 4q^{\frac{9}{2}+2j+n}[2\,(1+j)]_q\right)\right. \\ &+q^{-\frac{7}{2}-j}\left(A + q^{3+2j}B\right)(\alpha\lambda q^n A + 2q)\,[2]_q D_{\frac{A}{q^2}} \\ &\left.-q^{-\frac{7}{2}-j}\left(B + q^{3+2j}A\right)(\alpha\lambda q^n B - 2q)\,[2]_q D_{\frac{B}{q^2}}\right] F(A,B) \end{aligned} \tag{7.12}$$

Here we can insert the result for $F(A,B)$, which gives us for $H(A,B)$ the following explicit expression

$$H^I_{n,j}(A,B) = \frac{[2(j+\frac{1}{2})]_q!}{([j+\frac{1}{2}]_q!)^2} \sum_{a,b\geq 0}^{n+\frac{1}{2}-j} (-1)^b \left(\frac{\alpha\lambda}{2q}\right)^{a+b} \begin{bmatrix} n+\frac{1}{2}-j \\ b \end{bmatrix}_q \begin{bmatrix} n+\frac{1}{2}-j-b \\ a \end{bmatrix}_q \frac{[a+j+\frac{1}{2}]_q![b+j+\frac{1}{2}]_q!}{[2(j+\frac{1}{2})+a+b]_q!} A^a B^b$$

The normalisation is fixed by demanding $H^I_{n,j}(0,0) = 1$. If we compare this polynomial with the expression for $F^N_{n+1,j}(A,B)$, we see that they are very similar. The only difference is the q-factorial $\frac{1}{[2(j+\frac{1}{2})+a+b]_q!}$, compared to $\frac{1}{[2(j+1)+a+b]_q!}$ in $F^N_{n+1,j}(A,B)$. Using the standard derivatives we can correct for this factor by writing[4]:

$$H^I_{n,j}(A,B) = \frac{1}{N[2(j+1)]_q}[A\partial_A + B\partial_B + 2(j+1)]_q F^N_{n+1,j}(A,B)$$

Because of equation (7.10) the normalisation constant N for the function $F^N_{n,j}(A,B)$ is now also fixed. We get

$$N^I_{h=n+1} = q^{-(j+\frac{1}{2})}\frac{\alpha\lambda}{[2]_q}\frac{[n+\frac{1}{2}-j]_q}{[2(j+1)]_q}$$

and write $F^I_{n,j}(A,B)$ to indicate this normalisation.

In the case $h = n-1$, we can proceed in the same way. We eliminate $H(\frac{A}{q^2},B)$ and $H(A,\frac{B}{q^2})$ from the equations (7.9) and (7.11) and find

$$H^{II}_{n,j}(A,B) = \frac{[j-n-\frac{1}{2}]_q}{[j+n+\frac{3}{2}]_q}\left[O_H + \frac{q^{j-\frac{1}{2}}[2]_q}{2}\frac{[n+1]_q}{[n+\frac{1}{2}-j]_q}(B-A)\right] F^N_{n,j}(A,B)$$

where O_H is the operator from (7.12). Therefore, we get

$$H^{II}_{n,j}(A,B) = H^I_{n,j}(A,B) + \frac{q^{j-\frac{1}{2}}[2]_q}{2}\frac{[n+1]_q}{[n+\frac{3}{2}+j]_q}(A-B)F^{II}_{n,j}(A,B)$$

Again, we have chosen the normalisation such that $H^{II}_{n,j}(0,0) = 1$, which then also determines the prefactor N^{II} of $F^{II}_{n,j}(A,B)$:

$$N^{II}_{h=n-1} = -q^{-(j+\frac{1}{2})}\frac{\alpha\lambda}{[2]_q}\frac{[n+\frac{3}{2}+j]_q}{[2(j+1)]_q}$$

[4]remember: $x\partial_x x^n = nx^n$ and therefore $[x\partial_x]_q x^n = [n]_q x^n$

7.2.1.4 The eigenvalue of ∂^2

Using the ansatz (7.6) as the starting point for our calculations, we were able to fix the state vectors up to an overall normalisation constant without considering the eigenvector equation of ∂^2 at all. Because the space generated by functions of the form of the ansatz is closed under the action of ∂^2 and because ∂^2 commutes with the other observables, the unique eigenvectors from above must also be eigenvectors of ∂^2. The only thing left to check is whether these solutions have the correct eigenvalue with respect to ∂^2. But if we inspect the state vectors, we see that the part containing the spinor component x is of the same form as the solution of the Klein-Gordon equation

$$\begin{aligned}|-\frac{\alpha^2}{q[2]_q}, h, n, j, j\rangle \;\; &\propto \;\; e_q^{-q^n \frac{\alpha}{2} A} e_q^{q^n \frac{\alpha}{2} B}\, F(A,B)\,(X^+)^{j+\frac{1}{2}} \otimes x \\ &\quad + \text{const.}\, G_1(A,B,X^{0/3})(X^+)^{j-\frac{1}{2}} \otimes y\end{aligned}$$

Therefore, we know from the previous chapter that this will yield the desired factor $-\frac{\alpha^2}{q[2]_q}$ when we act with ∂^2 on the state.

7.2.2 The representation on $\mathcal{M}_q \otimes \mathbf{D}^{(\frac{1}{2},0)}$

In the same way we have done it for the Klein-Gordon equation, we obtain the eigenvalues with $m < j$ by successively applying the T^- operator, which lowers at each step the magnetic quantum number by one. This time we have to compute its action on tensor products of $(X^+)^l$ with the spinor components:

$$(T^-)^k \triangleright ((X^+)^l \otimes x) \;=\; \mathcal{Y}^l_{l-k} \otimes x \tag{7.13}$$

$$(T^-)^k \triangleright ((X^+)^l \otimes y) \;=\; \mathcal{Y}^l_{l-k} \otimes y + q^{k-1-2l}[k]_q \mathcal{Y}^l_{l-k+1} \otimes x \tag{7.14}$$

$$\begin{aligned}(T^-)^k \triangleright (X^{0/3}(X^+)^l \otimes y) \;&=\; \left(X^{0/3}\mathcal{Y}^l_{l-k} - q^k \sqrt{q[2]_q}[k]_q X^- \mathcal{Y}^l_{l-k+1} \right) \otimes y \\ &\quad + [k]_q \left(q^{k-1-2l} X^{0/3} \mathcal{Y}^l_{l-k+1} - q^{2(k-1-l)} \sqrt{q[2]_q}[k-1]_q X^- \mathcal{Y}^l_{l-k+2} \right) \otimes x\end{aligned}$$

Here the polynomials $\mathcal{Y}^j_m = (T^-)^{j-m} \triangleright (X^+)^j$ are the basis vectors of the irreducible $U_q(su_2)$ representation, as they were defined in equation (6.14).

Putting all things together we can write down the result for the common eigenvectors of the observables $(\partial)^2, \partial_0, T^3, \vec{T}^2$ and H on the space $\mathcal{M}_q \otimes \mathrm{D}^{(\frac{1}{2},0)}$:

For the two possible cases $h = n \pm 1$ we have:

$$|-\frac{\alpha^2}{q[2]_q}, h = n\pm 1, n, j, m\rangle_{(\frac{1}{2},0)} = \frac{q^{\frac{-3n}{2}+n^2}\alpha^n}{[2]_q{}^{\frac{n}{2}}} e_q^{-q^n\frac{\alpha}{2}A} e_q^{q^n\frac{\alpha}{2}B} \tag{7.15}$$
$$\left[\left(F^{I/II}_{n,j}(A,B)\,\mathbf{C}^{\pm} + H^I_{n,j}(A,B)\right)\mathcal{Y}^{j-\frac{1}{2}}_{m-\frac{1}{2}} - q^{j-m}\sqrt{q[2]_q}[j-m]_q F^{I/II}_{n,j}(A,B)X^-\mathcal{Y}^{j-\frac{1}{2}}_{m+\frac{1}{2}}\right]\otimes y$$
$$+q^{-(j+2m+1)}\left[q^{m+1}[j-m]_q\left(F^{I/II}_{n,j}(A,B)\,\mathbf{C}^{\pm} + H^I_{n,j}(A,B)\right)\mathcal{Y}^{j-\frac{1}{2}}_{m+\frac{1}{2}}\right.$$
$$\left.+q^j\sqrt{q[2]_q}F^{I/II}_{n,j}(A,B)\left(q^{2(j+m)}\mathcal{Y}^{j+\frac{1}{2}}_{m+\frac{1}{2}} - [j-m]_q[j-m-1]_q X^-\mathcal{Y}^{j-\frac{1}{2}}_{m+\frac{3}{2}}\right)\right]\otimes x$$

with the abbreviation

$$\mathbf{C}^+ = X^{0/3} \qquad \text{for } h = n+1 \tag{7.16}$$
$$\mathbf{C}^- = X^{0/3} + \frac{q^{j-\frac{1}{2}}[2]_q}{2}\frac{[n+1]_q}{[n+\frac{3}{2}+j]_q}(A-B) \qquad \text{for } h = n-1$$

7.2.3 The representation on $\mathcal{M}_q \otimes \mathbf{D}^{(0,\frac{1}{2})}$

The representation for the conjugate spinors can simply be obtained from the previous one, when we replace in (7.15) x with $-q\overline{y}$ and y with $\overline{x}$. The reason for this is, that this map constitutes an isomorphism between the module algebras ${}_{U_q(su_2)}\mathrm{D}^{(\frac{1}{2},0)}$ and ${}_{U_q(su_2)}\mathrm{D}^{(0,\frac{1}{2})}$, where $U_q(su_2)$ is the Hopf algebra generated by the rotations. This can easily be seen when you inspect the relations in (B.8). If we extend this map to the tensor product $\mathcal{M}_q \otimes \mathrm{D}^{(\frac{1}{2},0)}$, simply by setting it $id \otimes i$, if i denotes the map on $\mathrm{D}^{(\frac{1}{2},0)}$ and id is the identity map, we see that it also commutes with the action of the observables, because these operators are built up solely by the derivatives, which only act on the first tensor factor, and the generators of the rotations. So this replacement immediately gives us the set of common eigenvectors in the space $\mathcal{M}_q \otimes \mathrm{D}^{(0,\frac{1}{2})}$:

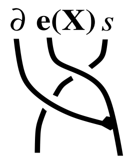
$$|-\frac{\alpha^2}{q[2]_q}, h = n\pm 1, n, j, m\rangle_{(0,\frac{1}{2})} = (id\otimes i)\left(|-\frac{\alpha^2}{q[2]_q}, h = n\pm 1, n, j, m\rangle_{(\frac{1}{2},0)}\right) \tag{7.17}$$

7.3 The solution of the q-Dirac equation

In this section, we will assemble the irreducible representations on the tensor products $\mathcal{M}_q \otimes \mathrm{D}^{(\frac{1}{2},0)}$ and $\mathcal{M}_q \otimes \mathrm{D}^{(0,\frac{1}{2})}$ to a solution of the q-Dirac equation on the space $(\mathrm{D}^{(\frac{1}{2},0)} \oplus \mathrm{D}^{(0,\frac{1}{2})}) \otimes \mathcal{M}_q$.

The first thing we have to do is to restore the original order of the tensor factors in the spinor wave function. We started with spinor fields, where the space functions were located in the first tensor factor, because in this case we don't have to take care of the spin part, if we calculate the action of the derivatives. To get a solution were the spin degrees of freedom appear in the first tensor factor, we just swap the two factors with the braiding, which means that we use one of the possible relations shown in (B.12, B.13). Because of our definition for the action of the derivative in (7.2) and the consistency condition (7.4), this swapping commutes with the action of the derivatives. So for any function $e(X)$ and spinor s we have:

$$\partial \triangleright (\mathcal{R}_{(2)} \triangleright s \otimes \mathcal{R}_{(1)} \triangleright e(X)) =$$



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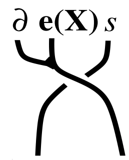

$$= (\mathcal{R}_{(2)} \triangleright s \otimes \mathcal{R}_{(1)} \triangleright (\partial \triangleright e(X))) \tag{7.18}$$

Of course, this swapping is also covariant and therefore it establishes an isomorphism between the irreducible representations on $\mathcal{M}_q \otimes \mathrm{D}^{(\frac{1}{2},0)}$, resp. $\mathcal{M}_q \otimes \mathrm{D}^{(0,\frac{1}{2})}$ and $\mathrm{D}^{(\frac{1}{2},0)} \otimes \mathcal{M}_q$, resp. $\mathrm{D}^{(0,\frac{1}{2})} \otimes \mathcal{M}_q$.

Now we can easily find the irreducible subspace in the direct summand $(\mathrm{D}^{(\frac{1}{2},0)} \oplus \mathrm{D}^{(0,\frac{1}{2})}) \otimes \mathcal{M}_q$, which contains the solutions of the Dirac equation (7.1). If $|-\frac{\alpha^2}{q[2]_q}, 1, 0, \frac{1}{2}, \frac{1}{2}\rangle_{(\frac{1}{2},0)} = e_q^{-\frac{\alpha}{2}A} e_q^{\frac{\alpha}{2}B} \otimes y$, resp. $|-\frac{\alpha^2}{q[2]_q}, 1, 0, \frac{1}{2}, \frac{1}{2}\rangle_{(0,\frac{1}{2})} = e_q^{-\frac{\alpha}{2}A} e_q^{\frac{\alpha}{2}B} \otimes \overline{x}$, is the rest state in $\mathcal{M}_q \otimes \mathrm{D}^{(\frac{1}{2},0)}$, resp. $\mathcal{M}_q \otimes \mathrm{D}^{(0,\frac{1}{2})}$, we make the following ansatz for the rest state in $(\mathrm{D}^{(\frac{1}{2},0)} \oplus \mathrm{D}^{(0,\frac{1}{2})}) \otimes \mathcal{M}_q$:

$$|-\frac{\alpha^2}{q[2]_q}, 1, 0, \frac{1}{2}, \frac{1}{2}\rangle_{\mathrm{Dirac}} = \tau \circ \mathcal{R} \triangleright \left[e_q^{-\frac{\alpha}{2}A} e_q^{\frac{\alpha}{2}B} \otimes (y + w\,\overline{x}) \right] \tag{7.19}$$

We just take a linear combination of the states and map them via the braiding into $(\mathrm{D}^{(\frac{1}{2},0)} \oplus \mathrm{D}^{(0,\frac{1}{2})}) \otimes \mathcal{M}_q$. Note that although we fix the linear combination only for the rest states, the factor w will of course be valid for the other states of the representation, too. Next, we insert this ansatz into the Dirac equation, giving us

$$(m - \alpha\gamma^0)|-\frac{\alpha^2}{q[2]_q}, 1, 0, \frac{1}{2}, \frac{1}{2}\rangle_{\mathrm{Dirac}} = 0 \tag{7.20}$$

because due to (7.18) the derivatives can immediately be evaluated. To solve this matrix equation, we now have to calculate $|-\frac{\alpha^2}{q[2]_q}, 1, 0, \frac{1}{2}, \frac{1}{2}\rangle_{\mathrm{Dirac}}$ explicitly.

As you can see from the relations in (B.12,B.13), there are various possible ways to commute the spinors with the coordinates and therefore we also get several expressions for $|-\frac{\alpha^2}{q[2]_q},1,0,\frac{1}{2},\frac{1}{2}\rangle_{\text{Dirac}}$. Consider first the swapping for the states $|-\frac{\alpha^2}{q[2]_q},1,0,\frac{1}{2},\frac{1}{2}\rangle_{(0,\frac{1}{2})}$ and $|-\frac{\alpha^2}{q[2]_q},1,0,\frac{1}{2},\frac{1}{2}\rangle_{(\frac{1}{2},0)}$ separately. We find:

$$\tau\circ\mathcal{R}\triangleright\left(e_q^{-\frac{\alpha}{2}A}e_q^{\frac{\alpha}{2}B}\otimes y\right) = \begin{cases} \text{if} \quad (a,b)=(1,0): \qquad y\otimes e_q^{-\frac{\alpha}{2}A}e_q^{\frac{\alpha}{2}B} \\ \\ \text{if} \quad (a,b)=(\frac{[4]_q}{[2]_q^2},-\frac{q\lambda}{[2]_q}): \\ \frac{1}{q^2[2]_q}y\otimes e_q^{-\frac{q\alpha}{2}A}e_q^{\frac{q\alpha}{2}B}\left(q^2[2]_q-q\alpha\lambda\left(X^0+q^2X^3\right)\right)+\frac{\alpha\lambda}{q\sqrt{q[2]_q}}x\otimes e_q^{-\frac{q\alpha}{2}A}e_q^{\frac{q\alpha}{2}B}X^+ \end{cases}$$

$$\tau\circ\mathcal{R}\triangleright\left(e_q^{-\frac{\alpha}{2}A}e_q^{\frac{\alpha}{2}B}\otimes \overline{x}\right) = \begin{cases} \text{if} \quad (c,d)=(1,0): \qquad \overline{x}\otimes e_q^{-\frac{\alpha}{2}A}e_q^{\frac{\alpha}{2}B} \\ \\ \text{if} \quad (c,d)=(\frac{[4]_q}{[2]_q^2},\frac{\lambda}{q[2]_q}): \\ \frac{1}{q^2[2]_q}\overline{x}\otimes e_q^{-\frac{q\alpha}{2}A}e_q^{\frac{q\alpha}{2}B}\left(q^2[2]_q-q\alpha\lambda\left(X^0+q^2X^3\right)\right)-\frac{\alpha\lambda}{\sqrt{q[2]_q}}\overline{y}\otimes e_q^{-\frac{q\alpha}{2}A}e_q^{\frac{q\alpha}{2}B}X^+ \end{cases}$$

where the parameters a,b,c and d are the coefficients from (B.12,B.13) and we have used the relations (B.14,B.15) to be able to commute the spinors with the function $e_q^{-\frac{\alpha}{2}A}e_q^{\frac{\alpha}{2}B}$. We dropped the cases with negative parameters a and d because they would not give the correct classical limit. These possibilities can then be combined to four different linear combinations in (7.19), but only two of them will give us solutions of the Dirac equation:

$$|-\frac{\alpha^2}{q[2]_q},1,0,\frac{1}{2},\frac{1}{2}\rangle_{\text{Dirac}} = \begin{cases} \left(y\pm\frac{i}{\sqrt{q}}\,\overline{x}\right)\otimes e_q^{-\frac{\alpha}{2}A}e_q^{\frac{\alpha}{2}B} \\ \text{or} \\ \frac{1}{q^2[2]_q}\left(y\pm\frac{i}{\sqrt{q}}\,\overline{x}\right)\otimes e_q^{-\frac{q\alpha}{2}A}e_q^{\frac{q\alpha}{2}B}\left(q^2[2]_q-q\alpha\lambda\left(X^0+q^2X^3\right)\right) \\ +\frac{\alpha\lambda}{q\sqrt{q[2]_q}}\left(x\mp i\sqrt{q}\,\overline{y}\right)\otimes e_q^{-\frac{q\alpha}{2}A}e_q^{\frac{q\alpha}{2}B}X^+ \end{cases} \tag{7.21}$$

where $\alpha=\mp im$. Only for the cases $(a,b,c,d)=(1,0,1,0)$ and $(a,b,c,d)=(\frac{[4]_q}{[2]_q^2},-\frac{q\lambda}{[2]_q},\frac{[4]_q}{[2]_q^2},\frac{\lambda}{q[2]_q})$ the coordinate functions in the second tensor factor cancel, when we insert the linear combinations into (7.20). If we compare this result with the list (7.5), we see that this combination of the coefficients forces us to use the $\mathcal{R}_I$, resp. $\mathcal{R}_I^{-1}$, matrix in the γX-$\gamma\partial$-relations and, to ensure consistency, also for the sX-$s\partial$-relations.

Having found how to combine the states of $\mathcal{M}_q \otimes \mathrm{D}^{(\frac{1}{2},0)}$ and $\mathcal{M}_q \otimes \mathrm{D}^{(0,\frac{1}{2})}$ in the right way to give a solution of the q-Dirac equation on the space $(\mathrm{D}^{(\frac{1}{2},0)} \oplus \mathrm{D}^{(0,\frac{1}{2})}) \otimes \mathcal{M}_q$, the only thing that remains is to calculate the braiding between the spinors and the space functions for the general solution:

$$|-\frac{\alpha^2}{q[2]_q}, h, n, j, m\rangle_{\mathrm{Dirac}} = \tau \circ \mathcal{R} \triangleright \left[| - \frac{\alpha^2}{q[2]_q}, h, n, j, m\rangle_{(\frac{1}{2},0)} \pm \frac{i}{\sqrt{q}} | - \frac{\alpha^2}{q[2]_q}, h, n, j, m\rangle_{(0,\frac{1}{2})} \right]$$

We will do this for the $\mathcal{R}_I$-matrix, which corresponds to the first solution in (7.21) and not for $\mathcal{R}_I^{-1}$. The reason for this is, that in this case the spinors and the coordinates A, B commute and therefore the only functions appearing in (7.15), resp. (7.17), for which we have to compute the braiding with the spinors are the $\mathcal{Y}^j_m$.

Let us do this first for the barred spinor $\overline{x}$ and $\overline{y}$. If we apply to the equations (7.13) and (7.14) the isomorphism $id \otimes i$ from subsection 7.2.3 we get

$$(T^-)^k \triangleright ((X^+)^j \otimes \overline{y}) = \mathcal{Y}^j_{j-k} \otimes \overline{y} \tag{7.22}$$

$$(T^-)^k \triangleright ((X^+)^j \otimes \overline{x}) = \mathcal{Y}^j_{j-k} \otimes \overline{x} - q^{k-2j}[k]_q \mathcal{Y}^j_{j-k+1} \otimes \overline{y} \tag{7.23}$$

and furthermore we will need

$$\begin{aligned} (T^-)^k \triangleright \left(\overline{y} \otimes (X^+)^j\right) &= q^k\, \overline{y} \otimes \mathcal{Y}^j_{j-k} \\ (T^-)^k \triangleright \left(\overline{x} \otimes (X^+)^j\right) &= q^{-k}\, \overline{x} \otimes \mathcal{Y}^j_{j-k} - q[k]_q\, \overline{y} \otimes \mathcal{Y}^j_{j-k+1} \end{aligned} \tag{7.24}$$

Using these actions and due to the simple commutation relations of the barred spinors with X^+, $X^+\overline{x} = q\overline{x}X^+$ and $X^+\overline{y} = q^{-1}\overline{y}X^+$, we now easily can compute:

$$\begin{aligned} \tau \circ \mathcal{R}_I \triangleright \left(\mathcal{Y}^j_{j-k} \otimes \overline{y}\right) &= \tau \circ \mathcal{R}_I \triangleright \left[\left((T^-)^k \triangleright (X^+)^j\right) \otimes \overline{y}\right] \\ &\overset{(7.22)}{=} \tau \circ \mathcal{R}_I \triangleright \left[(T^-)^k \triangleright \left((X^+)^j \otimes \overline{y}\right)\right] \\ &= (T^-)^k \triangleright \left[\tau \circ \mathcal{R}_I \triangleright \left((X^+)^j \otimes \overline{y}\right)\right] \\ &\overset{(B.17)}{=} q^{-j}(T^-)^k \triangleright \left[\overline{y} \otimes (X^+)^j\right] \\ &= q^{k-j}\, \overline{y} \otimes \mathcal{Y}^j_{j-k} \end{aligned}$$

and in the same way

$$\begin{aligned} \tau \circ \mathcal{R}_I \triangleright \left(\mathcal{Y}^j_{j-k} \otimes \overline{x}\right) &\overset{(7.23)}{=} (T^-)^k \triangleright \tau \circ \mathcal{R}_I \triangleright ((X^+)^j \otimes \overline{x}) \\ &\qquad + q^{k-2j}[k]_q\, \tau \circ \mathcal{R}_I \triangleright \left(\mathcal{Y}^j_{j-k+1} \otimes \overline{y}\right) \\ &= q^j (T^-)^k \triangleright \left(\overline{x} \otimes (X^+)^j\right) + q^{2k-3j-1}[k]_q \overline{y} \otimes \mathcal{Y}^j_{j-k+1} \\ &\overset{(7.24)}{=} q^{j-k}\, \overline{x} \otimes \mathcal{Y}^j_{j-k} - \lambda q^{k-j}[k]_q[2j+1-k]_q \overline{y} \otimes \mathcal{Y}^j_{j-k+1} \end{aligned}$$

The calculation for x and y can be done in analogous manner, if we use $\widetilde{\mathcal{Y}}^j_m = (T^+)^{j+m} \triangleright (X^-)^j$ instead of $\mathcal{Y}^j_m$, see equation (6.15). We do this because we can utilise the simple commutation relations with X^-, $X^-x = q^{-1}xX^-$ and $X^-y = q\,yX^-$, instead of the more complicated one for X^+. We have

$$
\begin{aligned}
(T^+)^k \triangleright ((X^-)^j \otimes y) &= \widetilde{\mathcal{Y}}^j_{k-j} \otimes y && (7.25)\\
(T^+)^k \triangleright ((X^-)^j \otimes x) &= \widetilde{\mathcal{Y}}^j_{k-j} \otimes x + q^{2j+1-k}[k]_q \widetilde{\mathcal{Y}}^j_{k-j-1} \otimes y && (7.26)\\
(T^+)^k \triangleright (y \otimes (X^-)^j) &= q^{-k}\, y \otimes \widetilde{\mathcal{Y}}^j_{k-j} && (7.27)\\
(T^+)^k \triangleright (x \otimes (X^-)^j) &= q^k\, x \otimes \widetilde{\mathcal{Y}}^j_{k-j} + [k]_q\, y \otimes \widetilde{\mathcal{Y}}^j_{k-j-1} && (7.28)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\tau \circ \mathcal{R}_I \triangleright \left(\mathcal{Y}^j_{k-j} \otimes y\right) &\overset{(6.15)}{=} q^{(k-j)^2 - j(j-2)} \frac{[2j-k]!_q}{[k]!_q}\, \tau \circ \mathcal{R}_I \triangleright \left(\widetilde{\mathcal{Y}}^j_{k-j} \otimes y\right)\\
&\overset{(7.25)}{=} q^{(k-j)^2 - j(j-2)} \frac{[2j-k]!_q}{[k]!_q}\, (T^+)^k \triangleright \left(\tau \circ \mathcal{R}_I \triangleright \left((X^-)^j \otimes y\right)\right)\\
&= q^{(k-j)^2 - j(j-2)} \frac{[2j-k]!_q}{[k]!_q}\, q^j\, (T^+)^k \triangleright (y \otimes (X^-)^j)\\
&\overset{(7.27),(6.15)}{=} q^{j-k}\, y \otimes \mathcal{Y}^j_{k-j}
\end{aligned}
$$

and

$$
\begin{aligned}
\tau \circ \mathcal{R}_I \triangleright \left(\mathcal{Y}^j_{k-j} \otimes x\right) &\overset{(6.15)}{=} q^{(k-j)^2 - j(j-2)} \frac{[2j-k]!_q}{[k]!_q}\, \tau \circ \mathcal{R}_I \triangleright \left(\widetilde{\mathcal{Y}}^j_{k-j} \otimes x\right)\\
&\overset{(7.26)}{=} q^{(k-j)^2 - j(j-2)} \frac{[2j-k]!_q}{[k]!_q} \Big[(T^+)^k \triangleright \left(\tau \circ \mathcal{R}_I \triangleright \left((X^-)^j \otimes x\right)\right)\\
&\quad - q^{2j+1-k}[k]_q\, \tau \circ \mathcal{R}_I \triangleright \left(\widetilde{\mathcal{Y}}^j_{k-j-1} \otimes y\right)\Big]\\
&\overset{(7.28),(6.15)}{=} q^{k-j} x \otimes \mathcal{Y}^j_{k-j} - \lambda q^{k-j}\, y \otimes \mathcal{Y}^j_{k-j-1}
\end{aligned}
$$

In the end we can summarise the results and find for the two chiral parts of the

Dirac spinor the following explicit expressions:

$$\tau \circ \mathcal{R}_I \triangleright \left[| -\frac{\alpha^2}{q[2]_q}, h, n, j, m\rangle_{(\frac{1}{2},0)} \right] \quad = \quad \frac{q^{\frac{-3n}{2}+n^2}\alpha^n}{[2]_q{}^{\frac{n}{2}}} e_q^{-q^n \frac{\alpha}{2} A} e_q^{q^n \frac{\alpha}{2} B}$$

$$\Bigg(y \otimes \Bigg[q^{\frac{1}{2}-2j+m} \left(F(A,B)\, \mathbf{C}^{\pm} + H(A,B) \right) \mathcal{Y}^{-\frac{1}{2}+j}_{-\frac{1}{2}+m}$$

$$- F(A,B) \sqrt{q[2]_q} \left(q^{-\frac{1}{2}+2j+m} \lambda \mathcal{Y}^{\frac{1}{2}+j}_{-\frac{1}{2}+m} + q^{\frac{1}{2}-j}[j-m]_q X^- \mathcal{Y}^{-\frac{1}{2}+j}_{\frac{1}{2}+m} \right) \Bigg]$$

$$+ x \otimes \Bigg[q^{\frac{1}{2}-j}[j-m]_q \left(F(A,B)\, \mathbf{C}^{\pm} + H(A,B) \right)^{-\frac{1}{2}+j}_{\frac{1}{2}+m}$$

$$+ \sqrt{q[2]_q} F(A,B) \left(q^{-\frac{1}{2}+2j+m} \mathcal{Y}^{\frac{1}{2}+j}_{\frac{1}{2}+m} - q^{-\frac{1}{2}-m}[-1+j-m]_q [j-m]_q X^- \mathcal{Y}^{-\frac{1}{2}+j}_{\frac{3}{2}+m} \right) \Bigg] \Bigg)$$

$$\tau \circ \mathcal{R}_I \triangleright \left[| -\frac{\alpha^2}{q[2]_q}, h, n, j, m\rangle_{(0,\frac{1}{2})} \right] \quad = \quad \frac{q^{\frac{-3n}{2}+n^2}\alpha^n}{[2]_q{}^{\frac{n}{2}}} e_q^{-q^n \frac{\alpha}{2} A} e_q^{q^n \frac{\alpha}{2} B}$$

$$\Bigg(\overline{x} \otimes \Bigg[q^{-\frac{1}{2}+m} \left(F(A,B)\, \mathbf{C}^{\pm} + H(A,B) \right) \mathcal{Y}^{-\frac{1}{2}+j}_{-\frac{1}{2}+m} - q^{-\frac{1}{2}+j} \sqrt{q[2]_q}[j-m]_q F(A,B) X^- \mathcal{Y}^{-\frac{1}{2}+j}_{\frac{1}{2}+m} \Bigg]$$

$$+ \overline{y} \otimes \Bigg[q^{\frac{1}{2}+j}[-j+m]_q H(A,B)$$

$$+ \frac{q^{-\frac{3}{2}+j}}{2} \Big(q^2 \left(2\, \mathbf{C}^{\pm} + \lambda [2]_q \left(A + 2X^{0/3} \right) \right) - q^2 \lambda [2]_q B \Big) [-j+m]_q F(A,B) \mathcal{Y}^{-\frac{1}{2}+j}_{\frac{1}{2}+m}$$

$$+ \sqrt{q[2]_q} F(A,B) \Big(q^{-\frac{3}{2}+m} \lambda X^+ \mathcal{Y}^{-\frac{1}{2}+j}_{-\frac{1}{2}+m} - q^{-\frac{1}{2}+2j-m} \mathcal{Y}^{\frac{1}{2}+j}_{\frac{1}{2}+m}$$

$$+ q^{\frac{3}{2}+2j-m}[j-1-m]_q [j-m]_q \mathcal{Y}^{-\frac{1}{2}+j}_{\frac{3}{2}+m} X^- \Big) \Bigg] \Bigg)$$

where $\mathbf{C}^{\pm}$ are defined in (7.16).

Appendix A

Representations

A.1 Representation of $U_q(su_2)$

The standard D^j representation of $U_q(su_2)$ is given by

$$\begin{aligned}
\vec{T}^2 \triangleright |j,m\rangle &= q[j]_q[j+1]_q|j,m\rangle \\
T^3 \triangleright |j,m\rangle &= q^{-2m}[2m]_q|j,m\rangle \\
T^+ \triangleright |j,m\rangle &= q^{-m-\frac{3}{2}}\sqrt{[j+m+1]_q[j-m]_q}\,|j,m+1\rangle \\
T^- \triangleright |j,m\rangle &= q^{-m+\frac{3}{2}}\sqrt{[j+m]_q[j-m+1]_q}\,|j,m-1\rangle \\
\tau^3 \triangleright |j,m\rangle &= q^{-4m}|j,m\rangle
\end{aligned} \tag{A.1}$$

A.2 The representation of the coordinates

<u>The case l=0:</u>

$$\begin{aligned}
C|0,n,\nu,m,h\rangle &= \lambda\sqrt{q\,[2]_q}q^{\nu}\,[n-\nu]_q\,\tau_0\,|0,n,\nu,m,h\rangle \\
D|0,n,\nu,m,h\rangle &= \sqrt{q\,[2]_q}q^{2\nu-n}\tau_0|0,n,\nu,m,h\rangle \\
A|0,n,\nu,m,h\rangle &= q^{\nu-n}\sqrt{q\,[2]_q}\sqrt{q^{n+\nu+1}\lambda\,[n-\nu-1]_q}\tau_0\,|0,n,\nu+1,m+1,h\rangle \\
B|0,n,\nu,m,h\rangle &= q^{\nu-n-1}\sqrt{q\,[2]_q}\sqrt{q^{n+\nu}\lambda\,[n-\nu]_q}\tau_0\,|0,n,\nu-1,m-1,h\rangle
\end{aligned}$$

The time-like case:

$$
\begin{aligned}
C|l,n,\nu,m,h\rangle &= \frac{q^{-n}\left(1+q^{n+1+\nu}\lambda\left[n+1-\nu\right]_q\right)}{\sqrt{q\left[2\right]_q}}t_0\left|l,n,\nu,m,h\right\rangle \\
D|l,n,\nu,m,h\rangle &= \frac{q^{2\nu-n}}{\sqrt{q\left[2\right]_q}}t_0\left|l,n,\nu,m,h\right\rangle \\
A|l,n,\nu,m,h\rangle &= \frac{q^{\nu-\frac{n}{2}}\lambda\sqrt{\left[1+\nu\right]_q\left[n-\nu\right]_q}}{\sqrt{\left[2\right]_q}}t_0\left|l,n,\nu+1,m+1,h\right\rangle \\
B|l,n,\nu,m,h\rangle &= \frac{q^{\nu-\frac{n}{2}-1}\lambda\sqrt{\left[\nu\right]_q\left[n+1-\nu\right]_q}}{\sqrt{\left[2\right]_q}}t_0\left|l,n,\nu-1,m-1,h\right\rangle
\end{aligned}
$$

The space-like case:

$$
\begin{aligned}
C|l,n,\nu,m,h\rangle &= \frac{q^{1-n}}{\sqrt{q\left[2\right]_q}}\left(q^{n+\nu}\lambda\left[n-\nu\right]_q-1\right)l_0\left|l,n,\nu,m,h\right\rangle \\
D|l,n,\nu,m,h\rangle &= \frac{q^{2\nu+1-n}}{\sqrt{q\left[2\right]_q}}l_0|l,n,\nu,m,h\rangle \\
A|l,n,\nu,m,h\rangle &= \frac{q^{\nu-\frac{n-1}{2}}\sqrt{\lambda}}{\sqrt{\left[2\right]_q}}\sqrt{\left[n-\nu-1\right]_q\left\{\nu+1\right\}_q}l_0\left|l,n,\nu+1,m+1,h\right\rangle \\
B|l,n,\nu,m,h\rangle &= \frac{q^{\nu-\frac{n+1}{2}}\sqrt{\lambda}}{\sqrt{\left[2\right]_q}}\sqrt{\left[n-\nu\right]_q\left\{\nu\right\}_q}l_0\left|l,n,\nu-1,m-1,h\right\rangle
\end{aligned}
$$

A.3 The representation of the rotations

$$
\begin{aligned}
\tau^3|l,n,\nu,m,h\rangle &= q^{-4m}|l,n,\nu,m,h\rangle \\
T^+|l,n,\nu,m,h\rangle &= t^+\left(n,\nu\right)\left|l,n,\nu+1,m+1,h\right\rangle+\gamma\left(n,\nu,m,h\right)\left|l,n,\nu,m+1,h\right\rangle \\
T^-|l,n,\nu,m,h\rangle &= q^2t^+\left(n,\nu-1\right)\left|l,n,\nu-1,m-1,h\right\rangle \\
&\quad +q^2\gamma\left(n,\nu,m-1,h\right)\left|l,n,\nu,m-1,h\right\rangle
\end{aligned}
$$

<u>The case l=0</u>:

$$\begin{aligned} t^+(n,\nu) &= \frac{q^{\frac{1}{2}(n-\nu-3)}}{\sqrt{\lambda}}\sqrt{[n-\nu-1]_q} \\ \gamma(n,\nu,m,h) &= \frac{q^{\frac{1}{4}(h+n-2(3n+\nu+4))}}{\sqrt{\lambda}}\sqrt{\left[\frac{1}{2}(h+n+2(m-\nu))\right]_q} \end{aligned}$$

<u>The time-like case:</u>

$$\begin{aligned} t^+(n,\nu) &= q^{\frac{1}{2}(n-2\nu-3)}\sqrt{[\nu+1]_q\,[n-\nu]_q} \\ \gamma(n,\nu,m,h) &= q^{\frac{1}{2}(n-2(m+\nu)-3)}\sqrt{\left[\frac{1}{2}(h+n+2(m-\nu+1))\right]_q\left[\frac{1}{2}(h-n-2(m-\nu))\right]_q} \end{aligned}$$

<u>The space-like case</u>:

$$\begin{aligned} t^+(n,\nu) &= \frac{q^{\frac{n-2(\nu+2)}{2}}}{\sqrt{\lambda}}\sqrt{[n-\nu-1]_q\,\{\nu+1\}_q} \\ \gamma(n,\nu,m,h) &= \frac{q^{\frac{n-2(m+\nu+2)}{2}}}{\sqrt{\lambda}}\sqrt{\left[\frac{1}{2}(h+n+2(m-\nu))\right]_q\{\frac{1}{2}(h-n-2(m-\nu))\}_q} \end{aligned}$$

A.4 Intermediate results for τ^1

The rules following from the relations of τ^1 with the coordinates:

$$\begin{aligned} \tau_1(0,n,1+\nu,1+m,h,h') &= q^2\tau_1(0,n,\nu,m,h,h') \qquad \text{(A.2)} \\ \tau_2(0,n,1+\nu,1+m,h,h') &= q^{\frac{3}{2}}\sqrt{\frac{[-1+n-\nu]_q}{[n-\nu]_q}}\tau_2(0,n,\nu,m,h,h') \\ \tau_1(-\frac{{t_0}^2}{q[2]_q},n,1+\nu,1+m,h,h') &= q^{\frac{3}{2}}\sqrt{\frac{[1+\nu]_q}{[\nu]_q}}\tau_1(-\frac{{t_0}^2}{q[2]_q},n,\nu,m,h,h') \\ \tau_2(-\frac{{t_0}^2}{q[2]_q},n,1+\nu,1+m,h,h') &= q^{\frac{3}{2}}\sqrt{\frac{[n-\nu]_q}{[1+n-\nu]_q}}\tau_2(-\frac{{t_0}^2}{q[2]_q},n,\nu,m,h,h') \\ \tau_1(\frac{{l_0}^2}{q[2]_q},n,1+\nu,1+m,h,h') &= q^{\frac{3}{2}}\sqrt{\frac{\{1+\nu\}_q}{\{\nu\}_q}}\tau_1(\frac{{l_0}^2}{q[2]_q},n,\nu,m,h,h') \\ \tau_2(\frac{{l_0}^2}{q[2]_q},n,1+\nu,1+m,h,h') &= q^{\frac{3}{2}}\sqrt{\frac{[-1+n-\nu]_q}{[n-\nu]_q}}\tau_2(\frac{{l_0}^2}{q[2]_q},n,\nu,m,h,h') \end{aligned}$$

The rules following from the relation $\tau^1 T^+ - T^+\tau^1 - \lambda T^2 = 0$:

$$\tau^1_{1/2}(0,n,\nu,1+m,h,h') = q^{\frac{1-h+h'}{4}}\sqrt{\frac{[\frac{1+2m+n-2\nu+h'}{2}]_q}{[\frac{h+2m+n-2\nu}{2}]_q}}\tau^1_{1/2}(0,n,\nu,m,h,h') \tag{A.3}$$

$$\tau^1_{1/2}(\frac{-t_0{}^2}{q[2]_q},n,\nu,1+m,h,h') = \sqrt{q\frac{[\frac{3+2m+n-2\nu+h'}{2}]_q[\nu+\frac{-1-2m-n+h'}{2}]_q}{[\frac{2+h+2m+n-2\nu}{2}]_q[\frac{h-2m-n}{2}+\nu]_q}}\,\tau^1_{1/2}(\frac{-t_0{}^2}{q[2]_q},n,\nu,m,h,h')$$

$$\tau_{1/2}(\frac{l_0{}^2}{q[2]_q},n,\nu,1+m,h,h') = \sqrt{q\frac{[\frac{1+2m+n-2\nu+h'}{2}]_q\{\frac{1+2m+n-2\nu-h'}{2}\}_q}{[\frac{h+2m+n-2\nu}{2}]_q\{\frac{-h+2m+n-2\nu}{2}\}_q}}\,\tau^1_{1/2}(\frac{l_0{}^2}{q[2]_q},n,\nu,m,h,h')$$

τ^1_2 in terms of τ^1_1:

$$\tau^1_2(0,n,\nu,m,h,h') = q^{\frac{2-h-2m+n-8\nu}{4}}\sqrt{\frac{[n-\nu]_q}{[\frac{h+2m+n-2\nu}{2}]_q}} \tag{A.4}$$
$$\left(q^{\frac{1}{2}+2\nu} - q^{\frac{h+4m+2n+h'}{2}}[\frac{1-h+h'}{2}]_q\right)\tau^1_1(0,1+n,\nu,m,h',h)$$

$$\tau^1_2(-\frac{t_0{}^2}{q[2]_q},n,\nu,m,h,h') = \frac{1}{\lambda}q^{\frac{1-h-2m-2\nu-h'}{2}}\sqrt{\frac{[1+n-\nu]_q}{[\frac{2+h+2m+n-2\nu}{2}]_q[\nu]_q[\frac{h-2m-n}{2}+\nu]_q}}$$
$$\left(q^{\frac{1+h+4\nu+h'}{2}} + q^{2m+n}\left([\frac{h'-1-h}{2}]_q - q^{2+h+h'}[\frac{1-h+h'}{2}]_q\right)\right)\tau^1_1(-\frac{t_0{}^2}{q[2]_q},1+n,\nu,m,h',h)$$

$$\tau^1_2(\frac{l_0{}^2}{q[2]_q},n,\nu,m,h,h') = q^{\frac{-h-2m-2\nu-h'}{2}}\sqrt{\frac{[n-\nu]_q}{[\frac{h+2m+n-2\nu}{2}]_q\{\nu\}_q\{\frac{h-2m-n}{2}+\nu\}_q}}$$
$$\left(q^{\frac{2+h+4\nu+h'}{2}} - q^{\frac{1}{2}+2m+n}\left([\frac{h'-1-h}{2}]_q + q^{h+h'}[\frac{1-h+h'}{2}]_q\right)\right)\tau^1_1(\frac{l_0{}^2}{q[2]_q},1+n,\nu,m,h',h)$$

Shift of h translates into a shift of n:

$$\tau^1_{1,1}(0,n,\nu,m,1+h) = q^{-\frac{5}{2}}\sqrt{\frac{[\frac{1+h+2m+n-2\nu}{2}]_q}{[\frac{h-1+2m+n-2\nu}{2}]_q}}\tau^1_{1,1}(0,n-1,\nu,m,h) \tag{A.5}$$

$$\tau^1_{1,1}(\frac{l_0{}^2}{q[2]_q},n,\nu,m,1+h) =$$
$$\sqrt{\frac{\{h\}_q\{n-2\}_q[\frac{1+h+2m+n-2\nu}{2}]_q}{q\,\{2+h\}_q\{n\}_q[\frac{-1+h+2m+n-2\nu}{2}]_q}}\,\tau^1_{1,1}(\frac{l_0{}^2}{q[2]_q},n-1,\nu,m,h)$$

$$\tau^1_{1,1}(-\frac{t_0{}^2}{q[2]_q},n,\nu,m,1+h) =$$
$$\sqrt{\frac{[1+h]_q[n-1]_q[\frac{3+h+2m+n-2\nu}{2}]_q}{q\,[3+h]_q[1+n]_q[\frac{1+h+2m+n-2\nu}{2}]_q}}\,\tau^1_{1,1}(-\frac{t_0{}^2}{q[2]_q},n-1,\nu,m,h)$$

$$\tau_{1,-1}(0,n,\nu,m,1+h) \quad = \quad q\tau_{1,-1}(0,1+n,\nu,m,h), \tag{A.6}$$

$$\tau_{1,-1}(\frac{{l_0}^2}{q[2]_q},n,\nu,m,1+h) \quad = \quad \sqrt{q\frac{\{h-1\}_q\{n+1\}_q\{\frac{-1-h+2m+n-2\nu}{2}\}_q}{\{h+1\}_q\{n-1\}_q\{\frac{1-h+2m+n-2\nu}{2}\}_q}}\,\tau_{1,-1}(\frac{{l_0}^2}{q[2]_q},1+n,\nu,m,h)$$

$$\tau_{1,-1}(-\frac{{t_0}^2}{q[2]_q},n,\nu,m,1+h) \quad = \quad \sqrt{q\frac{[h]_q[2+n]_q[\frac{1+h-2m-n+2\nu}{2}]_q}{[2+h]_q[n]_q[\frac{-1+h-2m-n+2\nu}{2}]_q}}\,\tau_{1,-1}(-\frac{{t_0}^2}{q[2]_q},1+n,\nu,m,h)$$

Recursion relations resulting from the relation $1=\sigma^2\tau^1-\lambda^2S^1T^2$ for the case $l\neq 0$:

$$\begin{aligned}1 \quad = \quad & \frac{\{h\}_q\left(q^{-1-h}\{n\}_q+q^{-n}\left(q^h\lambda+q^{-2m+2\nu}\{1+n\}_q\right)\right)}{q^\nu\lambda^2[h-n]_q\{n\}_q\{\nu\}_q}R(l_0,n) \\ & +\frac{q^{-1-2h-2m-n-\nu}\{h\}_q}{\lambda^3[-2+h-n]_q[h-n]_q\{n\}_q\{\nu\}_q} \\ & \Big[q^{2(1+m+n)}\{n\}_q-q^{1+2h+2\nu}\lambda[2]_q[-1+h-n]_q\{1+n\}_q \\ & +q^{2(h+m)+n}\left(q^2\lambda[2]_q-\{2\}_q-\{2\,(1+n)\}_q\right)+q^{2(2h+m)}\{2+n\}_q\Big]R(l_0,n+2) \\ & -\frac{q^{\frac{h-2m-n}{2}}[\frac{h+2m+n-2\nu}{2}]_q\{h\}_q\{1+n\}_q}{\lambda[-2+h-n]_q\{n\}_q\{\nu\}_q}R(l_0,n+4)\end{aligned} \tag{A.7}$$

$$\begin{aligned}1 \quad = \quad & \frac{q^{-2-h-2m-n-\nu}[1+h]_q\left(q^{2(1+h+m)}+q^{2m+n}[1+n]_q-q^{1+h+2\nu}[2+n]_q\right)}{\lambda^2[h-n]_q[1+n]_q[\nu]_q}R(t_0,n) \\ & -\frac{q^{-4-h-2m-2n-\nu}}{\lambda^4[2]_q[2+2h]_q[-2+h-n]_q[h-n]_q[1+n]_q[\nu]_q} \\ & \Bigg[-q^{2m}\left(q^{2h}+q^{6+4n}\right)[2]_q{[2+2h]_q}^2+q^{3+h+n}[1+h]_q\Bigg(q^{2+2m+n}[2]_q[4+4h]_q \\ & -[2+2h]_q\Big(q^{2m+n}\left(q^2\lambda{[2]_q}^2-[4]_q\right)+q^{2\nu}\lambda^2{[2]_q}^2[-1+h-n]_q[2+n]_q\Big)\Bigg)\Bigg]R(t_0,n+2) \\ & +\frac{q^{\frac{h-2m-n}{2}}[1+h]_q[2+n]_q[\frac{2+h+2m+n-2\nu}{2}]_q}{\lambda[-2+h-n]_q[1+n]_q[\nu]_q}R(t_0,n+4)\end{aligned} \tag{A.8}$$

A.5 The Pauli-Lubanski vector

Expressed in terms of our set of generators the components of the Pauli-Lubanski vector are given by

$$\begin{aligned}
\mathfrak{P}_A &= \frac{q^2}{\lambda}A - qCT^2\tau^1(\tau^3)^{\frac{1}{2}} - q^2DT^+ + q^3DT^2\tau^1(\tau^3)^{\frac{1}{2}} - \frac{q^2}{\lambda}A\,(\tau^1)^2(\tau^3)^{\frac{1}{2}} \\
&\quad +\lambda B(T^2)^2(\tau^3)^{\frac{1}{2}} \\
\mathfrak{P}_B &= \frac{1}{q^2\lambda}B + q^{-3}CS^1\sigma^2(\tau^3)^{-\frac{1}{2}} - q^{-2}DT^- - \frac{1}{q}DS^1\sigma^2(\tau^3)^{-\frac{1}{2}} \\
&\quad -\frac{1}{q^2\lambda}B(\sigma^2)^2(\tau^3)^{-\frac{1}{2}} + \frac{\lambda}{q^2}A(S^1)^2(\tau^3)^{-\frac{1}{2}} \\
\mathfrak{P}_C &= -\frac{1}{\lambda}C - AT^-(\tau^3)^{-\frac{1}{2}} + \frac{1}{q}BT^2\sigma^2 - qAS^1\tau^1 - q^2BT^+(\tau^3)^{-\frac{1}{2}} + \frac{1}{\lambda}C(\tau^3)^{-\frac{1}{2}} \\
&\quad -\lambda CS^1T^2 + q^2\lambda DS^1T^2 + q^2\lambda DT^-T^+(\tau^3)^{-\frac{1}{2}} \\
\mathfrak{P}_D &= -\frac{1}{\lambda}D - \frac{1}{q}BT^2\sigma^2 + qAS^1\tau^1 + \frac{1}{\lambda}D(\tau^3)^{\frac{1}{2}} + \lambda CS^1T^2 - q^2\lambda DS^1T^2
\end{aligned} \tag{A.9}$$

The representation of the Pauli-Lubanski vector

The case $l = 0$:

$$\begin{aligned}
&\mathfrak{P}_A|0,n,\nu,m,h\rangle \quad = \\
&\quad -\frac{q^{-\frac{1}{2}-2h-2m-n+2\nu}\tau_0}{\lambda}\sqrt{\left(q^{2+h+n} - C(0,h+n)\right)C(0,h+n)[2]_q}|0,n,\nu,1+m,-2+h\rangle \\
&\quad -q^{\frac{-10-7h-6m-3n+6\nu}{4}}\tau_0\sqrt{\frac{[2]_q[\frac{h+2m+n-2\nu}{2}]_q}{\lambda}} \\
&\quad \left(C(0,h+n)[2]_q + q^{\frac{6+3h+n}{2}}\lambda[\frac{h-n}{2}]_q\right)|0,n,\nu,1+m,h\rangle + \\
&\quad +q^{-4-\frac{3h}{2}-m-\frac{n}{2}+\nu}\tau_0\sqrt{[2]_q[\frac{h+2m+n-2\nu}{2}]_q[\frac{2+h+2m+n-2\nu}{2}]_q} \\
&\quad \sqrt{\left(q^{4+h+n} - C(0,h+n)\right)C(0,h+n)}|0,n,\nu,1+m,2+h\rangle
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{P}_B|0,n,\nu,m,h\rangle &= \\
&-\frac{q^{-\frac{9}{2}-2h-2m-n+2\nu}\tau_0}{\lambda}\sqrt{\left(q^{4+h+n}-(0,h+n)\right)C(0,h+n)[2]_q}|0,n,\nu,-1+m,2+h\rangle \\
&-q^{\frac{-7h-3(4+2m+n-2\nu)}{4}}\tau_0\sqrt{\frac{[2]_q[\frac{-2+h+2m+n-2\nu}{2}]_q}{\lambda}} \\
&\left(C(0,h+n)[2]_q+q^{\frac{6+3h+n}{2}}\lambda[\frac{h-n}{2}]_q\right)|0,n,\nu,-1+m,h\rangle \\
&+q^{-2-\frac{3h}{2}-m-\frac{n}{2}+\nu}\tau_0\sqrt{[2]_q[\frac{-4+h+2m+n-2\nu}{2}]_q[\frac{-2+h+2m+n-2\nu}{2}]_q} \\
&\sqrt{\left(q^{2+h+n}-C(0,h+n)\right)C(0,h+n)}\,|0,n,\nu,-1+m,-2+h\rangle \\
\mathfrak{P}_C|0,n,\nu,m,h\rangle &= \\
&q^{\frac{-4-7h-6m-3n+6\nu}{4}}\tau_0\sqrt{\frac{[2]_q}{\lambda}[\frac{-2+h+2m+n-2\nu}{2}]_q} \\
&\sqrt{\left(q^{2+h+n}-C(0,h+n)\right)C(0,h+n)}|0,n,\nu,m,-2+h\rangle \\
&+q^{\frac{-7h-3(6+2m+n-2\nu)}{4}}\tau_0\sqrt{\frac{[2]_q}{\lambda}[\frac{h+2m+n-2\nu}{2}]_q} \\
&\sqrt{\left(q^{4+h+n}-CC(0,h+n)\right)CC(0,h+n)}|0,n,\nu,m,2+h\rangle \\
&-\tau_0\frac{q^{-\frac{7}{2}-2h-2m-n+\nu}}{\lambda}\sqrt{[2]_q} \\
&\left[C(0,h+n)\left(q^{2+\nu}-q^{\frac{h+2m+n}{2}}\lambda[\frac{h+2m+n-2\nu}{2}]_q\right)\right. \\
&\left.-q^{4+2h+m+n}\lambda^2[\frac{h-n}{2}]_q[\frac{h+2m+n-2\nu}{2}]_q\right]|0,n,\nu,m,h\rangle \\
\mathfrak{P}_D|0,n,\nu,m,h\rangle &= \\
&-q^{\frac{-4-7h-6m-3n+6\nu}{4}}\tau_0\sqrt{\frac{[2]_q}{\lambda}[\frac{-2+h+2m+n-2\nu}{2}]_q} \\
&\sqrt{\left(q^{2+h+n}-C(0,h+n)\right)C(0,h+n)}|0,n,\nu,m,-2+h\rangle \\
&-q^{\frac{-7h-3(6+2m+n-2\nu)}{4}}\tau_0\sqrt{\frac{[2]_q}{\lambda}[\frac{h+2m+n-2\nu}{2}]_q} \\
&\sqrt{\left(q^{4+h+n}-C(0,h+n)\right)C(0,h+n)}|0,n,\nu,m,2+h\rangle \\
&+\frac{q^{-\frac{7}{2}-2h-2m-n+\nu}\tau_0}{\lambda}\sqrt{[2]_q}\left[q^{\frac{8+3h+n+2\nu}{2}}\lambda[\frac{h-n}{2}]_q\right. \\
&\left.+\left(q^{2+\nu}-q^{\frac{h+2m+n}{2}}\lambda[\frac{h+2m+n-2\nu}{2}]_q\right)C(0,h+n)\right]|0,n,\nu,m,h\rangle
\end{aligned}
$$

The case $l<0$:

$$\mathfrak{P}_A|-\frac{{t_0}^2}{q[2]_q},n,\nu,m,h\rangle =$$

$$\frac{q^{\frac{2+h-2m-n}{2}+\nu}\lambda}{[2+h]_q}\sqrt{\frac{[\frac{2+h+2m+n-2\nu}{2}]_q[\frac{4+h+2m+n-2\nu}{2}]_q}{\lambda^2[2]_q[1+h]_q[3+h]_q}}t_0$$

$$\sqrt{\left(C(t_0,h+n)-[\tfrac{1+h-n}{2}]_q{}^2\right)\left([\tfrac{3+h+n}{2}]_q{}^2-C(t_0,h+n)\right)}|\frac{-t_0^2}{q[2]_q},n,\nu,1+m,2+h\rangle$$

$$-\frac{q^{\frac{-h}{2}-m-\frac{n}{2}+\nu}\lambda t_0}{[h]_q}\sqrt{\frac{[\frac{-2+h-2m-n}{2}+\nu]_q[\frac{h-2m-n}{2}+\nu]_q}{\lambda^2[2]_q[-1+h]_q[1+h]_q}}$$

$$\sqrt{\left(C(t_0,h+n)-[\tfrac{-1+h-n}{2}]_q{}^2\right)\left([\tfrac{1+h+n}{2}]_q{}^2-C(t_0,h+n)\right)}|\frac{-t_0^2}{q[2]_q},n,\nu,1+m,-2+h\rangle$$

$$+\frac{q^{-2-\frac{3h}{2}-m-n+\nu}t_0}{\lambda^2[h]_q[2+h]_q}\sqrt{\frac{[\frac{2+h+2m+n-2\nu}{2}]_q[\frac{h-2m-n}{2}+\nu]_q}{[2]_q}}$$

$$\Bigg[\frac{q^{\frac{3}{2}(2+h+n)}[2\,(1+h)]_q}{[1+h]_q}+q^{\frac{3h+n}{2}}\left(-2q-2q^3-q^2\lambda^2C(t_0,h+n)[2]_q+\frac{q^2[4]_q}{[2]_q}\right)$$

$$+\lambda\left([\frac{h-n}{2}]_q-q^{3(1+h)}[\frac{2+h+n}{2}]_q\right)\Bigg]|-\frac{{t_0}^2}{q[2]_q},n,\nu,1+m,h\rangle$$

$$\mathfrak{P}_B|-\frac{{t_0}^2}{q[2]_q},n,\nu,m,h\rangle =$$

$$\frac{q^{\frac{-3+h-2m-n}{2}+\nu}t_0}{[h]_q}\sqrt{\frac{q[\frac{-2+h+2m+n-2\nu}{2}]_q[\frac{h+2m+n-2\nu}{2}]_q}{[2]_q[-1+h]_q[1+h]_q}}$$

$$\sqrt{\left(C(t_0,h+n)-[\tfrac{-1+h-n}{2}]_q{}^2\right)\left([\tfrac{1+h+n}{2}]_q{}^2-C(t_0,h+n)\right)}|\frac{-t_0^2}{q[2]_q},n,\nu,-1+m,-2+h\rangle$$

$$-\frac{q^{\frac{-5-h-2m-n+2\nu}{2}}t_0}{[2+h]_q}\sqrt{\frac{q[\frac{2+h-2m-n}{2}+\nu]_q[\frac{4+h-2m-n}{2}+\nu]_q}{[2]_q[1+h]_q[3+h]_q}}$$

$$\sqrt{\left(C(t_0,h+n)-[\tfrac{1+h-n}{2}]_q{}^2\right)\left(-C(t_0,h+n)+[\tfrac{3+h+n}{2}]_q{}^2\right)}|\frac{-t_0^2}{q[2]_q},n,\nu,-1+m,2+h\rangle$$

$$+\frac{q^{-3-\frac{3h}{2}-m-n+\nu}t_0}{\lambda^2[h]_q[2+h]_q}\sqrt{\frac{[\frac{h+2m+n-2\nu}{2}]_q[\frac{2+h-2m-n}{2}+\nu]_q}{[2]_q}}$$

$$\Bigg[\frac{q^{\frac{3}{2}(2+h+n)}[2\,(1+h)]_q}{[1+h]_q}+q^{\frac{3h+n}{2}}\left(-2q-2q^3-q^2\lambda^2C(t_0,h+n)[2]_q+\frac{q^2[4]_q}{[2]_q}\right)$$

$$+\lambda\left([\frac{h-n}{2}]_q-q^{3(1+h)}[\frac{2+h+n}{2}]_q\right)\Bigg]|-\frac{{t_0}^2}{q[2]_q},n,\nu,-1+m,h\rangle$$

$$\mathfrak{P}_C| - \frac{{t_0}^2}{q[2]_q}, n, \nu, m, h) \quad = \quad \frac{q^{-\frac{1}{2}-m-\frac{n}{2}+\nu} t_0}{[h]_q} \sqrt{\frac{[\frac{h+2m+n-2\nu}{2}]_q [\frac{h-2m-n}{2}+\nu]_q}{[2]_q[-1+h]_q[1+h]_q}}$$
$$\sqrt{\left(C(t_0,h+n) - [\tfrac{-1+h-n}{2}]_q^{\;2}\right)\left([\tfrac{1+h+n}{2}]_q^{\;2} - C(t_0,h+n)\right)} | - \tfrac{{t_0}^2}{q[2]_q}, n, \nu, m, -2+h\rangle$$
$$+ \frac{q^{-\frac{1}{2}-m-\frac{n}{2}+\nu} t_0}{[2+h]_q} \sqrt{\frac{[\frac{2+h+2m+n-2\nu}{2}]_q [\frac{2+h-2m-n}{2}+\nu]_q}{[2]_q[1+h]_q[3+h]_q}}$$
$$\sqrt{\left(C(t_0,h+n) - [\tfrac{1+h-n}{2}]_q^{\;2}\right)\left([\tfrac{3+h+n}{2}]_q^{\;2} - C(t_0,h+n)\right)} | - \tfrac{{t_0}^2}{q[2]_q}, n, \nu, m, 2+h\rangle$$
$$\frac{q^{-2(1+h+m)-\frac{3n}{2}} | t_0}{\lambda^3 \sqrt{q[2]_q} [h]_q [2+h]_q} \Bigg[- q^{\frac{4+4h+n+4\nu}{2}} \left(2[2]_q \lambda^2 [2+h]_q [h]_q\right) + q^{1+h+2m+\frac{n}{2}}$$
$$\left\{ q^n \left(2 - q^2\lambda + q^{2(1+h)}(2+\lambda)\right) + q\lambda\left([2+h]_q + q^{1+h+n}\left(q[2+n]_q + \lambda[3+2h]_q[h]_q\right)\right) \right\}$$
$$+ q^{1+\frac{3h}{2}+m+\nu}\lambda \Bigg\{ - \tfrac{q^{3+h+2n}}{[1+h]_q}[2(1+h)]_q [\tfrac{2+h+2m+n-2\nu}{2}]_q - q^{1+h}[\tfrac{4+3h+2m+n-2\nu}{2}]_q$$
$$+ q^n \left(\lambda^2 C(t_0,h+n)\left(q^{2+h}[\tfrac{h+2m+n-2\nu}{2}]_q - [\tfrac{h-2m-n}{2}+\nu]_q\right) + q[\tfrac{3h-2m-3n+2\nu}{2}]_q\right) \Bigg\} \Bigg]$$
$$| - \frac{{t_0}^2}{q[2]_q}, n, \nu, m, h)$$

$$\mathfrak{P}_D| - \frac{{t_0}^2}{q[2]_q}, n, \nu, m, h\rangle \quad = \quad -\frac{q^{-\frac{1}{2}-m-\frac{n}{2}+\nu} t_0}{[h]_q} \sqrt{\frac{[\frac{h+2m+n-2\nu}{2}]_q [\frac{h-2m-n}{2}+\nu]_q}{[2]_q[-1+h]_q[1+h]_q}}$$
$$\sqrt{\left(C(t_0,h+n) - [\tfrac{-1+h-n}{2}]_q^{\;2}\right)\left([\tfrac{1+h+n}{2}]_q^{\;2} - C(t_0,h+n)\right)} | - \frac{{t_0}^2}{q[2]_q}, n, \nu, m, -2+h\rangle$$
$$- \frac{q^{-\frac{1}{2}-m-\frac{n}{2}+\nu} t_0}{[2+h]_q} \sqrt{\frac{[\frac{2+h+2m+n-2\nu}{2}]_q [\frac{2+h-2m-n}{2}+\nu]_q}{[2]_q[1+h]_q[3+h]_q}}$$
$$\sqrt{\left(C(t_0,h+n) - [\tfrac{1+h-n}{2}]_q^{\;2}\right)\left([\tfrac{3+h+n}{2}]_q^2 - C(t_0,h+n)\right)} | - \frac{{t_0}^2}{q[2]_q}, n, \nu, m, 2+h\rangle$$
$$+ \frac{q^{\frac{-5-3h-4m-3n}{2}} t_0}{\lambda^3\sqrt{[2]_q}[h]_q[2+h]_q} \Bigg[\frac{2q^{\frac{4+3h+n}{2}}\left(q^{2\nu}[2]_q[1+h]_q - q^{2m+n}[2(1+h)]_q\right)}{[1+h]_q} + q^{1+n+2\nu}\lambda\left(q^{3(1+h)}[\tfrac{h-n}{2}]_q - [\tfrac{2+h+n}{2}]_q\right)$$
$$+ q^{h+m+\nu}\lambda \Bigg\{ [\frac{h+2m+n-2\nu}{2}]_q + q^{\frac{2+h+3n}{2}}\left(q^3[m+n-\nu]_q + [1+m+n-\nu]_q\right)$$
$$- q^{1+n}\lambda^2 C(t_0,h+n)\left(q^{2+h}[\tfrac{h+2m+n-2\nu}{2}]_q - [\tfrac{h-2m-n}{2}+\nu]_q\right) - q^{2+h}[\tfrac{h-2m-n}{2}+\nu]_q \Bigg\} \Bigg]$$
$$| - \frac{{t_0}^2}{q[2]_q}, n, \nu, m, h\rangle$$

the case $l > 0$:

$$\mathfrak{P}_A|\frac{{l_0}^2}{q[2]_q}, n, \nu, m, h\rangle \;=$$

$$\frac{l_0 q^{\frac{2+h-2m-n+2\nu}{2}}}{\{h+1\}_q}\sqrt{\frac{[\frac{h+2m+n-2\nu}{2}]_q[\frac{2+h+2m+n-2\nu}{2}]_q}{[2]_q\{h\}_q\{h+2\}_q}}$$

$$\sqrt{\left(2C(l_0,h+n)+\lambda^2[\tfrac{1+h-n}{2}]_q^2\right)\left(-2C(l_0,h+n)+\{\tfrac{1+h+n}{2}\}_q^2\right)}|\frac{{l_0}^2}{q[2]_q}, n, \nu, 1+m, 2+h\rangle$$

$$-\frac{l_0 q^{1-\frac{h}{2}-m-\frac{n}{2}+\nu}}{\lambda\{h-1\}_q}\sqrt{\frac{\{\frac{-2+h-2m-n}{2}+\nu\}_q\{\frac{h-2m-n}{2}+\nu\}_q}{[2]_q\{h\}_q\{h-2\}_q}}$$

$$\sqrt{\left(2C(l_0,h+n)+\lambda^2[\tfrac{-1+h-n}{2}]_q^2\right)\left(-2C(l_0,h+n)+\{\tfrac{-1+h+n}{2}\}_q^2\right)}|\frac{{l_0}^2}{q[2]_q}, n, \nu, 1+m, -2+h\rangle$$

$$-\frac{l_0 q^{\frac{5}{2}-m-\frac{n}{2}+\nu}\sqrt{\frac{[\frac{h+2m+n-2\nu}{2}]_q\{\frac{h-2m-n+2\nu}{2}\}_q}{q\lambda}}}{(q[2]_q)^{\frac{3}{2}}[2h]_q\{-1+h\}_q\{1+h\}_q}$$

$$\left([2]_q[4h]_q+[2h]_q\left(-2{[2]_q}^2+2C(l_0,h+n){[2]_q}^2+[4]_q-\lambda^2[2]_q[h]_q[n]_q\right)\right)|\frac{{l_0}^2}{q[2]_q}, n, \nu, 1+m, h\rangle$$

$$\mathfrak{P}_B|\frac{{l_0}^2}{q[2]_q}, n, \nu, m, h\rangle \;=$$

$$\frac{l_0 q^{\frac{-2+h-2m-n+2\nu}{2}}}{\{h-1\}_q}\sqrt{\frac{[\frac{-4+h+2m+n-2\nu}{2}]_q[\frac{-2+h+2m+n-2\nu}{2}]_q}{[2]_q\{-2+h\}_q\{h\}_q}}$$

$$\sqrt{\left(2C(l_0,h+n)+\lambda^2[\tfrac{-1+h-n}{2}]_q^2\right)\left(-2C(l_0,h+n)+\{\tfrac{-1+h+n}{2}\}_q^2\right)}|\frac{{l_0}^2}{q[2]_q}, n, \nu, -1+m, -2+h\rangle$$

$$-\frac{l_0 q^{-3-\frac{h}{2}-m-\frac{n}{2}+\nu}\left(\left(1+q^3[2]_q\right)[2n]_q[3n]_q+q^2[n]_q[6n]_q\right)}{\lambda[2n]_q[3n]_q\{1+h\}_q\{-1+n\}_q\{1+n\}_q}$$

$$\sqrt{\frac{\{\frac{2+h-2m-n}{2}+\nu\}_q\{\frac{4+h-2m-n}{2}+\nu\}_q}{[2]_q\{h\}_q\{2+h\}_q}}$$

$$\sqrt{\left(2C(l_0,h+n)+\lambda^2[\tfrac{1+h-n}{2}]_q^2\right)\left(-2C(l_0,h+n)+\{\tfrac{1+h+n}{2}\}_q^2\right)}|\frac{{l_0}^2}{q[2]_q}, n, \nu, -1+m, 2+h\rangle$$

$$-\frac{l_0 q^{\frac{3}{2}-m-\frac{n}{2}+\nu}}{(q[2]_q)^{\frac{3}{2}}[2h]_q\{-1+h\}_q\{1+h\}_q}\sqrt{\frac{[\frac{-2+h+2m+n-2\nu}{2}]_q\{1+\frac{h}{2}-m-\frac{n}{2}+\nu\}_q}{q\lambda}}$$

$$\left([2]_q[4h]_q+[2h]_q\left(-2{[2]_q}^2+2C(l_0,h+n){[2]_q}^2+[4]_q-\lambda^2[2]_q[h]_q[n]_q\right)\right)|\frac{{l_0}^2}{q[2]_q}, n, \nu, -1+m, h\rangle$$

$$\mathfrak{P}_C|\frac{{l_0}^2}{q[2]_q},n,\nu,m,h\rangle \quad = $$
$$\frac{l_0 q^{-\frac{1}{2}-2m-n}\lambda^2}{(\lambda^2[2]_q)^{\frac{3}{2}}[2h]_q\{-1+h\}_q\{1+h\}_q}$$
$$\Big[2C(l_0,h+n)[2]_q\left(-q^{1+2\nu}[2]_q+q^{2m+n}\lambda[h]_q\right)[2h]_q$$
$$-q^{2m+n}\lambda\left([h]_q\left((2[2]_q-q[4]_q)[2h]_q-q[2]_q[4h]_q\right)+q[2]_q\left(q\lambda[2h]_q+[4h]_q\right)[n]_q\right)$$
$$+q^{1+2\nu}\left(-[2]_q[4h]_q+[2h]_q\left(2{[2]_q}^2-[4]_q+\lambda^2[2]_q[h]_q[n]_q\right)\right)\Big]|\frac{{l_0}^2}{q[2]_q},n,\nu,m,h\rangle$$
$$+\frac{l_0 q^{\frac{1}{2}-m-\frac{n}{2}+\nu}}{\{h-1\}_q}\sqrt{\frac{[\frac{-2+h+2m+n-2\nu}{2}]_q\{\frac{h-2m-n}{2}+\nu\}_q}{(-1+q)(1+q)[2]_q\{h\}_q\{h-2\}_q}}$$
$$\sqrt{\left(2C(l_0,h+n)+\lambda^2[\tfrac{-1+h-n}{2}]_q^2\right)\left(-2C(l_0,h+n)+\{\tfrac{-1+h+n}{2}\}_q^2\right)}|\frac{{l_0}^2}{q[2]_q},n,\nu,m,-2+h\rangle$$
$$+\frac{l_0 q^{\frac{1}{2}-m-\frac{n}{2}+\nu}[1+h]_q\sqrt{\frac{[\frac{h+2m+n-2\nu}{2}]_q\{\frac{2+h-2m-n}{2}+\nu\}_q}{(-1+q)(1+q)[2]_q\{h\}_q\{2+h\}_q}}}{[2+2h]_q}$$
$$\sqrt{\left(2C(l_0,h+n)+\lambda^2[\tfrac{1+h-n}{2}]_q^2\right)\left(-2C(l_0,h+n)+\{\tfrac{1+h+n}{2}\}_q^2\right)}|\frac{{l_0}^2}{q[2]_q},n,\nu,m,2+h\rangle$$

$$\mathfrak{P}_D|\frac{{l_0}^2}{q[2]_q},n,\nu,m,h\rangle \quad = $$
$$\frac{l_0 q^{-\frac{1}{2}-2m-n}\lambda^2}{(\lambda^2[2]_q)^{\frac{3}{2}}[2h]_q\{-1+h\}_q\{1+h\}_q}$$
$$\Big[2C(l_0,h+n)[2]_q\left(q^{1+2\nu}[2]_q-q^{2m+n}\lambda[h]_q\right)[2h]_q-q^{2m+n}\lambda[2]_q[2h]_q\left(-2[h]_q+[2]_q[n]_q\right)$$
$$+q^{1+2\nu}\left([2]_q[4h]_q+[2h]_q\left(-2{[2]_q}^2+[4]_q-\lambda^2[2]_q[h]_q[n]_q\right)\right)\Big]|\frac{{l_0}^2}{q[2]_q},n,\nu,m,h\rangle$$
$$-\frac{l_0 q^{\frac{1}{2}-m-\frac{n}{2}+\nu}[-1+h]_q}{[-2+2h]_q}\sqrt{\frac{[\frac{-2+h+2m+n-2\nu}{2}]_q\{\frac{h-2m-n}{2}+\nu\}_q}{(-1+q)(1+q)[2]_q\{-2+h\}_q\{h\}_q}}$$
$$\sqrt{\left(2C(l_0,h+n)+\lambda^2[\tfrac{-1+h-n}{2}]_q^2\right)\left(-2C(l_0,h+n)+\{\tfrac{-1+h+n}{2}\}_q^2\right)}|\frac{{l_0}^2}{q[2]_q},n,\nu,m,-2+h\rangle$$
$$-\frac{l_0 q^{\frac{1}{2}-m-\frac{n}{2}+\nu}[1+h]_q\sqrt{\frac{[\frac{h+2m+n-2\nu}{2}]_q\{\frac{2+h-2m-n}{2}+\nu\}_q}{(-1+q)(1+q)[2]_q\{h\}_q\{2+h\}_q}}}{[2+2h]_q}$$
$$\sqrt{\left(2C(l_0,h+n)+\lambda^2[\tfrac{1+h-n}{2}]_q^2\right)\left(-2C(l_0,h+n)+\{\tfrac{1+h+n}{2}\}_q^2\right)}|\frac{{l_0}^2}{q[2]_q},n,\nu,m,2+h\rangle$$

A.6 The final result for the representation of the boosts

The case $l = 0$:

$$\tau^1|0,n,\nu,m,h\rangle \quad = \quad q^{-h-2n+2\nu}\sqrt{q^{2+h+n}}|0,-1+n,-1+\nu,m,-1+h\rangle - q^{\frac{-h+2m-3n+4\nu}{4}}\lambda\sqrt{[\frac{h+2m+n-2\nu}{2}]_q[n-\nu]_q}|0,1+n,\nu,m,1+h\rangle$$

$$T^2|0,n,\nu,m,h\rangle \quad = \quad \frac{1}{\lambda}q^{\frac{1-h-2n+3\nu}{2}}\sqrt{\lambda[-1+n-\nu]_q}|0,-1+n,\nu,1+m,-1+h\rangle + \frac{1}{\lambda}q^{\frac{-h+2m-5n+6\nu}{4}}\sqrt{\lambda[\frac{h+2m+n-2\nu}{2}]_q}|0,1+n,1+\nu,1+m,1+h\rangle$$

$$S^1|0,n,\nu,m,h) \quad = \quad -\frac{1}{\lambda}q^{\frac{6-h-6m-5n+6\nu}{4}}\sqrt{\lambda[\frac{-2+h+2m+n-2\nu}{2}]_q}|0,-1+n,-1+\nu,-1+m,-1+h\rangle - q^{\frac{2-h-4m-2n+3\nu}{2}}\sqrt{\frac{[n-\nu]_q}{\lambda}}|0,1+n,\nu,-1+m,1+h\rangle$$

$$\sigma^2|0,n,\nu,m,h\rangle \quad = \quad -q^{\frac{4-h-6m-3n+4\nu}{4}}\lambda\sqrt{[\frac{-2+h+2m+n-2\nu}{2}]_q[-1+n-\nu]_q}|0,-1+n,\nu,m,-1+h\rangle + q^{-1-h-2m-2n+2\nu}\sqrt{q^{4+h+n}}|0,1+n,1+\nu,m,1+h\rangle$$

The case $l < 0$:

$$\tau^1|-\frac{{t_0}^2}{q[2]_q},n,\nu,m,h) \ = $$

$$q^{\frac{-2-h+2m-3n+4\nu}{4}}\sqrt{\frac{[\frac{h}{2}+\frac{n}{2}-s]_q[\frac{2+h+n+2s}{2}]_q[\nu]_q[\frac{h-2m-n+2\nu}{2}]_q}{[h]_q[1+h]_q[n]_q[1+n]_q}}$$
$$|-\frac{{t_0}^2}{q[2]_q},-1+n,-1+\nu,m,-1+h\rangle$$
$$+q^{\frac{h+2m-3n+4\nu}{4}}\sqrt{\frac{[1+\frac{h}{2}-\frac{n}{2}+s]_q[\frac{-h}{2}+\frac{n}{2}+s]_q[\frac{2+h+2m+n-2\nu}{2}]_q[\nu]_q}{[1+h]_q[2+h]_q[n]_q[1+n]_q}}$$
$$|-\frac{{t_0}^2}{q[2]_q},-1+n,-1+\nu,m,1+h\rangle$$
$$+q^{\frac{-h+2m-n+4\nu}{4}}\sqrt{\frac{[\frac{h}{2}-\frac{n}{2}+s]_q[1-\frac{h}{2}+\frac{n}{2}+s]_q[1+n-\nu]_q[\frac{h-2m-n+2\nu}{2}]_q}{[h]_q[1+h]_q[1+n]_q[2+n]_q}}$$
$$|-\frac{{t_0}^2}{q[2]_q},1+n,\nu,m,-1+h\rangle$$
$$-q^{\frac{2+h+2m-n+4\nu}{4}}\sqrt{\frac{[1+\frac{h}{2}+\frac{n}{2}-s]_q[\frac{4+h+n+2s}{2}]_q[\frac{2+h+2m+n-2\nu}{2}]_q[1+n-\nu]_q}{[1+h]_q[2+h]_q[1+n]_q[2+n]_q}}$$
$$|-\frac{{t_0}^2}{q[2]_q},1+n,\nu,m,1+h\rangle$$

$$T^2|-\frac{{t_0}^2}{q[2]_q},n,\nu,m,h) \ = $$

$$\frac{q^{\frac{-2-h+2m-n+4\nu}{4}}}{\lambda}\sqrt{\frac{[\frac{h}{2}+\frac{n}{2}-s]_q[\frac{2+h+n+2s}{2}]_q[n-\nu]_q[\frac{h-2m-n+2\nu}{2}]_q}{[h]_q[1+h]_q[n]_q[1+n]_q}}$$
$$|-\frac{{t_0}^2}{q[2]_q},-1+n,\nu,1+m,-1+h\rangle$$
$$+\frac{q^{\frac{h+2m-n+4\nu}{4}}}{\lambda}\sqrt{\frac{[1+\frac{h}{2}-\frac{n}{2}+s]_q[\frac{-h}{2}+\frac{n}{2}+s]_q[\frac{2+h+2m+n-2\nu}{2}]_q[n-\nu]_q}{[1+h]_q[2+h]_q[n]_q[1+n]_q}}$$
$$|-\frac{{t_0}^2}{q[2]_q},-1+n,\nu,1+m,1+h\rangle$$
$$-\frac{q^{\frac{-4-h+2m-3n+4\nu}{4}}}{\lambda}\sqrt{\frac{[\frac{h}{2}-\frac{n}{2}+s]_q[1-\frac{h}{2}+\frac{n}{2}+s]_q[1+\nu]_q[\frac{h-2m-n+2\nu}{2}]_q}{[h]_q[1+h]_q[1+n]_q[2+n]_q}}$$
$$|-\frac{{t_0}^2}{q[2]_q},1+n,1+\nu,1+m,-1+h\rangle$$
$$+\frac{q^{\frac{-2+h+2m-3n+4\nu}{4}}}{\lambda}\sqrt{\frac{[1+\frac{h}{2}+\frac{n}{2}-s]_q[\frac{4+h+n+2s}{2}]_q[\frac{2+h+2m+n-2\nu}{2}]_q[1+\nu]_q}{[1+h]_q[2+h]_q[1+n]_q[2+n]_q}}$$
$$|-\frac{{t_0}^2}{q[2]_q},1+n,1+\nu,1+m,1+h\rangle$$

$$S^1| - \frac{{t_0}^2}{q[2]_q}, n, \nu, m, h) \ =$$

$$\frac{-q^{\frac{2+h-6m-3n+4\nu}{4}}}{\lambda}\sqrt{\frac{[\frac{h}{2}+\frac{n}{2}-s]_q[\frac{2+h+n+2s}{2}]_q[\frac{h+2m+n-2\nu}{2}]_q[\nu]_q}{[h]_q[1+h]_q[n]_q[1+n]_q}}$$

$$| - \frac{{t_0}^2}{q[2]_q}, -1+n, -1+\nu, -1+m, -1+h\rangle$$

$$+\frac{q^{\frac{-h-6m-3n+4\nu}{4}}}{\lambda}\sqrt{\frac{[1+\frac{h}{2}-\frac{n}{2}+s]_q[\frac{-h}{2}+\frac{n}{2}+s]_q[\nu]_q[1+\frac{h}{2}-m-\frac{n}{2}+\nu]_q}{[1+h]_q[2+h]_q[n]_q[1+n]_q}}$$

$$| - \frac{{t_0}^2}{q[2]_q}, -1+n, -1+\nu, -1+m, 1+h\rangle$$

$$-\frac{q^{\frac{4+h-6m-n+4\nu}{4}}}{\lambda}\sqrt{\frac{[\frac{h}{2}-\frac{n}{2}+s]_q[1-\frac{h}{2}+\frac{n}{2}+s]_q[\frac{h+2m+n-2\nu}{2}]_q[1+n-\nu]_q}{[h]_q[1+h]_q[1+n]_q[2+n]_q}}$$

$$| - \frac{{t_0}^2}{q[2]_q}, 1+n, \nu, -1+m, -1+h\rangle$$

$$-\frac{q^{\frac{2-h-6m-n+4\nu}{4}}}{\lambda}\sqrt{\frac{[+1+\frac{h}{2}+\frac{n}{2}-s]_q[\frac{4+h+n+2s}{2}]_q[1+n-\nu]_q[1+\frac{h}{2}-m-\frac{n}{2}+\nu]_q}{[1+h]_q[2+h]_q[1+n]_q[2+n]_q}}$$

$$| - \frac{{t_0}^2}{q[2]_q}, 1+n, \nu, -1+m, 1+h\rangle$$

$$\sigma^2| - \frac{{t_0}^2}{q[2]_q}, n, \nu, m, h\rangle =$$

$$-q^{\frac{2+h-6m-n+4\nu}{4}}\sqrt{\frac{[\frac{h}{2}+\frac{n}{2}-s]_q[\frac{2+h+n+2s}{2}]_q[\frac{h+2m+n-2\nu}{2}]_q[n-\nu]_q}{[h]_q[1+h]_q[n]_q[1+n]_q}}$$

$$| - \frac{{t_0}^2}{q[2]_q}, -1+n, \nu, m, -1+h\rangle$$

$$+q^{\frac{-h-6m-n+4\nu}{4}}\sqrt{\frac{[1+\frac{h}{2}-\frac{n}{2}+s]_q[\frac{-h}{2}+\frac{n}{2}+s]_q[n-\nu]_q[1+\frac{h}{2}-m-\frac{n}{2}+\nu]_q}{[1+h]_q[2+h]_q[n]_q[1+n]_q}}$$

$$| - \frac{{t_0}^2}{q[2]_q}, -1+n, \nu, m, 1+h\rangle$$

$$+q^{\frac{h-6m-3n+4\nu}{4}}\sqrt{\frac{[\frac{h}{2}-\frac{n}{2}+s]_q[1-\frac{h}{2}+\frac{n}{2}+s]_q[\frac{h+2m+n-2\nu}{2}]_q[1+\nu]_q}{[h]_q[1+h]_q[1+n]_q[2+n]_q}}$$

$$| - \frac{{t_0}^2}{q[2]_q}, 1+n, 1+\nu, m, -1+h\rangle$$

$$+q^{\frac{-2-h-6m-3n+4\nu}{4}}\sqrt{\frac{[1+\frac{h}{2}+\frac{n}{2}-s]_q[\frac{4+h+n+2s}{2}]_q[1+\nu]_q[1+\frac{h}{2}-m-\frac{n}{2}+\nu]_q}{[1+h]_q[2+h]_q[1+n]_q[2+n]_q}}$$

$$| - \frac{{t_0}^2}{q[2]_q}, 1+n, 1+\nu, m, 1+h\rangle$$

The case $l>0$:

$$\tau^1|\frac{{l_0}^2}{q[2]_q},n,\nu,m,h\rangle \;=$$

$$q^{\frac{2-h+2m-3n+4\nu}{4}}\sqrt{\frac{\{1-\frac{h}{2}-\frac{n}{2}+s\}_q\{\frac{h+n+2s}{2}\}_q\{\nu\}_q\{\frac{h-2m-n+2\nu}{2}\}_q}{\{-1+h\}_q\{h\}_q\{-1+n\}_q\{n\}_q}}$$
$$|\frac{{l_0}^2}{q[2]_q},-1+n,-1+\nu,m,-1+h\rangle$$
$$+q^{\frac{2+h+2m-3n+4\nu}{4}}\sqrt{\frac{\lambda^3[1+\frac{h}{2}-\frac{n}{2}+s]_q[\frac{h}{2}-\frac{n}{2}-s]_q[\frac{h+2m+n-2\nu}{2}]_q\{\nu\}_q}{\{h\}_q\{1+h\}_q\{-1+n\}_q\{n\}_q}}$$
$$|\frac{{l_0}^2}{q[2]_q},-1+n,-1+\nu,m,1+h\rangle$$
$$+q^{\frac{2-h+2m-n+4\nu}{4}}\lambda^2\sqrt{\frac{[\frac{h}{2}-\frac{n}{2}+s]_q[-1+\frac{h}{2}-\frac{n}{2}-s]_q[n-\nu]_q\{\frac{h-2m-n+2\nu}{2}\}_q}{\lambda\{-1+h\}_q\{h\}_q\{n\}_q\{1+n\}_q}}$$
$$|\frac{{l_0}^2}{q[2]_q},1+n,\nu,m,-1+h\rangle$$
$$-q^{\frac{2+h+2m-n+4\nu}{4}}\lambda\sqrt{\frac{[\frac{h+2m+n-2\nu}{2}]_q[n-\nu]_q\{\frac{-h}{2}-\frac{n}{2}+s\}_q\{\frac{2+h+n+2s}{2}\}_q}{\{h\}_q\{1+h\}_q\{n\}_q\{1+n\}_q}}$$
$$|\frac{{l_0}^2}{q[2]_q},1+n,\nu,m,1+h\rangle$$

$$T^2|\frac{{l_0}^2}{q[2]_q},n,\nu,m,h\rangle \;=$$

$$\frac{q^{\frac{-h+2m-n+4\nu}{4}}}{\lambda}\sqrt{\frac{\lambda[-1+n-\nu]_q\{1-\frac{h}{2}-\frac{n}{2}+s\}_q\{\frac{h+n+2s}{2}\}_q\{\frac{h-2m-n+2\nu}{2}\}_q}{\{-1+h\}_q\{h\}_q\{-1+n\}_q\{n\}_q}}$$
$$|\frac{{l_0}^2}{q[2]_q},-1+n,\nu,1+m,-1+h\rangle$$
$$+q^{\frac{h+2m-n+4\nu}{4}}\lambda\sqrt{\frac{[1+\frac{h}{2}-\frac{n}{2}+s]_q[\frac{h}{2}-\frac{n}{2}-s]_q[\frac{h+2m+n-2\nu}{2}]_q[-1+n-\nu]_q}{\{h\}_q\{1+h\}_q\{-1+n\}_q\{n\}_q}}$$
$$|\frac{{l_0}^2}{q[2]_q},-1+n,\nu,1+m,1+h\rangle$$
$$-q^{\frac{-h+2m-3n+4\nu}{4}}\sqrt{\frac{[\frac{h}{2}-\frac{n}{2}+s]_q[-1+\frac{h}{2}-\frac{n}{2}-s]_q\{1+\nu\}_q\{\frac{h-2m-n+2\nu}{2}\}_q}{\{-1+h\}_q\{h\}_q\{n\}_q\{1+n\}_q}}$$
$$|\frac{{l_0}^2}{q[2]_q},1+n,1+\nu,1+m,-1+h\rangle$$
$$+\frac{q^{\frac{h+2m-3n+4\nu}{4}}}{\lambda}\sqrt{\frac{\lambda[\frac{h+2m+n-2\nu}{2}]_q\{\frac{-h}{2}-\frac{n}{2}+s\}_q\{\frac{2+h+n+2s}{2}\}_q\{1+\nu\}_q}{\{h\}_q\{1+h\}_q\{n\}_q\{1+n\}_q}}$$
$$|\frac{{l_0}^2}{q[2]_q},1+n,1+\nu,1+m,1+h\rangle$$

$$S^1|\frac{l_0^2}{q[2]_q}, n, \nu, m, h\rangle \quad =$$

$$-\frac{q^{\frac{4+h-6m-3n+4\nu}{4}}}{\lambda}\sqrt{\frac{\lambda[\frac{-2+h+2m+n-2\nu}{2}]_q\{1-\frac{h}{2}-\frac{n}{2}+s\}_q\{\frac{h+n+2s}{2}\}_q\{\nu\}_q}{\{-1+h\}_q\{h\}_q\{-1+n\}_q\{n\}_q}}$$

$$|\frac{{l_0}^2}{q[2]_q}, -1+n, -1+\nu, -1+m, -1+h\rangle$$

$$+q^{\frac{4-h-6m-3n+4\nu}{4}}\sqrt{\frac{[1+\frac{h}{2}-\frac{n}{2}+s]_q[\frac{h}{2}-\frac{n}{2}-s]_q\{\nu\}_q\{1+\frac{h}{2}-m-\frac{n}{2}+\nu\}_q}{\{h\}_q\{1+h\}_q\{-1+n\}_q\{n\}_q}}$$

$$|\frac{{l_0}^2}{q[2]_q}, -1+n, -1+\nu, -1+m, 1+h\rangle$$

$$-q^{\frac{4+h-6m-n+4\nu}{4}}\lambda\sqrt{\frac{[\frac{h}{2}-\frac{n}{2}+s]_q[-1+\frac{h}{2}-\frac{n}{2}-s]_q[\frac{-2+h+2m+n-2\nu}{2}]_q[n-\nu]_q}{\{-1+h\}_q\{h\}_q\{n\}_q\{1+n\}_q}}$$

$$|\frac{{l_0}^2}{q[2]_q}, 1+n, \nu, -1+m, -1+h\rangle$$

$$-\frac{q^{\frac{4-h-6m-n+4\nu}{4}}}{\lambda}\sqrt{\frac{\lambda[n-\nu]_q\{\frac{-h}{2}-\frac{n}{2}+s\}_q\{\frac{2+h+n+2s}{2}\}_q\{1+\frac{h}{2}-m-\frac{n}{2}+\nu\}_q}{\{h\}_q\{1+h\}_q\{n\}_q\{1+n\}_q}}$$

$$|\frac{{l_0}^2}{q[2]_q}, 1+n, \nu, -1+m, 1+h\rangle$$

$$\sigma^2|\frac{{l_0}^2}{q[2]_q}, n, \nu, m, h\rangle \quad =$$

$$-q^{\frac{2+h-6m-n+4\nu}{4}}\lambda\sqrt{\frac{[\frac{-2+h+2m+n-2\nu}{2}]_q[-1+n-\nu]_q\{1-\frac{h}{2}-\frac{n}{2}+s\}_q\{\frac{h+n+2s}{2}\}_q}{\{-1+h\}_q\{h\}_q\{-1+n\}_q\{n\}_q}}$$

$$|\frac{{l_0}^2}{q[2]_q}, -1+n, \nu, m, -1+h\rangle$$

$$+q^{\frac{2-h-6m-n+4\nu}{4}}\lambda^2\sqrt{\frac{[1+\frac{h}{2}-\frac{n}{2}+s]_q[\frac{h}{2}-\frac{n}{2}-s]_q[-1+n-\nu]_q\{1+\frac{h}{2}-m-\frac{n}{2}+\nu\}_q}{\lambda\{h\}_q\{1+h\}_q\{-1+n\}_q\{n\}_q}}$$

$$|\frac{{l_0}^2}{q[2]_q}, -1+n, \nu, m, 1+h\rangle$$

$$+q^{\frac{2+h-6m-3n+4\nu}{4}}\sqrt{\frac{\lambda^3[\frac{h}{2}-\frac{n}{2}+s]_q[-1+\frac{h}{2}-\frac{n}{2}-s]_q[\frac{-2+h+2m+n-2\nu}{2}]_q\{1+\nu\}_q}{\{-1+h\}_q\{h\}_q\{n\}_q\{1+n\}_q}}$$

$$|\frac{{l_0}^2}{q[2]_q}, 1+n, 1+\nu, m, -1+h\rangle$$

$$+q^{\frac{2-h-6m-3n+4\nu}{4}}\sqrt{\frac{\{\frac{-h}{2}-\frac{n}{2}+s\}_q\{\frac{2+h+n+2s}{2}\}_q\{1+\nu\}_q\{1+\frac{h}{2}-m-\frac{n}{2}+\nu\}_q}{\{h\}_q\{1+h\}_q\{n\}_q\{1+n\}_q}}$$

$$|\frac{{l_0}^2}{q[2]_q}, 1+n, 1+\nu, m, 1+h\rangle$$

A.7 The representation of the derivatives

The case $l < 0$:

$$\begin{aligned}
\hat{\partial}^D|M,n,\nu,m\rangle &= \frac{q^{-3-M-n+2\nu}\sqrt{q[2]_q}}{t^0\lambda}|M,n,\nu,m\rangle \\
&\quad -\frac{q^{\frac{-5+m-2M-2n}{2}+\nu}}{t^0\lambda}\sqrt{\frac{q[2]_q[n-\nu]_q[m+n-\nu]_q}{[n]_q[1+n]_q}}|M-1,n-1,\nu,m\rangle \\
&\quad +\frac{q^{\frac{-3+m-2M}{2}+\nu}}{t^0\lambda}\sqrt{\frac{q[2]_q[1+\nu]_q[1-m+\nu]_q}{[1+n]_q[2+n]_q}}|M-1,n+1,\nu+1,m\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{\partial}^A|M,n,\nu,m\rangle &= -\frac{q^{\frac{-5+m-2M+n-2\nu}{2}}}{t^0\lambda}\sqrt{\frac{q[2]_q[n-\nu]_q[-m+\nu]_q}{[n]_q[1+n]_q}}|-1+M,-1+n,\nu,1+m\rangle \\
&\quad -\frac{q^{\frac{-5+m-2M-n}{2}}}{t^0}\sqrt{\frac{q[2]_q[-1+n-\nu]_q[n-\nu]_q[m+n-\nu]_q[1+\nu]_q}{[n]_q[1+n]_q}}|-1+M,-1+n,1+\nu,1+m\rangle \\
&\quad -\frac{q^{\frac{-5+m-2M+n-2\nu}{2}}}{t^0\lambda}\sqrt{\frac{q[2]_q[1+m+n-\nu]_q[1+\nu]_q}{[1+n]_q[2+n]_q}}|-1+M,1+n,1+\nu,1+m\rangle \\
&\quad +\frac{q^{\frac{-3+m-2M+n}{2}}}{t^0}\sqrt{\frac{q[2]_q[n-\nu]_q[1+\nu]_q[2+\nu]_q[1-m+\nu]_q}{[1+n]_q[2+n]_q}}|-1+M,1+n,2+\nu,1+m\rangle \\
&\quad +\frac{q^{-\frac{5}{2}-M-\frac{n}{2}+\nu}}{t^0}\sqrt{q[2]_q[n-\nu]_q[1+\nu]_q}|M,n,1+\nu,1+m\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{\partial}^B|M,n,\nu,m\rangle &= +\frac{q^{-\frac{7}{2}-M-\frac{n}{2}+\nu}}{t^0}\sqrt{q[2]_q[1+n-\nu]_q[\nu]_q}|M,n,-1+\nu,-1+m\rangle \\
&\quad +\frac{q^{\frac{-3+m-2M+n}{2}+\nu}}{t^0\lambda}\sqrt{\frac{q[2]_q[1+n-\nu]_q[1-m+\nu]_q}{[1+n]_q[2+n]_q}}|-1+M,1+n,\nu,-1+m\rangle \\
&\quad \frac{q^{\frac{-7+m-2M-3n}{2}+\nu}}{t^0\lambda}\sqrt{\frac{q[2]_q[m+n-\nu]_q[\nu]_q}{[n]_q[1+n]_q}}|-1+M,-1+n,-1+\nu,-1+m\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{\partial}^C|M,n,\nu,m) &= \frac{q^{\frac{-3+m-2M-2\nu}{2}}}{t^0\lambda}\sqrt{\frac{q[2]_q[\nu]_q[-m+\nu]_q}{[n]_q[1+n]_q}}|-1+M,-1+n,-1+\nu,m\rangle \\
&\quad +\frac{q^{\frac{-5+m-2M-2n}{2}}[\nu]_q}{t^0}\sqrt{\frac{q[2]_q[n-\nu]_q[m+n-\nu]_q}{[n]_q[1+n]_q}}|-1+M,-1+n,\nu,m\rangle \\
&\quad -\frac{q^{\frac{-1+m-2M+2n-2\nu}{2}}}{t^0\lambda}\sqrt{\frac{q[2]_q[1+n-\nu]_q[1+m+n-\nu]_q}{[1+n]_q[2+n]_q}}|-1+M,1+n,\nu,m\rangle \\
&\quad +\frac{q^{\frac{-1+m-2M}{2}+n}[1+n-\nu]_q}{t^0}\sqrt{\frac{q[2]_q[1+\nu]_q[1-m+\nu]_q}{[1+n]_q[2+n]_q}}|-1+M,1+n,1+\nu,m\rangle \\
&\quad +\frac{q^{-3-M-n}\left(1+q^{1+n+\nu}\lambda[1+n-\nu]_q\right)}{t^0\lambda}\sqrt{q[2]_q}|M,n,\nu,m\rangle
\end{aligned}$$

The case $l > 0$:

$$\begin{aligned}
\hat{\partial}^{D}|M,n,\nu,m) &= -\frac{q^{-2-M-n+2\nu}}{l^{0}\lambda}\sqrt{q[2]_q}|M,n,\nu,m\rangle \\
&\quad -\frac{q^{\frac{-3+m-2M}{2}+\nu}}{l^{0}\lambda}\sqrt{\frac{q[2]_q\{\nu+1\}_q\{1-m+\nu\}_q}{\{n\}_q\{n+1\}_q}}|-1+M,1+n,1+\nu,m\rangle \\
&\quad -\frac{q^{\frac{-3+m-2M-2n}{2}+\nu}}{l^{0}}\sqrt{\frac{q[2]_q[-1+n-\nu]_q[-1+m+n-\nu]_q}{\{n\}_q\{n-1\}_q}}|-1+M,-1+n,\nu,m\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{\partial}^{A}|M,n,\nu,m) &= -\frac{q^{-\frac{3}{2}-M-\frac{n}{2}+\nu}}{l^{0}\lambda}\sqrt{\lambda[2]_q\{\nu+1\}_q[-1+n-\nu]_q}|,M,n,1+\nu,1+m\rangle \\
&\quad -\frac{q^{\frac{-5+m-2M+n-2\nu}{2}}}{l^{0}}\sqrt{\frac{[2]_q\{\nu-m\}_q[-1+n-\nu]_q}{\lambda\{n-1\}_q\{n\}_q}}|-1+M,-1+n,\nu,1+m\rangle \\
&\quad -\frac{q^{\frac{-3+m-2M-n}{2}}}{l^{0}}\sqrt{\frac{\lambda[2]_q\{\nu+1\}_q[-2+n-\nu]_q[-1+n-\nu]_q[-1+m+n-\nu]_q}{\{n\}_q\{n-1\}_q}} \\
&\qquad |-1+M,-1+n,1+\nu,1+m\rangle \\
&\quad -\frac{q^{\frac{-5+m-2M+n-2\nu}{2}}}{l^{0}}\sqrt{\frac{[2]_q\{\nu+1\}_q[m+n-\nu]_q}{\lambda\{n\}_q\{n+1\}_q}}|-1+M,1+n,1+\nu,1+m\rangle \\
&\quad -\frac{q^{\frac{-3+m-2M+n}{2}}}{l^{0}}\sqrt{\frac{q[2]_q\{\nu+1\}_q\{\nu+2\}_q\{1-m+\nu\}_q[-1+n-\nu]_q}{q\lambda\{n\}_q\{n+1\}_q}} \\
&\qquad |-1+M,1+n,2+\nu,1+m\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{\partial}^{B}|M,n,\nu,m) &= -\frac{q^{-\frac{5}{2}-M-\frac{n}{2}+\nu}}{l^{0}\lambda}\sqrt{\lambda[2]_q\{\nu\}_q[n-\nu]_q}|M,n,-1+\nu,-1+m\rangle \\
&\quad \frac{q^{\frac{m-2M-3(1+n)}{2}+\nu}}{l^{0}}\sqrt{\frac{[2]_q\{\nu\}_q[-1+m+n-\nu]_q}{\lambda\{n\}_q\{n-1\}_q}}|-1+M,-1+n,-1+\nu,-1+m\rangle \\
&\quad -\frac{q^{\frac{-3+m-2M+n}{2}+\nu}}{l^{0}}\sqrt{\frac{[2]_q\{1-m+\nu\}_q[n-\nu]_q}{\lambda\{n\}_q\{n+1\}_q}}|-1+M,1+n,\nu,-1+m\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{\partial}^{C}|M,n,\nu,m\rangle &= \frac{q^{-2-M-n}\left(1-q^{n+\nu}\lambda[n-\nu]_q\right)}{l^{0}\lambda}\sqrt{q[2]_q}|M,n,\nu,m\rangle \\
&\quad +\frac{q^{\frac{-3+m-2M-2\nu}{2}}}{l^{0}\lambda}\sqrt{\frac{q[2]_q\{\nu\}_q\{\nu-m\}_q}{\{n\}_q\{n-1\}_q}}|-1+M,-1+n,-1+\nu,m\rangle \\
&\quad +\frac{q^{\frac{-3+m-2M-2n}{2}}[2\nu]_q}{l^{0}[\nu]_q}\sqrt{\frac{q[2]_q[-1+n-\nu]_q[-1+m+n-\nu]_q}{\{n\}_q\{n-1\}_q}}|-1+M,-1+n,\nu,m\rangle \\
&\quad -\frac{q^{\frac{-3+m-2M+2n-2\nu}{2}}}{l^{0}}\sqrt{\frac{q[2]_q[n-\nu]_q[m+n-\nu]_q}{\{n\}_q\{n+1\}_q}}|-1+M,1+n,\nu,m\rangle \\
&\quad -\frac{q^{\frac{-3+m-2M}{2}+n}[n-\nu]_q}{l^{0}}\sqrt{\frac{q[2]_q\{\nu+1\}_q\{1-m+\nu\}_q}{\{n\}_q\{n+1\}_q}}|-1+M,1+n,1+\nu,m\rangle
\end{aligned}$$

A.8 The representation of the momenta

The case $l < 0$

$$
\begin{aligned}
P^A|M,n,\nu,m,n\rangle \;=\;& \\
& iq^{\frac{m-2M-3n-2\nu}{2}}\frac{1}{2t\lambda}\sqrt{\frac{[2]_q}{[n]_q[1+n]_q[2+n]_q}} \\
& \Big[q^{2(1+n)}\sqrt{[2+n]_q[n-\nu]_q[-m+\nu]_q}|-1+M,-1+n,\nu,1+m,-1+n\rangle \\
& +q^{2+n+\nu}\lambda\sqrt{[2+n]_q[-1+n-\nu]_q[n-\nu]_q[m+n-\nu]_q[1+\nu]_q} \\
& \qquad\qquad |-1+M,-1+n,1+\nu,1+m,-1+n\rangle \\
& +q^{2(1+n)}\sqrt{[n]_q[1+m+n-\nu]_q[1+\nu]_q}|-1+M,1+n,1+\nu,1+m,1+n\rangle \\
& -q^{3+2n+\nu}\lambda\sqrt{[n]_q[n-\nu]_q[1+\nu]_q[2+\nu]_q[1-m+\nu]_q} \\
& \qquad\qquad |-1+M,1+n,2+\nu,1+m,1+n\rangle \\
& +q^{2(1+n+\nu)}\sqrt{[2+n]_q[n-\nu]_q[-m+\nu]_q}|1+M,-1+n,\nu,1+m,-1+n\rangle \\
& +q^{2\nu}\sqrt{[n]_q[1+m+n-\nu]_q[1+\nu]_q}|1+M,1+n,1+\nu,1+m,1+n)\Big]
\end{aligned}
$$

$$
\begin{aligned}
P^B|M,n,\nu,m,n\rangle \;=\;& \\
& -iq^{\frac{m-2M-3n-2\nu}{2}}\frac{1}{2t\lambda}\sqrt{\frac{[2]_q}{[n]_q[1+n]_q[2+n]_q}} \\
& \Big[q^{1+2\nu}\sqrt{[2+n]_q[m+n-\nu]_q[\nu]_q}|-1+M,-1+n,-1+\nu,-1+m,-1+n\rangle \\
& +q^{1+2n+2(1+\nu)}\sqrt{[n]_q[1+n-\nu]_q[1-m+\nu]_q}|-1+M,1+n,\nu,-1+m,1+n\rangle \\
& -q^{1+2n+\nu}\lambda\sqrt{[2+n]_q[1+n-\nu]_q[-1+\nu]_q[\nu]_q[-m+\nu]_q} \\
& \qquad\qquad |1+M,-1+n,-2+\nu,-1+m,-1+n\rangle \\
& +q^{1+2n}\sqrt{[2+n]_q[m+n-\nu]_q[\nu]_q}|1+M,-1+n,-1+\nu,-1+m,-1+n\rangle \\
& +q^{n+\nu}\lambda\sqrt{[n]_q[1+n-\nu]_q[2+n-\nu]_q[1+m+n-\nu]_q[\nu]_q} \\
& \qquad\qquad |1+M,1+n,-1+\nu,-1+m,1+n\rangle \\
& +q^{1+2n}\sqrt{[n]_q[1+n-\nu]_q[1-m+\nu]_q}|1+M,1+n,\nu,-1+m,1+n\rangle\Big]
\end{aligned}
$$

$$
\begin{aligned}
&P^C|M,n,\nu,m,n\rangle = \\
&\quad -iq^{\frac{m-2(M+n+\nu)}{2}}\frac{1}{2t\lambda}\sqrt{\frac{[2]_q}{[n]_q[1+n]_q[2+n]_q}} \\
&\quad \Big[q^{3+n}\sqrt{[2+n]_q[\nu]_q[-m+\nu]_q}|-1+M,-1+n,-1+\nu,m,-1+n\rangle \\
&\quad +q^{2+\nu}\lambda\sqrt{[2+n]_q[n-\nu]_q[m+n-\nu]_q[\nu]_q}|-1+M,-1+n,\nu,m,-1+n\rangle \\
&\quad -q^{4+2n}\sqrt{[n]_q[1+n-\nu]_q[1+m+n-\nu]_q}|-1+M,1+n,\nu,m,1+n\rangle \\
&\quad +q^{2+2(1+n)+\nu}\lambda[1+n-\nu]_q\sqrt{[n]_q[1+\nu]_q[1-m+\nu]_q}|M-1,1+n,1+\nu,m,1+n\rangle \\
&\quad -q^{2(1+n)+\nu}\lambda[1+n-\nu]_q\sqrt{[2+n]_q[\nu]_q[-m+\nu]_q}|1+M,n-1,\nu-1,m,n-1\rangle \\
&\quad +q^{2+2n}\sqrt{[2+n]_q[n-\nu]_q[m+n-\nu]_q}|1+M,-1+n,\nu,m,-1+n\rangle \\
&\quad -q^{\nu}\lambda\sqrt{[n]_q[1+n-\nu]_q[1+m+n-\nu]_q[\nu]_q}|1+M,1+n,\nu,m,1+n\rangle \\
&\quad -q^{1+n}\sqrt{[n]_q[1+\nu]_q[1-m+\nu]_q}|1+M,1+n,1+\nu,m,1+n\rangle\Big]
\end{aligned}
$$

$$
\begin{aligned}
P^D|M,n,\nu,m,n) \quad = \\
&-iq^{\frac{m}{2}-M-n+\nu}\frac{1}{2t\lambda}\sqrt{\frac{[2]_q}{[n]_q[1+n]_q[2+n]_q}} \\
&\Big[-q^2\sqrt{[2+n]_q[n-\nu]_q[m+n-\nu]_q}|-1+M,-1+n,\nu,m,-1+n\rangle \\
&+q^{3+n}\sqrt{[n]_q[1+\nu]_q[1-m+\nu]_q}|-1+M,1+n,1+\nu,m,1+n\rangle \\
&-q^{1+n}\sqrt{[2+n]_q[\nu]_q[-m+\nu]_q}|1+M,-1+n,-1+\nu,m,-1+n\rangle \\
&+\sqrt{[n]_q[1+n-\nu]_q[1+m+n-\nu]_q}|1+M,1+n,\nu,m,1+n\rangle\Big]
\end{aligned}
$$

The case $l > 0$

$$
\begin{aligned}
P^A|M,n,\nu,m,n\rangle \;=\; & \\
& iq^{\frac{m-2M-3n-2\nu}{2}}\frac{1}{2l}\sqrt{\frac{[2]_q}{\lambda\{-1+n\}_q\{n\}_q\{1+n\}_q}} \\
& \Bigg[q^{\frac{3}{2}+2n}\sqrt{[m+n-\nu]_q\{-1+n\}_q\{1+\nu\}_q}|-1+M,1+n,1+\nu,1+m,1+n\rangle \\
& +q^n q^{\frac{3}{2}+n}\sqrt{[-1+n-\nu]_q\{1+n\}_q\{-m+\nu\}_q}|-1+M,-1+n,\nu,1+m,-1+n\rangle \\
& +q^{\frac{5}{2}+\nu}\lambda\sqrt{[-2+n-\nu]_q[-1+n-\nu]_q[-1+m+n-\nu]_q\{1+n\}_q\{1+\nu\}_q} \\
& \qquad\qquad |-1+M,-1+n,1+\nu,1+m,-1+n\rangle \\
& +q^{\frac{5}{2}+n+\nu}\sqrt{[-1+n-\nu]_q\{-1+n\}_q\{1+\nu\}_q\{2+\nu\}_q\{1-m+\nu\}_q} \\
& \qquad\qquad |-1+M,1+n,2+\nu,1+m,1+n\rangle \\
& -q^{\frac{3}{2}+n+2\nu}\sqrt{[-1+n-\nu]_q\{1+n\}_q\{-m+\nu\}_q}|1+M,-1+n,\nu,1+m,-1+n) \\
& +q^{\frac{3}{2}+2\nu}\sqrt{[m+n-\nu]_q\{-1+n\}_q\{1+\nu\}_q}|1+M,1+n,1+\nu,1+m,1+n\rangle\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
P^B|M,n,\nu,m,n\rangle \;=\; & \\
& -iq^{\frac{1+m-2M-3n-2\nu}{2}}\frac{1}{2l}\sqrt{\frac{[2]_q}{\lambda\{-1+n\}_q\{n\}_q\{1+n\}_q}} \\
& \Bigg[q^{2(1+\nu)}\sqrt{[-1+m+n-\nu]_q\{1+n\}_q\{\nu\}_q}|M-1,n-1,\nu-1,m-1,n-1\rangle \\
& -q^{2n+2(1+\nu)}\sqrt{[n-\nu]_q\{-1+n\}_q\{1-m+\nu\}_q}|M-1,1+n,\nu,m-1,1+n\rangle \\
& +q^{2n+\nu}\sqrt{[n-\nu]_q\{1+n\}_q\{-1+\nu\}_q\{\nu\}_q\{-m+\nu\}_q} \\
& \qquad\qquad |1+M,-1+n,-2+\nu,-1+m,-1+n\rangle \\
& +q^{2n}\sqrt{[-1+m+n-\nu]_q\{1+n\}_q\{\nu\}_q}|1+M,n-1,\nu-1,-1+m,n-1\rangle \\
& +q^{n+\nu}\lambda\sqrt{[n-\nu]_q[1+n-\nu]_q[m+n-\nu]_q\{-1+n\}_q\{\nu\}_q} \\
& \qquad\qquad |1+M,1+n,-1+\nu,-1+m,1+n\rangle \\
& +q^{2n}\sqrt{[n-\nu]_q\{-1+n\}_q\{1-m+\nu\}_q}|1+M,1+n,\nu,-1+m,1+n\rangle\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
P^C|M,n,\nu,m,n\rangle &= \\
& iq^{\frac{m-2(-1+M+n+\nu)}{2}}\frac{1}{2l\lambda}\sqrt{\frac{[2]_q}{\{-1+n\}_q\{n\}_q\{1+n\}_q}} \\
& \Big[-q^{2+n}\sqrt{\{1+n\}_q\{\nu\}_q\{-m+\nu\}_q}|-1+M,-1+n,-1+\nu,m,-1+n\rangle \\
& -q^{2+\nu}\lambda\sqrt{[-1+n-\nu]_q[-1+m+n-\nu]_q\{1+n\}_q}\{\nu\}_q \\
& \qquad |-1+M,-1+n,\nu,m,-1+n\rangle \\
& +q^{2+2n}\lambda\sqrt{[n-\nu]_q[m+n-\nu]_q\{-1+n\}_q}|-1+M,1+n,\nu,m,1+n\rangle \\
& +q^{2+2n+\nu}\lambda[n-\nu]_q\sqrt{\{-1+n\}_q\{1+\nu\}_q\{1-m+\nu\}_q} \\
& \qquad |-1+M,1+n,1+\nu,m,1+n\rangle \\
& -q^{2n+\nu}\lambda[n-\nu]_q\sqrt{\{1+n\}_q\{\nu\}_q\{-m+\nu\}_q}|1+M,n-1,\nu-1,m,n-1\rangle \\
& -q^{2n}\lambda\sqrt{[-1+n-\nu]_q[-1+m+n-\nu]_q\{1+n\}_q}|1+M,n-1,\nu,m,n-1\rangle \\
& +q^{\nu}\lambda\sqrt{[n-\nu]_q[m+n-\nu]_q\{-1+n\}_q\{\nu\}_q}|1+M,1+n,\nu,m,1+n\rangle \\
& +q^{n}\sqrt{\{-1+n\}_q\{1+\nu\}_q\{1-m+\nu\}_q}|1+M,1+n,1+\nu,m,1+n\rangle\Big]
\end{aligned}
$$

$$
\begin{aligned}
P^D|M,n,\nu,m,n\rangle &= \\
& iq^{\frac{m}{2}-M-n+\nu}\frac{1}{2l\lambda^2}\sqrt{\frac{[2]_q}{\{-1+n\}_q\{n\}_q\{1+n\}_q}} \\
& \Big[q^3\lambda^2\sqrt{[-1+n-\nu]_q[-1+m+n-\nu]_q\{1+n\}_q}|M-1,n-1,\nu,m,n-1\rangle \\
& +q^{3+n}\lambda\sqrt{\{-1+n\}_q\{1+\nu\}_q\{1-m+\nu\}_q}|-1+M,1+n,1+\nu,m,1+n\rangle \\
& -q^{1+n}\lambda\sqrt{\{1+n\}_q\{\nu\}_q\{-m+\nu\}_q}|1+M,-1+n,-1+\nu,m,-1+n\rangle \\
& -q\lambda^2\sqrt{[n-\nu]_q[m+n-\nu]_q\{-1+n\}_q}|1+M,1+n,\nu,m,1+n\rangle\Big]
\end{aligned}
$$

Appendix B

Commutation relations

B.1 The $\mathcal{R}$-matrices

In the following we use light-cone coordinates. All the matrices are block diagonal in the basis that is labelled by:

$$AA, BB, \qquad DA, CA, AD, AC, \qquad CB, DB, BC, BD, \qquad BA, DD, DC, CD, CC, AB$$

The projector decomposition of the $\mathcal{R}$-matrices is:

$$\begin{aligned} \mathcal{R}_I &= P_S + P_T - q^2 P_+ - \frac{1}{q^2} P_- \\ \mathcal{R}_{II} &= \frac{1}{q^2} P_S + q^2 P_T - P_+ - P_- \end{aligned} \tag{B.1}$$

with

$$\begin{aligned} 1 &= P_S + P_T + P_+ + P_- \\ P_A &= P_+ + P_- \end{aligned} \tag{B.2}$$

Every projector decomposes into four blocks:

$$P = P^{(1)} + P^{(2)} + P^{(3)} + P^{(4)}$$

Explicitly the blocks are gives by

$$P_+^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_-^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_+^{(2)} = \frac{1}{[2]_q}\begin{pmatrix} \frac{1}{q} & 0 & -q & 0 \\ \lambda & 0 & -q^2\lambda & 0 \\ -\frac{1}{q} & 0 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_-^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{q} & \lambda & -\frac{1}{q} \\ 0 & 0 & 0 & 0 \\ 0 & -q & -q^2\lambda & q \end{pmatrix}$$

$$P_+^{(3)} = \frac{1}{[2]_q}\begin{pmatrix} q & -q^2\lambda & -q & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{q} & \lambda & \frac{1}{q} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_-^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & 0 & -\frac{1}{q} \\ 0 & -q^2\lambda & 0 & \lambda \\ 0 & -q & 0 & \frac{1}{q} \end{pmatrix}$$

$$P_+^{(4)} = \frac{1}{[2]_q^2}\begin{pmatrix} 1 & q\lambda & q^{-2} & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & q\lambda & q^{-2} & -1 & 0 & -1 \\ -q^2 & -q^3\lambda & -1 & q^2 & 0 & q^2 \\ q\lambda & q^2\lambda^2 & \frac{\lambda}{q} & -q\lambda & 0 & -q\lambda \\ -1 & -q\lambda & -q^{-2} & 1 & 0 & 1 \end{pmatrix}$$

$$P_-^{(4)} = \frac{1}{[2]_q^2}\begin{pmatrix} 1 & q\lambda & -1 & q^{-2} & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -q^2 & -q^3\lambda & q^2 & -1 & 0 & q^2 \\ 1 & q\lambda & -1 & q^{-2} & 0 & -1 \\ q\lambda & q^2\lambda^2 & -q\lambda & \frac{\lambda}{q} & 0 & -q\lambda \\ -1 & -q\lambda & 1 & -q^{-2} & 0 & 1 \end{pmatrix}$$

$$P_S^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_T^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_S^{(2)} = \frac{1}{[2]_q} \begin{pmatrix} q & 0 & q & 0 \\ -\lambda & q & q\lambda^2 & \frac{1}{q} \\ \frac{1}{q} & 0 & \frac{1}{q} & 0 \\ 0 & q & q^2\lambda & \frac{1}{q} \end{pmatrix} \quad P_T^{(2/3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_S^{(3)} = \frac{1}{[2]_q} \begin{pmatrix} \frac{1}{q} & q^2\lambda & q & 0 \\ 0 & \frac{1}{q} & 0 & \frac{1}{q} \\ \frac{1}{q} & q\lambda^2 & q & -\lambda \\ 0 & q & 0 & q \end{pmatrix}$$

$$P_S^{(4)} = \frac{1}{[2]_q^2} \begin{pmatrix} q^2 & -\left(2+q^{-2}\right) q\lambda & 1 & 1 & 0 & 1 \\ 0 & {[2]_q}^2 & 0 & 0 & 0 & 0 \\ q^2 & q^3\lambda & 1 & 1 & 0 & 1 \\ q^2 & q^3\lambda & 1 & 1 & 0 & 1 \\ -q\lambda & -q^2\lambda^2 & -\frac{\lambda}{q} & -\frac{\lambda}{q} & {[2]_q}^2 & q\left(2+q^2\right)\lambda \\ 1 & q\lambda & q^{-2} & q^{-2} & 0 & q^{-2} \end{pmatrix}$$

$$P_T^{(4)} = \frac{1}{[2]_q^2} \begin{pmatrix} q^{-2} & \frac{\lambda}{q} & -q^{-2} & -q^{-2} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -q\lambda & 1 & 1 & 0 & -q^2 \\ -1 & -q\lambda & 1 & 1 & 0 & -q^2 \\ -q\lambda & -q^2\lambda^2 & q\lambda & q\lambda & 0 & -q^3\lambda \\ 1 & q\lambda & -1 & -1 & 0 & q^2 \end{pmatrix}$$

The four-dimensional metric tensor g_{ij} can be derived from P_T , the projector on a singlet:

$$g_{ij} = \begin{pmatrix} 0 & q^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & q\lambda \end{pmatrix}$$

Written out, using the q-relations for the coordinates the Minkowski length of a four vector is

$$(X)^2 = \frac{1}{q[2]_q} g_{ij} X^i X^j = AB - q^{-2}CD$$

B.2 The vectorial form of the q-Lorentz algebra

The RS-form of the q-Lorentz algebra consists of two $\mathcal{U}_q(su_2)$ vector operators $\vec{R}$ and $\vec{S}$, see [32]. Explicitly the defining commutation relations are [19],

$$
\begin{aligned}
0 &= R^3R^+ - q^2R^+R^3 - \frac{U^1R^+}{q[2]_q}\\
0 &= -q^2R^3R^- + R^-R^3 - \frac{U^1R^-}{q[2]_q}\\
0 &= -q\left(\lambda R^3R^3 - R^-R^+ + R^+R^-\right) - \frac{U^1R^3}{q[2]_q}\\
0 &= S^3S^+ - q^2S^+S^3 + \frac{U^2S^+}{q[2]_q}\\
0 &= S^3S^- - \frac{U^2S^- + qS^-S^3[2]_q}{q^3[2]_q}\\
0 &= -\lambda S^3S^3 + S^-S^+ - S^+S^- + \frac{U^2S^3}{q^2[2]_q}
\end{aligned}
$$

$$
\begin{aligned}
0 &= R^+S^+ - q^2S^+R^+\\
0 &= -\frac{1}{q^2}S^-R^+ + R^+S^-\\
0 &= -S^3R^+ + R^+S^3\\
0 &= -q^2R^-S^+ + S^+R^-\\
&\quad +q\lambda[2]_q\left(S^3R^3 + \lambda S^-R^+\right)\\
0 &= R^-S^- - q^2S^-R^-\\
0 &= -S^3R^- + R^-S^3 - \lambda S^-R^3[2]_q\\
0 &= R^3S^+ - S^+R^3 - \lambda S^3R^+[2]_q\\
0 &= R^3S^- - S^-R^3\\
0 &= R^3S^3 - S^3R^3 - \frac{\lambda S^-R^+[2]_q}{q}
\end{aligned}
$$

$$
\begin{aligned}
0 &= U^1R^+ - R^+U^1\\
0 &= U^1R^- - R^-U^1\\
0 &= -R^3U^1 + U^1R^3\\
0 &= U^2S^+ - S^+U^2\\
0 &= U^2S^- - S^-U^2\\
0 &= -S^3U^2 + U^2S^3\\
0 &= U^1U^2 - U^2U^1\\
0 &= R^3R^3 - \frac{R^-R^+}{q} - qR^+R^- + \frac{1 - U^1U^1}{q^4\lambda^2{[2]_q}^2}
\end{aligned}
$$

The semidirect product with the q-Minkowski space is defined via the relations

$$
\begin{aligned}
R^3X^0 &= -\frac{1}{q{[2]_q}^2}\left(X^3U^1 + q\lambda\left(\lambda X^3R^3 - X^-R^+ + X^+R^-\right)[2]_q - qX^0R^3[4]_q\right)\\
R^-X^0 &= \frac{1}{q{[2]_q}^2}\left(-X^-U^1 - q^2\lambda X^3R^-[2]_q + \lambda X^-R^3[2]_q + qX^0R^-[4]_q\right)\\
R^+X^0 &= \frac{1}{q{[2]_q}^2}\left(-X^+U^1 + \lambda\left(X^3R^+ - q^2X^+R^3\right)[2]_q + qX^0R^+[4]_q\right)\\
R^-X^- &= qX^-R^-\\
R^+X^+ &= qX^+R^+\\
R^+X^- &= \frac{1}{q^2{[2]_q}^2}\left(X^0U^1 - X^3U^1 + q[2]_q\left(q\lambda\left(X^0R^3 - X^3R^3\right) + X^-R^+[2]_q\right)\right)\\
R^-X^+ &= \frac{1}{q^4{[2]_q}^2}\Big[X^0U^1 + q^2\Big(X^3U^1 +\\
&\quad +[2]_q\left(-q^2\lambda X^0R^3 + X^3R^3\left(2q^3 - [2]_q\right) + q\left(\lambda^2X^-R^+ + X^+R^-\right)[2]_q\right)\Big)\Big]\\
R^3X^3 &= -\frac{1}{q^3{[2]_q}^2}\Big[X^0U^1 + q\lambda X^3U^1 +\\
&\quad +q[2]_q\left(-q^2\left(\lambda^2X^0R^3 + 4X^3R^3 + 2qX^-R^+ - \lambda X^+R^-\right) + X^-R^+[2]_q\right) + q^3X^3R^3[4]_q\Big]\\
R^3X^- &= \frac{1}{q{[2]_q}^2}\left(-X^-U^1 + q[2]_q\left(q\lambda X^0R^- + 2X^-R^3 + qX^3R^-\left(-2q + [2]_q\right)\right)\right)\\
R^+X^3 &= \frac{1}{q{[2]_q}^2}\left(-X^+U^1 + q[2]_q\left(q\lambda X^0R^+ + 2X^3R^+ + qX^+R^3\left(-2q + [2]_q\right)\right)\right)\\
R^3X^+ &= \frac{1}{q^3{[2]_q}^2}\left(X^+U^1 + q[2]_q\left(-\left(q\lambda X^0R^+\right) + 2q^2X^+R^3 + X^3R^+\left(-2 + q^3[2]_q\right)\right)\right)\\
R^-X^3 &= \frac{1}{q^3{[2]_q}^2}\left(X^-U^1 + q[2]_q\left(-\left(q\lambda X^0R^-\right) + 2q^2X^3R^- + X^-R^3\left(-2 + q^3[2]_q\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
S^3X^0 &= \frac{1}{q^3{[2]_q}^2}\left(-X^3U^2+q^3\left(-\left(\lambda\left(\lambda X^3S^3-X^-S^++X^+S^-\right)[2]_q\right)+X^0S^3[4]_q\right)\right)\\
S^-X^0 &= \frac{1}{q^3{[2]_q}^2}\left(-X^-U^2+q^2\left(-\left(q^2\lambda X^3S^-[2]_q\right)+\lambda X^-S^3[2]_q+qX^0S^-[4]_q\right)\right)\\
S^+X^0 &= \frac{1}{q^3{[2]_q}^2}\left(-X^+U^2+q^2\left(\lambda\left(X^3S^+-q^2X^+S^3\right)[2]_q+qX^0S^+[4]_q\right)\right)\\
S^-X^- &= \frac{1}{q}X^-S^-\\
S^+X^+ &= \frac{1}{q}X^+S^+\\
S^-X^+ &= qX^+S^-+\frac{1}{q^2{[2]_q}^2}\left(X^0U^2-X^3U^2+q^2\lambda\left(-X^0S^3+X^3S^3\right)[2]_q\right)\\
S^3X^+ &= \frac{1}{q^3{[2]_q}^2}\left(-X^+U^2+q^2\left(-\left(\lambda X^0S^+\right)+\lambda X^3S^++2qX^+S^3\right)[2]_q\right)\\
S^-X^3 &= \frac{1}{q^3{[2]_q}^2}\left(-X^-U^2+q^2\left(-\left(\lambda X^0S^-\right)+2qX^3S^-+\lambda X^-S^3\right)[2]_q\right)\\
S^+X^- &= \frac{1}{q^2{[2]_q}^2}\Big[q^2X^0U^2+X^3U^2\\
&\quad +q[2]_q\left(q\lambda X^0S^3+X^3S^3\left(2-q^3[2]_q\right)+q^2\left(X^-S^+[2]_q+X^+S^-\left(-2[2]_q+[4]_q\right)\right)\right)\Big]\\
S^3X^3 &= \frac{1}{q^2{[2]_q}^2}\Big[\lambda X^3U^2+q\Big(-X^0U^2+q\lambda^2X^0S^3[2]_q\\
&\quad +[2]_q\left(q\lambda X^-S^++X^+S^-\left(2-q^3[2]_q\right)\right)+qX^3S^3\left(4[2]_q-[4]_q\right)\Big)\Big]\\
S^3X^- &= \frac{1}{q{[2]_q}^2}\left(X^-U^2+[2]_q\left(q^2\lambda X^0S^-+2qX^-S^3+X^3S^-\left(-2q^3+[2]_q\right)\right)\right)\\
S^+X^3 &= \frac{1}{q{[2]_q}^2}\left(X^+U^2+[2]_q\left(q^2\lambda X^0S^++2qX^3S^++X^+S^3\left(-2q^3+[2]_q\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
U^1X^0 &= \lambda^2\left(-qX^3R^3+X^-R^++q^2X^+R^-\right)+\frac{X^0U^1[4]_q}{[2]_q^2}\\
U^1X^+ &= -q\lambda^2\left(X^3R^++q^2\left(X^0R^+-X^+R^3\right)\right)+\frac{X^+U^1[4]_q}{[2]_q^2}\\
U^1X^- &= -q\lambda^2\left(q^2\left(X^0R^--X^3R^-\right)+X^-R^3\right)+\frac{X^-U^1[4]_q}{[2]_q^2}\\
U^1X^3 &= q^2\lambda^2\left(-qX^0R^3+\lambda X^3R^3-X^-R^++X^+R^-\right)+\frac{X^3U^1[4]_q}{[2]_q^2}\\
U^2X^0 &= q^2\lambda^2\left(-qX^3S^3+X^-S^++q^2X^+S^-\right)+\frac{X^0U^2[4]_q}{[2]_q^2}\\
U^2X^+ &= -q\lambda^2\left(X^0S^+-X^3S^++q^2X^+S^3\right)+\frac{X^+U^2[4]_q}{[2]_q^2}\\
U^2X^- &= -q\lambda^2\left(X^0S^-+q^2X^3S^--X^-S^3\right)+\frac{X^-U^2[4]_q}{[2]_q^2}\\
U^2X^3 &= -q\lambda^2\left(X^0S^3+q\left(\lambda X^3S^3-X^-S^++X^+S^-\right)\right)+\frac{X^3U^2[4]_q}{[2]_q^2}
\end{aligned}
$$

The realisation of these operators in terms of our set of seven generators is given by the relations

$$
\begin{aligned}
R^+ &= \frac{q}{\sqrt{q^3{[2]_q}^3}}T^2,\\
R^- &= -\frac{1}{q\sqrt{q^3{[2]_q}^3}}\left(qS^1+\tau^1T^-\right)\\
R^3 &= \frac{1}{q^2\lambda{[2]_q}^2}\left(q\sigma^2-q\tau^1+\lambda^2T^2T^-\right)\\
S^+ &= \frac{1}{\sqrt{q^3{[2]_q}^3}}\left(q\left(\tau^3\right)^{\frac{1}{2}}T^2-\left(\tau^3\right)^{-\frac{1}{2}}\sigma^2T^+\right)\\
S^- &= -\frac{1}{\sqrt{q^3{[2]_q}^3}}\left(\tau^3\right)^{-\frac{1}{2}}S^1\\
S^3 &= \frac{1}{q^3\lambda{[2]_q}^2}\left(\left(\tau^3\right)^{-\frac{1}{2}}\sigma^2-\left(\tau^3\right)^{\frac{1}{2}}\tau^1+q^3\lambda^2\left(\tau^3\right)^{-\frac{1}{2}}S^1T^+\right)\\
U^1 &= -\frac{1}{q[2]_q}\left(q^2\sigma^2+\tau^1+q\lambda^2T^2T^-\right)\\
U^2 &= -\frac{1}{q[2]_q}\left(q^2\left(\tau^3\right)^{-\frac{1}{2}}\sigma^2+\left(\tau^3\right)^{\frac{1}{2}}\tau^1-q^3\lambda^2\left(\tau^3\right)^{-\frac{1}{2}}S^1T^+\right)
\end{aligned}
$$

and thereby establish an isomorphism between these two different forms. The inversion of this isomorphism is given by [19]:

$$
\begin{aligned}
T^2 &= \sqrt{q[2]_q{}^3}R^+ \\
\tau^1 &= -U^1 - q^2 R^3 \lambda [2]_q \\
S^1 &= -\sqrt{q^3[2]_q{}^3}\left(\tau^3\right)^{\frac{1}{2}} S^- \\
\sigma^2 &= -\left(\tau^3\right)^{\frac{1}{2}} U^2 + q^2\lambda[2]_q\left(\tau^3\right)^{\frac{1}{2}} S^3 \\
T^+ &= q^{\frac{5}{2}}\sqrt{[2]_q}\left(\tau^3\right)^{\frac{1}{2}} L^+ \\
T^- &= -q^{\frac{7}{2}}\sqrt{[2]_q}\left(\tau^3\right)^{\frac{1}{2}} L^- \\
\left(\tau^3\right)^{-\frac{1}{2}} &= W - L^3 q^3 \lambda
\end{aligned}
\tag{B.3}
$$

The vector $\vec{L} = (L^+, L^3, L^-)$ together with the operator W generate the $\mathcal{U}_q(su_2)$ subalgebra of rotations. Written in terms of the $\vec{R}$ and $\vec{S}$ they are [31]:

$$
\begin{aligned}
L^3 &= \frac{[2]_q}{q}\left(U^1 S^3 - U^2 R^3 + q^2 Z^3 \lambda [2]_q\right) \\
L^+ &= \frac{[2]_q}{q}\left(U^1 S^+ - U^2 R^+ + q^2 \lambda [2]_q Z^+\right) \\
L^- &= \frac{[2]_q}{q}\left(U^1 S^- - U^2 R^- + q^2 \lambda [2]_q Z^-\right) \\
W &= U^1 U^2 + q^5 \lambda^2 [2]_q{}^2\left(-q R^3 S^3 + R^- S^+ + q^2 R^+ S^-\right) \\
Z^3 &= -q\left(\lambda R^3 S^3 - R^- S^+ + R^+ S^-\right) \\
Z^+ &= R^3 S^+ - q^2 R^+ S^3 \\
Z^- &= -q^2 R^3 S^- + R^- S^3
\end{aligned}
$$

If $\vec{L} \circ \vec{X} = L^3 X^3 - q L^+ X^- - \frac{1}{q} X^- Y^+$ denotes the 3-dimensional scalar product of the two 3-vectors $\vec{L}$ and $\vec{X}$, we the helicity operator is defined by

$$
\begin{aligned}
H &= \vec{L} \circ \vec{X} - X^0 W \qquad \text{(B.4)} \\
&= \frac{1}{\sqrt{q[2]_q}}\Bigg[C(\tau^3)^{-\frac{1}{2}} + D(\tau^3)^{\frac{1}{2}} - \lambda\, A T^-(\tau^3)^{-\frac{1}{2}} - q^2\lambda\, B T^+(\tau^3)^{-\frac{1}{2}} \\
&\quad + q^2\lambda^2\, D T^- T^+(\tau^3)^{-\frac{1}{2}}\Bigg]
\end{aligned}
$$

It commutes with the coordinates and the rotations, whereas the relations with the boost generators are

$$
\begin{aligned}
H\tau^1 &= q\tau^1 H - \tfrac{q\lambda}{\sqrt{q[2]_q}} D\tau^1(\tau^3)^{\frac{1}{2}} - \tfrac{q^3\lambda^2}{\sqrt{q[2]_q}} S^1 A(\tau^3)^{-\frac{1}{2}} + \tfrac{q^4\lambda^3}{\sqrt{q[2]_q}} DS^1T^+(\tau^3)^{-\frac{1}{2}} \\
HT^2 &= qT^2 H - \tfrac{q^2}{\sqrt{q[2]_q}} A\sigma^2(\tau^3)^{-\frac{1}{2}} - \tfrac{q\lambda}{\sqrt{q[2]_q}} DT^2(\tau^3)^{\frac{1}{2}} + \tfrac{q^2\lambda}{\sqrt{q[2]_q}} DT^+\sigma^2(\tau^3)^{-\frac{1}{2}} \\
HS^1 &= \tfrac{1}{q} S^1 H - \tfrac{1}{\sqrt{q[2]_q}} B\tau^1(\tau^3)^{\frac{1}{2}} + \tfrac{\lambda}{\sqrt{q[2]_q}} DT^-\tau^1(\tau^3)^{\frac{1}{2}} + \tfrac{q\lambda}{\sqrt{q[2]_q}} DS^1(\tau^3)^{\frac{1}{2}} \\
H\sigma^2 &= \tfrac{1}{q}\sigma^2 H + \tfrac{\lambda}{q\sqrt{q[2]_q}} D\sigma^2(\tau^3)^{\frac{1}{2}} - \tfrac{\lambda^2}{\sqrt{q[2]_q}} BT^2(\tau^3)^{\frac{1}{2}} + \tfrac{\lambda^3}{\sqrt{q[2]_q}} DT^-T^2(\tau^3)^{\frac{1}{2}}
\end{aligned}
$$

We can realize the RS-form of the q-Lorentz algebra by means of coordinates and derivatives in the following way:

$$
\begin{aligned}
R^3 &= -\frac{\Lambda^{\frac{1}{2}}}{q^2{[2]_q}^2}\left(X^0\hat{\partial}^3 + q\left(-qX^3\hat{\partial}^0 + \lambda X^3\hat{\partial}^3 - X^-\hat{\partial}^+ + X^+\hat{\partial}^-\right)\right) \\
R^+ &= \frac{\Lambda^{\frac{1}{2}}}{q^2{[2]_q}^2}\left(-X^0\hat{\partial}^+ + X^3\hat{\partial}^+ + q^2\left(X^+\hat{\partial}^0 - X^+\hat{\partial}^3\right)\right) \\
R^- &= \frac{\Lambda^{\frac{1}{2}}}{q^2{[2]_q}^2}\left(-X^0\hat{\partial}^- - q^2X^3\hat{\partial}^- + q^2X^-\hat{\partial}^0 + X^-\hat{\partial}^3\right) \\
S^3 &= \frac{\Lambda^{\frac{1}{2}}}{q^2{[2]_q}^2}\left(-q^2X^0\hat{\partial}^3 + X^3\hat{\partial}^0 + q\left(\lambda X^3\hat{\partial}^3 - X^-\hat{\partial}^+ + X^+\hat{\partial}^-\right)\right) \\
S^+ &= \frac{\Lambda^{\frac{1}{2}}}{q^2{[2]_q}^2}\left(-q^2X^0\hat{\partial}^+ - X^3\hat{\partial}^+ + X^+\hat{\partial}^0 + q^2X^+\hat{\partial}^3\right) \\
S^- &= \frac{\Lambda^{\frac{1}{2}}}{q^2{[2]_q}^2}\left(-q^2X^0\hat{\partial}^- + q^2X^3\hat{\partial}^- + X^-\hat{\partial}^0 - X^-\hat{\partial}^3\right) \\
U^1 = U^2 &= \frac{\Lambda^{\frac{1}{2}}}{[2]_q}\left(-q^2\lambda X^0\hat{\partial}^0 + q^2\lambda X^3\hat{\partial}^3 - q\lambda X^-\hat{\partial}^+ - q^3\lambda X^+\hat{\partial}^- + [2]_q\right)
\end{aligned}
\tag{B.5}
$$

Rewriting these relations in terms of our set of generators, we obtain the equations

$$
\begin{aligned}
0 &= T^2 - q^2 A\hat{\partial}^D \Lambda^{\frac{1}{2}} + D\hat{\partial}^A \Lambda^{\frac{1}{2}} && \text{(B.6)} \\
0 &= -q^2 B\hat{\partial}^D \Lambda^{\frac{1}{2}} + q^4 D\hat{\partial}^B \Lambda^{\frac{1}{2}} + S^1 \left(\tau^3\right)^{-\frac{1}{2}} \\
0 &= -A\hat{\partial}^C \Lambda^{\frac{1}{2}} + q\lambda A\hat{\partial}^D \Lambda^{\frac{1}{2}} + C\hat{\partial}^A \Lambda^{\frac{1}{2}} + T^2 \left(\tau^3\right)^{\frac{1}{2}} - qT^+ \sigma^2 \left(\tau^3\right)^{-\frac{1}{2}} \\
0 &= -qS^1 + q^3 \left(B\hat{\partial}^C - C\hat{\partial}^B + q\lambda D\hat{\partial}^B\right) \Lambda^{\frac{1}{2}} - T^- \tau^1 \\
0 &= q^3 \lambda \left(q^2 A\hat{\partial}^B + B\hat{\partial}^A - C\hat{\partial}^D - D\hat{\partial}^C + q\lambda D\hat{\partial}^D\right) \Lambda^{\frac{1}{2}} - q[2]_q \left(U - \Lambda^{\frac{1}{2}}\right) \\
0 &= q\left(\sigma^2 - \tau^1\right) && \text{(B.7)} \\
&\quad + \lambda \left(D\hat{\partial}^C \Lambda^{\frac{1}{2}} + q^2 \left(-A\hat{\partial}^B + B\hat{\partial}^A - C\hat{\partial}^D + q\lambda D\hat{\partial}^D\right) \Lambda^{\frac{1}{2}} + \lambda T^2 T^-\right) \\
0 &= -\sigma^2 \left(\tau^3\right)^{-\frac{1}{2}} + \tau^1 \left(\tau^3\right)^{\frac{1}{2}} \\
&\quad + q\lambda \left(C\hat{\partial}^D \Lambda^{\frac{1}{2}} + q^2 \left(\left(-A\hat{\partial}^B + B\hat{\partial}^A - D\hat{\partial}^C + q\lambda D\hat{\partial}^D\right) \Lambda^{\frac{1}{2}} - \lambda S^1 T^+ \left(\tau^3\right)^{-\frac{1}{2}}\right)\right)
\end{aligned}
$$

B.3 The spinors

B.3.1 The relations between the spinors

The spinor algebra is defined by the following relations

$$xy = qyx \qquad \bar{y}\bar{x} = q\bar{x}\bar{y}$$

$$\begin{aligned} x\bar{x} &= \bar{x}x - q\lambda\bar{y}y \\ x\bar{y} &= q\bar{y}x \\ y\bar{x} &= q\bar{x}y \\ y\bar{y} &= \bar{y}y \end{aligned}$$

B.3.2 The action of the symmetry generators on the spinors

$$\begin{aligned} T^+x &= y + qxT^+ & T^+\bar{x} &= \tfrac{1}{q}\bar{x}T^+ \\ T^+y &= \tfrac{1}{q}yT^+ & T^+\bar{y} &= -\tfrac{1}{q}\bar{x} + q\bar{y}T^+ \\ T^-x &= qxT^- & T^-\bar{x} &= \tfrac{1}{q}\bar{x}T^- - q\bar{y} \\ T^-y &= x + \tfrac{1}{q}yT^- & T^-\bar{y} &= q\bar{y}T^- \\ \tau^3x &= q^2x\tau^3 & \tau^3\bar{y} &= q^2\bar{y}\tau^3 \\ \tau^3y &= q^{-2}y\tau^3 & \tau^3\bar{x} &= q^{-2}\bar{x}\tau^3 \end{aligned} \tag{B.8}$$

$$\begin{aligned} \tau^1x &= \tfrac{1}{q}x\tau^1 & \tau^1\bar{x} &= \bar{x}\tau^1 + (\tfrac{1}{q} - 2q + q^3)\bar{y}T^2 \\ \tau^1y &= qy\tau^1 & \tau^1\bar{y} &= \bar{y}\tau^1 \\ T^2x &= xT^2 + y\tau^1 & T^2\bar{x} &= q\bar{x}T^2 \\ T^2y &= yT^2 & T^2\bar{y} &= \tfrac{1}{q}\bar{y}T^2 \\ S^1x &= xS^1 & S^1\bar{x} &= \bar{y}\sigma^2 + \tfrac{1}{q}\bar{x}S^1 \\ S^1y &= yS^1 & S^1\bar{y} &= q\bar{y}S^1 \\ \sigma^2x &= qx\sigma^2 + (\tfrac{1}{q} - 2q + q^3)yS^1 & \sigma^2\bar{x} &= \bar{x}\sigma^2 \\ \sigma^2y &= \tfrac{1}{q}y\sigma^2 & \sigma^2\bar{y} &= \bar{y}\sigma^2 \end{aligned}$$

from these relations we get on $(\mathrm{D}^{(\frac{1}{2},0)} \oplus \mathrm{D}^{(0,\frac{1}{2})})$ the following matrix representation, if we choose the basis $x = (1,0,0,0)$, $y = (0,1,0,0)$, $\overline{x} = (0,0,1,0)$, $\overline{y} = (0,0,0,1)$:

$$T^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{q} \\ 0 & 0 & 0 & 0 \end{pmatrix}, T^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -q & 0 \end{pmatrix}, \tau^3 = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q^{-2} & 0 & 0 \\ 0 & 0 & q^{-2} & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix} \tag{B.9}$$

$$T^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, S^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tau^1 = \begin{pmatrix} \frac{1}{q} & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \sigma^2 = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \frac{1}{q} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

B.3.3 The Clifford algebra

The commutation relations of the q-gamma matrices are

$$\begin{aligned} \gamma^-\gamma^- &= 0 \\ \gamma^+\gamma^+ &= 0 \\ \gamma^0\gamma^+ &= \frac{\lambda}{q}\gamma^+\gamma^3 - \gamma^+\gamma^0 \\ \gamma^3\gamma^+ &= -q^{-2}\gamma^+\gamma^3 \\ \gamma^-\gamma^0 &= \frac{\lambda}{q}\gamma^3\gamma^- - \gamma^0\gamma^- \\ \gamma^-\gamma^3 &= -q^{-2}\gamma^3\gamma^- \\ \gamma^-\gamma^+ &= \frac{1}{q^2\lambda} - \frac{1}{\lambda}\gamma^3\gamma^3 \\ \gamma^+\gamma^- &= \frac{1}{\lambda}\gamma^3\gamma^3 - \frac{q^2}{\lambda} \\ \gamma^0\gamma^0 &= -1 \\ \gamma^0\gamma^3 &= \gamma^3\gamma^3 - \gamma^3\gamma^0 - 1 \end{aligned} \tag{B.10}$$

The action on the spinors is given by

$$\begin{aligned} \gamma^0 \triangleright x = \bar{y},\ \gamma^0 \triangleright y = -\frac{1}{q}\bar{x},\ \gamma^0 \triangleright \bar{x} = qy,\ \gamma^0 \triangleright \bar{y} = -x, \\ \gamma^3 \triangleright x = -q^2\bar{y},\ \gamma^3 \triangleright y = -\frac{1}{q}\bar{x},\ \gamma^3 \triangleright \bar{x} = -\frac{1}{q}y,\ \gamma^3 \triangleright \bar{y} = -x, \\ \gamma^+ \triangleright x = q\sqrt{q[2]_q}\bar{x},\ \gamma^+ \triangleright y = 0,\ \gamma^+ \triangleright \bar{x} = 0,\ \gamma^+ \triangleright \bar{y} = -\sqrt{q[2]_q}y, \\ \gamma^- \triangleright x = 0,\ \gamma^- \triangleright y = \frac{\sqrt{q[2]_q}}{q}\bar{y},\ \gamma^- \triangleright \bar{x} = -\frac{\sqrt{q[2]_q}}{q^2}x,\ \gamma^- \triangleright \bar{y} = 0 \end{aligned} \tag{B.11}$$

B.3.4 The commutation relations with the coordinates and derivatives

For (V_+, V_-, V_3, V_0) equal to the coordinates (X_+, X_-, X_3, X_0) or the derivatives $(\partial_+, \partial_-, \partial_3, \partial_0)$ we have:

$$\begin{aligned}
V_0 x &= a x V_0 + \frac{b}{q^2} x V_3 + \frac{b[2]_q}{\sqrt{q[2]_q}} y V_+ \\
V_3 x &= b x V_0 + (a + \frac{b\lambda}{q}) x V_3 + \frac{\sqrt{q[2]_q}}{q^3} \left(b + q^3\lambda\,(a+b)\right) y V_+ \\
V_- x &= q\,(a+b)\,x V_- - \frac{b[2]_q}{\sqrt{q[2]_q}} y V_0 + \frac{\sqrt{q[2]_q}}{q^3} \left(b + q^3\lambda\,(a+b)\right) y V_3 \\
V_+ x &= \frac{1}{q^3}(q^2 a - b) x V_+ \\
V_0 y &= a y V_0 - b y V_3 - \frac{\sqrt{q[2]_q}\,b}{q^2} x V_- \\
V_3 y &= -\frac{b}{q^2} y V_0 + (a + \frac{b\lambda}{q}) y V_3 + \frac{b\sqrt{q[2]_q}}{q^2} x V_- \\
V_- y &= \frac{1}{q^3}(q^2 a - b) y V_- \\
V_+ y &= (a+b)\,q y V_+ + \frac{b\sqrt{q[2]_q}}{q^2} x\,(V_0 + V_3)
\end{aligned} \tag{B.12}$$

with four possible choices for the constants a and b: $(a, b) = (\pm 1, 0)$ or $(a, b) = \pm(\frac{[4]_q}{[2]_q^2}, -\frac{q\lambda}{[2]_q})$

$$\begin{aligned}
V_0 \bar{x} &= c \bar{x} V_0 + q^2 d \bar{x} V_3 - q\sqrt{q[2]_q}\, d \bar{y} V_- \\
V_3 \bar{x} &= d \bar{x} V_0 + (c + d - dq^2) \bar{x} V_3 - \frac{[2]_q}{\sqrt{q[2]_q}} \left(d - cq\lambda + dq^3\lambda\right) \bar{y} V_- \\
V_- \bar{x} &= q(c - q^2 d) \bar{x} V_- \\
V_+ \bar{x} &= \frac{c+d}{q} \bar{x} V_+ - \frac{[2]_q}{\sqrt{q[2]_q}} \left(d - cq\lambda + dq^3\lambda\right) \bar{y} V_3 + dq\sqrt{q[2]_q}\, \bar{y} V_0 \\
V_0 \bar{y} &= c \bar{y} V_0 - d \bar{y} V_3 + d\sqrt{q[2]_q}\, \bar{x} V_+ \\
V_3 \bar{y} &= -q^2 d \bar{y} V_0 + (c + d - dq^2) \bar{y} V_3 - d\sqrt{q[2]_q}\, d \bar{x} V_+ \\
V_- \bar{y} &= \frac{c+d}{q} \bar{y} V_- - \sqrt{q[2]_q}\, d\, \bar{x}\,(V_0 + V_3) \\
V_+ \bar{y} &= q(c - q^2 d) \bar{y} V_+
\end{aligned} \tag{B.13}$$

again we have four possible choices: $(c,d) = (\pm 1, 0)$ or $(c,d) = \pm(\frac{[4]_q}{[2]_q^2}, \frac{\lambda}{q[2]_q})$.

How to commute the spinors with space functions

To deduce commutation relations with space functions, we proceed in the same way as we have done it in section 5.1 for the derivatives. We express the commutation relations in a matrix notation and then make use of the fact, that these matrices fulfil characteristic equations. Explicitly we write

$$X\begin{pmatrix} x \\ y \end{pmatrix} = (x,y)L^s_X \qquad X\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = (\overline{x},\overline{y})L^{s}_X$$

with $X \in \{X^0, X^{0/3}, X^+, X^-\}$ and the L-matrices are in the case $(a,b) = (\frac{[4]_q}{[2]_q^2}, -\frac{q\lambda}{[2]_q})$ and $(c,d) = (\frac{[4]_q}{[2]_q^2}, \frac{\lambda}{q[2]_q})$ given by

$$\begin{aligned}
L^s_{X^0} &= \begin{pmatrix} qX^0 - \frac{\lambda}{q[2]_q}X^{0/3} & \frac{\lambda}{q\sqrt{q[2]_q}}X^+ \\ -\frac{q^2\lambda}{\sqrt{q[2]_q}}X^- & \frac{1}{q}X^0 + \frac{q\lambda}{[2]_q}X^{0/3} \end{pmatrix} &\qquad L^{\overline{s}}_{X^0} &= \begin{pmatrix} \frac{1}{q}X^0 + \frac{q\lambda}{[2]_q}X^{0/3} & \frac{q\lambda}{\sqrt{q[2]_q}}X^- \\ -\frac{\lambda}{\sqrt{q[2]_q}}X^+ & qX^0 - \frac{\lambda}{q[2]_q}X^{0/3} \end{pmatrix} \\
L^s_{X^+} &= \begin{pmatrix} X^+ & 0 \\ \frac{q^2\lambda}{\sqrt{q[2]_q}}X^{0/3} & X^+ \end{pmatrix} &\qquad L^{\overline{s}}_{X^+} &= \begin{pmatrix} X^+ & -\frac{q\lambda}{\sqrt{q[2]_q}}X^{0/3} \\ 0 & X^+ \end{pmatrix} \\
L^s_{X^-} &= \begin{pmatrix} X^- & -\frac{\lambda}{q\sqrt{q[2]_q}}X^{0/3} \\ 0 & X^- \end{pmatrix} &\qquad L^{\overline{s}}_{X^-} &= \begin{pmatrix} X^- & 0 \\ \frac{\lambda}{\sqrt{q[2]_q}}X^{0/3} & X^- \end{pmatrix} \\
L^s_{X^{0/3}} &= \begin{pmatrix} \frac{1}{q}X^{0/3} & 0 \\ 0 & qX^{0/3} \end{pmatrix} &\qquad L^{\overline{s}}_{X^{0/3}} &= \begin{pmatrix} qX^{0/3} & 0 \\ 0 & \frac{1}{q}X^{0/3} \end{pmatrix}
\end{aligned}$$

They fulfil the following finiteness conditions

$$\begin{aligned}
(L^s_{X^0})^2 &= -(X^0)^2 + \frac{q\lambda^2}{[2]_q}(X)^2 + [2]_q L^s_{X^0}X^0, \\
(L^s_{X^{0/3}})^2 &= -(X^{0/3})^2 + [2]_q L^s_{X^{0/3}}X^{0/3} \\
(L^s_{X^+})^2 &= -q^{-2}(X^+)^2 + \frac{[2]_q}{q}L^s_{X^+}X^+ \\
(L^s_{X^-})^2 &= -q^2(X^-)^2 + q[2]_q L^s_{X^-}X^-
\end{aligned}$$

The characteristic equations for the $L^{\overline{s}}$-matrices are the same.

The commutation with a space function is then given by

$$F(X)\begin{pmatrix} x \\ y \end{pmatrix} = (x,y)F(L^s_X) \qquad F(X)\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = (\overline{x},\overline{y})F(L^{\overline{s}}_X) \tag{B.14}$$

with

$$
\begin{aligned}
F(L^{s/\bar{s}}_{X^0}) &= \frac{\left(Aq^2 - B + 2qL^{s/\bar{s}}_{X^0}\right) F(\frac{Bq^2-A}{2q}) - \left(A - Bq^2 + 2qL^{s/\bar{s}}_{X^0}\right) F(\frac{B-Aq^2}{2q})}{(A+B)\, q\lambda} \\
F(L^{s/\bar{s}}_{X^{0/3}}) &= \frac{q^2 F(\frac{X^{0/3}}{q}) - F(qX^{0/3})}{q\lambda} + L^{s/\bar{s}}_{X^{0/3}} \frac{F(qX^{0/3}) - F(\frac{X^{0/3}}{q})}{X^{0/3}\lambda} \\
F(L^{s/\bar{s}}_{X^+}) &= \frac{q^2 F(\frac{X^+}{q^2}) - F(X^+)}{q\lambda} + L^{s/\bar{s}}_{X^+} \frac{q\left(F(X^+) - F(\frac{X^+}{q^2})\right)}{\lambda X^+} \\
F(L^{s/\bar{s}}_{X^-}) &= \frac{q^2 F(X^-) - F(q^2 X^-)}{q\lambda} + L^{s/\bar{s}}_{X^-} \frac{(F(q^2X^-) - F(X^-))}{q\lambda X^-} \\
F(L^{s/\bar{s}}_{A}, L^{s/\bar{s}}_{B}) &= \frac{(Aq^2 - B)\, F(\frac{A}{q}, Bq) + (Bq^2 - A)\, F(Aq, \frac{B}{q})}{(A+B)\, q\lambda} \\
&\quad + L^{s/\bar{s}}_{X^0} \frac{2\left(F(\frac{A}{q}, Bq) - F(Aq, \frac{B}{q})\right)}{(A+B)\,\lambda}
\end{aligned}
\tag{B.15}
$$

If we choose $(a,b) = (1,0)$ and $(c,d) = (1,0)$ in the relations (B.12,B.13) the L-matrices are

$$
\begin{aligned}
L^s_{X^0} &= \begin{pmatrix} X^0 & 0 \\ 0 & X^0 \end{pmatrix} & L^{\bar{s}}_{X^0} &= \begin{pmatrix} X^0 & 0 \\ 0 & X^0 \end{pmatrix} \\
L^s_{X^+} &= \begin{pmatrix} qX^+ & 0 \\ -q\lambda\sqrt{q[2]_q}X^3 & \frac{1}{q}X^+ \end{pmatrix} & L^{\bar{s}}_{X^+} &= \begin{pmatrix} qX^+ & 0 \\ 0 & \frac{1}{q}X^+ \end{pmatrix} \\
L^s_{X^-} &= \begin{pmatrix} \frac{1}{q}X^- & 0 \\ 0 & qX^- \end{pmatrix} & L^{\bar{s}}_{X^-} &= \begin{pmatrix} \frac{1}{q}X^- & 0 \\ -\frac{\lambda}{q}\sqrt{q[2]_q}\, X^3 & qX^- \end{pmatrix} \\
L^s_{X^{0/3}} &= \begin{pmatrix} X^{0/3} & 0 \\ q\lambda\sqrt{q[2]_q}\, X^- & X^{0/3} \end{pmatrix} & L^{\bar{s}}_{X^{0/3}} &= \begin{pmatrix} X^{0/3} & 0 \\ \frac{\lambda}{q}\sqrt{q[2]_q}\, X^- & X^{0/3} \end{pmatrix}
\end{aligned}
$$

The characteristic equation for $L_{X^{0/3}}$ is

$$
(L^s_{X^{0/3}})^2 = -\frac{1}{q^2}(X^{0/3})^2 + \frac{[2]_q}{q} L^s_{X^{0/3}}\, X^{0/3} \quad \text{and} \quad (L^{\bar{s}}_{X^{0/3}})^2 = -q^2 (X^{0/3})^2 + q[2]_q L^{\bar{s}}_{X^{0/3}}\, X^{0/3}
$$

and the calculation for $F(L^s_{X^+})$, resp. $F(L^{\bar{s}}_{X^-})$, is done directly by considering powers of L_{X^+}, resp. $L^{\bar{s}}_{X^-}$. We find:

$$\begin{aligned}
F(L^s_{X^0}) &= F(X^0) \\
F(L^s_{X^{0/3}}) &= \frac{q^2 F(\frac{X^{0/3}}{q^2}) - F(X^{0/3})}{q\lambda} + L^s_{X^{0/3}} \frac{q\left(F(X^{0/3}) - F(\frac{X^{0/3}}{q^2})\right)}{X^{0/3}\lambda} \\
F(L^s_{X^+}) &= \begin{pmatrix} & F(qX^+) & \\ \frac{q}{\sqrt{q[2]_q}} \frac{F(q^3X^+) - F(\frac{X^+}{q})}{X^+} X^{0/3} + \sqrt{q[2]_q} \frac{q\left(F(\frac{X^+}{q}) - F(qX^+)\right)}{X^+} X^0 & & F(\frac{X^+}{q}) \end{pmatrix} \\
F(L^s_{X^-}) &= \begin{pmatrix} F(\frac{1}{q}X^-) & \\ & F(q\,X^-) \end{pmatrix} \\
F(L^s_A, L^s_B) &= F(A, B)
\end{aligned} \tag{B.16}$$

$$\begin{aligned}
F(L^{\bar{s}}_{X^0}) &= F(X^0) \\
F(L^{\bar{s}}_{X^{0/3}}) &= \frac{q^2 F(X^{0/3}) - F(q^2 X^{0/3})}{q\lambda} + L^{\bar{s}}_{X^{0/3}} \frac{F(q^2 X^{0/3}) - F(X^{0/3})}{q\lambda X^{0/3}} \\
F(L^{\bar{s}}_{X^+}) &= \begin{pmatrix} F(qX^+) & \\ & F(\frac{1}{q}\,X^+) \end{pmatrix} \\
F(L^s_{X^-}) &= \begin{pmatrix} & F(\frac{1}{q}X^-) & \\ \frac{q}{\sqrt{q[2]_q}} \frac{F(qX^-) - F(\frac{1}{q^3}X^-)}{X^-} X^{0/3} + \sqrt{q[2]_q} \frac{F(\frac{X^-}{q}) - F(qX^-)}{qX^-} X^0 & & F(q\,X^-) \end{pmatrix} \\
F(L^s_A, L^s_B) &= F(A, B)
\end{aligned} \tag{B.17}$$

B.4 The coordinates and derivatives

B.4.1 The commutation relations of the coordinates

The algebra of the q-Minkowski space and its derivatives is defined by the relations

$$
\begin{array}{lllllll}
X^0X^3 &=& X^3X^0 & \qquad & \partial_0\partial_3 &=& \partial_3\partial_0 \\
X^0X^- &=& X^-X^0 & & \partial_0\partial_- &=& \partial_-\partial_0 \\
X^0X^+ &=& X^+X^0 & & \partial_0\partial_+ &=& \partial_+\partial_0 \\
X^3X^- &=& q^{-2}X^-X^3 + \frac{\lambda}{q}X^-X^0 & & \partial_3\partial_- &=& q^2\partial_-\partial_3 + q\lambda\partial_-\partial_0 \\
X^3X^+ &=& q^2X^+X^3 - q\lambda X^+X^0 & & \partial_3\partial_+ &=& q^{-2}\partial_+\partial_3 - \frac{\lambda}{q}\partial_+\partial_0 \\
X^-X^+ &=& X^+X^- - \lambda X^3X^0 + \lambda X^3X^3 & & \partial_-\partial_+ &=& \partial_+\partial_- - \lambda\partial_3\partial_0 - \lambda\partial_3\partial_3
\end{array}
$$

$$
\begin{array}{lll}
\partial_0X^0 &=& 1 - \frac{\lambda}{q^2[2]_q}X^0\partial_3 - \frac{\lambda}{q^2[2]_q}X^-\partial_- - \frac{\lambda}{q^2[2]_q}X^+\partial_+ + \frac{\lambda}{q^2[2]_q}X^{0/3}\partial_3 + \frac{[4]_q}{q[2]_q{}^2}X^0\partial_0 \\
\partial_3X^0 &=& \frac{2}{q[2]_q}X^0\partial_3 - \frac{\lambda}{[2]_q}X^0\partial_0 - \frac{\lambda}{[2]_q}X^-\partial_- + \frac{\lambda}{[2]_q}X^{0/3}\partial_0 + \frac{\lambda}{q^2[2]_q}X^+\partial_+ + \frac{\lambda^2}{q[2]_q}X^{0/3}\partial_3 \\
\partial_-X^0 &=& X^0\partial_- - \frac{\lambda}{q^2[2]_q}X^{0/3}\partial_- + \frac{\lambda}{q[2]_q}X^+\partial_0 + \frac{\lambda}{q[2]_q}X^+\partial_3 \\
\partial_+X^0 &=& q^{-2}X^0\partial_+ + \frac{\lambda}{[2]_q}X^{0/3}\partial_+ - \frac{\lambda}{q[2]_q}X^-\partial_3 + \frac{q\lambda}{[2]_q}X^-\partial_0
\end{array}
$$

$$
\begin{array}{lll}
\partial_0X^{0/3} &=& 1 - \frac{\lambda}{q}X^-\partial_- + \frac{\lambda}{[2]_q}X^{0/3}\partial_3 + \frac{[4]_q}{q[2]_q{}^2}X^{0/3}\partial_0 \\
\partial_3X^{0/3} &=& -1 + \frac{\lambda}{q}X^-\partial_- + \frac{2}{q[2]_q}X^{0/3}\partial_3 + \frac{\lambda}{q^2[2]_q}X^{0/3}\partial_0 \\
\partial_-X^{0/3} &=& q^{-2}X^{0/3}\partial_- \\
\partial_+X^{0/3} &=& X^{0/3}\partial_+ + \lambda X^-\partial_0 + \lambda X^-\partial_3
\end{array}
$$

$$
\begin{array}{lll}
\partial_0X^+ &=& \lambda X^0\partial_- + \frac{\lambda}{[2]_q}X^+\partial_3 - \frac{\lambda}{q[2]_q}X^{0/3}\partial_- + \frac{[4]_q}{q[2]_q{}^2}X^+\partial_0 \\
\partial_3X^+ &=& -\lambda X^0\partial_- + (q - \frac{2}{q^2[2]_q})X^{0/3}\partial_- + \frac{2}{q[2]_q}X^+\partial_3 + \frac{\lambda}{q^2[2]_q}X^+\partial_0 \\
\partial_-X^+ &=& X^+\partial_- \\
\partial_+X^+ &=& 1 + q^{-2}X^+\partial_+ - \frac{\lambda}{q}X^0\partial_0 - \frac{\lambda}{q}X^0\partial_3 - (q^{-2} + \frac{2q}{[2]_q})X^{0/3}\partial_3 \\
&& + \frac{\lambda}{[2]_q}X^{0/3}\partial_0 - (2 - \frac{[4]_q}{[2]_q})X^-\partial_-
\end{array}
$$

$$
\begin{array}{lll}
\partial_0X^- &=& -\frac{\lambda}{q^2[2]_q}X^-\partial_3 + \frac{\lambda}{q[2]_q}X^{0/3}\partial_+ + \frac{[4]_q}{q[2]_q{}^2}X^-\partial_0 \\
\partial_3X^- &=& \frac{2}{q[2]_q}X^-\partial_3 - \frac{\lambda}{[2]_q}X^-\partial_0 - \frac{\lambda}{q[2]_q}X^{0/3}\partial_+ \\
\partial_-X^- &=& 1 + q^{-2}X^-\partial_- - \frac{\lambda}{q^2[2]_q}X^{0/3}\partial_0 - \frac{\lambda}{q^2[2]_q}X^{0/3}\partial_3 \\
\partial_+X^- &=& X^-\partial_+
\end{array}
$$

A consistent definition of conjugation for the coordinates is

$$\overline{X^0} = X^0,\ \overline{X^3} = X^3,\ \overline{X^+} = -qX^-,\ \overline{X^-} = -\frac{1}{q}X^+$$

The definition of conjugates for the derivatives is more involved, because there is no linear relation between ∂ and $\overline{\partial}$ anymore, see [20], [17], [18]. Instead we have

$$\overline{\partial}_0 = -q^4\hat{\partial}_0,\ \overline{\partial}_3 = -q^4\hat{\partial}_3,\ \overline{\partial}_+ = q^5\hat{\partial}_-,\ \overline{\partial}_- = q^3\hat{\partial}_+ \tag{B.18}$$

and the hatted derivatives can be expressed algebraically in terms of X and ∂:

$$\hat{\partial}_a = \Lambda^{-1}\left[\partial_a - \frac{\lambda}{q^2[2]_q}X_a(\partial)^2\right] \tag{B.19}$$

Λ is the scaling operator with the properties

$$\Lambda X^a = \frac{1}{q^2}X^a\Lambda,\ \Lambda\partial_a = q^2\partial_a\Lambda,\ \Lambda\hat{\partial}_a = q^2\hat{\partial}_a\Lambda,\ \overline{\Lambda} = q^8\Lambda^{-1}$$

Similarly we can express the normal derivatives by the hatted ones:

$$\partial_a = \Lambda\left[\hat{\partial}_a + \frac{\lambda q^2}{[2]_q}X_a(\hat{\partial})^2\right]$$

The commutation relations for the hatted derivatives are

$$\hat{\partial}_i\hat{\partial}_j = \mathcal{R}_{I\,ji}^{lk}\hat{\partial}_k\hat{\partial}_l \qquad \partial_i\hat{\partial}_j = \mathcal{R}_{II\ \ ji}^{-1\,lk}\hat{\partial}_k\partial_l \qquad \hat{\partial}_i X^j = \delta_j^i + \mathcal{R}_{II\ \ il}^{-1\,ik}X^l\hat{\partial}_k$$

Another basis for the q-Minkowski space are light-cone coordinates, which are defined via the spinors by

$$\begin{array}{ll} A = \overline{x}y & C = \overline{x}x \\ B = \overline{y}x & D = \overline{y}y \end{array}$$

They satisfy the relations

$$\begin{array}{rclrcl} AB &=& BA - \frac{1}{q}\lambda CD + q\lambda D^2, & BC &=& CB - \frac{1}{q}\lambda BD \\ AC &=& CA + q\lambda AD, & BD &=& q^2 DB \\ AD &=& \frac{1}{q^2}DA, & CD &=& DC \end{array} \tag{B.20}$$

with the reality properties $\overline{A} = B$, $\overline{B} = A$, $\overline{C} = C$, $\overline{D} = D$.

The coordinates $\{X^0,\ \vec{X} = (X^+, X^3, X^-)\}$ constitute the decomposition into a singlet and a triplet with respect to the $\mathcal{U}_q(su_2)$ subalgebra of rotations: $D^{(\frac{1}{2},\frac{1}{2})} \longrightarrow D^0 \oplus D^1$

The basis transformation between these two bases is given by

$$A = \frac{1}{q}X^+, \quad B = -X^-, \quad C = \frac{1}{\sqrt{q[2]_q}}(q^2X^0 + X^3), \quad D = \frac{1}{\sqrt{q[2]_q}}(X^0 - X^3) \tag{B.21}$$

B.4.2 The L-matrices for the coordinates

$$L_{X^{0/3}} = \begin{pmatrix} \frac{[4]_q}{q{[2]_q}^2}X^{0/3} & \frac{\lambda}{[2]_q}X^{0/3} & 0 & -\frac{\lambda}{q}X^- \\ \frac{\lambda}{q^2[2]_q}X^{0/3} & \frac{2}{1+q^2}X^{0/3} & 0 & \frac{\lambda}{q}X^- \\ \lambda X^- & \lambda X^- & X^{0/3} & 0 \\ 0 & 0 & 0 & q^{-2}X^{0/3} \end{pmatrix} \quad \text{(B.22)}$$

$$L_{X^-} = \begin{pmatrix} \frac{[4]_q}{q{[2]_q}^2}X^- & -\frac{\lambda}{q^2[2]_q}X^- & \frac{\lambda}{q[2]_q}X^{0/3} & 0 \\ -\frac{\lambda}{[2]_q}X^- & \frac{2}{1+q^2}X^- & -\frac{\lambda}{q[2]_q}X^{0/3} & 0 \\ 0 & 0 & X^- & 0 \\ -\frac{\lambda}{q^2[2]_q}X^{0/3} & -\frac{\lambda}{q^2[2]_q}X^{0/3} & 0 & q^{-2}X^- \end{pmatrix}$$

$$L_{X^0} =$$

$$\begin{pmatrix} \frac{[4]_q}{q{[2]_q}^2}X^0 & -\frac{\lambda}{q^2[2]_q}X^0 + \frac{\lambda}{q^2[2]_q}X^{0/3} & -\frac{\lambda}{q^2[2]_q}X^+ & -\frac{\lambda}{q^2[2]_q}X^- \\ -\frac{\lambda}{[2]_q}X^0 + \frac{\lambda}{[2]_q}X^{0/3} & \frac{2}{q[2]_q}X^0 + \frac{\lambda^2}{q[2]_q}X^{0/3} & \frac{\lambda}{q^2[2]_q}X^+ & -\frac{\lambda}{[2]_q}X^- \\ \frac{q\lambda}{[2]_q}X^- & -\frac{\lambda}{q[2]_q}X^- & q^{-2}X^0 + \frac{\lambda}{[2]_q}X^{0/3} & 0 \\ \frac{\lambda}{q[2]_q}X^+ & \frac{\lambda}{q[2]_q}X^+ & 0 & X^0 - \frac{\lambda}{q^2[2]_q}X^{0/3} \end{pmatrix}$$

$$L_{X^+} =$$

$$\begin{pmatrix} \frac{[4]_q}{q{[2]_q}^2}X^+ & \frac{\lambda}{[2]_q}X^+ & 0 & \lambda X^0 - \frac{\lambda}{q[2]_q}X^{0/3} \\ \frac{\lambda}{q^2[2]_q}X^+ & \frac{2}{1+q^2}X^+ & 0 & -\lambda X^0 + (q - \frac{2}{q^2[2]_q})X^{0/3} \\ -\frac{\lambda}{q}X^0 + \frac{\lambda}{[2]_q}X^{0/3} & -\frac{\lambda}{q}X^0 - (q^{-2} - \frac{2q}{[2]_q})X^{0/3} & q^{-2}X^+ & \lambda^2 X^- \\ 0 & 0 & 0 & X^+ \end{pmatrix}$$

B.5 The commutation relations of $T^\pm, T^3$

The $\mathcal{U}_q(su_2)$ algebra of rotations is defined via the generators $T^\pm, T^3$ by the relations

$$\begin{aligned} q^{-1}T^+T^- - qT^-T^+ &= T^3 \\ q^2T^3T^+ - q^{-2}T^+T^3 &= (q+q^{-1})T^+ \\ q^2T^-T^3 - q^{-2}T^3T^- &= (q+q^{-1})T^- \end{aligned}$$

The semidirect product algebra with the q-Minkowski space is given by

$$
\begin{aligned}
T^3X^3 &= X^3T^3 \\
T^3X^+ &= q^{-4}X^+T^3 + q^{-1}(1+q^{-2})X^+ \\
T^3X^- &= q^4X^-T^3 - q(1+q^2)X^- \\
\\
T^+X^3 &= X^3T^+ + q^{-2}\sqrt{1+q^2}X^+ \\
T^+X^+ &= q^{-2}X^+T^+ \\
T^+X^- &= q^2X^-T^+ + q^{-1}\sqrt{1+q^2}X^3 \\
\\
T^-X^3 &= X^3T^- + q\sqrt{1+q^2}X^- \\
T^-X^+ &= q^{-2}X^+T^- + \sqrt{1+q^2}X^3 \\
T^-X^- &= q^2X^-T^- \\
\\
[T^{\pm,3}, X^0] &= 0
\end{aligned}
\tag{B.23}
$$

If we use the light-cone coordinates the relations are

$$
\begin{aligned}
T^+A &= q^{-2}AT^+ & T^+C &= CT^+ + q^{-1}A \\
T^+B &= q^2BT^+ + qD - q^{-1}C & T^+D &= DT^+ - q^{-1}A \\
\\
T^-A &= q^{-2}AT^- + q^{-1}C - qD & T^-C &= CT^- - q^{-1}B \\
T^-B &= q^2BT^- & T^-D &= DT^- + q^{-1}B \\
\\
\tau^3A &= q^{-4}A\tau^3 & \tau^3C &= C\tau^3 \\
\tau^3B &= q^4B\tau^3 & \tau^3D &= D\tau^3
\end{aligned}
\tag{B.24}
$$

with $\tau^3 = 1 - \lambda T^3$.

B.6 Commutation relations of $T^2, \tau^1, S^1, \sigma^2$

The relations for the boost generators are

$$\begin{aligned}
T^+T^- &= -\frac{q\tau^3}{\lambda} + q^2T^-T^+ + \frac{q}{\lambda} \\
T^+T^2 &= q^{-2}T^2T^+ \\
T^+S^1 &= q^2S^1T^+ - \frac{1}{\lambda}\sigma^2 + \frac{1}{\lambda}\tau^1\tau^3 \\
T^-T^2 &= T^2T^- - \frac{1}{\lambda}\tau^1 + \frac{1}{\lambda}\sigma^2 \\
T^-S^1 &= S^1T^- \\
\tau^3T^+ &= q^{-4}T^+\tau^3 \\
\tau^3T^- &= q^4T^-\tau^3 \\
\tau^3\tau^1 &= \tau^1\tau^3 \\
\tau^3\sigma^2 &= \sigma^2\tau^3 \\
\tau^3T^2 &= q^{-4}T^2\tau^3 \\
\tau^3S^1 &= q^4S^1\tau^3
\end{aligned}
\qquad
\begin{aligned}
\tau^1T^+ &= T^+\tau^1 + \lambda T^2 \\
\tau^1T^- &= q^{-2}T^-\tau^1 + -\lambda S^1 \\
\tau^1\sigma^2 &= \sigma^2\tau^1 + q\lambda^3S^1T^2 \\
\tau^1T^2 &= q^2T^2\tau^1 \\
\tau^1S^1 &= S^1\tau^1 \\
\sigma^2T^+ &= T^+\sigma^2 - \frac{\lambda}{q^2}T^2\tau^3 \\
\sigma^2T^- &= q^2T^-\sigma^2 + q^2\lambda S^1 \\
\sigma^2T^2 &= q^{-2}T^2\sigma^2 \\
\sigma^2S^1 &= S^1\sigma^2 \\
T^2S^1 &= S^1T^2
\end{aligned}$$

The relations defining the semidirect product with the q-Minkowski space are

$$\begin{aligned}
\tau^1X^- &= \frac{1}{q}X^-\tau^1 \\
\tau^1X^+ &= qX^+\tau^1 - \frac{q^2\lambda^2}{\sqrt{q[2]_q}}X^3T^2 + \frac{q^2\lambda^2}{\sqrt{q[2]_q}}X^0T^2 \\
\tau^1X^0 &= \frac{[4]_q}{[2]_q{}^2}X^0\tau^1 - \frac{q\lambda}{[2]_q}X^3\tau^1 - \frac{q\lambda^2}{\sqrt{q[2]_q}}X^-T^2 \\
\tau^1X^3 &= \frac{2}{[2]_q}X^3\tau^1 - \frac{\lambda}{q[2]_q}X^0\tau^1 - \frac{q\lambda^2}{\sqrt{q[2]_q}}X^-T^2 \\
\\
T^2X^- &= \frac{1}{q}X^-T^2 - \frac{1}{q\sqrt{q[2]_q}}X^0\tau^1 + \frac{1}{q\sqrt{q[2]_q}}X^3\tau^1 \\
T^2X^+ &= qX^+T^2 \\
T^2X^0 &= \frac{\lambda}{q[2]_q}X^3T^2 + \frac{1}{\sqrt{q[2]_q}}X^+\tau^1 + \frac{[4]_q}{[2]_q{}^2}X^0T^2 \\
T^2X^3 &= \frac{2}{[2]_q}X^3T^2 + \frac{q\lambda}{[2]_q}X^0T^2 + \frac{1}{\sqrt{q[2]_q}}X^+\tau^1 \\
\\
\sigma^2X^- &= qX^-\sigma^2 - \frac{q\lambda^2}{\sqrt{q[2]_q}}X^0S^1 + \frac{q\lambda^2}{\sqrt{q[2]_q}}X^3S^1 \\
\sigma^2X^+ &= \frac{1}{q}X^+\sigma^2 \\
\sigma^2X^0 &= \frac{\lambda}{q[2]_q}X^3\sigma^2 + \frac{\lambda^2}{\sqrt{q[2]_q}}X^+S^1 + \frac{[4]_q}{[2]_q^2}X^0\sigma^2 \\
\sigma^2X^3 &= \frac{2}{[2]_q}X^3\sigma^2 + \frac{q\lambda}{[2]_q}X^0\sigma^2 + \frac{\lambda^2}{\sqrt{q[2]_q}}X^+S^1 \\
\\
S^1X^- &= qX^-S^1 \\
S^1X^+ &= \frac{1}{q}X^+S^1 - \frac{1}{\sqrt{q[2]_q}}X^3\sigma^2 + \frac{1}{\sqrt{q[2]_q}}X^0\sigma^2 \\
S^1X^0 &= \frac{[4]_q}{[2]_q^2}X^0S^1 - \frac{q\lambda}{[2]_q}X^3S^1 - \frac{q}{\sqrt{q[2]_q}}X^-\sigma^2 \\
S^1X^3 &= \frac{2}{[2]_q}X^3S^1 - \frac{\lambda}{q[2]_q}X^0S^1 - \frac{q}{\sqrt{q[2]_q}}X^-\sigma^2
\end{aligned} \tag{B.25}$$

In terms of the light-cone coordinates we get

$$
\begin{aligned}
T^2A &= qAT^2 & T^2C &= qCT^2 + qA\tau^1 \\
T^2B &= q^{-1}BT^2 + q^{-1}D\tau^1 & T^2D &= q^{-1}DT^2
\end{aligned}
$$

$$
\begin{aligned}
S^1A &= q^{-1}AS^1 + q^{-1}D\sigma^2 & S^1C &= (q^{-1}C + q\lambda^2D)S^1 + qB\sigma^2 \\
S^1B &= qB^2S^1 & S^1D &= qDS^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\tau^1A &= qA\tau^1 + q\lambda^2DT^2 & \tau^1C &= (q^{-1}C + q\lambda^2D)\tau^1 + q\lambda^2BT^2 \\
\tau^1B &= q^{-1}B\tau^1 & \tau^1D &= qD\tau^1
\end{aligned}
$$

$$
\begin{aligned}
\sigma^2A &= q^{-1}A\sigma^2 & \sigma^2C &= qC\sigma^2 + q\lambda^2AS^1 \\
\sigma^2B &= qB\sigma^2 + q\lambda^2DS^1 & \sigma^2D &= q^{-1}D\sigma^2
\end{aligned}
$$

B.7 The boosts in A B basis

The action of the Lorentz generators on a function depending on the new coordinates defined in equation (6.7) and $X^{0/3}$:

$$
\begin{aligned}
S^1 f(A,B,X^{0/3}) &= \\
&\frac{\left(A+Aq^2+2q^2X^{0/3}\right) f(\frac{A}{q},Bq,qX^{0/3}) + \left(B+Bq^2-2q^2X^{0/3}\right) f(Aq,\frac{B}{q},qX^{0/3})}{q[2]_q\,(A+B)} S^1 \\
&-\frac{2q}{\lambda\sqrt{q[2]_q}\,(A+B)}\left(f(\frac{A}{q},Bq,qX^{0/3}) - f(Aq,\frac{B}{q},qX^{0/3})\right) X^-\sigma^2 \\
T^2 f(A,B,X^{0/3}) &= \\
&\frac{\left(B+Bq^2-2X^{0/3}\right) f(\frac{A}{q},Bq,\frac{X^{0/3}}{q}) + \left(A+Aq^2+2X^{0/3}\right) f(Aq,\frac{B}{q},\frac{X^{0/3}}{q})}{q[2]_q\,(A+B)} T^2 \\
&+\frac{2}{\lambda\sqrt{q[2]_q}\,(A+B)}\left(f(\frac{A}{q},Bq,\frac{X^{0/3}}{q}) - f(Aq,\frac{B}{q},\frac{X^{0/3}}{q})\right) X^+\tau^1 \\
\sigma^2 f(A,B,X^{0/3}) &= \\
&\frac{\left(B+Bq^2-2X^{0/3}\right) f(\frac{A}{q},Bq,\frac{X^{0/3}}{q}) + \left(A+Aq^2+2X^{0/3}\right) f(Aq,\frac{B}{q},\frac{X^{0/3}}{q})}{q[2]_q\,(A+B)} \sigma^2 \\
&+\frac{2\lambda\sqrt{q\lambda}}{q\sqrt{\lambda[2]_q}\,(A+B)}\left(f(\frac{A}{q},Bq,\frac{X^{0/3}}{q}) - f(Aq,\frac{B}{q},\frac{X^{0/3}}{q})\right) X^+S^1 \\
\tau^1 f(A,B,X^{0/3}) &= \\
&\frac{\left(A+Aq^2+2q^2X^{0/3}\right) f(\frac{A}{q},Bq,qX^{0/3}) + \left(B+Bq^2-2q^2X^{0/3}\right) f(Aq,\frac{B}{q},qX^{0/3})}{q[2]_q\,(A+B)} \tau^1 \\
&-\frac{2(q\lambda)^{\frac{3}{2}}}{q\sqrt{\lambda[2]_q}\,(A+B)}\left(f(\frac{A}{q},Bq,qX^{0/3}) - f(Aq,\frac{B}{q},qX^{0/3})\right) X^-T^2
\end{aligned}
\tag{B.26}
$$

$$
\begin{aligned}
\tau^3 f(A,B,X^{0/3}) &= f(A,B,X^{0/3})\tau^3 \\
T^- f(A,B,X^{0/3}) &= f(A,B,X^{0/3})T^- + \frac{\left(f(A,B,X^{0/3}) - f(A,B,q^2X^{0/3})\right)\sqrt{q[2]_q}}{X^{0/3}\lambda} X^- \\
T^+ f(A,B,X^{0/3}) &= f(A,B,X^{0/3})T^+ + \frac{\left(f(A,B,\frac{X^{0/3}}{q^2}) - f(A,B,X^{0/3})\right)[2]_q}{X^{0/3}\lambda\sqrt{q[2]_q}} X^+
\end{aligned}
$$

B.8 The derivative of $f(A,B)$ and $f(X^{0/3})$

The derivative of a function $f(A,B)$ or $f(X^{0/3})$ is given by:

$$
\begin{aligned}
\partial f(A,B) \;=\; & \frac{8q^3\left(f(A,\frac{B}{q^2})-f(\frac{A}{q^2},B)\right)}{AB\,(A+B)\,\lambda^2[2]_q}\begin{pmatrix}\frac{\frac{AB[4]_q}{q^2[2]_q}-(A^2+B^2)}{4}\\ \frac{(B-A)X^3}{2}\\ \frac{(A-B)qX^-}{2}\\ \frac{(A-B)X^+}{2q}\end{pmatrix}\\
&+\frac{4q\left((A+B)\,q\lambda f(A,B)+(A-Bq^2)\,f(A,\frac{B}{q^2})+(B-Aq^2)\,f(\frac{A}{q^2},B)\right)}{AB\,(A+B)\,\lambda^2[2]_q}\begin{pmatrix}\frac{A-B}{2}\\ X^3\\ -qX^-\\ -\frac{X^+}{q}\end{pmatrix}\\
&+\left[\frac{2q\left(f(\frac{A}{q^2},B)-f(A,\frac{B}{q^2})\right)}{(A+B)\,\lambda}L_{X^0}-\frac{(A-Bq^2)\,f(A,\frac{B}{q^2})+(B-Aq^2)\,f(\frac{A}{q^2},B)}{(A+B)\,q\lambda}\right]\partial
\end{aligned}
\tag{B.27}
$$

$$
\begin{aligned}
\partial_0 f(X^{0/3}) &= f(X^{0/3})\partial_0\\
&\quad +D_{\frac{X^{0/3}}{q^2}}f(X^{0/3})\left(1-\frac{\lambda}{q^2[2]_q}\left(X^{0/3}\left(\partial_0-q^2\partial_3\right)+q[2]_qX^-\partial_-\right)\right)\\
\partial_3 f(X^{0/3}) &= f(X^{0/3})\partial_3\\
&\quad -D_{\frac{X^{0/3}}{q^2}}f(X^{0/3})\left(1-\frac{\lambda}{q^2[2]_q}\left(X^{0/3}\left(\partial_0-q^2\partial_3\right)+q[2]_qX^-\partial_-\right)\right)\\
\partial_+ f(X^{0/3}) &= f(X^{0/3})\partial_+ +\lambda D_{q^2X^{0/3}}f(X^{0/3})X^-\left(\partial_0+\partial_3\right)\\
\partial_- f(X^{0/3}) &= f(\frac{X^{0/3}}{q^2})\partial_-
\end{aligned}
\tag{B.28}
$$

B.9 The action of H

The helicity operator is given by

$$
H = \frac{\lambda}{\sqrt{q[2]_q}}\left(\partial_-T^- - q\partial_+T^+\right)(\tau^3)^{-\frac{1}{2}}-\frac{\partial_0+\partial_3}{q[2]_q}\left((\tau^3)^{\frac{1}{2}}+q^2\lambda^2T^-T^+(\tau^3)^{-\frac{1}{2}}\right)+\frac{\partial_3-q^2\partial_0}{q[2]_q}(\tau^3)^{-\frac{1}{2}}
\tag{B.29}
$$

$$H \triangleright f(A,B,X^{3/0})(X^+)^k \otimes x \quad = \tag{B.30}$$

$$\frac{1}{(q[2]_q)^{\frac{5}{2}}}\left[\frac{q^{3-4k}\left(B+Bq^2-2q^2X^{0/3}\right)f(A,\frac{B}{q^2},X^{0/3})\left(2q^2X^{0/3}+Aq^{1+2k}[2]_q\right)}{B\,(A+B)}\right.$$
$$+\frac{q^{3-4k}\left(A+Aq^2+2q^2X^{0/3}\right)f(\frac{A}{q^2},B,X^{0/3})\left(-2q^2X^{0/3}+Bq^{1+2k}[2]_q\right)}{A\,(A+B)}$$
$$\left.-\frac{q^{3-4k}f(A,B,X^{0/3})\left(2q^2X^{0/3}+Aq^{1+2k}[2]_q\right)\left(-2q^2X^{0/3}+Bq^{1+2k}[2]_q\right)}{AB}\right](X^+)^{k-1}\otimes y$$
$$+\left[\frac{\left(A+Aq^2+\frac{2X^{0/3}}{q^{2k}}\right)f(A\frac{B}{q^2}\frac{X^{0/3}}{q^2})}{(A+B)\,qX^{0/3}\lambda[2]_q}+\frac{\left(B+Bq^2-\frac{2X^{0/3}}{q^{2k}}\right)f(\frac{A}{q^2},B,\frac{X^{0/3}}{q^2})}{(A+B)\,qX^{0/3}\lambda[2]_q}\right.$$
$$+\frac{f(A,B,X^{0/3})\left(4q^{6-2k}X^{0/3}\lambda-2\,(A-B)\,q^4[2]_q\right)}{ABq^3\lambda{[2]_q}^2}$$
$$-\frac{q^{-2-2k}\left(B+Bq^2-2q^2X^{0/3}\right)f(A,\frac{B}{q^2},X^{0/3})\left(-2q^3X^{0/3}\lambda+Aq^{1+2k}[2]_q\right)}{B\,(A+B)\,X^{0/3}\lambda{[2]_q}^2}$$
$$\left.-\frac{q^{-2-2k}\left(A+Aq^2+2q^2X^{0/3}\right)f(\frac{A}{q^2},B,X^{0/3})\left(2q^3X^{0/3}\lambda+Bq^{1+2k}[2]_q\right)}{A\,(A+B)\,X^{0/3}\lambda{[2]_q}^2}\right](X^+)^k\otimes x$$

$$H \triangleright f(A,B,X^{3/0})(X^+)^k \otimes y \quad = \tag{B.31}$$

$$\left[\frac{-4q^3f(A,B,X^{0/3})}{AB(q[2]_q)^{\frac{3}{2}}}+\frac{2q\left(A+Aq^2+2q^2X^{0/3}\right)f(\frac{A}{q^2},B,X^{0/3})}{A\,(A+B)\,X^{0/3}(q[2]_q)^{\frac{3}{2}}}+\frac{2qf(A,\frac{B}{q^2},\frac{X^{0/3}}{q^2})}{(A+B)\,X^{0/3}\sqrt{q[2]_q}}\right.$$
$$\left.-\frac{2qf(\frac{A}{q^2},B,\frac{X^{0/3}}{q^2})}{(A+B)\,X^{0/3}\sqrt{q[2]_q}}+\frac{f(A,\frac{B}{q^2},X^{0/3})\left(4q^3X^{0/3}-2Bq^2[2]_q\right)}{B\,(A+B)\,X^{0/3}(q[2]_q)^{\frac{3}{2}}}\right](X^+)^{1+k}\otimes x$$
$$+\left[\frac{q\left(A+Aq^2+\frac{2X^{0/3}}{q^{2k}}\right)f(A\frac{B}{q^2}\frac{X^{0/3}}{q^2})}{(A+B)\,X^{0/3}\lambda[2]_q}+\frac{q\left(B+Bq^2-\frac{2X^{0/3}}{q^{2k}}\right)f(\frac{A}{q^2},B,\frac{X^{0/3}}{q^2})}{(A+B)\,X^{0/3}\lambda[2]_q}\right.$$
$$+\frac{f(A,B,X^{0/3})\left(-4q^{4-2k}X^{0/3}\lambda-2\,(A-B)\,q^6[2]_q\right)}{ABq^3\lambda{[2]_q}^2}$$
$$-\frac{q^{-2-2k}\left(B+Bq^2-2q^2X^{0/3}\right)f(A,\frac{B}{q^2},X^{0/3})\left(2qX^{0/3}\lambda+Aq^{1+2(1+k)}[2]_q\right)}{B\,(A+B)\,X^{0/3}\lambda{[2]_q}^2}$$
$$\left.-\frac{q^{-2-2k}\left(A+Aq^2+2q^2X^{0/3}\right)f(\frac{A}{q^2},B,X^{0/3})\left(-2qX^{0/3}\lambda+Bq^{1+2(1+k)}[2]_q\right)}{A\,(A+B)\,X^{0/3}\lambda{[2]_q}^2}\right](X^+)^k\otimes y$$

Bibliography

[1] H. S. Snyder, "Quantized Space-Time," *Phys. Rev.* **71** (1947) 38–41.

[2] F. Baader and T. Nipkow, "Term rewriting and all that,". Cambridge University Press (1998).

[3] H. Hopf, "Über die Topologie der Gruppenmannigfaltigkeiten und ihre Verallgemeinerungen," *Ann. Math.* **42** (1941) 22.

[4] E. Abe, "Hopf algebras,". Cambridge University Press (1980).

[5] M. E. Sweedler, "Hopf algebras,". Benjamin (1980).

[6] V. G. Drinfel'd, "Quantum groups," in Proceedings of the International Congress of Mathematicians, edited by A.M. Gleason, pp. 789-820, *Amer. Math. Soc.* (1986).

[7] V. G. Drinfeld, "Hopf algebras and the quantum Yang-Baxter equation," *Sov. Math. Dokl.* **32** (1985) 254–258.

[8] M. Jimbo, "A q-analogue of the of $U(\mathfrak{g})$ and the Yang-Baxter equation," *Lett. Math. Phys* **10** (1985) 63.

[9] S. L. Woronowicz, "Compact matrix pseudogroups," *Commun. Math. Phys.* **111** (1987) 613–665.

[10] N. Reshetikhin, L. Takhtadzhyan, and L. Faddeev, "Quantization of Lie groups and Lie algebras," *Leningrad Math. J.* **1** (1990) 193.

[11] M. Takeuchi, "Matrix bialgebras and quantum groups," *Israel J. Math.* **72** (1990) 232.

[12] S. Majid, "Quantum groups and noncommutative geometry," *J. Math. Phys.* **41** (2000) 3892–3942, `hep-th/0006167`.

[13] Y. I. Manin, "Quantum groups and non-commutative geometry," *preprent Montreal University CRM-1561* (1988).

[14] U. Carow-Watamura, M. Schlieker, M. Scholl, and S. Watamura, "Tensor representation of the quantum group $SL_q(2, C)$ and quantum Minkowski space," *Z. Phys.* **C48** (1990) 159–166.

[15] U. Carow-Watamura, M. Schlieker, M. Scholl, and S. Watamura, "Quantum lorentz group," *Int. J. Mod. Phys.* **A6** (1991) 3081–3108.

[16] W. B. Schmidke, J. Wess, and B. Zumino, "A q deformed lorentz algebra," *Z. Phys.* **C52** (1991) 471–476.

[17] O. Ogievetsky, W. B. Schmidke, J. Wess, and B. Zumino, "q deformed Poincaré algebra," *Commun. Math. Phys.* **150** (1992) 495–518.

[18] O. Ogievetsky and B. Zumino, "Reality in the differential calculus on q euclidean spaces," *Lett. Math. Phys.* **25** (1992) 121–130, `hep-th/9205003`.

[19] M. Rohregger and Wess, "q-deformed Lorentz algebra in Minkowski phase space," *Eur. Phys. J.* **7** (1999) 177.

[20] A. Lorek, W. Weich, and J. Wess, "Non-commutative Euclidean and Minkowski structures," *Z. Phys.* **C76** (1997) 375–386, `q-alg/9702025`.

[21] P. Podles and S. L. Woronowicz, "Quantum deformation of lorentz group," *Commun. Math. Phys.* **130** (1990) 381–431.

[22] S. Majid, "Braided momentum in the q poincare group," *J. Math. Phys.* **34** (1993) 2045–2058, `hep-th/9210141`.

[23] J. Wess and B. Zumino, "Covariant differential calculus on the quantum hyperplane," *Nucl. Phys. Proc. Suppl.* **18B** (1991) 302–312.

[24] U. Carow-Watamura, M. Schlieker, and S. Watamura, "$SO_q(N)$ covariant differential calculus on quantum space and quantum deformation of Schrodinger equation," *Z. Phys.* **C49** (1991) 439–446.

[25] G. Fiore, "Realization of $U_q(so(N))$ within the differential algebra on $\mathbb{R}_q(N)$," *Commun. Math. Phys.* **169** (1995) 475–500, `hep-th/9403033`.

[26] E. Wigner, "On Unitary Representations of the Inhomogeneous Lorentz Group," *Annals Math.* **40** (1939) 149.

[27] W. Heisenberg, "Was ist ein Elementarteilchen ?," *Naturwissenschaften* **63** (1976) 1.

[28] M. Pilling, "Das q-deformierte relativistische Einteilchenproblem,". Ph.D. thesis, Ludwig-Maximilians-Universität München, Fakultät für Physik (1994).

[29] M. Pillin, W. B. Schmidke, and J. Wess, “q deformed relativistic one particle states,” *Nucl. Phys.* **B403** (1993) 223–237.

[30] B. L. Cerchiai and J. Wess, “q-deformed Minkowski Space based on a q-Lorentz Algebra,” *Eur. Phys. J.* **C5** (1998) 553–566, `math.qa/9801104`.

[31] B. Cerchiai, “Hilbert space representations of a q-deformed Minkowski algebra,”. Ph.D. thesis, Ludwig-Maximilians-Universität München, Fakultät für Physik (2001).

[32] C. Blohmann, “Spin Representations of the q-Poincaré Algebra,” `math.QA/0110219`. Ph.D. thesis, Ludwig-Maximilians-Universität München, Fakultät für Physik (2001).

[33] B. L. Cerchiai, J. Madore, S. Schraml, and J. Wess, “Structure of the three-dimensional quantum euclidean space,” *Eur. Phys. J.* **C16** (2000) 169–180, `math.qa/0004011`.

[34] U. Meyer, “Wave equations on q minkowski space,” *Commun. Math. Phys.* **174** (1995) 457–476, `hep-th/9404054`.

[35] P. Podles, “Solutions of klein-gordon and dirac equations on quantum minkowski spaces,” *Commun. Math. Phys.* **181** (1996) 569–586, `q-alg/9510019`.

[36] C. Blohmann, “Spin in the q-deformed poincare algebra,” `math.qa/0111008`.

[37] R. U. Sexl and H. K. Urbantke, “Relativity groups, particles. special theory of relativity as the basis of field and elementary particle physics. (in german),”. Wien 1976, 301p.

[38] C. Bauer and H. Wachter, “Operator Representations on Quantum Spaces,” `math-ph/0201023`.

[39] N. Z. Iorgov and A. U. Klimyk, “Harmonics on the quantum euclidean space related to the quantum orthogonal group,” `math.QA/0302119`.

[40] H. Wachter and M. Wohlgenannt, “$*$-Products on Quantum Spaces,” *Eur. Phys. J.* **C23** (2002) 761–767, `hep-th/0103120`.

[41] S. Schraml, “Untersuchung nichtkommutativer Räume als Grundlage für physikalische Modelle,”. Ph.D. thesis, Ludwig-Maximilians-Universität München, Fakultät für Physik (2000), www.theorie.physik.uni-muenchen.de/∼schraml/paper/thesis.ps.gz.

[42] C. Blohmann, "Free q-deformed relativistic wave equations by representation theory," hep-th/0111172.

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