Spectral and Dynamical Properties of Certain Quantum Hamiltonians in Dimension Two

Inauguraldissertation
zur Erlangung des akademischen Grades Dr. rer. nat.
an der Fakultät für Mathematik, Informatik und Statistik
der Ludwig-Maximilians-Universität München

vorgelegt von
Josef Georg Mehringer
aus Gmund am Tegernsee

am 20. Juli 2015

1.Gutachter: Prof. Dr. Edgardo Stockmeyer (PUC Chile)
2.Gutachter: Prof. Dr. Peter Müller (LMU München)
3.Gutachter: Prof. Søren Fournais, PhD (Aarhus University)

Tag der Disputation: 19. Oktober 2015

Zusammenfassung

Nachdem 2004 zum ersten Mal Graphen-Flocken isoliert werden konnten, hat sich das Interesse an der Quantenmechanik flacher Systeme merklich verstärkt. In Graphen, also einer einzigen Schicht von Kohlenstoffatomen, welche in einer regulären hexagonalen Gitterstruktur angeordnet sind, wird die Dynamik von Ladungsträgern nahe der Bandkante durch den masselosen Dirac-Operator in Dimension Zwei beschrieben.

Wir untersuchen das Spektrum des zweidimensionalen masselosen Dirac-Operators H_D, der an ein externes elektromagnetisches Feld gekoppelt ist. Genauer gesagt liegt unser Fokus auf der Charakterisierung des Spektrums $\sigma(H_D)$ für Feldstrukturen, die durch unbeschränkte elektrische und magnetische Potentiale beschrieben werden. Wir beobachten dabei, dass die Existenz von Lücken in $\sigma(H_D)$ abhängig ist vom Quotienten V^2/B im Unendlichen, also vom Verhältnis des elektrischen Potentials V zum magnetischen Feld B. Insbesondere erhalten wir eine scharfe Schranke an V^2/B, unterhalb welcher $\sigma(H_D)$ rein diskret ist. Darüberhinaus zeigen wir im Falle der Unbeschränktheit von V^2/B im Unendlichen, dass H_D für eine große Klasse von Feldern B und Potentialen V keine spektralen Lücken aufweist. Letzteres führt zu Beispielen von zweidimensionalen masselosen Dirac-Operatoren mit dichtem reinem Punktspektrum. Wir erweitern die Ideen, entwickelt für H_D, auf den klassischen Pauli-(und magnetischen Schrödinger-)Operator in Dimension Zwei. Es stellt sich heraus, dass auch solche nichtrelativistische Operatoren mit einem stark repulsiven Potential Bedingungen an B und V zulassen, unter denen Lücken im Spektrum auftreten oder nicht. Ähnlich wie im Falle des Dirac-Operators können wir zeigen, dass dort im Allgemeinen keine Lücken existieren, falls $|V|$ das Magnetfeld B im Unendlichen dominiert. Damit ist es insbesondere möglich, das Spektrum des Pauli-(und magnetischen Schrödinger-)Operators für einige grundlegende, rotationssymmetrische Felder komplett zu charakterisieren.

Betrachten wir beim Dirac-Operator H_D den Bereich eines wachsenden Quotienten V^2/B, so ereignet sich dort ein Übergang von Punktspektrum zu kontinuierlichem Spektrum. Da ein solches Phänomen insbesondere hinsichtlich der Dynamik interessant ist, behandeln wir in einem zweiten Teil dieser Arbeit die Frage, unter welchen Bedingungen ein ballistisches Ausbreiten von Wellenpaketen bei zweidimensionalen Dirac-Systemen möglich ist. Um die Fragestellung zu präzisieren: Reichen Aussagen über die Art des Spektrums aus, um festzustellen, ob das zeitliche Mittel

$$\frac{1}{T}\int_0^T \langle \psi(t), |\mathbf{x}|^2 \psi(t)\rangle \, \mathrm{d}t$$

sich wie T^2 verhält? Hierbei bezeichnet $\psi(t)$ die Zeitentwicklung eines Zustandes ψ unter dem Operator H_D. Wir können dies positiv beantworten, zumindest unter gewissen Symmetriebedingungen an das elektromagn. Feld.

Abstract

After 2004, when it was possible for the first time to isolate graphene flakes, the interest in quantum mechanics of plain systems has been intensified significantly. In graphene, that is a single layer of carbon atoms aligned in a regular hexagonal structure, the generator of dynamics near the band edge is the massless Dirac operator in dimension two.

We investigate the spectrum of the two-dimensional massless Dirac operator H_D coupled to an external electro-magnetic field. More precisely, our focus lies on the characterisation of the spectrum $\sigma(H_D)$ for field configurations that are generated by unbounded electric and magnetic potentials. We observe that the existence of gaps in $\sigma(H_D)$ depends on the ratio V^2/B at infinity, which is a ratio of the electric potential V and the magnetic field B. In particular, a sharp bound on V^2/B is given, below which $\sigma(H_D)$ is purely discrete. Further, we show that if the ratio V^2/B is unbounded at infinity, H_D has no spectral gaps for a huge class of fields B and potentials V. The latter statement leads to examples of two-dimensional massless Dirac operators with dense pure point spectrum. We extend the ideas, developed for H_D, to the classical Pauli (and the magnetic Schrödinger) operator in dimension two. It turns out that also such non-relativistic operators with a strong repulsive potential do admit criteria for spectral gaps in terms of B and V. Similarly as in the case of the Dirac operator, we show that those gaps do not occur in general if $|V|$ is dominating B at infinity. It should be mentioned that this leads to a complete characterisation of the spectrum of certain Pauli (and Schrödinger) operators with very elementary, rotationally symmetric field configurations.

Considering for the Dirac operator H_D the regime of a growing ratio V^2/B, there happens a transition from pure point to continuous spectrum. A phenomenon that is particularly interesting from the dynamical point of view. Therefore, we address in a second part of the thesis the question under which spectral conditions ballistic wave package spreading in two-dimensional Dirac systems is possible. To be more explicit, we study the following problem: Do statements on the spectral type of H_D already suffice to decide whether the time mean of the expectation value

$$\frac{1}{T}\int_0^T \langle \psi(t), |\mathbf{x}|^2 \psi(t) \rangle \, \mathrm{d}t$$

behaves like T^2? Here $\psi(t)$ denotes the time evolution of a state ψ under the corresponding Dirac operator. We can answer that question affirmatively, at least for certain electro-magnetic fields with symmetry.

Contents

Chapter 1

Introduction and Motivation

Originally, Paul Dirac introduced in [Dir28] his famous equation for describing free electrons consistently with the special theory of relativity, which had been established 23 years before. As in the Schrödinger theory this advanced equation ought to be of first order in time t, and therefore, by the equivalence principle, be also linear in the momentum operators p_1, p_2, p_3. In addition, it should concord with the relativistic energy-momentum relation

$$E^2(\mathbf{p}) = c^2\mathbf{p}^2 + m^2c^4$$

for a particle with mass m via first quantisation. Dirac saw that this was possible if one requires the wave function to be a four-component vector on which a matrix-valued wave equation acts. His observation resulted in the Dirac equation

$$-\mathrm{i}\,\partial_t\psi(\,\cdot\,,t) = D_0\psi(\,\cdot\,,t).$$

Here D_0 denotes the Dirac operator

$$D_0 = c\boldsymbol{\alpha}\cdot(-\mathrm{i}\,\nabla) + mc^2\beta$$

in dimension three, where c is the speed of light (often set to 1). For the $4\times 4-$Dirac matrices $\alpha_1, \alpha_2, \alpha_3, \beta$ we choose the representation

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i \in \{1,2,3\}, \qquad \beta = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix}$$

in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The four-component character of the wave function includes naturally inner degrees of freedom for the electron like the spin, which was introduced years before because of experimental phenomena. The immediate success of the theory might be traced back to the fact that the eigenvalue problem

$(D_0 + V)\psi = E\psi$, applied for the Coulomb potential V, leads directly to the fine structure of the spectrum of the hydrogen atom. This fine structure had been explained before by a more ad-hoc implemented spin-orbit coupling. As pointed out in [Dir28], such an interaction of the spin with the angular momentum is naturally incorporated in the matrix structure of D_0. Surprisingly, besides the spin, Dirac's ansatz comprises a further degree of freedom of an electron: In view of the symmetry of the spectrum of D_0 with respect to 0, the equation describes also particles with the same mass as electrons, but negative kinetic energy. In particular, this energy can be arbitrarily low. Initially, in [Dir28], Dirac disregarded these solutions as a bare mathematical artefact, than ascribing them a physical meaning. However, soon after the introduction of the equation this was seen as a problem since such an operator would model a particle that would become faster and faster by lowering its energy. As discussed in [Dir30], one can resolve this discrepancy between that property of D_0 and the classical quantum mechanical picture by arguing that the states assigned to the negative part of the spectrum are occupied. Then the Pauli principle for fermions would prohibit that electrons can accelerate by radiating energy. Implementing such a virtual or hidden "sea" of electrons[1] (the so-called Dirac sea) implies that an occupied state with negative energy can also get excited, leaving a hole. Such an unoccupied state should be perceptible by the absence of a negative charge. Although Dirac was hesitating[2] to postulate a new particle (as a missing electron of the sea), already four years later a "positive electron" was discovered, the so-called positron, the first anti-particle. Hence, the introduction of this matrix-valued differential equation can surely be seen as the birth of modern particle physics and it is still the corner stone for the description of fermions in quantum field theory. Subsequently, also mathematicians started to investigate properties of Dirac's operator, which turned out to be a powerful tool for tackling problems in pure mathematics[3].

While the interpretation of positrons as missing electrons of the virtual Dirac sea might be quite abstract, a similar concept in solid state physics is crucial for describing electronic properties in condensed matter. In solids, the energy levels of single atoms form energy bands that are usually separated by band gaps. At temperature zero, states of the bands are occupied by atomic electrons up to a certain energy, called the Fermi energy. The position of this Fermi energy relatively to those of the bands and gaps is decisive for the electronic features of the material and responsible for its classification as conductor, semi-conductor or isolator. By absorbing energy, an electron can occupy an energy level above this Fermi energy, now contributing to electronic transport, while it leaves behind an unoccupied site, considered as a hole. Such holes can, in analogy to those of the abstract

[1]Or more general a "sea" of fermions.

[2]He initially interpreted such a hole as a proton.

[3]A very well known example is the Atiyah-Singer index theory in global analysis.

Dirac sea, be seen as positive charge carriers. However, the description of a filled virtual sea of electrons used by Dirac, as well as the union of occupied states up to the Fermi energy in solid state physics (also called Fermi sea) remains problematic: Introducing a sea of electrons, i.e. a background source of particles, out of which one can generate the "relevant" charge carriers requires a mathematical language able to deal with an indefinite number of particles (and a corresponding indefinite number of holes in the background source). Since operators in classical quantum mechanics model systems with a fixed particle number, a further quantisation was introduced, the so-called second quantisation. This formalism is based on the construction of a Hilbert space, where one doesn't consider the indistinguishable particles itself, but merely the occupation numbers of certain quantum states, e.g. energy levels $\{E_n\}_{n\in\mathbb{N}}$. This Hilbert space is generated out of a single "vacuum" $|0\rangle$ by a set of creation and annihilation operators $\hat{a}_n^\dagger$ and $\hat{a}_n$ associated to the quantum number $n \in \mathbb{N}$, which are defined through their property to increase or decrease the particle number of the $n-$th state by one. In that framework the Dirac or Fermi sea is naturally incorporated as a state

$$|F\rangle = \prod_{E_n \leq E_F} \hat{a}_n^\dagger \, |0\rangle,$$

generated out of the vacuum by applying creation operators up to a zero point or Fermi energy E_F. Applying further creation operators $\hat{a}_n^\dagger$ on $|F\rangle$, with $E_n > E_F$, then results in the occupation of free electron states, while the annihilation operators $\hat{a}_n$, with $E_n \leq E_F$, create holes in the sea. Within that language it is also convenient to introduce positrons (or more general antiparticles) as proper particles by performing a particle-hole transformation, i.e. by redefining the creation/annihilation operators as well as the vacuum and the vacuum energy[4]. How important the concept of a background source of particles and the second quantisation in solid states physics is was highlighted in [BCS57]. In that theory of superconductivity one uses the creation of electron pairs out of a Fermi sea in order to describe the energetic favourability of a superconducting state. Also many other problems in condensed matter physics are treated nowadays in the language of second quantisation, as for example the tight-binding approximation for determining the band structure of a solid where the atoms are arranged in a periodic structure.

The tight-binding method for lattice problems assumes that the eigenstates of the corresponding Hamiltonian are strongly localised around the lattice atoms, hence "perceive" barely the neighbouring atoms. From the mathematical point of view, this means that the second quantised Hamiltonian of the lattice electrons has a nearly diagonal representation if one writes it in terms of creation and annihilation operators of the lattice sites $1, \ldots, N$.

[4] See Chapter 5 of [GR96] or Chapter 2 of [AS10] for a detailed explanation.

More precisely, suppressing spin and band indices, the Hamiltonian reads

$$\hat{H}_{\mathrm{tb}} = \sum_{i,j=1}^{N} \hat{c}^{\dagger}_{\mathbf{r}_i} t_{i,j} \hat{c}_{\mathbf{r}_j} \, .$$

Here the overlap or hopping coefficients $t_{i,j}$ are only non-zero for $i = j$, or if the lattice sites $\mathbf{r}_i$ and $\mathbf{r}_j$ are next neighbours. It means that the interaction between two states, localised in $\mathbf{r}_i$ and $\mathbf{r}_j$, is neglected if they are not close to each other. This extremely simplified ansatz for the description of periodic solids leads to energy-momentum relations, which depend only on the arrangement of the atoms. One of the exactly solvable plain lattice structures is the honeycomb lattice, first discussed in [Wal47]. This atomic arrangement has drawn great attention in the last ten years thanks to the discovery of isolated graphene flakes[5]. In the honeycomb lattice every atom has three nearest neighbours, whose directions differ by a rotational angle of 120° (see Figure 1). In order to write the Hamiltonian $\hat{H}_{\mathrm{tb}}$ in momentum space one needs a Bravais lattice, i.e. a periodic arrangement of points, which corresponds to a discrete group of translations that is generated itself (as a $\mathbb{Z}-$module) by two primitive vectors.

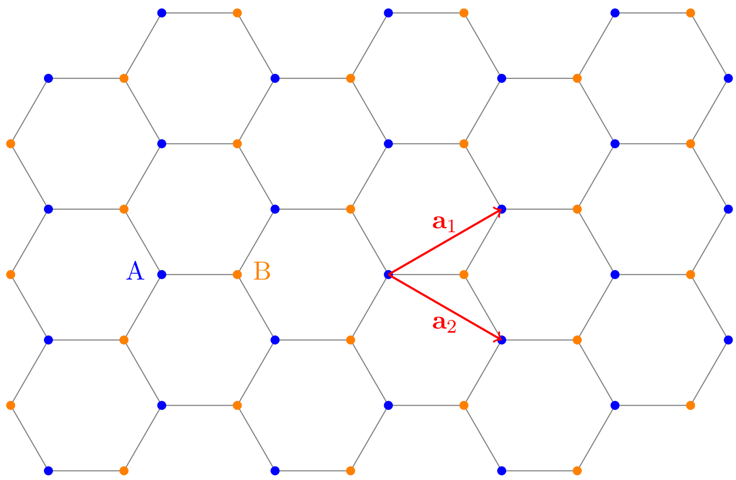

Figure 1: Honeycomb array with primitive vectors $\mathbf{a}_1, \mathbf{a}_2$

Since the honeycomb structure is not of Bravais type, one has to use a little gimmick: The arrangement can be divided in two sub-lattices, consisting of atoms A and of atoms B, such that pairs of atoms A-B form the basis of a Bravais lattice with primitive vectors $\mathbf{a}_1, \mathbf{a}_2$ (see Figure 1). Consequently, the creation and annihilation operators have to be considered as vector-valued with two components, that means

[5]First observations of electric properties of this flakes are described in [NGM$^+$04].

$$\hat{c}_{\mathbf{R}_i} = \left(\hat{a}^\dagger_{\mathbf{R}_i}\,,\,\hat{b}^\dagger_{\mathbf{R}_i}\right) \qquad \text{and} \qquad \hat{c}_{\mathbf{R}_i} = \begin{pmatrix} \hat{a}_{\mathbf{R}_i} \\ \hat{b}_{\mathbf{R}_i} \end{pmatrix},$$

where $\mathbf{R}_i$ denote the site of the $i-$th basis. In addition, the hopping elements $t_{i,j}$ then become $2 \times 2-$matrices. This two-component character of the basis implies an inner degree of freedom, like the spin of an electron, and is therefore called pseudo-spin. Keeping in mind that each atom A and each atom B is of the same sort, one assumes that the inner-atomic interactions, corresponding to the diagonal entries of $t_{i,j}$, have the same value (which has the meaning of a zero point energy ϵ_0). With a suitable Fourier transform (see Appendix A), we obtain the tight-binding Hamiltonian in momentum space

$$\hat{H}_{\text{tb}} \;=\; \sum_{\mathbf{k}\in\mathcal{V}_B} \left(\hat{a}^\dagger_{\mathbf{k}}\,,\,\hat{b}^\dagger_{\mathbf{k}}\right) \begin{pmatrix} \epsilon_0 & \zeta(\mathbf{k}) \\ \zeta^*(\mathbf{k}) & \epsilon_0 \end{pmatrix} \begin{pmatrix} \hat{a}_{\mathbf{k}} \\ \hat{b}_{\mathbf{k}} \end{pmatrix},$$

where the summation runs over the first Brillouin zone $\mathcal{V}_B$ and $\zeta(\mathbf{k})$ expresses the energy-momentum relation. Within the first Brillouin zone there are two non-equivalent roots $\mathbf{K}, \mathbf{K}'$ of the function ζ, which correspond to points where the two energy bands $E_\pm(\mathbf{k}) = \epsilon_0 \pm |\zeta(\mathbf{k})|$ meet each other. For momenta $\mathbf{k}$ near one of these roots, i.e. for $|\mathbf{k} - \mathbf{K}|$ small, one can expand ζ and neglect second order terms. This linearisation around the the points $\mathbf{K}, \mathbf{K}'$ results in a Hamiltonian, which describes lattice electrons with momenta near the band edge. Up to unitary equivalence, it has the form of the two-dimensional massless Dirac Hamiltonian

$$\hat{D}_0 = \sum_{\mathbf{k}\in\mathcal{V}_B} \hat{\Psi}^\dagger_{\mathbf{k}} \big(v_F \boldsymbol{\sigma}\cdot\mathbf{k} + \epsilon_0\big) \hat{\Psi}_{\mathbf{k}}$$

in the second quantised form. The positive real constant v_F is called the Fermi velocity and $\boldsymbol{\sigma}$ denotes the matrix-valued vector (σ_1, σ_2). The two components of $\hat{\Psi}_{\mathbf{k}}$ represent in that case not the spin degree of freedom, but the one due to the degeneration of the basis. Although this relation between the tight-binding Hamiltonian of the honeycomb lattice and the Dirac operator is known since the 1960s, a mathematically rigorous statement about the approximation was established only recently in [FW12] and in [FW14]. Using Floquet-Bloch theory for the ordinary two-dimensional Schrödinger operator $-\Delta + \epsilon V_h$ with an arbitrary[6] honeycomb-periodic potential V_h, there it was shown that for almost every $\epsilon \in \mathbb{R}$ the desired double-cone-like behaviour for the two band functions $E_\pm(\mathbf{k})$ holds around two points $\mathbf{K}, \mathbf{K}'$ within the Brioullin zone[7]. In addition, in [FW14] the authors demonstrated that in the case of a double-cone-like behaviour near the point $\mathbf{K}$, the following Dirac-like description of the time evolution is valid: For $\mathbf{K}$ one has

[6] With some constraints on the regularity and on a Fourier component of V_h.
[7] The precise statement is given in Theorem 5.1 of [FW12].

two linear independent Bloch states[8] Φ_1, Φ_2 associated to the eigenenergy $E_0(\mathbf{K})$, i.e. two linear independent solutions of the eigenvalue problem

$$\big(-\Delta + V_h\big)\Phi \;=\; E_0(\mathbf{K})\Phi\,, \qquad \Phi(\,\cdot\,+\mathbf{v}) = e^{\mathrm{i}\,\mathbf{K}\cdot\mathbf{v}}\Phi(\,\cdot\,) \;\text{ for all } \mathbf{v}\in\Lambda\,,$$

where $\Lambda = \mathbb{Z}\mathbf{a}_1 \oplus \mathbb{Z}\mathbf{a}_2$ denotes the $\mathbb{Z}$-module generated by the lattice vectors $\mathbf{a}_1, \mathbf{a}_2$. Since Φ_1 and Φ_2 span the space of Floquet-Bloch eigenstates associated to the energy $E_0(\mathbf{K})$ and the point $\mathbf{K}$ of the first Brioullin zone, one can consider a state $\phi \in L^2(\mathbb{R}^2,\mathbb{C})$ to be spectrally localised around $\mathbf{K}$ if it has the form $\phi = \sum_{j=1,2}\alpha_j\Phi_j$ with coefficients $\alpha_j \in \mathcal{S}(\mathbb{R}^2,\mathbb{C})$ of Schwartz type. Such a wave package has a Dirac-like behaviour in the sense that for δ sufficiently small the rescaled state

$$\phi^\delta(\mathbf{x}) = \sum_{j=1,2}\delta\alpha_j(\delta\mathbf{x})\Phi_j(\mathbf{x})$$

has approximately the time evolution

$$\phi^\delta(\,\cdot\,,t) := e^{-\mathrm{i}\,t(-\Delta+V_h)}\phi^\delta \;\approx\; e^{-\mathrm{i}\,tE_0(\mathbf{K})}\sum_{j=1,2}\delta\alpha_j(\delta\,\cdot\,,\delta t)\Phi_j(\,\cdot\,). \tag{1.1}$$

Here $\boldsymbol{\alpha}(\,\cdot\,,t) = (\alpha_1,\alpha_2)(\,\cdot\,,t)$ denotes the time evolution of the initial coefficients $\boldsymbol{\alpha} = (\alpha_1,\alpha_2) \in \mathcal{S}(\mathbb{R}^2,\mathbb{C}^2)$ under the two-dimensional massless Dirac operator

$$D_{0,\theta} = e^{-\mathrm{i}\,\theta/2\,\sigma_3}\,v\big[\sigma_1(-\mathrm{i}\,\partial_1) - \sigma_2(-\mathrm{i}\,\partial_2)\big]e^{\mathrm{i}\,\theta/2\,\sigma_3}$$

with a fixed phase $\theta \in [0,2\pi)$ and propagation speed $v > 0$. The approximation (1.1) is good for times of order $0 \le t \le \mathcal{O}(\delta^{-2+\epsilon})$ (with $\epsilon > 0$ arbitrarily small) in the sense that one has, for times $0 \le t \le \mathcal{O}(\delta^{-2+\epsilon})$, uniform Sobolev-bounds of order $\mathcal{O}(\delta^s)$ (for some $s > 0$) on the error term. Those ultimate results on a mathematically rigorous derivation of the Dirac-like behaviour of initial states, spectrally localized at the Fermi edge $E_0(\mathbf{K})$, so far only give a description of the charge carriers without any additional external electro-magnetic field. However, for the evidence of isolated graphene flakes the unusual phenomena of the charge carriers in presence of external fields were crucial.[9]

One remarkable feature of a particle that is described by the Dirac operator, moving under an external electric potential, is pointed out in [Kle29]. Essentially Klein observed that for Dirac particles a potential well V has in general no confining effect like in the classical case (i.e. in the Schrödinger theory). More explicitly, consider the one-dimensional Dirac operator with a step potential

$$h_{\text{well}} = \sigma_1(-\mathrm{i}\,\partial_x) + mc^2\sigma_3 + v_0\big(1 - \mathbb{1}_{[-x_0,x_0]}\big),$$

[8] They correspond to the degeneracy of the basis.

[9] As one can read off the observations in [NGM+04].

with positive constants m, v_0, x_0 and the characteristic function $\mathbb{1}_{[-x_0,x_0]}$ on the interval $[-x_0, x_0]$. Then the spectrum of h_{well} includes the union $(-\infty, v_0 - mc^2) \cup (v_0 + mc^2, \infty)$, on which it is purely absolutely continuous[10]. In particular, there are no bound states for energies $E \in (0, v_0 - mc^2)$, independent of how large $v_0 > mc^2$ might be. In the case $m = 0$ there are no bound states at all and the spectrum of h_{well} is purely absolutely continuous, which is obvious since h_{well} is then unitarily equivalent to the free operator $\sigma_1(-\mathrm{i}\,\partial_x)$ via the unitary map

$$U = \exp\left(\mathrm{i}\,\sigma_1 \int_0^{\cdot} v(s)\mathrm{d}s\right).$$

Here $v = v_0(1 - \mathbb{1}_{[-x_0,x_0]})$ denotes the potential function. Similarly, a classical trapping potential v, i.e. a potential with $v(x) \to \infty$ as $|x| \to \infty$, has no confining effect on a Dirac operator on the line or half-line (even with mass term). Indeed, various mathematical works as [Tit61], [Erd63] and [Sch97] discussed conditions for such trapping potentials, under which the spectrum of the one-dimensional Dirac operator is still purely absolutely continuous (and covers the whole real line). For a higher-dimensional Dirac operator D_0 coupled to a trapping potential V, a decomposition into one-dimensional operators can be used (if V satisfies certain symmetry, see Subsection 2.2.3) to deduce the continuity of the spectrum also in higher-dimensional cases. Even without such symmetries and a corresponding reduction, the absence of eigenvalues in the spectrum of $D_0 + V$ has been verified for a large class of trapping potentials in [Vog87] and [KŌY03] (treating only dimension 3).

Besides the effects of purely electric fields on charge carriers, a key evidence for the detection of single layer graphene was a peculiar quantum Hall effect (as highlighted in [NJZ+07]), induced by a magnetic field $B : \mathbb{R}^2 \to \mathbb{R}$ pointing in the direction perpendicular to the layer. To incorporate such a magnetic field into quantum mechanics of plane systems, one substitues the classical momentum operator $-\mathrm{i}\,\nabla$ by the magnetic momentum operator[11] $-\mathrm{i}\,\nabla - \mathbf{A}$, where the vector potential $\mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2$ generates the magnetic field B via $B = \operatorname{curl} \mathbf{A} = \partial_1 A_2 - \partial_2 A_1$. For the description of graphene it is common to perform this substitution directly in the approximated Hamiltonian for electrons near the band edge[12], i.e. one uses the magnetic Dirac Hamiltonian

$$D_{\mathbf{A}} := \boldsymbol{\sigma} \cdot \big(-\mathrm{i}\,\nabla - \mathbf{A} \big)$$

in $\mathbb{R}^2$ in order to describe their kinetic energy. Unlike electric potentials, vector potentials $\mathbf{A}$ do alter the spectrum of the two-dimensional Dirac operator significantly: "Sufficiently strong" magnetic fields B generate eigenvalues and even spectral gaps, which allows to compare the spectrum of $D_{\mathbf{A}}$

[10] This can be seen by considering generalized eigenfunctions, c.f. [Wei03] or [Tit61].
[11] In the physics literature known as Peierls substitution (due to [Pei33]).
[12] In order to explain effects like the integer QHE in graphene (see [GS05]).

directly with spectroscopic measurements[13]. In order to precise "sufficiently strong" let us recall some well-known properties of the spectrum $\sigma(D_{\mathbf{A}})$ of the magnetic Dirac Hamiltonian:

- Assume that $B(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$, then $\sigma(D_{\mathbf{A}}) = \mathbb{R}$. If B is rotationally symmetric and decays not too fast, then $\sigma(D_{\mathbf{A}})$ is pure point, i.e. $\sigma(D_{\mathbf{A}})$ has neither a singular continuous part nor an absolutely continuous part.[14]
- Assume that $B(\mathbf{x}) = B_0 = const.$, then $\sigma(D_{\mathbf{A}}) = \{\pm\sqrt{2nB_0} \mid n \in \mathbb{N}_0\}$. Each eigenvalue is infinitely degenerate.
- Assume that $B(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$, then $\sigma(D_{\mathbf{A}})\backslash\{0\}$ consists only of eigenvalues of finite multiplicity. The kernel $\ker(D_{\mathbf{A}})$ is in general infinite dimensional.

One may observe that magnetic fields, strong enough to generate eigenvalues, tend to widen the gaps between eigenvalues of $D_{\mathbf{A}}$ with increasing strength at infinity. If we couple now the magnetic Dirac Hamiltonian in dimension two to a trapping potential $V : \mathbb{R}^2 \to \mathbb{R}$, only little is known on how the spectrum changes. For getting an idea of how the spectrum of the operator

$$H_D = D_{\mathbf{A}} + V$$

depends unusually on the behaviour of the potential V at large values of $|\mathbf{x}|$, we discuss shortly some known results on this problem:

Assume that B and V are rotationally symmetric functions on $\mathbb{R}^2$, i.e. they can be written as $B(\mathbf{x}) = b(|\mathbf{x}|), V(\mathbf{x}) = v(|\mathbf{x}|)$ with functions b and v on the half-line. Since changing the gauge does not alter the spectrum[15] of the operator H_D, we may assume that $\mathbf{A}$ has the form

$$\mathbf{A}(\mathbf{x}) := \frac{A(r)}{r}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad A(r) = \frac{1}{r}\int_0^r b(s)s\mathrm{d}s.$$

Due to the rotational symmetry of $\mathbf{A}$ and V, we can decompose H_D as a direct sum of Dirac operators on the half-line, i.e. H_D is unitarily equivalent to the direct sum $\bigoplus_{k\in\mathbb{Z}+1/2} h_k$ of operators

$$h_k := -\mathrm{i}\,\sigma_1\partial_r + \sigma_1\left(\frac{k}{r} - A\right) + v.$$

For the operators h_k on the half-line it is rather easy to see that, roughly speaking[16], if $A(r) \to \infty$ as $r \to \infty$ and

$$\limsup_{r\to\infty} \left|v(r)/A(r)\right| < 1, \tag{1.2}$$

[13]Such comparison has been realized in [SMP+06].
[14] Originally pointed out by [MS80].
[15]See Section 2.2.1 for a detailed explanation.
[16] The precise statement can be found in Section 2.1.3.

then $\sigma(h_k)$ consists only of eigenvalues of finite multiplicity. On the other hand, in [SY98] it is shown that for strong scalar potentials V, i.e. if $v(r) \to \infty$ as $r \to \infty$ and

$$\limsup_{r\to\infty} \big|A(r)/v(r)\big| < 1, \tag{1.3}$$

the spectrum of each h_k is absolutely continuous and includes the whole real line. Consequently, since

$$\sigma_{\#}(D_{\mathbf{A}} + V) = \overline{\bigcup_{k\in\mathbb{Z}+\frac{1}{2}} \sigma_{\#}(h_k)}\,, \tag{1.4}$$

where # denotes the absolutely/singular continuous or the pure point part of the spectrum, also $\sigma(H_D)$ is purely absolutely continuous and covers the whole real line in case (1.3). As one may observe, classical trapping potentials do not only lack of a confining effect when coupled to D_0. They are even able to dissolve eigenvalues and spectral gaps of the magnetic Dirac Hamiltonian $D_{\mathbf{A}}$. Thus, such trapping potentials have (according to the RAGE theorem, see below) a deconfining effect on Dirac particles. In view of conditions (1.2), (1.3) and relation (1.4), apparently the ratio between A and v is crucial for the type of the spectrum of the operator H_D. A fact which also has been observed in [GMR09] in context of proposing devices for manipulating the electronic properties of graphene by varying the external field. Nonetheless, if V (and hence v) is small compared to A (case (1.2)), it is unclear how the spectrum of H_D, then given by the closure of the eigenvalues, looks like. A first guess is indicated by the discussion of $\sigma(D_{\mathbf{A}})$ as a function of the strength of B at infinity. Since strong magnetic fields tend to produce spectral gaps and eigenvalues of finite multiplicity, it is reasonable to investigate the spectrum of H_D with respect to the relation of B and V. One part of this thesis is dedicated to the latter question. Roughly speaking, our results[17] in Chapter 3 are stating the following:

- If $V^2/|B| < 2$ at infinity, then $\sigma(H_D)$ is purely discrete. Moreover, the bound 2 is necessary for the discreteness of the spectrum.
- If $V^2/2|B|$ is unbounded at infinity, then $\sigma(H_D)$ has no gap.

For trying to grasp this deconfining effect of a scalar potential V, we look at a non-relativistic model of a spin-$\frac{1}{2}$ particle in dimension two. Considering again a magnetic field B pointing in the perpendicular direction of a plane, the kinetic energy of a non-relativistic spin-$\frac{1}{2}$ particle within that plane is given by the Pauli operator $P_{\mathbf{A}} := D_{\mathbf{A}}^2$. The operator $P_{\mathbf{A}}$ has, as the square of the magnetic Dirac operator, a similar spectral dependency on the strength of B as $D_{\mathbf{A}}$. Thus, "hard" magnetic fields B also tend to produce and widen spectral gaps of $P_{\mathbf{A}}$. More precisely, as above, we have:

[17] Already published in a slightly different form in [MS14].

- Assume that $B(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$, then $\sigma(P_{\mathbf{A}}) = [0, \infty)$. If B is rotationally symmetric and decays not too fast, then $\sigma(P_{\mathbf{A}})$ is of pure point type.
- Assume that $B(\mathbf{x}) = B_0 = const.$, then $\sigma(P_{\mathbf{A}}) = \{2nB_0 \,|\, n \in \mathbb{N}_0\}$. Each eigenvalue is infinitely degenerate.
- Assume that $B(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$, then $\sigma(P_{\mathbf{A}}) \cap \mathbb{R}^+$ consists only of finitely degenerate eigenvalues. $\ker(P_{\mathbf{A}})$ is in general infinite dimens.

In the last case, i.e. if $B(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$, one may use the operator inequality[18] $P_{\mathbf{A}} \geq 2B$, holding on the orthogonal complement of $\ker(P_{\mathbf{A}})$, to observe that such strong magnetic fields have a trapping effect on wave functions orthogonal to $\ker(P_{\mathbf{A}})$. In presence of an additional electric potential $V : \mathbb{R}^2 \to \mathbb{R}$, the Hamiltonian of the non-relativistic spin-$\frac{1}{2}$ particle in the plane is then given by

$$H_P := P_{\mathbf{A}} + V = \big[\boldsymbol{\sigma} \cdot \big(-\mathrm{i}\,\nabla - \mathbf{A}\big)\big]^2 + V.$$

Positive (trapping) potentials V would only enhance this effect of the magnetic field B, leading also to a purely discrete spectrum (see, e.g. [KMS05]). If we consider an "anti-trapping" potential V instead, i.e. $V(\mathbf{x}) \to -\infty$ as $|\mathbf{x}| \to \infty$, the situation is quite different: The particle lowers its energy by staying in a region where the potential is small, thus it tends to escape any compact set of the plane. As a consequence, one expects that such a repulsive potential counteracts the trapping effect of a strong field B. It is reasonable to assume that for (sufficiently strong) negative potentials V this results in a similar spectral behaviour as that of $P_{\mathbf{A}}$ with a comparatively weaker magnetic field. In this thesis we confirm that guess by verifying that[19]:

- If $V \leq 0$ but $|V|/B < 2$ at infinity, the spectrum $\sigma(H_P)$ is purely discrete. In particular, there are gaps between the eigenvalues of finite multiplicity.
- If the quotient V/B converges to $-\infty$ (as $|\mathbf{x}|$ grows), at least in some direction, then there is no gap in $\sigma(H_P)$.

Within the classical frame of a spin-$\frac{1}{2}$ particle it is perspicuous that an anti-trapping potential V can destroy the trapping effect of strong magnetic fields; Mere energetic considerations suggest that. Nonetheless, one can use this non-relativistic picture for trying to interpret naively our results concerning the spectrum of the Dirac operator $H_D = D_{\mathbf{A}} + V$, coupled to a trapping potential V: The spectrum of the magnetic Dirac operator $D_{\mathbf{A}}$ is symmetric with respect to 0, meaning that also with magnetic field one can

[18] The precise statement is given by Proposition 2.28.

[19] Disregarding some additional constraints on B and V.

associate a Dirac sea to the operator $D_{\mathbf{A}}$. Interpreting a missing particle of this sea as a positive charge carrier, a potential well V would have a repulsive effect on such a hole. This point of view allows a similar interpretation of the deconfining effect of a classical trapping potential on $D_{\mathbf{A}}$ as of the deconfining effect of a repulsive potential on the Pauli operator $P_{\mathbf{A}}$.

For practical purposes, a considerable part of the research on graphene focuses on transport properties of charge carriers near the band edge. Especially the possibility of ballistic movement of these carriers over macroscopic distances has attracted much interest over last years (see [MGM+11], [KLR10] and [PSJWG07]). In mathematical physics, a way to classify the type of dynamics of a charge distribution $|\psi|^2(\,\cdot\,)$ is given by the scaling behaviour of the expectation value

$$\left\langle \psi(t), |\mathbf{x}|^2 \psi(t) \right\rangle = \left\langle e^{-\mathrm{i}\,tH}\psi, |\mathbf{x}|^2 e^{-\mathrm{i}\,tH}\psi \right\rangle \tag{1.5}$$

in time t. Here the Hamiltonian H, self-adjoint on a Hilbert space $\mathcal{H}$, describes the system in position space. Ballistic behaviour is then identified by

$$\left\langle \psi(t), |\mathbf{x}|^2 \psi(t) \right\rangle \gtrsim C_\psi t^2$$

for some constant $C_\psi > 0$. In general, spectral properties of an operator H do not suffice to characterise the dynamics of the system. The RAGE theorem states, though, that spectral information about H has certain implications on how a spatial charge distribution $|\psi|^2(\,\cdot\,)$ evolve in time (as long as H is locally compact[20]). This theorem establishes the following relations:

$$\begin{aligned}
\psi \in P_{\mathrm{pp}}(H)\mathcal{H} \quad &\Longleftrightarrow \quad \lim_{R\to\infty} \sup_{t\in\mathbb{R}} \left\| \mathbb{1}_{\{|\mathbf{x}|>R\}} \psi(t) \right\|^2 = 0, \\
\psi \in P_{\mathrm{ac}}(H)\mathcal{H} \quad &\Longrightarrow \quad \sup_{R>0} \lim_{t\to\infty} \left\| \mathbb{1}_{\{|\mathbf{x}|\le R\}} \psi(t) \right\|^2 = 0,
\end{aligned}$$

and

$$\psi \in P_{\mathrm{ac}}(H)\mathcal{H} \oplus P_{\mathrm{sc}}(H)\mathcal{H} \quad \Longleftrightarrow \quad \sup_{R>0} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left\| \mathbb{1}_{\{|\mathbf{x}|\le R\}} \psi(t) \right\|^2 \mathrm{d}t = 0,$$

where

$$\begin{aligned}
P_{\mathrm{pp}}(H) &= \text{orth. projection on the subspace associated to } \sigma_{\mathrm{pp}}(H), \\
P_{\mathrm{ac}}(H) &= \text{orth. projection on the subspace associated to } \sigma_{\mathrm{ac}}(H), \\
P_{\mathrm{sc}}(H) &= \text{orth. projection on the subspace associated to } \sigma_{\mathrm{sc}}(H),
\end{aligned}$$

and $\mathbb{1}_X$ denotes the characteristic function on a set X (c.f. Chapter 5.2 of [Tes09]). Hence, the spectral characteristics of H give us a first impression

[20] A property which is satisfied by the operators considered here (see Chapter 2).

on how a charge distribution behave in time: Does it stay locally concentrated or does it spread out into the whole space. In particular, the link between $P_{\mathrm{pp}}(H)\mathcal{H}$ and the time evolution of $\|\mathbb{1}_{\{|\mathbf{x}|>R\}}\psi(t)\|$ justifies the connotation of "confining system" for operators that satisfy $P_{\mathrm{pp}}(H) = \mathrm{id}_{\mathcal{H}}$. An interesting fact is that even if for $\psi \in P_{\mathrm{pp}}(H)\mathcal{H}$ the time evolved distribution $\psi(t)$ remains uniformly localised in some bounded region, this does not necessarily mean that the expectation value (1.5) is bounded in time[21]. Indeed, in [DRJLS95] there are pointed out examples of quantum mechanical operators in position space with only pure point spectrum (i.e. $P_{\mathrm{pp}}(H) = \mathrm{id}_{\mathcal{H}}$) admitting that $t^{-(2-\delta)}\langle\psi(t), |\mathbf{x}|^2\psi(t)\rangle \to \infty$ as $t \to \infty$ for some given ψ (here $\delta > 0$ can be arbitrarily small). This result shows that it is possible to have almost ballistic behaviour in systems without any continuous part in the spectrum. In particular, it manifests that the quality of the spectrum is by far not enough for a canonical categorisation of the dynamics. Nevertheless, at least ballistic behaviour can be ruled out for a large class of (discrete and continuous) Schrödinger operators without a continuous part in the spectrum, as pointed out in [Sim90]. In Section 4.2 we shall see that Simon's argument applies also for a huge class of Dirac operators, excluding dynamical propagation in absence of continuous spectrum. In [BMST15] there are obtained even time-independent bounds on (1.5) for two-dimensional massless Dirac operators with pure point spectrum. However, this result requires so far the rotational symmetry of the field configuration.

On the other hand, if the spectrum of an operator H has a non-trivial continuous part, i.e. $P_{\mathrm{sc}}(H)\mathcal{H} \oplus P_{\mathrm{ac}}(H)\mathcal{H} \neq \{0\}$, the RAGE theorem implies only that

$$\begin{aligned} 0 \neq \psi \in P_{\mathrm{ac}}(H)\mathcal{H} \quad &\Longrightarrow \quad \lim_{t\to\infty} \langle\psi(t), |\mathbf{x}|^2\psi(t)\rangle = \infty, \\ 0 \neq \psi \in P_{\mathrm{sc}}(H)\mathcal{H} \oplus P_{\mathrm{ac}}(H)\mathcal{H} \quad &\Longrightarrow \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T \langle\psi(t), |\mathbf{x}|^2\psi(t)\rangle \,\mathrm{d}t = \infty. \end{aligned}$$

The latter conclusion does not approach the limiting behaviour of the original expectation value (1.5), but the one of the Cesàro mean

$$\big\langle\langle\psi(t), |\mathbf{x}|^2\psi(t)\rangle\big\rangle_T = \frac{1}{T}\int_0^T \langle\psi(t), |\mathbf{x}|^2\psi(t)\rangle \,\mathrm{d}t, \tag{1.6}$$

which is only a time average. Surprisingly, for the time mean (1.6) lower bounds in terms of powers of T can be obtained by investigating the regularity of the spectral measure μ_ψ associated to $\psi \in P_{\mathrm{sc}}(H)\mathcal{H} \oplus P_{\mathrm{ac}}(H)\mathcal{H}$. As explained in [Gua89] and [Com93], such lower bounds rely on decay features of the Fourier transform of μ_ψ, originating from the regularity of μ_ψ. Afterwards, in [Las96], this idea was extended to a refined decomposition

[21] Assuming that one starts with $\langle\psi, |\mathbf{x}|^2\psi\rangle < \infty$.

of the (singular) continuous measure to obtain sharper results on dynamical quantities of type (1.6). Concerning the question of ballistical lower bounds for (1.6), it turns out that for $\psi \in P_{\mathrm{ac}}(H)\mathcal{H}$, i.e. for states associated to the most regular part of the spectrum, lower bounds of order $T^{2/d}$ (where d is the dimension of the position space) can be deduced. Even if this does not straightforwardly lead to the desired result for operators H in dimension two, it enables us to obtain a lower bound for (1.6) of order T^2 in some important cases: Under the assumption of certain symmetries on the electro-magnetic field, a decomposition of H into a family of one-dimensional operators allows to deduce a lower bound of the desired scale. Nonetheless, one problem remains in order to use the technique of [Gua89] and [Com93] for establishing the link between the continuity of the spectrum and the dynamical behaviour: A proper Hilbert-Schmidt bound for the operator product

$$\mathbb{1}_X \mathbb{1}_{[-E,E]}(H) = \mathbb{1}_X(\mathbf{x}) \mathbb{1}_{[-E,E]}(H), \qquad \text{for } E > 0, \tag{1.7}$$

in terms of the volume of the bounded set $X \subset \mathbb{R}^d$ is necessary. For free operators, i.e. operators of the form $H = f(-\mathrm{i}\,\nabla)$, this is a direct consequence of the Seiler-Simon inequality. As shown in [Sim82], also in many cases of (magnetic) Schrödinger operators with external potentials it is possible to obtain such bounds for (1.7) by using semi-group properties and perturbation theory. Since we focus on Dirac operators coupled to potentials that are allowed to have a singularity at infinity, simple perturbational arguments do not work here. Instead, we demonstrate that for one-dimensional Dirac operators of type

$$h_k = -\mathrm{i}\,\sigma_1 \partial_r + \sigma_2 \left(\frac{k}{r} - A \right) + v \quad \text{ on } L^2(\mathbb{R}^+, \mathbb{C}^2), \tag{1.8}$$

with $k \in \mathbb{Z} + \frac{1}{2}$, or

$$h = -\mathrm{i}\,\sigma_1 \partial_x - \sigma_2 A + v \quad \text{ on } L^2(\mathbb{R}, \mathbb{C}^2) \tag{1.9}$$

it is possible to employ operator transformations for deriving Hilbert-Schmidt bounds on (1.7) of the right scale. These transformations, applicable as long as $|A| < |v|$ at large values (or $|v| < |A|$ at large values), originate from a representation of Lorentz boosts that are capable to modify electro-magnetic fields by changing the reference frame (c.f. Chapter 11 of [Jac99]). To be physically more explicit, we may consider a one-dimensional Dirac operator like (1.9) as the description of a particle in the plane having no momentum in the second, perpendicular direction (in particular this momentum is a conserved quantity due to symmetry). In this second direction of symmetry one can perform a Lorentz boost into an inertial frame where the electro-magnetic field is described by (A', v'). One of the most important properties of the Dirac equation, namely the Lorentz covariance[22], states that in the new inertial frame the particle, now possessing a non-trivial momentum in the direction of the boost, is again described by the Dirac equation with potentials (A', v'). Since in the boosted frame the momentum in the direction of the boost is again conserved, the particle there is effectively modelled by a one-dimensional Dirac Hamiltonian h' with potentials (A', v'). The energies E and E' of the particle in the different inertial frames are of course not equal, hence the two Hamiltonians h and h' cannot simply be unitarily equivalent. However, there is an invertible map that maps E to E'. Therefore, also h and h' should be connected via an invertible operator deriving from a representation of the corresponding Lorentz transformation. Such invertible operators can be used to establish resolvent identities between h and h', even if we deal with a non-constant boost velocity $\beta(x)$ (which is of course no longer of Lorentz type). Deducing Hilbert-Schmidt bounds for

$$\mathbb{1}_I \mathbb{1}_{[-E,E]}(h),$$

with $E > 0$ and $I \subset \mathbb{R}$ bounded, from resolvent relations between h and h' is probably just one application of this idea. However, we restrict ourselves here to use this relation just for obtaining some interesting examples of two-dimensional Dirac Hamiltonians that admit ballistic wave package spreading for states associated to the absolutely continuous part of the spectrum.

Finally, we should mention that within this work several mathematical problems, which appeared throughout our project on the massless Dirac (and Pauli) operator in dimension two, could not be solved. Some of them, concerning the spectrum, are pointed out in the end of Chapter 3. Further questions on the dynamics of Dirac systems are addressed in [BMST15]. Also in the context of impurities in graphene, such as magnetic dots or point charges, there is an ongoing interest in mathematical issues affiliated to the magnetic Dirac operator in the plane with perturbations (see [EDMSS10], [MS12] and [KS12]). The huge motivation for the investigation of the Dirac operator during the last years can surely be traced back to the possibility of using graphene as a experimental playground for Dirac-like phenomenons. This was highlighted in the case of the Klein tunneling through potential barriers: Until direct measurements by [YK09] the effect was barely seen as a Gedankenexperiment of Klein.

Declaration concerning already published material: Many results presented within this thesis have already been published in a slightly different form in the research papers [MS14], [Meh15], [MS15] and [BMST15]. The introduction of each chapter provides more information on the relevance of those publications for the content therein. Further, the concerned lemmata and theorems are tagged with corresponding references.

[22]See Chapter 3.3 and 4.1 of [Tha92] for a detailed discussion.

Chapter 2

Basic Features of Dirac and Pauli Operators

In this chapter we discuss the mathematical setup, required for our investigations. Essential points of the discussion on Dirac operators on the (half-)line are results of [MS15], though, it is thematically convenient to place them here.

2.1 Dirac Operators on the Line and Half-Line

2.1.1 Self-Adjointness and Local Compactness

Let us define now the one-dimensional Dirac operators we are working with and discuss some of their basic properties.

For $A, v \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ we consider the operator

$$h = \sigma_1(-\mathrm{i}\,\partial_x) - \sigma_2 A + v \qquad \text{on } L^2(\mathbb{R}, \mathbb{C}^2), \tag{2.1}$$

which is densely defined and symmetric on

$$\mathcal{D}_0(h) := \{\varphi \in \mathcal{D}_{\max}(h) \mid \varphi \text{ has compact support in } \mathbb{R}\}, \tag{2.2}$$

where

$$\mathcal{D}_{\max}(h) := \{\varphi \in L^2(\mathbb{R}, \mathbb{C}^2) \mid \varphi \text{ abs. cont.}, h\varphi \in L^2(\mathbb{R}, \mathbb{C}^2)\}. \tag{2.3}$$

As it is known from Sturm-Liouville theory (see Chapter 15 of [Wei03]), the operator h is in the limit point case at $\pm\infty$. Hence, h is essentially self-adjoint on $\mathcal{D}_0(h)$ and the domain of the self-adjoint extension is given by $\mathcal{D}_{\max}(h)$. We denote this extension again by h.

Proposition 2.1. *The subspace $C_0^\infty(\mathbb{R}, \mathbb{C}^2) \subset L^2(\mathbb{R}, \mathbb{C}^2)$ is dense in $\mathcal{D}_0(h)$ with respect to the h-graph norm $\|\cdot\|_h$. Therefore, h is also essentially self-adjoint on $C_0^\infty(\mathbb{R}, \mathbb{C}^2)$.*

Proof. First note that $C_0^\infty(\mathbb{R},\mathbb{C}^2)$ is a subset of $\mathcal{D}_0(h)$. Let $\varphi \in \mathcal{D}_0(h)$ and let I be a compact interval that contains the support of φ. Since $\varphi \in C_0(\mathbb{R},\mathbb{C}^2)$ and $(v-\sigma_2 A)$ is locally square-integrable, we have

$$-\mathrm{i}\,\sigma_1\varphi' = h\varphi - (v-\sigma_2 A)\varphi \in L^2(\mathbb{R},\mathbb{C}^2),$$

implying that $\varphi \in H^1(\mathbb{R},\mathbb{C}^2)$. Let $(\varphi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R},\mathbb{C}^2)$ be a sequence of mollifiers of φ whose support is also contained in I. We estimate

$$\begin{aligned}\|\varphi-\varphi_n\|_h^2 &= \|\varphi-\varphi_n\|^2 + \|h(\varphi-\varphi_n)\|^2\\ &\le \|\varphi-\varphi_n\|_{H^1}^2 + \|(v-\sigma_2 A)(\varphi-\varphi_n)\|^2\\ &\le \|\varphi-\varphi_n\|_{H^1}^2 + \|(v-\sigma_2 A)\mathbb{1}_I\|_2^2\,\|\varphi-\varphi_n\|_\infty^2.\end{aligned}$$

By the H^1-convergence of mollifiers, we know that $\|\varphi-\varphi_n\|_{H^1}^2 \to 0$. Moreover, the Sobolev inequality in dimension one (c.f. Chapter 8 of [LL01]) implies that $\|\varphi-\varphi_n\|_\infty^2 \to 0$ as $n\to\infty$. Hence, φ_n converges to φ in the graph norm of h as claimed. □

Proposition 2.2. *Any multiplication operator $M \in L^2(\mathbb{R},\mathbb{C}^{2\times 2})$ with compact support is h-bounded. Moreover, h is locally compact, i.e. for any compact interval $I\subset\mathbb{R}$ the operator product $\mathbb{1}_I(h-\mathrm{i})^{-1}$ is compact.*

Proof. Let $\chi \in C_0^\infty(\mathbb{R},[0,1])$ be a smooth cutoff function, which equals 1 on the support of M. For $\varphi \in C_0^\infty(\mathbb{R}^+,\mathbb{C}^2)$ we have

$$\|M\varphi\| = \|M\chi\varphi\| \le \|M\|_2\|\chi\varphi\|_\infty \le \|M\|_2\sqrt{\|(\chi\varphi)'\|^2+\|\chi\varphi\|^2}, \tag{2.4}$$

where we used the Sobolev inequality in the last step. Since the function $W := (\sigma_2 A - v)\mathbb{1}_{\mathrm{supp}(\chi)}$ is square integrable, we obtain

$$\begin{aligned}\|-\mathrm{i}\,\sigma_1\partial_x(\chi\varphi)\| &\le \|h\chi\varphi\| + \|(\sigma_2 A - v)\chi\varphi\|\\ &\le \|h\varphi\| + \|\chi'\|_\infty\|\varphi\| + \|W\|_2\|\chi\varphi\|_\infty\\ &\le \|h\varphi\| + \|\chi'\|_\infty\|\varphi\| + \|W\|_2\sqrt{2\|(\chi\varphi)'\|\|\chi\varphi\|}\\ &\le \|h\varphi\| + \|\chi'\|_\infty\|\varphi\| + \|W\|_2\big(\epsilon\|(\chi\varphi)'\| + \epsilon^{-1}\|\chi\varphi\|\big)\end{aligned}$$

for any $\epsilon > 0$, implying that

$$\big(1-\epsilon\|W\|_2\big)\|(\chi\varphi)'\| \le \|h\varphi\| + \big(\|\chi'\|_\infty + \epsilon^{-1}\|W\|_2\big)\|\varphi\|. \tag{2.5}$$

By choosing ϵ small enough, (2.5) together with (2.4) proves the first claim. Further, let $I\subset\mathbb{R}$ be a compact interval and $\chi_I \in C_0^\infty(\mathbb{R},[0,1])$ be a smooth cutoff function on I, i.e. $\mathbb{1}_I\chi_I = \mathbb{1}_I$ on $\mathbb{R}$. By the Kato-Simon-Seiler inequality (see [SS75] or [Sim79]), we know that $((-\mathrm{i}\,\sigma_1\partial_x)-\mathrm{i})^{-1}\mathbb{1}_{\mathrm{supp}(\chi_I)}$ is compact. Hence, using the resolvent identity

$$\begin{aligned}\chi_I\frac{1}{h-\mathrm{i}} - \frac{1}{\sigma_1(-\mathrm{i}\,\partial_x)-\mathrm{i}}\,\chi_I &= \frac{1}{\sigma_1(-\mathrm{i}\,\partial_x)-\mathrm{i}}\big((-\mathrm{i}\,\sigma_1\partial_x)\chi_I - \chi_I h\big)\frac{1}{h-\mathrm{i}}\\ &= \frac{1}{\sigma_1(-\mathrm{i}\,\partial_x)-\mathrm{i}}\big((\sigma_2 A - v)\chi_I - \mathrm{i}\,\sigma_1\chi_I'\big)\frac{1}{h-\mathrm{i}}\end{aligned}$$

and the boundedness of

$$\big((\sigma_2 A - v)\chi_I - \mathrm{i}\,\sigma_1\chi_I'\big)\frac{1}{h-\mathrm{i}}\,,$$

as shown above, we deduce that also $\chi_I(h-\mathrm{i}\,)^{-1}$ is a compact operator. □

For establishing the Dirac operators on the half-line assume that $A, v \in L^2_{\mathrm{loc}}([0,\infty),\mathbb{R})$. If $k \in \left(\mathbb{Z}+\frac{1}{2}\right) \cup \{0\}$, we set

$$h_k = \sigma_1(-\mathrm{i}\,\partial_x) + \sigma_2\left(\frac{k}{x} - A\right) + v \qquad \text{on } L^2(\mathbb{R}^+,\mathbb{C}^2). \tag{2.6}$$

A priori h_k is defined as a symmetric operator on the dense subset

$$\mathcal{D}_0(h_k) := \{\varphi \in \mathcal{D}_{\max}(h_k) \mid \varphi \text{ has compact support in } \mathbb{R}^+\}, \tag{2.7}$$

with the maximal domain

$$\mathcal{D}_{\max}(h_k) = \{\varphi \in L^2(\mathbb{R}^+,\mathbb{C}^2) \mid \varphi \text{ abs. cont., } h_k\varphi \in L^2(\mathbb{R}^+,\mathbb{C}^2)\}. \tag{2.8}$$

Remark 2.3. *Analogously as for the operator* h*, one shows that* $C_0^\infty(\mathbb{R}^+,\mathbb{C}^2)$ *is a dense subset of* $\mathcal{D}_0(h_k)$ *with respect to the* h_k*-graph norm.*

As above, h_k is in the limit point case at $+\infty$ for any $k \in \left(\mathbb{Z}+\frac{1}{2}\right) \cup \{0\}$ due to Korollar 15.21 of [Wei03]. Concerning the endpoint 0 we distinguish two cases:

Proposition 2.4. *Let* $|k| \geq \frac{1}{2}$*, then* h_k *is in the limit point case at* 0*. Hence,* h_k *is essentially self-adjoint on* $\mathcal{D}_0(h_k)$ *and on* $C_0^\infty(\mathbb{R}^+,\mathbb{C}^2)$*.*

Proof. It suffices to show that there is a solution for the eigenvalue problem

$$h_k\varphi = \lambda\varphi, \tag{2.9}$$

with $\lambda \in \mathbb{R}$, which is not square-integrable at 0. According to Theorem 1 of [ES08] (see also [Tit61]), there exists for $k \geq \frac{1}{2}$ a unique solution τ of (2.9) with the asymptotic behaviour

$$\tau(x) = (o(1), 1 + o(1))^{\mathrm{T}} x^k \quad \text{as} \quad x \to 0.$$

(Let us note that [ES08, Theorem 1] is only stated for $A = 0$. However, the same argument works provided that A is integrable at zero.) Let $\tilde{\tau}$ be a linear independent solution of (2.9) such that the Wronski determinant $W(\tau,\tilde{\tau}) := \tau_1\tilde{\tau}_2 - \tau_2\tilde{\tau}_1 \equiv 1$. Assume that $\liminf_{x\to 0}|\tilde{\tau}(x)|x^k = 0$, then clearly $\liminf_{x\to 0} W(\tau,\tilde{\tau})(x) = 0$, which is a contradiction. Hence, τ can not be square-integrable at 0. An analogous argument holds if $k \leq -\frac{1}{2}$. □

Since for $|k| \geq \frac{1}{2}$ the operator h_k is in the limit point case at both endpoints, we know that $\mathcal{D}_{\max}(h_k)$ equals the domain $\mathcal{D}(h_k)$ of the self-adjoint extension. As usual, we consider h_k to be the extension on $\mathcal{D}(h_k)$.

If $k = 0$, we know that the operator h_0 is in the limit circle case at 0. Therefore, according to Sturm-Liouville theory (see Satz 15.12 of [Wei03]), one has a one-parameter family of self-adjoint realisations of h_0 with corresponding domains

$$\mathcal{D}^{\alpha}(h_0) = \big\{\varphi \in \mathcal{D}_{\max}(h_0) \,|\, \lim_{x\to 0}(\varphi_1(x)\cos\alpha - \varphi_2(x)\sin\alpha) = 0\big\}, \tag{2.10}$$

where $\alpha \in [0, 2\pi)$. For practical purposes, we work with the self-adjoint realisation on $\mathcal{D}(h_0) := \mathcal{D}^0(h_0)$ and denote, as before, the resulting operator by the same symbol h_0.

Remark 2.5. *Let $\chi \in C^\infty(\mathbb{R}^+, [0,1])$ be a smooth function, supported away from zero with bounded first derivative. By the characterisation of the domains of self-adjointness* (2.8) *and* (2.10)*, together with the fact that k/x is bounded on the support of χ, we observe that*

$$\chi\mathcal{D}(h_k) \subset \mathcal{D}(h_0) \tag{2.11}$$

whenever the two operators have the same potentials A and v.

Now we want to discuss conditions under which the operators h_k for $k \neq 0$ are locally compact. Technically, the main difference between h and h_k is the k/x singularity at 0, which is not square-integrable (not even integrable), hence it cannot simply be controlled by $-\mathrm{i}\,\sigma_1\partial_x$ via the Sobolev inequality. However, the Hardy inequality on the half-line

$$\int_0^\infty |\varphi'(x)|^2 \mathrm{d}x \geq \frac{1}{4}\int_0^\infty \frac{|\varphi(x)|^2}{x^2}\,\mathrm{d}x \tag{2.12}$$

enables us to treat the term $1/x$ as a perturbation. More precisely, consider h_k when $A = v = 0$, then for $\varphi \in C_0^\infty(\mathbb{R}^+, \mathbb{C}^2)$ we have

$$\begin{aligned} \|\varphi\|_{h_k}^2 &= \left\langle \left(\sigma_1(-\mathrm{i}\,\partial_x) + \sigma_2\frac{k}{x}\right)\varphi, \sigma_1\left((-\mathrm{i}\,\partial_x) + \sigma_2\frac{k}{x}\right)\varphi \right\rangle \\ &= \|\varphi'\|^2 + \|\varphi\|^2 - \left\langle \varphi, \sigma_3\frac{k}{x^2}\varphi \right\rangle + \left\|\frac{k}{x}\varphi\right\|^2 \\ &\geq \|\varphi'\|^2 + \|\varphi\|^2 + (k^2 - |k|)\left\|\frac{1}{x}\varphi\right\|^2. \end{aligned} \tag{2.13}$$

Thus, by applying (2.12) one concludes that if $|k| > \frac{1}{2}$, the H^1-norm on the half-line can be bounded by the h_k-graph norm. In particular, for $|k| > \frac{1}{2}$ any L^p-perturbation can be controlled by the h_k-graph norm whenever $p \geq 2$. The value $|k| = \frac{1}{2}$ is not covered by this argument since it corresponds to the critical constant of the Hardy inequality. To treat this critical value, we use the Hardy-Sobolev-Maz'ya inequality on the half-line, proven recently in [FL12]. This inequality allows us to control also L^p-perturbations for $p > 2$, meaning that we have to impose slight restrictions on the regularity of the potentials A, v in the following results.

Lemma 2.6 (Theorem 1 of [MS15])**.** *For $|k| \geq \frac{1}{2}$ consider h_k with $A = v = 0$. Then any multiplication operator $M \in L^p(\mathbb{R}^+, \mathbb{C}^{2\times 2})$, with $p > 2$, is infinitesimally h_k-bounded. In addition, any multiplication operator $M \in L^2(\mathbb{R}^+, \mathbb{C}^{2\times 2})$ is infinitesimally h_k-bounded whenever $|k| > \frac{1}{2}$.*

Proof. For $\varphi \in C_0^\infty(\mathbb{R}^+, \mathbb{C}^2)$ we estimate

$$\begin{aligned} \|\varphi\|_{h_k}^2 &= \|\varphi'\|^2 + \|\varphi\|^2 - \left\langle \varphi, \sigma_3 \frac{k}{x^2} \varphi \right\rangle + \left\| \frac{k}{x} \varphi \right\|^2 \\ &\geq \|\varphi'\|^2 + \|\varphi\|^2 + (k^2 - |k|) \left\| \frac{1}{x} \varphi \right\|^2 \\ &\geq \|\varphi'\|^2 + \|\varphi\|^2 - e^{-4(|k|-1/2)^2} \frac{1}{4} \left\| \frac{1}{x} \varphi \right\|^2 . \end{aligned} \tag{2.14}$$

Using the Sobolev inequality $\|\varphi\|_\infty^2 \leq \kappa \|\varphi\|^2 + \kappa^{-1} \|\varphi'\|^2$ on the half-line, valid for any $\kappa > 1$, we deduce

$$\begin{aligned} \|\varphi\|_{h_k}^2 \geq \big(1 - \mu(k)\big) \kappa \|\varphi\|_\infty^2 &+ (1 - \kappa^2) \|\varphi\|^2 \\ &+ \mu(k) \bigg(\|\varphi'\|^2 - \frac{1}{4} \left\| \frac{1}{x} \varphi \right\|^2 \bigg), \end{aligned} \tag{2.15}$$

where $\mu(k) := e^{-4(|k|-1/2)^2} \in (0, 1]$. Note that for $|k| = \frac{1}{2}$ the first term on the right hand side of (2.15) equals zero. By the Hardy-Sobolev-Maz'ya inequality on the half-line (c.f. Thm. 1.2 of [FL12]), we obtain for $q \in (2, \infty)$ and $\theta = \frac{1}{2}(1 - 2q^{-1})$ a constant c_θ (depending only on θ) such that

$$\begin{aligned} c_\theta \|\varphi\|_q^2 &\leq \bigg(\|\varphi'\|^2 - \frac{1}{4} \left\| \frac{1}{x} \varphi \right\|^2 \bigg)^\theta \big(\|\varphi\|^2 \big)^{1-\theta} \\ &\leq \epsilon \theta \bigg(\|\varphi'\|^2 - \frac{1}{4} \left\| \frac{1}{x} \varphi \right\|^2 \bigg) + (1 - \theta) \epsilon^{-\frac{\theta}{1-\theta}} \|\varphi\|^2 \end{aligned}$$

for any $\epsilon \in (0, 1)$ (here we apply the Young inequality in the last step). Combining this with (2.15), we conclude that

$$\mu(k) c_\theta \|\varphi\|_q^2 + (1 - \mu(k)) \kappa \epsilon \theta \|\varphi\|_\infty^2 \leq \epsilon \theta \|\varphi\|_{h_k}^2 + c(\epsilon, \kappa, \theta) \|\varphi\|^2. \tag{2.16}$$

For $M \in L^p(\mathbb{R}^+, \mathbb{C}^{2\times 2})$, with $p > 2$, we choose $\theta = p^{-1}$, hence $p^{-1} + q^{-1} = \frac{1}{2}$. Then (2.16) yields

$$\|M\varphi\|^2 \leq \|M\|_p^2 \|\varphi\|_q^2 \leq \|M\|_p^2 (\mu(k) c_\theta)^{-1} \left(\epsilon \theta \|\varphi\|_{h_k}^2 + c(\epsilon, \kappa, \theta) \|\varphi\|^2 \right)$$

with $\epsilon \in (0,1)$. If $|k| > \frac{1}{2}$ and M is an L^2-function, we use again (2.16) (dropping the first term) to conclude

$$\|M\varphi\|^2 \leq \|M\|_2^2 \|\varphi\|_\infty^2 \leq \frac{\kappa^{-1}}{(1-\mu(k))} \|M\|_2^2 \|\varphi\|_{h_k}^2 + \tilde{c}(\epsilon,\kappa,\theta)\|\varphi\|^2.$$

The last two inequalities imply the claim since h_k is essentially self-adjoint on $C_0^\infty(\mathbb{R}^+, \mathbb{C}^2)$ as remarked in Proposition 2.4. □

Corollary 2.7. *For $|k| \geq \frac{1}{2}$ consider h_k with $A, v \in L^p_{\mathrm{loc}}([0,\infty),\mathbb{R})$ for some $p > 2$. Then any multiplication operator $M \in L^s(\mathbb{R}^+, \mathbb{C}^{2\times 2})$ with bounded support is infinitesimally h_k-bounded whenever $s > 2$. If $|k| > \frac{1}{2}$, then the same holds true for $p, s \geq 2$.*

Proof. Let $\chi \in C^\infty(\mathbb{R}^+, [0,1])$ be a smooth cutoff function, which equals 1 on the support of M and vanishes for large x. By Lemma 2.6, we find for any $\epsilon \in (0,1)$ a constant c_ϵ such that for $\varphi \in C_0^\infty(\mathbb{R}^+, \mathbb{C}^2)$ it holds that

$$\|M\varphi\| = \|M\chi\varphi\| \leq \epsilon \left\| \left(-\mathrm{i}\,\sigma_1 \partial_x + \sigma_2 \frac{k}{x} \right) \chi\varphi \right\| + c_\epsilon \|\varphi\|. \tag{2.17}$$

Set $W := v - \sigma_2 A \in L^p_{\mathrm{loc}}([0,\infty),\mathbb{R})$. Using again Lemma 2.6, we find a constant $c > 0$ such that

$$\begin{aligned}
\left\| \left(-\mathrm{i}\,\sigma_1 \partial_x + \sigma_2 \frac{k}{x} \right) \chi\varphi \right\| &\leq \|h_k \chi\varphi\| + \|W\chi\varphi\| \\
&\leq \|h_k \chi\varphi\| + \frac{1}{2} \left\| \left(-\mathrm{i}\,\sigma_1 \partial_x + \sigma_2 \frac{k}{x} \right) \chi\varphi \right\| + c\|\varphi\| \\
&\leq \|h_k \varphi\| + \frac{1}{2} \left\| \left(-\mathrm{i}\,\sigma_1 \partial_x + \sigma_2 \frac{k}{x} \right) \chi\varphi \right\| + \tilde{c}\|\varphi\|,
\end{aligned}$$

where $\tilde{c} = c + \|\chi'\|_\infty$. Combining this with (2.17) results in the statement of the corollary. □

Theorem 2.8 (Corollary 4 of [MS15])**.** *For $|k| \geq \frac{1}{2}$ consider h_k with $A, v \in L^p_{\mathrm{loc}}([0,\infty),\mathbb{R})$ for some $p > 2$. Then h_k is a locally compact operator, i.e. for any bounded intervall $I \subset (0,\infty)$ the product $\mathbb{1}_I (h_k - \mathrm{i}\,)^{-1}$ is compact.*

Proof. Let $I \subset (0,\infty)$ be bounded and $\chi_I \in C^\infty([0,\infty),[0,1])$ be a smooth cutoff function on I, i.e. $\chi_I(x) = 1$ if $x \in I$, and $\chi_I(x) = 0$ if $\mathrm{dist}(x, I) > 1$. We use the reference operator

$$h_{\mathrm{ref}} = \sigma_1(-\mathrm{i}\,\partial_x) + \sigma_2 \frac{1}{x} \quad \text{on} \quad L^2(\mathbb{R}^+, \mathbb{C}^2),$$

which is known to be locally compact (see e.g. [Sch95]), and prove that the right hand side of the resolvent difference

$$\begin{aligned}\chi_I^2 \frac{1}{h_{\mathrm{ref}}-\mathrm{i}} - \frac{1}{h_k-\mathrm{i}}\chi_I^2 &= \frac{1}{h_k-\mathrm{i}}\big((h_k-\mathrm{i})\chi_I^2 - \chi_I^2(h_{\mathrm{ref}}-\mathrm{i})\big)\frac{1}{h_{\mathrm{ref}}-\mathrm{i}} \\ &= \frac{1}{h_k-\mathrm{i}}\big((v-\sigma_2 A)\chi_I^2 - 2\mathrm{i}\,\sigma_1\chi_I\chi_I'\big)\frac{1}{h_{\mathrm{ref}}-\mathrm{i}} \\ &\quad + (k-1)\frac{1}{h_k-\mathrm{i}}x^{-\frac{1}{4}}\chi_I^2\sigma_2 x^{-\frac{3}{4}}\frac{1}{h_{\mathrm{ref}}-\mathrm{i}}\end{aligned}$$

is a compact operator. Since h_{ref} is locally compact and $(v-\sigma_2 A)\chi_I$, $x^{-\frac{1}{4}}\chi_I$, and $2\mathrm{i}\,\sigma_1\chi_I'$ are relatively h_k-bounded (see Corollary 2.7), it suffices to show that

$$\chi_I x^{-\frac{3}{4}}\frac{1}{h_{\mathrm{ref}}-\mathrm{i}} \tag{2.18}$$

is compact. To this end recall that by (2.13) the singularity x^{-2} is bounded with respect to h_{ref}^2 in the sense of quadratic forms. Because exponentiating to the power $a \in (0,1)$ is operator monotone, we conclude that the composition $x^{-3/4}|h_{\mathrm{ref}}-\mathrm{i}\,|^{-3/4}$ is bounded. Therefore, the resolvent identity

$$\begin{aligned}\chi_I x^{-\frac{3}{4}}\frac{1}{h_{\mathrm{ref}}-\mathrm{i}} &= x^{-\frac{3}{4}}\frac{1}{h_{\mathrm{ref}}-\mathrm{i}}\chi_I + x^{-\frac{3}{4}}\frac{1}{h_{\mathrm{ref}}-\mathrm{i}}\big(-\mathrm{i}\,\sigma_1\chi_I'\big)\frac{1}{h_{\mathrm{ref}}-\mathrm{i}} \\ &= x^{-\frac{3}{4}}\left|\frac{1}{h_{\mathrm{ref}}-\mathrm{i}}\right|^{\frac{3}{4}}\operatorname{sgn}\left(\frac{1}{h_{\mathrm{ref}}-\mathrm{i}}\right)\left|\frac{1}{h_{\mathrm{ref}}-\mathrm{i}}\right|^{\frac{1}{4}}\chi_I \\ &\quad + x^{-\frac{3}{4}}\frac{1}{h_{\mathrm{ref}}-\mathrm{i}}\big(-\mathrm{i}\,\sigma_1\chi_I'\big)\frac{1}{h_{\mathrm{ref}}-\mathrm{i}}\end{aligned}$$

leads to the compactness of (2.18) as soon as $|h_{\mathrm{ref}}-\mathrm{i}\,|^{-1/4}\chi_I$ has that property. With the integral representation of the 8-th root, i.e. by writing

$$\begin{aligned}\left|\frac{1}{h_{\mathrm{ref}}-\mathrm{i}}\right|^{\frac{1}{4}}\chi_I &= \left(\frac{1}{h_{\mathrm{ref}}^2+1}\right)^{\frac{1}{8}}\chi_I \\ &= B\big(\tfrac{7}{8},\tfrac{1}{8}\big)^{-1}\int_0^\infty \frac{1}{h_{\mathrm{ref}}^2+s+1}\,\chi_I\,\frac{1}{s^{1/8}}\,\mathrm{d}s\end{aligned} \tag{2.19}$$

(here $B(x,y)$ denotes the beta function), this missing component is evident: Since the function

$$(0,\infty)\ni s\mapsto \frac{1}{h_{\mathrm{ref}}^2+1+s}\,\chi_I\,\frac{1}{s^{1/8}}$$

is continuous and absolutely integrable (with respect to the operator norm on $L^2(\mathbb{R}^+,\mathbb{C}^2)$), integral (2.19) can be seen as a Bochner integral in the Banach space of compact operators, equipped with the operator norm of $L^2(\mathbb{R}^+,\mathbb{C}^2)$ (see e.g. Appendix E of [Coh13]). □

Remark 2.9. *If $|k| > \frac{1}{2}$, then Theorem 2.8 is also valid for potentials $A, v \in L^2_{\mathrm{loc}}([0,\infty),\mathbb{R})$.*

2.1.2 Usual and Unusual Potential Transformations

In this subsection we want to discuss some unitary/non-unitary transformations on $L^2(\mathbb{R},\mathbb{C}^2)$ and on $L^2(\mathbb{R}^+,\mathbb{C}^2)$, which help to establish relations between different Dirac operators in dimension one. As in Subsection 2.1.1, we assume A, v to be at least locally square-integrable.

Proposition 2.10. *For $k \in \left(\mathbb{Z}+\frac{1}{2}\right) \cup \{0\}$ the operator h_k is unitarily equivalent to the operators*

$$\sigma_2(-\mathrm{i}\,\partial_x) + \sigma_1\left(A - \frac{k}{x}\right) + v \qquad \text{on } L^2(\mathbb{R}^+,\mathbb{C}^2) \tag{2.20}$$

and

$$\sigma_1(-\mathrm{i}\,\partial_x) + \sigma_3\left(A - \frac{k}{x}\right) + v \qquad \text{on } L^2(\mathbb{R}^+,\mathbb{C}^2), \tag{2.21}$$

with domains

$$\tfrac{1}{\sqrt{2}}(1-\mathrm{i}\,\sigma_3)\mathcal{D}(h_k) \quad \text{and} \quad \tfrac{1}{\sqrt{2}}(1+\mathrm{i}\,\sigma_1)\mathcal{D}(h_k)$$

respectively. The statement is also valid for h and corresponding operators on $L^2(\mathbb{R},\mathbb{C}^2)$.

Proof. Using the commutator properties of the Pauli matrices, we compute

$$\frac{1}{\sqrt{2}}(1-\mathrm{i}\,\sigma_3)\,\sigma_1\,\frac{1}{\sqrt{2}}(1+\mathrm{i}\,\sigma_3) = \frac{1}{2}(\mathrm{i}\,\sigma_1\sigma_3 - \mathrm{i}\,\sigma_3\sigma_1) = \sigma_2,$$
$$\frac{1}{\sqrt{2}}(1-\mathrm{i}\,\sigma_3)\,\sigma_2\,\frac{1}{\sqrt{2}}(1+\mathrm{i}\,\sigma_3) = \frac{1}{2}(\mathrm{i}\,\sigma_2\sigma_3 - \mathrm{i}\,\sigma_3\sigma_2) = -\sigma_1,$$

and

$$\frac{1}{\sqrt{2}}(1+\mathrm{i}\,\sigma_1)\,\sigma_2\,\frac{1}{\sqrt{2}}(1-\mathrm{i}\,\sigma_1) = \frac{1}{2}(\mathrm{i}\,\sigma_1\sigma_2 - \mathrm{i}\,\sigma_2\sigma_1) = -\sigma_3.$$

Consequently, we get

$$\frac{1}{\sqrt{2}}(1-\mathrm{i}\,\sigma_3)\,h_k\,\frac{1}{\sqrt{2}}(1+\mathrm{i}\,\sigma_3) = \sigma_2(-\mathrm{i}\,\partial_x) + \sigma_1\left(A - \frac{k}{x}\right) + v$$

on $\frac{1}{2}(1-\mathrm{i}\,\sigma_3)\mathcal{D}(h_k)$ and

$$\frac{1}{\sqrt{2}}(1+\mathrm{i}\,\sigma_1)\,h_k\,\frac{1}{\sqrt{2}}(1-\mathrm{i}\,\sigma_1) = \sigma_1(-\mathrm{i}\,\partial_x) + \sigma_3\left(A - \frac{k}{x}\right) + v$$

on $\frac{1}{2}(1+\mathrm{i}\,\sigma_1)\mathcal{D}(h_k)$. □

Remark 2.11. *Note that* (2.21) *means that h_k (as well as h) is unitarily equivalent to a one-dimensional Dirac operator with a mass term and a pure scalar potential.*

As already remarked in the introduction, the (massless) Dirac operators in dimension one with pure scalar potentials are completely solvable:

Proposition 2.12. *Assume that $A = 0$. Then h and h_0 are unitarily equivalent to the corresponding free operators, i.e. to $\sigma_1(-\mathrm{i}\,\partial_x)$ on the corresponding domains, via the map*

$$[U\varphi](x) = \exp\left(\mathrm{i}\,\sigma_1 \int_0^x v(s)\mathrm{d}s\right)\varphi(x)$$

on $L^2(\mathbb{R},\mathbb{C}^2)$ and on $L^2(\mathbb{R}^+,\mathbb{C}^2)$ respectively.

Proof. We first note that

$$\left[U^*\varphi\right](x) = \left[U^{-1}\varphi\right](x) = \exp\left(-\mathrm{i}\,\sigma_1 \int_0^x v(s)\mathrm{d}s\right)\varphi(x)$$

holds on $L^2(\mathbb{R},\mathbb{C}^2)$ and on $L^2(\mathbb{R}^+,\mathbb{C}^2)$ respectively. Hence, U is a unitary operator on both Hilbert spaces. We only proceed with the case of h_0 since the one of h is completely analogous. First note that the (matrix-valued) function

$$\exp\left(-\mathrm{i}\,\sigma_1 \int_0^x v(s)\mathrm{d}s\right)$$

is absolutely continuous, so for $\varphi \in \mathcal{D}(h_0)$ we know that $U^*\varphi$ is absolutely continuous (see e.g. Chapter VII of [Els05]), i.e. one can compute

$$U^*\sigma_1(-\mathrm{i}\,\partial_x)U\varphi = \sigma_1(-\mathrm{i}\,)\varphi' + \sigma_1(-\mathrm{i}\,)(\mathrm{i}\,\sigma_1)v\varphi = h_0\varphi. \tag{2.22}$$

Therefore, $\sigma_1(-\mathrm{i}\,\partial_x)U\varphi \in L^2(\mathbb{R}^+,\mathbb{C}^2)$ and because $[U\varphi]_1(0) = 0$ we have $U\varphi \in \mathcal{D}(\sigma_1(-\mathrm{i}\,\partial_1))$. Similarly we get the inclusion $U^*\mathcal{D}(\sigma_1(-\mathrm{i}\,\partial_1)) \subset \mathcal{D}(h_0)$. Further, by (2.22), the equation $U^*\sigma_1(-\mathrm{i}\,\partial_x)U = h_0$ holds on $\mathcal{D}(h_0)$. □

Corollary 2.13. *Assume that $A = 0$. Then for bounded intervals $I \subset \mathbb{R}$ and $I_0 \subset \mathbb{R}^+$ the operators $\mathbb{1}_I(h-\mathrm{i}\,)^{-1}$ and $\mathbb{1}_{I_0}(h_0-\mathrm{i}\,)^{-1}$ are Hilbert-Schmidt with Hilbert-Schmidt norms*

$$\left\|\mathbb{1}_I\frac{1}{h-\mathrm{i}}\right\|_{\mathrm{HS}} \le \frac{1}{\sqrt{2}}\sqrt{|I|}\,, \tag{2.23}$$

$$\left\|\mathbb{1}_{I_0}\frac{1}{h_0-\mathrm{i}}\right\|_{\mathrm{HS}} \le \sqrt{|I_0|}\,. \tag{2.24}$$

Proof. Note first that due to (the proof of) Proposition 2.12 we have the identity

$$\mathbb{1}_I\frac{1}{h-\mathrm{i}} = \mathbb{1}_I U\frac{1}{\sigma_1(-\mathrm{i}\,\partial_x)-\mathrm{i}}\,U^* = U\mathbb{1}_I\frac{1}{\sigma_1(-\mathrm{i}\,\partial_x)-\mathrm{i}}\,U^*,$$

which holds analogously for h_0. Thus, using the Kato-Seiler-Simon inequality (see [SS75] or Chapter 4 of [Sim79]), we deduce that $\mathbb{1}_I(h-\mathrm{i})^{-1}$ is Hilbert-Schmidt with

$$\left\|\mathbb{1}_I\frac{1}{h-\mathrm{i}}\right\|_{\mathrm{HS}}=\left\|\mathbb{1}_I\frac{1}{\sigma_1(-\mathrm{i}\,\partial_x)-\mathrm{i}}\right\|_{\mathrm{HS}}\le\frac{1}{\sqrt{2\pi}}\|\mathbb{1}_I\|_2\left\|\frac{1}{\sigma_1x-\mathrm{i}}\right\|_2\le\frac{1}{\sqrt{2}}\sqrt{|I|}\,.$$

For the operator h_0 we use the resolvent kernel of $\sigma_2(\mathrm{i}\,\partial_x)$ on $L^2(\mathbb{R}^+,\mathbb{C}^2)$, which can, due to Section 15.5 of [Wei03], be computed as follows: If one has a fundamental system $\tau_1(z,\,\cdot\,),\tau_2(z,\,\cdot\,)$ of the ODE $[\sigma_2(\mathrm{i}\,\partial_x)-z]\tau=0$, then, for $z\in\mathbb{C}\backslash\mathbb{R}$, the kernel is given by

$$\frac{1}{\sigma_2(\mathrm{i}\,\partial_x)-z}(x_1,x_2)=\begin{cases}\sum_{j,k=1}^2 m_{j,k}^+(z)\overline{\tau_j(\overline{z},x_1)}\tau_k^{\mathrm{T}}(z,x_2) & \text{if } x_1>x_2>0,\\ \sum_{j,k=1}^2 m_{j,k}^-(z)\overline{\tau_j(\overline{z},x_1)}\tau_k^{\mathrm{T}}(z,x_2) & \text{if } x_2>x_1>0.\end{cases}$$

Here the 2×2 matrices $m^+(z),m^-(z)$ are given in terms of complex coefficients $m_a(z),m_b(z)$, chosen such that $m_a(z)\tau_1(z,\,\cdot\,)+\tau_2(z,\,\cdot\,)$ satisfies the boundary condition at 0 and that $m_b(z)\tau_1(z,\,\cdot\,)+\tau_2(z,\,\cdot\,)$ is square-integrable at ∞. Using the fundamental system

$$\tau_1(z,x)=\begin{pmatrix}\cos zx\\ \sin zx\end{pmatrix},\qquad \tau_2(z,x)=\begin{pmatrix}-\sin zx\\ \cos zx\end{pmatrix},$$

we obtain that $m_a(\mathrm{i})=0$, $m_b(\mathrm{i})=\mathrm{i}$ and therefore (by the formula of [Wei03]) the explicit resolvent kernel

$$\frac{1}{\sigma_2(\mathrm{i}\,\partial_x)-\mathrm{i}}(x_1,x_2)=\begin{cases}e^{-x_1}\begin{pmatrix}\mathrm{i}\sinh x_2 & -\cosh x_2\\ \sinh x_2 & \mathrm{i}\cosh x_2\end{pmatrix} & \text{for } x_1>x_2\ge0,\\[2ex] e^{-x_2}\begin{pmatrix}\mathrm{i}\sinh x_1 & \sinh x_1\\ -\cosh x_1 & \mathrm{i}\cosh x_1\end{pmatrix} & \text{for } x_2>x_1\ge0.\end{cases}$$

Since $\mathbb{1}_{I_0}(x_1)(\sigma_2(\mathrm{i}\,\partial_x)-\mathrm{i})^{-1}(x_1,x_2)$ is square-integrable as a function on $\mathbb{R}^+\times\mathbb{R}^+$, we deduce with Proposition 2.10 that $\mathbb{1}_{I_0}(h_0-\mathrm{i})^{-1}$ is Hilbert-Schmidt with HS norm

$$\begin{aligned}\left\|\mathbb{1}_{I_0}\frac{1}{h_0-\mathrm{i}}\right\|_{\mathrm{HS}}^2&=\left\|\mathbb{1}_{I_0}\frac{1}{\sigma_2(\mathrm{i}\,\partial_x)-\mathrm{i}}\right\|_{\mathrm{HS}}^2\\&=\int_0^\infty\int_0^\infty|\mathbb{1}_{I_0}(x_1)|^2\left\|\frac{1}{\sigma_2(\mathrm{i}\,\partial_x)-\mathrm{i}}(x_1,x_2)\right\|_{M_2(\mathbb{C})}^2\mathrm{d}x_1\mathrm{d}x_2\\&=\int_0^\infty|\mathbb{1}_{I_0}(x_1)|^2\int_0^\infty e^{-2|x_1-x_2|}\,\mathrm{d}x_2\mathrm{d}x_1\\&\le|I_0|.\end{aligned}$$

□

Now we want to present a method to connect (massless) one-dimensional Dirac operators with different potentials (A, v) and $(\tilde{A}, \tilde{v})$. More explicitly, we consider classes of potentials (A, v) for which one can establish a relation to a one-dimensional Dirac operator with a (almost) pure scalar potential $\tilde{v}/$ (almost) pure magnetic potential $\tilde{A}$. We first discuss the possibility to reduce h/h_k to a Dirac operator with (almost) pure scalar potential, for what we introduce the classes:

Assumption 2.14 (AS1)**.** *$A, v \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ such that $A = A_1 + A_2$, $v = v_1 + v_2$, where A_1, v_1 have compact support and $A_2, v_2 \in C^1(\mathbb{R}, \mathbb{R})$ fulfill*

i) *v_2 is supported away from* 0 *and* $\mathrm{supp}(A_2) \subset \mathrm{supp}(v_2)$,

ii) $\|A_2/v_2\|_\infty < 1$,

iii) *the derivative $(A_2/v_2)'$ is bounded on $\mathbb{R}$.*

Assumption 2.15 (AS1′)**.** *$A, v \in L^p_{\mathrm{loc}}([0, \infty), \mathbb{R})$ for some $p > 2$ such that $A = A_1 + A_2$, $v = v_1 + v_2$, where A_1, v_1 have compact support and $A_2, v_2 \in C^1(\mathbb{R}^+, \mathbb{R})$ fulfill*

i) *v_2 is supported away from* 0 *and* $\mathrm{supp}(A_2) \subset \mathrm{supp}(v_2)$,

ii) $\|A_2/v_2\|_\infty < 1$,

iii) *the derivative $(A_2/v_2)'$ is bounded on $[0, \infty)$.*

We remark that condition ii) is essential for the transformation we use. As we already mentioned in the introduction, Lorentz boosts can manipulate two-dimensional electromagnetic field configurations which one may describe by (A, v). Since $L_\Lambda = e^{\mathbf{n}\cdot\boldsymbol{\sigma}\theta/2}$ represents the Lorentz boost in direction $\mathbf{n} \in \mathbb{R}^2$, we study the behaviour of h and h_k under the transformation $e^{\sigma_2\theta/2}$ on the corresponding Hilbert spaces:

Let $a < \infty$, then for $\theta \in C^1((a, \infty), \mathbb{R})$ we obtain

$$e^{-\sigma_2\theta/2}\sigma_1 e^{\sigma_2\theta/2} = e^{-\sigma_2\theta}\sigma_1 = (\cosh\theta - \sigma_2\sinh\theta)\sigma_1$$

by the commutator and multiplication properties of the Pauli matrices. Therefore, we have

$$\begin{aligned} e^{-\sigma_2\theta/2}\sigma_1(-\mathrm{i}\,\partial_x)e^{\sigma_2\theta/2} &= e^{-\sigma_2\theta/2}\sigma_1 e^{\sigma_2\theta/2}e^{-\sigma_2\theta/2}(-\mathrm{i}\,\partial_x)e^{\sigma_2\theta/2} \\ &= (\cosh\theta - \sigma_2\sinh\theta)\sigma_1\big(-\mathrm{i}\,\partial_x - \mathrm{i}\,\sigma_2\tfrac{\theta'}{2}\big) \qquad (2.25) \\ &= \cosh\theta(1 - \sigma_2\tanh\theta)\sigma_1\big(-\mathrm{i}\,\partial_x - \mathrm{i}\,\sigma_2\tfrac{\theta'}{2}\big) \end{aligned}$$

on $C_0^\infty((a, \infty), \mathbb{C}^2)$. For potentials (v, A) belonging to class (AS1) or (AS1′), condition ii) allows one to define the bounded function

$$\theta = \tanh^{-1}\beta := \tanh^{-1}\left(\frac{A_2}{v_2}\right) = \tanh^{-1}\left(\frac{A_2}{v}\right) \quad \in C^1((a, \infty), \mathbb{R}) \quad (2.26)$$

for $a = -\infty$ or $a = 0$ respectively. We set, as common in physics literature, $\gamma = \cosh\theta$, i.e. $\gamma^{-1} = \sqrt{1-\beta^2}$. For a Dirac operator h on $L^2(\mathbb{R},\mathbb{C}^2)$ with (A, v) satisfying (AS1) we use (2.26) and obtain

$$\begin{aligned} e^{-\sigma_2\theta/2} h e^{\sigma_2\theta/2} &= M\sigma_1\Big(-\mathrm{i}\,\partial_x - \mathrm{i}\,\sigma_2\frac{\theta'}{2}\Big) + \Big(1-\sigma_2\frac{A_2}{v}\Big)v - \sigma_2 A_1 \\ &= M\Big[\sigma_1(-\mathrm{i}\,\partial_x) + v/\gamma - \gamma(1+\sigma_2\beta)\sigma_2 A_1 + \sigma_3\frac{\theta'}{2}\Big] \end{aligned} \tag{2.27}$$

on $C_0^\infty(\mathbb{R},\mathbb{C}^2)$, where the multiplication operator $M := \gamma(1-\sigma_2\beta)$ is bounded on $L^2(\mathbb{R},\mathbb{C}^2)$ with inverse $M^{-1} = \gamma(1+\sigma_2\beta)$. Respectively, the operator h_k, with potentials (A, v) satisfying (AS1′), transforms as

$$e^{-\sigma_2\theta/2} h_k e^{\sigma_2\theta/2} = M\Big[\sigma_1(-\mathrm{i}\,\partial_x) + v/\gamma + \gamma(1+\sigma_2\beta)\sigma_2\Big(\frac{k}{x} - A_1\Big) + \sigma_3\frac{\theta'}{2}\Big]$$

on $C_0^\infty(\mathbb{R}^+,\mathbb{C}^2)$. As above, we use the abbreviation $M = \gamma(1-\sigma_2\beta)$, though, in this case the operator M acts on $L^2(\mathbb{R}^+,\mathbb{C}^2)$. Summarising, we have the identities

$$e^{-\sigma_2\theta/2} h e^{\sigma_2\theta/2} = M\tilde{h} \quad \text{and} \quad e^{-\sigma_2\theta/2} h_k e^{\sigma_2\theta/2} = M\tilde{h}_k \tag{2.28}$$

on $C_0^\infty(\mathbb{R},\mathbb{C}^2)$ and on $C_0^\infty(\mathbb{R}^+,\mathbb{C}^2)$ respectively. The operator

$$\tilde{h} := \sigma_1(-\mathrm{i}\,\partial_x) - \gamma(1+\sigma_2\beta)\sigma_2 A_1 + v/\gamma + \sigma_3\frac{\theta'}{2} \quad \text{on } L^2(\mathbb{R},\mathbb{C}^2)$$

is of type (2.1) with a bounded perturbation term $\sigma_3\frac{\theta'}{2}$, hence essentially self-adjoint on $C_0^\infty(\mathbb{R},\mathbb{C}^2)$. Similarly,

$$\tilde{h}_k := \sigma_1(-\mathrm{i}\,\partial_x) + \gamma(1+\sigma_2\beta)\sigma_2\Big(\frac{k}{x} - A_1\Big) + v/\gamma + \sigma_3\frac{\theta'}{2} \quad \text{on } L^2(\mathbb{R}^+,\mathbb{C}^2)$$

is of type (2.6) (since $A_2 = 0$ in a vicinity of zero and therefore β), perturbed by a bounded operator. So for $k \in \mathbb{Z} + \frac{1}{2}$ also $\tilde{h}_k$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^+,\mathbb{C}^2)$. As a consequence, the operators $M\tilde{h}$ and $M\tilde{h}_k$ are closed on the corresponding domains $\mathcal{D}(\tilde{h})$ and $\mathcal{D}(\tilde{h}_k)$.

Theorem 2.16 (Lemma 1 of [MS15])**.** *Consider the operator h with potentials satisfying* (AS1)*, and for $k \in \mathbb{Z} + \frac{1}{2}$ the operators h_k with potentials satisfying* (AS1′)*. Then the domain relations $\mathcal{D}(\tilde{h}) = \mathcal{D}(M\tilde{h}) = e^{-\sigma_2\theta/2}\mathcal{D}(h)$ and $\mathcal{D}(\tilde{h}_k) = \mathcal{D}(M\tilde{h}_k) = e^{-\sigma_2\theta/2}\mathcal{D}(h_k)$ hold true. Further, we have for any $z \in \varrho(h) = \varrho(M\tilde{h})$ that*

$$(M\tilde{h} - z)^{-1} = e^{-\sigma_2\theta/2}(h-z)^{-1}e^{\sigma_2\theta/2},$$

as well as for any $z \in \varrho(h_k) = \varrho(M\tilde{h}_k)$ that

$$(M\tilde{h}_k - z)^{-1} = e^{-\sigma_2\theta/2}(h_k-z)^{-1}e^{\sigma_2\theta/2}.$$

Proof. We give the proof only for the operator h. The argument for h_k is completely analogous since in this case we have $C_0^\infty(\mathbb{R}^+, \mathbb{C}^2)$ as a common core for h_k and $M\tilde{h}_k$. The equality $\mathcal{D}(\tilde{h}) = \mathcal{D}(M\tilde{h})$ is a direct consequence of the bounded invertibility of M, following from ii) of (AS1). We note that $C_0^1(\mathbb{R}, \mathbb{C}^2)$, consisting of C^1-functions with compact support, is also contained in $\mathcal{D}(h)$ and in $\mathcal{D}(\tilde{h})$, hence it is also an operator core for h and $\tilde{h}$. Further, the first relation of (2.28) is also valid on $C_0^1(\mathbb{R}, \mathbb{C}^2)$ (which is an invariant subspace under the transformation $e^{\pm\theta/2\sigma_2}$). Therefore, we deduce, for $\varphi \in C_0^1(\mathbb{R}, \mathbb{C}^2)$, that

$$\left\|e^{-\sigma_2\theta/2}\varphi\right\|_{M\tilde{h}} \le \left\|e^{-\sigma_2\theta/2}\right\|_\infty \|\varphi\|_h\,, \tag{2.29}$$

$$\left\|e^{\sigma_2\theta/2}\varphi\right\|_h \le \left\|e^{\sigma_2\theta/2}\right\|_\infty \|\varphi\|_{M\tilde{h}}\,. \tag{2.30}$$

If $\varphi \in \mathcal{D}(h)$, we choose a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C_0^1(\mathbb{R}, \mathbb{C}^2)$ that converges to φ in the h-graph norm. By (2.29), the sequence $(e^{-\sigma_2\theta/2}\varphi_n)_{n\in\mathbb{N}} \subset C_0^1(\mathbb{R}, \mathbb{C}^2)$ is Cauchy in the $M\tilde{h}$-graph norm. Hence,

$$\lim_{n\to\infty} e^{-\sigma_2\theta/2}\varphi_n = e^{-\sigma_2\theta/2}\varphi \in \mathcal{D}(M\tilde{h}).$$

The inclusion $\mathcal{D}(M\tilde{h}) \subset e^{-\sigma_2\theta/2}\mathcal{D}(h)$ follows similarly by using estimate (2.30) instead of (2.29) . To derive $\varrho(h) = \varrho(M\tilde{h})$ and the resolvent identity for $z \in \varrho(h)$, first note that for any $z \in \mathbb{C}$ we have the operator identity

$$e^{-\sigma_2\theta/2}(h - z)e^{\sigma_2\theta/2} = (M\tilde{h} - z) \quad \text{on } \mathcal{D}(M\tilde{h}) = e^{-\sigma_2\theta/2}\mathcal{D}(h). \tag{2.31}$$

Recall that $z \in \varrho(h)$ iff $h - z : \mathcal{D}(h) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is bijective with bounded inverse, which holds (due to (2.31)) iff $M\tilde{h} - z : \mathcal{D}(M\tilde{h}) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is bijective with bounded inverse. Hence, the resolvent sets are equal. In this case, i.e. $z \in \varrho(h)$, it is easy to see that

$$(M\tilde{h} - z)^{-1} = e^{-\theta/2\sigma_2}(h - z)^{-1}e^{\theta/2\sigma_2}.$$

□

Now we want to discuss the second class of potentials (A, v), for which one can establish similar relations as in Theorem 2.16, but to operators with an almost vanishing scalar potential. As one may guess, this is possible when $v < A$ at large values in order to use a similar transformation as above with a reciprocal "boost speed" $\theta_* \approx v/A$. We introduce:

Assumption 2.17 (AS2). *$A, v \in L^2_{\text{loc}}(\mathbb{R}, \mathbb{R})$ such that $A = A_1 + A_2$, $v = v_1 + v_2$, where A_1, v_1 are compactly supported and $A_2, v_2 \in C^1(\mathbb{R}, \mathbb{R})$ fulfill*

i) *A_2 is supported away from* 0 *and* $\operatorname{supp}(v_2) \subset \operatorname{supp}(A_2)$,

ii) $\|v_2/A_2\|_\infty < 1$,

iii) *the derivative $(v_2/A_2)'$ is bounded on $\mathbb{R}$.*

Assumption 2.18 (AS2$'$)**.** *$A, v \in L^p_{\mathrm{loc}}([0,\infty),\mathbb{R})$ for some $p > 2$ such that $A = A_1 + A_2$, $v = v_1 + v_2$, where A_1, v_1 are compactly supported and $A_2, v_2 \in C^1(\mathbb{R}^+,\mathbb{R})$ fulfill*

i) *A_2 is supported away from* 0 *and* $\mathrm{supp}(v_2) \subset \mathrm{supp}(A_2)$*,*

ii) $\|v_2/A_2\|_\infty < 1$*,*

iii) *the derivative $(v_2/A_2)'$ is bounded on* $[0,\infty)$*.*

Analogously as in (2.26), we define for potentials (A, v) satisfying (AS2) or (AS2$'$)

$$\theta_* = \tanh^{-1}\beta_* := \tanh^{-1}\left(\frac{v_2}{A_2}\right) = \tanh^{-1}\left(\frac{v_2}{A}\right) \quad \in C^1((a,\infty),\mathbb{R}),$$

with $a = -\infty$ or $a = 0$ respectively. Note that θ_* is bounded. Further, we set $\gamma_*^{-1} = \sqrt{1-\beta_*^2}$ and $M_* = \gamma_*(1-\sigma_2\beta_*)$. The latter is again an invertible matrix function with inverse $M_*^{-1} = \gamma_*(1+\sigma_2\beta_*)$. For an operator h with potentials of class (AS2) we obtain

$$e^{-\sigma_2\theta_*/2} h e^{\sigma_2\theta_*/2} = M_*\left[\sigma_1(-\mathrm{i}\,\partial_x) - \sigma_2 A/\gamma_* + \gamma_*(1+\sigma_2\beta_*)v_1 + \sigma_3\frac{\theta_*'}{2}\right]$$

on $C_0^1(\mathbb{R},\mathbb{C}^2)$. Note that

$$\hat{h} := \sigma_1(-\mathrm{i}\,\partial_x) - \sigma_2 A/\gamma_* + \gamma_*(1+\sigma_2\beta_*)v_1 + \sigma_3\frac{\theta_*'}{2}$$

is again essentially self-adjoint on $C_0^\infty(\mathbb{R},\mathbb{C}^2)$ and has (beside the residual terms $\gamma_*(1+\sigma_2\beta_*)v_1$, $\sigma_3\theta_*'/2$) only the magnetic potential A/γ_*. If h_k is coupled to potentials of type (AS2$'$), we obtain

$$e^{-\sigma_2\theta_*/2} h_k e^{\sigma_2\theta_*/2} = M_*\hat{h}_k$$

on $C_0^1(\mathbb{R}^+,\mathbb{C}^2)$, with the operator

$$\hat{h}_k = \sigma_1(-\mathrm{i}\,\partial_x) + \sigma_2\left(\frac{k}{x} - A/\gamma_*\right) + M_*^{-1}v_1 + \sigma_2(M_*^{-1}-1)\frac{k}{x} + \sigma_3\frac{\theta_*'}{2}$$

on $L^2(\mathbb{R}^+,\mathbb{C}^2)$. Since $M_* = 1$ in a vicinity of 0, we see that $\sigma_2(M_*^{-1}-1)\frac{k}{x}$ is bounded. So h_k is again essentially self-adjoint on $C_0^\infty(\mathbb{R}^+,\mathbb{C}^2)$ for every $k \in \mathbb{Z}+\frac{1}{2}$. Analogously to Theorem 2.16, one proves

Theorem 2.19. *Consider the operator h with potentials satisfying* (AS2)*, and for $k \in \mathbb{Z}+\frac{1}{2}$ the operators h_k with potentials satisfying* (AS2$'$)*. Then the domain relations $\mathcal{D}(\hat{h}) = \mathcal{D}(M_*\hat{h}) = e^{-\sigma_2\theta_*/2}\mathcal{D}(h)$ and $\mathcal{D}(\hat{h}_k) = \mathcal{D}(M_*\hat{h}_k) = e^{-\sigma_2\theta_*/2}\mathcal{D}(h_k)$ hold true. Further, we have for any $z \in \varrho(h) = \varrho(M_*\hat{h})$ that*

$$(M_*\hat{h} - z)^{-1} = e^{-\sigma_2\theta_*/2}(h-z)^{-1}e^{\sigma_2\theta_*/2},$$

as well as for any $z \in \varrho(h_k) = \varrho(M_*\hat{h}_k)$ that

$$(M_*\hat{h}_k - z)^{-1} = e^{-\sigma_2\theta_*/2}(h_k-z)^{-1}e^{\sigma_2\theta_*/2}.$$

2.1.3 Spectral Properties

In this last part of the first section we shortly present some (partially known) facts about Dirac operators on the line and half-line. In particular, we focus on spectral features of h and of the h_k's for potential functions A and v that are allowed to grow at large values. Some of them are immediate consequences of the previous theorems on potential transformations.

1) First of all, we consider the case $A = 0$, i.e. the operators h and h_k with purely scalar potentials v. Since the operators $\sigma_1(-\mathrm{i}\,\partial_x)$ on $L^2(\mathbb{R}, \mathbb{C}^2)$ and $\sigma_1(-\mathrm{i}\,\partial_x)$ on $L^2(\mathbb{R}^+, \mathbb{C}^2)$ have purely absolutely continuous spectrum covering the whole real line (see Satz 15.31 of [Wei03]), we deduce from Proposition 2.12 that

$$\sigma_{\mathrm{ac}}(h) = \sigma_{\mathrm{ac}}(h_0) = \mathbb{R}, \quad \sigma_{\mathrm{sc}}(h) = \sigma_{\mathrm{sc}}(h_0) = \emptyset, \quad \sigma_{\mathrm{pp}}(h) = \sigma_{\mathrm{pp}}(h_0) = \emptyset.$$

   For $k \neq 0$ the spectrum of h_k has not that canonical structure. Indeed, even for potentials v with bounded support eigenvalues can occur if $|k| > \frac{1}{2}$ as shown in [Sch10]. However, assuming certain additional regularity on v, this can only happen at points of $\sigma(h_k) = \mathbb{R}$ that lie in the limit range of v, i.e. in the range of v at ∞; Outside that limit range of v the spectrum remains purely absolutely continuous (see Theorem 4 in [Sch10] for the precise statement). In particular, if $v(x) \to \infty$ as $x \to \infty$ this implies again the absence of pure point and singular continuous spectrum for h_k.

2) Now assume that $v = 0$. Because the potential function A describes in our considerations a magnetic field, we are particularly interested in statements when A^2 grows at $(\pm)\infty$.
   For h assume that A satisfies the regularity conditions of Assumption 2.17. Then for $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{C}^2)$ with $\operatorname{supp}(\varphi) \cap \operatorname{supp}(A_1) = \emptyset$ we compute

$$\begin{aligned}
\|h\varphi\|^2 &= \big\langle \big(\sigma_1(-\mathrm{i}\,\partial_x) - \sigma_2 A_2\big)\varphi, \big(\sigma_1(-\mathrm{i}\,\partial_x) - \sigma_2 A_2\big)\varphi \big\rangle \\
&= \langle \varphi', \varphi' \rangle + \langle A_2\varphi, A_2\varphi \rangle + \langle -\mathrm{i}\,\varphi', -\mathrm{i}\,\sigma_3 A_2 \varphi \rangle - \langle -\mathrm{i}\,\sigma_3 A_2\varphi, -\mathrm{i}\,\varphi' \rangle \\
&= \langle \varphi', \varphi' \rangle + \langle A_2\varphi, A_2\varphi \rangle - \langle \varphi, \sigma_3 A_2' \varphi \rangle \\
&\geq \langle A_2\varphi, A_2\varphi \rangle - \langle \varphi, \sigma_3 A_2' \varphi \rangle.
\end{aligned}$$

   In the case of h_k assume that A satisfies the regularity conditions of Assumption 2.18. Then for $\varphi \in C_0^\infty(\mathbb{R}^+, \mathbb{C}^2)$ with $\operatorname{supp}(\varphi) \cap \operatorname{supp}(A_1) = \emptyset$ we obtain (c.f. Chapter 7.3 of [Tha92])

$$\|h_k\varphi\|^2 \geq \langle A_2\varphi, A_2\varphi \rangle - \langle \varphi, \sigma_3 A_2' \varphi \rangle - 2\Big\langle \frac{k}{x}\varphi, A_2\varphi \Big\rangle.$$

   By Remark B.2 of the Appendix, we conclude that if

- $A_2^2(x) \longrightarrow \infty \quad \text{as } |x| \to \infty,$
- $A_2'(x)/A_2^2(x) \longrightarrow 0 \quad \text{as } |x| \to \infty,$

then the operators h and h_k, with $k \neq 0$, have purely discrete spectrum, i.e. $\sigma(h)$, $\sigma(h_k)$ consist only of eigenvalues of finite multiplicity.

3) Unlike scalar potentials v, strong off-diagonal potential functions $\sigma_2 A$ do produce eigenvalues and spectral gaps. In view of the discussion in the previous subsection for preparing Theorem 2.19, it is plausible that the quality of the spectrum of h and h_k does not change as long as v is small compared to A at $(\pm)\infty$. Let us make that precise:
Consider h and h_k (with $k \neq 0$) coupled to potentials (A, v) of class (AS2) and (AS2′) respectively. Then, by Theorem 2.19, we know that (using the notation of Subsection 2.1.2)

$$(h - \mathrm{i}\,)^{-1} \quad \text{is compact} \quad \Longleftrightarrow \quad (M_* \hat{h} - \mathrm{i}\,)^{-1} \quad \text{is compact},$$

which analogously holds for h_k and $\hat{h}_k$. In view of $\mathcal{D}(\hat{h}) = \mathcal{D}(M_*\hat{h})$, the closed graph Theorem (see Chapter 4 of [Wei00]) implies that the operators $(\hat{h} - \mathrm{i}\,)(M_*\hat{h} - \mathrm{i}\,)^{-1}$ and $(M_*\hat{h} - \mathrm{i}\,)(\hat{h} - \mathrm{i}\,)^{-1}$ are bounded (here $\hat{h}$ can be replaced by $\hat{h}_k$). Hence, we deduce that

$$\begin{aligned} (h - \mathrm{i}\,)^{-1} \quad &\text{is compact} \quad \Longleftrightarrow \quad (\hat{h} - \mathrm{i}\,)^{-1} \quad \text{is compact},\\ (h_k - \mathrm{i}\,)^{-1} \quad &\text{is compact} \quad \Longleftrightarrow \quad (\hat{h}_k - \mathrm{i}\,)^{-1} \quad \text{is compact}. \end{aligned}$$

The operators $\hat{h}$ and $\hat{h}_k$ have, beside the bounded residual terms $\sigma_3\theta_*'/2$ and $\sigma_2(M_*^{-1} - 1)\frac{1}{x}$, only a magnetic potential. As in 2), we see that if

- $(A_2/\gamma_*)^2(x) \longrightarrow \infty \quad \text{as } |x| \to \infty,$
- $(A_2/\gamma_*)'(x)/(A_2/\gamma_*)^2(x) \longrightarrow 0 \quad \text{as } |x| \to \infty,$

the operators $\hat{h}$ and $\hat{h}_k$ have compact resolvent. This enables us to generalise the case $v = 0$ to

Corollary 2.20. *Consider the operators h and h_k, with $k \neq 0$, coupled to potentials (A, v) that satisfy Assumptions 2.17 and 2.18 respectively. Assume that*

a) $A_2^2(x) \longrightarrow \infty \quad \text{as } |x| \to \infty,$
b) $A_2'(x)/A_2^2(x) \longrightarrow 0 \quad \text{as } |x| \to \infty,$

then h and h_k have compact resolvent, i.e. $\sigma(h) = \sigma_{\mathrm{disc}}(h)$ and $\sigma(h_k) = \sigma_{\mathrm{disc}}(h_k)$.

We remark that one can also obtain Corollary 2.20 by a more direct argument (see Appendix A of [MS14]). This second argument bypasses potential transformations, so one can see that the statement of Corollary 2.20 holds true, even if one weakens the regularity condition on v_2 and drop the boundedness of $(v_2/A_2)'$.

4) Finally, we have a look on the quality of the spectrum of h and h_k when v is large compared to A. The discussion on potential transformations suggests that for potentials (A, v) with $|A| < |v|$ at ∞ (c.f. Assumptions 2.14 and 2.15), the operators h and h_k should also be spectrally related to operators with an almost pure scalar potential. Deducing this from the resolvent identities of Theorem 2.16 is not that canonical as in case 3) above. However, there exist already results confirming that if v dominates A at large values, the spectrum of h and h_k is again purely absolutely continuous. These are immediate consequences of the Gilbert-Pearson theory (see e.g. Section 15.7 of [Wei03]), which gives the absolute continuity of the spectrum due to the boundedness of the solutions of the corresponding ODE. In our context this technique leads to

Proposition 2.21. *Consider the operators h and h_k, with $k \neq 0$, coupled to potentials (A, v) that satisfy Assumptions 2.14 and 2.15 respectively. Assume that*

a) $|v_2(x)| \longrightarrow \infty$ as $x \to \infty$,

b) $(A_2/v_2)'$ *is integrable at* ∞,

then $\sigma(h) = \sigma(h_k) = \mathbb{R}$ is purely absolutely continuous, i.e. $\sigma_{\mathrm{sc}}(h) = \sigma_{\mathrm{pp}}(h) = \emptyset$ and $\sigma_{\mathrm{sc}}(h_k) = \sigma_{\mathrm{pp}}(h_k) = \emptyset$.

A proof of this statement is given in [SY98] for even more general conditions and also for Dirac operators on the (half-)line with mass terms. We remark that the additional condition on the integrability of $(A_2/v_2)'$ also means that the term $\sigma_3\theta'/2$ of $\tilde{h}$ and $\tilde{h}_k$ can be seen as an L^1-perturbation. In addition, one also needs condition a) of Proposition 2.21 to exclude eigenvalues for h_k with pure scalar potential (i.e. $A = 0$).

2.2 Dirac and Pauli Operators in Dimension Two

2.2.1 Self-Adjointness and Gauge Invariance

In this section we establish the mathematical setup for the massless Dirac operator and the Pauli operator in dimension two.

Let $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and $V \in L^q_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$, with $p, q \in (2, \infty)$. The magnetic Dirac operator on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ is a priori given by

$$D_{\mathbf{A}}\psi := \big[\boldsymbol{\sigma} \cdot (-\mathrm{i}\,\nabla - \mathbf{A})\big]\psi, \qquad \psi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2), \tag{2.32}$$

and the massless Dirac Hamiltonian on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ by

$$H_D\psi := \big[D_{\mathbf{A}} + V\big]\psi, \qquad \psi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2). \tag{2.33}$$

For such weak regularity conditions on $\mathbf{A}$ the magnetic field

$$B = \operatorname{curl}\mathbf{A} := \partial_1 A_2 - \partial_2 A_1 \tag{2.34}$$

is in general defined in distributional sense. Due to [Che77], the operators $D_{\mathbf{A}}$ and H_D, as defined in (2.32) and (2.33), are essentially self-adjoint. As usual, we denote the self-adjoint extensions of $D_{\mathbf{A}}$ and H_D by the same symbols and their domains by $\mathcal{D}(D_{\mathbf{A}})$ and $\mathcal{D}(H_D)$ respectively.

Remark 2.22. *For any given $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and $V \in L^q_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$, with $p, q \in (2, \infty)$, the self-adjoint operator H_D on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ is locally compact.*

Proof. For $R > 0$ let $\chi_R \in C_0^\infty(\mathbb{R}^2, [0,1])$ be a smooth characteristic function on the ball of radius R, i.e.

$$\chi_R(\mathbf{x}) = \begin{cases} 1 & \text{for } |\mathbf{x}| \leq R, \\ 0 & \text{for } |\mathbf{x}| \geq R+1. \end{cases}$$

The right hand side of the resolvent identity

$$\frac{1}{D_{\mathbf{A}} + V - \mathrm{i}}\,\chi_R - \chi_R\,\frac{1}{D_0 - \mathrm{i}} = \frac{1}{D_{\mathbf{A}} + V - \mathrm{i}}\left[(\boldsymbol{\sigma}\cdot\mathbf{A} - V)\chi_R - \boldsymbol{\sigma}\cdot(-\mathrm{i}\,\nabla\chi_R)\right]\frac{1}{D_0 - \mathrm{i}}$$

is compact since due to the Kato-Seiler-Simon inequality, we know that the product $W(D_0 - \mathrm{i}\,)^{-1}$ is compact whenever $W \in L^p(\mathbb{R}^2, \mathbb{C}^{2\times 2})$ for some $p > 2$. Consequently, also the operator $(D_{\mathbf{A}} + V - \mathrm{i}\,)^{-1}\chi_R$ is compact. □

Since the physics of the described system should depend on the magnetic field B and not on the vector potential $\mathbf{A}$, we briefly discuss the gauge invariance of H_D: Assume that the two vector potentials $\mathbf{A}, \tilde{\mathbf{A}} \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, with $p > 2$, generate the same magnetic field, i.e.

$$\operatorname{curl}\mathbf{A} = \operatorname{curl}\tilde{\mathbf{A}} = B.$$

Then, as shown in [Lei83], there exists a gauge function $\Gamma \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$ such that

$$\tilde{\mathbf{A}} = \mathbf{A} + \nabla\Gamma.$$

Let $V \in L^q_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$ be a given scalar potential. Using a local approximation of Γ by smooth functions (with respect to the $W^{1,p}$-norm), one can verify that $e^{-\mathrm{i}\,\Gamma}C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \subset \mathcal{D}(D_{\mathbf{A}} + V)$ and that

$$e^{\mathrm{i}\,\Gamma}(D_{\mathbf{A}} + V)e^{-\mathrm{i}\,\Gamma}\psi = (D_{\tilde{\mathbf{A}}} + V)\psi \tag{2.35}$$

holds for $\psi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ (see [KS12] for a detailed calculation). Because $D_{\tilde{\mathbf{A}}} + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$, we conclude that

$$e^{-\mathrm{i}\,\Gamma}\,\mathcal{D}(D_{\tilde{\mathbf{A}}} + V) \subset \mathcal{D}(D_{\mathbf{A}} + V) \tag{2.36}$$

and that (2.35) is also valid for $\psi \in \mathcal{D}(D_{\tilde{\mathbf{A}}} + V)$. It is easy to see that even "=" holds in (2.36), thus the two operators $D_{\mathbf{A}} + V$ and $D_{\tilde{\mathbf{A}}} + V$ are unitarily equivalent.

The Pauli operator without an electric potential can, for $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and $p > 2$, be simply defined as the square of the magnetic Dirac operator, i.e. $P_{\mathbf{A}} = D^2_{\mathbf{A}}$. Clearly, it is a self-adjoint operator on the natural domain $\{\psi \in \mathcal{D}(D_{\mathbf{A}}) \,|\, D_{\mathbf{A}}\psi \in \mathcal{D}(D_{\mathbf{A}})\}$, and since $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ is an operator core of $D_{\mathbf{A}}$, it is also a form core for $P_{\mathbf{A}}$. We remark that it is possible to define the magnetic Pauli operator $P_{\mathbf{A}}$ for even more singular vector potentials than L^2_{loc} by using proper quadratic forms (see [EV02]). However, we deal with strong negative scalar potentials V and therefore we cannot use quadratic form methods to define the full Pauli Hamiltonian. Instead, we establish the self-adjoint operator $P_{\mathbf{A}} + V$ on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ by using essential self-adjointness on a nice core:

Proposition 2.23. *Assume that $\mathbf{A} \in C^{1,\alpha}(\mathbb{R}^2, \mathbb{R}^2)$ for some $\alpha \in (0, 1)$, i.e. assume that the derivatives of $\mathbf{A}$ are locally (uniformly) α-Hölder continuous. In addition, let $V \in C^\alpha(\mathbb{R}^2, \mathbb{R})$ satisfy the lower bound condition*

$$V(\mathbf{x}) \;\geq\; -c|\mathbf{x}|^2 + d\,, \qquad \mathbf{x} \in \mathbb{R}^2, \tag{2.37}$$

for some constants $c > 0$, $d \in \mathbb{R}$. Then the densely defined operator

$$H_P\psi = \big[[\boldsymbol{\sigma} \cdot (-\mathrm{i}\,\nabla - \mathbf{A})]^2 + V\big]\psi, \qquad \psi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$$

is essentially self-adjoint.

A proof of this Proposition can be found in [Iwa90] or [Meh15]. Note that condition (2.37) on the maximal growth rate of the negative part of V is the same as for the classical Schrödinger operator $-\Delta + V$. Therefore, (2.37) cannot be relaxed or dropped. The regularity conditions on $\mathbf{A}$ and V are quite strong compared to those in the magnetic Schrödinger case (see, e.g. Chapter X of [RS75] or Chapter 1 of [CFKS87]). This results from the fact that the proof of Proposition 2.23 does not rely on quadratic form techniques, which involve the diamagnetic inequality for $-\mathrm{i}\,\nabla - \mathbf{A}$. Instead, one proves Proposition 2.23 by arguing that $\ker(H_P^* \pm \mathrm{i}\,) = \{0\}$, requiring more regularity on the potential functions. One might be able to extend this method up to potentials $\mathbf{A} \in L^\infty_{\mathrm{loc}}$, with $\operatorname{div}\mathbf{A}, \operatorname{curl}\mathbf{A} \in L^\infty_{\mathrm{loc}}$, and $V \in L^\infty_{\mathrm{loc}}$, but still that is much compared to the regularity conditions on the magnetic Schrödinger operator. Anyway, for our purposes we are contented with Proposition 2.23, even if this remains an interesting problem.

Remark 2.24. *For $V = 0$ the self-adjoint operator H_P, given by Proposition 2.23, coincides with the operator square of $D_{\mathbf{A}}$, i.e. also $D^2_{\mathbf{A}}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$.*

Remark 2.25. *The self-adjoint operator H_P on $L^2(\mathbb{R}^2, \mathbb{C}^2)$, given by Proposition 2.23, is locally compact.*

The statement of the last remark can easily be verified using a comparison argument as in the proof of Remark 2.22: the local compactness of $D_{\mathbf{A}}$ implies immediately the one of $D_{\mathbf{A}}^2$, hence one may compare the resolvents of H_P and $D_{\mathbf{A}}^2$.

Remark 2.26. *If $\mathbf{A}, \tilde{\mathbf{A}} \in C^{1,\alpha}(\mathbb{R}^2, \mathbb{R}^2)$ are two different gauges of the same magnetic field B, then the operators $H_P = P_{\mathbf{A}} + V$ and $\tilde{H}_P = P_{\tilde{\mathbf{A}}} + V$ on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ are unitarily equivalent.*

2.2.2 Supersymmetry, Zero Modes and Landau Levels

In this subsection we want to point out some particular features of the magnetic Dirac operator $D_{\mathbf{A}}$ and of its square $P_{\mathbf{A}}$, the magnetic Pauli operator. As in the definition of $D_{\mathbf{A}}$, assume that $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p > 2$. First, let us briefly recapitulate the relation between the positive and negative part of $D_{\mathbf{A}}$ and describe the structure of $\ker(D_{\mathbf{A}})$. To this aim, we introduce the following orthogonal projectors on $L^2(\mathbb{R}^2, \mathbb{C}^2)$:

$$\Pi_- := \mathbb{1}_{(-\infty,0)}(D_{\mathbf{A}}), \quad \Pi_0 := \mathbb{1}_{\{0\}}(D_{\mathbf{A}}), \quad \Pi_+ := \mathbb{1}_{(0,\infty)}(D_{\mathbf{A}}). \tag{2.38}$$

Further, we set $\Pi_0^{\perp} := \mathrm{id} - \Pi_0$, which denotes the projection onto $\ker(D_{\mathbf{A}})^{\perp}$.

Since the self-adjoint operator $D_{\mathbf{A}}$ on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ has an off-diagonal structure, we can write

$$D_{\mathbf{A}} \restriction_{C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)} \;=\; \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix}, \tag{2.39}$$

with the closable operators

$$d := (-\mathrm{i}\,\partial_1 - A_1) + \mathrm{i}\,(-\mathrm{i}\,\partial_2 - A_2) \qquad \text{on } C_0^\infty(\mathbb{R}^2, \mathbb{C}),$$

and

$$d^* := (-\mathrm{i}\,\partial_1 - A_1) - \mathrm{i}\,(-\mathrm{i}\,\partial_2 - A_2) \qquad \text{on } C_0^\infty(\mathbb{R}^2, \mathbb{C}).$$

As usual, we denote also their closures by d and d^*. The notation already indicates that d^* is the adjoint operator of d, which is a direct consequence of the essential self-adjointness of $D_{\mathbf{A}} \restriction_{C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)}$ (see Section 5.2.2 of [Tha92]). We extend identity (2.39) to

$$D_{\mathbf{A}} \;=\; \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} \qquad \text{on } \mathcal{D}(D_{\mathbf{A}}) = \mathcal{D}(d) \oplus \mathcal{D}(d^*), \tag{2.40}$$

and remark that

$$\ker(D_{\mathbf{A}}) = \ker(d) \oplus \ker(d^*), \qquad \ker(D_{\mathbf{A}})^{\perp} = \ker(d)^{\perp} \oplus \ker(d^*)^{\perp},$$

as well as

$$\Pi_0 = \begin{pmatrix} \pi & 0 \\ 0 & \pi_* \end{pmatrix}, \qquad \Pi_0^\perp = \begin{pmatrix} \pi^\perp & 0 \\ 0 & \pi_*^\perp \end{pmatrix}.$$

Obviously, π, π_* denote the orthogonal projections onto $\ker(d)$, $\ker(d^*)$, and analogously $\pi^\perp$, $\pi_*^\perp$ the orthogonal projections onto $\ker(d)^\perp$, $\ker(d^*)^\perp$. Due to the off-diagonal matrix structure of $D_{\mathbf{A}}$, one has the same structure for the signum of $D_{\mathbf{A}}$, i.e.

$$\operatorname{sgn}(D_{\mathbf{A}}) := \frac{D_{\mathbf{A}}}{|D_{\mathbf{A}}|} = \begin{pmatrix} 0 & s^* \\ s & 0 \end{pmatrix} \qquad \text{on } \ker(D_{\mathbf{A}})^\perp.$$

Since $\operatorname{sgn}(D_{\mathbf{A}})$ is an involution on $\ker(D_{\mathbf{A}})^\perp$, the maps

$$s: \ \ker(d)^\perp \to \ker(d^*)^\perp, \qquad s^*: \ \ker(d^*)^\perp \to \ker(d)^\perp \tag{2.41}$$

are unitary and adjoint to each other. Taking the square of (2.40) results in the diagonal operator

$$D_{\mathbf{A}}^2 = \begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} \qquad \text{on } \mathcal{D}(D_{\mathbf{A}}^2) = \{\psi \in \mathcal{D}(D_{\mathbf{A}}) \,|\, D_{\mathbf{A}}\psi \in \mathcal{D}(D_{\mathbf{A}})\}.$$

The diagonal entries d^*d and dd^* are self-adjoint on the domains $\mathcal{D}(d^*d) = \{\phi \in \mathcal{D}(d) \,|\, d\phi \in \mathcal{D}(d^*)\}$ and $\mathcal{D}(dd^*) = \{\phi \in \mathcal{D}(d^*) \,|\, d^*\phi \in \mathcal{D}(d)\}$ respectively. Considering the operator identity $D_{\mathbf{A}}^2 = \operatorname{sgn}(D_{\mathbf{A}}) D_{\mathbf{A}}^2 \operatorname{sgn}(D_{\mathbf{A}})$ on $\ker(D_{\mathbf{A}})^\perp$ by components, we get

$$\begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} \psi = \begin{pmatrix} s^*dd^*s & 0 \\ 0 & sd^*ds^* \end{pmatrix} \psi, \tag{2.42}$$

for $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$ with $\psi_1 \in \mathcal{D}(d^*d) \cap \ker(d)^\perp$ and $\psi_2 \in \mathcal{D}(dd^*) \cap \ker(d^*)^\perp$. In particular, $d^*d\restriction_{\ker(d)^\perp}$ and $dd^*\restriction_{\ker(d^*)^\perp}$ are unitarily equivalent and satisfy

$$\sigma(d^*d)\backslash\{0\} = \sigma(dd^*)\backslash\{0\} \subset (0, \infty).$$

The operators d^*d and dd^* are useful for diagonalising the matrix structure of $D_{\mathbf{A}}$. Indeed, using the Foldy-Wouthuysen transformation (see, e.g. [KS12] for an explicit computation), one can see that $D_{\mathbf{A}}$ is unitarily equivalent to

$$\begin{pmatrix} \sqrt{d^*d} & 0 \\ 0 & -\sqrt{dd^*} \end{pmatrix}.$$

We conclude

Proposition 2.27. *The positive and negative part of the self-adjoint operator $D_{\mathbf{A}}$ admit the unitarily equivalent representation*

$$\Pi_+ D_{\mathbf{A}} \Pi_+ \cong \sqrt{d^*d}\restriction_{\ker(d)^\perp} \quad \text{and} \quad \Pi_- D_{\mathbf{A}} \Pi_- \cong -\sqrt{dd^*}\restriction_{\ker(d^*)^\perp}$$

in terms of d^*d and dd^*. In particular, $-\Pi_- D_{\mathbf{A}} \Pi_-$ and $\Pi_+ D_{\mathbf{A}} \Pi_+$ are unitarily equivalent, hence the spectrum of $D_{\mathbf{A}}$ is symmetric with respect to the origin.

One can even gain a bit more information on the spectrum of $D_{\mathbf{A}}$ if one looks at the (formal) commutator of d and d^*: For $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R}^2, \mathbb{C})$ we obtain

$$\langle d^*\phi_1, d^*\phi_2\rangle = \sum_{j=1}^{2} \big\langle(-\mathrm{i}\,\partial_j - A_j)\phi_1, (-\mathrm{i}\,\partial_j - A_j)\phi_2\big\rangle + \langle\phi_1, B\phi_2\rangle,$$

$$\langle d\phi_1, d\phi_2\rangle = \sum_{j=1}^{2} \big\langle(-\mathrm{i}\,\partial_j - A_j)\phi_1, (-\mathrm{i}\,\partial_j - A_j)\phi_2\big\rangle - \langle\phi_1, B\phi_2\rangle,$$

with $B = \operatorname{curl}\mathbf{A}$, as defined in (2.34), considered as a distribution. Combining both identities, we deduce that

$$\langle d^*\phi_1, d^*\phi_2\rangle - \langle d\phi_1, d\phi_2\rangle = \langle\phi_1, 2B\phi_2\rangle \tag{2.43}$$

for $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R}^2, \mathbb{C})$ and observe

Proposition 2.28. *For $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, with $p > 2$, assume that $B = \operatorname{curl}\mathbf{A} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$ is bounded from below. Then*

$$\|D_{\mathbf{A}}\Pi_0^\perp\psi\|^2 = \|d^* s\pi^\perp\psi_1\|^2 + \|d^*\pi_*^\perp\psi_2\|^2 \geq \big\langle S\Pi_0^\perp\psi, 2BS\Pi_0^\perp\psi\big\rangle$$

for $\psi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$, hence also for $\psi \in \mathcal{D}(D_{\mathbf{A}})$. Here S denotes the map

$$S = \begin{pmatrix} s & 0 \\ 0 & \mathrm{id} \end{pmatrix} \qquad \text{on } \ker(d)^\perp \oplus L^2(\mathbb{R}^2, \mathbb{C}).$$

This proposition states that the operator $D_{\mathbf{A}}^2$ on $\Pi_0^\perp L^2(\mathbb{R}^2, \mathbb{C}^2)$ can be considered as bounded from below by $2B$ (up to unitary equivalence). Similarly, we deduce the lower bound $-2B$ on $\Pi_0^\perp L^2(\mathbb{R}^2, \mathbb{C}^2)$. As a consequence, the next corollary is also true if the assumptions hold for $-B$ instead of B.

Corollary 2.29. *For $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, with $p > 2$, assume that $B = \operatorname{curl}\mathbf{A} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$ is bounded from below. If $B \geq B_0 > 0$ on $\mathbb{R}^2$, then*

$$\big(-\sqrt{2B_0}, 0\big) \cup \big(0, \sqrt{2B_0}\big) \subset \varrho(D_{\mathbf{A}}).$$

Further, if $B \to \infty$ as $|\mathbf{x}| \to \infty$, then $D_{\mathbf{A}}\Pi_0^\perp$ on $\Pi_0^\perp L^2(\mathbb{R}^2, \mathbb{C}^2)$ has compact resolvent, i.e. $\sigma(D_{\mathbf{A}})\backslash\{0\}$ consists only of eigenvalues of finite multiplicity.

Now we take a closer look on the kernel $\ker(D_{\mathbf{A}}) = \ker(D_{\mathbf{A}}^2)$ of the operator $D_{\mathbf{A}}$. To this aim, it is convenient to work with the notation of complex calculus, i.e. we set $z = x_1 + \mathrm{i}\,x_2$ and use the differential operators

$$\partial_z := \frac{1}{2}(\partial_1 - \mathrm{i}\,\partial_2), \qquad \partial_{\bar{z}} := \frac{1}{2}(\partial_1 + \mathrm{i}\,\partial_2).$$

We can write d and d^* as

$$d = -2\mathrm{i}\,\partial_{\bar{z}} - (A_1 + \mathrm{i}\,A_2), \qquad d^* = -2\mathrm{i}\,\partial_z - (A_1 - \mathrm{i}\,A_2).$$

To employ this compact notation, suppose that $\operatorname{curl}\mathbf{A} \in L^q_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R})$ for some $q \in (1,2]$. Then, as shown in [EV02], one can find a solution $\Phi \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R})$ of the Laplace equation

$$\Delta\Phi = \operatorname{curl}\mathbf{A}. \tag{2.44}$$

Since $\tilde{\mathbf{A}} := (-\partial_2\Phi, \partial_1\Phi) \in L^2_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R})$ (see, e.g. [KS12]) satisfies $\operatorname{curl}\mathbf{A} = \operatorname{curl}\tilde{\mathbf{A}}$, we may assume that $\mathbf{A} = \tilde{\mathbf{A}}$ in view of the gauge invariance of $D_{\mathbf{A}}$. A simple calculation shows that

$$d = e^{-\Phi}(-2\mathrm{i}\,\partial_{\bar z})e^{\Phi} \qquad \text{and} \qquad d^* = e^{\Phi}(-2\mathrm{i}\,\partial_z)e^{-\Phi}$$

hold on $C_0^\infty(\mathbb{R}^2,\mathbb{C}^2)$. So the essential self-adjointness of $D_{\mathbf{A}}$ allows us to rewrite $\ker(D_{\mathbf{A}})$ in the following way:

$$\begin{aligned}
\psi \in \ker(D_{\mathbf{A}}) \;&\Leftrightarrow\; \langle D_{\mathbf{A}}\hat\psi, \psi\rangle = 0 \quad \forall\, \hat\psi \in C_0^\infty(\mathbb{R}^2,\mathbb{C}^2)\\
&\Leftrightarrow\; \langle e^{\Phi}(-2\mathrm{i}\,\partial_z)e^{-\Phi}\phi, \psi_1\rangle = 0 \quad \forall\, \phi \in C_0^\infty(\mathbb{R}^2,\mathbb{C}),\\
&\qquad \langle e^{-\Phi}(-2\mathrm{i}\,\partial_{\bar z})e^{\Phi}\phi, \psi_2\rangle = 0 \quad \forall\, \phi \in C_0^\infty(\mathbb{R}^2,\mathbb{C})\\
&\Leftrightarrow\; \partial_{\bar z}e^{\Phi}\psi_1 = 0, \qquad \partial_z e^{-\Phi}\psi_2 = 0,
\end{aligned}$$

where the equations in the last line have to be seen in the sense of distributions. We obtain

$$\ker(D_{\mathbf{A}}) = \left\{ \begin{pmatrix} \omega_1 e^{-\Phi} \\ \overline{\omega_2}\, e^{\Phi} \end{pmatrix} \in L^2(\mathbb{R}^2,\mathbb{C}^2) \,\middle|\, \omega_1,\omega_2 \text{ are entire in } x_1 + \mathrm{i}\,x_2 \right\}. \tag{2.45}$$

Note that the right hand side of (2.45) does not depend on the solution Φ of the Laplace equation (2.44). This representation of $\ker(D_{\mathbf{A}})$ is useful to prove (see, e.g. [RS06])

Proposition 2.30. *Let $\Phi \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R})$, with $q \in (1,2]$, be a potential function for the vector potential $\mathbf{A}$, i.e. assume that $\mathbf{A} = (-\partial_2\Phi, \partial_1\Phi)$. If the positive and negative part of $B = \operatorname{curl}\mathbf{A}$ satisfy*

$$\int_{\mathbb{R}^2} [B]_+ \,\mathrm{d}^2x = \infty, \qquad \int_{\mathbb{R}^2} [B]_- \,\mathrm{d}^2x < \infty,$$

then the space $\ker(d) = \{\omega e^{-\Phi} \in L^2(\mathbb{R}^2,\mathbb{C}) \,|\, \omega \text{ is entire in } x_1 + \mathrm{i}\,x_2\}$ *is infinite dimensional.*

To close this paragraph, we want to discuss an example of $D_{\mathbf{A}}$ that is completely solvable: The Landau-Dirac Hamiltonian denotes $D_{\mathbf{A}}$ in the case $B = B_0 \equiv$ *const*. Let us only discuss the case $B_0 > 0$. Using the potential function $\Phi(\mathbf{x}) = \frac{B_0}{4}(x_1^2 + x_2^2)$, generating the vector potential

$$\mathbf{A}(\mathbf{x}) = \frac{B_0}{2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

we can write (with the complex calculus notation)

$$d = -2\mathrm{i}\left(\partial_{\bar z} + \tfrac{B_0}{4} z\right), \qquad d^* = -2\mathrm{i}\left(\partial_z - \tfrac{B_0}{4}\bar z\right). \tag{2.46}$$

The commutator relation (2.43) reads as $[d, d^*] = dd^* - d^*d = 2B_0$ on $C_0^\infty(\mathbb{R}^2, \mathbb{C})$, hence the self-adjoint operators dd^* and d^*d differ only by the constant $2B_0$. Therefore, we have

$$\sigma(dd^*) = \sigma(d^*d) + 2B_0 \subset [2B_0, \infty),$$

and by the unitary equivalence of $d^*d\restriction_{\ker(d)^\perp}$ and $dd^*\restriction_{\ker(d^*)^\perp}$ we know that $\sigma(d^*d) \subset \{0\} \cup [2B_0, \infty)$. Iterating this argument leads to

$$\sigma(d^*d) \subset \{2nB_0 \,|\, n \in \mathbb{N}_0\}, \qquad \sigma(dd^*) \subset \{2nB_0 \,|\, n \in \mathbb{N}\}. \tag{2.47}$$

For showing equality in (2.47), note that any function

$$\Omega_m(z) := z^m e^{-\frac{B_0}{4} z\bar z} = (x_1 + \mathrm{i}\, x_2)^m e^{-\frac{B_0}{4}(x_1^2 + x_2^2)},$$

with $m \in \mathbb{N}_0$, satisfies $\Omega_m \in \ker(d)$. In addition, using (2.46), we have

$$\big[(d^*)^n \Omega_m\big](z) = (\mathrm{i}\, B_0)^n \bar z^n z^m e^{-\frac{B_0}{4} z\bar z}, \quad z \in \mathbb{C},$$

for any $n, m \in \mathbb{N}_0$. Obviously, $(d^*)^{n+1}\Omega_m \in \mathcal{D}(d)$ and

$$dd^*(d^*)^n \Omega_m = 2(n+1)B_0(d^*)^n \Omega_m \tag{2.48}$$

holds for any $n, m \in \mathbb{N}_0$, i.e. $(d^*)^n\Omega_m$ is an eigenfunction of the operator dd^* for the eigenvalue $2(n+1)B_0$. Consequently,

$$\sigma(dd^*) = \sigma(d^*d) + 2B_0 = \{2nB_0 \,|\, n \in \mathbb{N}\}$$

and each eigenvalue is infinitely degenerate. Thus, Proposition 2.27 implies that

$$\sigma(D_{\mathbf{A}}) = \sigma_{\mathrm{ess}}(D_{\mathbf{A}}) = \big\{ \pm \sqrt{2nB_0} \;|\; n \in \mathbb{N}_0 \big\}.$$

A simple computation shows that for any $m \in \mathbb{N}_0$

$$\psi^-_{n,m} = \begin{pmatrix} (d^*)^n\Omega_m \\ -\sqrt{2nB_0}(d^*)^{n-1}\Omega_m \end{pmatrix} \quad \text{and} \quad \psi^+_{n,m} = \begin{pmatrix} (d^*)^n\Omega_m \\ \sqrt{2nB_0}(d^*)^{n-1}\Omega_m \end{pmatrix}$$

are (not normalised) eigenfunctions of $D_{\mathbf{A}}$ associated to the values $-\sqrt{2nB_0}$ and $\sqrt{2nB_0}$ respectively.

2.2.3 Two-Dimensional Systems with Symmetry

There are two important types of symmetries in dimension two that allow a decomposition of the operators H_D and H_P in a family of (Dirac and Schrödinger) operators on the (half-)line: The invariance of the system under rotations of the plane and the invariance of the system under translations in one distinct direction (for example the x_2-direction). To discuss a few consequences of this two symmetries, we assume here that $\mathbf{A} \in C^{1,\alpha}(\mathbb{R}^2, \mathbb{R}^2)$ (and therefore $\operatorname{curl} \mathbf{A} = B \in C^\alpha(\mathbb{R}^2, \mathbb{R})$), and that $V \in C^\alpha(\mathbb{R}^2, \mathbb{R})$ for some $\alpha \in (0, 1)$, even if a part of the following considerations is also valid for less regularity.

Assume that the considered system in dimension two is invariant under rotations, i.e. the electro-magnetic field configuration is invariant under rotations of the plane. Then we can write $B(\mathbf{x}) = b(|\mathbf{x}|)$ and $V(\mathbf{x}) = v(|\mathbf{x}|)$ for $\mathbf{x} \in \mathbb{R}^2$, where $b, v : \mathbb{R}^+ \to \mathbb{R}$ are sufficiently regular. To perform the polar decomposition, suppose that $\mathbf{A}$ is in the radial gauge

$$\mathbf{A}(\mathbf{x}) = \frac{A(|\mathbf{x}|)}{|\mathbf{x}|} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \qquad A(r) = \frac{1}{r} \int_0^r b(s) s \mathrm{d}s. \tag{2.49}$$

Let H_D be the self-adjoint operator given by (2.33). As carried out in Chapter 7.3 of [Tha92] (or in Section 6 of [KS12]), one can find a unitary map

$$U : \; L^2(\mathbb{R}^2, \mathbb{C}^2) \longrightarrow \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} L^2(\mathbb{R}^+, \mathbb{C}^2)$$

such that

$$U H_D U^* = \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} Q h_k Q^* \cong \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} h_k,$$

where, for $k \in \mathbb{Z} + \frac{1}{2}$,

$$h_k = \sigma_1(-\mathrm{i}\, \partial_r) + \sigma_2 \left(\frac{k}{r} - A \right) + v \quad \text{ on } L^2(\mathbb{R}^+, \mathbb{C}^2)$$

and $Q = 1/\sqrt{2}\,(1 - \mathrm{i}\,\sigma_3)$ (c.f. Proposition 2.10). Recall that we have already discussed the half-line operators h_k in Section 2.1. The decomposition is also valid for bounded functions of H_D, i.e. for $\chi \in L^\infty(\mathbb{R}, \mathbb{C})$ we have

$$\chi(H_D) \cong \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} \chi(h_k). \tag{2.50}$$

Hence, the spectral family of H_D satisfies

$$\mathbb{1}_{(-\infty, \lambda]}(H_D) \cong \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} \mathbb{1}_{(-\infty, \lambda]}(h_k),$$

with $\lambda \in \mathbb{R}$, resulting in

$$\sigma_\#(H_D) = \overline{\bigcup_{k \in \mathbb{Z} + \frac{1}{2}} \sigma_\#(h_k)} \qquad \text{for } \# \in \{\mathrm{pp}, \mathrm{sc}, \mathrm{ac}\}.$$

From the spectral properties of the h_k (see Subsection 2.1.3) we deduce

Corollary 2.31. *Let $b, v \in C^\alpha([0,\infty),\mathbb{R})$, with $\alpha \in (0,1)$, be the radii-dependent notation of B, V. Suppose that $A(r) = \frac{1}{r}\int_0^r b(s)s\mathrm{d}s$ and v satisfy Assumption 2.18 and, in addition, that*

a) $A^2(r) \longrightarrow \infty \quad \text{as } r \to \infty,$

b) $A'(r)/A^2(r) \longrightarrow 0 \quad \text{as } r \to \infty.$

Then $\sigma_{\mathrm{sc}}(H_D) = \sigma_{\mathrm{ac}}(H_D) = \emptyset$, i.e. H_D has only pure point spectrum.

Corollary 2.32. *Let $b, v \in C^\alpha([0,\infty),\mathbb{R})$, with $\alpha \in (0,1)$, be the radii-dependent notation of B, V. Suppose that $A(r) = \frac{1}{r}\int_0^r b(s)s\mathrm{d}s$ and v satisfy Assumption 2.15 and, in addition, that*

a) $|v(r)| \longrightarrow \infty \quad \text{as } r \to \infty,$

b) $(A/v)'$ *is an integrable function.*

Then $\sigma_{\mathrm{sc}}(H_D) = \sigma_{\mathrm{pp}}(H_D) = \emptyset$ and $\sigma(H_D) = \sigma_{\mathrm{ac}}(H_D) = \mathbb{R}$.

Analogously, in the case of rotationally symmetric fields we decompose the two-dimensional Pauli operator H_P (defined in Proposition 2.23) as $UH_PU^* = \bigoplus_{j\in\mathbb{Z}} g_j$, with

$$g_j := \begin{pmatrix} -\partial_r^2 + \frac{j^2-\frac{1}{4}}{r^2} & 0 \\ 0 & -\partial_r^2 + \frac{(j+1)^2-\frac{1}{4}}{r^2} \end{pmatrix} + A^2 - \frac{j+\frac{1}{2}}{r}A + \sigma_3 A' + v$$

on $L^2(\mathbb{R}^+,\mathbb{C}^2)$ (see Chapter 7.3 of [Tha92]). Again we conclude

$$\sigma_\#(H_P) = \overline{\bigcup_{j\in\mathbb{Z}} \sigma_\#(g_j)}\,, \qquad \text{for } \# \in \{\mathrm{pp}, \mathrm{sc}, \mathrm{ac}\}.$$

Note that the half-line operators g_j are well-studied in Sturm-Liouville theory. For our purposes we just want to point out:

Remark 2.33. *Let $b, v \in C^\alpha([0,\infty),\mathbb{R})$, with $\alpha \in (0,1)$, be the radii-dependent notation of B, V. Suppose that $A(r) = \frac{1}{r}\int_0^r b(s)s\mathrm{d}s$ and v satisfy*

a) $A^2(r) \longrightarrow \infty \quad \text{as } r \to \infty,$

b) $A'(r)/A^2(r) \longrightarrow 0 \quad \text{as } r \to \infty,$

c) $\limsup_{r\to\infty} |v(r)|/A^2(r) < 1.$

Then $\sigma_{\mathrm{ac}}(H_P) = \sigma_{\mathrm{sc}}(H_P) = \emptyset$, i.e. H_P has only pure point spectrum.

This remark can easily be proven by using the results from Chapter 13.4 of [Wei03]. Further, one can apply the Gilbert-Pearson theory to deduce that if (roughly speaking, for precise conditions see [Sto92])

- $v(r) \to -\infty$ as $r \to \infty$,
- $\limsup_{r\to\infty} A^2(r)/|v(r)| < 1$,

then $\sigma_{\rm sc}(H_P) = \sigma_{\rm pp}(H_P) = \emptyset$ and $\sigma_{\rm ac}(H_P) = \mathbb{R}$.

Now suppose that the electro-magnetic field configuration obeys a translational symmetry in one distinct direction. For $B(\mathbf{x}) = B(x_1)$ we choose the Landau gauge

$$\mathbf{A}(\mathbf{x}) = \begin{pmatrix} 0 \\ A(x_1) \end{pmatrix}, \qquad A(x_1) = \int_0^{x_1} B(s)\mathrm{d}s. \tag{2.51}$$

If V has the same direction of symmetry, meaning that $V(\mathbf{x}) = V(x_1)$, we can perform a Fourier transform in the x_2-direction. This results in the direct integral representations

$$H_D \cong \int_{\mathbb{R}}^{\oplus} h(\xi)\,\mathrm{d}\xi, \qquad H_P \cong \int_{\mathbb{R}}^{\oplus} g(\xi)\,\mathrm{d}\xi \tag{2.52}$$

with the one-dimensional fiber Hamiltonians

$$h(\xi) = \sigma_1(-\mathrm{i}\,\partial_x) + \sigma_2(\xi - A) + V \quad \text{on} \quad L^2(\mathbb{R}, \mathbb{C}^2) \tag{2.53}$$

and

$$g(\xi) = -\partial_x^2 + (\xi - A)^2 + \sigma_3 B + V \quad \text{on} \quad L^2(\mathbb{R}, \mathbb{C}^2).$$

Again, this fiber decomposition holds for any bounded function of H_D and H_P, so for any $\chi \in L^\infty(\mathbb{R}, \mathbb{C})$ we know that

$$\chi(H_D) \cong \int_{\mathbb{R}}^{\oplus} \chi(h(\xi))\,\mathrm{d}\xi, \qquad \chi(H_P) \cong \int_{\mathbb{R}}^{\oplus} \chi(g(\xi))\,\mathrm{d}\xi. \tag{2.54}$$

The investigation of $h(\xi)$ and $g(\xi)$, with fixed $\xi \in \mathbb{R}$, is rather elementary (as already seen in Chapter 2.1.3). Hence, one can use Theorems XIII.85 and XIII.86 of [RS78] to find intervalls of purely absolutely continuous spectrum of H_D and H_P. For applications later on we just give one specific example (which follows from the discussion in Section 2.1.3).

Corollary 2.34. *Let $B, V \in C^\alpha(\mathbb{R}, \mathbb{R})$, with $\alpha \in (0,1)$, be translationally invariant in the direction of x_2. Suppose that $A = \int_0^{\cdot} B(s)\mathrm{d}s$ and V satisfy Assumption 2.14 and, in addition, that*

a) $|V(x)| \longrightarrow \infty$ as $|x| \to \infty$,

b) $(A/V)'$ *is integrable at* $\pm\infty$.

Then $\sigma_{\rm sc}(H_D) = \sigma_{\rm pp}(H_D) = \emptyset$ and $\sigma(H_D) = \sigma_{\rm ac}(H_D) = \mathbb{R}$.

There are other examples of translationally symmetric field configurations, where the fiber-decomposition is helpful to deduce absolute continuity of the spectrum. We only mention the examples discussed in [Iwa85] and [BMR11] for the magnetic Schrödinger operator; Configurations which may also be of interest for the Dirac/Pauli operator (in dimension two).

Chapter 3

Results on the Spectrum of Two-Dimensional Dirac and Pauli Operators

In this Chapter we present new results on the structure of $\sigma(H_D)$ and $\sigma(H_P)$ for potentials V and magnetic fields $B = \operatorname{curl} \mathbf{A}$ that do not decay at infinity. As already discussed in the introduction, we show that the existence of spectral gaps in $\sigma(H_D)$ and $\sigma(H_P)$ depend on the relation of the functions B and V at ∞. These results do not require certain symmetries such that a decomposition of the operator (as illustrated in Subsection 2.2.3) would be possible. However, the theorems we present in Section 3.1 and 3.2 are useful to determine completely the spectra $\sigma(H_D)$, $\sigma(H_P)$ for very basic, rotationally symmetric field configurations. To demonstrate this we discuss in Section 3.3 very natural examples of two-dimensional Dirac and Pauli operators with dense pure point spectrum. Before we start, let us remark that the main results, presented in this Chapter, have already been published in [MS14] and [Meh15] in a slightly different form.

3.1 The Essential Spectrum of Two-Dimensional Dirac Operators with Potential Wells

For $B, V \in C(\mathbb{R}^2, \mathbb{R})$ and $\mathbf{A} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ such that $\operatorname{curl} \mathbf{A} = B$, the two-dimensional magnetic Dirac operator $D_{\mathbf{A}}$ is given by (2.32) and the massless Dirac operator H_D by (2.33). In this section we focus on the dependency of $\sigma_{\text{ess}}(H_D)$ on the quotient V^2/B (at large values) for growing potentials V, i.e. for potentials V such that $|V(\mathbf{x})| \to \infty$ as $|\mathbf{x}| \to \infty$. Unlike for non-relativistic operators (see next section), the results do not depend on the sign of V.

Theorem 3.1 (Theorem 1 of [MS14]). *For* $\mathbf{A} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ *suppose that* $B = \operatorname{curl} \mathbf{A} \in C(\mathbb{R}^2, \mathbb{R})$ *and* $V \in C^1(\mathbb{R}^2, \mathbb{R})$ *satisfy*

$$|V(\mathbf{x})| \longrightarrow \infty \qquad \text{as } |\mathbf{x}| \to \infty, \tag{3.1}$$

$$\left| \frac{\nabla V(\mathbf{x})}{V(\mathbf{x})} \right| \longrightarrow 0 \qquad \text{as } |\mathbf{x}| \to \infty, \tag{3.2}$$

$$\limsup_{|\mathbf{x}| \to \infty} \frac{V^2(\mathbf{x})}{2|B(\mathbf{x})|} < 1. \tag{3.3}$$

Then H_D *has compact resolvent, i.e.* $\sigma_{\text{ess}}(H_D) = \emptyset$.

This theorem states that if V remains small compared to B, the eigenvalues do not accumulate. The notion "V remains small to B" is expressed by condition (3.3) quantitatively. A similar statement on the discreteness of the spectrum of H_D was already proven in [Suz00]. However, the result there requires the constant in (3.3) to be $\frac{1}{4}$ instead of 1 since the proofs are based on different methods. We discuss the sharpness of this constant after the proof of Theorem 3.1.

In order to prove this first theorem, we note that by (3.1)−(3.3) and the local boundedness of B and V, we may assume that both satisfy the global conditions

$$|V(\mathbf{x})| \geq 1/\delta, \tag{3.4}$$

$$|\nabla V(\mathbf{x})| \leq \delta |V(\mathbf{x})|, \tag{3.5}$$

$$V^2(\mathbf{x}) \leq 2(1-\eta)|B(\mathbf{x})| \tag{3.6}$$

for $\mathbf{x} \in \mathbb{R}^2$ (see Appendix B) . Here $\eta \in (0,1)$ is a constant and $\delta \in (0, \frac{1}{2})$ is fixed, but can be made arbitrarily small (by altering B and V inside a sufficiently large compact set). The purpose of this global assumptions is that 0 becomes an isolated eigenvalue of $D_{\mathbf{A}}$ since (due to Corollary 2.29) the operator $D_{\mathbf{A}}$ then has the spectral gap $(-2\beta_0, 0) \cup (0, 2\beta_0) \subset \varrho(D_{\mathbf{A}})$, with

$$\beta_0 := \left(2\delta\sqrt{1-\eta}\,\right)^{-1}. \tag{3.7}$$

This spectral gap enables us to express the operators Π_0 and $\operatorname{sgn}(D_{\mathbf{A}})$ in terms of contour integrals of the resolvent of $D_{\mathbf{A}}$, which is required for the following estimates on commutators of V and functions of $D_{\mathbf{A}}$:

Lemma 3.2 (Lemma 3 of [MS14]). *Let* $V \in C^1(\mathbb{R}^2, \mathbb{R})$, $B \in C(\mathbb{R}^2, \mathbb{R})$ *and* $\mathbf{A} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ *such that* $B = \operatorname{curl} \mathbf{A}$. *Assume that* (3.4)−(3.6) *hold for* $\delta \in (0, \frac{1}{2})$ *and* $\eta \in (0,1)$. *Then we have:*

a) *The operators* $\left[\Pi_0^\perp, V^{-1}\right] V$, $V\left[\Pi_0^\perp, V^{-1}\right]$ *are well-defined on* $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ *and they extend to bounded operators on* $L^2(\mathbb{R}^2, \mathbb{C}^2)$ *with*

$$\left\| V\left[\Pi_0^\perp, V^{-1}\right] \right\|, \left\| \left[\Pi_0^\perp, V^{-1}\right] V \right\| \leq 4\delta^2. \tag{3.8}$$

The same holds true if we replace $\Pi_0^\perp$ *by* Π_0.

b) $\Pi_0\mathcal{D}(V), \Pi_0^\perp\mathcal{D}(V) \subset \mathcal{D}(V)$.

c) *The operator* $V\left[\Pi_0^\perp, V^{-1}\right]$ *maps* $C_0^\infty(\mathbb{R}^2,\mathbb{C}^2)$ *into* $\mathcal{D}(D_\mathbf{A})$. *Moreover, we have*

$$\left\|D_\mathbf{A} V\left[\Pi_0^\perp, V^{-1}\right]\right\| \leq 4\delta. \tag{3.9}$$

Lemma 3.3 (Lemma 4 of [MS14])**.** *Let* $V \in C^1(\mathbb{R}^2,\mathbb{R})$*,* $B \in C(\mathbb{R}^2,\mathbb{R})$ *and* $\mathbf{A} \in C^1(\mathbb{R}^2,\mathbb{R}^2)$ *such that* $B = \operatorname{curl}\mathbf{A}$. *Assume that* (3.4)$-$(3.6) *hold for* $\delta \in (0,\frac{1}{2})$ *and* $\eta \in (0,1)$. *Then* $\left[\operatorname{sgn}(D_\mathbf{A})P_0^\perp, V^{-1}\right]$ *maps* $L^2(\mathbb{R}^2,\mathbb{C}^2)$ *into* $\mathcal{D}(V)$ *and*

$$\left\|V\left[\operatorname{sgn}(D_\mathbf{A})\Pi_0^\perp, V^{-1}\right]\right\| \leq 4\delta^2. \tag{3.10}$$

The proofs of these claims can be found in Appendix B. Both lemmata are necessary to reduce criteria (B.1) (from the Appendix) to hold only on each of the subspaces $\Pi_0\mathcal{D}(H_D)$ and $\Pi_0^\perp\mathcal{D}(H_D)$.

Proof of Theorem 3.1. In view of the discussion above we assume, w.l.o.g., that the global conditions (3.4)$-$(3.6) hold for B and V. By (3.4) and the continuity of V, we know that V is either strictly positive or strictly negative. To simplify the notation, suppose that V is positive, even if the proof is valid for both signs. In addition, it suffices to show the statement for positive B. Otherwise, if B is negative, one has to interchange the role of d and d^* in the following lines.

For $\psi \in C_0^\infty(\mathbb{R}^2,\mathbb{C}^2)$ we know that $\Pi_0\psi, \Pi_0^\perp\psi \in \mathcal{D}(V)$, hence we compute

$$\begin{aligned}\|H_D\psi\|^2 &= \left\|(D_\mathbf{A}+V)(\Pi_0+\Pi_0^\perp)\psi\right\|^2\\ &= \left\|V\Pi_0\psi + (D_\mathbf{A}+V)\Pi_0^\perp\psi\right\|^2\\ &= \left\|(D_\mathbf{A}+V)\Pi_0^\perp\psi\right\|^2 + 2\mathrm{Re}\langle(D_\mathbf{A}+V)\Pi_0^\perp\psi, V\Pi_0\psi\rangle + \|V\Pi_0\psi\|^2\\ &= \left\|(D_\mathbf{A}+V)\Pi_0^\perp\psi\right\|^2 - \delta\|V\Pi_0^\perp\psi\|^2\\ &\quad + 2\mathrm{Re}\langle V\Pi_0\psi, D_\mathbf{A}\Pi_0^\perp\varphi\rangle + \|V\psi\|^2 - (1-\delta)\left\|V\Pi_0^\perp\psi\right\|^2.\end{aligned} \tag{3.11}$$

Let us estimate the terms of (3.11) in blocks. Lemma 3.2 c) yields

$$\begin{aligned}|\langle V\Pi_0\psi, D_\mathbf{A}\Pi_0^\perp\psi\rangle| &= |\langle V\Pi_0V^{-1}V\psi, D_\mathbf{A}\Pi_0^\perp\psi\rangle|\\ &= \left|\langle\Pi_0V\psi, D_\mathbf{A}\Pi_0^\perp\psi\rangle + \left\langle V\left[\Pi_0, V^{-1}\right]V\psi, D_\mathbf{A}\Pi_0^\perp\psi\right\rangle\right|\\ &= \left|\left\langle V\left[\Pi_0, V^{-1}\right]V\psi, D_\mathbf{A}\Pi_0^\perp\psi\right\rangle\right|\\ &\leq 4\delta\,\|V\psi\|\,\|\psi\|\\ &\leq 4\delta^2\,\|V\psi\|^2,\end{aligned} \tag{3.12}$$

where (3.4) is used in the last inequality. Further, employing part a) of the same Lemma results in

$$\left\|V\Pi_0^\perp\psi\right\| \leq \|V\psi\| + \left\|V\left[\Pi_0^\perp, V^{-1}\right]V\psi\right\| \leq (1+4\delta^2)\|V\psi\|,$$

and therefore

$$\|V\psi\|^2 - (1-\delta)\big\|V\Pi_0^\perp\psi\big\|^2 \geq (\delta - 12\delta^2)\|V\psi\|^2. \tag{3.13}$$

For the remaining terms of (3.11) we use the weighted Young inequality (for numbers) to obtain

$$\begin{aligned}\|(D_{\mathbf{A}} + V)\Pi_0^\perp\psi\|^2 - \delta\big\|V\Pi_0^\perp\psi\big\|^2 \\ \geq (1-\varepsilon)\|D_{\mathbf{A}}\Pi_0^\perp\psi\|^2 + (1-\varepsilon^{-1}-\delta)\|V\Pi_0^\perp\psi\|^2,\end{aligned}$$

where $\varepsilon \in (0,1)$ can be chosen arbitrarily. Hence, in view of inequalities (3.12) and (3.13), it suffices to show that

$$\big\|D_{\mathbf{A}}\Pi_0^\perp\psi\big\|^2 + \tfrac{(1-\varepsilon^{-1}-\delta)}{(1-\epsilon)}\big\|V\Pi_0^\perp\psi\big\|^2 \geq 0 \tag{3.14}$$

for $\delta > 0$ sufficiently small and for a corresponding $\varepsilon \in (0,1)$. We set

$$c_{\epsilon,\delta} := -\tfrac{(1-\varepsilon^{-1}-\delta)}{(1-\epsilon)} > 0.$$

Recall the statement of Proposition 2.28, i.e.

$$\big\|D_{\mathbf{A}}\Pi_0^\perp\psi\big\|^2 - c_{\epsilon,\delta}\big\|V\Pi_0^\perp\psi\big\|^2 \geq \big\|\sqrt{2B}S\Pi_0^\perp\psi\big\|^2 - c_{\epsilon,\delta}\big\|V\Pi_0^\perp\psi\big\|^2, \tag{3.15}$$

with

$$S = \begin{pmatrix} s & 0 \\ 0 & \mathrm{id}\end{pmatrix}.$$

Note that $\ker(d^*) = \{0\}$ since B is strictly positive, which means that the isometry

$$S^* = \begin{pmatrix} s^* & 0 \\ 0 & \mathrm{id}\end{pmatrix}$$

is defined on full $L^2(\mathbb{R}^2, \mathbb{C}^2)$. In particular, the operator

$$V\big[S^*, V^{-1}\big] = \begin{pmatrix} V\big[s^*, V^{-1}\big] & 0 \\ 0 & 0\end{pmatrix} \quad \text{on } L^2(\mathbb{R}^2, \mathbb{C}^2)$$

is well-defined and operator bounded by

$$V\big[\mathrm{sgn}(D_{\mathbf{A}})\Pi_0^\perp, V^{-1}\big] = \begin{pmatrix} 0 & V\big[s^*, V^{-1}\big] \\ V\big[s\pi^\perp, V^{-1}\big] & 0\end{pmatrix}.$$

Using Lemma 3.3, we conclude that

$$\big\|V\big[S^*, V^{-1}\big]\big\| \leq \big\|V\big[\mathrm{sgn}(D_{\mathbf{A}})\Pi_0^\perp, V^{-1}\big]\big\| \leq 4\delta^2. \tag{3.16}$$

With this commutator bound we can proceed at (3.15) and obtain

$$\begin{aligned}\big\|D_{\mathbf{A}}\Pi_0^{\perp}\psi\big\|^2 &- c_{\epsilon,\delta}\big\|V\Pi_0^{\perp}\psi\big\|^2\\ &\geq \big\|\sqrt{2B}S\Pi_0^{\perp}\psi\big\|^2 - c_{\epsilon,\delta}\big\|VS^*V^{-1}VS\Pi_0^{\perp}\psi\big\|^2\\ &\geq \big\|\sqrt{2B}S\Pi_0^{\perp}\psi\big\|^2 - c_{\epsilon,\delta}\big\|\big(S^* + V\big[S^*, V^{-1}\big]\big)VS\Pi_0^{\perp}\psi\big\|^2\\ &\geq \big\|\sqrt{2B}S\Pi_0^{\perp}\psi\big\|^2 - c_{\epsilon,\delta}(1+4\delta^2)^2\big\|VS\Pi_0^{\perp}\psi\big\|^2\\ &\geq \big[1-(1-\eta)c_{\epsilon,\delta}(1+12\delta^2)\big]\big\|\sqrt{2B}S\Pi_0^{\perp}\psi\big\|^2,\end{aligned}$$

where condition (3.6) is applied in the last line. Choosing $\varepsilon = 1-\delta^{\frac{1}{2}}$ results in

$$(1-\eta)c_{\epsilon,\delta}(1+12\delta^2) = (1-\eta)\frac{1}{1-\delta^{\frac{1}{2}}}\big(1+\delta^{\frac{1}{2}}-\delta\big)(1+12\delta^2) < 1$$

for δ sufficiently small. Putting all together, we have

$$\|H_D\psi\|^2 \geq (\delta - 20\delta^2)\|V\psi\|^2, \qquad \psi \in C_0^{\infty}(\mathbb{R}^2, \mathbb{C}^2)$$

for $\delta > 0$ sufficiently small. Hence, Lemma B.1 yields the statement. □

One may read off the proof of Theorem 3.1 that the growth restriction (3.2) on V is of technical necessity. Therefore, in what follows we only focus on condition (3.3). It is quite obvious that if the quotient $V^2/2B$ is not strictly smaller than 1 at ∞, the spectrum of H_D can not longer assumed to be discrete: For a suitable $\psi \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ we compute

$$\begin{aligned}(D_{\mathbf{A}}+V)(D_{\mathbf{A}}-V)\psi &= (D_{\mathbf{A}}^2 - V^2)\psi + o(V)\psi\\ &\approx \begin{pmatrix} d^*d + 2B - V^2 & 0\\ 0 & dd^* - 2B - V^2\end{pmatrix}\psi.\end{aligned}$$

Hence, if $\ker(d)$ or $\ker(d^*)$ is large enough, we may obtain points in the essential spectrum. Let us concretise that in the case $B > 0$ with

Theorem 3.4 (Theorem 2 of [MS14])**.** *For $\mathbf{A} \in C^3(\mathbb{R}^2, \mathbb{R}^2)$ suppose that $B = \operatorname{curl}\mathbf{A} \in C^2(\mathbb{R}^2, \mathbb{R})$, with $B > B_0 > 0$, and $V \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfy*

$$|V(\mathbf{x})| \longrightarrow \infty \qquad \text{as } |\mathbf{x}| \to \infty, \tag{3.17}$$

$$\|\nabla V/V\|_{\infty} < \infty \quad \text{and} \quad \|\Delta B/B\|_{\infty} < \infty, \tag{3.18}$$

$$\|(V^2 - 2B)/V\|_{\infty} < \infty. \tag{3.19}$$

Then $\sigma_{\text{ess}}(H_D) \neq \emptyset$.

Remark 3.5. *Note that conditions* (3.17) *and* (3.19) *are equivalent to*

$$\left|\frac{V^2(\mathbf{x})}{2B(\mathbf{x})} - 1\right| \leq \frac{c}{|V(\mathbf{x})|} \longrightarrow 0 \quad \text{as} \quad |\mathbf{x}| \to \infty,$$

for some constant $c > 0$.

Lemma 3.6 (Lemma 5 of [MS14]). *Assume that the conditions of Theorem 3.4 are satisfied. Let $\Phi \in C^2(\mathbb{R}^2, \mathbb{R})$ be a potential function for $\mathbf{A}$, i.e. $\mathbf{A} = (-\partial_2\Phi, \partial_1\Phi)$ and $B = \Delta\Phi$. Define the subspace*

$$\mathcal{Y} := \big\{\omega e^{-\Phi} \in L^2\big(\mathbb{R}^2, \mathbb{C}; Bd^2x\big) \mid \omega \text{ is entire in } x_1 + \mathrm{i}\, x_2\big\}. \tag{3.20}$$

Then, for any $\Omega \in \mathcal{Y}$ we have

a) $\Omega \in \mathcal{D}(d^*) \cap \mathcal{D}(V)$ *and* $\|d^*\Omega\| = \big\|\sqrt{2B}\Omega\big\|$.

b) $(d^*\Omega, -V\Omega)^T \in \mathcal{D}(H_D)$.

Proof. By the definition of $\mathcal{Y}$ and Remark 3.5, it is easy to see that $\mathcal{Y} \subset \mathcal{D}(V)$. Further, $\mathcal{Y} \subset \ker(d)$ since B is bounded from below by some positive constant. Let $\chi \in C_0^\infty(\mathbb{R}^2, [0,1])$ be a smooth cutoff function, i.e.

$$\chi(x) = \begin{cases} 1, & \text{for } |x| \le 1, \\ 0, & \text{for } |x| \ge 2. \end{cases}$$

For $\Omega \in \mathcal{Y}$ and $n \in \mathbb{N}$ set $\Omega_n := \chi(\frac{\cdot}{n})\Omega \in C_0^2(\mathbb{R}^2, \mathbb{C})$. Note that $\Omega_n \to \Omega$ in $L^2\big(\mathbb{R}^2, \mathbb{C}; B\mathrm{d}^2x\big)$ as $n \to \infty$ by dominated convergence. Using the commutator identity (2.43), we get

$$\|d^*\Omega_n\|^2 = \|d\Omega_n\|^2 + 2\big\|\sqrt{B}\Omega_n\big\|^2 = \frac{1}{n^2}\left\|(i\partial_1\chi - \partial_2\chi)\left(\frac{\cdot}{n}\right)\Omega\right\|^2 + 2\big\|\sqrt{B}\Omega_n\big\|^2$$

for any $n \in \mathbb{N}$. In particular, the sequence $(d^*\Omega_n)_{n\in\mathbb{N}}$ is Cauchy with respect to the $L^2(\mathbb{R}^2, \mathbb{C})$-norm. The closedness of d^* implies that $\Omega \in \mathcal{D}(d^*)$. Clearly, after taking the limit we have that $\|d^*\Omega\| = \big\|\sqrt{2B}\Omega\big\|$.

To verify part b), we pick $\Omega \in \mathcal{Y}$ and set $\psi := (d^*\Omega, -V\Omega)^T$. For $\hat{\psi} \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ we compute

$$\begin{aligned}\langle H_D\hat{\psi}, \psi\rangle &= \left\langle \begin{pmatrix} V\hat{\psi}_1 \\ d\hat{\psi}_1 \end{pmatrix}, \begin{pmatrix} d^*\Omega \\ -V\Omega \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} d^*\hat{\psi}_2 \\ V\hat{\psi}_2 \end{pmatrix}, \begin{pmatrix} d^*\Omega \\ -V\Omega \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \end{pmatrix}, \begin{pmatrix} -(-\mathrm{i}\,\partial_1 V - \partial_2 V)\Omega \\ (2B - V^2)\Omega \end{pmatrix} \right\rangle.\end{aligned}$$

Since, due to assumptions (3.18) and (3.19), the functions $(\mathrm{i}\,\partial_1 V + \partial_2 V)\Omega$ and $(2B - V^2)\Omega$ are square-integrable, we deduce from the definition of the adjoint operator that $\psi \in \mathcal{D}(H_D^*) = \mathcal{D}(H_D)$. □

Proof of Theorem 3.4. Let $\mathcal{Y}$ be defined as in Lemma 3.6. We use Proposition 2.30 to show that $\dim \mathcal{Y} = \infty$. To do so we work with the modified potential function $\hat{\Phi} := \Phi - \frac{1}{2}\ln B$. Since

$$\Delta\hat{\Phi} = B + \frac{(\nabla B)^2}{2B^2} - \frac{\Delta B}{2B} > B - \frac{\Delta B}{2B}\,,$$

Proposition 2.30 implies that

$$\hat{\mathcal{Y}} := \{\omega e^{-\hat{\Phi}} \in L^2(\mathbb{R}^2, \mathbb{C}) \mid \omega \text{ is entire in } x_1 + \mathrm{i}\, x_2\}$$

is infinite dimensional. By the isomorphism $\mathcal{Y} \ni \Omega \mapsto \sqrt{B}\Omega \in \hat{\mathcal{Y}}$, we deduce that also $\mathcal{Y}$ is infinite dimensional. Let us consider the subspace

$$\mathcal{X} := \left\{ \begin{pmatrix} d^*\Omega \\ -V\Omega \end{pmatrix} \in L^2(\mathbb{R}^2, \mathbb{C}^2) \mid \Omega \in \mathcal{Y} \right\} \subset \mathcal{D}(H_D).$$

Then from Lemma 3.6 follows that

$$\left\| \begin{pmatrix} d^*\Omega \\ -V\Omega \end{pmatrix} \right\| = \|d^*\Omega\| + \|V\Omega\| \geq \sqrt{2B_0}\|\Omega\|,$$

i.e. the linear map $\mathcal{Y} \ni \Omega \mapsto (d^*\Omega, -V\Omega)^{\mathrm{T}} \in \mathcal{X}$ is injective and therefore an isomorphism again. In particular, it holds that $\dim \mathcal{X} = \infty$. For $\psi = (d^*\Omega, -V\Omega)^T \in \mathcal{X}$ we estimate

$$\begin{aligned} \|H_D\psi\| &= \left\| \begin{pmatrix} V & d^* \\ d & V \end{pmatrix} \begin{pmatrix} d^*\Omega \\ -V\Omega \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} (\mathrm{i}\,\partial_1 V + \partial_2 V)\Omega \\ (2B - V^2)\Omega \end{pmatrix} \right\| \leq \left(\left\| \frac{\nabla V}{V} \right\|_\infty + \left\| \frac{2B - V^2}{V} \right\|_\infty \right) \|V\Omega\|. \end{aligned}$$

Hence, $\|H_D\psi\| \leq c\|\psi\|$ for $\psi \in \mathcal{X}$ and since $\dim \mathcal{X} = \infty$, we conclude that H_D cannot have only eigenvalues of finite multiplicity. □

We want to emphasise that for Theorem 3.4 we require $2B \approx V^2$ to be satisfied globally (c.f. condition (3.19)). It is even possible to weaken this global condition, i.e. it suffices that $2B \approx V^2$ holds only around certain points in $\mathbb{R}^2$, in order to obtain $\sigma_{ess}(H_D) \neq \emptyset$. To demonstrate this we need some further restrictions on the class of admissible magnetic fields B and potentials V.

Definition 3.7. *A function $f : \mathbb{R}^2 \to \mathbb{R}$ varies with rate $\nu \in [0, 1]$ on a set $X \subset \mathbb{R}^2$ if there is a constant $C > 0$ such that for all $\mathbf{x} \in X$ it holds that*

$$|f(\mathbf{x} + \mathbf{y})| \leq C|f(\mathbf{x})|,$$

whenever $\mathbf{y} \in \mathbb{R}^2$ satisfies $|\mathbf{y}| \leq \frac{1}{2}|\mathbf{x}|^\nu$.

Very natural examples of this definition are power-functions, which vary with any rate $\nu \in [0, 1]$ on certain sets:

Example 3.8. *Functions of type $f_1(\mathbf{x}) = c|\mathbf{x}|^s$ and $f_2(\mathbf{x}) = c|x_1|^s$, with $c, s \in \mathbb{R}$, vary with any rate $\nu \in [0, 1]$ on $\mathbb{R}^2 \backslash B_1(0)$ and on $\mathbb{R}^2 \backslash [-1, 1] \times \mathbb{R}$ respectively.*

Theorem 3.9 (Theorem 3 of [MS14]). *Let $B, V \in C(\mathbb{R}^2, \mathbb{R})$ be continuously differentiable outside some compact set X. Assume that there is a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset \mathbb{R}^2 \backslash X$, with $|\mathbf{x}_n| \to \infty$ as $n \to \infty$, such that $|\nabla V|, |\nabla B|$ vary with rate 0 on $(\mathbf{x}_n)_{n\in\mathbb{N}}$. If there exist constants $k \in \mathbb{N}$ and $\varepsilon \in (0,1)$ for which holds that*

$$|V(\mathbf{x}_n)| \longrightarrow \infty, \tag{3.21}$$

$$\frac{\nabla B(\mathbf{x}_n)}{|B(\mathbf{x}_n)|^{1-\varepsilon}}, \frac{\nabla V(\mathbf{x}_n)}{|V(\mathbf{x}_n)|^{1-\varepsilon}} \longrightarrow 0, \tag{3.22}$$

$$\frac{V^2(\mathbf{x}_n) - 2k|B(\mathbf{x}_n)|}{V(\mathbf{x}_n)} \longrightarrow 0 \tag{3.23}$$

as $n \to \infty$, then $0 \in \sigma_{\text{ess}}(H_D)$.

Remark 3.10. *Note that* (3.21) *and* (3.22) *imply*

$$\left| \frac{V^2(\mathbf{x}_n)}{2|B(\mathbf{x}_n)|} - k \right| \leq \frac{c}{|V(\mathbf{x}_n)|} \longrightarrow 0 \qquad \text{as } n \to \infty,$$

for some constant $c > 0$, meaning that a certain rate of convergence is required for $V^2/(2|B|)$. However, we see that condition (3.3) *in Theorem 3.1 is quite sharp.*

In Theorem 3.9 the condition $V^2 \approx 2|B|$ needs to hold only locally because the proof is more constructive than the one of Theorem 3.4. More precisely, for proving Theorem 3.9 we construct directly a Weyl sequence for 0 out of functions that are localised around the points of the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$. Let us concisely describe how the construction of a Weyl sequence originates from the fact that $V^2/(2|B|)$ attains integer values on the $\mathbf{x}_n$'s:

Supposing that our approximated eigenfunctions (which form the Weyl sequence) are strongly localised, they "perceive" B and V only locally, hence we can treat them like constants $V_n = V(\mathbf{x}_n)$, $B_n = B(\mathbf{x}_n)$. Now recall that we can determine the spectrum and eigenfunctions of the Dirac-Landau Hamiltonian, i.e. of the Hamiltonian $D_{\mathbf{A}_n}$ with the constant magnetic field $B_n = \operatorname{curl} \mathbf{A}_n$, explicitly (see Section 2.2.2). For a Gaussian-like localised eigenfunction ψ_k associated to the Landau eigenvalue $l_k = \operatorname{sgn}(k)\sqrt{2|kB_n|}$ we then obtain

$$\big[D_{\mathbf{A}} + V\big]\psi_k \approx \big[D_{\mathbf{A}_n} + V_n\big]\psi_k = \big[\operatorname{sgn}(k)\sqrt{2|kB_n|} + V_n\big]\psi_k \approx 0.$$

This idea is flexible enough to find also other points in the essential spectrum: Suppose that we want to construct a Weyl sequence for an arbitrary value $E \in \mathbb{R}$. Then the analogous crossing condition for the values of B_n and V_n would be $\operatorname{sgn}(k)\sqrt{2|kB_n|} = E - V_n$, or written in a different way

$$\frac{V_n^2}{2|B_n|}\left(1 - \frac{E}{V_n}\right)^2 \quad \in \mathbb{N} \tag{3.24}$$

for $n \in \mathbb{N}$. Since we consider potentials V growing at ∞, the value of E is irrelevant whenever $(V^2/2|B|)(\mathbf{x})$ passes through "sufficiently many" intervalls of type $(k_n - \delta, k_n + \delta)$ as $|\mathbf{x}| \to \infty$ (where $\delta > 0$ and $(k_n)_{n\in\mathbb{N}}$ is a sequence of natural numbers). In particular, if V^2 grows stronger than $2|B|$ at infinity, this considerations result in

Theorem 3.11 (adapted from Corollary 1 of [MS14])**.** *Let $B, V \in C(\mathbb{R}^2, \mathbb{R})$ be continuously differentiable outside some compact set X. Assume that there is a continuous path $\gamma : \mathbb{R}^+ \to \mathbb{R}^2 \backslash X$, with $|\gamma(t)| \to \infty$ as $t \to \infty$, such that $|\nabla V|, |\nabla B|$ vary with rate $\nu \in [0, 1]$ on $\mathrm{Im}(\gamma)$. If there exist constants $\varepsilon, \lambda, \kappa, B_0 > 0$ for which holds that*

$$\frac{V^2(\gamma(t))}{2|B(\gamma(t))|} \longrightarrow \infty, \tag{3.25}$$

$$\left(\frac{|\nabla B(\gamma(t))|}{|B(\gamma(t))|} + \frac{|\nabla V(\gamma(t))|}{|V(\gamma(t))|}\right)\left(\frac{V^2(\gamma(t))}{2|B(\gamma(t))|}\right)^{1+\varepsilon} \longrightarrow 0, \tag{3.26}$$

$$\frac{1}{|\gamma(t)|^{2\nu}}\left(\frac{V^2(\gamma(t))}{B^2(\gamma(t))}\right)^{1+\varepsilon} \longrightarrow 0 \tag{3.27}$$

as $t \to \infty$ and, in addition,

$$B_0 \leq |B(\gamma(t))| \leq \lambda \exp\left(\kappa \, \frac{V^2(\gamma(t))}{|B(\gamma(t))|}\right) \tag{3.28}$$

for $t \in (0, \infty)$, then $\sigma(H_D) = \sigma_{\mathrm{ess}}(H_D) = \mathbb{R}$.

Remark 3.12. *One may observe that* (3.26) *and* (3.27) *impose constraints on the growth rate of the quotient $V^2/|B|$ along the path γ. They arise from the local approximation of B and V by constants. Anyway, they give us a scope, which leads to some very interesting examples (see Section 3.3).*

Remark 3.13. *To expect from the derivatives of B and V to vary with some rate ν on certain sets may look unnatural. However, also the moduli of the gradients of the functions given in Example 3.8 vary with any rate $\nu \in [0, 1]$ on the corresponding sets. Hence, this should not be considered as a strong impediment.*

For the proofs of Theorem 3.9 and 3.11 we have to cement our idea of considering B and V to be locally constant: For a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset \mathbb{R}^2$ we use the notation $V_n := V(\mathbf{x}_n)$ and $B_n := B(\mathbf{x}_n)$. Since it is necessary to compare also magnetic vector potentials locally, we define, for any $n \in \mathbb{N}$,

$$\mathbf{A}_n(\mathbf{x}) := \int_0^1 B_n \wedge (\mathbf{x} - \mathbf{x}_n) s \mathrm{d}s = \tfrac{1}{2} B_n \wedge (\mathbf{x} - \mathbf{x}_n),$$

$$\tilde{\mathbf{A}}_n(\mathbf{x}) := \int_0^1 B(\mathbf{x}_n + s(\mathbf{x} - \mathbf{x}_n)) \wedge (\mathbf{x} - \mathbf{x}_n) s \mathrm{d}s.$$

Here the wedge product $a \wedge \mathbf{v}$ of a scalar $a \in \mathbb{R}$ and a vector $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ is given by $a \wedge \mathbf{v} := a(-v_2, v_1)$. The vector potential $\tilde{\mathbf{A}}_n$ is a generator of the given magnetic field $B = \operatorname{curl} \mathbf{A}$. As pointed out in Subsection 2.2.1, we therefore find for each $n \in \mathbb{N}$ a gauge function $\Gamma_n \in C^2(\mathbb{R}^2, \mathbb{R})$ such that

$$\nabla \Gamma_n = \mathbf{A} - \tilde{\mathbf{A}}_n \qquad \text{on } \mathbb{R}^2. \tag{3.29}$$

On the other hand, the potential $\mathbf{A}_n$ generates the field

$$\operatorname{curl} \mathbf{A}_n = B_n \quad \text{on } \mathbb{R}^2,$$

i.e. the operator $D_{\mathbf{A}_n}$ is (up to gauge) the Dirac-Landau Hamiltonian for the constant field B_n. For any $n \in \mathbb{N}$ we obtain the corresponding off-diagonal components d_n and d_n^* through

$$D_{\mathbf{A}_n} =: \begin{pmatrix} 0 & d_n^* \\ d_n & 0 \end{pmatrix}.$$

For the explicit form of d_n and d_n^* we refer to Subsection 2.2.2.

By the discussion above, we need for a sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers eigenfunctions for the (positive and negative) k_n-th Dirac-Landau level of the operator $D_{\mathbf{A}_n}$. For $B_n > 0$ we already know how they look like (c.f. end of Subsection 2.2.2): If $B_n > 0$, we set

$$\hat{\psi}_n(\mathbf{x}) := \begin{pmatrix} (d_n^*)^{k_n} e^{-\frac{B_n}{4}|\mathbf{x}-\mathbf{x}_n|^2} \\ E_n (d_n^*)^{k_n - 1} e^{-\frac{B_n}{4}|\mathbf{x}-\mathbf{x}_n|^2} \end{pmatrix}, \qquad \mathbf{x} \in \mathbb{R}^2, \tag{3.30}$$

with $E_n := -\operatorname{sgn}(V_n)\sqrt{2k_n B_n}$. Note that $\hat{\psi}_n$ has the explicit form

$$\hat{\psi}_n(\mathbf{x} + \mathbf{x}_n) = \begin{pmatrix} (\mathrm{i}\, B_n(x_1 - \mathrm{i}\, x_2))^{k_n} e^{-\frac{B_n}{4}|\mathbf{x}|^2} \\ E_n (\mathrm{i}\, B_n(x_1 - \mathrm{i}\, x_2))^{k_n - 1} e^{-\frac{B_n}{4}|\mathbf{x}|^2} \end{pmatrix}. \tag{3.31}$$

It holds that

$$\big[D_{\mathbf{A}_n} + V_n\big]\hat{\psi}_n = \big(-\operatorname{sgn}(V_n)\sqrt{2k_n B_n} + V_n\big)\hat{\psi}_n.$$

If $B_n < 0$, the roles of d_n and d_n^* interchange, i.e. we must use

$$\hat{\psi}_n(\mathbf{x}) := \begin{pmatrix} E_n (d_n)^{k_n - 1} e^{\frac{B_n}{4}|\mathbf{x}-\mathbf{x}_n|^2} \\ (d_n)^{k_n} e^{\frac{B_n}{4}|\mathbf{x}-\mathbf{x}_n|^2} \end{pmatrix}, \qquad \mathbf{x} \in \mathbb{R}^2, \tag{3.32}$$

with $E_n := -\operatorname{sgn}(V_n)\sqrt{2k_n |B_n|}$. The explicit form of $\hat{\psi}_n$ in this case can easily be calculated with (2.46) and is given by

$$\hat{\psi}_n(\mathbf{x} + \mathbf{x}_n) = \begin{pmatrix} E_n(-\mathrm{i}\, B_n(x_1 + \mathrm{i}\, x_2))^{k_n - 1} e^{\frac{B_n}{4}|\mathbf{x}|^2} \\ (-\mathrm{i}\, B_n(x_1 + \mathrm{i}\, x_2))^{k_n} e^{\frac{B_n}{4}|\mathbf{x}|^2} \end{pmatrix}. \tag{3.33}$$

Consequently, also in the case $B_n < 0$ we obtain

$$\big[D_{\mathbf{A}_n} + V_n\big]\hat{\psi}_n = \big(-\operatorname{sgn}(V_n)\sqrt{2k_n|B_n|} + V_n\big)\hat{\psi}_n. \tag{3.34}$$

So (3.34) holds in both cases, i.e. for any $n \in \mathbb{N}$. Since we need Weyl functions with compact support, pick $\chi \in C_0^\infty(\mathbb{R}^2, [0,1])$ such that $\chi(\mathbf{x}) = 1$ for $|\mathbf{x}| \leq 1$ and $\chi(\mathbf{x}) = 0$ for $|\mathbf{x}| \geq 2$. We rescale our cutoff functions and set

$$\chi_n(\mathbf{x}) := \chi\left(\frac{\mathbf{x} - \mathbf{x}_n}{r_n}\right),$$

with (not yet chosen) localisation radii $r_n > 0$. Then our Weyl functions are given, for every $n \in \mathbb{N}$, by

$$\psi_n(\mathbf{x}) := e^{\mathrm{i}\,\Gamma_n(\mathbf{x})}\chi_n(\mathbf{x})\hat{\psi}_n(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2. \tag{3.35}$$

The choice of the radii r_n depends on the mass concentration of the $\hat{\psi}_n$'s around their centers $\mathbf{x}_n$. More precisely, the r_n's ought to be large enough such that $\|\psi_n\|$ does not vanish as $n \to \infty$.

Lemma 3.14. *For $n \in \mathbb{N}$ let $\hat{\psi}_n$ be given by* (3.30) *or* (3.32) *(depending on the sign of B_n), and ψ_n be given by* (3.35)*. Then we have the norm estimates*

$$\|\psi_n\|^2 \leq \|\hat{\psi}_n\|^2 = 2^{k_n+2}\pi|B_n|^{k_n-1}k_n!\,, \tag{3.36}$$

$$\|\psi_n\|^2 \geq \frac{1}{2}\|\hat{\psi}_n\|^2\left(1 - \frac{1}{k_n!}\int_{|B_n|r_n^2/2}^\infty s^{k_n}e^{-s}\mathrm{d}s\right). \tag{3.37}$$

Proof. Using the explicit representations (3.31) and (3.33), we compute

$$\begin{aligned}\|\psi_n\|^2 \leq \|\hat{\psi}_n\|^2 &= 2\pi|B_n|^{2k_n}\int_0^\infty s^{2k_n}e^{-\frac{|B_n|}{2}s^2}s\mathrm{d}s \\ &\qquad + 4\pi k_n|B_n|^{2k_n-1}\int_0^\infty s^{2(k_n-1)}e^{-\frac{|B_n|}{2}s^2}s\mathrm{d}s \\ &= 2^{k_n+1}\pi|B_n|^{k_n-1}\left(\int_0^\infty s^{k_n}e^{-s}\mathrm{d}s + k_n\int_0^\infty s^{k_n-1}e^{-s}\mathrm{d}s\right) \\ &= 2^{k_n+2}\pi|B_n|^{k_n-1}k_n!.\end{aligned}$$

On the other hand, we can bound $\|\psi_n\|^2$ form below like

$$\begin{aligned}\|\psi_n\|^2 = \|\psi_{n,1}\|^2 + \|\psi_{n,2}\|^2 &\geq 2\pi|B_n|^{2k_n}\int_0^{r_n} s^{2k_n}e^{-\frac{|B_n|}{2}s^2}s\mathrm{d}s \\ &= 2^{k_n+1}\pi|B_n|^{k_n-1}\left(k_n! - \int_{|B_n|r_n^2/2}^\infty s^{k_n}e^{-s}\mathrm{d}s\right) \\ &\geq \frac{1}{2}\|\hat{\psi}_n\|^2\left(1 - \frac{1}{k_n!}\int_{|B_n|r_n^2/2}^\infty s^{k_n}e^{-s}\mathrm{d}s\right).\end{aligned}$$

□

Proof of Theorem 3.9. For the given sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset \mathbb{R}^2$ let us agree the short hand notation $B_n = B(\mathbf{x}_n) \neq 0$ and $V_n = V(\mathbf{x}_n)$ as above. Here we employ the ψ_n's (defined in (3.35)) as Weyl sequence with $k_n = k$ and $r_n = |B_n|^{(\varepsilon-1)/2}$ for $n \in \mathbb{N}$. Note that there is an $N > 0$ such that $r_n < \frac{1}{4}$ for $n \geq N$, while $|\mathbf{x}_n| \to \infty$ as $n \to \infty$. So we can consider the ψ_n's to be linearly independent (otherwise extract a subsequence with mutually disjoint support). Since $|B_n| \to \infty$ as $n \to \infty$ (c.f. Remark 3.10), we observe that

$$\frac{1}{k_n!}\int_{|B_n|r_n^2/2}^{\infty} s^{k_n} e^{-s}\mathrm{d}s = \frac{1}{k!}\int_{|B_n|^{\varepsilon}/2}^{\infty} s^{k} e^{-s}\mathrm{d}s \longrightarrow 0 \tag{3.38}$$

as $n \to \infty$. Due to Lemma 3.14, we may assume $N > 0$ to be so large that for $n \geq N$ we have

$$\|\hat{\psi}_n\|^2 \leq 4\|\psi_n\|^2. \tag{3.39}$$

Applying H_D to ψ_n results in

$$\begin{aligned} e^{-\mathrm{i}\Gamma_n}(D_{\mathbf{A}} + V)\psi_n &= (D_{\tilde{\mathbf{A}}_n} + V)\chi_n\hat{\psi}_n \\ &= \chi_n(D_{\mathbf{A}_n} + V_n)\hat{\psi}_n - \mathrm{i}\,(\boldsymbol{\sigma}\cdot\nabla\chi_n)\hat{\psi}_n \\ &\quad + \boldsymbol{\sigma}\cdot(\mathbf{A}_n - \tilde{\mathbf{A}}_n)\chi_n\hat{\psi}_n + (V - V_n)\chi_n\hat{\psi}_n \end{aligned} \tag{3.40}$$

for $n \in \mathbb{N}$. We have to estimate the r.h.s of (3.40). First of all, the eigenvalue equation (3.34) implies

$$\|(D_{\mathbf{A}_n} + V_n)\hat{\psi}_n\|^2 = \big||V_n| - \sqrt{2k|B_n|}\big|\|\hat{\psi}_n\|^2 \leq c\big|(V_n^2 - 2k|B_n|)/V_n\big|\|\hat{\psi}_n\|^2$$

for some $c > 0$. Thus, using (3.39) and condition (3.23), we conclude that $\|(D_{\mathbf{A}_n} + V_n)\hat{\psi}_n\|^2/\|\psi_n\|^2$ vanishes as $n \to \infty$. Further, we apply (3.36) and (3.39) to estimate

$$\begin{aligned} \|(\boldsymbol{\sigma}\cdot\nabla\chi_n)\hat{\psi}_n\|^2 &\leq r_n^{-2}\|\nabla\chi\|_\infty \int_{r_n\leq|\mathbf{x}-\mathbf{x}_n|\leq 2r_n} |\hat{\psi}_n(\mathbf{x})|^2\,\mathrm{d}^2\mathbf{x} \\ &\leq |B_n|^{1-\varepsilon}\|\nabla\chi\|_\infty 2^{k+2}\pi|B_n|^{k-1}\int_{|B_n|^{\epsilon}/2}^{\infty} s^k e^{-s}\mathrm{d}s \\ &\leq 4\|\psi_n\|^2\|\nabla\chi\|_\infty|B_n|^{1-\varepsilon}\frac{1}{k!}\int_{|B_n|^{\epsilon}/2}^{\infty} s^k e^{-s}\mathrm{d}s \end{aligned}$$

for $n \geq N$. It is easy to see that

$$|B_n|^{1-\varepsilon}\int_{|B_n|^{\epsilon}/2}^{\infty} s^k e^{-s}\mathrm{d}s \longrightarrow 0$$

as $n \to \infty$ and therefore $\|(\boldsymbol{\sigma}\cdot\nabla\chi_n)\hat{\psi}_n\|/\|\psi_n\| \to 0$ as $n \to \infty$. For the two remaining terms of (3.40) we need that $|\nabla B|$ and $|\nabla V|$ vary with rate 0 on

$(\mathbf{x}_n)_{n\geqslant N}$. Since $|\nabla B|$ has this property, we find a constant $C > 0$ such that, whenever $n \geq N$ and $|\mathbf{x} - \mathbf{x}_n| \leq 2r_n \leq \frac{1}{2}$, we have

$$\big|(\mathbf{A}_n - \tilde{\mathbf{A}}_n)(\mathbf{x})\chi_n(\mathbf{x})\big|^2 \leq |\nabla B(\boldsymbol{\xi}_{\mathbf{x}})|^2(2r_n)^4 \leq 16C^2|\nabla B(\mathbf{x}_n)|^2 r_n^4,$$

where $|\boldsymbol{\xi}_{\mathbf{x}} - \mathbf{x}_n| \leq 2r_n$. In particular, we obtain

$$\frac{\big\|\boldsymbol{\sigma}\cdot(\mathbf{A}_n - \tilde{\mathbf{A}}_n)\chi_n\hat{\psi}_n\big\|^2}{\|\psi_n\|^2} \leq 64C^2|\nabla B(\mathbf{x}_n)|^2 r_n^4 = 64C^2\frac{|\nabla B(\mathbf{x}_n)|^2}{|B_n|^{2-2\varepsilon}}$$

for $n \geq N$. Similarly, we use that V varies with rate 0 on $(\mathbf{x}_n)_{n\geqslant N}$ for estimating

$$\frac{\|(V - V_n)\chi_n\hat{\psi}_n\|^2}{\|\psi_n\|^2} \leq 4\hat{C}^2|\nabla V(\mathbf{x}_n)|^2 r_n^2 \leq 4\hat{C}^2\frac{|\nabla V(\mathbf{x}_n)|}{|V_n|^{2-2\varepsilon}}\left(\frac{V_n^2}{|B_n|}\right)^{1-\varepsilon}.$$

In view of (3.22) we deduce, after dividing through $\|\psi_n\|^2$, that also the last two terms of the right hand side of (3.40) vanish as $n \to \infty$. Therefore, $(\psi_n)_{n\in\mathbb{N}}$ is a Weyl sequence for 0. □

Proof of Theorem 3.11. Let $E \in \mathbb{R}$. We conclude from (3.25) that there is a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset \mathbb{R}^2$ such that $(V(\mathbf{x}_n) - E)^2 = 2n|B(\mathbf{x}_n)|$ for $n \geq N$, where $N \in \mathbb{N}$ is assumed to be sufficiently large. Let us abbreviate again $V(\mathbf{x}_n)$ by V_n and $B(\mathbf{x}_n)$ by B_n. Consider ψ_n, given by (3.35), with $k_n = n$ and

$$r_n = \sqrt{\frac{2n^{1+\varepsilon}}{|B_n|}} \tag{3.41}$$

for $n \geq N$. Since the function $s^n e^{-s/2}$ is bounded by $(2n)^n e^{-n}$, we obtain

$$\frac{1}{n!}\int_{n^{1+\varepsilon}}^{\infty} s^n e^{-s}\mathrm{d}s \leq 2\,\frac{(2n)^n e^{-\frac{1}{2}n^{1+\varepsilon}-n}}{n!} \leq \exp\left(n\ln(2n) - \frac{1}{2}n^{1+\varepsilon} - n\right).$$

Therefore, we may assume N to be so large that

$$\frac{1}{n!}\int_{n^{1+\varepsilon}}^{\infty} s^n e^{-s}\mathrm{d}s \leq \frac{1}{2} \tag{3.42}$$

for all $n \geq N$, meaning that (in view of Lemma 3.14)

$$\|\hat{\psi}_n\|^2 \leq 4\|\psi_n\|^2 \tag{3.43}$$

whenever $n \geq N$. Applying H_D to ψ_n yields

$$\begin{aligned} e^{-\mathrm{i}\Gamma_n}[D_{\mathbf{A}} + V - E]\psi_n &= \big[D_{\tilde{\mathbf{A}}_n} + V - E\big]\chi_n\hat{\psi}_n \\ &= \chi_n\big[D_{\mathbf{A}_n} + V_n - E\big]\hat{\psi}_n - \mathrm{i}\,(\boldsymbol{\sigma}\cdot\nabla\chi_n)\hat{\psi}_n \\ &\quad + \boldsymbol{\sigma}\cdot(\mathbf{A}_n - \tilde{\mathbf{A}}_n)\chi_n\hat{\psi}_n + (V - V_n)\chi_n\hat{\psi}_n \end{aligned} \tag{3.44}$$

for $n \in \mathbb{N}$. We have to show that, after dividing through $\|\psi_n\|$, the terms on the right hand side of (3.44) vanish as $n \to \infty$:

Since $|V_n| \to \infty$ as $n \to \infty$, we know that (at least after increasing N) $\operatorname{sgn}(V_n) = \operatorname{sgn}(V_n - E)$ for $n \geq N$. As a consequence, (3.34) implies that

$$\big[D_{\mathbf{A}_n} + V_n - E\big]\hat{\psi}_n = \operatorname{sgn}(V_n)\big(|V_n - E| - \sqrt{2k|B_n|}\big)\hat{\psi}_n = 0$$

for $n \geq N$. For the second term of (3.44) we need a bound on the radii r_n. Keeping in mind that $E/|V_n| \to 0$ as $n \to \infty$, we may assume that

$$\frac{1}{2} \leq \frac{V_n^2}{2n|B_n|} \leq 2 \tag{3.45}$$

for all $n \geq N$, hence (3.28) leads to

$$r_n^{-2} \leq n^{-(1+\varepsilon)}\lambda e^{4\kappa n} \leq \lambda e^{4\kappa n}.$$

This estimate, together with (3.36), results in

$$\begin{aligned}\|(\boldsymbol{\sigma}\cdot\nabla\chi_n)\hat{\psi}_n\|^2 &\leq \lambda e^{4\kappa n}\|\nabla\chi\|_\infty 2^{n+2}\pi|B_n|^{n-1}\int_{n^{1+\varepsilon}}^{\infty} s^n e^{-s}\mathrm{d}s \\ &\leq 4\lambda\|\nabla\chi\|_\infty\|\psi_n\|^2\exp\left(n\ln(2n) - \frac{1}{2}n^{1+\varepsilon} - n + 4\kappa n\right)\end{aligned}$$

for $n \geq N$. Thus, $\|(\boldsymbol{\sigma}\cdot\nabla\chi_n)\hat{\psi}_n\|^2/\|\psi_n\|^2 \to 0$ as $n \to \infty$. For the remaining terms of (3.44) we observe that, due to the definition of r_n and bound (3.45), we have

$$\frac{r_n^2}{|\mathbf{x}_n|^{2\nu}} = \frac{2}{|B_n|}\frac{n^{1+\varepsilon}}{|\mathbf{x}_n|^{2\nu}} \leq \frac{2^{2+\varepsilon}}{|B_n||\mathbf{x}_n|^{2\nu}}\left(\frac{V_n^2}{2|B_n|}\right)^{1+\varepsilon}.$$

Here $\nu \in [0,1]$ is given by the theorem. In view of condition (3.27) we may consider N to be so large that $r_n/|\mathbf{x}_n|^\nu \leq \frac{1}{4}$ for $n \geq N$. Since $|\nabla V|$ and $|\nabla B|$ vary with rate $\nu \in [0,1]$ on $(\mathbf{x}_n)_{n\geqslant N}$, we deduce (by applying the mean value theorem) that

$$\begin{aligned}\big\|\boldsymbol{\sigma}\cdot(\mathbf{A}_n - \tilde{\mathbf{A}}_n)\chi_n\hat{\psi}_n\big\|^2 &\leq C^2|\nabla B(\mathbf{x}_n)|^2 r_n^4\|\hat{\psi}_n\|^2 \\ &\leq 2^{6+2\varepsilon}C^2\left[\frac{|\nabla B(\mathbf{x}_n)|}{|B_n|}\left(\frac{V_n^2}{2|B_n|}\right)^{1+\varepsilon}\right]^2\|\psi_n\|^2\end{aligned}$$

for n large enough. Note that we applied again (3.43) and (3.45). Analogously, we argue that for n sufficiently large it holds that

$$\begin{aligned}\|(V - V_n)\chi_n\hat{\psi}_n\|^2 &\leq C^2|\nabla V(\mathbf{x}_n)|^2 r_n^2\|\hat{\psi}_n\|^2 \\ &\leq 2^{5+2\varepsilon}C^2\left[\frac{|\nabla V(\mathbf{x}_n)|}{V_n}\left(\frac{V_n^2}{2B_n}\right)^{1+\varepsilon}\right]^2\|\psi_n\|^2.\end{aligned}$$

Hence, condition (3.26) implies that last two terms of (3.44) are of order $o(\|\psi_n\|^2)$ as $n \to \infty$, i.e. $\|H_D\psi_n\|/\|\psi_n\| \to 0$ as $n \to \infty$. □

3.2 The Essential Spectrum of Two-Dimensional Pauli Operators with Repulsive Potentials

Now we consider the non-relativistic Pauli operator H_P in dimension two coupled to a magnetic field B that points in the direction perpendicular to the plane and a potential V such that $|V|$ grows at ∞. As in the definition of the Pauli operator (see Proposition 2.23), let us assume that $\mathbf{A} \in C^{1,\alpha}(\mathbb{R}^2, \mathbb{R}^2)$, hence $B = \operatorname{curl} \mathbf{A} \in C^{\alpha}(\mathbb{R}^2, \mathbb{R})$, and $V \in C^{\alpha}(\mathbb{R}^2, \mathbb{R})$ for some $\alpha \in (0, 1)$. In addition, let the growth condition (2.37) on the negative part of the potential V be satisfied.

Unlike in the case of the Dirac operator H_D in dimension two, the operator $P_{\mathbf{A}}$, describing the the kinetic energy in $H_P = P_{\mathbf{A}} + V$, is non-negative. This designated sign of the kinetic energy operator implies that the properties of H_P depend crucially on the sign of V. Trapping potentials, i.e. potentials with $V(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$, have a classical confining effect: The expectation value $\langle \psi, H_P \psi \rangle$ is growing if the localised state ψ is removed more and more from the origin. In terms of $\sigma(H_P)$ this manifests in the discreteness of the spectrum, i.e. one obtains, independently of B, that $\sigma(H_P)$ consists only of eigenvalues of finite multiplicity with spectral gaps in between them. The latter is an immediate consequence of the local compactness of H_P and a simple resolvent comparison argument (alternatively one can argue with Proposition B.1). However, if we consider an "anti-trapping" potential, i.e. a potential V with $V(\mathbf{x}) \to -\infty$ as $|\mathbf{x}| \to \infty$, such an elementary argumentation is not possible to deduce the type of $\sigma(H_P)$. In particular, one cannot ignore the influence of the magnetic field B: Recalling Proposition 2.28, strong magnetic fields can raise the value

$$\langle \psi, H_P \psi \rangle = \langle \psi, P_{\mathbf{A}} \psi \rangle + \langle \psi, V \psi \rangle, \tag{3.46}$$

while V has the contrary effect. Whether (3.46) necessarily has to leave a fixed energy interval for states localised far away from the origin depends on the interplay between B and V. Again, there is a dependency of $\sigma(H_P)$ on the quotient V/B, similarly as in the case of the Dirac operator H_D.

Theorem 3.15 (Theorem 1 of [Meh15]). *Assume that V is continuously differentiable outside some compact set X and that*

$$V(\mathbf{x}) \longrightarrow -\infty \qquad \text{as } |\mathbf{x}| \to \infty, \tag{3.47}$$

$$\left| \frac{\nabla V(\mathbf{x})}{V(\mathbf{x})} \right| \longrightarrow 0 \qquad \text{as } |\mathbf{x}| \to \infty, \tag{3.48}$$

$$\limsup_{|\mathbf{x}| \to \infty} \left| \frac{V(\mathbf{x})}{2B(\mathbf{x})} \right| < 1. \tag{3.49}$$

Then $\sigma_{\mathrm{ess}}(H_P) = \emptyset$, i.e. H_P has only discrete spectrum.

For the next theorems recall Definition 3.7 from the previous section.

Theorem 3.16 (Theorem 2 of [Meh15]). *Let $B, V \in C^\alpha(\mathbb{R}^2, \mathbb{R})$, with $\alpha \in (0,1)$, be continuously differentiable outside some compact set X. Assume that there is a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset \mathbb{R}^2 \backslash X$, with $|\mathbf{x}_n| \to \infty$ as $n \to \infty$, such that $|\nabla V|, |\nabla B|$ vary with rate 0 on $(\mathbf{x}_n)_{n\in\mathbb{N}}$. If there are constants $k \in \mathbb{N}$ and $\varepsilon \in (0,1)$ for which holds that*

$$V(\mathbf{x}_n) \longrightarrow -\infty, \tag{3.50}$$

$$\frac{|\nabla B(\mathbf{x}_n)|^2}{|B(\mathbf{x}_n)|^{1-\varepsilon}}, \frac{|\nabla V(\mathbf{x}_n)|^2}{|V(\mathbf{x}_n)|^{1-\varepsilon}} \longrightarrow 0, \tag{3.51}$$

$$V(\mathbf{x}_n) + 2k|B(\mathbf{x}_n)| \longrightarrow 0 \tag{3.52}$$

as $n \to \infty$, then $0 \in \sigma_{\text{ess}}(H_P)$.

Theorem 3.17 (Theorem 3 of [Meh15]). *Let $B, V \in C^\alpha(\mathbb{R}^2, \mathbb{R})$, with $\alpha \in (0,1)$, be continuously differentiable outside some compact set X. Assume that there is a continuous path $\gamma : \mathbb{R}^+ \to \mathbb{R}^2 \backslash X$, with $|\gamma(t)| \to \infty$ as $t \to \infty$, such that $|\nabla V|, |\nabla B|$ vary with rate $\nu \in [0,1]$ on $\operatorname{Im}(\gamma)$. If there are constants $\epsilon, \lambda, \kappa, B_0 > 0$ for which holds that*

$$\frac{V(\gamma(t))}{2|B(\gamma(t))|} \longrightarrow -\infty, \tag{3.53}$$

$$\left(\frac{|\nabla V(\gamma(t))|}{|V(\gamma(t))|} + \frac{|\nabla B(\gamma(t))|}{|B(\gamma(t))|}\right)\left(\frac{|V(\gamma(t))|^3}{B^2(\gamma(t))}\right)^{\frac{1+\epsilon}{2}} \longrightarrow 0, \tag{3.54}$$

$$\frac{1}{|\gamma(t)|^{2\nu}}\left(\frac{|V(\gamma(t))|}{B^2(\gamma(t))}\right)^{1+\epsilon} \longrightarrow 0 \tag{3.55}$$

as $t \to \infty$, and, in addition,

$$B_0 \leq |B(\gamma(t))| \leq \lambda \exp\left(\kappa \left|\frac{V(\gamma(t))}{B(\gamma(t))}\right|\right) \tag{3.56}$$

for all $t \in \mathbb{R}^+$, then $\sigma_{\text{ess}}(H_P) = \mathbb{R}$.

The proofs of Theorems 3.15−3.17 are given in the mentioned reference. They are based on the ideas that were already used to prove Theorems 3.1, 3.9 and 3.11. However, since the Pauli operator H_P is a second-order operator, the commutator and localisation estimates for H_P are much more laborious. Further, approximating B and V locally by constants within the second-order operator H_P leads to the constraints (3.51) and (3.54) on the growth rate of V/B, which are different to those for the operator H_D (c.f. (3.22) and (3.26)). As we will see in the next section, (3.51) and (3.54) pose an interesting limit on the applicability of Theorem 3.16 and 3.17: Unlike in the case of H_D, one can apply Theorem 3.16 and 3.17 for power-like growth only up to the order $|\mathbf{x}|^2$, which corresponds to restriction (2.37) on the growth of the negative part of V for assuring essential self-adjointness.

Having a look on the proof of Theorem 3.17, one observes that the constraints on the growth of V/B (or on V^2/B in Theorem 3.11) are due to the fact that one only uses a zero-order approximation for B and V. A question arising from that observation is, whether one can weaken (3.54) by considering higher order Taylor expansions of B and V. It would mean to construct Weyl functions that are closer to the solutions of the eigenvalue Problem $H_P\psi = E\psi$. A first-order approximation of V, i.e. writing locally $V(\mathbf{x}) \approx V_0(\mathbf{x}_0) + \nabla V(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$, leads to

Theorem 3.18 (Theorem 4 of [Meh15]). *Let $B = B_0 > 0$, and $V \in C^\alpha(\mathbb{R}^2, \mathbb{R})$, with $\alpha \in (0,1)$, be two times continuously differentiable outside some compact set X. Assume that there is a continuous path $\gamma : \mathbb{R}^+ \to \mathbb{R}^2$, with $|\gamma(t)| \to \infty$ as $t \to \infty$, such that the matrix norm of the Hessian matrix $\|\mathrm{Hess}(V)\|_{2\times 2} \colon \mathbb{R}^2 \to \mathbb{R}$ vary with rate $\nu \in [0,1]$ on $\mathrm{Im}(\gamma) \subset \mathbb{R}^2 \backslash X$. If there is a constant $\epsilon > 0$ for which holds that*

$$V(\gamma(t)) \longrightarrow -\infty, \tag{3.57}$$

$$\|\mathrm{Hess}(V)\|_{2\times 2}(\gamma(t))|V(\gamma(t))|^{1+\epsilon} \longrightarrow 0, \tag{3.58}$$

$$\frac{1}{|\gamma(t)|^{2\nu}}\, |V(\gamma(t))|^{1+\epsilon} \longrightarrow 0 \tag{3.59}$$

as $t \to \infty$, and if

$$\limsup_{t\to\infty} \frac{|\nabla V(\gamma(t))|^2}{|V(\gamma(t))|} < (2B_0)^2, \tag{3.60}$$

then $\sigma_{\mathrm{ess}}(H_P) = \mathbb{R}$.

Remark 3.19. *Even if conditions* (3.58) *and* (3.54) *(for $B = B_0$) are not comparable in general, our main examples (see Section 3.3) show that Theorem 3.18 is indeed an improvement of Theorem 3.17.*

For the proof of Theorem 3.18 we first have to discuss the Pauli operator with a constant magnetic field B_0 and a constant electric field $\mathcal{E}_0$ in a certain direction. Since $B(\mathbf{x}) = B_0$, we assume that $\mathbf{A}$ is in the rotational gauge, i.e.

$$\mathbf{A}(\mathbf{x}) = \frac{B_0}{2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

With this gauge $P_\mathbf{A}$ is invariant under rotations of the plane. More precisely, for $\mathcal{R} \in SO(2,\mathbb{R})$ consider the unitary map

$$U_\mathcal{R} : L^2(\mathbb{R}^2, \mathbb{C}^2) \to L^2(\mathbb{R}^2, \mathbb{C}^2), \qquad \psi(\,\cdot\,) \mapsto \psi(\mathcal{R}^{-1}\cdot\,),$$

which represents the rotation $\mathcal{R}$ in $L^2(\mathbb{R}^2, \mathbb{C}^2)$. Then $U_\mathcal{R}^{-1} P_\mathbf{A} U_\mathcal{R} = P_\mathbf{A}$, so we get

$$U_\mathcal{R}^{-1}(P_\mathbf{A} + V)U_\mathcal{R} = P_\mathbf{A} + V_\mathcal{R},$$

with $V_{\mathcal{R}}(\,\cdot\,) = V(\mathcal{R}\,\cdot\,)$. Such a rotation enables us to change the direction of the constant electric field. Therefore, we assume that V has the form $V(\mathbf{x}) = V_0 + \mathcal{E}_0(x_1 - \zeta)$, with constants $V_0, \mathcal{E}_0, \zeta \in \mathbb{R}$. To construct approximative solutions of the eigenvalue problem $H_P\psi = E\psi$, we want to use the symmetry of V (and therefore of the system) in the x_2-direction. Hence, we consider the Landau gauge $\hat{\mathbf{A}}$ for B_0, given by $\hat{\mathbf{A}}(\mathbf{x}) = B_0 x_1 \hat{e}_2$. Coupled to the vector potential $\hat{\mathbf{A}}$, the Hamiltonian H_P has the form

$$\begin{aligned} P_{\hat{\mathbf{A}}} + V &= -\partial_1^2 + (-\mathrm{i}\,\partial_2 - B_0 x_1)^2 - \sigma_3 B_0 + V \\ &= \hat{d}^*\hat{d} + B_0 - \sigma_3 B_0 + V_0 + \mathcal{E}_0(x_1 - \zeta), \end{aligned} \tag{3.61}$$

where $\hat{d}, \hat{d}^*$ read

$$\hat{d} = -\mathrm{i}\,\partial_1 + \mathrm{i}\,(-\mathrm{i}\,\partial_2 - B_0 x_1), \qquad \hat{d}^* = -\mathrm{i}\,\partial_1 - \mathrm{i}\,(-\mathrm{i}\,\partial_2 - B_0 x_1).$$

Observe that one can rewrite H_P as

$$\begin{aligned} P_{\hat{\mathbf{A}}} + V &= -\partial_1^2 + (-\mathrm{i}\,\partial_2 - B_0 x_1)^2 - \sigma_3 B_0 + V_0 + \mathcal{E}_0(x_1 - \zeta) \\ &= -\partial_1^2 + B_0^2\big(x_1 - \tfrac{1}{B_0}\big(-\mathrm{i}\,\partial_2 - \tfrac{\mathcal{E}_0}{2B_0}\big)\big)^2 \\ &\qquad + \tfrac{\mathcal{E}_0}{B_0}\big(-\mathrm{i}\,\partial_2 - \tfrac{\mathcal{E}_0}{2B_0}\big) - \mathcal{E}_0\zeta - \sigma_3 B_0 + V_0 + \big(\tfrac{\mathcal{E}_0}{2B_0}\big)^2. \end{aligned} \tag{3.62}$$

Because of the symmetry in x_2-direction, we are able to perform a Fourier transformation with respect to that direction and obtain

$$P_{\hat{\mathbf{A}}} + V \cong \int_{\mathbb{R}}^{\oplus} h(\xi)\mathrm{d}\xi,$$

i.e. a direct integral representation on $L^2(\mathbb{R}_\xi, L^2(\mathbb{R}, \mathbb{C}^2))$, with

$$\begin{aligned} h(\xi) &= -\partial_1^2 + B_0^2\big(x_1 - \tfrac{1}{B_0}\big(\xi - \tfrac{\mathcal{E}_0}{2B_0}\big)\big)^2 + \tfrac{\mathcal{E}_0}{B_0}\big(\xi - \tfrac{\mathcal{E}_0}{2B_0}\big) \\ &\qquad\qquad - \mathcal{E}_0\zeta - \sigma_3 B_0 + V_0 + \big(\tfrac{\mathcal{E}_0}{2B_0}\big)^2 \\ &= -\partial_1^2 + B_0^2\big(x_1 - \tilde{\zeta}\big)^2 + \mathcal{E}_0\tilde{\zeta} - \mathcal{E}_0\zeta - \sigma_3 B_0 + V_0 + \big(\tfrac{\mathcal{E}_0}{2B_0}\big)^2. \end{aligned}$$

Here we use the notation $\tilde{\zeta} = \frac{1}{B_0}\big(\xi - \frac{\mathcal{E}_0}{2B_0}\big)$. Since $h(\xi)$ is the Hamiltonian of the harmonic oscillator (up to a shift), we set

$$\phi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}}\,\vartheta_n(x) e^{-\frac{1}{2}x^2}, \quad x \in \mathbb{R},$$

where ϑ_n, for $n \in \mathbb{N}_0$, denotes the n-th Hermite polynomial. The normalised functions

$$\varphi_{\mathcal{E}_0,n,\xi}(x_1) := \sqrt[4]{B_0}\begin{pmatrix} \phi_n\big(\sqrt{B_0}\big(x_1 - \frac{1}{B_0}\big(\xi - \frac{\mathcal{E}_0}{2B_0}\big)\big)\big) \\ 0 \end{pmatrix}$$

satisfy the equation

$$h(\xi)\varphi_{\mathcal{E}_0,n,\xi} = \Big(2nB_0 + \mathcal{E}_0\big(\tfrac{1}{B_0}\big(\xi - \tfrac{\mathcal{E}_0}{2B_0}\big) - \zeta\big) + V_0 + \big(\tfrac{\mathcal{E}_0}{2B_0}\big)^2\Big)\varphi_{\mathcal{E}_0,n,\xi}\,.$$

To obtain a wave function with momentum ξ in the direction of x_2, we set

$$\psi_{\mathcal{E}_0,n,\xi}(x_1,x_2) := e^{\mathrm{i}\,\xi x_2}\varphi_{\mathcal{E}_0,n,\xi}(x_1). \tag{3.63}$$

Applying H_P to $\psi_{\mathcal{E}_0,n,\xi}$ leads to the differential equation

$$\begin{aligned}\big[P_{\hat{\mathbf{A}}} + V\big]\psi_{\mathcal{E}_0,n,\xi}\\ = \Big(2nB_0 + \mathcal{E}_0\big(\tfrac{1}{B_0}\big(\xi - \tfrac{\mathcal{E}_0}{2B_0}\big) - \zeta\big) + V_0 + \big(\tfrac{\mathcal{E}_0}{2B_0}\big)^2\Big)\psi_{\mathcal{E}_0,n,\xi},\end{aligned} \tag{3.64}$$

with $\xi \in \mathbb{R}$ and $n \in \mathbb{N}_0$. In addition, for every $\xi \in \mathbb{R}$ and $n \in \mathbb{N}_0$, it holds that

$$\begin{aligned}\hat{d}\psi_{\mathcal{E}_0,n,\xi} &= \Big[-\mathrm{i}\,\partial_1 - \mathrm{i}\,B_0\big(x_1 - \tfrac{1}{B_0}\big(\xi - \tfrac{\mathcal{E}_0}{2B_0}\big)\big) + \mathrm{i}\,\tfrac{\mathcal{E}_0}{2B_0}\Big]\psi_{\mathcal{E}_0,n,\xi}\\ &= -\mathrm{i}\,\sqrt{2nB_0}\,\psi_{\mathcal{E}_0,n-1,\xi} + \mathrm{i}\,\tfrac{\mathcal{E}_0}{2B_0}\psi_{\mathcal{E}_0,n,\xi}\,,\end{aligned} \tag{3.65}$$

$$\begin{aligned}\hat{d}^*\psi_{\mathcal{E}_0,n,\xi} &= \Big[-\mathrm{i}\,\partial_1 + \mathrm{i}\,B_0\big(x_1 - \tfrac{1}{B_0}\big(\xi - \tfrac{\mathcal{E}_0}{2B_0}\big)\big) - \mathrm{i}\,\tfrac{\mathcal{E}_0}{2B_0}\Big]\psi_{\mathcal{E}_0,n,\xi}\\ &= \mathrm{i}\,\sqrt{2(n+1)B_0}\,\psi_{\mathcal{E}_0,n+1,\xi} - \mathrm{i}\,\tfrac{\mathcal{E}_0}{2B_0}\psi_{\mathcal{E}_0,n,\xi}\,.\end{aligned} \tag{3.66}$$

Proof of Theorem 3.18. It suffices to find a Weyl sequence for $E = 0$ since conditions (3.57)−(3.60) hold also for the potential $V_E = V - E$. Applying (3.57) and (3.60), we find a sequence $\{\mathbf{y}_n\}_{n\in\mathbb{N}} \subset \mathrm{Im}(\gamma)$ satisfying

$$V(\mathbf{y}_n) = -2nB_0 - \Big(\tfrac{|\nabla V(\mathbf{y}_n)|}{2B_0}\Big)^2 \tag{3.67}$$

for $n \geq N$, with $N \in \mathbb{N}$ large enough. In addition, we can find rotations $\mathcal{R}_n \in SO(2,\mathbb{R})$ such that $\nabla V_{\mathcal{R}_n}(\mathbf{x}_n) = |\nabla V_{\mathcal{R}_n}(\mathbf{x}_n)|\hat{e}_1$, where $\mathbf{x}_n := \mathcal{R}_n^{-1}\mathbf{y}_n$. Let us introduce the short hand notation

$$V_n := V(\mathbf{y}_n) = V_{\mathcal{R}_n}(\mathbf{x}_n), \tag{3.68}$$

$$\mathcal{E}_n := |\nabla V(\mathbf{y}_n)| = |\nabla V_{\mathcal{R}_n}(\mathbf{x}_n)|, \tag{3.69}$$

$$\xi_n := B_0 x_{n,1} + \tfrac{\mathcal{E}_n}{2B_0}. \tag{3.70}$$

Pick $\chi \in C_0^\infty(\mathbb{R},[0,1])$, with $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. To localise around the points $\mathbf{x}_n = (x_{n,1}, x_{n,2})^T$, we set

$$\chi_{n,j}(x) := \chi\Big(\tfrac{x - x_{n,j}}{r_n}\Big), \qquad j = 1,2\,,$$

with $r_n = \sqrt{n^{1+\epsilon}/B_0}$, and define

$$\begin{aligned}\psi_n(\mathbf{x}) &:= \chi_{n,1}(x_1)\chi_{n,2}(x_2)\psi_{\mathcal{E}_n,n,\xi_n}(x_1,x_2)\\ &= \chi\Big(\tfrac{x_2 - x_{n,2}}{r_n}\Big)\, e^{-\mathrm{i}\,\xi_n x_2}\chi\Big(\tfrac{x_1 - x_{n,1}}{r_n}\Big)\begin{pmatrix}\sqrt[4]{B_0}\,\phi_n\big(\sqrt{B_0}(x_1 - x_{n,1})\big)\\ 0\end{pmatrix}.\end{aligned}$$

The representation

$$\vartheta_n(x) = (-1)^n \sum_{k_1+2k_2=n} \frac{n!}{k_1!k_2!}(-1)^{k_1+k_2}(2x)^{k_1}$$

of the $n-$th Hermite polynomial (see [AS64]) leads to the estimate $|\vartheta_n(x)| \leq (n+1)2^n n!|x|^n$ for $x \in \mathbb{R}$, implying that

$$\int_{\sqrt{n^{1+\epsilon}}}^{\infty} |\phi_n(x)|^2 \mathrm{d}x \leq \frac{(n+1)^2}{4} 2^n n! \int_{\sqrt{n^{1+\epsilon}}}^{\infty} x^{2n} e^{-x^2} \mathrm{d}x \longrightarrow 0 \quad \text{as } n \to \infty.$$

In particular, in view of the symmetry of $|\phi_n|^2$, we have

$$\begin{aligned} r_n \leq 2r_n \int_{-\sqrt{n^{1+\epsilon}}}^{\sqrt{n^{1+\epsilon}}} |\phi_n(x)|^2 \mathrm{d}x &\leq \|\psi_n\|^2 \\ &\leq 4r_n \int_{-2\sqrt{n^{1+\epsilon}}}^{2\sqrt{n^{1+\epsilon}}} |\phi_n(x)|^2 \mathrm{d}x \leq 4r_n \end{aligned} \tag{3.71}$$

for $n \in \mathbb{N}$ sufficiently large. By setting $\Gamma(\mathbf{x}) = \frac{B_0}{2} x_1 x_2$, equations (3.61), (3.64) and (3.67) imply, together with the choice of ξ_n, that

$$\begin{aligned} H_P U_{\mathcal{R}_n} e^{-\mathrm{i}\,\Gamma}\psi_n &= U_{\mathcal{R}_n} e^{-\mathrm{i}\,\Gamma}\big[P_{\hat{\mathbf{A}}} + V_{\mathcal{R}_n}\big]\psi_n \\ &= U_{\mathcal{R}_n} e^{-\mathrm{i}\,\Gamma}\big(\hat{d}^*\hat{d}\psi_n - \chi_{n,1}\chi_{n,2}\hat{d}^*\hat{d}\psi_{\mathcal{E}_n,n,\xi_n}\big) + \\ &\quad\; U_{\mathcal{R}_n} e^{-\mathrm{i}\,\Gamma}\big(V_{\mathcal{R}_n} - V_n - \mathcal{E}_n(x_1 - x_{1,n})\big)\psi_n. \end{aligned} \tag{3.72}$$

To estimate the localisation error

$$\begin{aligned} \hat{d}^*\hat{d}\psi_n - \chi_{n,1}\chi_{n,2}\hat{d}^*\hat{d}\psi_{\mathcal{E}_n,n,\xi_n} \;&=\; \big(-\mathrm{i}\,\chi_{n,2}\partial_1\chi_{n,1} + \chi_{n,1}\partial_2\chi_{n,2}\big)\hat{d}^*\psi_{\mathcal{E}_n,n,\xi_n} + \\ &\quad\; \big(-\mathrm{i}\,\chi_{n,2}\partial_1\chi_{n,1} - \chi_{n,1}\partial_2\chi_{n,2}\big)\hat{d}\psi_{\mathcal{E}_n,n,\xi_n} + \\ &\quad\; \big(-\chi_{n,2}\partial_1^2\chi_{n,1} - \chi_{n,1}\partial_2^2\chi_{n,2}\big)\psi_{\mathcal{E}_n,n,\xi_n}, \end{aligned}$$

we apply (3.65) and (3.66), resulting in

$$\begin{aligned} &\big\|\big(-\mathrm{i}\,\chi_{n,2}\partial_1\chi_{n,1} + \chi_{n,1}\partial_2\chi_{n,2}\big)\hat{d}^*\psi_{\mathcal{E}_n,n,\xi_n}\big\| \\ &\quad\leq\; \sqrt{2(n+1)B_0}\,\big\|\big(-\mathrm{i}\,\chi_{n,2}\partial_1\chi_{n,1} + \chi_{n,1}\partial_2\chi_{n,2}\big)\psi_{\mathcal{E}_n,n+1,\xi_n}\big\| \\ &\qquad\qquad + \tfrac{\mathcal{E}_n}{2B_0}\,\big\|\big(-\mathrm{i}\,\chi_{n,2}\partial_1\chi_{n,1} + \chi_{n,1}\partial_2\chi_{n,2}\big)\psi_{\mathcal{E}_n,n,\xi_n}\big\| \\ &\quad\leq 2\sqrt{2(n+1)B_0}\,r_n^{-1}\|\chi'\|_\infty\sqrt{2r_n}\,\|\phi_{n+1}\| + \tfrac{\mathcal{E}_n}{B_0}r_n^{-1}\|\chi'\|_\infty\sqrt{2r_n}\,\|\phi_n\| \\ &\quad\leq 2\sqrt{2}\|\chi'\|_\infty\Big(B_0\sqrt{\tfrac{2n+2}{n^{1+\epsilon}}} + \tfrac{\mathcal{E}_n}{2}\sqrt{\tfrac{B_0}{n^{1+\epsilon}}}\Big)\sqrt{r_n}\,, \end{aligned}$$

$$\begin{aligned} &\big\|\big(-\mathrm{i}\,\chi_{n,2}\partial_1\chi_{n,2} - \chi_{n,1}\partial_2\chi_{n,2}\big)\hat{d}\psi_{\mathcal{E}_n,n,\xi_n}\big\| \\ &\qquad\qquad\leq \sqrt{2nB_0}\,\big\|\big(-\mathrm{i}\,\chi_{n,2}\partial_1\chi_{n,1} - \chi_{n,1}\partial_2\chi_{n,2}\big)\psi_{\mathcal{E}_n,n-1,\xi_n}\big\| \\ &\qquad\qquad\quad + \tfrac{\mathcal{E}_n}{2B_0}\,\big\|\big(-\mathrm{i}\,\chi_{n,2}\partial_1\chi_{n,1} - \chi_{n,1}\partial_2\chi_{n,2}\big)\psi_{\mathcal{E}_n,n,\xi_n}\big\| \\ &\qquad\qquad\leq 2\sqrt{2}\|\chi'\|_\infty\Big(B_0\sqrt{\tfrac{2n}{n^{1+\epsilon}}} + \tfrac{\mathcal{E}_n}{2}\sqrt{\tfrac{B_0}{n^{1+\epsilon}}}\Big)\sqrt{r_n}\,. \end{aligned}$$

The last term of the localisation error can be bounded like

$$\big\|\big(-\chi_{n,2}\partial_1^2\chi_{n,1}-\chi_{n,1}\partial_2^2\chi_{n,2}\big)\psi_{\mathcal{E}_n,n,\xi_n}\big\| \le 2\sqrt{2}\,\|\chi''\|_\infty r_n^{-2}\sqrt{r_n}\,.$$

Since, due to (3.67) and (3.60), we have

$$V_n = 2nB_0\bigg(1-\frac{\mathcal{E}_n^2}{V_n 4B_0^2}\bigg)^{-1} \tag{3.73}$$

for $n \in \mathbb{N}$ large enough, and therefore $\mathcal{E}_n/\sqrt{n^{1+\epsilon}} \to 0$ as $n \to \infty$, we conclude with norm estimate (3.71) that

$$\big\|\hat{d}^*\hat{d}\psi_n - \chi_{n,1}\chi_{n,2}\hat{d}^*\hat{d}\psi_{\mathcal{E}_n,n,\xi_n}\big\|/\|\psi_n\| \longrightarrow 0 \quad \text{as } n\to\infty. \tag{3.74}$$

For the remaining term of (3.72) we expand $V_{\mathcal{R}_n}$ up to second order: The definition of V_n and $\mathcal{E}_n$ (see (3.68) and (3.69)) implies, for any $\mathbf{x}\in\mathbb{R}^2$, that

$$\begin{aligned}\big|\big[V_{\mathcal{R}_n}(\mathbf{x}) - V_n - \mathcal{E}_n(x_1 - x_{1,n})\big]\psi_n(\mathbf{x})\big| \\ \le \big\|\mathrm{Hess}(V_{\mathcal{R}_n})\big\|_{2\times2}(\boldsymbol{\eta}_{\boldsymbol{x},\boldsymbol{x_n}})|\mathbf{x}-\mathbf{x}_n|^2|\psi_n(\mathbf{x})|,\end{aligned}$$

where $\boldsymbol{\eta}_{\boldsymbol{x},\boldsymbol{x_n}} \in [\mathbf{x},\mathbf{x}_n]$. Due to the fact that $\mathcal{R}_n$ are rotations, we have $\|\mathrm{Hess}(V_{\mathcal{R}_n})\|_{2\times2}(\,\cdot\,) = \|\mathrm{Hess}(V)\|_{2\times2}(\mathcal{R}_n\,\cdot\,)$ for any $n\in\mathbb{N}$. Now recall that $\|\mathrm{Hess}(V)\|_{2\times2}$ varies with rate ν along $\mathrm{Im}(\gamma)$ and observe that (by (3.59) and (3.73)) $r_n/|\mathbf{x}_n|^\nu \to 0$ as $n\to\infty$. Thus, we find a constant $C>0$ such that for $n\in\mathbb{N}$ large enough it holds that

$$\|\mathrm{Hess}(V_{\mathcal{R}_n})\|_{2\times2}(\boldsymbol{\eta}) \le C\|\mathrm{Hess}(V_{\mathcal{R}_n})\|_{2\times2}(\mathbf{x}_n)$$

whenever $\boldsymbol{\eta}$ satisfies $|\boldsymbol{\eta}-\mathbf{x}_n| \le 2r_n$. Assuming that $N\in\mathbb{N}$ is large enough, we conclude

$$\begin{aligned}\big\|U_{\mathcal{R}_n}e^{-\mathrm{i}\Gamma}\big[V_{\mathcal{R}_n} - V_n - \mathcal{E}_n(x_1-x_{1,n})\big]\psi_n\big\| \\ \le 4C\|\mathrm{Hess}(V)\|_{2\times2}(\mathcal{R}_n\mathbf{x}_n)\,r_n^2\|\psi_n\| \\ \le 4C\|\mathrm{Hess}(V)\|_{2\times2}(\mathbf{y}_n)\left(\tfrac{|V_n|}{B_0}\right)^{1+\epsilon}\|\psi_n\|\end{aligned}$$

for $n \ge N$. Now condition (3.58) states that the right hand side of the last inequality vanishes as $n\to\infty$, i.e. $(U_{\mathcal{R}_n}e^{-\mathrm{i}\Gamma}\psi_n)_{n\in\mathbb{N}}$ is a Weyl sequence for the value 0. □

Remark 3.20. *One may observe that in the proof of Theorem 3.18 we actually show that the upper component of H_P satisfies $\sigma(d^*d+V)=\mathbb{R}$. It is easy to see that also $\sigma(dd^*+V)=\mathbb{R}$ under the conditions of the Theorem.*

Remark 3.21. *The idea of the proof of Theorem 3.18 is also useful to improve the statement of 3.9 for H_D in certain directions. However, we do not precise the statement in this work.*

3.3 Applications and Open Questions

3.3.1 Systems with Rotational or Translational Symmetry

In this small paragraph we want to point out basic examples of electromagnetic field configurations, where our Theorems lead to new insights. Even if there are far more examples, for which the results are applicable, we concentrate on the most important ones.

First recall that for every $s, c \in \mathbb{R}$ the power functions $f_1(\mathbf{x}) = c|\mathbf{x}|^s$ and $f_2(\mathbf{x}) = c|x_1|^s$ vary with an arbitrary rate $\nu \in [0,1]$ along the half-line $L := \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 > 2, x_2 = 0\}$ (c.f. Example 3.8). Further, we know that

$$
\begin{aligned}
|\nabla f_1(\mathbf{x})| &= cs|\mathbf{x}|^{s-1}, \qquad \|\mathrm{Hess}(f_1)\|_{2\times 2}(\mathbf{x}) = cs\max(1, s-1)|\mathbf{x}|^{s-2},\\
|\nabla f_2(\mathbf{x})| &= cs|x_1|^{s-1}, \qquad \|\mathrm{Hess}(f_2)\|_{2\times 2}(\mathbf{x}) = cs(s-1)|x_1|^{s-2},
\end{aligned}
$$

for $\mathbf{x} \in \mathbb{R}^+ \times \mathbb{R}$. Hence, also $|\nabla f_1|$, $\|\mathrm{Hess}(f_1)\|_{2\times 2}$, $|\nabla f_2|$ and $\|\mathrm{Hess}(f_2)\|_{2\times 2}$ vary with rate $\nu \in [0,1]$ on the set L. Since, in addition, if $c \neq 0$ one has

$$
\frac{|\nabla f_1(\mathbf{x})|}{f_1(\mathbf{x})} = s\frac{1}{|\mathbf{x}|}, \qquad \frac{|\nabla f_2(\mathbf{x})|}{f_2(\mathbf{x})} = s\frac{1}{|x_1|}
$$

for $\mathbf{x} \in L$, we can use Theorems 3.1, 3.9 and 3.11 to conclude

Corollary 3.22. *Let $s, t \geq 0$ and $b_0, v_0 \neq 0$. Assume that B and V are of form $B(\mathbf{x}) = b_0|\mathbf{x}|^s$, $V(\mathbf{x}) = v_0|\mathbf{x}|^t$. Then we have:*

a) *$\sigma(H_D)$ is purely discrete, i.e $\sigma_{\mathrm{ess}}(H_D) = \emptyset$, if $0 < 2t < s$, or if $0 < 2t = s$ and $|v_0|^2 < 2|b_0|$.*

b) *$0 \in \sigma_{\mathrm{ess}}(H_D)$ if $0 < 2t = s$ and $|v_0|^2 = 2b_0k$ for some $k \in \mathbb{N}$.*

c) *$\sigma(H_D) = \mathbb{R}$ if $0 \leq s < 2t < s+1$. Together with Corollary 2.31 we conclude that in this case H_D has dense pure point spectrum.*

Other examples are magnetic fields B and potentials V that are translationally symmetric with respect to the x_2-axis:

Corollary 3.23. *Let $s, t \geq 0$ and $b_0, v_0 \neq 0$. Assume that B and V are of form $B(\mathbf{x}) = b_0|x_1|^s$, $V(\mathbf{x}) = v_0|x_1|^t$. Then we have:*

a) *$0 \in \sigma_{\mathrm{ess}}(H_D)$ if $0 < 2t = s$ and $|v_0|^2 = 2b_0k$ for some $k \in \mathbb{N}$.*

b) *$\sigma(H_D) = \mathbb{R}$ if $0 \leq s < 2t < s+1$.*

Note that the second statement on systems with translational symmetry carries even more information: If we investigate the fiber decomposition of H_D (see (2.52)), we may deduce that $\sigma_{\mathrm{ac}}(H_D) = \mathbb{R}$ in case b) of the last corollary (c.f. Theorem XIII.86 of [RS78]).

Let us proceed with the Pauli operator H_P. Analogously as above, we conclude from Theorems 3.15−3.18:

Corollary 3.24. *Let $0 \le s, t \le 2$ and $v_0 < 0 < |b_0|$. Assume that B and V are of form $B(\mathbf{x}) = b_0|\mathbf{x}|^s$, $V(\mathbf{x}) = v_0|\mathbf{x}|^t$. Then we have:*

a) *$\sigma(H_P)$ is purely discrete, i.e $\sigma_{\mathrm{ess}}(H_P) = \emptyset$, if $0 < t < s$, or if $0 < t = s$ and $|v_0| < 2b_0$.*

b) *$0 \in \sigma_{\mathrm{ess}}(H_P)$ if $0 < t = s$ and $|v_0| = 2b_0 k$ for some $k \in \mathbb{N}$.*

c) *$\sigma(H_P) = \mathbb{R}$ if $0 \le 3s < 3t < 2(s+1)$. Using Remark 2.33, we conclude that in this case H_P has dense pure point spectrum.*

d) *$\sigma(H_P) = \mathbb{R}$ if $s = 0$ and $0 < t < 1$. Using Remark 2.33, we conclude that in this case H_P has dense pure point spectrum.*

Corollary 3.25. *Let $0 \le s, t \le 2$ and $v_0 < 0 < |b_0|$. Assume that B and V are of form $B(\mathbf{x}) = b_0|x_1|^s$, $V(\mathbf{x}) = v_0|x_1|^t$. Then we have:*

a) *$0 \in \sigma_{\mathrm{ess}}(H_P)$ if $0 < t = s$ and $|v_0| = 2b_0 k$ for some $k \in \mathbb{N}$.*

b) *$\sigma(H_P) = \mathbb{R}$ if $0 \le 3s < 3t < 2(s+1)$.*

c) *$\sigma(H_P) = \mathbb{R}$ if $s = 0$ and $0 < t \le 1$.*

Remark 3.26. *We observe that in part* c) *of Corollaries 3.24 and 3.25 the condition $3s < 3t < 2(s+1)$ does necessarily require that $t < 2$, which is exactly the critical exponent for assuring the essential self-adjointness of the operator H_P.*

3.3.2 Magnetic Schrödinger Operators

We briefly discuss another important operator from quantum mechanics, where our results can be applied: The magnetic Schrödinger operator in dimension two. Our results allow us to point out new, very simple examples of rotationally symmetric electro-magnetic field configurations, where this operator has dense pure point spectrum. Some further examples with periodic potentials V are discussed in [Hoe97].

For the definition of the operator let $B, V \in C(\mathbb{R}^2, \mathbb{R})$. Assume that $\mathbf{A} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ generates the magnetic field B, i.e. $B = \operatorname{curl} \mathbf{A}$, and that V satisfies the lower bound condition (2.37) as for the Pauli operator. We set

$$H_S\psi := \big[(-\mathrm{i}\,\nabla - \mathbf{A})^2 + V\big]\psi = \big[H_P - \sigma_3 B\big]\psi$$

for $\psi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$. The operator H_S is essentially self-adjoint on the given core (see [CFKS87]), so let us use the same symbol for the self-adjoint extension on $L^2(\mathbb{R}^2, \mathbb{C}^2)$. For pointing out our examples of magnetic Schrödinger operators with dense pure point spectrum, resulting from our discussion on H_D and H_P, we consider the analogue of Theorem 3.18 for H_S:

Theorem 3.27. *Let $B = B_0 > 0$, and $V \in C(\mathbb{R}^2, \mathbb{R})$ be two times continuously differentiable outside some compact set X. Assume that there is a continuous path $\gamma : \mathbb{R}^+ \to \mathbb{R}^2$, with $|\gamma(t)| \to \infty$ as $t \to \infty$, such that the matrix norm of the Hessian matrix $\|\mathrm{Hess}(V)\|_{2\times 2} : \mathbb{R}^2 \to \mathbb{R}$ varies with rate $\nu \in [0, 1]$ on $\mathrm{Im}(\gamma) \subset \mathbb{R}^2 \backslash X$. If there is a constant $\epsilon > 0$ for which holds that*

$$V(\gamma(t)) \longrightarrow -\infty, \tag{3.75}$$

$$\|\mathrm{Hess}(V)\|_{2\times 2}(\gamma(t))|V(\gamma(t))|^{1+\epsilon} \longrightarrow 0, \tag{3.76}$$

$$\frac{1}{|\gamma(t)|^{2\nu}}\, |V(\gamma(t))|^{1+\epsilon} \longrightarrow 0 \tag{3.77}$$

as $t \to \infty$, and if

$$\limsup_{t\to\infty} \frac{|\nabla V(\gamma(t))|^2}{|V(\gamma(t))|} < (2B_0)^2, \tag{3.78}$$

then $\sigma_{\mathrm{ess}}(H_S) = \mathbb{R}$.

We note that this theorem is a direct consequence of Remark 3.20 since $H_S = d^*d + B_0 + V$ in this case. Also for non-constant magnetic fields one can obtain statements similar to those of Theorems 3.15−3.17. However, we leave this branch of applications with:

Corollary 3.28. *Assume that $B(\mathbf{x}) = B_0 > 0$ and that V has the form $V(\mathbf{x}) = v_0|\mathbf{x}|^t$, with $v_0 < 0$ and $t \in (0, 1)$. Then $\sigma(H_S) = \mathbb{R}$ and the operator H_S has dense pure point spectrum.*

3.3.3 Further Questions on the Spectrum

To close this chapter, let us propose some further questions, which raised along our studies of the spectra of quantum Hamiltonians in dimension two.

One of them one may notice while going through Corollaries 3.22 and 3.25 of the previous subsection: Our results lead to a complete determination of the spectrum of H_D and H_P in the case of magnetic fields and potentials of the form $B(\mathbf{x}) = b_0|\mathbf{x}|^s$, $V(\mathbf{x}) = v_0|\mathbf{x}|^t$, but only for a certain scope of exponents s and t. While one can read of Corollary 2.31 and Remark 2.33 that $\sigma(H_D)$ and $\sigma(H_P)$ are of pure point type when $t \in [0, s+1)$ and when $t \in [0, 2)$ respectively, our results do only state that:

- The set of eigenvalues of H_D is dense in $\mathbb{R}$ if $2t \in (s, s+1)$.
- The set of eigenvalues of H_P is dense in $\mathbb{R}$ if $3t \in (3s, 2(s+1))$ and if $s = 0$, $t \in (0, 1)$.

So far, we still cannot say whether this holds also true for the powers $t \in [(s+1)/2, s+1)$ in the case of H_D, and for the powers $t \in [2(s+1)/3, 2)$ in the case of H_P. In view of the results on $\sigma(H_D)$ for $t > s+1$ (see Corollary 2.32) and in view of the argumentation in the introduction, one may suggest that the answer of this questions is positive.

A further problem, targeting another direction, is how sensitive these examples of dense pure point spectrum are with respect to small perturbations of V (that are not spherically symmetric). It is known that for certain operators of this spectral type even rank one perturbation do suffice to alter the spectrum completely (c.f. [DRMS94]). Hence, it would be interesting to see whether already a destruction of the rotational symmetry of V on a compact set can generate a continuous part in the spectrum of these operators.

Chapter 4

Time Evolution of Dirac Systems in Dimension Two

In this chapter we investigate how the density distribution $|\psi|^2(\,\cdot\,)$ of a state ψ spreads with time assuming that the time evolution is governed by the Dirac wave equation in dimension $2+1$. In particular, we are interested in the relation between the spectrum of the corresponding Dirac operator and the possibility of ballistic behaviour.

To be more precise, we assume, as in the definition of the Dirac operator in Subsection 2.2.1, that $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and $V \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$ for some $p > 2$. Since we only consider Dirac systems in this chapter, let us use the notation

$$H := H_D = \boldsymbol{\sigma} \cdot (-\mathrm{i}\,\nabla - \mathbf{A}) + V \qquad \text{on } L^2(\mathbb{R}^2, \mathbb{C}^2).$$

As usual, we assume that the time evolution $\psi(t)$ of a state $\psi \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ is given by the Dirac wave equation

$$-\mathrm{i}\,\partial_t \psi(t) = H\psi(t), \qquad \psi(0) = \psi.$$

Due to the self-adjointness of H, we can define the unitary time evolution operator $e^{-\mathrm{i}\,tH}$ on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ and obtain the time evolved state via

$$\psi(t) = e^{-\mathrm{i}\,tH}\psi.$$

As mentioned in the introduction, the RAGE theorem already states that the time evolution of the expectation value

$$\big\langle \psi(t), \mathbb{1}_{\{|\mathbf{x}| \leq R\}} \psi(t) \big\rangle$$

depends sensitively on whether one starts with $\psi \in P_{\mathrm{pp}}(H) L^2(\mathbb{R}^2, \mathbb{C}^2)$ or with $\psi \in P_{\mathrm{ac}}(H) L^2(\mathbb{R}^2, \mathbb{C}^2)$, i.e. whether ψ belongs to the subspace associated to the point spectrum of H or to the one associated to the absolutely continuous spectrum of H. The results of this chapter confirm that the occurrence of ballistic behaviour, meaning that

$$\big\langle \mathbf{x}\psi(t), \mathbf{x}\psi(t)\big\rangle \approx C_\psi t^2 \tag{4.1}$$

holds for some $C_\psi > 0$, also depends on whether $\psi \in P_{\mathrm{pp}}(H)L^2(\mathbb{R}^2,\mathbb{C}^2)$ or $\psi \in P_{\mathrm{ac}}(H)L^2(\mathbb{R}^2,\mathbb{C}^2)$. However, while (4.1) can be ruled out in general for initial states that satisfy $\psi \in P_{\mathrm{pp}}(H)L^2(\mathbb{R}^2,\mathbb{C}^2)$ (see Section 4.2), we obtain ballistic dynamics for $\psi \in P_{\mathrm{ac}}(H)L^2(\mathbb{R}^2,\mathbb{C}^2)$ only in the Cesàro mean and only for certain electro-magnetic potentials. The precise statements on the latter point, presented in Section 4.3, were already published in [MS15].

4.1 Heisenberg Picture and Bounds on the Propagation Velocity

We shortly recapitulate some facts concerning the time evolution of the multiplication operator $\mathbf{x}$ on $L^2(\mathbb{R}^2,\mathbb{C}^2)$. To do this we work with the regularised operator

$$\mathbf{x}_\lambda := \frac{\mathbf{x}}{1+\lambda|\mathbf{x}|^2} \qquad \text{on } L^2(\mathbb{R}^2,\mathbb{C}^2),$$

with $\lambda > 0$. Note that for any $\psi \in \mathcal{D}(\mathbf{x})$ we have $\mathbf{x}_\lambda\psi \to \mathbf{x}\psi$ in L^2-sense as $\lambda \to 0$. Since $\mathbf{x}_\lambda$ and

$$\mathrm{i}\,[H,\mathbf{x}_\lambda] \;=\; \frac{\boldsymbol{\sigma}}{1+\lambda|\mathbf{x}|^2} \;-\; \lambda\,\frac{\mathbf{x}}{1+\lambda|\mathbf{x}|^2}\,\frac{(\boldsymbol{\sigma}\cdot\mathbf{x})}{1+\lambda|\mathbf{x}|^2} \tag{4.2}$$

are bounded, we know that for $\psi \in \mathcal{D}(H)$ both components of $\mathbf{x}_\lambda\psi$ are in the domain of H. Hence, for $\psi \in C_0^\infty(\mathbb{R}^2,\mathbb{C}^2)$ we can compute

$$\frac{d}{dt}\,e^{\mathrm{i}\,tH}\mathbf{x}_\lambda e^{-\mathrm{i}\,tH}\psi = e^{\mathrm{i}\,tH}[\mathrm{i}\,H,\mathbf{x}_\lambda]e^{-\mathrm{i}\,tH}\psi,$$

or equivalently

$$e^{\mathrm{i}\,tH}\mathbf{x}_\lambda e^{-\mathrm{i}\,tH}\psi - \mathbf{x}_\lambda\psi = \int_0^t e^{\mathrm{i}\,sH}[\mathrm{i}\,H,\mathbf{x}_\lambda]e^{-\mathrm{i}\,sH}\psi\,\mathrm{d}s$$

for any $t \in \mathbb{R}$. Formula (4.2) implies that

$$\int_0^t e^{\mathrm{i}\,sH}[\mathrm{i}\,H,\mathbf{x}_\lambda]e^{-\mathrm{i}\,sH}\mathrm{d}s \longrightarrow \int_0^t e^{\mathrm{i}\,sH}\boldsymbol{\sigma}e^{-\mathrm{i}\,sH}\mathrm{d}s \qquad \text{as } \lambda \to 0,$$

where $t \in \mathbb{R}$, which leads to

Proposition 4.1. *The operator $e^{-\mathrm{i}\,tH}$ maps $\mathcal{D}(\mathbf{x})$ into $\mathcal{D}(\mathbf{x})$. In addition, for $\psi \in \mathcal{D}(\mathbf{x})$ one has the representation*

$$\mathbf{x}(t)\psi := e^{\mathrm{i}\,tH}\mathbf{x}e^{-\mathrm{i}\,tH}\psi \;=\; \mathbf{x}\psi + \int_0^t e^{\mathrm{i}\,sH}\boldsymbol{\sigma}e^{-\mathrm{i}\,sH}\psi\,\mathrm{d}s.$$

Similarly, one can use a regularisation of the multiplication operator $|\mathbf{x}|^n$ on $L^2(\mathbb{R}^2, \mathbb{C}^2)$, with $n \in \mathbb{N}$, to deduce

Proposition 4.2 (c.f. Theorem 8.5 of [Tha92])**.** *For any $n \in \mathbb{N}$ the operator $e^{-\mathrm{i}tH}$ maps $\mathcal{D}(|\mathbf{x}|^n)$ into $\mathcal{D}(|\mathbf{x}|^n)$. In addition, for any $\psi \in \mathcal{D}(|\mathbf{x}|^n)$ one has*

$$e^{\mathrm{i}tH}|\mathbf{x}|^n e^{-\mathrm{i}tH}\psi \;=\; |\mathbf{x}|^n\psi + n\int_0^t e^{\mathrm{i}sH}|\mathbf{x}|^{n-2}(\boldsymbol{\sigma}\cdot\mathbf{x})\, e^{-\mathrm{i}sH}\psi\,\mathrm{d}s.$$

We can derive immediately an upper bound on the propagation speed by applying the last proposition inductively.

Corollary 4.3. *Let $n \in \mathbb{N}$ and $\psi \in \mathcal{D}(|\mathbf{x}|^{n/2})$. Then one can find a constant $C_n(\psi) > 0$ such that*

$$\big\||\mathbf{x}|^{n/2}e^{-itH}\psi\big\|^2 \le C_n(\psi)(1+|t|)^n$$

for all $t \in \mathbb{R}$.

This corollary states that the wave package spreading under the Dirac time evolution cannot be super-ballistic.

4.2 Pure Point Spectrum and Absence of Ballistic Behaviour

We show in this section that the expectation value

$$\langle \mathbf{x}\psi(t), \mathbf{x}\psi(t)\rangle \tag{4.3}$$

is at most of order $o(t^2)$ if H has only point spectrum (within the energy interval where ψ is supported). Of course, this upper bound is quite weak compared to the ones which can be achieved for more specific situations (see discussion at the end of this section). However, it is valid without any further conditions. As pointed out in [Sim90], one can use the representation

$$\mathbf{x}(t) \;=\; \mathbf{x} + \int_0^t e^{\mathrm{i}sH}\boldsymbol{\sigma}e^{-\mathrm{i}sH}\,\mathrm{d}s \;:=\; \mathbf{x} + \int_0^t \boldsymbol{\sigma}(s)\,\mathrm{d}s$$

on $\mathcal{D}(\mathbf{x})$ to show

Theorem 4.4. *Let $I \subset \mathbb{R}$ be an open interval. Assume that $\sigma(H)\cap I = \sigma_{\mathrm{pp}}(H)\cap I$. Then for any $\psi \in \mathcal{D}(\mathbf{x}) \cap \mathbb{1}_I(H)L^2(\mathbb{R}^2,\mathbb{C}^2)$ we have*

$$\frac{1}{t^2}\big\langle \psi(t), \mathbf{x}^2\psi(t)\big\rangle \longrightarrow 0 \qquad \text{as } t \to \infty.$$

In view of the integral representation of $\mathbf{x}(t)$ we can write

$$\begin{aligned}\frac{1}{t^2}\big\langle \mathbf{x}\psi(t), \mathbf{x}\psi(t)\big\rangle \;&=\; \frac{1}{t^2}\big\langle \mathbf{x}\psi, \mathbf{x}\psi\big\rangle + 2\mathrm{Re}\left(\frac{1}{t^2}\int_0^t \big\langle \mathbf{x}\psi, \boldsymbol{\sigma}(s)\psi\big\rangle \,\mathrm{d}s\right)\\ &\quad + \frac{1}{t^2}\int_0^t\int_0^t \big\langle \boldsymbol{\sigma}(u)\psi, \boldsymbol{\sigma}(s)\psi\big\rangle \,\mathrm{d}u\,\mathrm{d}s, \quad (4.4)\end{aligned}$$

so we have to show that (4.4) vanishes as $t \to \infty$. To do so, we first consider this quadratic form on eigenfunctions of H.

Lemma 4.5. *Let $I \subset \mathbb{R}$ be an open interval such that $\sigma(H)\cap I = \sigma_{\mathrm{pp}}(H)\cap I$. If $\psi, \tilde{\psi}$ are eigenstates of H associated to an eigenvalue $E \in I$, we have*

$$\big\langle \psi, \boldsymbol{\sigma}\tilde{\psi}\big\rangle = 0.$$

Proof. Let $\boldsymbol{\mathcal{X}} \in C^\infty\big(\mathbb{R}^2, \mathbb{R}^2\big)$ be such that

$$\boldsymbol{\mathcal{X}}(\mathbf{x}) = \begin{cases}\mathbf{x} & \text{if } |\mathbf{x}| \le 1,\\ 2\,\frac{\mathbf{x}}{|\mathbf{x}|} & \text{if } |\mathbf{x}| \ge 2\,.\end{cases}$$

For $R > 0$ we define the bounded multiplication operator $\mathbf{x}_R := R\,\boldsymbol{\mathcal{X}}(\mathbf{x}/R)$ and

$$\boldsymbol{\sigma}_R := \boldsymbol{\sigma}\cdot\nabla\mathbf{x}_R = \big(\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\mathcal{X}}\big)\Big(\frac{\cdot}{R}\Big) = \begin{cases}\boldsymbol{\sigma} & \text{if } |\mathbf{x}| \le R\\ 0 & \text{if } |\mathbf{x}| \ge 2R\,.\end{cases}$$

Since for $\hat{\psi} \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ one can estimate

$$\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_R)\hat{\psi}\| \le M\|\chi_{\{|\mathbf{x}|>R\}}\hat{\psi}\| \;\longrightarrow 0 \qquad \text{as } R \to \infty,$$

we conclude that

$$\big\langle \psi, \boldsymbol{\sigma}\tilde{\psi}\big\rangle \;=\; \lim_{R\to\infty}\big\langle \psi, \boldsymbol{\sigma}_R\tilde{\psi}\big\rangle \;=\; \lim_{R\to\infty}\mathrm{i}\,\big\langle \psi, [H, \mathbf{x}_R]\tilde{\psi}\big\rangle = 0.$$

□

Lemma 4.6. *Let $I \subset \mathbb{R}$ be an open interval such that $\sigma(H)\cap I = \sigma_{\mathrm{pp}}(H)\cap I$. Denote by $\{E_m\}_{m\in\mathbb{N}} \subset I$ the set of eigenvalues of H within I. If ψ_k and ψ_l are eigenstates of H associated to eigenvalues $E_k \in I$ and $E_l \in I$, we have*

$$\lim_{t\to\infty}\frac{1}{t^2}\int_0^t\int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, \boldsymbol{\sigma}(s)\psi_l\big\rangle \,\mathrm{d}u\,\mathrm{d}s = 0.$$

Proof. Using that $\mathrm{id} = \mathbb{1}_I(H) + \mathbb{1}_{\mathbb{R}\setminus I}(H)$ on $L^2(\mathbb{R}^2, \mathbb{C}^2)$, we can split

$$\begin{aligned}&\frac{1}{t^2}\int_0^t\int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, \boldsymbol{\sigma}(s)\psi_l\big\rangle \,\mathrm{d}u\,\mathrm{d}s\\ &\qquad = \frac{1}{t^2}\int_0^t\int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, \mathbb{1}_I(H)\boldsymbol{\sigma}(s)\psi_l\big\rangle \,\mathrm{d}u\,\mathrm{d}s\\ &\qquad\quad + \frac{1}{t^2}\int_0^t\int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, \mathbb{1}_{\mathbb{R}\setminus I}(H)\boldsymbol{\sigma}(s)\psi_l\big\rangle \,\mathrm{d}u\,\mathrm{d}s.\end{aligned} \qquad (4.5)$$

Note that

$$\begin{aligned}
&\int_0^t \int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, \mathbb{1}_{\mathbb{R}\setminus I}(H)\boldsymbol{\sigma}(s)\psi_l \big\rangle \, \mathrm{d}u \, \mathrm{d}s \\
&\qquad = \int_0^t \int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, e^{\mathrm{i}\, s(H-E_l)} \mathbb{1}_{\mathbb{R}\setminus I}(H)\boldsymbol{\sigma}\psi_l \big\rangle \, \mathrm{d}u \, \mathrm{d}s \\
&\qquad = \int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, \big(e^{\mathrm{i}\, t(H-E_l)} - \mathrm{id}\big)(H-E_l)^{-1} \, \mathbb{1}_{\mathbb{R}\setminus I}(H)\boldsymbol{\sigma}\psi_l \big\rangle \, \mathrm{d}u,
\end{aligned}$$

hence

$$\frac{1}{t^2} \int_0^t \int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, \mathbb{1}_{\mathbb{R}\setminus I}(H)\boldsymbol{\sigma}(s)\psi_l \big\rangle \, \mathrm{d}u \, \mathrm{d}s \;\longrightarrow\; 0 \quad \text{as } t \to \infty\,.$$

For the other term of (4.5) we use the representation $\mathbb{1}_I(H) = \sum_{m=1}^{\infty} |\psi_m\rangle\langle\psi_m|$, where ψ_m is the eigenstate associated to E_m, to obtain

$$\begin{aligned}
&\int_0^t \int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, \mathbb{1}_I(H)\boldsymbol{\sigma}(s)\psi_l \big\rangle \, \mathrm{d}u \, \mathrm{d}s \\
&\quad = \sum_{m=1}^{\infty} \int_0^t \int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, \psi_m \big\rangle \big\langle \psi_m, \boldsymbol{\sigma}(s)\psi_l \big\rangle \, \mathrm{d}u \, \mathrm{d}s \\
&\quad = \sum_{m=1}^{\infty} \int_0^t \int_0^t e^{\mathrm{i}\, u(E_k-E_m)} e^{\mathrm{i}\, s(E_m-E_l)} \, \mathrm{d}u \, \mathrm{d}s \, \big\langle \boldsymbol{\sigma}\psi_k, \psi_m \big\rangle \big\langle \psi_m, \boldsymbol{\sigma}\psi_l \big\rangle. \quad (4.6)
\end{aligned}$$

Now, if $E_m = E_k$ or $E_m = E_l$, we know by Lemma 4.5 that $\big\langle \boldsymbol{\sigma}\psi_k, \psi_m \big\rangle = 0$ or $\big\langle \psi_m, \boldsymbol{\sigma}\psi_l \big\rangle = 0$. If $E_m \neq E_k$ and $E_m \neq E_l$, we can integrate, getting

$$\begin{aligned}
&\int_0^t \int_0^t e^{\mathrm{i}\, u(E_k-E_m)} e^{\mathrm{i}\, s(E_m-E_l)} \, \mathrm{d}u \, \mathrm{d}s \\
&\qquad\qquad = \frac{1}{(E_k-E_m)} \frac{1}{(E_l-E_m)} \big(e^{\mathrm{i}\, t(E_k-E_m)} - 1\big)\big(e^{\mathrm{i}\, t(E_m-E_l)} - 1\big)
\end{aligned}$$

and therefore

$$\frac{1}{t^2} \int_0^t \int_0^t e^{\mathrm{i}\, s(E_k-E_m)} e^{\mathrm{i}\, u(E_m-E_l)} \, \mathrm{d}u \, \mathrm{d}s \longrightarrow 0 \quad \text{as } t \to \infty.$$

Summarising, we see that for any $m \in \mathbb{N}$ it holds that

$$\frac{1}{t^2} \int_0^t \int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, \psi_m \big\rangle \big\langle \psi_m, \boldsymbol{\sigma}(s)\psi_l \big\rangle \, \mathrm{d}u \, \mathrm{d}s \longrightarrow 0 \quad \text{as } t \to \infty.$$

The inequality

$$\left| \frac{1}{t^2} \int_0^t \int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, \psi_m \big\rangle \big\langle \psi_m, \boldsymbol{\sigma}(s)\psi_l \big\rangle \, \mathrm{d}u \, \mathrm{d}s \right| \leq \big|\big\langle \boldsymbol{\sigma}\psi_k, \psi_m \big\rangle\big| \, \big|\big\langle \psi_m, \boldsymbol{\sigma}\psi_l \big\rangle\big|$$

allows us to apply the dominated convergence theorem in (4.6) since the right hand side of it is summable in m. We deduce that

$$\frac{1}{t^2}\int_0^t\int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, \mathbb{1}_I(H)\boldsymbol{\sigma}(s)\psi_l\big\rangle \,\mathrm{d}u\,\mathrm{d}s \longrightarrow 0 \quad \text{as } t\to\infty.$$

□

Proof of Theorem 4.4. As in Lemma 4.6, let us denote by $\{E_m\}_{m\in\mathbb{N}}$ the set of eigenvalues of H within I (counting multiplicity). The corresponding normalised eigenfunctions $\{\psi_m\}_{m\in\mathbb{N}}$ form an orthonormal basis of the subspace $\mathbb{1}_I(H)L^2(\mathbb{R}^2,\mathbb{C}^2)$. We write $\psi = \sum_{l=1}^\infty c_l\psi_l \in \mathbb{1}_I(H)L^2(\mathbb{R}^2,\mathbb{C}^2)$ and set $\psi^N = \sum_{l=1}^N c_l\psi_l$ for $N\in\mathbb{N}$. From Lemma 4.6 follows that

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t^2}\int_0^t\int_0^t \big\langle \boldsymbol{\sigma}(u)\psi^N, \boldsymbol{\sigma}(s)\psi^N\big\rangle \,\mathrm{d}u\,\mathrm{d}s\\
&\qquad\qquad = \sum_{k=1}^N\sum_{l=1}^N \bar{c}_k c_l \lim_{t\to\infty}\frac{1}{t^2}\int_0^t\int_0^t \big\langle \boldsymbol{\sigma}(u)\psi_k, \boldsymbol{\sigma}(s)\psi_l\big\rangle \,\mathrm{d}u\,\mathrm{d}s = 0
\end{aligned}$$

for any $N\in\mathbb{N}$. We choose $\epsilon>0$. Then there exist $N\in\mathbb{N}$ and $T>0$ such that $\|\psi-\psi^N\|\le\epsilon/3$ and

$$\left|\frac{1}{t^2}\int_0^t\int_0^t \big\langle \boldsymbol{\sigma}(u)\psi^N, \boldsymbol{\sigma}(s)\psi^N\big\rangle \,\mathrm{d}u\,\mathrm{d}s\right| \le \epsilon/3$$

for all $t>T$. Hence,

$$\begin{aligned}
&\left|\frac{1}{t^2}\int_0^t\int_0^t \big\langle \boldsymbol{\sigma}(u)\psi, \sigma(s)\psi\big\rangle \,\mathrm{d}u\,\mathrm{d}s\right|\\
&\qquad \le \left|\frac{1}{t^2}\int_0^t\int_0^t \big\langle \boldsymbol{\sigma}(u)\psi^N, \boldsymbol{\sigma}(s)\psi^N\big\rangle \,\mathrm{d}u\,\mathrm{d}s\right|\\
&\qquad\quad + \frac{1}{t^2}\int_0^t\int_0^t \big|\big\langle \boldsymbol{\sigma}(u)\big(\psi-\psi^N\big), \boldsymbol{\sigma}(s)\psi\big\rangle\big| \,\mathrm{d}u\,\mathrm{d}s\\
&\qquad\quad + \frac{1}{t^2}\int_0^t\int_0^t \big|\big\langle \boldsymbol{\sigma}(u)\psi^N, \boldsymbol{\sigma}(s)\big(\psi-\psi^N\big)\big\rangle\big| \,\mathrm{d}u\,\mathrm{d}s\\
&\qquad \le \frac{\epsilon}{3} + \frac{1}{t^2}\int_0^t\int_0^t \big\|\boldsymbol{\sigma}(u)\big(\psi-\psi^N\big)\big\|\big\|\boldsymbol{\sigma}(s)\psi\big\|\mathrm{d}u\,\mathrm{d}s\\
&\qquad\quad + \frac{1}{t^2}\int_0^t\int_0^t \big\|\boldsymbol{\sigma}(u)\psi^N\big\|\big\|\boldsymbol{\sigma}(s)\big(\psi-\psi^N\big)\big\|\mathrm{d}u\,\mathrm{d}s\\
&\qquad \le \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon
\end{aligned}$$

for $t>T$. Thus, we see that

$$\frac{1}{t^2}\int_0^t\int_0^t \big\langle \boldsymbol{\sigma}(u)\psi, \boldsymbol{\sigma}(s)\psi\big\rangle \,\mathrm{d}u\,\mathrm{d}s \longrightarrow 0 \qquad \text{as } t\to\infty.$$

□

Let us briefly point out that if H has pure point spectrum, there exists also the possibility to use (uniform) decay properties of the eigenfunctions to obtain upper bounds on (4.3). If the decay is strong enough, these bounds can even be time-independent. Within our framework, let us consider the case of a rotationally symmetric field configuration. Then a decomposition of H into operators h_k on the half-line enables us to gain information on the eigenfunction-decay via the eigenfunctions of the h_k's. This suffices to deduce (almost) dynamical localisation, i.e. time-independent bounds on (4.3), for many cases in which $\sigma(H)$ is of pure point type. For the precise statement recall that if $B \in C(\mathbb{R}^2, \mathbb{R})$ and $V \in C^1(\mathbb{R}^2, \mathbb{R})$ are rotationally symmetric, we can write $B(\,\cdot\,) = b(|\cdot|)$, $V(\,\cdot\,) = v(|\cdot|)$ and use the gauge (2.49) to obtain the decomposition

$$Q^* U H U^* Q = \bigoplus_{k \in \mathbb{Z}+\frac{1}{2}} h_k,$$

with the operators h_k as defined in (2.6). Recall that $Q = 1/\sqrt{2}(1 - \mathrm{i}\,\sigma_3)$ and $U : L^2(\mathbb{R}^2, \mathbb{C}^2) \longrightarrow \bigoplus_{j \in \mathbb{Z}+\frac{1}{2}} L^2(\mathbb{R}^+, \mathbb{C}^2)$ are unitary maps. Due to Corollary 2.31, we know that if

$$|A(r)| \longrightarrow \infty \quad \text{as} \;\; r \to \infty, \tag{4.7}$$

$$\limsup_{r \to \infty} \left| \frac{v(r)}{A(r)} \right| < 1, \quad \left(\frac{v}{A} \right)' \text{ is bounded,} \tag{4.8}$$

$$\frac{A'(r)}{A^2(r)} \longrightarrow 0 \quad \text{as} \;\; r \to \infty, \tag{4.9}$$

then H has pure point spectrum. For such a system we even have

Theorem 4.7 (Theorem 1.1 of [BMST15])**.** *Consider the Dirac operator H with rotationally symmetric $B(\,\cdot\,) = b(|\cdot|)$ and $V(\,\cdot\,) = v(|\cdot|)$. Suppose that the potentials $A(r) = r^{-1} \int_0^r B(s) s \mathrm{d}s$ and v fulfill (4.7)–(4.9). If for given constants $E, m > 0$ the state $\psi \in \mathbb{1}_{[-E,E]}(H) L^2(\mathbb{R}^2, \mathbb{C}^2)$, with decomposition $Q^* U \psi = \oplus_{k \in \mathbb{Z}+\frac{1}{2}} \varphi_k$, satisfies*

$$\sum_{k \in \mathbb{Z}+\frac{1}{2}} |k|^m \|\varphi_k\|^2 < \infty, \tag{4.10}$$

then

$$\sup_{t \geq 0} \left\| |\mathbf{x}|^{\frac{m}{2}} e^{-\mathrm{i}\,tH} \psi \right\|^2 < \infty. \tag{4.11}$$

The proof of this theorem can be found in the given reference. Keeping in mind that U is essentially a Fourier transformation in the angular variable, condition (4.10) is a mild constraint on the regularity in the angular variable of ψ (considered in polar coordinates). It is necessary since the decay of the eigenfunctions of the operators h_k, which is exploited in the proof, is not uniform in k. Condition (4.10) is therefore required to compensate this missing uniformity in the Fourier-transformed angular variable.

4.3 Ballistic Dynamics for certain Systems with Absolutely Continuous Spectrum

In this section we show that it is possible to obtain ballistic dynamics for massless Dirac operators with an absolutely continuous part in the spectrum in the sense that

$$\big\langle\big\langle \mathbf{x}\psi(\cdot), \mathbf{x}\psi(\cdot)\big\rangle\big\rangle_T \approx C_\psi T^2. \tag{4.12}$$

Here we use the notation

$$\langle \Theta \rangle_T := \frac{1}{T}\int_0^T \Theta(t)\,\mathrm{d}t \tag{4.13}$$

for the Cesàro-mean of a function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$. Against intuition, deducing (4.12) for states with support in the absolutely continuous part of the spectrum is not trivial and we can only prove it for systems with rotational or translational symmetry. To do so, we use ideas originating from [Gua89] and [Com93]. A mathematically more refined elaboration (especially from the measure theoretic point of view) can be found in [Las96]. Let us briefly recall the connection between the time evolution of a state and the continuity of the spectral measure associated to it.

Definition 4.8. *Let $\mu : \mathfrak{B}(\mathbb{R}) \to [0,\infty]$ be a Borel measure on $\mathbb{R}$. We say that μ is uniformly Lipschitz continuous (with respect to the Lebesgue measure) if there is a constant $C > 0$ such that for any interval $I \subset \mathbb{R}$ with $|I| \le 1$ it holds that*

$$\mu(I) \le C|I|.$$

Example 4.9. *Let μ be an absolutely continuous Borel measure on $\mathbb{R}$ with bounded Radon-Nikodym derivative. Then μ is uniformly Lipschitz continuous.*

As pointed out by [Str90] and authors cited therein, the continuity of a measure μ implies a certain decay of its Fourier transform $\widehat{\mu}$. For our results we only need the following special case:

Proposition 4.10. *Let μ be a uniformly Lipschitz continuous Borel measure. Then there exists a constant $C > 0$ such that for any function $\varrho \in L^2(\mathbb{R}, \mathbb{C}; \mathrm{d}\mu)$ it holds that*

$$\big\langle |\widehat{\varrho\mu}|^2(\cdot)\big\rangle_T := \frac{1}{T}\int_0^T \left| \int_{\mathbb{R}} e^{-\mathrm{i}\,ts}\varrho(s)\,\mathrm{d}\mu(s)\right|^2 \mathrm{d}t \le CT^{-1}\int_{\mathbb{R}} |\varrho(s)|^2 \mathrm{d}\mu(s).$$

Now assume that G is a self-adjoint operator on a Hilbert space $\mathcal{H}$ and that the Borel measure

$$\mu_\varphi : \mathfrak{B}(\mathbb{R}) \to [0,\infty], \qquad I \mapsto \big\langle \varphi, \mathbb{1}_I(G)\varphi\big\rangle$$

associated to $\varphi \in \mathcal{H}$ is Lipschitz continuous. Then, due to the spectral theorem, the overlap of $\varphi(t)$ with another state $\tilde{\varphi} \in \mathcal{H}$ can be written as

$$\langle \tilde{\varphi}, \varphi(t) \rangle = \langle \tilde{\varphi}, e^{-\mathrm{i}tG}\varphi \rangle = \int_{\mathbb{R}} e^{-\mathrm{i}ts} \varrho_{\tilde{\varphi}}(s) \mathrm{d}\mu_\varphi(s),$$

where $\varrho_{\tilde{\varphi}} \in L^2(\mathbb{R}, \mathbb{C}; \mathrm{d}\mu)$ satisfies $\int_{\mathbb{R}} |\varrho_{\tilde{\varphi}}|^2 \mathrm{d}\mu_\varphi(s) \leq \|\tilde{\varphi}\|^2$. Therefore, Proposition 4.10 states that

$$\big\langle \big|\langle \tilde{\varphi}, \varphi(t) \rangle\big|^2 \big\rangle_T \leq C_\varphi T^{-1} \|\tilde{\varphi}\|^2, \tag{4.14}$$

i.e. the considered overlap vanishes in the Cesàro-mean with a rate proportional to T^{-1}. We now demonstrate that (4.14) provides a tool to estimate how fast the bulk of a state φ leaves a given compact region under time evolution (assuming that μ_φ is sufficiently regular).

Proposition 4.11. *Let G be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and suppose that $K : \mathcal{H} \to \mathcal{H}$ is a Hilbert-Schmidt operator. If for $\varphi \in \mathcal{H}$ the Borel measure $\mu_\varphi(\,\cdot\,) = \langle \varphi, \mathbb{1}_{\cdot}(G)\varphi \rangle$ is uniformly Lipschitz continuous, then there exists a constant C_φ such that*

$$\big\langle \langle K e^{-\mathrm{i}tG}\varphi, K e^{-\mathrm{i}tG}\varphi \rangle \big\rangle_T \leq C_\varphi \|K\|_{HS}^2 T^{-1}.$$

Proof. Using the singular value decomposition of the compact operator K on $\mathcal{H}$, we can write

$$K^*K = \sum_{n=1}^{\infty} \lambda_n \bar{\lambda}_n \langle \phi_n, \,\cdot\, \rangle \phi_n,$$

with an orthonormal set $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$. Here $\{\lambda_n\}_{n\in\mathbb{N}}$ is a square-summable sequence that satisfies

$$\|K\|_{HS}^2 = \sum_{n=1}^{\infty} |\lambda_n|^2.$$

Using (4.14), we estimate

$$\big\langle \langle K e^{-\mathrm{i}tG}\varphi, K e^{-\mathrm{i}tG}\varphi \rangle \big\rangle_T = \sum_{n=1}^{\infty} |\lambda_n|^2 \big\langle \big|\langle \phi_n, e^{-\mathrm{i}tG}\varphi \rangle\big|^2 \big\rangle_T \leq C_\varphi \|K\|_{HS}^2 T^{-1}$$

for $T > 0$. □

Applied to local compact operators later on, this rather general proposition leads to a bound of order T^{-1} on the quantity

$$\frac{1}{T} \int_0^T \big\| \mathbb{1}_{\{|\mathbf{x}| \leq R\}} \varphi(t) \big\|^2 \mathrm{d}t \tag{4.15}$$

if we assume that μ_φ is uniformly Lipschitz continuous. Recall that the RAGE theorem merely states that (4.15) is vanishing as $T \to \infty$ if μ_φ is continuous. To deduce finally the ballistic wave package spreading for our two-dimensional Dirac operator H, it would require Hilbert-Schmidt bounds of type

$$\left\| \mathbb{1}_{\{|\mathbf{x}|\le R\}} \frac{1}{H-\mathrm{i}} \right\|_{HS}^2 \le cR$$

for some $c > 0$. Even in the free case ($\mathbf{A} = 0$ and $V = 0$) this is not possible.

However, if one allows the electro-magnetic field to be rotationally symmetric or translationally symmetric (see Subsection 2.2.3) one can think of a decomposition

$$e^{\mathrm{i}tH}|\mathbf{x}|^2 e^{-\mathrm{i}tH} \cong \bigoplus_{k\in\mathbb{Z}+\frac{1}{2}} e^{\mathrm{i}th_k} r^2 e^{-\mathrm{i}th_k}$$

or

$$e^{\mathrm{i}tH}|x_1|^2 e^{-\mathrm{i}tH} \cong \int_{\mathbb{R}}^{\oplus} e^{\mathrm{i}th(\xi)}|x_1|^2 e^{-\mathrm{i}th(\xi)}\,\mathrm{d}\xi$$

respectively, to reduce the problem to a question on the dynamics of Dirac operators on the (half-)line. We therefore aim to apply Proposition 4.11 to the operators h_k on $L^2(\mathbb{R}^+, \mathbb{C}^2)$ and $h(\xi)$ on $L^2(\mathbb{R}, \mathbb{C}^2)$, given by (2.6) and by (2.53) respectively. The following two lemmata state that the potential transformations, introduced in Subsection 2.1.2, enable us to establish the desired Hilbert-Schmidt bounds for these one-dimensional operators in the case when $|A| < |v|$ at ∞ (c.f. Assumptions 2.14 and 2.15).

Lemma 4.12 (Theorem 3 of [MS15])**.** *Let h_k, for $k \in \mathbb{Z} + \frac{1}{2}$, be defined as in* (2.6)*, with A, v satisfying Assumption 2.15. Then there is a constant $C_k > 0$ such that for any bounded intervall $I \subset [1, \infty)$ we have*

$$\left\| \mathbb{1}_I \frac{1}{h_k - \mathrm{i}} \right\|_{\mathrm{HS}} \le C_k \sqrt{|I|}\,.$$

Proof. In view of Theorem 2.16, we can write

$$\mathbb{1}_I \frac{1}{h_k - \mathrm{i}} = e^{\sigma_2\theta/2}\, \mathbb{1}_I \frac{1}{\tilde{h}_k - \mathrm{i}} \Big[(\tilde{h}_k - \mathrm{i}\,)(M\tilde{h}_k - \mathrm{i}\,)^{-1}\Big] e^{-\sigma_2\theta/2},$$

with $\theta = \tanh^{-1}\beta = \tanh^{-1}(A_2/v)$ and

$$\tilde{h}_k = \sigma_1(-\mathrm{i}\,\partial_x) + \gamma(1+\sigma_2\beta)\sigma_2\left(\frac{k}{x} - A_1\right) + v_1/\gamma + v_2/\gamma + \sigma_3\frac{\theta'}{2}.$$

Note that Theorem 2.16 also implies that $(\tilde{h}_k - \mathrm{i}\,)(M\tilde{h}_k - \mathrm{i}\,)^{-1}$ is bounded, hence it remains to find the right Hilbert-Schmidt bound for

$$\mathbb{1}_I \frac{1}{\tilde{h}_k - \mathrm{i}}.$$

Let us simplify the notation by writing

$$\tilde{h}_k = \sigma_1(-\mathrm{i}\,\partial_x) + \sigma_2 \frac{k}{x} + W + v_2/\gamma,$$

where $W = W_1 + W_2$ denotes the sum of the potentials

$$W_1 := v_1/\gamma - \gamma(1+\sigma_2\beta)\sigma_2 A_1 \quad \in L^p(\mathbb{R}^+, \mathbb{C}^{2\times 2}),$$

with $p > 2$, and

$$W_2 := \gamma(1+\sigma_2\beta)\sigma_2 \frac{k}{x} - \sigma_2\frac{k}{x} + \sigma_3\frac{\theta'}{2} = \big((\gamma-1)+\gamma\beta\sigma_2\big)\sigma_2\frac{k}{x} + \sigma_3\frac{\theta'}{2}.$$

Due to the support properties of A_2, the function β is supported away from zero, meaning that W_2 is uniformly bounded on $\mathbb{R}^+$. In particular, there exists a constant $c_1 > 0$ for which holds that

$$\left\| \mathbb{1}_I \frac{1}{\tilde{h}_k - \mathrm{i}} \right\|_{\mathrm{HS}} \leq c_1 \left\| \mathbb{1}_I \frac{1}{\tilde{h}_k - W_2 - \mathrm{i}} \right\|_{\mathrm{HS}}. \tag{4.16}$$

Since W_1 has bounded support, this term is an infinitesimally small perturbation with respect to $\tilde{h}_k - W = \sigma_1(-\mathrm{i}\,\partial_x) + \sigma_2\frac{k}{x} + v_2/\gamma$ (see Corollary 2.7), i.e. we obtain the estimate

$$\left\| \mathbb{1}_I \frac{1}{\tilde{h}_k - W_2 - \mathrm{i}} \right\|_{\mathrm{HS}} \leq c_2 \left\| \mathbb{1}_I \frac{1}{\tilde{h}_k - W - \mathrm{i}} \right\|_{\mathrm{HS}}, \tag{4.17}$$

with $c_2 > 0$. Finally, we compare $\tilde{h}_k - W$ with the half-line operator

$$h_0 := \sigma_1(-\mathrm{i}\,\partial_x) + v_2/\gamma \qquad \text{on } L^2(\mathbb{R}^+, \mathbb{C}^2),$$

which is self-adjoint on $\mathcal{D}(h_0)$ (given by (2.10) with $\alpha = 0$), as follows: We choose $\chi \in C^\infty(\mathbb{R}^+, [0,1])$, with

$$\chi(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2}, \\ 1 & \text{if } x \geq 1, \end{cases}$$

and write

$$\begin{aligned} \mathbb{1}_I \frac{1}{\tilde{h}_k - W - \mathrm{i}} &= \mathbb{1}_I \chi \frac{1}{\tilde{h}_k - W - \mathrm{i}} \\ &= \mathbb{1}_I \frac{1}{h_0 - \mathrm{i}} \Big[(h_0 - \mathrm{i}) \chi \big(\tilde{h}_k - W - \mathrm{i}\big)^{-1} \Big]. \end{aligned}$$

Here the operator in [...] is well defined and bounded in view of Remark 2.5 and the Closed Graph Theorem (however, the operator norm depends on $|k|$). So there is a $c_{|k|} > 0$ such that

$$\left\| \mathbb{1}_I \frac{1}{\tilde{h}_k - W - \mathrm{i}} \right\|_{\mathrm{HS}} \leq c_{|k|} \left\| \mathbb{1}_I \frac{1}{h_0 - \mathrm{i}} \right\|_{\mathrm{HS}} \leq c_{|k|} \sqrt{|I|}. \tag{4.18}$$

Here we used Proposition 2.13 for the second inequality. Putting together (4.16)–(4.18), the claim follows. □

Lemma 4.13 (Theorem 1 of [MS15]). *Let $h(\xi)$, for $\xi \in \mathbb{R}$, be defined as in* (2.53)*, with A, v satisfying Assumption 2.14. Then there is a constant $C_\xi > 0$ such that for any bounded interval $I \subset \mathbb{R}$ we have*

$$\left\| \mathbb{1}_I \, \frac{1}{h(\xi) - \mathrm{i}} \right\|_{\mathrm{HS}} \leq C_\xi \sqrt{|I|}.$$

Proof. Since $h(\xi) = h(0) + \sigma_2 \xi$, it suffices to show the statement for the operator $h = h(0)$. In view of Theorem 2.16, we write

$$\begin{aligned} \mathbb{1}_I \, \frac{1}{h - \mathrm{i}} &= \mathbb{1}_I \, e^{\sigma_2 \theta/2} \, \frac{1}{M\tilde{h} - \mathrm{i}} \, e^{-\sigma_2 \theta/2} \\ &= e^{\sigma_2 \theta/2} \, \mathbb{1}_I \, \frac{1}{\tilde{h} - \mathrm{i}} \big[(\tilde{h} - \mathrm{i})(M\tilde{h} - \mathrm{i})^{-1} \big] e^{-\sigma_2 \theta/2}, \end{aligned}$$

with

$$\tilde{h} = -\mathrm{i}\,\sigma_1 \partial_x - \gamma(1 + \sigma_2 \beta)\sigma_2 A_1 + v_1/\gamma + v_2/\gamma + \sigma_3 \frac{\theta'}{2},$$

where $\theta = \tanh^{-1} \beta = \tanh^{-1}(A_2/v)$ has a uniformly bounded derivative. The operator $(\tilde{h} - \mathrm{i})(M\tilde{h} - \mathrm{i})^{-1}$ is bounded by Theorem 2.16 and the Closed Graph Theorem, hence we have

$$\left\| \mathbb{1}_I \, \frac{1}{h - \mathrm{i}} \right\|_{\mathrm{HS}} \leq c \left\| \mathbb{1}_I \, \frac{1}{\tilde{h} - \mathrm{i}} \right\|_{\mathrm{HS}}$$

for some $c > 0$. Setting $W_1 = -\gamma(1 + \sigma_2 \beta)\sigma_2 A_1 + v_1/\gamma$, we use the second resolvent identity to obtain

$$\begin{aligned} \big(\tilde{h} - \mathrm{i}\big)^{-1} &= \big(-\mathrm{i}\,\sigma_1 \partial_x + W_1 + v_2/\gamma - \mathrm{i} \big)^{-1} \left[1 - \sigma_3 \frac{\theta'}{2} \big(\tilde{h} - \mathrm{i}\big)^{-1} \right] \\ &= U^* \big(-\mathrm{i}\,\sigma_1 \partial_x + W - \mathrm{i} \big)^{-1} U \left[1 - \sigma_3 \frac{\theta'}{2} \big(\tilde{h} - \mathrm{i}\big)^{-1} \right], \end{aligned}$$

where $W := U W_1 U^*$ and U is the unitary transformation from Proposition 2.12. Note that $|W| \in L^p$ for some $p \geq 2$. Hence, the Kato-Seiler-Simon inequality yields that W is infinitesimally bounded with respect to $-\mathrm{i}\,\sigma_1 \partial_x$. In particular,

$$W \big(-\mathrm{i}\,\sigma_1 \partial_x + W - \mathrm{i} \big)^{-1}$$

is a bounded operator. Therefore, we use the resolvent identity

$$\big(-\mathrm{i}\,\sigma_1 \partial_x + W - \mathrm{i} \big)^{-1} = \big(-\mathrm{i}\,\sigma_1 \partial_x - \mathrm{i} \big)^{-1} \big[1 - W \, (-\mathrm{i}\,\sigma_1 \partial_x + W - \mathrm{i})^{-1} \big]$$

to deduce that

$$\left\| \mathbb{1}_I \, \frac{1}{\tilde{h} - \mathrm{i}} \right\|_{\mathrm{HS}} \leq C \left\| \mathbb{1}_I \, \frac{1}{-\mathrm{i}\,\sigma_1 \partial_x - \mathrm{i}} \right\|_{\mathrm{HS}}$$

for some $C > 0$. The claim is then a direct consequence of Proposition 2.13. □

With these two lemmata we can prove the main theorems of this section:

Theorem 4.14 (Theorem 4 of [MS15]). *Consider the operator h_k, where $k \in \mathbb{Z} + \frac{1}{2}$, with potentials A, v satisfying Assumption 2.15. For $E > 0$ let $\varphi \in P_{\mathrm{ac}}(h_k)\mathbb{1}_{[-E,E]}(h_k)L^2(\mathbb{R}^+, \mathbb{C}^2)$ be non-zero. Then for each $m > 0$ there exists a constant $C_k(\varphi, E, m) > 0$ such that*

$$\left\langle \left\| x^{\frac{m}{2}} e^{-\mathrm{i}\, t h_k} \varphi \right\|^2 \right\rangle_T \geq C_k(\varphi, E, m)\, T^m \tag{4.19}$$

for $T > 0$.

Theorem 4.15 (Theorem 2 of [MS15]). *Consider the operator $h(\xi)$, where $\xi \in \mathbb{R}$, with potentials A, v satisfying Assumption 2.14. For $E > 0$ let $\varphi \in P_{\mathrm{ac}}(h(\xi))\mathbb{1}_{[-E,E]}(h(\xi))L^2(\mathbb{R}, \mathbb{C}^2)$ be non-zero. Then for each $m > 0$ there exists a constant $C_\xi(\varphi, E, m) > 0$ such that*

$$\left\langle \left\| |x|^{\frac{m}{2}} e^{-\mathrm{i}\, t h(\xi)} \varphi \right\|^2 \right\rangle_T \geq C_\xi(\varphi, E, m) T^m \tag{4.20}$$

for $T > 0$.

We only give the proof of Theorem 4.14. The one of Theorem 4.15 is completely analogous, but uses Lemma 4.13 instead of Lemma 4.12.

Proof of Theorem 4.14. As above, let us denote by

$$\mu_\varphi : \mathfrak{B}(\mathbb{R}) \to [0, \infty], \qquad O \mapsto \langle \varphi, \mathbb{1}_O(h_k)\varphi \rangle$$

the Borel measure associated to $\varphi \in L^2(\mathbb{R}^+, \mathbb{C}^2)$ (with respect to h_k). Due to the absolute continuity (with respect to the Lebesgue measure), the measure μ_φ has an L^1- Radon-Nikodym derivative, i.e. we can write

$$\mu_\varphi(O) = \int_O \mu'_\varphi(s) \mathrm{d}s$$

with $\mu'_\varphi \in L^1(\mathbb{R}, \mathbb{R}^+)$. For $\alpha > 0$, so large that

$$\mu_\varphi(\{\mu'_\varphi > \alpha\}) \leq \frac{1}{4} \|\varphi\|^2,$$

we define $O_a = \{\mu'_\varphi \leq \alpha\}$ and $O_b = \{\mu'_\varphi > \alpha\}$. Then the measures

$$\begin{aligned} \mu_{\varphi,a}(O) &:= \mu_\varphi(O \cap O_a) = \int_O \mu'_\varphi(s)\, \mathbb{1}_{O_a}(s) \mathrm{d}s, \qquad O \in \mathfrak{B}(\mathbb{R}) \\ \mu_{\varphi,b}(O) &:= \mu_\varphi(O \cap O_b) = \int_O \mu'_\varphi(s)\, \mathbb{1}_{O_b}(s) \mathrm{d}s, \qquad O \in \mathfrak{B}(\mathbb{R}) \end{aligned}$$

are singular to each other and satisfy $\mu_\varphi = \mu_{\varphi,a} + \mu_{\varphi,b}$. In addition, $\mu_{\varphi,a}$ is uniformly Lipschitz continuous (with respect to the Lebesgue measure) since it has the Radon-Nikodym derivative $\mu'_\varphi\, \mathbb{1}_{O_a}$. Observe that the states

$\varphi_j := \mathbb{1}_{O_j}(h_k)\varphi$, with $j \in \{a,b\}$, generate the measures $\mu_{\varphi,j}$, i.e. for any $O \in \mathfrak{B}(\mathbb{R})$ we have

$$\begin{aligned}\mu_{\varphi_j}(O) &= \langle \varphi_j, \mathbb{1}_O(h_k)\varphi_j\rangle = \langle \varphi, \mathbb{1}_{O\cap O_j}(h_k)\varphi\rangle \\ &= \mu_\varphi(O\cap O_j) = \mu_{\varphi,a}(O\cap O_j) + \mu_{\varphi,b}(O\cap O_j) = \mu_{\varphi,j}(O).\end{aligned}$$

Summarising, we can decompose φ in a sum of mutually orthogonal states φ_a and φ_b such that $\mu_\varphi = \mu_{\varphi_a} \oplus \mu_{\varphi_b}$, where μ_{φ_a} is uniformly Lipschitz continuous. In particular, we can apply Proposition 4.11 to φ_a, which satisfies

$$\|\varphi_a\|^2 = \|\varphi\|^2 - \|\varphi_b\|^2 = \|\varphi\|^2 - \mu_{\varphi,b}(\mathbb{R}) \geq \frac{3}{4}\|\varphi\|^2 > 0.$$

To obtain (4.19), we estimate

$$\begin{aligned}\big\|x^{\frac{m}{2}} e^{-\mathrm{i}\, h_k t}\varphi\big\|^2 &\geq \big\|R^{\frac{m}{2}}\mathbb{1}_{(R,\infty)} e^{-\mathrm{i}\, h_k t}\varphi\big\|^2 \\ &\geq R^m\big(\|\varphi\|^2 - \big\|\mathbb{1}_{(0,R)} e^{-\mathrm{i}\, h_k t}\varphi\big\|^2\big) \\ &\geq R^m\big(\|\varphi\|^2 - 2\big\|\mathbb{1}_{(0,R)} e^{-\mathrm{i}\, h_k t}\varphi_a\big\|^2 - 2\|\varphi_b\|^2\big) \\ &\geq R^m\big(\tfrac{1}{2}\|\varphi\|^2 - 2\big\|\mathbb{1}_{(0,R)} e^{-\mathrm{i}\, h_k t}\varphi_a\big\|^2\big) \end{aligned} \tag{4.21}$$

for $R > 1$. Using that $\varphi_a \in \mathbb{1}_{[-E,E]} L^2(\mathbb{R}^+, \mathbb{C}^2)$ and the local compactness of h_k, we deduce from the RAGE theorem that

$$\big\langle \big\|\mathbb{1}_{(0,1]} e^{-\mathrm{i}\, h_k t}\varphi_a\big\|^2\big\rangle_T \leq \tfrac{1}{8}\|\varphi\|^2$$

for $T \geq T_0$ (where T_0 has to be chosen large enough). So we can proceed with estimate (4.21) to obtain

$$\big\langle \big\|x^{\frac{m}{2}} e^{-\mathrm{i}\, h_k t}\varphi\big\|^2\big\rangle_T \geq R^m\big(\tfrac{1}{4}\|\varphi\|^2 - 2\big\langle\big\|\mathbb{1}_{(1,R)} e^{-\mathrm{i}\, h_k t}\varphi_a\big\|^2\big\rangle_T\big) \tag{4.22}$$

for $T \geq T_0$. For the second term on the right hand side of (4.22) we apply Proposition 4.10 and Lemma 4.12, resulting in

$$\begin{aligned}\big\langle \big\|\mathbb{1}_{(1,R)} e^{-\mathrm{i}\, h_k t}\varphi_a\big\|^2\big\rangle_T &= \big\langle \big\|\mathbb{1}_{(1,R)}\mathbb{1}_{[-E,E]}(h_k) e^{-\mathrm{i}\, h_k t}\varphi_a\big\|^2\big\rangle_T \\ &\leq C_{\varphi_a} T^{-1}\left\|\mathbb{1}_{(1,R)}\frac{1}{(h_k - \mathrm{i}\,)}\right\|_{\mathrm{HS}}^2 \big\|(h_k - \mathrm{i}\,)\mathbb{1}_{[-E,E]}(h_k)\big\|^2 \\ &\leq C_{\varphi_a} C_k (E^2+1) R T^{-1}.\end{aligned}$$

Combining this with (4.22), we get

$$\begin{aligned}\big\langle \big\|x^{\frac{m}{2}} e^{-\mathrm{i}\, h_k t}\varphi\big\|^2\big\rangle_T &\geq R^m\big(\tfrac{1}{4}\|\varphi\|^2 - C_{\varphi_a} C_k (E^2+1) R T^{-1}\big) \\ &= \frac{1}{8}(8C_{\varphi_a} C_k (E^2+1))^{-m}\|\varphi\|^{2m+2}\, T^m,\end{aligned} \tag{4.23}$$

for $T > T_0$, if we choose

$$R \equiv R(T) = \frac{\|\varphi\|^2}{8C_{\varphi_a} C_k (E^2+1)}\, T.$$

After lowering the constant in front of T^m, one can even assume that (4.23) holds also for $T \in [0, T_0]$. □

We come back to our original problem: The occurrence of ballistic dynamics for Dirac particles in dimension two, under the assumption that the magnetic vector potential is small compared to the scalar potential.

Corollary 4.16. *Consider the massless Dirac operator H on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ with rotationally symmetric $B(\,\cdot\,) = b(|\cdot|)$ and $V(\,\cdot\,) = v(|\cdot|)$. Assume that $A(r) = r^{-1}\int_0^r b(s)s\mathrm{d}s$ and v satisfy Assumption 2.15. For $E > 0$ let*

$$\psi \in P_{\mathrm{ac}}(H)\mathbb{1}_{[-E,E]}(H)L^2(\mathbb{R}^2, \mathbb{C}^2)$$

be non-trivial. Then for each $m > 0$ there exists a constant $C(\psi, E, m) > 0$ such that

$$\big\langle \big\| |\mathbf{x}|^{\frac{m}{2}} e^{-\mathrm{i}tH}\psi \big\|^2 \big\rangle_T \geq C(\psi, E, m)\, T^m$$

holds for all $T > 0$.

Proof. In view of the discussion in Subsection 2.2.3, we have

$$U\psi = \oplus_{k\in\mathbb{Z}+\frac{1}{2}}\varphi_k \in \bigoplus_{k\in\mathbb{Z}+\frac{1}{2}} P_{\mathrm{ac}}(h_k)L^2(\mathbb{R}^+, \mathbb{C}^2). \tag{4.24}$$

Since $U|\mathbf{x}|^{\frac{m}{2}}U^*$ is diagonal, with $\big[U|\mathbf{x}|^{\frac{m}{2}}U^*\big]_{k,k} = r^{\frac{m}{2}}$, we can estimate

$$\big\| |\mathbf{x}|^{\frac{m}{2}} e^{-\mathrm{i}tH}\psi \big\|^2 = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \big\| r^{\frac{m}{2}} e^{-\mathrm{i}th_k}\varphi_k \big\|^2 \geq \sum_{k=-l-\frac{1}{2}}^{l+\frac{1}{2}} \big\| r^{\frac{m}{2}} e^{-\mathrm{i}th_k}\varphi_k \big\|^2,$$

where $l \in \mathbb{N}$ is chosen so large that $\sum_{k=-l-1/2}^{l+1/2} \|\varphi_k\|^2 \geq \frac{1}{2}\|\psi\|$. Observe that (2.50) implies $\mathbb{1}_{[-E,E]}(H) = \bigoplus_{k\in\mathbb{Z}+\frac{1}{2}} \mathbb{1}_{[-E,E]}(h_k)$. Hence, due to (4.24), we can apply Theorem 4.14 to deduce

$$\big\| |\mathbf{x}|^{\frac{m}{2}} e^{-\mathrm{i}tH}\psi \big\|^2 \geq \sum_{k=-l-\frac{1}{2}}^{l+\frac{1}{2}} C_k(\varphi_k, E, m)\, T^m = C(\psi, E, m)\, T^m.$$

□

Corollary 4.17. *Consider the massless Dirac operator H on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ with translationally symmetric B and V (w.r.t. the x_2-axis). Assume that the potential functions $A(x) := \int_0^x B(s)ds$ and V satisfy Assumption 2.14. Let $E, m > 0$. Then for any $\psi \in \mathbb{1}_{[-E,E]}(H)L^2(\mathbb{R}^2, \mathbb{C}^2)$ such that the set*

$$\big\{\xi \in \mathbb{R} \,|\, \widehat{\psi}(\,\cdot\,, \xi) \neq 0,\ \widehat{\psi}(\,\cdot\,, \xi) \in P_{\mathrm{ac}}(h(\xi))L^2(\mathbb{R}, \mathbb{C}^2)\big\} \tag{4.25}$$

has non-trivial Lebesgue measure, there exists a constant $C(\psi, E, m) > 0$ such that

$$\big\langle \big\| |x_1|^{\frac{m}{2}} e^{-\mathrm{i}tH}\psi \big\|^2 \big\rangle_T \geq C(\psi, E, m)\, T^m$$

for all $T > 0$. Here $\widehat{\psi}$ denotes the Fourier-transform of ψ in the x_2-variable, i.e. we use the notation $\widehat{\psi}(x_1, \,\cdot\,) = \mathcal{F}_{x_2}\psi(x_1, \,\cdot\,)$.

Proof. If B and V are translationally symmetric with respect to the x_2-direction, we may assume that H is in the Landau gauge, i.e. that $\mathbf{A}$ satisfies (2.51). Performing a Fourier transform in the direction of x_2, we obtain

$$\big\langle \big\| |x_1|^{\frac{m}{2}} e^{-\mathrm{i}tH}\psi \big\|^2 \big\rangle_T = \int_{-\infty}^{\infty} \big\langle \big\| |x_1|^{\frac{m}{2}} e^{-\mathrm{i}th(\xi)} \widehat{\psi}(\,\cdot\,,\xi) \big\|^2 \big\rangle_T \,\mathrm{d}\xi.$$

Due to the assumption on (4.25), we find a $\xi_0 > 0$, so large that

$$\hat{O} := \big\{ \xi \in [-\xi_0, \xi_0] \,|\, \widehat{\psi}(\,\cdot\,,\xi) \neq 0,\ \widehat{\psi}(\,\cdot\,,\xi) \in P_{\mathrm{ac}}(h(\xi)) L^2(\mathbb{R}, \mathbb{C}^2) \big\}$$

has positive Lebesgue measure. In view of (2.54), we have

$$\mathbb{1}_{[-E,E]}(H) = \int_{\mathbb{R}}^{\oplus} \mathbb{1}_{[-E,E]}(h(\xi)) \mathrm{d}\xi$$

and thus we conclude that $\widehat{\psi}(\,\cdot\,,\xi) \in \mathbb{1}_{[-E,E]}(h(\xi)) L^2(\mathbb{R}, \mathbb{C}^2)$ for a.e. $\xi \in \mathbb{R}$. Applying Theorem 4.14 yields

$$\big\langle \big\| |x_1|^{\frac{m}{2}} e^{-\mathrm{i}tH}\psi \big\|^2 \big\rangle_T \geq \int_{\hat{O}} \big\langle \big\| |x_1|^{\frac{m}{2}} e^{-\mathrm{i}th(\xi)} \widehat{\psi}(\,\cdot\,,\xi) \big\|^2 \big\rangle_T \,\mathrm{d}\xi \geq C(\psi, E, m) T^m,$$

with $C(\psi, E, m) := \int_{\hat{O}} C_\xi(\widehat{\psi}(\,\cdot\,,\xi), E, m) \mathrm{d}\xi > 0$. □

Let us close this section with an example that can easily be verified with Corollary 2.32.

Example 4.18. *Consider the massless Dirac operator H on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ with magnetic field $B(\mathbf{x}) = b_0 |\mathbf{x}|^s$ and electric potential $V(\mathbf{x}) = v_0 |\mathbf{x}|^t$, where $t, s \geq 0$ and $|v_0|, |b_0| \neq 0$. Suppose that $t > s+1$, or $t = s+1$ and $|v_0| > |b_0|/(s+2)$. Then for every $m > 0$ and*

$$\psi \in P_{\mathrm{ac}}(H) \mathbb{1}_{[-E,E]}(H) L^2(\mathbb{R}^2, \mathbb{C}^2),$$

there is a constant $C(\psi, E, m) > 0$ such that

$$\big\langle \big\| |\mathbf{x}|^{\frac{m}{2}} e^{-\mathrm{i}tH}\psi \big\|^2 \big\rangle_T \geq C(\psi, E, m)\, T^m$$

holds for all $T > 0$.

In the case of translationally symmetric electro-magnetic fields, analogous examples can be derived from Corollary 4.17 and Corollary 2.34.

Appendix A

Tight-Binding Ansatz for the Honeycomb Lattice

In this part of the Appendix we demonstrate how the ordinary tight-binding ansatz for a single band leads to the description of charge carriers near the band edge by the Dirac operator: We consider the honeycomb lattice with $2N = 2n^2$ atoms. Then, according to [AS10], the tight-binding Hamiltonian in second quantised form is given by

$$\hat{H}_{\text{tb}} = \sum_{i,j=1}^{2N} \hat{c}^\dagger_{\mathbf{r}_i} t_{i,j} \hat{c}_{\mathbf{r}_j} = \sum_{|\mathbf{r}_i - \mathbf{r}_j| \leq d} \hat{c}^\dagger_{\mathbf{r}_i} t_{i,j} \hat{c}_{\mathbf{r}_j} \, .$$

Here $\mathbf{r}_i$ denotes the position of the $i-$th lattice atom, $\hat{c}^\dagger_{\mathbf{r}_i}$ and $c_{\mathbf{r}_i}$ the corresponding creation and annihilation operators of the lattice site $\mathbf{r}_i$, and d the nearest-neighbour distance. Note that the spin index of the lattice electrons is suppressed in $\hat{H}_{\text{tb}}$. Let us briefly describe the geometry we have to deal with:

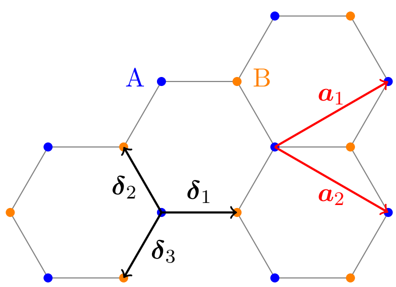

Figure 2

If $a = |\boldsymbol{a}_1| = |\boldsymbol{a}_2|$ is the lattice constant, we choose (as indicated in the picture) the primitive lattice vectors

$$\boldsymbol{a}_1 = \frac{a}{2}\begin{pmatrix}\sqrt{3}\\ 1\end{pmatrix}, \qquad \boldsymbol{a}_2 = \frac{a}{2}\begin{pmatrix}\sqrt{3}\\ -1\end{pmatrix}.$$

The nearest-neighbour vectors are given by

$$\boldsymbol{\delta}_1 = \frac{a}{2\sqrt{3}}\begin{pmatrix}2\\0\end{pmatrix}, \qquad \boldsymbol{\delta}_2 = \frac{a}{2\sqrt{3}}\begin{pmatrix}-1\\\sqrt{3}\end{pmatrix}, \qquad \boldsymbol{\delta}_3 = \frac{a}{2\sqrt{3}}\begin{pmatrix}-1\\-\sqrt{3}\end{pmatrix}.$$

Since there sits a carbon atom in each point of the considered honeycomb lattice, we suppose that $t_{i,i}$ has the same value for each $1 \le i \le 2N$, i.e. $t_{i,i} = t^*_{i,i} = \epsilon_0$ for $1 \le i \le 2N$. Further, the symmetric arrangement of the nearest neigbours around the atom in $\mathbf{r}_i$ justifies the assumption that $t_{i,j} = t^*_{j,i} =: \zeta_0$ for $\mathbf{r}_j = \mathbf{r}_i + \boldsymbol{\delta}_l$ and $l = 1, 2, 3$. In what follows, we consider N to be so large that boundary effects are negligible.

For obtaining the second quantised Hamiltonian in momentum space (and determining the band function $\epsilon(\mathbf{k})$), we work with a unit cell spanned by the primitive vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. Since the basis then consists of two atoms (see Figure 2), we divide the set of lattice points in a set $\mathscr{A}$ of lattice points $\mathbf{A}$ (blue), and a set $\mathscr{B} = \mathscr{A} + \boldsymbol{\delta}_1$ of lattice points $\mathbf{B}$ (orange). Let us denote by $\mathbf{R}_i$ the position of the $i-$th unit cell, which is (per definition) the position of atom $\mathbf{A}$ within the cell. Then $\mathbf{R}_i + \boldsymbol{\delta}_1$ is the position of the atom $\mathbf{B}$ in the $i-$th unit cell. It is convenient to use the notation

$\hat{a}^{(\dagger)}_{\mathbf{R}_i}$	creation/annihilation operator of the lattice site $\mathbf{R}_i$
$\hat{b}^{(\dagger)}_{\mathbf{R}_i}$	creation/annihilation operator of the lattice site $\mathbf{R}_i + \boldsymbol{\delta}_1$

With that convention our tight-binding Hamiltonian reads

$$\begin{aligned}\hat{H}_{\text{tb}} \;=\; & \sum_{i=1}^{N} \big(\epsilon_0 \hat{a}^\dagger_{\mathbf{R}_i}\hat{a}_{\mathbf{R}_i} \;+\; \zeta_0 \hat{a}^\dagger_{\mathbf{R}_i}\hat{b}_{\mathbf{R}_i} \;+\; \zeta_0 \hat{a}^\dagger_{\mathbf{R}_i}\hat{b}_{\mathbf{R}_i-\mathbf{a}_1} \;+\; \zeta_0 \hat{a}^\dagger_{\mathbf{R}_i}\hat{b}_{\mathbf{R}_i-\mathbf{a}_2}\big)\\ & + \sum_{i=1}^{N} \big(\epsilon_0 \hat{b}^\dagger_{\mathbf{R}_i}\hat{b}_{\mathbf{R}_i} \;+\; \zeta_0^* \hat{b}^\dagger_{\mathbf{R}_i}\hat{a}_{\mathbf{R}_i} \;+\; \zeta_0^* \hat{b}^\dagger_{\mathbf{R}_i}\hat{a}_{\mathbf{R}_i+\mathbf{a}_1} \;+\; \zeta_0^* \hat{b}^\dagger_{\mathbf{R}_i}\hat{a}_{\mathbf{R}_i+\mathbf{a}_2}\big).\end{aligned}$$

Note that the first Brillouin zone of the finite lattice, spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$, is also the first Brillouin zone of the sub-lattices $\mathscr{A}$ and $\mathscr{B}$. Therefore, as pointed out in [Mah00], we have the representations

$$\hat{a}^\dagger_{\mathbf{R}_i} = \frac{1}{n}\sum_{\mathbf{k}\in\mathcal{V}_B} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{R}_i}\hat{a}^\dagger_{\mathbf{k}}\,, \qquad \hat{b}^\dagger_{\mathbf{R}_i} = \frac{1}{n}\sum_{\mathbf{k}\in\mathcal{V}_B} e^{\mathrm{i}\,\mathbf{k}\cdot(\mathbf{R}_i+\boldsymbol{\delta}_1)}\hat{b}^\dagger_{\mathbf{k}},$$

where $\mathcal{V}_B$ denotes the first Brillouin zone

$$\mathcal{V}_B = \big\{\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 \,|\, \lambda_1, \lambda_2 \in \big\{0, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}\big\}\big\},$$

generated by the reciprocal lattice vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Further, $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ are creation operators of electrons with momentum $\mathbf{k} \in \mathcal{V}_B$. Because of the identities

$$\frac{1}{n^2}\sum_{i=1}^{n^2} e^{-\mathrm{i}\,\mathbf{R}_i\cdot(\mathbf{k}-\mathbf{k}')} = \delta_{\mathbf{k},\mathbf{k}'} \quad \text{and} \quad \frac{1}{n^2}\sum_{\mathbf{k}\in\mathcal{V}_B} e^{-\mathrm{i}\,\mathbf{k}\cdot(\mathbf{R}_i-\mathbf{R}_j)} = \delta_{i,j}\,,$$

the invers transformations are given by

$$\hat{a}^\dagger_{\mathbf{k}} = \frac{1}{n}\sum_{i=1}^{n^2} e^{-\mathrm{i}\,\mathbf{R}_i\cdot\mathbf{k}}\,\hat{a}^\dagger_{\mathbf{R}_i}\,, \qquad \hat{b}^\dagger_{\mathbf{k}} = \frac{1}{n}\sum_{i=1}^{n^2} e^{-\mathrm{i}\,(\mathbf{R}_i+\boldsymbol{\delta}_1)\cdot\mathbf{k}}\,\hat{b}^\dagger_{\mathbf{R}_i}.$$

Using them, we can write $\hat{H}_{\mathrm{tb}}$ in terms of $\hat{a}^{(\dagger)}_{\mathbf{k}}$ and $\hat{b}^{(\dagger)}_{\mathbf{k}}$, resulting in

$$\begin{aligned}\hat{H}_{\mathrm{tb}} &= \epsilon_0 \sum_{\mathbf{k}\in\mathcal{V}_B} \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;+\; \epsilon_0 \sum_{\mathbf{k}\in\mathcal{V}_B} \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \\ &\quad + \zeta_0^* \sum_{\mathbf{k}\in\mathcal{V}_B} \big(e^{\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{\delta}_1} + e^{\mathrm{i}\,\mathbf{k}\cdot(\boldsymbol{\delta}_1-\mathbf{a}_1)} + e^{\mathrm{i}\,\mathbf{k}\cdot(\boldsymbol{\delta}_1-\mathbf{a}_2)}\big)\,\hat{b}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \\ &\quad + \zeta_0 \sum_{\mathbf{k}\in\mathcal{V}_B} \big(e^{-\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{\delta}_1} + e^{-\mathrm{i}\,\mathbf{k}\cdot(\boldsymbol{\delta}_1-\mathbf{a}_1)} + e^{-\mathrm{i}\,\mathbf{k}\cdot(\boldsymbol{\delta}_1-\mathbf{a}_2)}\big)\,\hat{a}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}.\end{aligned}$$

With the abbreviation

$$\begin{aligned}\zeta(\mathbf{k}) &:= \zeta_0\big(e^{-\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{\delta}_1} + e^{-\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{\delta}_3} + e^{-\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{\delta}_2}\big) \\ &= \zeta_0\big(e^{-\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{\delta}_1} + e^{-\mathrm{i}\,\mathbf{k}\cdot(\boldsymbol{\delta}_1-\mathbf{a}_1)} + e^{-\mathrm{i}\,\mathbf{k}\cdot(\boldsymbol{\delta}_1-\mathbf{a}_2)}\big)\end{aligned}$$

our Hamiltonian takes on the compact form

$$\hat{H}_{\mathrm{tb}} \;=\; \sum_{\mathbf{k}\in\mathcal{V}_B} \big(\hat{a}^\dagger_{\mathbf{k}}\,,\,\hat{b}^\dagger_{\mathbf{k}}\big)\begin{pmatrix}\epsilon_0 & \zeta(\mathbf{k})\\ \zeta^*(\mathbf{k}) & \epsilon_0\end{pmatrix}\begin{pmatrix}\hat{a}_{\mathbf{k}}\\ \hat{b}_{\mathbf{k}}\end{pmatrix}. \tag{A.1}$$

Note that the constant ϵ_0 can be removed by subtracting the diagonal term $\epsilon_0\sum_{\mathbf{k}\in\mathcal{V}_B}\big(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\big)$ from $\hat{H}_{\mathrm{tb}}$, i.e. by redefining the Fermi energy of the considered band. Therefore, we set $\epsilon_0 = 0$.

Let us only remark that in the large particle limit ($N\to\infty$) one usually replaces

$$\sum_{\mathbf{k}\in\mathcal{V}_B} \qquad \text{by} \qquad \int_{\mathcal{V}_B}\frac{\mathrm{d}^2\mathbf{k}}{|\mathcal{V}_B|}\,,$$

where $|\mathcal{V}_B| = (2\pi)^2/|\mathbf{a}_1\times\mathbf{a}_2|$ is the volume of the Brillouin zone (see [BF04]). However, since we deal with second quantised operators, it requires some effort to do that properly. We just mention that this leads to the same Hamiltonian as one gets when starting with an infinite lattice (see [Sem84]).

Now let us consider the Hamiltonian $\hat{H}_{\mathrm{tb}}$ near the band edge $\epsilon_0 = 0$. Note that the band function $\epsilon(\mathbf{k})$ has the form

$$\begin{aligned}\epsilon(\mathbf{k}) &= \epsilon_0 \pm \sqrt{\zeta(\mathbf{k})\zeta^*(\mathbf{k})} \\ &= \pm\,|\zeta_0|\sqrt{3 + 2\cos(\mathbf{k}\cdot\mathbf{a}_1) + 2\cos(\mathbf{k}\cdot\mathbf{a}_2) + 2\cos(\mathbf{k}\cdot(\mathbf{a}_2-\mathbf{a}_1))}\,.\end{aligned}$$

For finding the roots of $\epsilon(\mathbf{k})$, we work with the primitive vectors

$$\mathbf{v}_1 = \frac{2\pi}{a\sqrt{3}}\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}, \qquad \mathbf{v}_2 = \frac{2\pi}{a\sqrt{3}}\begin{pmatrix}1\\ -\sqrt{3}\end{pmatrix}$$

of the reciprocal lattice. By writing

$$\zeta(\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2) \;=\; e^{-\frac{2\pi\mathrm{i}}{3}(\lambda_1+\lambda_2)}\big(1 + e^{2\pi\mathrm{i}\,\lambda_1} + e^{2\pi\mathrm{i}\,\lambda_2}\big),$$

one can read off directly the roots

$$\mathbf{K} = \tfrac{2}{3}\mathbf{v}_1 + \tfrac{1}{3}\mathbf{v}_2 = \frac{2\pi}{a3\sqrt{3}}\begin{pmatrix}3\\ \sqrt{3}\end{pmatrix}, \qquad \mathbf{K}' = \tfrac{1}{3}\mathbf{v}_1 + \tfrac{2}{3}\mathbf{v}_2 = \frac{2\pi}{a3\sqrt{3}}\begin{pmatrix}3\\ -\sqrt{3}\end{pmatrix}$$

of $\zeta(\mathbf{k})$ in the first Brillouin zone. Within a small neighbourhood of $\mathbf{K}$ and $\mathbf{K}'$ one can linearise the function $\zeta(\mathbf{k})$, i.e. we can write

$$\begin{aligned}\zeta(\mathbf{K}+\mathbf{k}) &= \zeta(\mathbf{K}) + (\nabla\zeta)(\mathbf{K})\cdot\mathbf{k} + \mathcal{O}((a|\mathbf{k}|)^2)\\ &= \zeta_0\frac{a}{2}\sqrt{3}(-\mathrm{i}\,)e^{-\frac{2\pi\mathrm{i}}{3}}(k_1 - \mathrm{i}\,k_2) + \mathcal{O}((a|\mathbf{k}|)^2),\\ \zeta(\mathbf{K}'+\mathbf{k}) &= \zeta(\mathbf{K}') + (\nabla\zeta)(\mathbf{K}')\cdot\mathbf{k} + \mathcal{O}((a|\mathbf{k}|)^2)\\ &= \zeta_0\frac{a}{2}\sqrt{3}(-\mathrm{i}\,)e^{-\frac{2\pi\mathrm{i}}{3}}(k_1 + \mathrm{i}\,k_2) + \mathcal{O}((a|\mathbf{k}|)^2).\end{aligned}$$

With the abbreviation $v_F = \zeta_0 a\sqrt{3}/2$, we therefore see that

$$\begin{aligned}&\sum_{|\mathbf{K}-\mathbf{k}|\ll a^{-1}} (\hat a^\dagger_{\mathbf{k}}\,,\,\hat b^\dagger_{\mathbf{k}})\begin{pmatrix}0 & \zeta(\mathbf{k})\\ \zeta^*(\mathbf{k}) & 0\end{pmatrix}\begin{pmatrix}\hat a_{\mathbf{k}}\\ \hat b_{\mathbf{k}}\end{pmatrix}\\ &\qquad\approx \sum_{|\mathbf{k}|\ll a^{-1}} (\hat a^\dagger_{\mathbf{K}+\mathbf{k}}\,,\,\hat b^\dagger_{\mathbf{K}+\mathbf{k}})U^*(v_F\boldsymbol{\sigma}\cdot\mathbf{k})U\begin{pmatrix}\hat a_{\mathbf{K}+\mathbf{k}}\\ \hat b_{\mathbf{K}+\mathbf{k}}\end{pmatrix},\end{aligned}$$

$$\begin{aligned}&\sum_{|\mathbf{K}'-\mathbf{k}|\ll a^{-1}} (\hat a^\dagger_{\mathbf{k}}\,,\,\hat b^\dagger_{\mathbf{k}})\begin{pmatrix}0 & \zeta(\mathbf{k})\\ \zeta^*(\mathbf{k}) & 0\end{pmatrix}\begin{pmatrix}\hat a_{\mathbf{k}}\\ \hat b_{\mathbf{k}}\end{pmatrix}\\ &\qquad\approx \sum_{|\mathbf{k}|\ll a^{-1}} (\hat a^\dagger_{\mathbf{K}'+\mathbf{k}}\,,\,\hat b^\dagger_{\mathbf{K}'+\mathbf{k}})U^*(v_F\boldsymbol{\sigma}^*\cdot\mathbf{k})U\begin{pmatrix}\hat a_{\mathbf{K}'+\mathbf{k}}\\ \hat b_{\mathbf{K}'+\mathbf{k}}\end{pmatrix},\end{aligned}$$

where $U = e^{\frac{7\pi\mathrm{i}}{12}\sigma_3}$ is a unitary matrix. The approximated Hamiltonians on the right hand sides are well known from $(2+1)-$dimensional QED. More precisely, if we take the massless two-dimensional Dirac operator in the second quantised form

$$\hat D_0 = \int_{\mathcal{V}} \hat\Psi^\dagger(\mathbf{x})\big[v_F\boldsymbol{\sigma}\cdot(-\mathrm{i}\,\nabla)\big]\hat\Psi(\mathbf{x})\,\mathrm{d}^2\mathbf{x}$$

(see Chapter 13.3 of [Sch05]) and transform the field operators via

$$\hat\Psi^\dagger(\mathbf{x}) = \frac{1}{\sqrt{|\mathcal{V}|}}\sum_{\mathbf{k}} e^{-\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}\hat\Psi^\dagger_{\mathbf{k}}\,, \qquad \hat\Psi(\mathbf{x}) = \frac{1}{\sqrt{|\mathcal{V}|}}\sum_{\mathbf{k}} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}\hat\Psi_{\mathbf{k}}\,,$$

we obtain (up to unitary equivalence and re-labeling) a Hamiltonian of the same form as in the small momenta approximation of (A.1). This analogy is justifiable if the momenta are small compared to a^{-1}. However, we should remark that the analysis presented here is not rigorous, in particular one does not have any error estimates.

Appendix B

Supplementary Arguments and Results

B.1 On Local Compactness and Variation of Potential Functions

Proposition B.1. *Let $O = \mathbb{R}^n$ or $O = \mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n \,|\, x_1 > 0\}$ for some $n \in \mathbb{N}$. Assume that H is a locally compact, self-adjoint operator on $L^2(O, \mathbb{C}^2)$. If there exists a function $V \in L^\infty_{\mathrm{loc}}(O, [0, \infty))$, with $V(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$, such that*

$$\|H\psi\| \geq \|V\psi\| \qquad \text{for } \psi \in \mathcal{D}(H), \tag{B.1}$$

then $\sigma_{\mathrm{ess}}(H) = \emptyset$, i.e. H has purely discrete spectrum.

Proof. If $\lambda \in \sigma_{\mathrm{ess}}(H) \subset \mathbb{R}$, we can find a normalised sequence $(\psi_k)_{k\in\mathbb{N}} \subset \mathcal{D}(H)$ such that $\psi_k \rightharpoonup 0$ and $\|(H - \lambda)\psi_k\| \to 0$ as $k \to \infty$. For any $R > 0$ we estimate

$$\begin{aligned}\|\mathbb{1}_{\{|\mathbf{x}|\leq R\}}\psi_k\| &= \left\| \mathbb{1}_{\{|\mathbf{x}|\leq R\}} \frac{1}{H - \lambda - \mathrm{i}} (H - \lambda - \mathrm{i})\psi_k \right\| \\ &\leq \left\| \mathbb{1}_{\{|\mathbf{x}|\leq R\}} \frac{1}{H - \lambda - \mathrm{i}} \psi_k \right\| + C\|(H - \lambda)\psi_k\|,\end{aligned}$$

where $C > 0$ denotes the norm of the operator

$$\mathbb{1}_{\{|\mathbf{x}|\leq R\}} \frac{1}{H - \lambda - \mathrm{i}}. \tag{B.2}$$

Using the compactness of (B.2), this estimate implies that $\|\chi_{\{|\mathbf{x}|\leq R\}}\psi_k\| \to 0$ as $k \to \infty$ for any $R > 0$. By the assumption on V, we can choose $R > 0$ so large that $V(\mathbf{x}) \geq 5|\lambda| + 1$ for $|\mathbf{x}| \geq R$. If $N \in \mathbb{N}$ is sufficiently big, we get

$$\begin{aligned}
\|(H-\lambda)\psi_k\| &\geq \|V\psi_k\| - |\lambda| \\
&\geq \big\|V\mathbb{1}_{\{|\mathbf{x}|>R\}}\psi_k\big\| - \big\|V\mathbb{1}_{\{|\mathbf{x}|\leq R\}}\big\|_\infty \big\|\mathbb{1}_{\{|\mathbf{x}|\leq R\}}\psi_k\big\| - |\lambda| \\
&\geq (5|\lambda|+1)\big\|\mathbb{1}_{\{|\mathbf{x}|>R\}}\psi_k\big\| - \big\|V\mathbb{1}_{\{|\mathbf{x}|\leq R\}}\big\|_\infty \big\|\mathbb{1}_{\{|\mathbf{x}|\leq R\}}\psi_k\big\| - |\lambda| \\
&\geq (|\lambda| + \tfrac{1}{2}) - \big\|V\mathbb{1}_{\{|\mathbf{x}|\leq R\}}\big\|_\infty \big\|\mathbb{1}_{\{|\mathbf{x}|\leq R\}}\psi_k\big\|
\end{aligned}$$

whenever $k \geq N$. In particular, $\|(H-\lambda)\varphi_n\| \nrightarrow 0$ as $k \to \infty$, which is a contradiction. □

Remark B.2. *From the proof of Proposition B.1 one may read off that Condition* (B.1) *can be relaxed to hold only on*

$$\big\{\psi \in \mathcal{D}(H) \mid \operatorname{supp}(\psi) \subset \{|\mathbf{x}| > R\}\big\},$$

where $R > 0$ is arbitrary but fixed.

Proposition B.3. *Assume that $B \in C(\mathbb{R}^2, \mathbb{R})$ and $V \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfy the conditions*

$$|V(\mathbf{x})| \longrightarrow \infty \qquad \text{as } |\mathbf{x}| \to \infty, \tag{B.3}$$

$$\left|\frac{\nabla V(\mathbf{x})}{V(\mathbf{x})}\right| \longrightarrow 0 \qquad \text{as } |\mathbf{x}| \to \infty, \tag{B.4}$$

$$\limsup_{|\mathbf{x}|\to\infty} \frac{V^2(\mathbf{x})}{2|B(\mathbf{x})|} < 1. \tag{B.5}$$

Then there is an $\eta \in (0,1)$ such that for any $\delta \in (0,1)$ one can find functions $\hat{B} \in C(\mathbb{R}^2, \mathbb{R})$ and $\hat{V} \in C^1(\mathbb{R}^2, \mathbb{R})$, with

$$|\hat{V}(\mathbf{x})| \geq 1/\delta, \tag{B.6}$$

$$|\nabla \hat{V}(\mathbf{x})| \leq \delta |\hat{V}(\mathbf{x})|, \tag{B.7}$$

$$\hat{V}^2(\mathbf{x}) \leq 2(1-\eta)|\hat{B}(\mathbf{x})| \tag{B.8}$$

for $\mathbf{x} \in \mathbb{R}^2$, and which differ from V and B only on a compact set.

Proof. We only consider the case $B(\mathbf{x}), V(\mathbf{x}) \to \infty$ as $|\mathbf{x}| \to \infty$ since the other sign combinations can be treated completely analogously. Let us start with the construction of $\hat{V}$: Due to assumption (B.5), we find constants $r_1 \geq 1$ and $\eta \in (0,1)$ such that

$$V^2(\mathbf{x}) < 2(1-\eta)B(\mathbf{x}) \tag{B.9}$$

if $|\mathbf{x}| \geq r_1$. Let $\delta \in (0,1)$ be given, then we deduce from (B.3) and (B.4) that there is an $r_2 > r_1$ such that

$$2\delta^{-1} \leq V(\mathbf{x}) \qquad \text{and} \qquad |\nabla V(\mathbf{x})| \leq \frac{\delta}{4} V(\mathbf{x}) \tag{B.10}$$

hold if $|\mathbf{x}| \geq r_2$. For $\chi \in C^\infty(\mathbb{R}^2,\ [0,1])$ satisfying

$$\chi(\mathbf{x}) = \begin{cases} \frac{1}{2} & \text{for } |\mathbf{x}| \leq 1, \\ 1 & \text{for } |\mathbf{x}| \geq 2, \end{cases}$$

we set $\chi_r(\,\cdot\,) := \chi\left(\frac{\cdot}{r}\right)$, where $r > 0$. Now consider for $r \geq r_2$ the C^1-function

$$\hat{V}_r(\mathbf{x}) := \chi_r(\mathbf{x})V(\mathbf{x}) + (1 - \chi_r(\mathbf{x}))4\delta^{-1}(1 + c_1 + c_2)$$

on $\mathbb{R}^2$, with

$$c_1 := \sup\left\{\max(-V(\mathbf{x}), 0) \mid \mathbf{x} \in \mathbb{R}^2\right\},$$
$$c_2 := \sup\left\{|\nabla V(\mathbf{x})| \mid |\mathbf{x}| \leq r_2\right\}.$$

Observe that $\hat{V}_r(\mathbf{x}) = V(\mathbf{x})$ whenever $|\mathbf{x}| > 2r$. By the choice of r_2, we have

$$\delta^{-1} \leq \hat{V}_r(\mathbf{x}), \qquad |V(\mathbf{x})| \leq 2\hat{V}_r(\mathbf{x}) \tag{B.11}$$

for any $\mathbf{x} \in \mathbb{R}^2$. The second inequality of (B.11) yields

$$|\nabla \hat{V}_r(\mathbf{x})| \leq |\nabla V(\mathbf{x})| + \frac{1}{r}\left(2\|\nabla\chi\|_\infty + 4\|\nabla\chi\|_\infty\delta^{-1}(1 + c_1 + c_2)\right)\hat{V}_r(\mathbf{x})$$

for $\mathbf{x} \in \mathbb{R}^2$, where $r \geq r_2$. In view of the second estimate of (B.10), we know that $|\nabla V(\mathbf{x})| \leq \max(c_2, \delta V(\mathbf{x})/4)$ holds globally. So for any $r \geq r_2$ we get

$$|\nabla V(\mathbf{x})| \leq \frac{\delta}{2}\hat{V}_r(\mathbf{x})$$

on $\mathbb{R}^2$. Note that we also applied (B.11) and the definition of $\hat{V}_r$ for the previous conclusion. Therefore, we can choose $r_3 > r_2$ so large that

$$|\nabla \hat{V}_{r_3}(\mathbf{x})| \leq \delta\hat{V}_{r_3}(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \mathbb{R}^2. \tag{B.12}$$

Set $\hat{V} = \hat{V}_{r_3}$ and observe that $\hat{V}$ satisfies (B.6) and (B.7) due to (B.11) and (B.12). With $r_4 = 2r_3$ let us define $\hat{B}$ through

$$\hat{B}(\mathbf{x}) := (2\chi_{r_4}(\mathbf{x}) - 1)\,B(\mathbf{x}) + 2\,(1 - \chi_{r_4}(\mathbf{x}))\,\frac{1}{2(1-\eta)}\,\hat{V}^2(\mathbf{x})$$

for $\mathbf{x} \in \mathbb{R}^2$. By (B.9) and since $\hat{V}$ coincides with V on $\{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| \geq 2r_3\}$, we have

$$\hat{V}^2(\mathbf{x}) \leq 2(1-\eta)\hat{B}(\mathbf{x})$$

on $\mathbb{R}^2$, i.e. $\hat{V}$ and $\hat{B}$ satisfy also condition (B.8). □

Corollary B.4. *Assume that the magnetic field* $B \in C(\mathbb{R}^2, \mathbb{R})$ *and the electric potential* $V \in C^1(\mathbb{R}^2, \mathbb{R})$ *satisfy* (B.3)−(B.5). *Let* $\mathbf{A} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ *be a vector potential for* B*, i.e.* $B = \operatorname{curl}\mathbf{A}$*. Then there exist* $\hat{B} \in C(\mathbb{R}^2, \mathbb{R})$ *and* $\hat{V} \in C^1(\mathbb{R}^2, \mathbb{R})$*, for which the global conditions* (B.6)−(B.8) *hold true, and such that*

$$(D_{\mathbf{A}} + V - \mathrm{i}\,)^{-1} \text{ is compact} \qquad \text{iff} \qquad (D_{\hat{\mathbf{A}}} + \hat{V} - \mathrm{i}\,)^{-1} \text{ is compact.}$$

Here $\hat{\mathbf{A}} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ *denotes a vector potential for* $\hat{B}$.

Proof. For B and V satisfying (B.3)−(B.5) consider $\hat{B}$ and $\hat{V}$, given by Proposition B.3. Since the difference $B - \hat{B}$ has compact support, there exist a bounded function $\mathbf{A}_0 \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ with $\operatorname{curl} \mathbf{A}_0 = B - \hat{B}$ on $\mathbb{R}^2$. Observe that $\hat{\mathbf{A}} := \mathbf{A} - \mathbf{A}_0$ is a vector potential for $\hat{B}$. Due to the construction of $\hat{\mathbf{A}}$ and $\hat{V}$, we know that the operator difference $(D_{\mathbf{A}} + V) - (D_{\hat{\mathbf{A}}} + \hat{V})$ is bounded on $L^2(\mathbb{R}^2, \mathbb{C}^2)$. In particular, the difference

$$(D_{\mathbf{A}} + V - \mathrm{i}\,)^{-1} - (D_{\hat{\mathbf{A}}} + \hat{V} - \mathrm{i}\,)^{-1}$$

is compact if already one of the resolvents itself is compact. □

B.2 Proofs of Lemma 3.2 and Lemma 3.3

In this section we provide the proofs of the two lemmas, required for proving Theorem 3.1 (and for proving Theorem 3.15). Alternatively, one can find them in [MS14].

For $\mathbf{A} \in L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, with $p > 2$, let $D_{\mathbf{A}}$ be given by (2.32). We use the abbreviation

$$R_{\mathbf{A}}(z) := (D_{\mathbf{A}} - z)^{-1} \tag{B.13}$$

for $z \in \varrho(D_{\mathbf{A}})$. If $V \in C^1(\mathbb{R}^2, \mathbb{R})$ is strictly positive or strictly negative and satisfies $\|\nabla V / V\|_\infty < \infty$, we define the operator

$$K := [D_{\mathbf{A}}, V^{-1}]V = \frac{\mathrm{i}\,\boldsymbol{\sigma} \cdot \nabla V}{V} \qquad \text{on } C_0^\infty(\mathbb{R}^2, \mathbb{C}^2).$$

Note that K is bounded on the dense subset $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ of $L^2(\mathbb{R}^2, \mathbb{C}^2)$, hence extends to a bounded operator on the considered Hilbert space.

Lemma B.5. *Let* $B \in C(\mathbb{R}^2, \mathbb{R})$ *and* $\mathbf{A} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ *with* $B = \operatorname{curl} \mathbf{A}$. *Assume that* $V \in C^1(\mathbb{R}^2, \mathbb{R})$ *is strictly positive or negative and satisfies* $\|V^{-1}\|_\infty, \|\nabla V / V\|_\infty < \infty$. *Then for any* $z \in \varrho(D_{\mathbf{A}})$ *with* $\|K\|\|R_{\mathbf{A}}(z)\| < 1$ *we have*

$$\big[R_{\mathbf{A}}(z), V^{-1}\big] = -V^{-1} R_{\mathbf{A}}(z) K R_{\mathbf{A}}(z) L_1(z) \tag{B.14}$$

$$= L_2(z) R_{\mathbf{A}}(z) K R_{\mathbf{A}}(z) V^{-1}. \tag{B.15}$$

The operators

$$L_1(z) := \big(1 + K R_{\mathbf{A}}(z)\big)^{-1}, \qquad L_2(z) := \big(1 - R_{\mathbf{A}}(z) K\big)^{-1}$$

are bounded with

$$\|L_1(z)\|, \|L_2(z)\| \le \big(1 - \|K\|\|R_{\mathbf{A}}(z)\|\big)^{-1}. \tag{B.16}$$

Proof. For $z \in \varrho(D_{\mathbf{A}})$ we compute

$$\begin{aligned}\left[R_{\mathbf{A}}(z), V^{-1}\right] &= R_{\mathbf{A}}(z)\left[V^{-1}, D_{\mathbf{A}}\right]R_{\mathbf{A}}(z) \\ &= -R_{\mathbf{A}}(z)V^{-1}KR_{\mathbf{A}}(z) \\ &= -V^{-1}R_{\mathbf{A}}(z)KR_{\mathbf{A}}(z) - \left[R_{\mathbf{A}}(z), V^{-1}\right]KR_{\mathbf{A}}(z).\end{aligned}$$

It follows that

$$\left[R_{\mathbf{A}}(z), V^{-1}\right]\left(1 + KR_{\mathbf{A}}(z)\right) = -V^{-1}R_{\mathbf{A}}(z)KR_{\mathbf{A}}(z).$$

If $\|K\|\|R_{\mathbf{A}}(z)\| < 1$, we can invert $1 + KR_{\mathbf{A}}(z)$ by writing the invers as a Neumann series, which results in (B.14). The bound on the norm of $L_1(z)$ is a direct consequence of the representation as a Neumann series. In the same way we obtain formula (B.15) and the norm bound on $L_2(z)$. □

Proof of Lemma 3.2. a) Let $\psi, \tilde{\psi} \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ with $\|\psi\| = \|\tilde{\psi}\| = 1$. If β_0 is defined as in (3.7), we know that $(-2\beta_0, 0) \cup (0, 2\beta_0) \subset \varrho(D_{\mathbf{A}})$. In particular, we have the representation

$$\Pi_0 = -\frac{1}{2\pi\mathrm{i}}\int_{|z|=\beta_0} R_{\mathbf{A}}(z)\,\mathrm{d}z$$

for the orthogonal Projection on $\ker(D_{\mathbf{A}})$. Since $\|R_{\mathbf{A}}(z)\| \le \beta_0^{-1}$ for $|z| = \beta_0$ and since $\|\nabla V/V\|_\infty \le \delta$, we can estimate

$$\|K\|\|R_{\mathbf{A}}(z)\| \le \delta\beta_0^{-1} = 2\delta^2\sqrt{1-\eta} \le \tfrac{1}{2}$$

for $|z| = \beta_0$. Thus, Lemma B.5 yields

$$\begin{aligned}\left|\left\langle V\tilde{\psi}, \left[\Pi_0^\perp, V^{-1}\right]\psi\right\rangle\right| &= \left|\left\langle V\tilde{\psi}, \left[\Pi_0, V^{-1}\right]\psi\right\rangle\right| \\ &\le \frac{1}{2\pi}\int_{|z|=\beta_0} \left|\left\langle V\tilde{\psi}, \left[R_{\mathbf{A}}(z), V^{-1}\right]\psi\right\rangle\right|\mathrm{d}z \\ &= \frac{1}{2\pi}\int_{|z|=\beta_0} \left|\langle\tilde{\psi}, R_{\mathbf{A}}(z)KR_{\mathbf{A}}(z)L_1(z)\psi\rangle\right|\mathrm{d}z \\ &\le \frac{1}{2\pi}\int_{|z|=\beta_0} \frac{\|K\|\|R_{\mathbf{A}}(z)\|^2}{1 - \|K\|\|R_{\mathbf{A}}(z)\|}\,\mathrm{d}z \qquad \le 4\delta^2.\end{aligned}$$

Analogously, we get the same bound on the norm of $\left[\Pi_0^\perp, V^{-1}\right]V$.

b) One may observe from the previous calculation that the operator $\left[\Pi_0^\perp, V^{-1}\right]$ maps $L^2(\mathbb{R}^2, \mathbb{C}^2)$ into $\mathcal{D}(V)$. By writing

$$\Pi_0\psi = V^{-1}\Pi_0 V\psi + \left[\Pi_0, V^{-1}\right]\psi$$

for $\psi \in \mathcal{D}(V)$, the claim follows immediately.

c) In view of the Spectral Theorem, we deduce, for $|z| = \beta_0$, that

$$\|D_{\mathbf{A}}R_{\mathbf{A}}(z)\| = \sup_{\lambda\in\sigma(D_{\mathbf{A}})}\left|\frac{\lambda}{\lambda - z}\right| \le 2.$$

Using again the integral representation of Π_0, we can estimate

$$\big|\big\langle D_{\mathbf{A}}\tilde{\psi}, V\big[\Pi_0^\perp, V^{-1}\big]\psi\big\rangle\big| \\ \leq \frac{1}{2\pi}\int_{|z|=\beta_0} |\langle\tilde{\psi}, D_{\mathbf{A}}R_{\mathbf{A}}(z)KR_{\mathbf{A}}(z)L_1(z)\psi\rangle|\,\mathrm{d}z \leq 4\delta$$

for normalised functions $\psi, \tilde{\psi} \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$. □

Proof of Lemma 3.3. We write

$$\operatorname{sgn}(D_{\mathbf{A}})\Pi_0^\perp \;=\; \mathbb{1}_{(\beta_0,\infty)}(D_{\mathbf{A}}) - \mathbb{1}_{(-\infty,-\beta_0)}(D_{\mathbf{A}}),$$

with β_0 as in (3.7). By Lemma VI-5.6 of [Kat95], one can express the orthogonal projections $\mathbb{1}_{(\beta_0,\infty)}(D_{\mathbf{A}})$ and $\mathbb{1}_{(-\infty,-\beta_0)}(D_{\mathbf{A}})$ through

$$\mathbb{1}_{(\beta_0,\infty)}(D_{\mathbf{A}}) = \frac{1}{2}(\mathrm{id} + U(\beta_0)),$$
$$\mathbb{1}_{(-\infty,-\beta_0)}(D_{\mathbf{A}}) = \frac{1}{2}(\mathrm{id} - U(-\beta_0)),$$

where the partial isometry $U(\lambda)$ admits the integral representation

$$U(\lambda) = \mathrm{s} - \lim_{R\to\infty}\int_{-R}^{R} R_{\mathbf{A}}(\lambda + \mathrm{i}\,t)\frac{\mathrm{d}t}{\pi} =: \int_{-\infty}^{\infty} R_{\mathbf{A}}(\lambda+\mathrm{i}\,t)\frac{\mathrm{d}t}{\pi}$$

for $\lambda \in \varrho(D_{\mathbf{A}})$. Here $\mathrm{s} - \lim$ means that the limit has to be taken in the weak operator topology. The commutator relation

$$\big[\operatorname{sgn}(D_{\mathbf{A}})\Pi_0^\perp, V^{-1}\big] \;=\; \frac{1}{2}\big[U(-\beta_0), V^{-1}\big] + \frac{1}{2}\big[U(\beta_0), V^{-1}\big]$$

reduces the problem to a proper estimate on $V\big[U(\lambda), V^{-1}\big]$ for $\lambda = \pm\beta_0$. Since

$$\|R_{\mathbf{A}}(\beta_0 + \mathrm{i}\,t)\|\|K\| \leq \delta\beta_0^{-1} \leq \tfrac{1}{2},$$

one can deduce from Lemma B.5 that

$$\big[R_{\mathbf{A}}(\beta_0+\mathrm{i}\,t), V^{-1}\big] = -V^{-1}R_{\mathbf{A}}(\beta_0+\mathrm{i}\,t)KR_{\mathbf{A}}(\beta_0+\mathrm{i}\,t)L_1(\beta_0+\mathrm{i}\,t)$$

for all $t \in \mathbb{R}$. Hence, for $\psi, \tilde{\psi} \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ with $\|\psi\| = \|\tilde{\psi}\| = 1$ we have

$$\begin{aligned}\big|\big\langle V\tilde{\psi}, \big[U(\beta_0), V^{-1}\big]\psi\big\rangle\big| &\leq \frac{1}{\pi}\int_{-\infty}^{\infty} \big|\big\langle V\tilde{\psi}, \big[R_{\mathbf{A}}(\beta_0+\mathrm{i}\,t), V^{-1}\big]\psi\big\rangle\big|\,\mathrm{d}t \\ &\leq \frac{1}{\pi}\int_{-\infty}^{\infty} \|R_{\mathbf{A}}(\beta_0+\mathrm{i}\,t)\|^2\|K\|\|L_1(\beta_0+\mathrm{i}\,t)\|\,\mathrm{d}t \\ &\leq \frac{2\delta}{\pi}\int_{-\infty}^{\infty}\frac{1}{\beta_0^2+t^2}\,\mathrm{d}t \\ &\leq 4\delta^2.\end{aligned}$$

Note that we used, for $t \in \mathbb{R}$, the bound

$$\|L_1(\beta_0 + \mathrm{i}\, t)\| \leq \big(1 - \|R_{\mathbf{A}}(\beta_0 + \mathrm{i}\, t)\| \|K\|\big)^{-1} \leq 2.$$

Clearly, we obtain the same bound for $\big\langle V\tilde{\psi}, \big[U(-\beta_0), V^{-1}\big]\psi\big\rangle$, implying that $\big[U(\pm\beta_0), V^{-1}\big]$ maps $L^2(\mathbb{R}^2, \mathbb{C}^2)$ into $\mathcal{D}(V)$ and that

$$\big\|V\big[U(-\beta_0), V^{-1}\big]\big\|, \big\|V\big[U(\beta_0), V^{-1}\big]\big\| \leq 4\delta^2.$$

□

Bibliography

[AS64] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* National Bureau of Standards Applied Mathematics Series, vol. 55. U.S. Government Printing Office, Washington, D.C., 1964.

[AS10] A. Altland and B. D. Simons. *Condensed Matter Field Theory.* Cambridge University Press, 2010.

[BMST15] J.-M. Barbaroux, J. Mehringer, E. Stockmeyer, and A. Taarabt. Dynamical localization of Dirac particles in electromagnetic fields with dominating magnetic potentials. *arXiv preprint*, arXiv:1504.04077, 2015.

[BCS57] J. Bardeen, L. N. Cooper, and J. R. Schrieffer. Theory of superconductivity. *Phys. Rev.*, 108:1175–1204, 1957.

[BMR11] V. Bruneau, P. Miranda, and G. Raikov. Discrete spectrum of quantum Hall effect Hamiltonians I. Monotone edge potentials. *J. Spectr. Theory*, 1(3):237–272, 2011.

[BF04] H. Bruus and K. Flensberg. *Many-body Quantum Theory in Condensed Matter Physics: An Introduction.* Oxford University Press, 2004.

[Che77] P. R. Chernoff. Schrödinger and Dirac operators with singular potentials and hyperbolic equations. *Pacific J. Math.*, 72(2):361–382, 1977.

[Coh13] D. L. Cohn. *Measure Theory.* Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, New York, 2nd edition, 2013.

[Com93] J.-M. Combes. Connections between quantum dynamics and spectral properties of time-evolution operators. In *Differential equations with applications to mathematical physics*, volume 192 of *Math. Sci. Engrg.*, pages 59–68. Academic Press, Boston, MA, 1993.

[CFKS87] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon. *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry.* Texts and Monographs in Physics. Springer-Verlag, Berlin, 1987.

[DRJLS95] R. Del Rio, S. Jitomirskaya, Y. Last, and B. Simon. What is localization? *Phys. Rev. Lett.*, 75(1):117, 1995.

[DRMS94] R. Del Rio, N. Makarov, and B. Simon. Operators with singular continuous spectrum. II. Rank one operators. *Comm. Math. Phys.*, 165(1):59–67, 1994.

[Dir28] P. A. M. Dirac. The quantum theory of the electron. *Proc. R. Soc. Lond. A*, 117(778):610–624, 1928.

[Dir30] P. A. M. Dirac. A theory of electrons and protons. *Proc. R. Soc. Lond. A*, 126(801):360–365, 1930.

[ES08] M. S. P. Eastham and K. M. Schmidt. Asymptotics of the spectral density for radial Dirac operators with divergent potentials. *Publ. Res. Inst. Math. Sci.*, 44(1):107–129, 2008.

[EDMSS10] R. Egger, A. De Martino, H. Siedentop, and E. Stockmeyer. Multiparticle equations for interacting Dirac fermions in magnetically confined graphene quantum dots. *J. Phys. A: Math. Theor.*, 43(21):215202, 2010.

[Els05] J. Elstrodt. *Maß- und Integrationstheorie.* Springer-Lehrbuch [Springer Textbook]. Springer-Verlag, Berlin, 4th edition, 2005.

[Erd63] A. Erdélyi. Note on a paper by Titchmarsh. *Quart. J. Math. Oxford Ser. (2)*, 14:147–152, 1963.

[EV02] L. Erdős and V. Vougalter. Pauli operator and Aharonov-Casher theorem for measure valued magnetic fields. *Comm. Math. Phys.*, 225(2):399–421, 2002.

[FW12] C. L. Fefferman and M. I. Weinstein. Honeycomb lattice potentials and Dirac points. *J. Amer. Math. Soc.*, 25(4):1169–1220, 2012.

[FW14] C. L. Fefferman and M. I. Weinstein. Wave packets in honeycomb structures and two-dimensional Dirac equations. *Comm. Math. Phys.*, 326(1):251–286, 2014.

[FL12] R. L. Frank and M. Loss. Hardy-Sobolev-Maz'ya inequalities for arbitrary domains. *J. Math. Pures Appl. (9)*, 97(1):39–54, 2012.

[GMR09] G. Giavaras, P. A. Maksym, and M. Roy. Magnetic field induced confinement–deconfinement transition in graphene quantum dots. *J. Phys. Condens. Matter*, 21:102201, 2009.

[GR96] W. Greiner and J. Reinhardt. *Field Quantization*. Springer-Verlag, Berlin, 1996.

[Gua89] I. Guarneri. Spectral properties of quantum diffusion on discrete lattices. *Europhys. Lett.*, 10(2):95–100, 1989.

[GS05] V. Gusynin and S. Sharapov. Unconventional integer quantum hall effect in graphene. *Phys. Rev. Lett.*, 95:146801, 2005.

[Hoe97] G. Hoever. On the spectrum of two-dimensional Schrödinger operators with spherically symmetric, radially periodic magnetic fields. *Comm. Math. Phys.*, 189(3):879–890, 1997.

[Iwa85] A. Iwatsuka. Examples of absolutely continuous schrödinger operators in magnetic fields. *Publ. Res. Inst. Math. Sci.*, 21(2):385–401, 1985.

[Iwa90] A. Iwatsuka. Essential selfadjointness of the Schrödinger operators with magnetic fields diverging at infinity. *Publ. Res. Inst. Math. Sci.*, 26(5):841–860, 1990.

[Jac99] J. D. Jackson. *Classical Electrodynamics*. Wiley, New York, NY, 3rd edition, 1999.

[KŌY03] H. Kalf, T. Ōkaji, and O. Yamada. Absence of eigenvalues of Dirac operators with potentials diverging at infinity. *Math. Nachr.*, 259:19–41, 2003.

[KLR10] H. C. Kao, M. Lewkowicz, and B. Rosenstein. Ballistic transport, chiral anomaly, and emergence of the neutral electron-hole plasma in graphene. *Phys. Rev. B*, 82:035406, 2010.

[Kat95] T. Kato. *Perturbation Theory for Linear Operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[Kle29] O. Klein. Die Reflexion von Elektronen an einem Potentialsprung nach der relativistischen Dynamik von Dirac. *Z. Phys.*, 53(3-4):157–165, 1929.

[KMS05] V. Kondratiev, V. Maz'ya, and M. Shubin. Discreteness of spectrum and strict positivity criteria for magnetic schrödinger operators. *Comm. Part. Diff. Eq.*, 29(3-4):489–521, 2005.

[KS12] M. Könenberg and E. Stockmeyer. Localization of two-dimensional massless dirac fermions in a magnetic quantum dot. *J. Spectr. Theory*, 2(2):115–146, 2012.

[Las96] Y. Last. Quantum dynamics and decompositions of singular continuous spectra. *J. Funct. Anal.*, 142(2):406–445, 1996.

[Lei83] H. Leinfelder. Gauge invariance of Schrödinger operators and related spectral properties. *J. Operator Theory*, 9(1):163–179, 1983.

[LL01] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2nd edition, 2001.

[Mah00] G. D. Mahan. *Many-Particle Physics*. Kluwer Academic/ Plenum, New York, 3rd edition, 2000.

[MS12] T. Maier and H. Siedentop. Stability of impurities with Coulomb potential in graphene with homogeneous magnetic field. *J. Math. Phys.*, 53(9):095207, 2012.

[MGM+11] A. S. Mayorov, R. V. Gorbachev, S. V. Morozov, L. Britnell, R. Jalil, L. A. Ponomarenko, P. Blake, K. S. Novoselov, K. Watanabe, and T. Taniguchi. Micrometer-scale ballistic transport in encapsulated graphene at room temperature. *Nano letters*, 11(6):2396–2399, 2011.

[Meh15] J. Mehringer. On the essential spectrum of two-dimensional Pauli operators with repulsive potentials. *Ann. Henri Poincaré*, online first, DOI 10.1007/s00023-015-0406-0, 2015.

[MS14] J. Mehringer and E. Stockmeyer. Confinement-deconfinement transitions for two-dimensional Dirac particles. *J. Funct. Anal.*, 266(4):2225–2250, 2014.

[MS15] J. Mehringer and E. Stockmeyer. Ballistic dynamics of Dirac particles in electro-magnetic fields. *J. London Math. Soc.*, 92(2):465–482, 2015.

[MS80] K. Miller and B. Simon. Quantum magnetic Hamiltonians with remarkable spectral properties. *Phys. Rev. Lett.*, 44:1706–1707, 1980.

[NGM+04] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, Y. Zhang, S. V. Dubonos, I. V. Grigorieva, and A. A. Firsov. Electric field effect in atomically thin carbon films. *Science*, 306(5696):666–669, 2004.

[NJZ+07] K. S. Novoselov, Z. Jiang, Y. Zhang, S. V. Morozov, H. L. Stormer, U. Zeitler, J. C. Maan, G. S. Boebinger, P. Kim, and A. K. Geim. Room-temperature quantum hall effect in graphene. *Science*, 315(5817):1379, 2007.

[Pei33] R. Peierls. Zur theorie des diamagnetismus von leitungselektronen. *Z. Phys.*, 80(11-12):763–791, 1933.

[PSJWG07] E. Prada, P. San Jose, B. Wunsch, and F. Guinea. Pseudodiffusive magnetotransport in graphene. *Phys. Rev. B*, 75:113407, 2007.

[RS75] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-adjointness.* Academic Press, New York - London, 1975.

[RS78] M. Reed and B. Simon. *Methods of Modern Mathematical Physics IV. Analysis of Operators.* Academic Press, New York, 1978.

[RS06] G. Rozenblum and N. Shirokov. Infiniteness of zero modes for the Pauli operator with singular magnetic field. *J. Funct. Anal.*, 233(1):135–172, 2006.

[SMP+06] M. L. Sadowski, G. Martinez, M. Potemski, C. Berger, and W. A. de Heer. Landau level spectroscopy of ultrathin graphite layers. *Phys. Rev. Lett.*, 97:266405, 2006.

[Sch95] K. M. Schmidt. Dense point spectrum and absolutely continuous spectrum in spherically symmetric Dirac operators. *Forum Math.*, 7(4):459–475, 1995.

[Sch97] K. M. Schmidt. Absolutely continuous spectrum of Dirac systems with potentials infinite at infinity. *Math. Proc. Cambridge Philos. Soc.*, 122(2):377–384, 1997.

[Sch10] K. M. Schmidt. Spectral properties of rotationally symmetric massless Dirac operators. *Lett. Math. Phys.*, 92(3):231–241, 2010.

[SY98] K. M. Schmidt and O. Yamada. Spherically symmetric Dirac operators with variable mass and potentials infinite at infinity. *Publ. Res. Inst. Math. Sci.*, 34(3):211–227, 1998.

[Sch05] F. Schwabl. *Advanced Quantum Mechanics.* Springer-Verlag, Berlin, 3rd edition, 2005.

[SS75] E. Seiler and B. Simon. Bounds in the Yukawa2 quantum field theory: upper bound on the pressure, Hamiltonian bound and linear lower bound. *Comm. Math. Phys.*, 45(2):99–114, 1975.

[Sem84] G. W. Semenoff. Condensed-matter simulation of a three-dimensional anomaly. *Phys. Rev. Lett.*, 53(26):2449–2452, 1984.

[Sim79] B. Simon. *Trace Ideals and their Applications*, volume 35 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York, 1979.

[Sim82] B. Simon. Schrödinger semigroups. *Bull. Amer. Math. Soc. (N.S.)*, 7(3):447–526, 1982.

[Sim90] B. Simon. Absence of ballistic motion. *Comm. Math. Phys.*, 134(1):209–212, 1990.

[Sto92] G. Stolz. Bounded solutions and absolute continuity of Sturm-Liouville operators. *J. Math. Anal. Appl.*, 169(1):210–228, 1992.

[Str90] R. S. Strichartz. Fourier asymptotics of fractal measures. *J. Funct. Anal.*, 89(1):154–187, 1990.

[Suz00] N. Suzuki. Discrete spectrum of electromagnetic Dirac operators. *Proc. Amer. Math. Soc.*, 128(3):819–825, 2000.

[Tes09] G. Teschl. *Mathematical Methods in Quantum Mechanics. With Applications to Schrödinger Operators*, volume 157 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.

[Tha92] B. Thaller. *The Dirac Equation*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.

[Tit61] E. C. Titchmarsh. On the nature of the spectrum in problems of relativistic quantum mechanics. *Quart. J. Math. Oxford Ser. (2)*, 12:227–240, 1961.

[Vog87] V. Vogelsang. Absence of embedded eigenvalues of the Dirac equation for long range potentials. *Analysis*, 7(3-4):259–274, 1987.

[Wal47] P. R. Wallace. The band theory of graphite. *Phys. Rev.*, 71:622–634, 1947.

[Wei00] J. Weidmann. *Lineare Operatoren in Hilberträumen. Teil I: Grundlagen.* Mathematische Leitfäden. B. G. Teubner, Stuttgart, 2000.

[Wei03] J. Weidmann. *Lineare Operatoren in Hilberträumen. Teil II: Anwendungen.* Mathematische Leitfäden. B. G. Teubner, Stuttgart, 2003.

[YK09] A. F. Young and P. Kim. Quantum interference and Klein tunnelling in graphene heterojunctions. *Nature Physics*, 5(3):222–226, 2009.

Acknowledgments

Here I would like to point out those persons, who also contributed to this thesis. First of all, I thank my parents and my three sisters for all the support they gave me throughout the time of my PhD studies. I also have to mention my colleagues and fellows at the Ludwig-Maximilians-Universität München, as well as those at the Ponificia Universidad Catòlica de Chile for the nice fellowship and for finding always the time to answer questions, however they were subject related or not. In particular, I want to point out Verena von Conta, my office mate at the LMU, as well as Ignacio Reyes and Hanne van den Bosch, my campañeros de oficina at the PUC Chile. Further, I want to express my gratitude to Rafael Benguria and Maria Christina Depassier, who put a lot of effort in giving me a pleasant research stay in Santiago de Chile. In addition, I am delighted that the topic of this thesis engendered a lot of interest and feedback on the part of Jean-Marie Barbaroux, Peter Müller, Constanza Rojas-Molina and Amal Taarabt. Let me emphasise that without the PhD-scholarship of the project TR12 of the Deutsche Forschungsgemeinschaft this work would never have been realised. Finally, it is very special to me to thank my PhD-supervisor Edgardo Stockmeyer. His dedication and his passion for our common project impressed me as much as his personality.

Agradecimientos

Acá querría recordar a algunas personas, que también han contribuido a esta tesis. Por primero, quiero agradecer a mis padres y a mis tres hermanas por el apoyo que me otorgaron durante mis estudios doctorales. Asimismo, tengo que mencionar a mis compañeros de la Ludwig-Maximilians-Universität München, así como de la Ponificia Universidad Católica de Chile. Ellos han formado una gran compañía y siempre se tomaron el tiempo para reflexionar sobre mis preguntas, fuesen relacionadas con la investigación o no. En particular, quiero destacar a Verena von Conta, mi compañera de oficína en la LMU, así como a Ignacio Reyes y Hanne van den Bosch, mis compañeros de oficína en la PUC Chile. Además, les doy muchas gracias a Rafael Benguria y Maria Christina Depassier por sus esfuerzos por amenizar mi estadía en Santiago de Chile. Me alegro que el tema de esta tesis haya generado mucho interés y comentarios por parte de de Jean-Marie Barbaroux, Constanza Rojas-Molina, Peter Müller y Amal Taarabt. Permítanme mencionar que este trabajo no hubiese sido posible sin la beca de investigación del proyecto TR12 de la Deutsche Forschungsgemeinschaft. Finalmente es de gran importancia para mí agradecer a mi director de tesis, Edgardo Stockmeyer. Su entusiasmo y su pasión por nuestro proyecto me impresionaron tanto como su personalidad.

Danksagung

An dieser Stelle möchte ich noch an diejenigen Personen erinnern, die auch ihren Teil zu dieser Arbeit beigetragen haben. Zunächst will ich meinem Eltern und meinen drei Schwestern für die Unterstützung danken, die ich während meiner Promotionszeit erfuhr. Ebenso sind meine Kommilitonen der Ludwig-Maximilians-Universität München sowie der Ponificia Universidad Catòlica de Chile zu nennen, die eine tolle Gesellschaft waren und die bei fachlichen, aber auch bei persönlichen Fragen stets die Zeit fanden, um über eine passende Antwort nachzugrübeln. Dabei möchte ich vor allem Verena von Conta, meine Bürokollegin an der LMU, sowie Ignacio Reyes und Hanne van den Bosch, meine compañeros de oficina an der PUC Chile, hervorheben. Ein besonderer Dank gebührt auch Rafael Benguria und Maria Christian Depassier für ihre Anstrengungen, um meinen Forschungsaufenthalt in Santiago de Chile so angenehm wie möglich zu gestalten. Desweiteren bin ich froh, dass die Thematik auf Interesse und Anregungen von Seiten Jean-Marie Barbaroux, Peter Müller, Constanza Rojas-Molina und Amal Taarabt gestoßen ist. Lassen Sie mich noch darauf hinweisen, dass ohne das Forschungsstipendium des Projekts TR12 der Deutschen Forschungsgemeinschaft diese Arbeit nie zustande gekommen wäre. Zu guter Letzt liegt es mir besonders am Herzen, meinem Doktorvater Edgardo Stockmeyer zu danken. Sein Einsatz und seine Passion für dieses Projekt haben mich genau so sehr beeindruckt wie seine Persönlichkeit.

Eidesstattliche Versicherung

(Nach Promotionsordnung vom 12. 7. 2011, §8, Abs. 2 Pkt. 5)

Hiermit erkläre ich an Eides statt, dass die Dissertation von mir selbständig, ohne unerlaubte Beihilfe angefertigt wurde.

Josef Mehringer

Ort, Datum	Unterschrift Doktorand