Mean field limits for charged particles

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Hiermit erkläre ich an Eides statt, dass ich die Dissertation selbstständig und ohne unerlaubte Beihilfe angefertigt habe. Das Kapitel 3 enthält Ergebnisse, die in Zusammenarbeit mit einem Mitautor erzielt wurden. Die entsprechende Veröffentlichung ist am Kapitelanfang vermerkt.

Dustin Lazarovici,

München, den 26.01.2016

Contents

Abstract

The aim of this thesis is to provide a rigorous mathematical derivation of the Vlasov-Poisson equation and the Vlasov-Maxwell equations in the large N limit of interacting charged particles. We will extend a method previously proposed by Boers and Pickl to perform a mean field limit for the Vlasov-Poisson equation with the full Coulomb singularity and an N-dependent cut-off decreasing as $N^{-1/3+\epsilon}$. We will then discuss an alternative approach, deriving the Vlasov-Poisson equation as a combined mean field and point-particle limit of an N-particle Coulomb system of extended charges. Finally, we will combine both methods to prove a mean field limit for the relativistic Vlasov-Maxwell system in 3+1 dimensions. In each case, convergence of the empirical measures to solutions of the corresponding mean field equation can be shown for typical initial conditions. This implies, in particular, the propagation of chaos for the respective dynamics.

Zusammenfassung (Translation of the Abstract)

Ziel dieser Arbeit ist eine mathematische präzise Herleitung der Vlasov-Poisson Gleichung und der Vlasov-Maxwell Gleichungen als *mean field* Limes wechselwirkender Ladungen. Zu diesem Zweck erweitern wir zunächst eine Methode von Boers und Pickl auf den Coulomb-Fall mit einer N-abhändigen Regularisierung, die wie $N^{-1/3+\epsilon}$ abfällt. Damit beweisen wir einen mean field Limes für das Vlasov-Poisson System. Anschließend präsentieren wir einen alternativen Beweis und leiten das Vlasov-Poisson System als kombinierten mean field und Punktteilchen-Limes eines N-Teilchen Coulomb-System ausgeschmierter Ladungen her. Schließlich kombinieren wir beide Methoden, um den mean field Limes für das relativistische Vlasov-Maxwell Systems in 3+1 Dimensionen durchzuführen. Die Konvergenz der empirischen Dichten gegen Lösungen der entsprechenden kinetischen Gleichung wird jeweils für typische Anfangsbedingungen gezeigt. Dies impliziert inbesondere molekulares Chaos für die jeweiligen Dynamiken.

Keywords: Derivation of kinetic equations. Particle methods. Particle approximation. Vlasov equations. Validity problem. Molecular chaos. Propagation of chaos. Electrodynamics. Rigid charges. Typicality.

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Chapter 1

Introduction

This work is about consistency between the microscopic and macroscopic level of physical description. More precisely, it is about the microscopic justification of kinetic equations with electromagnetic interactions, particularly the Vlasov-Poisson and Vlasov-Maxwell equations.

Both these equations have been known and successfully used in physics for many decades to provide an effective, macroscopic description of a collisionless plasma of charged (or gravitating) particles. They are the kind of equations that a well-trained physicist could easily guess. Yet, a precise mathematical derivation from first principles has been an open problem, so far. In this work, we will present some results with a suitable microscopic regularization that vanishes in the limit of large particle numbers.

That a rigorous treatment of large particle systems – at least in some relevant limiting cases – is possible at all, is ultimately a testimony to the power and beauty of probabilities. It is relatively easy to provide an analytic description of 2 interacting particles. It is extremely difficult for 3 particles. It is practically impossible for 10 particles. However, as soon as we consider systems consisting of billions or trillions of particles, we begin to discover typical regularities that give rise to new kinds of effective laws.

1.1 The microscopic equations

The starting point of our investigation are the classical Newtonian dynamics of N identical particles in d dimensions given by

$$\begin{cases} \dot{q}_i = \frac{1}{m}\, p_i \\ \dot{p}_i = \alpha \sum\limits_{j\neq i} k(q_i - q_j). \end{cases} \tag{1.1}$$

Here, q_i and p_i denote the position and momentum of the i'th particle, m is the particle mass and $\alpha > 0$ a coupling constant comprising all other relevant constants. The force k describes a pair-interaction among particles. We will consider, in particular, the Coulomb kernel

$$k(q) = -\nabla \frac{\sigma}{|q|^{d-2}} = \sigma \frac{q}{|q|^d}, \quad \sigma \in \{\pm 1\} \tag{1.2}$$

where $\sigma = +1$ corresponds to the electrostatic force between equally charged particles, while the attractive case $\sigma = -1$ describes Newtonian gravitation. Here, we have introduced the potential

$$V(q) = \frac{\sigma}{|q|^{d-2}}. \tag{1.3}$$

Of course, much more general interactions are conceivable, in principle: mixed species of particles, force-kernels depending on three or more particles, forces depending on the particle velocities, stochastic terms, external potentials, first order dynamics, etc. In particular, when we discuss the relativistic Vlasov-Maxwell system, k will be replaced by the Lorentz force generated by an electromagnetic field which enters the physical description as independent degrees of freedom. Also, more general velocitiy-momentum relations $\dot{q} = v(p)$ can be considered, in particular $v(p) = \frac{p}{\sqrt{c^2+p^2}}$ for special relativity, where c is the speed of light. However, for the sake of simplicity, only the standard Newtonian setting will be discussed in this introduction.

The mean field scaling. As we want to approximate a kinetic equation of the Vlasov type, we consider the system (1.1) in the so-called *mean field scaling* where $\alpha \sim \frac{1}{N}$, so that the total mass / charge of the system remains of order 1. This requires a corresponding rescaling of time, position and momentum. To ensure that the initial data $Z = (x_i(0), p_i(0))_{1\leq i\leq N}$ remains of order 1, it is convenient to consider rescaled time- and momentum coordinates such that $p_i = N^{1/2}\overline{p}_i$, $t_i = N^{-1/2}\overline{t}_i$. Setting all physical constants (including the particle mass) to 1, the microscopic equations thus read

$$\begin{cases} \dot{q}_i = p_i \\ \dot{p}_i = \frac{1}{N} \sum\limits_{j\neq i} k(q_i - q_j). \end{cases} \tag{1.4}$$

One motivation for this particular scaling is the *Virial theorem* which states that, for homogeneous k, the long-time averages of the total kinetic energy $E_{kin} = \frac{1}{2}\sum\limits_{i=1}^{N} p_i^2$ and the potential energy $E_{pot} = \sum\sum\limits_{1\leq i<j\leq N} V(q_i - q_j)$ are of the same order (see e.g. [38, Ch. I §10]). Moreover, as, for instance, Jabin [32] explains, this is the simplest scaling of the system for which one would expect an interesting behavior in the limit $N \to \infty$. If $\alpha \ll \frac{1}{N}$, the force term becomes very small and the time-evolution will be essentially free for large N. If $\alpha \gg \frac{1}{N}$, the force term becomes more and more dominating and one expects a highly complex (and possibly singular) behavior that might be heavily dependent on the details of the microscopic interactions.

Nevertheless, the $\frac{1}{N}$-scaling is just one of many possible choices and we can only hope for the large N limit to capture *some* relevant traits of the system. Other interesting scalings - which are not going to be discusses here - include, in particular, the Boltzmann-Grad limit, leading to the famous Boltzmann equation [31].

1.2 The Vlasov equation

In classical mechanics, a set of equations of the form (1.1) is generally assumed to provide a complete description of the physical system. The only problem with these equations is that they become extremely complex for large N. The number of atoms in a macroscopic system is typically of the order of Avogadro's constant, i.e. $N \sim 10^{23}$. If we want to describe a galaxy or a small cluster of galaxies (the "particles" here are stars) the order of N may still be 10^9 or higher. Solving equation (1.1) with so many degrees of freedom is virtually impossible – or at least extremely resource-intensive.

The basic idea, going back to Boltzmann, is to consider instead of the N-particle micro-state a continuous distribution function $f(t, q, p)$ that provides an efficient description of the most important (macroscopic) characteristics of the system. More precisely, for any "observable" $H(q, p)$ on the reduced phase-space $\mathbb{R}^d \times \mathbb{R}^d$ and any time t, the distribution function yields an expectation value

$$\langle H \rangle_t = \int\int H(q,p) f(t,q,p)\, \mathrm{d}q \mathrm{d}p.$$

More simply put, $f(t, q, p)$ can be thought of as a coarse-grained density of particles with position (close to) q and momentum (close to) p.

The Jeans-Vlasov equation. The *Vlasov equation* or *Jeans-Vlasov* equation is a non-linear partial differential equation defining an autonomous time-evolution for this continuous model. It was introduced by A.A. Vlasov for his work in plasma physics [69, 70] and even earlier by J.H. Jeans in the context of Newtonian stellar dynamics [33]. In the physical literature, it is also referred to as *collisionless Boltzmann equation*, see e.g. [27].

The Vlasov equation for the distribution function f_t reads:

$$\begin{aligned}
&\partial_t f + p \cdot \nabla_q f + K \cdot \nabla_p f = 0, \\
&K(t,x) = k * \rho(t,x) := \int k(x-y)\rho(t,y), \mathrm{d}y \\
&\rho(t,q) = \int f(t,q,p)\, \mathrm{d}p.
\end{aligned} \tag{1.5}$$

The marginal $\rho_t = \rho[f_t]$ is the charge- (or mass) density induced by the distribution f_t and K is the mean field force generated by this density. If k is the Coulomb kernel, the corresponding Vlasov equation is known as the *Vlasov-Poisson equation* (or Vlasov-Newton in the gravitational case.)

While the Vlasov equation may look complicated at first, its physical meaning is easy to understand. The Vlasov equation is a transport equation. An initial distribution f_0 is transported with an effective flow on the reduced phase-space $\mathbb{R}^d \times \mathbb{R}^d$, which, in turn, is generated by the mean field force $K = K[f_t]$. More precisely, let f_t be a solution of (1.5) and $\varphi_{t,s} = (Q_{t,s}, P_{t,s})$ the solution of

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} Q_{t,s} = P_{t,s} \\ \frac{\mathrm{d}}{\mathrm{d}t} P_{t,s} = k * \rho[f_t](Q_{t,s}) \\ Q(s,s,q_0,p_0) = q_0 \\ P(s,s,q_0,p_0) = p_0. \end{cases} \tag{1.6}$$

Then it holds that

$$f_t = \varphi_{t,s} \# f_s, \ \forall 0 \leq s \leq t \leq T, \tag{1.7}$$

where, $\varphi(\cdot)\#f$ denotes the image-measure of f under φ, defined by $\varphi\#f(A) = f(\varphi^{-1}(A))$ for any Borel set $A \subseteq \mathbb{R}^{2d}$. $\varphi_{t,s}$ is called the *characteristic flow* of the Vlasov equation.

Since the vector field $(p, k * \rho(q))$ is divergence free on (q,p)-space, the Vlasov evolution has several nice conservation properties along strong solutions. In particular, all L^p norms are conserved, that is $\|f(t)\|_p = \|f_0\|_p$. For $p = 1$, this is the conservation of mass: $\int \rho_t \,\mathrm{d}q = \int \rho_0 \,\mathrm{d}q = \int f_0(p,q)\mathrm{d}q\mathrm{d}p$, where f_0 is usually normalized to total mass one.

Of course, when the kernel k contains a singularity, the existence of solutions to either the mean field or the characteristic equation is anything but obvious. We will cite the pertinent results in due course.

1.3 Deriving mean field equations

Now, what does it mean to *derive* a Vlasov equation? That is, in what sense can we *prove* that the function f_t, evolving according to (1.5), provides a good effective description of the N-particle system (1.4)?

Convergence of empirical density. One possibility to make this precise, is to consider the *microscopic* or *empirical* density corresponding to the N-particle micro-state. That is, let $Z(t) = \big(q_1(t), p_1(t), ..., q_N(t), p_N(t)\big) \in \mathbb{R}^{6N}$ denote the configuration of the N-particle system at time t. Then the empirical density is given by

$$\mu_t^N = \mu^N[Z(t)] := \frac{1}{N} \sum_{i=1}^N \delta(\cdot - q_i(t))\delta(\cdot - p_i(t)). \tag{1.8}$$

(Note that the particles here are assumed to be "identical", i.e. the microscopic density is invariant under permutation of the particles.) A sequence of such microscopic densities can approximate a continuous density f_t in the sense that

$$\lim_{N\to\infty} \int\int H(q,p)\, \mu_t^N(q,p)\mathrm{d}q\mathrm{d}p = \int\int H(q,p)\, f_t(q,p)\mathrm{d}q\mathrm{d}p$$

for any bounded and continuous H. We then say that μ_t^N converges weakly to f_t and write $\mu_t^N \rightharpoonup f_t^N$. This weak convergence gives precise meaning to the continuum limit of a singular measure as (1.8).

Now, we may hope to prove a statement of the following kind:

If at time $t = 0$ we consider a sequence of initial configurations $Z = (q_i, p_i)_{1\leq i\leq N}$ such that $\mu^N[Z] = \frac{1}{N}\sum\limits_{i=1}^{N} \delta_{q_i}\delta_{p_i} \rightharpoonup f_0$ then at later times $t > 0$, it holds that $\mu^N[Z(t)] \rightharpoonup f_t$, where $Z(t)$ is the solution of the rescaled microscopic dynamics (1.4) and f_t a solution of the corresponding Vlasov equation (1.5).

$$\begin{tikzcd} \mu_0 \arrow[r, "N\to\infty"] \arrow[d, "\substack{\text{microscopic}\\ \text{time-evolution}}"'] & f_0 \arrow[d, "\substack{\text{mean field}\\ \text{time-evolution}}"] \\ \mu_t \arrow[r, "N\to\infty"] & f_t \end{tikzcd} \tag{1.9}$$

This hope is further sustained by the following observation: Suppose that the microscopic dynamics contain no self-interaction, i.e. $k(0) = 0$. Then, the N-particle force in the mean field scaling can be written as

$$\frac{1}{N}\sum_{i\neq j} k(q_i(t) - q_j(t)) = k * \rho[\mu^N[Z(t)]](q_i(t)).$$

It is then straightforward to check that $Z(t)$ is a solution of (1.4) if and only if $\mu_t^N = \mu^N[Z(t)]$ solves (1.5) in the sense of distributions. One can thus expect that the density still satisfies the same equation as one passes to the continuum limit in the sense explained above.

Molecular Chaos. The second point of view is concerned with random initial conditions rather than deterministic ones. In other words, it is concerned with distributions on the N-particle phase-space, corresponding to ensembles of systems, rather than distributions on the reduced phase-space, pertaining to the description of one particular system.

Suppose that at $t = 0$ the particles are identically and independently distributed according to the law f_0, that is, we consider the product-measure $F_0^N = \otimes^N f_0$ on $\mathbb{R}^{6N}$. Let $\Psi_{t,0}$ be the N-particle flow generated by the microscopic dynamics (1.4) and $F_t^N := \Psi_{t,0}\#F_0^N$. Then $F_t^N(q_1, p_1, ..., q_N, p_N)$ describes the distribution of (ensembles of) particles on phase space at time t. One checks that it is a solution to the *Liouville equation*

$$\partial_t F_t^N + \sum_{i=1}^{N} p_i \cdot \nabla_{q_i} F_t^N + \sum_{i=1}^{N} \frac{1}{N}\sum_{i\neq j} k(q_i - q_j)\cdot \nabla_{p_i} F_t^N. \tag{1.10}$$

Now one would like to show that under this time-evolution, the particles remain "approximately independent" with $F_t^N \approx \otimes^N f_t$, where f_t is the solution of the Vlasov equation. Formally, this approximation is understood in terms of the convergence of marginals. Writing $z_i = (q_i, p_i)$, we define for $k \in \mathbb{N}$ the reduced k-particle marginal

$${}^{(k)}F_t^N(z_1, ..., z_k) := \int F_t^N(Z)\, \mathrm{d}^3 z_{k+1} ... \mathrm{d}^3 z_N. \tag{1.11}$$

Then we want to show that

$$^{(k)}F^N_t \rightharpoonup \otimes^k f_t, \quad N \to \infty. \tag{1.12}$$

(In fact, it suffices that the convergence holds for $k \leq 2$). This property is known as *molecular chaos* or *Kac's chaos*. By a well-known result of probability theory, molecular chaos is equivalent to the convergence *in law* of the empirical measures $\mu^N_t[Z] = \mu^N[\Psi_{t,0}(Z)]$ against the constant variable f_t. (E.g. Kac, 1956 [34], Grünbaum, 1971 [23], Sznitman, 1991, [66, Prop. 2.2], see [48] for recent quantitative results). In other words, molecular chaos is equivalent to convergence of the empirical measures for *typical* initial conditions. In particular, it holds that

$$\lim_{N\to\infty} \mathbb{P}^N_0\Big[Z \in \mathbb{R}^{6N} : \Big|\int h(z)\mu^N_t[Z](z)\mathrm{d}z - \int h(z) f_t(z)\mathrm{d}z\Big| > \epsilon\Big] = 0 \tag{1.13}$$

for any $\epsilon > 0$ and any bounded, continuous test-function h, where the probability $\mathbb{P}^N_0$ is defined in terms of $F^N_0 = \otimes^N f_0$.

1.4 Classical results

To my knowledge, the first paper to discuss a mathematically rigorous derivation of Vlasov equations is Neunzert and Wick, 1974 [50]. Better known are the publications of Braun and Hepp, 1977 [10] and Dobrushin, 1979 [15], as well as the later exposition of Neunzert, 1984 [49]. For a general overview of the topic, we refer the reader to the book of Spohn [65], the survey article of Kiessling [35], as well as the lecture notes of Jabin [32] and Golse [22] that I have found to be very helpful.

The results of Neunzert, Braun-Hepp and Dobrushin are all of the first, deterministic kind. Rather than the Vlasov-Poisson equation, they treat simplified models with Lipschitz-continuous forces $k \in W^{1,\infty} = \{k \in C^1(\mathbb{R}^d) : \|k\|_\infty + \|\nabla k\|_\infty < \infty\}$. For instance, one can think of replacing the singular Coulomb potential (1.3) by a regularized variant like (in the 3-dimensional case)

$$V(x) = \frac{\sigma}{\sqrt{x^2+\epsilon^2}}, \ \epsilon > 0. \tag{1.14}$$

The strategy of proof can then be summarized as follows: choose an appropriate distance metrizing weak convergence of probability measures (e.g. the bounded Lipschitz metric in [10] or the Wasserstein metric in [15]) and establish a bound of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} dist(\mu^N_t, f_t) \leq C \ dist(\mu^N_t, f_t). \tag{1.15}$$

Then one concludes with Gronwall's lemma that

$$dist(\mu^N_t, f_t) \leq e^{tC} dist(\mu^N_0, f_0), \tag{1.16}$$

so that convergence of the empirical measure at the initial time implies convergence of the empirical measure at later times.

On the one hand, these proofs capture well the basic intuition behind the scheme (1.9): As long as $\mu^N_t \approx f_t$ in the weak topology, one hopes that $K[\mu_t] \approx K[f_t]$, in some (stronger) sense. Hence, microscopic time-evolution and mean field time evolution will be "close" in some sense. Hence, μ_t^N and f_t remain close as probability measures - and so on and so forth. On the other hand, the mollified interactions studied in the aforementioned papers also turned out to be deceptive, in some respect. Formally, the problem is that the Lipschitz constant in (1.15) will depend on the size of the cut.off, $C = C(\epsilon)$, in such a way that $C(\epsilon) \to \infty$ as $\epsilon \to 0$.

Note that it is always possible to choose an N-dependent cut-off like $\epsilon(N) \sim \log(N)^{-1}$ (if $C(\epsilon) \sim \epsilon$) with sufficiently small constants so that the right-hand-side (1.16) converges, see e.g. [19]. However, the scale of the regularization is then so large compared to the typical distance between neighboring particles ($\sim N^{-1/d}$) that it captures only very long-range characteristics of the original dynamics.

In any more satisfying sense, generalizing the results of Neunzert, Braun-Hepp and Dobrushin to more realistic systems proved to be problematic. Indeed, the understanding that has grown in recent years is that such deterministic statements are actually too strong for singular interactions, the reason being that there exist "bad" initial conditions leading to clustering of particles and hence to significant deviations from the typical mean field behavior. The best we can hope for is to prove convergence of the empirical measure for *typical* initial conditions, i.e. the propagation of chaos. This becomes apparent, for instance, in the works of Hauray and Jabin [25, 26] and is also one of the basic insight behind the present thesis.

1.5 Recent results for singular forces

On a conceptual note, I believe it's important to appreciate the fact that the probabilistic character of such results is not primarily a matter of ignorance or limited accuracy of observation. Certainly, whether parts of nature can be described – at least approximately – by a particular mathematical equation cannot depend on what we know or don't know about the respective systems. Rather, the validity of the macroscopic equation is ultimately *explained* by the fact that the mean field approximation is applicable to *typical* systems and fails only for extremely special configurations of particles. See [40] for a detailed conceptual discussion of typicality.

For this reason, results for particles initially arranged on a regular mesh (e.g. [73], [3]) are relevant to certain numerical experiments, but less for explaining the validity of the mean field approximation as referring to real-life physical systems. The situation is somewhat similar with respect to the recent result of Kiessling, 2014 [36], who proves a (non-quantitative) approximation for mean field equations including the Coulomb singularity under the assumption of a uniform bound on the microscopic forces, but leaves open whether or not this assumption is satisfied for a statistically relevant subset of initial conditions.

In contrast, the strategy employed by Hauray and Jabin in [26] is to impose additional constraints on the initial configurations, subsequently showing that the set of "good" initial conditions, for which these constraints are satisfied, approaches measure 1 as $N \to \infty$. In this way, the authors are able to treat systems with singular potentials up to – but not including – the Coulomb case.

More precisely, they consider force kernels bounded like $|k(q)| \leq \frac{C}{|q|^\alpha}$ with $\alpha < d-1$ in dimension $d \geq 3$. For $1 < \alpha < d-1$ they require an N-dependent cut-off which can be chosen as small as $N^{-1/2d}$ for $\alpha \nearrow d-1$, while for $\alpha < 1$, they are able to prove molecular chaos with no cut-off at all. The results of Hauray and Jabin marked a significant advancement in the derivation of Vlasov-type equations and it is a pitty that the method fails precisely at the Coulomb threshold $\alpha = d - 1$.

Recently, Boers and Pickl proposed a new method for deriving mean field equations which is designed for stochastic initial conditions, thus aiming directly at a typicality result [6]. This method captures nicely the intuition of mean field approximations as *law of large numbers* results and allowed to improve the cut-off width for $\alpha < d - 1$ to $N^{-1/d}$, corresponding to the typical nearest-neighbor distance in d-dimensional space.

1.6 Aim of this work

One of the main goals of this thesis is to generalize the method of Boers and Pickl to include the Coulomb singularity, thus proving a mean field limit for the Vlasov-Poisson equation. In brief, this will be achieved by exploiting the second order nature of the equation, introducing an anisotropic N-dependent metric that weighs spatial- and momentum coordinates differently.

Afterwards, we will propose an alternative approach, deriving the Vlasov-Poisson equation as a combined mean field and point-particle limit of an N-particle Coulomb system of extended charges. This proof is based on controlling the Wasserstein distance between microscopic density and mean field density, thus showing how Dobrushin's method can, after all, be extended to singular forces with an N-dependent cut-off decreasing much faster than logarithmic. Moreover, this alternative approximation of the Vlasov-Poisson dynamics is interesting in view of the Vlasov-Maxwell problem, because it treats, as a microscopic model, the nonrelativistic analogue of the Abraham model of rigid charges that we are going to use as a regularization of the field dynamics in the relativistic case. In the end, we want to combine both methods, developed and tested for Vlasov-Poisson, into a derivation of the 3-dimensional relativistic Vlasov-Maxwell system.

The Vlasov-Maxwell system is, simply put, the electrodynamic Vlasov theory, including the Vlasov-Poisson equation as its nonrelativistic limit. It describes a collisionless plasma of identical charged particles, interacting through a self-consistent electromagnetic field. The analogue for gravitational interactions are the Vlasov-Einstein equations, not treated in this thesis. Explicitly, the Vlasov-Maxwell system consist in the following set of equations:

$$\begin{aligned} &\partial_t f + v(\xi) \cdot \nabla_x f + K(t,x,\xi) \cdot \nabla_\xi f = 0, \\ &\partial_t E - \nabla_x \times B = -j, \qquad \nabla_x \cdot E = \rho, \\ &\partial_t B + \nabla_x \times E = \ 0, \qquad \nabla_x \cdot B = 0, \end{aligned} \tag{1.17}$$

where

$$v(\xi) = \frac{\xi}{\sqrt{1 + |\xi|^2}} \tag{1.18}$$

is the relativistic velocity of a particle with momentum ξ,

$$\rho(t,x) = \int f(t,x,\xi)\,\mathrm{d}\xi, \quad j(t,x) = \int v(\xi) f(t,x,\xi)\,\mathrm{d}\xi \tag{1.19}$$

are the charge- and current density induced by the distirbution f_t and

$$K(t,x,\xi) = E(t,x) + v(\xi) \times B(t,x) \tag{1.20}$$

is the Lorentz force acting at time t on a particle at x with velocity $v(\xi)$.

The mean field limit for Vlasov-Maxwell is considerably more complex than the electrostatic case, as it involves relativistic (retarded) interactions and the electromagnetic field as additional degrees of freedom. However, we will show that the basic insights and techniques developed for the Vlasov-Poisson equation carry over to the relativistic regime. In view of the rigid charges model, the cut-off parameter has a straightforward physical interpretation in terms of a finite electron-radius which will formally decrease with N.

A previous result for the Vlasov-Maxwell system was recently obtained by Golse, who performed the mean field limit for a *regularized* version dynamics, i.e. with a fixed cut-off, similar to what Braun-Hepp, Dobrushin and Neunzert did for the Vlasov-Poisson equation [21]. In the spirit of the recent developments in the Vlasov-Poisson case, outlined above, our aim is to proof a mean field limit for the actual Vlasov-Maxwell equations by using an N-dependent cut-off which decreases as $N^{-1/12}$. I thus believe that this result constitutes significant progress in regard to the microscopic justification of the Vlasov-Maxwell dynamics.

Chapter 2

The Wasserstein distances

In this chapter, we recall the definition as well as some basic facts and applications of the *Wasserstein distances*, also known as *Monge-Kantorovich-Rubinstein* distances. The Wasserstein distances are intimately connected to the problem of *optimal transpartation*; in the context of kinetic equations, they were first introduced by Dobrushin [15]. For more details and proofs, we refer the reader to the book of Villani [68]. This chapter is only preparatory and does not include new results.

2.1 Definition and basic properties

We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of probability measures on $\mathbb{R}^n$ equipped with its Borel algebra. If $(\mu_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{P}(\mathbb{R}^n)$ and μ another element, we denote by $\mu_k \rightharpoonup \mu$ the weak convergence of probability measures, meaning

$$\int \phi(x)\,\mathrm{d}\mu_k(x) \to \int \phi(x)\,\mathrm{d}\mu(x), \quad k \to \infty,$$

for all bounded and continuous functions $\phi : \mathbb{R}^n \to \mathbb{R}$.

Definition 2.1.1. For given $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$ let $\Pi(\mu,\nu)$ be the set of all probability measures $\mathbb{R}^n \times \mathbb{R}^n$ with marginal μ and ν respectively, i.e.

$$\int \phi_1(x)\pi(\mathrm{d}x,\mathrm{d}y) = \int \phi_1(x)\mathrm{d}\mu(x), \ \int \phi_2(y)\pi(\mathrm{d}x,\mathrm{d}y) = \int \phi_2(y)\mathrm{d}\mu(y)$$

for ϕ_1, ϕ_2 bounded and continuous. The elements of $\Pi(\mu,\nu)$ are called *couplings* or *transference plans* between μ and ν.

For $p \in [1,\infty)$ we define the *Wasserstein distance* of order p by

$$W_p(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \Big(\int\limits_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^p \,\mathrm{d}\pi(x,y) \Big)^{1/p}. \tag{2.1}$$

The value might be infinite, unless one demands that μ and ν have finite p'th moments.

In the context of *optimal transportation*, $|x-y|^p$ is called the *cost function* and could be replaced by a more general expression $c(x,y)$. The problem of minimizing the right-hand-side of (2.1) then corresponds to finding an *optimal transference plan* for shifting a distribution (of mass or goods of some sort) μ to a distribution ν if the cost of transportation is given by $c(x,y)$.

In view of (2.1), a direct application of Hölder's inequality yields the relation

$$p \leq q \;\Rightarrow\; W_p(\mu,\nu) \leq W_q(\mu,\nu).$$

In general, a higher order means that large distances in $\mathbb{R}^n$ become more and more "costly". We can complete the analogy to the L^p-hierarchy by introducing the *infinite Wasserstein distance* defined as

$$W_\infty(\mu,\nu) = \inf\big\{\pi - esssup\,|x-y| \mid \pi \in \Pi(\mu,\nu)\big\}. \tag{2.2}$$

Turning back to the cases $p \in [1,\infty)$, a central result is the *Kantorovich-Rubinstein duality.* For $p \in [1,\infty)$:

$$\begin{aligned} W_p^p(\mu,\nu) = \sup\Big\{ \int \phi_1(x)\,\mathrm{d}\mu(x) - \int \phi_2(y)\,\mathrm{d}\nu(y) : \\ (\phi_1,\phi_2) \in L^1(\mu)\times L^1(\nu), \phi_1(y)-\phi_2(x) \leq |x-y|^p \Big\}. \end{aligned} \tag{2.3}$$

Much of the power of the Wasserstein distance lies in this duality formula. It establishes two equivalent characterizations of W_p, one involving an infimum and one involving a supremum. This allows us to switch between one and the other, depending on whether we want to establish upper or lower bounds.

For any integrable function ϕ and $p \in [1,\infty)$ we define its *c-conjugate* by

$$\phi^c(y) := \sup_x\{\phi(x) - |x-y|^p\}. \tag{2.4}$$

One easily verifies that this is the smallest function satisfying $\phi(x)-\phi^c(y) \leq |x-y|^p$, $\forall x,y \in \mathbb{R}^n$. Hence, the Kantorovich duality formula becomes

$$W_p^p(\mu,\nu) = \sup_{\phi\in L^1(\mu)}\Big\{ \int \phi(x)\,\mathrm{d}\mu(x) - \int \phi^c(y)\,\mathrm{d}\nu(y)\Big\}. \tag{2.5}$$

The most common variant is the first Wasserstein distance, for which the problem further reduces to

$$W_1(\mu,\nu) = \sup_{\|\phi\|_{Lip}\leq 1}\Big\{\int \phi(x)\,\mathrm{d}\mu(x) - \int \phi(x)\,\mathrm{d}\nu(x)\Big\}, \tag{2.6}$$

where $\|\phi\|_{Lip} := \sup\limits_{x\neq y} \frac{\phi(x)-\phi(y)}{|x-y|}$, to be compared with the *bounded Lipschitz distance*

$$d_{BL}(\mu,\nu) = \sup\Big\{\int \phi(x)\,\mathrm{d}\mu(x) - \int \phi(x)\,\mathrm{d}\nu(x) \;:\; \|\phi\|_{Lip} = \|\phi\|_\infty = 1\Big\}. \tag{2.7}$$

Since the class of test-functions is smaller for the bounded Lipschitz metric, we have $d_{BL} \leq W_1$, which shows that the Wasserstein distances (with respect to the Euclidean norm) are relatively strong. Indeed, convergence in Wasserstein distance implies not only weak convergence, but also convergence of the first p moments, that is:

$$W_p(\mu_k, \mu) \to 0 \iff$$
$$\mu_k \rightharpoonup \mu \text{ and } \lim_{k\to\infty} \int |x|^p \, \mathrm{d}\mu_k(x) = \int |x|^p \, \mathrm{d}\mu(x).$$

More formally, one can introduce the $p-$th *Wasserstein space* $\mathcal{P}_p(\mathbb{R}^n) \subset \mathcal{P}(\mathbb{R}^n)$ as the set of probability measures with finite p-th moment, that is

$$\mathcal{P}_p(\mathbb{R}^n) = \big\{\mu \in \mathcal{P}(\mathbb{R}^n) : \int |x|^p \, \mathrm{d}\mu < \infty\big\}. \tag{2.8}$$

We say that μ_k converges weakly to μ in $\mathcal{P}_p(\mathbb{R}^n)$ if $\int \varphi(x)\mathrm{d}\mu_k(x) \to \int \varphi(x)\mathrm{d}\mu(x)$ for all continuous φ with $\varphi(x) \leq (1 + |x|^p)$. Then W_p metrizes the topology of weak convergence on $\mathcal{P}_p(\mathbb{R}^n)$. In particular, one checks with a little bit of effort that W_p is indeed a (finite) metric on $\mathcal{P}_p(\mathbb{R}^n)$. (On $\mathcal{P}(\mathbb{R}^n)$, the Wasserstein distances also satisfy all properties of a metric, except they can take the value $+\infty$.)

Sometimes it is also convenient to replace the Euclidean norm by a bounded metric on $\mathbb{R}^n$, e.g. $d(x, y) := \min\{1, |x - y|\}$. Then the Wasserstein distances for this metric d generate the usual weak* topology in $\mathcal{P}(\mathbb{R}^n)$ and the first Wasserstein distance is equivalent to the bounded Lipschitz distance.

2.2 Large deviations

A question that we will repeatedly encounter throughout our further discussion is the following: Suppose we pick N points $x_1, ..., x_N$ randomly and independently according to the law f. How fast will the empirical density $\mu^N[X] = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$ typically approximate f, if the difference is measured in a Wasserstein distance?

It is a classical result - known as the empirical law of large numbers, Varadarajan's theorem or Glivenko-Cantelli theorem - that $\mu^N[X]$ converges to f in probability. Establishing quantitative bounds on large deviations (concentration estimates) is, however, a longstanding problem in probability theory with a vast amount of literature. To my knowledge, one of the first paper to address this question in the context of Wasserstein metrics was Bolley, Guillin, Villani, 2007 [8]. Subsequently, other authors have derived stronger concentration estimates, see, in particular, [7] and [13]. Very recently, great progress has been made in the paper of Fournier and Guillin, 2014 [17] which considerably improves upon previous results, both in strength and generality. In fact, the results can be shown to be almost optimal in many cases. Maybe more importantly for us, the assumptions on the law f are much weaker and easier to check than in the aforementioned publications. We will cite here the concentration estimates of Fournier and Guillin [17, Thm. 2] and apply them on various occasions throughout our further discussion.

Theorem 2.2.1 (Fournier and Guillin). *Let $f \in \mathcal{P}(\mathbb{R}^n)$ and $p \in (0,\infty)$. For $q > 0, \kappa > 0$, and $\gamma > 0$, consider*

$$M_q(f) := \int_{\mathbb{R}^n} |x|^q \mathrm{d}f(x); \quad E_{\kappa,\gamma}(f) := \int e^{\gamma|x|^\kappa} \mathrm{d}f(x).$$

Assume one of the following three conditions:

(1) $\exists \kappa > p, \gamma > 0 : E_{\kappa,\gamma}(f) < +\infty$

(2) $\exists \kappa \in (0,p), \gamma > 0 : E_{\kappa,\gamma}(f) < +\infty$

(3) $\exists q > 2p : M_q(f) < +\infty$

Let $(x_i)_{i=1,\dots,N}$ be a sample of independent variables, distributed according to the law f and $\mu^N[X] := \sum_{i=1}^N \delta_{x_i}$. Then, for all $N \geq 1$ and $\xi \in (0,\infty)$:

$$\mathbb{P}\big[W_p^p(\mu^N[X], f) > \xi\big] \leq a(N,\xi)\mathbb{1}_{\xi\leq 1} + b(N,\xi)$$

with

$$a(N,\xi) := C \begin{cases} \exp(-cN\xi^2) & \text{if } p > n/2 \\ \exp(-cN(\frac{\xi}{\ln(2+1/\xi)})^2) & \text{if } p = n/2 \\ \exp(-cN\xi^{n/p}) & \text{if } p \in [1, n/2) \end{cases}$$

and

$$b(N,\xi) := C \begin{cases} \exp(-cN\xi^{\frac{\kappa}{p}})\mathbb{1}_{\xi>1} & \text{under } (1) \\ \exp(-c(N\xi)^{\frac{\kappa-\epsilon}{p}})\mathbb{1}_{\xi\leq 1} + \exp(-c(N\xi)^{\frac{\kappa}{p}})\mathbb{1}_{\xi>1} & \forall \epsilon \in (0,\kappa) \text{ under } (2) \\ N(N\xi)^{-\frac{q-\epsilon}{p}} & \forall \epsilon \in (0,q) \text{ under } (3) \end{cases}$$

The positive constants C and c depend only on p, n and either $\kappa, \gamma, E_{\kappa,\gamma}(f)$ (under assumption (1)) or $\kappa, \gamma, E_{\kappa,\gamma}(f), \epsilon$ (under (2)) or on $q, M_q(f), \epsilon$ (under (3)).

Discussing the rather intricate proof in more detail would go far beyond the scope of this thesis. Very briefly put, the strategy of Fournier and Guillin involves 3 steps. First, large deviation estimates are derived for the case were f has compact support and the fixed sample size N is replaced by a Poisson(N)-distributed random variable with intensity measure Nf, which yields some useful independence properties. In a second step, one removes this randomization by using the fact that, for large N, a Poisson(N)-distributed random variable is concentrated around N with high probability. Finally, one has to extend the estimates to the non-compact case by summing over a sequence of nested, disjoint sets with increasing support and exploiting the decay properties of f.

2.3 Stability of the Coulomb force

As mentioned in the introduction, the classical mean field results are essentially stability results of the form

$$\|k * \rho_1 - k * \rho_2\| \lesssim dist(\rho_1, \rho_2), \tag{2.9}$$

where $dist(\cdot,\cdot)$ represents an appropriate distance between probability measures and $\|\cdot\|$ some stronger (usually L^p) norm. In particular, if the kernel k has bounded derivative, one immediately concludes with (2.6) that $\|k * \rho_1 - k * \rho_2\|_\infty \leq \|k\|_{Lip} W_1(\rho_1, \rho_2)$. This inequality is at the core of Dobrushin's mean field approximations [15], which amount to the Gronwall bound

$$W_1(\mu_t^N, f_t) \leq e^{t(1+2\|\nabla k\|_\infty)} W_1(\mu_0^N, f_0) \tag{2.10}$$

for μ_t^N the microscopic density and f_t the Vlasov density solving the corresponding Vlasov equation.

Unfortunately, generalization to less benign interactions is difficult and will in general require additional regularity assumptions on ρ_i. For instance, when $k = \frac{x}{|x|^d}$ is the Coulomb kernel (and ρ_1, ρ_2 have compact support) one would maybe like to exploit an inequality of the form $\|k * (\rho_1 - \rho_2)\|_2 \leq C\|\rho_1 - \rho_2\|_2$.[1] So we try:

$$\int (\rho_1 - \rho_2)(\rho_1 - \rho_2)\mathrm{d}q \leq \|\rho_1 - \rho_2\|_{Lip}\, W_1(\rho_1, \rho_2)$$

and thus

$$\|\rho_1 - \rho_2\|_2 \leq \max\{\|\nabla\rho_1\|_\infty, \|\nabla\rho_2\|_\infty\}^{1/2} \sqrt{W_1(\rho_1, \rho_2)}, \tag{2.11}$$

which is not good enough to derive a Gronwall bound similar to (2.10) (even if we assumed that ρ_1 and ρ_2 had bounded derivatives – which is emphatically not the case for a point charge density).

However, if we exploit the fact that the Coulomb kernel k is generated by a potential solving Poisson's equation, we gain just enough regularity to derive a linear bound with respect to the second Wasserstein distance. This is due to an ingenious argument by Loeper [42, Theorem 2.9], which we recall in the following.

Definition 2.3.1. For any measurable function T, we denote by $T\#\mu$ the *push-forward* (image meausure) of μ by T defined by $T\#\mu(A) = \mu(T^{-1}(A))$ for any Borel set $A \subseteq \mathbb{R}^n$. A measurable function $T : \mathbb{R}^d \to \mathbb{R}^d$ is called a *deterministic coupling* or *transference map* between μ and ν if $T\#\mu = \nu$.

Now we have the following theorem:

Theorem 2.3.2. *If μ is absolutely continuous with respect to the Lebesgue measure, there exists a unique deterministic coupling such that $(Id, T)\#\mu \in \Pi(\mu, \nu)$ is optimal with respect to the quadratic cost-function, i.e.*

$$W_2(\mu, \nu) := \Big(\int_{\mathbb{R}^d} |T(x) - x|^2 \,\mathrm{d}\mu(x)\Big). \tag{2.12}$$

The original theorem is due to Brenier [11], the proof was later simplified and generalized by Gangbo and McCann [18, Theorem 1.2]. See also [68, Chapter 10] for a comprehensive discussion.

[1] In particular, one would have liked to apply such an estimate to the Vlasov-Maxwell system, exploiting the relation $\|(E_1, B_1) - (E_2, B_2)(t)\|_2 \leq \|(E_1, B_1) - (E_2, B_2)(0)\|_2 + \int \|(j_1 - j_2)(s)\|_2 \,\mathrm{d}s$.

Definition 2.3.3. Let f a bounded linear functional on the Sobolev space $H^1(\mathbb{R}^n)$. Then we consider the norm

$$\|f\|_{H^{-1}} := \sup\Big\{\int fg\,\mathrm{d}x : g \in C_c^\infty(\mathbb{R}^d),\ \int |\nabla g|^2 \mathrm{d}x \le 1\Big\}. \tag{2.13}$$

As a motivation for introducing this somewhat more abstract norm, we note that a) it is weaker than the L^2-norm and b) the test-functions $g \in H^1(\mathbb{R}^n)$ come with some bound on their variation, $\|\nabla g\|_2 \le 1$, to be compared with $\|\nabla g\|_\infty \le 1$ in case of W_1. This should give us some hope that it is possible to establish a bound of the form (2.9) in terms of an appropriate Wasserstein metric.

Let $\rho_1, \rho_2 \in L^1(\mathbb{R}^n)$. Let T be the optimal coupling between ρ_1 and ρ_2 with respect to the second Wasserstein distance and consider the interpolation

$$\rho_\theta = ((\theta - 1)T + (2-\theta)Id)\#\rho_1,\ \theta \in [1,2]. \tag{2.14}$$

This path has some interesting properties. For instance, the *displacement convexity* (see [47], [42, Thm. 2.6 and Cor. 2.7]) implies that

$$\|\rho_\theta\|_\infty \le \max\{\|\rho_1\|_\infty, \|\rho_2\|_\infty\},\ \forall \theta \in [1,2]. \tag{2.15}$$

Now Loeper proves the following:

Proposition 2.3.4. *Let $\rho_1, \rho_2 \in H^{-1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and ρ_θ the interpolant defined above. Then*

$$\|\rho_1 - \rho_2\|_{H^{-1}} \le \{\|\rho_1\|_\infty, \|\rho_2\|_\infty\}^{1/2}\, W_2(\rho_1, \rho_2). \tag{2.16}$$

Proof. Note that $\frac{\mathrm{d}}{\mathrm{d}\theta}\rho_\theta = \rho_2 - \rho_1$ and that for all $g \in C_c^\infty(\mathbb{R}^n)$:

$$\int \rho_\theta(x) g(x) \mathrm{d}x = \int \rho_1(x) g((\theta-1)T(x) + (2-\theta)x))\mathrm{d}x.$$

Differentiating with respect to θ yields:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\int \rho_\theta(x) g(x)\mathrm{d}x = \int \rho_1(x) \nabla g((\theta-1)T(x) + (2-\theta)x)(T(x) - x)\mathrm{d}x.$$

Applying the Cauchy-Schwartz inequality w.r.t the measure ρ_1 yields

$$\int (\rho_2 - \rho_1)(x) g(x) \mathrm{d}x \le \Big(\int \rho_\theta(x)|\nabla g(x)|^2\Big)^{1/2}\Big(\int \rho_1(x)|T(x)-x|^2\mathrm{d}x\Big)^{1/2}. \tag{2.17}$$

The second term on the right-hand side is identical to $W_2(\rho_1, \rho_2)$. Using (2.15) and taking the supremum over all $g \in C_c^\infty(\mathbb{R}^n)$ with $\|\nabla g\|_2 \le 1$, the statement follows. □

Furthermore, we have the following estimate:

Lemma 2.3.5. *Let Φ_i, $i = 1, 2$ be the solution of*

$$-\Delta\Phi_i = \rho_i,$$
$$\Phi_i(x) \to 0, |x| \to \infty,$$

and $E_i = -\nabla\Phi_i$. Then it holds that

$$\|E_1 - E_2\|_2 \le \|\rho_1 - \rho_2\|_{H^{-1}}. \tag{2.18}$$

Proof. We compute

$$\begin{aligned} \int (\rho_1 - \rho_2)(x) g(x) \, \mathrm{d}x \ &= \int \operatorname{div}(E_1 - E_2)(x) g(x) \, \mathrm{d}x \\ = - \int (E_1 - E_2)(x) \nabla g(x) \mathrm{d}x &= \int \nabla (\Phi_1 - \Phi_2)(x) \nabla g(x) \mathrm{d}x. \end{aligned}$$

Taking the supremum over all $g \in C_c^\infty(\mathbb{R}^n)$ with $\|\nabla g\|_2 \leq 1$, the inequality follows. $\square$

In total, we have derived the following result that we will use on several occasions.

Proposition 2.3.6 (Loeper). *Let k the d-dimensional Coulomb kernel and $\rho_1, \rho_2 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ two (probability) densities. Then*

$$\|k * \rho_1 - k * \rho_2\|_2 \leq \left[\max\{\|\rho_1\|_\infty, \|\rho_2\|_\infty\}\right]^{1/2} W_2(\rho_1, \rho_2). \tag{2.19}$$

Note: This result can be generalized to the less singular kernels, see e.g. [24].

Chapter 3

A mean field limit for the Vlasov-Poisson system

This chapter presents joint work with Prof. Dr. Peter Pickl and is largely copied from the paper *D. Lazarovici, P. Pickl: A mean field limit for the Vlasov-Poisson equation* which is currently under review for publication. For a preprint, see [39]. Some parts of the discussion have been modified or expended.

3.1 The Vlasov-Poisson equation

We are interested in a microscopic derivation of the nonrelativistic Vlasov-Poisson system. This equation describes a plasma of identical, charged particles with Coulomb interactions. For simplicity, we shall focus on the 3-dimensional case. Generalization to higher dimensions is straightforward and will be included in the next chapter. The Vlasov-Poisson equation reads:

$$\partial_t f + p \cdot \nabla_q f + (k * \rho_t) \cdot \nabla_p f = 0, \tag{3.1}$$

where k is the Coulomb kernel

$$k(q) := \sigma \frac{q}{|q|^3}, \quad \sigma = \{\pm 1\} \tag{3.2}$$

and

$$\rho_t(q) = \rho[f_t](q) = \int \mathrm{d}^3 p \; f(t, q, p) \tag{3.3}$$

is the charge density induced by the distribution $f(t, p, q) \geq 0$.
Units are chosen such that all constants, in particular the mass and charge of the particles, are equal to 1. The case $\sigma = +1$ corresponds to electrostatic (repulsive) interactions while $\sigma = -1$ describes gravitational (attractive) interactions. In the latter case, (3.1) is also known as the *Vlasov-Newton* equation.

While the the existence theory of the Vlasov-Poisson dynamics is pretty well understood – we will cite the pertinent results below – its microscopic derivation has been an open problem. As discussed in more detail the introductory chapter, the last few years have seen great progress in treating mean field limits for singular forces – up to, but not including the Coulomb case, see in particular Hauray and Jabin, 2013 [26] and Boers and Pickl, 2015 [6]. The aim of this chapter is to extend the method of Boers and Pickl to include the Coulomb singularity, thus aiming at a microscopic justification of the Vlasov-Poisson dynamics. The Coulomb case is qualitatively different from the previously treated interactions since the mean field force $k * \rho$ is no longer Lipschitz, in general, even if the density is bounded. However, we will show how it can be included by exploiting the second order nature of the dynamics and introducing an appropriate scaling of the relevant metrics.

3.2 The microscopic model

Since the Coulomb kernel is strongly singular at the origin, we will require a regularization on the microscopic level. We shall introduce a force kernel with an N-dependent cut-off, approximating the Coulomb interaction in the limit $N \to \infty$. Of course, the N-dependence of the force thus introduced is a technical necessity rather than a realistic physical model, though similar regularizations are commonly used in numerical computations. For $N \in \mathbb{N}$ and $\delta \geq 0$, let

$$k_\delta^N(q) := \sigma \begin{cases} \frac{q}{|q|^3} & \text{, if } |q| \geq N^{-\delta} \\ qN^{3\delta} & \text{, else.} \end{cases} \tag{3.4}$$

For $N \to \infty$ and any $\delta > 0$ this converges point-wise to the Coulomb kernel on $\mathbb{R}^3 \setminus \{0\}$, which justifies the notation $k^\infty(q) := k(q) = \frac{\sigma q}{|q|^3}$. Moreover, we note that $|k_\delta^N(q)| \leq N^{2\delta}$ and $k_\delta^N(0) = 0$. In the mean field scaling, the equations of motion for the regularized N-particle system are given by

$$\begin{cases} \dot{q}_i(t) = p_i(t) \\ \dot{p}_i(t) = \frac{1}{N} \sum\limits_{j=1}^{N} k_\delta^N(q_i - q_j), \end{cases} \tag{3.5}$$

for $i \in 1, ..., N$. Since the vector field is Lipschitz for fixed δ, N, we have global existence and uniqueness of solutions and hence an N-particle Hamiltonian flow, which we denote by ${}^N\Psi_{t,s}(Z) = \big({}^N\Psi_{t,s}^1(Z), {}^N\Psi_{t,s}^2(Z)\big) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N}$. We will often omit the index N when the particle number is fixed. Introducing the N-particle force field $K : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$ given by

$$(K(q_1, .., q_N))_i := \frac{1}{N} \sum_{j=1}^{N} k_\delta^N(q_i - q_j), \quad i = 1, .., N, \tag{3.6}$$

we can also characterize $\Psi_{t,s}$ as the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\big({}^N\Psi_{t,s}^1(Z), {}^N\Psi_{t,s}^2(Z)\big) = \big({}^N\Psi_{t,s}^2(Z), K({}^N\Psi_{t,s}^1(Z))\big), \ {}^N\Psi_{s,s}(Z) = Z. \tag{3.7}$$

Finally, if ${}^N\Psi_{t,0}(Z) = (q_i(t), p_i(t))_{i=1,..,N}$ we define the corresponding *microscopic* or *empirical density* by

$$\mu_t^N[Z] := \mu^N[\Psi_{t,0}(Z)] = \frac{1}{N} \sum_{i=1}^{N} \delta(\cdot - q_i(t))\delta(\cdot - p_i(t)). \tag{3.8}$$

Of course, more general cut-offs can be considered. In the literature, the following nomenclature has been established (see e.g. [26, 32]).

Definition 3.2.1. A pair-interaction defined by a kernel $k : \mathbb{R}^d \to \mathbb{R}^d$ satisfies a S^α-*condition*, if

$$(S^\alpha) \qquad \exists c > 0, \forall q \in \mathbb{R}^d \setminus \{0\} \;\; |k(q)| \leq \tfrac{c}{|q|^\alpha}, \;\; |\nabla k| \leq \tfrac{c}{|q|^{\alpha+1}}.$$

Introducing a cut-off of order $N^{-\delta}$ near the origin, the regularized force kernel k_δ^N satisfies a (S_δ^α)-*condition if*

$$(S_\delta^\alpha) \qquad \begin{aligned} &i) \;\; k \text{ satisfies a } (S^\alpha) \text{ condition,} \\ &ii) \;\; k_\delta^N(q) = k(q) \text{ for } |q| \geq N^{-\delta}, \\ &iii) \; |k_\delta^N(q)| \leq N^{-\delta\alpha} \text{ for all } |q| < N^{-\delta}. \end{aligned}$$

In addition, we shall require that

$$iv) \; |\nabla k_\delta^N(q)| \leq N^{-\delta(\alpha+1)} \text{ for all } |q| < N^{-\delta} \tag{3.9}$$

which merely assures that the regularization around the origin is not somehow erratic.

Within this setting, we thus consider 3-dimensional force kernels satisfying a (S_δ^α) condition with $\alpha = 2$ and the additional assumption (3.9). The lower bound on the cut-off will later be determined as $\delta < \frac{1}{3}$. Moreover, we shall adopt the convention $k_\delta^N(0) = 0$, meaning that the microscopic dynamics do not contain self-interactions. The reader is free to think of (3.4) as defining the microscopic model or consider another regularization of his liking that satisfies the assumptions above.

3.2.1 The mean field flow

For any $\delta > 0$ and $N \in \mathbb{N} \cup \{\infty\}$, we also consider the corresponding mean field equation

$$\partial_t f + p \cdot \nabla_q f + \left(k_\delta^N * \rho_t\right) \cdot \nabla_p f = 0. \tag{3.10}$$

For (formally) $N = \infty$, this reduces to the Vlasov-Poisson equation (3.1). For a fixed initial distribution $f_0 \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ with $f_0 \geq 0$ and $\int f = 1$ we denote by f_t^N the unique solution of (3.10) with initial datum $f_t^N(0, \cdot, \cdot) = f_0$.

As mentioned in the introduction, it is convenient to consider the *characteristic flow* of the mean field system. For $N \in \mathbb{N}, \delta > 0$ and $\rho \in L^1(\mathbb{R}^3)$, we define $\widehat{K}_\delta^N(\cdot; \rho) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ by

$$\widehat{K}_\delta^N(q, p; \rho) := \big(p, k_\delta^N * \rho\,(q)\big). \tag{3.11}$$

Then, the (regularized) Vlasov-Poisson equation (3.10) with initial f_0 is equivalent to the following system of integro-differential equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\varphi_{t,s}^N(z; f_0) = \widehat{K}_\delta^N\big(\varphi_{t,s}^N(z; f_0); \rho_t^N\big) \\ \rho_t^N(q) = \int f^N(t, q, p)\, \mathrm{d}^3 p \\ f^N(t, \cdot) = \varphi_{t,s}^N(\cdot\,; f_0) \# f_s^N \\ \varphi_{s,s}^N(z; f_0) = z. \end{cases} \tag{3.12}$$

In other words, we have non-linear time-evolution in which $\varphi_{t,s}(\cdot\,; f_0)$ is the one-particle flow induced by the mean field dynamics with initial distribution f_0, while, in turn, f_0 is transported with the flow $\varphi^N_{t,s}$. Due to the semi-group property $\varphi^N_{t,s'} \circ \varphi^N_{s',s} = \varphi^N_{t,s}$ it generally suffices to consider the initial time $s = 0$.

The method of characteristics can be though of as establishing a kind of duality between the (rescaled) Newtonian dynamics (3.5) and the Vlasov equation (3.10). Indeed, observing that the microscopic force can be written as

$$\frac{1}{N}\sum_{j=1}^{N} k^N_\delta(q_i - q_j) = k^N_\delta * \mu^N_t[Z](q_i) \tag{3.13}$$

one easily checks that $\Psi_{t,0}(Z)$ solves (3.5) with $\Psi_0(Z) = 0$ if and only if $g_t = \mu^N[\Psi_{t,0}(Z)]$ is a weak solution of (3.10) with $g_0 = \mu^N[Z]$. This is often used to translate the microscopic dynamics into a Vlasov equation, allowing to treat $\mu^N_t[Z]$ and f_t on the same footing. Here, we will go the opposite way, so to speak, and translate the mean field dynamics – for continuous f_0 – into corresponding N particle dynamics.
To this end, we consider the lift of $\varphi^N_{t,s}(\cdot)$ to the N-particle phase-space, which we denote by ${}^N\Phi_{t,s}$. That is, for $f_0 \in L^1(\mathbb{R}^6)$ and $Z = (q_i, p_i)_{1\leq i\leq N}$, we define

$${}^N\Phi_{t,s}(Z; f_0) := \big(\varphi^N_{t,s}(q_1, p_1; f_0), ..., \varphi^N_{t,s}(q_N, p_N; f_0)\big). \tag{3.14}$$

We shall often omit the index N and the initial distribution f_0, unless necessary. Denoting by $\overline{K} : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$ the lift of the mean field force to the N-particle phase-space, i.e.

$$(\overline{K}_t(Z))_i := k^N_\delta * \rho[f^N_t](z_i), \quad Z = (z_1, ..., z_N), \tag{3.15}$$

the flow ${}^N\Phi_{t,s}(Z) = \big({}^N\Phi^1_{t,s}(X), {}^N\Phi^2_{t,s}(X)\big)$ can also be characterized as the solution of the non-autonomous differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big({}^N\Phi^1_{t,s}(Z), {}^N\Phi^2_{t,s}(Z)\big) = \big({}^N\Phi^2_{t,s}(Z), \overline{K}_t({}^N\Phi^1_{t,s}(Z))\big),\ {}^N\Phi_{s,s}(Z) = Z \tag{3.16}$$

to be compared with (3.7). Finally, we introduce the corresponding empirical density

$$\mu^N[\Phi_{t,0}(Z)] = \varphi^N_{t,0}\#\mu^N[Z]. \tag{3.17}$$

The N-point process $Z \to {}^N\Phi_{t,0}(Z)$ can be called a "quantization" of the Vlasov equation, which has nothing to do with quantum mechanics, but refers to the fact that we sample the characteristic flow along N trajectories with random initial condition Z. In summary, for fixed f_0 and $N \in \mathbb{N}$, we consider for any initial configuration $Z \in \mathbb{R}^{6N}$ two different time-evolutions: $\Psi_{t,0}(Z)$, given by the microscopic equations (3.5) and $\Phi_{t,0}(Z)$, given by the time-dependent mean field force generated by f^N_t. Our aim is to show that for typical Z, the two time-evolutions or close in an appropriate sense.

3.3 Existence of solutions

For the regularized Vlasov-Poisson equation (3.10), all forces are Lipschitz and the solution theory is fairy standard, see e.g. [10, 15]. In the Coulomb case, the issue is more subtle.

Fortunately, we can rely on various results establishing global existence and uniqueness of (strong) solutions under fairly mild conditions on the initial configuration f_0. (Pfaffelmoser, 1990 [53], Schaeffer, 1991 [59], Lions and Perthame, 1991 [41], Horst, 1993 [30].) For our purposes, the following existence result due to Lions and Perthame will prove to be particularly useful:

Theorem 3.3.1 (Lions and Perthame). *Let $f_0 \geq 0, f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ satisfy*

$$\int |p|^m f_0(q,p)\, \mathrm{d}q\, \mathrm{d}p < +\infty, \tag{3.18}$$

for all $m < m_0$ and some $m_0 > 3$.

a) Then, the Vlasov-Poisson system defined by equations (1–3) has a continuous, bounded solution $f(t,\cdot,\cdot) \in C(\mathbb{R}^+; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ for $1 \leq p < \infty$ satisfying

$$\sup_{t \in [0,T]} \int |p|^m f(t,q,p)\, \mathrm{d}p\, \mathrm{d}p < +\infty, \tag{3.19}$$

for all $T < \infty, m < m_0$.

b) If, in fact, $m_0 > 6$ and we assume that f_0 satisfies

$$\begin{aligned} \operatorname{supess}\{f_0(q' + pt, p') : |q - q'| \leq Rt^2, |p - p'| < Rt\} \\ \in L^\infty\big((0,T) \times \mathbb{R}^3_q; L^1(\mathbb{R}^3_p)\big) \end{aligned} \tag{3.20}$$

for all $R > 0$ and $T > 0$, then

$$\sup_{t \in [0,T]} \|\rho_t(q)\|_\infty < +\infty,\ \forall T \in (0, +\infty). \tag{3.21}$$

Under the assumption of part b) of the theorem, the result of Loeper, 2006 [42] then shows that for any $T > 0$, said f is the *unique* solution in the set of bounded, positive measures on $[0,T) \times \mathbb{R}^6$ satisfying $f\big|_{t=0} = f_0$ in the sense of distributions. Moreover, it has been long known that as long as the charge density is bounded, solutions with smooth initial data remain smooth (see e.g. in [28]).

As Lions and Perthame remark – and as one can easily verify by following the proof – part b) of the theorem actually yields a bound on the charge density that is uniform in N if one considers a sequence of regularized time-evolutions as (for instance) in (3.10). We will note this important fact in the following Lemma.

Lemma 3.3.2. *Let $f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and f^N_t be the solution of the regularized Vlasov-Poisson equation* (3.10) *(with corresponding cut-off) and initial datum $f^N(0,\cdot,\cdot) = f_0$. If f_0 satisfies assumption* (3.20) *of the above theorem, there exists a constant $C_\rho > 0$ such that*

$$\|\rho^N_t\|_\infty \leq C_\rho,\ \forall N \in \mathbb{N} \cup \{\infty\},\ \forall t > 0, \tag{3.22}$$

where formally $\rho^\infty_t = \rho[f_t]$.

Since condition (3.20) is rather abstract, we want to state a more intuitive sufficient condition.

Lemma 3.3.3. *Let $f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3),\ f \geq 0$. Suppose there exist functions $\rho \in L^\infty(\mathbb{R}^3)$ and $\vartheta(|p|) \in L^1(\mathbb{R}^3)$ with ϑ monotonously decreasing and an $S > 0$ such that for all $|p| > S$*

$$f_0(q,p) \leq \rho(q)\vartheta(|p|).$$

Then f_0 satisfies assumption (3.20). Special cases:

- *f_0 has compact support in the p-variables.*
- *f_0 is a thermal state of the form $\rho(q)\, e^{-\beta p^2}$ with $\|\rho\|_\infty < \infty, \beta > 0$.*

Proof. For given $R, t > 0$ we have to consider the function

$$\tilde{f}(t,q,p) := \operatorname{supess}\{f_0(q' + pt, p') : |q - q'| \leq Rt^2, |p - p'| < Rt\}.$$

Choosing $R' > S + RT$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \tilde{f}(t,q,p)\,\mathrm{d}^3p = \int_{|p|\leq R'} + \int_{|p|>R'} \tilde{f}(t,q,p)\,\mathrm{d}^3p \\
&\leq \frac{4}{3}\pi R'^3 \|\tilde{f}(t,\cdot,\cdot)\|_\infty + \|\rho\|_\infty \int \sup_{|p-p'|<Rt} \vartheta(|p'|)\,\mathrm{d}^3p \\
&\leq \frac{4}{3}\pi R'^3 \|f_0\|_\infty + \|\rho\|_\infty \int \vartheta(|p| - Rt)\,\mathrm{d}^3p \\
&\leq C\|f_0\|_\infty + \|\rho\|_\infty \|\vartheta\|_1 < \infty,
\end{aligned}$$

where in the second to last line we used the monotonicity of $\vartheta(|p|)$ and the fact that $\|\tilde{f}\|_\infty = \|f_0\|_\infty$. □

One important consequence of the bounded density is that the mean field force remains bounded, as well.

Lemma 3.3.4. *Let k be the Coulomb kernel, and $\rho \in L^1 \cap L^\infty(\mathbb{R}^3; \mathbb{R}^+)$. Then there exists $C > 0$ such that*

$$\|k * \rho\|_\infty \leq C \|\rho\|_1^{1/3} \|\rho\|_\infty^{2/3}. \tag{3.23}$$

Proof. For $R > 0$, we compute:

$$\begin{aligned}
\|k * \rho\|_\infty &\leq \Big\| \int_{|y|<R} k(y)\rho(x-y)\,\mathrm{d}^3y \Big\|_\infty + \Big\| \int_{|y|>R} k(y)\rho(x-y)\,\mathrm{d}^3y \Big\|_\infty \\
&\leq \|\rho\|_\infty \int_{|y|<R} \frac{1}{|y|^2}\mathrm{d}^3y + R^{-2}\|\rho\|_1 = 4\pi R \|\rho\|_\infty + R^{-2}\|\rho\|_1.
\end{aligned}$$

This last expression is optimized by setting $R = (4\pi)^{-1/3}\|\rho\|_\infty^{-1/3}\|\rho\|_1^{1/2}$, which yields $\|k * \rho\|_\infty \leq 2(4\pi)^{2/3}\|\rho\|_1^{1/3}\|\rho\|_\infty^{2/3}$.

□

3.4 Statement of the results

In the following, all probabilities and expectation values are meant with respect to the product measure given at a certain time. That is, for any random variable $H : \mathbb{R}^{6N} \to \mathbb{R}$ and any element A of the Borel-algebra

$$\mathbb{P}_t^N(H \in A) = \int_{H^{-1}(A)} \prod_{j=1}^N f_t^N(z_j) dZ \tag{3.24}$$

$$\mathbb{E}_t^N(H) = \int_{\mathbb{R}^{6N}} H(Z) \prod_{j=1}^N f_t^N(z_j) dZ \ . \tag{3.25}$$

Note that since ${}^N\Phi_{t,s}$ leaves the measure invariant,

$$\begin{aligned}\mathbb{E}_s^N(H \circ {}^N\Phi_{t,s}) &= \int_{\mathbb{R}^{6N}} H({}^N\Phi_{t,s}(Z)) \prod_{j=1}^N f_s^N(z_j) dZ \\ &= \int_{\mathbb{R}^{6N}} H(Z) \prod_{j=1}^N f_s^N(\varphi_{s,t}^N(z_j)) dZ \\ &= \int_{\mathbb{R}^{6N}} H(Z) \prod_{j=1}^N f_t^N(z_j) dZ = \mathbb{E}_t^N(H).\end{aligned}$$

In particular:

$$\mathbb{P}_t^N(Z \in A) = \mathbb{P}_0^N({}^N\Phi_{t,0}(Z) \in A). \tag{3.26}$$

We will often omit the index N when the particle number is fixed and write only $\mathbb{P}_t, \mathbb{E}_t$. To quantify the convergence of probability measures, we will use the Wasserstein distances introduced in Chapter 2. We can now state our precise results in the following theorem.

Theorem 3.4.1 (Particle approximation of the Vlasov-Poisson system). *Let $f_0 \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ a probability measure satisfying the assumptions of Theorem 3.3.1 a) and b). Let $p \in [1,2]$ and assume that, in addition, there exists $m > 2p$ such that $\int |q|^m f_0(q,p) \,\mathrm{d}q \,\mathrm{d}p < +\infty$. For $N > 3$ and $\delta > 0$ let $\Psi_{t,s}$ be the N-particle flow solving (3.5) with cut-off width $N^{-\delta}$, $\delta < \frac{1}{3}$. Then, the empirical density $\mu_t^N[Z] := \mu^N[\Psi_{t,0}(Z)]$ typically converges to the solution of the Vlasov-Poisson equation in the following sense:*

For $\delta < \frac{1}{3}$, $\gamma < \min\{\frac{1}{6}, \delta\}$, and every $T > 0$ there exists constants c, C depending on m, p, γ and a constant C_0 depending on f_0 and T such that for all $N \geq 4$:

$$\begin{aligned}\mathbb{P}_0\Big[\exists t \in [0,T] : W_p(\mu_t^N[Z], f_t) > (3\sqrt{\log(N)}) N^{-\gamma}\, e^{t(C_0+1)\sqrt{\log(N)}}\Big] \\ \leq 2N^{-1+3\delta} e^{TC_0\sqrt{\log(N)}} + C\big(e^{-cN^{1-6\gamma}} + N^{-1+2p\gamma}\big),\end{aligned} \tag{3.27}$$

where f_t is the unique solution of the Vlasov-Poisson system (3.1) on $[0,T]$ with $f(0,\cdot) = f_0$.

Remarks 3.4.2.

1) Note that since $\exp\big[\sqrt{\log(N)}\big] = \exp\big[\log(N)/\sqrt{\log(N)}\big] = N^{\frac{1}{\sqrt{\log(N)}}}$, we have $e^{\sqrt{\log N}} = o(N^{\epsilon})$ for arbitrary small $\epsilon > 0$. Thus $\mu_t^N[Z]$ converges to f_t in probability. However, (3.27) yields good error bounds only for $N > \exp[(\frac{t(C_0+1)}{\gamma})^2]$.

2) Without the additional assumption of spatial moments, molecular chaos still holds, albeit without the quantitative bounds stated in the theorem (see Proposition 3.5.5).

3) Our result allows to choose the width of the cut-off arbitrary close to $N^{-1/3}$, which corresponds to the scale of the typical distance between a particle and its closest neighbor.

3.5 A new measure of chaos

The strategy of the proof, following Boers and Pickl [6], is to control the deviation of the microscopic time-evolution from the mean field time evolution in terms of the following N-dependent quantity:

Definition 3.5.1. Let ${}^N\Phi_{t,0}$ the mean field flow defined in (3.14) and ${}^N\Psi_{t,0}$ the microscopic flow defined in (3.6). We denote by ${}^N\Phi^1_{t,0} = (q_i(t))_{1\leq i\leq N}$ and ${}^N\Phi^2_{t,0} = (p_i(t))_{1\leq i\leq N}$ the projection onto the spatial, respectively the momentum coordinates.

Let $J(t)$ be the stochastic process given by

$$J_t^N(Z) := \min\Big\{1, \lambda(N)N^{\delta}\sup_{0\leq s\leq t}|{}^N\Psi^1_{t,0}(Z) - {}^N\Phi^1_{t,0}(Z)|_\infty \\ + N^{\delta}\sup_{0\leq s\leq t}|{}^N\Psi^2_{t,0}(Z) - {}^N\Phi^2_{t,0}(Z)|_\infty\Big\}, \tag{3.28}$$

where $|Z|_\infty = \max\{|z_i| : 1 \leq i \leq N\}$ denotes the maximum-norm on $\mathbb{R}^{3N}$ and $\lambda(N) \geq 1$ is a scaling factor that we will fix as $\lambda(N) := \max\{1, \sqrt{\log(N)}\}$.

The small but crucial innovation with respect to [6] is that distances in spatial and momentum coordinates are weighted differently by a factor $\lambda(N)$, exploiting the second-order nature of the dynamics.

Our aim is to derive a Gronwall estimate for the time-evolution of $\mathbb{E}_0(J_t^N)$, showing that $\mathbb{E}_0^N(J_t^N) \xrightarrow{N\to\infty} 0, \forall 0 \leq t \leq T$. The relevance of this statement for the proof of the theorem is grounded in the following observations.

Lemma 3.5.2. *For $X = (x_1, ..., x_n) \in \mathbb{R}^n$ let $\mu^N[X] := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} \in \mathcal{P}(\mathbb{R}^n)$. Then we have for all $p \in [1,\infty]$:*

$$W_p(\mu^N[X], \mu^N[Y]) \leq \big|X - Y\big|_\infty. \tag{3.29}$$

Proof. Since $W_p \leq W_q$ for $p \leq q$, it suffices to consider the infinite Wasserstein distance defined by

$$W_\infty(\mu,\nu) = \inf\{\pi - esssup\,|x-y| \,\big|\, \pi \in \Pi(\mu,\nu)\}.$$

We then observe that $\pi_0 = \sum_{i=1}^{N} \delta_{x_i}\delta_{y_i} \in \Pi(\mu^N[Z], \mu^N[Y])$ with $\pi_0 - esssup\,|x-y| = \max_{1\leq i\leq N} |x_i - y_i| = |X - Y|_\infty$. □

With this Lemma, we immediately conclude the following:

Proposition 3.5.3. *For all $p \in [1, \infty]$ it holds that*

$$\mathbb{P}_0\Big[\sup_{0\leq s\leq t} W_p(\mu^N[\Psi_{s,0}(Z)], \mu^N[\Phi_{s,0}(Z)]) \geq N^{-\delta}\Big] \leq \mathbb{E}_0(J_t^N). \tag{3.30}$$

Proof. Observe that $J_t^N(Z) = 1$ if there exists $s \in [0,t]$ with $|^N\Psi_{s,0}(Z) - {}^N\Phi_{s,0}(Z)|_\infty \geq N^{-\delta}$. Hence, we have $\mathbb{P}_0\Big[Z \in \mathbb{R}^{6N} : \sup_{0\leq s\leq t} |^N\Psi_{s,0}(Z) - {}^N\Phi_{s,0}(Z)|_\infty \geq N^{-\delta}\Big] \leq \mathbb{E}_0(J_t^N)$ and since $W_p(\mu^N[\Psi_{s,0}(Z)], \mu^N[\Phi_{s,0}(Z)]) \leq |^N\Psi_{s,0}(Z) - {}^N\Phi_{s,0}(Z)|_\infty$ according to the previous lemma, the proposition follows. □

In total, we will split our approximation into

$$W_p(\mu_t^N[Z], f_t) \leq W_p(\mu^N[\Psi_{t,0}(Z)], \mu^N[\Phi_{t,0}(Z)]) \tag{3.31}$$
$$+ W_p(\mu^N[\Phi_{t,0}(Z)], f_t^N) \tag{3.32}$$
$$+ W_p(f_t^N, f_t). \tag{3.33}$$

The first term (3.31) is the most interesting one, concerning the difference between microscopic time-evolution and mean field time-evolution. It will be controlled by $\mathbb{E}_0(J_t^N)$, by virtue of Proposition 3.5.3.

The second term $W_p(\mu^N[\Phi_{t,0}(Z)], f_t^N) = W_p(\varphi_{t,0}^N \# \mu_0^N[Z], \varphi_{t,0}^N \# f_0)$ concerns the sampling of the mean field dynamics by discrete particle trajectories. We will use the large-deviation estimate of Fournier and Guillin, Thm. 2.2.1, to determine the typical rates of convergence for the initial distribution. The challenge is then to control the growth of (3.32) uniformly in N. This will be achieved with the stability result of Loeper, discussed in Chapter 2.3.

Convergence of (3.33) is a purely deterministic result: solutions of the regularized Vlasov-Poisson equation (3.10) approximate solutions of the proper Vlasov-Poisson equation (3.1) as the width of the cut-off goes to zero. Concretely, we will show that $W_2(f_t^N, f_t) \to 0$.

The key conceptual innovation with respect to previous approaches is that we first sample the (regularized) mean field dynamics along trajectories with random initial conditions, i.e. approximate f_t^N by $\mu^N[\Phi_{t,0}(Z)]$ and then control the difference between the mean field trajectories and the "true" microscopic trajectories in terms of the expectation value $\mathbb{E}_0(J_t^N)$. The virtues of this method, first proposed in [6], are manifold:

1. The method is designed for stochastic initial conditions, thus allowing for law-of-large number estimates that turn out to be very powerful. (Note that the particles evolving with the mean field flow remain statistically independent at all times.)

2. The metric $|^N\Psi_{t,0}(Z) - {}^N\Phi_{t,0}(Z)|_\infty$ is much stronger than usual weak distances between probability measures, thus allowing for better stability estimates.

3. Since $\frac{\mathrm{d}}{\mathrm{d}t}J_t^N(Z) = 0$ if $\exists 0 \leq s \leq t : |{}^N\Psi_{s,0}(Z) - {}^N\Phi_{s,0}(Z)|_\infty \geq N^\delta$ we only have to consider situations in which mean field trajectories and microscopic trajectories are still close together.

4. Exploiting the second-order nature of the dynamics, we weigh distances in x-space and momentum space differently, with an N-dependent factor $\lambda(N)$. Note that as we compare microscopic trajectories to characteristic curves of the mean field equation, the growth the *spatial* distance is trivially bounded by the difference of the respective momenta. The idea is thus to be a little more strict on deviations in space, so to speak, and use this to obtain better control on fluctuations of the force.

3.5.1 Convergence of marginals

As mentioned in the introduction, it is a classic result that convergence of the empirical density in the sense of Theorem 3.4.1 implies molecular chaos in the sense of (1.12). Nevertheless, for completeness, we want to show that convergence of the k-particle marginals can be straightforwardly concluded from the convergence of $\mathbb{E}_0(J_t^N) \to 0$.

Definition 3.5.4 (Bounded Lipschitz distance)**.** Let $\mathcal{L}$ be the space of functions $g : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\|g\|_\infty := \sup_x |g(x)| = 1, \ \|g\|_{Lip} := \sup_{x,y} \frac{g(x) - g(y)}{|x-y|} = 1. \tag{3.34}$$

For two probability densities μ, ν on $\mathbb{R}^k$, the *bounded Lipschitz distance* is defined by

$$d_{BL}(\mu, \nu) := \sup_{g \in \mathcal{L}} \left| \int g(x) \mathrm{d}\mu(x) - \int g(x) \mathrm{d}\nu(x) \right| .$$

The bounded Lipschitz distance metrizes weak convergence in $\mathcal{P}(\mathbb{R}^n)$.

Proposition 3.5.5. *Suppose that* $\lim_{N\to\infty} \mathbb{E}_0^N(J_t^N) = 0$. *Then, the reduced k-particle marginal given by*

$$^{(k)}F_t^N(z_1, ..., z_k) := \int F_t^N(Z)\, \mathrm{d}^3 z_{k+1} ... \mathrm{d}^3 z_N \tag{3.35}$$

converges weakly to $\otimes^k f_t^N$ *as* $N \to \infty$ *for all* $k \in \mathbb{N}$. *More precisely, we have:*

$$d_{BL}(^{(k)}F_s^N, \otimes^k f_s^N) \leq \mathbb{E}_0(J_t) + N^{-\delta}, \forall s \leq t. \tag{3.36}$$

Proof. Let $g : \mathbb{R}^{6k} \to \mathbb{R}$ be a test-function with $\|g\|_{Lip} = \|g\|_\infty = 1$. Let $\mathcal{A}_t \subset \mathbb{R}^{6N}$ be given by $Z \in \mathcal{A}_t \iff J_t(Z) < 1$. Then $Z \in \mathcal{A}_t$ implies in particular $\left|\Psi_{s,0}(Z) - \Phi_{s,0}(Z)\right|_\infty \leq N^{-\delta}$, $\forall s \in [0,t]$, while $\lim_{N\to\infty} \mathbb{E}_0^N(J_t) = 0$ implies $\lim_{N\to\infty} \mathbb{P}_0^N(\mathcal{A}_t^c) = 0$. Thus, we find for all $s \leq t$:

$$
\begin{aligned}
&d_{BL}({}^{(k)}F_s^N, \otimes^k f_s^N) \\
&\quad = \sup_{g\in\mathcal{L}} \Big| \int \big({}^{(k)}F_s^N - \otimes^k f_s^N\big) g(z_1,...,z_k) \mathrm{d}^3 z_1 ... \mathrm{d}^3 z_k \Big| \\
&\quad = \sup_{g\in\mathcal{L}} \Big| \int \big(F_s^N(Z) - \otimes^N f_s^N(Z)\big) g(z_1,...,z_k) \mathrm{d}^3 z_1 ... \mathrm{d}^3_{z_k} ... \mathrm{d}^3 z_N \Big| \\
&\quad = \sup_{g\in\mathcal{L}} \Big| \int \big(\Psi_{0,s}\#F_0(Z) - \Phi_{0,s}\#F_0(Z)\big) g(z_1,...,z_k) \, \mathrm{d}^{6N} x \Big| \\
&\quad = \sup_{g\in\mathcal{L}} \Big| \int F_0(Z) \big(g(P_k\Psi_{s,0}(Z)) - g(P_k\Phi_{s,0}(Z))\big) \, \mathrm{d}^{6N} z \Big| \\
&\quad = \sup_{g\in\mathcal{L}} \Big| \int_{\mathcal{A}_t} F_0(Z) \big(g(P_k\Psi_{s,0}(Z)) - g(P_k\Phi_{s,0}(Z))\big) \, \mathrm{d}^{6N} z \Big| \qquad (3.37) \\
&\quad + \sup_{g\in\mathcal{L}} \Big| \int_{\mathcal{A}_t^c} F_0(Z) \big(g(P_k\Psi_{s,0}(Z)) - g(P_k\Phi_{s,0}(Z))\big) \, \mathrm{d}^{6N} z \Big| \qquad (3.38)
\end{aligned}
$$

where $P_k : \mathbb{R}^N \to \mathbb{R}^k, (z_1, ..., z_N) \mapsto (z_1, ..., z_k)$ is the projection onto the first k coordinates. Since g and F_0 are bounded by 1, we have (3.38) $\leq \mathbb{P}_0(\mathcal{A}_t^c) \leq \mathbb{E}_0(J_t)$.

Using that $\|g\|_{Lip} = 1$, we obtain

$$
\sup_{Z\in\mathcal{A}_t} |g(P_k\Psi_{s,0}(Z)) - g(P_k\Phi_{s,0}(Z))| \leq |\Psi_{s,0} - \Phi_{s,0}|_\infty \leq N^{-\delta}, \; \forall 0 \leq s \leq t. \tag{3.39}
$$

Hence, also (3.37) $\leq N^{-\delta}$ and the proposition follows. □

Since we will also prove that $f_t^N \rightharpoonup f_t$, this implies molecular chaos for the Vlasov-Poisson system. Note that this result holds without further assumptions on f_0, but is much weaker than the approximation stated in Theorem 3.4.1.

3.6 Local Lipschitz bound

If all forces were Lipschitz continuous with a Lipschitz constant L independent of N, we could readily conclude that $\frac{\mathrm{d}}{\mathrm{d}t}|{}^N\Psi_{t,0}(Z) - {}^N\Phi_{t,0}(Z)|_\infty \leq (1+L)|{}^N\Psi_{t,0}(Z) - {}^N\Phi_{t,0}(Z)|_\infty$. The desired convergence for $\mathbb{E}_0^N(J_t^N)$ would then immediately follow by a simple application of Gronwall's Lemma. However, the forces considered here become singular in the limit $N \to \infty$ and hence do not satisfy a uniform Lipschitz bound. Nevertheless, we observe that, for the mean field force $k^N * \rho_t$, the global Lipschitz constant $\|k^N * \rho_t\|_{Lip}$ diverges only logarithmically as the cut-off is lifted with increasing N. Setting $\lambda(N) = \max\{1, \sqrt{\log{(N)}}\}$ in Definition 3.5.1, the particular anisotropic scaling of our metric will allow us to "trade" part of this divergence for a tighter control on spatial fluctuations. This will suffice to establish the desired convergence of $\mathbb{E}_0(J_t)$ by virtue of $\mathbb{E}_0(J_{t+\Delta t}) - \mathbb{E}_0(J_t) \sim \sqrt{\log{(N)}}\, \mathbb{E}_0(J_t)\, \Delta t + o(\Delta t)$.

We summarize our first observation in the following Lemma.

Lemma 3.6.1. *Let $l =: \mathbb{R}^3 \to \mathbb{R}^k$ satisfy*

$$|l(q)| \leq c \cdot \min\{N^{3\delta}, |q|^{-3}\} \tag{3.40}$$

for some $c > 0$. Then there exists a constant $C_l > 0$ such that

$$\|l * \rho_t(x)\|_\infty \leq C_l \max\{1, \sqrt{\log(N)}\} \left(\|\rho_t\|_1 + \|\rho_t\|_\infty\right). \tag{3.41}$$

Proof.

$$\begin{aligned}\|l * \rho_t(x)\|_\infty =& \Big\| \int l(x-y)\rho_t(y)\,\mathrm{d}^3y \Big\|_\infty \\ \leq& \Big\| \int\limits_{|x-y|<N^{-\delta}} l(x-y)\rho_t(y)\,\mathrm{d}^3y \Big\|_\infty + \Big\| \int\limits_{N^{-\delta}<|x-y|<1} l(x-y)\rho_t(y)\,\mathrm{d}^3y \Big\|_\infty \\ &+ \Big\| \int\limits_{|x-y|>1} l(x-y)\rho_t(y)\,\mathrm{d}^3y \Big\|_\infty .\end{aligned}$$

The first term is bounded by

$$\Big\| \int\limits_{|x-y|<N^{-\delta}} l(x-y)\rho_t(y)\,\mathrm{d}^3y \Big\|_\infty \leq \|\rho_t\|_\infty N^{3\delta} |\mathrm{B}(N^{-\delta})| \leq \frac{4}{3}\pi \, \|\rho_t\|_\infty,$$

where $\mathrm{B}(r)$ denotes the ball with radius r. The last term is bounded by

$$\Big\| \int\limits_{|x-y|>1} l(x-y)\rho_t(y)\,\mathrm{d}^3y \Big\|_\infty \leq c \, \|\rho_t\|_1.$$

Finally, the second term yields

$$\begin{aligned}\Big\| \int\limits_{N^{-\delta}<|x-y|<1} g(x-y)\rho_t(y)\,\mathrm{d}^3y \Big\|_\infty &\leq \|\rho_t\|_\infty \int\limits_{N^{-\delta}<|y|<1} \frac{c}{|y|^3}\,\mathrm{d}^3y \\ &\leq 4\pi c \|\rho_t\|_\infty \log(N^\delta) = 4\pi c\delta \, \|\rho_t\|_\infty \log(N).\end{aligned}$$

□

One immediate application of the Lemma is to $l(q) = \nabla k_\delta^N(q)$, showing that the mean field force for the regularized system is Lipschitz continuous with a constant proportional to $\log(N)$. Our goal is now to prove that for typical initial conditions, the fluctuations in the *microscopic* forces can be bound in a similar fashion, as long as $\Psi_{t,0}(Z)$ and $\Phi_{t,0}(Z)$ are close. Following [6], we thus introduce a function controlling the difference $|k(q) - k(q+\xi)|$, for $|\xi| < 2N^{-\delta}$.

Definition 3.6.2. Let

$$l_\delta^N(q) := \begin{cases} \frac{54}{|q|^3} & , \text{if } |q| \geq 3N^{-\delta} \\ N^{3\delta} & , \text{else} \end{cases} \tag{3.42}$$

and $L : \mathbb{R}^{6N} \to \mathbb{R}^N$ be defined by $(L(Z))_i := \frac{1}{N} \sum\limits_{j \neq i} l_\delta^N(q_i - q_j)$. Furthermore, for given f_t, we define $\overline{L}_t(Z)$ by $(\overline{L}_t(Z))_i := l_\delta^N * \rho_t(q_i) = \int l_\delta^N *_q f(t, q_i, p)\,\mathrm{d}p$.

Lemma 3.6.3. *For any $\xi \in \mathbb{R}^3$ with $|\xi|_\infty < 2N^{-\delta}$, it holds that*

$$|k_\delta^N(q) - k_\delta^N(q+\xi)|_\infty \le l_\delta^N(q)|\xi|_\infty. \tag{3.43}$$

Proof. First note that by assumption the derivative of k^N is bounded by $N^{3\delta}$, so that (3.43) holds for $|q| < 3N^{-\delta}$. For $|q| \ge 3N^{-\delta}$, there exists $s \in [0,1]$ such that $|k_\delta^N(q) - k_\delta^N(q+\xi)| \le |\nabla k_\delta^N(q+s\xi)|_\infty |\xi|_\infty$, where

$$|\nabla k_\delta^N(q+s\xi)|_\infty \le 2|q+s\xi|^{-3}. \tag{3.44}$$

The expression on the right-hand-side takes its greatest value if ξ is antiparallel to q and $s = 1$. Hence, we have

$$|k_\delta^N(q) - k_\delta^N(q+\xi)|_\infty \le 2\left|q(1-\frac{|\xi|}{|q|})\right|^{-3}|\xi|_\infty. \tag{3.45}$$

Since $|q| \ge 3N^{-\delta}$ and $|\xi| < 2N^{-\delta}$, it follows that $\frac{|\xi|}{|q|} < \frac{2}{3}$. Hence, as claimed, $|k_\delta^N(q) - k_\delta^N(q+\xi)|_\infty \le 2\left(\frac{3}{|q|}\right)^3 |\xi|_\infty \le \frac{54}{|q|^3}|\xi|_\infty$. □

3.7 Law of large numbers

In order to control the evolution of $\mathbb{E}_0(J_t^N)$, we will require as an intermediate step that the mean field force (and its derivative) can be approximated by the analogous expressions for the discrete measure $\mu^N[\Phi_{t,0}(Z)]$ with random Z. The key observation here is that if the N-particle configuration evolves with the mean field flow ${}^N\Phi_{t,0}$, the particles remain statistically independent for all t, thus giving rise to a law-of-large-numbers estimate.

Definition 3.7.1. For any $t > 0$ and fixed $\delta < \frac{1}{3}$, we consider the (time-dependent) sets $\mathcal{A}_t, \mathcal{B}_t, \mathcal{C}_t$ defined by

$$\begin{aligned}
Z \in \mathcal{A}_t &\iff |J_t(Z)| < 1\\
Z \in \mathcal{B}_t &\iff \left|K(\Phi_{t,0}(Z)) - \overline{K}(\Phi_{t,0}(Z))\right|_\infty < N^{-1+2\delta}\\
Z \in \mathcal{C}_t &\iff \left|L(\Phi_{t,0}(Z)) - \overline{L}(\Phi_{t,0}(Z))\right|_\infty < 1
\end{aligned}$$

where $\overline{K}$ is the mean field force (3.15) and $\overline{L}$ as in Definition 3.6.2.

We now want to show that for any t, initial conditions in $\mathcal{B}_t \cap \mathcal{C}_t$ are *typical* with respect to the product measure $F_0 := \otimes^N f_0$ on $\mathbb{R}^{6N}$.

Proposition 3.7.2. *Let $\rho_t \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ with $\|\rho_t\|_1 = 1$ as before. Let $h : \mathbb{R}^3 \to \mathbb{R}$ and suppose that for given $\delta > 0$ and $N \in \mathbb{N}$ there exists $c > 0$ and an exponent $2 \le \alpha \le 3$ such that $|h(x)| \le c \cdot \min\{N^{\alpha\delta}, |q|^{-\alpha}\}$, $\forall q \in \mathbb{R}^3$. Assume furthermore that*

$$\delta < \min\left\{\frac{1-2\beta}{2\alpha-3}, \frac{1-\beta}{\alpha}\right\}. \tag{3.46}$$

Then there exists for all $\gamma > 0$ a constant $C_\gamma > 0$ such that

$$\mathbb{P}_t\Big[\sup_{1\le i\le N}\Big|\frac{1}{N}\sum_{j\ne i}^N h(q_i - q_j) - h * \rho_t(q_i)\Big| \ge N^{-\beta}\Big] \le \frac{C_\gamma}{N^\gamma}. \tag{3.47}$$

Proof. Let

$$D_i := \Big\{ Z \in \mathbb{R}^6 : \Big| \frac{1}{N} \sum_{j \neq i}^{N} h(q_i - q_j) - h * \rho_t(q_i) \Big| \geq N^{-\beta} \Big\} \tag{3.48}$$

and $D := \bigcup_{i=1}^{N} D_i$. Then $\mathbb{P}(D) \leq \sum_{i=1}^{N} \mathbb{P}(D_i) = N\mathbb{P}(D_1)$.

By Markov's inequality, we have for every $M \in \mathbb{N}$:

$$\begin{aligned} \mathbb{P}_t(D_1) \leq & \mathbb{E}_t \Big[N^{2M\beta} \Big| \frac{1}{N} \sum_{j=1}^{N} h(q_1 - q_j) - h * \rho_t(q_1) \Big|^{2M} \Big] \\ = & \frac{1}{N^{2M(1-\beta)}} \, \mathbb{E} \Big[\Big(\sum_{j=1}^{N} \big(h(q_1 - q_j) - h * \rho_t(q_i) \big) \Big)^{2M} \Big]. \end{aligned} \tag{3.49}$$

Let $\mathcal{M} := \{ \mathbf{k} \in \mathbb{N}_0^N \mid |\mathbf{k}| = 2M \}$ the set of multiindices $\mathbf{k} = (k_1, ..., k_N)$ with $\sum_{j=1}^{N} k_j = 2M$. Let

$$G^{\mathbf{k}} := \prod_{j=1}^{N} \big(h(q_j - q_1) - h * \rho_t(q_1) \big)^{k_j}. \tag{3.50}$$

Then:

$$\mathbb{E} \Big[\Big(\sum_{j=1}^{N} \big(h(q_1 - q_j) - h * \rho_t(q_1) \big) \Big)^{2M} \Big] = \sum_{\mathbf{k} \in \mathcal{M}} \binom{2M}{\mathbf{k}} \mathbb{E}_t(G^{\mathbf{k}}). \tag{3.51}$$

Now we note that $\mathbb{E}_t(G^{\mathbf{k}}) = 0$ whenever there exists a $1 \leq j \leq N$ such that $k_j = 1$. This can be seen by integrating the j'th variable first.

For the remaining terms, we have for any $1 \leq m \leq M$:

$$\int |h(q_1 - q_j)|^m f_t(q_j, p_j) \, \mathrm{d}^3 p_j \, \mathrm{d}^3 p_j = \int |h|^m (q_1 - q_j) \rho_t(q_j) \, \mathrm{d}^3 q_j.$$

Now for $2 \leq \alpha < 3$ and $m = 1$ we estimate

$$\begin{aligned} & |h * \rho_t(q_1)| \leq \int |h|(q_1 - y) \rho_t(y) \, \mathrm{d}^3 y \\ & \leq c \int_{|y|<1} |y|^{-\alpha} \, \rho_t(q_1 - y) \, \mathrm{d}^3 y + c \int_{|y| \geq 1} |q_j|^{-\alpha} \rho_t(q_1 - y) \, \mathrm{d}^3 y \\ & \leq c \left(4\pi \|\rho_t\|_\infty + \|\rho_t\|_1 \right), \end{aligned}$$

while for $\alpha = 3$, we find:

$$\begin{aligned}
|h * \rho_t(q_1)| &\leq \int |h|(q_1 - y)\rho_t(y)\, \mathrm{d}^3 y \\
&\leq c\Big(\int\limits_{|y|\leq N^{-\delta}} + \int\limits_{N^{-\delta}<|y|<1} + \int\limits_{|y|\geq 1} \Big) |h(y)|\, \rho_t(q_1 - y)\, \mathrm{d}^3 y \\
&\leq c\|\rho_t\|_\infty \int\limits_{|y|\leq N^{-\delta}} N^{3\delta}\, \mathrm{d}^3 y + c\|\rho_t\|_\infty \int\limits_{N^{-\delta}<|y|<1} \frac{1}{|y|^3} \mathrm{d}^3 y + c \int\limits_{|y|\geq 1} \rho_t(q_1 - y)\, \mathrm{d}^3 y \\
&\leq c\left(4\pi\|\rho_t\|_\infty(\frac{1}{3} + \log(N^\delta)) + \|\rho_t\|_1\right).
\end{aligned}$$

For $m \geq 2$, we find in any case

$$\begin{aligned}
&\int |h|^m(q_1 - y)\rho_t(y)\, \mathrm{d}^3 y = \int |h|^m(y)\rho_t(q_1 - y)\, \mathrm{d}^3 y \\
\leq& \int\limits_{|y|<N^{-\delta}} |h|^m(y)\rho_t(q_1 - y)\, \mathrm{d}^3 y + \int\limits_{|y|\geq N^{-\delta}} |h|^m(y)\rho_t(q_1 - y)\, \mathrm{d}^3 y \\
\leq& c\|\rho_t\|_\infty \Big(4\pi N^{-3\delta} N^{\alpha\delta m} + \int\limits_{|y|\geq N^{-\delta}} \frac{1}{|y|^{\alpha m}}\, \mathrm{d}^3 y\Big) \leq 8\pi c\|\rho_t\|_\infty\, N^{(\alpha m - 3)\delta}.
\end{aligned}$$

Hence, setting $C_\alpha := 16\pi c\|\rho_t\|_\infty\big(1 + \mathbb{1}_{\{\alpha=3\}}\log(N)\big)$ we can conclude that $\forall\, m \geq 2$:

$$\big|h(q_j - q_i) - h(q_i)\big|^m \leq C_\alpha^m N^{(\alpha m - 3)\delta}. \tag{3.52}$$

Now, for $\mathbf{k} = (k_1, k_2, ..., k_N) \in \mathcal{M}$, let $\#\mathbf{k}$ denote the number of k_j with $\alpha k_j \neq 0$. Note that if $\#\mathbf{k} > M$, we must have $k_j = 1$ for at least one $1 \leq j \leq N$, so that $\mathbb{E}_t(G^{\mathbf{k}}) = 0$. For the other multiindices, we get (using that the particles are statistically independent):

$$\begin{aligned}
\mathbb{E}_t(G^{\mathbf{k}}) &= \mathbb{E}_t\Big[\prod_{j=1}^N \big(k_\delta(q_j - q_i) - k * \rho_t(q_i)\big)^{k_j}\Big] \\
&\leq \prod_{j=1}^N \mathbb{E}_t\Big[\big(|h(q_j - q_i)| + |h * \rho_t(q_i)|\big)^{k_j}\Big] \\
&\leq \prod_{j=1}^N C_\alpha^{k_j}\, N^{(\alpha k_j - 3)\delta} \\
&\leq C_\alpha^{2M} N^{2M\alpha\delta} N^{-3\delta\#\mathbf{k}}.
\end{aligned} \tag{3.53}$$

Finally, we observe that for any $l \geq 1$, the number of multiindices $\mathbf{k} \in \mathcal{M}$ with $\#\mathbf{k} = l$ is bounded by

$$\sum_{\#\mathbf{k}=l} 1 \leq \binom{N}{l}(2M)^l \leq (2M)^{2M} N^l.$$

Thus:

$$\begin{aligned}
\mathbb{P}_t(D_1) &\leq \frac{1}{N^{2M(1-\beta)}} \sum_{\mathbf{k}\in\mathcal{M}} \binom{2M}{\mathbf{k}} \mathbb{E}_t(G^{\mathbf{k}}) \\
&\leq C_\alpha^{2M} C_M \frac{N^{2M\alpha\delta}}{N^{2M(1-\beta)}} \sum_{l=1}^{M} N^{(1-3\delta)l} \\
&\leq C_\alpha^{2M} M C_M N^{2M(\alpha\delta+\beta-1)} \max\{N^{M(1-3\delta)}, 1\} \\
&\leq C_\alpha^{2M} M C_M N^{-\epsilon M},
\end{aligned}$$

where C_M is some constant depending on M and

$$\epsilon := \begin{cases} 1 - 2\beta + \delta(3-2\alpha) & \text{if } 3\delta < 1 \\ 2(1-\beta-\alpha\delta) & \text{if } 3\delta \geq 1. \end{cases} \tag{3.54}$$

$\epsilon \geq 0$ according to (3.46). For $2 \leq \alpha < 3$ we conclude the proof by noting that

$$\mathbb{P}_t(D) \leq N\,\mathbb{P}_t(D_1) \leq C_\alpha^{2M} M C_M\, N^{-(\epsilon M+1)}, \tag{3.55}$$

and choosing M so large that $(\epsilon M - 1) = \gamma$. For $\alpha = 3$, however, (3.55) becomes

$$\mathbb{P}_t(D) \leq C'(M)(1+\log(N))^{2M} N^{-(\epsilon M-1)}, \tag{3.56}$$

where $C'(M)$ is some constant depending on M and $\|\rho_t\|_\infty$. This can be rewritten as

$$(1+\log(N))^{2M} N^{-\epsilon M+1} = \Big(\frac{1+\log(N)}{N^{\epsilon/4}}\Big)^{2M} N^{-\frac{\epsilon}{2}M+1}. \tag{3.57}$$

The function $g(x) = \frac{1+\log(x)}{x^{\epsilon/4}}$, $x \in [1,\infty)$ is continuous with $\lim\limits_{x\to\infty} g(x) = 0$. Hence, it has a maximum $C < +\infty$. In particular, $\frac{1+\log(N)}{N^{\epsilon/4}} \leq C$ independent of N and the announced result holds for $\alpha = 3$, as well. □

Corollary 3.7.3. *Let $\mathcal{B}_t, \mathcal{C}_t$ as in Definition 3.7.1. Then we find for any $\gamma > 0$ a constant C_γ such that*

$$\begin{aligned}
\mathbb{P}_0(\mathcal{B}_t) &\geq 1 - \frac{C_\gamma}{N^\gamma}, \\
\mathbb{P}_0(\mathcal{C}_t) &\geq 1 - \frac{C_\gamma}{N^\gamma}.
\end{aligned}$$

In other words, for any fixed t, initial conditions in $\mathcal{B}_t \cap \mathcal{C}_t$ are typical with the measure of "bad" initial conditions decreasing faster than any inverse power of N.

Proof. Note that

$$\begin{aligned}
Z \in \Phi_{t,0}(\mathcal{B}_t) &\iff \big|K(Z) - \overline{K}(Z)\big|_\infty < N^{-1+2\delta} \\
&\iff \max_{1\leq i\leq N} \Big| \frac{1}{N} \sum_{j\neq i}^{N} k_\delta^N(q_i - q_j) - k_\delta^N * \rho_t(q_i) \Big| \geq N^{-1+2\delta}
\end{aligned}$$

and similarly

$$
\begin{aligned}
Z \in \Phi_{t,0}(\mathcal{C}_t) &\iff \big| L(Z) - \overline{L}(Z) \big|_\infty < 1 \\
&\iff \max_{1 \le i \le N} \Big| \frac{1}{N} \sum_{j \neq i}^{N} l_\delta^N (q_i - q_j) - l_\delta^N * \rho_t(q_i) \Big| \ge 1.
\end{aligned}
$$

Applying the previous result once for k_δ^N with $\alpha = 2$ and $\beta = 1 - 2\delta$ and once for l_δ^N with $\alpha = 3$ and $\beta = 0$, we get

$$
\mathbb{P}_t \Big[\max_{1 \le i \le N} \Big| \frac{1}{N} \sum_{j \neq i}^{N} k_\delta^N (q_i - q_j) - k_\delta^N * \rho_t(q_i) \Big| \ge N^{-1+2\delta} \Big] \le \frac{C_\gamma}{N^\gamma}, \tag{3.58}
$$

$$
\mathbb{P}_t \Big[\max_{1 \le i \le N} \Big| \frac{1}{N} \sum_{j \neq i}^{N} l_\delta^N (q_i - q_j) - l_\delta^N * \rho_t(q_i) \Big| \ge 1 \Big] \le \frac{C_\gamma}{N^\gamma}. \tag{3.59}
$$

Observing that $\mathbb{P}_0(\mathcal{B}_t) = \mathbb{P}_t(\Phi_{t,0}(\mathcal{B}_t))$ and $\mathbb{P}_0(\mathcal{C}_t) = \mathbb{P}_t(\Phi_{t,0}(\mathcal{C}_t))$, the statement follows. $\square$

3.8 A Gronwall estimate

The following proposition contains the core of the proof or our main theorem, a Gronwall estimate for the growth of $\mathbb{E}_0(J_t^N)$.

Proposition 3.8.1. *Under the assumptions of Thm. 3.4.1, we find for all $\delta < \frac{1}{3}$ and $t > 0$*

$$
\mathbb{E}_0(J_t^N) \le 2N^{-1+3\delta} \exp\Big[2C_l \lambda(N) \int_0^t (\|\rho_s^N\|_\infty + 1)\,\mathrm{d}s\Big]. \tag{3.60}
$$

In particular, $\mathbb{E}_0(J_t^N) \le 2N^{-1+3\delta} e^{t2C_l(C_\rho+1)\lambda(N)}$ with C_ρ as in (3.22).

In order to control the evolution of $J_t^N(Z)$, we will need the following Lemma.

Lemma 3.8.2. *For a function $g : \mathbb{R} \to \mathbb{R}$, we denote by*

$$
\partial_t^+ g(t) := \lim_{\Delta t \searrow 0} \frac{g(t + \Delta t) - g(t)}{\Delta t} \tag{3.61}
$$

the right-derivative with respect to t. Let $g \in C^1(\mathbb{R})$ and $h(t) := \sup\limits_{0 \le s \le t} g(t)$. Then $\partial_t^+ h(t)$ exists and $\partial_t^+ h(t) \le \min\{0, g'(t)\}$ for all t.

Proof. We have to distinguish 3 cases.
1) If $g'(t) \le 0$, there exists $\Delta t > 0$ such that $g(s) \le g(t), \forall s \in [t, t + \Delta t)$. Thus for all $t' \in [t, t + \Delta t)$ we have $h(t') := \sup\limits_{0 \le s \le t'} g(s) = \sup\limits_{0 \le s \le t} g(s) = h(t)$ and $\partial_t^+ h(t) = 0$.
2) If $g(t) < h(t)$, there exists $\Delta t > 0$ such that $g(s) \le h(t) \forall s \in (t - \Delta t, t + \Delta t)$. This means that h is constant on $(t - \Delta t, t + \Delta t)$ so that, in particular, $\partial_t^+ h(t) = 0$.
3) If $g(t) = h(t)$ and $g'(t) > 0$, there exists $\Delta t > 0$ such that g is monotonously increasing on $(t - \Delta t, t + \Delta t)$. Hence, we have $h(t') = \sup\limits_{0 \le s \le t'} g(s) = g(t')$ for all $t' \in [t + \Delta t)$ and thus $\partial_t^+ h(t) = g'(t)$. $\square$

Proof of Proposition 3.8.1. Recall from Definition 3.5.1

$$\begin{aligned} J_t^N(Z) := \min\Big\{1, \lambda(N) N^\delta \sup_{0\le s\le t} |{}^N\Psi^1_{t,0}(Z) - {}^N\Phi^1_{t,0}(Z)|_\infty \\ + N^\delta \sup_{0\le s\le t} |{}^N\Psi^2_{t,0}(Z) - {}^N\Phi^2_{t,0}(Z)|_\infty\Big\}. \end{aligned}$$

We split the expectation $\mathbb{E}_0(J_t)$ in the following way:

$$\mathbb{E}_0(J_t) = \mathbb{E}_0(J_t \mid \mathcal{A}_t^c) + \mathbb{E}_0(J_t \mid \mathcal{A}_t \setminus \mathcal{B}_t \cap \mathcal{C}_t) + \mathbb{E}_0(J_t \mid \mathcal{A}_t \cap \mathcal{B}_t \cap \mathcal{C}_t) \tag{3.62}$$

where $J_t \mid \mathcal{A}_t$ denotes the restriction of J_t to the set $\mathcal{A}_t \subset \mathbb{R}^{6N}$.

1) On $\mathcal{A}_t^c$, we have $\frac{\mathrm{d}}{\mathrm{d}t} J_t = 0$, since $J_t(Z)$ is already maximal and thus also

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \mathbb{E}_t(J_t \mid \mathcal{A}_t^c) \le 0. \tag{3.63}$$

2) For $Z \in \mathcal{A}_t$, we have to consider

$$\begin{aligned} \partial_t^+ \sup_{0\le s\le t} |\Psi^1_{s,0}(Z) - \Phi^1_{s,0}(Z)|_\infty &\le |\partial_t(\Psi^1_{t,0}(Z) - \Phi^1_{t,0}(Z))|_\infty \\ &\le |\Psi^2_{t,0}(Z) - \Phi^2_{t,0}(Z)|_\infty \le \sup_{0\le s\le t} |\Psi^2_{s,0}(Z) - \Phi^2_{s,0}(Z)|_\infty \end{aligned} \tag{3.64}$$

and

$$\begin{aligned} \partial_t^+ \sup_{0\le s\le t} |\Psi^2_{s,0}(Z) - \Phi^2_{s,0}(Z)|_\infty &\le |\partial_t(\Psi^2_{t,0}(Z) - \Phi^2_{t,0}(Z))|_\infty \\ &\le |K(\Psi^1_{t,0}(Z)) - \overline{K}_t(\Phi^1_{t,0}(Z))|_\infty. \end{aligned} \tag{3.65}$$

We begin by controlling the contribution of "bad" initial conditions *not* contained in $\mathcal{B}_t$ and $\mathcal{C}_t$. Since k_δ^N is bounded by $N^{2\delta-1}$, the total force acting on each particle is bounded as $|K(Z)|_\infty \le N^{2\delta}$. The mean field force $\overline{K}$ is of order 1, according to Lemma 3.3.4 and $N^\delta |\Psi^2_{t,0}(Z) - \Phi^2_{t,0}(Z)|_\infty \le 1$ since $Z \in \mathcal{A}_t$.

According to Proposition 3.7.2, the probability for $Z \in \mathcal{B}_t^c \cup \mathcal{C}_t^c$ decreases faster than any power of N. Hence, we can find for any $s > 0$ a constant C_s, such that

$$\partial_t^+ \mathbb{E}_t(J_t \mid \mathcal{A}_t \setminus (\mathcal{B}_t \cap \mathcal{C}_t)) \le \sup\{|J_t^N(Z)| : Z \in \mathcal{A}_t\}\, \mathbb{P}_0\big((\mathcal{A}_t \cap \mathcal{B}_t)^c\big) \le \frac{C_s}{N^s}. \tag{3.66}$$

3) It remains to control the change of J_t for typical initial conditions, i.e. $Z \in \mathcal{A}_t \cap \mathcal{B}_t \cap \mathcal{C}_t$. To this end, we consider:

$$|K(\Psi^1_{t,0}(Z)) - \overline{K}_t(\Phi^1_{t,0}(Z))|_\infty \le |K(\Psi^1_{t,0}(Z)) - K(\Phi^1_{t,0}(Z))|_\infty \tag{3.67}$$

$$+ |K(\Phi^1_{t,0}(Z)) - \overline{K}_t(\Phi^1_{t,0}(Z))|_\infty. \tag{3.68}$$

Since $Z \in \mathcal{B}_t$, it follows that

$$|K(\Phi^1_{t,0}(Z)) - \overline{K}_t(\Phi^1_{t,0}(Z))|_\infty < N^{-1+2\delta}, \tag{3.69}$$

which controls (3.68). Now, by triangle inequality, we get for any $1 \le i \le N$:

$$\Big|\big(K(\Psi^1_{t,0}(Z)) - K(\Phi^1_{t,0}(Z))\big)_i\Big|_\infty \le \Big|\sum_{j=1}^{N} k^N_\delta(\Psi^1_j - \Psi^1_i) - k^N_\delta(\Phi^1_j - \Phi^1_i)\Big|_\infty$$
$$\le \sum_{j=1}^{N} \big|k^N_\delta(\Psi^1_j - \Psi^1_i) - k^N_\delta(\Phi^1_j - \Phi^1_i)\big|_\infty.$$

Thus, with Lemma 3.6.3:

$$\big|k^N_\delta(\Psi^1_j - \Psi^1_i) - k^N_\delta(\Phi^1_j - \Phi^1_i)\big|_\infty \le l^N_\delta(\Phi^1_j - \Phi^1_i)|(\Psi^1_j - \Psi^1_i) - (\Phi^1_j - \Phi^1_i)|_\infty$$
$$\le 2\, l^N_\delta(\Phi^1_j - \Phi^1_i)|\Psi^1_{t,0} - \Phi^1_{t,0}|_\infty.$$

Since $Z \in \mathcal{C}_t$, it follows that

$$\sum_{j=1}^{N} l^N_\delta(\Phi^1_j - \Phi^1_i) = \big(L^N_\delta(\Phi_{t,0}(Z)\big)_i \le \|l^N_\delta * \rho^N_t(q)\|_\infty + 1 \le 2C_l \max\{1, \sqrt{\log(N)}\}(1 + \|\rho^N_t\|_\infty),$$

where $\rho^N_t = \rho[f^N_t]$ and we applied Lemma 3.6.1 to l^N_δ. Hence, we have found for $Z \in \mathcal{A}_t \cap \mathcal{B}_t \cap \mathcal{C}_t$:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Psi^2_t(Z) - \Phi^2_{t,0}(Z)|_\infty \le 2C_l \max\{1, \sqrt{\log(N)}\}(1 + \|\rho^N_t\|_\infty)\,\big|\Psi^1_{t,0}(Z) - \Phi^1_{t,0}(Z)\big|_\infty. \tag{3.70}$$

Together with (3.64), this yields:

$$\begin{aligned}\partial^+_t J_t\Big|_{\mathcal{A}_t\cap\mathcal{B}_t\cap\mathcal{C}_t} &\le \lambda(N)N^\delta \frac{\mathrm{d}}{\mathrm{d}t}|\Psi^1_{t,0}(Z) - \Phi^1_{t,0}(Z)|_\infty + N^\delta \frac{\mathrm{d}}{\mathrm{d}t}|\Psi^2_{t,0}(Z) - \Phi^2_{t,0}(Z)|_\infty \\ &\le \lambda(N)N^\delta|\Psi^2_{t,0}(Z) - \Phi^2_{t,0}(Z)|_\infty \\ &+N^\delta\Big[2C_l \max\{1, \sqrt{\log(N)}\}\big(1 + \|\rho^N_t\|_\infty\big)|\Psi^1_{t,0}(Z) - \Phi^1_{t,0}(Z)|_\infty + N^{-1+2\delta}\Big].\end{aligned}$$

Hence, fixing $\lambda(N) := \max\{1, \sqrt{\log(N)}\}$,we have found

$$\partial^+_t J_t\Big|_{\mathcal{A}_t\cap\mathcal{B}_t\cap\mathcal{C}_t} \le 2C_l\big(1 + \|\rho^N_t\|_\infty\big)\,\lambda(N)J_t(Z) + N^{-1+3\delta}. \tag{3.71}$$

Together with (3.66) and (3.64) and observing that $\mathbb{E}_0(J_0) = 0$, we have for any $t \ge 0$ and some $s > 1$:

$$\mathbb{E}_0(J^N_t) \le \int_0^t \Big(CN^{-s} + 2C_l(1 + \|\rho^N_s\|_\infty)\,\lambda(N)\,\mathbb{E}_0(J^N_s) + N^{-1+3\delta}\Big)\,\mathrm{d}s.$$

With Gronwall's Lemma and choosing s large enough, we conclude

$$\mathbb{E}_0(J^N_t) \le 2N^{-1+3\delta}\exp\big[2C_l\lambda(N)\int_0^t (\|\rho^N_s\|_\infty + 1)\,\mathrm{d}s\big].$$

$\square$

3.9 Controlling the mean field dynamics

The previous proposition contains our main approximation result for the mean field dynamics. However, as explained in Section 3.5, two more steps remain in order to complete the proof of Theorem 3.4.1 and show that the empirical density converges to solutions of the Vlasov-Poisson equation for typical initial conditions. First, we have to show that the solutions f_t^N of the regularized Vlasov-Poisson equation converge to a solution of the proper Vlasov-Poisson equation as the cut-off is lifted with $N \to \infty$. Second we have to prove the approximation of the continuous Vlasov-density by the discretized version $\mu^N[\Phi_{t,0}(Z)]$, in (3.32). To this end, we recall from Section 2.3.

Proposition (Loeper). Let $k(q)$ be the Coulomb-kernel and $\rho_1, \rho_2 \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ two (probability) densities. Then we have the stability result

$$\|k * \rho_1 - k * \rho_2\|_2 \leq \left[\max\{\|\rho_1\|_\infty, \|\rho_2\|_\infty\}\right]^{1/2} W_2(\rho_1, \rho_2). \tag{3.72}$$

From this we derive the following approximation result:

Proposition 3.9.1. *Let f_0 satisfy the assumptions of Theorem 3.4.1. For $N > 3$, let f_t^N and f_t be the solution of the regularized, respectively the unregularized Vlasov-Poisson equation with initial datum f_0. Then we have for $p \in [1, 2]$:*

$$W_p(f_t^N, f_t) \leq N^{-\delta}\ e^{tC_0\lambda(N)}, \tag{3.73}$$

with $\lambda(N) = \max\{1, \sqrt{\log(N)}\}$ and $C_0 := C_l(1+C_\rho)$ depending on $\sup_{t,N}\{\|\rho_t^N\|_\infty, \|\rho_t^f\|_\infty\}$.

Proof. Let $\rho_t^N := \rho[f_t^N]$ and $\rho_t^f := \rho[f_t]$ denote the charge density induced by f_t^N and f_t, respectively. Let $\varphi_t^N = (Q_t^N, P_t^N)$ be the characteristic flow of f_t^N. For the (unregularized) Vlasov-Poisson equation, the corresponding vector-field is not Lipschitz. However, as we assume the existence of a solution f_t with bounded density ρ_t, the mean field force $k * \rho_t$ does satisfy a Log-Lip bound of the form $|k * \rho_t(x) - k * \rho_t(y)| \leq C|x-y|(1+\log^-(|x-y|))$, where $\log^-(x) = \max\{0, -\log(x)\}$. This is sufficient to ensure the existence of a characteristic flow $\psi_{t,s}^f = (Q_{t,s}^f, P_{t,s}^f)$ such that $f_t = \psi_{t,s}\# f_s$.

Now we consider $\pi_0(x,y) := f_0(x)\delta(x-y) \in \Pi(f_0, f_0)$, which is the optimal coupling yielding $W_2(f_t^N, f_t)|_{t=0} = W_2(f_0, f_0) = 0$ and define $\pi_t = (\varphi_{t,0}^N, \psi_{t,0})\#\pi_0$. Then $\pi_t \in \Pi(f_t^N, f_t)$, $\forall t \in [0, T)$. Set

$$\begin{aligned} D(t) &:= \Big[\int\limits_{\mathbb{R}^6\times\mathbb{R}^6} \Big(\lambda(N)\,|x^1 - y^1| + |x^2 - y^2|\Big)^2 \,\mathrm{d}\pi_t(x,y)\Big]^{1/2} \\ &= \Big[\int\limits_{\mathbb{R}^6\times\mathbb{R}^6} \Big(\lambda(N)\,|Q_t^N(x) - Q_t^f(y)| + |P_t^N(x) - P_t^f(y)|\Big)^2 \,\mathrm{d}\pi_0(x,y)\Big]^{1/2}. \end{aligned} \tag{3.74}$$

Note that $W_2(f_t^N, f_t) \leq D(t)$ for any $\pi_0 \in \Pi(f_0, f_0)$ and $N \in \mathbb{N}$. Now we compute:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} D^2(t) = \\
&\qquad 2 \int \Big(\lambda(N)\, |Q_t^N(x) - Q_t^f(y)| + |P_t^N(x) - P_t^f(y)| \Big) \\
&\qquad\qquad \Big(\lambda(N)\ |P_t^N(x) - P_t^f(y)| + \left| k_\delta^N * \rho_t^N (Q_t^N(x)) - k * \rho_t^f (Q_t^f(y)) \right| \Big)\, \mathrm{d}\pi_0(x,y).
\end{aligned} \tag{3.75}$$

The interesting term to control is the interaction term

$$\begin{aligned}
&\left| k_\delta^N * \rho_t^N \left(Q_t^N(x)\right) - k * \rho_t^f \left(Q_t^f(y)\right) \right| \\
&\leq \left| k_\delta^N * \rho_t^N \left(Q_t^N(x)\right) - k_\delta^N * \rho_t^N \left(Q_t^f(y)\right) \right. && (3.76) \\
&+ \left| k_\delta^N * \rho_t^N \left(Q_t^f(y)\right) - k * \rho_t^f \left(Q_t^f(y)\right) \right|. && (3.77)
\end{aligned}$$

We begin with (3.76) and find with Lemma 3.6.1:

$$\begin{aligned}
&\left| k_\delta^N * \rho_t^N \left(Q_t^N(x)\right) - k_\delta^N * \rho_t^N \left(Q_t^f(y)\right) \right| \\
&\leq C_l \ \max\{1, \log(N)\} \left(\|\rho_t^N\|_\infty + 1\right) \left| Q_t^N(x) - Q_t^f(y) \right| \\
&= C_l \, \lambda(N) (\|\rho_t^N\|_\infty + 1) \left| Q_t^N(x) - Q_t^f(y) \right|
\end{aligned} \tag{3.78}$$

Using this in (3.75) we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} D^2(t) = \\
&\qquad 2 \int \Big(\lambda(N)\, |Q_t^N(x) - Q_t^f(y)| + |P_t^N(x) - P_t^f(y)| \Big) && (3.79) \\
&\qquad\qquad \Big(\lambda(N)\ |P_t^N(x) - P_t^f(y)| + \lambda(N) C_l \,(C_\rho + 1) \left| Q_t^N(x) - Q_t^f(y) \right| \Big) \mathrm{d}\pi_0(x,y) \\
&\qquad + 2 \int \Big(\lambda(N)\, |Q_t^N(x) - Q_t^f(y)| + |P_t^N(x) - P_t^f(y)| \Big) && (3.80) \\
&\qquad\qquad \left| k_\delta^N * \rho_t^N \left(Q_t^f(y)\right) - k * \rho_t^f \left(Q_t^f(y)\right) \right| \mathrm{d}\pi_0(x,y)
\end{aligned}$$

where we used the uniform bound (3.22) on the charge densities. The first term (3.79) can be bounded as

$$(3.79) \leq 2 C_l (C_\rho + 1) \lambda(N) D^2(t) \tag{3.81}$$

while for (3.80) we use the Cauchy-Schwartz inequality to get

$$\begin{aligned}
(3.80) \leq 2 &\Big[\int \Big(\lambda(N)\, |Q_t^N(x) - Q_t^f(y)| + |P_t^N(x) - P_t^f(y)| \Big)^2 \mathrm{d}\pi_0(x,y) \Big]^{1/2} && (3.82) \\
&\Big[\int \left| k_\delta^N * \rho_t^N \left(Q_t^f(y)\right) - k * \rho_t^f \left(Q_t^f(y)\right) \right|^2 \mathrm{d}\pi_0(x,y) \Big]^{1/2}. && (3.83)
\end{aligned}$$

We identify the first factor in (3.82) as $2D(t)$. Hence, it remains to estimate

$$
\begin{aligned}
&\Big[\int \big|k_\delta^N * \rho_t^N\,(Q_t^f(y)) - k * \rho_t^f\,(Q_t^f(y))\big|^2 \mathrm{d}\pi_0(x,y)\Big]^{1/2}\\
=&\Big[\int \big|k_\delta^N * \rho_t^N\,(Q_0^f(y)) - k * \rho_t^f\,(Q_0^f(y))\big|^2 \mathrm{d}\pi_t(x,y)\Big]^{1/2}\\
=&\Big[\int |k_\delta^N * \rho_t^N - k * \rho_t^f|^2(q)\; f(t,q,p)\,\mathrm{d}^3q\mathrm{d}^3p\Big]^{1/2}\\
=&\Big[\int |k_\delta^N * \rho_t^N - k * \rho_t^f|^2(q)\,\rho_t^f(q)\,\mathrm{d}^3q\Big]^{1/2}\\
\leq&\|\rho_t^f\|_\infty^{1/2}\Big[\int |k_\delta^N * \rho_t^N - k * \rho_t^f|^2(q)\,\mathrm{d}^3q\Big]^{1/2}\\
\leq&C_\rho^{1/2}\,\big\|k_\delta^N * \rho_t^N - k * \rho_t^f\big\|_2.
\end{aligned}
$$

We split this into:

$$
\|k_\delta^N * \rho_t^N - k * \rho_t^f\|_2 \leq \big\|k * \rho_t^N - k * \rho_t^f\big\|_2 + \big\|k_\delta^N * \rho_t^N - k * \rho_t^N\big\|_2.
$$

According to Proposition 2.3.6, the first summand is bounded by

$$
\big\|k * \rho_t^N - k * \rho_t^f\big\|_2 \leq C_\rho^{1/2}\, W_2(\rho_t^N, \rho_t^f) \leq C_\rho^{1/2}\, W_2(f_t^N, f_t) \leq C_\rho^{1/2}\, D(t).
$$

For the second term, we get with Young's inequality:

$$
\begin{aligned}
\big\|(k-k_\delta^N) * \rho_t^N\big\|_2 \leq \big\|\rho_t^N\big\|_2\big\|k-k_\delta^N\big\|_1 &\leq \big(\|\rho_t^N\|_\infty\,\big\|\rho_t^N\big\|_1\big)^{1/2}\big\|k-k_\delta^N\big\|_1\\
&\leq C_\rho^{1/2}\int\limits_{|q|<N^{-\delta}} \frac{1}{|q|^2}\,\mathrm{d}^3q = 4\pi\, C_\rho^{1/2}\, N^{-\delta},
\end{aligned}
\tag{3.84}
$$

where we used the fact that k_δ^N and k differ only in the ball $\{|q| \leq N^{-\delta}\}$. Putting everything together, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}D^2(t) \leq 2C_l\,(C_\rho+1)\lambda(N)\,D^2(t) + 2D(t)\big(4\pi C_\rho\, N^{-\delta} + C_\rho D(t)\big)
$$

or, with $C_0 := 2(C_l+1)(C_\rho+1)$,

$$
\frac{\mathrm{d}}{\mathrm{d}t}D(t) \leq C_0\lambda(N)\,D(t) + C_0 N^{-\delta}. \tag{3.85}
$$

Using Gronwall's inequality and the fact that $D(0)=0$, we conclude

$$
W_2(f_t^N, f_t) \leq D(t) \leq\; N^{-\delta}\, e^{tC_0\lambda(N)}.
$$

The case $p<2$ is included since $W_p \leq W_2$, $\forall p \leq 2$. □

A more detailed discussion of this method will be given in Chapter 4.

A similar, but simpler Gronwall estimate yields the following result:

Lemma 3.9.2. *Let $\varphi_t^N = (Q(t,\cdot), P(t,\cdot))$ the characteristic flow of f_t^N defined by* (3.12) *and $\Phi_{t,s}$ the lift to the N-particle phase-space defined in* (3.14)*. Then we have for all $p \in [1,\infty)$:*

$$W_p(\mu^N[\Phi_{t,0}(Z)], f_t^N) \leq \lambda(N)\, W_p(\mu_0^N[Z], f_0)\, e^{tC_0\lambda(N)}. \tag{3.86}$$

Proof. For $Z \in \mathbb{R}^{6N}$ let $\pi_0(x,y) \in \Pi(\mu_0^N, f_0)$ and define $\pi_t = (\varphi_t^N, \varphi_t^N)\#\pi_0 \in \Pi(\mu^N[\Phi_{t,0}(Z)], f_t^N)$. Note that both measures are now transported with the same flow. Set

$$\begin{aligned} D_p(t) &:= \Big[\int\limits_{\mathbb{R}^6\times\mathbb{R}^6} \Big(\lambda(N)\,|x^1 - y^1| + |x^2 - y^2|\Big)^p \,\mathrm{d}\pi_t(x,y)\Big]^{1/p} \\ &= \Big[\int\limits_{\mathbb{R}^6\times\mathbb{R}^6} \Big(\lambda(N)\,|Q_t(x) - Q_t(y)| + |P_t(x) - P_t(y)|\Big)^p \,\mathrm{d}\pi_0(x,y)\Big]^{1/p}. \end{aligned} \tag{3.87}$$

Using again the Lipschitz bound as in (3.78), a standard argument yields

$$D(t) \leq D(0) + C_0\lambda(N) \int D(s)\,\mathrm{d}s, \tag{3.88}$$

and hence by Gronwall's inequality:

$$W_p(\mu^N[\Phi_{t,0}(Z)], f_t^N) = W_p(\varphi_t^N\#\mu_0^N, \varphi_t^N\#f_t) \leq D(t) \leq D(0)e^{tC_0\lambda(N)}. \tag{3.89}$$

Taking on the right-hand side the infimum over all $\pi_0(x,y) \in \Pi(\mu_0^N, f_0)$,

$$W_p(\mu^N[\Phi_{t,0}(Z)], f_t^N) \leq \lambda(N)\, W_p(\mu_0^N[Z], f_0)\, e^{tC_0\lambda(N)}, \tag{3.90}$$

so that the announced statement follows. □

In view of (3.86), it remains to establish an upper bound on the typical rate of convergence for $W_p(\mu_0^N[Z], f_0) \to 0$. (Note that, other than that, the result of Lemma 3.9.2 is actually deterministic.) Fortunately, we can rely for this purpose on the large deviation estimates of Fournier and Guillin, Thmeorem 2.2.1, that we cited in Chapter 2.

Proposition 3.9.3. *Let $p \in [1,2]$ and $\gamma < \frac{1}{6}$. Then there exists a constants $c, C > 0$ such that*

$$\mathbb{P}_0\Big[\exists t \in [0,T] : W_p(\mu^N[\Phi_{t,0}(Z)], f_t^N) \geq \lambda(N)N^{-\gamma}e^{tC_0\lambda(N)}\Big] \leq C\big(e^{-cN^{1-6\gamma}} + N^{-1+2p\gamma}\big) \tag{3.91}$$

Proof. By assumption of Thm. 3.4.1, there exists $m > 2p$ such that $\int |q|^m f_0(q,p)\,\mathrm{d}q\,\mathrm{d}p < +\infty$. Applying Thm. 2.2.1 with $\xi = N^{-p\gamma}, \epsilon = m - 2p$ and the finite-moment condition (1), we get constants $C, c > 0$ such that

$$\mathbb{P}_0\Big[W_p(\mu_0^N[Z], f_0) > N^{-\gamma}\Big] \leq C\big(e^{-cN^{1-6\gamma}} + N^{-1+2p\gamma}\big).$$

Thus with Lemma 3.9.2, the statement follows. □

Now we have everything in place to complete the proof of our main theorem.

3.9.1 Proof of the main theorem

Let $p \in [1,2]$ and $\gamma < \frac{1}{6}$. We split the approximation into

$$\begin{aligned} W_p(\mu_t^N[Z], f_t) \leq\, & W_p(\mu^N[\Psi_{t,0}(Z)], \mu^N[\Phi_{t,0}(Z)]) \\ & + W_p(\mu^N[\Phi_{t,0}(Z)], f_t^N) \\ & + W_p(f_t^N, f_t). \end{aligned}$$

According to Proposition 3.9.3, we have constants $c, C > 0$ such that

$$\mathbb{P}_0\Big[\exists t \in [0,T] : W_p(\mu^N[\Phi_{t,0}(Z)], f_t^N) \geq N^{-\gamma}\lambda(N)e^{tC_0\lambda(N)}\Big] \leq C\big(e^{-cN^{1-6\gamma}} + N^{-1+2p\gamma}\big).$$

According to Proposition 3.9.1, we have

$$W_p(f_t^N, f_t) \leq N^{-\delta}\, e^{tC_0\lambda(N)}. \tag{3.92}$$

From Proposition 3.5.3:

$$\mathbb{P}_0\Big[\exists t \in [0,T] : W_p(\mu^N[\Psi_{t,0}(Z)], \mu^N[\Phi_{t,0}(Z)]) \geq N^{-\delta}\Big] \leq \mathbb{E}_0(J_T). \tag{3.93}$$

Putting everything together and choosing $\gamma < \min\{\frac{1}{6}, \delta\}$ we have found

$$\begin{aligned} \mathbb{P}_0\Big[\exists t \in [0,T] : W_p(\mu_t^N[Z], f_t) \geq 3\lambda(N)N^{-\gamma}\, e^{t(C_0+1)\lambda(N)}\Big] & \\ \leq \mathbb{E}_0(J_T) + C\big(e^{-cN^{1-6\gamma}} + N^{-1+2p\gamma}\big). & \end{aligned} \tag{3.94}$$

Recalling Proposition 3.8.1 and the fact that $\mathbb{E}_0(J_T) < 2N^{-1+3\delta}e^{TC_0\lambda(N)}$, the theorem is proven. For simplicity, we demand $N \geq 4$ so that $\lambda(N) = \sqrt{\log(N)}$.

□

3.10 Weaker singularities, open questions

While the present paper focuses on the Vlasov-Poisson equation, the method presented here can, of course, be applied to interactions with milder singularities (see [6]). For better comparison with other approaches, in particular the reference paper [26], we shall state here the corresponding results without further proof. Generalization to higher dimensions would be straight-forward, as well.
We use the characterization of force kernels introduced in Definition 3.2.1.

Theorem 3.10.1. *Let $\alpha < 2$. Let k satisfy a (S^α) condition and k_δ^N satisfy a (S_δ^α) condition with the additional assumption* (3.9) *and*

$$\delta < \frac{1}{1+\alpha}. \tag{3.95}$$

Assume (for simplicity) that $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R}^+)$, normalized to $\int f_0 = 1$ has compact support and let f_t the unique solution of the Vlasov equation with force kernel k. For $Z \in \mathbb{R}^{6N}$, let $\mu_t^N[Z]$ the unique weak solution of the (regularized) Vlasov equation with *force* k_δ^N *and initial data* $\mu_0^N[Z]$. *Then we have molecular chaos in the sense that for all* $\gamma \leq \min\{\frac{1}{6}, \delta\}$ *and all* $T > 0$ *and sufficiently large* N*:*

$$\mathbb{P}_0\big[\exists t \in [0,T] : W_1(\mu_t^N[Z], f_t) \geq e^{C_1 t} N^{-\gamma}\big] \leq 2e^{C_1 T} N^{-1+(\alpha+1)\delta} + C_2 N^{-1+2\gamma} \tag{3.96}$$

with constants C_1, C_2 *depending on* f_0 *and* α.

This can be compared to the results in Hauray and Jabin, 2013 [26], where a statement similar to (3.96) is derived for the case $1 \leq \alpha < 2$ with a cut-off of order

$$\delta < \frac{1}{6} \min\Big\{\frac{1}{\alpha - 1}, \frac{5}{\alpha}\Big\}. \tag{3.97}$$

For $\alpha \in [1, 2)$, the upper bound on δ given by (3.97) ranges between $\frac{5}{6}$ and $\frac{1}{6}$, while our upper bound from (3.95) ranges between $\frac{1}{2}$ and $\frac{1}{3}$. In particular, it is interesting to note that the cut-off required in [26] is smaller than ours for $\alpha < \frac{7}{5}$ but larger for $\frac{7}{5} < \alpha < 2$.

This suggests that the purely probabilistic estimates presented here fare better for strong singularities – in the sense of admitting a significantly smaller cut-off – while the method proposed in [26] provides better controls for mild singularities. In particular, Hauray and Jabin are able to treat the case $0 < \alpha < 1$ without cut-off, by providing an explicit control on the minimal particle distance (in (p, q)-space, strictly speaking, while integrating the forces over small time-intervals). As it stands, our method requires some microscopic regularization even for very mild singularities. Since it proves quite effective in this settings, it would be interesting to investigate if it can be improved – or combined with the approach of [26] – to further reduce the cut-off or dispense with it altogether in some cases. We will expand on this discussion in the final chapter.

Chapter 4

Vlasov-Poisson as a mean field limit of extended charges

In this chapter, we are going to propose an alternative derivation of the Vlasov-Poisson system, based on a variant of the Wasserstein distance and the stability result of Loeper discussed in 2.3. As microscopic regularization, we will consider an N-particle Coulomb system of extended charges with an N-dependent radius that goes to 0 in the limit $N \to \infty$. This model can be understood as the nonrelativistic limit of the Abraham model of rigid charges that we are going use as a regularization of the Maxwell-Lorentz dynamics when we discuss the Vlasov-Maxwell system in the next chapter.

While so far, we restricted our discussion to the Vlasov-Poisson system in 3-dimensional space in order to keep the presentation more simple, we will now opt for generality and formulate our results in dimensions $d \geq 2$. The result presented here is weaker than the one in the previous chapter, in the sense that it requires a significantly larger cut-off of order $N^{-\delta}$ with $\delta < \frac{1}{d(2+d)}$, to be compared with $\delta < \frac{1}{d}$ in Theorem 3.4.1, but yields better rates of convergence depending on integrability properties of the initial Vlasov density f_0. It is also interesting in view of the alternative techniques and in preparation for our discussion of the Vlasov-Maxwell equations.

4.1 The d-dimensional Vlasov-Poisson equation

The d-dimensional Vlasov-Poisson equations ($d \geq 2$) reads

$$\partial_t f + p \cdot \nabla_q f + (k * \rho_t) \cdot \nabla_p f = 0 \tag{4.1}$$

where

$$\rho_t(q) = \rho[f_t](q) = \int \mathrm{d}^3 p \; f(t, q, p) \tag{4.2}$$

is, as usual, the charge density induced by the distribution $f(t, p, q)$. The Coulomb kernel takes the form

$$k(q) := \sigma \frac{q}{|q|^d}, \quad \sigma = \{\pm 1\} \tag{4.3}$$

where $\sigma = +1$ corresponds to the electrostatic (repulsive) case and $\sigma = -1$ to the gravitational (attractive) case.

More generally, the Coulomb kernel in arbitrary dimensions can be derived from Poisson's equation - which is the simplest rotational invariant differential equation, determining how a source generates a potential. The unique solution of

$$\begin{aligned} -\Delta\Phi &= \sigma\rho \\ \lim_{|q|\to+\infty} \Phi(q) &= 0 \;\text{ on } \mathbb{R}^d \end{aligned} \tag{4.4}$$

is given by

$$\Phi(q) = \frac{1}{\alpha(d)} \int \frac{\sigma}{|q'-q|^{d-2}}\, \rho(q')\, \mathrm{d}^d q'; \;\text{ if } d \geq 3$$

with $\alpha(d) = d(d-2)|\mathrm{B}^d(1)|$, $|\mathrm{B}^d(1)|$ the volume of the d-dimensional unit ball, or

$$\Phi(q) = -\frac{1}{2\pi} \int \log(|q-q'|)\rho(q')\, \mathrm{d}q'; \;\text{ for } d = 2.$$

Then, the force is given by

$$-\nabla\Phi(q) = k * \rho(q) = \frac{\sigma}{\alpha(d)} \int \frac{q-q'}{|q-q'|^d}\, \rho(q')\, \mathrm{d}^d q'.$$

(For convenience, one shifts the constant $\alpha(d)$ to the right-hand-side of (4.4), so it doesn't appear in (4.3).)

4.2 The microscopic model

As a microscopic model, we consider a system of N charges, smeared out by a smooth, non-negative, spherically symmetric form factor $\chi \in C_0^\infty(\mathbb{R}^d)$. We shall assume that χ satisfies:

i) $\mathrm{supp}(\chi) \subseteq \mathrm{B}^d(1;0) = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$

ii) $\|\chi\|_\infty = \sup_{x\in\mathbb{R}} |\chi(x)| = 1$

iii) $\|\chi\|_1 = \int \chi(x)\mathrm{d}x = 1$

For the point-particle limit, we define a *rescaled* form factor as follows:

Definition 4.2.1. We call a sequence $(r_N)_{N\in\mathbb{N}}$ of positive real numbers a *rescaling sequence* if it is monotonously decreasing with $r_1 = 1$ and $\lim\limits_{N\to\infty} r_N = 0$. For any $N \in \mathbb{N}$, we then define

$$\chi^N(x) := \frac{1}{r_N^d}\, \chi\Big(\frac{x}{r_N}\Big). \tag{4.5}$$

This rescaled form factor satisfies:

i') $\mathrm{supp}\big(\chi^N\big) \subseteq B(r_N;0)$

ii') $\|\chi^N\|_\infty = r_N^{-d}$

iii') $\|\chi^N\|_1 = \int \chi^N(x)\mathrm{d}x = 1$

The cut-off parameter r_N can be interpreted as a finite electron radius, which is formally sent to 0 in the limit $N \to 0$.

We denote the configuration of the microscopic system by $Z(t) = (q_i(t), p_i(t))_{1\leq i\leq N}$, where $q_i(t)$ is the center of mass of particle i, and $p_i(t)$ the corresponding momentum at time t. For fixed $N \in \mathbb{N}$, the equations of motion in the mean field scaling read:

$$\begin{cases} \dot{q}_i(t) = p_i(t) \\ \dot{p}_i(t) = K^N(q_i; q_1, ..., q_N) \end{cases} \tag{4.6}$$

with

$$K^N(q_i\,; q_1, ..., q_N) := \frac{1}{N}\sum_{j=1}^{N} \int\int \chi^N(q_j - y)k(z-y)\chi^N(q_i - z)\,\mathrm{d}^d y\,\mathrm{d}^d z. \tag{4.7}$$

The N-particle force (4.7) can be rewritten in the following way: Given the microscopic distribution

$$\mu_t^N[Z] = \mu^N[Z(t)] = \frac{1}{N}\sum_{i=1}^{N} \delta_{q_i(t)}\delta_{p_i(t)}, \tag{4.8}$$

one easily checks that

$$K^N(\cdot\,; q_1, ..., q_N) = \chi^N * k * \chi^N * \rho[\mu_t^N] =: \tilde{k} * \tilde{\rho}[\mu_t^N],$$

where we introduce the notation

$$\tilde{\varphi} := \chi^N * \varphi, \text{ for } \varphi : \mathbb{R}^3 \to \mathbb{R}^k. \tag{4.9}$$

The smeared charge density, i.e. the charge density of the extended particles thus corresponds to

$$\tilde{\rho}_t(q) := \frac{1}{N}\sum_{i=1}^{N} \chi^N(q - q_i(t)) \tag{4.10}$$

to be compared with the point-charge density

$$\rho_t(q) := \frac{1}{N}\sum_{i=1}^{N} \delta(q - q_i(t)). \tag{4.11}$$

In the limit $N \to \infty, r_N \to 0$, we have $\chi^N(\cdot - q_i) \rightharpoonup \delta(\cdot - q_i)$ in the sense of distributions (see Lemma 4.2.2 below), so that (4.10) approximates (4.11).

Except for the scaling factor N^{-1}, these equations describe the regular Coulomb dynamics for smeared charges with form factor χ^N. The double-convolution results from the fact that the charge enters the interaction-term quadratically; In other words, the charges acting and the charge being acted upon are both smeared out. Note that this system is Hamiltonian for

$$H(q_i, p_i) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2N} \sum_{i,j} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \chi(y - q_i) \frac{\sigma}{|z - y|^{d-1}} \chi(z - q_j) \mathrm{d}y \, \mathrm{d}z$$

and thus conserves total energy. Note also that – in contrast to the previous chapter – these dynamics contain self-interactions.

In view of (4.10), we note the following Lemma concerning the 'smearing' of measures.

Lemma 4.2.2.
Let $\chi \in C_0^\infty(\mathbb{R}^d)$, $(r_N)_N$ a rescaling sequence and χ^N the rescaled form factor as in (4.5). For a probability measure $\nu \in \mathcal{P}(\mathbb{R}^d)$, we define $\tilde{\nu} = \chi^N *_x \nu$, i.e. $\int h(x) \mathrm{d}\tilde{\nu}(x) = \int \tilde{h}(x) \mathrm{d}\nu(x)$ for all bounded, continuous h. Then we have for all $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and $1 \leq p < \infty$

i) $W_p(\tilde{\nu}, \nu) \leq r_N$

ii) $W_p(\tilde{\mu}, \tilde{\nu}) \leq W_p(\mu, \nu)$

where W_p denotes the Wasserstein distance of order p.

Proof. i) Define $\pi'(x, y) := \nu(x)\chi^N(x-y)$ and observe that $\int \mathrm{d}x \, \pi'(x, y) = \tilde{\nu}(y)$, $\int \mathrm{d}y \, \pi'(x, y) = \nu(x)$, hence $\pi' \in \Pi(\tilde{\nu}, \nu)$. π' has support in $\{|x - y| < r_N\}$. Thus, we conclude

$$\begin{aligned} W_p(\tilde{\nu}, \nu) &= \inf_{\pi \in \Pi(\nu, \tilde{\nu})} \Big(\int\limits_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, \mathrm{d}\pi'(x, y) \Big)^{1/p} \\ &\leq \Big(\int\limits_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, \mathrm{d}\pi'(x, y) \Big)^{1/p} \leq r_N. \end{aligned}$$

ii) In view of the Kantorovich duality (2.3), we find for $(\phi_1, \phi_2) \in L^1(\mu) \times L^1(\nu)$ with $\phi_1(y) - \phi_2(x) \leq |x - y|^p$:

$$\int \phi_1(x) \, \mathrm{d}\tilde{\mu}(x) - \int \phi_2(y) \, \mathrm{d}\tilde{\nu}(y) = \int (\chi * \phi_1)(x) \, \mathrm{d}\mu(x) - \int (\chi * \phi_2)(y) \, \mathrm{d}\nu(y).$$

But $\chi^N * \phi_1$ and $\chi^N * \phi_2$ also satisfy

$$\begin{aligned} \big|\chi^N * \phi_1(x) - \chi * \phi_2(y)\big| &= \Big| \int \chi^N(z)\phi_1(x - z) \, \mathrm{d}z - \int \chi^N(z)\phi_2(y - z) \, \mathrm{d}z \Big| \\ \leq \int \chi^N(z) \, \big|\phi_1(x - z) - \phi_2(y - z)\big| \, \mathrm{d}z &\leq \int \chi^N(z) \, |x - y|^p \, \mathrm{d}z = |x - y|^p. \end{aligned}$$

Hence, we have

$$\int \phi_1 \, \mathrm{d}\tilde{\mu} - \int \phi_2 \, \mathrm{d}\tilde{\nu} \leq W_p(\mu, \nu),$$

and taking the supremum over all (ϕ_1, ϕ_2) yields the desired inequality. □

4.2.1 A note on the regularization.

From a purely formal point of view, the regularization thus introduced is merely a special case of the ones treated in the previous chapter with $k^N := \chi^N * \chi^N * k$ satisfying a (S^α_δ)-condition with $\alpha = 2$ and δ depending on the rescaling sequence r_N. However, the approach that we want to present now will take the physical picture of smeared charges seriously in a certain sense. Recall that the stability result for the Coulomb force, Proposition 2.3.6, applies to bounded charge densities. In particular, it does not apply to point-charge densities of the form $\rho^\mu = \frac{1}{N}\sum\limits_{i=1}^{N} \delta_{q_i}$ which are singular measures. The idea is thus to consider smeared densities of the form (4.10), corresponding to extended charges with N-dependent form factor. We will then take the mean field limit together with the point-particle limit $r_N \to 0$ in a such a way as to assure that the charge density typically remains bounded. Intuitively, this describes a situation in which a large number of small, extended particles blur into a continuous *charge cloud.*

While the smearing of charges is a natural way of regularizing point-interactions, the cut-off thus introduced must still be considered a technical necessity rather than a realistic physical model. In the context of the relativistic field theory, considered in the next chapter, the issue is a bit more subtle and will be discussed in due course.

4.2.2 The regularized Vlasov-Poisson equation

For the microscopic model described above, we introduce a corresponding mean field equation:

$$\begin{aligned} &\partial_t f + p\cdot\nabla_q f + (\tilde{k} * \tilde{\rho})\cdot\nabla_p f = 0,\\ &\tilde{k} := \chi^N * k; \quad \tilde{\rho}_t = \int \chi^N *_q f(t,\cdot,p)\,\mathrm{d}^d p. \end{aligned} \tag{4.12}$$

We call this the *regularized Vlasov-Poisson* system with cut-off parameter r_N.

Definition 4.2.3 (Characteristic flow)**.** Let $\nu = (\nu_t)_{t\in[0,T]}$ a continuous family of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$. Let $\rho_t[\nu](q) = \int \nu(q,p)\,\mathrm{d}^d p$ the induced (charge-)distribution on $\mathbb{R}^d$. We denote by $\varphi^\nu_{t,s}(q_0,p_0) = \big(Q^\nu(t,s,q_0,p_0), P^\nu(t,s,q_0,p_0)\big)$ the one-particle flow on $\mathbb{R}^d\times\mathbb{R}^d$ solving:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}Q = P\\ \frac{\mathrm{d}}{\mathrm{d}t}P = \chi^N * k * \chi^N * \rho(Q)\\ Q(s,s,q_0,p_0) = q_0\\ P(s,s,q_0,p_0) = p_0. \end{cases} \tag{4.13}$$

This flow exists and is well-defined since the vector-field is Lipschitz for all N. If $f^N(t,q,p)$ is a solution of (4.12), it is straight-forward to check that

$$f^N_t = \varphi^{f^N}_{t,s}\#f^N_s, \quad \forall t,s \geq 0. \tag{4.14}$$

Conversely, if f_t is a fixed-point of $(\nu_t) \to \varphi^\nu_t\# f_0$, it is a solution (4.12) with initial datum f_0. In particular, one observes that $Z(t) = (q_i(t),p_i(t))_{i=1,..,N}$ is a solution of (4.6) if and only if $\mu_0^N[Z(t)] = \frac{1}{N}\sum\limits_{i=1}^{N} \delta_{q_i(t)}\delta_{p_i(t)}$ solves (4.12) in the sense of distributions. Basically, our aim is thus to show that this relation carries over to the limit $N \to \infty$.

For the (unregularized) Vlasov-Poisson equation, the corresponding vector-field is not Lipschitz, in general. However, if we assume the existence of a solution f_t with $\rho \in L^\infty([0,T] \times \mathbb{R}^d)$, the mean field force $k * \rho_t$ does satisfy a Log-Lip bound of the form $|k*\rho_t(x) - k*\rho_t(y)| \leq C|x-y||\log(|x-y|)|$ (for $|x-y| < \frac{1}{2}$, let's say, see e.g. [46, Ch. 7]). This is sufficient to ensure the existence of a characteristic flow $\psi_{t,s} = (Q_{t,s}, P_{t,s})$ solving

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} Q_{t,s} = P_{t,s} \\ \frac{\mathrm{d}}{\mathrm{d}t} P_{t,s} = k * \rho[f_t](Q_{t,s}) \\ Q(s,s,q_0,p_0) = q_0 \\ P(s,s,q_0,p_0) = p_0 \end{cases} \tag{4.15}$$

such that $f_t = \psi_{t,s}\# f_s$, for all $0 \leq s \leq t \leq T$.

4.2.3 Existence of solutions

As in the previous chapter, existence and uniqueness of solutions for the regularized Vlasov-Poisson equations (4.12) is standard, since all forces are Lipschitz. In the Coulomb case, the issue is more delicate, in particular with respect to the higher-dimensional problem. For the rest of this chapter, we shall work under the following assumption:

Assumption 4.2.4. Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}_0^+)$ with total mass one. We assume that there exists $T^* > 0$ such that the Vlasov-Poisson system (1-3) has a unique solution $f(t,x,p)$ on $[0,T^*)$ with $f(0,\cdot,\cdot) = f_0$. Moreover, as we consider the sequence of solutions of the regularized equations, the charge density remains bounded uniformly in N and t, i.e. $\forall T < T^* \; \exists C_0 < +\infty$ such that

$$\|\rho[f_t^N]\|_\infty \leq C_0, \; \forall t \in [0,T], \; \forall N \in \mathbb{N} \cup \{+\infty\}, \tag{4.16}$$

where, with a slight abuse of notation, $f_t^\infty := f_t$.

In fact, as stated in the previous chapter, under the assumption of a bounded charge density, the uniqueness of the solution (in the space of bounded positive measures) was proven by Loeper [42]. Existence of weak solutions in arbitrary dimensions was already proven e.g.in [2,15]. Apart from this, the status of the assumption is the following: In the physically most relevant, 3-dimensional case, we can rely on the various results cited in the previous chapter. In particular, the theorem of Lions and Perthame, Thm. 3.3.1, ensures that (4.16) is satisfied for a reasonably large class of initial distributions and $T^* = +\infty$. The situation is similar in the 2-dimensional case, which is treated in [67,73]. In dimensions $d \geq 4$, where blow-up might occur, there exists at least some $T^* > 0$, depending only on f_0, such that (4.16) is satisfied if we assume that f_0 has compact support. This is ensured by the following Lemma.

Lemma 4.2.5 (Local existence of solutions)**.** *Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^3)$ with compact support and f a (local) solution to* (4.1) *with $f|_{t=0} = f_0$. Let*

$$D(t) := \sup\{|q| : \exists p \in \mathbb{R}^d : f(t,q,p) \neq 0\} \tag{4.17}$$

$$R(t) := \sup\{|p| : \exists q \in \mathbb{R}^d : f(t,q,p) \neq 0\} \tag{4.18}$$

the size of the support in the q-, respectively p-coordinates. Then there exists a constant $C > 0$ such that

$$D(t) \leq D(0) + \int_0^t R(s)\,\mathrm{d}s \tag{4.19}$$

$$R(t) \leq R(0) + C\,\|f_0\|_\infty \|f_0\|_1^{1/d} \int_0^t R^{d-1}(s)\,\mathrm{d}s. \tag{4.20}$$

These estimates hold independent of N as we consider the sequence f^N of solutions to the regularized equation (4.12) *with $f^N|_{t=0} = f_0$.*

Note that since $\rho_t(q) = \int f(t,q,p)\,\mathrm{d}^d p \leq |\mathrm{B}^d(1)| R(t)^d \|f_0\|_\infty$, a (uniform) bound on the momentum support implies a (uniform) bound on the charge density.

Proof. Given a solution f_t of (4.1), let $\varphi_{t,s}(z) = (Q,P)(t,s,z)$ the corresponding solution of the characteristic system (4.13). Then, $f(t,q,p) = f_0(Q(0,t,q,p), P(0,t,q,p))$ and hence

$$\frac{\mathrm{d}}{\mathrm{d}t} D(t) \leq \sup_{q,p} \left|\frac{\mathrm{d}}{\mathrm{d}t} Q(t,0,q,p)\right| = \sup_{q,p} |P(t,0,q,p)| = R(t), \tag{4.21}$$

which proves the first inequality, and

$$\frac{\mathrm{d}}{\mathrm{d}t} R(t) \leq \sup_{q,p} \left|\frac{\mathrm{d}}{\mathrm{d}t} P(t,0,q,p)\right| \leq \|\tilde{k} * \rho_t\|_\infty \leq \|k * \rho_t\|_\infty \tag{4.22}$$

which implies

$$R(t) \leq R(0) + \int_0^t \|k * \rho_t(s)\|_\infty\,\mathrm{d}s. \tag{4.23}$$

Moreover,

$$\rho_t(q) = \int f(t,q,p)\,\mathrm{d}^d p \leq C_1 \|f_t\|_\infty\, R^d(t) = C_1 \|f_0\|_\infty\, R^d(t) \tag{4.24}$$

with $C_1 = |\mathrm{B}^d(1)|$. Now, we estimate:

$$\begin{aligned} |k * \rho_t(x)| = \int |k|(y) \rho_t(x-y)\,\mathrm{d}y &\leq \int_{|y| \leq r} \frac{1}{|y|^{d-1}} \rho_t(x-y)\,\mathrm{d}^d y + \int_{|q'|>r} \frac{1}{|y|^{d-1}} \rho_t(x-y)\,\mathrm{d}^d y \\ &\leq C_2 \|\rho_t\|_\infty r + r^{-(d-1)} \|\rho_t^N\|_1 \end{aligned} \tag{4.25}$$

where $C_2 = |S^{d-1}|$, the surface area of the unit sphere in $\mathbb{R}^d$. The optimal choice is $r = \|\rho_t\|_\infty^{-1/d}\|\rho_t\|_1^{1/d}$ for which we get

$$\|k * \rho_t[f]\|_\infty \leq (C_2 + 1)\|f_0\|_1^{1/d}\|\rho_t\|_\infty^{\frac{d-1}{d}}. \tag{4.26}$$

Together with (4.24), it follows that:

$$R(t) \leq R(0) + C_1(C_2+1)\|f_0\|\|f_0\|_1^{1/d}\int_0^t R^{d-1}(s)\,\mathrm{d}s, \tag{4.27}$$

independent of N. Thus, a standard Gronwall argument yields the bound:

$$R(t) \leq \frac{R(0)}{(1 - CR(0)^{d-2}t)^{\frac{1}{d-2}}}, \tag{4.28}$$

with $C = (d-2)C_1(C_2+1)\|f_0\|\|f_0\|_1^{1/d}$ which is finite for all $t < \frac{1}{CR(0)^{d-2}}$ and all $N \in \mathbb{N}$. Since $\|\tilde{k} * \rho_t\|_\infty \leq \|k * \rho_t\|_\infty$ for any N, these estimates hold independent of N as we consider the sequence f^N of solutions to the regularized equation. □

4.3 Statement of the results

We now sate our precise results in the following two theorems. The approximation of the Vlasov density is again formulated in terms of the Wasserstein distances.

Proposition 4.3.1 (Deterministic Result). *Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, $f \geq 0$. Let $(r_N)_{N\in\mathbb{N}}$ be a rescaling sequence and f_t^N the unique solution of the regularized Vlasov-Poisson equation* (4.12) *with $f^N(0,\cdot,\cdot) = f_0$. Assume that on $[0,T]$ the sequence $(f_N)_N$ satisfies the uniform bound* (4.16) *on the induced charge-densities. Suppose we have a sequence of initial conditions $Z \in \mathbb{R}^{6N}$ such that*

$$\lim_{N\to\infty} r_N^{-(1+\frac{d}{2}+\epsilon)} W_2(\mu_0^N[Z], f_0) = 0 \tag{4.29}$$

for some $\epsilon > 0$. Then we have

$$\lim_{N\to\infty} r_N^{-(1+\frac{d}{2})} W_2(\mu_t^N[Z], f_t^N) = 0,\ \forall 0 \leq t \leq T. \tag{4.30}$$

Since we will also show that $W_2(f_t^N, f_t) = o(r_N^{1-\epsilon})$ (Prop. 4.4.5) this establishes a particle approximation of the Vlasov-Poisson equation for initial conditions satisfying (4.29).

Theorem 4.3.2 (Typicality Result). *Let $f_0 \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ a probability measure such that the Vlasov-Poisson equation* (4.1) *has a unique solution on $[0,T^*), T^* \in \mathbb{R}^+ \cup \{+\infty\}$ with $f(0,\cdot,\cdot) = f_0$. Assume that the sequence $(f^N)_N$ of solutions to the regularized Vlasov-Poisson equation* (4.12) *with the same initial data satisfies the uniform bound* (4.16) *on the induced charge-densities. Assume, in addition, that there exists $k > 4$ such that*

$$M_k(f_0) := \int (|q| + |p|)^k f_0(q,p)\,\mathrm{d}q\,\mathrm{d}p < +\infty. \tag{4.31}$$

Suppose that $r_N \geq N^{-\delta}$ *with*

$$\delta = \frac{1-\epsilon}{d(2+d+2\epsilon)}, \ \epsilon > 0.$$

Then there exist constants C_1, C_2, C_3 *such that for all* $T < T^*$ *and* N *large enough that* $r_N \leq \exp[-(\frac{2C_1T}{\epsilon})^2]$ *it holds that*

$$\mathbb{P}_0\Big[\sup_{t\in[0,T]} W_2(\mu_t^N[Z], f_t) > r_N^{1-\epsilon}\Big] \leq C_2\big(e^{-C_3N^{\epsilon}} + N^{1-\frac{k}{2}+\frac{k}{2d}}\big), \tag{4.32}$$

where the probability $\mathbb{P}_0$ *is defined in terms of the product measure* $\otimes^N f_0$ *on* $(\mathbb{R}^d \times \mathbb{R}^d)^N$*. The constant* C_1 *depends on* d, χ *and* C_0 *as in* (4.16)*, while* C_2, C_3 *depend on* d*,* k *and* $M_k(f_0)$*.*

Remarks 4.3.3.

1. In dimension 3, the necessary cut-off is of order $N^{-\delta}$ with $\delta < \frac{1}{15}$.
2. In view of Thm. 2.2.1, if the finite moment condition (4.31) is replaced by the assumption of a finite exponential moment $\int e^{\gamma|x|^{\kappa}} \mathrm{d}f_0(x)$, the rate of convergence becomes exponential, as well. This holds, in particular, for compactly supported f_0.

4.3.1 Sketch of the proof

We give here a brief sketch of our derivation and the central concepts and ideas on which it is based.

1. To control the distance between microscopic density and mean field density, we introduce a variant of the second Wasserstein distance W_2^N defined with respect to the N-dependent metric:

   $$d^N\big((q_1,p_1),(q_2,p_2)\big) := (1 \vee \sqrt{|\log(r_N)|})\,|q_1 - q_2| + |p_1 - p_2|,$$

   where $1 \vee \sqrt{|\log(r_N)|} := \max\{1, \sqrt{|\log(r_N)|}\}$.
2. We use Loeper's stability estimate, Proposition 2.3.6, to control the L^2 norm of the difference between mean field force and microsocpic force in terms of the quadratic Wasserstein distance.
3. The regularization yields a Lipschitz bound on the microscopic force that diverges logarithmically with N. In terms of the modified Wasserstein distance, this leads to a Gronwall estimate of the form

   $$\frac{\mathrm{d}}{\mathrm{d}t} W_2^N(\mu_t^N, f_t^N) \leq C\sqrt{|\log(r_N)|}W_2^N(\mu_t^N, f_t^N).$$
4. The previous bounds can be applied if the (smeared out) microscopic charge density $\tilde{\rho}^{\mu} = \chi^N * \rho[\mu_t]$ remains bounded uniformly in N. We show that this can be assured as long as $W_2(\mu_t^N[Z], f_t^N) = o(r_N^{-(1+d/2)})$. Given a sufficiently fast rate of convergence at $t = 0$, i.e. assumption (4.29), we conclude with 3. that this bound propagates.

5. It remains to check that the constraints so imposed on the initial data are satisfied for typical Z, if the initial configuration is chosen randomly according to the law f_0. This is achieved with the large deviation estimate of Fournier and Guilin, Thm. 2.2.1. This result also determines how fast r_N can go to zero in the limit $N \to \infty$.

4.4 A Gronwall argument

We recall:

Proposition (Loeper) Let k the Coulomb kernel and $\rho_1, \rho_2 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ two (probability) densities. Then

$$\|k * \rho_1 - k * \rho_2\|_{L^2(\mathbb{R}^d)} \leq \left[\max\{\|\rho_1\|_\infty, \|\rho_2\|_\infty\}\right]^{1/2} W_2(\rho_1, \rho_2). \tag{4.33}$$

Moreover, we will use the following estimates on the mean field force:

Lemma 4.4.1. *Let k as before and $\rho \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then it holds that*

i) $\|k * \rho\|_\infty \leq |S^{d-1}|\, \|\rho\|_\infty + \|\rho\|_1$

ii) $\|\chi^N * k * \rho\|_{Lip} \leq C_L(1 \vee |\log(r_N)|) \left(\|\rho\|_1 + \|\rho\|_\infty\right)$

where we used the notation $a \vee b := \max\{a, b\}$. $|S^{d-1}|$ denotes the surface area of the unit sphere and C_L is a constant depending on χ.

Proof. i) For the first inequality, we compute

$$\begin{aligned}\|k * \rho\|_\infty &\leq \Big\| \int\limits_{|y|<1} k(y)\rho(x-y)\, \mathrm{d}^d y \Big\|_\infty + \Big\| \int\limits_{|y|>1} k(y)\rho(x-y)\, \mathrm{d}^d y \Big\|_\infty \\ &\leq \|\rho\|_\infty \int\limits_{|y|<1} \frac{1}{|y|^{d-1}} \mathrm{d}^d y + \|\rho\|_1 = |S^{d-1}| \|\rho\|_\infty + \|\rho\|_1.\end{aligned}$$

ii)We split the expression as

$$\begin{aligned}\left\|\nabla(\chi * k * \rho)\right\|_\infty &\leq \left\|\nabla(\chi * k|_{x \geq r_N^{d+1}} * \rho)\right\|_\infty + \left\|\nabla(\chi * k|_{x < r_N^{d+1}} * \rho)\right\|_\infty \\ &\leq \left\|\chi^N\right\|_1 \left\| \nabla k|_{x \geq r_N^{d+1}} * \rho\right\|_\infty + \left\|\nabla \chi^N\right\|_\infty \left\|k|_{x<r_N^{d+1}}\right\|_1 \left\|\rho\right\|_\infty.\end{aligned}$$

Now, we have:

$$\begin{aligned}\left|\nabla k|_{x \geq r_N^{d+1}} * \rho\,(x)\right| &\leq \int\limits_{|y| \geq r_n^{d+1}} \frac{1}{|y|^d}\, \rho(x-y)\, \mathrm{d}^d y \\ &\leq \int\limits_{r_N^{d+1} \leq |y| \leq 1} \frac{1}{|y|^d} \rho(x-y) \mathrm{d}^d y + \int\limits_{|y|>1} \frac{1}{|y|^d} \rho(x-y) \mathrm{d}^d y \\ &\leq (d+1) C \left\|\rho\right\|_\infty \log(r_N^{-1}) + \|\rho\|_1.\end{aligned}$$

Furthermore:

$$\|\nabla\chi^N\|_\infty = r_N^{-(d+1)}\|\nabla\chi\|_\infty$$

and

$$\big\|k|_{x<r_N^{d+1}}\big\|_1 = \int\limits_{|y|<r_N^{d+1}} \frac{1}{|y|^{d-1}}\,\mathrm{d}^d y = |S^{d-1}|\,r_N^{d+1}.$$

Putting everything together, the statement follows. □

For the solutions f_t^N to the (regularized) Vlasov-Poisson equation, the corresponding charge-densities $\rho_t = \rho[f_t^N]$ are bounded by assumption. The challenge is to provide a bound on the microscopic charge density that holds uniformly in N, i.e. as the electron radius decreases and the forces become more singular. The idea is to show that as long as μ_t^N and f_t^N are close in Wasserstein distance, the L^∞-norm of $\tilde{\rho}[f_t^N]$ provides a bound on the L^∞-norm of $\tilde{\rho}[\mu_t^N]$. A simple such estimate can be obtained as follows (c.f. [8, Prop. 2.1]).

Lemma 4.4.2. *Let ρ_1, ρ_2 two probability measures on $\mathbb{R}^d$ and $\tilde{\rho}_i := \chi^N * \rho_i$. Then there exists a constant C depending on χ such that*

$$\|\tilde{\rho}_1\|_\infty \le \|\tilde{\rho}_2\|_\infty + C\,r_N^{-(d+1)}\,W_1(\rho_1,\rho_2). \tag{4.34}$$

Proof. For all $q\in\mathbb{R}^d$ we have

$$|(\tilde{\rho}_1-\tilde{\rho}_2)(q)| = \big|\chi^N * (\rho_1-\rho_2)(q)\big| \le \|\chi^N\|_{Lip}\,W_1(\rho_1,\rho_2).$$

Since $\|\chi^N\|_{Lip} \le \|\nabla\chi^N\|_\infty \le r_n^{-(d+1)}\|\nabla\chi\|_\infty$, the lemma follows. □

In view of the general Kantorovich-Rubinstein duality, we generalize this result to Wasserstein distances of higher order.

Lemma 4.4.3. *Let ρ_1, ρ_2 two probability measures on $\mathbb{R}^d$ and $\rho_2 \in L^\infty(\mathbb{R}^d)$. Then:*

$$\|\tilde{\rho}_1\|_\infty \le |\mathrm{B}^d(2)|\,\|\rho_2\|_\infty + r_N^{-(p+d)}\,W_p^p(\rho_1,\rho_2). \tag{4.35}$$

Proof. For any integrable function Φ, we consider the c-conjugate

$$\Phi^c(y) := \sup_x\{\Phi(x) - |x-y|^p\}$$

as introduced in equation (2.4). Now, we write

$$\tilde{\rho}_1(x) = r_N^{-(d+p)}\Big[\int r_N^{d+p}\chi^N(x-y)\rho_1(y)\mathrm{d}y - \int (r_N^{d+p}\chi^N(x-\cdot))^c(z)\rho_1(z)\,\mathrm{d}z \\ + \int (r_N^{d+p}\chi^N(x-\cdot))^c(z)\,\rho_1(z)\mathrm{d}z\Big].$$

By the Kantorovich duality theorem (2.5),

$$\int r_N^{d+p}\chi^N(x-y)\,\rho_1(y)\mathrm{d}y - \int (r_N^{d+p}\chi^N(x-\cdot))^c(z)\,\rho_2(z)\mathrm{d}z \le W_p^p(\rho_1,\rho_2).$$

It remains to estimate
$$\int (r_N^{d+p}\chi^N(x-\cdot))^c(z)\,\rho_2(z)\,\mathrm{d}z.$$
Recalling that $\|\chi^N\|_\infty = r_N^{-d}$, we find
$$(r_N^{d+p}\chi^N(x-\cdot))^c(z) = \sup_{y\in\mathbb{R}^3}\{r_N^{d+p}\chi^N(x-y) - |y-z|^p\} \leq r_N^{d+p}\|\chi^N\|_\infty = r_N^p.$$
Moreover, we observe that
$$\operatorname{supp}(r_N^{d+p}\chi^N(x-\cdot))^c \subseteq \mathrm{B}(2r_N;x) := \{z\in\mathbb{R}^3 : |z-x|\leq 2r_N\}, \tag{4.36}$$
since $|z-x| > 2r_N$ implies that $\chi^N(x-y)=0$, unless $|y-z|\geq r_N$. But then: $r_N^{d+p}\chi^N(x-y) - |y-z|^p \leq r_N^{d+p}r_N^{-d} - r_N^p = 0$. Hence,
$$\int (r_N^{d+p}\chi^N(x-\cdot))^c(z)\rho_2(z)\mathrm{d}z \leq \|\rho_2\|_\infty\, r_N^p\,|\mathrm{B}(2r_N;x)| \leq 2^d|\mathrm{B}^d(1)|\,\|\rho_2\|_\infty\, r_N^{d+p}.$$
In total:
$$\|\tilde\rho_1\|_\infty \leq r_N^{-(p+d)}\,W_p^p(\rho_1,\rho_2) + |B^d(2)|\|\rho_2\|_\infty$$
as announced. □

We shall apply the previous Lemma to $\rho_1 := \rho[\mu_t^N(Z)]$ and $\rho_2 := \rho[f_t^N]$ using $\|\rho[f_t^N]\| \leq C_\rho$ and $W_2(\rho[\mu_t^N(Z)],\rho[f_t^N]) \leq W_2(\mu_t^N(Z), f_t^N)$ to get a bound on the (smeared) microscopic charge density.

4.4.1 Modified Wasserstein distance

As we want to establish a Gronwall estimate for the distance between empirical density and Vlasov density, we aim for a bound of the form:
$$\operatorname{dist}(\mu_{t+\Delta t}^N, f_{t+\Delta t}^N) - \operatorname{dist}(\mu_t^N, f_t^N) \propto \operatorname{dist}(\mu_t^N, f_t^N)\,\Delta t + o(\Delta t).$$
The choice of a metric giving precise meaning to $\operatorname{dist}(\mu_t^N, f_t^N)$ is thus a balancing act. While a stronger metric is, in general, more difficult to control, it also yields stronger bounds as it appears on the right hand side of the Gronwall inequality.

If we compare the characteristic flow of the mean field dynamics with the flow corresponding to the “true”, i.e. microscopic, dynamics, the growth in the *spatial* distance is trivially bounded by the distance of the respective momenta. The only problem lies in controlling fluctuations in the force, i.e. the growth of the distance in *momentum* space. The idea (that we already employed in the previous chapter) is thus to be more rigid on deviations in the q-coordinates, weighing them with an appropriate N-dependent factor, and use this to obtain better control on the forces.

Definition 4.4.4. Let $(r_N)_{N\in\mathbb{N}}$ be a rescaling sequence. On $\mathbb{R}^d\times\mathbb{R}^d$ we introduce the (N-dependent) metric:
$$d^N\big((q_1,p_1),(q_2,p_2)\big) := (1\vee\sqrt{|\log(r_N)|})\,|q_1-q_2| + |p_1-p_2|. \tag{4.37}$$

Now let $W_p^N(\cdot,\cdot)$ be the p'th Wasserstein metric with respect to d^N, i.e.:

$$W_p^N(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \Big(\int\limits_{\mathbb{R}^d\times\mathbb{R}^d} d^N(x,y)^p \,\mathrm{d}\pi(x,y) \Big)^{1/p}. \tag{4.38}$$

Note that $W_p(\mu,\nu) \leq W_p^N(\mu,\nu) \leq (1 \vee \sqrt{|\log(r_N)|})\, W_p(\mu,\nu)$, $\forall \mu,\nu \in \mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d)$. Finally, we define

$$W^*(\mu,\nu) := \min\Big\{1,\, r_N^{-(1+\frac{d}{2})}\, W_2^N(\mu,\nu)\Big\}. \tag{4.39}$$

Obviously, convergence with respect to W^* is much stronger than convergence with respect to W_2. Concretely, we have for any sequence $(\nu_N)_{N\in\mathbb{N}}$ and $\nu \in \mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d)$:

$$W^*(\nu_N,\nu) \to 0 \Rightarrow W_2(\nu_N,\nu) = o\big(r_N^{1+\frac{d}{2}}\big).$$

4.4.2 Deterministic result

We now come to the central part of our argument.

Proof of Proposition 4.3.1. Let $N \in \mathbb{N}$ and $\pi_0 \in \Pi(\mu_0^N, f_0)$. Let $\varphi_t^\mu = (Q_t^\mu, P_t^\mu)$ and $\varphi_t^f = (Q_t^f, P_t^f)$ the flow induced by the characteristic equation (4.13) for μ_t^N and f_t^N, respectively. For any $t \in [0,T]$, $T < T^*$, define the (N-dependent) measure π_t on $\mathbb{R}^{6N}\times\mathbb{R}^{6N}$ by $\pi_t = (\varphi_t^\mu, \varphi_t^f)\#\pi_0$. Then $\pi_t \in \Pi(\mu_t^N, f_t)$, $\forall t \in [0,T]$. We set

$$\begin{aligned}
D(t) :=& \Big[\int\limits_{\mathbb{R}^6\times\mathbb{R}^6} d^N(x,y)^2 \,\mathrm{d}\pi_t(x,y)\Big]^{1/2}\\
=& \Big[\int\limits_{\mathbb{R}^6\times\mathbb{R}^6} \Big((1 \vee \sqrt{|\log(r_N)|})\, |x^1-y^1| + |x^2-y^2|\Big)^2 \,\mathrm{d}\pi_t(x,y)\Big]^{1/2}\\
=& \Big[\int\limits_{\mathbb{R}^6\times\mathbb{R}^6} \Big((1 \vee \sqrt{|\log(r_N)|})\, |Q_t^\mu(x)-Q_t^f(y)| + |P_t^\mu(x)-P_t^f(y)|\Big)^2 \,\mathrm{d}\pi_0(x,y)\Big]^{1/2}.
\end{aligned}$$

Note that $W_2^N(\mu_t^N, f_t^N) < D(t)$ for any $\pi_0 \in \Pi(f_0, f_0)$. Now we consider:

$$D^*(t) := \min\big\{1,\, r_N^{-(1+\frac{d}{2})}\, D(t)\big\}. \tag{4.40}$$

Obviously, $\frac{\mathrm{d}}{\mathrm{d}t} D^*(t) \leq 0$ whenever $D(t) \geq r_N^{1+\frac{d}{2}}$ since then $D^*(t)$ is already maximal. For $D(t) < r_N^{1+\frac{d}{2}}$, we compute:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} D^2(t) =\\
&\quad 2\int \Big((1 \vee \sqrt{|\log(r_N)|})\, |Q_t^\mu(x)-Q_t^f(y)| + |P_t^\mu(x)-P_t^f(y)|\Big) \cdot\\
&\qquad \Big((1 \vee \sqrt{|\log(r_N)|})\, |P_t^\mu(x)-P_t^f(y)| + \big|\tilde{k}*\tilde{\rho}_t^\mu(Q_t^\mu(x)) - \tilde{k}*\tilde{\rho}_t^f(Q_t^f(y))\big|\Big) \,\mathrm{d}\pi_0(x,y).
\end{aligned}$$

The interesting term to control is the interaction term

$$\begin{aligned}
&\left|\tilde{k} * \tilde{\rho}_t^{\mu}\left(Q_t^{\mu}(x)\right) - \tilde{k} * \tilde{\rho}_t^{f}\left(Q_t^{f}(y)\right)\right| \\
&\leq \left|\tilde{k} * \tilde{\rho}_t^{\mu}\left(Q_t^{\mu}(x)\right) - \tilde{k} * \tilde{\rho}_t^{\mu}\left(Q_t^{f}(y)\right)\right| && (4.41)\\
&+ \left|\tilde{k} * \tilde{\rho}_t^{\mu}\left(Q_t^{f}(y)\right) - \tilde{k} * \tilde{\rho}_t^{f}\left(Q_t^{f}(y)\right)\right|. && (4.42)
\end{aligned}$$

We begin with (4.41) and find with Lemma 4.4.1:

$$\begin{aligned}
&\left|\tilde{k} * \tilde{\rho}_t^{\mu}\left(Q_t^{\mu}(x)\right) - \tilde{k} * \tilde{\rho}_t^{\mu}\left(Q_t^{f}(y)\right)\right| \\
&\leq C_L(1 \vee |\log(r_N)|)(1 + \|\rho_t^{\mu}\|_{\infty}) \left|Q_t^{\mu}(x) - Q_t^{f}(y)\right|.
\end{aligned} \tag{4.43}$$

Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} D^2(t) \leq J_1(t) + J_2(t) \tag{4.44}$$

with

$$\begin{aligned}
J_1(t) := 2 \int &\Big((1 \vee \sqrt{|\log(r_N)|})\, |Q_t^{\mu}(x) - Q_t^{f}(y)| + |P_t^{\mu}(x) - P_t^{f}(y)|\Big) \cdot \\
&\Big((1 \vee \sqrt{|\log(r_N)|})\, |P_t^{\mu}(x) - P_t^{f}(y)| + C_L(1 \vee |\log(r_N)|)(1 + \|\rho_t^{\mu}\|_{\infty}) \left|Q_t^{\mu}(x) - Q_t^{f}(y)\right|\Big)\, \mathrm{d}\pi_0(x, y)
\end{aligned} \tag{4.45}$$

$$\begin{aligned}
J_2(t) := 2 \int &\Big((1 \vee \sqrt{|\log(r_N)|})\, |Q_t^{\mu}(x) - Q_t^{f}(y)| + |P_t^{\mu}(x) - P_t^{f}(y)|\Big) \cdot \\
&\left|\tilde{k} * \tilde{\rho}_t^{\mu}\left(Q_t^{f}(y)\right) - \tilde{k} * \tilde{\rho}_t^{f}\left(Q_t^{f}(y)\right)\right| \mathrm{d}\pi_0(x, y).
\end{aligned} \tag{4.46}$$

Now we observe that

$$J_1(t) \leq C_L(1 \vee |\log(r_N)|)(1 + \|\rho_t^{\mu}\|_{\infty}) D^2(t), \tag{4.47}$$

while for the second term, we find with Hölders inequality

$$\begin{aligned}
J_2(t) \leq 2 &\Big[\int \Big((1 \vee \sqrt{|\log(r_N)|})\, |Q_t^{\mu}(x) - Q_t^{f}(y)| + |P_t^{\mu}(x) - P_t^{f}(y)|\Big)^2 \mathrm{d}\pi_0(x, y)\Big]^{1/2} && (4.48)\\
&\Big[\int \left|\tilde{k} * \tilde{\rho}_t^{\mu}\left(Q_t^{f}(y)\right) - \tilde{k} * \tilde{\rho}_t^{f}\left(Q_t^{f}(y)\right)\right|^2 \mathrm{d}\pi_0(x, y)\Big]^{1/2}. && (4.49)
\end{aligned}$$

We identify (4.48) as $D(t)$, while for (4.49) we get

$$\begin{aligned}
&\Big[\int \left|\tilde{k} * (\tilde{\rho}_t^{\mu} - \tilde{\rho}_t^{f})\big(Q_t^{f}(y)\big)\right|^2 \mathrm{d}\pi_0(x, y)\Big]^{1/2} = \Big[\int \left|\tilde{k} * (\tilde{\rho}_t^{\mu} - \tilde{\rho}_t^{f})\big(Q_0(y)\big)\right|^2 \mathrm{d}\pi_t(x, y)\Big]^{1/2} \\
&\leq \Big[\int \big(\tilde{k} * \tilde{\rho}_t^{\mu} - \tilde{k} * \tilde{\rho}_t^{f}\big)^2 f(t, y)\, \mathrm{d}^{2d}y\Big]^{1/2} = \Big[\int \big(\tilde{k} * \tilde{\rho}_t^{\mu} - \tilde{k} * \tilde{\rho}_t^{f}\big)^2(q)\, \rho_t^{f}(q)\, \mathrm{d}^d q\Big]^{1/2} \\
&\leq \|\rho_t^{f}\|_{\infty}^{1/2} \|\tilde{k} * (\tilde{\rho}_t^{\mu} - \tilde{\rho}_t^{f})\|_2 \leq C_0^{1/2} \|k * (\tilde{\rho}_t^{\mu} - \tilde{\rho}_t^{f})\|_2.
\end{aligned} \tag{4.50}$$

From Lemma 4.4.3, we know that as long as $D(t) \le r_N^{1+\frac{d}{2}}$, i.e. $D^*(t) \le 1$, the microscopic charge density is bounded by

$$\begin{aligned}
\|\rho_t^\mu\|_\infty \le & |B^d(2)| \|\rho[f_t^N]\|_\infty + r_N^{-(d+2)}\, D^2(t) \\
\le & |B^d(2)| \sup_{N\in\mathbb{N}} \|\rho[f_t^N]\|_\infty + 1 \\
\le & |B^d(2)| C_0 + 1 =: C_\rho .
\end{aligned} \tag{4.51}$$

Note that this bound holds independent of N. Hence, we can use Loeper's stability result, Proposition 2.3.6 in (4.50) and get:

$$\|k * (\tilde{\rho}_t^\mu - \tilde{\rho}_t^f)\|_2 \le \left[\max\{\|\tilde{\rho}_t^\mu\|_\infty, \|\tilde{\rho}_t^f\|_\infty\}\right]^{\frac{1}{2}} W_2(\tilde{\rho}_t^\mu, \tilde{\rho}_t^f) \le C_\rho^{\frac{1}{2}}\, D(t). \tag{4.52}$$

Putting everything together and setting $C_1 := 2C_\rho C_L$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} D^2(t) \le 2\, C_1 (1 \vee \sqrt{|\log(r_N)|})\, D^2(t)$$

or, after dividing by $2D(t)$ and multiplying both sides by $r_N^{-(1+\frac{d}{2})}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} D^*(t) \le C_1 (1 \vee \sqrt{|\log(r_N)|}) D^*(t).$$

By an application of Gronwall's Lemma, we conclude that:

$$D^*(t) \le D^*(0)\, e^{t\, C_1 (1 \vee \sqrt{|\log(r_N)|})}.$$

Finally, taking on the right hand side the infimum over all $\pi_0 \in \Pi(\mu_0^N, f_0)$, $D^*(0)$ becomes $W^*(\mu_0^N[Z], f_0)$ and we get for all $t \in T$:

$$W^*(\mu_t^N, f_t^N) \le W^*(\mu_0^N, f_0)\, e^{t\, C_1 (1 \vee \sqrt{|\log(r_N)|})}. \tag{4.53}$$

If there exists an $\epsilon > 0$ such that $\lim\limits_{N\to\infty} \frac{W_2(\mu_0^N, f_0)}{r_N^{1+d/2+\epsilon}} = 0$, the right hand side converges to 0, so that, in particular, $\lim\limits_{N\to\infty} r_N^{1+\frac{d}{2}} W_2(\mu_t^N, f_t^N) = 0$.

□

To show convergence to solutions of the (unregularized) Vlasov-Poisson equation, we also require the following:

Proposition 4.4.5. *Let f_0 satisfy the assumptions of Theorem 4.3.2. Let f_t^N and f_t be the solution of the regularized, respectively the proper Vlasov-Poisson equation with initial data f_0. Then:*

$$W_2(f_t^N, f_t) \le r_N\, e^{tC_1 (1 \vee \sqrt{|\log(r_N)|})}. \tag{4.54}$$

Proof. Let $\rho_t^N := \rho[f_t^N]$ and $\rho_t^\infty := \rho[f_t]$ be the charge density induced by f_t^N and f_t, respectively. Let $\varphi_t^N = (Q_t^N, P_t^N)$ the characteristic flow of f_t^N and $\psi_t = (Q_t, P_t)$ the characteristic flow of f_t. We consider $\pi_0(x, y) := f_0(x)\delta(x - y) \in \Pi(f_0, f_0)$, which is already the optimal coupling yielding $W_2^N(f_t^N, f_t)|_{t=0} = W_2^N(f_0, f_0) = 0$ and set $\pi_t = (\varphi_t^N, \psi_t)\#\pi_0 \in \Pi(f_t^N, f_t)$. As above, we define

$$D(t) := \Big[\int\limits_{\mathbb{R}^6\times\mathbb{R}^6} \Big((1 \vee \sqrt{|\log(r_N)|})\, |x^1 - y^1| + |x^2 - y^2| \Big)^2 \, \mathrm{d}\pi_t(x,y) \Big]^{1/2} \tag{4.55}$$

and compute

$$\frac{\mathrm{d}}{\mathrm{d}t} D^2(t) \leq 2 \int \Big((1 \vee \sqrt{|\log(r_N)|})\, |Q^N(t,x) - Q(t,y)| + |P^N(t,x) - P(t,y)| \Big)$$
$$\Big((1 \vee \sqrt{|\log(r_N)|})\, |P^N(t,x) - P(t,y)| + \big|\tilde{k} * \tilde{\rho}_t^N(Q^N(x)) - k * \rho_t^f(Q_t(y))\big| \Big) \, \mathrm{d}\pi_0(x,y).$$

The proof proceeds analogous to Prop. 4.3.1, simplified by the fact that the charge densities remain bounded by assumption. The only noteworthy difference is in eq. (4.52). Observing that $\tilde{k} * \tilde{\rho} = k * \tilde{\tilde{\rho}}$, we use Lemma 4.2.2 to conclude:

$$W_2(\tilde{\tilde{\rho}}_t^N, \rho_t) \leq W_2(\rho_t^N, \rho_t) + 2r_N \leq W_2(f_t^N, f_t) + 2r_N \leq D(t) + 2r_N. \tag{4.56}$$

In total, we find:

$$\frac{\mathrm{d}}{\mathrm{d}t} D^2(t) \leq 2C_0 C_L\, (1 \vee \sqrt{|\log r_N|})\, D^2(t) + 2C_0 D(t)(D(t) + 2r_N)$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t} D(t) \leq C_1 (1 \vee \sqrt{|\log r_N|})\, D(t) + 2C_0 r_N$$

with $C_1 > 2C_0(C_L + 1)$ as defined in the previous proof. Using Gronwall's inequality and the fact that $D(0) = 0$, we have

$$W_2(f_t^N, f_t) \leq D(t) \leq r_N\, e^{tC_1(1\vee\sqrt{|\log r_N|})},$$

from which the desired statement follows.

□

4.4.3 Typicality

In the previous sections, we performed the mean field limit for the Vlasov-Poisson system under the assumption of a sufficiently fast convergence of the initial distribution. How strong this result is, now depends on two questions:

1) How restrictive is the condition $W_2(\mu_0^N, f_0) = o(r_N^{1+\frac{d}{2}+\epsilon})$?

2) How fast can we let the electron radius (i.e. the cut-off parameter) r_N go to zero?

If we found that only very special sequences of initial distributions $\mu^N[Z], Z \in \mathbb{R}^{6N}$ achieve the necessary rate of convergence, the result would not be very satisfying from a physical point of view. If we observe a globular cluster, let's say, we cannot pretend that someone has arranged the galaxies in precisely such a way as to ensure the validity of Proposition 4.3.1. If, on the other hand, we can show that the good initial configurations are *typical*, it would mean that, on the contrary, the mean field approximation fails only for very special (in this sense "conspiratorial") initial conditions.

Hence, in order to complete the proof of Theorem 4.3.2, it remains to show that the assumptions of Proposition 4.3.1 are satisfied for generic initial data, i.e. with probability approaching 1 as N tends to infinity. To this end, we apply again the large deviation estimate from Theorem 2.2.1, which will also determine the lower bound for the N-dependent cut-off r_N.

Proof of Theorem *4.3.2*. Let $r_N \geq N^{-\delta}$ and $\epsilon > 0$. Let $\mathcal{A} \subseteq \mathbb{R}^{2d}$ be the (N-dependent) set defined by

$$Z \in \mathcal{A} \iff W_2(\mu_0^N[Z], f_0) > r_N^{1+\frac{d}{2}+\epsilon}. \tag{4.57}$$

We apply Theorem 2.2.1 in $n = 2d$ dimensions with $\xi = N^{-\delta(2+d+2\epsilon)} \leq r_N^{2(1+\frac{d}{2}+\epsilon)}$ and condition (4.31) (stating that f_0 has a finite k'th moment for $k > 4$). We find:

$$\mathbb{P}_0(\mathcal{A}) \leq C\big(\exp(-cNN^{-\delta(2+d+2\epsilon)d}) + N^{1-\frac{k-\epsilon}{2}(1-\delta(2+d+2\epsilon))}\big)$$

where the probability is defined with respect to $\otimes^N f_0$. Choosing

$$\delta = \frac{1-\epsilon}{(2+d+2\epsilon)d} \tag{4.58}$$

we have

$$\mathbb{P}_0(\mathcal{A}) \leq C\big(\exp(-cN^{\epsilon}) + N^{1-\frac{k}{2}+\frac{k}{2d}}\big) \to 0, \ N \to \infty.$$

For the typical initial conditions $Z \in \mathcal{A}^c$, we have according to Proposition 4.3.1 and, in particular, equation (4.53):

$$\begin{aligned} W^*(\mu_t^N, f_t^N) &\leq W^*(\mu_0^N, f_0)\, e^{t\,C_1(1\vee\sqrt{|\log(r_N)|})} \\ &\leq (1 \vee \sqrt{|\log(r_N)|})\, r_N^{-(1+\frac{d}{2})} W_2(\mu_0^N, f_0)\, e^{t\,C_1(1\vee\sqrt{|\log(r_N)|})} \\ &\leq (1 \vee \sqrt{|\log(r_N)|})\, r_N^{\epsilon}\, e^{T\,C_1(1\vee\sqrt{|\log(r_N)|})} \end{aligned} \tag{4.59}$$

for all $t \leq T$. Observing that $e^{\sqrt{|\log r_N|}} = \big(e^{-\log r_N}\big)^{\frac{1}{\sqrt{|\log r_N|}}} = (r_N)^{\frac{-1}{\sqrt{|\log r_N|}}}$, there exists $N_0 \in \mathbb{N}$ such that (4.59) < 1 for all $N \geq N_0$. More precisely, it suffices to choose N_0 large enough that $r_{N_0} < e^{-(\frac{2C_1T}{\epsilon})^2}$. Then we find:

$$W^*(\mu_t^N, f_t^N) < 1 \Rightarrow W_2(\mu_t^N, f_t^N) < r_N^{1+\frac{d}{2}} W^*(\mu_t^N, f_t^N) < r_N^{1+\frac{d}{2}}. \tag{4.60}$$

Now we recall from Proposition 4.4.5 that

$$W_2(f_t^N, f_t) \leq r_N\, e^{tC_1(1\vee\sqrt{|\log(r_N)|})},$$

which is smaller than $\frac{1}{2}r_N^{1-\epsilon}$ for $N \geq N_0$. We conclude the proof by noting that

$$W_2(\mu_t^N[Z], f_t) \leq W_2(\mu_t^N[Z], f_t^N) + W_2(f_t^N, f_t) \leq r_N^{1+\frac{d}{2}} + \frac{1}{2}r_N^{1-\epsilon} \leq r_N^{1-\epsilon}$$

for all $Z \in \mathcal{A}^c$ and $t \in [0, T]$.

□

Chapter 5

A mean field limit for the Vlasov-Maxwell system

5.1 The Vlasov-Maxwell equations

In this chapter, we are going to propose a microscopic derivation of the three dimensional relativistic Vlasov-Maxwell system. This is a set of differential equations describing a collisionless plasma of identical charged particles interacting through a self-consistent electromagnetic field:

$$\begin{aligned} &\partial_t f + v(\xi) \cdot \nabla_x f + K(t,x,\xi) \cdot \nabla_\xi f = 0, \\ &\partial_t E - \nabla_x \times B = -j, \qquad \nabla_x \cdot E = \rho, \\ &\partial_t B + \nabla_x \times E = \ 0, \qquad \nabla_x \cdot B = 0. \end{aligned} \tag{5.1}$$

As usual, units are chosen such that all physical constants, in particular the speed of light, are equal to 1. The distribution function $f(t,x,\xi) \geq 0$ describes the density of particles with position $x \in \mathbb{R}^3$ and relativistic momentum $\xi \in \mathbb{R}^3$. The other quantities figuring in the Vlasov-Maxwell equations are the relativistic velocity of a particle with momentum ξ, given by

$$v(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}} \tag{5.2}$$

and the charge and current density entering Maxwell's equations, given by

$$\rho(t,x) = \int f(t,x,\xi)\,\mathrm{d}\xi, \quad j(t,x) = \int v(\xi) f(t,x,\xi)\,\mathrm{d}\xi. \tag{5.3}$$

The function

$$K(t,x,\xi) = E(t,x) + v(\xi) \times B(t,x) \tag{5.4}$$

thus describes the Lorentz force acting at time t on a particle at x moving with momentum ξ. In contrast to the previous chapters, we denote the coordinates by (x,ξ) to emphasize that we are now working in a special-relativistic setting.

As in the case of Vlasov-Poisson, our aim is perform a mean field limes and show that solutions of the Vlasov-Maxwell equations can typically be approximated by the empirical density of a well-defined microscopic system in the large N limit.

As a microscopic theory, we will consider an N-particle Coulomb system of extended, rigid charges, also known as the *Abraham model* (after [1], see [64] for a recent discussion). Size and shape of the particles will be described by an N-dependent form factor that approximates a δ-distribution in the limit $N \to \infty$. The cut-off parameter thus has a straightforward physical interpretation in terms of a finite electron-radius. Our approximation of the Vlasov-Maxwell dynamics will be a combination of mean field limit and point-particle limit, very much like in the previous chapter where we treated, in fact, the nonrelativistic limit of the Abraham model.

One should note, however, that the status of the regularization is quite different in the context of Vlasov-Maxwell than with respect to the nonrelativistic Coulomb interactions. In the latter case, the correct microscopic dynamics are known and (relatively) well understood. Any regularization thereof is first and foremost a simplification of the mathematical problem, with the width of the cut-off essentially quantifying the deviation from the true microscopic theory. When it comes to the relativistic regime, though, the standard Maxwell-Lorentz dynamics are *not* well defined for point-particles and it is not clear what the "true" microscopic theory approximating the Vlasov-Maxwell dynamics is even supposed to be. The study of rigid charges (and their point-particle limit) thus seems like a natural way to make sense of the Maxwell-Lorentz equations, with a longstanding tradition in the physical literature, see e.g. Lorentz 1892 [43], 1904 [44], Sommerfeld, 1904 [61]. Nevertheless, at least with the particular scalings considered here, the regularization thus imposed remains a technical expedient rather than a realistic physical theory.

Compared to the discussion of the Vlasov-Poisson equation in the previous chapters, the derivation of the Vlasov-Maxwell dynamics is much more complicated for several reasons. First, we are now dealing with a relativistic (on the microscopic level: semi-relativistic) theory with retarded interactions. Second, this theory involves the electromagnetic field as additional degrees of freedom. Finally, the known results about existence and uniqueness of classical solutions are far less conclusive for Vlasov-Maxwell than for Vlasov-Poisson (we will give some relevant references below).

Nevertheless, since we have already addressed the electrostatic problem – including the nonrelativistic limit of the rigid charges model – one should not expect any *fundamentally* new difficulties in the relativistic case. Indeed, we will show how the methods developed in the previous two chapters can be combined and extended into a particle approximation for the Vlasov-Maxwell dynamics. Another essential ingredient is a decomposition of the electromagnetic field in terms of Liénard-Wiechert potentials that was proven, for instance, by Bouchut, Golse and Pallard in [9].

To my knowledge, the only previous mean field result for the Vlasov-Maxwell system is the paper of Golse [21], which uses the same rigid-charges model with a fixed (but arbitrarily small) radius to derive a *mollified* version of the equations (i.e. the smearing persists in the limiting equation). As the author notes (see [21, Prop. 6.2]), this result can be applied to approximate the actual Vlasov-Maxwell system, but only in a very weak sense, basically corresponding to choosing an N-dependent cut-off decreasing as $\sim \log(N)^{-\frac{1}{2}}$. We will considerably improve upon this result, allowing the cut-off to decrease as $N^{-\frac{1}{12}}$.

5.1.1 Structure of the chapter

The chapter is structured as follows:

We will first recall a representation of the electromagnetic field in terms of Liénard-Wiechert distributions that was derived, for instance, in [9]. The key advantage of this representation is that it does not depend on derivatives of the current-density, thus allowing for better control of fluctuations in terms of the Vlasov density.

In Section 5.3, we introduce the Abraham model of rigid charges as our microscopic theory and define a corresponding regularized mean field equation. By introducing an appropriate N-dependent rescaling, we will take the mean field limit together with a point-particle limit, in which the electron-radius goes to 0 and the particle form factor approximates a δ-distribution. This will allow us to approximate the actual Vlasov-Maxwell dynamics in the large N limit.

In section 5.4 we recall some known results about existence of (strong) solutions to the Vlasov-Maxwell equations.

After stating our precise results in Section 5.5, we derive a few simple but important corollaries from the solutions theory of the Vlasov-Maxwell equations in Section 5.6.

In Section 5.7, we briefly recall the stochastic process J_t^N defined in Chapter 3 and its relevance for proving molecular chaos.

In Section 5.8 we derive some global bounds on the (smeared) microscopic charge density and the corresponding fields.

Section 5.10 then contains the more detailed law-of-large number estimates for the difference between mean field dynamics and microscopic dynamics. These estimates are derived from the Liénard-Wiechert decomposition of the fields and are somewhat similar to the bounds proven in [9] for the regularity of solutions.

Finally, we combine all estimates into a proof of the mean field limes for the Vlasov-Maxwell dynamics.

5.2 Field representation

The Vlasov-Maxwell system contains in particular Maxwell's equations

$$\begin{aligned} \partial_t E - \nabla_x \times B &= -j, \qquad \nabla_x \cdot E = \rho, \\ \partial_t B + \nabla_x \times E &= \ 0, \qquad \nabla_x \cdot B = 0, \end{aligned} \tag{5.5}$$

where charge- and current-density are induced by the Vlasov density $f(t,x,\xi)$. In general, Maxwell's equations can be solved by introducing a scalar potential Φ and a vector potential A, satisfying

$$\Box_{t,x}\Phi = \rho, \qquad \Box_{t,x}A = j, \tag{5.6}$$

in terms of which the electric and magnetic fields are given by

$$E(t,x) = -\nabla_x\Phi(t,x) - \partial_t A(t,x); \qquad B(t,x) = \nabla \times A(t,x). \tag{5.7}$$

It is convenient to split the potential into a homogeneous and an inhomogeneous part, i.e. $A = A_0 + A_1$ with

$$\Box_{t,x} A_0 = 0,\ \partial_t A_0 \mid_{t=0} = -E_{in} \tag{5.8}$$

$$\Box_{t,x} A_1 = j,\ A_1 \mid_{t=0} = \partial_t A_1 \mid_{t=0} = 0. \tag{5.9}$$

We recall that the retarded fundamental solution of the d'Alembert operator $\Box_{t,x} = \partial_t^2 - \Delta_x$ (in $3+1$ dimensions) is given by the distribution

$$Y(t,x) = \frac{\mathbb{1}_{t>0}}{4\pi t}\delta(|x| - t). \tag{5.10}$$

Hence, in the Vlasov-Maxwell system, a solution of (5.9) is given by

$$A_1 = Y *_{t,x} j = \int v(\xi) Y *_{t,x} f(\cdot,\cdot,\xi)\mathrm{d}\xi. \tag{5.11}$$

Similarly, we set

$$\Phi = \Phi_1 = Y *_{t,x} \rho = \int Y *_{t,x} f(\cdot,\cdot,\xi)\mathrm{d}\xi. \tag{5.12}$$

The solution of the homogeneous wave-equation is given by (see e.g. [60, Thm. 4.1])

$$A_0(t,\cdot) = Y(t,\cdot) *_x E_{in}, \tag{5.13}$$

where the initial field has to satisfy the constraint

$$\mathrm{div} E_{in} = \rho_0 = \int f(0,\cdot,\xi)\mathrm{d}\xi. \tag{5.14}$$

Hence,

$$E_{in} = -\nabla_x G *_x \rho_0 + E'_{in} \tag{5.15}$$

with

$$G(x) = \frac{1}{4\pi|x|},\ x \in \mathbb{R}^3, \text{ and } \mathrm{div} E'_{in} = 0. \tag{5.16}$$

In total, for a given distribution function f_t, the Lorentz force-field $K(t,x,\xi) = E(t,x) + v(\xi) \times B(t,x)$ is given by

$$K[f] = -\int \partial_t \nabla_x \left(Y(t,\cdot) *_x G *_x f_0(\cdot,\eta)\right)\mathrm{d}\eta \tag{5.17}$$

$$-\int \left(\nabla_x + v(\eta)\partial_t\right) Y * f(\cdot,\cdot,\eta)\mathrm{d}\eta \tag{5.18}$$

$$-\int v(\xi) \times \left(v(\eta) \times \nabla_x\right) Y * f(\cdot,\cdot,\eta)\mathrm{d}\eta, \tag{5.19}$$

where we have set $E'_{in} = 0$, for simplicity. In more detail, this formulation of the field equations can be found e.g. in [21]. Note that equations (5.17 - 5.19) still allow for various representation in terms of f, depending on how one evaluates the derivatives.

5.2.1 Liénard-Wiechert distributions

A particularly useful representation of the electromagnetic field can be given as a superposition of Liénard-Wiechert fields (see, in particular, [9, Lemma 3.1].) For a given distribution f_t, the induced electric field can be written as

$$E(t,x) = E_0(t,x) + E_0'(t,x) + E_1(t,x) + E_2(t,x)$$

where

$$E_0[f_0] = \quad -\partial_t Y(t,\cdot) *_x E_{in} \tag{5.20}$$

$$E_0'[f_0] = \quad \int (\alpha^0 Y)(t,\cdot,\xi) *_{t,x} f_0 \,\mathrm{d}\xi \tag{5.21}$$

$$E_1[f] = \quad \int (\alpha^{-1} Y) *_{t,x} (\mathbb{1}_{t\geq 0} f) \,\mathrm{d}\xi \tag{5.22}$$

$$E_2[f] = -\int (\nabla_\xi \alpha^0 Y) *_{t,x} (K \mathbb{1}_{t\geq 0} f) \,\mathrm{d}\xi \tag{5.23}$$

with

$$\alpha^0(t,x,\xi) = \frac{x - tv(\xi)}{t - v(\xi)x}; \quad \alpha^{-1}(t,x,\xi) = \frac{(1 - v(\xi)^2)(x - tv(\xi))}{(t - v(\xi)x)^2}. \tag{5.24}$$

Hence

$$(\nabla_\xi \alpha^0)^i_j(t,x,\xi) = \frac{t(t - v\cdot x)(v_j v^i - \delta^i_j) + (x_j - tv_j)(x^i - (v\cdot x)v^i)}{\sqrt{1+|\xi|^2}(t - v\cdot x)^2}. \tag{5.25}$$

Here, we follow the notation from [9]; The upper index in α^j, $j = 0, -1$, refers to the degree of homogeneity in (t,x).

E_2 is called the radiation or acceleration term. It dominates in the far-field and depends on the acceleration of the particles.

E_1 corresponds to a relativistic Coulomb term and grows like the inverse square distance in the vicinity of a point source.

E_0' are "shock waves", depending only on the initial data and propagating with speed of light.

E_0 is the homogeneous field generated by the potential (5.13). It depends only on E_{in} and thus on the initial charge distribution via the constraint (5.14).

Similar expressions hold for the magnetic field. One finds that

$$B(t,x) = B_0(t,x) + B_0'(t,x) + B_1(t,x) + B_2(t,x)$$

with

$$B_0'[f_0] = \int (n \times \alpha^0 Y)(t,\cdot,\xi) *_x f_0 \,\mathrm{d}\xi \tag{5.26}$$

$$B_1[f] = \int (n \times \alpha^{-1} Y) *_{t,x} (\mathbb{1}_{t\geq 0} f) \,\mathrm{d}\xi \tag{5.27}$$

$$B_2[f] = -\int (\nabla_\xi (n \times \alpha^0 Y)) *_{t,x} (K \mathbb{1}_{t\geq 0} f) \,\mathrm{d}\xi \tag{5.28}$$

where we introduced the normal vector $n(x) := \frac{x}{|x|}$.

Remark 5.2.1. In the physical literature, the Liénard-Wiechert field is usually written in terms of the particle acceleration $\dot{v}$ rather than the force $\dot{\xi}$. Since $v(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}$, the two expressions are related as $\dot{v} = \sqrt{1-|v|^2}(K - (v \cdot K)v)$.

5.3 Microscopic theory (Abraham model)

Consider a system of N identical point-charges with phase-space trajectories $(x_i(t), \xi_i(t))_{i=1,..,N}$. The corresponding charge- and current-densities are then given by

$$\rho(t,x) = \sum_{i=1}^{N} \delta(x - x_i(t)); \quad j(t,x) = \sum_{i=1}^{N} v(\xi_i(t))\delta(x - x_i(t)) \tag{5.29}$$

and generate an electromagnetic field $(E, B)(t, x)$ according to Maxwell's equations. However, together with the Lorentz-force equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_i(t) = v(\xi_i(t)) \\ \frac{\mathrm{d}}{\mathrm{d}t} \xi_i(t) = E(t, x_i(t)) + v(\xi_i(t)) \times B(t, x_i(t)) \end{cases} \tag{5.30}$$

this does not yield a consistent theory due to the *self-interaction singularity*: The fields generated by (5.29) are singular precisely at the location of the particles, where they would have to be evaluated according to (5.30).

A classical way to regularize the Maxwell-Lorentz theory is to consider instead of point-particles a system of extended, rigid bodies to which the charge is permanently attached. This is also known as the Abraham model. Shape and size of the rigid charges are given by a smooth, compactly supported, spherically symmetric *form factor* χ satisfying:

$$\chi \in C_c^\infty(\mathbb{R}^3); \ \chi(x) = \chi(|x|); \ \chi(x) = 0 \text{ for } |x| > r = 1; \ \int \chi(x)\,\mathrm{d}x = 1. \tag{5.31}$$

The corresponding charge- and current-densities are then given by

$$\rho(t,x) = \frac{1}{N}\sum_{i=1}^{N} \chi(x - x_i(t)); \quad j(t,x) = \frac{1}{N}\sum_{i=1}^{N} v(\xi_i(t))\chi(x - x_i(t)), \tag{5.32}$$

where $x_i(t)$ now denotes the center of mass of particle i. In order to approximate the Vlasov-Maxwell equations, we shall perform the mean field limit together with a point-particle limit, introducing an N-dependent electron-radius r_N which tends to zero as $N \to \infty$. We thus define a *rescaled form factor* χ^N by

$$\chi^N(x) := r_N^{-3}\,\chi\Big(\frac{x}{r_N}\Big), \ N \in \mathbb{N}, \tag{5.33}$$

where $(r_N)_N$ is a decreasing sequence with $r_N = 1$, $\lim\limits_{N\to\infty} r_N = 0$, to be specified later. This rescaled form factor satisfies

$$\|\chi^N\|_\infty = r_N^{-3}; \ \chi^N(x) = 0 \text{ for } |x| > r_N; \ \int \chi^N(x)\,\mathrm{d}x = 1 \tag{5.34}$$

and approximates a δ-measure in the sense of distributions.
In the so-called *mean field scaling*, the new field equations read

$$\begin{cases} \partial_t E - \nabla_x \times B = -\frac{1}{N}\sum\limits_{i=1}^{N} v(\xi_i(t))\chi^N(x - x_i(t)), \\ \nabla_x \cdot E = \frac{1}{N}\sum\limits_{i=1}^{N} \chi^N(x - x_i(t)), \\ \partial_t B + \nabla_x \times E = \ 0, \qquad \nabla_x \cdot B = 0. \end{cases} \tag{5.35}$$

The particles move according to the equation of motion

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_i(t) = v(\xi_i(t)) \\ \frac{\mathrm{d}}{\mathrm{d}t}\xi_i(t) = \int \chi^N(x - x_i(t))\big[E(t,x) + v(\xi_i(t)) \times B(t,x)\big]\,\mathrm{d}x. \end{cases} \tag{5.36}$$

An equivalent regularization was used by Rein [55] to prove the existence of weak solutions to the Vlasov-Maxwell equations, and by Golse [21] to prove the mean field limit for the regularized Vlasov-Maxwell system. For any fixed r_N, initial particle configuration $Z = (x_i, \xi_i)_{1\leq i\leq N}$ and initial field configuration $(E_{in}, B_{in}) \in C^2(\mathbb{R}^3)$ satisfying the constraints

$$\mathrm{div} E_{in}(x) = \frac{1}{N}\sum \chi^N(x - x_i),\ \mathrm{div} B_{in}(x) = 0, \tag{5.37}$$

the system of equations defined by (5.35) and (5.36) has a unique strong solution as proven in [4] and [37].

Note that the Abraham model is only semi-relativistic, because the charges are assumed to maintain their shape in any frame of reference, neglecting the relativistic effect of Lorentz-contraction. Rotations of the rigid particles are neglected, as well (though one may expect that these degrees of freedom can be separated anyway due to spherical symmetry of the form factor). On the other hand, one important virtue of this theory is that the total energy

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N}\sqrt{1 + |\xi_i(t)|^2} + \frac{1}{2}\int E^2(t,x) + B^2(t,x)\,\mathrm{d}x \tag{5.38}$$

is a constant of motion, as we will verify with a simple computation.

Proof of energy conservation. On the one hand, we compute:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{N}\sum_{i=1}^{N}\sqrt{1 + |\xi_i(t)|^2} \\ &= \frac{1}{N}\sum_{i=1}^{N} v(\xi_i(t)) \cdot \Big(\int \chi^N(x - x_i(t))(E(t,x) + v(\xi_i(t)) \times B(t,x))\mathrm{d}x\Big) \qquad (5.39) \\ &= \frac{1}{N}\sum_{i=1}^{N} v(\xi_i(t)) \cdot \Big(\int \chi^N(x - x_i(t))E(t,x)\mathrm{d}x\Big) \\ &= \frac{1}{N}\sum_{i=1}^{N}\int v(\xi_i(t))\chi^N(x - x_i(t))E(t,x)\mathrm{d}x. \qquad (5.40) \end{aligned}$$

On the other hand, the usual computation for energy conservation in the Maxwell fields yields (with 5.35)

$$\begin{aligned}
&\partial_t\Big(\frac{1}{2}\int E^2(t,x)+B^2(t,x)\,\mathrm{d}x\Big)\\
&\quad=\int E(t,x)\partial_t E(t,x)+B(t,x)\partial_t E(t,x)\,\mathrm{d}x\\
&\quad=-\frac{1}{N}\sum_{i=1}^{N}\int v(\xi_i(t))\chi^N(x-x_i(t))E(t,x)\,\mathrm{d}x &(5.41)\\
&\quad+\int E(t,x)\cdot\nabla\times B(t,x)-B(t,x)\cdot\nabla\times E(t,x)\,\mathrm{d}x. &(5.42)
\end{aligned}$$

Assuming $\lim\limits_{|x|\to\infty}(E(t,x),B(t,x))=0$, the last integral vanishes, since

$$E\cdot\nabla\times B-B\cdot\nabla\times E(t,x)=\mathrm{div}(B\times E).$$

By comparison with (5.40), this shows that (5.38) is a constant of motion.

□

5.3.1 The regularized Vlasov-Maxwell system

In view of the extended charges model defined by equations (5.35) and 5.36, we introduce a corresponding mean field equation. For a given form factor $\chi\in C_c^\infty$ and a rescaling sequence $(r_N)_N$, we consider the set of equations

$$\begin{aligned}
&\partial_t f+v(\xi)\cdot\nabla_x f+\widetilde{K}(t,x,\xi)\cdot\nabla_\xi f=0,\\
&\partial_t E-\nabla_x\times B=-\tilde{j},\qquad \nabla_x\cdot E=\tilde{\rho},\\
&\partial_t B+\nabla_x\times E=\ 0,\qquad \nabla_x\cdot B=0.
\end{aligned}\tag{5.43}$$

$$\tilde{\rho}=\chi^N *_x\int f(t,\cdot,\xi)\,\mathrm{d}\xi,\quad \tilde{j}=\chi^N *_x\int v(\xi)f(t,\cdot,\xi)\,\mathrm{d}\xi.\tag{5.44}$$

$$\widetilde{K}(t,x,\xi)=\chi^N *_x\big(E+v(\xi)\times B\big)(t,x)\tag{5.45}$$

where χ^N is the rescaled form factor defined in (5.33). We call this set of equations the *regularized Vlasov-Maxwell system* with cut-off parameter r_N.

Since the L^1 norm of ρ propagates along any local solution and $\|D^\alpha\tilde{\rho}_t\|_\infty\leq\|D^\alpha\chi^N\|_\infty\|\rho_t\|_1$ all spatial derivatives of $\tilde{\rho}$ and $\tilde{j}$ are bounded uniformly in time. This is enough to show global existence of classical solutions for compact initial data $f_0\in C_c^1(\mathbb{R}^3\times\mathbb{R}^3)$, $\tilde{E}_{in},\tilde{B}_{in}\in C_c^2(\mathbb{R}^3)$ satisfying the constraints $\mathrm{div}\tilde{E}_{in}=\tilde{\rho}_0$, $\mathrm{div}\tilde{B}_{in}=0$, see [29, 54] for more details.

Remark 5.3.1. The regularized Vlasov-Maxwell system defined above is not exactly the same as the one considered by Golse [21] or Rein [54], at least not a priori. In those publications, a double convolution is applied to the charge/current density, that is, the fields solve Maxwell's equation for $\rho=\chi^N*\chi^N*\int f(t,\cdot,\xi)\mathrm{d}\xi,\ j=\chi^N*\chi^N*\int v(\xi)f(t,\cdot,\xi)\mathrm{d}\xi$. Here, only one mollifier is used in (5.44) to regularize the charge/current density, a second convolution with χ^N is applied as the fields act back on f_t, mirroring the form of the rigid charges model defined by eqs. (5.35,5.36). However, by using the uniqueness of solutions to Maxwell's equation and the fact that convolutions commute with each other and with derivatives, one checks that both formulations of the regularized Vlasov-Maxwell dynamics are actually equivalent.

5.4 Existence of solutions

While the 3-dimensional Vlasov-Poisson equation is very well understood from a PDE point of view, the state of research is less satisfying when it comes to the Vlasov-Maxwell equations. Existence of global weak solutions was first proven in DiPerna, Lions, 1989 [14]. Concerning existence and uniqueness of classical solutions, no conclusive answer has been given, so far. The central result is the paper of Glassey and Strauss, 1986, aptly titled “singularity formation in a collisionless plasma could occur only at high velocities” [20]. We recall their main theorem in the following.

Theorem 5.4.1 (Glassey-Strauss, 1986)**.** *Let* $f_0 \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ *and* $E_{in}, B_{in} \in C_c^2(\mathbb{R}^3)$ *satisfying* $\mathrm{div} E^{in} = \rho[f_0]$, $\mathrm{div} B_0 = 0$. *Let* (f_t, E_t, B_t) *be a (weak) solution of the Vlasov-Maxwell System* (5.1) *with initial datum* (f_0, E_{in}, B_{in}). *Suppose there exists* $T \in [0, +\infty]$ *and* $C > 0$ *such that*

$$R(t) = \sup\{|\xi| : \exists x \in \mathbb{R}^3\, f(t,x,\xi) \neq 0\} < C, \ \forall t < T \tag{5.46}$$

Then:

$$\sup_{0 \le t < T^*} \{\|f_t\|_{W^{1,\infty}_{x,\xi}}, \|(E_t, B_t)\|_{W^{1,\infty}_x}\} < \infty \tag{5.47}$$

where $\|f\|_{W^{1,\infty}_{x,\xi}} = \|f\|_\infty + \|\nabla_{x,\xi} f\|_\infty$ *etc. Hence,* (f_t, E_t, B_t) *is the unique classical solution on* $[0, T)$ *with initial data* (f_0, E_{in}, B_{in}).

Simply put, the theorem states that singularity formation can occur in finite time only if particles get accelerated to velocities arbitrarily close to the speed of light. Subsequently, seemingly weaker conditions have been identified that ensure the boundedness of the momentum support and thus the existence of strong solutions. For instance, Sospedra-Alfonso and Illner [62] prove:

$$\limsup_{t \to T^-} R(t) = +\infty \quad \Rightarrow \quad \limsup_{t \to T^-} \|\rho[f_t]\|_\infty = +\infty. \tag{5.48}$$

Most recently, Pallard [52] showed that

$$\limsup_{t \to T^-} R(t) = +\infty \quad \Rightarrow \quad \limsup_{t \to T^-} \|\rho[f_t]\|_{L^6(\mathbb{R}^3)} = +\infty. \tag{5.49}$$

Unfortunately, the criteria thus established are still far away from the known a priori bounds (the strongest, in L^p-sense, being the kinetic-energy bound on $\|\rho[f_t]\|_{L^{4/3}(\mathbb{R}^3)}$, see e.g. [54]) so that well-posedness of the Vlasov-Maxwell system is still considered an open problem. Note that the conditions (5.48) and (5.49) are actually necessary and sufficient for (5.46), because $\rho_t(x) = \int f(t,x,\xi)\mathrm{d}\xi \le \frac{4\pi}{3} R^3(t) \|f_0\|_\infty$.

We will also need the following theorem of Rein [55], who used the regularization introduced above to establish the existence of global weak solutions to the Vlasov-Maxwell system, simplifying the original proof of DiPerna and Lions [14].

Theorem 5.4.2 (Rein, 2004)**.** *Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and $E_{in}, B_{in} \in L^2(\mathbb{R}^3)$ satisfying the compatibility condition* (5.50)*. Let (f_t^N, E_t^N, B_t^N) be a solution of the regularized Vlasov-Maxwell system* (5.43) *with initial data $(f_0, \tilde{E}_{in}, \tilde{B}_{in})$. Then there exist functions $f \in L^\infty(\mathbb{R}; L^1 \cap L^\infty(\mathbb{R}^6)), E, B \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^3))$ such that, along a subsequence,*

$$f^N \rightharpoonup f \text{ in } L^\infty([0,T] \times \mathbb{R}^6); \quad E^N, B^N \rightharpoonup E, B \text{ in } L^2([0,T] \times \mathbb{R}^3), k \to \infty$$

for any bounded time-interval $[0,T]$, $T > 0$ and (f, E, B) is a global weak solution of the Maxell-Vlasov system (5.1) *with $\lim\limits_{t \to 0}(f_t, E_t, B_t) = (f_0, E_{in}, B_{in})$ and $\|f_t\|_{L^p(\mathbb{R}^6)} = \|f_0\|_{L^p(\mathbb{R}^6)}$ for all $p \in [1, \infty]$, $t > 0$.*

5.5 Statement of the results

In the previous sections, we have introduced three kinds of dynamics: The Vlasov-Maxwell system (5.1), the regularized Vlasov-Maxwell system (5.43) and the microscopic Abraham model of extended charges (5.35,5.36) which, in fact, can be viewed as a special case of (5.43) with $f_0 = \mu^N[Z]$. In order to approximate one solution by the other, it does not suffice to assume that the respective distributions are (in some sense) close at $t = 0$. We also have to fix the incoming fields in an appropriate manner, otherwise free fields can be responsible for large deviations between mean field dynamics and microscopic dynamics. We will note our respective convention in the following definition.

Definition 5.5.1. Let $f_0 \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ with $f_0 \geq 0, \int f_0(x,\xi)\mathrm{d}x\mathrm{d}\xi = 1$ and $E_{in}, B_{in} \in C_c^2(\mathbb{R}^3)$ satisfying the Gauss constraints

$$\mathrm{div} E_{in} = \rho[f_0] = \int f_0(\cdot, \xi)\mathrm{d}\xi, \quad \mathrm{div} B_{in} = 0. \tag{5.50}$$

Such (f_0, E_{in}, B_{in}) are the admissible initial data for the Vlasov-Maxwell system (5.1).

1) For the regularized Vlasov-Maxwell system, we fix initial data for the fields as

$$E_{in}^N := \chi^N * E_{in}, \quad B_{in}^N := \chi^N * B_{in}, \tag{5.51}$$

for any $N \geq 1$. These fields satisfy: $\mathrm{div} E_{in}^N = \tilde{\rho}[f_0]$ and $\mathrm{div} B_{in}^N = 0$. We denote by (f^N, E^N, B^N) the unique solution of (5.43) with initial data $(f_0, E_{in}^N, B_{in}^N)$.

2) For the microscopic system with initial configuration $Z = (x_1, \xi_1, ..., x_N, \xi_N) \in \mathbb{R}^{6N}$, the charge distribution can be written as $\tilde{\rho}[\mu^N[Z]](x) = \frac{1}{N}\sum\limits_{i=1}^N \chi^N(x - x_i)$. Given a renormalizing sequence $(r_N)_{N \geq 1}$ we fix compatible initial fields (E_{in}^μ, B_{in}^μ) such that

$$E_{in}^\mu := E_{in}^N - \nabla G * (\tilde{\rho}[\mu_0^N[Z]] - \tilde{\rho}[f_0]), \quad B_{in}^\mu := B_{in}^N. \tag{5.52}$$

Note that E^μ_{in} and B^μ_{in} depend on N and E^μ_{in} also on Z. For any $N \in \mathbb{N}$ and $Z = (x_i, \xi_i) \in \mathbb{R}^{6N}$ we then denote by $\big((x_i^*, \xi_i^*)_{1\leq i\leq N}, E^\mu, B^\mu\big)$ the unique solution of (5.35, 5.36) with initial data $(Z, E^\mu_{in}, B^\mu_{in})$. We call

$$^N\Psi_{t,0} = \mathbb{R}^{6N} \to \mathbb{R}^{6N},\ ^N\Psi_{t,0}(Z) = (x_i^*(t), \xi_i^*(t))_{i=1,..,N} \tag{5.53}$$

the *microscopic flow* and

$$\mu_t^N[Z] := \mu^N[\Psi_{t,0}(Z)] = \frac{1}{N}\sum_{i=1}^N \delta_{x_i^*(t)}\delta_{\xi_i^*(t)} \tag{5.54}$$

the *microscopic density* of the system with initial configuration Z.

Note: The macroscopic fields (E^N_{in}, B^N_{in}) are compactly supported, though the microscopic field E^μ_{in}, determined by (5.51), is not.

We now state our precise result in the following theorem. Our approximation of the Vlasov-Maxwell dynamics is formulated in terms of the Wasserstein distances W_p discussed in Chapter 2. Probabilities and expectation values referring to initial data $Z \in \mathbb{R}^{6N}$ are meant with respect to the product measure $\otimes^N f_0$ for a given probability density $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. That is, for any random variable $H : \mathbb{R}^{6N} \to \mathbb{R}$ and any element A of the Borel-algebra we write

$$\mathbb{P}_0^N(H \in A) = \int_{H^{-1}(A)} \prod_{j=1}^N f_0(z_j)dZ, \tag{5.55}$$

$$\mathbb{E}_t^N(H) = \int_{\mathbb{R}^{6N}} H(Z)\prod_{j=1}^N f_0(z_j)dZ\ . \tag{5.56}$$

When the particle number N is fixed, we will usually omit the index and write only $\mathbb{P}_0$, respectively $\mathbb{E}_0$.

Theorem 5.5.2. *Let $f_0 \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R}^+_0)$ with total mass one and $(E_{in}, B_{in}) \in C^2_c(\mathbb{R}^3)$ satisfying the constraints* (5.50)*. Let $\gamma < \frac{1}{12}$ and r_N a rescaling sequence with $r_N \geq N^{-\gamma}$. For $N \in \mathbb{N}$, let (f^N, E^N, B^N) the solution of the renormalized Vlasov-Maxwell equation* (5.43) *and $(\Psi_{t,0}(Z), E^\mu, B^\mu)$ the solution of the microscopic equations (5.35 5.36) with initial data as in Def. 5.5.1. Let $\mu_t^N[Z] := \mu^N[\Psi_{t,0}(Z)]$ the empirical density corresponding the the microscpic flow $\Psi_{t,0}(Z)$. Suppose there exists $T > 0$ and constant $C_0 > 0$ such that*

$$\|\rho[f_t^N]\|_\infty \leq C_0,\ \forall N \in \mathbb{N},\ 0 \leq t \leq T. \tag{5.57}$$

a) Then we have molecular chaos in the sense that for all $p \in [1, \infty)$ and $\epsilon > 0$:

$$\forall 0 \leq t \leq T:\ \lim_{N\to\infty} \mathbb{P}_0^N\Big[W_p(\mu_t^N[Z], f_t) \geq \epsilon\Big] = 0 \tag{5.58}$$

where (f_t, E_t, B_t) is the unique classical solution of the Vlasov-Maxwell system (5.1) *on $[0, T]$ with initial data (f_0, E_{in}, B_{in}).*

b) *For the regularized dynamics, we have the following quantitative approximation result: Let $p \geq 1$, $\alpha < \min\{\frac{1}{6}, \frac{1}{2p}\}$ and $\gamma < \delta < \frac{1}{4}$. Then there exist constants L, C depending on T, C_0 and the initial data such that for all $t \in [0,T]$ and $N \geq 4$:*

$$\mathbb{P}_0\Big[\sup_{0\leq s\leq t} W_p(\mu_s^N[Z], f_s^N) \geq N^{-\delta} + e^{tL}N^{-\alpha}\Big] \leq e^{tC\sqrt{\log(N)}}N^{-\frac{1}{4}+\delta} + a(N,p,\alpha) \quad (5.59)$$

where

$$a(N,p,\alpha) = c' \cdot \begin{cases} \exp(-cN^{1-2p\alpha}) & \text{if } p > 3 \\ \exp(-c\frac{N^{1-6\alpha}}{\log(2+N^{3\alpha})^2}) & \text{if } p = 3 \\ \exp(-cN^{1-6\alpha}) & \text{if } p \in [1,3). \end{cases} \quad (5.60)$$

The constant $c', c > 0$ depend only on p, α and f_0.

c) *For the fields, we have the following approximation results: For any compact region $M \subset \mathbb{R}^3$ there exists a constant $C_1 > 0$ such that for any $0 \leq t \leq T$ and $N \geq 4$:*

$$\mathbb{P}_0\Big[\|(E_t^N, B_t^N) - (E_t^\mu, B_t^\mu)\|_{L^\infty(M)} \geq C_1\sqrt{\log(N)}N^{-\delta}\Big] \leq e^{tC\sqrt{\log(N)}}N^{-\frac{1}{4}+\delta}. \quad (5.61)$$

Remarks 5.5.3.

1) The result implies propagation of molecular chaos in the sense of (1.12).

2) We do not have a quantitative result for the convergence $f_t^N \rightharpoonup f_t$, i.e. we do not know how fast $W_p(f_t^N, f_t)$ converges to 0 for any p.

3) Assumption (5.57) can be replaced by equivalent conditions, e.g. a uniform bound on $\|\rho[f_t^N]\|_{L^6(\mathbb{R}^3)}$ or on the momentum-support. Of course, it would be much more desirable to have a sufficient condition on f_0 only. However, such a condition would likely have to come out of the existence theory for Vlasov-Maxwell.

4) The constants C and C_0 blow up as the maximal velocity $\overline{v}$ approaches 1 (speed of light).

5.6 Corollaries from solution theory

We will first conclude some corollaries from the existence theorems cited above. Fix $f_0 \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R}_0^+)$ and $T > 0$ as in Theorem 5.5.2. By assumption, there exists C_0 such that

$$\|\rho[f_t^N]\|_\infty \leq C_0,\ \forall N \geq 1,\ 0 \leq t \leq T. \quad (5.62)$$

By the theorem of Sospedra-Alfonso and Illner [62], there thus exists a $\mathcal{R} > 0$ such that

$$R[f^N](t) = \sup\{|\xi| : \exists x \in \mathbb{R}^3\ f^N(t,x,\xi) \neq 0\} < \mathcal{R}, \quad (5.63)$$

for all $N \geq 1$ and $0 \leq t \leq T$. We define

$$\overline{\xi} := \mathcal{R} + 1 \text{ and } \overline{v} := |v(\overline{\xi})|, \quad (5.64)$$

which will serve us as an upper bound on the velocity of the particles. By the Glassey-Strauss theorem, there thus exists a constant $L' > 0$ such that

$$\|(E^N_t, B^N_t)\|_\infty + \|\nabla_x(E^N_t, B^N_t)\|_\infty \leq L', \tag{5.65}$$

for all $N \geq 1$, $0 \leq t \leq T$. In particular, observing that

$$\nabla_\xi v(\xi) = \nabla_\xi \frac{\xi}{\sqrt{1+\xi^2}} = \frac{\delta^{i,j}}{\sqrt{1+\xi^2}} - \frac{\xi^i \xi^j}{(\sqrt{1+\xi^2})^3}, \tag{5.66}$$

with $|\nabla_\xi v(\xi)| \leq 2$, we have

$$\|K[f^N](t,\cdot,\cdot)\|_{W^{1,\infty}(\mathbb{R}^3\times\mathbb{R}^3)} \leq \max\{L', 2\} =: L. \tag{5.67}$$

Note that the theorems of Glassey/Strauss und Sospedra-Alfonso/Illner are formulated for the unregularized Vlasov-Maxwell system (5.1), so one has to check that they actually yield bounds that are uniform in N as one considers the sequence of regularized solutions f^N_t. We refer, in particular, to the simplified proof of the Glasey-Strauss theorem proposed by Bouchut, Golse and Pallard [9]. For instance, the $W^{1,\infty}$-bound on the fields is derived from estimates of the form

$$\|K(t)\|_{W^{1,\infty}_{x,\xi}} \leq C_2 e^{TC_2}\big(1 + \log_+(\|\nabla_x f\|_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)})\big),$$

$$\sup_{s\leq t}\|\nabla_{x,\xi} f(s)\|_\infty \leq \|\nabla_{x,\xi} f_0\|_\infty + C_1 \int_0^t (1 + \log_+(\sup_{s'\leq s}\|\nabla_{x,\xi} f(s')\|_\infty)) \sup_{s'\leq s}\|\nabla_{x,\xi} f(s')\|_\infty \mathrm{d}s,$$

where $\log_+(x) := \max\{0, \log(x)\}$ and the constants C_1, C_2 depend only on T, f_0 and $\mathcal{R}$ (see [9, Section 5.4]). Hence, one readily sees that the bounds hold independent of N.

Since the velocity of the particles is bounded by 1, the support in the space-variables remains bounded, as well, for compact initial data. We set

$$\overline{r} = \sup\big\{|x| : \exists \xi \in \mathbb{R}^3\ f_0(x,\xi) \neq 0\big\} + T + 1. \tag{5.68}$$

Then we have, in particular, $\operatorname{supp}\tilde{\rho}[f_t] \subseteq \mathrm{B}(\overline{r};0) = \{x \in \mathbb{R}^3 : |x| \leq \overline{r}\}$ for all $0 \leq t \leq T$ as well as $|\Psi^1_{t,0}(Z)|_\infty < \overline{r}$ if $Z \in \operatorname{supp} \otimes^N f_0$.

Now we recall from Theorem 5.4.2 that, along a subsequence,

$$(f^N_t, E^N, B^N) \rightharpoonup (f'_t, E'_t, B'_t), \tag{5.69}$$

where (f', E', B') is a global weak solution of the Vlasov-Maxwell system (5.1) with initial data (f_0, E_{in}, B_{in}) and weak convergence of the fields is understood in L^2 sense. However, for any $t \in [0,T]$ and any test-function $\varphi \in C^\infty_c(\mathbb{R}^3\times\mathbb{R}^3)$ with $|\xi| < \mathcal{R} \Rightarrow \varphi(x,\xi) = 0$,

$$\int \varphi(x,\xi) f'_t(x,\xi)\mathrm{d}\xi\mathrm{d}x = \lim_{N\to\infty} \int \varphi(x,\xi) f^N_t(x,\xi)\mathrm{d}\xi\mathrm{d}x = 0.$$

This means that the momentum-support of f' remains bounded by $\mathcal{R}$ and according to the Glassey-Strauss theorem, (f', E', B') is actually a strong solution on $[0,T]$. Thus, under the assumptions of the theorem, we can conclude that

$$(f_t^N, E_t^N, B_t^N) \rightharpoonup (f_t, E_t, B_t),\ \forall 0 \leq t \leq T, \tag{5.70}$$

where (f_t, E_t, B_t) is the *unique classical* solution on $[0,T]$ with initial data (f_0, E_{in}, B_{in}) and the convergence holds for any subsequence (otherwise one could extract a convergent subsubsequence) and thus for the sequence itself.

Finally, note that since we can restrict all measures to the compact space $\mathrm{B}(\overline{r}) \times \mathrm{B}(\overline{\xi})$, weak convergence is equivalent to convergence in Wasserstein distance so that, in particular, $W_p(f_t^N, f_t) \to 0$ for all $p \in [1, \infty)$.

5.7 Strategy of proof

Definition 5.7.1. Let f_0, E_{in}, B_{in} as above. Let f_t^N the solution of the regularized Vlasov-Maxwell system with initial datum f_0. Let $K[\tilde{f}^N]$ the Lorentz-force field corresponding to the charge- and current-density induced by $\tilde{f}^N = \chi^N * f^N$. We denote by $\varphi_{t,s}^N$ the characteristic flow of the regularized Vlasov-Maxwell system (5.43), i.e. the solution of

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} y(t) = v(\eta(t)) \\ \frac{\mathrm{d}}{\mathrm{d}t} \eta(t) = \tilde{K}[\tilde{f}^N](t, y, \eta) \end{cases} \tag{5.71}$$

with $\varphi_{s,s}^N(z) = z$. We denote by ${}^N\Phi_{t,s}$ the lift of $\varphi_{t,s}^N(\cdot)$ to the N-particle phase-space, that is ${}^N\Phi_{t,s}(Z) := (\varphi_{t,s}^N(z_1), ..., \varphi_{t,s}^N(z_N))$. In other words, ${}^N\Phi_{t,s}$ is the N-particle flow generated by the (regularized) mean field force induced by f_t^N. We will often omit the index N.

The strategy of the proof is very similar to the one in Chapter 3 in the Vlasov-Poisson case. We recall the J function which we introduced as our measure of chaos to control the difference between mean field dynamics and microscopic dynamics.

Definition 5.7.2. Let ${}^N\Phi_{t,0}$ the mean field flow defined above and ${}^N\Psi_{t,0}$ the microscopic flow solving (5.36). We denote by ${}^N\Psi_{t,0}^1 = (x_i^*(t))_{1 \leq i \leq N}$ and ${}^N\Psi_{t,0}^2 = (\xi_i^*(t))_{1 \leq i \leq N}$ the projection onto the spatial, respectively the momentum coordinates.
Let $J(t)$ be the stochastic process given by

$$\begin{aligned} J_t^N(Z) := \min\Big\{1, \lambda(N) N^\delta \sup_{0 \leq s \leq t} |{}^N\Psi_{t,0}^1(Z) - {}^N\Phi_{t,0}^1(Z)|_\infty \\ + N^\delta \sup_{0 \leq s \leq t} |{}^N\Psi_{t,0}^2(Z) - {}^N\Phi_{t,0}^2(Z)|_\infty \Big\}, \end{aligned} \tag{5.72}$$

where $|Z|_\infty = \max\{|x_i| : 1 \leq i \leq N\}$ denotes the maximum-norm on $\mathbb{R}^{3N}$ and $\lambda(N) := \max\{1, \sqrt{\log(N)}\}$.

Our aim is to derive a Gronwall estimate for the time-evolution of $\mathbb{E}_0^N(J_t^N)$, showing that $\mathbb{E}_0^N(J_t^N) \xrightarrow{N \to \infty} 0,\ \forall 0 \leq t \leq T$. This will be achieved by using the Liénard-Wiechert representation of the fields introduced in section 5.2.1. The field corresponding to the (regularized) Vlasov-Maxwell dynamics is generated by the smeared Vlasov-density $\tilde{f}^N$, while the field corresponding to the microscopic dynamics of the rigid charges is generated by the smeared microscopic density $\tilde{\mu}^N[Z] := \chi^N *_x \mu[Z]$. For a given space-time point $(t,x) \in \mathbb{R} \times \mathbb{R}^3$, we will estimate the difference as:

$$\begin{aligned}
&\left|E_i[\tilde{f}^N](t,x) - E_i[\tilde{\mu}^N](t,x)\right| \\
&\leq \left|E_i[\tilde{f}^N](t,x) - E_i[\tilde{\mu}^N[\Phi_{s,0}(Z)]](t,x)\right| && (5.73)\\
&+ \left|E_i[\tilde{\mu}^N[\Phi_{s,0}(Z)]](t,x) - E_i[\tilde{\mu}^N[\Psi_{s,0}(Z)]](t,x)\right| && (5.74)
\end{aligned}$$

for $i = 1,2,3$ and similarly for the magnetic field components. Here, we have introduced as an intermediate, the field corresponding to the (smeared) point-charge density $\mu^N[\Phi_{s,0}(Z)]$ of the mean field flow $\Phi_{s,0}(Z)$. We will use a law-of-large number estimate to show that terms of the form (5.73) are typically small, because the particles evolving with the mean field flow are at all times i.i.d. with law f^N. For the terms of the form (5.74), we will derive a local Lipschitz bound in terms of $J_t^N(Z)$, the (weighted) maximal distance between the respective mean field and microscopic trajectories.

In total, the approximation of the solution to the Vlasov-Maxwell system will be split as:

$$\begin{aligned}
W_p(\mu_t^N[Z], f_t) \leq\ & W_p(\mu^N[\Psi_{t,0}(Z)], \mu^N[\Phi_{t,0}(Z)]) && (5.75)\\
&+ W_p(\mu^N[\Phi_{t,0}(Z)], f_t^N) && (5.76)\\
&+ W_p(f_t^N, f_t). && (5.77)
\end{aligned}$$

The first term is the most interesting one, concerning the difference between microscopic time-evolution and mean field time-evolution. We recall from Proposition 3.5.3 that

$$\mathbb{P}_0\big[\sup_{0\leq s\leq t} W_p(\mu^N[\Psi_{s,0}(Z)], \mu^N[\Phi_{s,0}(Z)]) \geq N^{-\delta}\big] \leq \mathbb{E}_0(J_t^N). \tag{5.78}$$

Convergence of $\mathbb{E}_0(J_t^N)$ will thus yield the bound on (5.75).

Convergence of (5.77) is a purely deterministic statement and follows from Theorem 5.4.2 cited above. The proof of Rein, however, is based on a compactness argument and does not yield quantitative bounds. Hence, we do not know at what rate (5.77) goes to zero. Based on the corresponding result in the Vlasov-Poisson case, Prop. 3.9.1, we conjecture that $W_p(f_t^N, f_t) \sim r_N^{1-\epsilon}$ for any $\epsilon > 0$ and $p \leq 2$, though we were not yet able to prove this.

The second term $W_p(\mu^N[\Phi_{t,0}(Z)], f_t^N) = W_p(\varphi_{t,0}^N \# \mu_0^N[Z], \varphi_{t,0}^N \# f_0)$ concerns the sampling of the mean field dynamics by discrete particle trajectories. Since the mean field forces satisfy a Lipschitz bound uniformly in N according to (5.67), we have the following standard result:

Lemma 5.7.3. *Under the assumptions of Theorem 5.5.2, it holds that*

$$W_p(\mu^N[\Phi_{t,0}(Z)], f_t^N) = W_p(\varphi_{t,0}^N \# \mu_0^N[Z], \varphi_{t,0}^N \# f_0) \leq e^{tL} W_p(\mu_0^N[Z], f_t^N)$$

for all $0 \leq t \leq T$, *where* L *is the uniform Lipschitz constant defined in* (5.67).

Proof. We will give a somewhat non-standard proof of this standard result. Since $\varphi^N_{t,0}(x,\xi)$ is the solution of (5.71) with $\varphi^N_{0,0}(x,\xi) = (x,\xi)$, it is a classical result that the Jacobian of the flow satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}t} D_{x,\xi}\varphi^N_{t,0} = D_{x,\xi}(v(\xi), K(t,x,\xi)) D_{x,\xi}\varphi^N_{t,0}, \; D_{x,\xi}\varphi^N_{0,0} = \mathbb{1}_{6\times 6}, \tag{5.79}$$

and since $|D_{x,\xi}(v(\xi), K(t,x,\xi))|_\infty < L$ it follows that $|D_{x,\xi}\varphi_{t,0}|_\infty < e^{tL}$.

Now for any $Z \in \mathbb{R}^{6N}$ let $\pi_0(x,y) \in \Pi(\mu^N_0, f_0)$ and define $\pi_t = (\varphi^N_t, \varphi^N_t)\#\pi_0 \in \Pi(\mu^N_0[\Phi_{t,0}(Z)], f^N_t)$. Then

$$\begin{aligned} W_p(\mu^N[\Phi_{t,0}(Z)], f^N_t) &\leq \Big(\int\limits_{\mathbb{R}^6\times\mathbb{R}^6} |x-y|^p \,\mathrm{d}\pi_t(x,y) \Big)^{1/p} \\ = \Big(\int\limits_{\mathbb{R}^6\times\mathbb{R}^6} |\varphi^N_t(x) - \varphi^N_t(y)|^p \,\mathrm{d}\pi_0(x,y) \Big)^{1/p} &\leq e^{tL} \Big(\int\limits_{\mathbb{R}^6\times\mathbb{R}^6} |x-y|^p \,\mathrm{d}\pi_t(x,y) \Big)^{1/p}. \end{aligned}$$

We conclude by taking on the right-hand-side the infimum over all $\pi_0(x,y) \in \Pi(\mu^N_0, f_0)$. $\square$

Hence, it remains to check that if the initial configuration Z is chosen randomly with law $\otimes^N f_0$, the microscopic density $\mu^N_0[Z]$ approximates f_0 in Wasserstein distance. To this end, we will once again rely on the large deviation estimates from Fournier and Gullin, Theorem 2.2.1. Since f_0 here is compactly supported, we can use the result with the exponential moment condition. This yields the following:

Lemma 5.7.4. *Applying Thm. 5.59 in dimension $d = 6$ with $\epsilon = N^{\alpha p}$ we get*

$$\mathbb{P}\Big[W_p(\mu^N_0[Z], f_0) > N^{-\alpha} \Big] \leq a(N,p,\alpha) = c' \cdot \begin{cases} \exp(-cN^{1-2p\alpha}) & \text{if } p > 3 \\ \exp(-c\frac{N^{1-6\alpha}}{\log(2+N^{3\alpha})^2}) & \text{if } p = 3 \\ \exp(-cN^{1-6\alpha}) & \text{if } p \in [1,3). \end{cases}$$

5.8 Global estimates

By assumption, there exists a constant $C_0 > 0$ such that $\|\rho[f^N]\|_{L^\infty([0,T]\times\mathbb{R}^3)} \leq C_0$ for all $N \in \mathbb{N}\cup\{+\infty\}$. Using the methods introduced in Chapter 4, we will now show that as long as mean field dynamics and microscopic dynamics are sufficiently close, this implies certain bounds on the microscopic density and fields. As we have to deal with singular kernels, the necessary regularizations come from the smearing with the N-dependent mollifier χ^N.

Notation / Definition: Following [52] we introduce the shorthand notation

$$g \lesssim h :\Longleftrightarrow \exists C > 0 : g \leq C\, h, \tag{5.80}$$

where $C \in \mathbb{R}$ is a constant that may depend only on T and initial data.
Moreover, for fixed $N \geq 1$ and any measurable function h on $\mathbb{R}^n$, $n = 3$ or $n = 6$, we introduce the notation $\tilde{h} := \chi^N *_x h$. For a probability measure $\mathcal{P}(\mathbb{R}^n)$ we define $\tilde{v} \in \mathcal{P}(\mathbb{R}^n)$

by $\int h \mathrm{d}\tilde{\nu} := \int \tilde{h} \mathrm{d}\nu$ for all measureable h. Note that if $\rho(x) = \frac{1}{N}\sum_{i=1}^{N}\delta(x - x_i)$ for $x_i \in \mathbb{R}^3$, we have $\tilde{\rho} = \frac{1}{N}\sum_{i=1}^{N}\chi^N(x - x_i)$, consistent with the notation of Section 5.3.

Lemma 5.8.1. *Let $h : \mathbb{R}^3 \to \mathbb{R}^n$ a measurable function satisfying $|h(x)| \leq \frac{1}{|x|^2}$. Then:*

$$i)\quad |\chi^N * h(x)| \quad \lesssim \min\{r_N^{-2}, \frac{1}{|x|^2}\}, \tag{5.81}$$

$$ii)\quad |\nabla\chi^N * h(x)| \lesssim \min\{r_N^{-3}, \frac{1}{|x|^3}\}. \tag{5.82}$$

Proof. Recalling that $\|\chi^N\|_\infty = r_N^{-3}\|\chi\|_\infty$ and $\|\chi^N\|_1 = 1$, we compute:

$$\begin{aligned}
|\chi^N * h(x)| &\leq \int |k(y)|\chi^N(x-y)\mathrm{d}^3y \leq \int \frac{1}{|y|^2}\,\chi^N(x-y)\,\mathrm{d}^3y \\
&\leq \int\limits_{|y|\leq r_N} + \int\limits_{|y|>r_N} \frac{1}{|y|^2}\chi^N(x-y)\mathrm{d}^3y \\
&\leq \|\chi^N\|_\infty \int\limits_{|y|\leq r_N} \frac{1}{|y|^2}\mathrm{d}^3y + \frac{1}{r_N^2}\int \chi^N(x-y)\mathrm{d}^3y \lesssim r_N^{-2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|\nabla(\chi^N * h)(x)| \leq |\nabla\chi^N| * |k|(x) &\leq \int\limits_{|y|\leq r_N} + \int\limits_{|y|>r_N} \frac{1}{|y|^2}|\nabla\chi^N(x-y)|\mathrm{d}^3y \\
&\leq \|\nabla\chi^N\|_\infty \int\limits_{|y|\leq r_N} \frac{1}{|y|^2}\mathrm{d}^3y + \frac{1}{r_N^2}\int |\nabla\chi^N(x-y)|\mathrm{d}^3y \\
&\leq r_N^{-4}\|\nabla\chi\|_\infty\, 4\pi r_N + r_N^{-2}r_N^{-1}\|\nabla\chi\|_1 \leq r_N^{-3}(4\pi\|\nabla\chi\|_\infty + \|\nabla\chi\|_1).
\end{aligned}$$

Finally, if $|x| > 2r_N$, the mean-value theorem of integration yields for $s \geq 1$:

$$\chi^N * \frac{1}{|y|^s}(x) = \int \frac{1}{|x-y|^s}\chi^N(y)\mathrm{d}^3y \leq \sup\{|x-y|^{-s} \mid y \in \operatorname{supp}\chi^N\} \leq \frac{2^s}{|x|^s},$$

where we used the fact that $\int \chi^N = 1$ and $|y| \leq r_N \leq \frac{1}{2}|x|$, $\forall y \in \operatorname{supp}(\chi^N)$. $\square$

5.8.1 Bounds on the charge density

Proposition 5.8.2. *Suppose there exists a $p \in [1, \infty)$ such that*

$$W_p(\mu_0^N[Z], f_0) \leq r_N^{3+p}. \tag{5.83}$$

Then there exists a constant C_ρ depending on T such that

$$|{}^N\Psi_{t,0}(Z) - {}^N\Phi_{t,0}(Z)|_\infty < r_N \Rightarrow \|\tilde{\rho}[\mu_t^N[Z]]\|_\infty \leq C_\rho. \tag{5.84}$$

Corollary 5.8.3. *Under the conditions of the proposition, we also have*

$$|^N\Psi_{t,0}(Z) - {}^N\Phi_{t,0}(Z)|_\infty < r_N \Rightarrow \|D^\alpha \tilde{\rho}[\mu_t^N[Z]]\|_\infty \lesssim r_N^{-|\alpha|}. \tag{5.85}$$

Proof. Note that $D^\alpha \tilde{\rho}[\mu_t^N] = D^\alpha(\chi^N * \rho[\mu_t^N]) = (D^\alpha \chi^N) * \rho[\mu_t^N]$, and

$$D^\alpha \chi^N(x) = D_x^\alpha r_N^{-3} \chi(\frac{x}{r_N}) = r_N^{-|\alpha|} r_N^{-3} (D^\alpha \chi)(\frac{x}{r_N}).$$

Let $\overline{\chi} := \frac{D^\alpha \chi}{\|D^\alpha \chi\|_1}$. This $\overline{\chi}$ satisfies (5.31) and can thus be used as a form factor instead of χ. The previous proposition then yields $|^N\Psi_{t,0}(Z) - {}^N\Phi_{t,0}(Z)|_\infty < r_N \Rightarrow \|\overline{\chi}^N * \rho[\mu_t^N]\|_\infty \leq C$, and thus

$$\|D^\alpha \tilde{\rho}[\mu_t^N]\|_\infty = \|D^\alpha \chi\|_1 \, r_N^{-|\alpha|} \|\overline{\chi}^N * \rho[\mu_t^N]\|_\infty \lesssim r_N^{-|\alpha|}.$$

□

Remark 5.8.4. In the end, we will have to show that assumption (5.83) is satisfied for *typical* initial conditions, as the initial particle configurations are chosen randomly and independently with law f_0. This (and only this) requirement will set the lower bound on the cut-off to $r_N \sim N^{-\gamma}$ with $\gamma < \frac{1}{12}$.

The proof of Proposition 5.8.2 is based on Lemma 4.4.2, derived in Chapter 4. We recall:

Lemma Let $\rho_1, \rho_2 \in \mathcal{P}(\mathbb{R}^3)$ two probability measures. Then

$$\|\chi^N * \rho_1\|_\infty \leq \frac{32\pi}{3} \|\rho_2\|_\infty + r_N^{-(3+p)} \, W_p^p(\rho_1, \rho_2). \tag{5.86}$$

Proof of Proposition 5.8.2. As an intermediate step, we introduce the density $\mu^N[\Phi_{t,0}(Z)]$ corresponding to the mean field flow defined in 5.7.1. Since the mean field force is Lipschitz continuous with a constant L independent of N, we have according to Lemma 5.7.3

$$W_p^p(\mu^N[\Phi_{t,0}(Z)], f_t^N) \leq e^{tL} W_p^p(\mu_0^N[Z], f_0).$$

Moreover, by assumption, $\|\tilde{\rho}[f_t^N]\|_\infty \leq \|\rho[f_t^N]\|_\infty \leq C_0$, $\forall N$. Applying the previous Lemma with $\rho_1 = \rho[\mu^N[\Phi_{t,0}(Z)]]$, $\rho_2 = \rho[f_t^N]$, we get

$$\|\tilde{\rho}[\mu^N[\Phi_{t,0}(Z)]]\|_\infty \lesssim C_0 + e^{tL}.$$

Now, recall from Lemma 3.5.2 that $W_\infty(\mu[\Phi_{t,0}(Z)], \mu[\Psi_{t,0}(Z)]) \leq \left|\Phi_{t,0}(Z) - \Psi_{t,0}(Z)\right|_\infty$, where W_∞ is the infinity Wasserstein distance. If $\left|\Phi_{t,0}(Z) - \Psi_{t,0}(Z)\right|_\infty < r_N$, there exists $q > 0$ such that $\left|\Phi_{t,0}(Z) - \Psi_{t,0}(Z)\right|_\infty \leq r_N^{1+\frac{3}{q}}$. We thus have

$$\begin{aligned} r_N^{-(q+3)} W_q^q(\mu^N[\Phi_{t,0}(Z)], \mu^N[\Psi_{t,0}(Z)]) &\leq r_N^{-(q+3)} (W_\infty(\mu[\Phi_{t,0}(Z)], \mu[\Psi_{t,0}(Z)]))^q \\ &\leq r_N^{-(q+3)} \left|\Phi_{t,0}(Z) - \Psi_{t,0}(Z)\right|_\infty^q \leq 1. \end{aligned}$$

Applying once more Lemma 4.4.2 with $\rho_1 = \rho[\mu^N[\Psi_{t,0}(Z)]]$, $\rho_2 = \rho[\mu^N[\Phi_{t,0}(Z)]]$ and the Wasserstein metric of order q, we get the announced result.

□

5.8.2 Bounds on the field derivatives

Proposition 5.8.5. *Under the conditions of Proposition 5.8.2, the microscopic fields satisfy*

$$\|\nabla_x E_t[\tilde{\mu}^N]\|_\infty,\, \|\nabla_x B_t[\tilde{\mu}^N]\|_\infty \lesssim r_N^{-2}. \tag{5.87}$$

Proof. We begin with the homogeneous field

$$E_0(t,x) = \partial_t Y(t,\cdot) * E_{in}(x) = \partial_t\Big(\frac{t}{4\pi}\int\limits_{S^2} E_{in}(y+\omega t)\mathrm{d}\omega\Big). \tag{5.88}$$

From this representation, one reads of the bounds

$$\|E_0(t,\cdot)\|_{W_x^{k-1,\infty}} \leq \|E_{in}\|_{W_x^{k-1,\infty}} + t\|E_{in}\|_{W_x^{k,\infty}}. \tag{5.89}$$

In particular, for $E_{in} = -\nabla G * \rho_0$, we have

$$\|D^\alpha E_{in}(t,\cdot)\|_\infty \lesssim \|D^\alpha \rho_0\|_\infty + \|D^\alpha \rho_0\|_1,\ |\alpha| = 0,1,2,$$

where we used

$$\int \frac{1}{|y|^2}|D^\alpha\rho_0|(x-y)\mathrm{d}^3y = \int\limits_{|y|\leq 1} + \int\limits_{|y|>1} \frac{1}{|y|^2}|D^\alpha\rho_0|(x-y)\mathrm{d}^3y \leq 4\pi\|D^\alpha\rho_0\|_\infty + \|D^\alpha\rho_0\|_1.$$

For the inhomogeneous parts, we can use equation (5.17) to write

$$\begin{aligned} E(t,x) &= -\int (\nabla_x + v(\eta)\partial_t)Y * f(\cdot,\cdot,\eta)\mathrm{d}\eta \\ &= -\int (\nabla_x + v(\eta)\partial_t)\int\limits_0^t\int\limits_{S^2}(t-s)f(s,x+\omega(t-s),\eta)\mathrm{d}\eta, \\ B(t,x) &= -\int (v(\eta)\times\nabla_x)Y * f(\cdot,\cdot,\eta)\mathrm{d}\eta \\ &= -\int (v(\eta)\times\nabla_x)\int\limits_0^t\int\limits_{S^2}(t-s)f(s,x+\omega(t-s),\eta)\mathrm{d}\eta, \end{aligned}$$

from which we read off the bounds

$$\|\nabla E\|_\infty,\, \|\nabla B\|_\infty \leq 4\pi(1+T)T\sup_{s\leq T}\sum_{|\alpha|\leq 2}\|D^\alpha\rho[f(s)]\|_\infty. \tag{5.90}$$

Applying this to $f(t) = \tilde{\mu}_t^N = \chi^N *_x \mu_t^N[Z]$ and using (5.85), the desired statement follows. $\square$

5.8.3 Bound on the total force

While we will show that for typical initial conditions, the microscopic time-evolution will be close to the mean field time-evolution, we also need to control how "bad" initial conditions contribute to the growth of $\mathbb{E}_0(J_t)$. To this end, we require a bound on the total microscopic force, although a rather coarse one will suffice.

Proposition 5.8.6. *The total microscopic force is bounded as*

$$\|\tilde{K}_t[\tilde{\mu}^N]\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \leq \|\tilde{E}_t[\tilde{\mu}^N]\|_{L^\infty(\mathbb{R}^3)} + \|\tilde{B}_t[\tilde{\mu}^N]\|_{L^\infty(\mathbb{R}^3)} \lesssim r_N^{-2},\ \forall t \geq 0. \tag{5.91}$$

Note that this holds independently of assumption (5.83).

Proof. Recall that the total energy

$$\varepsilon(t) = \frac{1}{N}\sum_{i=1}^{N}\sqrt{1+|\xi_i(t)|^2} + \frac{1}{2}\int E_t^2(x) + B_t^2(x)\mathrm{d}x$$

is a constant of motion. At $t=0$, we thus have:

$$\varepsilon(0) \leq \frac{1}{2}\big(\|E_{in}\|_2^2 + \|B_{in}\|_2^2\big) + \sqrt{1+\bar{\xi}^2}.$$

For the microscopic system, we have according to our convention, equation (5.52),

$$E_{in}^{\mu} := E_{in}^N - \nabla G * (\tilde{\rho}[\mu_0^N[Z]] - \tilde{\rho}[f_0]), \quad B_{in}^{\mu} := B_{in}^N.$$

Since $E_{in}^N = \chi^N * E_{in}$, we have $\|E_{in}^N\|_2 \leq \|E_{in}\|_2$ uniformly in N. The same holds for $B_{in}^{\mu} = B_{in}^N$. It remains to estimate $\|\nabla G * \tilde{\rho}[\mu_0^N[Z]]\|_2$ and $\|\nabla G * \tilde{\rho}[f_0]\|_2$.
Since $|\nabla G(x)| = \frac{1}{4\pi|x|^2}$, Lemma 5.8.1 yields $|\chi^N *_x \nabla G| \lesssim \min\{r_N^{-2}, |x|^{-2}\}$ and we compute

$$\begin{aligned}\|\chi^N * \nabla G\|_2^2 &\leq \int\limits_{|y|\leq r_N} |\chi^N *_x \nabla G|^2(x) + \int\limits_{|y|>r_N} |\chi^N *_x \nabla G|^2(x)\\ &\lesssim r_N^{-4}\int\limits_{|x|<r_N} \mathrm{d}^3x + \int\limits_{|x|\geq r_N} |x|^{-4}\mathrm{d}^3x\\ &\lesssim r_N^{-4}\,r_N^3 + r_N^{-1} = 2r_N^{-1}.\end{aligned} \tag{5.92}$$

This yields, on the one hand,

$$\|\nabla G * \tilde{\rho}[\mu_0^N[Z]]\|_2^2 = \Big\|\frac{1}{N}\sum_{i=1}^{N}\nabla G * \chi^N(\cdot - x_i(0))\Big\|_2^2 \leq \|\chi^N * \nabla G\|_2^2 \lesssim r_N^{-1}, \tag{5.93}$$

and, on the other hand,

$$\|\nabla G * \tilde{\rho}[f_0]\|_2 = \|\chi^N * \nabla G * \rho[f_0]\|_2 \leq \|\chi^N * \nabla G\|_2 \|\rho[f_0]\|_1 \lesssim r_N^{-1/2}. \tag{5.94}$$

In total, we have found that

$$\|E(t,\cdot)\|_2 + \|B(t,\cdot)\|_2 \leq \sqrt{2\varepsilon + 1 + \bar{\xi}^2} \lesssim r_N^{-1/2}. \tag{5.95}$$

Finally, by Young's inequality, we have for $\tilde{K}(t,x,\xi) = \chi^N *_x (E_t + v(\xi) \times B_t)(t,x)$:

$$\|\widetilde{K}[\tilde{\mu}^N](t,\cdot,\cdot)\|_\infty \leq \|\chi^N\|_2\big(\|E[\tilde{\mu}^N](t,\cdot)\|_2 + \|B[\tilde{\mu}^N](t,\cdot)\|_2\big) \lesssim r_N^{-3/2} r_N^{-1/2} = r_N^{-2},$$

where we used

$$\|\chi^N\|_2^2 = \int (\chi^N(x))^2 \mathrm{d}^3x = \int (r_N^{-3}\chi(x/r_N))^2 \mathrm{d}^3x = r_N^{-3} \int \chi(y)^2 \mathrm{d}^3y = r_N^{-3}\|\chi\|_2^2.$$

□

It might be interesting to note that – in contrast to the other mean field results presented or referenced in this thesis – we actually use an energy bound here, exploiting the conservation of energy in the Abraham model. Also note that this is the only bound for which we have to use both mollifiers appearing in (5.43).

5.9 Light cone structure

The Maxwell theory as well as the Vlasov-Maxwell approximation are relativistic. Particle interactions – mediated by the electromagnetic field – are retarded, with influences "propagating" with the speed of light. More precisely, the field value at a given space-time point $(t,x) \in \mathbb{R} \times \mathbb{R}^3$ depends on the particle trajectories only at their intersection with the backwards light cone $\{(s,y) \mid (t-s)^2 - (x-y)^2 = 0, t-s \geq 0\}$. Formally, this light cone structure is manifested in the d'Alembert kernel $Y(t,x)$ defined in (5.10), which has support in $\{t = |x|, t > 0\}$. The regularized Vlasov-Maxwell system (5.43) is only semi-relativistic (because of the rigid form factor), but inherits this light-cone structure. Integral expressions of the form (5.22, 5.23), determining the inhomogeneous field components, evaluate the mean field density on the backwards light cone. Since the Vlasov density is transported with the characteristic flow, the respective integrals can be pulled-back to the $t = 0$ hypersuface in a canonical way. The respective field components at a space-time point (t,x) then depend on the initial distribution f_0 on $\mathrm{B}_t(x) \times \mathbb{R}^3$ where $\mathrm{B}_t(x) = \mathrm{B}(t;x)$ is the ball around x with radius t. In the following, we make these observations more precise.

Definition 5.9.1 (Retarded time)**.** Fix a spacetime point $(t,x) \in \mathbb{R} \times \mathbb{R}^3$. Let f_t a solution of (5.43) and $\varphi_{s,0}(z) = (y^*(s,z), \eta^*(s,z))$ the characteristic flow, i.e. the solution of (5.71) with $(y^*(0), \eta^*(0)) = z$. Then we denote by $t_{ret}(z)$ the unique solution of

$$(t-s)^2 - (x - y^*(s,z))^2 = 0;\ (t-s) > 0. \tag{5.96}$$

$t_{ret}(z) = t_{ret}(y^*(s,z);t,x)$ is the time at which the trajectory $y^*(s)$ crosses the backward light cone with origin (t,x). We have $t_{ret}(z) \geq 0 \iff y_0 \in \mathrm{B}_t(x) = \{y \in \mathbb{R}^3 : |x-y| \leq t\}$.

Lemma 5.9.2 (Distributions on the light cone)**.** *Let f_t a solution of* (5.43) *and $\varphi_{s,0}(z) = (y^*(s,z), \eta^*(s,z))$ as above. For a fixed space-time point $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$ consider the diffeomorphism*

$$\begin{aligned} \phi : \mathrm{B}_t(x) \times \mathbb{R}^3 &\to \mathrm{B}_t(x) \times \mathbb{R}^3 \\ z = (x,\xi) &\mapsto (y^*(t_{ret}(z),z), \eta^*(t_{ret}(z),z)). \end{aligned} \tag{5.97}$$

1) For $a \in C(\mathbb{R}^3 \times \mathbb{R}^3)$, we have (with $n(x-y) = \frac{x-y}{|x-y|}$):

$$\int_{\mathrm{B}_t(x)\times\mathbb{R}^3} a(\phi(z))\, f_0(z)\,\mathrm{d}z = \int_{\mathrm{B}_t(x)\times\mathbb{R}^3} a(y,\eta)\,(1 - n(x-y)v(\eta))\, f(t-|x-y|, y, \eta)\mathrm{d}y\,\mathrm{d}\eta. \tag{5.98}$$

2) For $\alpha \in C(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$:

$$\int (\alpha Y) *_{t,x} (\mathbb{1}_{t\geq 0} f)(t,x,\eta)\,\mathrm{d}\eta = \int_{\mathrm{B}_t(x)\times\mathbb{R}^3} \left.\frac{\alpha(t-s, x-y^*(s,z), \eta^*(s,z))}{|x-y^*(s,z)|(1-n(x-y^*(s,z))\cdot v(\eta^*(s,z))}\right|_{s=tret(z)} f_0(z)\,\mathrm{d}z. \tag{5.99}$$

Proof. Since $f_t = \varphi_{t,0}\# f_0$, we compute

$$\begin{aligned}
&\int_{\mathrm{B}_t(x)\times\mathbb{R}^3} a(y,\eta)\, f(t-|x-y|, y,\eta)\mathrm{d}y\,\mathrm{d}\eta \\
&= \int_{[0,t]\times\mathrm{B}_t(x)\times\mathbb{R}^3} a(y,\eta)\,\delta(|x-y|-(t-s))\, f(s,y,\eta)\mathrm{d}s\mathrm{d}y\mathrm{d}\eta \\
&= \int a(y,\eta)\,\delta(|x-y|-(t-s))\,\varphi_{s,0}\# f_0(y,\eta)\mathrm{d}s\mathrm{d}y\mathrm{d}\eta \\
&= \int a(y^*(s;y,\eta),\eta^*(s;y,\eta))\,\delta(|x-y^*(s;y,\eta)|-(t-s))\, f_0(y,\eta)\mathrm{d}s\mathrm{d}y\mathrm{d}\eta.
\end{aligned}$$

Now we use: If $h \in C^1$ has a unique root ζ, then $\delta(h(x)) = \delta(x-\zeta)h'(\zeta)$ in the sense of distributions. The function $h(s) = |x - y^*(s;y,\eta)| - (t-s)$ is differentiable with $h'(s) = 1 - \frac{(x-y^*(s))\cdot v(\eta^*(s))}{|x-y^*(s)|} = 1 - n(x-y^*(s))\cdot v(\eta^*(s))$. If $y^*(0) \in \mathrm{B}_t(x)$, it has a unique positive root $t_{ret} = t_{ret}(z)$. Hence, we get:

$$\int a(y,\eta)\,\delta(t-s-|x-y|)\, f(s,y,\eta)\,\mathrm{d}s\mathrm{d}y\mathrm{d}\eta = \int \frac{a(y^*(t_{ret}(z),z),\eta^*(t_{ret}(z),z))}{1-n(x-y^*(t_{ret}(z)))\cdot v(\eta^*(t_{ret}(z)))}\, f_0(z)\,\mathrm{d}z \tag{5.100}$$

and the identity follows. For (5.99), we have

$$\begin{aligned}
&\int (\alpha Y) *_{t,x} (\mathbb{1}_{t\geq 0} f)\mathrm{d}\eta(t,x) \\
&= \int_{\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3} \alpha(t-s, x-y, \eta) Y(|x-y|-(t-s))\mathbb{1}_{\{s\geq 0\}} f(s,y,\eta)\mathrm{d}s\mathrm{d}y\mathrm{d}\eta.
\end{aligned}$$

Now observe that on the support of Y, we have $\mathbb{1}_{\{s\geq 0\}} = \mathbb{1}_{\{y\in B_t(x)\}}$ and $(t-s) = |x-y|$ and apply part 1) of the Lemma to $a(y,\eta) = |x-y|^{-1}\alpha(|x-y|, x-y, \eta)$. □

Furthermore, in order to compare the fields generated by the mean field trajectories with those generated by the microscopic trajectories, we will require the following lemma.

Lemma 5.9.3. *Let $x_1^*(s), x_2^*(s)$ two trajectories with velocity bounded by $\overline{v} < 1$. Fix a space-time point $(t,x) \in \mathbb{R} \times \mathbb{R}^3$ and denote by t^i_{ret}, $i = 1,2$ the time at which trajectory i intersects the backward light cone with origin (t,x). Then we have:*

$$|x_1^*(t^1_{ret}) - x_2^*(t^2_{ret})| \leq \frac{1}{1-\overline{v}}\, |x_1^*(t^1_{ret}) - x_2^*(t^1_{ret})|. \tag{5.101}$$

Similarly, if we denote that respective momenta by $\xi_1(s), \xi_2(s)$ and assume that the force $\dot{\xi}_2$ is bounded by $L < \infty$, then

$$|\xi_1^*(t^1_{ret}) - \xi_2^*(t^2_{ret})| \leq |\xi_1^*(t^1_{ret}) - \xi_2^*(t^1_{ret})| + \frac{L}{1-\overline{v}}\, |x_1^*(t^1_{ret}) - x_2^*(t^1_{ret})|. \tag{5.102}$$

Proof. Suppose w.l.o.g. that

$$\begin{aligned} (t - t^1_{ret}) - |x - x_1^*(t^1_{ret})| &= 0, \\ (t - t^1_{ret}) - |x - x_2^*(t^1_{ret})| &> 0. \end{aligned}$$

Set $r := |x_1^*(t^1_{ret}) - x_2^*(t^1_{ret})|$ and $\tau = \min\{t, t^1_{ret} + \frac{r}{1-\overline{v}}\}$. Obviously, if $\tau = t$, we have

$$(t-\tau) - |x - x_2^*(\tau)| = -|x - x_2^*(\tau)| \leq 0.$$

If $\tau = t^1_{ret} + \frac{r}{1-\overline{v}} < t$, we estimate

$$\begin{aligned} |x - x_2^*(\tau)| &\geq |x - x_1^*(t^1_{ret})| - |x_1^*(t^1_{ret}) - x_2^*(t^1_{ret})| - |x_2^*(t^1_{ret}) - x_2^*(s_2)| \\ &\geq (t - t^1_{ret}) - r - \overline{v}(\tau - t^1_{ret}) \\ &= (t-\tau) + (\tau - t^1_{ret}) - r - \overline{v}(\tau - t^1_{ret}) \\ &= (t-\tau) + (1-\overline{v})(\tau - t^1_{ret}) - r \end{aligned}$$

and therefore also

$$(t-\tau) - |x - x_2^*(\tau)| \leq r - (1-\overline{v})(\tau - t^1_{ret}) = 0.$$

By continuity, there thus exists $s \in (t^1_{ret}, \tau]$ with $(t-s) - |x - x_2^*(s)| = 0$. Hence, $s = t^2_{ret}$ and we found

$$\begin{aligned} |x_2^*(t^2_{ret}) - x_1^*(t^1_{ret})| &\leq |x_2^*(t^1_{ret}) - x_1^*(t^1_{ret})| + |x_2^*(t^2_{ret}) - x_2^*(t^1_{ret})| \\ &\leq r + \overline{v}(t^2_{ret} - t^1_{ret}) \leq \frac{r}{1-\overline{v}} = \frac{|x_2^*(t^1_{ret}) - x^*(t^1_{ret})|}{1-\overline{v}}, \end{aligned}$$

as well as

$$\begin{aligned} |\xi_2^*(t^2_{ret}) - \xi_1^*(t^1_{ret})| &\leq |\xi_2^*(t^1_{ret}) - \xi_1^*(t^1_{ret})| + |\xi_2^*(t^2_{ret}) - \xi_2^*(t^1_{ret})| \\ &\leq |\xi_2^*(t^1_{ret}) - \xi_1^*(t^1_{ret})| + L|t^2_{ret} - t^1_{ret}| \\ &\leq |\xi_2^*(t^1_{ret}) - \xi_1^*(t^1_{ret})| + \frac{L}{1-\overline{v}}|x_1^*(t^1_{ret}) - x_2^*(t^1_{ret})|. \end{aligned}$$

□

Remark 5.9.4. The previous lemma has a simple geometric proof. Consider the projection onto a 2-dimensional (x,t)-plane and set $(x_1^*(t_{ret}^1), t_{ret}^1) = (0,0)$. Then, the light ray crossing the trajectory of the first particle corresponds to the line $x = s$. In the worst case, the second trajectory moves away from $x = 0$ with constant velocity $\overline{v}$. This corresponds to the straight line $x(s) = r + s\overline{v}$. The point of intersection with the light ray is then $s = r + s\overline{v} \Rightarrow x = s = \frac{r}{1-\overline{v}}$. If $s > t$, this line intersects the other side of the light cone first.

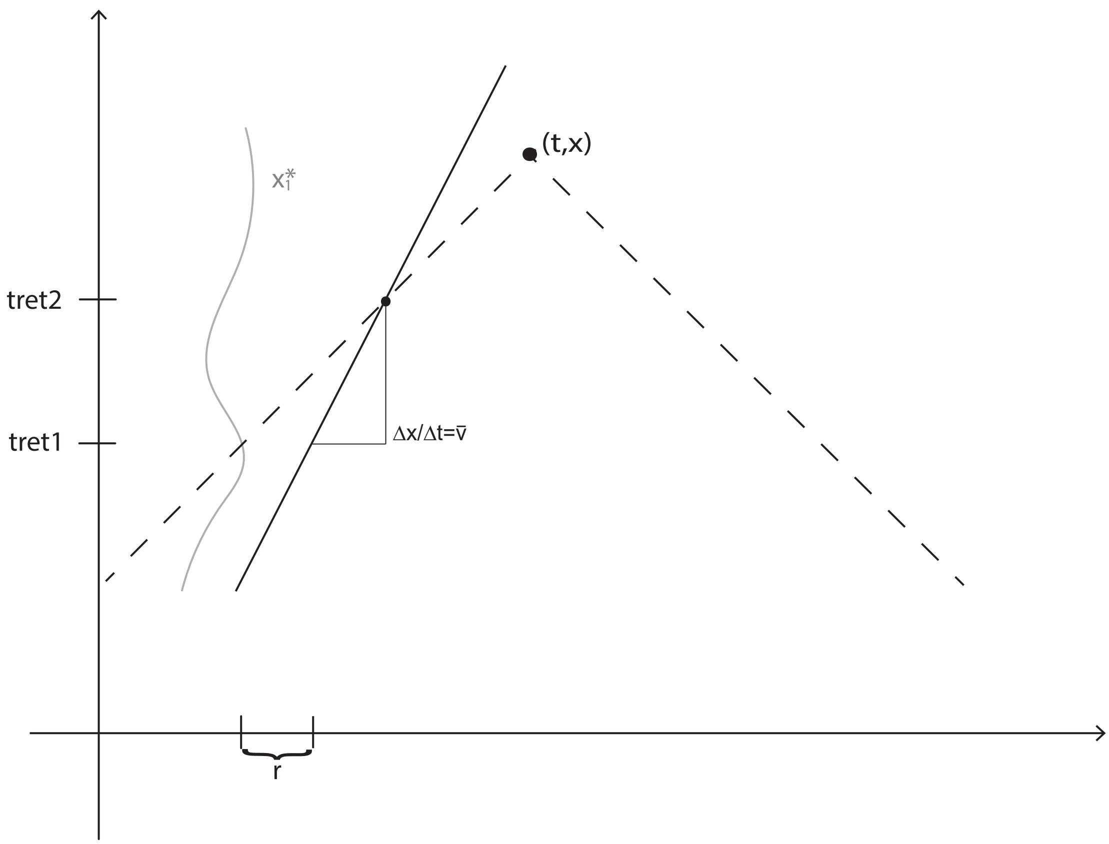

Figure 5.1: Fig. 1: Intersections of the light cone

5.9.1 Law of large numbers

Part of our proof consists in sampling the mean field dynamics along (random) trajectories, i.e. approximating the mean field distribution f_t^N with the discrete measure $\mu^N[\Phi_{t,0}(Z)]$, where $\Phi_{t,0}$ is the mean field flow defined in (5.7.1) and $Z \in \mathbb{R}^{6N}$ is random with distribution $\otimes^N f_0$. One advantage of this approach is that the N particles evolving with the mean field flow remain i.i.d. with law f_t^N for all times, thus allowing for law of large numbers estimates. We will work with the following (more or less standard) result:

Proposition 5.9.5. *Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ a probability density. Let $\alpha, \beta > 0$ with $\alpha + \beta < \frac{1}{2}$. Let $h : \mathbb{R}^6 \to \mathbb{R}$ such that $|h(z)| \lesssim N^\alpha$. Let $\phi : \mathbb{R}^6 \to \mathbb{R}^6$ a diffeomorphism with bounded derivative. Then, for all $\gamma > 0$ there exists a $C_\gamma > 0$ such that*

$$\mathbb{P}_0\Big[\Big|\frac{1}{N}\sum_{i=1}^N h(\phi(z_i)) - \int h(\phi(z))f_0(z)\Big| \geq N^{-\beta}\Big] \leq \frac{C_\gamma}{N^\gamma}. \tag{5.103}$$

Note: Finer estimates, exploiting decay-properties of h, were proven in Proposition 3.7.2.

Proof. Let

$$A := \Big\{Z \in \mathbb{R}^{6N} : \Big|\frac{1}{N}\sum_{i=1}^N h(\phi(z_i)) - \int h(\phi(z))f_0(z)\Big| \geq N^{-\beta}\Big\}. \tag{5.104}$$

By Markov's inequality, we have for every $M \geq 2$:

$$\begin{aligned}\mathbb{P}_0(A) \leq & \mathbb{E}_0\Big[N^{2M\beta}\Big|\frac{1}{N}\sum_{1=1}^N h(\phi(z_i)) - \int h(\phi(z))f_0(z)\Big|^{2M}\Big]\\ = & \frac{1}{N^{2M(1-\beta)}}\,\mathbb{E}\Big[\Big(\sum_{i=1}^N\big[h(\phi(z_i)) - \int h(\phi(z))f_0(z)\big]\Big)^{2M}\Big].\end{aligned} \tag{5.105}$$

Let $\mathcal{M} := \{\mathbf{k} \in \mathbb{N}_0^N \mid |\mathbf{k}| = 2M\}$ the set of multiindices $\mathbf{k} = (k_1, k_2, ..., k_N)$ with $\sum_{j=1}^N k_j = 2M$. Let

$$G^{\mathbf{k}} := \prod_{i=1}^N\big[h(\phi(z_i) - \int h(\phi(z))f_0(z)\big]^{k_j}.$$

Then:

$$\mathbb{E}_0\Big[\Big(\sum_{i=1}^N\big[h(\phi(z_i)) - \int h(\phi(z))f_0(z)\big]\Big)^{2M}\Big] = \sum_{\mathbf{k}\in\mathcal{M}}\binom{2M}{\mathbf{k}}\,\mathbb{E}_t(G^{\mathbf{k}}).$$

Now we observe that $\mathbb{E}_0(G^{\mathbf{k}}) = 0$ whenever there exists a $1 \leq j \leq N$ such that $k_j = 1$. This can be seen by integrating the j'th variable first.

For the remaining terms, we have the bound

$$\int |h(\phi(z))|^m f_0(z)\,\mathrm{d}z \lesssim N^{\alpha m}\|f_0\|_\infty. \tag{5.106}$$

Now, for $\mathbf{k} = (k_1, k_2, ..., k_N) \in \mathcal{M}$, let $\#\mathbf{k}$ denote the number of k_i with $k_i \neq 0$. Note that if $\#\mathbf{k} > M$, we must have $k_i = 1$ for at least one $1 \leq i \leq N$, so that $\mathbb{E}_0(G^{\mathbf{k}}) = 0$. For the other multiindices, we get:

$$\mathbb{E}_0(G^{\mathbf{k}}) = \mathbb{E}_0\Big[\prod_{i=1}^N\big(h(\phi(q_i)) - \int h(\phi(z))f_0(z)\big)^{k_i}\Big] \lesssim N^{2M\alpha}. \tag{5.107}$$

Finally, for any $k \geq 1$, the number of multiindices $\mathbf{k} \in \mathcal{M}$ with $\#\mathbf{k} = j$ is bounded by

$$\sum_{\#\mathbf{k}=j} 1 \leq \binom{N}{j}(2M)^j \leq (2M)^{2M} N^j.$$

Thus:

$$\mathbb{P}_0(A) \lesssim \frac{N^M N^{2M\alpha}}{N^{2M(1-\beta)}} = N^{M(2(\alpha+\beta)-1)}$$

and the proposition follows. □

We have formulated the proposition with ϕ for convenience. The relevant examples for us will be $\phi(z) = z$ and ϕ the diffeomorphism defined in (5.97).

In the next section, we will use the law of large numbers to sample the fields on a regular lattice that we introduce on the following definition.

Definition 5.9.6. Let $\overline{r}$ as defined in (5.68). For $N \in \mathbb{N}$ let $\mathcal{G}^N$ be the regular lattice in $[-\overline{r}, \overline{r}]^3$ with side length $\frac{d}{N}$. $\mathcal{G}^N$ contains a total of $(3N)^3$ lattice points and for any $x \in [-\overline{r}, \overline{r}]^3$, the maximal distance to the next lattice point is at most $\frac{\sqrt{3}}{2}\frac{\overline{r}}{N}$.

5.10 Pointwise estimates

We will now go deeper into the details of the dynamics to control the difference between mean field and microscopic time-evolution. To this end, we have to control the differences in the electromagnetic fields generated by the (regularized) mean field density $\tilde{f}^N_t$ and the (smeared) microscopic density $\tilde{\mu}^N_t[Z] = \mu^N[\Psi_{t,0}(Z)]$ (recall that in view of (5.43)m the distributions are "smeared out" with χ^N as they enter the field equations.) We will use the decomposition of the fields in terms of Liénard-Wiechert distributions introduced in Section 5.2.1. We will denote by $E_i[\tilde{f}]$ and $E_i[\tilde{\mu}]$, $i = 0, 1, 2$ the respective field component generated by $\tilde{f}^N$, respectively $\tilde{\mu}^N_t[Z]$.

5.10.1 Controlling the Coulomb term

We begin by controlling the contribution of the Coulombic term (5.22):

$$\left|E_1[\tilde{f}^N](t,x) - E_1[\tilde{\mu}^N](t,x)\right| = \left|\int (\alpha^{-1}Y) *_{t,x} (\mathbb{1}_{t\geq 0}\tilde{f}^N)\,\mathrm{d}\xi - \int (\alpha^{-1}Y) *_{t,x} (\mathbb{1}_{t\geq 0}\tilde{\mu}^N_{(\cdot)}[Z])\,\mathrm{d}\xi\right|$$

with the kernel α^{-1} defined in (5.24). The expression on the r.h.s. is to be evaluated at (t, x). Since convolutions commute, we may write

$$\begin{aligned}
&\left|E_1[\tilde{f}^N](t,x) - E_1[\tilde{\mu}^N](t,x)\right| \\
=&\left|\chi^N * \left(\int (\alpha^{-1}Y) * (\mathbb{1}_{t\geq 0} f^N)\,\mathrm{d}\xi - \int (\alpha^{-1}Y) * (\mathbb{1}_{t\geq 0}\mu^N[\Psi_{s,0}(Z)])\,\mathrm{d}\xi\right)\right| \\
\leq&\left|\chi^N * \left(\int (\alpha^{-1}Y) * (\mathbb{1}_{t\geq 0} f^N)\,\mathrm{d}\xi - \int (\alpha^{-1}Y) * (\mathbb{1}_{t\geq 0}\mu^N[\Phi_{s,0}(Z)])\,\mathrm{d}\xi\right)\right| &(5.108)\\
+&\left|\chi^N * \left(\int (\alpha^{-1}Y) * (\mathbb{1}_{t\geq 0}\mu^N[\Phi_{s,0}(Z)])\,\mathrm{d}\xi - \int (\alpha^{-1}Y) * (\mathbb{1}_{t\geq 0}\mu^N[\Psi_{s,0}(Z)])\,\mathrm{d}\xi\right)\right| &(5.109)
\end{aligned}$$

where we have inserted the density $\mu^N[\Phi_{s,0}(Z)]$ corresponding to the mean field flow $\Phi_{s,0}(Z) = {}^N\Phi_{s,0}(Z)$, in addition to the actual microsocpic density $\mu_s^N[Z] = \mu^N[\Psi_{s,0}(Z)]$.

A law of large numbers bound for (5.108)**.** Recall from Definition 5.7.1, that $\mu^N[\Phi_{t,0}(Z)] = \varphi_{t,0}^N \# \mu[Z]$, where $\varphi_{t,0}^N$ is the characteristic flow of f_t^N. More explicitly, with $\varphi_{t,0}^N(z_i) = (y^*, \eta^*)(t, z_i)$, we have

$$\mu^N[\Phi_{t,0}(Z)] = \frac{1}{N} \sum_{i=1} \delta(x - y^*(t, z_i))\delta(\xi - \eta^*(t, z_i)).$$

We shall also use the shorthand $y_i^*(t) = y^*(t, z_i)$, $\eta_i^*(t) = \eta^*(t, z_i)$. Now we observe that,

$$\begin{aligned} f^N(t, x, \xi) = (\varphi_{t,0}^N \# f_0)(x, \xi) &= \int \delta(x - y)\delta(\xi - \eta)(\varphi_{t,0}^N \# f_0)(y, \eta) \mathrm{d}y \mathrm{d}\eta \\ &= \int \delta(x - y^*(t, z))\delta(\xi - \eta^*(t, z)) f_0(z) \mathrm{d}z. \end{aligned}$$

Inserting this into (5.108) and performing the z-integration last (assuming, for the moment, that the order of integration can be exchanged), we see that

$$\mathbb{E}_0\Big[\chi^N * \Big(\int (\alpha^{-1}Y) * (\mathbb{1}_{t\geq 0} f^N)\, \mathrm{d}\xi - \int (\alpha^{-1}Y) * (\mathbb{1}_{t\geq 0}\mu^N[\Phi_{s,0}(Z)])\, \mathrm{d}\xi\Big)\Big] = 0,$$

where the expectation value is defined with respect to $\otimes^N f_0$. The idea is thus to use the law of large numbers to show that (5.108) goes to 0 in probability.

Recall from (5.24) that:

$$\alpha^{-1}(t, x, \xi) = \frac{(1 - v(\xi)^2)(x - tv(\xi))}{(t - v(\xi)x)^2}.$$

Hence, we compute

$$\begin{aligned} &\int (\alpha^{-1}Y) *_{t,x} (\mathbb{1}_{t\geq 0}\mu^N[\Phi_{s,0}(Z)])(t, x)\, \mathrm{d}\xi \\ &= \frac{1}{N} \sum_{i=1}^N \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_0^t \mathrm{d}s \mathrm{d}y \mathrm{d}\xi\ \delta(y - y_i^*(s))\delta(\xi - \eta_i^*(s)) \\ &\qquad \frac{(1 - v(\xi)^2)(x - y - (t - s)v(\eta))}{(t - s - v(\eta)(x - y))^2}\ \frac{\delta(|x - y| - (t - s))}{4\pi|x - y|} \\ &= \frac{1}{N} \sum_{i=1}^N \int_0^t \frac{(1 - v(\eta_i^*)^2)(n(x - y_i^*) - v(\eta_s^*))}{4\pi(1 - v(\eta_i^*)n(x - y_i^*))^2 |x - y_i^*(s)|^2}\, \delta(|x - y_i^*(s)| - (t - s))\, \mathrm{d}s. \end{aligned}$$

The function $h : s \to |x - y_i^*(s)| - (t - s)$ is differentiable with $h'(s) = 1 - v^*(\eta^*(s))n(x - y_i^*(s))$. If it has a root in $[0, t]$, we denote it by $t_{ret,i}$, otherwise the integral is zero. Recall that $t_{ret,i} \geq 0 \iff z_i \in B_t(x) \times \mathbb{R}^3$. Hence, we find:

$$\begin{aligned}
&\int (\alpha^{-1} Y) *_{t,x} (\mathbb{1}_{t\geq 0} \mu^N[\Phi_{s,0}(Z)])(t,x)\,\mathrm{d}\xi \\
&= \frac{1}{N} \sum_{i=1}^N \frac{(1-v(\eta_i^*)^2)(n(x-y_i^*)-v(\eta_i^*))}{4\pi(1-v(\eta_i^*)n(x-y_i^*))^3|x-y_i^*(s)|^2} \mathbb{1}_{\{s\geq 0\}} \Bigg|_{s=t_{ret,i}} && (5.110)\\
&= \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{z_i \in B_t(x)\times\mathbb{R}^3\}} k\big(x - y^*(t_{ret,i}, z_i), \eta^*(t_{ret,i}, z_i)\big), && (5.111)
\end{aligned}$$

where we have introduced the kernel

$$k(x,\xi) := \frac{(1-v(\xi)^2)(n(x)-v(\xi))}{4\pi(1-v(\xi)\cdot n(x))^3|x|^2}. \tag{5.112}$$

Furthermore, according to Lemma 5.9.2,

$$\begin{aligned}
&\int (\alpha Y) *_{t,x} (\mathbb{1}_{t\geq 0} f^N)(t,x,\eta)\,\mathrm{d}\eta \\
&= \int_{B_t(x)\times\mathbb{R}^3} \frac{\alpha^{-1}(t-s, x-y^*(s,z), \eta^*(s,z))}{|x-y^*(s,z)|(1-n(x-y^*(s,z))\cdot v(\eta^*(s,z))}\Bigg|_{s=tret(z)} f_0(z)\,\mathrm{d}z \\
&= \int_{B_t(x)\times\mathbb{R}^3} \frac{(1-v(\eta^*(s,z))^2)(n(x-y^*(s,z))-v(\eta^*(s,z)))}{4\pi(1-v(\eta^*(s,z))n(x-y^*(s,z)))^3|x-y^*(s,z)|^2}\Bigg|_{s=t_{ret(z)}} f_0(z)\,\mathrm{d}z \\
&= \int_{B_t(x)\times\mathbb{R}^3} k(x-y^*(t_{ret(z)}, z), \eta^*(t_{ret}(z), z))\, f_0(z)\,\mathrm{d}z.
\end{aligned}$$

(In fact, we could have also applied the same identity (5.99) to $\mu^N[\Phi_{t,0}(Z)]$).
Now note that on the support of f, we have

$$|k(x,\xi)| \leq \frac{1}{2\pi(1-\overline{v})^3|x|^2}, \tag{5.113}$$

and thus, according to Lemma 5.8.1,

$$|\tilde{k}(x,\xi)| = |\chi^N *_x k(x,\xi)| \lesssim r_N^{-2}, \quad \forall x \in \mathbb{R}^3, |\xi| \leq \overline{\xi} \tag{5.114}$$

where we have applied the mollifier χ^N. In total, we have found that (5.108) is of the form

$$\left| \frac{1}{N} \sum_{i=1}^N h(\phi(z_i)) - \int h(\phi(z))\,\mathrm{d}f_0(z) \right|$$

with $h(y,\eta) = \tilde{k}(x-y,\eta)$ and ϕ the diffeomorphism defined in Lemma 5.9.2 and f_0 restricted to $\mathrm{B}(t;x) \times \mathbb{R}^3$. Hence, we can use the law of large numbers in the form of Proposition 5.9.5 to conclude the following:

Lemma 5.10.1. *Let A_t^1 be the (N and t dependent) set defined by*

$$A_t^1 := \{Z \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (5.108) < N^{-1/3} \text{ for all } x \in \mathcal{G}^N\}. \tag{5.115}$$

Then there exists $C_1 > 0$ such that $\mathbb{P}_0(A_t^1) \geq 1 - \frac{C_1}{N^1}$.

Proof. Let $\mathcal{G}^N$ the lattice defined in 5.9.6 and $x_k \in \mathcal{G}^N$. We want to apply Proposition 5.9.5 with $h(y,\eta) = \tilde{k}(x_k - y, \eta)$ and ϕ as in (5.97). Since $|h| \lesssim r_N^{-2} \leq N^{2\gamma}$, with $\gamma < \frac{1}{12}$, we can choose $\beta = \frac{1}{3}$. Thus, by Prop. 5.9.5, there exists a constant $C > 0$ such that

$$\mathbb{P}_0\Big[\Big|\chi^N * \Big(\int (\alpha^{-1}Y) * \mathbb{1}_{t\geq 0}(f^N - \mu^N[\Phi_{s,0}(Z)])\, \mathrm{d}\xi\Big)(t, x_k)\Big| \geq N^{-\frac{1}{3}}\Big] \leq \frac{C}{N^4}.$$

Since the lattice $\mathcal{G}^N$ contains $(3N)^3$ points, we have

$$\begin{aligned}
&\mathbb{P}_0\big[\exists x_k \in C^N : (5.108) \geq N^{-\frac{1}{3}}\big] \\
&\leq \sum_{x_k \in \mathcal{G}^N} \mathbb{P}_0\Big[\Big|\chi^N * \Big(\int (\alpha^{-1}Y) * (\mathbb{1}_{t\geq 0} f^N - \mathbb{1}_{t\geq 0}\mu^N[\Phi_{t,0}(Z)])\, \mathrm{d}\xi\Big)(t, x_k)\Big| \geq N^{-\frac{1}{3}}\Big] \\
&\leq (3N)^3 \frac{C}{N^4} \leq \frac{27C}{N}.
\end{aligned}$$

□

A Lipschitz bound bound for (5.109)**.** We now have to control (5.109), i.e. the difference of the field components E_1 generated by the mean field trajectories $(y_i^*, \eta_i^*)_{i=1,..,N}$ on the one hand and the true microscopic trajectories $(x_i^*, \xi_i^*)_{i=1,..,N}$ on the other hand. To this end, we want to establish a local Lipschitz bound for the kernel (5.112).

Lemma 5.10.2 (Local Lipschitz bound)**.** *There exists constants $b_1, b_2 > 0$ and functions*

$$g_1(x) := \frac{b_1}{(1-\overline{v})^3}\begin{cases} r_N^{-3} & ; |x| < \frac{2r_N}{1-\overline{v}} \\ |x|^{-3} & ; |x| \geq \frac{2r_N}{1-\overline{v}} \end{cases}, \quad g_2(x) := \frac{b_2}{(1-\overline{v})^4}\begin{cases} r_N^{-2} & ; |x| < r_N \\ |x|^{-2} & ; |x| \geq r_N \end{cases}. \tag{5.116}$$

such that for all $z_1 = (x_1, \xi_1), z_2 = (x_2, \xi_2)$ with $|\xi_1|, |\xi_2| \leq \overline{\xi}$ and $|x_1 - x_2| < \frac{r_N}{1-\overline{v}}$, $\overline{v} = |v(\overline{\xi})|$:

$$|\tilde{k}(x_1,\xi_1) - \tilde{k}(x_2,\xi_2)|_\infty \leq g_1(x_1)\,|x_1 - x_2|_\infty + g_2(x_1)\,|\xi_1 - \xi_2|_\infty. \tag{5.117}$$

Proof. We have

$$|\tilde{k}(x_1,\xi_1) - \tilde{k}(x_2,\xi_2)|_\infty \leq |\tilde{k}(x_1,\xi_2) - \tilde{k}(x_2,\xi_2)|_\infty + |\tilde{k}(x_1,\xi_1) - \tilde{k}(x_1,\xi_2)|_\infty,$$

hence, there exists y between x_1 and x_2 and ζ between ξ_1 and ξ_2 such that

$$|\tilde{k}(x_1,\xi_1) - \tilde{k}(x_2,\xi_2)|_\infty \leq |\nabla_x \tilde{k}(y,\xi_2)|_\infty |x_1 - x_2|_\infty + |\nabla_\xi \tilde{k}(x_1,\zeta)|_\infty |\xi_1 - \xi_2|_\infty.$$

Now one checks that

$$|\nabla_\xi k(x,\xi)|_\infty \leq \frac{18}{(1-\overline{v})^4 |x|^2},$$

so that according to Lemma 5.8.1, there exists $b_2 > 0$ such that

$$|\nabla_\xi \tilde{k}(x,\xi)|_\infty \leq \frac{b_1}{(1-\overline{v})^4} \min\{r_N^{-2}, |x|^{-2}\}. \tag{5.118}$$

For the difference in the x-coordinates, we get from (5.113) and Lemma 5.8.1 a constant $b > 0$ such that

$$|\nabla_x \tilde{k}(x,\xi)|_\infty \leq \frac{b}{(1-\overline{v})^3} \min\{r_N^{-3}, |x|^{-3}\}. \tag{5.119}$$

Thus, for $|x_1| < \frac{2r_N}{1-\overline{v}}$, a bound of the form (5.117) certainly holds, since the derivative is bounded by $\frac{b}{(1-\overline{v})^3} r_N^{-3}$. For $|x_1| > \frac{2r_N}{1-\overline{v}}$ and $|x_1 - x_2| < \frac{r_N}{1-\overline{v}}$ we observe that $|sx_1 + s(x_2 - x_1)| \geq \frac{|x_1|}{2}, \forall s \in [0,1]$, so that $\frac{1}{|sx_1+s(x_2-x_1)|^3} \leq \frac{8}{|x_1|^3}$. Setting $b_1 := 8b$, the statement follows. □

Now recall that as long as $J_t^N(Z) < 1$, the trajectories are close as per (5.7.2). More precisely, $J_t^N(Z) < 1 \Rightarrow \sup\limits_{0\leq s\leq t} |{}^N\Phi_{t,0}(Z) - {}^N\Psi_{t,0}(Z)|_\infty < N^{-\delta} \leq N^{-\gamma} \leq r_N$. This implies, in particular, $|x^*(s,z_i) - y^*(s,z_i)| < r_N$ as well as $|\xi^*(s,z_i)| < \overline{\xi}$ for $0 \leq s \leq t$ and all $1 \leq i \leq N$. Moreover, with Lemma 5.9.3 we have for any fixed $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$:

$$|x_i^*(t_{ret,i}^x) - y_i^*(t_{ret,i}^y)| \leq \frac{r_N}{1-\overline{v}}, \tag{5.120}$$

where $t_{ret,i}^x$ and $t_{ret,i}^y$ denote the retarded time of the trajectory $x_i^*(s)$, respectively $y_i^*(s)$, with respect to the space-time point (t,x). Hence, we can apply the previous Lemma and find that (5.109) is bounded by

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{t_{ret}\geq 0\}} \Big|\tilde{k}\big(x - x^*(t_{ret,i}^x, z_i), \xi^*(t_{ret,i}^x, z_i)\big) - \tilde{k}\big(x - y^*(t_{ret,i}^y, z_i), \eta^*(t_{ret,i}^x, z_i)\big)\Big| \\
\leq &\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{t_{ret}\geq 0\}} \Big(g_1(x - y_i^*(t_{ret,i}^y))\, |x_i^*(t_{ret,i}^x) - y_i^*(t_{ret,i}^y)|_\infty + g_2(x - y_i^*(t_{ret,i}^y))\, |\xi_i^*(t_{ret,i}^x) - \eta_i^*(t_{ret,i}^y)|_\infty\Big) \\
\leq &\Big(\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{t_{ret}\geq 0\}} g_1(x - y_i^*(t_{ret,i}^y))\Big) \frac{1}{1-\overline{v}} \sup_{0\leq s\leq t} |{}^N\Phi_{s,0}^1(Z) - {}^N\Psi_{s,0}^1(Z)|_\infty \qquad (5.121) \\
+ &\Big(\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{t_{ret}\geq 0\}} g_2(x - y_i^*(t_{ret,i}^y))\Big) \sup_{0\leq s\leq t} \Big(|{}^N\Phi_{s,0}^2(Z) - {}^N\Psi_{s,0}^2(Z)|_\infty + \frac{L}{1-\overline{v}}\, |{}^N\Phi_{s,0}^1(Z) - {}^N\Psi_{s,0}^1(Z)|_\infty\Big).
\end{aligned} \tag{5.122}$$

For the last inequality, we used Lemma 5.9.3 and the bound (5.67) on the mean field force to account for the fact that the distance $|x_i^*(t_{ret,i}^x) - y_i^*(t_{ret,i}^y)|$, respectively $|\xi_i^*(t_{ret,i}^x) - \eta_i^*(t_{ret,i}^y)|$, involves to different retarded times. Now, we want to estimate $\frac{1}{N}\sum\limits_{i=1}^N \mathbb{1}_{\{t_{ret}\geq 0\}} g_j(x -$

$y_i^*(t^y_{ret,i})), j=1,2$ by its expectation value w.r.to f_0. In view of Lemma 5.9.2, we write:

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{t_{ret}\geq 0\}} g_j(x-y_i^*(t^y_{ret,i}))\\
\leq &\Big|\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{t_{ret}\geq 0\}} g_j(x-y_i^*(t^y_{ret,i})) - \int\limits_{\mathrm{B}_t(x)\times\mathbb{R}^3} g_j(x-y)(1-n(x-y)v(\eta))f^N(t-|x-y|,y,\eta)\Big|\\
+&\Big|\int\limits_{\mathrm{B}_t(x)\times\mathbb{R}^3} g_j(x-y)(1-n(x-y)v(\eta))f^N(t-|x-y|,y,\eta)\mathrm{d}y\mathrm{d}\eta\Big|.
\end{aligned}$$

For the last term, we recall the bounds from (5.116) and estimate, using $|1-n\cdot v|\leq 2$,

$$\begin{aligned}
&\Big|\int\limits_{\mathrm{B}_t(x)\times\mathbb{R}^3} g_1(x-y)(1-n(x-y)v(\eta))f^N(t-|x-y|,y,\eta)\mathrm{d}y\mathrm{d}\eta\Big|\\
&\quad\lesssim \int\limits_{|x-y|\leq t} g_1(x-y)\rho[f^N](t-|x-y|,y)\mathrm{d}y\\
&\quad\leq \sup_{0\leq s\leq t}\|\rho[f^N](s,\cdot)\|_\infty\Big(\int\limits_{|y|\leq\frac{2r_N}{1-\bar v}} g_1(y)\,\mathrm{d}^3y + \int\limits_{\frac{2r_N}{1-\bar v}<|y|\leq t} g_1(y)\,\mathrm{d}^3y\Big)\\
&\quad\lesssim C_0\Big(\int\limits_{|y|\leq\frac{2r_N}{1-\bar v}} r_N^{-3}\,\mathrm{d}^3y + \int\limits_{\frac{2r_N}{1-\bar v}<|y|\leq t} |y|^{-3}\,\mathrm{d}^3y\Big)\\
&\quad\lesssim C_0\,(1+\log(r_N^{-1})+\log(T)),
\end{aligned}\tag{5.123}$$

and for g_2:

$$\begin{aligned}
&\Big|\int\limits_{\mathrm{B}_t(x)\times\mathbb{R}^3} g_2(x-y)(1-n(x-y)v(\eta))f^N(t-|x-y|,y,\eta)\mathrm{d}y\mathrm{d}\eta\Big|\\
&\quad\lesssim \int\limits_{|x-y|\leq t} g_2(x-y)\rho[f^N](t-|x-y|,y)\mathrm{d}y\\
&\quad\lesssim \sup_{0\leq s\leq t}\|\rho[f^N](s,\cdot)\|_\infty \int\limits_{|y|\leq t}|y|^{-2}\mathrm{d}^3y\\
&\quad\lesssim C_0T.
\end{aligned}\tag{5.124}$$

It remains to show that the difference

$$\Big|\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{t_{ret}\geq 0\}} g_j(x-y_i^*(t_{ret,i})) - \int g_j(x-y)(1-nv)f^N(t-|x-y|,y,\xi)\Big| \tag{5.125}$$

is typically small. According to part 1) of Lemma 5.9.2, (5.125) can be written as

$$\Big|\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{z_i\in B_t(x)\times\mathbb{R}^3\}} g_j(x-\pi_x\phi(z_i)) - \int \mathbb{1}_{\{z\in B_t(x)\times\mathbb{R}^3\}} g_j(x-z)\phi\# f_0(z)\mathrm{d}z\Big|,$$

where $\pi_x(x,\xi) = x$ is the projection on the spatial coordinates and we used the fact that $t_{ret}(z) \geq 0 \iff z \in B(t,x) \times \mathbb{R}^3$. Hence, we can apply again the law of large numbers.

For any $x \in \mathcal{G}^N$, we consider $h : \mathbb{R}^6 \to \mathbb{R}, z \mapsto \mathbb{1}_{\{\phi^{-1}(z) \in B_t(x) \times \mathbb{R}^3\}} g_j(x - \pi_x z)$. This function is bounded as $|h| \lesssim r_N^{-3} \leq N^{3\gamma}$ with $\gamma < \frac{1}{12}$. Applying Proposition 5.9.5 with ϕ as in (5.97), $\alpha = 3\gamma$ and $\beta = 0$, we find

$$\mathbb{P}_0\Big[\Big|\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{t_{ret}\geq 0\}} g_j(x - y_i^*(t_{ret,i})) - \int g_j(x-y)(1-nv) f^N(t - |x-y|, y, \xi)\Big| > 1\Big] \lesssim N^{-4}$$

and thus $\mathbb{P}_0\big[\exists x_k \in \mathcal{G}^N \mid (5.125) > 1\big] \lesssim N^{-1}$, for $j = 1, 2$, since the grid $\mathcal{G}^N$ consists of $(3N)^3$ points. We define the (N and t dependent) set

$$A_t^2 := \{Z \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (5.125) \leq 1, j = 1, 2\ \forall x \in \mathcal{G}^N\}. \tag{5.126}$$

Then there exists $C_2 > 0$ such that $\mathbb{P}(A_t^2) \geq 1 - \frac{C_2}{N}$.

For the magnetic field component B_1, the proof works analogously, since the corresponding kernel $n \times \alpha^{-1}$ has the same bounds and regularity properties.

5.10.2 Controlling the radiation term

We now consider the contribution of the radiation term E_2. The corresponding kernel is less singular in the near-field, but depends on the acceleration of the particles. From (5.23):

$$\begin{aligned}
&|E_2[\tilde{f}^N](t,x) - E_2[\tilde{\mu}^N](t,x)| \\
&= \Big|\int (\nabla_\xi \alpha Y) * (\tilde{K}[\tilde{f}^N]\mathbb{1}_{t\geq 0}\tilde{f}^N)\,\mathrm{d}\xi - \int (\nabla_\xi \alpha Y) * (\tilde{K}[\tilde{\mu}^N]\mathbb{1}_{t\geq 0}\tilde{\mu}^N)\,\mathrm{d}\xi\Big| \\
&\leq \Big|\int (\nabla_\xi \alpha Y) * (\tilde{K}[\tilde{f}^N]\mathbb{1}_{t\geq 0}\tilde{f}^N)\,\mathrm{d}\xi - \int (\nabla_\xi \alpha Y) * (\tilde{K}[\tilde{f}^N]\mathbb{1}_{t\geq 0}\tilde{\mu}^N[\Psi_{s,0}(x)])\,\mathrm{d}\xi\Big| \qquad (5.127) \\
&+ \Big|\int (\nabla_\xi \alpha Y) * (\tilde{K}[\tilde{f}^N] - \tilde{K}[\tilde{\mu}^N])(\mathbb{1}_{t\geq 0}\,\tilde{\mu}^N[\Psi_{s,0}(x)])\,\mathrm{d}\xi\Big|, \qquad (5.128)
\end{aligned}$$

where we use the regularized distributions and the corresponding regularized forces $K[\tilde{f}^N]$, respectively $K[\tilde{\mu}^N]$ in view of (5.43). The integrals on the r.h.s. are to be evaluated at (t,x). For the second term (5.128):

$$\begin{aligned}
&\Big|\int (\nabla_\xi \alpha\, Y) * (\tilde{K}[\tilde{f}^N] - \tilde{K}[\tilde{\mu}^N])(\mathbb{1}_{t\geq 0}\,\tilde{\mu}^N)\,\mathrm{d}\xi\Big| \\
&= \Big|\frac{1}{N}\sum_{i=1}^N \int_0^t \int_{S^2} (t-s)\nabla_\xi \alpha(t-s, \omega(t-s), \xi_i^*(s)) \\
&\qquad (\tilde{K}[\tilde{f}^N] - \tilde{K}[\tilde{\mu}^N])(s, x - \omega(t-s), \xi_i^*(s))\chi^N(x - \omega(t-s) - x_i^*(s))\,\mathrm{d}\omega\mathrm{d}s\Big| \\
&\leq \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{S^2} \Big|(t-s)\nabla_\xi \alpha(t-s, \omega(t-s), \xi_i^*(s))\Big| \\
&\qquad \Big|(\tilde{K}[\tilde{f}^N] - \tilde{K}[\tilde{\mu}^N])(s, x - \omega(t-s), \xi_i^*(s))\Big|\chi^N(x - \omega(t-s) - x_i^*(s))\,\mathrm{d}\omega\mathrm{d}s.
\end{aligned}$$

Now, recall from (5.25):

$$(\nabla_\xi \alpha^0)^i_j(t,x,\xi) = \frac{t(t-v\cdot x)(v_j v^i - \delta^i_j) + (x_j - tv_j)(x^i - (v\cdot x)v^i)}{\sqrt{1+|\xi|^2}(t-v\cdot x)^2}$$

and thus

$$(t-s)\nabla_\xi \alpha(t-s,\omega(t-s),\xi^*) = \frac{(1-v\cdot\omega)(v_j v^i - \delta^i_j) + (\omega_j - v_j)(\omega^i - (v\cdot\omega)v^i)}{\sqrt{1+|\xi|^2}(1-v\cdot\omega)^2}.$$

Since the vectors appearing in the nominator are all of norm 1 or smaller, we can estimate

$$|(t-s)\nabla_\xi \alpha(t-s,\omega(t-s),\xi^*)| \leq \frac{8}{(1-\overline{v})^2}. \tag{5.129}$$

Moreover, we observe that $\frac{1}{N}\sum\limits_{i=1}^{N} \chi^N(x-\omega(t-s)-x_i^*(s))$ is nothing else than the (smeared) microscopic charge density $\tilde{\rho}[\mu^N[Z]](s,x-\omega(t-s))$. In total, we can thus write

$$\begin{aligned}
&\Big|\int (\nabla_\xi \alpha\, Y) * (\tilde{K}[\tilde{f}^N] - \tilde{K}[\tilde{\mu}^N])(\mathbb{1}_{t\geq 0}\,\tilde{\mu})\,\mathrm{d}\xi\Big| \\
&\leq \frac{8}{(1-\overline{v})^2}\int_0^t \int_{S^2} \big|E[\tilde{f}^N](s,x-\omega(t-s)) - E[\tilde{\mu}^N](s,x-\omega(t-s))\big| \\
&\qquad + \big|B[\tilde{f}^N](s,x-\omega(t-s)) - B[\tilde{\mu}^N](s,x-\omega(t-s))\big|\,\tilde{\rho}[\mu](s,x-\omega(t-s))\mathrm{d}\omega\mathrm{d}s \\
&\lesssim \frac{\|\tilde{\rho}[\mu]\|_{L^\infty([0,T]\times\mathbb{R}^3)}}{(1-\overline{v})^2}\int_0^t \|E[\tilde{f}^N](s) - E[\tilde{\mu}^N](s)\|_{L^\infty(\mathrm{B}(\overline{r}))} + \|B[\tilde{f}^N](s) - B[\tilde{\mu}^N](s)\|_{L^\infty(\mathrm{B}(\overline{r}))}\mathrm{d}s,
\end{aligned} \tag{5.130}$$

where in the last line, we used the fact that $\operatorname{supp}\tilde{\rho}[\mu](s) \subseteq \mathrm{B}(\overline{r};0)$, $\forall s \leq T$.

For (5.127) we write

$$\Big|\int (\nabla_\xi \alpha\, Y) * (\tilde{K}[\tilde{f}^N]\mathbb{1}_{t\geq 0}\tilde{f}^N)\,\mathrm{d}\xi - \int (\nabla_\xi \alpha\, Y) * (\tilde{K}[\tilde{f}^N]\mathbb{1}_{t\geq 0}\tilde{\mu}^N[\Psi_{t,0}(Z)])\,\mathrm{d}\xi\Big|$$

$$\leq \Big|\int (\nabla_\xi \alpha\, Y) * (\tilde{K}[\tilde{f}^N]\mathbb{1}_{t\geq 0}\tilde{f})\,\mathrm{d}\xi - \int (\nabla_\xi \alpha\, Y) * (\tilde{K}[\tilde{f}^N]\mathbb{1}_{t\geq 0}\tilde{\mu}^N[\Phi_{t,0}(Z)])\,\mathrm{d}\xi\Big| \tag{5.131}$$

$$+ \Big|\int (\nabla_\xi \alpha\, Y) * (\tilde{K}[\tilde{f}^N]\mathbb{1}_{t\geq 0}\tilde{\mu}^N[\Phi_{t,0}(Z)])\,\mathrm{d}\xi - \int (\nabla_\xi \alpha\, Y) * (\tilde{K}[\tilde{f}^N]\mathbb{1}_{t\geq 0}\tilde{\mu}^N[\Psi_{t,0}(Z)])\,\mathrm{d}\xi\Big|. \tag{5.132}$$

We evaluate

$$\int (\nabla_\xi \alpha\, Y) * (\tilde{K}[\tilde{f}^N]\mathbb{1}_{t\geq 0}\tilde{\mu}^N[\Phi_{t,0}(Z)]) = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{t_{ret,i}>0\}}\kappa(t_{ret,i}, y^*(t_{ret,i}), \eta^*(t_{ret,i}))$$

with kernel

$$\kappa(s,y,\eta) := \frac{(\tilde{K}[\tilde{f}](s,y,\eta)\cdot v(\eta))v(\eta) - \tilde{K}[\tilde{f}](s,y,\eta))}{\sqrt{1+\eta^2}(1-v(\eta)\cdot n(x-y))^2|x-y|} + \frac{\tilde{K}[\tilde{f}](s,y,\eta)\cdot(n(x-y)-v(\eta))\big(n(x-y)-(v\cdot n)v(\eta)\big)}{\sqrt{1+\eta^2}(1-v(\eta)\cdot n(x-y))^2|x-y|}. \tag{5.133}$$

With L as in (5.67), the function κ satisfies

$$|\kappa(s,y,\eta)| \lesssim \frac{|\tilde{K}[\tilde{f}^N](s,y,\eta)|}{(1-\overline{v})^2|x-y|} \leq \frac{L}{(1-\overline{v})^2|x-y|} \tag{5.134}$$

$$\begin{aligned}|\nabla_{x,\xi}\kappa(s,y,\eta)| &\lesssim \frac{|\nabla_{x,\xi}\tilde{K}[\tilde{f}^N](s,y,\eta)|}{(1-\overline{v})^3|x-y|} + \frac{|\tilde{K}[\tilde{f}^N](s,y,\eta)|}{(1-\overline{v})^2|x-y|^2}\\ &\leq \frac{L}{(1-\overline{v})^3}\Big(\frac{1}{|x-y|}+\frac{1}{|x-y|^2}\Big).\end{aligned} \tag{5.135}$$

Now we proceed along the lines of section 5.10.1, simplified by the fact that the kernel is homogeneous of degree -1 (rather than -2) in x.

Let A_t^3 be the (N and t dependent) set defined by

$$A_t^3 := \{Z\in\mathbb{R}^3\times\mathbb{R}^3 \mid (5.131) \leq N^{-1/4} \text{ for all } x\in\mathcal{G}^N\}. \tag{5.136}$$

Then there exists $C_3>0$ such that $\mathbb{P}(A_t^3) \geq 1-\frac{C_3}{N}$.

For (5.132), we introduce a function $g_3 \lesssim \min\{r_N^{-2}, |x|^{-1}+|x|^{-2}\}$ such that

$$|\tilde{\kappa}(t,x_1,\xi_1)-\tilde{\kappa}(t,x_2,\xi_2)|_\infty \leq g_3(x_1)\,|(x_1,\xi_1)-(x_2,\xi_2)|_\infty, \tag{5.137}$$

for all $t\leq T$, $|\xi_1|,|\xi_2|\leq\overline{\xi}$ and $|x_1-x_2| < \frac{r_N}{1-\overline{v}}$ (c.f. Lemma 5.10.2). With this, we find that

$$(5.132) \leq \Big(\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{t_{ret}\geq 0\}} g_3(x-y_i^*(t_{ret,i}^y))\Big)\,\frac{L}{1-\overline{v}}\sup_{0\leq s\leq t}|^N\Phi_{s,0}(Z) - {}^N\Psi_{s,0}(Z)|_\infty. \tag{5.138}$$

In contrast to 5.10.1, we do not have to treat distances in physical space and momentum space separately, other than that, the argument is the same. We estimate the g_3 term by

$$\Big|\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{t_{ret}\geq 0\}}\, g_3(x-y_i^*(t_{ret,i}^y))\Big|$$

$$\leq \Big|\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{z_i\in B_t(x)\times\mathbb{R}^3\}}\, g_3(x-\pi_x\phi(z_i)) - \int \mathbb{1}_{\{z\in B_t(x)\times\mathbb{R}^3\}} g_3(x-z)\phi\#f_0(z)\mathrm{d}z\Big| \tag{5.139}$$

$$+\Big|\int \mathbb{1}_{\{z\in B_t(x)\times\mathbb{R}^3\}}\, g_3(x-z)\phi\#f_0(z)\mathrm{d}z\Big|. \tag{5.140}$$

Since $g_3 \lesssim \min\{r_N^{-2}, |x|^{-1} + |x|^{-2}\}$, one checks that (5.140) $\lesssim C_0(1+T^2)$. Now we define the (N and t dependent) set

$$A_t^4 := \{Z \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (5.139) \leq 1 \text{ for all } x \in \mathcal{G}^N\}. \tag{5.141}$$

According to Proposition 5.9.5, there exists a constant $C_4 > 0$ such that $\mathbb{P}_0(A) \geq 1 - \frac{C_4}{N}$. For $Z \in A_t^4, J_t^N(Z) < 1$, we thus have (5.132) $\lesssim \sup\limits_{0 \leq s \leq t} |^N\Phi^1_{s,0}(Z) - {}^N\Psi^1_{s,0}(Z)|_\infty$.

For the magnetic field component B_2, the proof works analogously, since the corresponding kernel $\nabla_\xi n \times \alpha^0$ has the same bounds and regularity properties.

5.10.3 Controlling shock waves

We now consider the term (5.21). We compute

$$\begin{aligned} E_0'(t,x) &= \int (\alpha^0\, Y)(t,\cdot,\xi) *_x \chi^N *_x f_0(x,\xi) \mathrm{d}\xi \\ &= \frac{t}{4\pi} \int \frac{\omega - v}{1 - v\cdot\omega}\, \chi^N(x - y - wt)\, f_0(y,\xi)\, \mathrm{d}w \mathrm{d}y \mathrm{d}\xi \\ &= \int h(t, x-y)\, f_0(y,\xi) \mathrm{d}y \mathrm{d}\xi, \end{aligned}$$

with

$$h(t,x,\xi) = \frac{t}{4\pi} \int_{S^2} \frac{\omega - v}{1 - v\cdot\omega}\, \chi^N(x - wt). \tag{5.142}$$

This function satisfies

$$|h(t,x,\xi)| \lesssim \frac{t}{1-\overline{v}}\, r_N^{-3}. \tag{5.143}$$

We have to control the difference

$$\begin{aligned} &\big|E_0'[\tilde{\mu}_0^N[Z]](t,x) - E_0'[\tilde{f}_0](t,x)\big| \\ &= \Big|\frac{1}{N}\sum_{i=1}^N h(t, x - x_i, \xi_i) - \int h(t, x-y, \xi)\, f_0(y,\xi)\Big|, \end{aligned} \tag{5.144}$$

which depends only on initial data. Applying Proposition 5.9.5 (with $\phi(z) = z$ and $\alpha = 3\gamma$, $\beta = \frac{1}{4}$) we have for any (t,x):

$$\mathbb{P}_0\Big[\Big|\frac{1}{N}\sum_{i=1}^N h(t, x - x_i, \xi_i) - \int h(t, x-y,\xi)\, f_0(y,\xi)\Big| > N^{-\frac{1}{4}}\Big] \lesssim N^{-4}$$

and thus $\mathbb{P}_0\big[\exists x \in \mathcal{G}^N \mid (5.144) > N^{-\frac{1}{4}}\big] \lesssim N^{-1}$. We conclude:

Let A_t^5 be the (N and t dependent) set defined by

$$A_t^5 := \{Z \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (5.144) \leq N^{-\frac{1}{4}} \text{ for all } x \in \mathcal{G}^N\}. \tag{5.145}$$

Then there exists $C_5 > 0$ such that $\mathbb{P}(A_t^5) \geq 1 - \frac{C_5}{N}$.

Remark: Without regularization, the kernel (5.142) would have the form $t \int\limits_{S^2} \frac{\omega - v}{1 - v\cdot\omega}\, \delta(x - wt)$, which is not only unbounded, but distribution valued, reflecting the fact that $E_0'(t,x)$ depends on the initial charge distribution only via $\rho_0\big|_{\{|x-y|=t\}}$. However, after smearing with χ^N, the term is relatively harmless. The width of the necessary cut-off for the law of large number estimate could be further reduced by exploiting the fact that $h(t,x,\xi) = 0$ unless $t - r_n < |x| < t + r_N$.

For the magnetic field component B_0', the proof works analogously, since the corresponding kernel satisfies the same bound (5.143).

5.10.4 Controlling the homogeneous fields

It remains to control the contribution of the homogeneous fields (5.20), which depend only on the initial data via the Gauss constraint $\mathrm{div} E_0\big|_{t=0} = \rho_0$. The solution of the homogeneous field-equation is given by

$$E_0(t,x) = \partial_t Y(t,\cdot) * E_{in}(x) = \partial_t\Big(\frac{t}{4\pi} \int\limits_{S^2} E_{in}(x + \omega t)\mathrm{d}\omega\Big).$$

If $E_{in}(x) = -\nabla G * \rho_0(x) = \int \frac{x-y}{|x-y|^3}\rho_0(y)\,\mathrm{d}y$ is the Coulomb field, we compute:

$$\begin{aligned}
&-\partial_t \nabla_x \int G *_x Y(t,\cdot) *_x \tilde{f}_0(x,\xi)\mathrm{d}\xi \\
&= \frac{1}{4\pi}\int \int\limits_{S^2} \Big[\frac{x - y + 2\omega t}{|x-y+\omega t|^3} - \frac{t\omega\cdot(x-y+\omega t)(x-y+\omega t)}{|x-y+\omega t|^5}\Big]\mathrm{d}\omega\, \tilde{\rho}_0(y)\,\mathrm{d}y \\
&= \frac{1}{4\pi}\int \int\limits_{S^2} h'(t\omega, x-y)\,\mathrm{d}\omega\, \tilde{\rho}_0(y)\mathrm{d}y,
\end{aligned}$$

with $h'(t\omega, x) := \frac{1}{4\pi}\big(\frac{x+2\omega t}{|x+\omega t|^3} - \frac{t\omega\cdot(x+\omega t)(x+\omega t)}{|x+\omega t|^5}\big)$. Shifting the mollifier to the kernel, we get:

$$|\chi^N * h'| \lesssim r_N^{-2} + t\, r_N^{-3},$$

where we used again Lemma 5.8.1, and thus

$$E_0(t,x) = \int\int h_0(t, x-y)\, f_0(y,\xi)\,\mathrm{d}y\mathrm{d}\xi, \tag{5.146}$$

with

$$h_0(t,x) := \int\limits_{S_1} \chi^N * h'(x, \omega t)\mathrm{d}\omega, \quad |h_0(t,x)| \lesssim r_N^{-2} + t\, r_N^{-3}. \tag{5.147}$$

Now, by (5.51), the incoming fields are fixed such that $E_{in}^N - E_{in}^\mu = -\nabla G * (\rho_0[f] - \rho_0[\mu[Z]])$. Hence, we have to control the difference

$$\Big|\frac{1}{N}\sum_{i=1}^N h_0(t, x - x_i) - \int h_0(t, x-y)\, f_0(y,\xi)\mathrm{d}y\mathrm{d}\xi\Big|. \tag{5.148}$$

As before, an application of the law of large numbers in form of Proposition 5.9.5 yields the following: Let A_t^6 be the (N and t dependent) set defined by

$$A_t^6 := \{Z \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (5.146) \leq N^{-\frac{1}{4}} \text{ for all } x \in \mathcal{G}^N\}. \tag{5.149}$$

Then there exists $C_6 > 0$ such that $\mathbb{P}_0(A_t^6) \geq 1 - \frac{C_6}{N}$.

For the magnetic field, $B_0^N - B_0^\mu = 0$ since, by assumption, $B_{in}^N = B_{in}^\mu$.

For every t, our law of large numbers estimates yield bounds on a finite number on points, that we have chosen to lie on the grid $\mathcal{G}^N$ covering the interval $[-\overline{r}, \overline{r}]$ which contains the support of f^N and μ^N. However, combined with the bound on the field derivatives from Proposition 5.8.5, this can be used to derive a L^∞-bound. We give an example in the following lemma.

Lemma 5.10.3. *Let $\overline{r}$ as defined in* (5.68)*. In view of the assumptions of Propositions 5.8.2 and 5.8.5, we fix some $p \geq 1$ and consider the set $M = M(p)$ defined by*

$$Z \in M \iff W_p^p(\mu_0^N[Z], f_0) \leq r_N^{3+p}. \tag{5.150}$$

Let E_{in}^N and $E_{in}^\mu = E_{in}^\mu[Z]$ as fixed in (5.52)*. Then there exists a constant $C > 0$ such that*

$$\mathbb{P}_0\Big[\|E_{in}^N - E_{in}^\mu\|_{L^\infty(\mathrm{B}(\overline{r}))} \lesssim N^{-\frac{1}{4}} \Big| M\Big] \geq 1 - \frac{C}{N}. \tag{5.151}$$

Proof. Above, we have proven that

$$\mathbb{P}_0\Big[\exists x_k \in \mathcal{G}^N : |E_{in}^N(x_k) - E_{in}^\mu(x_k)| \geq N^{-\frac{1}{4}}\Big] \lesssim N^{-1}. \tag{5.152}$$

Furthermore, according to Proposition 5.8.5, we have $\|\nabla_x(E^N - E^\mu)\|_\infty \lesssim r_N^{-2}$ for $Z \in M$. By construction:

$$\sup\{\min_{x_i \in \mathcal{G}^N} |x - x_i| : x \in \mathrm{B}(\bar{\mathrm{r}})\} \leq \frac{\sqrt{3}}{2}\frac{\overline{r}}{N}. \tag{5.153}$$

Hence, $|E_{in}^N(x_k) - E_{in}^\mu(x_k)| \leq N^{-\frac{1}{4}}$ $\forall x_k \in \mathcal{G}$ implies $|E_{in}^N(x) - E_{in}^\mu(x)| \lesssim N^{-\frac{1}{4}} + \frac{r_N^{-2}}{N} \leq N^{-\frac{1}{4}} + N^{-1+2\gamma}$ for all $x \in \mathrm{B}(\bar{\mathrm{r}})$. Since $\gamma < \frac{1}{12}$, we conclude

$$\mathbb{P}_0\Big[\|E_{in}^N - E_{in}^\mu\|_{L^\infty(\mathrm{B}(\overline{r}))} \lesssim N^{-\frac{1}{4}} \Big| Z \in M\Big] \lesssim N^{-1}.$$

□

5.11 A Gronwall argument

We are finally ready to combine the results of the previous sections into a prove of the main theorem. Our aim is to establish a Gronwall bound for the quantity $\mathbb{E}_0(J_t^N)$ defined in 5.7.2, thus proving the mean field limit for typical initial conditions.

5.11.1 Good initial conditions

Let $\gamma < \frac{1}{12}$ and $r_N \geq N^{-\gamma}$. Fix an initial distribution f_0 with compact support as in Theorem 5.5.2. We begin by noting the (time-independent) conditions that the initial configuration $Z \in \mathbb{R}^{6N}$ has to satisfy. All probabilities are meant with respect to the product-measure $\otimes^N f_0$ on $\mathbb{R}^{3N}$. Consider the sets $\mathrm{C}_1, \mathrm{C}_2$ defined by

$$Z \in \mathrm{C}_1 \iff z_i \in \operatorname{supp}(f_0), \forall 1 \leq i \leq N. \tag{5.154}$$

$$Z \in \mathrm{C}_2 \iff \|(E^N_{in}, B^N_{in}) - (E^\mu_{in}, B^\mu_{in})\|_{L^\infty(\mathrm{B}(\overline{r}))} \leq N^{-\frac{1}{4}}. \tag{5.155}$$

Moreover, setting $p := \frac{1}{4\gamma}$, we consider the set $\mathrm{C}_3 \subset \mathbb{R}^{6N}$ defined by

$$Z \in \mathrm{C}_3 \iff W_p^p(\mu^N[Z], f_0) \leq r_N^{3+p}. \tag{5.156}$$

Obviously, $\mathbb{P}_0(Z \notin \mathrm{C}_1) = 0$ and according to Lemma 5.10.3, $\mathbb{P}_0(Z \notin \mathrm{C}_2) \lesssim N^{-1}$. For C_3, we apply the large deviation estimate, Theorem 2.2.1, with $d = 6$, $p := \frac{1}{4\gamma}$ and $\xi = r_N^{3+p} \geq N^{-(3+p)\gamma} = N^{-(3\gamma+1/4)}$. This yields constants $c, c' > 0$ such that

$$\mathbb{P}_0\Big(W_p^p(\mu_0^N[Z], f_0) > r_N^{3+p}\Big) \leq c' e^{-cN^s}, \tag{5.157}$$

where

$$s = 1 - 2(3\gamma + 1/4) = \frac{1}{2}(1 - 12\gamma) > 0. \tag{5.158}$$

In total, setting

$$\mathcal{C} := \mathrm{C}_1 \cap \mathrm{C}_2 \cap \mathrm{C}_3 \tag{5.159}$$

there exists a constant C_7 such that $\mathbb{P}_0(\mathcal{C}) \geq 1 - \frac{C_7}{N}$. Note that the requirement $\gamma < \frac{1}{12}$ for the width of the cut-off comes from (5.158).

5.11.2 Evolution of J_t^N

For $t > 0$ we have to control the growth of $\mathbb{E}_0(J_t^N)$. Recall from Def. 5.7.2:

$$\begin{aligned} J_t^N(Z) := \min\Big\{1, \lambda(N) N^\delta \sup_{0 \leq s \leq t} |{}^N\Psi^1_{t,0}(Z) - {}^N\Phi^1_{t,0}(Z)|_\infty \\ + N^\delta \sup_{0 \leq s \leq t} |{}^N\Psi^2_{t,0}(Z) - {}^N\Phi^2_{t,0}(Z)|_\infty \Big\}, \end{aligned}$$

with $\lambda(N) := \max\{1, \sqrt{\log(N)}\}$. For fixed $t > 0$ we denote by $\mathcal{B}_t$ the set

$$\mathcal{B}_t := \{Z \in \mathbb{R}^3 \times \mathbb{R}^3 : J_t^N(Z) < 1\}. \tag{5.160}$$

Moreover, we define the set

$$\mathcal{A}_t := A_t^1 \cap A_t^2 \cap A_t^3 \cap A_t^4 \cap ... \cap A_t^{12}, \tag{5.161}$$

where $A_t^1, A_t^2, A_t^3, A_t^4, A_t^5, A_t^6$ are defined in Section 5.10 and $A_t^7, .., A_t^{12}$ are the analogous sets for the magnetic field components.

We split $\mathbb{E}_0(J_t^N)$ into

$$\mathbb{E}_0(J_t^N) = \mathbb{E}_0(J_t^N \mid \mathcal{A}_t \cap \mathcal{B}_t \cap \mathcal{C}) + \mathbb{E}_0(J_t^N \mid \mathcal{B}_t \cap (\mathcal{A}_t \cap \mathcal{C})^c) + \mathbb{E}_0(J_t^N \mid \mathcal{B}_t^c).$$

Now, we first observe that if $Z \in \mathcal{B}_t^c$, we have $\frac{\mathrm{d}}{\mathrm{d}t} J_t^N = 0$, since $J_t^N(Z) = 1$ is already maximal. In particular,

$$\partial_t \, \mathbb{E}_0(J_t^N \mid \mathcal{B}_t^c) = 0. \tag{5.162}$$

Hence, we only need to consider the case $J_t^N(Z) < 1$ for which, in particular,

$$\sup_{0 \le s \le t} |{}^N\Psi_{s,0}(Z) - {}^N\Phi_{s,0}(Z)|_\infty < N^{-\delta} \le N^{-\gamma} \le r_N. \tag{5.163}$$

We have to control the evolution of

$$\lambda(N) N^\delta \sup_{0 \le s \le t} |{}^N\Psi^1_{s,0}(Z) - {}^N\Phi^1_{s,0}(Z)|_\infty + N^\delta \sup_{0 \le s \le t} |{}^N\Psi^2_{s,0}(Z) - {}^N\Phi^2_{s,0}(Z)|_\infty.$$

We will denote by $E^N = E^N[\tilde{f}^N]$ and $B^N = B^N[\tilde{f}^N]$ the macroscopic fields, generated by the (regularized) Vlasov density, and by $E^\mu = E^\mu[\tilde{\mu}^N[Z]], B^\mu = B^\mu[\tilde{\mu}^N[Z]]$ the microscopic fields, generated by the rigid charges.

Recalling Lemma 3.8.2 and denoting by ∂_t^+ the derivative from the right w.r.t. t, we find:

$$\begin{aligned}
&\partial_t^+ \sup_{0 \le s \le t} |{}^N\Psi^1_{s,0}(Z) - {}^N\Phi^1_{s,0}(Z)|_\infty \\
&\le \left|\partial_t({}^N\Psi^1_{t,0}(Z) - {}^N\Phi^1_{t,0}(Z))\right|_\infty = \max_{1 \le i \le N} |v(\xi_i^*(t)) - v(\eta_i^*(t))| \\
&\le 2 \max_{1 \le i \le N} |\xi_i^*(t) - \eta_i^*(t)| = 2 |{}^N\Psi^2_{t,0}(Z) - {}^N\Phi^2_{t,0}(Z)|_\infty,
\end{aligned} \tag{5.164}$$

as well as

$$\begin{aligned}
&\partial_t^+ \sup_{0 \le s \le t} |{}^N\Psi^2_{s,0}(Z) - {}^N\Phi^2_{s,0}(Z)|_\infty \\
\le& \left|\partial_t({}^N\Psi^2_{t,0}(Z) - {}^N\Phi^2_{t,0}(Z))\right|_\infty = \max_{1 \le i \le N} |\tilde{K}[\tilde{\mu}](t, x_i^*, \xi_i^*) - \tilde{K}[\tilde{f}](t, y_i^*, \eta_i^*)| \\
\le& \max_{1 \le i \le N} |\tilde{K}[\tilde{f}](t, x_i^*, \xi_i^*) - \tilde{K}[\tilde{f}](t, y_i^*, \eta_i^*)| + \max_{1 \le i \le N} |\tilde{K}[\tilde{\mu}](t, y_i^*, \eta_i^*) - \tilde{K}[\tilde{f}](t, y_i^*, \eta_i^*)| \\
\le& L |{}^N\Psi_{t,0}(Z) - {}^N\Phi_{t,0}(Z)|_\infty + \|\tilde{E}^N(t) - \tilde{E}^\mu(t)\|_{L^\infty(\mathrm{B}(\overline{r}))} + \|\tilde{B}^N(t) - \tilde{B}^\mu(t)\|_{L^\infty(\mathrm{B}(\overline{r}))}
\end{aligned} \tag{5.165}$$

In the last line, we used the uniform Lipschitz bound on the mean field force (5.67) and the fact that $|x_i^*|, |y_i^*| < \overline{r}$ for all $i = 1, .., N$ and $t \le T$.

It remains to control the term

$$\begin{aligned}
&\|\tilde{E}^N(t, \cdot) - \tilde{E}^\mu(t, \cdot)\|_{L^\infty(\mathrm{B}(\overline{r}))} + \|\tilde{B}^N(t, \cdot) - \tilde{B}^\mu(t, \cdot)\|_{L^\infty(\mathrm{B}(\overline{r}))} \\
\le& \|E^N(t, \cdot) - E^\mu(t, \cdot)\|_{L^\infty(\mathrm{B}(\overline{r}))} + \|B^N(t, \cdot) - B^\mu(t, \cdot)\|_{L^\infty(\mathrm{B}(\overline{r}))}.
\end{aligned} \tag{5.166}$$

Now, $Z \in (\mathcal{A}_t \cap \mathcal{C})^c$ are the "bad" initial conditions that may lead to large fluctuations in the fields or a blow-up of the microscopic charge density. However, the Vlasov fields $(\tilde{E}^N, \tilde{B}^N)$

are bounded uniformly in N according to (5.67), while the (smeared) microscopic fields $(\tilde{E}^\mu, \tilde{B}^\mu)$ diverge at most as $\|(\tilde{E}^\mu, \tilde{B}^\mu)\|_\infty \lesssim r_N^{-2}$ according to Prop. 5.8.5. Therefore:

$$\begin{aligned} &\|\partial_t^+ J_t^N(\cdot)\|_{L^\infty(\mathbb{R}^{6N})} \\ &\leq (2\lambda(N) + L) J_t^N + \|\tilde{E}_t^N\|_\infty + \|\tilde{E}_t^\mu\|_\infty + \|\tilde{B}_t^N\|_\infty + \|\tilde{B}_t^\mu\|_\infty \lesssim r_N^{-2}. \end{aligned} \tag{5.167}$$

Hence, there exists a constant C' such that

$$\begin{aligned} &\partial_t^+ \mathbb{E}_0(J_t^N \mid \mathcal{B}_t \cap (\mathcal{A}_t \cap \mathcal{C})^c)) = \mathbb{E}_0(\partial_t^+ J_t^N \mid \mathcal{B}_t \cap (\mathcal{A}_t \cap \mathcal{C})^c) \\ &\leq \|\partial_t^+ J_t^N\|_{L^\infty(\mathbb{R}^{6N})} \mathbb{P}_0(\mathcal{A}_t^c \cup \mathcal{C}^c) \leq C' r_N^{-2} \frac{1}{N} \leq C' N^{-1+2\gamma}. \end{aligned} \tag{5.168}$$

$Z \in \mathcal{A}_t \cap \mathcal{B}_t \cap \mathcal{C}$ are the "good" initial conditions, for which we have derived various nice properties:

$$\begin{aligned} &|x_i^*(t)| < \overline{r},\ |\xi_i^*(t)| < \overline{\xi},\ \forall t \in [0,T] && \text{(from eq. 5.163)} \\ &\|\rho[\mu_t^N[Z]]\|_\infty \leq C_\rho,\ \forall N \geq 1, t \in [0,T] && \text{(from Proposition 5.8.2)} \\ &\|(\nabla_x E^\mu, \nabla_x B^\mu)\|_\infty \lesssim r_N^{-2} && \text{(Proposition 5.8.5)} \\ &\|(E_{in}^N, B_{in}^N) - (E_{in}^\mu, B_{in}^\mu)\|_{L^\infty(\mathrm{B}(\overline{r}))} \leq N^{-1/4} && \text{(since } Z \in \mathrm{C}_2) \end{aligned}$$

In particular, combining the results of Section 5.10, we have:

$$\begin{aligned} &\max\big\{|E^N(t,x_i) - E^\mu(t,x_i)|_\infty + |B^N(t,x_i) - B^\mu(t,x_i)|_\infty : x_i \in \mathcal{G}^N\big\} \\ \lesssim\ & \underbrace{N^{-\frac{1}{4}}}_{from\,(5.115,5.136,5.145,5.149)} + \underbrace{\frac{C_0}{(1-\overline{v})^4}(1 + \log(r_N^{-1})) \sup_{0\leq s\leq t} |{}^N\Phi_{s,0}^1(Z) - {}^N\Psi_{s,0}^1(Z)|_\infty}_{from\,(5.121,5.123\,,5.141)} \\ &+ \underbrace{\frac{LC_0T}{(1-\overline{v})^5} \sup_{0\leq s\leq t} |{}^N\Phi_{s,0}^1(Z) - {}^N\Psi_{s,0}^1(Z)|_\infty + \frac{C_0T}{(1-\overline{v})^4} \sup_{0\leq s\leq t} |{}^N\Phi_{s,0}^2(Z) - {}^N\Psi_{s,0}^2(Z)|_\infty}_{from(5.122,5.124\,,5.141)} \\ &+ \underbrace{\frac{LC_0(1+T^2)}{(1-\overline{v})^4}\Big(\sup_{0\leq s\leq t} |{}^N\Phi_{s,0}^1(Z) - {}^N\Psi_{s,0}^1(Z)|_\infty + \sup_{0\leq s\leq t} |{}^N\Phi_{s,0}^2(Z) - {}^N\Psi_{s,0}^2(Z)|_\infty\Big)}_{from(5.138-5.140)} \\ &+ \underbrace{\frac{C_\rho}{(1-\overline{v})^2} \int_0^t \|E^N(s) - E^\mu(s)\|_{L^\infty(\mathrm{B}(\overline{\mathrm{r}}))} + \|B^N(s) - B^\mu(s)\|_{L^\infty(\mathrm{B}(\overline{\mathrm{r}}))}\, \mathrm{d}s.}_{from\,(5.130)} \end{aligned}$$

We simplify this expression to:

$$\begin{aligned} &\max\big\{|E^N(t,x_i) - E^\mu(t,x_i)|_\infty + |B^N(t,x_i) - B^\mu(t,x_i)|_\infty : x_i \in \mathcal{G}^N\big\} \\ &\lesssim N^{-\frac{1}{4}} + \frac{C_0 \log(r_N^{-1})}{(1-\overline{v})^4} \sup_{0\leq s\leq t} |{}^N\Phi_{s,0}^1(Z) - {}^N\Psi_{s,0}^1(Z)|_\infty + \frac{LC_0(1+T^2)}{(1-\overline{v})^5} \sup_{0\leq s\leq t} |{}^N\Phi_{s,0}(Z) - {}^N\Psi_{s,0}(Z)|_\infty \\ &+ \frac{C_\rho}{(1-\overline{v})^2} \int_0^t \|E^N(s) - E^\mu(s)\|_{L^\infty(\mathrm{B}(\overline{\mathrm{r}}))} + \|B^N(s) - B^\mu(s)\|_{L^\infty(\mathrm{B}(\overline{\mathrm{r}}))}\, \mathrm{d}s. \end{aligned} \tag{5.169}$$

According to Proposition 5.8.5 and equation (5.67), we have $\|(E^N, B^N) - (E^\mu, B^\mu)\|_{Lip} \lesssim r_N^{-2}$. Moreover, by construction: $\sup\{\min_{x_i \in \mathcal{G}^N} |x - x_i| : x \in \mathrm{B}(\bar{\mathrm{r}}, 0)\} \leq \frac{\sqrt{3}}{2}\frac{\bar{r}}{N}$. Hence, by the same argument as in Lemma 5.10.3,

$$\|E^N(t,\cdot) - E^\mu(t,\cdot)\|_{L^\infty(\mathrm{B}(\bar{\mathrm{r}}))} + \|B^N(t,\cdot) - B^\mu(t,\cdot)\|_{L^\infty(\mathrm{B}(\bar{\mathrm{r}}))}$$
$$\lesssim \max\{|E^N(t,x_i) - E^\mu(t,x_i)|_\infty + |B^N(t,x_i) - B^\mu(t,x_i)|_\infty : x_i \in \mathcal{G}^N\} + \frac{r_N^{-2}}{N},$$

where $\frac{r_N^{-2}}{N} \leq N^{-1+2\gamma} \leq N^{-\frac{1}{4}}$. Together with (5.169), we thus have:

$$\|E^N(t,\cdot) - E^\mu(t,\cdot)\|_{L^\infty(\mathrm{B}(\bar{\mathrm{r}}))} + \|B^N(t,\cdot) - B^\mu(t,\cdot)\|_{L^\infty(\mathrm{B}(\bar{\mathrm{r}}))}$$
$$\lesssim N^{-\frac{1}{4}} + \frac{C_0 \log(r_N^{-1})}{(1-\overline{v})^4} \sup_{0\leq s\leq t} |{}^N\Phi^1_{s,0}(Z) - {}^N\Psi^1_{s,0}(Z)|_\infty + \frac{LC_0(1+T^2)}{(1-\overline{v})^5} \sup_{0\leq s\leq t} |{}^N\Phi_{s,0}(Z) - {}^N\Psi_{s,0}(Z)|_\infty$$
$$+ \frac{C_\rho}{(1-\overline{v})^2} \int_0^t \|E^N(s) - E^\mu(s)\|_{L^\infty(\mathrm{B}(\bar{\mathrm{r}}))} + \|B^N(s) - B^\mu(s)\|_{L^\infty(\mathrm{B}(\bar{\mathrm{r}}))} \,\mathrm{d}s.$$

By Gronwall's inequality, there exists a constant $C'' > 0$ depending on $\overline{v}$ and C_ρ such that

$$\begin{aligned} &\|E^N(t,\cdot) - E^\mu(t,\cdot)\|_{L^\infty(\mathrm{B}(\bar{\mathrm{r}}))} + \|B^N(t,\cdot) - B^\mu(t,\cdot)\|_{L^\infty(\mathrm{B}(\bar{\mathrm{r}}))} \\ &\leq e^{tC''}\Big(N^{-\frac{1}{4}} + \frac{C_0 \log(r_N^{-1})}{(1-\overline{v})^4} \sup_{0\leq s\leq t} |{}^N\Phi^1_{s,0}(Z) - {}^N\Psi^1_{s,0}(Z)|_\infty \\ &\qquad + \frac{LC_0(1+T^2)}{(1-\overline{v})^5} \sup_{0\leq s\leq t} |{}^N\Phi_{s,0}(Z) - {}^N\Psi_{s,0}(Z)|_\infty \\ &\qquad + \|E^N(0,\cdot) - E^\mu(0,\cdot)\|_{L^\infty(\mathrm{B}(\bar{\mathrm{r}}))} + \|B^N(0,\cdot) - B^\mu(0,\cdot)\|_{L^\infty(\mathrm{B}(\bar{\mathrm{r}}))}\Big) \end{aligned} \tag{5.170}$$

and with (5.155):

$$\begin{aligned} &\|E^N(t,\cdot) - E^\mu(t,\cdot)\|_{L^\infty(\mathrm{B}(\bar{r}))} + \|B^N(t,\cdot) - B^\mu(t,\cdot)\|_{L^\infty(\mathrm{B}(\bar{r}))} \\ &\leq e^{TC''}\frac{C_0 \log(r_N^{-1})}{(1-\overline{v})^4} \sup_{0\leq s\leq t} |{}^N\Phi^1_{s,0}(Z) - {}^N\Psi^1_{s,0}(Z)|_\infty \\ &+ e^{TC''}\frac{LC_0(1+T^2)}{(1-\overline{v})^5} \sup_{0\leq s\leq t} |{}^N\Phi_{s,0}(Z) - {}^N\Psi_{s,0}(Z)|_\infty + e^{TC''} 2N^{-\frac{1}{4}}. \end{aligned} \tag{5.171}$$

Plugging this into (5.165), we get:

$$\begin{aligned}
&\partial_t^+ \big(N^\delta \sup_{0\le s\le t} |{}^N\Psi^2_{s,0}(Z) - {}^N\Phi^2_{s,0}(Z)|_\infty\big)\\
&\le N^\delta L|{}^N\Psi_{t,0}(Z) - {}^N\Phi_{t,0}(Z)|_\infty + 2e^{TC''}N^{-\frac{1}{4}+\delta}\\
&+ e^{TC''}\frac{LC_0(1+T^2)}{(1-\overline{v})^5}N^\delta \sup_{0\le s\le t}|{}^N\Phi_{s,0}(Z) - {}^N\Psi_{s,0}(Z)|_\infty\\
&+ e^{TC''}\frac{C_0\log(N)}{(1-\overline{v})^4}\,N^\delta \sup_{0\le s\le t}|{}^N\Phi^1_{s,0}(Z) - {}^N\Psi^1_{s,0}(Z)|_\infty.
\end{aligned} \tag{5.172}$$

Note, in particular, that the last summand can be rewritten as:

$$\frac{\sqrt{\log(N)}}{(1-\overline{v})^4}\Big(\sqrt{\log(N)}N^\delta \sup_{0\le s\le t}|{}^N\Phi^1_{s,0}(Z) - {}^N\Psi^1_{s,0}(Z)|_\infty\Big),$$

so that, together with (5.164) and $\lambda(N) = \max\{1, \sqrt{\log(N)}\}$:

$$\begin{aligned}
\partial_t^+ J_t^N(Z) &\le 2\lambda(N)N^\delta|{}^N\Psi^2_{t,0}(Z) - {}^N\Phi^2_{t,0}(Z)|_\infty + N^\delta L|{}^N\Psi_{t,0}(Z) - {}^N\Phi_{t,0}(Z)|_\infty + 2e^{TC''}N^{-\frac{1}{4}+\delta}\\
&+ e^{TC''}\frac{C_0\sqrt{\log(N)}}{(1-\overline{v})^4}\Big(\sqrt{\log(N)}N^\delta \sup_{0\le s\le t}|{}^N\Psi^1_{s,0}(Z) - {}^N\Phi^1_{s,0}(Z)|_\infty\Big)\\
&+ e^{TC''}\frac{LC_0(1+T^2)}{(1-\overline{v})^5}N^\delta \sup_{0\le s\le t}|{}^N\Phi_{s,0}(Z) - {}^N\Psi_{s,0}(Z)|_\infty\\
&\le \frac{e^{TC''}LC_0(3+T^2)}{(1-\overline{v})^5}\lambda(N)\,J_t^N(Z) + 2e^{TC''}N^{-\frac{1}{4}+\delta}.
\end{aligned}$$

Together with (5.168), we have found:

$$\begin{aligned}
&\mathbb{E}_0(J^N_{t+\Delta t}) - \mathbb{E}_0(J^N_{t,0})\\
&\quad\le \Big(\frac{e^{TC''}LC_0(3+T^2)}{(1-\overline{v})^5}\lambda(N)\,J_t^N(Z) + 2e^{TC''}N^{-\frac{1}{4}+\delta} + C'N^{-1+2\delta}\Big)\Delta t + o(\Delta t).
\end{aligned}$$

Finally, using Gronwalls inequality and the fact that $J_0^N(Z) = 0\ \forall Z$ we get

$$\mathbb{E}_0(J_t^N) \le e^{tC\lambda(N)}N^{-\frac{1}{4}+\delta}, \tag{5.173}$$

with

$$C(T, C_0, f_0) = \max\Big\{\frac{e^{TC''}LC_0(3+T^2)}{(1-\overline{v})^5}, C'\Big\}. \tag{5.174}$$

Together with the results of Section 5.7, Proposition 3.5.3 and Lemma 5.7.4, this concludes the proof of the theorem. For simplicity, we demand $N \ge 4$, so that $\lambda(N) = \sqrt{\log(N)}$.

The approximation result for the fields, i.e. part c) of the theorem, can be read off equation (5.171) using $\mathbb{P}_0\big[\sqrt{\log(N)}\sup_{0\le s\le t}|{}^N\Phi^1_{s,0}(Z) - {}^N\Psi^1_{s,0}(Z)|_\infty \ge N^{-\delta}\big] \le \mathbb{E}_0(J_t^N)$ and $\mathbb{P}_0\big[\sup_{0\le s\le t}|{}^N\Phi_{s,0}(Z) - {}^N\Psi_{s,0}(Z)|_\infty \ge N^{-\delta}\big] \le \mathbb{E}_0(J_t^N)$. By choosing the grid $\mathcal{G}^N$ accordingly, $\mathrm{B}(\overline{r})$ can be replaced by any compact set $M \subset \mathbb{R}^3$.

□

Chapter 6

Discussion

We have presented two alternative approximations of the Vlasov-Poisson equation and one approximation of the Vlasov-Maxwell equations as mean field limits of regularized N-particle dynamics. To my knowledge, these are the first such results concerning the actual Vlasov-Poisson and Vlasov-Maxwell equations used in physics with generic initial data and an N-dependent cut-off decreasing much faster than logarithmic. Hence, I believe that they constitute significant progress with regard to the microscopic justification of these equations. They can give us some confidence that consistency between the fundamental microscopic theory and the mesoscopic kinetic theory can be established in a rigorous fashion. Nevertheless, the results we obtained are, of course, just one step towards a conclusive derivation and leave room for improvement in various respects. In particular, one would like to further reduce the size of the cut-off or, ideally, dispense with the microscopic regularization altogether.

6.1 Vlasov-Maxwell: On the status of the regularization

However, as already noted in the introduction to Chapter 5, the status of the regularization is more subtle in the context of Vlasov-Maxwell than with respect to the Vlasov-Poisson case. In the context of Vlasov-Poisson, the correct particle dynamics are clear and relatively well understood and skeptical individuals must insist that we have only conclusively proven the mean field approximation, once we derive the Vlasov equation from an N-particle Coulomb systems with no cut-off at all. We will discuss the prospects of this ambitious endeavour in the next section.

When it comes to the relativistic theory, though, the standard Maxwell-Lorentz equations are not well defined for point-particles due to the self-interaction singularity, and there is no universal agreement on what the "correct" microscopic theory is supposed to be. (In fact, a successful derivation of the Vlasov-Maxwell equations would seem to justify or corroborate the respective microscopic model just as much as the other way round.) Personally, I would advocate that the optimal result in this case would be a derivation of the Vlasov-Maxwell equations on the basis of *Wheeler-Feynman electrodynamics*, which is a time-symmetric version of classical electrodynamics that contains no fields and no self-interactions and hence no (a priori) singularities ([71, 72], see [5] for a recent mathematical discussion). However, the Wheeler-Feynman theory is still so little understood from a mathematical point of view, that the investigation of its mean field limit seems very far away.

Against this backdrop, the point-particle limit of the rigid charges model, that was considered here, seems like a natural – though still rather pragmatic – way to understand both classical electrodynamics and its mean field limit. Colloquially speaking, if the equations do not make sense for *infinitely* small particles, we read them as referring to *arbitrarily* small particles. In fact, there are also physicists willing to entertain the idea of rigid charges on a more fundamental level, see e.g. Lyle, 2010 [45], though the main objection remains the break of fundamental Lorentz invariance.[1]

To be clear, none of this is to say that the microscopic model considered in Chapter 5 amounts to a realistic physical theory. It certainly does not. However, as other authors have pointed out before (see e.g. [16,21]), any more satisfying microscopic approach to the Vlasov-Maxwell dynamics will most likely require a satisfying solution to the self-interaction problem first. Given the current state of affairs, I believe that the approach taken here is very reasonable, not only from a mathematical but also from a physical point of view.

Vlasov-Maxwell: Outlook and related questions. In any case, though, our result leaves much room for improvement as far as the size of the cut-off is concerned. Note that the lower bound on the cut-off, $r_N \sim N^{-\delta}$ with $\delta < \frac{1}{12}$, comes only from the Wasserstein bound on the charge density, Proposition 5.8.2, which assures that the microscopic charge density will typically remain bounded uniformly in N and t. This is a relatively powerful, but rather coarse way to prevent a blow-up of the microscopic dynamics. All the other estimates would allow the cut-off (electron radius) to decrease at least with $\delta < \frac{1}{4}$, even with the rough law of large number estimates used here. Hence, it seems likely that the width of the cut-off could be significantly decreased by a more detailed analysis of the microscopic dynamics, in particular the so-called "acceleration" or "radiation" component of the electromagnetic field.

There are other ways in which our approximation result for the Vlasov-Maxwell system could be improved. In particular, one would like to get rid of assumption (5.57) – the uniform bound on the charge density for the sequence of solutions to the regularized Vlasov-Maxwell equation – and replace it with a condition on f_0, preferably one that can be easily checked. However, such a condition would likely have to come out of the existence theory for Vlasov-Maxwell. The same might be true with respect to a possible extension of the results to a larger class of initial data.

On a different note, it might be interesting to include rotational degrees of freedom and study the rigid charges model with spin. Moreover, it would be interesting to see whether the methods employed here for the Vlasov-Maxwell system can also be applied, in an appropriate sense, to the Vlasov-Einstein equations. A first step in that direction was already made by Elskens, Kiessling and Ricci [16], who studied a relativistic version of the gravitational Vlasov-Poisson system coupled to a linear wave equation.

[1] Empirically, experiments currently put the upper bound on the electron radius to $10^{-22}m$ [12].

6.2 Vlasov-Maxwell: A note on the existence theory

Concerning the existence of classical solutions to the Vlasov-Maxwell systems, the efforts in recent years have largely focused on proving sufficient conditions for the result of Glassey and Strauss ([20], see our Thm. 5.4.1). Most recently, Pallard [52] proved that for compact initial data, singularity formation can only happen in finite time T, if $\lim\limits_{t\to T^-} \|\rho(t)\|_6 = +\infty$. Unfortunately, this is still far away from the best known a priori bound on the charge density, which is $\|\rho(t)\|_{4/3} \leq C$, for a constant C depending on initial data. This estimate comes from the conservation of the energy

$$\varepsilon(t) := \int\int \sqrt{1+|\xi|^2} f(t,x,\xi)\mathrm{d}x\mathrm{d}\xi + \frac{1}{2}\int |E(t,x)|^2 + |B(t,x)|^2\mathrm{d}x, \tag{6.1}$$

more precisely, from the resulting upper bound in the kinetic energy term. From a physical point of view, it would seem that the relevant bounds on singularity formation should come from the potential / field energy rather than the kinetic energy. However, to my knowledge, so far no one has been able to extract valuable estimates from the L^2-bounds on E and B.

In fact, the following observation might indicate that no satisfying results are to be expected. Let's consider, as a rough estimate, a stationary, spherically symmetric charge distribution with

$$\rho(x) = \rho(|x|) = \begin{cases} |x|^{-\beta}, & |x| < R_1 \\ 0, & |x| > R_2 \end{cases},$$

where $0 < R_1 < R_2 < \infty$ and $\beta < 3$. According to Gauß' law, the Coulomb field of this charge distribution is then given by (with $r = |x|$):

$$E(r) \sim r^{-2}\int_0^r r'^2 \rho(r')\,\mathrm{d}r' = r^{1-\beta}, \quad \text{for } r < R_1 \tag{6.2}$$

and $E(r) \sim \frac{1}{r^2}$ for $r > R_2$. The corresponding field energy is $\|E\|_2^2 \approx \int_0^{R_1} r^2 r^{2(1-\beta)}\mathrm{d}r + Const$. Now the integral is finite if and only if $\beta < \frac{2}{5}$. But with $\beta < \frac{2}{5}$ we have $\rho \in L^p(\mathbb{R}^3)$ if and only if $p \leq \frac{6}{5}$. In other words, an upper bound on $\|E\|_2$ does not preclude a singularity for which the L^p norms of ρ are already infinite for $p > \frac{6}{5}$.

Of course, this electrostatic situation does not correspond to a consistent solution of a Vlasov-Maxwell equations and it says nothing about the *formation* of singularities. However, it might suggest that the upper bound on the field energy is unlikely to produce stronger a priori estimates for the charge density.

6.3 Vlasov-Poisson: Comparison of recent results

Let us now turn to the Vlasov-Poisson system, for which the difficulties are of less fundamental nature. In this thesis, we have presented two alternative approximations of the Vlasov-Poisson equation – one based on the method of Boers and Pickl and a regularization of the force (Chapter 3) and one based on the stability result of Loeper and a smearing of the charge density (Chapter 4). It is thus interesting to compare both approaches with each other as well as with the results of Hauray and Jabin [26] that must be viewed as the reference for mean field limits with singular forces. Of course, we have to emphasize that the results of Hauray and Jabin do not include the Coulomb singularity, which is the main focus of our work, while our work does not include results without cut-off, which is the main focus of theirs. Nevertheless, it is instructive to compare the various approaches and techniques, in particular with regard to possible future improvements. In this spirit, we want to highlight some important similarities and differences.

The role of probability. What all recent results have in common, is that they are probabilistic in the sense that the mean field limit can be performed for *typical* initial conditions. As mentioned in the introduction, this is in contrast to the classical results of Braun-Hepp and Dobrushin, which are, in effect, deterministic, allowing arbitrary sequences of initial configurations approximating a macroscopic profile f_0. The reason is that for unbounded forces, there exist "bad" initial conditions leading to clustering and/or strong correlations between the particles and thus to significant deviations from the typical mean field behavior.

The strategy employed in [26] – as well as in our proof from Chapter 4 – is thus to impose additional constraints on the initial conditions, subsequently showing that these constraints are satisfied with probability 1 in the limit $N \to \infty$. In [26], the necessary bounds are imposed on the concentration of particles at $t = 0$, while in our proof, the probabilistic element enters through the requirement of a sufficiently fast convergence of the initial distribution. In any case, these assumptions assure that at $t = 0$, the particles are "well-placed" so to speak, preventing, in particular, a blow-up of the microscopic dynamics.

In contrast, the method introduced in [6] and refined in Chapter 3 is designed for stochastic initial conditions. The relevant quantity to control is a stochastic process on the N-particle phase-space, rather than distributions pertaining to the description of an individual system. Indeed, recognizing the need for a probability result, it is tempting to work with the N-particle distribution F_t^N defined in Section 1.3 rather than empirical densities $\mu_t^N[Z]$. In the past, this has usually lead to the study of the BBKGY hierarchy which, however, has not produced particularly strong results for the mean field scaling (see [63], for instance). The method of Boers and Pickl – while also taking the ensemble point of view – seems to be more flexible and more powerful in the mean field context.

The quantities to control. An interesting distinction between the three methods lies in the way they control the difference between mean field dynamics and microscopic dynamics. [26] uses the infinite Wasserstein distance. As the authors explain:

> "The use of the infinite MKW distance is important. We were not able to perform our calculations with other MKW distances of order $p < +\infty$ as the infinite distance is the only MKW distance with which we can handle a localized singularity in the force and Dirac masses in the empirical distribution." [26, p.17]

This is in contrast to the situation in Chapter 4 of this thesis, where we could use the more common and much weaker Wasserstein distance of order 2. The reason is that we apply the microscopic regularization on the level of the charge density, so that we deal with bounded densities rather than Dirac masses.

The method used in Chapter 3 takes the opposite approach, so to speak. Instead of smearing the microscopic density, we approximate the Vlasov density by singular measures by sampling the mean field flow along random initial conditions. Controlling the difference between microscopic dynamics and mean field dynamics then comes down to controlling the distance between two sets of particle trajectories rather than two probability measures, which allows for relatively strong estimates.

Admissible initial distributions. The result of Hauray and Jabin requires f_0 with compact support. At least in dimension 3, both of our results for the Vlasov-Poisson equation allows a significantly larger class of initial data which includes physically relevant examples such as Boltzmann distributions.

The size of cut-off. The three results differ significantly with respect to the scale of the required microscopic cut-off. The comparison has to be taken with a grain of salt, since the results in [26] do not include the Coulomb case, while our result from Chapter 4 would have to be adapted to singularities weaker than Coulomb. However, if we consider inverse power laws of order α, i.e. $k_\alpha(x) = \pm \frac{x}{|x|^{\alpha+1}}$, $x \in \mathbb{R}^d$ molecular chaos can be proven with a cut-off of order $N^{-\delta}$ for any $\delta < \kappa$, where

- $\kappa \to \frac{1}{2d}$, as $\alpha \nearrow 2$ with the method of Hauray and Jabin [26]
- $\kappa = \frac{1}{d}$, for $\alpha = 2$ with for the method of Boers and Pickl (Ch.3)
- $\kappa = \frac{1}{d(d+2)}$, for $\alpha = 2$ with the method of smeared charges (Ch.4)

Moreover, we note that the last two results hold in dimension $d \geq 2$ while the result in [26] requires $d \geq 3$.

The method of smeared charges, presented in Chapter 4, is arguably the simplest one. It avoids any detailed analysis of the microscopic dynamics by propagating the L^∞-bound (3.21) on the microscopic charge density with $W_2(\mu_t^N, f_t)$. Similar estimates can be used to carry over stronger regularity properties from the Vlasov density to the regularized microscopic density. The price for this simplification is a relatively large cut-off, in particular in higher dimensions. Moreover, we observe that there is no immediate connection between the size of the required cut-off and the strength of the singularity. This is in contrast to the situation in [26] and [6], where the lower bound on the cut-off decreases with α.

Turning to the result of Chapter 3, a cut-off of order $\sim N^{-\frac{1}{3}}$ is already quite satisfying, as this corresponds to the scale of the average distance between two neighboring particles. In other words, while a particle interacts with $N-1$ other particles at any given time, the number of interactions affected by the cut-off is typically of order 1. One reason for the relative strength of the result – as far as the size of the cut-off is concerned – is that all non-trivial estimates take place in d-dimensional physical space, rather than $2d$-dimensional (p, q)-space.

Finally, in [26], the necessary cut-off for singularities near the Coulomb case is of order $N^{-\frac{1}{2d}}$, corresponding to the typical distance between two neighboring particle states in (p, q)-space.

Results without cut-off? Probably the more important result in the paper of Hauray and Jabin concerns weak singularities, for which molecular chaos is proven with no cut-off at all. For $\alpha < 1$, the authors are able to provide an explicit control of the minimal particle distance – in (p, q)-space, strictly speaking, while integrating the force over short time intervals. More precisely, they show that, for typical initial conditions,

$$\inf_{i \neq j} |(q_i, p_i) - (q_j, p_j)| \geq N^{-\gamma}, \;\; \gamma < \frac{\alpha + d}{2d} \tag{6.3}$$

which provides the necessary bound on close encounters to prove molecular chaos. If and how these results can be extended to $\alpha \geq 1$ is an open question.

Concerning our method developed in Chapter 4, there is probably much room for improvement as far as the size of the cut-off is concerned. However, it is clear that this particular method is by all means committed to a microscopic regularization, i.e. a smearing of the point-charges.

As far as the method of Boers and Pickl is concerned, the issue is a bit more subtle. While this approach is not a priori committed to a regularization, it seems unlikely that the cut-off can be removed completely – even for very weak singularities – without a more detailed analysis of the N-particle dynamics. So far, our handle on the microscopic trajectories comes merely from their closeness to the mean field trajectories: only those Z contribute to the growth of $\mathbb{E}_0(J_t^N)$ for which $|\Psi_{s,0}(Z) - \Phi_{s,0}(Z)|_\infty < N^{-\delta}$, $\forall 0 \leq s \leq t$. This defines a scale beyond which we have no control on close encounters of particles and the cut-off takes over. The method, however, is very flexible. In particular, it is possible to include additional quantities in the definition of J_t^N – for instance, something along the lines of the minimal particle distance considered in [26] – to get better control on the clustering of particles. Continued efforts along these lines seems like a promising project.

6.4 Related questions and remarks

It might be interesting to observe that all the results discussed here hold equally in the repulsive and the attractive (gravitational) case, while physical intuition would tell us that repulsive interaction might help to prevent close encounters and mitigate the influence of the singularity. The situation is quite similar with respect to the solution theory, where most results do not distinguish between Vlasov-Newton and Vlasov-Poisson. However, as already argued in [26, Section 6.2], individual two-particle interactions become so weak in the $\frac{1}{N}$-scaling that the difference between repulsive and attractive forces are relevant only at extremely short distances.

For instance, the potential energy bound for a repulsive potential $V(x) \sim \frac{1}{|x|^{\alpha-1}}$, $\alpha > 1$ yields a lower bound on the minimal distance between particles in physical space which is of order $\sim N^{-\frac{2}{\alpha-1}}$, i.e. $\sim N^{-2}$ in the 3-dimensional Coulomb case $\alpha = 2$. This is far beyond the scale of the cut-off in any of the available results and presumably even far beyond the bounds that could be obtained on purely probabilistic grounds. (For the free dynamics, the typicality bound on close encounters is of order $N^{-\frac{1}{d-2}}$, that is, for $\gamma > -\frac{1}{d-2}$, the probability of any two particles coming closer than $N^{-\gamma}$ over a compact time-interval goes to 0 for $N \to \infty$). Nevertheless, it will be interesting to see whether future improvements of the existing results can exploit the repulsive character of the dynamics.

Singularity formation in Newtonian gravity. In view of possible results without cut-off, there is of course a difference between attractive and repulsive interactions as far as the existence of the microscopic dynamics is concerned. In our discussions, this has not been an issue, since we always considered regularized dynamics on the microscopic level, for which the existence theory is standard. The same holds for singular forces (with $\alpha > 1$) in the repulsive case, provided they are generated by a potential. Since the energy conservation yields a bound on the minimal particle distance, the standard Picard-Lindelöff theory applies for every fixed N.

Concerning singularity formation in Newtonian gravity, the state of the art is summarized in the book of Saari [58], to whom we also owe many of the pertinent results. In particular, it is known that for the Newtonian N-body problem in 2 or 3 dimensions, initial conditions leading to a collision of two or more particles form a set of first Baire category and Lebesgue measure zero [56, 57]. However, there is also the possibility of non-collision singularities, where particles go off to infinity in finite time. These non-collision singularities are known to exist for $N \geq 5$ [74], but not for $N \leq 3$ [51]. Concerning their likelihood, what has been proven so far is that for $N = 4$, initial conditions leading to non-collision singularities (if they exist at all) form a set of first Baire category and Lebesgue measure zero. Saari conjectures that this holds true for all $N \geq 4$ [58, p. 221] and intuitively, it seems clear that only extremely conspiratorial behavior could lead to particles being accelerated to infinity in finite time. However, as far as I known, no rigorous proof has been given so far.

Bibliography

[1] M. Abraham. Prinzipien der Dynamik des Elektrons. *Annalen der Physik*, 315(1):105–179, 1903.

[2] A. A. Arsen'ev. Existence in the large of a weak solution of Vlasov's system of equations. *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki*, 15(1):136–147, 1975.

[3] J. Batt. N-particle approximation to the nonlinear Vlasov-Poisson system. *Nonlinear Analysis: Theory, Methods & Applications*, 47(3):1445 – 1456, 2001. Proceedings of the Third World Congress of Nonlinear Analysts.

[4] G. Bauer, D.-A. Deckert, and D. Dürr. Maxwell-Lorentz dynamics of rigid charges. *Communications in Partial Differential Equations*, 38(9):1519–1538, 2013.

[5] G. Bauer, D.-A. Deckert, and D. Dürr. On the existence of dynamics in Wheeler-Feynman electromagnetism. *Zeitschrift für angewandte Mathematik und Physik*, 64(4):1087–1124, 2013.

[6] N. Boers and P. Pickl. On mean field limits for dynamical systems. *Journal of Statistical Physics*, pages 1–16, 2015.

[7] E. Boissard. Simple bounds for the convergence of empirical and occupation measures in 1-Wasserstein distance. *Electronic Journal of Probability*, 16:2296–2333, 2011.

[8] F. Bolley, A. Guillin, and C. Villani. Quantitative concentration inequalities for empirical measures on non-compact spaces. *Probability Theory and Related Fields*, 137(3):541–593, 2007.

[9] F. Bouchut, F. Golse, and C. Pallard. Classical solutions and the Glassey-Strauss theorem for the 3d Vlasov-Maxwell system. *Archive for Rational Mechanics and Analysis*, 170(1):1–15, 2003.

[10] W. Braun and K. Hepp. The Vlasov dynamics and its fluctuations in the 1/N limit of interacting classical particles. *Communications in Mathematical Physics*, 56(2):101–113, 1977.

[11] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Communications on Pure and Applied Mathematics*, 44(4):375–417, 1991.

[12] H. Dehmelt. A single atomic particle forever floating at rest in free space: New value for electron radius. *Physica Scripta*, 1988(T22):102–110, 1988.

[13] S. Dereich, M. Scheutzow, and R. Schottstedt. Constructive quantization: approximation by empirical measures. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 49, pages 1183–1203. Institut Henri Poincaré, 2013.

[14] R. J. DiPerna and P. L. Lions. Global weak solutions of Vlasov-Maxwell systems. *Communications on Pure and Applied Mathematics*, 42(6):729–757, 1989.

[15] R. L. Dobrushin. Vlasov equations. *Functional Analysis and Its Applications*, 13(2):115–123, 1979.

[16] Y. Elskens, M.K.-H. Kiessling, and V. Ricci. The Vlasov limit for a system of particles which interact with a wave field. *Communications in Mathematical Physics*, 285(2):673–712, 2009.

[17] N. Fournier and A. Guillin. On the rate of convergence in wasserstein distance of the empirical measure. *Probability Theory and Related Fields*, 162:1–32, 2014.

[18] W. Gangbo and R.J. McCann. The geometry of optimal transportation. *Acta Mathematica*, 177(2):113–161, 1996.

[19] K. Ganguly, J. T. Lee, and Jr. Victory, H. D. On simulation methods for Vlasov-Poisson systems with particles initially asymptotically distributed. *SIAM Journal on Numerical Analysis*, 28(6):pp. 1574–1609, 1991.

[20] R. T. Glassey and W. A. Strauss. Singularity formation in a collisionless plasma could occur only at high velocities. *Archive for Rational Mechanics and Analysis*, 92(1):59–90, 1986.

[21] F. Golse. The mean-field limit for a regularized Vlasov-Maxwell dynamics. *Communications in Mathematical Physics*, 310(3):789–816, 2012.

[22] F. Golse. On the dynamics of large particle dystems in the mean field limit. arXiv:1301.5494 [math.AP], 2013.

[23] F.A. Grünbaum. Propagation of chaos for the Boltzmann equation. *Archive for Rational Mechanics and Analysis*, 42:323–345, 1971.

[24] M. Hauray. Wasserstein distances for vortices approximation of Euler-type equations. *Mathematical Models and Methods in Applied Sciences*, 19(8):1357–1384, 2009.

[25] M. Hauray and P.-E. Jabin. N particles approximation of the Vlasov equations with singular potential. *Archive for Rational Mechanics and Analysis*, 183(3):489–524, 2007.

[26] M. Hauray and P.-E. Jabin. Particles approximations of Vlasov equations with singular forces : Propagation of chaos. To appear in *Annales scientifiques de l'École Normale Supérieure*, 2013.

[27] D. Heggie and P. Hut. *The gravitational million-body problem*. Cambridge University Press, Cambridge, 2003.

[28] E. Horst. Global strong solutions of Vlasov's equation – necessary and sufficient conditions for their existence. *Banach Center Publications*, 19(1):143–153, 1987.

[29] E. Horst. *Global solutions of the relativistic Vlasov-Maxwell system of plasma physics*. Dissertationes Mathematicae. Instytut Matematyczny Polskiej Akademi Nauk, 1990.

[30] E. Horst. On the asymptotic growth of the solutions of the Vlasov-Poisson system. *Mathematical Methods in the Applied Sciences*, 16:75–85, 1993.

[31] O.E. Lanford III. Time evolution of large classical systems. In J. Moser, editor, *Dynamical Systems, Theory and Applications*, volume 38 of *Lecture Notes in Physics*, pages 1–111. Springer, Berlin, Heidelberg, 1975.

[32] P.-E. Jabin. A review of the mean field limits for Vlasov equations. `http://www2.cscamm.umd.edu/~jabin/review_MF.pdf`, 2014.

[33] J. H. Jeans. On the theory of star-streaming and the structure of the universe. *Monthly Notices of the Royal Astronomical Society*, 76(2):70–84, 1915.

[34] M. Kac. Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954-1955*, volume vol III, pages 171–197. University of California Press, 1956.

[35] M. K.-H. Kiessling. Microscopic derivations of Vlasov equations. *Communications in Nonlinear Science and Numerical Simulation*, 13(1):106 – 113, 2008. Vlasovia 2006: The Second International Workshop on the Theory and Applications of the Vlasov Equation.

[36] M. K.-H. Kiessling. The microscopic foundations of Vlasov theory for jellium-like Newtonian N-body systems. *Journal of Statistical Physics*, 155(6):1299–1328, 2014.

[37] A. Komech and H. Spohn. Long-time asymptotics for the coupled Maxwell-Lorentz equations. *Communications in Partial Differential Equations*, 25(3-4):559–584, 2000.

[38] L.D. Landau and E.M. Lifshitz. *Mechanics*, volume 1 of *Course of Theoretical Physics*. Butterworth-Heinemann, 1976.

[39] D. Lazarovici and P. Pickl. A mean-field limit for the Vlasov-Poisson system. Preprint: arXiv:1502.04608, 2015.

[40] D. Lazarovici and P. Reichert. Typicality, irreversibility and the status of macroscopic laws. *Erkenntnis*, 80(4):689–716, 2015.

[41] P.-L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Inventiones Mathematicae*, 105:415–430, 1991.

[42] G. Loeper. Uniqueness of the solution to the Vlasov-Poisson system with bounded density. *Journal de Mathématiques Pures et Appliquées*, 86:68–79, 2006.

[43] H.A. Lorentz. La théorie électromagnétique de Maxwell et son application aux corps mouvants. *Archives Néerlandaises des Sciences Exactes et Naturelles*, 25:363–552, 1892.

[44] H.A. Lorentz. Weiterbildung der Maxwell'schen Theorie: Elektronentheorie. In *Enzyklopädie der Mathematischen Wissenschaften*, volume 5, T.2, pages 145–280, 1904.

[45] S. Lyle. *Self-Force and Inertia: Old Light on New Ideas*, volume 796 of *Lecture Notes in Physics*. Springer, Berlin, 2010.

[46] A.J. Majda and A.L. Bertozzi. *Vorticity and Incompressible Flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[47] R. J. McCann. A convexity principle for interacting gases. *Advances in Mathematics*, 128(1):153–179, 1997.

[48] S. Mischler and M. Hauray. On Kac's chaos and related problems. *Journal of Functional Analysis*, 266(10):6055–6157, 2014.

[49] H. Neunzert. An introduction to the nonlinear Boltzmann-Vlasov equation. In C. Cercignani, editor, *Kinetic Theories and the Boltzmann Equation*, volume 1048 of *Lecture Notes in Mathematics*, pages 60–110. Springer, Berlin, Heidelberg, 1984.

[50] H. Neunzert and J. Wick. Die Approximation der Lösung von Integro-Differentialgleichungen durch endliche Punktmengen. In R. Ansorge and W. Törnig, editors, *Numerische Behandlung nichtlinearer Integrodifferential - und Differentialgleichungen*, volume 395 of *Lecture Notes in Mathematics*, pages 275–290. Springer, Berlin, Heidelberg, 1974.

[51] P. Painlevé. *Leçons sur la théorie analytique des équations différentielles*. A. Hermann, Paris, 1897.

[52] C. Pallard. A refined existence criterion for the relativistic Vlasov-Maxwell system. *Communications in Mathematical Sciences*, 13(2):347–354, 2015.

[53] K. Pfaffelmoser. Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. *Journal of Differential Equations*, 95(2):281–303, 1990.

[54] G. Rein. Selfgravitating systems in Newtonian theory - the Vlasov-Poisson system. In *Proceedings of the Minisemester on Mathematical Aspects of Theories of Gravitation*, number 41, pages 179–194. Banach Center Publications, 1997.

[55] G. Rein. Global weak solutions of the relativistic Vlasov-Maxwell system revisited. *Communications in Mathematical Sciences*, 2:145–158, 2004.

[56] D. G. Saari. Improbability of collisions in Newtonian gravitational systems. ii. *Transactions of the American Mathematical Society*, 181:351–368, 1973.

[57] D.G. Saari. Improbability of collisions in Newtonian gravitational systems. *Transactions of the American Mathematical Society*, 162:267–271, 1971.

[58] D.G. Saari. *Collisions, Rings, and Other Newtonian N-body Problems*. Number Nr. 104 in CBMS Regional Conference Series in Mathematics. American Mathematical Society, 2005.

[59] J. Schaeffer. Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. *Communications in Partial Differential Equations*, 16(8-9):1313–1335, 1991.

[60] J.M.I. Shatah and M. Struwe. *Geometric Wave Equations*. Courant lecture notes in mathematics. Courant Institute of Mathematical Sciences, 1998.

[61] A. Sommerfeld. Simplified deduction of the field and the forces of an electron mowing in any given way. *Koninklijke Nederlandse Akademie van Wetenschappen Amsterdam*, 7:346–367, 1904.

[62] R. Sospedra-Alfonso and R. Illner. Classical solvability of the relativistic Vlasov-Maxwell system with bounded spatial density. *Mathematical Methods in the Applied Sciences*, 33(6):751–757, 2010.

[63] H. Spohn. On the Vlasov hierarchy. *Mathematical Methods in the Applied Sciences*, 3(1):445–455, 1981.

[64] H. Spohn. *Large scale dynamics of interacting particles*. Springer, Berlin, Heidelberg, 1991.

[65] H. Spohn. *Dynamics of Charged Particles and their Radiation Field*. Cambridge University Press, Cambridge, 2004.

[66] A.-S. Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX – 1989*, volume 1464 of *Lecture Notes in Mathematics*, pages 165–251. Springer, Berlin, 1991.

[67] S. Ukai and T. Okabe. On classical solutions in the large in time of two-dimensional Vlasov's equation. *Osaka Journal of Mathematics*, 15(2):245–261, 1978.

[68] C. Villani. *Optimal Transport Old and New*, volume 338 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, 2009.

[69] A. A. Vlasov. On vibration properties of electron gas. *Journal of Experimental and Theoretical Physics*, 8(3):291, 1938.

[70] A. A. Vlasov. The vibrational properties of an electron gas. *Soviet Physics Uspekhi*, 10(6):721, 1968.

[71] J. A. Wheeler and R. P. Feynman. Interaction with the absorber as the mechanism of radiation. *Reviews of Modern Physics*, 17:157–181, 1945.

[72] J. A. Wheeler and R. P. Feynman. Classical electrodynamics in terms of direct interparticle action. *Reviews of Modern Physics*, 21:425–433, 1949.

[73] S. Wollman. Global-in-time solutions of the two-dimensional Vlasov-Poisson systems. *Communications on Pure and Applied Mathematics*, 33(2):173–197, 1980.

[74] Z. Xia. The existence of noncollision singularities in Newtonian systems. *Annals of Mathematics*, 135(3):411–468, 1992.