On Geometric Corrections to Effective Actions of String Theory

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München 2015

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Dissertation an der Fakultät für Physik der Ludwig-Maximilians-Universität München,
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München

2015

Erstgutachter: Prof. Dr. Dieter Lüst

Zweitgutachter: PD Dr. Ralph Blumenhagen

Datum der mündlichen Prüfung: 23.09.2015

Zusammenfassung

Diese Doktorarbeit befasst sich mit dem Studium von geometrischen Korrekturen zu den effektiven Wirkungen von String Theorie. Im speziellen werden Korrekturen zu dreidimensionaler, $\mathcal{N} = 2$ supersymmetrischer und vier dimensionaler, $\mathcal{N} = 1$ supersymmetrischer Supergravitation hergeleitet. Letztere Theorie ist von besonderem Interesse für die Phänomenologie, da es in ihrem Rahmen möglich ist, chirale Teilchenspektren zu etablieren. Die prinzipielle Methodik, die der Herleitung dieser Korrekturen zu den Kopplungen der oben genannten Theorien zu Grunde liegt, ist das Konzept der dimensionalen Reduktion von höher-dimensionalen Theorien auf kompakten Mannigfaltigkeiten, nach drei bzw. vier Dimensionen. Hierbei wird ausgehend von in diesem konkreten Fall, M-Theorie bzw. ihre Niederenergiewirkung, gegeben durch elfdimensionale Supergravitation, durch Reduktion auf dem internen kompakten Raum, bestehend aus einer achtdimensionalen Calabi-Yau Mannigfaltigkeit, eine dreidimensionale, $\mathcal{N} = 2$ Supergravitationstheorie hergeleitet. Deren Kopplungen sind durch geometrische, bzw. topologische Grössen der Calabi-Yau Mannigfaltigkeit bestimmt. Im Rahmen der M/F-Theorie Dualität kann man die dreidimensionale Theorie kontrolliert auf eine vierdimensionale Theorie abbilden. Hierzu verlangt man, dass die Calabi-Yau Mannigfaltigkeit eine elliptische Faser besitzt, d.h. lokal aus dem Produkt eines Torus und einer sechsdimensionalen Kähler Mannigfaltigkeit besteht. In Limes von verschwindendem Torus Volumen erhält man unter T-Dualität eine weitere ausgedehnte Dimension, und dadurch eine vierdimensionale, $\mathcal{N} = 1$ Supergravitationstheorie.

Zusätzlich zu der niedrigsten Ordnung von elfdimensionaler Supergravitation die zwei Ableitungsterme besitzt, haben wir uns dem Studium von durch String Streuamplituden induzierten, und nach elf Dimensionen gehobenen Korrekturen mit acht Ableitungen, im Rahmen der oben geschilderten Prozedur, gewidmet. Ein Beispiel für solche Korrekturen sind Kontraktionen von vier Riemanntensoren.

Um diese Korrekturen konsistent nach drei Dimensionen zu reduzieren, muss man den Hintergrund, im klassischen Fall bestehend aus dem direkten Produktraum aus zwei externen Raumdimensionen und einer Zeitdimension, und einer internen Calabi-Yau Mannigfaltigkeit, abändern. Dies is der erste wichtige Schritt, mit dem wir uns befassen. Hierbei wird der externe Raum mit einer exponentiellen Abhängigkeit, dem sogenannten Warp-Faktor, versehen, der wiederum von dem internen Raum abhängt. Wir finden eine explizite Lösung der elfdimensionalen Bewegungsgleichungen für den metrischen Hintergrund. Des Weiteren schlagen wir korrigierte elfdimensionale Gravitinio Variationen vor, die die supersymmetrie der Lösung andeuten. In einem weiteren Schritt kompaktifizieren wir alle bekannten Korrekturen mit acht Ableitungen, zu der bosonischen elfdimensionalen Supergravitationswirkung, auf diesem Hintergrund, um eine dreidimensionale Theorie zu erhalten. Hierbei stellt sich

das Zusammenspiel von dem Warp-Faktor und den höheren Ableitungstermen als besonders wichtig heraus. Um $\mathcal{N} = 2$ Strukturen in der reduzierten Theorie zu entdecken, vergleichen wir diese mit der kanonischen Form von dreidimensionaler, $\mathcal{N} = 2$ Supergravitation. Im speziellen zeigen wir, dass das Reduktionsresultat kompatibel mit einem vorgeschlagenen Kähler potential und den dazugehörigen komplexen Koordinaten ist. Die Korrekturen zum Kählerpotnetial bestehen aus dem Warpfaktor und einem Term der proportional zur dritten Chern-Form der klassischen Calabi-Yau Mannigfaltigkeit ist. Die komplexen Koordinaten sind als Integrale über Divisoren definiert und enthalten neben einer Warpfaktor Abhängigkeit, ebenfalls einen Teil der mit der Nichtharmonizität der vierten Chern-From korreliert ist. Diese Korrekturen bestehen damit aus geometrischen Grössen der Mannigfaltigkeit.

Im ersten Teil dieser Arbeit haben wir in einem vereinfachten Modell eine Untermenge der bekannten Acht-Ableitungskorrekturen auf dem klassischen Hintergrund der Calabi-Yau Mannigfaltigkeit ohne den Warpfaktor reduziert, und konnten die Korrekturen unter Benutzung der M/F-Theorie Dualität nach vier Dimensionen heben. Die Analyse bei schwacher Stringkopplung zeigt, dass diese von Selbstschnittkurven von $D7$-Branen herrühren.

Abstract

In this thesis we study geometric corrections to the low-energy effective actions of string theory. More concretely, we compute higher-derivative corrections to the couplings of three-dimensional, $\mathcal{N} = 2$ supergravity theories and interpret the induced α'-corrections in $\mathcal{N} = 1$, minimal supergravity theories in four dimensions, in the framework of F-theory. These allow for chiral spectra and are therefore phenomenological relevant. We analyzed higher-derivative corrections to M-theory, accessible through its low-energy effective theory, given by eleven-dimensional supergravity. The next to leading order terms to eleven-dimensional supergravity carry eight-derivatives, and are suppressed by l_M^6 compared to the classical terms, with l_M being the eleven-dimensional Planck-Length - the only scale in eleven dimensions. These corrections are lifted from IIA supergravity corrections, which are derived from string scattering amplitudes.

The common theme of this thesis is to compactify the bosonic sector of the eleven-dimensional supergravity action, including all known eight-derivative corrections, on a supersymmetric background to find a $3d$, $\mathcal{N} = 2$ theory, which then can be lifted to a $4d$, $\mathcal{N} = 1$ theory. This goal is approached in several steps.

In the classical reduction of eleven-dimensional supergravity the metric background is a direct product of the external space, consisting of two space and one time dimension and the internal eight spacelike-dimensional Calabi-Yau manifold. However, when considering higher-derivative corrections the background has to be altered by introducing a dependence of the external space on the warp-factor, which is a function of the internal space. We find an explicit warped background solution to the eleven-dimensional E.O.M.'s including non-vanishing flux. To check the background for its supersymmetry features one would need to consider the eleven-dimensional gravitino variations at this order in l_M. However, these are not known, which leads us to propose higher-order l_M-corrected gravitino variations consistent with our background solution.

As a next step we dimensionally reduce the bosonic sector of the eleven-dimensional supergravity action including all eight-derivative terms on this warped background and analyze the resulting three-dimensional theory. In this context the interplay of the warp-factor and the higher-derivative terms is of crucial importance. To identify the $\mathcal{N} = 2$ properties of the resulting three-dimensional theory obtained by dimensional reduction, we compare it to the canonical from of three-dimensional $\mathcal{N} = 2$ supergravtiy. We conclude that the reduced action is compatible with $\mathcal{N} = 2$ supersymmetry and give a proposal for the Kähler potential and the complex coordinates, which receive l_M^6 corrections. Besides a warp-factor contribution, the Kähler potential receives a correction proportional to the third Chern-

form of the zeroth order internal background, being the Calabi-Yau fourfold. The complex coordinates are defined as divisor-integrals and are corrected by a warp-factor dependent term as well as one related to the non-harmonic part of the fourth Chern-form, of the zeroth order Calabi-Yau manifold. Thus the couplings of the resulting theory receive besides the warp-factor, in particular geometric corrections of order l_M^6.

In the first part of this thesis we study a simplified setup, only considering a subset of the relevant eight-derivative corrections in eleven dimensions. Furthermore, we do compactify on the classical background, consisting of the internal Calabi-Yau fourfold without warping and fluxes, to gain a three-dimensional theory. However, we use the M/F-theory duality to uplift the yielded corrections, which results in corrections to the couplings of the four-dimensional theory. In the weak coupling limit we find that these are sourced by the self-intersection curves of $D7$-branes.

Acknowledgments

First of all, I would like to deeply thank Thomas Grimm for his valuable supervision during my PhD, and for providing me the means to conduct research in the scientifically productive and friendly environment he has created. He has guided me through the first steps in academia and shared invaluable lessons on the way.

Special thanks go to Dieter Lüst, who has generously offered his support as my official supervisor at the Ludwig-Maximilians-Universität and as first referee for the PhD dissertation, while sustaining to encourage us for an academic career and establishing the fruitful exchange of knowledge in between the university and the Max Planck Institute. Furthermore, I would like to thank Ralph Blumenhagen, who has kindly agreed to be my second referee and whose presence at the Max Planck Institute significantly benefits the quality of the conducted research.

I am particularly grateful to my fellow colleagues at the Max Planck Institute: Federico Bonetti, Ioannis Florakis, Andreas Kapfer, Jan Keitel, Severin Lüst, Noppadol Mekareeya, Tom Pugh, Diego Regalado, and Raffaele Savelli, for sharing with me their passion for physics and for perfecting the scientific education with numerous stimulating discussions. Furthermore, I would also like to express my gratitude to all my fellow PhD students at the Max Planck Institute, and in particular to Michael Fuchs, Sebastian Halter and Felix Rennecke, for completing these years with enjoyable moments outside of physics.

Finally, I will always be greatly indebted to all my friends, to my family and to my lovely girlfriend for their support and encouragement.

Contents

CHAPTER I

Introduction

"See now the power of truth; the same experiment which at first glance seemed to show one thing, when more carefully examined, assures us of the contrary." *Galileo Galilei*

Refined experiments teach us that our theoretical models at hand lack completeness. If one wishes to address dark matter, dark energy or inflation there are various theoretical candidates available, but their validity shall be verified only by future experiments. From a theoretical perspective it seems more intriguing to wish for a theory, which presents various answers at once and moreover provides us with an understanding of why nature favors certain structures over others.

Let me start with an assumption - there exists a unified theory of nature - one theory describing all the interactions of particles, their masses, and their effect on space and time and vice versa. Its existence is commonly believed and the quest for unification of the four known forces of nature, the electro-magnetic force and gravity being long ranging in contrast to the weak and strong nuclear force, is an active field of research. String theory being its most promising candidate forces us to rethink the concept of particles as fundamental objects, replacing them by one-dimensional open and closed strings. However, quantum field theory sourced by particles provides an elegant way to describe the quantum structure of nature as long as gravity is weak. The Glashow-Salam-Weinberg model - the renowned Standard Model - is a quantum field theory describing the electro-magnetic the strong and the weak force based on gauge symmetry, and has been tested to hold with remarkable precision up to energies of order $10^3 GeV$. Its most recent success due to experimental verification was the seemingly discovery of the Higgs boson [1]. In quantum field theory, however, it is completely arbitrary to choose the Standard Model gauge group of $SU(3) \times SU(2) \times U(1)$ among the infinite set of renormalizable theories, which is most famously argued for with the anthropic principle. Its main statement condenses to that our observation of nature is originated by the fact that it allows for life, and thus physical constants and theories will lie in the narrow range of habitable Universes. One circumvents the need for an universal argument, which in return implies that one may not be able to see behind this horizon, i.e. nature does not favor any gauge group of the standard model over another. Let us pick up our working assumption - there exists a unified theory of nature. String theory manages to remove our incapabilities when it comes to quantum gravity but also might serve to explain why certain laws of nature are favored over others. It is highly constraint by consistency conditions, e.g. engineering a particular gauge group like $SU(3) \times SU(2) \times U(1)$ comes at a price. But this prize as hard to pay as it might be, may shed some light on the question why certain theories are preferable over others by looking at various string vacua, each representing a different Universe with different laws of nature. Very remarkably universality schemes emerge from these studies, e.g. axion decay constants seem to be constraint to be smaller than the Planck mass $M_p = \sqrt{\hbar c/G} \approx 1.2 \cdot 10^{19}\ GeV/c^2$, [2, 3].

Due to our current incapability to explain experimental data [4], one might hope that string theory provides natural theoretical models, which simply emerge from a subsector of the theory. Only 5% of the total energy content of the Universe are accounted for by known particles, the remaining part consists of 27% dark matter and 68% so called dark energy. The standard model of cosmology also referred to as the ΛCDM model, successfully describes observation on macroscopic lengths scales, providing a coherent description of the Universe from the Big-Bang to the present day, by requiring only six parameters to be fixed. One necessary ingredient of the ΛCDM model is a novel kind of matter, which mainly couples to the ordinary particles gravitationally, referred to as dark matter. Moreover, to explain the observation of a positive cosmological constant the introduction of dark energy is required, which can be interpreted as the vacuum energy. However, no microscopic description for these two constituents is provided. Many theoretical candidates for dark matter particles have been proposed, while the quest for a microscopic characterization of dark energy remains more mysterious - both constitute interesting open problems of modern physics, see e.g. [5]. Besides these fundamental obstacles certain parameters of the ΛCDM model and the Standard Model of particle physics need to be fine-tuned, which is considered unnatural. In this category one desires to resolve the hirachy and the CP problem, and moreover the flatness problem in cosmology, which most prominently might be resolved by Inflation, see e.g. [6].

In particular string theory is obligated to reproduce the Standard Model of particle physics and of cosmology, at the same time, which has not been accomplished so far. Note that these two theories span a hirachy of length scales from particle physics at $\sim 10^{-19}m$ to the Hubble radius $\sim 10^{26}m$. Since string theory provides a description of quantum gravity the span of hierarchies becomes even larger reaching down to the Planck length, $l_P \sim 10^{-35}m$. To describe physics at these distant length scales within string theory one considers different limits of the theory. To connect string theory to the well tested field theoretical models one takes the limit of vanishing string length $l_s \to 0$, which results in ten-dimensional supergravity theories, one for each of the five unique superstring theories: type I, Het $SO(32)$, Het $E_8 \times E_8$, type IIA and IIB. Type IIA supergravity in the limit of strong string coupling $g_s \to \infty$ is described by a dual theory, given by unique eleven-dimensional supergravity, the low-energy effective action of M-theory, subordinate to solely the Planck length l_M, which is the only scale in eleven dimensions. Although a microscopic formulation of M-theory is still missing plenty of indirect evidence hints towards its existence. One can study higher-order l_s, or in more common notation $\alpha' = l_s^2$ corrections to the various supergravity theories in ten-dimensions, given by higher-derivative terms in fields of the supergravity multiplets, hence representing an imprint of the finiteness of the string length on the field theory side. One can lift the α'-corrections of IIA supergravity to M-theory[1], where they become higher-order l_M corrections.

The main focus of this work is the dimensional reduction of M-theory including eight-derivative corrections of order l_M^6, on an eight-dimensional internal manifold to gain a $3d$, $\mathcal{N} = 2$ supergravity theory. The higher-derivative corrections in eleven dimensions naturally give rise to corrections of the couplings of the two derivative $3d$, $\mathcal{N} = 2$ supergravity theory, but also to higher external derivative terms. However, we will discuss latter only marginally. Such reductions have not been performed consistently before and it is a novel and interesting endeavor of conceptional as well as phenomenological interest. We focus on the vectors and the real scalar Kähler deformations, which form a vector multiplet in $3d$. The main phenomenological motivation to discuss these reductions of M-theory is the so called M/F-theory duality, which allows to lift the $3d$, $\mathcal{N} = 2$ theory to a $4d$, $\mathcal{N} = 1$ supergravity theory, that is of superior interest since it can incorporate chiral spectra. Of special interest are α'-corrections to the Kähler potential, which unlike the super potential in a $3d$, $\mathcal{N} = 2$ and a $4d$, $\mathcal{N} = 1$ theory can receive perturbative corrections. It is therefore particularly intriguing to answer the question if in the light of the M/F-theory duality the higher-derivative corrections give rise to novel α'-corrections to the Kähler potential and complex coordinates in $3d$, and thus eventually lift to corrections in the $4d$ theory, which will constitute the common theme of this work.

Let us address a particular difficulty when building phenomenology relevant string models. Dimensionally reducing a higher-dimensional theory, certain deformations of the internal geometry appear as massless scalars in the effective lower-dimensional field theory, see section 2.4. These scalars are not observed in nature, hence need to gain a VEV via some mechanism, which goes under the name of moduli stabilization. In [7] one achieves this with the help of a α'^3 correction to the Kähler potential [8].

This thesis starts by illustrating introductory material focusing on certain concepts particularly related to the following work. The common theme visible throughout this chapter is the notion of an effective field theory and how it arises in the context of the limit of vanishing string length. In section 1 the concept of a string is introduced and it is argued for its effective field theory action, for the bosonic as well as for the supersymmetric string, focusing on type IIB and type IIA supergravity. In section 2 we then discuss how to derive higher-derivative corrections to the effective type II supergravity theories from string scattering amplitudes. Furthermore, we introduce the concept of dimensional reduction. Section 5 intends to give a short introduction to F-theory, which we define via M-theory.

In the second chapter we look at specific eight-derivative corrections at order l_M^6 to the eleven-dimensional supergravity action. In particular we consider corrections of the schematic form $\hat{R}^4$ and $\hat{G}^2\hat{R}^3$, with $\hat{R}$ the Riemann tensors and $\hat{G}$ the four-form field strength, which upon dimensional

[1]We commonly refer to eleven-dimensional supergravity as M-theory in this work.

reduction on a Calabi-Yau fourfold $M_3 \times Y_4$ and the F-theory lift to $4d$, give α'^2-corrections to the Kähler potential and the Kähler coordinates. However, the Kähler potential expressed as a function of the Kähler coordinates is of the same analytic form as in the classical case, thus these corrections do not alter the $4d$ physics, which is equivalently shown in $3d$. This is to be considered a toy model since we do not consider all relevant higher-derivative correction and we compactly on a non-supersymmetric background.

In the third and last chapter, we approach the main challenge of a warped reduction including the full set of bosonic eight-derivative corrections to the eleven-dimensional supergravity action. Taking into account higher-derivative corrections, the space $M_3 \times Y_4$ is no longer a supersymmetric background, but one needs to consider a warped metric and turn on background fluxes on the internal space. In section 6 we take the first step by proposing a supersymmetric background solution for the metric in the presence of non-vanishing flux. Since the $11d$ gravitino variations are not known at order l_M^6, we give necessary conditions for the background to be supersymmetric and propose higher order l_M-corrections to the gravitino variations. In section 7 we reduce the known set of eight-derivative corrections of schematic form $\hat{R}^4$, $\hat{G}^2\hat{R}^3$ and $(\hat{\nabla}\hat{G})^2\hat{R}^2$ on the warped background including fluxes, derived in section 6, and compute the resulting three-dimensional two derivative action. We consider solely Kähler moduli deformations of the metric parametrized by harmonic $(1,1)$-forms, and do not allow for complex structure deformations. Furthermore we premise that $h^{2,1} = 0$. We vary the higher-derivative terms according to the Kähler deformations of the metric, which embodies the first reduction of this kind. An indirect proof of supersymmetry of the background derived in section 6, is the observation that the derived $3d$ couplings upon reduction give a $\mathcal{N} = 2$ supergravity theory. In particular the couplings of the real scalars and of the vectors are given in terms of a Kähler metric originating from a Kähler potential, which will be discussed in the dual picture with propagating complex multiplets, referred to as complex coordinates. It is the focus of section 8 to suggest a proposal for the Kähler potential and complex coordinates consistent with the reduction results.

1 A short story on strings

The following introduction to string theory indents to emphasize certain beautiful aspects, which are worth being discussed in more detail. This is of cost of providing a more complete story.

1.1 The paradigm string

By incorporating gravity in a consistent quantum theory the success story of string theory suggests that one might need to abandon the idea of particles being the fundamental objects of nature. One may be bound to introduce strings instead, one-dimensional lines, open and closed to loops. Deriving a consistent theory of strings leads us to consider also other fundamental objects, referred to as branes, extended hypersurfaces of various dimensions where open strings are bound to end. At the same time starting off by first introducing branes one is led to incorporate strings - it is intriguing that these two notions imply each other.

String theory provides a framework in which all the four forces can be discussed at the quantum level. One may argue with the following intuitive picture. The natural length scale of string theory is the string length l_s. The extended nature of strings smears out the interaction vertices in contrast to point particle interactions, which intuitively provides a natural momentum cut-off since length scales smaller than l_s cannot be probed. Although string theory's ultraviolet finiteness is not rigorously proven it is has been made highly plausible. Explicit results of super-string perturbation theory show finiteness up to two loops, but further observation can be made for higher loops, which leads to the conjecture, see e.g. [9].

Before we continue let us pause for a moment and comment on the underlying assumed paradigm. We start with a classical object the particle or the string and think of the quantum theory as the path integral of the classical action. Thus averaging the classical action over all classical paths in which the classical object may travel. We know that our Universe is fundamentally quantum and classical physics only appears due to a separation of scales. It is quite remarkable that starting with a classical idea one is able to describe intrinsically non classical quantum behaviors. One should be always aware that this kind of mindset might be holding us back. At some point we may be led to introduce new notions to better capture the nature of quantum physics. String theory provides a window into such new lands - indeed we know of the existence of theories which we cannot be described via the Lagrangian principles [10].

1.1.1 The classical relativistic string

Before we introduce the action of the bosonic string let us review some aspects of the relativistic point particle, which will provide a natural choice for the string action. Note that it is desirable to have a limit of the theory in which the string effectively looks like a point particle to be able to compare it to field theory. This is the case when the length of the string shrinks to zero $\alpha' \to 0$.

The relativistic free particle obeys the well known energy and momentum relations

$$E = \sqrt{m^2 + \vec{p}^2} \quad , \quad \vec{p} = \frac{m\dot{\vec{x}}}{1 - \dot{\vec{x}}^2} \quad . \tag{1.1}$$

To find a covariant expression for the action we first choose coordinates $X^\mu = (t, \vec{x})$ in d-dimensional Minkowski space-time $\mathbb{R}^{d-1,1}$, with signature $\eta_{\mu\nu} = diag(-1, 1, \ldots, 1)_{\mu\nu}$. We can reveal (1.1) from the action

$$S = -m \int d\tau \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} \ , \tag{1.2}$$

by choosing $\tau = X^0(\tau) \equiv t$, with τ being the eigen-time of the particle. This is always possible since (1.2) is invariant under transformations of $\tau \to \tau'(\tau)$. $X^\mu(\tau)$ describes the particle trajectory in Minkowski space-time, but via leaving the covariant picture and restricting ourselves to $\tau = X^0(\tau) \equiv t$, one makes the interesting encounter that (1.2) is nothing else but the length of the particles trajectory in the Euclidian space $\mathbb{R}^{d-1}$. For a more detailed introduction to string theory, see e.g. [11, 12, 13, 14, 15, 16, 17].

This observation is the guideline to introduce the action for the classical relativistic string - its action being the area of its two-dimensional trajectory surface, the so called world-sheet. The string is described by a map $X^\mu(\tau,\sigma)$ from the world-sheet coordinates $\sigma^i = (\tau,\sigma) =: \sigma$[2] to the space-time, called target space with $\tau \in \mathbb{R}$, and $\sigma \in [0,2\pi)$ or $\sigma \in [0,\pi]$ for the closed and open string, respectively. In the following the target space will constitute a d-dimensional Minkowski space. The area of the world-sheet is given by the pullback of the Minkowski metric onto the world-sheet

$$S = -T \int d^2\sigma \sqrt{-\det X^*\eta} \quad , \quad (X^*\eta)_{ij} = \frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial X^\nu}{\partial \sigma^j}\eta_{\mu\nu} \quad , \tag{1.3}$$

well known from basic differential geometry referred to as the Nambu-Goto action in physics literature. The pre-factor T is the string tension carrying the dimensions energy or mass per unit length. This is easily seen by a dimensional analysis recalling that the area of the world-sheet carries dimension space×time. Since $[T] = 2$ and $[X] = -1$ in unit-mass dimensions one can associate a one over length squared with the string tension, which is naturally written in terms of the only length scale involved being the string length l_s as

$$T = \frac{1}{2\pi l_s^2} = \frac{1}{2\pi\alpha'} \quad , \tag{1.4}$$

where we have introduced the common notation $\alpha' = l_s^2$. Naively one would expect the string length to be of the order of the Planck length thus $1/\sqrt{\alpha'} \approx M_P$, however, it can be shown that it could be as low as the TeV scale and thus in reach for collider experiments, see e.g. [18, 19]. The square root in (1.3) gives an obstruction to the path integral quantization, which led to the discovery of the Polyakov action,

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}g^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} \quad , \tag{1.5}$$

with $g \equiv \det g$. This gives the same E.O.M.'s as (1.3), but with the difference that the world-sheet metric g_{ij} is now a dynamical degree of freedom. This action is manifestly symmetric under Poincare transformations $X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu$ constituting a global symmetry in the world-sheet perspective. Furthermore, it obeys reparametrization invariance $\sigma^i \to \sigma'(\sigma)$, which is a non-physical gauge invariance of the action, and it is invariant under Weyl transformations

$$g_{ij} \to \Phi^2(\sigma) g_{ij}. \tag{1.6}$$

One can use the reparametrization invariance to gauge-fix the metric on the world-sheet to be flat $g_{ij} = \eta_{ij}$, thus the Polyakov action takes the beautifully simple form

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \eta^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} \quad . \tag{1.7}$$

For closed strings the embedding map needs to be periodic

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi) \quad , \tag{1.8}$$

[2] This is an common abuse of notation and it will be clear form the context which coordinates it refers to.

but is not subject to any further constrains. One the other hand, the open string endpoints need to be constraint by boundary conditions since they are different in nature from the rest of the open string, which is locally described via the same action as the closed string (1.7). It is easy to see that by varying (1.7), in the open string case one picks up boundary contributions, which need to vanish and thus give rise to the constraint $\partial_i X^\mu \delta X_\mu = 0$ at the endpoints $\sigma = 0,\ \pi$. This condition can either be fulfilled by Dirichlet $\delta X^\mu = 0$ or Neumann boundary conditions $\partial_i X^\mu = 0$ at $\sigma = 0,\ \pi$, respectively. The Dirichlet boundary condition restricts the endpoints of the string to fix positions in the target space whereas Neumann boundary conditions allow the string endpoints to move freely with the speed of light. One can allow the string to move freely in $p+1$ direction obeying Neumann conditions while fixing it via Dirichlet conditions in the other $d-p-1$ directions -this $p+1$-dimensional hypersurface is called *Dp*-brane. This boundary conditions seem ad hoc and historically it took a while before the string community realized that branes are natural dynamical objects equally fundamental as the string itself [20]. Let us return to the closed string for the reminder of this section. We can analyze the E.O.M.'s of (1.7) or (1.5) respectively. The E.0.M. for g_{ij} given by

$$T_{ij} = 0 \ , \qquad\qquad T_{ij} = -\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\delta(\mathcal{L}\sqrt{-g})}{\delta g^{ij}} \ , \tag{1.9}$$

expressed via the strew-energy tensor T_{ij}[3]. From (1.7) we find for X^μ the free wave equation

$$\partial_i \partial^i X^\mu = 0 \ . \tag{1.10}$$

The relativistic string obeys bizarre kinematic features from a classical point of view, which we analyze by choosing $x^0 = r_0\tau \equiv t$ and discuss the strings motion in $d-1$ - dimensional Euclidean space. The free wave equation (1.10) and the two constraints following (1.9) in this new coordinates result in

$$\ddot{\vec{x}} - \vec{x}'' = 0 \ , \quad \dot{\vec{x}}\vec{x}' = 0 \ , \ \text{ and } \ \dot{\vec{x}}^2 + \vec{x}'^2 = r_0^2 \ . \tag{1.11}$$

The second equation in (1.11) tells us that the motion of the string is perpendicular to the string itself. In particular this implies that if the center of mass of the string is moving any further oscillations of the string have to be perpendicular to the direction of this movement, which is easily seen since the conditions (1.11) are valid for any point (τ, σ), in other words the moving relativistic string is infinitely stiff in the direction of the center of mass velocity. Let us now choose the string such that the perpendicular directions to the movement are in the $x_1 - x_2$ - plane. Note that the string needs to be a circle if it shall have any radial contraction or expansion in the $x_1 - x_2$ directions, due to the second condition (1.11). One can now use polar coordinates to solve (1.11), which results in

$$r(t) = c_1 \sin t + c_2 \cos t \ , \qquad\qquad c_1^2 + c_2^2 = r_0^2 \ , \tag{1.12}$$

describing an oscillating circular string with maximal radial expansion r_0, thus the notation. Note that the string contracts to zero size thus to a point before expanding again until its tension stops the expansion, and the cyclic process is reinitiated.

[3] With $\mathcal{L}$ the Lagrangian density as usually $S = \int d\sigma^2 \sqrt{-g}\mathcal{L}$.

1.1.2 Strings and Einstein gravity

Einstein gravity, more commonly known as the theory of general relativity is a "prediction" of string theory. Let us shortly sketch the idea before going into more detail. By naively quantizing the relativistic closed string action (1.7) one derives its massless spectrum and furthermore infers that Lorentz invariance of these states imply the target space to be 26-dimensional, referred to as the critical dimension. Furthermore, by demanding conformal invariance of the quantum theory at one loop, among other effective terms one yields the Einstein-Hilbert term of general relativity. Thus natural symmetry arguments predict Einstein gravity. This is also the first time we will encounter an effective field theory action arising from string theory.

The quantization of the bosonic closed string is a straightforward procedure, which we will not address in detail. The most convenient way of naively quantizing the bosonic closed string is the light-cone quantization. The main point is to Fourier expand the classical X^μ in the periodic coordinate σ. One chooses new coordinates $\sigma^\pm = \tau \pm \sigma$ on the world-sheet, which is a symmetry of (1.7) since it can be undone by a Weyl transformation. The physical degrees of freedom are solutions to the free wave equation and the two constraints on X^μ following from (1.9), which can be solved by a Fourier Ansatz

$$X^\mu = X^\mu_L + X^\mu_R \quad \text{with} \quad X^\mu_L \propto \sum_{n>0} \frac{1}{n} a_n^{L\mu} e^{-in\sigma^+} \quad \text{, and} \quad X^\mu_R \propto \sum_{n>0} \frac{1}{n} a_n^{R\mu} e^{-in\sigma^-} \quad , \tag{1.13}$$

decomposed in left and right-moving modes on the world-sheet, as easily seen by the sign of $\sigma^\pm$. Furthermore, one breaks manifest space-time Lorentz symmetry by making the string propagate in a specific direction $X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^1\right)$, which leaves us with $i, j = 1, .., d-2$ transverse oscillation of the string. We will have to restore Lorentz invariance later on. We can now naively quantize the string by promoting X^μ and its related conjugate momentum $\Pi_\mu = \frac{1}{2\pi\alpha'}\dot{X}_\mu$, to operator valued fields and enforce equal-time commutation relations upon them

$$\begin{aligned} \left[X^\mu(\sigma), \Pi_\nu(\sigma')\right]|_{\tau=\tau'} = &\quad i\delta(\sigma-\sigma')\delta^\mu{}_\nu \\ \left[X^\mu(\sigma), X^\nu(\sigma')\right]|_{\tau=\tau'} = &\quad \left[\Pi^\mu(\sigma), \Pi^\nu(\sigma')\right]|_{\tau=\tau'} = 0 \quad . \end{aligned} \tag{1.14}$$

This implies that also the Fourier coefficients have to be promoted to operators. Focusing on the $d-2$ transverse left and right-moving modes, (1.14) gives rise to their commutation relations of generation and annihilation operators as

$$[a_n^{L\,i}, a_m^{L\,j}] = [a_n^{R\,i}, a_m^{R\,j}] = n\,\delta^{ij}\,\delta_{n,-m} \quad \text{and} \quad [a_n^{L\,i}, a_m^{R\,j}] = 0 \quad . \tag{1.15}$$

The classical equation of motions (1.9) and (1.10) translate into operator equations in creation $a_{-n}^{R/L i}$ and annihilation operators $a_n^{R/L\,i}$ with $n > 0$. One of them being the relativistic rest mass $M^2 = -p^\mu p_\mu$, which translates after normal ordering of operators in

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{d-2}{24}\right) \quad , \tag{1.16}$$

with N being the number-operator of the excitations of the left or the right-moving modes, respectively, which for consistency need to have the same excitation number.[4] The first observation is that the ground state for $N \equiv 0$ has a negative mass squared. This state is referred to as tachyonic and indicates that we have expanded the theory at an inappropriate point, stabilizing the tachyon to negative mass squared, which motivates the introduction of supersymmetry later in section 1.2. Let us first look at the excited state $a^{Ri}_{-1} a^{Lj}_{-1} |0\rangle$, giving $(d-2)^2$ degrees of freedom $i, j = 1, \ldots, d-2$. These degrees of freedom have to transform under the full Lorentz group $SO(1, d-1)$ for the quantum theory to be Lorentz invariant. Considering Wigner's classification of representations of the Poincare group one finds that this is only possible if the first excited state is massless, thus one infers from (1.16) that $d = 26$ giving the critical dimension of the bosonic string. Thus the first excited state sits in a $24 \otimes 24$ representation of the symmetry group $SO(24)$, which can be shown to decompose into a traceless symmetric $\oplus$ an antisymmetric $\oplus$ a singlet irreducible representation. It is natural to assign fields according these degrees of freedom, which we denote by $G_{\mu\nu}(X), B_{\mu\nu}(X)$ and $\Phi(X)$, respectively. $G_{\mu\nu}$ is a massless spin 2 particle. According to an argument of Feynman and Weinberg, any theory of interacting massless spin 2 particles must be Einstein gravity, [5] argued for as following. One can show that any theory of massless spin 2 particles must have a gauge symmetry to remove negative norm states from the spectrum. To ensure that this still works when interactions are taken into account the theory must be diffeomorphism invariant, which is strong enough such that the claim follows. $B_{\mu\nu}$ is the so called Kalb-Ramond field mathematically a 2-form field and the scalar field Φ is the dilaton. Note that the string gives rise to the massless spectrum (B, G, Φ) at the microscopic level. Similar to electrodynamics where one considers charges moving in background fields, one is not concerned about the microscopic generation of these background fields due to other charged particles. Analogously one may write down a theory of the strings propagating in the macroscopic background fields G, B, Φ. To argue in which limit it is justified to consider a string propagating in a classical background, let us assume we knew how to build the macroscopic background fields in terms of microscopic states $G(X), B(X), \Phi(X)$, which gives the non-linear sigma model

$$S_\sigma = \frac{1}{4\pi\alpha'} \int d^2\sigma \big(G_{\mu\nu}(X) g^{ij} \partial_i X^\mu \partial_j X^\nu + i B_{\mu\nu}(X) \epsilon^{ij} \partial_i X^\mu \partial_j X^\nu + \alpha' \Phi(X) R(\sigma) \big) \ , \tag{1.17}$$

with $R(\sigma)$ being the Ricci-scalar of the world-sheet. The couplings to the metric $G_{\mu\nu}$ and the 2-form field $B_{\mu\nu}$ retain reparametrization and Weyl invariance, whereas the coupling to the dilaton superficially breaks Weyl invariance at tree-level. Note that it comes with an additional power of α'. Indeed at one-loop level Weyl invariance is restored by cancelation of the anomalous Weyl contributions generated from the couplings $G_{\mu\nu}, B_{\mu\nu}$. Let us consider a string fluctuating around a point in a classical background metric solution $X^\mu \to \xi^\mu + \sqrt{\alpha'} X^\mu(\sigma)$. Furthermore, we assume we knew the fully back-reacted background of this string sitting at ξ^μ and Taylor expand it as

$$G_{\mu\nu} = \alpha' \big(G_{\mu\nu}|_{\xi^\mu} + \sqrt{\alpha'} (G^{(1)}_{\mu\nu})_{\rho_1}|_{\xi^\mu} X^{\rho_1} + \frac{\alpha'}{2} (G^{(2)}_{\mu\nu})_{\rho_1\rho_2}|_{\xi^\mu} X^{\rho_1} X^{\rho_2} + \ldots \big) \ , \tag{1.18}$$

[4] We have used the so called level matching constraint - in this form telling us that $N = N^L = N^R$, simply understood by the fact that the invariant mass can be described by left and right-moving modes alike. Note that thus anything else but equality of these operators is physically excluded.

[5] Possible including higher-derivative terms.

which gives rise to interaction terms with the formal couplings $(G^{(i)}_{\mu\nu})_{\rho_1\ldots\rho_i} = \frac{\partial}{\partial X^{\rho_1}} \cdots \frac{\partial}{\partial X^{\rho_i}} G_{\mu\nu}(X) \approx \frac{\partial}{\partial x^{\rho_1}} \cdots \frac{\partial}{\partial x^{\rho_i}} G_{\mu\nu}(x)$. The approximation is valid in the limit where the string doesn't see the ripples in space-time. Thus we note that (1.18) is a good pertubative approach if the space-time metric only varies on scales much greater that the string scale $\sqrt{\alpha'}$. An intuitive picture is provided by a locally curved space of radius r_l and a string propagating in it, which must have a length l_s that is much smaller than $\frac{1}{r_l}$.

Since we have now seen that the non-linear sigma model possess an expansion in α' let us make a short a side in loop corrections arising from (1.17). Analogously to the quantization of field theory one can compute stringy quantum-loop corrections to the tree-level result given by the classical solution of (1.17). Note that, since corrections in α' measure the extendedness of the string, they are no loop corrections in a string scattering process in the usual sense, which is instead given by the so called genus expansion. Analogously to field theory loops we can write a sum over world-sheet topologies, see figure I.1. Let us start from (1.17) by rewriting the dilaton term as the variation around its vacuum

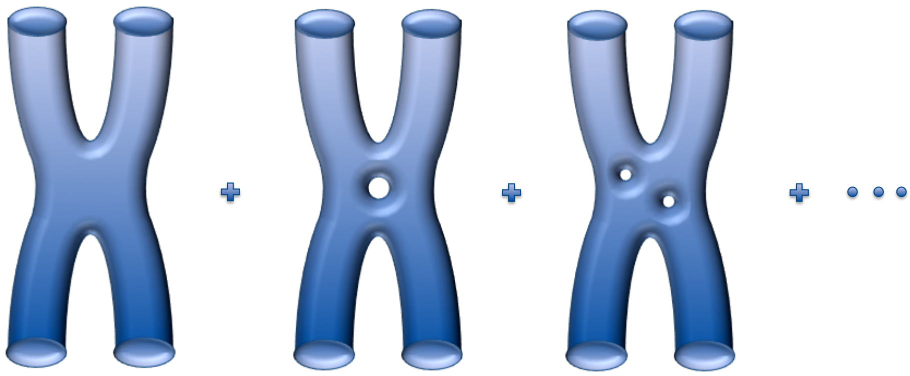


Figure I.1: Pictorial Escheresque representation of the sum over world sheet topologies.

expectation value $\Phi \to \langle\Phi\rangle + \Phi'$, and furthermore apply the Gauss-Bonnet theorem of differential geometry, which states that the integral of the Ricci scalar over a Riemannian surface - in this case the world-sheet - is proportional to its Euler-characteristic $\chi_g = 2 - 2g$ [6], with g the so called genus of the surface. The sphere has genus zero, the torus genus one, and so on counting the number of holes in a perforated donut. Thus we find that

$$\int \alpha' \Phi(X) R(\sigma) \to \alpha' \langle\Phi\rangle\; \chi_g + \alpha' \int \Phi(X) R(\sigma) \quad . \tag{1.19}$$

To quantize this expression we proceed by evaluating its path integral

$$\int \mathcal{D}X\mathcal{D}g e^{-\langle\Phi\rangle\chi_g - \int S_\sigma[X,g,G,B,\Phi]} = \sum_{g\in\mathbb{N}_+} e^{-\langle\Phi\rangle\,\chi_g} \int \mathcal{D}X\mathcal{D}g e^{-\int S_\sigma[X,g,G,B,\Phi]} \quad , \tag{1.20}$$

where we have used that the path integral of χ_g simply becomes the sum over all different topologies of the world-sheet. From (1.20) one infers that the string coupling constant is given by $g_s = e^{\langle\Phi\rangle}$. This

[6] $\int \sqrt{-g} d\sigma^2 R = 4\pi\,\chi_g$.

is very beautiful, the string couples to itself via the vacuum expectation value of a field it sources. The incoming and outgoing states necessary to compute string scattering amplitudes are represented by so called vertex operators $V_i(\Lambda_i, p_i)$, corresponding to the string states Λ_i, respectively, and can be computed by using methods of conformal field theory. These are inserted as sources in the path integral resulting in

$$\mathcal{A}(\Lambda_1,\ldots,\Lambda_n,p_1,\ldots,p_n) = \sum_{g\in\mathbb{N}_+} {g_s}^{-\chi_g}\frac{1}{V}\int \mathcal{D}X\mathcal{D}g e^{-\int S_\sigma[X,g,G,B,\Phi]}\prod_{i=1}^{n} V_{\Lambda_i,p_i} \quad . \tag{1.21}$$

Where we have divided out physical equivalent gauge-orbits via the pre-factor $1/V$.[7] The sum over the world-sheet topologies can now in the light of (1.21), be viewed in a different way, see figure I.2.

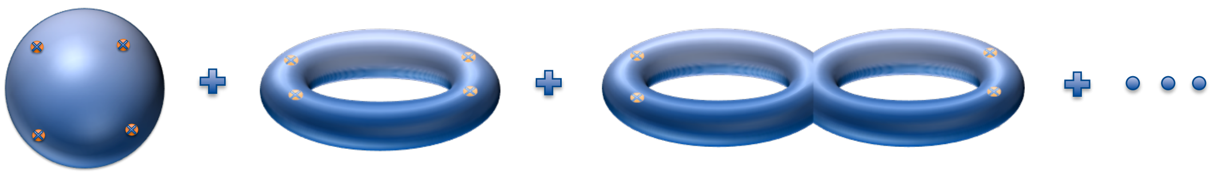


Figure I.2: Pictorial representation of the genus expansion with four vertex operator insertions representing the external states.

This concludes the aside on the g_s expansion, and we continue by focusing once more on the α' expansion of (1.17). The action (1.17) is a two-dimensional theory with $d = 26$ bosonic fields interacting via couplings given by the space-time fields G, B, Φ.[8] Thus alternatively to the interpretation of the 26-dimensional space-time the string propagates in, one can view it as a two-dimensional theory with conformal symmetry. Reversely one treats (1.21) as the S-matrix of an 26-dimensional theory, although one computes correlation functions among 26 bosonic fields living in two dimensions, subordinate to various interactions. Let us next consider the field theory in $d = 26$ dimensions, arising from (1.17) and (1.21) in the so called field theory limit $\alpha' \to 0$. The resulting field theory knows only little about the extended nature of the string, however needs to reproduce the same S-Matrix elements as (1.21) in the limit $\alpha' \to 0$. This implies that the theory is only accurate at low energies where the string can effectively be treated as a point particle. Generically, one can construct this theory via the matching of scattering amplitudes, as we comment on in section 2.2. Here we will take a somewhat more elegant approach by using a field theory symmetry argument. As mentioned previously we desire the quantum theory of (1.17) to obey Weyl invariance, which needs to be checked at one-loop level, done by computing the one-loop in α' counter-terms of he various couplings. Note that one-loop in α' constitutes an ordinary field theory loop rather than the genus expansion in g_s. To understand, which of the appearing counter-terms may violate Weyl invariance let us compute the trace of the energy momentum tensor, that vanishes ${T_\mu}^\mu = 0$, if the action posses Weyl invariance.

Loop α'-corrections can violate Weyl invariance. Applying dimensional regularization $d \to d + \epsilon$,

[7] Specifying inequivalent gauge-orbits is a nontrivial task and we refer the interested reader to [15].

[8] Note that we can choose the world-sheet metric such that $g_{ij} = \eta_{ij}$.

at one-loop level one encounters counter-terms, which generically alter the action by renormalisation of the couplings and the fields, introducing divergences in ϵ. By performing the Weyl rescaling of the resulting action by $\eta_{ij} \to e^{\lambda}\eta_{ij}$, a factor $e^{2\lambda\epsilon}$ arises since the Weyl rescaling from the inverse metric and its determinant do not cancel identically in two dimensions. However, one should recover Weyl invariance in the limit $\epsilon \to 0$. Due to the counter-terms added to (1.17) one encounters new terms which do not scale with ϵ at all, e.g. combining the linear term from the expansion of $e^{2\lambda\epsilon}$ and the inverse power in the counter-terms. This hints towards the fact that the counter-terms violate Weyl invariance and one has to impose equations on these new couplings to restore it. Computing the one-loop corrected action starting from (1.17) one finds according to [21], for the trace of the stress-energy tensor that

$$T_i{}^i = -\frac{1}{2\alpha'}\beta^G_{\mu\nu}\partial_i X^\mu \partial^i X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{ij}\partial_i X^\mu \partial_j X^\nu - \frac{1}{2\alpha'}\beta^\Phi R(\sigma) \quad , \tag{1.22}$$

with

$$\begin{aligned} \beta^G_{\mu\nu} &= \alpha' R_{\mu\nu} - \frac{\alpha'}{4}H_{\mu\rho\gamma}H_\nu{}^{\rho\gamma} + 2\alpha'\nabla_\mu\nabla_\nu\Phi \quad , \\ \beta^B_{\mu\nu} &= -\frac{\alpha'}{2}\nabla^\gamma H_{\gamma\mu\nu} + \alpha'\Phi\nabla^\gamma H_{\gamma\mu\nu} \quad , \\ \beta^\Phi &= -\frac{\alpha'}{2}\Delta\Phi + \alpha'\nabla_\mu\Phi\nabla^\mu\Phi - \frac{\alpha'}{24}H_{\mu\nu\rho}H^{\mu\nu\rho} \quad , \end{aligned} \tag{1.23}$$

with $H = dB$, the field strength of the Kalb-Ramond field. To restore Weyl invariance at one-loop α' quantum-level one needs to demand that

$$\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0 \quad . \tag{1.24}$$

This is very intriguing, the consistency condition subjects the classical fields $G_{\mu\nu}, B_{\mu\nu}, \phi$ to the differential equations (1.24). These can be then considered as E.O.M.'s for the fields $G_{\mu\nu}, B_{\mu\nu}, \phi$, and one can hope to find an action which originates them. We will argue for this action after a short aside, which intends to convey the idea of how the previous result was derived, in particular how the Ricci tensor arose in (1.23) . Since the derivation of (1.22) is involved, we will briefly discuss the case of the string coupling to $G_{\mu\nu}$ only. To extract the explicit interaction terms we expand (1.17) around a classical background (1.18) and compute the couplings $G^{(1)}_{\mu\nu,\rho_1}, G^{(2)}_{\mu\nu,\rho_1\rho_2}$. Since this expansion is around a point ξ we can choose normal coordinates, see e.g. A.6, to simplify the computation. One finds that a single partial derivative of the metric vanishes, thus $G^{(1)}_{\mu\nu,\rho_1} = 0$. To compute $G^{(2)}$, we express the partial derivatives on the metric in terms of Christoffel symbols, which in normal coordinates yields

$$G^{(2)}_{\mu\nu,\rho_1\rho_2} = \frac{\partial}{\partial X^{\rho_1}}\frac{\partial}{\partial X^{\rho_2}}G_{\mu\nu}(X)|_\xi \approx \frac{\partial}{\partial x^{\rho_1}}\frac{\partial}{\partial x^{\rho_2}}G_{\mu\nu}(X)|_\xi = 2\,\partial_{\rho_1}\Gamma_{(\mu\nu)\rho_2} \quad . \tag{1.25}$$

One expresses the partial derivative of the Christoffel symbols in terms of Riemann tensors (A.39), such that $G^{(2)}_{\mu\nu,\rho_1\rho_2} = -2R_{\mu(\rho1|\nu|\rho2)}$. Thus we find the action at his order in the α' expansion to be

$$S_\sigma = \frac{1}{4\pi\alpha'}\int d^2\sigma\big(G_{\mu\nu}\eta^{ij}\partial_i X^\mu\partial_j X^\nu - \alpha' R_{\mu\rho_1\nu\rho_2}X^{\rho_1}X^{\rho_2}\eta^{ij}\partial_i X^\mu \partial_j X^\nu\big) \quad , \tag{1.26}$$

as we can always choose $g_{ij} = \eta_{ij}$. We can now compute the one-loop in α'-correction to the $G_{\mu\nu}$ by computing the diagram I.3, using the vertex arising from the interaction therm in (1.26) proportional to $R_{\mu\rho\nu\gamma}p_i^{\rho}p^{i\gamma}$. Note that the space-time index here is merely a label denoting the momentum

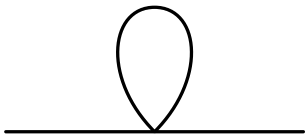
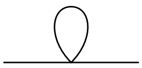


Figure I.3: Feynman diagram of the one-loop correction to the propagator of the massless bosnonic fields X^{μ}.

corresponding to the $d = 26$ bosonic fields X^{ρ}. Thus the one-loop diagram, using the well known Feynman rules for scalar fields is given by



$$= -\int \frac{dp^2}{2\pi^2}\sum_{\gamma=\rho}\frac{1}{p_i p^i}\delta(p^{(\rho)},p)\alpha' R_{\mu\rho\nu\gamma}p_i^{\rho}p^{i\gamma} \;=\; -\lim_{\mu\to\infty}\alpha' R_{\mu\rho\nu\gamma}\delta^{\rho\gamma}\frac{\mu^2}{4\pi}\quad , \tag{1.27}$$

where we have used polar coordinate to evaluate the integral,[9] and introduced the mass cut off μ. Note that it is quadratic in the cut off scale μ, and furthermore that in flat normal coordinates $R_{\mu\rho\nu\gamma}\delta^{\rho\gamma} = R_{\mu\nu}$, the Ricci tensor in normal coordinates. Thus one needs to introduce the counter-term $\frac{\alpha'\mu^2}{4\pi}R_{\mu\nu}\eta^{ij}\partial_i X^{\mu}\partial_j X^{\nu}$, which can be absorbed by a wave function renormalisation and a redefinition of the coupling as

$$G_{\mu\nu} \to G_{\mu\nu}(\mu) := G_{\mu\nu} + \frac{\alpha'\mu^2}{4\pi}R_{\mu\nu}\quad . \tag{1.29}$$

We can now compute the renormalisation group beta-function, which is defined as $\beta_{\mu\nu} \propto \mu\frac{\partial G_{\mu\nu}(\mu)}{\partial\mu} = \frac{\alpha'\mu^2}{2\pi}R_{\mu\nu}$.[10] Thus we see that the coupling runs quadratically with cut-off scale. We want the non-linear sigma model coupled to gravity to obey conformal symmetry, which is the invariance of change of coordinates $\sigma^i \to {\sigma'}^i(\sigma)$ such that the world-sheet metric changes as

$$g_{ij}(\sigma) \to \Omega^2(\sigma)g_{ij}(\sigma). \tag{1.30}$$

From (1.30) one infers that the physics arising from a conformally invariant theory is not dependent on the length scale, which is furthermore equivalent to saying that its couplings cannot run with any energy scale. Thus we conclude from our previous analysis that in order to restore conformal invariance at one-loop level we need to demand that $R_{\mu\nu} = 0$. Hence the space-time has to be Ricci-flat in order for the non-linear sigma model to be conformally invariant. Note that this is the E.O.M.

[9]Since the Riemann tensor does not depend on the integration variable one is left with the same integrand for the $d = 26$ bosonic fields running in the loop. Note that since naturally only one field can run in the loop the sum in (1.27) is restricted to $\gamma = \rho$ and $\delta(p^{(\rho)}, p)$. We choose polar coordinates $p^0 = p\cos\phi, p^1 = p\sin\phi$, with the radial coordinate $p = \sqrt{p_i p^i}$, to find

$$\int \frac{dp^2}{(2\pi)^2}e^{ip(\sigma-\sigma')} = \int_{\mathbb{R}_+}\int_0^{2\pi}\frac{dpd\phi}{(2\pi)^2}\,p\,e^{ip(\sigma-\sigma')} = \lim_{\mu\to\infty}\frac{-1+e^{i\mu(\sigma-\sigma')}(1-i\mu(\sigma-\sigma'))}{2\pi(\sigma-\sigma')^2} \overset{\sigma-\sigma'\to 0}{\longrightarrow} \lim_{\mu\to\infty}\frac{\mu^2}{4\pi}\quad , \tag{1.28}$$

were we have used the cut off in the momentum integration μ.

[10]Note that strictly speaking this beta-function is different from the one obtained in (1.23), since we have used a different renormalization scheme.

of free Einstein gravity sourced by the Einstein-Hilbert action

$$S \sim \int R *^{26} 1 \ . \tag{1.31}$$

Thus remarkably, we were led to incorporate the dynamics of the gravity upon the background field $G_{\mu\nu}$ of the string. Ricci-flatness is a sufficient condition to guarantee the vanishing of the beta-function at two and three-loop level [22], but will be altered at four loops as we will discuss in more detail in section 2.2. Let us note that Ricci-flatness is a local statement and does not imply that the entire space is equivalent to Minkowski space. In fact a major part of this work will deal with Ricci-flat Calabi-Yau manifolds, which are topologically far from trivial.

Let us return to the previous discussion where string couples to all background fields (1.17). We found that the background fields $G_{\mu\nu}, B_{\mu\nu}$ and Φ are subordinate to differential equations (1.23) and (1.24), which following [21] can be derived as the E.O.M.'s from the 26-dimensional action

$$S = \frac{1}{\kappa_0} \int e^{-2\Phi} \Big(R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4 \nabla_\mu \Phi \nabla^\mu \Phi \Big) *^{26} 1 \ , \tag{1.32}$$

with $\kappa_0 \propto l_s^{24}$. Among other things we encounter the Einstein-Hilbert term in (1.32), which seeds classical gravity. We conclude that one derives (1.32) in a perturbative regime in terms of α', by expanding the couplings of the non-linear sigma model to leading order in α'. For this to be a reasonable approach the ripples in space-time need to be much bigger than the string length. In other words the energy scale at which (1.32) describes physics with high precision, is much smaller than M_p. Therefore one considers (1.32) to be the low-energy effective action of the bosonic string describing the low-energy dynamics of the fields G, B, Φ, which in return correspond to the massless first excited state of the bosonic quantum string. Let me emphasize the beauty in this, we started with the classical relativistic free string in flat Minkowski space and merely just due to symmetry constraints of the arising quantum theory of the string, we were led to (1.32), which among other things describes gravity.

1.2 The supersymmetric string

Supersymmetry relates particles of different spin, namely it exchanges bosons to fermions and vice versa [23]. Since their theoretical introduction in the 1970 's, supersymmetric quantum field theories, in particular extensions of the standard model have been a flourishing field of research, although there has not been found any experimental evidence for its realization in nature up to now. In supersymmetric theories the mass of the bosons and their supersymmetric partners given by fermions are equal, which is clearly not observed in nature thus this symmetry needs to be broken at some higher scale. However, supersymmetric extensions of the Standard Model help to solve many problems of the Standard Model, which we know lacks completeness. Among other things it solves the hirachy problem and improves gauge coupling unification, see e.g. [24].

In this section we will discuss supersymmetry on the world-sheet of the string and its effect on the resulting supersymmetric effective field theories in analogy to the discussion of 1.1, given by supergravity theories. We will then comment on dualities of the various effective superstring theories.

1.2.1 World-sheet fermions and supersymmetry

Let us introduce world-sheet supersymmetry by adding fermionic degrees of freedom to the bosonic string action (1.7). Note that the world-sheet is embedded into d-dimensional space-time, thus the fermionic degrees of freedom are d massless Majorana fermions, whose dynamics on the world-sheet is described by the two-dimensional Dirac action as

$$S = \frac{1}{4\pi} \int d^2\sigma \big(\frac{1}{\alpha'} \partial_i X^\mu \partial^i X_\mu + \bar{\psi}_\mu \rho^i \partial_i \psi^\mu \big) \ , \tag{1.33}$$

where ψ is a two component spinor arranged with its cousins to transform in the vector representation ψ_μ of $SO(1, d-1)$, and ρ^i are the two-dimensional Dirac matrices such that $\{\rho^i, \rho^j\} = 2\eta^{ij}$.[11] The components of the spinor are given by Grassmann numbers obeying

$$\{\psi^\mu, \psi^\nu\} = 0 \quad , \tag{1.35}$$

where $\mu = 0, .., d-1$. The action (1.33) is invariant under the supersymmetry transformations

$$\delta X^\mu = \alpha' \bar{\epsilon} \psi^\mu \ , \tag{1.36}$$
$$\delta \psi^\mu = \rho^i \partial_i X^\mu \epsilon \ , \tag{1.37}$$

where ϵ is the infinitesimal supersymmetry parameter, being a constant Majorana spinor. Physical observables of bosons are different in nature than that of fermions, which need to be quadratic in the fermionic field ψ. It is therefore consistent for closed world-sheet fermions to be periodic $\psi^\mu(\sigma, \tau) = \psi^\mu(\sigma + 2\pi, \tau)$ or anti-periodic $\psi^\mu(\sigma, \tau) = -\psi^\mu(\sigma + 2\pi, \tau)$. These are called Ramond and Neveu-Schwarz boundary conditions, respectively, which in principle could give rise to different physical theories. However, it turns out that these constitute two sectors of the same theory. In fact, more precisely, after quantization one needs to combine states from both fermionic sectors and the bosonic one to build physical consistent theories. Furthermore, one concludes that there are exactly five different ways to do so, which result in the five unique superstring theories IIB , IIA , Het $E_8 \times E_8$, Het $SO(32)$ and type I.

1.2.2 The five unique supersymmetric strings

Since we have introduced new fermionic fields to the string theory action we will perform a discussion in analogy to the one of the bosonic string, in section 1.1. We start by deriving the massless excitations

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[11]

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{1.34}$$

arising from the fermionic world-sheet string (1.33), to then proceed by analyzing which fields they correspond to, and in particular study the effective field theory action that governs their dynamics for low energies.

The naive quantization of (1.33) proceeds as in section 1.1 for the bosonic string. One can solve the E.O.M.'s of the world-sheet by decomposing the d-fermionic fields in right and left-moving components $\psi^\mu = \psi^\mu_L + \psi^\mu_R$, and then by Fourier expanding the periodic (anti-periodic) directions in new coordinates $\sigma^\pm = \tau \pm \sigma$, which results in

$$\psi^{L/R\mu}(\sigma^\pm) \propto \sum_m \psi_n^{L/R\mu} e^{-im\sigma^\pm} \quad , \quad m = n \ \ (m = n + \frac{1}{2}) \ \ , \ \ n \in \mathbb{Z} \ . \tag{1.38}$$

The periodic and anti-periodic boundary conditions constrain the mode expansion index to be integer and half-integer, respectively. By going to light-cone gauge one breaks manifest space-time Lorentz invariance due to singling out one preferred space direction in which the string moves, thus one is left with the $i = 2, .., d$ transverse oscillations. To quantize the fermionic string one promotes the spinor fields to operator-valued fields obeying the equal-time anti-commutation relations

$$\{\psi^\mu_a(\sigma,\tau), \psi^\nu_b(\sigma',\tau)\} = 0 \quad , \quad \text{and} \quad \{\psi^\mu_a(\sigma,\tau), \Pi^b_\nu(\sigma',\tau)\} = \delta_a{}^b \delta(\sigma - \sigma') \ \ , \tag{1.39}$$

with the conjugate momentum $\Pi^a_\mu = \frac{\delta \mathcal{L}}{\delta \partial_0 \psi^\mu_a}$. These translate into anti-commutation relations of the mode expansion coefficients (1.38), which promote to operators and obey

$$\{\psi^{R/Li}_m, \psi^{L/Rj}_{m'}\} = 0 \quad \{\psi^{L/Ri}_m, \psi^{L/Rj}_{m'}\} = \delta_{ij}\delta_{m,-m'} \ \ , \tag{1.40}$$

with $m, m' = n \in \mathbb{Z}$ for the R-sector, and $m, m' = n + \frac{1}{2}$, $n \in \mathbb{Z}$ for the NS-sector. We are interested in the spectrum of (1.33), which combines now the bosonic and fermionic side. The invariant mass of the left and right-moving sector needs to be equal for consistency $M^2 = 2M_L^2 = 2M_R^2$, which results in

$$M^2_{NS} = \frac{2}{\alpha'}\Big(N_{NS} + N_B - \frac{d-2}{16}\Big) \quad , \quad M^2_R = \frac{2}{\alpha'}\Big(N_R + N_B\Big) \ , \tag{1.41}$$

for the NS and the R-sector, respectively. The number operators are

$$N_B = \sum_{n\in\mathbb{Z}} a^{L/Ri}_{-n} a^{L/Ri}_n \ \ , \ \ N_{NS} = \sum_{n\in\mathbb{Z}} \Big(n + \frac{1}{2}\Big) \psi^{L/Ri}_{-(n+\frac{1}{2})} \psi^{L/Ri}_{(n+\frac{1}{2})}, \ \ \text{and} \ \ N_R = \sum_{n\in\mathbb{Z}} n \psi^{L/Ri}_{-n} \psi^{L/Ri}_n \ \ , \tag{1.42}$$

with creation $a^{L/Ri}_{-n}, \psi^{L/Ri}_{-(n+\frac{1}{2})}, \psi^{L/Ri}_{-n}$ and annihilation operators $a^{L/Ri}_n, \psi^{L/Ri}_{n+\frac{1}{2}}, \psi^{L/Ri}_n$.

The ground state of the NS-sector (1.41) has negative mass squared thus is tachyonic - as for the bosonic string. This is problematic but in contrast to the bosonic string it can be removed from the spectrum via a consistent truncation as we will argue for in a moment. Let us first consider the first excited state of the NS-sector, which is given via $\psi^{L/Ri}_{-\frac{1}{2}} |0\rangle$. The $d-2$ degrees of freedom need to form a representation of the $SO(d-2)$ Lorentz group, if Lorentz symmetry shall be obeyed. This is the Lorentz group for massless particles in d dimensions. Since this state thus needs to be massless we find from (1.41), that the critical dimension of the superstring is $d = 10$. Obviously the ground state of the R-sector is massless and thus well behaved. The first excited states in the NS-sector are $a^{L/Ri}_{-n} |0\rangle$

and $\psi^{L/R\,i}_{-\frac{1}{2}}\psi^{L/Rj}_{-\frac{1}{2}}|0\rangle$, which is anti-symmetry in i,j following from (1.40). Thus this state is traceless and anti-symmetric and has $\frac{1}{2}(d-2)(d-3)$ degrees of freedom, respectively. In ten dimensions these become $8 \oplus 28$ degrees of freedom, which sit in a massive representation of the Lorentz group $SO(9)$, which reversely could have been used to derive the critical dimension $d = 10$. A similar argument can be applied to the first excitation of the R-sector $\psi^{L/R\,i}_{-1}|0\rangle$, which can be equally arranged into representations of $SO(9)$. There is a subtlety arising, the ground state of the R-sector is massless but degenerate, since acting upon it with fermionic generators $\psi_0^{R/Li}$ results in a state with the same energy, thus zero mass. It can be shown that the ground state therefore can be represented by a non-chiral 16 component spinor representation of $SO(8)$, which is reducible into two chiral spinor representations $8_s \oplus 8_c$ of opposite chirality corresponding to the left and right-moving modes.

Observe that all states originating from the operators a^i_n related to the bosonic world-sheet action are massive, thus will not give rise to any light degrees of freedom on the field theory side. The graviton, the Kalb-Ramond field and the dilaton indeed will originate from the operators ψ_n, which might seem a bit bizarre from the viewpoint of the bosonic string. Since the vacuum for the bosonic string 1.1 and for the superstring are not identical, also the first excited states under action with the operator a^i_{-n} are different. Moreover, the observation that all light degrees of freedom originate from the operators ψ_n is merely a remnant of the light cone gauge quantization - note that supersymmetry transformations exchange bosons and fermions.

However, the two sectors of the theory NS and R cannot be combined naively since these leads to inconsistencies. Furthermore, one needs to truncate the tachyonic ground state of the NS-sector. One should only perform such a projection if there is a plausible physical argument for its validity. The guiding principle of this consistent truncation was introduced by Gliozzi, Scherk and Olive (GSO), [25]. There are two equivalent ways of formulating such an argument. Firstly by introducing a new quantum number represented by the operator $(-1)^{\mathcal{F}}$, under which the Fermi fields ψ^μ are odd and the bosonic fields X^μ are even. More precisely it anti-commutes with NS and R-fermion generators $\{(-1)^{\mathcal{F}}, \psi^i_m\} = 0$, which implies that states with odd and even fermion generators have eigenvalue -1 and $+1$, respectively. In the NS-sector the projector $P_{NS} = \frac{1}{2}\big(1-(-1)^{\mathcal{F}}\big)$, removes the tachyonic ground state and all even fermionic generator states. Hence the massless state $\psi^{L/R\,i}_{-\frac{1}{2}}|0\rangle$ remains in the spectrum. In the R-sector there is an ambiguity of how to define the projector $P^{\pm}_R = \frac{1}{2}\big(1 \pm (-1)^{\mathcal{F}}\big)$ according to the two different chiralities of the ground state $8_s, 8_c$. But these two projections are related via space-time parity which exchanges the ground states $8_s \leftrightarrow 8_c$.

Secondly, from the observation that the closed string partition function[12] arising from (1.33) can only be modular invariant when string states from both sectors NS and R co-exist, which naturally implements a GSO projection and removes the tachyonic mode from the NS-sector. Furthermore, one needs to ensure that the one-loop corrections to the supersymmetric partition function vanish, which implies target space supersymmetry. Ensuring these two constrains one can find five inequivalent consistent combinations of states from the NS and R-sector. In other words since we naturally start

[12] We refer the reader to, e.g. [15, 17], for a more detailed discussion.

with all excitations of the bosonic and fermionic string, this amounts to the implementation of five different consistent truncations. These results in the five supersymmetric string theories: type IIB, type IIA , Het $E_8 \times E_8$, Het $SO(32)$ and type I. Type IIB superstring theory arises when applying the GSO Projection $P_{NS}, P_R^{\pm}$ to the left and right-moving modes yielding a modular invariant partition function and space-time supersymmetry. There is another combination of these projectors acting different on the left and right-moving sector, $P_{Ns}, P_R^{\pm}$ on the left and $P_{Ns}, P_R^{\mp}$ on the right-moving modes, which leaves us with IIA superstring theory.[13] We refer the reader to [15] for a detailed discussion of these truncations, especially for the Heterotic and type I theories.

Analogous to the discussion of the bosonic string we are now - after a bit more work - in a position to discuss the massless field-content of the superstring theories, focusing in the following on type IIA and type IIB. The table 1.1 gives the massless states of the type IIB string, composed of its left and right-moving components, where e.g. $|8_v\rangle_{L,NS}$ is a state in the 8_v vector representation of $SO(8)$, which arises from the left moving NS-sector. The allocation to the left or right-moving sector will be simply denoted by the left and right side of the tensor product in the following, where $\Gamma_{MN}, B_{NM}, \Phi$ are the graviton, the Kalb-Ramond field and the dilation, respectively, as for the bosonic string, and $C^{(0)}, C^{(2)}_{MN}$, and $C^{(3)}_{MNO}$ are anti-symmetric form fields, completing the bosonic field-content of IIB. The

$\vert\ \rangle_L \otimes \vert\ \rangle_R$	**representation of** $SO(8)$	$10d$ **field-content**
$\vert 8_v\rangle_{NS} \otimes \vert 8_v\rangle_{NS}$	$1 \oplus 24_v \oplus 35_v$	$G_{MN},\, B_{NM},\, \Phi$
$\vert 8_v\rangle_{NS} \otimes \vert 8_c\rangle_R$	$8_s \oplus 56_s$	$\lambda^1_a,\, \psi^1_{M\,a}$
$\vert 8_c\rangle_R \otimes \vert 8_v\rangle_{NS}$	$8_s \oplus 56_s$	$\lambda^2_a,\, \psi^2_{M\,a}$
$\vert 8_c\rangle_R \otimes \vert 8_c\rangle_R$	$1 \oplus 24_c \oplus 35_c$	$C^{(0)},\, C^{(2)}_{MN},\, C^{(4)}_{MNOP}$

Table 1.1: Massless bosonic and fermionic field-content of type IIB superstring theory.

fermionic field-content is given by $\lambda_a^{1/2}, \psi_{M\,a}^{1/2}$ two gravitinos and dilatinos, respectively. The light states of the type IIA superstring and their representations can be studied from table 1.2. The spectrum

$\vert\ \rangle_L \otimes \vert\ \rangle_R$	**representation of** $SO(8)$	$10d$ **field-content**
$\vert 8_v\rangle_{NS} \otimes \vert 8_v\rangle_{NS}$	$1 \oplus 24_v \oplus 35_v$	G_{MN}, B_{NM}, Φ
$\vert 8_v\rangle_{NS} \otimes \vert 8_S\rangle_R$	$8_C \oplus 56_C$	$\lambda^1_a, \psi^1_{M\,a}$
$\vert 8_c\rangle_R \otimes \vert 8_v\rangle_{NS}$	$8_S \oplus 56_S$	$\lambda^2_a, \psi^2_{M\,a}$
$\vert 8_c\rangle_R \otimes \vert 8_s\rangle_R$	$24_V \oplus 35_V$	$C^{(1)}_M, C^{(3)}_{MNO}$

Table 1.2: Massless bosonic and fermionic field-content of type IIA superstring theory.

differs solely in the form fields $C^{(1)}, C^{(3)}$, which have odd number of indices, in contrast to type IIB

[13] In both cases, IIB and IIA one is left with two physically equivalent theories according to the projection $P_R^{\pm}$.

where they are even $C^{(0)}, C^{(2)}, C^{(4)}$.

The mass of the excited states of the bosonic and superstring are of order α', which could be as high as $\sim 10^{16} GeV$.[14] Thus all the relevant particles in the standard model and quite far beyond originate from the massless string excitations, which are given mass via some mechanism.

From a classical string action with commuting and anti-commuting world-sheet variables X^μ, ψ^μ respectively, we found the massless quantum excitations give rise to the ten-dimensional fields content given in table 1.1 and table 1.2. We are now at a stage where it is intriguing to ask for the E.O.M.'s for the classical background fields of type IIA and type IIB. One can do so by solely using symmetry constraints and scattering amplitudes. For the moment we will just state the result and focus on dualities between these different theories for the remainder of this section, we refer the reader to section 2.2, where we elaborate more on the connection between amplitudes and effective actions. Let us emphasize again that the effective action is valid in an energy regime where the extendedness of the string $\alpha' \to 0$ is not seen and it can be considered a point particle, which restricts the energy scale of validity to $E \ll \frac{1}{l_s}$. In this limit the dynamics of the II string is governed by the II supergravity theories. With the field strength of the form fields $G^{(i+1)} = dC^{(i)}, i = 0, 2, 4$ the IIB supergravity action is given by

$$\begin{aligned} S^{IIB} = \frac{1}{2\kappa_{10}} \int_{\mathcal{M}_{10}} e^{-2\phi} R * 1 \quad &+ \quad e^{-2\phi} d\phi \wedge * d\phi - \frac{1}{2} e^{-2\phi} H^{(3)} \wedge * H^{(3)} - \frac{1}{2} G^{(1)} \wedge * G^{(1)} - \frac{1}{2} \tilde{G}^{(3)} \wedge * \tilde{G}^{(3)} \\ &- \quad \frac{1}{2} \tilde{G}^{(5)} \wedge * \tilde{G}^{(5)} - \frac{1}{2} C^{(4)} \wedge H^{(3)} \wedge G^{(3)} \quad , \end{aligned} \tag{1.43}$$

where $\kappa_{10} = (2\pi)^3 \sqrt{\pi} \alpha'^2 g_s$, and

$$\tilde{G}^{(3)} = G^{(3)} - C^{(0)} H^{(3)} \quad , \qquad \tilde{G}^{(5)} = G^{(5)} - \frac{1}{2} C^{(2)} \wedge H^{(3)} + \frac{1}{2} B^{(2)} \wedge G^{(3)} \quad . \tag{1.44}$$

The five-form field strength is self-dual, which cannot be implemented in the Lagrangian directly thus has to be imposed as a separate equation $G^{(5)} = * G^{(5)}$.

The IIA supergravity theory is given by

$$\begin{aligned} S^{IIA} = \frac{1}{2\kappa_{10}} \int_{\mathcal{M}_{10}} e^{-2\phi} R * 1 \quad &+ \quad e^{-2\phi} d\phi \wedge d\phi - \frac{1}{2} e^{-2\phi} H^{(3)} \wedge * H^{(3)} - \frac{1}{2} G^{(2)} \wedge * G^{(2)} \\ &- \quad \frac{1}{2} \tilde{G}^{(4)} \wedge * \tilde{G}^{(4)} - \frac{1}{2} B^{(2)} \wedge G^{(4)} \wedge G^{(4)} \quad , \end{aligned} \tag{1.45}$$

with

$$\tilde{G}^{(4)} = G^{(4)} - C^{(1)} \wedge H^{(3)} \quad . \tag{1.46}$$

Not surprisingly the two effective actions (1.43) and (1.45) are rather similar and only differ from each other in respect to the field contents in the form fields, see table 1.1 and table 1.2 .

[14] Note that in some scenarios it can be much lower [18, 19].

1.2.3 Dualities and M-theory

Let us next comment on one of the most remarkable encounters of superstring theories. Additionally to the high uniqueness of only five different theories originating from the superstring, these theories are furthermore dual to each other. Rising in prominence since the second half of the last century, dualities among different physical theories have become a very powerful tool to understand nature, its most beautiful example might be the AdS/CFT duality [26]. The notion of a duality describes the fact that two seemingly different physical theories can under certain circumstances describe the same physics in a regime of their validity. Thus what is hard to derive in one theory might be an easy question to answer in the language of the other and vice versa. Let us propagate the IIB superstring in the space $M^{8,1} \times S^1$, where the circle S^1 has the radius r_s. One can show that this is the same physical theory as the IIA string propagating in a the space $M^{8,1} \times S^1$ but now with radius of the circle $\frac{1}{r_s}$. Thus the difference in the target space geometry reflects onto the massless spectrum of the string.

This is the first time we have met the influence of geometry on effective physics in this work. The described duality goes under the name T-duality and will play a crucial role in the discussion of F-theory in section 3. A long with the massless closed string spectrum, which changes under T-duality the non perturbative objects Dp-branes are mapped from type IIB $\leftrightarrow$ IIA and thus the open string spectrum, see e.g. [27, 28, 29, 30]. Furthermore, type IIB supergravity is invariant under a weak to strong coupling transformation, thus one can go from an asymptotically free region of the theory $g_s \ll 1$ to a strongly coupled regime without any change of the effective physics, which is quite remarkable. This strong weak coupling duality $g_s \to \frac{1}{g_s}$ more commonly goes under the name S-duality. We can redefine the fields in the type IIB action (1.43) such that its $SL(2,\mathbb{Z})$ invariance becomes manifest, which in particular captures the S-duality invariance. To make $SL(2,\mathbb{Z})$ symmetry manifest firstly, we need to bring (1.43) in canonical Einstein-Hilbert form, which is done via a Weyl rescaling

$$g_{MN} \to e^{\frac{\Phi}{2}} g_{MN} \quad , \tag{1.47}$$

which according to (A.40) exactly cancels the Φ dependence. Furthermore, one rewrites the action by packaging Φ and $C^{(0)}$ into a new field referred to as the axio-dilaton

$$\tau := C^{(0)} + i\, e^{-\Phi} \quad , \tag{1.48}$$

which gives

$$S^{IIB} = \frac{1}{2\kappa_{10}} \int_{\mathcal{M}_{10}} R{*}1 + \frac{1}{\mathrm{Im}\tau^2} d\tau \wedge d\bar{\tau} + \frac{1}{\mathrm{Im}\tau} \tilde{G}^{(3)} \wedge {*}\tilde{G}^{(3)} - \frac{1}{2}\tilde{G}^{(5)} \wedge {*}\tilde{G}^{(5)} - \frac{1}{2} C^{(4)} \wedge H^{(3)} \wedge G^{(3)} \quad , \tag{1.49}$$

being equivalent to (1.43). This action is now evidently invariant under $SL(2,\mathbb{Z})$, [15] which acts on

[15] It is straightforward to show that

$$d\tau \wedge {*}d\bar{\tau} \to \frac{1}{\mathrm{Im}\frac{a\tau+b}{c\tau+d}^2} d\left(\frac{a\tau+b}{c\tau+d}\right) \wedge {*}d\left(\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = d\tau \wedge {*}d\bar{\tau} \tag{1.50}$$

for any $a, b, c, d \in \mathbb{R}$, thus is given by $SL(2,\mathbb{R})$. The $SL(2,\mathbb{Z})$ arises due to non-perturbative instanton effects, which break the classical $SL(2,\mathbb{R})$ symmetry to a $SL(2,\mathbb{Z})$ symmetry.

(1.49) as

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad , \quad \begin{pmatrix} H^{(3)} \\ F^{(3)} \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H^{(3)} \\ F^{(3)} \end{pmatrix} \quad , \quad \tilde{G}^{(5)} \to \tilde{G}^{(5)} \quad , \quad \text{and} \quad g_{MN} \to g_{MN} \ ,$$
$$\text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad \Leftrightarrow \quad a, b, c, d \in \mathbb{Z} \quad \text{with} \quad ad - bc = 1 \ . \tag{1.51}$$

The strong-weak symmetry map is an element of $SL(2, \mathbb{Z})$, as seen by chossing the element $a = d = 0, b = c = 1$, which transforms $\tau \to \frac{1}{\tau}$. Note that the type IIB string coupling is given by $g_{IIB} = e^{\Phi}$ and by using (1.48) one infers that $g_{IIB} \to \frac{1}{g_{IIB}}$, thus strong to weak coupling and vice versa.

The five different superstring theories are connected by S and T-dualities. However, to form a connected chain amongst all of them one misses a link, which is filled in by M-theory. Since the

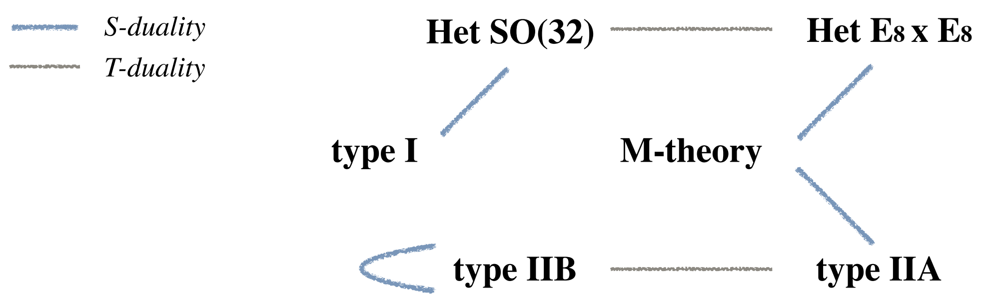


Figure I.4: A diagrammatic depiction of the chain of dualitites among the five different string theories, centered around M-theory.

microscopic description of M-theory is not known it is thus best accessible through its low-energy effective action given by eleven-dimensional supergravity. But there is plenty of undeniable indirect evidence for the existence of M-theory. Various attempts have been made to describe its fundamental degrees of freedom, the most promising proposal for a microscopic description is based on matrix theory, describing it by taking a certain kinematic limit of a stack of $D0$-branes [31, 32, 33]. The dualitites between the string theories hold in the world-sheet picture, but in particular they are valid in the various low-energy effective actions. When applying S-duality to type IIA supergravity the strong coupling side is described by a theory living in eleven instead of ten dimensions, being eleven-dimensional supergravity, which is appealing on its own right due to its high uniqueness being the only supergravity theory in eleven dimensions. M-theory, in the following used as a synonym for eleven-dimensional supergravity, when compactified on $\mathcal{M}_{10} \times S^1$ yields type IIA supergravity. The bosonic part of the $11d$ supergravity action is

$$S^{(0)} = \frac{1}{2\kappa_{11}^2} \int \left[\hat{R} \hat{*} 1 - \frac{1}{2} \hat{G} \wedge \hat{*} \hat{G} - \frac{1}{6} \hat{C} \wedge \hat{G} \wedge \hat{G} \right] \ , \tag{1.52}$$

first worked out in [34], with $\hat{R}$ the Ricci scalar evaluated with conventions introduced in appendix A, and $\kappa_{11} = \frac{1}{\sqrt{2}} (2\pi)^4 \, l_M^{\frac{9}{2}}$, with the eleven-dimensional Planck length l_M. The hat denotes eleven-dimensional quantities in the following, such as $\hat{C}$ the three-form field and $\hat{G} = d\hat{C}$ its four-form field strength. The fermionic side of eleven-dimensional supergravity as its supersymmetry variations will be discussed in section 2.2.1, where we also comment on the construction of (1.52). We will only focus on the bosonic side in the following.

The paradigm of dimensional reduction of a theory is elaborated on in more detail in section 2.4. For now let us state that the principal idea. If one is only interested in the theory living in the subspace of the total space, here $\mathcal{M}_{10}$ of $\mathcal{M}_{11} = \mathcal{M}_{10} \times S^1$, one needs to consistently integrate out the fields living on S^1. One does this by expanding the fields living on the internal space - referring to S^1 here - in the most general fashion, thus in this case one Fourier expands them on S^1. This gives rise to a whole tower of so called Kaluza-Klein (KK) states, where only the massless modes shall be kept. After expanding the eleven-dimensional action one needs to integrate out the whole massive Kaluza-Klein tower, which leaves one with two scenarios. Firstly, the remaining light modes do not couple to the massive modes, which means that for any energy or size of the internal space the resulting lower-dimensional theory is unaltered, one refers to this as a consistent truncation. Secondly, this is not the case, and in principle we can not separate the light modes from the massive ones in a clean way, which implies that one needs a separation of energy scales to argue that the massive modes are not present, or in other words can not be excited. Since the masses of the heavy Kaluza-Klein modes are of order of the size of the internal space, the energy scale at which the resulting lower dimensional inconsistently truncated theory is valid, is much smaller than the so called KK-scale. The most general ansatz giving rise to massless fields for the eleven-dimensional metric is

$$\hat{G}_{MN} = e^{-\frac{2\Phi}{3}} \begin{pmatrix} g_{\mu\nu} + e^{2\Phi} C^{(1)}_\mu C^{(1)}_\nu & e^{2\Phi} C^{(1)}_\mu \\ e^{2\Phi} C^{(1)}_\nu & e^{2\Phi} \end{pmatrix} , \tag{1.53}$$

with $\mu, \nu = 0, 1, \ldots, 9$, chosen such that the $10d$ scalar and vector appear already in the right decomposition of Φ and $C^{(1)}$, anticipating the matching with the dilaton and one-form field of type IIA supergravity. The various components of the eleven-dimensional field strength gives rise to $10d$ fields, where we again anticipate the identification with the well known type IIA supergravity spectrum. One finds that

$$\hat{C}_{\mu\nu\rho} = C^{(3)}_{\mu\nu\rho} \quad \text{and} \quad \hat{C}_{\mu\nu 11} = B^{(2)}_{\mu\nu} \quad , \tag{1.54}$$

with the corresponding field strengths $\hat{G}_{\mu\nu\rho\gamma} = G^{(3)}_{\mu\nu\rho\gamma}$ and $\hat{G}_{\mu\nu\rho 11} = H^{(3)}_{\mu\nu\rho}$. The reduction on a circle is best performed transforming the components of the eleven-dimensional four-form field strength to co-tangent space coordinates using the eleven and ten-dimensional vielbeins, $\hat{e}^{\hat{a}}{}_M$, $e^a{}_M$ with $\hat{a}$ and a denoting eleven and ten-dimensional flat tangent space indices, respectively. [16] From

$$\hat{G}_{\hat{a}\hat{b}\hat{c}\hat{d}} = e_{\hat{a}}{}^M e_{\hat{b}}{}^N e_{\hat{c}}{}^O e_{\hat{d}}{}^P \hat{G}_{MNOP} , \tag{1.56}$$

[16] With the Vielbein $\hat{e}^a{}_M$ and the inverse vielbein $\hat{e}_a{}^M$ derived from (1.53) where $\hat{e}^{\hat{a}}{}_M \hat{e}^{\hat{b}}{}_N \hat{\eta}_{\hat{a}\hat{b}} = \hat{G}_{MN}$ and $\hat{e}_{\hat{a}}{}^M \hat{e}_{\hat{b}}{}^N \hat{G}_{MN} = \hat{\eta}_{\hat{a}\hat{b}}$, with $\hat{\eta}$ the eleven-dimensional flat Minkowski metric.

$$\hat{e}^{\hat{a}}{}_M = \begin{pmatrix} e^{-\frac{\Phi}{3}} e^a{}_\mu & 0 \\ e^{\frac{2\Phi}{3}} C^{(1)}_\mu & e^{\frac{2\Phi}{3}} \end{pmatrix} \quad \text{and} \hat{e}_{\hat{a}}{}^M = \begin{pmatrix} e^{\frac{\Phi}{3}} e_a{}^\mu & 0 \\ e^{-\frac{\Phi}{3}} C^{(1)}_\mu e_a{}^\mu & e^{-\frac{2\Phi}{3}} \end{pmatrix} . \tag{1.55}$$

one infers upon using (1.55) and (1.54) that

$$\hat{G}_{abcd} = e^{\frac{4\Phi}{3}}\left(G^{(4)}_{abcd} + 4C^{(1)}_{[a}H^{(3)}_{bcd]}\right) =: e^{\frac{4\Phi}{3}}\tilde{G}^{(4)}_{abcd} \quad \text{and} \quad \hat{G}_{abc11} = e^{\frac{\Phi}{3}}H^{(3)}_{abc} \,. \tag{1.57}$$

By going to tangent space indices in (1.57) one notices that the degrees of freedom from the metric $C^{(1)}$ and Φ combine with the eleven-dimensional field strength do give the type IIA four-form field $\tilde{G}^{(4)}$. To perform the reduction at this point boils down to plugging the field identifications (1.57),(1.55) and (1.53) into the action (1.52), which in component notion reads

$$S^{(0)} = \frac{1}{2\kappa_{11}^2}\int\left[\hat{R} - \frac{1}{2}\frac{1}{4!}\hat{G}_{MNOP}\hat{G}^{MNOP} - \frac{1}{6}\frac{1}{3!4!4!}\epsilon^{N_1\ldots N_{11}}\hat{C}_{N_1N_2N_3}\hat{G}_{N_4N_5N_6N_7}\hat{G}_{N_8N_9N_{10}N_{11}}\right]\hat{*}1 \,. \tag{1.58}$$

By transforming (1.58) to tangent space indices analogous to (1.56) one can naively split each index $\hat{a} \to (a, 11)$ and replace the components according to (1.57). Reducting the Einstein-Hilbert term is computationally a bit more involved but conceptually one proceeds analogously. Note that $\hat{R} = \hat{R}^{M}{}_{NM}{}^{N} = \hat{R}^{\hat{a}}{}_{\hat{b}\hat{a}}{}^{\hat{b}}$, which decomposes into Christoffel symbols, that can be written in flat tangent space coordinates to give the various components. Performing the substitution of the lower-dimensional fields into (1.58), one indeed arrives at the type IIA supergravity action (1.45), with string coupling given in terms of the radius of the circle as $g_s = \frac{R}{\sqrt{\alpha'}}$. By using that the eleven and ten-dimensional Newton's constant are related as

$$G_{11} = 2\pi R\, G_{10} \ , \tag{1.59}$$

and with the identification $16\pi G_{11} = 2\kappa_{11}^2 = (2\pi)^8 l_M^9$ in eleven dimensions and $16\pi G_{10} = 2\kappa_{10}^2 = (2\pi)^7 l_s^8 g_s^2$ in ten dimensions, one finds that

$$l_M^9 = R\, l_s^8\, g_s^2 \ . \tag{1.60}$$

Thus naturally, one can express the ten-dimensional couplings in terms of the eleven-dimensional ones.

2 Effective field theory and strings

In section 1.1 we have introduced the notion of an effective field theory of a string theory when describing its effective physics in the limit $\alpha' \to 0$. This description is valid when the considered energy scale is much smaller than the string scale. Furthermore, we noted that for inconsitent truncations in dimensional reductions of theories the fields that are integrated out are suppressed by the volume of the internal space setting a cut-off scale, referred to as the KK-scale. Hence the resulting effective theory is only valid in regimes where the energy is much smaller than the KK-scale. The paradigm of scale separation is in fact maybe the the most fundamental concept inherent to all physical theories. Classical mechanics is only accurate and thus valid for velocities much smaller than the speed of light. The movement of planets around the sun can be described neglecting their extendedness, thus going to a limit *volume* $\to 0$. It is quite remarkable that the anomalous magnetic moment of the electron is known to ten significant digits without involving any TeV scale physics. It is not important to consider heavy states in these computations since they decouple. This separation of scales is the reason why we need to build bigger colliders instead of low energy precision measurements. This paradigm can be perfectly implemented in the language of quantum field theories as we will show in section 2.1, and can be applied to renormalizable as well as non-renormalizable quantum field theories. Note that supergravity theories are non-renormalizable because they incorporate gravity via the Einstein-Hilbert term.[17]

This concept derives a low energy field theory from a not necessarily ultraviolet complete quantum field theory valid at higher scales. Hence, this is in principal somewhat different from the limit $\alpha' \to 0$ where the extendedness of the string and massive states thereof are washed away. But nevertheless, these ideas rely on the same notion of deriving low energy physics by consistently neglecting higher energy or mass-states. In section 2.2 we then show how one can derive string effective field theory actions from the matching with superstring scattering amplitudes and taking the limit $\alpha' \to 0$. This results in ten-dimensional supergravity theories and corrections thereof. We also give an introduction to supergravity theories since their interplay with string theory is fundamental. In section 2.4 we focus on the paradigm of dimensional reduction of supergravity theories. Since the superstring lives in ten-dimensional space-time one needs to establish a connection to reality, thus six of the dimensions need to be small and curled-up. The idea first introduced by Kaluza and Klein is discussed in detail, with a focus on supersymmetry preserving reductions.

2.1 The notion of an effective field theory

In the following we will simple consider a light scalar ϕ and a heavy scalar field Φ in four space-time dimensions, to show the main techniques and principles. What quantitatively distinguishes a light particle from a heavy particle is that at the energy we want the theory to be applicable, the light particle can be generated on shell whereas the heavy ones cannot. Formally we can integrate out the heavy states by performing a path integral over the heavy states only

$$\int \mathcal{D}\Phi e^{iS(\phi,\Phi)} = e^{iS_{eff}(\phi)} \quad . \tag{2.1}$$

Since one is not capable to perform the path integral analytically in the most cases one is limited to a perturbative approach by computing Feynman diagrams. The effective Lagrangian density can be expanded in a finite number of terms, which are not suppressed by the cut-off scale Λ and an infinite tower of higher-dimensional local operators $\mathcal{O}_n$ as

$$\mathcal{L}_{eff}(\phi) = \mathcal{L}^0 + \sum_n \frac{1}{\Lambda^{\dim \mathcal{O}_n - 4}} \mathcal{O}_n \quad . \tag{2.2}$$

In practice, one can truncate the infinite sum over higher-dimensional operators since at higher orders the effects to the action and therefore to the low energy observables become smaller. Note that one anyway desires an approximate picture. Since one truncates the sum of higher-dimensional operators

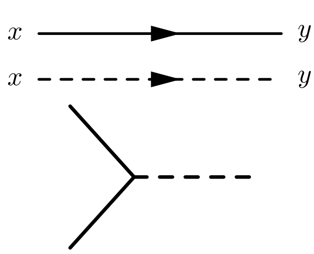
[17] Recent work shows that any four-particle amplitude in $\mathcal{N} = 8$ maximal supergravity in four and five dimensions is ultraviolet finite at four loops [35].

this process is equally valid for renormalizable as well as non-renormalizable theories. The breakdown scale Λ of the theory corresponds to the on shell contribution from heavy states.

One can formally predict the magnitude of different operators $\mathcal{O}_i$ by analyzing how they scale with energy simply done by power counting. Using that $[mass] = [length^{-1}] = 1$ one infers that the Lagrangian density has $[\mathcal{L}] = 4$ since the action $\int d^4x\mathcal{L}$ is dimensionless. The kinetic energy of the scalar in four space-time dimensions $\partial_\mu\Phi\partial^\mu\Phi$ implies that $[\Phi] = 1$. The toy-model Lagrangian of the high energy theory of a massless scalar ϕ and a massive scalar Φ shall be

$$\mathcal{L}(\phi, \Phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{M^2}{2}\Phi^2 - \frac{\lambda}{2}\phi^2\Phi \ , \tag{2.3}$$

with $m \ll M$, and $[\lambda] = 1$, thus the interaction terms correspond to relevant and marginal operators, respectively.[18] This results in the following Feynman rules 2.1 using the full and the dashed line for the light and heavy field, respectively. We next consider tree-level effects from the full path integral



$\Delta_\phi(x-y) = \frac{e^{i(x-y)}}{p^2}$

$\Delta_\Phi(x-y) = \frac{e^{i(x-y)}}{p^2-M^2}$

$-i\lambda$

Table 2.1: Feynman rules of the action (2.3).

(2.1) of the action (2.3).

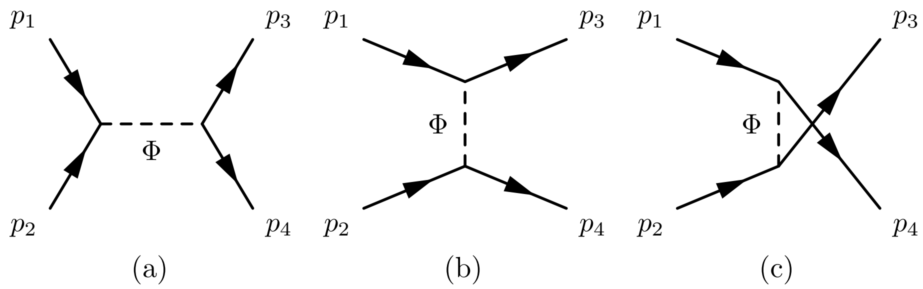
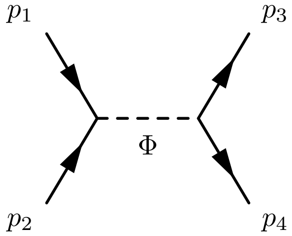


Table 2.2: Tree-level $\phi\phi \to \phi\phi$ scattering diagrams for the s, t, u channel.

At tree-level the Feynman diagrams we need to compute are given in table 2.2. The first diagram

[18]Note that in three dimensions we find that $[\phi] = [\Phi] = 1/2$ and thus $[\lambda] = 2$.

in table 2.2, which corresponds to the s-channel results in



$$= (-i\lambda)^2 \frac{1}{(p_1+p_2)^2 - M^2} = -\frac{\lambda^2}{M^2}\frac{1}{\frac{(p_1+p_2)^2}{M^2}-1} = \frac{\lambda^2}{M^2}\Big(1 + \frac{(p_1+p_2)^2}{M^2} + \mathcal{O}(\frac{p^4}{M^4})\Big) \ , \tag{2.4}$$

where we formed an expansion in $p \ll M$, which is a valid choice since the energies for the on-shell fields ϕ are assumed to be much smaller than M. The $u = (p_1 - p_3)^2$ and $t = (p_1 - p_4)^2$ diagram (b) and (c), respectively, contribute the same momentum independent contribution to the amplitude, the next to leading order amplitude depending on the momenta vanishes since $s + t + u = 0$. We want to reproduce the amplitude (2.4) for the $\phi\phi \to \phi\phi$ scattering to leading and next to leading order in the momentum expansion without involving the heavy states, which determines the effective Lagrangian [19] to be

$$L^{(0)}_{eff}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda^2}{8M^2}\phi^4 - \frac{\lambda^2}{2M^4}\partial_\mu\phi\partial^\mu\phi\phi^2 \ . \tag{2.5}$$

For the more involved discussion of one-loop amplitude matching we refer the reader to [36, 37]. However, a perturbative approach for the toy model (2.3) is not necessary since one can exactly integrate out the massive scalar in the path integral as we will show next. One may use the exact formula for path integrals for operators A and the functions $B(x), C(x)$ given by

$$\int \mathcal{D}\phi Exp\Big[-\int \big(\frac{1}{2}\phi(x)A\phi(x) + B(x)\phi(x) + C(x)\big) *_d 1\Big] = \frac{e^{\int\big(\frac{1}{2}B(x)A^{-1}B(x) - C(x)\big)*_d 1}}{\sqrt{\det A}} \ . \tag{2.6}$$

We compute the effective theory by evaluating the path integral of the massive states[20]

$$\int \mathcal{D}\phi\mathcal{D}\Phi e^{-i\int L(\phi,\Phi)} = (\det A)^{-\frac{1}{2}} \int \mathcal{D}\phi e^{-i\int\big(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}B(x)A^{-1}B(x) - C(x)\big)*_4 1} \tag{2.7}$$

with $A(x) = \Box + M^2 - i\epsilon$, and $B(x) = -\lambda/2\phi(x)^2$ and $C(x) = 0$, and with

$$(\det A)^{-\frac{1}{2}} = \int \mathcal{D}\Phi e^{-\frac{i}{2}\int \Phi(x)(\Box + M^2 - i\epsilon)\Phi(x) *_4 1} \ , \tag{2.8}$$

which we can absorb in the normalization since it is just the path integral of a free theory. Note that we performed a wick rotation in the path integral to be in Euclidean space. One thus finds the exact effective theory to be

$$\mathcal{L}_{eff}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda^2}{8}\phi^2 \frac{1}{-\Box + M^2 - i\epsilon}\phi^2 \ , \tag{2.9}$$

[19] We have used that a coupling of the form $g/4\partial_\mu\phi\partial^\mu\phi\phi^2$ gives rise to a vertex $-ig(p_1\cdot p_2 + p_1\cdot p_3 + p_1\cdot p_4 + p_2\cdot p_3 + p_2\cdot p_4 + p_3\cdot p_4)$. Furthermore, that $s = (p_1 + p_2) = 2p_1\cdot p_2$ etc. , and that $s + t + u = 0$ since the external particles are massless. Note that the amplitudes in the ultraviolet and effective theory vanish for the next to leading order momentum expansion, thus we fixed the pre-factor by demanding that the factor of the vanishing kinematic combination matches between the two theories.

[20] Note that we are working in a flat metric background here.

where the minus in the denominator is due to the wick rotation back to Lorentzian signature. We are considering only external momenta at energies much smaller than the mass M of the particle Φ, which was integrated out. Thus $\frac{\Box}{M^2}\phi \ll 1$, which can be seen by going to momentum space where morally this gives $\frac{p^2}{M^2}\phi(p) \ll 1$. Hence we can expand the inverse operator in powers of $\frac{\Box}{M^2}$, which gives

$$\mathcal{L}_{eff}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda^2}{8M^2}\phi^2\left(1 + \frac{\Box}{M^2} - \left(\frac{\Box}{M^2}\right)^2 + \ldots ..\right)\phi^2 \quad . \tag{2.10}$$

We obtain by partial integration that the leading order contributions of (2.10) can be written as

$$\mathcal{L}_{eff}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda^2}{8M^2}\phi^4 - \frac{\lambda^2}{2M^4}\phi^2\partial_\mu\phi\partial^\mu\phi + \ldots \quad . \tag{2.11}$$

Comparing (2.5) and (2.10) one notes that in the perturbative approach one naturally only infers the leading order terms of the expansion (2.10). The leading order action given by (2.11) matches the low-energy effective action obtained by the perturbative approach (2.5).

2.2 String effective actions

In the previous section we have discussed how to integrate out high-energy degrees of freedom from a field theory, yielding an effective low-energy theory, which governs certain remnants of the high-energy dynamics. Morally, this is exactly what happens when taking the field theory limit $\alpha' \to 0$ of string theory, discussed already in section 2.1. Alternatively to the vanishing of the sigma model beta-function in section 1.1 we will in the following give a different systematic approach for deriving the effective action. Starting from string scattering amplitudes we construct an effective field theory such that the amplitudes of the field theory mimic the ones of the string theory to a certain accuracy. This is morally the procedure we applied in section 2.1, matching the scattering amplitudes of the high energy field theory. The non-linear sigma model in $1+1$ dimensions with Ricci-flat target space does not violate conformal and Weyl-invariance for one, two and three loops in α', as commented on in section 2.1. If this would hold to all order in α', any Ricci-flat space-time would constitute a solution to type II string theory, and furthermore would imply that gravitational wave scattering as described by Einsteins equation $R_{MN} = 0$, is governed by the same law in string theory. This is not to be expected since the tree-level graviton-graviton scattering in string theory is fundamentally different than gravitational wave scattering. Indeed, it turns out that at fourth order in α' one needs to add further counter-terms to the non-linear sigma model to guarantee the vanishing of the beta-function, which in return modifies the effective action. Equivalently this is necessary for the effective action to reproduce the string scattering amplitude at this order. Following the procedure of section 2.1 to derive the effective action one writes down all couplings allowed by symmetry for the entire light field-content, adding higher-dimensional operators. The standard ten-dimensional $\mathcal{N} = 2$ type II supergravity actions (1.43) and (1.45) are enough to match the one, two and three loop scattering amplitudes in α', these results are at tree-level in g_s. Thus there will not be any higher-order operators in the effective field theory containing only one, two or three fields. However, it is expected that to match the four loop result one requires terms containing four Riemann tensors, which is exactly what we will encounter soon. The first non-vanishing correction thus carries eight derivatives and was derived in literature via vanishing of the beta-function [38, 39, 40, 41, 42, 43] as well as from tree-level four graviton scattering [44, 45]. Before discussing the derivation of the correction to the effective action let us state the result. The relevant corrections arising from tree-level in g_s four graviton scattering are

$$\delta\mathcal{L} \propto \left(t_8 t_8 R^4 + \frac{1}{8}\epsilon_{10}\epsilon_{10}R^4\right) \quad . \tag{2.12}$$

With the tensor t_8 defined in (A.35) and ϵ_{10} the ten-dimensional Levi-Civita tensor. The explicit index structure of $t_8 t_8 R^4$and $\epsilon_{10}\epsilon_{10}R^4$ are given in (2.67). One-loop in g_s string scattering of four gravitons is given by the torus diagrams with four graviton vertex operator insertions, which results in a contribution to the effective action given by

$$\delta\mathcal{L} \propto \left(t_8 t_8 R^4 \mp \frac{1}{8}\epsilon_{10}\epsilon_{10}R^4\right) \ , \tag{2.13}$$

with the minus and plus sign for type IIA and IIB respectively. The torus diagram with four graviton and one Kalb-Ramond field vertex operator insertions yields the correction

$$\delta\mathcal{L} \propto B_2 \wedge \left(trR^4 - \frac{1}{4}(trR^2)^2\right) \ , \tag{2.14}$$

where the index structure of the traces is given in (2.3).

The four graviton string amplitudes at tree and one-loop level have been first computed in [46] and [47] respectively, see as well e.g. [11, 12]. Let us next review the derivation of the tree-level amplitude and the construction of the correction (2.12) to the effective action. The four graviton scattering amplitude tree diagram is a closed string diagram. KLT [48] showed that any closed string diagram can be written as a product of left and right-moving open string diagrams, thus one needs to evaluate the open string diagram and secondly apply the KLT relation. This is particularly simple for a three particle scattering, which reproduces an open string tree amplitude $\mathcal{A}_3^{op}(p_1,p_2,p_3) = g\zeta_1^A\zeta_2^B\zeta_3^C V_{ABC}(p_1,p_2,p_3)$. With V_{ABC} depending on the external momenta p_1,p_2,p_3 and carrying information of the kinematics of the scattering process and ζ^A the polarization vector or spinor, depending on the nature of the external state. The corresponding closed string amplitude derives to

$$\mathcal{A}_3^{cl}(p_1,p_2,p_3) = \kappa\zeta_1^{A\bar{A}}\zeta_2^{B\bar{B}}\zeta_3^{C\bar{C}}V_{ABC}(\frac{p_1}{2},\frac{p_2}{2},\frac{p_3}{2})V_{\bar{A}\bar{B}\bar{C}}(\frac{p_1}{2},\frac{p_2}{2},\frac{p_3}{2}) \ , \tag{2.15}$$

with $\zeta^{AB} = \zeta^A \otimes \zeta^B$. When computing four-particle scattering it is convenient to introduce Mandelstam variables to encode the external momenta $s = (p_1+p_2)^2, t = (p_1-p_3)^2, u = (p_1-p_4)^2$.[21] The four-particle open string tree diagrams can be written as

$$\mathcal{A}_4^{op}(p_i,s,t,u) = g^2\zeta_1^A\zeta_2^B\zeta_3^C\zeta_4^D K_{ABCD}(p_i)\frac{\Gamma(\frac{s}{2})\Gamma(\frac{t}{2})}{\Gamma(1-\frac{u}{2})} \ , \tag{2.16}$$

with p_i , $i = 1,2,3,4$ and $K_{ABCD}(p_i)$ being the kinematic factor of the scattering. The closed string amplitude can be shown to take the form

$$\mathcal{A}_4^{cl}(p_i,s,t,u) = \sin\left(\frac{\pi t}{8}\right)\mathcal{A}_4^{op}\left(\frac{p_i}{2},\frac{s}{4},\frac{t}{4},\frac{u}{4}\right)\mathcal{A}_4^{op}\left(\frac{p_i}{2},\frac{t}{4},\frac{u}{4},\frac{s}{4}\right) \ , \tag{2.17}$$

[21] Note that $s+t+u = \sum_{i=1}^4 m_i^2$ as seen by using conservation of momentum $p_1+p_2 = p_3+p_4$, which gives $s+t+u=0$ for massless external states.

which using $\Gamma(x)\Gamma(1-x)\sin(\pi x) = \pi$ results in

$$\mathcal{A}_4^{cl}(s,t,u) = -\pi\kappa^2 \zeta_1^{A\bar{A}}\zeta_2^{B\bar{B}}\zeta_3^{C\bar{C}}\zeta_4^{D\bar{D}} K_{ABCD}\Big(\frac{p_i}{2}\Big) K_{\bar{A}\bar{B}\bar{C}\bar{D}}\Big(\frac{p_i}{2}\Big) \frac{\Gamma(\frac{s}{8})\Gamma(\frac{t}{8})\Gamma(\frac{u}{8})}{\Gamma(1-\frac{s}{8})\Gamma(1-\frac{t}{8})\Gamma(1-\frac{u}{8})} \ . \tag{2.18}$$

We are interested in solely gravitons as external states thus we restrict the further discussion to the case $\zeta^{MN} = h^{MN}$, where h_{MN} is the graviton polarization tensors, see figure I.5. Due to KLT (2.18)

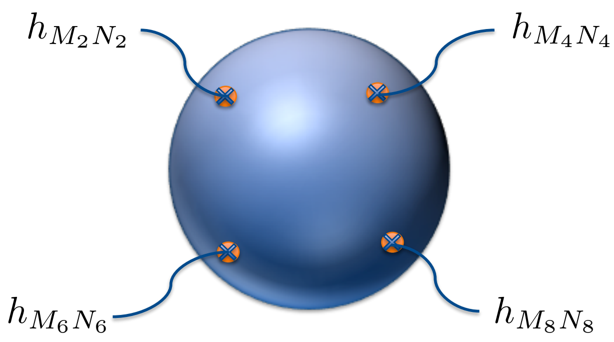


Figure I.5: Schematic depiction of the tree-level scattering amplitude with four graviton vertex operator insertions with external polarization tensors h_{MN}. The indices are chosen such that it reproduces the index structure of the external graviton polarization tensors in (2.21).

one only needs to compute the tree-level scattering, note that also $\zeta^M = h^M$ in the following. The details of the computation are beyond the scope of this text but can be found in literature [11]. The kinematic factor derives to

$$h_1^L h_2^M h_3^N h_4^O K_{LMNO}(p_i) = \Big(t_{8\,N_1N_2N_3N_4N_5N_6N_7N_8} - \frac{1}{2}\epsilon_{N_1N_2N_3N_4N_5N_6N_7N_8}\Big) p_1^{N_1} p_2^{N_3} p_3^{N_5} p_4^{N_7} h_1^{N_2} h_2^{N_4} h_3^{N_6} h_4^{N_8} \ , \tag{2.19}$$

with the tensor t_8 defined in (A.35), and ϵ totally antisymmetric in all its indices. By plugging this result (2.19) in (2.16) and by expanding the Gamma function

$$\frac{\Gamma(\frac{s}{8})\Gamma(\frac{t}{8})\Gamma(\frac{u}{8})}{\Gamma(1-\frac{s}{8})\Gamma(1-\frac{t}{8})\Gamma(1-\frac{u}{8})} = \frac{2^9}{stu} - 2\zeta(3) + \mathcal{O}(s,u,t) \ , \tag{2.20}$$

with $\zeta(3)$ the Riemann zeta-function one finds for the closed string four graviton amplitude

$$\mathcal{A}_{4\,grav} = \Big(\frac{2^9}{stu} - 2\zeta(3) + \dots\Big)\Big(t_{8\,N_1N_2N_3N_4N_5N_6N_7N_8} - \frac{1}{2}\epsilon_{N_1N_2N_3N_4N_5N_6N_7N_8}\Big)\Big(t_{8\,M_1M_2M_3M_4M_5M_6M_7M_8} - \frac{1}{2}\epsilon_{M_1M_2M_3M_4M_5M_6M_7M_8}\Big) p_1^{N_1} p_2^{N_3} p_3^{N_5} p_4^{N_7} p_1^{M_1} p_2^{M_3} p_3^{M_5} p_4^{M_7} h_1^{N_2M_2} h_2^{N_4M_4} h_3^{N_6M_6} h_4^{N_8M_8} \ . \tag{2.21}$$

The expansion (2.20) is the core of the field theory limit, we expand the full string amplitude in a regime where the external moment s,u,t are small, thus the on-shell energy of the scattered states, such that we can describe the leading terms by a field theory. Note that in this conventions the Mandelstam variables are dimensionless variables inside (2.20).

The part of the amplitude (2.21) proportional to $\frac{2^9}{stu}$ is reproduced by $d = 10$ supergravity. To reproduce the part of the amplitude (2.21) proportional to $\zeta(3)$ one needs to add an higher-dimensional operator to the effective theory. We will now argue for its structure. In linearized gravity $g_{MN} = \eta_{MN} + h_{MN}$ we can identify the external polarization representing the graviton with the linear fluctuation, where h_{MN} in (2.21) is taken to be transverse $p^M h_{MN} = 0$ and traceless $h_M{}^M = 0$, see figure I.5. Furthermore in this linearized approximation one can identify the Riemann tensor from the combination

$$R^{N_1 M_1}{}_{N_2 M_2} = 4h^{[N_1}{}_{[M_1} p^{N_2]} p_{M_2]} \ , \tag{2.22}$$

where indices are raised and lowered with η. Evaluating (2.21) in the linearized gravity limit only the combination $t_8 - t_8$ contributes to the amplitude and thus can be reproduced by adding the term

$$\frac{\zeta(3)}{3 \cdot 2^{11}} t_8 t_8 R^4 \ , \tag{2.23}$$

with

$$t_8 t_8 R^4 = t_8^{M_1 \ldots M_8} t_{8\,N_1 \ldots N_8} R^{N_1 N_2}{}_{M_1 M_2} R^{N_3 N_4}{}_{M_3 M_4} R^{N_5 N_6}{}_{M_5 M_6} R^{N_7 N_8}{}_{M_7 M_8} \ , \tag{2.24}$$

to the effective supergravity theory, which when expanded in $g \to \eta + h$ reproduces (2.21) at order h^4.[22] Although the $\epsilon_{10} - \epsilon_{10}$ part seems to appear in (2.21) by reversing the logic and expanding $\epsilon_{10}\epsilon_{10}R^4$ in linearized gravity the first non-vanishing contribution is at order h^5. Thus it can not be seen by the four-point amplitude (2.21) and one would need to compute the five-point amplitude. Alternatively this term has been derived via vanishing of the σ model β function [42, 43]. The tree-level correction to the effective action then reads

$$\mathcal{L}^{\alpha'^3}_{tree} = \alpha'^3 \frac{\zeta(3)}{3 \cdot 2^{11}} e^{-2\Phi} \Big(t_8 t_8 R^4 + \frac{1}{8} \epsilon_{10}\epsilon_{10} R^4 \Big) *_{10} 1 \ . \tag{2.25}$$

Note that we added a dilaton dependent term e^{Φ}, which could only arise at the level of the five-point function but is needed to guarantee $SL(2, \mathbb{Z})$ symmetry in type IIB thus it can be inferred by symmetry arguments. Interestingly, computing the one-loop four graviton scattering, see figure I.6, one finds

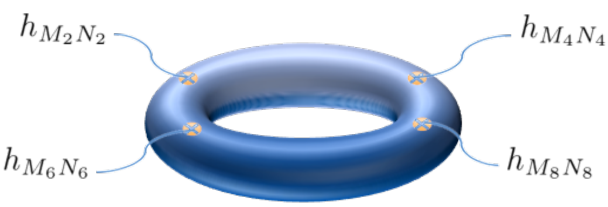


Figure I.6: Schematic depiction of the one-loop scattering amplitude with four graviton vertex operator insertions with external polarization tensors h_{MN}.

that the same structure $t_8 t_8 R^4$ appears. The term $\epsilon_{10}\epsilon_{10}R^4$ appears in the effective action only at the

[22] The factor $3 \cdot 2^{11}$ is a combination of (2.22) and the combinatorics of the four linearized graviton tree-level term $4^4 \cdot 4! = 3 \cdot 2^{11}$.

level of the five-point function when computing five graviton scattering. Thus in total one finds for the purely Riemann terms

$$\mathcal{L}^{\alpha'^3}_{R^4} = \frac{\alpha'^3}{3\cdot 2^{11}}\Big(\frac{\zeta(3)}{g_s^2} + \frac{\pi^2}{3}\Big)e^{-2\Phi}\Big(t_8 t_8 R^4 + \frac{1}{8}\epsilon_{10}\epsilon_{10}R^4\Big) *_{10} 1 \ . \tag{2.26}$$

When computing the one-loop five-point function of four gravitons and one Kalb-Ramond field, see

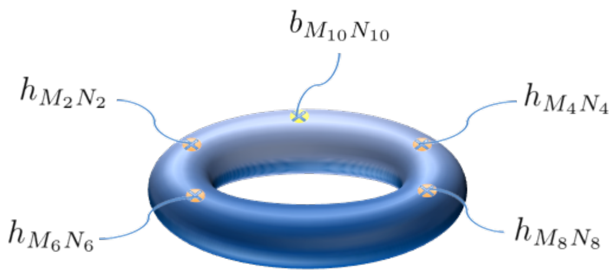


Figure I.7: Schematic depiction of the one-loop scattering amplitude with four graviton vertex operator insertions and one Kalb-Ramond field vertex operator insertion, with external polarization tensors h_{MN} and b_{MN}, respectively.

figure I.7, one finds another contribution to the effective action, given by

$$\mathcal{L}^{\alpha'^3}_{BR^4} = \frac{1}{3\cdot 2^{11}} B_2 \wedge \Big(trR^4 - \frac{1}{4}\big(trR^2\big)^2\Big). \tag{2.27}$$

This concludes our discussion of the derivation of α'-corrections to the type II supergravity theories. In the next section 2.2.1 we will take a different perspective, starting from local supersymmetry one can deduce the supergravity actions, which from a string theory point of view are its low-energy effective actions. Not all higher-derivative corrections can be easily inferred from amplitude computations and supersymmetry is a powerful tool to complete the picture.

2.2.1 Introduction to supergravity

The paradigm, which leads to supergravity is to promote global supersymmetry to be a local symmetry, where the spinor parameters $\epsilon(x)$ are arbitrary functions of space-time. In fact realizing supersymmetry in a theory of gravity enforces it to be local, conversely starting off with local supersymmetry equally enforces us to think of the local translation parameters as diffeomorphism inferring gravity [49]. Every supergravity theory contains interaction terms of the fields. The core of supergravity is given by the gravity or gauge multiplet consisting of the frame field $e^a_M(x)$, being the vielbein related to the metric describing the graviton, and a number of $\mathcal{N}$ vector-spinor fields $\psi^i_M(x)$ representing the gravitinos where $i = 1, .., \mathcal{N}$ determines the amount of supersymmetry. In a $4d$, $\mathcal{N} = 1$ theory one can add additional vector $\big(\phi, A^{\mathcal{V}}_M, \lambda^{\mathcal{V}}\big)$ and chiral multiplets $\big(z^\alpha, \chi^\alpha\big)$ to the theory and promote the well known global supersymmetry obeying couplings to local ones, where $\lambda^{\mathcal{V}}, \chi^\alpha$ are represented by Majorana spinors. Local supersymmetry can be established in $d \leq 11$ with a specific type of spinor for each dimension, as for $d = 4$ it is Majorana or Weyl. As well as supergravity can not be engineered for every space-time dimension also the amount of supersymmetry is not arbitrary, as for $d = 4$ we find $\mathcal{N} = 1, 2, 4, 8$. It is intriguing and vividly to see how local supersymmetry works, therefore we will devote the next part of this work to the discussion of the universal part of supergravity, solely containing the gravity multiplet and thus $\mathcal{N} = 1$. The action is

$$S^d_{SUGRA} = \frac{1}{2\kappa} \int R_{MNab}(\omega) e^{aM} e^{aN} *_d 1 - \bar{\psi}_\mu \gamma^{MNO} D_M \psi_O *_d 1 \quad , \tag{2.28}$$

where the volume element written in terms of vielbeins is $*_d 1 = e\, d^d x$, with $e \equiv \det e^a_M$ and $\gamma^{M_1 \ldots M_n} \equiv \gamma^{[M_1} \cdots \gamma^{M_n]}$. The gravitino covariant derivative is given by

$$D_M \psi_N \equiv \partial_M \psi_N + \frac{1}{4} \omega_{Mab} \gamma^{ab} \psi_N \quad , \tag{2.29}$$

where ω_{Mab} is the torsion free spin connection. We now want to show that the action (2.28) is invariant at linear order in the graviton and gravitino variation. Restrictions on the dimensions appear when considering higher-order variations. One finds that the following variations leave the action invariant at linear order

$$\delta e^a_M = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_M \quad , \tag{2.30}$$

$$\delta \psi_M = D_M \epsilon(x) = \partial_M \epsilon(x) + \frac{1}{4} \omega_{Mab} \gamma^{ab} \epsilon(x) \quad . \tag{2.31}$$

We will show that this is the right choice in a moment. Note that (2.31) suggests that the gravitino is the gauge field of local supersymmetry. At higher-order in variations (2.28) is not invariant under (2.30) and (2.31). In $\mathcal{N} = 1$ and $d = 4$ one needs to add a four fermion term or equivalently introduce a connection with torsion such that (2.28) stays formally invariant, but this discussion is beyond the scope of this text. For other dimensions one must add other fields to enlargen the gravity multiplet as we will see in the case of eleven dimensions later. Varying the action (2.28) by using (2.31) and (2.33) one finds

$$\delta S_{SUGRA} = \frac{1}{2\kappa^2} \int \Big(- \bar{\epsilon} \gamma^M \psi^N \big(R_{MN} - \frac{1}{2} g_{MN} R \big) + \frac{1}{4} \bar{\epsilon} \gamma^{MNO} R_{MNab} \gamma^{ab} \psi_O + \delta_{high} S_{SUGRA} \Big) *_d 1 \quad , \tag{2.32}$$

where the first term in (2.32) originates from the graviton part and the remaining terms from the gravitino part of (2.28), including the cubic variation which we denote as $\delta_{high} S_{SUGRA}$.[23] For (2.28) to possess supersymmetry we need $\delta S_{SUGRA} = 0$. Focusing on the linear fluctuation in (2.32), we need to evaluate the product of γ-matrices using some γ-matrix algebra. Let us note that

$$\gamma^{MNO} \gamma^{ab} = e^{aL} e^{aP} \gamma^{MNO} \gamma_{LP} = e^{aL} e^{aP} \Big(\gamma^{MNO}{}_{LP} + 6 \gamma^{[MN}{}_{[P}\, \delta^{O]}{}_{L]} + 6 \gamma^{[M} \delta^{N}{}_{[P} \delta^{O]}{}_{L]} \Big) \quad , \tag{2.34}$$

which using the symmetries of the Riemann tensor leads to

$$\gamma^{MNO} \gamma^{ab} R_{MNab} = \gamma^{MNO}{}_{LP} R_{MN}{}^{LP} + 2 R_{MN}{}^{O}{}_{P} \gamma^{MNP} + 4 \gamma^M R_{MN}{}^{ON} + 2 \gamma^O R_{MN}{}^{MN} \quad . \tag{2.35}$$

[23]To derive (2.32) we use

$$\delta e^M_a = -\frac{1}{2} \bar{\epsilon} \gamma^M \psi_a \quad , \qquad \delta e = \frac{1}{2} e \big(\bar{\epsilon} \gamma^M \psi_M \big) \quad . \tag{2.33}$$

Using the first Bianchi identity (A.3) for the Levi-Civita connection without torsion, the first two terms in (2.35) vanish. Thus using (2.35) we infer for the variation of the gravitino in (2.32) that

$$\frac{\bar{\epsilon}}{4}\gamma^{MNO}R_{MNab}\gamma^{ab}\psi_O = \bar{\epsilon}\gamma^M\psi^N\left(R_{MN} - \frac{1}{2}g_{MN}R\right) \ , \qquad (2.36)$$

which exactly cancels the variation originating from the graviton at linear order.

Let us now turn to the study of eleven-dimensional supergravity, the low energy limit of M-theory. We already noted that the maximal dimension where supergravity can be realized is eleven, furthermore eleven-dimensional supergravity is very simple compared to its cousins in lower dimensions. Following [50] one can argue by contradiction that there are no supergravity theories in more than eleven dimensions, moreover it is a unique theory without any cousins. Assume there exists a supergravity theory in $d \geq 11$ dimensions than one can compactify it on a torus $M_4 \times T^{d-4}$ down to four dimensions. The concept of KK reductions will be elaborated in section 2.4, for now let us note that in this compactification all the supersymmetry is preserved in the lower dimensional theory. To start with an eleven-dimensional supergravity theory we need the minimal content of a graviton and one gravitino (2.28). The gravitino is represented by an eleven-dimensional 32-component Majorana spinor, which gives rise to 8 Majorana gravitinos and 56 Majorana graviphotons upon reduction to four dimensions, which saturate the representation of maximal spin of two, and thus the field-content of the maximal $\mathcal{N} = 8$ supergravity algebra. Thus by adding another graviton in eleven dimensions, one has $\mathcal{N} = 2$ supersymmetry, or alternatively by increasing the dimension to $d \geq 12$ the arising degrees of freedom upon reduction need to sit in a representation of spin $\geq \frac{5}{2}$, of which there do not exist consistent interactions obeying local supersymmetry, thus the claim follows.

This discussion is quite illuminating and already hints in the direction of a simple theory at least containing the graviton $\hat{g}_{MN}$ and a gravitino $\hat{\psi}$. We have seen already in section 1.2.3 that the spectrum is completed by the 3-form field $\hat{A}_{MNO}$. Taking a more constructive perspective here we follow the argument of [34]. The graviton in eleven dimensions is represented by a traceless symmetric

field	**off-shel d.o.f**	**on-shell d.o.f**
ϕ	1	1
λ	$2^{\frac{d}{2}\rfloor}$	$\frac{1}{2}2^{\frac{d}{2}\rfloor}$
A_M	$d-1$	$d-2$
$A_{M_1\ldots M_p}$	$\begin{pmatrix} d-2 \\ p \end{pmatrix}$	$\begin{pmatrix} d-2 \\ p \end{pmatrix} = \frac{(d-2)!}{p!(d-2-p)!}$
ψ_M	$(d-1)2^{\frac{d}{2}\rfloor}$	$\frac{1}{2}(d-3)2^{\frac{d}{2}\rfloor}$
g_{MN}	$\frac{1}{2}d(d-1)$	$\frac{1}{2}d(d-3)$

Table 2.3: On-shell and off-shell degrees of freedom of various bosonic and fermionic fields relevant in supergravity theories, with the symbol $\rfloor$ denoting the integer part of this number. Furthermore, note that since A is antisymmetric in all its indices $p \leq d$.

tensor according to 2.2.1 incorporating 44 on shell bosonic states. The gravitino represented by a Majorana spinor transforms under a representation of $SO(d-2)$ according to 2.2.1, represented by 128 real fermionic states. Since supersymmetry demands an equal number of fermionic and bosonic states in the theory we miss 84 bosonic degrees of freedom, which are delivered by the three-form field A_{MNO} carrying exactly this amount of bosonic states. This is a necessary but not sufficient requirement for $(g_{MN}, A_{MNO}, \psi_M)$ being the complete field content. Let us now argue for the $d = 11, \mathcal{N} = 1$ supergravity action, whose construction is very technical. Hence let us be very brief here. Incorporating the field-content $(g_{MN}, A_{MNO}, \psi_M)$, we minimally need to have the action

$$S_0 = \frac{1}{2\kappa_{11}^2} \int \hat{R}_{MNab}(\omega)\hat{e}^{aM}\hat{e}^{aN}\hat{*}1 - \bar{\hat{\psi}}_\mu \gamma^{MNO} D_M \hat{\psi}_O \hat{*}1 - \frac{1}{2\cdot 4!}\hat{G}_{MNOP}\hat{G}^{MNOP}\hat{*}1 \ , \tag{2.37}$$

where we have added to (2.28) a kinetic term of the three-form field with $\hat{G} = d\hat{C}$ its four-form field strength. Surely, this action is not invariant under local supersymmetry. Let us postulate the supersymmetry transformation rules altered by terms with coefficients c_1, c_2, c_3 compared to (2.30) and (2.31)

$$\delta \hat{e}^a_M = \frac{1}{2}\bar{\epsilon}\gamma^a \hat{\psi}_M \ , \tag{2.38}$$

$$\delta\hat{\psi}_M = D_M \epsilon(x) + \left(c_1 \gamma^{NLOP}{}_M + c_2 \gamma^{LOP}\delta^N{}_M\right)\hat{G}_{NLOP}\,\epsilon \ , \tag{2.39}$$

$$\delta\hat{C}_{MNO} = -c_3\,\bar{\epsilon}\gamma_{[MN}\,\hat{\psi}_{O]} \ . \tag{2.40}$$

From (2.39) one may infer that the variation of the gravitino dependent term in (2.37) will solely give rise to $\hat{G}$ dependent terms in δS_0. Thus by considering global supersymmetry variations of (2.37) and by neglecting the Einstein-Hilbert term one infers that $c_1 = \frac{c_3}{216\sqrt{2}}$ and $c_2 = -\frac{c_3}{27\sqrt{2}}$. To fix $c_3 = \frac{3}{2^{3/2}}$ one computes the commutator of two global supersymmetry transformations $[\delta_1, \delta_2]A_{MNOP}$ and matches this with the local supersymmetry algebra. By promoting $\epsilon \to \epsilon(x)$ to the actual case of interest, one varies (2.37) and finds a piece $\mathcal{J}^M D_M\epsilon$ with the coefficient being the supercurrent

$$\hat{\mathcal{J}}^M = \frac{1}{3\cdot 2^4}\left(\gamma^{NLOPMQ} + 3\cdot 2^2 \gamma^{NL}\hat{g}^{MO}\hat{g}^{PQ}\right)\hat{G}_{NLOP}\,\hat{\psi}_Q \ . \tag{2.41}$$

Constructing δS_0 one finds terms proportional to $\gamma^{M_1\dots M_9} \propto \epsilon^{M_1\dots M_{11}}\gamma_{M_{10}M_{11}}$, which do not cancel and lead to the introduction of the Chern-Simons term to the action given by[24]

$$S_{CS} = -\frac{1}{6}\frac{1}{2\kappa^2}\int \hat{C}\wedge\hat{G}\wedge\hat{G} \ . \tag{2.45}$$

[24] The remaining uncanceled term hase the structure

$$\bar{\epsilon}\,\gamma^{M_1\dots M_9}F_{M_1M_2M_3M_4}F_{M_5M_6M_7M_8}\psi_{M_9} \propto \bar{\epsilon}\,\epsilon^{M_1\dots M_{11}}\gamma_{M_{10}M_{11}}F_{M_1M_2M_3M_4}F_{M_5M_6M_7M_8}\psi_{M_9} \tag{2.42}$$

$$= \bar{\epsilon}\,\delta(A_{M_9M_{10}M_{11}})\epsilon^{M_1\dots M_{11}}F_{M_1M_2M_3M_4}F_{M_5M_6M_7M_8} \ . \tag{2.43}$$

Furthermore the variation of the Chern-Simons term derives to

$$\delta\int\hat{C}\wedge\hat{G}\wedge\hat{G} = \int 2(d\delta C)\wedge\hat{G}\wedge\hat{C} + \hat{\delta}C\wedge\hat{G}\wedge\hat{G} = 3\int\hat{\delta}C\wedge\hat{G}\wedge\hat{G} \ , \tag{2.44}$$

which uses the Bianchi identity $d\hat{G} = 0$. By comparing (2.42) and (2.44) one infers that the missing term in the Lagrangian is exactly the Chern-Simons term (2.45).

This completes the main ingredients to the $11d$, $\mathcal{N}=1$ supergravity action. However, some small additional modifications are needed to indeed establish local supersymmetry. Since we will only be concerned with the bosonic field-content of the supergravity theories from now on, we present the bosonic and fermionic parts of the action separately. The bosonic part was already stated in (1.52), but for completeness we give it again

$$S_B^{(0)} = \frac{1}{2\kappa_{11}^2} \int \left[\hat{R} \hat{*} 1 - \frac{1}{2} \hat{G} \wedge \hat{*} \hat{G} - \frac{1}{6} \hat{C} \wedge \hat{G} \wedge \hat{G} \right] \; . \tag{2.46}$$

The fermionic part results in

$$S_F^{(0)} = \frac{1}{2\kappa_{11}^2} \int \big(\hat{R}(\omega^T) - \hat{R}\big) \hat{*} 1 - \bar{\hat{\psi}}_\mu \gamma^{MNO} \big[D_M(\omega^T) + \frac{1}{2^6} \bar{\hat{\psi}}_P \gamma^{PQ}{}_{Mab} \hat{\psi}_Q \gamma^{ab}\big] \hat{\psi}_O \hat{*} 1 - \bar{\hat{\psi}}_M \big[\hat{\mathcal{J}}^M + \frac{3}{\sqrt{2}} \hat{\mathcal{J}}^M_{\hat{\psi}}\big] \; , \tag{2.47}$$

with $D_M(\omega^T)$ the connection with torsion given by

$$\omega^T_{Mab} = \omega_{Mab} + K_{Mab} \qquad \text{with} \qquad K_{Mab} = -\frac{1}{4} \Big(\bar{\hat{\psi}}_M \gamma_b \hat{\psi}_a - \bar{\hat{\psi}}_a \gamma_M \hat{\psi}_b + \bar{\hat{\psi}}_b \gamma_a \hat{\psi}_M \Big) \; , \tag{2.48}$$

and

$$\hat{\mathcal{J}}_{\hat{\psi}}{}^M = \frac{1}{3 \cdot 2^4} \big(\gamma^{NLOPMQ} + 3 \cdot 2^2 \gamma^{NL} \hat{g}^{MO} \hat{g}^{PQ} \big) \bar{\hat{\psi}}_{[N} \gamma_{LO} \hat{\psi}_{P]} \hat{\psi}_Q \; . \tag{2.49}$$

[25] To accomplish local supersymmetry came at the cost of adding the term K_{Mab} to the gravitino variation (2.39), by using the connection with torsion $D \to D(\omega^T)$. We will next discuss the bosonic part of the $3d$, $\mathcal{N}=1$ and $3d$, $\mathcal{N}=2$ supergravity action coupled to chiral and vector multiplets. The reason for dropping the fermionic sector is due to the complication arising from the necessity of γ-matrices. As seen in the example of (2.46) and (2.47) the appearing structures in the bosonic sector are generically simpler. Local supersymmetry gives very strong constraints on the couplings in three and four dimensions, such that in practice it provides sufficient evidence for local supersymmetry to verify that these constraints are met, e.g. when deriving an action from a Kaluza-Klein reduction including α' corrections where supersymmetry of the background can not be shown directly, see e.g. section 8.

2.2.2 Relevant supergravity theories in F-theory

In (2.28) we gave the minimal field-content of $4d$, $\mathcal{N}=1$ supergravity. To describe nature one needs to couple these fields to vector and chiral multiplets also referred to as matter multiplets. In fact it turns out that for theories with more than eight real supercharges the couplings of the kinetic terms are strongly fixed by the field content. However, $4d$, $\mathcal{N}=1$ supergravity omits four real supercharges and the couplings can thus depend on more general functions of the fields in a systematic way, given by

$$S^{(4)}_{\mathcal{N}=1} = \frac{1}{\kappa_4^2} \int_{M_4} \left(\frac{1}{2} R * 1 - K_{M\bar{N}} \nabla M^M \wedge * \nabla \bar{M}^{\bar{N}} - \frac{1}{2} \text{Re} f_{IJ} F^I \wedge * F^J - \frac{1}{2} \text{Im} f_{IJ} F^I \wedge F^J - V * 1 \right) \; . \tag{2.50}$$

[25] Note that the name $\hat{\mathcal{J}}_{\hat{\psi}}$ is due to the similarity to $\hat{\mathcal{J}}$ under the exchange of $\hat{G}_{NLOP} \leftrightarrow \bar{\hat{\psi}}_{[N} \gamma_{LO} \hat{\psi}_{P]}$.

With the scalar potential given as the sum of the F- and D-term potential

$$V = V_D + V_F \text{ with } V_D = -\frac{1}{2}\mathrm{Re}f^{-1\,IJ} D_I D_J \text{ and } V_F = e^K \left(K^{M\bar{N}} D_M W D_{\bar{N}} \bar{W} - 3|W|^2\right) . \tag{2.51}$$

The complex bosonic fields M^N are in the chiral multiplet and may be gauged along the direction of a Killing vector using the covariant derivative

$$\nabla M^N = dM^N + X_I^N A^I \ , \tag{2.52}$$

with constant Killing vectors X_I^N. The couplings of these complex scalar are completely determined by a function called the Kähler potential $K(M, \bar{M})$, as $K_{M\bar{N}} = \partial_M \partial_{\bar{M}} K(M\bar{M})$. Whereas the gauge couplings are given by the gauge kinetic function $f(M)$. Note that the scalar potential (2.51) is written in terms of the Kähler-covariant derivative $D_M W = \partial_M W + (\partial_M K)W$.

As the $4d$, $\mathcal{N} = 1$ also the $3d$, $\mathcal{N} = 2$ theory admits four supercharges. In general the gravity multiplet can be coupled to a number of complex scalars N^A in chiral multiplets, which are coupled to non-dynamical vectors. In the following, we will only consider the ungauged case and can hence start with a three-dimensional theory with only gravity and chiral multiplets.[26]

The bosonic part of the $3d$, $\mathcal{N} = 2$ action then reads [52]

$$S^{(3)}_{\mathcal{N}=2} = \frac{1}{\kappa_3^2} \int \frac{1}{2} R_3 *_3 1 - K_{N^A \bar{N}^B}\, dN^A \wedge *_3 d\bar{N}^B - (V_F + V_D) *_3 1\,. \tag{2.53}$$

Supersymmetry ensures that the metric $K_{N^A\bar{N}^B}$ is actually encoded in a real Kähler potential $K(N, \bar{N})$ as $K_{N^A\bar{N}^B} = \partial_{N^A}\partial_{\bar{N}^{\bar{B}}} K$. A scalar F-term potential can arise from a holomorphic superpotential $W(N)$ and takes the form

$$V_F = e^K \left(K^{N^A\bar{N}^B} D_{N^A} W \overline{D_{N^B} W} - 4|W|^2\right) , \tag{2.54}$$

where $K^{N^A\bar{N}^B}$ is the inverse of $K_{N^A\bar{N}^B}$ and $D_{N^A} W = \partial_{N^A} W + (\partial_{N^A} K)W$ is the Kähler covariant derivative. The scalar D-term potential is given by

$$V_D = \left(K^{N^A\bar{N}^B} \partial_{N^A}\mathcal{N} \partial_{\bar{N}^B}\mathcal{N} - \mathcal{N}^2\right) \ , \tag{2.55}$$

where $\mathcal{N}$ is real function in the fields N_A.

In order to match the action (2.53) with the dimensional reduction of eleven-dimensional supergravity, it turns out to be useful to dualize some of the complex scalars N^A in the chiral multiplets into $3d$ vectors. Therefore, we decompose $N^A = \{M^I, T_i\}$ and split the index as $A = (I, i)$. Note that for our purposes in chapter II and III it would suffice to only consider (2.53) with propagating $\{T_i\}$, since we only consider the vector multiplet arising form the vectors and the Kähler fluctuations but do not consider complex structure deformation, which form chiral multiplets $\{M^I\}$. However, let us emphasize that in $3d$ one can indeed dualize the vector multiplet to a chiral multiplet as we

[26] Let us stress that most of the derivation presented in the following can be generalized to the case with non-trivial gaugings in a straightforward fashion [51].

will show in the following, for its bosonic part. This is the case if the real scalars $\text{Im}T_i$ have shift symmetries, since then it is possible to dualize them to vectors A^i. The real parts of T_i are redefined to real scalars L^i that naturally combine with the vectors A^i into the bosonic components of $3d$, $\mathcal{N}=2$ vector multiplets. The dual $3d$, $\mathcal{N}=2$ action reads

$$\begin{aligned} S^{(3)}_{\mathcal{N}=2} \;=\; & \frac{1}{\kappa_3^2}\int \frac{1}{2}R_3 *_3 1 - \tilde{K}_{I\bar{J}}\, dM^I \wedge *_3 d\bar{M}^{\bar{J}} + \frac{1}{4}\tilde{K}_{ji} dL^j \wedge *_3 dL^i \\ & + \frac{1}{4}\tilde{K}_{ji} F^j \wedge *_3 F^i + \text{Im}[\tilde{K}_{Ij} dM^I] \wedge F^j - V_F *_3 1\,. \end{aligned} \tag{2.56}$$

The new couplings can now be derived from a real function $\tilde{K}(L, M, \bar{M})$ known as the kinetic potential according to

$$\tilde{K}_{ji} = \partial_{L^j}\delta_{L^i}\tilde{K}\ , \qquad \tilde{K}_{I\bar{J}} = \partial_{M^I}\delta_{\bar{M}^{\bar{J}}}\tilde{K}\ , \qquad \tilde{K}_{Ij} = \partial_{M^I}\delta_{L^j}\tilde{K}\ . \tag{2.57}$$

The Kähler potential K and kinetic potential $\tilde{K}$ as well as the fields $\text{Re}T_i$ and L^i are related by a Legendre transform. Explicitly, the relations are given by

$$\tilde{K}(L, M, \bar{M}) = K(T, \bar{T}, M, \bar{M}) + \text{Re}T_i\, L^i\ , \qquad L^i = -\frac{\delta K}{\delta \text{Re}T_i}\ . \tag{2.58}$$

In reverse, one finds that

$$\text{Re}T_i = \frac{\delta \tilde{K}}{\delta L^i}\ . \tag{2.59}$$

2.3 Higher-derivative corrections to M-theory

By introducing eleven-dimensional supergravity in section 2.2.1 and discussing higher-derivative or α'-corrections to II supergravities in section 2.2, we have laid the foundation to discuss the main topic of this work, higher-derivative corrections to M-theory. Note that higher l_M corrections to the M-theory effective action, given by eleven-dimensional supergravity can not be computed by a similar procedure as in string theory, due to a lack of knowledge of the theory itself. However, this is not necessary since there exists a duality between IIA supergravity, which in the strong coupling regime $g_s \to \infty$ can be described by $11d$ supergravity, see e.g. figure I.4, which in return compactified on S^1 gives back IIA supergravity. One can thus lift the corrections from IIA to covariant expression in M-theory, and simple check their correctness due to a circular reduction. This and other techniques were used to derive l_M-corrections to eleven-dimensional supergravity [53, 54, 55, 56, 57, 58, 59, 60]. The first non-vanishing higher-derivative corrections to M-theory carry eight derivatives and are of order l_M^6.

Note that the most recent result of [60] claims to describe the complete eight-derivative bosonic sector of M-theory. The fermionic higher-derivative corrections as well as the corrections to the gravitino variations are not known at order l_M^6, thus a supersymmetric completion seems out of reach. However, we propose in section 6 corrections to the gravitino variations based on an indirect argument. We next introduce certain higher-derivative corrections relevant for the discussion in chapter II and III. Although these corrections only represent a subset of the known eight-derivative corrections to eleven-dimensional supergravity, they are complete in a sense that upon reduction the neglected terms would give rise to higher-order corrections in the dimensionful expansion parameter

$$\alpha^2 = \frac{(4\pi\kappa_{11}^2)^{\frac{2}{3}}}{(2\pi)^4 3^2 2^{13}} \,. \tag{2.60}$$

Note that $\alpha^2 \propto l_M^6$, and thus our discussion will involve terms of order α^2. Our starting point will be the eleven-dimensional two-derivative, $\mathcal{N}=1$ supergravity action (1.52). Recall that the dynamical fields of this supergravity theory arrange in an $\mathcal{N}=1$ gravity multiplet, with bosonic fields being the eleven-dimensional metric $\hat{g}_{NM}$ and a three-form $\hat{C}_{MNP}$ with field strength $\hat{G}_{QMNP} = \partial_{[Q}\hat{C}_{MNP]}$.

The full relevant action at eighth order in derivatives, which is crucial for our studies in chapter II and III, takes the form

$$S = S^{(0)} + \alpha^2 S^{(2)}_{\hat{R}^4} + \alpha^2 S^{(2)}_{\hat{G}^2\hat{R}^3} + \alpha^2 S^{(2)}_{(\hat{\nabla}\hat{G})^2\hat{R}^2} + \mathcal{O}(\hat{G}^3\alpha^2) + \mathcal{O}(\alpha^3) \,, \tag{2.61}$$

with the zeroth order action and the eight-derivative terms given by

$$S^{(0)} = \frac{1}{2\kappa_{11}^2}\int \Big[\hat{R}\hat{*}1 - \frac{1}{2}\hat{G}\wedge\hat{*}\hat{G} - \frac{1}{6}\hat{C}\wedge\hat{G}\wedge\hat{G}\Big] \,, \tag{2.62}$$

$$S^{(2)}_{\hat{R}^4} = \frac{1}{2\kappa_{11}^2}\int \Big[(\hat{t}_8\hat{t}_8 - \frac{1}{24}\hat{\epsilon}_{11}\hat{\epsilon}_{11})\hat{R}^4\hat{*}1 - 3^2 2^{13}\hat{C}\wedge\hat{X}_8\Big] \,, \tag{2.63}$$

$$S^{(2)}_{\hat{G}^2\hat{R}^3} = \frac{1}{2\kappa_{11}^2}\int \Big[-(\hat{t}_8\hat{t}_8 + \frac{1}{96}\hat{\epsilon}_{11}\hat{\epsilon}_{11})\hat{G}^2\hat{R}^3\hat{*}1\Big] \,, \tag{2.64}$$

$$S^{(2)}_{(\hat{\nabla}\hat{G})^2\hat{R}^2} = \frac{1}{2\kappa_{11}^2}\int \hat{s}_{18}(\hat{\nabla}\hat{G})^2\hat{R}^2\hat{*}1 \,. \tag{2.65}$$

The terms at higher-order in $\hat{G}$ and α will not be needed in what follows as their contribution is higher-order in α when evaluated for both the ansatz in chapter II as well as in chapter III.

Let us now discuss the various couplings in (2.63)-(2.65) in more detail. In (2.63) we used the definition

$$\hat{X}_8 = \frac{1}{192}\Big(\mathrm{Tr}\hat{\mathcal{R}}^4 - \frac{1}{4}(\mathrm{Tr}\hat{\mathcal{R}}^2)^2\Big) \,, \tag{2.66}$$

where $\hat{\mathcal{R}}$ is the eleven-dimensional curvature two-from $\hat{\mathcal{R}}^M{}_N = \frac{1}{2}\hat{R}^M{}_{NPQ}dx^P\wedge dx^Q$, and

$$\begin{aligned}
\hat{\epsilon}_{11}\hat{\epsilon}_{11}\hat{R}^4 &= \epsilon^{R_1R_2R_3M_1\ldots M_8}\epsilon_{R_1R_2R_3N_1\ldots N_8}\hat{R}^{N_1N_2}{}_{M_1M_2}\hat{R}^{N_3N_4}{}_{M_3M_4}\hat{R}^{N_5N_6}{}_{M_5M_6}\hat{R}^{N_7N_8}{}_{M_7M_8} \,, \\
\hat{t}_8\hat{t}_8\hat{R}^4 &= \hat{t}_8^{M_1\ldots M_8}\hat{t}_{8\,N_1\ldots N_8}\hat{R}^{N_1N_2}{}_{M_1M_2}\hat{R}^{N_3N_4}{}_{M_3M_4}\hat{R}^{N_5N_6}{}_{M_5M_6}\hat{R}^{N_7N_8}{}_{M_7M_8} \,,
\end{aligned} \tag{2.67}$$

where ϵ_{11} is the eleven-dimensional totally anti-symmetric epsilon tensor and t_8 is given explicitly in (A.35) in appendix A. Using ϵ_{11} and t_8 the explicit form for the terms in (2.64) is given by

$$\begin{aligned}
\hat{\epsilon}_{11}\hat{\epsilon}_{11}\hat{G}^2\hat{R}^3 &= \hat{\epsilon}^{RM_1\ldots M_{10}}\hat{\epsilon}_{RN_1\ldots N_{10}}\hat{G}^{N_1N_2}{}_{M_1M_2}\hat{G}^{N_3N_4}{}_{M_3M_4}\hat{R}^{N_5N_6}{}_{M_5M_6}\hat{R}^{N_7N_8}{}_{M_7M_8}\hat{R}^{N_9N_{10}}{}_{M_9M_{10}} \,, \\
\hat{t}_8\hat{t}_8\hat{G}^2\hat{R}^3 &= \hat{t}_8^{M_1\ldots M_8}\hat{t}_{8N_1\ldots N_8}\hat{G}^{N_1}{}_{M_1R_1R_2}\hat{G}^{N_2}{}_{M_2}{}^{R_1R_2}\hat{R}^{N_3N_4}{}_{M_3M_4}\hat{R}^{N_5N_6}{}_{M_5M_6}\hat{R}^{N_7N_8}{}_{M_7M_8} \,.
\end{aligned} \tag{2.68}$$

Finally, we need to introduce the tensor $\hat{s}_{18}^{N_1 \dots N_{18}}$ appearing in (2.65), which will be only relevant for the discussion in chapter III. Unfortunately, the precise form of $\hat{s}_{18}$ is not known. However, one can fix significant parts of it following [61]. In order to express these parts we use the basis $B_i,\ i = 1, ..., 24$ of [61], that labels all unrelated index contractions in $\hat{s}_{18}(\hat{\nabla}\hat{G})^2\hat{R}^2$. The B_i are explicitly given in (A.36). The result can then be expressed in terms of a four-point amplitude contribution $\mathcal{A}$ and a linear combination of six contributions $\mathcal{Z}_i$ which do not affect the 4-point amplitude as

$$\hat{s}_{18}(\hat{\nabla}\hat{G})^2\hat{R}^2 = \hat{s}_{18}^{N_1 \dots N_{18}} \hat{R}_{N_1 \dots N_4} \hat{R}_{N_5 \dots N_8} \hat{\nabla}_{N_9} \hat{G}_{N_{10} \dots N_{13}} \hat{\nabla}_{N_{14}} \hat{G}_{N_{15} \dots N_{18}} = \mathcal{A} + \sum_n a_n \mathcal{Z}_n \,. \tag{2.69}$$

The combinations $\mathcal{A}$ and $\mathcal{Z}_n$ are then given in terms of the basis elements as

$$\begin{aligned}
\mathcal{A} &= -24B_5 - 48B_8 - 24B_{10} - 6B_{12} - 12B_{13} + 12B_{14} + 8B_{16} - 4B_{20} + B_{22} + 4B_{23} + B_{24} \,, \\
\mathcal{Z}_1 &= 48B_1 + 48B_2 - 48B_3 + 36B_4 + 96B_6 + 48B_7 - 48B_8 + 96B_{10} \\
&\quad + 12B_{12} + 24B_{13} - 12B_{14} + 8B_{15} + 8B_{16} - 16B_{17} + 6B_{19} + 2B_{22} + B_{24} \,, \\
\mathcal{Z}_2 &= -48B_1 - 48B_2 - 24B_4 - 24B_5 + 48B_6 - 48B_8 - 24B_9 - 72B_{10} - 24B_{13} + 24B_{14} - B_{22} + 4B_{23} \,, \\
\mathcal{Z}_3 &= 12B_1 + 12B_2 - 24B_3 + 9B_4 + 48B_6 + 24B_7 - 24B_8 + 24B_{10} \\
&\quad + 6B_{12} + 6B_{13} + 4B_{15} - 4B_{17} + 3B_{19} + 2B_{21} \,, \\
\mathcal{Z}_4 &= 12B_1 + 12B_2 - 12B_3 + 9B_4 + 24B_6 + 12B_7 - 12B_8 + 24B_{10} + 3B_{12} + 6B_{13} + 4B_{15} - 4B_{17} + 2B_{20} \,, \\
\mathcal{Z}_5 &= 4B_3 - 8B_6 - 4B_7 + 4B_8 - B_{12} - 2B_{14} + 4B_{18} \,, \\
\mathcal{Z}_6 &= B_4 + 2B_{11} \,.
\end{aligned} \tag{2.70}$$

However, we will show in chapter III that the terms $\mathcal{Z}_3$ to $\mathcal{Z}_6$ vanish both on the considered background solution and their perturbed cousins to the order in α we are considering.

2.4 The paradigm of compactifications & $4d$ effective physics

The paradigm of compactifications in the context of string theory originates the experimental fact that our Universe persists of four extended space-time dimensions, in contrast to the ten dimensions predicted by superstring theory. To make these two sides compatible six space-like dimensions need to be curled-up or in other words compact. Note that the size of the internal space needs to be large compared to the string length for the effective field theory description in ten dimensions to be a good approximation, see the discussion in section 2.1, which we always assume henceforth when talking about compactifications. The metric describing the Lorentzian space-time and the internal six-dimensional Euclidean space is referred to as background. Estimates on the upper bound of the size of the internal space are given by experiments. These "extra dimensions" as referred to in this context, have escaped detection so far, thus must lie below the length scales probed by existing particle detectors $\sim 10^{-19}m$. The goal of the procedure of compactification is to derive a lower dimensional - in this case $4d$ - theory, that solely describes the dynamics of the lower-dimensional fields, which is induced by the action of the higher-dimensional fields. The concept of deriving the dynamics of a theory sourced by a higher-dimensional theory amongst compactification is interesting on its own right, despite its natural application in string theory. Reaching back to Kaluza [62] and Klein [63], whose original attempt to unify general relativity and electro-magnetism, introduced the idea that space-time could be of higher dimension than the extended ones we observe.

The procedure due to Kaluza and Klein is quite different for theories in which gravity is dynamical compared to those where it is not. In the case that gravity is non-dynamical, one simply premises a background consisting of an external and an internal space, and uses the existence of an eigenfunction expansion along the internal directions to expand the various fields and ingrate them out. When gravity is dynamical one needs to be proceed more carefully for three reasons. Firstly, not every background will be a solution to Einsteins equations. Secondly, since every field couples to gravity and thus back-reacts on the geometry one can only consider small fluctuations around their background field configurations. Finally, one can fluctuate the geometry itself around its background field configuration, which gives rise to additional dynamical degrees of freedom in the lower-dimensional theory.

2.4.1 Preliminaries of geometry

Let us briefly introduce a few mathematical concepts. For a more exclusive discussion we suggest, e.g. [64, 65, 66, 67, 68]. On a manifold with an associated k-vector bundle and equipped with a connection one can parallel transport a vector along closed curves over the manifold. The vector which was transported from p_0 along a closed curve back to the point p_0, is generically different from the original one, such that one can get the final vector by acting on the original vector with a linear transformation in $GL(k,\mathbb{R})$. In other words the parallel transport along a certain curve defines a linear transformation on the vector space at p_0. It is intuitive that the set of all linear transformations obtained by linear transport may not span the full set of linear transformations $GL(k,\mathbb{R})$ but a subgroup thereof - the so called holonomy group of the connection. One drops the dependence of the base point p_0 since on connected manifolds the holonomy groups at different points are conjugate in $GL(k,\mathbb{R})$ w.r.t. to each other.

We restrict ourselves to the case of Riemannian manifolds (M,g) of dimension n. Its holonomy group is the one of the Levi-Civita connection ∇ on the tangent bundle TM.

We will make the above statements more precise in the following. The parallel transport of tangent vectors along a curve $\gamma(t)$ in M form a section $X\in TM$, which needs to obey $\nabla_{\dot\gamma(t)}X=0,\forall t$. Thus we can define the holonomy group at a point $p\in M$ via the linear and invertible map $g_\gamma: TM_p\to TM_p$, given by the parallel transport along a closed curve γ to be

$$Hol(\nabla,p)=\{g_\gamma\in GL(n,\mathbb{R})|\ \text{closed}\ \ \gamma\in M\ \ \text{with}\ \ \gamma(0)=p\}\ . \tag{2.71}$$

If we restrict the curves to be contractible one can then define the restricted holonomy $Hol^0(\nabla,p)$, analogously. We drop the base point dependence for the further discussion. On a Riemannian manifold without further structures the holonomy group is $O(n)$ or $SO(n)$ if it is orientible. According to Berger [69] there exists a classification of the holonomy groups on Riemannian manifolds depending of to their additional properties, see table 2.4. We continue by discussing maybe the most relevant manifold in

class of manifold	**defining properties**	**dimension**	**holonomy**
Riemannian	(∇, g)	n	$SO(n)$
Kaehler	(∇, g, J) $dJ = 0$	$2n$	$U(n)$
Calabi-Yau	(∇, g, J), $dJ = 0$, $R_{mn} = 0$	$2n$	$SU(n)$
Quaternion-Kähler	$(\nabla, g, J_{i=1,2,3})$, $R_{mn} = \kappa g_{mn}, \kappa \neq 0$	$4n$	$Sp(n) \times Sp(1)$
Hyper-Kähler	$(\nabla, g, J_{i=1,2,3})$, $dJ_1 = 0$, $R_{mn} = \kappa g_{mn}, \kappa = 0$	$4n$	$Sp(n)$
G_2	$R_{mn} = 0$, $G_2 = \{g \in GL(7, \mathbb{R})\|$preserve non-degenerate 3-from$\}$	7	G_2
Spin-Seven	$R_{mn} = 0$, $\exists$ parallel Cayley four-form	8	$Spin(7)$

Table 2.4: Berger's classification of simply connected Riemannian manifolds whose holonomy group acts irreducible on the tangent bundle and which are not locally symmetric.

string theory - the Calabi-Yau manifold. Let us start by noting that a Kähler manifold (X, g, J, ∇) is a complex manifold of complex dimension n, endowed with a Hermitian metric g such that its Kähler form is closed.[27] The components of metric and Kähler form can be written in holomorphic and antiholomorphic coordinates $m, \bar{m} = 1, \dots, n$ as related to the complex structure on X

$$g_{m\bar{n}} = g_{\bar{n}m} \ , \qquad g_{mn} = g_{\bar{m}\bar{n}} = 0 \ , \qquad J_{m\bar{n}} = i g_{m\bar{n}} \ , \tag{2.72}$$

and, in particular the only non-vanishing components of the Riemann tensor are given by

$$R_{m\bar{m}n\bar{n}} = -R_{\bar{m}mn\bar{n}} = -R_{\bar{m}m\bar{n}n} \ . \tag{2.73}$$

The first Bianchi identity (A.3) implies a further symmetry of the Riemann tensors, such that

$$R_{m\bar{m}n\bar{n}} = R_{n\bar{m}m\bar{n}} = R_{m\bar{n}n\bar{m}} \ . \tag{2.74}$$

While the second Bianchi identity (A.3) then furthermore results in the relations

$$\nabla_r R_{m\bar{m}n\bar{n}} = \nabla_m R_{r\bar{m}n\bar{n}} = \nabla_n R_{m\bar{m}r\bar{n}} \ , \qquad \nabla_{\bar{r}} R_{m\bar{m}n\bar{n}} = \nabla_{\bar{m}} R_{m\bar{r}n\bar{n}} = \nabla_{\bar{n}} R_{m\bar{m}n\bar{r}} \ . \tag{2.75}$$

Thus the Bianchi identities become implemented as simple symmetries on the Riemann tensors.

Let us proceed with the definition of a Calabi-Yau manifold [67].

Definition: A Calabi-Yau manifold is a compact Kähler manifold (X, J, g, ∇), of dimensions $n \geq 2$ with $Hol(\nabla) = SU(n)$, or for $n = 1$ uniquely given by the Torus T^2.

Note that on a compact Kähler manifold (X, J, g, ∇), $Hol(\nabla) = SU(n)$ is equivalent to the existence of a nowhere vanishing holomorphic $(n, 0)$-form Ω. Which is furthermore equivalent to the

[27] For an introduction to complex Riemannian geometry see [70, 71] or [68].

existence of a nowhere vanishing covariantly constant spinor η on X, which defines the almost complex structure and the holomorphic n-form as described in A.4.

Proposition: Every compact Kähler manifold (X, J, g, ∇) with $Hol(\nabla) = SU(n)$, is Ricci-flat and has $c_1 = 0$, thus vanishing first Chern class. Note that the converse need not be true.

However note that on a compact Kähler manifold (X, J, g, ∇) with vanishing first Chern class $c_1 = 0$, one can always find a Ricci-flat metric. Furthermore, one can show that any Ricci-flat space does only admit parallel Killing vector fields, which is a local statement proofed by the Weizenböck formula. The only example that allows for locally parallel vector fields is the torus, thus it follows that there are no other continuous isometries on Ricci-flat manifolds, in particular this holds for Calabi-Yau manifolds. Let us next discuss the Dolbeault cohomology groups $H^{p,q}_{\bar\partial}(X)$ of a Calabi-Yau manifold, which measure the topological non triviality of the manifold. The hodge numbers $h^{p,q}$ give the dimension of $H^{p,q}_{\bar\partial}(X)$, we refer the reader to [68] for a detailed discussion. On any compact Kähler manifold one finds that

$$h^{p,q} = h^{q,p} \ , \qquad\qquad h^{p,q} = h^{n-q,n-p}, \qquad\qquad h^{p,p} > 0 \ , \tag{2.76}$$

which follows from complex conjugation, the hodge duality and the unique always existing closed but not exact (p,p)-form J^p. Furthermore, since one discusses connected manifolds one finds that $h^{0,0} = 1$ and any Calabi-Yau has a nowhere vanishing holomorphic $(n,0)$-form, thus

$$h^{n,0} = h^{0,n} = 1 \ , \qquad\qquad h^{0,p} = h^{0,n-p} \ , \ \ p \neq 0, n \ . \tag{2.77}$$

Let us specialize in the following to the case of a Calabi-Yau fourfold. One then finds using (2.76) and (2.77) that the only independent hodge numbers are

$$h^{1,1} \ , \ h^{2,1} \ , \ h^{3,1} \ , \tag{2.78}$$

where we further used the relation valid for Calabi-Yau fourfolds

$$h^{2,2} = 2(22 + 2h^{1,1} - h^{2,1} + 2h^{3,1}) \ . \tag{2.79}$$

These arrange in the so called Hodge star as

$$\begin{array}{ccccccccc} & & & & h^{0,0} & & & & \\ & & & h^{1,0} & & h^{0,1} & & & \\ & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\ & h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} & \\ h^{4,0} & & h^{3,1} & & h^{2,2} & & h^{1,3} & & h^{0,4} \\ & h^{4,1} & & h^{3,2} & & h^{2,3} & & h^{1,3} & \\ & & h^{4,2} & & h^{3,3} & & h^{2,4} & & \\ & & & h^{4,3} & & h^{3,4} & & & \\ & & & & h^{4,4} & & & & \end{array} = \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & h^{1,1} & & 0 & & \\ & 0 & & h^{1,2} & & h^{1,2} & & 0 & \\ 1 & & h^{1,3} & & h^{2,2} & & h^{1,3} & & 1 \\ & 0 & & h^{1,2} & & h^{1,2} & & 0 & \\ & & 0 & & h^{1,1} & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array} \ . \tag{2.80}$$

Note that, in particular using (2.76) one sees that $h^{1,2} = h^{2,1}$ and $h^{1,3} = h^{3,1}$. One can express other topological quantities like the Euler characteristic for the Calabi-Yau fourfold in terms of the hodge numbers as

$$\chi = 6(8 + h^{1,1} - h^{2,1} + h^{3,1}) \ . \tag{2.81}$$

The Hodge diamond (2.80) is symmetric under reflection of the vertical and horizontal axis due to the relations (2.76). Le us shortly comment on mirror symmetry [72], which is given by reflection with respect to the main diagonal of the Hodge diamond. This statement can be generalized, to every Calabi-Yau n-fold there exists a mirror Calabi-Yau manifold constructed upon reflection of the main diagonal, which in the case of $n = 4$ amounts to exchanging $h^{1,1}$ and $h^{1,3}$ in (2.80).

Note that one finds $\omega_i\,, i = 1, \ldots, h^{1,1}$ holomorphic $(1,1)$-forms. Thus one can define the intersection numbers

$$\mathcal{K}_{ijkl} = \textstyle\int_{Y_4} \omega_i \wedge \omega_j \wedge \omega_k \wedge \omega_l$$

$$\mathcal{V} = \tfrac{1}{4!} K_{ijkl} v^i v^j v^k v^l \,, \qquad \mathcal{K}_i = \mathcal{K}_{ijkl} v^k v^l v^j \ , \qquad \mathcal{K}_{ik} = \mathcal{K}_{iklj} v^l v^j \ , \qquad \mathcal{K}_{ikl} = \mathcal{K}_{iklj} v^j \ . \tag{2.82}$$

These quantities can be expressed as integrals including powers of J using $J = v^i \omega_i$. Note that $\mathcal{V}$ is the volume of the Calabi-Yau fourfold.

2.4.2 Supersymmetry preserving backgrounds

This section is devoted to the derivation of backgrounds, which preserve a certain amount of supersymmetry. The dynamics of the fields describing the fluctuations around the vacuum and their compactification is discussed in section 2.4.3.

The most general metric, which is maximally symmetric and Poincare invariant in the extended space-time is

$$G_{MN} = \begin{pmatrix} e^{w(y)} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{pmatrix} \ , \tag{2.83}$$

or equivalently the line element is given by

$$ds^2 = e^{w(y)} g_{\mu\nu}(x) dx^\mu dx^\nu + g_{mn}(y) dy^m dy^n \ , \tag{2.84}$$

where the dimension of the external space is $d - n$, where n is the dimension of the internal space. The external space metric may either describe a Minkowski, de Sitter or anti de Sitter space, and $w(y)$ is referred to as the warp-factor. If (2.84) solves the higher-dimensional E.O.M.'s, the theory admits a spontaneous compactification. Note that by compactifying M-theory supergravity on a supersymmetric background of the from (2.84) one finds that $w = 0$, unless one turns on fluxes and considers α' corrections, see 6. One can allow for background fluxes and break supersymmetry, however, we will restrict ourselves to vanishing fluxes. Furthermore, one can argue that non-zero background values of any field, which is not a scalar under the Lorentz group $SO(1, d-1)$ reduces the symmetries of the extend dimensions, the exception being the the p-form fields that are allowed to be $C_{m_1 \ldots m_p} \propto Q\epsilon_{m_1 \ldots m_p}$ along the internal directions. Note that this implies zero background values for all other fields e.g. spinors. It has proven to be very hard to engineer compactifications with de Sitter solutions in supergravity [73]. Let us examine the Einstein equations of IIB/IIA and M-theory supergravtiy theories (1.43),(1.45) and (2.62), for vanishing internal flux and using that the vacuum configuration of any other field needs to be zero, and furthermore using (2.84) with $w = 0$ one finds

$$R_{MN} - \frac{1}{2} R g_{MN} = 0 \;\; \Rightarrow \;\; R_{\mu\nu} - \frac{1}{2}(R^{ext} + R^{int}) g_{\mu\nu} = 0 \;\; \& \;\; R_{mn} - \frac{1}{2}(R^{ext} + R^{int}) g_{mn} = 0 \;\; . \quad (2.85)$$

These equations solve for $R_{mn} = 0$ and external Minkowski space with $R^{ext} = 0$. One can derive this condition also from the requirement that the compactification shall preserve supersymmetry, which is stronger since solutions to the supersymmetry condition are always solutions to the E.O.M.'s but not vice versa. For the vacuum to be supersymmetric it needs to be annihilated by the supersymmetry generator $\mathcal{Q}$. More generally any state of the theory which obeys $\mathcal{Q}\,|\psi\rangle = 0$ preserves supersymmetry while state $\mathcal{Q}\,\langle\psi'\rangle \neq 0$ are said to break supersymmetry spontaneously. Since $\delta_\epsilon \psi \propto [\epsilon \mathcal{Q}, \psi]$ one infers that $\langle \delta_\epsilon \psi \rangle \propto \langle 0 | [\epsilon \mathcal{Q}, \psi] | 0 \rangle = 0$ for any field of the theory which preserves supersymmetry. Alternatively, this is to be understood connecting to the previous comment on the vanishing of all fields in the background, that non-trivially transform under $SO(1, d-1)$. Thus also their supersymmetry variations better vary to zero. The variation of generic bosonic fields Φ are fermionic and thus need to be zero $\delta_\epsilon \Phi|_{background} = 0$. Hence one is left to consider the fluctuations of the fermionic fields. Let us consider e.g. the gravitino variation (2.39), which for various theories is of the generic form

$$\delta_\epsilon \psi_M = \nabla_M \epsilon + f(\text{bosonic fields}, \epsilon) \;\; \text{with} \;\; f(\text{bosonic fields}, \epsilon)|_{\text{background}} = 0 \;\; , \quad (2.86)$$

which implies that[28]

$$\nabla_M \epsilon|_{\text{background}} = 0 \;\; \Rightarrow \;\; \nabla_\mu \epsilon|_{\text{background}} = 0 \;\; , \;\; \nabla_m \epsilon|_{\text{background}} = 0 \;\; . \quad (2.87)$$

Thus the existence of a covariantly constant spinor also referred to as parallel Killing spinor (2.87) is a necessary and sufficient requirement for the compactification to preserve supersymmetry in this setup. Note that maximally symmetric spaces of dimensions d have $\frac{d}{2}(d+1)$ Killing vectors, which in Minkowski space give the boosts, rotations and translations that are all symmetries of the metric. Thus for maximally symmetric spaces the requirement (2.87) is automatically satisfied. This is not true for any manifold, where the existence of a no-where vanishing spinor requires a certain topological structure of the manifold, namely that the bundles of orthogonal frames can be patched together using a proper subgroup of $SO(n)$. In the case of six internal dimensions this is $SU(3) \subset SO(6)$, while for eight-dimensional internal manifolds one has e.g. $SU(4) \subset SO(8)$, see table 2.4. Nota that (2.87) poses a requirement on the connection and thus on the differentiable structure of the manifold, to guarantee the existence of a covariantly constant spinor, which is reflected int the holonomy group of the manifold (2.71). Under parallel transport along closed loops spinors are rotated analogously to vectors , however, if a spinor is covariantly constant it is not rotated. Thus it transforms trivially or in other words as a singlet under $Hol(\nabla, M)$. Hence one reduces the problem of finding covariantly

[28]Note that we used again that we do not allow for background fluxes.

constant spinors on the internal space to counting the singlet representations of $Hol(\nabla, M)$. In the following we give an derivation of this counting procedure in the case of $Hol(\nabla, M) = SU(n)$, which is relevant for internal manifolds considered in this work. So far the only fermion we have fluctuated is the gravitino, which in the case of eleven-dimensional supergravity is the only fermion but for instance in II supergravity theories there are two dilatinos whose variations vanish due to the above arguments. In the heterotic and type I string there additionally exist gauginos whose treatment is a bit more subtle due to the gauge fields, see e.g. [15]. Let us assume the existence of a nowhere vanishing covariantly constant spinor $\nabla_M \epsilon = 0$ and derive the implications on the metric background. This implies the integrability condition

$$[\nabla_M, \nabla_N]\epsilon = \frac{1}{4} R_{MNRS} \gamma^{[R} \gamma^{S]} \epsilon = 0 \ . \tag{2.88}$$

By using gamma matrix algebra and building spinor bilinears on derives from (2.88) that $g_{\mu\nu} = \eta_{\mu\nu}$, thus Minkowski space-time and that

$$R_{mn} = 0 \ . \tag{2.89}$$

Hence the internal space is Ricci-flat. This is a necessary condition for the existence of a covariantly constant spinor. Note that we have already seen in the world-sheet perspective that Weyl invariance of the world-sheet CFT requires the background to be Ricci-flat.

To proceed it is of crucial interest to study the spinor representations of the groups $SO(n)$ and $SU(n)$. We will introduce a few concepts of representation theory and their hands on application for the purpose of dimensional reduction. We start with spinors in the d-dimensional space-time with block diagonal metric

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n \quad , \tag{2.90}$$

where η is the Minkowski metric. The Lorentz symmetry group of the higher-dimensional space decomposes in this product space as

$$SO(1, d-1) \to SO(1, d-n-1) \times SO(n) \ \ . \tag{2.91}$$

Thus the spinor representations of $SO(1, d-1)$ will form new representation under the product group. In the case that the internal manifold has more structure, see table 2.4, e.g. the holonomy group can reduces to $SU(n)$, one furthermore needs to consider how the spinor representations of $SO(n)$ decompose to those of $SU(n)$. From the number of singlet representations on the internal space one can infer the preserved amount of supersymmetry. To see this one parallel transports a spinor a long a loop in the internal space, and it will differ upon acting with an element of the holonomy group.

We give a hands on procedure for this steps in the following. The groups need different treatment for even and odd dimensions $n = 2r$ and $n = 2r + 1$, respectively, with r the rank of the group. Although we are mainly concerned with spinors in this section let us for completeness make a small detour to vector representations. Vector representations of $SO(n)$ in even dimensions $n = 2r$ can be characterized by the r-dimensional vector

$$(\underline{\pm, 0, 0, \ldots, 0}) \ \ , \tag{2.92}$$

where $\pm, 0$ can be chosen at any position denoted by the underline. The number of different permutations gives the dimension of the vector representation, for a more detailed discussion, see e.g. [17, 74]. Let us look at the example of $SO(1,9) \to SO(1,3) \times SO(6)$, the sourced by the product metric of the corresponding geometry. The vector representations can be characterized by vectors of length $r = 5, r = 2$ and $r = 3$ and thus dimensions $10, 4$ and 6 respectively. For the dimension of the representation in the split one finds

$$(\underline{\pm,0,0,0,0}) \to \left.\begin{array}{c} (\underline{+,0}|0,0,0) \\ (\underline{-,0}|0,0,0) \\ (0,0|\underline{+,0,0}) \\ (0,0|\underline{-,0,0}) \end{array}\right\} \begin{array}{c} (\frac{\underline{+,0}}{\underline{-,+}}|0,0,0) = (4_v, 1_v) \\ (0,0|\frac{\underline{+,0,0}}{\underline{-,0,0}}) = (1_v, 6_v) \ , \end{array} \tag{2.93}$$

where $\pm$ denotes ± 1 and we have used the simple fact, that one finds four and six different permutations in the first and second line, respectively. Thus we find the decomposition

$$10_v \to (4_v, 1_v) \oplus (1_v, 6_v) \ . \tag{2.94}$$

The counting illustrated in the previous simple example can be applied for spinors analogously. In the even-dimensional case there are two irreducible spinor representations of dimension 2^{r-1} carrying different chirality, which are represented by the vector of weights of length r

$$\text{s:} \left(\underline{\pm\frac{1}{2},\ldots,\pm\frac{1}{2}}\right)\Big|_{\text{even}+} \ \& \ \text{c:} \left(\underline{\pm\frac{1}{2},\ldots,\pm\frac{1}{2}}\right)\Big|_{\text{odd}+} \ , \tag{2.95}$$

with the even or odd number giving the chirality denoted by s and c, respectively. Let us consider the spinor representations of the previous example of ten-dimensional space-time with product metric (2.83) and six internal dimensions. One then finds the split $SO(1,9) \to SO(1,3) \times SO(6)$. Table 2.5 gives an overview of the spinors representations in various dimensions focusing on Lorentz groups of space relevant for the compactifications at hand.[29] Note that a Dirac spinor in ten dimensions can be written as the sum of a left and right-handed Weyl spinor $16_s \oplus 16_c$, see table 2.5. Type II supergravity has maximal supersymmetry in ten dimension $\mathcal{N} = 2$, choosing two Majorana-Weyl spinors of equal chirality $(2,0)$ one describes type IIB, while in the case of IIA supergravity one has two Majorana-Weyl spinors of opposite chirality $(1,1)$. When compactifying IIA supergravity on the product space one needs to consider 16_c and 16_s separately. Since the treatment is analogous we will only consider the latter case of 16_s now, using (2.95) and $r = 5$ one infers

$$\frac{1}{2}\left(\underline{\pm,\pm,\pm,\pm,\pm}\right)\Big|_{\text{even}+} \to \frac{1}{2}(\underline{+,+,+,+,-}) \ , \ \frac{1}{2}(\underline{+,+,-,-,-}) \ \text{and} \ \frac{1}{2}(-,-,-,-,-) \ , \tag{2.96}$$

which decompose under the split to

$$\tfrac{1}{2}(\underline{+,+,+,+,-}) \to \begin{array}{c} \frac{1}{2}(+,+|\underline{+,+,-}) \\ \frac{1}{2}(\underline{+,-}|+,+,+) \end{array} \ , \quad \frac{1}{2}(\underline{+,+,-,-,-}) \to \begin{array}{c} \frac{1}{2}(+,+|-,-,-) \\ \frac{1}{2}(\underline{+,-}|\underline{+,-,-}) \\ \frac{1}{2}(-,-|\underline{+,+,-}) \end{array} \ , \tag{2.97}$$

$$\tfrac{1}{2}(-,-,-,-,-) \to \tfrac{1}{2}(-,-|-,-,-) \ . \tag{2.98}$$

[29] A related discussion can be found in [75].

Metric Signature	left,right - handed Weyl (complex)	Dirac (complex)	left,right - handed Majorana-Weyl (real)	Majorana (real)
$SO(1,2)$	-	2	-	2
$SO(1,3)$	2_s , 2_c	4	-	4
$SO(1,9)$	16_s , 16_c	32	16 , 16	32
$SO(1,10)$	-	32	-	32
$SO(2)$	1_s , 1_c	2	-	2
$SO(6)$	4_s , 4_c	8	-	8
$SO(7)$	-	8	-	8
$SO(8)$	8_s , 8_c	16	8 , 8	16

Table 2.5: Spinor representations of orthogonal special groups relevant for string compactifications.

Note that for notational simplicity we pulled the weight $\frac{1}{2}$ out of the bracket. We next want to rearrange this permutations into different sets, such that we can fit them into representations of the four-dimensional external and the six-dimensional internal space, given by $2_s, 2_c$ and $4_s, 4_c$, respectively. Written in terms of the weights as $\frac{1}{2}(\underline{\pm,\pm})|_{even+}$, $\frac{1}{2}(\underline{\pm,\pm})|_{odd+}$ and $\frac{1}{2}(\underline{\pm,\pm,\pm})|_{even+}$, $\frac{1}{2}(\underline{\pm,\pm,\pm})|_{odd+}$. One repackages the vectors in (2.97) to

$$\left.\begin{array}{l} \frac{1}{2}(+,+|\frac{\underline{+,+,-}}{\underline{-,-,-}}) \\ \frac{1}{2}(-,-|\frac{\underline{+,+,-}}{\underline{-,-,-}}) \end{array}\right\} = \frac{1}{2}(\frac{+,+}{-,-}|\frac{\underline{+,+,-}}{\underline{-,-,-}}) = (2_s, 4_s) \quad \& \quad \frac{1}{2}(\underline{+,-}|\frac{\underline{+,-,-}}{+,+,+}) = (2_c, 4_c) \ . \tag{2.99}$$

Analogously for the 16_c one finds $\frac{1}{2}(\underline{+,-}|\frac{\underline{+,+,-}}{\underline{-,-,-}}) = (2_c, 4_s)$ and $\frac{1}{2}(\frac{+,+}{-,-}|\frac{\underline{+,-,-}}{+,+,+}) = (2_s, 4_c)$. Let us now introduce an additional structure on the internal space to promote it to a Calabi-Yau manifold, which according to table 2.4 has $SU(3)$ holonomy. Thus we are interested how the spinor representations of $SO(6)$ decompose into those of $SU(3)$. We use that $SO(6)$ is isomorphic to $SU(4)$, and that the complex left and right-handed Weyl spinor representations $4_s, 4_c$ correspond to the fundamental and anti-fundamental representation $4, \bar{4}$. Note that $SU(n)$ has rank $r = n-1$ and that the fundamental and anti-fundamental spinor representations can be represented by a n-dimensional vector as[30]

$$(\underline{+,0,\ldots,0}) \quad \text{and} \quad (\underline{-,0,\ldots,0}) \ , \tag{2.100}$$

respectively. Thus in the case of $n = 4$ one easily infers from (2.100) that the representations are 4 and $\bar{4}$. Instead of splitting the representation of $SO(6) \to SU(3)$ we derive $SO(6) \simeq SU(4) \to SU(3)$, which gives

$$\left.\begin{array}{ll} 4: & (\underline{+,0,0,0}) \\ \bar{4}: & (\underline{-,0,0,0}) \end{array}\right\} \to \left.\begin{array}{ll} (+,0,0,0)\ , & (0,\underline{+,0,0}) \\ (-,0,0,0)\ , & (0,\underline{-,0,0}) \end{array}\right\} = \begin{array}{l} (1+3) \\ (\bar{1}+\bar{3}) \end{array} \ . \tag{2.101}$$

[30]The following representation (2.100) represents $U(n)$. $SU(n)$ is represented by the vectors $(\underline{+,0,\ldots,0}) - \frac{1}{r}(1,\ldots,1)$, hence the difference lies in the second contribution, which is irrelevant for the following discussion, thus we drop it.

Thus combining (2.99) and (2.101) one finds that 16_s reduces to the spinor representations

$$\begin{aligned} 16_s &\rightarrow (2_s,1)\ ,\ \ (2_c,\bar{1})\ ,\ \ (2_s,3)\ ,\ \ (2_c,\bar{3})\ , \\ 16_c &\rightarrow (2_s,\bar{1})\ ,\ \ (2_c,1)\ ,\ \ (2_s,\bar{3})\ ,\ \ (2_c,3)\ . \end{aligned} \tag{2.102}$$

Note that the preserved supersymmetry can be read off from the number of singlets in the internal space, which combine to Majorana spinors of the external space, thus

$$\left(\begin{matrix} 2_s \\ 2_c \end{matrix},1\right) = (4,1)\ \ \&\ \ \left(\begin{matrix} 2_s \\ 2_c \end{matrix},\bar{1}\right) = (4,\bar{1})\ \ , \tag{2.103}$$

where we have used that $(2_s, 2_c)$ forms a single Majorana spinor of $SO(1,3)$ see table 2.5. Thus one arrives at the decomposition

$$16_s \oplus 16_c \ \ \rightarrow\ \ (4,1)\oplus(4,\bar{1})\oplus(4,3)\oplus(4,\bar{3})\ \ . \tag{2.104}$$

We conclude from (2.104) that we have $\mathcal{N}=2$ supersymmetry in four dimensions. Assume we had used a manifold with $Hol(\nabla, M) = SU(2)$ the representations of $SO(6) \simeq SU(4)$ reduce to those of $SU(2)$ analogously to (2.101) giving $4 \rightarrow 1 \oplus 1 \oplus 2$ and $\bar{4} \rightarrow \bar{1} \oplus \bar{1} \oplus \bar{2}$. Thus one finds via the same logic as in (2.103) that one yields four Majorana spinors $(4,1)\oplus(4,1)\oplus(4,\bar{1})\oplus(4,\bar{1})$, and thus $\mathcal{N}=4$ supersymmetry.

Let us us next examine the case where the internal manifold has trivial holonomy or in other words every spinor is covariantly constant. This is the case for S^1 and its higher-dimensional cousin the n-torus T^n. In the case of T^6, one finds the reduction of $SO(6) \simeq SU(4)$ to the trivial group, thus $4 \rightarrow 1\oplus1\oplus1\oplus1$ and $\bar{4} \rightarrow \bar{1}\oplus\bar{1}\oplus\bar{1}\oplus\bar{1}$, which by using (2.104) results in $\mathcal{N}=8$, maximal supersymmetry in four dimensions.

Two theories, which are related by circular reduction and crucial for the F-theory lift are $3d$, $\mathcal{N}=2$ and $4d$, $\mathcal{N}=1$ supergravity. Note that upon circular reduction of the one Majorana spinor in $4d$ and the fact that S^1 has trivial holonomy, it simply reduces to two component Majorana spinors in $3d$ as $4 \rightarrow 2 \oplus 2$, see table 2.5. Thus we find $\mathcal{N}=2$ supersymmetry in three dimensions.

Another case of interest is the reduction of M-theory on a Calabi-Yau fourfold with $SU(4)$ holonomy 2.4.1. The single gravitino is a 32 component Majorana spinor, see table 2.5. Thus the reduction from $SO(1,10) \rightarrow SO(1,2) \times SO(8)$ decomposes the representation as

$$32\ \ \rightarrow\ \ (2,8_s)\ \ \oplus\ \ (2,8_c)\ . \tag{2.105}$$

To show this on proceeds completely analogous to the previous case (2.97) with the exception not to differentiate between odd and even plus or minus weights in the total and external space, since there do not exist chiral or anti-chiral representations in these dimensions. Next one breaks these representation of $SO(8)$ to the spinor representations of $SU(4)$, which are $1, \bar{1}, 4, \bar{4}, 6$. The embedding in this case is more advanced since $SO(8)$ has a unique Dynkin diagram D_4, and can thus be shown not to be isomorphic to any other classical Lie group. We simple note that $8_s \rightarrow 4 \oplus \bar{4}$ and $8_c \rightarrow 1 \oplus \bar{1} \oplus 6$. Combining these results one finds that

$$32\ \ \rightarrow\ \ (2,1)\ \ \oplus\ \ (2,\bar{1})\oplus\ \ (2,4)\oplus\ \ (2,\bar{4})\oplus\ \ (2,\bar{6})\ \ . \tag{2.106}$$

Note that there are two Majorana spinors in $2+1$ dimensions and hence one infers $\mathcal{N}=2$ supersymmetry. We conclude that the reduction of eleven-dimensional supergravity on a Calabi-Yau fourfold gives a $3d$, $\mathcal{N}=2$ supergravity theory.

2.4.3 Kaluza-Klein recipe

The concept of dimensional reduction was introduced in the previous section 2.4.2. We noted that the procedure of dimensional reduction of theories is fundamentally different in the case where gravity is dynamical to the case where it is not, however, these two setups share some common features. To examine the principal idea we first consider the simpler case without the Einstein-Hilbert term by considering a d-dimensional theory, which lives on a space $M \times N$ where M, N are pseudo-Riemannian manifolds of dimensions m and n, respectively. The core of the dimensional reduction is to express the dependence of the fields on the internal coordinates in an expansion of eigenfunctions of the internal manifold. The existence proof of such expansions rely on differential operators such as the Laplace-Beltrami or the Hodge Laplacian, where latter is also defined for manifolds without connection. The eigenfunctions $\{\phi_i\}$ of the Laplace-Beltrami operator

$$\Delta = \nabla_M \nabla^M = \frac{1}{\sqrt{g}} \partial_M \left(\sqrt{g} \partial^M \right) \quad , \tag{2.107}$$

satisfy $\Delta\phi_i = -\lambda^i \phi_i$ such that $\lambda^i > 0$, and form a complete orthonormal [31] basis on any compact Riemannian manifold [76], such that

$$f = c_0 + \sum_i c_i \phi_i \ , \quad \text{with} \quad c_i = \int_N f \phi_i *_n 1 \ . \tag{2.108}$$

Note that certain eigenvalues posses multiplicities included in the notation of the index i, and furthermore that eigenfunctions ϕ_i can in principle chosen to be real, however, a complex representation involves some advantages. For the expansion of p-forms, which are anti-symmetric one naturally uses the eigenfunctions of the Hodge Laplacian $\Delta_H = d^\dagger d + d^\dagger d$ and the related Hodge decomposition [68]. For the decomposition of symmetric tensors it is convenient to use the eigenfunction of the Lichnerowicz operator [77], its action on symmetric two-tensors is

$$\Delta_L s_{MN} = -\Delta s_{MN} - 2R_{MPNQ}\ s^{PQ} + R_{(M}{}^P\ s_{N)P} \ . \tag{2.109}$$

When considering spinors one needs to restrict the manifold to be a spin-manifold, such that their existence is granted. As mentioned before this restricts the structure group and one expands in eigenspinors of the Dirac operator $D^{1/2} = \gamma^M \partial_M$ for spin 1/2 and the Rarita-Schwinger operator for spin 3/2, given by $D^{3/2\mu\nu} = \epsilon^{\mu\lambda\sigma\nu}\gamma_5\gamma_\lambda\partial_\sigma - im\gamma^{\mu\nu}$. Where it acts on the fermion as $D^{3/2\mu\nu}\psi_\nu$.

Let us start with the case of the free scalar field on $M \times N$, whose dynamics is governed by the action

$$S^{(d)} = \frac{1}{2} \int_{M\times N} -d\Phi \wedge *_d d\Phi \ . \tag{2.110}$$

[31] $\int \phi_i \phi_j *_i 1 = \delta_{ij}$.

Note that $\Phi = \Phi(x, y)$, which by using (2.108) can be expressed as[32]

$$\Phi(x,y) = \varphi_0(x) + \sum_i \varphi_i(x)\phi_i(y) \ , \tag{2.111}$$

where coefficients of the eigenfunction expansion depend on the space M. Substituted into (2.110) and by using the orthonormality of the eigenfunctions ϕ_i, one can perform the integral over the internal space to yield

$$S^{(d)} = \frac{1}{2}\int_M -d\varphi_0 \wedge *_d d\varphi_0 - \sum_i \left(d\varphi_i \wedge *_d d\varphi_i + \lambda^i \varphi_i^2 *_d 1\right) \ , \tag{2.112}$$

where we have performed a field redefinition $\varphi_0 \to \frac{1}{\mathcal{V}_N}\varphi_0$, with $\mathcal{V}_N$ the volume of the internal space N, to arrive at 2.112. This theory describes one massless scalar field φ_0 and an infinite tower of massive scalar fields φ_i, of mass $m^2 = \lambda^i$. Since the eigenvalues are positive there are no tachyons and thus the spectrum is stable. Let us now turn to the simple case of a circular reduction $N = S^1$. In this case the eigenfunctions are $\phi_j = \frac{e^{i(jRy)}}{\sqrt{2\pi R}}$, to the eigenvalue $\lambda^j = \frac{j^2}{R^2}$, with R the radius of the circle. Note that since we have chosen a complex representation of the internal eigenfunctions but the total field Φ is real, one finds that φ_0 is real and $\varphi_i^\star = \varphi_{-i}$, which renders (2.112) to be real. The mass of the i$^{\text{th}}$ KK-state is $m_i = \frac{i}{R}$. Note that the massless state does not couple to the massive ones and thus also their equation of motions are independent. One can safely truncate the massive tower of states and the dynamics of the massless state is unaltered.[33] This constitutes a consistent truncation. In the case of a circular reduction such a consistent truncation is always possible [78, 79]. However, one needs to argue that the KK-tower of massive states is not relevant for the lower dimensional physics. This is achieved in the limit where the radius of the circle is small, such that the masses become larger than the energy scale $m_i \ll E$ at which one considers the effective lower-dimensional theory to be valid. In other words if the size of the internal space is physical and cannot be tuned arbitrary small, one can only trust the effective theories at energies safely smaller than m_1, since elsewise these states become excited and alter the theory as dynamical degrees of freedoms.

Let us treat another example, with higher-dimensional gauge invariance on a non-dynamical background, namely the d-dimensional vector field $A_M(x, y)$ on $M \times N$,

$$S = \frac{1}{2}\int_{M\times N} -F \wedge *_d F \ , \tag{2.113}$$

with its fields strength $F = dA$.[34] The theory (2.113) is invariant under the gauge transformation

$$A \ \ \to \ \ A + d\Lambda \ , \tag{2.114}$$

with gauge space-time function $\Lambda = \Lambda(x, y)$. One can write the d-dimensional gauge field in terms of eigenfunctions of the scalar Laplacian $\phi_i(y)$ to the eigenvalue $\lambda^i > 0$, and eigenvectors of the vector

[32] Note that on a compact manifold the scalar zero mode of the Laplace-Beltrami operator is constant, denoted by $\varphi_0(x)$.

[33] Let us emphasize, that since this constitutes a free theory also the massive mode have no interaction terms among each other.

[34] Note that due to (A.8), in particular $F \wedge *_d F = \frac{1}{2}F_{MN}F^{MN} *_d 1$, and that $F = dA$ is equivalent to $F_{MN} = \nabla_M A_N - \nabla_N A_M$ due to the fact that the terms proportional to the Christoffel symbols cancel.

Laplacian on the internal space $\mathcal{A}^\alpha_m(y)$ to eigenvalues $\lambda^\alpha > 0$, where $\nabla^m \mathcal{A}^\alpha_m = 0$ and $\int_N \mathcal{A}^\alpha_m \mathcal{A}^{\beta m} *_n 1 = \delta^{\alpha\beta}$. The vector Laplacian is a cousin of the Hodge Laplacian that acts upon a vector as $\Delta_V \mathcal{A}_m = -\Delta \mathcal{A}_m + R_{mn}\mathcal{A}^n \Leftrightarrow (dd^\dagger + d^\dagger d)\mathcal{A}$. Note that generically multiplicity's to certain eigenvalues occur, i.e. multiple eigensolutions to the same eigenvalue. Those to eigenvalue zero are of special interest since they tend to give rise to the massless modes. We explicitly expresses the multiplicity's of the eigenvalue zero vectors, by the index α_0 thus by $\mathcal{A}^{\alpha_0}_m$. One expands

$$\begin{aligned} A_\mu(x,y) &= a^0_\mu(x) + \sum_i a^i_\mu(x)\phi_i(y) \\ A_m(x,y) &= \sum_i \varphi^i_0(x)\partial_m\phi_i(y) + \sum_{\alpha_0}\varphi^{\alpha_0}(x)\mathcal{A}^{\alpha_0}_m(y) + \sum_\alpha \varphi^\alpha(x)\mathcal{A}^\alpha_m(y) \ , \end{aligned} \tag{2.115}$$

and the gauge parameter field

$$\Lambda(x,y) = \Lambda^0(x) + \sum_i \Lambda^i(x)\phi_i(y) \ . \tag{2.116}$$

Replacing the field strength in (2.113) by the total derivative of (2.115), using the orthonormality relations and integrating over the internal space one arrives at

$$\begin{aligned} S^d = \frac{1}{2}\int_M &- F^{(0)} \wedge *_d F^{(0)} - \sum_i \left(F^{(i)} \wedge *_d F^{(i)} - \frac{\lambda^i}{2}\left(a^i_\mu - \partial_\mu\varphi^i_0\right)\left(a^{i\mu} - \partial^\mu\varphi^i_0\right) *_d 1 \right) \\ &- \frac{1}{2}\sum_{\alpha_0} d\varphi^{\alpha_0} \wedge *_d d\varphi^{\alpha_0} - \frac{1}{2}\sum_\alpha \left(d\varphi^\alpha \wedge *_d d\varphi^\alpha + \lambda^\alpha \varphi^{\alpha 2}\right) \ , \end{aligned} \tag{2.117}$$

with $F^{(0)} = da^0$ and $F^{(i)} = da^i = \partial_{[\mu}a^i_{\nu]}dx^\mu \wedge dx^\nu$. Furthermore, following from (2.114) one finds an infinite tower of gauge symmetries given by

$$a^0_\mu \to a^0_\mu + \partial_\mu \Lambda^0 \ , \ a^i_\mu \to a^i_\mu + \partial_\mu \Lambda^i \ \text{ and } \ \varphi^i_0 \to \varphi^i_0 + \Lambda^i \ , \tag{2.118}$$

where the scalars $\varphi^{\alpha_0}, \varphi^\alpha$ do not transform.[35] To bring (2.117) to a manifestly gauge invariant form note that $F^{(i)} = da^i = d(a^i + d\varphi^i_0)$, which allows us to rewrite (2.117) in terms of the gauge invariant combination $a^i_\mu + \partial_\mu\varphi^i_0$. Note that

$$\frac{1}{2}\int -F^{(i)} \wedge *_d F^{(i)} - \frac{\lambda^i}{2}\left(a^i_\mu - \partial_\mu\varphi^i_0\right)\left(a^{i\mu} - \partial^\mu\varphi^i_0\right) *_d 1 \ , \tag{2.119}$$

is the so called Stückelberg action. One can go to unitary gauge $\varphi^i_0 = 0$ to yield the standard action of massive vectors or in the original sense photons

$$\frac{1}{2}\int -F^{(i)} \wedge *_d F^{(i)} - \frac{\lambda^i}{2} a^i_\mu a^{i\mu} *_d 1 \ . \tag{2.120}$$

The original idea follows the revers logic, starting from an apparently non-gauge invariant massive vector action (2.120) on introduces the shift of the fields $a^i_\mu \to a^i_\mu + \partial_\mu\varphi^i_0$, which is not a gauge symmetry a priori but introduces a new degree of freedom to arrive at (2.119). However, using the

[35] By allowing for complex valued eigensolutions in the expansion, the restriction of $A(x,y)$ to be real implies relations of the various expansion fields $a^i_\mu, \varphi^i_0, \varphi^\alpha, \varphi^{\alpha_0}$.

manifest gauge symmetry of (2.119) one can go to unitary gauge and arrive at (2.120), which proves the physically equivalence of (2.119) and (2.120). The manifest gauge invariance of (2.119) is needed to understand the limit of vanishing mass of the vector field, which gives the theory of of a massless vector and a massless scalar. Comparing to (2.120) one would trivially arrive at the massless vector only, thus a degree of freedom would get lost This reflects the fact that a massive photon in four dimension has three degrees of freedom that in the limit of vanishing mass reproduce the two degrees of freedom of a massless vector plus the scalar field. Thus the action (2.117) carries a single massless vector field, and a set of massless scalars of the size of the degeneracy of the vector Laplacian to eigenvalue zero, which corresponds to the number of harmonic one-forms on N. And a tower of massive vectors of mass $m^i = \sqrt{\lambda^i}$ and moreover a tower of massive scalars of mass $m^a = \sqrt{\lambda^\alpha}$, that are never tachyonic due to the positivity of the eigenvalues.

In the case that $N = S^1$ of the circular reduction one does not find the same spectrum since on S^1 there do not exist harmonic one-forms except for the trivial constant one. Thus the tower of massless scalars is absent and reduces to a single massless scalar. The masses of the massive scalars are $m^a = \frac{\alpha^2}{R^2}, \alpha = 1, 2, 3. \dots,$ where R is the radius of the circle.

Let us begin the more important passage of the story of dimensional reduction, in which we allow for dynamical metric backgrounds, thus incorporate gravity. The first step of this procedure as outlined in the previous section 2.4.2, is to solve the higher-dimensional E.O.M.'s to find a valid background for the metric and the p-form fields.[36] Since the eigenfunction expansion of the fields is crucially related to the explicit form of the metric background, one needs to be careful when deforming it. Let us therefore look at a small deformation of the metric as

$$g_{MN} \quad \to \quad g_{MN} + \delta g_{MN} \ , \tag{2.121}$$

where g is the background metric, thus a solution to the E.O.M.'s of the higher-dimensional theory.[37] Let us return to the case of IIB/IIA or M-theory where we have discussed the background solution in section 2.4.2. The metric is block diagonal with external space being Minkowski $g_{\mu\nu} = \eta_{\mu\nu}$ and the internal space being Ricci-flat, thus g_{mn} such that $R_{mn} = 0$. Let us perform the small perturbation (2.121) of Einstein's equation in ten or eleven dimensions up to linear order in g_{MN}, which results in

$$\begin{aligned} 0 = \delta \left(R_{MN} - \frac{1}{2} R g_{MN} + \text{other fields} \right)\Bigg|_{background} &= \nabla_L \nabla_{(M} \delta g_{N)}{}^L + g_{MN} \nabla_L \nabla^{[L} \delta g_O{}^{O]} \\ &- \frac{1}{2} \nabla_L \nabla^L \delta g_{MN} - \frac{1}{2} \nabla_M \nabla_N \delta g_L{}^L \ , \end{aligned} \tag{2.122}$$

where the other fields in the theory vanish in the background, so do terms proportional to the Ricci tensors and Ricci scalar. One can infer three different sets of equations by choosing the free indices in (2.122) to be μm, $\mu\nu$ or mn. The complete set of perturbations, which leaves the Einstein equations

[36] Let us emphasize that in order to perturb around an appropriate background one also needs to solve for the e.g. the gauge fields in the Heterotic string. However, note that these fields live in space-time, which constitute the difference to solving for the background metric of the space.

[37] Note that the inverse metric is shifted by $g^{MN} \ \to \ g^{MN} - \delta g^{MN}$ which guarantees the inverse metric property up to higher-order perturbations.

invariant allow for $\delta g_{\mu m}, \delta g_{\mu\nu}$ and δg_{mn}. Let us solve the three sectors μm, $\mu\nu$ and mn of (2.122) for the three different types of variations, respectively, under the retrospectively right assumption that the equations decouple in each sector under the different variations $\delta g_{\mu m}, \delta g_{\mu\nu}$ and δg_{mn}. Let us start with the variation δg_{mn}, where only from the mixed free index equations (2.122) one finds

$$\nabla_\mu \nabla_{[m} \delta g_{n]}{}^n = 0 \quad \Rightarrow \quad \nabla_m \delta g_n{}^n = \nabla_n \delta g_m{}^n \quad . \tag{2.123}$$

Using this for the external free index sector of (2.122) one finds that

$$\delta g_m{}^m = 0 \;\; \& \;\; \nabla_n \delta g_m{}^n = 0 \;\; , \tag{2.124}$$

thus it is traceless. Finally applying (2.123) and (2.124) to the purely internal free index part of (2.122) one infers that

$$\nabla_r \nabla^r \delta g_{mn} - 2\nabla_{[r}\nabla_{m]}\delta g_n{}^r - 2\nabla_{[r}\nabla_{n]}\delta g_m{}^r + \nabla_\mu \nabla^\mu \delta g_{mn} = 0 \;\; . \tag{2.125}$$

Using the fact that $\nabla_{[m}\nabla_{n]}\delta g_{rs} = \frac{1}{2} R^t{}_{rmn}\delta g_{ts} + \frac{1}{2} R^t{}_{smn}\delta g_{rt}$ one can rewrite (2.125) in terms of the Lichnerowicz operator (2.109) as

$$\Delta_L^{int}\big|_{background}\, \delta g_{mn} = \nabla_\mu \nabla^\mu \delta g_{mn} \;\; . \tag{2.126}$$

Note that $\delta g_{mn} = \delta g_{mn}(x,y)$, thus to solve (2.126) we make the product Ansatz $\delta g_{mn}(x,y) = \delta v(x) w_{mn}(y)$, which gives

$$\Delta_L^{int}\big|_{background}\, w_{Imn} = \lambda^I\, w_{Imn} \quad \& \quad \Delta^{ext}\delta v = \lambda^I\, \delta v \; . \tag{2.127}$$

Let us emphasize that the notation $\delta\varphi$ keeps track of the fact that this is a small perturbation around the background. Furthermore, Δ_L^{int} and Δ^{ext} denote the Lichnerowicz Laplacian on the internal and the scalar Laplacian on the external space. We see from the second equation in (2.127) that we find scalar fields with mass $m_i = \sqrt{\lambda_i}$ for every eigensolution of the Lichnerowicz operator on the internal background metric to the eigenvalue λ^I. Hence we introduce the notation $\delta v \to \delta v^I$ in the following. This gives an infinite number of deformations, one for each eigenvalue solution of the Lichnerowicz operator on the internal background metric, which leave the higher-dimensional Einstein equation invariant when perturbed around to background to linear order. Note that the eigenvalues are in general degenerate.

The derivation for the external variations is analogously and result in

$$\delta g_\mu{}^\mu = 0 \quad , \quad \nabla_\nu \delta g_\mu{}^\nu = 0 \;\; \& \;\; \Delta_L^{ext}\big|_{background}\, \delta g_{\mu\nu} = \nabla_m \nabla^m \delta g_{\mu\nu} \;\; , \tag{2.128}$$

with Δ_L^{ext} the Lichnerowicz operator of the external space. We can solve (2.128) by making the Ansatz $\delta g_{\mu\nu} = \delta h_{\mu\nu}(x)\phi(y)$. which leaves us with

$$\Delta^{int}\phi^i = \lambda^i \phi^i \quad \& \quad \Delta_L^{ext}\big|_{background}\, \delta h^i_{\mu\nu} = \lambda^i \delta h^i_{\mu\nu} \;\; , \tag{2.129}$$

and furthermore with $\nabla^\mu \delta h^i_{\mu\nu} = 0$ thus transverse and traceless $\delta h^i_\mu{}^\mu = 0$. Hence a decomposition into internal space eigenfunctions of the scalar Laplacian gives rise to massive gravitons described by the

E.O.M given in (2.129), and which are of mass $m_i = \sqrt{\lambda^i}$ where λ^i are the eigenvalues of the internal scalar Laplacian. Since the internal space is compact the only solution to the scalar Laplacian with zero eigenvalue is constant. Furthermore, in the case of external Minkowski space (2.129) becomes $\Delta\delta h^0_{\mu\nu} = 0$, which is the usual homogeneous wave equation leading to gravitational waves. This modes are taken care of in the dimensional reduction of the action trivially, where one takes the external space metric to be an arbitrary tensor $g_{\mu\nu}$ and not the background value. However, then by substituting the external metric by its background value plus a deformation one would encounter exactly these massless excitations arising from the deformations $\delta h^0_{\mu\nu}$.

Finally, we discuss the mixed deformations $\delta g_{\mu m}$. From the free index sector mn, $\mu\nu$ of (2.122) one infers

$$\nabla^m \delta g_{m\mu} = 0 \qquad \& \qquad \nabla^\mu \delta g_{m\mu} = 0 \ , \tag{2.130}$$

and by using (2.130) from the mixed free index sector of (2.122) that

$$\Delta_H^{int}\big|_{background} \delta g_{m\mu} = \nabla^\nu \nabla_{[\nu} \ , \delta g_{m\mu]} \ , \tag{2.131}$$

with $\Delta_H^{int}, \Delta_H^{ext}$ the internal and external vector or Hodge Laplacian, respectively. We solve (2.131) by the product ansatz $\delta g_{m\mu} = A_\mu(x)\mathcal{A}_m(y)$, which results in

$$\nabla^\nu \nabla_{[\nu} A^\alpha_{\mu]} = \lambda^\alpha A^\alpha_\mu \qquad \& \qquad \Delta_H^{int}\big|_{background} \mathcal{A}^\alpha_m = \lambda^\alpha \mathcal{A}^\alpha_m \ . \tag{2.132}$$

This leads to vectors with mass $m_\alpha = \sqrt{\lambda^\alpha}$ in the external space, where λ^α are the eigenvalues of the internal vector Laplacian.[38] Thus there is a massless vector for every harmonic one-form on the internal space.[39]

We are interested in the massless deformations only. One can show that these leave the internal space Ricci tensor unchanged $R_{mn}(g + \delta g) = 0$, which follows from the above discussion trivially. To be able to solve for explicit solutions one needs to specify the background metric more concretely, which leads us to restrict to the case of the Calabi-Yau fourfold in the following. This is relevant for this work since it turns the focus on the discussion of the reduction of eleven-dimensional supergravity to a $3d$, $\mathcal{N} = 2$ supergravity theory. However, the discussion of the reduction of IIB/IIA - supergravity on a Calabi-Yau threefold, is analogous. Note that since there do not exist harmonic one-forms on Calabi-Yau manifolds, see (2.80), the only massless deformations arise due to the metric variations δg_{mn}. These massless deformations are particularly special since in the case of the Calabi-Yau it has been shown that one can extend the infinitesimal shift of the metric along the massless direction, to become finite without changing the Ricci-flatness and thus the "Calabi-Yau 'ness" of the manifold. This gives rise to the so called moduli space of the theory, which we will discuss below [80].

One can show that the equation $\Delta_L|_{Y_4} w_{mn} = 0$ has to different sectors of solutions. One is given in terms of the harmonic $(1,1)$ forms

$$\{\omega_{im\bar{n}}\} \ , \quad i = 1, \dots, h^{1,1} \quad \Rightarrow \quad \delta g_{m\bar{n}} = \sum_i i\delta v^i \omega_{im\bar{n}} \ , \tag{2.133}$$

[38] In (2.132) we used the well known form of the equation of motion of vectors. Note that we could have written the differential operator in terms of the Hodge or vector Laplacian on the external space $\nabla^\nu \nabla_{[\nu} A^\alpha_{\mu]} = \Delta_H^{ext}\big|_{background} A^\alpha_\mu$.

[39] Note that $\nabla^\mu A_\mu = 0$, thus the vectors are in Lorentz gauge and that $\nabla^m \mathcal{A}^\alpha_m = 0$.

which gives rise to the so called Kähler deformations of the metric where $\delta v^i = \delta v^i(x)$ are real scalar fields on the external space. The other one being the complex structure deformations written in terms of the harmonic $(1,3),(3,1)$-forms and the holomorphic four-from as

$$\{\xi_{Im\bar{n}\bar{r}\bar{s}}\}\ , \quad I = 1,\dots,h^{1,3} \quad \Rightarrow \quad \delta g_{mn} = \frac{1}{3|\Omega|^2}\sum_I \delta d\bar{z}^I \xi_{Im\bar{n}\bar{r}\bar{s}} \Omega_n{}^{\bar{n}\bar{r}\bar{s}} \quad , \tag{2.134}$$

with $|\Omega|^2 = \frac{1}{4!}\Omega_{mnrs}\bar{\Omega}^{mnrs}$ and $\delta z^I(x)$ complex scalars [80]. Note that $\delta g_{\bar{m}n}$ and $\delta g_{\bar{m}\bar{n}}$ are given by the complex conjugates of (2.133) and (2.134), respectively, which in the Kähler case gives the identical expression. The Poincare dual of $(1,1)$-forms are divisors of the Calabi-Yau space, thus the Kähler deformations correspond to changing the overall size along these directions, in contrast to the complex structure deformations which change their shape.

From (2.127) we know that the deformations (2.133) and (2.134) give rise to massless real and complex scalars in three dimensions. The dimensional reduction is performed by plugging (2.133) and (2.134) into the M-theory action (2.62), which then sheds light on the couplings of these fields in the $3d$ action. We will present the result of the reduction of the Einstein-Hilbert term, which after a $3d$ Weyl rescaling to the Einstein form gives

$$\int_{M_3\times Y_4} R *_{11} 1 \quad \rightarrow \quad \int_{M_3} R *_3 1 \; - K_{ij} d\delta v^i \wedge *_3 d\delta v^j - G_{I\bar{J}} d\delta z^I \wedge *_3 d\delta \bar{z}^{\bar{J}} \quad , \tag{2.135}$$

where the couplings are given by

$$K_{ij} = G_{ij} + \frac{1}{\mathcal{V}^2}\mathcal{K}_i\mathcal{K}_j \quad ,\text{with} \quad G_{ij} = \frac{1}{2\mathcal{V}}\int_{Y_4}\omega_i \wedge *\omega_j \;\; \text{and} \;\; G_{I\bar{J}} = -\frac{\int_{Y_4}\xi_I \wedge \bar{\xi}_{\bar{J}}}{\int_{Y_4}\Omega\wedge\bar{\Omega}} \quad , \tag{2.136}$$

with $\mathcal{V} = \int_{Y_4} J\wedge J\wedge J\wedge J$ the volume of the Calabi-Yau fourfold. We comment on the reduction results of the Kähler sector in more detail in section 7.3. Note that since a Calabi-Yau manifold in particular is Kähler, the Kähler form and the metric relate via $J_{m\bar{n}} = i g_{m\bar{n}}$. Moreover, one can express them in terms of a basis expansion of harmonic $(1,1)$-forms as $g_{m\bar{n}} = -i v_0^i\, \omega_{im\bar{n}}$ and thus $J_{m\bar{n}} = v_0^i\, \omega_{im\bar{n}}$, which fixes the constants v_0^i according to the Calabi-Yau background metric. Note that the fluctuation of the background gives the fluctuated metric $g'_{m\bar{n}}(x,y) = -i(v_0^i + \delta v^i)\,\omega_{im\bar{n}}$ and $g'_{mn}(x,y) \sim \delta z^I \xi_{Im\bar{r}\bar{s}\bar{t}} \Omega_n{}^{\bar{r}\bar{s}\bar{t}}$. Due to the non trivial mathematical result that the infinitesimal shift can be replaced by a finite shift [80], one can make the replacement $v_0^i + \delta v^i(x) \rightarrow v^i(x)$ and $\delta z^I(x), \delta\bar{z}^{\bar{I}}(x) \rightarrow \bar{z}^{\bar{I}}(x)$. This is the manifestation of the statement that one can go from one vacuum configuration of the Calabi-Yau v_0^i to another $v^{i\prime}{}_0$, - with $v_0^i, v^{i\prime}{}_0$ constant - without the use of energy. Which in return can be seen from the fact that these massless scalars do not admit any potential thus $\langle v^i \rangle = v_0^i$ for any constant v_0^i. The space describing the different configurations of the internal space is called Moduli space. One notices from (2.136) that the geometry of the Moduli space is locally a product of complex structure and an Kähler moduli space

$$\mathcal{M}(Y_4) = \mathcal{M}^{3,1}(Y_4) \times \mathcal{M}^{1,1}(Y_4) \quad . \tag{2.137}$$

Furthermore, one notes that the metric on the space of $\mathcal{M}^{3,1}$ is given by $K_{I\bar{J}}$ in (2.136), which arises from a Kähler potential

$$G_{I\bar{J}} = \partial_{z^I}\partial_{\bar{z}^J}\mathcal{K}^{3,1} \ , \qquad \mathcal{K}^{3,1} = -\ln\Big(\int_{Y_4}\Omega\wedge\bar{\Omega}\Big) \ , \tag{2.138}$$

while the metric on the Kähler moduli space $\mathcal{M}^{1,1}$ given by G_{ij} in (2.136) analogously is sourced by

$$G_{ij} = \frac{1}{2\mathcal{V}^2}\partial_{L^i}\partial_{L^j}\mathcal{K}^{1,1} \ , \qquad \mathcal{K}^{1,1} = -3\ln\big(\mathcal{V}\big) \ , \tag{2.139}$$

where $\mathcal{V}$ is the volume of Y_4 and we have introduced the scalar fields $L^i = \frac{v^i}{\mathcal{V}}$.[40] Let us emphasize the beauty of this interpretation, on the one hand one has a description of a metric on the Moduli space of the theory, which on the other hand give the kinetic couplings, which due to $3d, \mathcal{N} = 2$ supersymmetry arise from a Kähler potential.

Let us return to the dimensional reduction of the only dynamical bosonic field of eleven-dimensional supergravity besides the metric, the 3-from field $\hat{C}$. Note that we inferred above $\hat{C}|_{background} = 0$. One can expand the components of $\hat{C}$ on the background metric as we did in the example (2.110) and (2.113). It is convenient to expand the p-form field in the eigentensors of the Hodge Laplacian, which gives a convergent expansion. The different components can be expanded as

$$A_{mnr} = \sum_{\Lambda} N^{\Lambda}(x)w_{\Lambda\, mnr}(y) \ , \ A_{\mu mn} = \sum_{i} A^i_{\mu}(x)w_{i\, mn}(y) \ \text{ and } \ A_{\mu\nu m} = \sum_{\alpha} A_{\alpha\mu\nu}(x)w_{\alpha\, m}(y) \ , \tag{2.141}$$

with $w_{i\,mnr}, w_{I\,mn}, w_{\alpha\, m}$ eigenforms of the Hodge Laplacian $\Delta_H w = \lambda w$ to the eigenvalues $\lambda^i, \lambda^j, \lambda^\alpha$, respectively. Note that the since we reduce to three dimensions $A_{\mu\nu\rho} \propto \epsilon_{\mu\nu\rho}$ does not give rise to any dynamical massless field. Since one knows that these expansions (2.141) converge on the metric background $M^{2,1}\times Y_4$ one can simply plug (2.141) in the M-theory action (1.52) and derive the spectrum of the reduced theory. The eigentensors to zero eigenvalue give rise to light modes. To see this one can also approach this by solving the field equations for $\hat{C}$ in the background, which read

$$d *_{11} \hat{G} = \frac{1}{3}\hat{G}\wedge\hat{G} \ , \tag{2.142}$$

where we have used that the covariant derivative acting on forms is equivalent to the exterior derivative. However, the discussion of massive modes is more involved due to the gauge invariance of $\hat{C}$. Hence let us just note that the massless modes are given by the zero modes of the Hodge Laplacian of $\Delta_H w_{i\,mnr} = \Delta_H w_{I\,mn} = \Delta_H w_{\alpha\, m} = 0$. Thus the massless fields are given in terms of harmonic forms, and since there are no harmonic one-forms on Calabi-Yau manifolds the decomposition $A_{\mu\nu m}$ is trivial. Furthermore $A_{\mu mn}$ is written in terms of harmonic $(1,1)$ forms giving rise to vectors A^i_μ in three dimension and A_{mnr} in terms of harmonic $(1,2),(2,1)$ forms, which gives rise to complex scalars

[40] This relation can easily be verified using the chain rule and the inverse definition of the intersection numbers (2.82) as

$$\partial_{L^i}\partial_{L_j}\mathcal{K}^{1,1} = \Big(\frac{\partial v^k}{\partial L^i}\partial_{v^k}\big(\frac{\partial v^m}{\partial L^j}\big)\Big)\partial_{v^m}\mathcal{K}^{1,1} + \frac{\partial v^k}{\partial L^i}\frac{\partial v^m}{\partial L^j}\partial_{v^k}\partial_{v^m}\mathcal{K}^{1,1} = -\mathcal{K}_i\mathcal{K}_j + \mathcal{K}_{ij}\mathcal{V} \ . \tag{2.140}$$

Where we have use that $\frac{\partial v^i}{\partial L^j} = \delta^i_j\mathcal{V} - \frac{1}{3}\mathcal{K}_j v^i$.

$N^\Lambda, \bar{N}^{\bar{\Lambda}}$. Combing the reduction of the metric moduli with the one of the three-form field one finds the $3d$ field-content of the $\mathcal{N} = 2$ supergravity theory, inherited from the eleven-dimensional fields, as depicted in the table 2.6. The vectors combine with the real scalars into a vector multiplet and the complex scalars form chiral multiplets. The explicit form of the reduction for the different sectors can

Fields	**Multiplet**	**Dimension**
$(N^\Lambda, \bar{N}^{\bar{\Lambda}})$	chiral	$h^{2,1}$
$(z^I, \bar{z}^{\bar{I}})$	chiral	$h^{3,1}$
(A^i_μ, v^i)	vector	$h^{1,1}$

Table 2.6: $3d$ field-content obtained from dimensional reduction of eleven-dimensional supergravity on a Calabi-Yau fourfold.

be reviewed in [81]. We discuss the reduction of the Kähler moduli and the vectors in section 7.

3 F-theory vs. IIB orientifolds

F-theory is a formulation of Type IIB string theory that incorporates seven-branes in a fully back reacted fashion [82] and at varying string coupling g_s, which is encoded in the geometry of an elliptically fibered higher-dimensional manifold. While this is a pragmatic definition, F-theory may have the potential to be more, however due to inconclusive evidence we stick to it for this discussion. Let us shortly give the M/F-theory recipe before diving into more detail. Starting with eleven-dimensional supergravity (2.62) upon compactification on a Calabi-Yau fourfold one finds a $3d$ $\mathcal{N} = 2$ supergravity theory. In F-theory one chooses the Calabi-Yau fourfold to be elliptically fibered, which can be written locally as the product of a Kähler manifold and a torus as $B_3 \times T_2$. Note furthermore that $4d$ $\mathcal{N} = 1$ supergravity compactified on a circle S^1, which in this context is taken to be one of the two circles of the torus $T^2 = S^1 \times S^1$, gives a $3d$ $\mathcal{N} = 2$ supergravity theory. Since the other circular dimension is already compactified it seems however hard to decompactify it to yield a $4d$ $\mathcal{N} = 1$ theory. However, this can be done using T-duality, which identifies a theory on a circle with radius R with another theory one on a circle of radius $1/R$. Note that M-theory reduced along one circle of the torus gives type IIA supergravity and one can then T-dualize the other circle to yield type IIB supergravity. By taking the limit $v \to 0$, where v is the volume of the torus, in the T-dual picture one sends the radius of the circle to infinity and a fourth extended dimensions appears. We will next discuss this in more detail focusing mainly on the reduction on Calabi-Yau fourfolds to $3d$ and the F-theory lift to $4d$. The following discussion is closely tied to [83, 84, 85, 86].

3.1 M/F-theory duality

The origin of F-theory is the manifest $SL(2,\mathbb{Z})$ S-duality invariance of type IIB supergravity described in section 2.2, with the axio-dilaton given by

$$\tau = C^{(0)} + i\, e^{-\Phi} \ \text{ transforming as } \ \tau \to \frac{a\tau + b}{c\tau + d} \ , \tag{3.1}$$

under an element in $SL(2,\mathbb{Z})$, thus $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$. This mirrors the complex structure modulus of a torus in every respect. Thus the axio-dilaton of type IIB supergravity can straightforwardly arise from a compactification of some twelve-dimensional theory on a torus. So do the $H^{(3)}$ and $F^{(3)}$ three-form field strengths, which transform as

$$\begin{pmatrix} H^{(3)} \\ F^{(3)} \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H^{(3)} \\ F^{(3)} \end{pmatrix} \ . \tag{3.2}$$

Thus can arise from a twelve-dimensional four-form $\hat{F}^{(4)}$ as $\hat{F}^{(4)}_{xMNO} = H^{(3)}_{MNO}$ and $\hat{F}^{(4)}_{yMNO} = F^{(3)}_{MNO}$, where x, y correspond to the two different one-cycles of the torus. Let us call them x, y-circle. However, as we have shown in section 2.2.1, supergravity in eleven dimensions is the highest possible dimensions in which one can engineer an ordinary supergravity multiplet. Thus there is no twelve-dimensional appropriate low-energy theory that upon dimensional reduction gives type IIB. However, one can use the detour via M-theory as explained in the beginning of the section. When allowing for higher-derivative corrections to eleven-dimensional supergravity (2.61) or in the presence of $M2$-branes the background metric needs to be warped, as we will see in section 6. Hence we write the background metric as

$$ds^2_{11} = e^{-2W^{(2)}} \eta_{\mu\nu} dx^\mu dx^\nu + 2e^{W^{(2)}} g_{m\bar{n}} dz^m d\bar{z}^{\bar{n}} \ , \tag{3.3}$$

where $\eta_{\mu\nu}$, $\mu = 0, 1, 2$ is the external space metric with Lorentzian signature and $g_{m\bar{n}} m, \bar{n} = 1, 2, 3, 4$ is the Calabi-Yau fourfold metric and $W^{(2)} = W^{(2)}(z, \bar{z})$ is the warp-factor being dependent on the internal space. Since we assume the Calabi-Yau fourfold to be elliptically fibered we can write it locally as $B_3 \times T^2$, where B_3 is a three complex dimensional Kähler manifold referred to as the base manifold. The torus is as well a Kähler manifold, thus one can locally express the product metric of the Calabi-Yau fourfold as

$$g_{m\bar{n}} dz^m d\bar{z}^{\bar{n}} = g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^{\bar{\beta}} + \frac{v}{2\tau_2} dz d\bar{z} \ , \tag{3.4}$$

where $\alpha, \bar{\alpha} = 1, 2, 3$ the indices on the base B_3 with metric $g_{\alpha\bar{\beta}}$ and $\frac{v}{\tau_2} dz d\bar{z}$ the metric on the torus. Note that a torus can be represented by a parallelogram, where opposite sides are identified and the base lies on the real axis in $\mathbb{C}$. The complex structure of the torus is given by a complex number $\tau = \tau_1 + i\tau_2$, while τ_1 representing the upward side of the parallelogram, the length of the base is given by a real number l, which parametrizes the overall size of the torus. Since the area of the torus v equals the area of the parallelogram on finds that $v = \tau_2 \cdot l$. Thus the information of the complex structure of the torus metric is hidden in the choice of complex coordinates $z, \bar{z}$, while the pre-factor in (3.4) reduces to l, which gives the size of the torus. Note that since we consider fibrations of the torus, $\tau = \tau(z^\alpha)$ is a holomorphic function over base manifold B_3, while v remains constant. It is convenient to go to real coordinates on the torus via the identification

$$\begin{array}{c} z = x + (\tau_1 + i\tau_2)y \\ \bar{z} = x + (\tau_1 - i\tau_2)y \end{array} \quad \Rightarrow \quad \begin{array}{rcl} x &=& \frac{1}{2\tau_2}\big(i\tau_1(z-\bar{z}) + \tau_2(z+\bar{z})\big) \ , \\ y &=& \frac{1}{2i\tau_2}\big(z-\bar{z}\big) \ , \end{array} \tag{3.5}$$

where x, y are periodic in $(0,1)$ and one finds[41]

$$\frac{v}{\tau_2} dz d\bar{z} \ \rightarrow \ \frac{v}{\tau_2}\big((dx + \tau_1 dy)^2 + \tau_2^2 dy^2\big) \ , \tag{3.6}$$

thus we can write the eleven-dimensional background metric as

$$ds_{11}^2 = e^{-2W^{(2)}}\eta_{\mu\nu}dx^\mu dx^\nu + e^{W^{(2)}}\Big(\frac{v}{\tau_2}\big((dx+\tau_1 dy)^2 + \tau_2^2 dy^2\big) + 2g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^{\bar{\beta}}\Big) \ . \tag{3.7}$$

By comparison with the metric of the circular compactification (1.53) of M-theory we can write (3.7) in terms of the IIA fields, resulting in

$$ds^2 = R^2 e^{\frac{4\Phi}{3}}(dx + C^{(1)})^2 + e^{-\frac{2\Phi}{3}} ds_{IIA}^2 \ , \tag{3.8}$$

where we have introduced the length scale R in (1.53), thus $x \in (0, R)$ the periodic length of the circle. By comparison of (3.7) and (3.8) one infers that

$$C^{(1)} = \tau_1\, dy \quad , \qquad\qquad e^{4\Phi/3} = \frac{v}{R^2\tau_2}e^{W^{(2)}} \ . \tag{3.9}$$

Furthermore by using $e^{-2\Phi/3} = R\sqrt{\frac{\tau_2}{v}}e^{-W^{(2)}/2}$ one finds that

$$ds_{IIA}^2 = \frac{\sqrt{v}}{R\sqrt{\tau_2}}\Big(e^{-3W^{(2)}/2}\eta_{\mu\nu}dx^\mu dx^\nu + e^{3W^{(2)}/2}\big(v\,\tau_2\, dy^2 + 2g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^{\bar{\beta}}\big)\Big) \ . \tag{3.10}$$

Thus upon circular reduction of (3.7) on the x-circle expressed in the correct type IIA variables one finds the type IIA background to be (3.10). As outlined in the introduction of this section the next step is to T-dualize the geometry along the other circle - thus the y-circle. Since T-duality maps type IIA to type IIB string theory one has

$$R^{IIB} = \frac{l_s}{R_{IIA}}, \qquad C^{(0)} = C^{(1)}_y \ , \qquad g_{s,IIB} = \frac{l_s}{R_{IIA}} g_{s,IIA} \ , \tag{3.11}$$

where R_{IIA} is the length of the y-circle. Since in (3.10) the periodicity of $y \in (0,1)$ we find the length of the circle $R_{IIA}^2 \sim e^{3W^{(2)}/2}$, which gives under T-duality $R_{IIB}^2 \sim e^{-3W^{(2)}/2}$. By matching the scales appropriately using reductions of $M2$ and $D2$-probe brane actions on the background (3.8) and by setting the length scale $R = \sqrt{v}$ one infers

$$\tau = C_0 + i(g_{s,IIB})^{-1} \quad , \qquad ds_{IIB}^2 = e^{-3W^{(2)}/2}\big(\eta_{\mu\nu}dx^\mu dx^\nu + d\tilde{y}^2\big) + 2e^{3W^{(2)}/2} g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^{\bar{\beta}} \ . \tag{3.12}$$

This is the metric in the Einstein frame and we have defined a new coordinate for the T-dual circle $\tilde{y}$ of periodicity $\tilde{y} \in (0, \frac{l_s^2}{\sqrt{v}})$. One can now take the limit of shrinking the torus volume to zero $v \rightarrow 0$ but

[41] Note that $\int_{T^2} dz \wedge d\bar{z} = \tau_2$, which is easily shown in real coordinates using (3.6).

keeping l_s finite in which the periodicity of $\tilde{y}$ goes to infinity and one gains another extended fourth dimension. Thus one can write the previous equation as

$$ds^2_{IIB} = e^{-3W^{(2)}/2}\eta_{\tilde{\mu}\tilde{\nu}}dx^{\tilde{\mu}}dx^{\tilde{\nu}} + 2e^{3W^{(2)}/2}g_{\alpha\bar{\beta}}dz^{\alpha}d\bar{z}^{\bar{\beta}} \quad , \tag{3.13}$$

where now $\tilde{\mu}, \tilde{\nu} = 0, 1, 2, 3$. Since we have now commented on the behavior of the background in the limit $v \to 0$ let us next discuss the scaling behavior of the $3d$ effective couplings of the theory depending on the geometry of the elliptically fibered Calabi-Yau fourfold. To be able to proceed in this discussion let us refine our notion of F-theory by comparing M-theory on elliptically fibered Calabi-Yau fourfold giving a $3d$, $\mathcal{N} = 2$ theory, to a $4d$, $\mathcal{N} = 1$ theory with non-Abelian gauge groups compactified on S^1. Let us preempt a result of the next section 3.2 namely that a stack of (p, q) seven-banes is identified with singular fibers in the Calabi-Yau fourfolds. Thus by concretely choosing singular fibers one can engineer gauge groups in the effective theory. Furthermore this allows to extend the collection of gauge groups $U(N), SO(n), Sp(n)$ of type IIB orientifold setups by exceptional gauge groups E_6, E_7 and E_8, which are of particular interest for GUT model building. However, it turns out to be difficult to establish $U(1)$ symmetries, which are needed e.g. to forbid proton decay operators, see e.g. [87, 88, 89, 90, 91, 92]. Let us shortly comment on this intriguing interplay between gauge theory and geometry. All possible singularities in the fiber of a elliptically fibered Calabi-Yau fourfold have been classified by Kodaira [93, 94]. Although there have been proposals how to extract the physics directly from singular Calabi-Yau spaces [95], the best understood procedure is to resolve the geometric singularities locally by smearing out the singularity by replacing it with a smooth patch, intuitively speaking. More precisely, to every singular fibered Calabi-Yau fourfold Y_4 one can find a smooth Calabi-Yau fourfold $\tilde{Y}_4$ and a map which smoothy transforms $\tilde{Y}_4 \to Y_4$ called the blow-down, reversely one uses the terminology blow-up to resolve singularities. One can depict the blow-up as following, the fiber of the Calabi-Yau be singular along a divisor of the base B_3. One can resolve this singularity by fibering appropriate intersections of $\mathbb{P}^1 \simeq S^3/U(1) \simeq S^2$ spaces over the singular divisor and hence replacing the singularity in the fiber by a smooth geometry to gain $\tilde{Y}_4$, e.g. see figure I.8.

Reversely the blow-down map simply shrinks the intersection of $\mathbb{P}^1$'s to zero and one recovers the

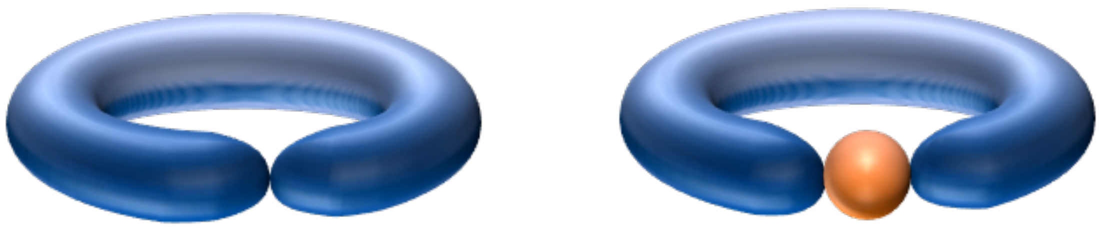


Figure I.8: Pictorial depiction of the singular fiber on the left characterized by the pinching of the torus and the resolved geometry on the right, where the singularity is removed due to the replacement by a smooth $\mathbb{P}_1$ geometry. The intersection pattern corresponds to the Dynkin diagram of $SU(2)$.

original singular geometry Y_4. Note that the geometry, that is patched in, consists of a fibration of a two-dimensional space over a four-curve thus gives new divisors of the total Calabi-Yau manifold, referred to as exceptional divisors. Remarkably, the intersection pattern of the $\mathbb{P}^1$'s, according to Kodaira's singularity classification mimics exactly the affine Dynkin diagrams of Lie Algebras of the A,D,E series. May the intersection pattern reflect the Dynkin Diagram of a group G then the number of exceptional divisors D_I coincides with the number of generators of the Cartan subalgebra of G thus on can write

$$D_I, \quad I = 1, \dots, \text{rank } G \ . \tag{3.14}$$

One furthermore can split the divisors D_i in three types according to their Poincaré-dual two-forms ω_0, ω_α, and ω_I, where ω_i, $i = 1, \dots, h^{1,1}(Y_4)$. The two-form ω_0 corresponds to the holomorphic zero-section and has two legs in the fiber, while the two-forms ω_α with $\alpha = 1, \dots, h^{1,1}(B_3)$ have two legs in the base and under Poincare duality correspond to divisors that are elliptic fibrations over divisors of the base. Finally the two-forms ω_I correspond to the blow-up or exceptional divisors.[42] Upon dimensional reduction of the M-theory three-form one finds a massless vector for every harmonic ω_i as discussed in detail in chapter II, identifying

$$\hat{G}_{\mu\nu m\bar{n}} = F^i_{\mu\nu}\omega_{im\bar{n}} \ , \tag{3.15}$$

where $F = dA$.

The locus of vanishing fiber being a four-curve in the base B_3 of an elliptically fibered Calabi-Yau fourfold Y_4 when resolved mimics a group G. One can thus interpret it as wrapped by a stack of space-time filling (p, q) seven-branes, and G as the gauge group living on the branes. Since we only treat the resolved geometry where the group is Abelianized the vectors A^I are interpreted as the $U(1)$ factors in the Cartan subalgebra of the non-Abelian gauge group. To restore the non-Abelian gauge group one needs more degrees of freedom encoded in $M2$-branes wrapping the resolution $\mathbb{P}^1$'s. These can be interpreted as type IIA strings stretching in-between parallel $D6$-branes, due to their separation these are massive and are interpreted as the massive "W" bosons of the gauge group G in the Coulomb branch, encoding the degrees of freedom of the roots of G. By blowing down the geometry these string states become massless, since the $D6$-branes approach each other during the down-lift to become coincident when the singular geometry $\tilde{Y}_4$ is recovered. This enhances the gauge group to the full non-Abelian group G. Note that since it is not known how to proceed to the singular limit this provides just an explanation an hence a justification for the interpretation of A^I being the $U(1)$ factors in the Cartan subalgebra of the non-Abelian gauge group G.

Furthermore one can thus expand the Kähler form of the Calabi-Yau fourfold as

$$J = v^0\omega_0 + v^\alpha\omega_\alpha + v^I\omega_I \, , \tag{3.16}$$

where v^0 represents the volume of the elliptic fiber. By comparing M-theory on elliptically fibered Calabi-Yau fourfolds giving a $3d\,, \mathcal{N} = 2$ theory obtained by circular reduction of radius r of a $4d, \mathcal{N} = 1$ one obtains the relation between the eleven-dimensional Planck length l_M and the string length l_s to be

$$l_s = l_M(v^0)^{-1/4} \, , \qquad r = (v^0)^{-3/4} \, . \tag{3.17}$$

[42] It is common notation that ω_I also denotes extra sections in the Calabi-Yau fourfold, which give rise to Abelian $U(1)$ factors.

In the F-theory limit one sends $v^0 \to 0$ and thus decompactifies the fourth dimension by sending $r \to \infty$. Then all volumes of the base B_3 become expressed in units of l_s. One introduces a small parameter ϵ to express the scaling of the dimensionless fields by writing $v^0 \sim \epsilon$. As explained in [96, 97] one finds $v^\alpha \sim \epsilon^{-1/2}$ and infers the scaling behavior of the classical volume of Y_4 to be $\mathcal{V}_4 \sim \epsilon^{-1/2}$. In the following we use the subscript $_b$ to denote quantities of the base that are finite in the limit $\epsilon \to 0$. In particular one has

$$v_b^\alpha = \sqrt{v^0} v^\alpha \ , \tag{3.18}$$

which holds in the strict $\epsilon \to 0$ limit. Note that v_b^α are the volumes of two-cycles of the base in the Einstein frame.

3.2 On branes and elliptic curves

Incorporating $D7$ and $D3$-branes in a $O7/O3$-planes background in type IIB is a non-trivial endeavor since the branes back-react on the geometry. This back-reaction is usually neglected by going to a large volume limit, where one can treat the branes as being far away and thus knowledge of the local solution around the brane becomes negligible. Although this allows to study some main properties of the background one has to keep in mind that this is not a fully consistent background solution. This approximation works for p-branes with spatial normal dimensions bigger than two thus for $p = 1, \dots, 6$ in ten-dimensional space-time, since they source a back-reaction to the metric and the $C^{(p+1)}$ from field, which decay as

$$C^{(p+1)} \sim N \frac{1}{r^{7-p}} \ , \qquad ds^2 = \Big(1 + \frac{Ne^{\Phi_0}}{r^{7-p}}\Big)^{-1} \eta_{\mu\nu} dx^\mu dx^\nu + \Big(1 + \frac{Ne^{\Phi_0}}{r^{7-p}}\Big) \delta_{ij} dx^i dx^j \ , \tag{3.19}$$

where this precise scaling holds for supergravity BPS solutions of stacks of N p-branes [98]. Where $\mu = 0, 1, \dots, p$ are the dimensions along the brane and $i, j = 9 - p$ the transverse ones, e^{Φ_0} the asymptotic value of e^Φ, which can be taken as the string coupling g_s, and r the spatial distance from the brane. Heuristically this can be understood by describing the fields of the branes as solutions of the Laplace equation sourced by a delta functional, which then has a solution depending on the normal spatial dimensions of the support of the delta functional proportional to $1/r^{7-p}$. Thus a six-brane behaves qualitatively like an electrostatic point particle, while lower dimensional branes give rise to even faster decaying fields, and a large volume limit gives a sensible approximation. This argument breaks down in the case of $D7$-branes, which source fields that decay not only logarithmically but even more severely have monodromies as we will argue in the following. In co-dimension two a complex coordinate $z = z_0$ describes the position of the 7-brane, which electrically sources a $C^{(8)}$ form field dual to a magnetically sourced $C^{(0)}$ axion, obeying the Poisson equation

$$d *_{10} F^{(9)} = \delta^{(2)}(z - z_0) \qquad \Rightarrow \qquad \int d *_{10} F^{(9)} = \oint_{S^1} dC^{(0)} = 1 \ . \tag{3.20}$$

Note that the axio-dilaton $\tau = C^{(0)} + ie^{-\Phi}$ due to supersymmetry is a holomorphic function of z thus one can solve the previous equation by

$$\tau = \tau_0 - \frac{i}{2\pi} \ln(z - z_0) + F_{\text{regular}}(z) \ . \tag{3.21}$$

Note that by choosing $z_0 = 0$ at the origin and taking a counterclockwise path around the brane due to the branch-cut the imaginary part of the logarithm $f(\varphi) := \ln(r_0 \cos\varphi + ir_0 \sin\varphi)$, for any radius r_0 increases at at rate of 2π per closed circle, thus $f(\varphi + 2\pi) = f(\varphi) + 2\pi$. This in return implies that using (3.21) the axion and axio-dilaton jump as

$$C^{(0)} \to C^{(0)} + 1 \,, \qquad\qquad \tau \to \tau + 1 \,, \tag{3.22}$$

when following it once around a single $D7$-brane. Note that in (3.21) for $z \to z_0$ the imaginary part arising due to the real part of the logarithm diverges thus $g_s \to 0$ and perturbation theory is valid in the vicinity of the brane. Note that due to the $SL(2,\mathbb{Z})$ invariance of type IIB string theory the monodromy of the axio-dilaton (3.22) is a symmetry of the theory and thus this does not pose a problem. Furthermore one can think of objects which generate more complicated monodromies of τ but are nevertheless locally related to $D7$-branes by a more complicated $SL(2,\mathbb{Z})$ transformation, see (1.51). These objects will be manifestly non-perturbative and are called (p,q)-branes, extended hypersurfaces where (p,q)-strings are bound to end on. By setting the notation that $(1,0)$ represents the $F1$-string giving rise to B_2 while $(0,1)$ is the $D1$-string charged under C_2, they form a $SL(2,\mathbb{Z})$ doublet. This sets a notation which manifests the mixing of B_2, C_2 under (1.51). A string that carries p units of electric B_2-charge and q units of electric C_2-charge is referred to as a (p,q)-string.

Note that in F-theory the geometry gives us a varying τ of the torus over the base manifold B_3, which becomes the axio-dilaton of type IIB compactified on the base. In the generic setup one finds regions of strong and weak coupling and the monodromies of the branes identify them as (p,q)-branes. Locally one can always find a transformation of $SL(2,\mathbb{Z})$ to transform them back to $D7$-branes, however, globally this is not possible in general and one has to apply a refined procedure to go to weak coupling as we discuss among other things in the following. Note that when the x-cycle in the torus fiber collapses along a divisor in the base B_3, in the F-theory lift this becomes a space-time filling $D7$-brane. Similarly, when a $p\,x + q\,y$-cycle of the torus collapses one gains (p,q)-brane. Since this work does not intend to give a self contained introduction to algebraic geometry let us just study the geometrization of physics in F-theory in an hands on example, which shows several features found in more complicated phenomenological more relevant compactifications. One can explicitly construct geometries by studying vanishing loci of polynomials, very similar to $ax + b + y = 0$ in $\mathbb{R}^2$ defines the geometry of a straight line. Elliptic curves are given of the vanishing loci of homogenous polynomials in a weighted projective space $\mathbb{P}_{231}$, where

$$\mathbb{P}_{231} = \{(x,y,z) \sim (\lambda^2 x, \lambda^3 y, \lambda z) \in \mathbb{C}^3/\{(0,0,0)\} | \lambda \in \mathbb{C}^*\} \ , \tag{3.23}$$

where the equivalence relation denotes that all points which are equivalent under this transformation contribute as a single point to $\mathbb{P}_{231}$. One can now form homogenous polynomials of the so called homogenous coordinates x, y, z, which define elliptic curves such as the Weierstrass form

$$y^2 = x^2 + f(u,v)xz^4 + g(u,v)z^6 \ . \tag{3.24}$$

Let us now turn to the example of M-theory on $\mathbb{R}^{1,6} \times K3$, where $K3$ is a complex two-dimensional Calabi-Yau manifold. Note that we want to describe an elliptically fibered $K3$, which can be done by subjecting the homogenous coordinates of (3.24) to a further equivalence relation

$$(u, v, x, y, z) \sim (u, v, \lambda^2 x, \lambda^3 y, \lambda z) \sim (\mu u, \mu v, \mu^4 x, \mu^6 y, z) \ , \tag{3.25}$$

where $\mu, \lambda \in \mathbb{C}/\{0\}$ and $(u, v, x, y, z) \in \mathbb{C}/\{(0, 0, x, y, z) \cup (u, v, 0, 0, 0)\}$, with f, g polynomials of degree eight and twelve in u, v. This describes a two-dimensional complex surface, since we added two coordinates u, v but have two equivalence relations (3.25) and one defining equation (3.24). Note that the sum of weights, hence the some of powers of μ, λ in (3.25) is equal to 6 in the first relation and in the second to 12. While counting the highest power of coordinates u, v, x, y, z in (3.24), which is subject to a weighting in the equivalence relations is as well 6 in the first and 12 in the second, defining the degree of the polynomial. One can show that since the degrees of the both countings agree the hypersurface is Calabi-Yau. Let us look at the local picture by fixing u, v and thus neglecting the newly introduced equivalence relation, one has thus an one-complex dimensional hypersurface and counting the degrees as before one finds from the first relation in (3.25) 6 vs. 6, thus the torus T^2. One can now choose coordinates in the full setup as $z \equiv 1, v \equiv 1$ such that the equation simplifies to

$$y^2 + x^3 + f(u)x + g(u) \ , \tag{3.26}$$

with f, g polynomials of degree eight and twelve. This describes an elliptic fibration of T^2 over the base $\mathbb{P}^1 \simeq S^2$. This algebraic definition of an elliptic fibration needs to be connected to the more familiar one of differential geometry, where one can explicitly see the complex structure τ of the torus and its metric. We note that this can be done by computing period integrals and one arrives at

$$j(\tau) = \frac{4(24f(u))^3}{\Delta} \ , \qquad \Delta = 27g^2 + 4f(u)^3 \tag{3.27}$$

$$j(\tau) = e^{-2\pi i \tau} + 744 + \dots \ , \qquad ds^2 \sim \frac{\tau_2 |\eta(\tau)|^4}{|\Delta|^6} dz d\bar{z} \ . \tag{3.28}$$

where $\eta(\tau)$ is the Dedekind eta-function, $j(\tau)$ is the $SL(2, \mathbb{Z})$ modular invariant j-function, and Δ is referred to as discriminant. Note that in this expression τ_2, τ, Δ vary over the base due to their dependence on u. Furthermore Δ is a homogeneous polynomial of degree 24 with generic zero points u_i, $i - 1, \dots, 24$, which implies the elliptic curve becomes singular at this points. let us explicitly compute τ in the vicinity of this singular fibers thus in the vicinity of the $D7$-branes, one has

$$j(\tau(u)) \sim \frac{1}{u - u_i} \qquad \Rightarrow \qquad \tau(u) \sim -\frac{i}{2\pi} \ln(u - u_i) \ . \tag{3.29}$$

Where this implication holds up to $SL(2, \mathbb{Z})$ transformations and one has to be more careful when going to the weak coupling limit. Equation (3.29) coincides with the analysis from (3.21) for the axio-dilaton profile near a $D7$-brane. Reversely, not knowing the above result one would identify the $D7$-brane by the induced monodromy (3.29). Thus we naively would find 24 $D7$-branes, one for each locus of Δ, which violates charge conservation in the compact space. However, as we commented on above one cannot simply go to the weak coupling limit globally. The procedure of allowing for a global weak coupling solution is due to Sen [99, 100] and is done by pushing the computation to a point in complex structure moduli space of $K3$ where $\tau(u)$ is constant and has large imaginary part, thus $g_s \ll 1$. Rewriting (3.27), one infers that $f(u)^3/g(u)^2 = \text{const.}$ is sufficient to guarantee that the axio-dilaton is constant, hence one finds

$$\frac{f(u)^3}{g(u)^2} = \text{const} \quad \Rightarrow \quad g(u) = h(u)^3 \ , f(u) = a \cdot h(u)^2 \quad \text{with} \quad h(u) = \prod_{i=1}^{4} (u - u_i) \ , \tag{3.30}$$

with a a constant and $h(u)$ a homogenous polynomial of degree four, where we set its coefficients to one by a rescaling of the homogenous coordinates x, y. Now it is straightforward to plug this into (3.27) and derive

$$j(\tau) \sim \frac{1}{27 + 4a^3} \quad \Rightarrow \quad a \approx -\frac{3}{4^{1/3}} \ , \quad \Delta \sim (27 + 4a^3) \prod_{i=1}^{4} (u - u_i)^6 \ , \tag{3.31}$$

which gives weak coupling on every point in the base. One notes that the fields B_2, C_2 are double-valued, namely flipping signs when circled around a zero of $h(u)$, which is seen by looking at the non-trivial monodromy element $a = 1, d = -1, b = c = 0$, see (1.51) sending $\tau \to \tau$. We will next show that this arises due to the presence of an $O7$-plane. Let us construct the double cover X of the base $\mathbb{P}^1$ expressed by one equation and one variable, which are then added to (3.24) and (3.25), and are given by

$$w^2 = h(u, v) \ , \quad (u, v, w) \in \mathbb{C}/(0, 0, 0) \ , \quad (u, v, w) \sim (\lambda u, \lambda v, \lambda^2 w) \ , \tag{3.32}$$

where this describes a one-complex dimensional space of weights 4 *vs.* 4 since $h(u, v)$ is of degree 4. Thus it represents the only compact one-dimensional Calabi-Yau, namely the torus. By adding another relation, which identifies $(w) \sim (-w)$ on finds the original base $\mathbb{P}^1$, this is more commonly denoted in type IIB language as the orientifold projection qoutienting the Calabi-Yau manifold as

$$\mathbb{P}^1 = X/\sigma \ , \quad \sigma : w \to -w \ . \tag{3.33}$$

This map can be compared to perturbative world-sheet results obtained by orientifolding X in type IIB to give justification of this identification.

Very intriguingly we can now look at fix points of the $\mathbb{Z}_2$ involution σ, where one expects the $O7$-planes. These are straightforwardly obtained from the loci of $h(u)$ where one finds four. Note that in the covering space one does not find any monodromies of $C^{(0)}$ since the fields are single-valued. Thus to cancel the charge of the $O7$-plane of -4, there must be four $D7$-branes sitting on top of each $O7$-planes, respectively.

Let us comment on the example of the previous section 3.1 where we have discussed the F-theory lift on an elliptically fibered Calabi-Yau fourfold Y_4 over the Kähler base B_3. The explicit singularity structure and therefore the gauge groups are model dependent, however, one encounters that in Sen's weak coupling limit the double cover of the base is the orientifolded Calabi-Yau threefold in the type IIB picture, thus

$$B_3 = X_3/\mathbb{Z}_2 \ . \tag{3.34}$$

Let us close this section by emphasizing again that in the realm of F-theory the back reaction of the $D7$-branes and $O7$-planes on the axio-dilaton τ is beautifully incorporated in the geometry.

CHAPTER II

Higher-derivative M-theory and α'-corrections to F-theory - a first step

The goal of this section is to perform a Kaluza-Klein reduction of M-theory including certain l_M^6-corrections on a Calabi-Yau fourfold down to three dimensions, and to subsequently apply the F-theory lift to gain a $4d$ theory. The line element of the background metric is

$$d\hat{s}^2 = \eta_{\mu\nu} dx^\mu dx\nu + 2g^{(0)}_{m\bar{n}} dy^m dy^{\bar{n}} \quad , \tag{3.1}$$

with $g^{(0)}_{m\bar{n}}$ the Calabi-Yau metric. Note that this background is not a solution to the $11d$ E.O.M.'s including higher-derivative terms, instead the background metric needs to be warped and non-vanishing background fluxes have to be considered, as discussed in section 6. Furthermore, for simplicity we do not allow for background fluxes in this chapter, although there is no principal objection. Let us summarize the required steps to perform a supersymmetric reduction including higher-derivative corrections. Following the procedure of dimensional reduction of section 2.4.3 one first needs to determine the supersymmetric background, and then perform the dimensional reduction of the M-theory action including the full set of l_M^6-corrections. Where the metric needs to varied w.r.t. the Kähler and complex structure moduli. This yields a $3d$, $\mathcal{N} = 2$ theory which can then be lifted to a $4d$, $\mathcal{N} = 1$ theory using F-theory. This is a vast endeavor which is performed to some extend in chapter III, however, this section reviews our early work [101, 102] on the subject - a first step, which serves as an excellent toy model to highlight the main features of this program.

We start with a simplified setting considering a subset of the 11- dimensional action

$$S = S^{(0)} + \alpha^2 S^{(2)}_{\hat{R}^4} + \alpha^2 S^{(2)}_{\hat{G}^2\hat{R}^3} + \alpha^2 S^{(2)}_{(\hat{\nabla}\hat{G})^2\hat{R}^2} + \mathcal{O}(\hat{G}^3\alpha^2) + \mathcal{O}(\alpha^3)\, , \tag{3.2}$$

at eight-derivative level given by

$$S_{\text{sub}} = \frac{1}{2\kappa_{11}^2} \int R *_{11} 1 - \frac{1}{2}\hat{G} \wedge *_{11} \hat{G} + \alpha^2 \Big[(\hat{t}_8 \hat{t}_8 - \frac{1}{24} \hat{\epsilon}_{11} \hat{\epsilon}_{11}) \hat{R}^4 - (\hat{t}_8 \hat{t}_8 + \frac{1}{96} \hat{\epsilon}_{11} \hat{\epsilon}_{11}) \hat{G}^2 \hat{R}^3 \Big] \hat{*} 1 + S_{CS} \ , \quad (3.3)$$

and

$$S_{CS} = -\frac{1}{2\kappa_{11}^2} \int \frac{1}{6} \hat{C} \wedge \hat{G} \wedge \hat{G} + 3^2 2^{13} \alpha^2 \hat{C} \wedge \hat{X}_8 \ , \quad (3.4)$$

where we reordered the terms in a more suitable way for this discussion. Note that we have dropped $S^{(2)}_{(\hat{\nabla}\hat{G})^2 \hat{R}^2}$ at level α^2, the terms of order $\mathcal{O}(\hat{G}^3 \alpha^2)$ are not relevant for this analysis since they would give only rise to more than two external derivatives.

The variations of the Calabi-Yau metric split into $h^{1,1}(Y_4)$ Kähler structure and $h^{3,1}(Y_4)$ complex structure deformations. For simplicity we will consider geometries with $h^{2,1}(Y_4) = 0$ in the following. Furthermore, we will not consider the complex structure deformations. In fact, one can check that the corrections analyzed in the following are indeed independent of the complex structure. In principle the Kähler deformations of the metric of the purely Riemann terms in (2.63) lead to kinetic terms for the real scalars and one expects that these kinetic couplings will receive corrections from the higher-derivative pieces as well, as we show in section 7. This approach is likely not to yield the physical complete answer since one neglects warping and does not take into account all relevant terms in the eleven-dimensional action. Nevertheless this is a pedagogical valuable toy model since one is confronted with all the conceptual steps and techniques required to approach the full problem. Performing the reduction on this background one breaks supersymmetry explicitly but perturbatively in l_M. Nevertheless, it turns out that the considered sector after reduction still obeys $3d, \mathcal{N} = 2$ supersymmetry properties, which are read off by comparing to canonical form of the $3d$, $\mathcal{N} = 2$ action. However, this only holds under the assumption that the third Chern form c_3 is harmonic with respect to the background Calabi-Yau metric. It is not clear if this assumption can ever be satisfied, however a more refined treatment in chapter III shows that indeed the non-harmonic part of c_3 plays an important role. In chapter III we exclusively study the general case of a non-harmonic third Chern form.

4 Unwarped reduction of higher-derivative M-theory

Let us start by commenting on the structure of $3d$, $\mathcal{N} = 2$ supergravity theories reviewed in 2.2.2, at which one arrives after reduction on a Calabi-Yau fourfold. In general the canonical form of the $3d$ $\mathcal{N} = 2$ action propagates a number of complex scalars N^A in chiral multiplets coupled to non-dynamical vectors. In the following, we will only consider the ungauged case and can hence start with a $3d$ theory with only gravity and chiral multiplets.[1] In order to match the action (2.53) with the dimensional reduction of M-theory, it turns out to be useful to dualize some of the scalar multiplets N^A into $3d$ vector multiplets. One decomposes $N^A = \{M^I, T_i\}$ and splits the index as $A = (I, i)$.

[1] Let us stress that most of the derivation presented in the following can be generalized to the case with non-trivial gaugings in a straightforward fashion [51].

If the real scalars ImT_i have shift symmetries, it is possible to dualize them to vectors A^i. The real parts of T_i are redefined to real scalars L^i that naturally combine with the vectors A^i into the bosonic components of $\mathcal{N}=2$ vector multiplets.

In the following we aim to read off the Kähler potential K and metric $\tilde{K}_{ij}$ from the dimensional reduction of the 11d action (3.3).

Neglecting higher-derivative terms, the $\mathcal{N}=2$ Kähler potential arising from a reduction on a Calabi-Yau fourfold Y_4 was derived in [103, 81]. For the Kähler structure moduli it was found to be

$$K = -3\log \mathcal{V}_0 \ , \qquad \mathcal{V}_0 = \frac{1}{4!}\int_{Y_4} J^{(0)} \wedge J^{(0)} \wedge J^{(0)} \wedge J^{(0)} \ , \tag{4.1}$$

where $\mathcal{V}_0$ is the classical volume of Y_4, and $J^{(0)}$ is the Kähler form on Y_4. Note that the quantity in the logarithm, i.e. the volume $\mathcal{V}_0$, appears in front of the 3d Einstein-Hilbert term after dimensional reduction. A Weyl rescaling of the metric $g_{\text{new}} = \mathcal{V}_0^2\, g_{\text{old}}$ transforms it to the Einstein frame. In fact, due to the Weyl rescaling also the scalar potential is rescaled and by comparison with the factor e^K in (2.54) one can heuristically infer (4.1).

Let us first give an overview of the logic of the computation before diving into more details in section 4.1. Including the higher-derivative terms present in $S_{\mathcal{R}}$ given by (2.63), one might expect a correction to the classical Kähler potential (4.1), sourced by a correction to the $3d$ Einstein-Hilbert term of the form

$$S_3 \supset \frac{1}{(2\pi)^8}\int \tilde{\mathcal{V}} R^{(3)}_{sc} *_3 \mathbf{1} \ , \tag{4.2}$$

with the corrected volume

$$\tilde{\mathcal{V}} = \frac{1}{4!}\int J^{(0)4} + \mathcal{V}_{R^4} \ . \tag{4.3}$$

The explicit form of the correction $\mathcal{V}_{R^4}$ induced by the higher-derivative terms is given in the next section 4.1. Applying the same strategy as above, one can then infer the corrected Kähler potential to be

$$K = -3\log \tilde{\mathcal{V}} \ . \tag{4.4}$$

Here we have used the conventions to avoid powers of l_M floating around, which we keep for this chapter that[2]

$$2\kappa_{11}^2 = (2\pi)^8 l_M^9 = (2\pi)^8 \equiv 2\kappa_3^2 \ , \qquad \alpha^2 \to \frac{\pi^2}{3^2\cdot 2^{11}} \tag{4.5}$$

It is important to emphasize that this derivation does not suffice to fix the $3d$ Kähler coordinates T_i. However, this can be achieved by reading off the metric $\tilde{K}_{ij}$ in front of the dynamical terms of the vectors in (2.56). More precisely, we perform the reduction of the terms $\hat{G}^2\hat{R}^3$ in S_{sub} given in (3.3) on a Calabi-Yau fourfold Y_4. The kinetic terms of the vectors arise as a subset of the terms induced by reduction of S_{G_4} and take the form

$$S_3 \supset \frac{1}{(2\pi)^8}\int G_{ij}\, F^i \wedge *_3 F^j \ . \tag{4.6}$$

[2]This corresponds to setting $\alpha' = g_S^{\text{IIA}} = 1$ in $l_M = (2\pi g_S^{\text{IIA}})^{1/3}\sqrt{\alpha'}$, when reducing to Type IIA string theory.

One arrives in the frame with dynamical vectors which thus can be compared to the canonical form of the action (2.56). Due to the corrected volume $\tilde{\mathcal{V}}$ in front of the Einstein-Hilbert term (4.2) one first Weyl rescales the action to to transform it to the Einstein frame. This process introduces a power of $\tilde{\mathcal{V}}$ in front of the kinetic term of the vectors and one finds

$$S_3 \supset \frac{1}{(2\pi)^8} \int R *_3 1 + \tilde{\mathcal{V}} G_{ij} F^j \wedge *_3 F^i \,. \tag{4.7}$$

After comparing to (2.56) and using (4.5), one infers that $\tilde{K}^{red}_{ij} = 2\tilde{\mathcal{V}} G_{ij}$. In order to find a consistent reduction, $\tilde{K}^{red}_{ij}$ has to be compatible with K as given in (4.4) and (4.13). This fixes the $3d$ Kähler coordinates T_i as we discuss in more detail in subsection 4.2.

4.1 Dimensional reduction of higher-curvature terms

In this subsection we present the reduction of (3.3) on a Calabi-Yau fourfold to three dimensions with focus on the Weyl rescaling factor of the Einstein-Hilbert term and the kinetic terms of the vectors which will allow us to infer the Kähler potential and the Kähler coordinates. We only consider the Kähler structure variations of the Calabi-Yau metric $h^{1,1}(Y_4)$ for the kinetic terms of the vectors and do not discuss the fluctuations of the purely gravitational terms in (3.3). The main part of this work discusses the analysis of corrections to the two derivative effective theory in three dimensions, induced by higher-derivative terms in the eleven-dimensional theory. Note that this makes use only of a part of the information contained in the higher-derivative terms which also give rise to higher-derivative external terms. In this section we also will derive a four-derivative correction to the effective action arising from the purely gravity part of (3.3). The Kähler structure deformations parametrize the variations of the Kähler form $J^{(0)}$ by expanding

$$J^{(0)} = v_0^i \omega_i^{(0)} \,, \tag{4.8}$$

where $\{\omega_i^{(0)}\}$ is a basis of harmonic $(1,1)$-forms on Y_4 w.r.t to the Calabi-Yau metric $g_{m\bar{n}}(v_0)$. After the lift $v_0^i \to v^i$ these correspond to real scalar fields in the $3d$ effective theory. Note that we have introduced the notation $\omega^{(0)}$ denoting that it is harmonic w.r.t. to the background Calabi-Yau metric, to distinguish it from $\omega^{(2)}$ which will become relevant in chapter III. Furthermore, we define the topological quantities

$$\mathcal{Z}_i^{(0)} = \int_{Y_4} c_3^{(0)} \wedge \omega_i^{(0)} \,, \qquad \mathcal{Z}^{(0)} = \mathcal{Z}_i^{(0)} v_0^i = \int_{Y_4} c_3^{(0)} \wedge J^{(0)} \,, \tag{4.9}$$

where $c_3^{(0)}$ is the third Chern class of the tangent bundle of Y_4 dependent on the background Calabi-Yau metric. Note that Z_i contains six internal derivatives.

Let us next analyze the result of the dimensional reduction on the background (3.1) of the terms in (3.3) separately,

$$\hat{R} *_{11} 1 \Big| = R *_3 1 *^{(0)} 1 \tag{4.10}$$

$$\hat{t}_8 \hat{t}_8 \hat{R}^4 *_{11} 1 \Big| = 192 \,\mathrm{Tr}\,(\mathcal{R} \wedge *_3 \mathcal{R})\; c_2^{(0)} \wedge J^{(0)2} + 1536 *_3 1 \;\; c_4^{(0)} \tag{4.11}$$

$$-\frac{1}{24}\, \hat{\epsilon}_{11} \hat{\epsilon}_{11} \hat{R}^4 *_{11} 1 \Big| = -768\, R *_3 1 \;\; c_3^{(0)} \wedge J^{(0)} + 1536 *_3 1 \;\; c_4^{(0)}, \tag{4.12}$$

where we denoted the evaluation on the background by the vertical line, $\mathcal{R}$ denotes the curvature two form of the external space (A.13), and $c_2^{(0)}, c_3^{(0)}, c_4^{(0)}$ the second, third and fourth Chern form of Y_4, respectively. The explicit relation of the Chern forms to Riemann tensors is given in (A.15). Thus, in particular we find the correction to the volume (4.13)

$$\mathcal{V}_{R^4} = \frac{\pi^2}{24} \ \mathcal{Z}^{(0)} \ . \tag{4.13}$$

The Chern-Simons term reduces to zero since we do not allow for fluxes and

$$3^2 2^{13} \alpha^2 \hat{C} \wedge \hat{X}_8 | = -3072 *_3 1 \ c_4^{(0)} \ . \tag{4.14}$$

Note that the scalar potential contributions that are proportional to the fourth Chern class cancel amongst (4.11), (4.12) and (4.14). There cannot be any contribution to the scalar potential arising from the other flux depend parts of (3.3). However we did not vary (4.11) and (4.12) with respect to the Kähler moduli, which gives rise to a non vanishing scalar potential, see section 7.4, and which is a manifestation of the caveat that we do not compactly on a supersymmetric background in this toy model.

Let us next tun our focus to the flux dependent parts of (3.3). In our reduction ansatz, the M-theory three-form $\hat{C}$ is expanded into the harmonic $(1,1)$-forms introduced in (4.8) with vector fields A^i as coefficients. Hence, the four-form field strength $\hat{G} = d\hat{C}$ upon reduction takes the form

$$\hat{G} \to F^i \wedge \omega_i^{(0)} = \frac{1}{2} F^i_{\mu\nu} (\omega_i^{(0)})_{m\bar{n}} \, dx^\mu \wedge dx^\nu \wedge dz^m \wedge d\bar{z}^{\bar{n}} \ , \tag{4.15}$$

where the $F^i = dA^i$ are the field strengths of the $3d$ vector fields with real external coordinates x^μ, $\mu = 0, 1, 2$ and complex internal coordinates $z^m, \bar{z}^{\bar{m}}$ $m, \bar{m} = 1, 2, 3, 4$.

Using (4.15), one performs the dimensional reduction of the classical part of (3.3), see [103, 81], and finds

$$-\frac{1}{2} \ \hat{G} \wedge *_{11} \hat{G} \, \Big| = -\frac{1}{2} F^i \wedge *_3 F^j \omega_i^{(0)} \wedge *^{(0)} \omega_j^{(0)} \, . \tag{4.16}$$

To rewrite expressions in terms of the quantities introduced in (2.82) and (4.9), one makes use of identities valid for the Hodge star $*_8$ evaluated on certain internal harmonic forms. One very relevant identity of this form is

$$*^{(0)} \omega_i^{(0)} = \frac{1}{3! \mathcal{V}_0} \mathcal{K}_i^{(0)} \, J^{(0)3} - \frac{1}{2} \omega_i^{(0)} \wedge J^{(0)2} \, . \tag{4.17}$$

We discuss further equations like this one in appendix A.7, where we also give the definition of the intersection numbers in the background $\mathcal{K}_i^{(0)}, \mathcal{K}_{ij}^{(0)}, \mathcal{K}_{ijk}^{(0)}, \mathcal{K}_{ijkl}^{(0)}$ and derive additional relations that straightforwardly follow from (4.17). These identities will be repeatedly used in the following. For example, applying the first equation in (4.17) one finds

$$\int_{Y_4} \omega_i^{(0)} \wedge *^{(0)} \omega_j^{(0)} = \frac{1}{\mathcal{V}_0} \mathcal{K}_i^{(0)} \mathcal{K}_j^{(0)} - \mathcal{K}_{ij}^{(0)} \ . \tag{4.18}$$

Note that since we do not discuss metric deformations in this chapter let us note that in this context by fluctuating the metric with Kähler deformations $g^{(0)}_{m\bar{n}} \to g^{(0)}_{m\bar{n}} + i\delta v^i \omega^{(0)}_{im\bar{n}}$ as done in chapter III, it becomes necessary to lift the fluctuations around the background metric to full fields $v_0 + \delta v \to v$ in the $3d$ theory. Since the intersection numbers are topological quantities they can be lifted safely, thus we find

$$\int_{Y_4} \omega_i^{(0)} \wedge *^{(0)} \omega_j^{(0)} \quad \to \frac{1}{\mathcal{V}_0} \mathcal{K}_i \mathcal{K}_j - \mathcal{K}_{ij} \ . \tag{4.19}$$

Let us now discuss the dimensional reduction of the higher-derivative corrections in (3.3) by applying the same logic as for the classical part discussed above. This requires us to use (4.15), (4.17) and related identities summarized in appendix A.7. We begin by discussing the reduction of $\hat{t}_8\hat{t}_8\hat{G}_4^2\hat{R}^3$ and then proceed with $\hat{\epsilon}_{11}\hat{\epsilon}_{11}\hat{G}^2\hat{R}^3$. We consider only terms that have two external derivatives and depend on the gauge fields A^i. Hence, $\hat{G}$ is of the form (4.15) and has two external and two internal indices. All other remaining summed indices are purely internal. The reduction of $\hat{t}_8\hat{t}_8\hat{G}^2\hat{R}^3$ then yields[3]

$$\hat{t}_8\hat{t}_8\hat{G}^2\hat{R}^3 *_{11} 1 \Big| \supset \operatorname{sgn}(\circ\cdots\circ)\, \hat{G}^{\circ\circ}{}_{\mu_1\mu_2} \hat{G}^{\mu_1\mu_2}{}_{\circ\circ} R^{\circ\circ}{}_{\circ\circ} R^{\circ\circ}{}_{\circ\circ} R^{\circ\circ}{}_{\circ\circ} *_{11} 1 \ := X_{t_8t_8} \ . \tag{4.20}$$

where $X_{t_8t_8}$ is given in (4.20) and the subset symbol suggest that on;y terms with two external derivatives are taken into account. Here, the symbols $\circ$ schematically represent all appearing permutations of internal indices dictated by the index structure of the t_8 tensor. Each of the 14 terms in $X_{t_8t_8}$ are of the general form

$$\left[F^i \wedge *_3 F^j\right] \left(\omega_i^{(0)}\right)^{\circ}{}_{\circ} \left(\omega_j^{(0)}\right)^{\circ}{}_{\circ} R^{\circ\circ}{}_{\circ\circ} R^{\circ\circ}{}_{\circ\circ} R^{\circ\circ}{}_{\circ\circ} *^{(0)} 1 \ . \tag{4.21}$$

Similarly, one reduces $\hat{\epsilon}_{11}\hat{\epsilon}_{11}\hat{G}^2\hat{R}^3$ and finds the following terms contributing to the kinetic terms of the vectors

$$\frac{1}{96}\, \hat{\epsilon}_{11}\hat{\epsilon}_{11}\hat{G}^2\hat{R}^3 *_{11} 1 \Big| \supset \operatorname{sgn}(\circ\cdots\circ)\, \hat{G}^{\circ\circ}{}_{\mu_1\mu_2} \hat{G}^{\mu_1\mu_2}{}_{\circ\circ} R^{\circ\circ}{}_{\circ\circ} R^{\circ\circ}{}_{\circ\circ} R^{\circ\circ}{}_{\circ\circ} *_{11} 1 \ = \ X_{\epsilon_{11}\epsilon_{11}} - X_{t_8t_8} \ . \tag{4.22}$$

with $X_{\epsilon_{11}\epsilon_{11}}$ is given (A.5) and its eight terms have the generic structure (4.21). The $X_{t_8t_8}$ term in the reductions of $\hat{t}_8\hat{t}_8\hat{G}^2\hat{R}^3$ and $\hat{\epsilon}_{11}\hat{\epsilon}_{11}\hat{G}^2\hat{R}^3$ cancels. The various index summations in (C.3) can be recast in terms of the following linear combination of top forms on the internal space, each containing the third Chern form $c_3^{(0)}(Y_4)$ and two $(1,1)$-forms $\omega_i^{(0)}$:

$$\begin{aligned}
-\Big(\hat{t}_8\hat{t}_8\hat{G}^2\hat{R}^3 + \frac{1}{96}\hat{\epsilon}_{11}\hat{\epsilon}_{11}\hat{G}^2\hat{R}^3\ \Big) *_{11} 1 \Big| \supset -X_{\epsilon_{11}\epsilon_{11}} = 3\cdot 2^7 \Big[F^i \wedge *_3 F^j\Big] \Big[*^{(0)} \left(\omega_i^{(0)} \wedge \omega_j^{(0)} \wedge J^{(0)}\right) \wedge c_3^{(0)} \\
- \frac{1}{2} *^{(0)} \left(\omega_i^{(0)} \wedge \omega_j^{(0)} \wedge J^{(0)2}\right) \wedge c_3^{(0)} \wedge J^{(0)} + \frac{1}{6}\, \omega_i^{(0)} \wedge J^{(0)3} \wedge *^{(0)}\big(c_3^{(0)} \wedge \omega_j^{(0)}\big) \\
+ \frac{1}{6}\, \omega_j^{(0)} \wedge J^{(0)3} \wedge *^{(0)}\big(c_3^{(0)} \wedge \omega_i^{(0)}\big) - \big(\omega_i^{(0)} \wedge *^{(0)}\omega_j^{(0)}\big) \wedge *^{(0)}\big(c_3^{(0)} \wedge J^{(0)}\big)\Big] \ .
\end{aligned} \tag{4.23}$$

One uses the identities (4.17) and (A.7) - (A.47) to express the result in terms of the basic building blocks (2.82) and (4.9). In the next step, we relate this result to the canonical form of the $3d$, $\mathcal{N}=2$ action (2.56) as already outlined in subsection 2.2. Taking into account the contribution arising from the reduction of the classical kinetic term (4.16) and performing the Weyl rescaling with the corrected

[3]These computations were performed in Mathematica using the X-tensor package http://xact.es/xTensor.

volume (4.13), one can read off the couplings $\tilde{K}^{red}_{ij}$ that arise from the reduction. Before we proceed let us comment on the non-harmonicity of c_3, which will be of crucial importance for the following integral splits. The third Chern form can be written in terms the curvature two-form $\mathcal{R}_{m\bar{n}} = R_{m\bar{n}r\bar{s}}dz^r \wedge d\bar{z}^{\bar{s}}$ on a Calabi-Yau manifold as

$$c_3 = -\frac{i}{3}Tr(\mathcal{R}^3) = -\frac{i}{3}\mathcal{R}_m{}^n \wedge \mathcal{R}_n{}^r \wedge \mathcal{R}_r{}^s \ . \tag{4.24}$$

Hence c_3 is real and one can easily explicitly verify that

$$dc_3 = 0 \ \text{ whilst } \ d *^{(0)} c_3 \neq 0 \ , \tag{4.25}$$

thus it is closed but not co-closed with respect to the Kähler metric $g_{m\bar{n}}$. This means that it may be expanded as

$$c_3 = Hc_3 + i\partial\bar{\partial}F_4 \ , \tag{4.26}$$

where H indicates the projection to the harmonic part with respect to the metric $g_{m\bar{n}}$. This equation defines a co-closed $(2,2)$-form F_4 which becomes relevant when using (A.47) to derive

$$\int_{Y_4} *^{(0)}(\omega_i^{(0)} \wedge \omega_j^{(0)} \wedge J^{(0)}) \wedge c_3^{(0)} = -\mathcal{V}_0\mathcal{K}^{(0)kl}\mathcal{K}^{(0)}_{kli}\mathcal{Z}^{(0)}_j + \frac{2}{3\mathcal{V}_0}\mathcal{K}^{(0)}_{ij}\mathcal{Z}^{(0)} + \int_{Y_4} *^{(0)}\partial\tilde{H} \wedge c_3^{(0)} \tag{4.27}$$

where $*^{(0)}\partial\tilde{H}$ is the co-closed part of the identity (A.47). Note that $\partial *^{(0)} c_3^{(0)} = 0$ is a sufficient criteria for the unwanted last term in (4.27) to vanish. It is not clear that one can find a Calabi-Yau geometry where $c_3^{(0)}$ is harmonic, however, we assume $c_3^{(0)}$ to be harmonic in the background Calabi-Yau for the remainder of this chapter. In case $c_3^{(0)}$ cannot be chosen harmonic it amounts to effectively dropping the term

$$\int_{Y_4} *^{(0)}(\omega_i^{(0)} \wedge \omega_j^{(0)} \wedge J^{(0)}) \wedge \partial\bar{\partial}F_4^{(0)} \tag{4.28}$$

from the following discussion, as we comment on it in more detail in section A.3. Discussing the warped background reduction in chapter III it becomes crucial to treat this issue with more care, as of we never assume c_3 to be harmonic. By using (A.7) one finds without any obstruction that

$$\int_{Y_4} -\frac{1}{2} *^{(0)} \left(\omega_i^{(0)} \wedge \omega_j^{(0)} \wedge J^{(0)2}\right) \wedge c_3^{(0)} \wedge J^{(0)} - \left(\omega_i^{(0)} \wedge *^{(0)}\omega_j^{(0)}\right) \wedge *^{(0)}\left(c_3^{(0)} \wedge J^{(0)}\right) = -\frac{1}{\mathcal{V}_0}\mathcal{K}_i^{(0)}\mathcal{K}_j^{(0)}\mathcal{Z}^{(0)} \tag{4.29}$$

since the terms $*^{(0)}\left(\omega_i^{(0)} \wedge \omega_j^{(0)} \wedge J^2\right) \wedge c_3^{(0)} \wedge J^{(0)}$ cancel without the necessity of splitting the integral, which would elsewise give rise to terms containing the non-harmonic part of c_3. Hence one could either take into account the dropped term (4.28) or assume c_3 to be harmonic thus $F = 0$, as we mentioned above we choose latter.

We find an overall factor of $3 \cdot 2^8 \cdot k_1 = \frac{\pi^2}{24}$ for the contributions from (4.23). This is the same factor that appeared in the corrected volume $\tilde{\mathcal{V}}$ given in (4.13). Due to the Weyl rescaling, the volume correction also contributes to $\tilde{K}^{red}_{ij}$ in linear order in $\mathcal{Z}_i^{(0)}$. Note that we will neglect quadratic corrections in $\mathcal{Z}_i^{(0)}$ to the Kähler metric in all of our computations. These corrections would contain six Riemann tensors of the internal space and would thus have twelve derivatives. Performing all outlined steps, we finally arrive at the result

$$\tilde{K}_{ij}^{red} = \tilde{K}_{ij}^{0} - \frac{\pi^2}{24}\left[\mathcal{V}_0\mathcal{K}^{(0)kl}\mathcal{K}_{kli}^{(0)}\mathcal{Z}_j^{(0)} - \frac{5}{3}\mathcal{K}_{ij}^{(0)}\mathcal{Z}^{(0)} - \mathcal{K}_i^{(0)}\mathcal{Z}_j^{(0)} - \mathcal{K}_j^{(0)}\mathcal{Z}_i^{(0)} + \frac{2}{\mathcal{V}_0}\mathcal{K}_i^{(0)}\mathcal{K}_j^{(0)}\mathcal{Z}^{(0)}\right] \ , \tag{4.30}$$

with the classical coupling function

$$\tilde{K}_{ij}^{0}(v_0) = \mathcal{V}_0\mathcal{K}_{ij}^{(0)} - \mathcal{K}_i^{(0)}\mathcal{K}_j^{(0)} = -\mathcal{V}_0\int_{Y_4}\omega_i^{(0)} \wedge *^{(0)}\omega_j^{(0)} \quad \rightarrow \quad K_{ij}^{0}(v) = \mathcal{V}\mathcal{K}_{ij} - \mathcal{K}_i\mathcal{K}_j \ , \tag{4.31}$$

Where we have performed the uplift to the full fields v^i for the classical part in (4.19). Let us next comment on the uplift of the higher-curvature terms in (4.9) which since there are of topological nature can as well simply lifted to obtain

$$\mathcal{Z}_i = \int_{Y_4} c_3 \wedge \omega_i \ , \qquad \mathcal{Z} = \mathcal{Z}_i\, v^i = \int_{Y_4} c_3 \wedge J \ , \tag{4.32}$$

and one thus finds for the uplifted total kinetic couplings of the vectors

$$\tilde{K}_{ij}^{red}(v) = \tilde{K}_{ij}^{0}(v) - \frac{\pi^2}{24}\left[\mathcal{V}\mathcal{K}^{kl}\mathcal{K}_{kli}\mathcal{Z}_j - \frac{5}{3}\mathcal{K}_{ij}\mathcal{Z} - \mathcal{K}_i\mathcal{Z}_j - \mathcal{K}_j\mathcal{Z}_i + \frac{2}{\mathcal{V}}\mathcal{K}_i\mathcal{K}_j\mathcal{Z}\right] \ , \tag{4.33}$$

where all the quantities depend on the full fluctuated fields v^i. This concludes the dimensional reduction of the action S_{sub} given in (3.3). In the next step, we will use this result to infer the $3d$, $\mathcal{N}=2$ Kähler coordinates. Let us stress that in order to derive the kinetic terms of the scalars in the vector multiplet (v^i, A^i) one needs to vary the purely gravitational l_M^6 terms w.r.t. the Kähler deformations of the metric $g_{m\bar{n}} \rightarrow g_{m\bar{n}} + i\delta v^i \omega_{im\bar{n}}$ as done in chapter III. However, as we will see next, the result (4.33) together with $3d$, $\mathcal{N}=2$ supersymmetry suffices to fix the Kähler coordinates.

4.2 Determining the $3d$, $\mathcal{N}=2$ coordinates and Kähler potential

As already noted above, from the reduction of the R^4 terms in (3.3) one infers the Kähler potential (4.4) but is unable to fix the Kähler coordinates T_i in the 3d, $\mathcal{N}=2$ action (2.53). The Kähler coordinates can however be determined by using the relation of the Kähler potential K given in (4.4) with the couplings $\tilde{K}_{ij}^{red}$ found in (4.33). As a first step, one computes the general form of $\tilde{K}_{ij}$ arising from a Kähler potential K by Legendre transform. If the Kähler metric separates w.r.t. the coordinates $N^A = \{M^I, T_i\}$, that is all mixed derivatives of K vanish, one can compute $\tilde{K}_{ij}$ using the identity

$$\tilde{K}_{ij} = -\frac{1}{4}\left(\frac{\partial^2 K}{\partial \bar{T}_j \partial T_i}\right)^{-1} . \tag{4.34}$$

In our reduction with $h^{2,1}(Y_4) = 0$, the separation into $N^A = \{M^I, T_i\}$ indeed takes place. Hence, one can compare the expression (4.34) to $\tilde{K}_{ij}^{red}$ in order to read off T_j.

The classical Kähler coordinates, which correspond to six-cycle volumes of the Calabi-Yau fourfold Y_4, are given by

$$\mathrm{Re}T_i = \mathcal{K}_i \, . \tag{4.35}$$

Performing the Legendre transform and using (4.34), one finds that the classical Kähler coordinates (4.35) together with the Kähler potential (4.4) do not suffice to arrive at the metric $\tilde{K}^{red}_{ji}$ given in (4.33). Indeed, it is necessary to correct the Kähler coordinates as

$$\mathrm{Re}T_i = \mathcal{K}_i\left(1 + \frac{\pi^2}{24\mathcal{V}}\mathcal{Z}\right) - \frac{\pi^2}{24}\mathcal{Z}_i\,, \tag{4.36}$$

to achieve the match $\tilde{K}_{ji} = \tilde{K}^{red}_{ji}$. This non-trivial field redefinition might also be interpreted as a correction to the six-cycle volumes. We stress that the last term in (4.36) is constant, since $\mathcal{Z}_i$ are topological quantities, and cannot be inferred by using (4.34). In fact, this term could be removed by a trivial holomorphic Kähler transformation. The reason for including this shift will be explained below.

Having determined both the Kähler potential in (4.4) and the Kähler coordinates in (4.36), one can now show that a $3d$ no-scale condition holds. More precisely, one derives that

$$K_{T_i}K^{T_i\bar{T}_j}K_{\bar{T}_j} = 4\;. \tag{4.37}$$

This implies that the term $-4|W|^2$ in the scalar potential (2.54) will cancel precisely if W is independent of T_j.

The coordinates T_i are the propagating complex scalars in the $3d$, $\mathcal{N} = 2$ action (2.53). If one changes to different propagating degrees of freedom by dualizing $\mathrm{Im}\,T_i$ and performing the Legendre transform for $\mathrm{Re}T_i$ as described in section 2.2.2, one arrives at propagating real scalars L^i in the dual version of the $3d$ $\mathcal{N} = 2$ action (2.56). It is convenient to perform all computations in this frame, since the Kähler potential K, the Kähler form J, and the geometric quantities (2.82) and (4.9) depend explicitly on the fields v^i. These are real scalars in the $3d$ action and correspond to 2-cycle volumes of the internal space. By definition of the Legendre transform one has the relation

$$L^i = -\frac{\partial K}{\partial \mathrm{Re}T_i} = -\frac{\partial K}{\partial v^j}\frac{\partial v^j}{\partial \mathrm{Re}T_i}\,. \tag{4.38}$$

To evaluate (4.38) one first needs to compute the partial derivative of the Kähler potential K and the Kähler coordinates T_i in (4.36) w.r.t. to the fields v^j. Then one inverts the matrix $\left(\frac{\partial \mathrm{Re}T}{\partial v}\right)^{-1,ij} = \frac{\partial v^j}{\partial \mathrm{Re}T_i}$. We neglect corrections that have more than six derivatives, hence being at least quadratic in $\mathcal{Z}_i$. This implies that we assume the corrections proportional to $\mathcal{Z}_i$ to be small compared to the classical contribution. Hence, we can expand the inverse matrix by using the formula $(A+B)^{-1}_{ij} = A^{-1}_{ij} - A^{-1}_{ij}B^{jj'}A^{-1}_{j'j} + \mathcal{O}(B^2)$. Using (4.36) and applying the above steps one arrives at

$$L^i = \frac{v^i}{\mathcal{V}} + \frac{\pi^2}{24}\left(-\frac{2}{3}\frac{\mathcal{Z}}{\mathcal{V}^2}v^i - \frac{1}{2\mathcal{V}}\mathcal{Z}_j\mathcal{K}^{ji}\right)\,. \tag{4.39}$$

Furthermore, one can compute

$$\mathrm{Re}T_iL^i = 4\,, \tag{4.40}$$

which is valid up to linear order in $\mathcal{Z}_i$. The dual kinetic potential then takes the form

$$\tilde{K} = \log\left(\frac{1}{4!}\mathcal{K}_{ijkl}L^iL^jL^kL^l\right) + 4\,. \tag{4.41}$$

Note that it is straightforward to evaluate the coordinates ReT_i given in (4.36) as a function of L^i given in (4.39) as

$$\mathrm{Re}\, T_i = \frac{1}{3!}\frac{\mathcal{K}_{ijkl}L^jL^kL^l}{\hat{\mathcal{V}}(L)}\ , \qquad \hat{\mathcal{V}}(L) = \frac{1}{4!}\mathcal{K}_{ijkl}L^iL^jL^kL^l\ . \tag{4.42}$$

This is clearly consistent with (2.59) when using (4.41).

Let us close this section with some further remarks. First of all, note that by using the field redefinition (4.39) one finds the same functional dependence of $\tilde{K}(L)$ w.r.t. L^i as in the classical reduction without higher-curvature terms. This is equally true when evaluating the Kähler potential K given in (4.4) as a function of the corrected T_i given in (4.36). Clearly, this implies the no-scale condition (4.37) to linear order in the correction $\mathcal{Z}_i$. Secondly, note that the redefinition of L^i in (4.38) does not change if one varies the coefficient of the last term in T_i given in (4.36). The convenient choice made in (4.36) implies that (4.40) and (4.41) do not have irrelevant linear terms of the form $\mathcal{Z}_iL^i$.

5 F-theory limit and the $4d$ effective action

In this section we examine the $4d$ effective theory obtained by taking the F-theory limit of the $3d$ results found in section 4. As in [102], we use the duality between M-theory and F-theory to lift the l_M-corrections to α'-corrections of the $4d$ effective action arising from F-theory compactified on Y_4. In subsection 5.1 we formulate the F-theory limit in terms of the corrected Kähler coordinates and discuss the resulting $4d$ Kähler potential. Next, in subsection 5.2 we derive the corrected expressions for the volume of the internal space and for the $4d$ Kähler coordinates in terms of two-cycle volumes. Analogously to the $3d$ case, the considered $4d$ effective couplings turn out to be identical to the classical ones when expressed in terms of the modified Kähler coordinates. We comment on the consequences of this observation.

5.1 F-theory limit and the $4d$, $\mathcal{N}=1$ effective action

To begin with, we require that Y_4 admits an elliptic fibration over a three-dimensional Kähler base B_3. We allow Y_4 to accommodate both non-Abelian and $U(1)$ gauge groups. A detailed discussion of its geometry will be given in subsection 5.3. The structure of the elliptic fibration allows us to split the divisors and Poincaré-dual two-forms ω_j, $j = 1, \dots, h^{1,1}(Y_4)$ into three types: ω_0, ω_α, and ω_I. The two-form ω_0 corresponds to the holomorphic zero-section, the two-forms ω_α to divisors obtained as elliptic fibrations over divisors of the base with $\alpha = 1, \dots, h^{1,1}(B_3)$, and the two-forms ω_I correspond to both the extra sections, i.e. Abelian $U(1)$ factors, and the blow-up divisors, i.e. $U(1)$ factors in the Cartan subalgebra of the non-Abelian gauge group. We can thus expand the Kähler form of the Calabi-Yau fourfold as

$$J = v^0\omega_0 + v^\alpha\omega_\alpha + v^I\omega_I\ , \tag{5.1}$$

where v^0 represents the volume of the elliptic fiber. Accordingly, one can also split the L^i and T_i introduced in (4.39) and (4.36) such that

$$L^i = \left(L^0 \equiv R,\ L^\alpha,\ L^I\right) , \qquad T_i = (T_0, T_\alpha, T_I) . \tag{5.2}$$

The field R will play a special role in the uplift from three to four dimensions. In fact, one finds that R is given by $R = r^{-2}$, where r is the radius of the circle compactifying the $4d$ theory to three dimensions.

In the F-theory limit one sends $v^0 \to 0$, which translates to sending $R \to 0$. Such an operation decompactifies the fourth dimension by sending the radius r of the $4d/3d$ circle in string units to infinity: $r \to \infty$. Henceforth, all volumes of the base B_3 will be expressed in units of l_s. In all $3d$ effective quantities one has to retain the leading order terms in such a limit. Therefore we introduce a small parameter ϵ and express the scaling of the dimensionless fields by writing $v^0 \sim \epsilon$. As explained in [96, 97] and in section 3.1, one shows that all v^I scale to zero in the limit of vanishing ϵ, whereas $v^\alpha \sim \epsilon^{-1/2}$. One then infers the scaling behavior of the classical and corrected volume of Y_4 to be $\mathcal{V} \sim \tilde{\mathcal{V}} \sim \epsilon^{-1/2}$. In the following we use the letter b to denote quantities of the base that are finite in the limit $\epsilon \to 0$.

When compactifying a general $4d$, $\mathcal{N} = 1$ supergravity theory on a circle, one can match the original $4d$ Kähler potential and gauge coupling functions with the $3d$ Kähler potential K or kinetic potential $\tilde{K}$. Since we have found that the dependence of K and $\tilde{K}$ on the modified coordinates T_i and L^i is the same as in the classical case, we can perform the limit by simply following [96]. Firstly, we recall that the fields T_α remain complex scalars in four dimensions, while the T_0, T_I should be dualized already in three dimensions into vector multiplets with (R, A^0) and (L^I, A^I) and then uplifted to four dimensions. In fact, (R, A^0) are parts of the $4d$ metric, while (L^I, A^I) form the Cartan gauge vectors of the $4d$ gauge group. In this mixed frame one finds a kinetic potential $\tilde{K}(R, L^I|T_\alpha, \bar{T}_\alpha)$, which can be computed for example by Legendre dualization of L^α starting from (4.41). This kinetic potential has to be matched with the one arising in a dimensional reduction from four to three dimensions, which has the form

$$\tilde{K}(r, L^I|T^b_\alpha) = -\log(r^2) + K^F(T^b_\alpha) - r^2 \mathrm{Re} f_{IJ} L^I L^J , \tag{5.3}$$

where the L^I are the Wilson line scalars from $4d$ Cartan vectors on a circle, and $f_{IJ}(T^b_\alpha)$ is the holomorphic $4d$ gauge coupling function. As a next step, one can implement the F-theory limit by identifying the $3d$ fields with appropriate $4d$ fields. In addition to $R = r^{-2}$ and identifying the L^I, we also set [4]

$$L^\alpha_b = L^\alpha|_{\epsilon=0} , \qquad T^b_\alpha = T_\alpha|_{\epsilon=0} , \tag{5.4}$$

which are the only L^i and T_i that are finite and non-zero in the limit $\epsilon \to 0$. This is the same limit as taken in [96], but with the modified coordinates L^i and T_i.

It is now straightforward to determine $K^F(T^b_\alpha)$, since in the modified coordinates this is just the classical analysis. First of all, one has to evaluate the intersection numbers $\mathcal{K}_{ijkl}$ for an elliptic

[4] One could speculate that also this identification is modified with terms depending on $\mathcal{Z}_i$. This would significantly change the conclusions of our analysis, but we found no further evidence that this should be the case.

fibration. One finds the always non-vanishing coupling $\mathcal{K}_{0\alpha\beta\gamma} = \mathcal{K}^b_{\alpha\beta\gamma}$, where we have introduced the base intersection numbers

$$\mathcal{K}^b_{\alpha\beta\gamma} = \int_{B_3} \omega_\alpha \wedge \omega_\beta \wedge \omega_\gamma \ . \tag{5.5}$$

Second of all, one can split the kinetic potential (4.41) and coordinates (4.42) for an elliptic fibration. The terms of leading order in ϵ are given by

$$\tilde{K}(L^i) = \log(R) + \log\Big(\frac{1}{3!}\mathcal{K}^b_{\alpha\beta\gamma} L^\alpha_b L^\beta_b L^\gamma_b + \dots\Big) + 4 \ , \tag{5.6}$$

$$\mathrm{Re}\, T_\alpha = \frac{1}{2!}\frac{\mathcal{K}^b_{\alpha\beta\gamma} L^\beta_b L^\gamma_b}{\hat{\mathcal{V}}^b(L_b)} + \dots \ , \qquad \hat{\mathcal{V}}^b(L_b) \equiv \frac{1}{3!}\mathcal{K}_{\alpha\beta\gamma} L^\alpha_b L^\beta_b L^\gamma_b \ , \tag{5.7}$$

where we have replaced the L^α with L^α_b by means of (5.4). Performing the Legendre transform in order to express everything in terms of T^b_α and comparing the result with (5.3) setting $R = r^{-2}$ one finds

$$K^F(T^b_\alpha) = \log\Big(\frac{1}{3!}\mathcal{K}^b_{\alpha\beta\gamma} L^\alpha_b L^\beta_b L^\gamma_b\Big) \ , \qquad \mathrm{Re}\, T^b_\alpha = \frac{1}{2!}\frac{\mathcal{K}^b_{\alpha\beta\gamma} L^\beta_b L^\gamma_b}{\hat{\mathcal{V}}^b(L_b)} \ , \tag{5.8}$$

where one has to solve T^b_α for $L^\alpha_b(T^b_\alpha)$ and insert the result into K^F. Analogously to the $3d$ case, one can compute

$$\mathrm{Re} T^b_\alpha L^\alpha_b = 3 \, . \tag{5.9}$$

In this case we also choose the constant shift in (5.14) in order to avoid irrelevant linear terms of the form $Z^b_\alpha L^\alpha_b$ in the kinetic potential.

The result (5.8) agrees with the classical result and hence, as in three dimensions, the functional dependence of K^F on T^b_α is not modified by the corrections. In particular one can trivially check that the no-scale condition

$$K^F_{T^b_\alpha} K^{F\, T^b_\alpha \bar{T}^b_\beta} K^F_{\bar{T}^b_\beta} = 3 \tag{5.10}$$

is satisfied by this Kähler potential and Kähler coordinates. It should be stressed that the modifications arise when expressing K^F and T^b_α in terms of the finite two-cycle volumes v^α_b as we discuss in detail in subsection 5.2.

Before closing this subsection we note that the gauge coupling function of the $4d$ gauge group can equally be determined by comparing (5.3) with the M-theory result (4.41). Clearly, one also just finds the classical result when working in the coordinates T^b_α. More precisely, if the seven-brane supporting the gauge theory wraps the divisor dual to $C^\alpha \omega_\alpha$ in B_3, the gauge coupling is proportional to $C^\alpha T^b_\alpha$. As we will see in the next subsection, also this result differs from the classical expression when written in terms the two-cycle volumes v^α_b of B_3.

5.2 Volume dependence of the $4d$, $\mathcal{N} = 1$ coordinates and Kähler potential

In this subsection we express the $4d$, $\mathcal{N} = 1$ coordinates T^b_α and Kähler potential K^F given in (5.8) in terms of finite two-cycle volumes v^α_b in the base B_3. In these coordinates the corrections will reappear and we can comment on their structure.

To begin with, we introduce some additional notation. The base Kähler form is denoted by $J_b = v_b^\alpha \omega_\alpha|_{B_3}$. The classical volume $\mathcal{V}_b^0$ of the base and the volume dependent matrix $\mathcal{K}^b_{\alpha\beta}$ are defined as

$$\mathcal{V}_b^0 = \frac{1}{3!}\int_{B_3} J_b^3\ , \qquad \mathcal{K}^b_{\alpha\beta} = \int_{B_3} \omega_\alpha \wedge \omega_\beta \wedge J_b = \mathcal{K}^b_{\alpha\beta\gamma} v_b^\gamma\ , \qquad \mathcal{K}^b_\alpha = \mathcal{K}^b_{\alpha\beta\gamma} v_b^\beta v_b^\gamma\ , \tag{5.11}$$

where $\mathcal{K}^b_{\alpha\beta\gamma}$ are the triple intersection numbers of B_3 defined in (5.5). All corrections to the $4d$ theory will be expressed in terms of the fundamental quantity

$$\mathcal{Z}_\alpha = \int_{Y_4} c_3(Y_4) \wedge \omega_\alpha \stackrel{!}{=} \int_{B_3} [\mathcal{C}] \wedge \omega_\alpha \equiv \mathcal{Z}^b_\alpha\ . \tag{5.12}$$

Since the ω_α are inherited from the base B_3 there always exists a curve $\mathcal{C}$ such that the middle equality in (5.12) is satisfied. An explicit expression for $\mathcal{C}$ is derived in subsection 5.3 starting from $c_3(Y_4)$ for numerous singular configurations with extra sections. Let us note that we have defined $\mathcal{Z}^b_\alpha = \mathcal{Z}_\alpha$ in order to more easily distinguish $\mathcal{Z} = v^j \mathcal{Z}_j$ and $\mathcal{Z}_b(J_b) = v_b^\alpha \mathcal{Z}^b_\alpha$.

We now can relate the two-cycle volumes v_b^α of B_3 to the two-cycle volumes v^i of Y_4. Since both v^0 and v^α scale with ϵ as discussed above, one is led to set

$$\sqrt{v^0} v^\alpha = 2\pi v_b^\alpha\ . \tag{5.13}$$

This is the classical relation between the different two-cycle volumes.[5] One can then evaluate the $\mathcal{N}=1$ Kähler coordinates $\mathrm{Re}T^b_\alpha$ and the real coordinates L_b^α in terms of the v_b^α. Inserting (5.13) into (4.36) and (4.39) one finds

$$\mathrm{Re}T^b_\alpha = (2\pi)^2 \frac{\mathcal{K}^b_\alpha}{2} + \frac{\pi^2}{24}\left(\frac{1}{2}\frac{\mathcal{K}^b_\alpha \mathcal{Z}_b(J_b)}{\mathcal{V}_b^0} - \mathcal{Z}^b_\alpha\right)\ , \tag{5.14}$$

$$L_b^\alpha = \frac{v_b^\alpha}{(2\pi)^2 \mathcal{V}_b^0} - \frac{1}{384\pi^2}\left(\frac{1}{2}\frac{v_b^\alpha \mathcal{Z}_b(J_b)}{\mathcal{V}_b^{0\,2}} + \frac{\mathcal{K}_b^{\alpha\beta}\mathcal{Z}^b_\beta}{\mathcal{V}_b^0}\right)\ . \tag{5.15}$$

The only non-trivial step in this computation is to relate the inverse $\mathcal{K}^{\alpha\beta}$ to the inverse $\mathcal{K}_b^{\alpha\beta}$ of $\mathcal{K}^b_{\alpha\beta}$ given in (5.11). We will discuss this in more detail momentarily. Before doing so, let us introduce the corrected base volume $\mathcal{V}_b$ by setting

$$R^{1/2}\mathcal{V}^{3/2} = (2\pi)^3 \mathcal{V}_b\ . \tag{5.16}$$

This equation can be viewed as an extension of the relation between the classical volumes of Y_4 and B_3 to a corrected $\mathcal{V}$ and $\mathcal{V}^b$. Inserting the identification (5.13) one finds

$$\mathcal{V}_b = \mathcal{V}_b^0 + \frac{\mathcal{Z}_b(J_b)}{96}\ . \tag{5.17}$$

Equation (5.16) also implies that the F-theory Kähler potential takes form

$$K^F = -2\log(2\pi)^3 \mathcal{V}_b\ . \tag{5.18}$$

[5] Note that one could have included further terms proportional to Z^b_α that would non-trivially mix the two-cycle volumes in a manifestly non-local way. It is straightforward to use such a more general ansatz in the following expressions. However, a string theory interpretation of such corrections would remain elusive and we refrain from including them in the following.

The identification of circle radius r with the coordinate R can equally be expressed in terms of the base volumes v_b^α and v_0. Using (5.13) and (5.24) in (4.39) one finds that

$$\frac{1}{r^2} = R = \frac{v_0^{3/2}}{(2\pi)^3}\frac{1}{\mathcal{V}_b} \,, \tag{5.19}$$

with $\mathcal{V}_b$ given in (5.17). Note that this implies the existence of a correction to the classical identification that only involved $\mathcal{V}_b^0$.

It remains to comment on the relation of $\mathcal{K}^{\alpha\beta}$ and $\mathcal{K}_b^{\alpha\beta}$. To this end, we need to determine the behavior of the matrix $\mathcal{K}^{ij}$ in the F-theory limit. Recall that here we are restricting our attention to corrections at order l_M^6, over which we have direct control through the higher-dimensional theory, and therefore we will only retain terms up to linear order in Z. By splitting the index i in $(0,\alpha)$, the equality $\mathcal{K}_{ij}\mathcal{K}^{j\Gamma} = \delta_i^\Gamma$ gives rise to the following conditions:

$$\mathcal{K}_{\alpha 0}\mathcal{K}^{0\beta} + \mathcal{K}_{\alpha\gamma}\mathcal{K}^{\gamma\beta} = \delta_\alpha^\beta \,, \tag{5.20}$$

$$\mathcal{K}_{00}\mathcal{K}^{0\alpha} + \mathcal{K}_{0\gamma}\mathcal{K}^{\gamma\alpha} = 0 \,, \tag{5.21}$$

$$\mathcal{K}_{00}\mathcal{K}^{00} + \mathcal{K}_{0\gamma}\mathcal{K}^{\gamma 0} = 1 \,. \tag{5.22}$$

It is easy to realize that $\mathcal{K}_{00}, \mathcal{K}_{0\alpha}, \mathcal{K}_{\alpha\beta}$ have leading terms which scale like $\epsilon^{-1}, \epsilon^{-1}, \epsilon^{1/2}$ respectively. This implies that, for (5.20) to be fulfilled in general, $\mathcal{K}^{\alpha\beta}$ must admit a term which scales like $\epsilon^{-1/2}$. Moreover, such a term is the leading one for $\epsilon \to 0$, as otherwise L^α would not stay finite in the limit. In contrast, $\mathcal{K}^{0\alpha}$ goes to zero at least as fast as ϵ, thus ensuring the right scaling behavior of R, i.e. $\epsilon^{3/2}$. Given the following ansatz for the leading term of $\mathcal{K}^{\alpha\beta}$

$$\sqrt{v^0}\mathcal{K}^{\alpha\beta} = \frac{1}{2(2\pi)}\left(\mathcal{K}_b^{\alpha\beta} - q\frac{v_b^\alpha v_b^\beta}{\mathcal{V}_b^0}\right) \,, \tag{5.23}$$

with q a yet to be determined coefficient, condition (5.20) at the zeroth order in ϵ implies after using (5.13) and neglecting higher-order terms that

$$\frac{\mathcal{K}^{0\alpha}}{v^0} = \frac{q}{(2\pi)^2}\frac{v_b^\alpha}{\mathcal{V}_b^0} \,. \tag{5.24}$$

Now, looking at condition (5.21), one realizes that there is a sum of divergent terms of order $\epsilon^{-3/2}$. Requiring this sum to be identically zero for every α fixes the coefficient q to be

$$q = \frac{1}{6} \,. \tag{5.25}$$

Note that if only one Type IIB modulus is present, the r.h.s. of equation (5.23) is identically zero, and thus $\mathcal{K}^{\alpha\beta}$ vanishes in the F-theory limit, as its leading term is of order ϵ. Let us remark here that the above result is not an artifact of the F-theory limit. In fact, one can alternatively infer equation (5.23) with q as in (5.25) by matching the inverse of the classical Kähler metrics in three and four dimensions.

To further discuss the result (5.14) we stress that in addition to the constant shift in $\mathrm{Re}T_\alpha^b$ one also finds a correction proportional to $\mathcal{Z}_b(J_b)$. Using (5.12) this implies that $\mathrm{Re}T_\alpha^b$ receives corrections depending on the volume of the curve $\mathcal{C}$. A priori this curve needs not to intersect the divisor dual to ω_α of which the classical part of $\mathrm{Re}T^b_\alpha$ parametrizes the volume. It would be interesting to understand the origin of this 'non-locality'. This becomes particularly apparent when interpreting $\mathrm{Re}T^b_\alpha$ as part of the seven-brane gauge coupling function as discussed at the end of subsection 5.1. In this case a local limit might exist in which one decouples gravity by sending the total classical volume $\mathcal{V}^0_b$ of B_3 to infinity. Note, however, that $\mathcal{Z}_b(J_b)$ is suppressed by $\mathcal{V}^0_b$ and the non-local correction disappears for $\mathcal{V}^0_b \to \infty$. This implies that the correction is consistent with the expected local behavior in the decompactification limit.

In summary, we found the corrected coordinates T^b_α given in (5.14) and Kähler potential (5.18) with (5.17). Both corrections appear when expressing the $4d$ results in terms of the geometrical two-cycle volumes v^α_b. We suggested that there are no further corrections to the map (5.13) in order that our results admit a reasonable string interpretation. To fully confirm this assertion, one should compute for example the D7-brane gauge coupling function. The relevant open string amplitude is at one-loop order in g_s and has been studied before in various Type II set-ups in [104, 105, 106, 107, 108]. It would be interesting to perform the match with our result.

5.3 Weak-coupling interpretation of the α' correction

In the previous sections, we found that the inclusion of higher-curvature terms in the M-theory reduction leads to a redefinition of the Kähler coordinates both in three and four dimensions. The main new object is

$$\mathcal{Z}_i = \int_{Y_4} c_3(Y_4) \wedge \omega_i \tag{5.26}$$

and in the following we will try to shed some light on its physical interpretation. In order to understand the physical quantities that $\mathcal{Z}_i$ and the related $Z = v^i \mathcal{Z}_i$ correspond to, we rewrite them in terms of geometrical objects in Sen's weak-coupling limit of F-theory [99, 100].

Let us first discuss the simple case of on non-singular elliptically fibered Calabi-Yau fourfolds for the corrections

$$\mathcal{Z} = v^i \mathcal{Z}_i = \int_{Y_4} c_3(Y_4) \wedge J \quad \& \quad \tilde{X} = \int_{Y_4} c_2(Y_4) \wedge J^2 \ . \tag{5.27}$$

In this case we can use adjunction formulæ to express Chern classes of Y_4 in terms of Chern classes of B_3. For simplicity, let us restrict to a smooth Weierstrass model, i.e. a geometry without non-Abelian singularities, that can be embedded in an ambient fibration with typical fibers being the weighted projective space $\mathrm{W}\mathbb{P}^2_{231}$. This implies having just two types of divisors D_j, $j = 1, \dots, h^{1,1}(Y_4)$. There is the horizontal divisor corresponding to the 0-section D_0, and the vertical divisors D_α, $\alpha = 1, \dots, h^{1,1}(B_3)$, corresponding to elliptic fibrations over base divisors. Denoting the Poincaré-dual two-forms to the divisors by $\omega_i = (\omega_0, \omega_\alpha)$, we expand

$$J = v^0 \omega_0 + v^\alpha \omega_\alpha \ , \tag{5.28}$$

where v^0 is the volume of the elliptic fiber. Using adjuction formulæ one derives

$$c_3(Y_4) = c_3 - c_1c_2 - 60c_1^3 - 60\,\omega_0 c_1^2\,, \tag{5.29}$$

$$c_2(Y_4) = c_2 + 11c_1^2 + 12\,\omega_0 c_1\,, \tag{5.30}$$

where the c_i on the r.h.s. of these expressions denote the Chern classes of B_3 pulled-back to Y_4.

In order to take the F-theory limit of the expression (5.27), we need the relation between the 11d Planck length l_M and the string length l_s, given in (3.17), and one sends $v^0 \to 0$ to decompactify the fourth dimension. One retains the leading order terms in (5.27) in the limit of vanishing fiber volume $v^0 \to 0$ upon using the scaling behavior of the various fields see 3.1 or section 5. One finds by inserting (3.17) , (5.13) and (5.29) into (5.27), and neglecting all terms that vanish for ϵ going to zero one finds

$$\mathcal{V}_0 = \int_{Y_4} J^4 \quad \to \quad \frac{1}{3!}\int_{B_3} J_b^3 \tag{5.31}$$

$$\mathcal{Z} = \int_{Y_4} c_3(Y_4)\wedge J \quad \to \quad -60\int_{B_3} c_1^2(B_3)\wedge J_b\,, \tag{5.32}$$

$$\tilde{X} = \int_{Y_4} c_2(Y_4)\wedge J^2 \quad \to \quad 12\int_{B_3} c_1(B_3)\wedge J_b^2\,. \tag{5.33}$$

This simple analysis allows us to comment on the behavior of the corrections in some special cases. First of all, when the elliptic fibration is trivial, i.e. $Y_4 = X \times T^2$ with X being a Calabi-Yau threefold, then $c_2(Y_4) = c_2(X)$ and $c_3(Y_4) = c_3(X)$. Since these have no components along the fiber, all corrections in (5.32) and (5.33) go to zero and the α' corrections are absent in the resulting $\mathcal{N} = 2$ theory.

Let us first look at the correction (5.32) and (5.33) in the light of the weak coupling limit by Sen [99]. This limit is performed in the complex structure moduli space of Y_4 and gives a weakly coupled description of F-theory in terms of Type IIB string theory on a Calabi-Yau threefold X with an O7-plane and D7-branes. The Calabi-Yau threefold is a double cover of the base B_3 branched along the O7-plane. The class of this branching locus is the pull-back of $c_1(B_3)$ to X.

As in the case we consider, the case where non-Abelian singularities are absent, the corresponding Sen limit contains a single recombined D7-brane wrapping a divisor of class $8c_1(B_3)$, as required by seven-brane tadpole cancellation. This D7-brane has the characteristic Whitney-umbrella shape [109, 110]. We first discuss the volume correction in (5.32). For this correction the intersection curve of the D7-brane with the O7-plane plays a crucial role. It is a double curve with additional pinch point singularities. However, all we need in the following is its volume in X given by

$$\mathcal{V}_{\mathrm{D7}\cap\mathrm{O7}} = 8\int_X c_1^2(B_3)\wedge J_b\ , \tag{5.34}$$

where we omitted the pullback map from B_3 to its double cover X in the integrand. Since the intersection numbers of X are twice the ones of B_3, we can immediately read off from (5.32) the induced correction to be

$$\mathcal{Z} = \int_{Y_4} c_3(Y_4)\wedge J \ \to\ -60\int_{B_3} c_1^2(B_3)\wedge J_b\,,\ \stackrel{g_s\ll 1}{\longrightarrow}\ -\frac{15}{2}\mathcal{V}_{\mathrm{D7}\cap\mathrm{O7}} \tag{5.35}$$

Note that the quantum correction in (5.35) can alternatively be expressed in terms of the volume of the self-intersection curve of the O7-plane by using tadpole cancellation. Note that the correction is of order α'^2 since two of the original six derivatives in M-theory have been absorbed by the integration on the elliptic fiber. Let us point out one possible obstruction to the procedure of pulling higher-derivative or l_M corrections depending on Riemann tensors through the F-theory lift. One could object that since in the limit of vanishing fiber the geometry becomes singular even higher l_M corrections become more relevant, however these corrections (5.32) and (5.33) are of topological nature and thus very well be protected.

The novel correction (5.35) should arise from a string amplitude. We first look at the $4d$ effective action in the string frame one infers that (5.35) has a string coupling dependence of g_S^{-1}. Recall that the power of the string coupling constant in a given amplitude coincides with $-\chi(\Sigma)$, where Σ is the string world-sheet. The general formula for the Euler number of Riemann surfaces, possibly non-orientable and with boundaries, is

$$\chi(\Sigma) = 2 - 2g - b - c\,, \tag{5.36}$$

where g, b, c denote the genus, the number of boundaries, and the number of cross caps, respectively. Therefore, we immediately see that the volume correction in (5.35) arises from a string amplitude that involves the sum over two topologies: The disk ($g = c = 0, b = 1$) and the projective plane ($g = b = 0, c = 1$). They correspond to the tree-level of orientable open strings and non-orientable closed strings, respectively. This is consistent with the fact that the existence of this correction relies on having D7-branes intersecting with an O7-plane. It would be interesting to perform a direct string derivation of this α'^2 correction.

Let us next discuss the correction (5.33) in the weak fouling limit and give it a string theory interpretation. In fact, at weak string coupling, the coefficient (5.33) can be easily written as

$$\tilde{X} = \int_{Y_4} c_2(Y_4) \wedge J^2 \;\;\rightarrow\;\; 12 \int_{B_3} c_1(B_3) \wedge J_b^2 \;\stackrel{g_s \lll 1}{\longrightarrow}\; \mathcal{V}_{\mathrm{D7}} + 4\mathcal{V}_{\mathrm{O7}}\,, \tag{5.37}$$

where $\mathcal{V}_{\mathrm{D7}}$ and $\mathcal{V}_{\mathrm{O7}}$ are the volumes of the D7-brane and the O7-plane in X, respectively. Both volumes are in the Einstein frame and in units of l_s. By tadpole cancellation one has $\mathcal{V}_{\mathrm{D7}} = 8\mathcal{V}_{\mathrm{O7}}$. However, in (5.37) we have split the volumes according to the appearance of the corresponding divisors in the F-theory discriminant. The relative factor in the volume split is in agreement with the relative factor in the higher-curvature terms of the Chern-Simons actions of D7-branes and O7-planes. These have been studied to derive the $4d$ higher-curvature term proportional to $\mathrm{Tr}\,(\mathcal{R}^{(4)} \wedge \mathcal{R}^{(4)})$ in [111], which is the supersymmetric partner of the $\mathrm{Tr}\,(\mathcal{R}^{(4)} \wedge *_4\mathcal{R}^{(4)})$ term in (4.11). Translated to the string frame the higher-derivative correction in (4.11) becomes

$$S_{(4)} \supset \frac{\pi^2}{192 g_{\mathrm{IIB}}\, (2\pi)^7} \big(\mathcal{V}^s_{\mathrm{D7}} + 4\mathcal{V}^s_{\mathrm{O7}}\big) \int \mathrm{Tr}\,(\mathcal{R}^{(4)} \wedge *^s_4 \mathcal{R}^{(4)})\,, \tag{5.38}$$

where we see that this correction has the same string loop order as the one in (4.12) correcting the volume. This term is expected to directly arise from a higher-curvature correction of the string-tree-level Dirac-Born-Infeld action on the D7-brane and O7-plane as discussed in [112]. This concludes the discussion of the corrections in the simple case of a non-singular fibered Calabi-Yau fourfold.

Let us return to the more generic case of elliptically fibered Calabi-Yau fourfolds focusing on the main object of interest $\mathcal{Z}_i$. Let us be clear about what we mean by analyzing its weak-coupling interpretation. We wish to find a geometric object inside X that contains the same information as $\mathcal{Z}_i$. More precisely, after taking the F-theory limit, all we are really interested in are the values $\mathcal{Z}^b_\alpha$ as defined in (5.12). This means that we are trying to find a curve $\mathcal{C} \subset X$ satisfying

$$\int_{B_3} [\mathcal{C}] \wedge \omega^b_\alpha = \mathcal{Z}^b_\alpha \qquad \forall \alpha \,. \tag{5.39}$$

Postponing a discussion of our methods to the following subsections, let us cut to the case and present our results. Restricting the gauge group to be

$$G = \prod_{i=1}^{n_{SU}} SU(N_i) \times \prod_{j=1}^{n_{USp}} USp(2M_j) \tag{5.40}$$

of which we believe to have a relatively decent weak-coupling understanding and embedding the elliptic fiber in $\mathbb{P}_{231}$ we suggest that $\mathcal{C}$ is given by [6]

$$\begin{aligned} \mathcal{C} &= -W \cdot (W - \frac{c_1}{2}) + \mathcal{C}_{non-Abelian} \\ &= -W \cdot (W - \frac{c_1}{2}) - \sum_{\bullet=+,-} \sum_{i=1}^{n_{SU}} N_i S^\bullet_i \cdot (S^\bullet_i + \frac{c_1}{2}) - \sum_i 2M_i T_i \cdot (T_i + \frac{c_1}{2}) \,. \end{aligned} \tag{5.41}$$

Here we denoted by W the class of the Whitney umbrella, by $S^\pm_i$ the brane stack and its orientifold image hosting the $SU(N_i)$ gauge group, and by T_i the brane stack on which the $USp(2N_i)$ gauge theory is located. For $U(1)$-restricted models with a simple non-Abelian gauge group, the Whitney umbrella splits into two pieces denoted by $W^\pm$ and we conjecture that the curve can be written as

$$\mathcal{C} = -W^+ \cdot (W^+ + \frac{c_1}{2}) - W^- \cdot (W^- + \frac{c_1}{2}) + \mathcal{C}_{non-Abelian} \,. \tag{5.42}$$

For the sake of brevity we used the abbreviation $c_1 = [\pi'^* c_1(B_3)]$ with $\pi' : Z \to B_3$ the projection from the double cover Z to the base manifold in the above formulas and will continue to do so from here on.

Given a clear geometric expression for $\mathcal{C}$, one can try and find a physical interpretation for the topological quantities $\mathcal{Z}^b_\alpha$ defined in (5.39). First of all, apart from some shifts proportional to c_1, $\mathcal{C}$ can roughly be interpreted as the curve over which the $D7$ branes intersect themselves in the manifold X. One explanation for the presence of the c_1 shifts might be that they correct effects of the orientifold planes, as the orientifold locus has class c_1. However, it is not entirely clear to us how this correction works. Let us denote the base divisor dual to $\omega^{(0)}_\alpha|_{B_3}$ by D^b_α. Then the topological quantities $\mathcal{Z}^b_\alpha$ clearly count the number of times that D^b_α intersects the curve $\mathcal{C}$. In the light of this piece of information, we can reconsider the shifts to T^b_α that were found in the previous section. While the term proportional to $\mathcal{Z}^b_\alpha$ is 'local' in the sense that it corresponds to intersections of the divisor D^b_α, the term linear in

[6] Here and in the following we use the notation $A \cdot B$, AB, and $[A] \wedge [B]$ interchangeably to denote the intersection product between two subvarieties A and B or, alternatively, the product of their Poincaré-dual forms.

$Z^b(J_b)$ is not. For generic values of v_b^α, J_b is a linear combination of all divisors D_α^b and hence the correction of the coordinate T_α^b also depends on the topology of divisors far away from D_α^b.

Before proceeding to the computations, let us be very clear about the class of models that we suggest our formulas apply to. In the absence of Abelian gauge groups, we believe that our result (5.41) holds very generally and depends neither on the total number of gauge group factors nor on the rank of the single factors.[7] As soon as one allows for Abelian gauge factors, things become more complicated and (5.42) only holds as long as the non-Abelian gauge group is simple and the $U(1)$ gauge group can be obtained by $U(1)$-restriction [114].

To this end, let us note here that not every F-theory model with a single $U(1)$-factor can be obtained by $U(1)$-restriction, or phrased differently, by embedding the elliptic curve inside the toric surface F_{11}, see [88] for notation. An easy way of seeing this uses the classification of tops [115]. Taking for example $SU(5)$, there exists only one top [88] with fiber F_{11}. However, since the top already fixes the matter split, i.e. imposes a condition on the $U(1)$ charges of the non-Abelian representations[8], one has that the $U(1)$-charge of **5** representation must satisfy

$$Q(\mathbf{5}) \equiv 2,3 \mod 5\,. \tag{5.43}$$

In more general models, this need not be the case and (5.42) does not apply to those. More generally, F-theory models obtained from Calabi-Yau manifolds with elliptic fibers embedded in other spaces than F_{11} appear to be described by (5.42) if and only if they have the matter split as the $U(1)$-restricted model with the same non-Abelian gauge group. In the examples we studied, all tops with generic fiber F_{11} that give rise to flat fibrations had the same matter split, namely the straightforward generalization of (5.43):

$$Q(\mathbf{N}) = \begin{cases} \frac{N}{2} & \text{for } N \text{ even} \\ \frac{N-1}{2}, \frac{N+1}{2} & \text{for } N \text{ odd} \end{cases} \tag{5.44}$$

It would be interesting to find a general proof that $U(1)$-restricted models always have this matter split.

Finally, we wish to remark that there does appear to be a similar logic for arbitrary splits and F-theory models with both Abelian and multiple non-Abelian gauge factors. While we would generally expect the same logic to hold for these more general cases, we currently do not have elegant expressions for $W^\pm$ in these scenarios. Studying those set-ups and improving our current understanding of the weak coupling limit for arbitrary gauge groups would be an interesting problem.

5.4 Conclusions

In this section we performed the classical Kaluza-Klein reduction of a seemingly controlled subset of the known higher-derivative couplings of 11d supergravity in an unwarped compactification on a

[7]Note, however, that an $SU(2)$ gauge group should be treated as $USp(2)$ as already observed for example in [113].
[8]See [116, 88] for a detailed discussion of the relation between tops and matter splits.

Calabi-Yau fourfold. In eleven dimensions, the two considered correction terms at order l_M^6 take the schematic form $\hat{R}^4$ and $\hat{G}^2\hat{R}^3$ in terms of the Riemann tensor and the M-theory four-form field strength $\hat{G}$. We analyzed the consequences for the resulting $3d$, $\mathcal{N}=2$ effective action and found that both the total volume of the Calabi-Yau fourfold and the $3d$, $\mathcal{N}=2$ Kähler coordinates are non-trivially corrected at order l_M^6. It is surprising that we have identified remnants of $\mathcal{N}=2$ supersymmetry since the background metric certainly is not a supersymmetric solution to the eleven-dimensional Einstein equations, as we discuss in section 6 in great detail. That these $\mathcal{N}=2$ supersymmetry features appear might be correlated to neglecting a sector of the relevant eight-derivative terms and assuming c_3 to be harmonic.

The first correction modifies the classical expression of the $3d$ Kähler potential in terms of two-cycle volumes, whereas the second is a shift of the classical volume of holomorphic six-cycles that also depends on the two-cycle volumes. The two corrections combine in such a way that the functional dependence of the $3d$ Kähler potential on the $3d$, $\mathcal{N}=2$ Kähler coordinates remains classical. Let us note that there actually exists a one-parameter family of $3d$, $\mathcal{N}=2$ Kähler coordinate deformations in terms of the considered basic geometric quantities of Y_4 under which the Kähler potential retains its classical functional dependence. The reduction of the 11d $\hat{G}^2\hat{R}^3$ coupling was therefore crucial to directly deduce the Kähler metric and to identify the correct $3d$, $\mathcal{N}=2$ Kähler coordinates.

Note that the physical implications have to be contrasted to the fact that we truncate away certain l_M^6 corrections in $11d$, compactify on a non-warped background and need to make the assumption about the harmonicity of the third Chern form c_3 to perform the match. Nevertheless it is intriguing that the integration into a Kähler potential can be performed, which might hint towards the fact that this sector of the theory is well controlled in some sense. Having these caveats in the back of our heads we can now conclude the physical implications which might or might not be washed away by a more refined future treatment.

After deriving the $3d$, $\mathcal{N}=2$ Kähler coordinates we examined the lift of such corrections to the $4d$, $\mathcal{N}=1$ effective theory obtained from an elliptically fibered Calabi-Yau fourfold compactification of F-theory by making use of the M-theory/F-theory duality. In doing so, we found a natural map between the $4d$ and the $3d$ Kähler coordinates and confirmed that the functional dependence of the Kähler potential remains classical also in four dimensions. Furthermore, we expressed the $4d$ Kähler potential as well as the Kähler coordinates and their Legendre dual variables in terms of two-cycle volumes and intersection numbers of the base manifold. Written in this form, both the Kähler potential and Kähler coordinates receive non-trivial α'^2 corrections depending on the volume and intersections of a specific curve $\mathcal{C}$ in the base of Y_4. This curve is defined by using the third Chern class of Y_4 and shown to crucially depend on the seven-brane configuration present in the compactification.

One could object that in the limit of vanishing fiber the geometry becomes singular and corrections carrying Riemann tensors as $\mathcal{Z}, \mathcal{Z}_i$, are thus not well behaved in the limit, and thus the lifted results written in terms of $\mathcal{C}$ cannot be trusted. However, these corrections are of topological nature and thus might be somehow protected. Furthermore in the case of the higher-curvature correction in (4.11) proportional to the second Chern form have been lifted in the simple case of non-singular elliptically fibered Calabi-Yau fourfolds and in the weak fouling limit (5.37) can be interpreted as arising as the string-tree-level Dirac-Born-Infeld action on the D7-brane and O7-plane as discussed in [112]. This natural interpretation gives confidence to the applicability of the procedure.

In order to gain a deeper understanding of the corrections parametrized by $\mathcal{C}$ we examined the $4d$ F-theory reduction in the Type IIB weak string coupling limit. The resulting set-up admits space-time filling D7-branes and O7-planes. We suggested the simple geometric expressions (5.41) and (5.42) for the curve $\mathcal{C}$ in terms of the D7-brane and O7-plane locations. In order to test these expressions we developed an algorithm to systemically perform this computation for a range of examples with multiple Abelian and non-Abelian gauge group factors. We infer that the self-intersection curve of each D7-brane present in the weakly coupled background contributes to $\mathcal{C}$ and hence induces an α'^2 correction. In particular, these corrections are due to open string diagrams and they rely on having D7-branes which have proper self-intersections. Indeed, not only do the corrections vanish in the absence of D7-branes, but also in $\mathcal{N} = 2$ compactifications in which the D7-branes are parallel.

In the presented general F-theory reduction a linear combination of the $4d$, $\mathcal{N} = 1$ Kähler coordinates is found to be the seven-brane gauge coupling function in the effective theory. This is also the case when including the eight-derivative couplings of M-theory and performing the duality to F-theory. The correction we find non-trivially shifts the gauge coupling function from its classical value, represented by the Einstein frame volume of the divisor wrapped by the seven-brane gauge stack. As the Kähler coordinates themselves, the shift depends on the volume and intersections of the curve $\mathcal{C}$. In particular, this shift can contain volumes of curves that do not meet the seven-brane with the considered gauge coupling function. This seemingly 'non-local' contribution does, however, vanish in the decompactification limit corresponding to decoupling gravity. Considered at weak string coupling a simple counting of powers of the string coupling shows that the relevant amplitude which computes such a shift is at one-loop order. Since it would be interesting to have an independent string derivation of this correction, let us mention here that gauge coupling corrections were computed for certain F-theory set-ups in [117, 118] and for general classes of Type IIA torus orientifolds for example in [104, 105, 106, 107, 108], see also [119] for a comprehensive review of orientifold set-ups. Naturally, finding a map between those string corrections and the one we found would be gratifying. While constructing compactification manifolds in F-theory, which reduce to the class of orientifolds that are under computational control as far as world-sheet corrections are concerned, may turn out to be a non-trivial task, it seems plausible that the qualitative behavior of both corrections can be matched in certain limits.

Finally, let us comment on the implications on the search for new string vacua. As explained in section 5, the fact that the corrections to the Kähler coordinates T^b_α are non-holomorphic suggests that the functional dependence of the superpotential $W(T)$ remains uncorrected. A non-perturbative superpotential depending on the T^b_α can arise, for example, from seven-brane gaugino condensates or D3-brane instantons. Consistent with the above observations both the gauge coupling function of the seven-brane stack and the D3-brane instanton action need to receive corrections. Clearly, if both

$W(T)$ and the Kähler potential have the same functional dependence as in the classical reduction the search for vacua remains unmodified. The structure of the functional dependence is very sensitive and thus can easily be altered by introducing the warp-factor and by considering the additional l_M^6 corrections to the eleven-dimensional action, which we address in the next chapter. However, at this point we provide a spoiler for the next chapter where we will not able to express the Kähler potential as functions of the Kähler coordinates since more subtle issues arise, a further study of this would be of great interest.

CHAPTER III

Warped reduction of M-theory

In this chapter we study the Kaluza Klein reduction of eleven-dimensional supergravity including the full set of eight-derivative corrections

$$S = S^{(0)} + \alpha^2 S^{(2)}_{\hat{R}^4} + \alpha^2 S^{(2)}_{\hat{G}^2\hat{R}^3} + \alpha^2 S^{(2)}_{(\hat{\nabla}\hat{G})^2\hat{R}^2} + \mathcal{O}(\hat{G}^3\alpha^2) + \mathcal{O}(\alpha^3)\,, \tag{5.1}$$

with the explicit terms given in (2.62) and (2.63)-(2.65). The terms fourth-order in $\hat{\mathcal{R}}$ are known since the works [53, 54, 55, 56, 57, 58, 59], while recently the third-order terms involving $\hat{G}$ have been analyzed in [60]. Note that for a consistent treatment at order α^2 one a priori needs to consider all eight-derivative corrections to the bosonic part of the eleven-dimensional supergravity action, whose complete form was conjectured by [60]. However, the zeroth order supersymmetry conditions force the flux to vanish in the background thus $G \sim \mathcal{O}(\alpha)$. Hence in order to determine the background solution at order α^2 one can neglect all terms in (5.1) which carry an explicit flux and are of order α^2. However, we will find in this analysis that the background flux is exactly of order α, hence $G \sim \alpha + \mathcal{O}(\alpha^3)$. Thus when performing the reduction on this background one can neglect eight-derivative terms which have more than two explicit fluxes since they would give rise to either α^2 terms, which have more than two external derivatives or to two external derivative terms of order α^3 or higher. Thus (5.1) represents the full set of terms at order α^2 relevant to perform a reduction to three dimensions keeping only two external derivatives, which is the focus of this chapter.

This chapter is organized in three sections according to the canonical steps in the procedure of a dimensional reduction, representing the work done in [120, 121, 122]. We start by determining the supersymmetry preserving background in section 6, to then dimensionally reduce the action (5.1) on this background in section 7. Performing the dimensional reduction we focus on the Kähler deformations giving rise to real scalars and the vectors arising form $\hat{C}$, which form the bosonic part of a vector multiplet in $3d$, where we assume that $h^{(2,1)} = 0$. Furthermore we do not allow for complex structure deformations. In section 8 we then comment on the $\mathcal{N} = 2$ supersymmetry of the $3d$ theory.

Let us comment on these three steps in more detail. Given the action (5.1) we introduce an ansatz for the background metric and fluxes expanded in powers of $\alpha \propto \ell_M^3$, where ℓ_M is the eleven-dimensional Planck length. This ansatz includes a warp-factor as well as a shift of the internal metric at order α^2 [123]. The field equations pose second order differential constraints on the shifted internal metric which we are able to solve explicitly. The internal manifold turns out to have still vanishing first Chern class, but the metric background has to be chosen to be no longer Ricci-flat. At order α^2 the deviation from Ricci-flatness is measured by the warp-factor and the non-harmonic part of the third Chern form $c_3^{(0)}$ evaluated in the zeroth order, Ricci-flat metric. In order to systematically find an explicit solution and analyze its supersymmetry properties we also study the eleven-dimensional supersymmetry variations. Unfortunately, these are not known to the required order to give a complete check of the preservation of three-dimensional $\mathcal{N} = 2$ supersymmetry corresponding to four supercharges. We will argue for corrections to the eleven-dimensional gravitino variations involving certain seven-derivative couplings incorporating three Riemann curvature tensors. Evaluated for the background ansatz this induces modified Killing spinor equations for a globally defined spinor that has to exist in order to have a supersymmetric solution. We show that the integrability condition on these Killing spinor equations yields the modified Einstein equations at order α^2. Furthermore, we use the globally defined spinor to introduce a globally defined real two-form J and complex four-form Ω. The Killing spinor equations translate into first order differential constraints on these forms, which imply that the metric is (conformally) Kähler. In fact, this formulation allows us to give a simple derivation of the α^2 correction to the internal metric found by solving the Einstein equations. Our results can also be reformulated in terms of torsion classes on an $SU(4)$ structure manifold. We find that, upon separating the conformal rescaling of the internal metric, only the torsion form $\mathcal{W}_5$ in $d\Omega = \overline{\mathcal{W}}_5 \wedge \Omega$ is non-vanishing but exact. At the two-derivative level eleven-dimensional supergravity on $SU(4)$ structure manifolds has recently been studied in [124].

In section 7 we then compactify the action (5.1) on this background allowing for a finite number of Kähler deformations of the metric and vector deformations of the M-theory three-form $\hat{C}$. Before discussing the reduction let us stress that there are important terms of the structure $(\hat{\nabla}\hat{G})^2\hat{R}^3$, that have not been fully determined. We argued in section 2.3 that they are given by a number of building blocks of index contractions with 4-point amplitudes only determining part of the numerical pre factors. Remarkably, most of these unknown coefficients actually do not effect our computation and we are able to suggest a fixation of the unknown coefficients up to one constant. Clearly, the complete form of the $(\hat{\nabla}\hat{G})^2\hat{R}^3$ terms could also be determined by considering amplitudes with 5 and more external legs. Given the eleven-dimensional action (5.1) we systematically construct the perturbed background order by order in a scale parameter α. The metric ansatz is modified and accordingly the mode expansion for Kähler structure perturbations of the metric and vector perturbations of the M-theory three-form is described in terms of forms non-harmonic in the zeroth order Calabi-Yau metric. We carefully keep track of all such modifications, but show that most of these modifications eventually cancel in the final three-dimensional effective action. In fact, inserting the ansatz into the higher-derivative action, we find that the kinetic terms for the deformations and vectors in the three-dimensional effective theory can be expressed using a single higher-curvature building block $Z_{m\bar{m}n\bar{n}} = \frac{1}{4!}(\epsilon_8\epsilon_8 R^{(0)3})_{m\bar{m}n\bar{n}}$, where $R^{(0)}$ is the internal Riemann tensor in the zeroth order Calabi-Yau metric, see (5.3) for the precise form of Z. Let us note that $Z_{m\bar{m}n\bar{n}}$ has the same symmetries as the Riemann tensor. It contracts with $R^{\bar{m}m\bar{n}n}$ to the Hodge-dual of the fourth Chern-form, and contracting any of the index pairs with the metric one finds expressions in terms of the third Chern-form. The equivalent quantity on a Calabi-Yau threefold was found to be important in [125]. It would be interesting to examine if $Z_{m\bar{m}n\bar{n}}$ plays a special role in describing the topology of the compact eightfold.

In addition to the complications arising from reducing higher-derivative terms in the action, a proper treatment of the warp-factor turns out to be crucial. Warped compactifications of M-theory and Type IIB have been considered previously in [126, 127, 128, 129, 130, 131, 132, 133, 134, 135], and were argued to be crucial in a complete understanding of the M-theory to F-theory limit for minimally supersymmetric setups [118]. We perform the crucial generalization to include the higher-derivative terms, since warped compactifications with fluxes are inconsistent without these contributions. It turns out, that in this general case the modifications of the warp-factor to the lower-dimensional effective theory are significantly more involved then the ones discussed previously in the literature. Nevertheless we will be able to show that the effective theory permits a non-trivial scaling symmetry induced by rescaling the warp-factor by a field-dependent function.

We furthermore derive the scalar potential for the Kähler structure deformations by dimensional reduction. Interestingly, reducing the higher-curvature terms on the leading order Calabi-Yau background it appears that they become massive with a coupling purely depending on the geometry. However, we will show that these mass terms are precisely cancelled by the higher-order corrections in the solution arising as a back-reaction effect. The remaining scalar potential is only induced by background fluxes as in [81]. This gives a further test that the included fluctuations are indeed the relevant light degrees of freedom and highlights the interplay from back-reaction effects in the solution and the corrections to the effective theory.

Finally we study the supersymmetry properties of the three-dimensional effective theory in section 8. The classical leading order three-dimensional $\mathcal{N} = 2$ theory obtained from M-theory on a Calabi-Yau fourfold with background fluxes was first found in [103, 81], while recent derivations of $\mathcal{N} = 1$ effective theories arising from M-theory flux compactifications can be found in [136, 124, 137]. Let us note that previous works on warped compactifications of M-theory and Type IIB include [126, 127, 128, 129, 130, 131, 132, 133, 118, 134, 135]. In order to reveal the supersymmetry properties of the three-dimensional effective action we discuss its promotion into the standard $\mathcal{N} = 2$ form. In three space-time dimensions massless vectors are dual to scalars and the dynamics of the light modes therefore should be describable by a Kähler potential and a set of complex coordinates. We study the order by order expansion of the Kähler potential and complex coordinates in the Kähler structure fluctuations. The coefficients are deduced by comparison with the dimensionally reduced action. We infer compatibility with $\mathcal{N} = 2$ supersymmetry and argue that a no-scale condition can be implemented. Since the dimensional reduction only includes the leading-order terms in the fluctuations we are not able to completely fix all coefficients by the comparison alone. The fundamental 'all-order' expression, as it is known for the classical reduction without higher-curvature terms [103, 81], turns out to be even more difficult to find. We argue that this problem lies in fixing the complex coordinates and should be approached by introducing divisor integrals. These integrals should be matched with the actions of M5-branes wrapped on divisors. We make steps towards finding an all order expression for the complex coordinates and Kähler potential. An intriguing interplay between variations of warped divisor integrals and higher-curvature terms via the warp-factor equation allows the compatibility with the dimensional reduction to be shown. As a byproduct this suggests that the M5-brane action should receive higher-curvature corrections that parametrise the non-harmonicity of the fourth Chern-form of the background geometry.

Let us set the stage by giving the conventions for the complex indices $m, \bar{m} = 1,2,3,4$, which always refer to the zeroth order complex structure on the internal manifold. On a Calabi-Yau fourfold there exists a nowhere vanishing covariantly constant Kähler form $J^{(0)}$ and holomorphic $(4,0)$-form $\Omega^{(0)}$ satisfying

$$dJ^{(0)} = d\Omega^{(0)} = 0\,. \tag{5.2}$$

Let us next introduce new building blocks which feature the main players of this chapter

$$Z_{m\bar{m}n\bar{n}} = \frac{1}{4!}\epsilon^{(0)}_{m\bar{m}m_1\bar{m}_1m_2\bar{m}_2m_3\bar{m}_3}\epsilon^{(0)}_{n\bar{n}n_1\bar{n}_1n_2\bar{n}_2n_3\bar{n}_3}R^{(0)\bar{m}_1m_1\bar{n}_1n_1}R^{(0)\bar{m}_2m_2\bar{n}_2n_2}R^{(0)\bar{m}_3m_3\bar{n}_3n_3}\;, \tag{5.3}$$

and

$$Y_{ijm\bar{n}} = \frac{1}{4!}\epsilon^{(0)}_{m\bar{m}m_1\bar{m}_1m_2\bar{m}_2m_3\bar{m}_3}\epsilon^{(0)}_{n\bar{n}n_1\bar{n}_1n_2\bar{n}_2n_3\bar{n}_3}\nabla^{(0)n}\omega_i^{(0)\bar{m}_1m_1}\nabla^{(0)\bar{m}}\omega_j^{(0)\bar{n}_1n_1}R^{(0)\bar{m}_2m_2\bar{n}_2n_2}R^{(0)\bar{m}_3m_3\bar{n}_3n_3}\;. \tag{5.4}$$

It turns out that the tensor $Z_{m\bar{m}n\bar{n}}$ given in (5.3) plays a central role in the following and is related to the key topological quantities on Y_4. It satisfies the identities

$$Z_{m\bar{m}n\bar{n}} = Z_{n\bar{m}m\bar{n}} = Z_{m\bar{n}n\bar{m}}\;, \qquad \nabla^{(0)m}Z_{m\bar{m}n\bar{n}} = \nabla^{(0)\bar{m}}Z_{m\bar{m}n\bar{n}} = 0\;. \tag{5.5}$$

It is related to the third Chern-form $c_3^{(0)}$ via

$$Z_{m\bar{m}} = i2Z_{m\bar{m}n}{}^{n} = \frac{1}{2}(*^{(0)}c_3^{(0)})_{m\bar{m}}\,,$$

$$Z = i2Z_m{}^m = *^{(0)}(J^{(0)}\wedge c_3^{(0)})\,, \qquad *^{(0)}(c_3^{(0)}\wedge\omega_i^{(0)}) = -2Z_{m\bar{n}}\omega_i^{(0)\bar{n}m}\,, \tag{5.6}$$

and yields the fourth Chern-form $c_4^{(0)}$ by contraction with the Riemann tensor as

$$Z_{m\bar{m}n\bar{n}}R^{(0)\bar{m}m\bar{n}n} = *^{(0)}c_4^{(0)}\;. \tag{5.7}$$

We note that $Y_{ijm\bar{n}}$ is also related to $Z_{m\bar{m}n\bar{n}}$ upon integration as

$$\int_{Y_4} Y_{ijm}{}^{m} *^{(0)} 1 = -\frac{1}{6}\int_{Y_4}(iZ_{m\bar{n}}\omega_i^{(0)\bar{r}m}\omega_j^{(0)\bar{n}}{}_{\bar{r}} + 2Z_{m\bar{n}r\bar{s}}\omega_i^{(0)\bar{n}m}\omega_j^{(0)\bar{s}r}) *^{(0)} 1\,, \tag{5.8}$$

where the right hand side represents the same linear combination that will be relevant in (C.15).

6 Warped background solutions to eleven-dimensional supergravity

In this section we will determine a bosonic solution to eleven-dimensional Einstein equations in the presence of higher-curvature corrections and background fluxes. We will explicitly solve the Einstein equations finding a correction to the internal Calabi-Yau metric and comment on the supersymmetry properties of the solution. More concretely, in section 6.2 we present the ansatz for the metric and the background fluxes and give the equations satisfied by the appearing functions. We then solve the internal Einstein equations finding corrections to the metric. Supersymmetry properties of this solution and the gravitino variations will be analyzed in section 6.5. We derive the modified Killing spinor equations and translate the conditions into first order differential equations for J, Ω. We comment on the compatibility with the Einstein equations and the implications for supersymmetry. Useful identities and a summary of our conventions are supplemented in appendix A.

6.1 The eleven-dimensional action

The dynamics of the fields $\hat{C}$ and $\hat{g}_{MN}$ is determined by the bosonic part of the $\mathcal{N}=1$ supergravity action (2.61) given by

$$S = S^{(0)} + \alpha^2 S^{(2)}_{\hat{R}^4} + \alpha^2 S^{(2)}_{\hat{G}^2\hat{R}^3} + \alpha^2 S^{(2)}_{(\hat{\nabla}\hat{G})^2\hat{R}^2} + \mathcal{O}(\hat{G}^3\alpha^2) + \mathcal{O}(\alpha^3) + \dots \ . \tag{6.1}$$

with the expansion parameter α given in (2.60) being proportional to the third power of the eleven-dimensional Planck length. The detailed structures of the various terms of (6.1) is given in (2.63) -(2.65) .

However, for the purpose of determining the background solutions to order α^2 it suffices to solely consider the following relevant terms. Firstly, the classical two-derivative action (2.62) and secondly, $S_{\hat{R}^4}$ given in (2.63) and again here given by

$$S_{\hat{R}^4} = \frac{1}{2\kappa_{11}^2}\int (\hat{t}_8\hat{t}_8 - \frac{1}{24}\hat{\epsilon}_{11}\hat{\epsilon}_{11})\hat{R}^4 \hat{*}1 + 3^2 2^{13}\hat{C}\wedge\hat{X}_8 \ . \tag{6.2}$$

The explicit form of the various terms in (6.2) is given (2.66) and (2.67). It is believed that these are all terms quartic in the Riemann tensor at this order in α. The terms at higher-order in $\hat{G}$ and α, such as $S_{\hat{R}^3\hat{G}^2}$, will not be needed in what follows as their contribution is higher-order in α when evaluated on the ansatz we will make.

6.2 Ansatz for the vacuum solution

We now consider solutions for which the internal space is a compact eight-dimensional manifold $\mathcal{M}_8$ and the external space is $\mathbb{R}^{2,1}$. At lowest order in α the solution takes the form

$$d\hat{s}^2 = \hat{g}_{MN}dx^M dx^N = \eta_{\mu\nu}dx^\mu dx^\nu + g^{(0)}_{mn}dy^m dy^n + \mathcal{O}(\alpha) \ , \qquad \hat{G} = 0 + \mathcal{O}(\alpha) \ , \tag{6.3}$$

where $\mu = 0, \dots, 2$ the external space and $m = 1, \dots, 8$ the real internal space indices. Note that in contrast to the previous sections of this work we will not write down the internal components in the holomorphic and antiholomorphic indices $m, \bar{m} = 1, 2, 3, 4$. The main advantage there is to use the Kähler property of the Calabi-Yau metric to simplify the discussion. In this section, however, we aim solve for the metric of the internal space and it proves more convenient to proceed in real indices. The Einstein equations imply Ricci-flatness of the internal space $R^{(0)}_{mn} = 0$. In fact, together with the supersymmetry conditions requiring the preservation of four supercharges, one infers that the internal manifold is Calabi-Yau and thus admits a nowhere vanishing Kähler form $J^{(0)}_{mn}$ and a holomorphic (4,0)-form $\Omega^{(0)}_{mnrs}$ that are harmonic.

Having deduced this lowest order solution we can then work to second order in α by considering the field equations of the α-corrected action. To solve the corrected Einstein equations we make an ansatz for the metric [1]

$$d\hat{s}^2 = e^{\alpha^2 \Phi^{(2)}} (e^{-2\alpha^2 W^{(2)}} \eta_{\mu\nu} dx^\mu dx^\nu + e^{\alpha^2 W^{(2)}} g_{mn} dy^m dy^n) \, + \mathcal{O}(\alpha^3), \tag{6.4}$$

where

$$g_{mn} = g^{(0)}_{mn} + \alpha^2 g^{(2)}_{mn} + \mathcal{O}(\alpha^3) \,. \tag{6.5}$$

Here $\Phi^{(2)}$, $W^{(2)}$, $g^{(0)}_{mn}$ and $g^{(2)}_{mn}$ depend only on the internal coordinates y^m in the background. The function $\Phi^{(2)}$ is an overall Weyl rescaling that we will discuss in more detail below, while $W^{(2)}$ is known as the warp-factor. At this order in α a background four-form field strength must also be included. Following [123] we make the ansatz

$$\hat{G}_{mnrs} = \alpha G^{(1)}_{mnrs} + \mathcal{O}(\alpha^3) \,, \qquad \hat{G}_{\mu\nu\rho m} = \epsilon_{\mu\nu\rho} \partial_m e^{-3\alpha^2 W^{(2)}} + \mathcal{O}(\alpha^3) \,, \tag{6.6}$$

where $G^{(1)}$ is a background four-form flux on the internal manifold $\mathcal{M}_8$ that is harmonic with respect to $g^{(0)}_{mn}$. Let us note that the term linear in α appearing in $\hat{G}_{mnrs}$ has the correct mass dimensions such that the background flux $G^{(1)}_{mnrs}$ integrates to a dimensionless number. In fact $T_{\text{M2}} \int_{C_4} \hat{G}$ has to be dimensionless and the inverse M2-brane tension T^{-1}_{M2} is proportional to α. We do not include a α^2 term in the Ansatz for $\hat{G}_{mnrs}$, since it can be shown to either decouple or to give contributions at only $\mathcal{O}(\alpha^3)$ in the following evaluations.

6.3 Equations determining the solution

The functions appearing in our ansatz may then be constrained by substituting into the eleven-dimensional equations of motion. The solution is found by expanding each of the equations of motion in powers of α and inferring the respective constraints [123].

To begin with, we note that the equations of motion of $\hat{C}$ and the eleven-dimensional Einstein equations derived from (6.1) do not decouple at first. However, combining the $\hat{C}$ equation with the

[1] Note that an alternative ansatz with AdS external space can also be analysed. However, this is not compatible with the lowest order supersymmetry conditions on the flux combined with the second order equations of motion.

external Einstein equations one infers that $G^{(1)}$ in the ansatz (7.6) is self-dual in the Calabi-Yau background, i.e.

$$\alpha G^{(1)} = \alpha *^{(0)} G^{(1)} + \mathcal{O}(\alpha^3)\,, \tag{6.7}$$

where one uses that $\mathcal{M}_8$ is compact. By using (6.7) the second order equation of motion of $\hat{C}$ implies the warp-factor equation

$$\Delta e^{3\alpha^2 W^{(2)}} + \tfrac{1}{4!2}\alpha^2 G^{(1)}_{mnrs} G^{(1)mnrs} - \tfrac{3^2 2^{13}}{8!}\alpha^2 \epsilon^{m_1\ldots m_8} X_{8\,m_1\ldots m_8} + \mathcal{O}(\alpha^3) = 0\;, \tag{6.8}$$

where the Laplacian $\Delta = \nabla_m \nabla^m$, the X_8, and the contractions of $G^{(1)}_{mnrs}$ are evaluated using g_{mn} given in (7.3). We stress that with the above ansatz (7.6) the corrections to the $\hat{C}$ equation of motion (6.7) and (7.7) from $S_{\hat{R}^3\hat{G}^2}$ in (6.1) give contributions at least of order α^3. At this order not all higher curvature contributions are known. Therefore, these conditions give constraints only to order α^2. This indicates consistency of our ansatz for the warp-factor and implies that lower α powers in the solution to (7.7) would be constants. Moreover, at this order in α the metric used in (7.7) is only $g^{(0)}_{mn}$. Integrating (7.7) over the internal manifold $\mathcal{M}_8$ one infers that, in the absence of localized sources, a non-trivial background flux $\tilde{G}^{(1)}_{mnrs}$ is required by consistency for a manifold with $\int_{\mathcal{M}_8} X_8^{(0)} \neq 0$.

Next we use the ansatz (7.2) and (7.6), along with the the constraints (6.7) and (7.7), to rewrite the Einstein equations into a simple form. Firstly, we expand

$$R_{mn} \equiv R(g^{(0)}_{rs} + \alpha^2 g^{(2)}_{rs})_{mn} = R^{(0)}_{mn} + \alpha^2 R^{(2)}_{mn} \tag{6.9}$$

which defines $R^{(2)}_{mn}$. Using this abbreviation the internal part of the eleven-dimensional Einstein equations can be rewritten as

$$R^{(2)}_{mn} - \tfrac{1}{2} g^{(0)}_{mn} g^{(0)\,rs} R^{(2)}_{rs} + 768 J^{(0)r}_{m} J^{(0)s}_{n} \nabla_r \nabla_s Z - \tfrac{9}{2}\nabla_m \nabla_n \Phi^{(2)} + \tfrac{9}{2} g^{(0)}_{mn} g^{(0)\,rs} \nabla_r \nabla_s \Phi^{(2)} = 0\,, \tag{6.10}$$

where $J^{(0)n}_{m} = J^{(0)}_{mp} g^{(0)pn}$ is the complex structure on the underlying Calabi-Yau manifold. The conditions (6.7) and (7.7) are used to cancel all flux dependence in (6.10) and ensure that the Einstein equations involving $\hat{R}_{m\mu}$ are automatically satisfied at the order considered. The external part of the Einstein equations takes the form

$$R^{(2)}_{mn} g^{(0)\,mn} - 9 g^{(0)\,mn} \nabla_m \nabla_n \Phi^{(2)} = 0\,. \tag{6.11}$$

The derivation of (6.10) and (6.11) is rather lengthy and requires the use of the identities summarized in appendix A. Furthermore, we have used Ricci-flatness $R^{(0)}_{mn} = 0$ for the lowest order part of the Riemann tensor to simplify the result. In these expressions the scalar Z is proportional to the six-dimensional Euler density and is given by

$$Z = *^{(0)}(J^{(0)} \wedge c_3^{(0)}) = \tfrac{1}{12}(R^{(0)}_{mn}{}^{rs} R^{(0)tu}_{rs} R^{(0)mn}_{tu} - 2R^{(0)r}_{m}{}_{n}{}^{s} R^{(0)t}_{r}{}_{s}{}^{u} R^{(0)m}_{t}{}_{u}{}^{n})\,, \tag{6.12}$$

where $c_3^{(0)}$ the third Chern form evaluated in the metric $g^{(0)}_{mn}$ given explicitly in (A.16). Tracing the internal part of the Einstein equation and demanding compatibility with the external part then fixes

$$\Phi^{(2)} = -\tfrac{512}{3} Z\,, \qquad R^{(2)}_{mn} = -768(J^{(0)r}_{m} J^{(0)s}_{n} \nabla_r \nabla_s Z + \nabla_m \nabla_n Z)\;. \tag{6.13}$$

In other words, the solution indeed requires the presence of a non-trivial eleven-dimensional Weyl rescaling involving the higher-curvature terms.

6.4 Solving the modified Einstein equation

In order to solve (6.13) we follow a technique equivalent to that shown in [138]. We begin by noting that as $c_3^{(0)}$ is real and closed but not co-closed with respect to the Kähler metric $g^{(0)}_{mn}$. This means that it may be expanded as

$$c_3^{(0)} = Hc_3^{(0)} + i\partial^{(0)}\bar{\partial}^{(0)}F_4 \tag{6.14}$$

where H indicates the projection to the harmonic part with respect to the metric $g^{(0)}_{mn}$. This equation defines a co-closed $(2,2)$-form F that will be key to the following discussions.[2] Then by using (6.12) we see that

$$Z = *^{(0)}(J^{(0)} \wedge Hc_3^{(0)}) + \tfrac{1}{4}\Delta^{(0)} *^{(0)} (J^{(0)} \wedge J^{(0)} \wedge F_4) \tag{6.15}$$

where $*^{(0)}(J^{(0)} \wedge Hc_3^{(0)})$ is constant over the internal space as a result of the harmonic projection. We are now in the position to use these quantities to solve (6.13) for a metric correction at order α^2. The explicit solution is given by

$$g^{(2)}_{mn} = 384(J_m^{(0)\,r} J_n^{(0)\,s} \nabla^{(0)}_r \nabla^{(0)}_s + \nabla^{(0)}_m \nabla^{(0)}_n) *^{(0)} (J^{(0)} \wedge J^{(0)} \wedge F_4) \ , \tag{6.16}$$

where F is the four-form introduced in (A.18). Clearly, one can now explicitly check that (6.16) solves (6.13).[3] In the next section we will show by introducing globally defined forms on $\mathcal{M}_8$ how one is naturally lead to the solution (6.16).

6.5 Killing spinor equations and globally defined forms

In this section we comment on the supersymmetry properties of the solution introduced in section 6. This is a challenging task, since the supersymmetry variations are not fully known at the desired order α^2. Following a strategy used in [141, 142] we will be able to extract at least partial information about the supersymmetry properties by studying the Killing spinor equations at order α^2. Furthermore, we will then translate these equations into differential conditions on the globally defined forms J and Ω on $\mathcal{M}_8$. This will lead to a stepwise derivation of the correction (6.16).

To set the stage of our study, let us note that we assert that at quadratic order in α the eleven-dimensional gravitino variation is given by

$$\begin{aligned}\delta\hat{\psi}_M = {} & \hat{\nabla}_M\hat{\epsilon} - \tfrac{1}{288}\hat{G}_{NRST}\hat{\Gamma}_M{}^{NRST}\hat{\epsilon} + \tfrac{1}{36}\hat{G}_{MNRS}\hat{\Gamma}^{NRS}\hat{\epsilon} \\ & + \tfrac{128}{3}\alpha^2\hat{\nabla}_N\hat{Z}\hat{\Gamma}_M{}^N\hat{\epsilon} - 48\alpha^2\hat{\nabla}^N\hat{R}_{MRN_1N_2}\hat{R}_{NSN_3N_4}\hat{R}^{RS}{}_{N_5N_6}\hat{\Gamma}^{N_1\ldots N_6}\hat{\epsilon} + \mathcal{O}(\alpha^2) \ ,\end{aligned} \tag{6.17}$$

[2] The harmonicity of Chern forms has been also discussed in the mathematical literature and lead to the introduction of the Bando-Futaki character [139], which is however trivially vanishing in the Calabi-Yau case.

[3] Recently, it was pointed out in [140] that a redefinition of the metric background $g_{mn} = g^{(0)}_{mn} - 768\alpha^2 J_m^{(0)\,r}(*^{(0)}c_3^{(0)})_{rn}$ trivializes the kinetic terms for the vectors obtained from $\hat{G}$ in the three-dimensional effective action. This interesting observation, however, has to be contrasted with the fact that this shift is not a solution to the Einstein equations at order α^2.

where the remaining order α^2 terms vanish on the backgrounds we consider. Here $\hat{Z}$ is proportional to the six-dimensional Euler density in eleven dimensions and is given by

$$\hat{Z} = \tfrac{1}{12}(\hat{R}_{MN}{}^{RS}\hat{R}_{RS}{}^{TU}\hat{R}_{TU}{}^{MN} - 2\hat{R}_{M}{}^{R}{}_{N}{}^{S}\hat{R}_{R}{}^{T}{}_{S}{}^{U}\hat{R}_{T}{}^{M}{}_{U}{}^{N}) \ . \tag{6.18}$$

This form of the gravitino variation is compatible with the terms that are necessary in [141, 142]. In other words, we will see below that the Killing spinor equations derived from (6.17) are compatible with the Einstein equations up to order α^2. Remarkably, the terms in (6.17) also appear in the gravitino variations deduced by eleven-dimensional Noether coupling in [143].

6.6 Dimensional reduction of the supergravity variations

We next dimensionally reduce the supersymmetry variations (6.17) on the background introduced in section 6. To begin with, we decompose the eleven-dimensional supersymmetry parameter and gamma matrices in a way that is compatible with our ansatz as

$$\hat{\epsilon} = e^{-\frac{1}{2}\alpha^2 W^{(2)}}\epsilon \otimes \eta \ , \qquad \hat{\Gamma}_\mu = e^{\frac{1}{2}\alpha^2\Phi^{(2)} - \alpha^2 W^{(2)}}\gamma_\mu \otimes \gamma^9 \ , \qquad \hat{\Gamma}_m = e^{\frac{1}{2}\alpha^2\Phi^{(2)} + \frac{1}{2}\alpha^2 W^{(2)}}\mathbb{1} \otimes \gamma_m \ , \tag{6.19}$$

where ϵ is a spinor in the three-dimensional external space and η is a no-where vanishing spinor on $\mathcal{M}_8$. The spinor η is chosen to satisfy $\gamma^9\eta = \eta$, $\eta^\dagger\eta = 1$ and $\eta^T\eta = 0$.

Substituting this decomposition along with the reduction ansatz (7.2) and (7.6) into (6.17) we find for the internal gravitino variation

$$\begin{aligned}\delta\hat{\psi}_m = {}& e^{-\frac{1}{2}\alpha^2 W^{(2)}}\epsilon \otimes \nabla_m\eta - \tfrac{1}{288}\alpha G^{(1)}_{nrst}\epsilon \otimes \gamma_m{}^{nrst}\eta + \tfrac{1}{36}\alpha G^{(1)}_{mnpq}\epsilon \otimes \gamma^{npq}\eta \\ & - 48\alpha^2\nabla^n R_{mrm_1m_2}R_{nsm_3m_4}R^{rs}{}_{m_5m_6}\epsilon \otimes \gamma^{m_1\ldots m_6}\eta \\ & + \tfrac{128}{3}\alpha^2\nabla_n Z\epsilon \otimes \gamma_m{}^n\eta + \tfrac{1}{4}\alpha^2\nabla_n\Phi^{(2)}\epsilon \otimes \gamma_m{}^n\eta + \mathcal{O}(\alpha^3) = 0 \ ,\end{aligned} \tag{6.20}$$

and for the external gravitino variation

$$\begin{aligned}\delta\hat{\psi}_\mu = {}& e^{-\frac{1}{2}\alpha^2 W^{(2)}}\nabla_\mu\epsilon \otimes \eta - \alpha\tfrac{1}{288}G^{(1)}_{mnpq}\gamma_\mu\epsilon \otimes \gamma^{mnpq}\eta \\ & - \tfrac{128}{3}\alpha^2\nabla_n Z\gamma_\mu\epsilon \otimes \gamma^n\eta - \tfrac{1}{4}\alpha^2\nabla_n\Phi^{(2)}\gamma_\mu\epsilon \otimes \gamma^n\eta + \mathcal{O}(\alpha^3) = 0 \ .\end{aligned} \tag{6.21}$$

These equations can then be satisfied if at lowest order in α if the background is Calabi-Yau, as already noted at the beginning of section 6.2, and one has $\nabla_\mu\epsilon = 0$. At linear order in α one finds the condition

$$G^{(1)}_{mnrs}\gamma^{nrs}\eta = 0 \tag{6.22}$$

Finally, at second order in α one finds that (6.13) has to be satisfied and η obeys the Killing spinor equation

$$\nabla_m\eta = -384\alpha^2 J^{(0)}{}_m{}^n\nabla_n Z_{rs}\gamma^{rs}\eta + \mathcal{O}(\alpha^3) \ , \qquad\qquad Z_{rs} = \tfrac{1}{2}(*c_3^{(0)})_{rs} \tag{6.23}$$

where $J^{(0)rs}Z_{rs} = Z$.

6.7 Differential conditions on the globally defined forms

Using the spinor η one can introduce a globally defined no-where vanishing real two-form J and a complex four-form Ω. This is a familiar strategy for manifolds with reduced structure group. The case of having $SU(4)$ structure was discussed in [124, 144] and we refer the reader to the appendix A.4 for more details . Concretely, we use η to construct the forms

$$J_{mn} = i\eta^\dagger \gamma_{mn}\eta \ , \qquad \Omega_{mnrs} = \eta^T \gamma_{mnrs}\eta \ . \tag{6.24}$$

By using Fierz identities we see that these forms satisfy

$$J \wedge \Omega = 0 \ , \qquad\qquad J \wedge J \wedge J \wedge J = \tfrac{3}{2}\Omega \wedge \bar{\Omega} \ . \tag{6.25}$$

The Kähler form $J^{(0)}_{mn}$ corresponding to the Ricci-flat metric $g^{(0)}_{mn}$ is then the lowest order part of J_{mn}.

We can now rewrite the supersymmetry conditions (6.22) and (6.23) using J and Ω. The constraint on the flux (6.22) implies that

$$G^{(1)} \wedge J^{(0)} = 0 \ , \qquad\qquad G^{(1)} \text{ is of type (2,2) in } J^{(0)\,n}_m \tag{6.26}$$

where $J^{(0)\,n}_m$ is the complex structure of the underlying Calabi-Yau fourfold. Furthermore, the Killing spinor equation (6.23) satisfied by η translates to the differential conditions

$$\nabla_m J_{nr} = 0 + \mathcal{O}(\alpha^3) \ , \qquad \nabla_m \Omega_{nrst} = 6144\alpha^2 J^{(0)p}_m \nabla^{(0)}_p Z_{[n}{}^q \Omega^{(0)}_{rst]q} + \mathcal{O}(\alpha^3) \tag{6.27}$$

Antisymmetrising in the indices then gives

$$dJ = 0 + \mathcal{O}(\alpha^3) \ , \qquad\qquad d\Omega = -768\alpha^2 dZ \wedge \Omega^{(0)} + \mathcal{O}(\alpha^3) \ . \tag{6.28}$$

We can thus infer that the metric g_{mn} including α^2 corrections is still Kähler. In fact, the higher-curvature terms only amount to introducing the non-closedness of Ω with a result proportional to Ω itself. In fact, translated into torsion forms for an $SU(4)$-structure manifold (see, for example, [124, 144]), the only non-trivial torsion form is $\overline{\mathcal{W}}_5 = -768\alpha^2\bar{\partial}^{(0)}Z$, which is exact.

Let us stress that the derivation of the Killing spinor equation makes use of the full internal space metric $\hat{g}_{MN}$. However, the overall Weyl rescaling and warp-factor terms precisely cancel and the resulting equation (6.23) depends only on the metric g_{mn} appearing in (7.2). The J and Ω appearing(6.28) are thus related to the metric g_{mn}. Clearly one could introduce a alternative $\tilde{J}$ and $\tilde{\Omega}$ related to rescaled metric $\hat{g}_{mn}$. This would induce new terms proportional to $\tilde{J}$ in $d\tilde{J}$ and $\tilde{\Omega}$ in $d\tilde{\Omega}$ will then be induced, since the gamma-matrices in (6.24) are rescaled.

We can now use the condition that g_{mn} is a Kähler metric and study the integrability condition of (6.23). Here the commutator $[\nabla_m, \nabla_n]\eta = \frac{1}{4}R_{mnrs}\gamma^{rs}\eta$ can be compared with the result obtained form (6.23). This simply results in the condition

$$\frac{1}{4}R_{mnrs}\gamma^{rs}\eta - 768\alpha^2 J^{(0)}{}_{[m}{}^r \nabla^{(0)}_{n]} \nabla^{(0)}_r Z_{pq}\gamma^{pq}\eta + \mathcal{O}(\alpha^3) = 0 \ . \tag{6.29}$$

Contracting with $\eta^\dagger$ we see that this implies

$$\tfrac{1}{4}R_{mnrs}J^{rs} - 768\alpha^2 J^{(0)}{}_{[m}{}^{r}\nabla^{(0)}_{n]}\nabla^{(0)}_{r}Z + \mathcal{O}(\alpha^3) = 0\,. \tag{6.30}$$

As we know that $R_{mnrs}J^{rs} = 2R_{mrns}J^{rs}$ by the first Bianchi identity and that for a Kähler manifold $J_m{}^pR_{pnrs} = J_n{}^pR_{mprs}$ we then see that (6.30) implies $R^{(0)}_{mn} = 0$ at zeroth α order and the Einstein equations (6.13) at order α^2.

6.7.1 Solving the equations for J and Ω

We now wish to solve the equations (6.28) subject to the algebraic constraints (6.25). To do this we begin by expanding these equations in α to find

$$dJ^{(2)} = 0\,, \qquad\qquad d\Omega^{(2)} = -768 dZ \wedge \Omega^{(0)}\,. \tag{6.31}$$

We may solve the constraint on $\Omega^{(2)}$ by letting

$$\Omega^{(2)} = \phi\Omega^{(0)} + \rho\,, \qquad \text{where} \qquad d\phi = -768dZ\,, \qquad d\rho = 0\,. \tag{6.32}$$

The (4,0) part of ρ can be absorbed into $\phi\Omega^{(0)}$ so we may assume that $\rho\wedge\bar{\Omega}^{(0)} = 0$. Similarly as $J^{(2)}$ is a real d-closed 2-form on a Kähler manifold

$$J^{(2)} = \sigma + i\partial^{(0)}\bar{\partial}^{(0)}\psi\,, \qquad \text{where} \qquad d\sigma = d^{(0)\dagger}\sigma = 0\,. \tag{6.33}$$

Then considering the expansion of (6.25) we see that

$$4J^{(2)}\wedge J^{(0)}\wedge J^{(0)}\wedge J^{(0)} = \frac{3}{2}(\Omega^{(2)}\wedge\bar{\Omega}^{(0)} + \Omega^{(0)}\wedge\bar{\Omega}^{(2)})\,, \tag{6.34}$$

and substituting (6.32) and (6.33) into (6.34) we find

$$\frac{1}{3} * (\sigma\wedge J^{(0)}\wedge J^{(0)}\wedge J^{(0)}) - \Delta^{(0)}\psi = 2(\phi + \bar{\phi})\,, \tag{6.35}$$

which implies that $d\Delta^{(0)}\psi = 3072dZ$. Considering this along with (6.32) and using the expansion of Z given by (6.15) we see that we are lead to a solution for $J^{(2)}$ and $\Omega^{(2)}$ where

$$J^{(2)} = i786\,\partial^{(0)}\bar{\partial}^{(0)} *^{(0)} (F\wedge J^{(0)}\wedge J^{(0)})\,, \qquad \Omega^{(2)} = -192\,\Delta^{(0)} *^{(0)} (F\wedge J^{(0)}\wedge J^{(0)})\Omega^{(0)}\,. \tag{6.36}$$

This shows that the internal space Kähler potential is shifted by a term proportional to F. The remaining forms ρ and σ correspond to moduli. Expanding the relationship

$$g_{mn} = \tfrac{i}{48}\Omega_{(m|rpt}\bar{\Omega}_{|n)squ}J^{rs}J^{pq}J^{tu}\,, \tag{6.37}$$

which may be demonstrated using the results of Appendix A, we find

$$g^{(2)}_{mn} = -J^{(0)r}_{(m}J^{(2)}_{n)r} + \frac{1}{2}J^{(0)rs}J^{(2)}_{rs}g^{(0)}_{mn} - \frac{1}{48}\Omega^{(2)}_{(m|rst}\bar{\Omega}^{(0)\,rst}_{|n)} - \frac{1}{48}\bar{\Omega}^{(2)}_{(m|rst}\Omega^{(0)\,rst}_{|n)}\,, \tag{6.38}$$

and using this we see that the correction to J and Ω implies the metric correction (6.16) that solves (6.13).

The analysis presented here shows that the first order equations (6.28) on J and Ω, which are derived from the Killing spinor equations (6.23) are economically solved by (6.36). This then provides a solution to the second order equations (6.13) arising from the internal space Einstein equations. While we have no complete proof of the supersymmetry of this solution this result provides a necessary condition. Furthermore, as we expect that the lowest order supersymmetry carries over to the higher-order analysis and we have made a general analysis of the corrections to the eleven-dimensional field equations, it seems natural to expect that further corrections to the gravitino variation (6.17) vanish in the background presented. It would be interesting to continue to develop the Noether coupling analysis of [143] to find the complete expression for the gravitino variation at order α^2.

7 Warped Kaluza-Klein reduction to $3d$

In this section we will compactify the full set of relevant eight-derivative corrections at order α^2

$$S = S^{(0)} + \alpha^2 S^{(2)}_{\hat{R}^4} + \alpha^2 S^{(2)}_{\hat{G}^2\hat{R}^3} + \alpha^2 S^{(2)}_{(\hat{\nabla}\hat{G})^2\hat{R}^2} + \mathcal{O}(\hat{G}^3\alpha^2) + \mathcal{O}(\alpha^3)\,, \tag{7.1}$$

on the background solution derived in the previous section 6 down to three dimensions considering a finite number of Kähler deformations of the metric and vector deformations of the M-theory three-form. Note that terms in (7.1) carrying a higher-order in the flux $\mathcal{O}(\hat{G}^3\alpha^2)$ can be safely discarded from the following discussion since their contribution in the reduction is of higher order in α. The precise form of the terms in (7.1) is given in section 2.3. Let us stress that the $(\hat{\nabla}\hat{G})^2\hat{R}^3$ terms have not been fully determined as discussed in section 2.3. We argued them to be given by a number of building blocks of index contractions, in section 2.3, with 4-point amplitudes only determining part of the numerical pre-factors. Remarkably, most of these unknown coefficients actually do not effect our computation and we are able to suggest a fixation of the unknown coefficients up to one constant. This last constant might then be fixed by supersymmetry, as the results in section 8 suggest. Clearly, the complete form of the $(\hat{\nabla}\hat{G})^2\hat{R}^3$ terms could also be determined by considering amplitudes with 5 and more external legs.

The metric ansatz is modified according to section 6 and upon reducing the action (7.1) the mode expansion for Kähler structure perturbations of the metric and vector perturbations of the M-theory three-form need to be corrected at order α^2, which is done in terms of forms non-harmonic in the zeroth order Calabi-Yau metric. Inserting the ansatz into the higher-derivative action (7.1), we find that the kinetic terms for the deformations and vectors in the three-dimensional effective theory can be expressed using a single higher-curvature building block $Z_{m\bar{m}n\bar{n}} = \frac{1}{4!}(\epsilon_8\epsilon_8 R^{(0)3})_{m\bar{m}n\bar{n}}$, see (5.3).

In section 7.1 we review the eleven-dimensional effective action of M-theory including higher-derivative terms and the considered warped solutions that admit an eight-dimensional compact internal manifold and background fluxes and comment on the supersymmetry conditions. The considered perturbations of the background solutions are introduced in section 7.2 and consist of vector modes of the M-theory three-form and Kähler structure deformations. We also discuss the field-dependence of the warp-factor. The dimensional reduction yielding a three-dimensional effective action is carried out in section 7.3, where we present the results for the kinetic terms and Chern-Simons terms. More details on the dimensional reduction of the higher-derivative terms can be found in appendix C.

7.1 Compactifying warped solutions with background fluxes

Let us set the stage for the reduction of (7.1) by reviewing the warped background solutions following the previous section 6. The solution for the eleven-dimensional metric background is

$$d\hat{s}^2 = e^{\alpha^2\Phi^{(2)}}(e^{-2\alpha^2 W^{(2)}}\eta_{\mu\nu}dx^\mu dx^\nu + 2e^{\alpha^2 W^{(2)}}g_{m\bar{n}}dy^m dy^{\bar{n}}) \; + \mathcal{O}(\alpha^3), \tag{7.2}$$

where $\eta_{\mu\nu}$ is the three-dimensional Minkowski metric and

$$g_{m\bar{n}} = g^{(0)}_{m\bar{n}} + \alpha^2 g^{(2)}_{m\bar{n}} + \mathcal{O}(\alpha^3)\,. \tag{7.3}$$

with the lowest order metric $g^{(0)}_{m\bar{n}}$ Calabi-Yau and

$$g^{(2)}_{m\bar{n}} = 768\partial^{(0)}_m\bar{\partial}^{(0)}_{\bar{n}} *^{(0)} (J^{(0)}\wedge J^{(0)}\wedge F), \qquad \Phi^{(2)} = -\tfrac{512}{3}Z\,, \qquad Z = *^{(0)}(J^{(0)}\wedge c_3^{(0)})\;. \tag{7.4}$$

This implies that the metric $g_{m\bar{n}}$ introduced in (7.3) is still Kähler and that the internal part of the eleven-dimensional metric (7.2) is conformally Kähler.

At zeroth order in α the background is a direct product and $g^{(0)}_{m\bar{n}}$ is a Ricci-flat metric and supersymmetry of the background at lowest order in α demands that the metric $g^{(0)}_{m\bar{n}}$ must be Calabi-Yau. The complex indices $m, \bar{m} = 1,2,3,4$ always refer to the zeroth order complex structure on the internal manifold. On a Calabi-Yau fourfold there exists a nowhere vanishing covariantly constant Kähler form $J^{(0)}$ and holomorphic $(4,0)$-form $\Omega^{(0)}$ satisfying

$$dJ^{(0)} = d\Omega^{(0)} = 0\,. \tag{7.5}$$

In what follows we will work in conventions in which the internal space indices are raised and lowered with the lowest order internal space metric $g^{(0)}_{m\bar{n}}$.

The background also includes a flux for the four-form given by

$$\hat{G}_{m\bar{n}r\bar{s}} = \alpha G^{(1)}_{m\bar{n}r\bar{s}} + \mathcal{O}(\alpha^3)\,, \qquad \hat{G}_{mnrs} = \alpha G^{(1)}_{mnrs} + \mathcal{O}(\alpha^3)\,,$$
$$\hat{G}_{\mu\nu\rho m} = \epsilon_{\mu\nu\rho}\partial_m e^{-3\alpha^2 W^{(2)}} + \mathcal{O}(\alpha^3)\,. \tag{7.6}$$

In order that the eleven-dimensional field equations are solved to order α^2 by this background the flux $G^{(1)}$ must be self-dual in the lowest-order metric $g^{(0)}_{m\bar{n}}$. This condition allows $(2,2)$ and $(4,0)+(0,4)$ components of the flux with respect to the lowest order complex structure.

The field equations for the M-theory three-form $\hat{C}$ and the external space Einstein equations then constrain the warp-factor $W^{(2)}$ to satisfy

$$d^\dagger d e^{3\alpha^2 W^{(2)}} *_8 1 - \alpha^2 Q_8 + \mathcal{O}(\alpha^3) = 0 \ , \tag{7.7}$$

where

$$Q_8 = -\frac{1}{2} G\wedge G - 3^2 2^{13}\alpha^2 X_8(Y_4) = -\frac{1}{2} G\wedge G + 3072\, c_4^{(0)} \ . \tag{7.8}$$

where we have used that $X_8(Y_4) = -\frac{1}{24}\, c_4^{(0)}$. In this expression c_4 is the fourth Chern-form evaluated in the metric $g_{m\bar{n}}$ given by

$$c_4 = \frac{1}{8}(\mathcal{R}_m{}^n \mathcal{R}_n{}^m \mathcal{R}_r{}^s \mathcal{R}_s{}^r - 2\mathcal{R}_m{}^n \mathcal{R}_n{}^r \mathcal{R}_r{}^s \mathcal{R}_s{}^m) \ . \tag{7.9}$$

For a compact Y_4 the warp-factor equation (7.7) implies the global consistency condition

$$\frac{1}{3^2 2^{14}} \int_{Y_4} G^{(1)} \wedge G^{(1)} = \frac{\chi(Y_4)}{24} \ , \tag{7.10}$$

where $\chi(Y_4) = -4! \int_{Y_4} X_8$ is the Euler number of Y_4. Using self-duality of the fluxes $G^{(1)}$ one thus realizes that in higher-derivative terms cannot be consistently ignored if one allows for a background flux. The somewhat unusual numerical factor in (7.10) stems from our normalization of $G^{(1)}$ with α and can be removed when moving to quantized fluxes $G^{\text{flux}} = \frac{1}{3\,2^6\sqrt{2}} G^{(1)}$.

Let us close this section with a short discussion on supersymmetry, by stressing again that the full supersymmetric completion of the action (2.61) is not known and neither have the supersymmetry variations of the fermions been written down. The proposal for the gravitino variations (6.17) included novel terms at order α^2. At linear order in α the supersymmetry variations were unchanged and the condition on the flux is the vanishing of the $(4,0)+(0,4)$-component of $G^{(1)}$, i.e.

$$G^{(1)}_{mnrs} = 0 \ , \tag{7.11}$$

and the primitivity condition

$$G^{(1)} \wedge J^{(0)} = 0 \ . \tag{7.12}$$

7.2 Perturbations of the background

In subsection 7.1 we have reviewed a supersymmetric background with an internal compact space that is conformally Kähler. We will now examine a set of deformations that preserve the Kähler condition but change the chosen Kähler structure. Our whole discussion will be carried out at fixed complex structure, i.e. there are no complex structure deformations that will be switched on. In the following, the complex structure is chosen such that the supersymmetry condition (7.11) on the flux is satisfied. At lowest order in α the Kähler structure deformations are known to combine with vectors arising from the M-theory three-form $\hat{C}$ into three-dimensional $\mathcal{N}=2$ multiplets, as discussed e.g. in [103, 81]. We therefore need to study vectors arising from $\hat{C}$ taking into account higher α-corrections in subsection 7.2.1. The real scalars v^i that correspond to the deformations of the Kähler structure will be introduced in subsection 7.2.2. In this latter subsection we will also study the variations of the warp-factor equation with respect to the Kähler structure deformations.

7.2.1 Vector modes from the M-theory three-form

Let us first examine the vector which arises in perturbations of the M-theory three-form $\hat{C}$. These correspond to a extra terms in the expansion of $\hat{G}$ of the form

$$\delta\hat{G} = F^i \wedge \omega_i^{(\mathrm{v})} , \tag{7.13}$$

where $F^i = dA^i$ and so provides the field strength for a three-dimensional vector A^i, and $\omega_i^{(\mathrm{v})}$ are two-forms on the internal manifold. The tensor gauge symmetry of $\hat{G}$ translates to the $U(1)$ gauge symmetry of the A^i in the three-dimensional effective theory.

In order to make the meaning of (7.13) precise, we need to specify the two-forms $\omega_i^{(\mathrm{v})}$. Therefore, as with the background fields studied in subsection 7.1, we consider the expansion of $\omega_i^{(\mathrm{v})}$ to order α^2 as

$$\omega_i^{(\mathrm{v})} = \omega_i^{(0)\,(\mathrm{v})} + \alpha^2 \omega_i^{(2)\,(\mathrm{v})} . \tag{7.14}$$

By making use of the Bianchi identity $d\hat{G} = 0$ in the absence of localized sources we see that $d\omega_i^{(0)\,(\mathrm{v})} = d\omega_i^{(2)\,(\mathrm{v})} = 0$. The standard analysis of the lowest order reduction shows that only the harmonic part of $\omega_i^{(0)\,(\mathrm{v})}$ contributes in the effective action and therefore we may pick $\omega_i^{(0)\,(\mathrm{v})}$ to be harmonic. On a Calabi-Yau fourfold this implies that $\omega_i^{(0)\,(\mathrm{v})}$ is a $(1,1)$-form and one has $i = 1, \dots, \dim(H^{1,1}(Y_4))$, where $H^{1,1}(Y_4)$ is the $(1,1)$-form cohomology of Y_4 whose dimension is independent of the metric chosen on Y_4.

Let us next turn to $\omega_i^{(2)\,(\mathrm{v})}$. We first note that $\omega_i^{(0)\,(\mathrm{v})}$ can be redefined to absorb the harmonic part of $\omega_i^{(2)\,(\mathrm{v})}$. This implies that $\omega_i^{(2)\,(\mathrm{v})}$ must be exact and as it is a real two-form on a Kähler manifold the $\partial\bar{\partial}$-lemma implies that it can be obtained by a $\partial^{(0)}\bar{\partial}^{(0)}$ of a scalar $\rho_i^{(\mathrm{v})}$. In other words, one can write

$$\omega_i^{(0)\,(\mathrm{v})} = H^{(0)} \omega_i^{(0)\,(\mathrm{v})} , \qquad\qquad \omega_i^{(2)\,(\mathrm{v})} = \partial^{(0)}\bar{\partial}^{(0)} \rho_i^{(\mathrm{v})} . \tag{7.15}$$

The scalars $\rho_i^{(\mathrm{v})}$ parametrizes our ignorance in incorporating the higher-derivative corrections in the ansatz for the three-dimensional vector perturbations. Strictly speaking the indices i on the $\rho_i^{(\mathrm{v})}$ and hence $\omega_i^{(2)\,(\mathrm{v})}$ and $\omega_i^{(\mathrm{v})}$ are not restricted to the range $1, \dots, \dim(H^{1,1}(Y_4))$ as before. However, as we will see in the explicit derivation of the effective action, all $\rho_i^{(\mathrm{v})}$ actually drop out of the final expression and therefore cannot yield additional dynamical fields. Interestingly, there is also a particular choice $\rho_i^{(\mathrm{v})}$ one could imagine, where $\omega_i^{(\mathrm{v})}$ is harmonic with respect to the full internal space metric (7.2).

7.2.2 Kähler structure deformations and the warp-factor

We now turn to the study of Kähler structure deformations of the conformally Kähler metric in (7.2). In order to do that, we introduce variations

$$\delta g_{m\bar{n}} = i\delta v^i \omega_{i\,m\bar{n}}^{(\mathrm{s})} , \tag{7.16}$$

where $g_{m\bar{n}}$ is the Kähler metric given in (7.3). The δv^i correspond to scalars in the three-dimensional effective theory, while the $\omega^{(\mathrm{s})}_{i\,m\bar{n}}$ is a set of two-forms on Y_4. Despite the misuse of notation, the field-range of the index i is not yet restricted. The key point is to consider only $\omega^{(\mathrm{s})}_{i\,m\bar{n}}$ that preserve the Kähler condition. As before we can expand the forms $\omega^{(\mathrm{s})}_i$ in α as

$$\omega^{(\mathrm{s})}_i = \omega^{(0)\,(\mathrm{s})}_i + \alpha^2 \omega^{(2)\,(\mathrm{s})}_i \ . \tag{7.17}$$

Preserving the Kähler condition requires that we impose $d\omega^{(0)}_i = d\omega^{(2)}_i = 0$. As before, we recall that at zeroth order in the parameter α the fluctuations δv^i are the well-known Kähler structure deformations of the Calabi-Yau metric $g^{(0)}_{m\bar{n}}$ and the $\omega^{(0)\,(\mathrm{s})}_i$ can be chosen to be harmonic $(1,1)$-forms with $i = 1, \ldots, \dim(H^{1,1}(Y_4))$. We may then make a redefinition to absorb the harmonic part of $\omega^{(2)\,(\mathrm{s})}_i$ so that $\omega^{(2)\,(\mathrm{s})}_i = \partial^{(0)}\bar{\partial}^{(0)}\rho^{(\mathrm{s})}_i$. We may then redefine the δv^i such that the lowest order harmonic $(1,1)$-forms match those used in the vector case

$$\omega^{(0)\,(\mathrm{s})}_i = \omega^{(0)\,(\mathrm{v})}_i = \omega^{(0)}_i \ . \tag{7.18}$$

Importantly the range of the index on the $\rho^{(\mathrm{s})}_i$ is once again a priori not restricted and there could be many more δv^i than harmonic forms. However, we will again see that all the $\rho^{(\mathrm{s})}_i$ as well as $\tilde{F}$ appearing in (6.16) do not appear in the three-dimensional effective action. This implies that one can equally consider deformations of the form

$$\delta g^{(0)}_{m\bar{n}} = i\delta v^i \omega^{(0)}_{i\,m\bar{n}} \ , \tag{7.19}$$

while making sure that all other quantities in the ansatz that are built from $g^{(0)}_{m\bar{n}}$ shift accordingly. It will be also convenient to define scalars v^i containing the background value of $g^{(0)}_{m\bar{n}}$ by setting

$$g^{(0)}_{m\bar{n}} + \delta g^{(0)}_{m\bar{n}} = i v^i \omega^{(0)}_{i\,m\bar{n}} \tag{7.20}$$

There are two main complications that arise when discussing the Kähler structure deformations in a warped flux compactification. Firstly, they will in general not all be massless. Secondly, a change of Kähler structure will induce a shift in the warp-factor. The first of these points is seen at linear order in α. When the shift (7.19) is made we see that the primitivity condition $G^{(1)} \wedge J^{(0)} = 0$ given in (7.12) is not preserved by the full set of fluctuations. This means that for constant δv^i the field equations do not remain solved and so the full range of δv^i no longer represent massless moduli of the background. Instead the set of massless δv^i now becomes those that satisfy

$$\delta v^i \omega^{(0)}_i \wedge G^{(1)} = 0 \ . \tag{7.21}$$

These terms are responsible for the well known potential terms studied in the Calabi-Yau fourfold reductions with fluxes in [103, 81]. That this result for the potential is not effected by the higher-order corrections that result from higher-curvature terms is due to the fact that the supersymmetry conditions receive no linear modification in α and the potential is the square of this supersymmetry constraint.

Let us now focus on the warp-factor. Going to second order in α we find that in addition to (7.6) the fluctuations δv^i must also preserve the warp-factor equation (7.7). In order that this equation is preserved by the fluctuations we must now take the warp-factor to depend both on the internal space position and also the fields δv^i such that $W^{(2)} = W^{(2)}(y^m, v^i)$. When we perturb the background we will then find that the derivatives of $W^{(2)}$ with respect to v^i, denoted by $\partial_i W^{(2)}$, appear in these equations. We will only deduce the effective action for the fluctuations δv^i up to second order in δv^i and therefore it will suffice to consider $W^{(2)}$ to be described by the truncated Taylor series

$$W^{(2)}(y^m, v^i) = W^{(2)}| + \partial_i W^{(2)}|\delta v^i + \frac{1}{2}\partial_i \partial_j W^{(2)}|\delta v^i \delta v^j \ , \tag{7.22}$$

where $W^{(2)}|$ indicates the restriction of $W^{(2)}$ to the point in moduli space where $\delta v^i = 0$. Demanding that (7.7) is invariant up to second order in δv^i we find that at first order in δv^i one has to impose

$$\nabla^{(0)m}\nabla^{(0)\bar{n}}\big(g^{(0)}_{m\bar{n}}\partial_i W^{(2)}\big| - i\omega^{(0)}_{im\bar{n}}W^{(2)}| + i\omega_i^{(0)r}{}_r g_{m\bar{n}}W^{(2)}| - i2048\omega_i^{(0)\bar{s}r}Z_{m\bar{n}r\bar{s}}) = 0 \ , \tag{7.23}$$

while at second order one constrains

$$\begin{aligned}&\nabla^{(0)m}\nabla^{(0)\bar{n}}\big(g^{(0)}_{m\bar{n}}\partial_i\partial_j W^{(2)}\big| - 2i\omega^{(0)}_{(i|m\bar{n}}\partial_{|j)}W^{(2)}| - 2\omega^{(0)}_{(i|m\bar{s}}\omega^{\bar{s}}_{|j)\bar{n}}W^{(2)}| + \omega_i^{(0)r}{}_r\omega_j^{(0)s}{}_s g^{(0)}_{m\bar{n}}W^{(2)}| \\ &+ \omega_i^{(0)r}{}_s\omega_j^{(0)s}{}_r g^{(0)}_{m\bar{n}}W^{(2)}| - 4096\omega_i^{(0)\bar{s}r}\omega_i^{(0)\bar{t}}{}_{\bar{t}}Z_{m\bar{n}r\bar{s}} - 2048\omega_i^{(0)\bar{s}t}\omega_{it}^{(0)r}Z_{m\bar{n}r\bar{s}} + 6114Y_{ijm\bar{n}}) = 0 \ .\end{aligned} \tag{7.24}$$

The observation that both equations (7.23) and (7.24) can be represented as total derivatives in the internal space reflects the topological nature of the terms appearing in (7.7).

We will see in the next section that the three-dimensional effective action contains the various contractions of $Z_{m\bar{m}n\bar{n}}$. Interestingly, the analog quantity on Calabi-Yau threefolds has played a key role in the analysis of [125].

7.3 The three-dimensional effective action

In this section we derive the three-dimensional effective action for the scalar and vector fields introduced in section 7.2. The kinetic terms for the Kähler structure deformations and vector fields will be discussed. In a flux background also Chern-Simons terms are induced and will be included in our analysis.[4] We also study a non-trivial field-dependent scaling symmetry of the kinetic terms, which involves a rescaling of the warp-factor. Some of the technical details of the performed reduction are supplemented in appendix C.

Having identified the background of eleven-dimensional action in section 7.1 and a set of perturbations in section 7.2 we are now in a position to derive the three-dimensional effective action using a dimensional reduction. To systematically approach this task we will consider an expansion up to second order in the scalar fluctuations δv^i and vectors A^i. Furthermore, we will restrict our analysis to terms with only two external space derivatives and only retain terms up to order α^2.

[4]Note that these terms are topological in nature and key in the study of chiral F-theory spectra and anomalies [145, 146].

For the convenience of the reader we begin by summarising the full ansatz that we will use in the reduction. The perturbed eleven-dimensional metric takes the form

$$\begin{aligned} d\hat{s}^2 =& e^{-\frac{512}{3}\alpha^2(Z|+\partial_i Z|\delta v^i+\frac{1}{2}\partial_i\partial_j Z|\delta v^i\delta v^j)}\Big[e^{-2\alpha^2(W^{(2)}|+\partial_i W^{(2)}|\delta v^i+\frac{1}{2}\partial_i\partial_j W^{(2)}|\delta v^i\delta v^j)}g_{\mu\nu}dx^\mu dx^\nu \\ &+ 2e^{\alpha^2(W^{(2)}|+\partial_i W^{(2)}|\delta v^i+\frac{1}{2}\partial_i\partial_j W^{(2)}|\delta v^i\delta v^j)}\big(g^{(0)}_{m\bar{n}}+\omega^{(0)}_{i\ m\bar{n}}dv^i \\ &+ \alpha^2\partial_m\partial_{\bar{n}}(\tilde{F}|+\rho^{(\mathrm{s})}_i\delta v^i+\partial_i\tilde{F}|\delta v^i+\frac{1}{2}\partial_i\partial_j\tilde{F}|\delta v^i\delta v^j)\big)dy^m dy^{\bar{n}}\Big] + \mathcal{O}(\alpha^3)+\mathcal{O}(\delta v^{i3})\ , \end{aligned} \tag{7.25}$$

while the perturbed M-theory four-form field strength is given by

$$\begin{aligned} \hat{G} =& \alpha G^{(1)} + F^i\wedge\omega^{(0)}_i + \alpha^2 F^i\wedge\partial\bar{\partial}\rho^{(\mathrm{v})}_i \\ &+ *_3 1\wedge de^{-3\alpha^2(W^{(2)}|+\partial_i W^{(2)}|\delta v^i+\frac{1}{2}\partial_i\partial_j W^{(2)}|\delta v^i\delta v^j)} + \mathcal{O}(\alpha^3)+\mathcal{O}(\delta v^{i3})\,. \end{aligned} \tag{7.26}$$

The rather involved form of this ansatz reflects the fact that the quantities present are expanded in both α and δv^i. Recall that the symbol $|$ means evaluation at $\delta v^i = 0$, ∂_i are derivatives with respect to v^i, and ∂_m, $\partial_{\bar{n}}$ are space-time derivatives in the lowest-order complex structure of the internal manifold.

The quantities $Z|$, $\partial_i Z|$, $\partial_i\partial_j Z|$ are directly evaluated by using the definition of Z given in (5.6). Similarly one proceeds with the derivatives of $\tilde{F} = *(J\wedge J\wedge F_4)$ given in (6.16). In contrast, since the warp-factor $W^{(2)}$ is only known as a solution to the warp-factor equation (7.7) one would have to apply (7.23) and (7.24) to determine $\partial_i W^{(2)}|$ and $\partial_i\partial_j W^{(2)}|$. It turns out to be sufficient, however, to keep $\partial_i W^{(2)}|$ and $\partial_i\partial_j W^{(2)}|$ throughout the analysis. Remarkably, we will find that all contributions involving $\partial_i\partial_j W^{(2)}|$ precisely cancel, while the first derivatives $\partial_i W^{(2)}|$ appear in the correct way to ensure the presence of a v^i-dependent scaling symmetry involving the warp-factor. Before turning to the derivation, let us also note that one may include compensators in the effective action along the lines of the discussion presented in [147, 128, 131]. However these do not change the effective action at the studied order.

In this subsection we only discuss the kinetic terms that are present in the reduction. The reduction process is quite lengthy and makes use of the intermediate results listed in appendix C. One inserts the ansatz (7.25), (7.26) into the eleven-dimensional action (2.61). The dimensional reduction requires numerous partial integrations and uses multiple Schouten and Bianchi identities, which was only possible by using a computer algorithm. Our goal was to represent all three-dimensional terms using the higher-curvature tensor $Z_{m\bar{m}n\bar{n}}$ introduced in (5.3). Combining all terms of the computation we find the action

$$S_{\mathrm{kin}} = S^{(0)}_{\mathrm{kin}} + \alpha\, S^{(1)}_{\mathrm{CS}} + \alpha^2\, S^{(2)}_{\mathrm{kin}}\ , \tag{7.27}$$

where at zeroth order one has

$$\begin{aligned} S^{(0)}_{\mathrm{kin}} = \frac{1}{2\kappa_{11}}\int_{\mathcal{M}_3}\Big[&\Omega^{(0)}R*1 + d\delta v^i\wedge *d\delta v^j\int_{Y_4}\Big(\frac{1}{2}\omega^{(0)}_{im\bar{n}}\omega^{(0)\bar{n}m}_j - \omega^{(0)\,m}_{im}\omega^{(0)n}_{jn}\Big)*^{(0)}1 \\ &+\frac{1}{2}F^i\wedge *F^j\int_{Y_4}\omega^{(0)}_{im\bar{n}}\omega^{(0)\bar{n}m}_j *^{(0)}1\Big]\ , \end{aligned} \tag{7.28}$$

while at first order one finds the Chern-Simons terms

$$S^{(1)}_{\rm CS} = \frac{1}{2\kappa_{11}}\int_{\mathcal{M}_3} \Theta_{ij} A^j \wedge F^i \ , \qquad \Theta_{ij} = \frac{1}{2}\alpha \int_{Y_4} \omega^{(0)}_i \wedge \omega^{(0)}_j \wedge G^{(1)} \ , \tag{7.29}$$

and at second order

$$\begin{aligned} S^{(2)}_{\rm kin} =& \frac{1}{2\kappa_{11}}\int_{\mathcal{M}_3} \Big[\Omega^{(2)} R * 1 + d\delta v^i \wedge * d\delta v^j \int_{Y_4} \Big(3i\partial_i W^{(2)}|\omega^{(0)}_{jm}{}^{m} + 3W^{(2)}\big(\frac{1}{2}\omega^{(0)}_{im\bar{n}}\omega^{(0)\bar{n}m}_j - \omega^{(0)}_{im}{}^{m}\omega^{(0)}_{jn}{}^{n}\big) \\ & - 768 Z \omega^{(0)}_{im}{}^{m}\omega^{(0)}_{jn}{}^{n} + 3072 i Z_{m\bar{n}}\omega^{(0)\bar{n}m}_i \omega^{(0)}_{js}{}^{s} + 3072 Z_{m\bar{n}r\bar{s}}\omega^{(0)\bar{n}m}_i\omega^{(0)\bar{s}r}_j \Big) *^{(0)} 1 \\ & + F^i \wedge * F^j \int_{Y_4} \Big((\frac{3}{2}W^{(2)} + 256 Z)\omega^{(0)}_{im\bar{n}}\omega^{(0)\bar{n}m}_j + 192(-7+a_1) i Z_{m\bar{n}}\omega^{(0)\bar{r}m}_i \omega^{(0)\bar{n}}_j{}_{\bar{r}} \\ & + 384(1+a_1) Z_{m\bar{n}r\bar{s}}\omega^{(0)\bar{n}m}_i\omega^{(0)\bar{s}r}_j \Big) *^{(0)} 1 \Big] \ . \end{aligned} \tag{7.30}$$

Here we have abbreviated

$$\begin{aligned} \Omega^{(0)} =& \int_{Y_4} \Big[1 + i\delta v^i\, \omega^{(0)}_{im}{}^{m} + \frac{1}{2}\delta v^i \delta v^j (\omega^{(0)}_{im\bar{n}}\omega^{(0)\bar{n}m}_j - \omega^{(0)}_{im}{}^{m}\omega^{(0)}_{jn}{}^{n}) \Big] *^{(0)} 1 \ , \\ \Omega^{(2)} =& \int_{Y_4} \Big[3W^{(2)} + 3\delta v^i \big(\partial_i W^{(2)}| + i\omega^{(0)}_{im}{}^{m} W^{(2)}\big) + \delta v^j \delta v^i \Big(\frac{3}{2}\partial_i\partial_j W^{(2)}| \\ & + 3i\omega^{(0)}_{im}{}^{m}\partial_j W^{(2)}| + \frac{3}{2}W^{(2)}\big(\omega^{(0)}_{im\bar{n}}\omega^{(0)\bar{n}m}_j - \omega^{(0)}_{im}{}^{m}\omega^{(0)}_{jn}{}^{n}\big)\Big) \Big] *^{(0)} 1 \, . \end{aligned} \tag{7.31}$$

A few comments are in order. Firstly, we show in appendix C that among all the terms in (2.69) only $\mathcal{A}$, $\mathcal{Z}_1$ an $\mathcal{Z}_2$ contribute, while $\mathcal{Z}_3$ to $\mathcal{Z}_6$ vanish identically. This implies that the result should depend on two unknown parameters a_1, a_2 that appear in (2.69). It turns out that for the choice $a_1 = a_2$ the result simplifies significantly and only depends on $Z_{m\bar{m}n\bar{n}}$ as is equally true for the reduction of all other term in the eleven-dimensional action (2.61). We therefore have chosen $a_1 = a_2$ in (7.30). Secondly, we note that, as already mentioned before, the scalar functions $\tilde{F}$, $\rho^{(\rm s)}{}_i$ and $\rho^{(\rm v)}{}_i$ have totally dropped out of this expression. This justifies the use of $\dim(H^{1,1}(Y_4))$ deformations δv^i and vectors A^i.

The action (7.30) still depends on $\partial_i\partial_j W^{(2)}$, however, only through the coefficient of the three-dimensional Einstein-Hilbert term. We now wish to Weyl rescale this action to bring it to the Einstein frame and show that this dependence actually drops. From (A.40) one finds that one needs to redefine the external metric by $g_{\mu\nu} \to g'_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$ for

$$\Omega = \Omega^{(0)} + \alpha^2\, \Omega^{(2)} \ . \tag{7.32}$$

Performing the Weyl rescaling we find that the kinetic terms displayed in (7.28) and (7.30) become

$$\begin{aligned} S^{(0)}_{\rm kin} = \frac{1}{2\kappa_{11}}\int_{\mathcal{M}_3} \Big[& R * 1 + d\delta v^i \wedge * d\delta v^j \frac{1}{\mathcal{V}_0}\int_{Y_4} (\frac{1}{2}\omega^{(0)}_{im\bar{n}}\omega^{(0)\bar{n}m}_j + \omega^{(0)}_{im}{}^{m}\omega^{(0)}_{jn}{}^{n}) *^{(0)} 1 \\ & + F^i \wedge * F^j \frac{\mathcal{V}_0}{2}\int_{Y_4} \omega^{(0)}_{im\bar{n}}\omega^{(0)\bar{n}m}_j *^{(0)} 1 \Big] \ , \end{aligned} \tag{7.33}$$

and

$$
\begin{aligned}
S^{(2)}_{\text{kin}} =& \frac{1}{2\kappa_{11}} \int_{\mathcal{M}_3} \Bigg[d\delta v^i \wedge * d\delta v^j \Bigg(\frac{1}{\mathcal{V}_0} \int_{Y_4} \Big(-9i\partial_i W^{(2)}| \omega^{(0)\,m}_{jm} + \frac{3}{2} W^{(2)}| \omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j \\
& - 768 Z \omega^{(0)\,m}_{im} \omega^{(0)\,n}_{jn} + 3072 i Z_{m\bar{n}} \omega^{(0)\bar{n}m}_i \omega^{(0)\,s}_{is} + 3072 Z_{m\bar{n}r\bar{s}} \omega^{(0)\bar{n}m}_i \omega^{(0)\bar{s}r}_j \Big) *^{(0)} 1 \\
& - \frac{1}{\mathcal{V}_0^2} \int_{Y_4} \frac{3}{2} W^{(2)}| *^{(0)} 1 \int_{Y_4} \omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j *^{(0)} 1 \Bigg) \\
& + F^i \wedge * F^j \Bigg(\mathcal{V}_0 \int_{Y_4} \Big((\frac{3}{2} W^{(2)}| + 256 Z) \omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j + 192(-7+a_1) i Z_{m\bar{n}} \omega^{(0)\bar{r}m}_i \omega^{(0)\bar{n}}_j{}_{\bar{r}} \\
& + 384(1+a_1) Z_{m\bar{n}r\bar{s}} \omega^{(0)\bar{n}m}_i \omega^{(0)\bar{s}r}_j \Big) *^{(0)} 1 + \int_{Y_4} \frac{3}{2} W^{(2)}| *^{(0)} 1 \int_{Y_4} \omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j *^{(0)} 1 \Bigg) \Bigg] , \qquad (7.34)
\end{aligned}
$$

where here we have introduced the zeroth-order volume

$$
\mathcal{V}_0 = \int_{Y_4} *^{(0)} 1 \ . \qquad (7.35)
$$

The warp-factor dependence can be nicely captured by introducing the warped volume and warped metric

$$
\mathcal{V}_W = \int_{Y_4} e^{3\alpha^2 W^{(2)}} *^{(0)} 1 \ , \qquad G^W_{ij} = \frac{1}{2\mathcal{V}_W} \int_{Y_4} e^{3\alpha^2 W^{(2)}} \omega^{(0)}_i \wedge *^{(0)} \omega^{(0)}_j \ , \qquad (7.36)
$$

which at zeroth order in α reduce to $\mathcal{V}_0$ and $G_{ij} = \frac{1}{2\mathcal{V}_0} \int_{Y_4} \omega^{(0)}_i \wedge *^{(0)} \omega^{(0)}_j$. We also introduce

$$
\mathcal{K}^W_i = i \mathcal{V}_W \, \omega^{(0)\,m}_{im} + \frac{9}{2} \alpha^2 \int_{Y_4} \partial_i W^{(2)}| *^{(0)} 1 \ , \qquad (7.37)
$$

which at lowest order simply reduces to $\mathcal{K}^{(0)}_i = i\mathcal{V}_0 \, \omega^{(0)\,m}_{im} = \frac{1}{3!} \int_{Y_4} \omega^{(0)}_i \wedge J^{(0)} \wedge J^{(0)} \wedge J^{(0)}$. Note that we use the notation $\mathcal{K}^{(0)}_i$ to abbreviate the intersection number evaluated in the background. With these definitions one rewrites the action (7.27) for all kinetic terms into the form

$$
\begin{aligned}
S_{\text{kin}} =& \frac{1}{2\kappa_{11}} \int_{\mathcal{M}_3} \Bigg[R * 1 - (G^W_{ij} + \mathcal{V}_W^{-2} K^W_i K^W_j) dv^i \wedge * dv^j - \mathcal{V}_W^2 G^W_{ij} F^i \wedge * F^j + \Theta_{ij} A^i \wedge F^i \\
& - dv^i \wedge * dv^j \frac{\alpha^2}{\mathcal{V}_0} \int_{Y_4} \Big(768 Z \omega^{(0)\,m}_{im} \omega^{(0)\,n}_{jn} - 3072 i Z_{m\bar{n}} \omega^{(0)\bar{n}m}_i \omega^{(0)\,s}_{js} - 3072 Z_{m\bar{n}r\bar{s}} \omega^{(0)\bar{n}m}_i \omega^{(0)\bar{s}r}_j \Big) *^{(0)} 1 \\
& + F^i \wedge * F^j \alpha^2 \mathcal{V}_0 \int_{Y_4} \Big(256 Z \omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j + 192(-7+a_1) i Z_{m\bar{n}} \omega^{(0)\bar{r}m}_i \omega^{(0)\bar{n}}_j{}_{\bar{r}} \\
& \qquad + 384(1+a_1) Z_{m\bar{n}r\bar{s}} \omega^{(0)\bar{n}m}_i \omega^{(0)\bar{s}r}_j \Big) *^{(0)} 1 \Bigg] , \qquad (7.38)
\end{aligned}
$$

where we have replaced $d\delta v^i$ directly with dv^i. Expanding to order α^2 one indeed recovers the above result.

It is interesting to observe that the three-dimensional effective action permits a scaling symmetry involving the rescaling of the warp-factor. We begin by noting that the eleven-dimensional background ansatz given in subsection 7.1 has a symmetry under which

$$
W^{(2)} \to W^{(2)} + j \ , \qquad g_{m\bar{n}} \to e^{-\alpha^2 j} g_{m\bar{n}} \ , \qquad g_{\mu\nu} \to e^{2\alpha^2 j} g_{\mu\nu} \ , \qquad (7.39)
$$

for $j = j(x^\mu)$. This can be extended to a symmetry of the perturbed background (7.25) and (7.26) by requiring that

$$v^i \to e^{-\alpha^2 j} v^i \ . \tag{7.40}$$

This then implies that

$$dv^i \to e^{-\alpha^2 j} dv^i - \alpha^2 v^i \, \partial_j j dv^j \ , \tag{7.41}$$

if we further restrict $j = j(v^i)$. This symmetry carries over to the reduced effective action before the Weyl rescaling to move to the Einstein frame is performed. When the rescaling is performed the value of Ω in $g_{\mu\nu} \to g'_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$ transforms as $\Omega \to e^{-\alpha^2 W^{(2)}}\Omega$ so that the rescaled metric does not transform. The final form of the effective action coming from the dimensional reduction is then invariant under the symmetry

$$W^{(2)} \to W^{(2)} + j \ , \qquad\qquad v^i \to e^{-\alpha^2 j} v^i \ . \tag{7.42}$$

We note that the $\partial_i W^{(2)}$ terms in the δv^i kinetic terms are key to ensuring the symmetry of the action for j as a function of v^i, as they covariantize the derivatives which appear in the reduction. Indeed, this symmetry can be made manifest by introducing a covariant derivative for v^i. Furthermore we note that if we make the choice $a_1 = 7$ then using the definitions,

$$\begin{aligned}
G^T_{ij} &= G^W_{ij} + 256 \frac{1}{\mathcal{V}_0^2} \int_{Y_4} Z *^{(0)} 1 \int_{Y_4} \omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j *^{(0)} 1 \\
&\quad - 256 \frac{1}{\mathcal{V}_0} \int_{Y_4} \left[Z \omega^{(0)}_i{}_{m\bar{n}} \omega^{(0)\bar{n}m}_j + 12 Z_{m\bar{n}r\bar{s}} \omega^{(0)\bar{n}m}_j \omega^{(0)\bar{s}r}_i \right] *^{(0)} 1 \, , \\
\mathcal{K}^T_i &= \mathcal{K}^{(0)}_i + \alpha^2 \int_{Y_4} \left[\frac{\mathcal{K}^{(0)}_i}{\mathcal{V}_0} (3W^{(2)} - 128 Z) *^{(0)} 1 - 1536 Z_{m\bar{n}} \omega^{(0)\bar{n}m}_i *^{(0)} 1 \right] , \\
\mathcal{V}_T &= \mathcal{V}_W + \alpha^2 256 \int_{Y_4} Z *^{(0)} 1 \, ,
\end{aligned} \tag{7.43}$$

and the covariant derivative

$$Dv^i = dv^i + \alpha^2 \mathcal{W}_j dv^j \, v^i \ , \qquad\qquad \mathcal{W}_j = \frac{1}{\mathcal{V}} \int_{Y_4} \partial_j W * 1 \ , \tag{7.44}$$

the action takes the simple form[5]

$$S_{\text{eff}} = S_{\text{kin}} + S_{\text{CS}} + S_{\text{pot}} \ , \tag{7.46}$$

with

$$S_{\text{kin}} = \frac{1}{2\kappa_{11}} \int_{\mathcal{M}_3} \left[R * 1 - (\mathcal{G}^T_{ij} + \mathcal{V}_T^{-2} \mathcal{K}^T_i \mathcal{K}^T_j) Dv^i \wedge * Dv^j - \mathcal{V}_T^2 \mathcal{G}^T_{ij} F^i \wedge * F^j \right] , \tag{7.47}$$

[5]Note that in making this match we have used that

$$\begin{aligned}
\int_{\mathcal{M}_3} dv^i \wedge * dv^j \frac{1}{\mathcal{V}_0} \int_{Y_4} Z \omega^{(0)}_i \wedge *^{(0)} \omega^{(0)}_j &= \int_{\mathcal{M}_3} dv^i \wedge * dv^j \frac{1}{\mathcal{V}_0^2} \int_{Y_4} Z *^{(0)} * 1 \int_{Y_4} \omega^{(0)}_i \wedge *^{(0)} \omega^{(0)}_j \\
\int_{\mathcal{M}_3} dv^i \wedge * dv^j \frac{1}{\mathcal{V}_0} \int_{Y_4} W^{(2)} \omega^{(0)}_i \wedge *^{(0)} \omega^{(0)}_j &= \int_{\mathcal{M}_3} dv^i \wedge * dv^j \frac{1}{\mathcal{V}_0^2} \int_{Y_4} W^{(2)} *^{(0)} 1 \int_{Y_4} \omega^{(0)}_i \wedge *^{(0)} \omega^{(0)}_j
\end{aligned} \tag{7.45}$$

which can be demonstrated by taking using integration by parts in the external space.

and

$$S_{\rm CS} = \frac{1}{2\kappa_{11}} \int_{\mathcal{M}_3} \Theta_{ij} A^i \wedge F^i \ , \qquad S_{\rm pot} = -\frac{\alpha^2}{4\kappa_{11}^2} \int_{\mathcal{M}_3} *_3 1 \int_{Y_4} \frac{1}{2}\Big(G \wedge *' G - G \wedge G\Big) \ , \tag{7.48}$$

with $S_{\rm pot}$ discussed in the next section 7.4. Under the transformations (7.42) one finds for the couplings

$$\mathcal{G}^T_{ij} \to e^{2\alpha^2 j} \mathcal{G}^T_{ij} \,, \qquad \mathcal{V}_T \to e^{-\alpha^2 j} \mathcal{V}_T \,, \qquad \mathcal{K}^T_i \to e^{2\alpha^2 j} \mathcal{K}^T_i \,, \qquad Dv^i \to e^{-\alpha^2 j} Dv^i \,, \tag{7.49}$$

thus it becomes manifest that the action (7.46) remains invariant.

7.4 Scalar potential

In this subsection we discuss the derivation of the scalar potential for the Kähler structure fluctuation δv^i introduced in (7.19). We expect a flux-induced scalar potential for all fluctuations that do not respect the primitivity condition (7.12).

To begin with we consider the terms containing $\hat{C}$ without derivative. Considering the pure three-dimensional space-time part for $\hat{C}$ one easily sees

$$-\int \big(\frac{1}{6} \hat{C} \wedge \hat{G} \wedge \hat{G} + 3^2 2^{13} \hat{C} \wedge \hat{X}_8\big)\Big|_{\rm pot} = 0 \ , \tag{7.50}$$

which can be traced back to the fact that this combination is proportional to the tadpole constraint (7.10). A pure flux-induced potential term arises from the reduction

$$-\int \frac{1}{2} \hat{G} \wedge \hat{*} \hat{G}\Big|_{\rm pot} = -\alpha^2 \int_{\mathcal{M}_3} *_3 1 \int_{Y_4} \frac{1}{2} G \wedge *' G \ , \tag{7.51}$$

where $*'$ is the Hodge star of the perturbed internal metric (7.20). In order to derive the full flux-induced potential, however, we need to also dimensionally reduce the higher-curvature terms. Inserting the fluctuated ansatz into the $\hat{R}^4$-corrections to the eleven-dimensional action we find

$$\begin{aligned} \int \hat{t}_8 \hat{t}_8 \hat{R}^4 \hat{*} 1 &= \int_{\mathcal{M}_3} *_3 1 \int_{Y_4} \Big(1536\, c_4^{(0)} - 768\, \delta v^i \delta v^j (\nabla^{(0)}_a \nabla^{(0)a} Z)\, \omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j * 1\Big) \\ -\int \frac{1}{24} \hat{\epsilon}_{11} \hat{\epsilon}_{11} \hat{R}^4 \hat{*} 1 &= \int_{\mathcal{M}_3} *_3 1 \int_{Y_4} 1536\, c_4^{(0)} \ . \end{aligned} \tag{7.52}$$

We thus encounter the integral over the forth Chern-form $\int_{Y_4} c_4 = \chi(Y_4)$ and (7.10) can be used to replace these terms with a flux-dependent contribution proportional to $\int_{Y_4} G \wedge G$. Furthermore, there appears to be an additional mass term for the fluctuations δv^i involving the higher-curvature invariant Z. However, we still need to dimensional reduce the zeroth order action inserting the α^2-corrected background solution. Performing this reduction one finds

$$\int \hat{R} \hat{*} 1 = \alpha^2 \int_{\mathcal{M}_3} *_3 1 \int_{Y_4} 768\, \delta v^i \delta v^j (\nabla^{(0)}_a \nabla^{(0)a} Z)\, \omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j * 1 \ , \tag{7.53}$$

which precisely cancels the Z-dependent mass-term arising from the higher-curvature reduction in (7.52).

In summary, adding all terms (7.50)-(7.53) one finds the scalar potential term

$$S_{\text{pot}} = -\frac{\alpha^2}{4\kappa_{11}^2}\int_{\mathcal{M}_3} *_3 1 \int_{Y_4} \frac{1}{2}\Big(G\wedge *' G - G\wedge G\Big) \,. \tag{7.54}$$

This term has to be still Weyl-rescaled to bring the action into the three-dimensional Einstein frame. The rescaled result will be given in (8.3). As expected one can check that the scalar potential vanishes for primitive $(2,2)$-fluxes, i.e. for all $(2,2)$-fluxes satisfying $G_{m\bar{n}\rho\bar{s}}J'^{\bar{r}s}=0$. This condition generically fixes a number of deformations δv^i in the vacuum. Note that this is the only effect stabilising moduli at order α^2 in our setting.

8 $3d$, $\mathcal{N}=2$ Kähler potential and complex coordinates

In this section in order to reveal the supersymmetry properties of the three-dimensional effective action (7.46) we discuss its promotion into the standard $\mathcal{N}=2$ form. Note that in section 6 we could only give necessary conditions for the derived background to be supersymmetric. Thus upon the procedure of dimensional reduction carried out in the previous section 7 we arrive at (7.46), which may not necessarily be $\mathcal{N}=2$ supersymmetric. The procedure of identifying the correct $\mathcal{N}=2$ building blocks to capture the higher-derivative corrections in the couplings turns out to be a difficult endeavor. However, in this section we infer compatibility of (7.46) with $\mathcal{N}=2$ supersymmetry and argue that a no-scale condition can be implemented. Note that in $3d$ vectors can be dualized to scalars and the dynamics of the vector multiplet in (7.46) can thus be described in terms of a Kähler potential and a set of complex coordinates. We expand the Kähler potential and complex coordinates in the Kähler fluctuations where we deduce the coefficients by comparison with the dimensionally reduced action (7.46). We cannot fix all the parameters in this expansion since the reduction (7.46) only incorporates the leading order terms of order δv^2. To derive an expression, which is exact to all orders in the fluctuations δv analogous to the classical reduction [103, 81], when including higher-curvature terms becomes a more involved discussion. We argue that a possible approach to this problem lies in fixing the complex coordinates by introducing divisor integrals, which then should be matched with the actions of M5-branes wrapped on divisors. Comparing the variations of warped divisor integrals and higher-curvature terms by using the warp-factor equation allows us to show compatibility with the dimensional reduction, which furthermore suggests that the M5-brane action should receive corrections related to the non-harmonic part of the fourth Chern-form $c_4^{(0)}$.

We start this section in 8.1 by reviewing the dimensionally reduced effective action (7.46) and discuss its scaling symmetry in more detail. The $\mathcal{N}=2$ supersymmetric structure and the no-scale condition are discussed in section 8.2. We derive the Kähler potential and complex coordinates as an expansion in the fluctuations and later propose a definition using divisor integrals.

8.1 Symmetries of the reduced action

Let us start by reviewing the three-dimensional action for the Kähler fluctuations δv^i and the vectors A^i section 7.3 derived by dimensional reduction of the M-theory action including all relevant α^2 terms (2.61) on the warped background solution derived in section 6. It was shown that it takes the remarkably simple form

$$\kappa_{11}^2 S_{\text{eff}} = S_{\text{kin}} + S_{\text{CS}} + S_{\text{pot}} \ , \tag{8.1}$$

with kinetic terms given by

$$S_{\text{kin}} = \frac{1}{2\kappa_{11}} \int_{\mathcal{M}_3} \Big[R * 1 - (\mathcal{G}_{ij}^T + \mathcal{V}_T^{-2} \mathcal{K}_i^T \mathcal{K}_j^T) Dv^i \wedge * Dv^j - \mathcal{V}_T^2 \mathcal{G}_{ij}^T F^i \wedge * F^j \Big] . \tag{8.2}$$

and flux-induced Chern-Simons terms and scalar potential given by

$$S_{\text{CS}} = \int_{\mathcal{M}_3} \frac{1}{2} \Theta_{ij} A^i \wedge F^i \ , \qquad S_{\text{pot}} = -\alpha^2 \int_{\mathcal{M}_3} *_3 1 \int_{Y_4} \frac{1}{8\mathcal{V}_0^3} \Big(G \wedge *' G - G \wedge G \Big) \ , \tag{8.3}$$

with $\mathcal{G}_{ij}^T, \mathcal{K}_i^T$ and $\mathcal{V}_T$ defined in (7.43) and the covariant derivative Dv^i in (7.44). The Chern-Simons terms are dependent on the fluxes via $\Theta_{ij} = \frac{\alpha}{2} \int_{Y_4} G \wedge \omega_i \wedge \omega_j$.

Let us comment on the notation widely used in this section. Note that the intersection numbers (2.82) are used here with fully fluctuated v^i defined in (7.20) i.e. $\mathcal{V} = \frac{1}{4!} \mathcal{K}_{ijkl} v^i v^j v^k v^l$ while in the background they take the value v_0^i, thus $\mathcal{V}_0 = \frac{1}{4!} \mathcal{K}_{ijkl}^{(0)} v_0^i v_0^j v_0^k v_0^l$, thus $\mathcal{V}_0$ is simply the background zeroth-order volume of Y_4 also given by $\mathcal{V}_0 = \int_{Y_4} *^{(0)} 1$. Following this logic one writes the case with with fully fluctuated v^i as $\mathcal{K}_i$ and in the background as $\mathcal{K}_i^{(0)}$ as given in (A.42).

It is necessary to discuss lifted and background quantities at the same time thus we abuse our notation, e.g. the warp-factor in the background $W^{(2)}$ is written as $W^{(2)} = W^{(2)}(v)$. Note that in this notation $\omega_i^{(0)}$ is harmonic w.r.t. $g_{m\bar{n}}^{(0)}(v^0)$ while ω_i is harmonic w.r.t. $g_{m\bar{n}}(v)$. This introduces an ambiguity when discussing explicit corrections of higher-order in α where we then use e.g. $W^{(2)}|, T_i^{(2)}|, T_i^{(0)}|$ to denote their value in the unfluctuated background. For instance one writes $\int W^{(2)} \omega_i \wedge \omega_j \wedge J^2| = \int W^{(2)}| \omega_i^{(0)} \wedge \omega_j^{(0)} \wedge J^{(0)2}$, where J is the Kähler form related to the metric $g_{m\bar{n}}(v)$.

8.1.1 Warp-factor scaling symmetry and integration

To connect (8.1) to the canonical form of a $3d$, $\mathcal{N}=2$ theory we wish to integrate the Kähler metric into a Kähler potential with appropriate complex coordinates. We have seen in chapter II for the simplified setup that the main issue arises when splitting the integrals in the couplings. This was necessary to connect them to objects with two free indices, which can be obtained from a small set of building blocks upon twofold differentiation w.r.t. to v^i. Note that to do this one needs to uplift these objets to moduli space independent topological building blocks. This procedure, however, turns out not to be applicable in the full fletched setup (8.1). Let us illustrate the uplift at the example of the classical Kähler fluctuations where one finds up to order δv^3

$$d\delta v^i \wedge *_3 d\delta v^j \frac{1}{\mathcal{V}_0}\int \Big[\Big(\omega^{(0)\,m}_{im}\omega^{(0)\,n}_{jn} + \frac{1}{2}\omega^{(0)}_{im\bar n}\omega^{(0)\bar n m}_{j}\Big) - i\delta v^k\Big(\omega^{(0)\,n}_{im}\omega^{(0)\,r}_{jn}\omega^{(0)\,m}_{kr} + 2\omega^{(0)\;n}_{km}\omega^{(0)\,m}_{in}\omega^{(0)\,r}_{jr}\Big) + \mathcal{O}(\delta v^2)\Big] *^{(0)} 1 \ . \tag{8.4}$$

As a very standard result performing the uplift consitently absorbs all the higher-order fluctuations in the previous equation to arrive at

$$d\delta v^i \wedge *_3 d\delta v^j (G_{ij}(v^0) + \delta v^k G_{ijk}(v^0) + \mathcal{O}(\delta v^2)) \rightarrow dv^i \wedge *_3 dv^j G_{ij}(v) \tag{8.5}$$

with

$$G_{ij}(v) = \frac{1}{\mathcal{V}}\int \Big[\Big(\omega_{im}{}^{m}\omega_{jn}{}^{n} + \frac{1}{2}\omega_{im\bar n}\omega_{j}{}^{\bar n m}\Big) * 1 = -\frac{3}{2\mathcal{V}^2}\mathcal{K}_i\mathcal{K}_j + \frac{1}{2\mathcal{V}}\mathcal{K}_{ij} \ , \tag{8.6}$$

where $g_{m\bar n}(v) = iv^i\omega_{im\bar n}$. To perform this uplift to (8.6) one argues that the coupling can be written entirely via topological quantities which trivially uplift.

Let us present an alternative approach of how to think about the uplift $v_0 + \delta v \rightarrow v$. Reversely, by guessing the lifted result we should be able to reproduce (8.4) by making the replacement $v^i \rightarrow v^i_0 + \delta v^i$ on the right hand side of equation (8.5), which is not trivial as we will show now. Making the replacement $v \rightarrow v_0 + \delta v$ in (8.5) to match to the reduction (8.4) we want to express everything in terms of the harmonic forms w.r.t. to the metric depending on v^i_0. Which is done by noting that

$$g_{m\bar n}(v) = g_{m\bar n}(v^0) + i\delta v^i\omega^{(0)}_{i\,m\bar n} \ , \tag{8.7}$$

and

$$\omega_{im\bar n} = \omega^{(0)}_{im\bar n} + i\delta v^j\partial_m\bar\partial_{\bar n}\kappa_{ij} \ , \tag{8.8}$$

with $v^i_0\kappa_{ij} = v^j_0\kappa_{ij} = 0$. From $d{*}\omega_i = 0$ we find that $\nabla_r(g^{m\bar n}i\delta v^j\partial_m\bar\partial_{\bar n}\kappa_{ij} + \frac{1}{2}\omega^{(0)}_{im\bar n}\omega^{(0)\bar n m}_{j}) = 0$ and thus $i\delta v^j\partial_m\bar\partial_{\bar n}\kappa_{ij} = \frac{1}{2}\omega^{(0)}_{im\bar n}\omega^{(0)\bar n m}_{j} + C_{ij}$. The integration constant can be fixed by applying the algebraic constraints on κ_{ij} to $C_{ij} = \frac{1}{2\mathcal{V}_0}\int \omega^{(0)}_{im\bar n}\omega^{(0)\bar n m}_{j} *_8 1$, thus

$$\delta v^j \nabla_m\nabla^m\kappa_{ij} = -\frac{1}{2}\omega^{(0)}_{im\bar n}\omega^{(0)\bar n m}_{j} + \frac{1}{2\mathcal{V}_0}\int \omega^{(0)}_{im\bar n}\omega^{(0)\bar n m}_{j} *_8 1 \ . \tag{8.9}$$

Thus to down lift $v \rightarrow v^0 + \delta v$ one plugs in (8.7) and (8.8) in (8.6) and indeed derives the term proportional to δv^3 in (8.4). The terms proportional to κ drop out due the harmonicity of $\omega^{(0)}$ and only become relevant at $\mathcal{O}(\delta v^4)$. It would be desirable to also derive the linear fluctuations of the kinetic couplings arising form the higher-derivative terms in (7.38) by an in principal straightforward analogous analysis.

Let us next take a different angle to discuss the uplift of terms. The coefficients $\mathcal{G}^T_{ij}$,$\mathcal{K}^T_i$, and $\mathcal{V}_T$ in (7.43) are evaluated in the background v^i_0, however, let us investigate how far an uplift of the couplings

$$\mathcal{G}^T_{ij}(v_0),\ \mathcal{K}^T_i(v_0),\ \mathcal{V}_T(v_0) \quad \rightarrow \quad \mathcal{G}^T_{ij}(v),\ \mathcal{K}^T_i(v),\ \mathcal{V}_T(v)\ , \tag{8.10}$$

is feasible. Let us start the argument by noting that at first sight the couplings appear to be unrelated, however, they are in fact precisely taking values so as to ensure the identity

$$(G^T_{ij} + \mathcal{V}_T^{-2}\mathcal{K}^T_i\mathcal{K}^T_j) = G^T_{kl}(\delta_i{}^k - \frac{1}{\mathcal{V}_0}v_0^k\mathcal{K}^{(0)}_i)(\delta_j{}^l - \frac{1}{\mathcal{V}_0}v_0^l\mathcal{K}^{(0)}_j)\ , \tag{8.11}$$

which holds in the background v_0^i. As we will demonstrate in the this section, this identity is one of the crucial ingredients to ensure supersymmetry of the three-dimensional effective action. Furthermore, we have observed in section 7.3 that (8.1) turns out to be invariant under the symmetry

$$W^{(2)} \to W^{(2)} + j\ , \qquad\qquad v^i \to e^{-\alpha^2 j}v^i\ . \tag{8.12}$$

for any scalar function $j = j(v^i)$ that can be space-time dependent. It is conceivable that this scaling invariance persists beyond the α-order testable in the current reduction. It is also interesting to note that one can introduce a potential $\mathcal{W}$ for the connection in (7.44) as

$$\mathcal{W}_j = \partial_j\Big(\frac{\mathcal{W}}{\mathcal{V}}\Big)\ , \qquad\qquad \mathcal{W}(v^i) = \frac{1}{4!}\int_{Y_4} W^{(2)}J^4\ , \tag{8.13}$$

where $J = v^i\omega_i$ contains the fluctuated Kähler moduli.

The scaling symmetry fixes a number of the warp-factor dependent terms in (8.1) and one readily infers a potential $\mathcal{W}$ that appears in these couplings. However, there is one contribution proportional to $\int_{Y_4} W\omega^{(0)}_i \wedge \omega^{(0)}_j \wedge J \wedge J$ that appears to be special. It arises by expanding (7.36)

$$G^W_{ij} = -\frac{1}{2\mathcal{V}}\mathcal{K}_{ij} + \frac{1}{2\mathcal{V}^2}\mathcal{K}_i\mathcal{K}_j - \frac{3}{4\mathcal{V}}\int_{Y_4} W^{(2)}\omega_i \wedge \omega_j \wedge J \wedge J + \frac{3}{2\mathcal{V}^2}\mathcal{K}_{ij}\int W^{(2)} * 1\ , \tag{8.14}$$

where we have used[6]

$$*\,\omega_i = -\frac{1}{2}\omega_i \wedge J \wedge J + \frac{1}{6\mathcal{V}_0}\mathcal{K}_i J^3\ . \tag{8.15}$$

At first, one might have suspected that all terms in (8.14) arise as derivatives of $\mathcal{W}$ as well. However, evaluating[7]

$$\partial_j \int W^{(2)}\omega_i \wedge J^3 = 3!\mathcal{K}_i\mathcal{W}_j + 3!\frac{\mathcal{W}}{\mathcal{V}}\mathcal{K}_{ij}\ , \qquad\qquad \partial_j\mathcal{W}_i = \frac{1}{4!\mathcal{V}_0}\int_{Y_4}(\partial_i\partial_j W^{(2)})J^4\ , \tag{8.16}$$

one infers that there is no term proportional to $\int W^{(2)}\omega_i \wedge \omega_j \wedge J^2$. This is a first example of a situation where one can connect couplings with zero and one index, but new structures arise at the two-index level. We discuss similar issues arising in the higher-derivative sector next.

In order to integrate terms in the higher-derivative sector, one might want to start with the scalar function

$$\mathcal{Z}(v^i) = \frac{1}{4!}\int_{Y_4} ZJ^4 = \int_{Y_4} J \wedge c_3\ . \tag{8.17}$$

[6] Note that this relation only holds for harmonic forms ω_i.

[7] A simple way to show the first identity is to split the integral $\int W^{(2)}\omega_i \wedge J^3 \propto \mathcal{W}\mathcal{K}_i$, by using that $\omega_i \wedge J^3$ is harmonic.

where we have used (5.6) and view $\mathcal{Z}$ as a function of the fluctuated moduli v^i. It is then straightforward to derive

$$\mathcal{Z}_i = \partial_i \mathcal{Z} = \int_{Y_4} \omega_i \wedge c_3 = -2 \int_{Y_4} Z_{m\bar{n}} \omega_i{}^{\bar{n}m} * 1 \ , \tag{8.18}$$

where we again inserted (5.6). Note that when written with the Chern-form c_3 it is obvious that $\mathcal{Z}_i$ is actually constant such that $\partial_j \mathcal{Z}_i = 0$. Thus, in complete analogy to the warping terms, there appears to be no obvious potential that admits the two-index terms

$$\int_{Y_4} Z_{m\bar{n}r\bar{s}} \omega_j^{\bar{n}m} \omega_i^{\bar{s}r} * 1 \ , \qquad\qquad \int_{Y_4} Z \omega_i \wedge \omega_j \wedge J \wedge J \ , \tag{8.19}$$

as derivatives. We will have to address precisely these obstacles when showing the supersymmetry of the effective action in next section.

To close this section let us point out that the two terms in (8.19) are just part of a set of higher-derivative terms of the form

$$\mathcal{X}^{(r)}_{ijkl} = \int_{Y_4} \omega_i^{(0)} \wedge \mathcal{R}_{m_1\bar{m}_1} \wedge \mathcal{R}_{m_2\bar{m}_2} \wedge \mathcal{R}_{m_2\bar{m}_2}\, \omega_i^{(0)\bar{n}_1 n_1} \omega_k^{(0)\bar{n}_2 n_2} \omega_l^{(0)\bar{n}_3 n_3}\, (\mathcal{Y}_{(r)})^{m_1\bar{m}_1 m_2\bar{m}_2 m_3\bar{m}_3}_{n_1\bar{n}_1 n_2\bar{n}_2 n_3\bar{n}_3} \ , \tag{8.20}$$

where the $\mathcal{Y}_{(r)}$ are defined to encode all possible index contractions of m_p with n_q. The two terms in (8.19) arise when contracting a particular set of $\mathcal{X}^{(r)}_{ijkl}$ with v^k and v^l. It would be very interesting to study the properties of such $\mathcal{X}^{(r)}_{ijkl}$. In particular, the variation of these terms with the moduli v^i might uncover interesting relations. Furthermore, it is worth stressing that the terms $\mathcal{X}^{(r)}_{ijkl}$ including the contractions (8.19) depend on the chosen forms ω_i, i.e. not just on the class of ω_i, for all appearing two-forms. In our study the ω_i were always harmonic, but it would be interesting to check if there are linear combinations of the $\mathcal{X}^{(r)}_{ijkl}$ or its v^p contractions that only depend on the cohomology class of the two-forms.

8.2 Demonstrating the supersymmetric structure

In this section we determine the Kähler potential and complex coordinates compatible with $\mathcal{N}=2$ supersymmetry in three dimensions. Our starting point will be the three-dimensional effective action (8.1) obtained by dimensional reduction. We discuss its supersymmetric structure both in the frame when working with vectors A^i and in the dual frame when the vectors are replaced by scalars ρ_i.

8.2.1 Comparing the reduction result with $\mathcal{N}=2$ supergravity

It turns out to be convenient to first work with three-dimensional vector multiplets with bosonic fields (L^i, A^i) and only later switch to chiral multiplets with complex scalars T_i. The kinetic terms of an ungauged $\mathcal{N}=2$ supergravity theory (2.56) propagates the vector multiplets (L^i, A^i) The couplings of the real scalars are given by $\tilde{K}_{ij}$ can be determined from a so-called kinetic potential $\tilde{K}(L)$ via $\tilde{K}_{ij} = \partial_{L^i}\partial_{L^j}\tilde{K}$. Dualising the vector A^i in the vector multiplet one can translate the three-dimensional theory into an action for complex scalars T_i with kinetic terms given by a Kähler potential $K(T,\bar{T})$ given by the action (2.53) where $K_{T_i\bar{T}_j} = \partial_{T_i}\partial_{\bar{T}_j}K$ is the Kähler metric. Note that $\mathrm{Re}T_i$, K and L^i, $\tilde{K}$ are related by a Legendre transform as as discussed in section 2.2.2 and in section 4.2, by

$$T_i = \tilde{K}_{L^i} + i\rho_i \ , \qquad\qquad K = \tilde{K} - \frac{1}{2}(T_i + \bar{T}_i)L^i \ , \tag{8.21}$$

where ρ_i is the three-dimensional scalar dual to the vector A^i. One can now straightforwardly derive that $K_{T_i\bar{T}_j} = -\frac{1}{4}\tilde{K}^{ij}$, which uses the inverse of $\tilde{K}_{ij}$. Note that K is independent of the scalar ρ_i and thus a function $K(\mathrm{Re}T_i)$. It is useful to recall the inverse transformation

$$L^i = -2K_{T_i} \ , \tag{8.22}$$

where $K_{T_i} = \partial_{T_i}K$. The theory formulated in the T_i coordinates can admit a scalar F and D-term potential of the form (2.54) and (2.55).

To read off $\tilde{K}_{ij}$ we compare the action (2.56) with the result from the dimensional reduction (8.1). We first read off the coefficient of the $F^i \wedge *F^j$ term and identify

$$\tilde{K}_{ij}\big| = -\tfrac{1}{2}\mathcal{V}_T^2 G^T_{ij} \ . \tag{8.23}$$

Here we have used the notation $f(v^i)| = f(v^i_0)$, i.e. the vertical dash denotes evaluation in the background setting all fluctuations $\delta v^i = 0$. Supersymmetry implies that for the correct definition of L^i, this metric has to match the one in front of $dL^i \wedge *dL^j$. Applied to (8.1) this implies the relation

$$\mathcal{V}_T^2 G^T_{ij} = (G^T_{kl} + \mathcal{V}_T^{-2}\mathcal{K}^T_k\mathcal{K}^T_l)(\delta^k_h + v^k_0\mathcal{W}^{(0)}_h)(\delta^l_o + v^l_0\mathcal{W}^{(0)}_o)\frac{\partial v^h}{\partial L^i}\bigg|\frac{\partial v^o}{\partial L^j}\bigg| \ , \tag{8.24}$$

where $\mathcal{W}^{(0)}_i = \mathcal{W}_i|$ is defined in (7.44) being evaluated in the background. Then using (8.11) we find that

$$\partial_j L^i| \equiv \frac{\partial L^i}{\partial v^j}\bigg| = \frac{1}{\mathcal{V}_T}\Big(\delta^i_k - \frac{v^i_0}{\mathcal{V}_0}\mathcal{K}^{(0)}_k\Big)\big(\delta^k_j + v^k_0\mathcal{W}^{(0)}_j\big) \ , \tag{8.25}$$

where as above we abbreviate derivatives with respect to v^i as $\partial_i \equiv \frac{\partial}{\partial v^i}$ and $\partial_{i_1}\dots\partial_{i_n}K = K_{i_1,\dots,i_n}$. It turns out that it is complicated to integrate this condition. This can be traced back to the fact that there is an evaluation and, as we discuss below, the fundamental objects to define L^i itself might be more involved. Nevertheless, we can already make some interesting observations. Firstly, the higher-curvature corrections only appear through $\mathcal{V}_T$ in (8.25). One suspects that this can only be true in the background. In fact, we might imagine that $\partial_j L^i$ contains a term

$$\partial_j L^i \supset v^i \int_{Y_4}\big[Z_{m\bar{n}}\omega_j^{\bar{n}m} - 2Z_{m\bar{n}r\bar{s}}\omega_j^{\bar{n}m}\omega_k^{\bar{s}r}v^k\big] * 1 \ , \tag{8.26}$$

which trivially gives zero when evaluated at v^i_0.[8] Terms of this type, however, will turn out to be crucial in order to determine the underlying objects of the theory. In contrast, artificially switching

[8]Note that the Z quantities in (8.26) are dependent on the Riemann tensors build form $g_{m\bar{n}}$ rather of $g^{(0)}_{m\bar{n}}$

off the higher-curvature corrections in (8.25) one finds that the L^i in the presence of warping actually takes the simple form

$$L^i = \frac{v^i}{\mathcal{V}_W} \ , \tag{8.27}$$

where $\mathcal{V}_W$ is the warped volume (7.36) now evaluated as a function of the perturbed v^i.

As a second requirement of supersymmetry we note that (8.21) implies

$$\partial_i \mathrm{Re} T_j \big| = \tilde{K}_{jk} \partial_i L^k \big| \ . \tag{8.28}$$

Using (8.23) and (8.25) we conclude that

$$\begin{aligned} \partial_j \mathrm{Re} T_i \big| \ = \ & \mathcal{K}^{(0)}_{ij} + 3\alpha^2 \mathcal{K}^{(0)}_i \mathcal{W}^{(0)}_j + \frac{3}{2}\alpha^2 \int_{Y_4} W^{(2)} |\omega^{(0)}_i \wedge \omega^{(0)}_j \wedge J^{(0)} \wedge J^{(0)} \\ & -256\alpha^2 \frac{1}{\mathcal{V}_0} \mathcal{K}^{(0)}_{ij} \mathcal{Z}^{(0)} - 1536\alpha^2 \frac{1}{\mathcal{V}_0} \mathcal{K}^{(0)}_j \mathcal{Z}^{(0)}_i \\ & +256\alpha^2 \int_{Y_4} Z \omega^{(0)}_i \wedge \omega^{(0)}_j \wedge J^{(0)} \wedge J^{(0)} + 6144\alpha^2 \int_{Y_4} \omega_i^{(0)\bar{n}m} \omega_j^{(0)\bar{s}r} Z_{m\bar{n}r\bar{s}} *^{(0)} 1 \ , \end{aligned} \tag{8.29}$$

where $K^{(0)}_{ij}$ and $K^{(0)}_i$ are introduced in (2.82) and evaluated at v^i_0.

8.2.2 Kähler potential and coordinates as a δv expansion

In the previous section we have deduced the expressions for $\partial L^i/\partial v^j$ and $\partial \mathrm{Re} T_j/\partial v^i$ when evaluated in the background $v^i = v^i_0$. We will next try to infer directly the coordinates T_i and the Kähler potential K. In order to do this we view T_i and K as being given by an expansion both in α and δv^i by writing

$$\begin{aligned} \mathrm{Re} T_i = \mathrm{Re} T^{(0)}_i + \alpha^2 \mathrm{Re} T^{(2)}_i \,, \qquad & \mathrm{Re} T^{(2)}_i = \mathrm{Re} T^{(2)}_i| + \partial_j \mathrm{Re} T^{(2)}_i | \delta v^j + \frac{1}{2} \partial_j \partial_k \mathrm{Re} T^{(2)}_i | \delta v^j \delta v^k \,, \\ K = K^{(0)} + \alpha^2 K^{(2)} \,, \qquad & K^{(2)} = K^{(2)}| + \partial_j K^{(2)} | \delta v^j + \frac{1}{2} \partial_j \partial_k K^{(2)} | \delta v^j \delta v^k \ . \end{aligned} \tag{8.30}$$

In the following we derive as much information as possible about the coupling functions that appear in this expansion by comparing to the reduction result.

As a first step, recall that the zeroth order result in α was already determined in [103, 81]. With our above expressions one can check that

$$K^{(0)} = -3 \log(\mathcal{V}) \ , \qquad \mathrm{Re} T^{(0)}_i = \mathcal{K}_i \ , \tag{8.31}$$

where now $\mathcal{V}$ and K_i depend on the varying v^i. At the next order in α we note that there are only few objects with zero or one index i that are non-trivial in the background. More precisely, one can write

$$K^{(2)}| = \frac{\mu_1}{\mathcal{V}_0} \mathcal{Z}^{(0)} + \frac{\mu_2}{\mathcal{V}_0} \mathcal{W}^{(0)} \ , \tag{8.32}$$

where $\mathcal{Z}$ and $\mathcal{W}$ are defined in (8.17) and (8.13). The constants μ_1, μ_2 are undetermined at this point. Clearly, the constant shifts in K are unimportant for the derivation of the Kähler metric. However, the form of (8.32) might hint towards the fully moduli-dependent form of K. To fix the coefficients μ_2 one might be inclined to use the scaling symmetry (8.12). Together with the classical form of K one then infers that an invariant K requires $\mu_2 = -12$.

We can proceed similarly for the one-index quantities. We first make an ansatz using all one-index building blocks we have encountered so far by setting

$$\begin{aligned} \mathrm{Re}T_i^{(2)}| &= \tilde{\nu}_1 \mathcal{Z}_i^{(0)} + \tilde{\nu}_2 \mathcal{V}_0 \mathcal{W}_i^{(0)} + \tilde{\nu}_3 \mathcal{K}_i^{(0)} \mathcal{Z}^{(0)} + \tilde{\nu}_4 \mathcal{K}_i^{(0)} \mathcal{W}^{(0)} \ , \\ \partial_i \mathcal{K}^{(2)}| &= \frac{\tilde{\mu}_1}{\mathcal{V}_0} \mathcal{Z}_i^{(0)} + \tilde{\mu}_2 \mathcal{W}_i^{(0)} + \frac{\tilde{\mu}_3}{\mathcal{V}_0} \mathcal{K}_i^{(0)} \mathcal{Z}^{(0)} + \frac{\tilde{\mu}_4}{\mathcal{V}_0} \mathcal{K}_i^{(0)} \mathcal{W}^{(0)} \ . \end{aligned} \tag{8.33}$$

The constant coefficients $\tilde{\nu}_\alpha, \tilde{\mu}_\alpha$ are not determined at this point, since there are no direct relations fixing the background values of T_i and $\partial_i K$. To fix at least some of the coefficients in (8.33) one can again use the symmetry (8.12). Note that T_i are proper complex coordinates that should be invariant under (8.12). This suggests that $\tilde{\nu}_4 = 3$ and $\tilde{\nu}_2 = 0$, where we have used that the leading contribution to T_i is of third power in v^i as in (8.31). In contrast, we note that K should be invariant under (8.12), while $\partial_i K^{(2)}$ should transform as a derivative and therefore contain the connection $\mathcal{W}_i$. Using again the leading form (8.31) and the expression (8.13) one concludes $\tilde{\mu}_2 = -12$ and $\tilde{\mu}_4 = 0$.

In contrast to (8.32) and (8.33) the form of $\partial_j \mathrm{Re}T_i^{(0)}|$ and $\partial_j \mathrm{Re}T_i^{(2)}|$ are fully fixed by the reduction and are trivially read off from (8.30) with

$$\begin{aligned} \partial_j \mathrm{Re}T_i^{(2)}\big| \quad = \quad & 3\mathcal{K}_i^{(0)} \mathcal{W}_j^{(0)} + \frac{3}{2} \int_{Y_4} W^{(2)} | \omega_i^{(0)} \wedge \omega_j^{(0)} \wedge J^{(0)} \wedge J^{(0)} \\ & -256 \frac{1}{\mathcal{V}_0} \mathcal{K}_{ij}^{(0)} \mathcal{Z}^{(0)} - 1536 \frac{1}{\mathcal{V}_0} \mathcal{K}_j^{(0)} \mathcal{Z}_i^{(0)} \\ & +256 \int_{Y_4} Z \omega_i^{(0)} \wedge \omega_j^{(0)} \wedge J^{(0)} \wedge J^{(0)} + 6144 \int_{Y_4} \omega_i^{(0)\bar{n}m} \omega_j^{(0)\bar{s}r} Z_{m\bar{n}r\bar{s}} *^{(0)} 1 \ . \end{aligned} \tag{8.34}$$

All other remaining terms in the expansion (8.33) are also not fully determined by our results obtained from the reduction. However, we can use (8.22) to show that the general relation

$$L^i = -2 \frac{\partial K}{\partial T_i} = -\frac{\partial K}{\partial v^j} \frac{\partial v^j}{\partial \mathrm{Re}T_i} \ , \tag{8.35}$$

together with (8.33) gives

$$\begin{aligned} L^{(2)i} = & -\mathcal{K}^{ij} \partial_j \mathcal{K}^{(2)}| - \frac{1}{\mathcal{V}} v^j \mathcal{K}^{ik} \partial_k T_j^{(2)}| + \mathcal{K}_{jlm} \mathcal{K}^{il} \mathcal{K}^{km} \partial_k \mathcal{K}^{(2)} | \delta v^j \\ & - \mathcal{K}^{ik} \partial_j \partial_k \mathcal{K}^{(2)} | \delta v^j - \frac{1}{\mathcal{V}} \mathcal{K}^{ik} \partial_k T_j^{(2)} | \delta v^j + \frac{1}{\mathcal{V}^2} \mathcal{K}_j v^l \mathcal{K}^{ik} \partial_k T_l^{(2)} | \delta v^j \\ & + \frac{1}{\mathcal{V}} \mathcal{K}_{jmn} \mathcal{K}^{im} \mathcal{K}^{ln} v^k \partial_l T_k^{(2)} | \delta v^j - \frac{1}{\mathcal{V}} \mathcal{K}^{il} v^k \partial_j \partial_l T_k^{(2)} | \delta v^j + \mathcal{O}(\delta v^2) \ . \end{aligned} \tag{8.36}$$

From this it is straightforward to evaluate $\partial_i L^j$ and compare the result with (8.25) in the background

$v^i = v_0^i$. One then infers that the coefficients in (8.33) have to satisfy the relation

$$\begin{aligned}
&\partial_i\partial_j \mathrm{Re}T_k^{(2)} v^k| - \mathcal{V} K_{ijk} K^{kl} \partial_l K^{(2)}| + \mathcal{V}\partial_j\partial_k K^{(2)}| \\
&\quad = 9\frac{1}{\mathcal{V}_0}\mathcal{K}_{ij}^{(0)}\mathcal{W}^{(0)} + 18\mathcal{V}_0\mathcal{W}_{(i}^{(0)}\mathcal{K}_{j)}^{(0)} + 12\mathcal{V}_0\mathcal{K}_{ijk}^{(0)}\mathcal{K}^{(0)kl}\mathcal{W}_l^{(0)} - \frac{3}{2}\int W^{(2)}|\omega_i^{(0)}\wedge\omega_j^{(0)}\wedge J^{(0)}\wedge J^{(0)} \\
&\quad - 256\int Z\omega_i^{(0)}\wedge\omega_j^{(0)}\wedge J^{(0)}\wedge J^{(0)} + 3072\frac{1}{\mathcal{V}_0}\mathcal{K}_{(i}^{(0)}\mathcal{Z}_{i)}^{(0)} - 1536\frac{1}{\mathcal{V}_0^2}\mathcal{K}_i^{(0)}\mathcal{K}_j^{(0)}\mathcal{Z}^{(0)} \\
&\quad - 6144\int \omega_i^{(0)\bar{n}m}\omega_j^{(0)\bar{s}r} Z_{m\bar{n}r\bar{s}} *^{(0)} 1 + 1536\mathcal{K}_{ijk}^{(0)}\mathcal{K}^{(0)kl}\mathcal{Z}_l^{(0)}
\end{aligned} \tag{8.37}$$

Imposing these conditions then implies that we match the metric (8.23). Note that this analysis can be carried out independent of any gauge fixing of the scaling symmetry (8.12). Also note that our first-order analysis does neither uniquely fix the Kähler coordinates nor the Kähler metric. This can be traced back to the fact that we performed the dimensional reduction only to leading order in the fluctuations δv^i.

In order to fix the coefficients in (8.33) further, one can try to impose conditions that might hold also at the higher-derivative level. For example, one may suspect that a no-scale condition holds even when including α-corrections to the action. In three space-time dimensions such a condition reads

$$K_{T_i} K^{T_i\bar{T}_j} K_{\bar{T}_j} = 4\ . \tag{8.38}$$

It ensures that in the scalar potential (2.54) the negative $-4|W|^2$ term cancels for a superpotential independent of T_i. Using (8.22) and $K^{T_i\bar{T}_j} = -4\tilde{K}_{ij}$ one rewrites (8.38) as

$$L^i \tilde{K}_{ij} L^j = -4\ . \tag{8.39}$$

In the background this expression can be evaluated by using (8.36) together with (8.34) to yield the condition[9]

$$\partial_i K^{(2)} v^i| = 2304\frac{1}{\mathcal{V}_0}\mathcal{Z}^{(0)} - 12\mathcal{W}_i^{(0)} v_0^i\ . \tag{8.41}$$

Keeping in mind that we have few objects with zero or one index, one can use this condition as a further motivation to make an ansatz for the Kähler potential and match the coefficients. This will be considered in the following section.

8.2.3 Completing the Kähler potential and complex coordinates

In this final subsection we comment on the completion of the Kähler potential and complex coordinates as a closed expression in Kähler deformations. Our goal is to replace the δv^i-expansion (8.33) with an appropriate ansatz hinting towards the underlying structure of the higher-derivative reduction. It

[9] We note also that a similar set constraints $\mathrm{Re}T_i\mathrm{Re}T_j G^{ij}| = L^i L^j G^{-1}_{ij}| = L^i\mathrm{Re}T_i| = 4$ and $\partial_k(L^i\mathrm{Re}T_i)| = 0$ can all be satisfied if we demand (8.41) as well as

$$\mathrm{Re}T_i^{(2)}| = \partial_i K^{(2)}| - \frac{1}{3}\mathcal{K}_i\partial_j K^{(2)} v^j| + 12\mathcal{W}_i^{(0)} + 3\mathcal{K}_i^{(0)}\mathcal{W}^{(0)} - 4\frac{1}{\mathcal{V}_0}\mathcal{K}_i^{(0)}\mathcal{W}_j^{(0)} v_0^j + 256\frac{1}{\mathcal{V}}\mathcal{K}_i^{(0)}\mathcal{Z}^{(0)}\ . \tag{8.40}$$

should be stressed that we are only able to fully justify the leading terms. However, we will also discover an intriguing interplay between warping effects and higher-curvature terms.

To begin with, let us propose an ansatz for the Kähler potential. We have noted in (8.32) that there are only few objects without indices. Using the quantities introduced in (8.13) and (8.17) we suggest

$$\begin{aligned} K &= -3\log\Big(\int_{Y_4} e^{4\alpha^2 W} * 1 + 256\mu\alpha^2 \int_{Y_4} Z * 1\Big) \\ &= -3\log(\mathcal{V} + 256\mu\alpha^2\mathcal{Z} + 4\alpha^2\mathcal{W} + \mathcal{O}(\alpha^4)) \ , \end{aligned} \tag{8.42}$$

where the functions that appear are now viewed as being dependent on the fields v^i. In this expression we fixed the factor in front of $\mathcal{W}$ by the fact that K has to be invariant under the symmetry (8.12). The factor in front of the $\mathcal{Z}$ term is not fixed a priori and we have introduced the constant μ to capture this freedom. Let us stress that it is straightforward to compute the v^i derivatives of K as defined in (8.42). In particular, one finds

$$\partial_i K = -3\frac{1}{\mathcal{V}}K_i + 768\mu\alpha^2\frac{1}{\mathcal{V}}\mathcal{Z}K_i - 768\mu\alpha^2\frac{1}{\mathcal{V}}\mathcal{Z}_i - 12\alpha^2\mathcal{W}_i \ . \tag{8.43}$$

Clearly, in order to compute the actual Kähler metric we also have to supplement an ansatz for the complex coordinates T_i. The involved form of the Kähler metric determined from the dimensional reduction (8.1) and the rather simple form of the Kähler potential (8.42) as a function of the v^i suggests that the T_i have to capture most of the non-trivial information about the $\mathcal{N}=2$ system.

To get some intuitive information about T_i, we note that these coordinates are expected to linearise the action of M5-brane instantons on divisors D_i. In fact, as discussed in [148] a holomorphic superpotential of the schematic form $W \propto e^{-T_i}$ can be induced by such instanton effects. This implies that the T_i are expected to be integrals over divisors D_i. We therefore suggest that they take the form

$$T_i = \int_{D_i}\Big(\frac{1}{3!}e^{3\alpha^2 W^{(2)}} J\wedge J\wedge J + 1536\alpha^2 F_6\Big) + i\rho_i \ , \tag{8.44}$$

where D_i are $h^{1,1}(Y_4)$ divisors of Y_4 that span the homology $H_2(Y_4,\mathbb{R})$. The six-form F_6 in this expression is a function of degrees of freedom associated with the internal space metric and will be responsible for the more complicated higher-derivative structures (8.19). It is constrained by a relation to the fourth Chern form c_4 such that F_6 determines the non harmonic part of c_4 as

$$c_4 = Hc_4 + i\partial\bar{\partial}F_6 \ . \tag{8.45}$$

This is in analogy to the quantity F_4 introduced representing the non-harmonic part of c_3. Note that (8.45) leaves the harmonic and exact part of F_6 unfixed and we will discuss constraints on these pieces in more detail below. The justification of the first term in ReT_i is simpler. It captures the warped volume of an M5-brane wrapped on D_i. In fact, the power of the warp-factor turns out to be appropriate to ensure invariance under the scaling symmetry (8.12), in accord with the expectation that T_i is invariant under this symmetry. Remarkably, this definition of the Kähler coordinates as D_i integrals will help us to obtain the couplings $\int e^{3\alpha^2 W^{(2)}} J\wedge J\wedge\omega_i\wedge\omega_j$, which, as we stressed in subsection 8.1.1, cannot be obtained as v^i-derivatives of the considered Y_4-integrals. Note that the following discussion of the warping is inspired by [135]. Here we will adapt and extend the arguments of [135] and include the higher-curvature pieces. Interestingly they turn out to complete the analysis in an elegant and non-trivial fashion.

In order to evaluate the derivatives of T_i with respect to v^i and to make contact with the Kähler metric found in (8.1), we have to rewrite the integrals over D_i into integrals over Y_4. Due to the appearance of the warp-factor and the non-closed form F_6 in (8.44) this is not straightforward. In particular, one cannot simply use Poincaré duality and write T_i as an integral over Y_4 with inserted ω_i. Of course, it is always possible to write T_i as a Y_4 integral when inserting a delta-current localised on D_i, i.e.

$$\mathrm{Re}T_i = \int_{Y_4} \Big(\frac{1}{3!}e^{3\alpha^2 W^{(2)}} J\wedge J\wedge J + 1536\alpha^2 F_6\Big)\wedge \delta_i \ , \tag{8.46}$$

where δ_i is the (1,1)-form delta-current that restricts to the divisor D_i. Appropriately extending the notion of cohomology to include currents [64, 149], we can now ask how much δ_i differs from the harmonic form ω_i in the same class. In fact, any current δ_i is related to the harmonic element of the same class ω_i by a doubly exact piece as

$$\delta_i = \omega_i + i\partial\bar\partial\lambda_i \ . \tag{8.47}$$

This equation should be viewed as relating currents. Importantly, as we assume D_i and hence δ_i to be v^i-independent, the v^i dependence of the harmonic form ω_i and the current λ_i has to cancel such that $\partial_j\omega_i = -i\partial\bar\partial\partial_j\lambda_i$. Importantly, once we determine $\partial_j\mathrm{Re}T_j$ we can express the result as Y_4-integrals without invoking currents. We therefore need to understand how each part of T_i varies under a change of moduli. This will also fix the numerical factor in front of F_6 in (8.44).

In order to take derivatives of T_i we first use the fact that D_i and hence δ_i are independent of the moduli v^i, which implies

$$\partial_j\mathrm{Re}T_i = \int_{Y_4}\Big(\frac{1}{2}e^{3\alpha^2W^{(2)}}\omega_j\wedge J\wedge J + \frac{1}{2}\alpha^2\partial_j W^{(2)} J\wedge J\wedge J + 1536\alpha^2\partial_j F_6\Big)\wedge\delta_i \ . \tag{8.48}$$

We next claim that we can replace δ_i with ω_i such that finally

$$\partial_j\mathrm{Re}T_i = \frac{1}{2}\int_{Y_4} e^{3\alpha^2W^{(2)}}\omega_i\wedge\omega_j\wedge J\wedge J + \frac{1}{2}\alpha^2\int_{Y_4}\partial_j W^{(2)}\omega_i\wedge J\wedge J\wedge J + 1536\alpha^2\int_{Y_4}\omega_i\wedge\partial_j F_6 \ . \tag{8.49}$$

Note that by using (8.47) the two expressions (8.48) and (8.49) only differ by a term involving $\partial\bar\partial\lambda_i$. By partial integration this term is proportional to

$$\begin{aligned}&\int_{Y_4}\lambda_i\partial\bar\partial\Big(\frac{1}{2}e^{3\alpha^2W^{(2)}}\omega_j\wedge J\wedge J + \frac{1}{2}\alpha^2\partial_j W^{(2)} J\wedge J\wedge J + 1536\alpha^2\partial_j F_6\Big)\\ =\ &\int_{Y_4}\lambda_i\Big(\frac{1}{2}\partial\bar\partial(e^{3\alpha^2W^{(2)}})\omega_j\wedge J\wedge J + \frac{1}{2}\alpha^2\partial\bar\partial(\partial_j W^{(2)}) J\wedge J\wedge J + 1536\alpha^2\partial\bar\partial\partial_j F_6\Big) \ .\end{aligned} \tag{8.50}$$

It is now straightforward to see that the terms multiplying λ_i are simply the ∂_j derivative of the warp-factor equation (7.7). One first writes (7.7) as

$$d^\dagger d e^{3\alpha^2W^{(2)}} *_8 1 - \alpha^2 Q_8 = -\frac{1}{3}i\partial\bar\partial(e^{3\alpha^2W^{(2)}})\wedge J\wedge J\wedge J - \alpha^2 Q_8 \ . \tag{8.51}$$

Then one takes the v^j-derivative of (8.51) by using the fact that Q_8 is given via (7.8) and (8.45). The moduli dependence of Q_8 only arises from the term involving F_6, i.e. one has $\partial_i Q_8 = 3072\, i\, \partial\bar{\partial}\partial_i F_6$. Hence one finds exactly the terms in (8.50) such that this λ_i dependent part of the T_i variation vanishes due to the warp-factor equation (7.7).

The final expression (8.49) is written using (2.82) and (8.13) as

$$\partial_j \mathrm{Re} T_i = \frac{1}{2}\int_{Y_4} e^{3\alpha^2 W^{(2)}} \omega_i \wedge \omega_j \wedge J \wedge J + 3\alpha^2 \mathcal{K}_i \mathcal{W}_j + 1536\alpha^2 \int_{Y_4} \omega_i \wedge \partial_j F_6 \ . \tag{8.52}$$

The L^i coordinates are then computed using (8.35) by inserting (8.43) and (8.52). This gives the result

$$L^i = \frac{v^i}{\mathcal{V}} - \alpha^2 \frac{v^i}{\mathcal{V}^2}(3\mathcal{W} + 256\mu\mathcal{Z}) + 1536\alpha^2 \frac{K^{ij}}{\mathcal{V}}\Big(\mathcal{Z}_j - \int_{Y_4} J \wedge \partial_j F_6\Big) \ . \tag{8.53}$$

It is then straightforward to derive

$$\begin{aligned}
\partial_j L^i =& \frac{\delta^i_j}{\mathcal{V}} - \frac{v^i K_j}{\mathcal{V}^2} - \frac{\delta^i_j}{\mathcal{V}^2}(3\mathcal{W} + 256\mu\mathcal{Z}) - \frac{1}{\mathcal{V}} v^i (3\mathcal{W}_j + 256\mu\mathcal{Z}_j) + \frac{1}{\mathcal{V}^3}\mathcal{K}_j v^i (3\mathcal{W} + 512\mu\mathcal{Z}) \\
& - \alpha^2 \frac{1}{\mathcal{V}} 768\mu \mathcal{K}^{im}\mathcal{K}^{kn}\mathcal{K}_{mnj}\mathcal{Z}_k - \alpha^2 \frac{1}{\mathcal{V}_0^2} 768\mu\mathcal{K}^{ik}\mathcal{K}_j\mathcal{Z}_k \\
& + \alpha^2 \frac{1}{\mathcal{V}} 1536 K^{im}\mathcal{K}^{kn}\mathcal{K}_{mnj}\int_{Y_4} J \wedge \partial_k F_6 + \alpha^2 \frac{1}{\mathcal{V}^2} 1536\mathcal{K}^{ik}\mathcal{K}_j \int_{Y_4} J \wedge \partial_k F_6 \\
& - \alpha^2 \frac{1}{\mathcal{V}} 1536\mathcal{K}^{ik}\int_{Y_4} \omega^{(0)}_j \wedge \partial_k F_6 - \alpha^2 \frac{1}{\mathcal{V}} 1536 \mathcal{K}^{ik}\int_{Y_4} J \wedge \partial_j \partial_k F_6
\end{aligned} \tag{8.54}$$

This allows to determine the derivatives of F_6 by comparing (8.25) and (8.30) with (8.54) and (8.52). We find that

$$\begin{aligned}
\int_{Y_4} \omega_i \wedge \partial_j F_6| =& 4\int_{Y_4} Z_{m\bar{n}r\bar{s}}\omega_i^{(0)\bar{n}m}\omega_j^{(0)\bar{s}r} *^{(0)} 1 + \frac{1}{3!}\int_{Y_4} Z\omega^{(0)}_i \wedge \omega^{(0)}_j \wedge J^{(0)} \wedge J^{(0)} - \frac{K_{ij}}{3!\mathcal{V}_0}\mathcal{Z}^{(0)} - \frac{1}{\mathcal{V}_0}\mathcal{K}^{(0)}_j \mathcal{Z}^{(0)}_i \\
\int_{Y_4} J \wedge \partial_i \partial_j F_6| =& -4\int_{Y_4} Z_{m\bar{n}r\bar{s}}\omega_i^{(0)\bar{n}m}\omega_j^{(0)\bar{s}r} * 1 - \frac{1}{3!}\int_{Y_4} Z\omega^{(0)}_i \wedge \omega^{(0)}_j \wedge J^{(0)} \wedge J^{(0)} \\
& - \mu\frac{1}{3!\mathcal{V}_0}\mathcal{K}^{(0)}_{ij}\mathcal{Z}^{(0)} - (1-\mu)\frac{1}{\mathcal{V}_0^2}\mathcal{K}^{(0)}_i\mathcal{K}^{(0)}_j\mathcal{Z}^{(0)} + (2-\mu)\frac{1}{\mathcal{V}_0}\mathcal{K}^{(0)}_{(i}\mathcal{Z}^{(0)}_{j)} + \frac{1}{2}(2-\mu)\mathcal{K}^{(0)}_{ijk}\mathcal{K}^{(0)kl}\mathcal{Z}^{(0)}_l \ ,
\end{aligned} \tag{8.55}$$

in order for the results to match. This implies that the Kähler potential (8.42) and coordinates (8.44) yield the metric matching with the reduction result.

The result (8.55) still depends on the free parameter μ introduced in the Kähler potential (8.42). Clearly, one expects that such a freedom is not fundamental, but rather due to the fact that we are only able to partially check the result. A dimensional reduction including fluctuations to higher-order is likely fixing μ unambiguously. Alternatively, we can impose the no-scale condition (8.38), which we presume persists at higher-curvature level. This implies that $\mu = 1$.

Let us note that the definition contains two ambiguities. Firstly, we did not specify the divisor basis D_i spanning $H_2(Y_4, \mathbb{R})$. This can be shifted by a boundary of a seven-chain Γ_i without changing the class as

$$D_i \;\rightarrow\; D_i + \partial\Gamma_i \ . \tag{8.56}$$

This would result in a different choice for the currents δ_i and j_i in (8.47). The result is a modification of the $\mathcal{N}=2$ coordinates T_i given in (8.46). However, as we have shown above, only the harmonic representative of the class enters in the variation $\partial_j T_i$, while j_i drops out due to the warp-factor equation. In other words, the transformation (8.56) is actually a symmetry of the Kähler metric. Secondly, the constraint (8.45) is invariant under shifts of F_6 by six-forms η_6, which get annihilated by the derivatives. In other words, one might transform

$$F_6 \;\rightarrow\; F_6 + \eta_6 \ , \qquad \bar\partial\eta_6 = \partial\eta_6 = 0 \ . \tag{8.57}$$

Clearly, this transformation will in general not respect (8.55). These conditions, however, constrain only the harmonic part of F_6 and allow for the symmetry

$$F_6 \;\rightarrow\; F_6 + d\tilde\eta_4 \ . \tag{8.58}$$

It would be interesting to investigate the implication of the symmetries (8.56) and (8.58) in greater detail. This is particularly interesting when including a superpotential explicitly depending on the coordinates T_i.

The presence of the F_6 term in (8.44) implies, by the above relationship between T_i and the action of a probe M5-brane on D_i, that higher-derivative corrections are relevant in the M5-brane action. Corrections of this type are also required for gravitational anomaly cancellation [150, 151, 152] for an M5-brane in the background of eleven-dimensional supergravity. From this anomaly analysis additional metric dependent contributions to the M5-brane action that are related to certain topological classes are expected, in a way similar to the relationship between F_6 and c_4. In future work it would be interesting to see if this analysis can be used to infer a more direct definition of the F_6 part of the correction in (8.44) and so prove the constraints (8.55) that are necessary in our analysis.

9 Conclusions & outlook

9.1 Summay and conclusions

The study of higher-derivative corrections to M-theory and the induced corrections to $3d$ and $4d$ effective physics upon compactification is of phenomenological as of conceptual interest. The declared goal of this research program, which is the common theme throughout this work is to understand the compactification of the complete bosonic sector of eleven-dimensional supergravity including $l_M^6 \sim \alpha^2$ corrections, to three dimensions.[10] In a second step we aspire to lift the action to $4d$ using the M/F-theory duality. Let us conclude on the status of this endeavor by taking into consideration the results of this work. The α^2-corrections in eleven dimensions carry eight derivatives and thus upon reduction not only correct the couplings of the two derivative external quantities, but as well give rise to higher-derivative corrections in $3d$ and $4d$. However, the main focus was devoted to the induced corrections with two external derivatives, in particular the arising vector multiplet (v^i, A^i) in $3d$. We were led to consider geometric backgrounds, where the internal manifold is a Calabi-Yau or a conformally Calabi-Yau fourfold, restricting ourselves to the case $h^{2,1} = 0$. Our approach in this work was twofold. Firstly, in chapter II we considered a simplified setup by omitting $(\hat{\nabla}\hat{G})^2\hat{R}^2$ structures and compactifying on a direct product space $M_3 \times Y_4$, with Y_4 the internal Calabi-Yau manifold, which reflects our early work on the subject [101, 102]. Remarkably, the corrections to the Kähler metric read off from the couplings of the $3d$ vectors integrated into a Kähler potential by making the assumption that $c_3^{(0)}$, the third Chern form, is harmonic. However, this assumption might not hold in general such that this amounted to neglecting a single contribution proportional to the non-harmonic part of $c_3^{(0)}$. Finally, we lifted the corrections proportional to the third Chern class to $4d$ using the M/F-theory duality. However, although we were able to push the program until the end this setup has to be considered a toy model, and one can not draw definite conclusions from its results. Therefore we refer the reader to section 5.4, where we discuss the characteristics of the simplified setup, but will not comment on it here.

Secondly, chapter III represented the work done in a series of three publications [121, 120, 122], discussed in sections 6, 8 and 7, respectively. We presented the dimensional reduction of eleven-dimensional supergravity including the full set of eight-derivative terms on a seemingly supersymmetric background, giving rise to a $3d$ effective action compatible with $\mathcal{N} = 2$ supersymmetry.

When considering higher-derivative corrections to eleven-dimensional supergravity it can be inferred that the background $M_3 \times Y_4$ is not a supersymmetric solution, but instead it must be warped and include non-vanishing fluxes [153]. Checking a background solution of the eleven-dimensional E.O.M.'s for supersymmetry is done by comparing it to the solution of the Killing spinor equations derived from the gravitino variations. However, the gravitino variations in $11d$ are not known at the order α^2, which led us to argue indirectly for supersymmetry by giving necessary conditions. More concretely, in section 6 we solved for the $11d$ E.O.M.'s arising at order α^2 and found the internal metric

[10] As we pointed out the completeness is a conjecture by [60].

background to be a conformally Kähler manifold with vanishing first Chern class, but a non-Ricci-flat metric. The conformal rescaling includes the warp-factor, which in the background is of order α^2 as well as another α^2 correction $\mathcal{Z} = \int_{Y_4} c_3^{(0)} \wedge J^{(0)}$, where the third Chern form $c_3^{(0)}$ and the Kähler form were evaluated in terms of the Ricci-flat zeroth order Calabi-Yau metric of Y_4. The deviation at order α^2 from the zeroth order Ricci-flat metric is due to the in general non-harmonicity of the third Chern-form.

The argument for supersymmetry of this background was indirect. Let us shortly discuss a constructive approach towards it. A solution with four real supercharges requires the existence of a no-where vanishing background complex Weyl spinor. One can uniquely fix a Killing spinor equation by compatibility with the background solution and the E.O.M.'s, which reversely led us to infer a conjecture for the eleven-dimensional gravitino variations with higher curvature terms, based on [141, 142]. We then compactified the proposed gravitino variations on the warped background solution, which naturally yielded the desired Killing spinor equations and led us conclude that the warped background admits a globally defined real two-form J and complex four-form Ω. The Killing spinor equations translated into first order differential constraints on J, Ω, with only $d\Omega' = \overline{\mathcal{W}}_5 \wedge \Omega'$ non-vanishing for an exact one-form $\overline{\mathcal{W}}_5$, after separating the warp-factor. We conclude that the desirable check of supersymmetry of the proposed solution and the completeness of the gravitino variations is still missing, however it could be provided by a tedious Noether coupling procedure. However, dimensional reduction of M-theory at order α^2 on the warped background with fluxes to a $3d, \mathcal{N} = 2$ would give further evidence for its correctness. Note that a proof of supersymmetry of proposed background would not directly imply that the asserted gravitino variations in eleven dimensions are complete, since our discussion could miss terms which vanish on the background and are thus never seen by our indirect derivation.

In section 8 we discussed the dimensional reduction on the α^2-corrected background and included all relevant corrections at order α^2 to the eleven-dimensional supergravity. Note that we discarded certain known corrections from our analysis since they would have contributed to more than two external derivative terms at order α^2, or to $\mathcal{O}(\alpha^3)$ two derivative corrections. This was due to the fact that the background flux was exactly of order α, hence $G \sim \alpha + \mathcal{O}(\alpha^3)$ and thus $11d$ quantities with more than two $\hat{G}$-forms fields could be safely neglected. However, we did not discuss the complete $3d$ field-content but rather focused on the Kähler moduli fields and the vectors arising form the three-form field strength $\hat{C}$, forming a vector multiplet in $3d$. We as well did not vary the metric w.r.t. the complex structure deformations and we premised that $h^{2,1} = 0$. The first step in this procedure is to determine the light field content, which can be done by solving the higher-dimensional E.O.M.'s and detecting the corrected Ansatz for the decomposition of the higher-dimensional fields in terms of internal and external components. In the zeroth order in α analysis this can be done straightforwardly and one finds that one massless vector and one massless scalar arises for every harmonic $(1, 1)$ -form of the internal Calabi-Yau space from the three-form $\hat{C}$ and the Kähler deformations of the metric, respectively. However, as we argued the expansion in harmonic $(1, 1)$-forms for the vectors and scalars receive α^2-corrections in form of double exact pieces, which was inferred from their closure put upon them by the Bianchi identity for $\hat{G}$ and by closure of the Kähler form, respectively. Note that this was only a necessary condition for the corrected mode to give rise to massless scalars, nevertheless this could be verified by computing the scalar potential. We then performed the dimensional reduction, using this new α^2-corrected modes, carefully expanding the action order by order in α, we inferred the $3d$ effective action. Note that we varied the higher-curvature corrections w.r.t. to the Kähler deformations, which is a novel step in the literature of string theory reductions. We concluded that the α^2 contributions to the modified light modes eventually canceled in the effective action and hence did not contribute at all. Furthermore, we found that the kinetic couplings of the Kähler deformations and vectors in the three-dimensional effective theory at order α^2 could be expressed using a single higher-curvature building block $Z_{m\bar{m}n\bar{n}} = \frac{1}{4!}(\epsilon_8\epsilon_8 R^{(0)3})_{m\bar{m}n\bar{n}}$ and the warp-factor $W^{(2)}$. Note that $R^{(0)}$ is the internal Riemann tensor in the zeroth order Calabi-Yau metric. The introduced building block $Z_{m\bar{m}n\bar{n}}$ carries interesting semi-topological features as it has same symmetries as the Riemann tensor, it contracts with $R^{\bar{m}m\bar{n}n}$ to the Hodge-dual of the fourth Chern-form, and contracting any of the index pairs with the metric one finds expressions in terms of the third Chern-form. Despite its direct connection to these topological quantities it seems that this structure has not appeared in mathematics so far and it remains interesting to explore if it plays a special role in describing the topology of a compact eightfold.

Furthermore we consistently incorporated the warp-factor in this dimensional reduction taking into account its interplay with higher-derivative terms and fluxes. Since the warp-factor is sourced at α^2 from the higher-derivative corrections and the fluxes, this setup can solely be discussed correctly when considering all the ingredients at the same time. In the general discussion of section 8 the warp-factor modifications to the $3d$ effective theory were significantly more involved than it had previously been argued in literature. Nevertheless we managed to transform the effective action to a remarkably simple form by introducing a warped volume, which was simple the volume measure weighted with the exponential of the warp-factor, and a covariant derivative for the scalars involving the variation of the warp-factor w.r.t. to the Kähler deformations of the metric. In these new variables the non-trivial scaling symmetry of the effective action, induced by rescaling of the warp-factor by a field-dependent function, became manifest.

Note that the conditions on the mode expansion did actually not guarantee to give rise to massless fields and hence one might doubt if these are the appropriate light degrees of freedom. Thus finally, we provided an inquiry by deriving the scalar potential for the Kähler deformations. At first, by reducing the α^2 higher-curvature terms on the zeroth order Calabi-Yau background, mass terms were generated with a coupling depending on the scalar Laplacian of Z, the twofold metric contraction of the building block $Z_{m\bar{m}n\bar{n}}$. Intriguingly, this mass terms were precisely canceled by the contribution from the Einstein-Hilbert term arising from the α^2-corrections to the background as a back-reaction effect, which led us to conclude that the remaining scalar potential was only induced by background fluxes as known in literature [81]. We conclude that this strongly supports the claim that the discussed fluctuations indeed represent the relevant light degrees of freedom and as well provides an intriguing example of the interplay of α^2 back-reaction effects to the solution and α^2-corrections to the effective theory.

Finally, in section 8 we turned our focus to the study of the $\mathcal{N}=2$ characteristics of the resulting $3d$ effective theory for the vector multiplet (L^i, A^i), where we obtained $L^i(v^i)$. In three dimension one can dualize a vector to a real scalar $A^i \to \text{Im}T_i$, while the Kähler moduli are identified with the real part $L^i \to \text{Re}T_i$, which transforms the vector multiplets into complex scalars T_i in chiral multiplets, referred to as complex coordinates. We then determined compatibility with the reduction result with a proposal for the $\mathcal{N}=2$ Kähler potential K and complex coordinates T_i, which were exact to all orders in the fluctuations. Note that in order to do this one needs to uplift the couplings of the chiral fields, in principal done by expressing them in terms of topological building blocks, which thus results in moduli space independent quantities in the Kähler metric. Furthermore these need to arise from a Kähler potential upon twofold differentiation w.r.t. T_i and $\bar{T}^j$. At the classical level α^0 it is well-known that the Kähler metric, the Kähler potential, and the coordinates T_i can be expressed by intersection numbers of Y_4. The uplift of the α^2-corrected couplings was more involved.

In the case of the warp-factor $W^{(2)}$ dependent corrections to the kinetic terms we performed the uplift by expressing the complex coordinates in terms of divisor integrals. Let us comment on this in a bit more detail. The kinetic couplings of the three-dimensional effective action derived from dimensional reduction contain non topological integrals like $\int_{Y_4} W^{(2)}\omega_i^{(0)} \wedge \omega_j^{(0)} \wedge J \wedge J$, which depends on the actual choice of representative $(1,1)$-forms in the class $[\omega_i]$, where $\omega_i^{(0)}$ is the harmonic representative, w.r.t. the lowest order Calabi-Yau metric. It seemed hard to integrate this expression into complex coordinates or in a Kähler potential, since as we argued there exists no obvious integral over Y_4 with only one free Kähler-index that yields the above integral upon taking a v^i derivative. Remarkably, we could resolve this obvious obstruction by defining T_i to be given by integrals over divisors D_i. Our key observation was that the v^i-derivatives of the warp-factor equation allows us to express v^i-variations of the complex coordinates $\partial_j T_i$ as Y_4-integrals. Furthermore, this v^j-derivative of T_i was argued to only depend on the homology class of the divisor D_i and not the precise representative. This might suggest that for a proper treatment of effective actions resulting from warped reductions, one may be led to loosen the stringent constraint of considering only topological integrals to also allow for "semi-topological" integrals up to usage of the warp-factor equation. Let us emphasize that since the warp-factor equation naturally contains α^2 higher-curvature terms, the above treatment shows that the discussion of the warp-factor and higher-derivative terms cannot be disentangled.

However, treating the higher-derivative corrections to the couplings composed of objects built from $Z_{m\bar{m}n\bar{n}}$ analogously, was very challenging. As for the $W^{(2)}$ correction, not all the appearing structures were of topological nature but a further difficulty arose due to the explicit appearance of the metric in the Riemann tensors in $Z_{m\bar{m}n\bar{n}}$, which led us to follow an indirect route for these couplings since it seemed very challenging to integrate them into a Kähler potential. Note that the findings of section 8 were only at lowest order in the fluctuations δv^i, which thus naturally drew us to determine K, T_i in terms of a δv^i-expansion. The problematic non-topological metric-dependent integrals, which appeared in the couplings given by $\int_{Y_4} Z\omega_i^{(0)} \wedge \omega_j^{(0)} \wedge J \wedge J$ and $\int_{Y_4} \omega_i^{(0)\bar{n}m}\omega_j^{(0)\bar{s}r} Z_{m\bar{n}r\bar{s}} * 1$, should be arising from a Kähler potential. We conjectured an all order Kähler potential and complex coordinates T_i and then expanded it in δv^i variations around the background and showed compatibility with the reduction result. The proposed expression for the Kähler potential contained the warp-factor and

$\mathcal{Z} = \int c_3 \wedge J$. While the form of T_i was severely constraint by the warp-factor equation it was given in terms of integrals of the divisor D_i containing the warped volume of D_i, and in a second contribution a six-form F_6, which contains the information about the non-harmonicity of the fourth Chern-form c_4. The compatibility with the reduction results bottom-up fixed F_6 to obey the constraints (8.55). Note that the definition of F_6 (8.45) as parametrizing the non-harmonicity of c_4 allows for shifts of harmonic six-forms, which will in general modify T_i and cannot be a symmetry of the system. It would thus be desirable to give an independent top-down definition of F_6, which naturally needs to be compatible with its relation to c_4, but furthermore allows us to study its moduli dependence such that one can verify the constraints obtained by the match with the reduction. Furthermore, let us note that due to the relationship between T_i and the action of a $M5$-probe brane on the divisor D_i, the presence of the F_6 correction suggests the relevance of this higher-derivative correction to the $M5$-brane action.

Let us close the discussion of chapter III by providing a few further comments. The attempted program of a supersymmetric compactification of M-theory to three dimensions including the full set of α^2-corrections meets to major obstructions. Firstly, the lack of knowledge of the $11d$ gravitino variations at order α^2 and secondly, the integration of non-topological couplings in the $3d$ theory into a Kähler potential. By resolving the latter issue, which we reduced to the problem of understanding the moduli dependence of F_6, one may be able to complete the analysis and show $\mathcal{N} = 2$ features of the obtained effective $3d$ theory. One would then conclude that the proposed background in section 6 is indeed supersymmetric and give justification to the asserted gravitino variations. However, we should note that the study of higher-curvature corrections to the effective theory led us to introduce new higher-derivative building blocks and it cannot be excluded that our analysis has to be completed with a yet missing set of those, as we comment on in the next section 9.2.

Let us comment on the connection between the findings of chapter II where we have studied the simplified setup, and the remnants of $\mathcal{N} = 2$ supersymmetry obtained by considering the complete warped reduction including fluxes of chapter III. The connection between the correction to the real part of T_i given by $\mathcal{Z} = \int c_3 \wedge \omega_i$ and $\mathcal{K}_i\mathcal{Z}$, as discussed in chapter II, and the divisor integral formulation of the corrections to the complex coordinates T_i in chapter III, is not immediate. A further discussion is required to connect these results. However, in the full setup of chapter III one still obtains the $\mathcal{Z}$-correction to the Kähler potential. This program is at a too early stage to give the dependence of the Kähler potential in terms of the complex coordinates T_i. However, it occurs to us that a solution of this is in sight, which would allow to answer the phenomenological very relevant question of a functional change of the Kähler potential. In this context the F-theory lift of the correction $\mathcal{Z}$ may become very relevant. Nevertheless, it is anyway worthwhile to comment on the F-theory lift of the $\mathcal{Z}, \mathcal{Z}_i$-corrections, due to their role in chapter II. We inferred in section 5, that in the weak-coupling limit the α'^2-corrections are sourced by the self-intersection curve of each D7-brane present in the background. We further argued that the corrections are due to open string diagrams and that it relies on having D7-branes with proper self-intersections. Note that the corrections vanish in the absence of D7-branes as in the case of $4d$, $\mathcal{N} = 2$ compactifications with parallel D7-branes. Note that one could object to the uplift of curvature related objects such as $\mathcal{Z}, \mathcal{Z}_i$, that the geometry becomes singular in the F-theory lift and an uplift of these corrections can therefore not be trusted. Yet these corrections are of topological nature and thus might be protected, as we commented on in more detail in section 5.4.

It would furthermore be desirable to investigate the F-theory uplift of the full $3d$, "$\mathcal{N} = 2$" action obtained in chapter III to gain a $4d$, $\mathcal{N} = 1$ theory. However, in contrast to the discussion of topological structures on elliptically fibered Calabi-Yau fourfolds where one can relate the quantities in the fourfold to the corresponding ones in the base due to their topological nature, the treatment of non-topological terms as the warp-factor and F_6 which would be required, poses an interesting open problem.

9.2 Outlook

The limitations of our previous analysis eventually arose from the possibility that we did not identify all relevant higher-derivative building blocks. Among the various combinations of possible index contractions of the Riemann tensor only a few are topologically relevant e.g. c_1, c_2, c_3, c_4, while others seem to have a special role in physics, such as $Z_{m\bar{m}n\bar{n}}$. Our guiding principle to find new building blocks was dimensional reduction, once a new structure is identified one is able to construct identities and relate the various components to each other. In this sense an immediate extension subsequent to chapter III is to consider the dimensional reduction at next order in the fluctuations δv^i, thus looking at the fluctuations of the couplings of the scalars and vectors. This could moreover help to understand the uplift of the moduli and provide justification for the proposed Kähler potential and complex coordinates. A preliminary analysis suggests that indeed new building blocks are relevant to match the reduction result.

There are two further obvious follow-ups of this research program, firstly, to consider the complex structure deformations and secondly, to allow for $h^{2,1}(Y_4) \neq 0$. These discussions are very interesting but incorporate the complication that one has to consider the Kähler and the complex structure deformations at the same time, and then moreover, needs to expand $\hat{C}$ in harmonic $(2,1)$ and $(1,2)$-forms, which give rise to additional complex three-dimensional scalars. In fact, we have derived preliminary results of this study, where details can be found in section D.0.1. We have dimensionally reduced the term $(\hat{t}_8\hat{t}_8 - \frac{1}{24}\hat{\epsilon}_{11}\hat{\epsilon}_{11})\hat{R}^4$ on the warped background considering Kähler and complex structure deformations. Note that this is equivalent to the reduction on the zeroth order background consisting of a Calabi-Yau manifold, since the analyzed correction is already of order α^2. Despite α^2-corrections to the kinetic couplings of the complex scalars, we encounter peculiar mixing-terms between the Kähler deformations and the complex structure deformations of the form $d\delta v^i \wedge * d\delta z^I \int_{Y_4} Y_{Iim}{}^{m} *^{(0)} 1 + d\delta v^i \wedge * d\delta \bar{z}^I \int_{Y_4} \overline{Y_{Iim}{}^{m}} *^{(0)} 1$. Remarkably, one encounters a coupling which has the analogous structure as the already introduced building block $Y_{ijm\bar{m}}$. Note that a full discussion of the reduction is required to guarantee that the mixing is present in the effective theory and not subject to some non-trivial cancellation, for instance due to terms arising from α^2-corrections to the background when reducing the Einstein-Hilbert term, the α^2-modifications of the of the zero modes, or the $\hat{C} \wedge \hat{X}_8$ term. However, in the case that these mixing terms remain in the full study it will be very interesting to analyze their implications on the effective physics. We refer the reader to future work.

Furthermore, one can allow for $h^{2,1}(Y_4) \neq 0$, which implies that the $\hat{C}$ form field gives rise to $h^{2,1}(Y_4)$ propagating complex scalars in chiral multiplets. In a simplified setup we look at the terms arising from the dimensional reduction of $\hat{G}^2\hat{R}^3$ considering only the $h^{2,1}$-sector, see section D.0.2 for details. We find that the reduced result takes a very similar, analogous form compared to the one found in chapter III, and remarkably, can agian be written entirely in terms of our building block $Z_{m\bar{m}n\bar{n}}$. Comparing to the analysis of chapter III, the building block with all four indices contracted on the metric only arose when reducing the $(\hat{\nabla}\hat{G})^2\hat{R}^2$ terms, whereas in the $h^{2,1}$-case it originates from $\hat{G}^2\hat{R}^3$. This makes the $h^{2,1}$-reduction of the $(\hat{\nabla}\hat{G})^2\hat{R}^2$ terms a promising location for the search of new fundamental building blocks, since one might expect that objects with six free indices arise, i.e. a structure $Z^6_{m\bar{m}n\bar{n}r\bar{r}}$.

However, alternatively one can broaden the field of study to see how universal the identified structure $Z_{m\bar{m}n\bar{n}}$ is. For instance by analyzing if there exist others, which are equally relevant. This problem seems hard to approach, due to the enormous number of possible index contractions of three Riemann tensors, or two Riemann tensors and two covariant derivatives acting on e.g. the harmonic $(1,1)$-forms as in $Y_{ijm\bar{m}}$. The guiding principle to identify the physical relevant objects was dimensional reduction. However, one can go a different route by scanning over certain structures to determine the full set of different index contractions. As a first step in this direction one could be inspired by $Z_{m\bar{m}n\bar{n}} \sim (\epsilon_8\epsilon_8 R^3)_{m\bar{m}n\bar{n}}$, and derive all inequivalent contractions of the form $(\epsilon_8\epsilon_8 R^3)_{m\bar{m}n\bar{n}}$. It would be interesting to see if one can find a minimal basis of structures $Z^{\gamma}_{m\bar{m}n\bar{n}}$, such that contracted only with two harmonic $(1,1)$-forms and no metric, under a linear combination matches the reduction result. Then all the higher-derivative structures in the effective action would be at the same footing, carrying four explicit indices. A first preliminary study shows that these different structure $Z^{\gamma}_{m\bar{m}n\bar{n}}$ are highly related to each other by linear combinations. One may hope that this furthermore leads to novel identities, which help to rewrite the reduction result in a more appropriate form to connect it to expressions depending on the fields v^i, rather than v^i_0. As a consequence this could provide further evidence for the proposed Kähler potential and complex coordinates. Let us conclude by emphasizing that the outlined topics are certainly worth a more detailed investigation, and thus constitute interesting future directions.

Appendix

A Conventions, definitions, and identities

In this work we denote the eleven-dimensional space indices by capital Latin letters $M, N, R = 0, \ldots, 10$, the external ones by $\mu, \nu = 0, 1, 2$, and the internal complex ones by $m, n, p = 1, ..., 4$ and $\bar{m}, \bar{n}, \bar{p} = 1, \ldots, 4$. Eleven-dimensional quantities for which the indices are raised and lower with the total space metric carry a hat, for example the M-theory three-form is denoted by $\hat{G}$. Furthermore, the convention for the totally anti-symmetric tensor in Lorentzian space in an orthonormal frame is $\epsilon_{012\ldots10} = \epsilon_{012} = +1$. The epsilon tensor in d dimensions then satisfies

$$\epsilon^{R_1 \cdots R_p N_1 \ldots N_{d-p}} \epsilon_{R_1 \ldots R_p M_1 \ldots M_{d-p}} = (-1)^s (d-p)! p! \delta^{N_1}{}_{[M_1} \ldots \delta^{N_{d-p}}{}_{M_{d-p}]} \, , \tag{A.1}$$

where $s = 0$ if the metric has Riemannian signature and $s = 1$ for a Lorentzian metric.

We adopt the following conventions for the Christoffel symbols and Riemann tensor

$$\begin{aligned} \Gamma^R{}_{MN} &= \frac{1}{2} g^{RS} (\partial_M g_{NS} + \partial_N g_{MS} - \partial_S g_{MN}) \, , & R_{MN} &= R^R{}_{MRN} \, , \\ R^M{}_{NRS} &= \partial_R \Gamma^M{}_{SN} - \partial_S \Gamma^M{}_{RN} + \Gamma^M{}_{RT} \Gamma^T{}_{SN} - \Gamma^M{}_{ST} \Gamma^T{}_{RN} \, , & R &= R_{MN} g^{MN} \, , \end{aligned} \tag{A.2}$$

with equivalent definitions on the internal and external spaces. Written in components, the first and second Bianchi identity are

$$\begin{aligned} R^O{}_{PMN} + R^O{}_{MNP} + R^O{}_{NPM} &= 0 \\ (\nabla_L R)^O{}_{PMN} + (\nabla_M R)^O{}_{PNL} + (\nabla_N R)^O{}_{PLM} &= 0 \quad . \end{aligned} \tag{A.3}$$

Differential p-forms are expanded in a basis of differential one-forms as

$$\Lambda = \frac{1}{p!} \Lambda_{M_1 \ldots M_p} dx^{M_1} \wedge \ldots \wedge dx^{M_p} \quad . \tag{A.4}$$

The wedge product between a p-form $\Lambda^{(p)}$ and a q-form $\Lambda^{(q)}$ is given by

$$(\Lambda^{(p)} \wedge \Lambda^{(q)})_{M_1 \ldots M_{p+q}} = \frac{(p+q)!}{p!q!} \Lambda^{(p)}_{[M_1 \ldots M_p} \Lambda^{(q)}_{M_1 \ldots M_q]} \ . \tag{A.5}$$

Furthermore, the exterior derivative on a p-form Λ results in

$$(d\Lambda)_{NM_1 \ldots M_p} = (p+1)\partial_{[N}\Lambda_{M_1 \ldots M_p]} \ , \tag{A.6}$$

while the Hodge star of p-form Λ in d real coordinates is given by

$$(*_d \Lambda)_{N_1 \ldots N_{d-p}} = \frac{1}{p!} \Lambda^{M_1 \ldots M_p} \epsilon_{M_1 \ldots M_p N_1 \ldots N_{d-p}} \ . \tag{A.7}$$

Moreover,

$$\Lambda^{(1)} \wedge *\Lambda^{(2)} = \frac{1}{p!} \Lambda^{(1)}_{M_1 \ldots M_p} \Lambda^{(2) M_1 \ldots M_p} *1 \ , \tag{A.8}$$

which holds for two arbitrary p-forms $\Lambda^{(1)}$ and $\Lambda^{(2)}$.

A.1 Complex manifolds

Let M be a complex Hermitian manifold with $dim_{\mathbb{C}} M = n$, thus $2n$ real coordinates $\{\xi^1, ..., \xi^{2n}\}$. We define the complex coordinates to be

$$(z^1, ..., z^n) = \left(\frac{1}{\sqrt{2}}(\xi^1 + i\xi^2), \ldots, \frac{1}{\sqrt{2}}(\xi^{2n-1} + i\xi^{2n}) \right) \ . \tag{A.9}$$

Using these conventions one finds

$$\sqrt{g} d\xi^1 \wedge \ldots \wedge d\xi^{2n} = \sqrt{g}(-)^{\frac{(n-1)n}{2}} i^n dz^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^1 \wedge \ldots \wedge d\bar{z}^n = \frac{1}{n!} J^n \, , \tag{A.10}$$

with g the determinant of the metric in real coordinates and $\sqrt{\det g_{mn}} = \det g_{m\bar{n}}$. The Kähler form is given by

$$J = i g_{m\bar{n}} dz^m \wedge d\bar{z}^{\bar{n}} \, . \tag{A.11}$$

Let $\omega_{p,q}$ be a (p,q)-form, then

$$\begin{aligned} *\omega_{p,q} = & \frac{(-1)^{\frac{n(n-1)+2np}{2}} i^n}{p!q!(n-p)!(n-q)!} \omega_{m_1 \ldots m_p \bar{n}_1 \ldots \bar{n}_q} \epsilon^{m_1 \ldots m_p}{}_{\bar{r}_1 \ldots \bar{r}_{n-p}} \\ & \times \epsilon^{\bar{n}_1 \ldots \bar{n}_q}{}_{s_1 \ldots s_{n-q}} dz^{s_1} \wedge \cdots \wedge dz^{s_{n-q}} \wedge d\bar{z}^{\bar{r}_1} \wedge \cdots \wedge d\bar{z}^{\bar{r}^{n-p}} . \end{aligned} \tag{A.12}$$

A.2 Chern classes

We define the curvature two-form for Hermitian manifolds to be

$$\mathcal{R}^m{}_n = R^m{}_{nr\bar{s}} dz^r \wedge d\bar{z}^{\bar{s}} \ , \tag{A.13}$$

and

$$
\begin{aligned}
\mathrm{Tr}\,\mathcal{R} &= R^m{}_{mr\bar{s}} dz^r \wedge d\bar{z}^{\bar{s}} \ , \\
\mathrm{Tr}\,\mathcal{R}^2 &= R^m{}_{nr\bar{s}} R^n{}_{mr_1\bar{s}_1} dz^r \wedge d\bar{z}^{\bar{s}} \wedge dz^{r_1} \wedge d\bar{z}^{\bar{s}_1} \ , \\
\mathrm{Tr}\,\mathcal{R}^3 &= R^m{}_{nr\bar{s}} R^n{}_{n_1 r_1\bar{s}_1} R^{n_1}{}_{mr_2\bar{s}_2} dz^r \wedge d\bar{z}^{\bar{s}} \wedge dz^{r_1} \wedge d\bar{z}^{\bar{s}_1} \wedge dz^{r_2} \wedge d\bar{z}^{\bar{s}_2} \ .
\end{aligned} \tag{A.14}
$$

The Chern forms can be expressed in terms of the curvature two-form as

$$
\begin{aligned}
c_0 &= 1 \ , \\
c_1 &= i\mathrm{Tr}\,\mathcal{R} \ , \\
c_2 &= \frac{1}{2}\left(\mathrm{Tr}\,\mathcal{R}^2 - (\mathrm{Tr}\,\mathcal{R})^2\right) \ , \\
c_3 &= \frac{1}{3}c_1 c_2 + \frac{1}{3}c_1 \wedge \mathrm{Tr}\,\mathcal{R}^2 - \frac{i}{3}\mathrm{Tr}\,\mathcal{R}^3 \ , \\
c_4 &= \frac{1}{24}\left(c_1^4 - 6c_1^2 \mathrm{Tr}\,\mathcal{R}^2 - 8ic_1 \mathrm{Tr}\,\mathcal{R}^3\right) + \frac{1}{8}((\mathrm{Tr}\,\mathcal{R}^2)^2 - 2\mathrm{Tr}\,\mathcal{R}^4) \, .
\end{aligned} \tag{A.15}
$$

The Chern classes of the n-dimensional Calabi-Yau manifold Y_n reduce to

$$
c_3(Y_{n\geq 3}) = -\frac{i}{3}\mathrm{Tr}\,\mathcal{R}^3 \ \text{ and } \ c_4(Y_{n\geq 4}) = \frac{1}{8}((\mathrm{Tr}\,\mathcal{R}^2)^2 - 2\mathrm{Tr}\,\mathcal{R}^4) \ , \tag{A.16}
$$

with $\mathrm{Tr}\,\mathcal{R}^4$ defined as in (A.14).

A.3 Non-harmonicity of c_3

The third Chern form on a Calabi-Yau given in (A.16). It is real and one can easily explicitly verify that

$$
dc_3 = 0 \ \text{ whilst } \ d *^{(0)} c_3 \neq 0 \ \ , \tag{A.17}
$$

thus being closed but not co-closed with respect to the Kähler metric g_{mn}. This implies that it may be expanded as

$$
c_3 = Hc_3 + i\partial\bar{\partial} F_4 \tag{A.18}
$$

where H indicates the projection to the harmonic part with respect to the metric g_{mn}. This equation defines a co-closed $(2,2)$-form F. Let us note that since F is a co-closed (2,2) form it obeys

$$
\nabla_m F_{4\,n\bar{n}r}{}^m = 0 \ \ . \tag{A.19}
$$

Using this one shows that

$$
\partial_{[n}(*^{(0)}\partial\bar{\partial}F_4)_{m]\bar{n}} = -2\nabla_{[n|}\nabla_r\nabla^r F_{4\,m]\bar{n}s}{}^s + g_{[m|\bar{n}}\nabla_{n]}\nabla_r\nabla^r F_{4\,s}{}^s{}_t{}^t - 2R_{[n|\bar{n}r}{}^s\nabla_{m]}F_{4\,s}{}^r{}_o{}^o \tag{A.20}
$$

which upon contraction with the inverse metric and using (A.19) gives.

$$
\partial_{[n}(*^{(0)}\partial\bar{\partial}F_4)_{m]\bar{n}}g^{\bar{n}m} = -\partial_{[n}(*^{(0)}\partial\bar{\partial}F_4)_{m]\bar{n}}g^{\bar{n}n} = \nabla_n\nabla_r\nabla^r F_{4\,s}{}^s{}_t{}^t \ . \tag{A.21}
$$

Note that the non harmonicity of c_3 becomes relevant for the effective action discussion when performing integral splits as

$$\int_{Y_4} *^{(0)}\big(\omega_i^{(0)}\wedge\omega_j^{(0)}\wedge J^2\big)\wedge c_3\wedge J=\frac{1}{\mathcal{V}_0}\mathcal{Z}\mathcal{K}_{ij}+\int_{Y_4}\omega_i^{(0)}\wedge\omega_j^{(0)}\wedge J^2 *^{(0)}(\partial\bar{\partial}F_4\wedge J) \tag{A.22}$$

The last term in the previous equation is not zero and has to be taken into account when performing these kind of integral splits. If $*^{(0)}(\partial\bar{\partial}F_4\wedge J)$ were covariantly constant one could pull it out of the integral and then by integrating over it the second term would vanish due since $\int\partial\bar{\partial}F_4\wedge J=0$, one straightforwardly computes

$$*^{(0)}(\partial\bar{\partial}F_4\wedge J)=2\nabla_m\nabla^m F_n{}^n{}_r{}^r *^{(0)} 1 \tag{A.23}$$

which gives $\nabla_r\nabla_m\nabla^m F_{4n}{}^n{}_r{}^r\neq 0$. However one can show that if F where additionally closed thus harmonic then $\nabla_m F_n{}^n{}_r{}^r=0$, which is against its definition but nevertheless provides a nice cross check. Thus, in particular

$$d *^{(0)}(c_3\wedge J)=d *^{(0)}(Hc_3\wedge J)+i\,d *^{(0)}(\partial\bar{\partial}F_4\wedge J)=d *^{(0)}(\partial\bar{\partial}F_4\wedge J)\neq 0 \tag{A.24}$$

unless $\nabla_m\nabla^m F_n{}^n{}_r{}^r=0$ which we do not have any weaker arguments than additional closedness of F at hand.

Let us next comment on the integral split performed in (4.27) given by

$$\int_{Y_4} *^{(0)}\big(\omega_i^{(0)}\wedge\omega_j^{(0)}\wedge J\big)\wedge c_3=-2\mathcal{V}_0\mathcal{K}^{kl}\mathcal{K}_{kli}\mathcal{Z}_j+\frac{1}{3\mathcal{V}_0}\mathcal{K}_{ij}\mathcal{Z}+\int_{Y_4} *^{(0)}\partial\tilde{H}\wedge c_3 \tag{A.25}$$

We see that the last term can be recast to the form

$$\int_{Y_4} *^{(0)}\partial\tilde{H}\wedge c_3=\int_{Y_4}\tilde{H}\wedge\partial *^{(0)} c_3=\int_{Y_4}\tilde{H}\wedge\partial *^{(0)}\partial\bar{\partial}F_4 \tag{A.26}$$

We can alternatively the left hand side of (A.25) as

$$\int_{Y_4} *^{(0)}\big(\omega_i^{(0)}\wedge\omega_j^{(0)}\wedge J\big)\wedge(Hc_3+\partial\bar{\partial}F_4)=-2\mathcal{V}_0\mathcal{K}^{kl}\mathcal{K}_{kli}\mathcal{Z}_j+\frac{1}{3\mathcal{V}_0}\mathcal{K}_{ij}\mathcal{Z}+\int_{Y_4}\omega_i^{(0)}\wedge\omega_j^{(0)}\wedge J\wedge *^{(0)}\partial\bar{\partial}F_4 \tag{A.27}$$

where we see that we can perform the split of the integral with the piece Hc_3 and promote $\int Hc_3\wedge\omega_i^{(0)}=\int c_3\wedge\omega_i^{(0)}=\mathcal{Z}_i$ due to the harmonicity of ω_i, which is equivalent to (A.25).

A.4 Spinors on $SU(4)$ structure manifolds

The existence of a nowhere vanishing spinor η on X implies that the structure group of the manifold reduces to $SU(4)$. Equivalently J and Ω uniquely determine the $SU(4)$ structure obeying

$$J\wedge\Omega=0 \quad \text{and} \quad \Omega\wedge\bar{\Omega}=\frac{2}{3}J^4\,. \tag{A.28}$$

The forms J and Ω can be constructed from spinor bilinears in η. One can normalize a complex Weyl spinor in eight dimensions to obey $\eta^\dagger\eta = 1$ and $\eta^T\eta = 0$. One can construct bilinear quantities which coincide with the nowhere vanishing J and Ω obeying (A.28) as

$$J_{mn} = i\eta^\dagger\gamma_{mn}\eta \tag{A.29}$$

$$\Omega_{mnrs} = \eta^T\gamma_{mnrs}\eta \ . \tag{A.30}$$

In section 6 we made use of the following set of identities. Let us first define the projection operators

$$\Pi^{\pm}{}_m{}^n = \frac{1}{2}\left(\delta_m{}^n \mp J_m{}^n\right) \tag{A.31}$$

actin on Ω as

$$\Pi^{-}{}_m{}^i\Omega_{inrs} = \Omega_{mnrs} \quad , \qquad \Pi^{+}{}_m{}^i\Omega_{inrs} = 0 \ . \tag{A.32}$$

The bilienear expressions obey

$$\begin{array}{lll} \eta^T\eta = 0 \ , & \eta^T\gamma_{mnrs}\eta = \Omega_{mnrs} \ , & \eta^T\gamma_{p_1\ldots p_d}\eta = 0 \ , \quad \text{for } d \neq 4 \\ \eta^\dagger\eta = 1 \ , & \eta^\dagger\gamma_{mn}\eta = -iJ_{mn} \ , & \eta^\dagger\gamma_{mnrs}\eta = -3J_{[mn}J_{rs]} \ , \end{array}$$

and

$$\begin{aligned} \eta^\dagger\gamma_{mnrstu}\eta &= 15iJ_{[mn}J_{rs}J_{tu]} \ , \\ \eta^\dagger\gamma_{mnrstuvw}\eta &= 105J_{[mn}J_{rs}J_{tu}J_{vw]} \ , \\ \eta^\dagger\gamma_{p_1\ldots p_d}\eta &= 0 \ , \quad \text{for odd d} \ . \end{aligned} \tag{A.33}$$

A.5 Higher-Derivative Terms

In this subsection we discuss the explicit form of the terms in the action (2.61). Let us start by commenting in more detail on the $\hat{X}_8$ structure

$$\hat{X}_8 = \frac{1}{192}\left[\mathrm{Tr}\,\hat{\mathcal{R}}^4_{\mathbb{R}} - \frac{1}{4}\left(\mathrm{Tr}\,\hat{\mathcal{R}}^2_{\mathbb{R}}\right)^2\right] \ . \tag{A.34}$$

Let the subscript $_{\mathbb{R}}$ denote the curvature two-forms in real coordinates, i.e. $\hat{\mathcal{R}}_{\mathbb{R}} = \frac{1}{2}R^O{}_{PNM}dx^N \wedge dx^M$. The traces of curvature two-forms in real coordinates are defined analogously to those in complex coordinates as in (A.14), but with an additional factor $\frac{1}{2}$ for each curvature two-form. On a Calabi-Yau manifold one has $X_8(Y_4) = -\frac{1}{24}c_4(Y_4)$. This follows straightforwardly by using the transformation properties under coordinate transformation from real to complex coordinates, which are $\mathrm{Tr}\,\hat{\mathcal{R}}^4_{\mathbb{R}} \leftrightarrow 2\mathrm{Tr}\,\hat{\mathcal{R}}^4$ and $\mathrm{Tr}\,\hat{\mathcal{R}}^2_{\mathbb{R}} \leftrightarrow 2\mathrm{Tr}\,\hat{\mathcal{R}}^2$, and then by comparison to (A.15).

The terms $\hat{t}_8\hat{t}_8\hat{R}^4$ and $\hat{t}_8\hat{t}_8\hat{G}^2\hat{R}^3$ in (2.63) and (2.64) require the definition

$$\begin{aligned} \hat{t}_8^{N_1\ldots N_8} = \frac{1}{16}\Big(& -2\left(\hat{g}^{N_1N_3}\hat{g}^{N_2N_4}\hat{g}^{N_5N_7}\hat{g}^{N_6N_8} + \hat{g}^{N_1N_5}\hat{g}^{N_2N_6}\hat{g}^{N_3N_7}\hat{g}^{N_4N_8} + \hat{g}^{N_1N_7}\hat{g}^{N_2N_8}\hat{g}^{N_3N_5}\hat{g}^{N_4N_6}\right) \\ & + 8\left(\hat{g}^{N_2N_3}\hat{g}^{N_4N_5}\hat{g}^{N_6N_7}\hat{g}^{N_8N_1} + \hat{g}^{N_2N_5}\hat{g}^{N_6N_3}\hat{g}^{N_4N_7}\hat{g}^{N_8N_1} + \hat{g}^{N_2N_5}\hat{g}^{N_6N_7}\hat{g}^{N_8N_3}\hat{g}^{N_4N_1}\right) \\ & - (N_1 \leftrightarrow N_2) - (N_3 \leftrightarrow N_4) - (N_5 \leftrightarrow N_6) - (N_7 \leftrightarrow N_8)\Big) \ . \end{aligned} \tag{A.35}$$

In order to discuss the term $\hat{s}_{18}$ appearing in (2.65) and (2.69) we introduce the basis

$$\begin{aligned}
B_1 &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6N_7N_8}\hat{\nabla}^{N_5}\hat{G}^{N_1N_7N_8}{}_{N_9}\hat{\nabla}^{N_3}\hat{G}^{N_2N_4N_6N_9}\,, & B_{13} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5}{}^{N_1}{}_{N_6}{}^{N_3}\hat{\nabla}_{N_9}\hat{G}^{N_2N_6}{}_{N_7N_8}\hat{\nabla}^{N_9}\hat{G}^{N_4N_5N_7N_8}\,,\\
B_2 &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6N_7N_8}\hat{\nabla}^{N_5}\hat{G}^{N_1N_3N_7}{}_{N_9}\hat{\nabla}^{N_8}\hat{G}^{N_2N_4N_6N_9}\,, & B_{14} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5}{}^{N_1}{}_{N_6}{}^{N_3}\hat{\nabla}_{N_9}\hat{G}^{N_2N_4}{}_{N_7N_8}\hat{\nabla}^{N_9}\hat{G}^{N_5N_6N_7N_8}\,,\\
B_3 &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6N_7N_8}\hat{\nabla}^{N_5}\hat{G}^{N_1N_3N_7}{}_{N_9}\hat{\nabla}^{N_6}\hat{G}^{N_2N_4N_8N_9}\,, & B_{15} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5}{}^{N_1}{}_{N_6}{}^{N_3}\hat{\nabla}^{N_2}\hat{G}^{N_6}{}_{N_7N_8N_9}\hat{\nabla}^{N_5}\hat{G}^{N_4N_7N_8N_9}\,,\\
B_4 &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6N_7N_8}\hat{\nabla}_{N_9}\hat{G}^{N_3N_4N_7N_8}\hat{\nabla}^{N_6}\hat{G}^{N_9N_1N_2N_5}\,, & B_{16} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5}{}^{N_1}{}_{N_6}{}^{N_3}\hat{\nabla}^{N_2}\hat{G}^{N_4}{}_{N_7N_8N_9}\hat{\nabla}^{N_5}\hat{G}^{N_6N_7N_8N_9}\,,\\
B_5 &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6N_7}{}^{N_4}\hat{\nabla}^{N_1}\hat{G}^{N_2N_3}{}_{N_8N_9}\hat{\nabla}^{N_5}\hat{G}^{N_6N_7N_8N_9}\,, & B_{17} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5}{}^{N_1}{}_{N_6}{}^{N_3}\hat{\nabla}^{N_2}\hat{G}^{N_5}{}_{N_7N_8N_9}\hat{\nabla}^{N_4}\hat{G}^{N_6N_7N_8N_9}\,,\\
B_6 &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6N_7}{}^{N_4}\hat{\nabla}^{N_1}\hat{G}^{N_2N_5}{}_{N_8N_9}\hat{\nabla}^{N_3}\hat{G}^{N_6N_7N_8N_9}\,, & B_{18} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5}{}^{N_1}{}_{N_6}{}^{N_3}\hat{\nabla}_{N_9}\hat{G}^{N_5N_6}{}_{N_7N_8}\hat{\nabla}^{N_4}\hat{G}^{N_2N_7N_8N_9}\,,\\
B_7 &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6N_7}{}^{N_4}\hat{\nabla}^{N_1}\hat{G}^{N_2N_5}{}_{N_8N_9}\hat{\nabla}^{N_7}\hat{G}^{N_3N_6N_8N_9}\,, & B_{19} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6}{}^{N_3N_4}\hat{\nabla}_{N_9}\hat{G}^{N_1N_5}{}_{N_7N_8}\hat{\nabla}^{N_9}\hat{G}^{N_2N_6N_7N_8}\,,\\
B_8 &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6N_7}{}^{N_4}\hat{\nabla}^{N_1}\hat{G}^{N_3N_5}{}_{N_8N_9}\hat{\nabla}^{N_2}\hat{G}^{N_6N_7N_8N_9}\,, & B_{20} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6}{}^{N_3N_4}\hat{\nabla}^{N_1}\hat{G}^{N_5}{}_{N_7N_8N_9}\hat{\nabla}^{N_2}\hat{G}^{N_6N_7N_8N_9}\,,\\
B_9 &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6N_7}{}^{N_4}\hat{\nabla}^{N_1}\hat{G}^{N_3N_5}{}_{N_8N_9}\hat{\nabla}^{N_6}\hat{G}^{N_2N_7N_8N_9}\,, & B_{21} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6}{}^{N_3N_4}\hat{\nabla}^{N_1}\hat{G}^{N_5}{}_{N_7N_8N_9}\hat{\nabla}^{N_6}\hat{G}^{N_2N_7N_8N_9}\,,\\
B_{10} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6N_7}{}^{N_4}\hat{\nabla}_{N_9}\hat{G}^{N_3N_5N_7N_8}\hat{\nabla}^{N_9}\hat{G}^{N_1N_2N_6N_8}\,, & B_{22} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5}{}^{N_1N_3N_4}\hat{\nabla}^{N_2}\hat{G}_{N_6N_7N_8N_9}\hat{\nabla}^{N_5}\hat{G}^{N_6N_7N_8N_9}\,,\\
B_{11} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6N_7}{}^{N_4}\hat{\nabla}_{N_8}\hat{G}^{N_1N_2N_6}{}_{N_9}\hat{\nabla}^{N_9}\hat{G}^{N_3N_5N_7N_8}\,, & B_{23} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5}{}^{N_1N_3N_4}\hat{\nabla}_{N_9}\hat{G}^{N_2}{}_{N_6N_7N_8}\hat{\nabla}^{N_9}\hat{G}^{N_5N_6N_7N_8}\,,\\
B_{12} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}_{N_5N_6N_7}{}^{N_4}\hat{\nabla}^{N_3}\hat{G}^{N_5N_6}{}_{N_8N_9}\hat{\nabla}^{N_7}\hat{G}^{N_2N_1N_8N_9}\,, & B_{24} &= \hat{R}_{N_1N_2N_3N_4}\hat{R}^{N_1N_2N_3N_4}\hat{\nabla}_{N_5}\hat{G}_{N_6N_7N_8N_9}\hat{\nabla}^{N_6}\hat{G}^{N_5N_7N_8N_9}\,.
\end{aligned} \tag{A.36}$$

The contributions to $\hat{s}_{18}(\hat{\nabla}\hat{G})^2\hat{R}^2$ are then formed from the linear combinations described in (2.69).

A.6 Normal Coordinates

On a Riemannian manifold of real dimension n the affine connection defines a unique geodesic γ_v for every tangent vector $v_p \in T_pM$ through every point $p \in M$, such that $\gamma_v(0) = p$. This gives rise to a map from a small neighborhood of the origin in the tangent space T_pM to the manifold, the so called exponential map with $exp(tv) = \gamma_v(t)$. We can now use the inverse of this map to map points on the manifold to the tangent space, which in return is isomorphic to $\mathbb{R}^n$. Thus we can use this procedure to define flat coordinates locally around any point $p \in M$. In this coordinates one finds that the Christoffel symbols vanish for the Levi-Civita connection and the covariant derivative reduces to the partial derivative. Furthermore, one finds that derivatives of The Christoffel symbols obey

$$\partial_{(\mu_1} \dots \partial_{\mu_n} \Gamma^{\mu}{}_{\nu_1\nu_2)} = 0 \quad . \tag{A.37}$$

Moreover, since the metric is locally flat one yields $g_{\mu\nu} \to \delta_{\mu\nu}$ and also that the first derivative of the metric vanishes $\partial_\rho g_{\mu\nu} \to 0$. Furthermore, one can show that a Riemanntensor can be written as

$$R_{\mu\nu\rho\gamma} = \partial_{[\nu}\Gamma_{\gamma|\mu]\rho} \quad , \tag{A.38}$$

and via using (A.37) that

$$\partial_\gamma \Gamma_{\rho\mu\nu} = R_{(\nu|\gamma|\mu)\rho} \quad . \tag{A.39}$$

A.7 Identities

We note that by performing a Weyl rescaling of the three-dimensional external metric with $g'_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$ one finds that

$$\int_{\mathcal{M}_3} \Omega R' *'_3 1 = \int_{\mathcal{M}_3} (R *_3 1 - \frac{2}{\Omega^2} \nabla_\mu \Omega \nabla^\mu \Omega *_3 1) \, . \tag{A.40}$$

Let us next define the intersection numbers, where $\{\omega_i\}$ are harmonic w.r.t. to the Calabi- Yau metric $g_{m\bar{n}}$

$$\mathcal{K}_{ijkl} = \int_{Y_4} \omega_i \wedge \omega_j \wedge \omega_k \wedge \omega_l \, , \qquad \mathcal{K}_{ijk} = \mathcal{K}_{ijkl} v^l \, , \qquad \mathcal{K}_{ij} = \frac{1}{2} \mathcal{K}_{ijkl} v^k v^l \, ,$$
$$\mathcal{K}_i = \frac{1}{3!} \mathcal{K}_{ijkl} v^j v^k v^l \, , \qquad \mathcal{V} = \frac{1}{4!} \mathcal{K}_{ijkl} v^i v^j v^k v^l \, . \tag{A.41}$$

While in the background we introduce the notation $\{\omega_i^{(0)}\}$ denoting that it is harmonic w.r.t. $g^{(0)}_{m\bar{n}}$, the background metric. One thus finds for the intersection numbers that

$$\mathcal{K}^{(0)}_{ijkl} = \int_{Y_4} \omega_i^{(0)} \wedge \omega_j^{(0)} \wedge \omega_k^{(0)} \wedge \omega_l^{(0)} \, , \qquad \mathcal{K}^{(0)}_{ijk} = \mathcal{K}^{(0)}_{ijkl} v_0^l \, , \qquad \mathcal{K}^{(0)}_{ij} = \frac{1}{2} \mathcal{K}^{(0)}_{ijkl} v_0^k v_0^l \, ,$$
$$\mathcal{K}^{(0)}_i = \frac{1}{3!} \mathcal{K}^{(0)}_{ijkl} v_0^j v_0^k v_0^l \, , \qquad \mathcal{V}_0 = \frac{1}{4!} \mathcal{K}^{(0)}_{ijkl} v_0^i v_0^j v_0^k v_0^l \, . \tag{A.42}$$

In this section we prove some identities that are necessary to derive the result of subsection 4.1. By choosing coordinates and using (A.11) and (A.12), one can straightforwardly show that

$$* J^4 = 4! \qquad \text{and} \qquad * J^3 = 3! J \, . \tag{A.43}$$

Furthermore, one can show that

$$*\omega_i = \frac{1}{3! \mathcal{V}} \mathcal{K}_i \wedge J^3 - \frac{1}{2} \omega_i \wedge J^2 \, , \tag{A.44}$$
$$*\big(\omega_i \wedge J^2\big) = -2\omega_i + \frac{2}{\mathcal{V}} \mathcal{K}_i \wedge J \, , \tag{A.45}$$
$$*\big(\omega_i \wedge J^3\big) = \frac{6}{\mathcal{V}} \mathcal{K}_i * 1 \, , \tag{A.46}$$
$$*\big(\omega_i \wedge \omega_j \wedge J\big) = -\mathcal{V} \tilde{K}^{0\,kl} \omega_k \mathcal{K}_{lij} + H_{ij} \, . \tag{A.47}$$

where H_{ij} is a closed $(1,1)$ - form, $H_{ij} = *\partial \tilde{H}_{ij}$, where $\tilde{H}_{ij}$ is a $(2,3)$ form.

These identities follow from using the topological intersection numbers (A.41), (4.9), and $K^{0\,jj'}$, the inverse of

$$\tilde{K}^0_{ij} = \mathcal{V} \mathcal{K}_{ij} - \mathcal{K}_i \mathcal{K}_j = -\mathcal{V} \int \omega_i \wedge *\omega_j \, . \tag{A.48}$$

Explicitly, $K^{0\,jj'}$ reads

$$\tilde{K}^{0\,ij} = \frac{1}{\mathcal{V}} \mathcal{K}^{ij} - \frac{1}{3\mathcal{V}^2} v^i v^j \, , \tag{A.49}$$

with $\mathcal{K}^{ij}$ the inverse intersection numbers, which obey $\mathcal{K}^{ik}\mathcal{K}_{kj} = \delta^i_j$. Let $\{\tilde{\omega}^i\}$ be the dual basis of $(3,3)$ -forms, which fulfill the relation $\int \tilde{\omega}^i \wedge \omega_j = \delta^i_j$. Then one finds

$$\tilde{\omega}^i = -\mathcal{V}\tilde{K}^{0\,ij} * \omega_j \,. \tag{A.50}$$

In the following the identities (A.7) - (A.47) are derived under the assumption that the underlying space is a $4d$ Kähler manifold. In order to argue for whose analog for a $3d$ Kähler manifold was derived in [154]. One applies Hodge's theorem to $*\omega_i$, which is harmonic and thus decomposes in harmonic $(3,3)$ forms. In this particular case on can straightforwardly use (A.12) to derive

$$*\,\omega_i = -\frac{i}{3!}\mathrm{Tr}\,\omega_i J^3 - \frac{1}{2}\omega_i \wedge J^2 \,, \tag{A.51}$$

with $\mathrm{Tr}\,\omega_i = \omega_{im}{}^m$. If ω_i is harmonic, then $\mathrm{Tr}\,\omega_i$ is covariantly constant. Thus one can separate it from the integrand and evaluate the integral. One has $\omega_i \wedge J^3 = -6i\mathrm{Tr}\,\omega_i * 1$ and hence

$$\mathrm{Tr}\,\omega_i = \frac{i}{6\mathcal{V}}\int \omega_i \wedge J^3 = \frac{i}{\mathcal{V}}\mathcal{K}_i \,. \tag{A.52}$$

Combining the two previous equations one arrives at (A.7) which upon using the Hodge star gives (A.46). The identity (A.45) follows trivially from (A.7) by applying the Hodge star on both sides of the equation. It is left to show that $\mathrm{Tr}\,\omega$ is covariantly constant for a harmonic form, which shall be done later.

In order to show (A.47) one expands according to Hodge's theorem $*\,(\omega_i \wedge \omega_{i'} \wedge J) = \Omega^j_{ii'}\omega_j + H_{ii'}$ in a basis of harmonic $(1,1)$-forms $\{\omega_j\}$ and a co-closed $(1,1)$-form. Let $\{\tilde{\omega}^i\}$ be the dual basis of harmonic $(3,3)$ -forms, which span a space isomorphic to $H^{(1,1)}$ on a Kähler manifold, and thus also on a Calabi-Yau fourfold. By making use of $\tilde{\omega}^i = -\mathcal{V}\tilde{K}^{0\,ij} * \omega_j$ and applying (A.7) one finds

$$\int \tilde{\omega}^j \wedge *(\omega_i \wedge \omega_{i'} \wedge J) = \Omega^j_{ii'} = -\mathcal{V}\tilde{K}^{0jj'}\mathcal{K}_{j'ii'} \,. \tag{A.53}$$

Note that the co-closed part is not determined by this integral since

$$\int \tilde{\omega}^j \wedge H_{ii'} = \int \tilde{\omega}^j \wedge *\partial \tilde{H}_{ii'} = 0 \tag{A.54}$$

Next, we show that $\mathrm{Tr}\,\omega$ is covariantly constant if ω is a harmonic form. Recall that a form ω is called ∂-harmonic ($\bar{\partial}$-harmonic) if $\Delta_\partial \omega = 0$ ($\Delta_{\bar{\partial}}\omega = 0$). A ∂-harmonic ($\bar{\partial}$-harmonic) form satisfies $\partial\omega = 0$, and $- * \bar{\partial} * \omega = 0$ ($\bar{\partial}\omega = 0$, and $- * \partial * \omega = 0$). On a Kähler manifold $\Delta = 2\Delta_\delta = 2\Delta_{\bar{\delta}}$, which implies that any $\bar{\delta}$-harmonic form is automatically ∂-harmonic and vice versa. In particular any harmonic form satisfies $\partial\omega = 0$ and $\partial * \omega = 0$ due to the injectivity of the Hodge star operator. Additionally, on a Kähler manifold one can show that one can replace the partial derivative with the covariant one in certain cases like

$$\partial_{[r}\omega_{m]\bar{n}} = \nabla_{[r}\omega_{m]\bar{n}} \,, \tag{A.55}$$

where ω is a $(1,1)-$ form. Assuming ω to be a harmonic $(1,1)$-form, one uses its closure $d\omega = 0$ and replaces the partial derivative with a covariant one. Then one uses the fact that the metric commutes with the covariant derivative to arrive at

$$\nabla_n \omega_m{}^m = \nabla_m \omega_n{}^m \, . \tag{A.56}$$

From $d *_8 \omega = 0$ one finds that

$$\nabla_n \omega_m{}^n = 0 \quad , \tag{A.57}$$

and thus closure and coclosure imply

$$\nabla_n \omega_m{}^n = 0 \quad \& \quad \nabla_m \omega_n{}^n = 0 \quad . \tag{A.58}$$

B F-theory lift

In this section we present some additional material to the discussion of section 5, filling in some details.

B.1 Weak coupling with non-Abelian gauge groups

Let us begin by briefly reviewing the Sen limit of an elliptically fibered fourfold with fiber embedded in $\mathbb{P}_{231}$. In that case we can take its defining equation to be given in Tate form as

$$y^2 = x^3 + a_1 xyz + a_2 x^2 z^2 + a_3 y z^3 + a_4 x z^4 + a_6 z^6 \tag{B.1}$$

and the singularities of the elliptic curve are located at the zero locus of its discriminant

$$\Delta = -\frac{1}{4}\beta_2^2(\beta_2\beta_6 - \beta_4^2) - 8\beta_4^3 - 27\beta_6^2 + 9\beta_2\beta_4\beta_6 \, , \tag{B.2}$$

where β_i is given by

$$\beta_2 = a_1^2 + 4a_2, \quad \beta_4 = a_1 a_3 + 2a_4, \quad \beta_6 = a_3^2 + 4a_6 \, . \tag{B.3}$$

In order to take the weak-coupling limit, one sets [155] $\beta_2 = -12h$, $\beta_4 = 2\epsilon\eta$, and $\beta_6 = -\frac{\epsilon^2}{4}\chi$ and obtains

$$\Delta = -36\epsilon^2 h^2 (3h\chi - 4\eta^2) + \mathcal{O}(\epsilon^3) \, . \tag{B.4}$$

Next, one defines the Calabi-Yau threefold X as the double cover of B_3 branched over $h = 0$ as $Z : \xi^2 = h$. In the limit $\epsilon \to 0$, the F-theory model then reduces to Type IIB string theory compactified on the orientifold obtained by quotienting X by the orientifold involution $i : \xi \mapsto -\xi$. A careful analysis of the monodromies along the singular loci of the Calabi-Yau fourfold reveals the presence of $O7$ and $D7$ branes at

$$O7 : \xi = 0 \qquad W : 3h\chi - 4\eta^2 = 0 \, , \tag{B.5}$$

where the $D7$-brane takes the shape of a Whitney umbrella [109, 110]. From these expressions one easily reads off the cohomology classes of the forms dual to these divisors. They are

$$O7 = c_1 \qquad W = 8c_1\,. \tag{B.6}$$

Having concluded a discussion of the smooth case, we now begin to enforce singularities along certain divisors of the base and study the pullbacks of these divisors to the double cover X. Let

$$S : s = 0 \tag{B.7}$$

be a divisor in the base manifold B_3. According to the Tate algorithm, we can then generate a non-Abelian singularity along S by restricting the coefficients a_i in such a way that they vanish along S to a certain order. Since it will turn out to be the simplest case, we begin by considering USp singularities. To create an $USp(2N)$ singularity one must restrict a_i in such a way that they factor as [156]

$$a_1 = a_1, \qquad a_2 = a_2, \qquad a_3 = a_{3,N}s^N, \qquad a_4 = a_{4,N}s^N, \qquad a_6 = a_{6,2N}s^{2N}. \tag{B.8}$$

Plugging this form of a_i into (B.4), one finds that it factorizes as

$$\Delta_{USp(2N)} = s^{2N}\xi^4\Delta'_{USp(2N)}\,. \tag{B.9}$$

One can then take the Whitney umbrella to be defined by the remaining I_1 locus

$$W_{USp(2N)} : \Delta'_{USp(2N)} = 0\,. \tag{B.10}$$

Let us now take a closer look at the projection $\pi' : Z \to B_3$ and study the pullback π'^*S. In fact, for $USp(2N)$ singularities this is simply

$$\pi'^*S : \begin{cases} \xi^2 = a_1^2 + 4a_2 \\ s = 0\,, \end{cases} \tag{B.11}$$

and, in particular π'^*S is generically irreducible if there is a USp singularity along S.

Next, let us consider $SU(N)$ singularities. In this case, one must choose Tate coefficients a_i such that

$$a_1 = a_1, \qquad a_2 = a_{2,1}s, \qquad a_3 = a_{3,\lfloor N/2\rfloor}s^{\lfloor N/2\rfloor}, \qquad a_4 = a_{4,\lceil N/2\rceil}s^{\lceil N/2\rceil}, \qquad a_6 = a_{6,N}s^N \tag{B.12}$$

where $\lfloor N/2\rfloor$ denotes the greatest integer smaller than $N/2$ and $\lceil N/2\rceil$ the smallest integer greater than $N/2$. This implies that the discriminant must factor as

$$\Delta_{SU(N)} = s^N\xi^4\Delta'_{SU(N)}\,. \tag{B.13}$$

As before, we set

$$W_{SU(N)} : \Delta'_{SU(N)} = 0\,. \tag{B.14}$$

Now, however, we encounter the crucial difference between the symplectic and the unitary case. Unlike for USp, a_2 vanishes on S. Considering again the pullback of S to the double cover, one finds that

$$\pi'^* S : \begin{cases} \xi^2 = a_1^2 + 4a_{2,1}s \\ s = 0 \end{cases} \tag{B.15}$$

is not irreducible anymore. Instead, it clearly has two components

$$S^{\pm} : \begin{cases} s = 0 \\ \xi^{\pm} = 0\,, \end{cases} \tag{B.16}$$

where we introduced the short-hand

$$\xi^{\pm} = a_1 \pm \xi\,. \tag{B.17}$$

The factorization of a_2 creates a conifold singularity in X which cannot be resolved while keeping both the Calabi-Yau condition and the orientifold symmetry [155][11]. As done in [158], in what follows we will always restrict to base manifolds B_3 whose topology does not allow the curve $\{a_1 = a_{2,1} = 0\}$ to intersect the surface $\{s = 0\}$, thus assuring smoothness of the double cover. Plugging in the equations, one sees that S^+ and S^- intersect precisely on their respective intersection curve with the $O7$ plane. To see this explicitly, simply compare the defining equations:

$$S^+ \cdot S^- : \begin{cases} s = 0 \\ \xi^+ = 0 \\ \xi^- = 0 \end{cases} \simeq S^{\pm} \cdot \xi : \begin{cases} s = 0 \\ \xi^{\pm} = 0 \\ \xi = 0 \end{cases} \tag{B.18}$$

To summarize, the pullback of one of the base divisors S hosting an SU singularity to the double cover X of the base branched over the orientifold locus is given by

$$\pi'^*(S) = S^+ + S^- \tag{B.19}$$

and $SU(N)$ brane stacks intersect with their images stacks only on the orientifold plane, allowing us to interchange the following three terms at will:

$$S^+ \cdot S^- = S^+ \cdot c_1 = S^- \cdot c_1 \tag{B.20}$$

After dealing with the brane stacks hosting the non-Abelian gauge theories, we turn to the last remaining piece, the Whitney umbrella. From the equations given above one readily reads off that for Tate models with gauge group G as in 5.40 its homology class inside the double cover X is given by

$$W = 8c_1 - \sum_i^{n_{SU}} N_i(S_i^+ + S_i^-) - \sum_j^{n_{USp}} 2M_j T_j\,, \tag{B.21}$$

where we abbreviated $\pi'^* T_j$ as T_j and took it to be the divisor on which the $USp(2M_j)$ gauge singularity is located.

[11] See [157] for the definition of alternative weak coupling limits which avoid the conifold problem.

B.2 Weak coupling for $U(1)$-restricted models with non-Abelian gauge groups

We would now like to understand what happens to W after $U(1)$-restricting a given Tate model. To do so, recall that a $U(1)$-restriction amounts to enforcing $a_6 \equiv 0$. The additional divisor class introduced by resolving the singularity caused by this restriction gives a second section of the fibration, which in turn gives rise to an additional $U(1)$ gauge factor. In order to understand what happens to W upon such a restriction, we need to take a closer look at the Whitney umbrella part of the discriminant, which we denoted Δ' above.

Beginning with the simplest conceivable model, the one without any non-Abelian gauge singularities, one finds that

$$\Delta'|_{\epsilon\to 0} \sim \epsilon^2 \left[a_6\xi^2 - \left(a_4 + \frac{\xi^+}{2}a_3\right)\left(a_4 + \frac{\xi^-}{2}a_3\right)\right] + \mathcal{O}(\epsilon^3)\,, \tag{B.22}$$

where the term in square brackets denotes the familiar Whitney umbrella. At the level of the Tate form, it is easy to understand what it means to embed the elliptic fiber inside F_{11} as opposed to $\mathbb{P}_{231}$: It splits into the two pieces defined by

$$W^{\pm} : a_4 + \frac{\xi^{\pm}}{2}a_3 = 0\,, \tag{B.23}$$

which both have homology class

$$W^{\pm} = 4c_1\,. \tag{B.24}$$

One therefore clearly sees that a $U(1)$ restriction amounts to the Whitney umbrella splitting into a brane and image brane. Next, one needs to generalize this to models with additional non-Abelian gauge factors. As it turns out, this generalization is fairly straightforward for $SU(2N)$ and $USp(2N)$, while requiring a bit more care when defining the split Whitney umbrella for the case of $SU(2N+1)$.

We begin by discussing the split Whitney umbrella for $SU(2N)$. As before, we place the non-Abelian singularity on a divisor in the base manifold B_3 defined by the vanishing of a single coordinate s. In the weak coupling limit we see that the defining equation of the Whitney umbrella takes the form [158]

$$\begin{aligned}\Delta'_{SU(2N)} &\sim \left[a_{6,2N}\xi^2 - \left(a_{4,N} + \frac{\xi^+}{2}a_{3,N}\right)\left(a_{4,N} + \frac{\xi^-}{2}a_{3,N}\right)\right] \\ &\sim \left(a_{4,N} + \frac{\xi^+}{2}a_{3,N}\right)\left(a_{4,N} + \frac{\xi^-}{2}a_{3,N}\right)\end{aligned} \tag{B.25}$$

and we again find that W splits into two irreducible pieces $W^{\pm}$. Both of them have the same homology class, namely

$$W^{\pm}_{SU(2N)} = 4c_1 - N\pi'^{*}S = 4c_1 - N(S^+ + S^-)\,. \tag{B.26}$$

In the next step, we proceed with the case of $USp(2N)$. In fact, the only difference to the $SU(2N)$ case is that a_2 does not factorize. However, since both W^+ and W^- depend only on the invariant divisor class S, the discussion carries over immediately. We therefore find that the homology classes of the split Whitney umbrella are

$$W^{\pm}_{USp(2N)} = 4c_1 - N\pi^* S \,. \tag{B.27}$$

Last but not least, let us take care of $SU(2N+1)$. Due to the fact that the discriminant vanishes with an odd power of s, that is

$$\Delta|_{\epsilon \to 0} \sim s^{2N+1} \Delta' \,, \tag{B.28}$$

it is a bit more tricky to properly define the Whitney umbrella. In the local patch away from the D7-stack one now finds that

$$\Delta'_{SU(2N+1)} = \left[a_{6,2N+1}\xi^2 - s\left(a_{4,N+1} + \frac{\xi^+}{2s} a_{3,N}\right)\left(a_{4,N+1} + \frac{\xi^-}{2s} a_{3,N}\right)\right] \,, \tag{B.29}$$

where, as before, the first term vanishes after setting $a_6 \equiv 0$. In order to obtain the split Whitney umbrella one uses the same trick as in the previous subsection and notes that on the threefold X the divisor S splits into two irreducible components. As the example in [159] suggests, one may find an alternative way of defining X such that S^+ and S^- can separately be written as the complete intersection with X of a unique equation in the ambient space[12], unlike what happens for the above definition of X, where this is only true for $S^+ + S^-$. In other words, there may exist polynomials s^+, s^-, r^+, r^- such that

$$\begin{aligned} s &= s^+ s^- \,, \\ \xi^{\pm} &= s^{\pm} r^{\mp} \,, \end{aligned} \tag{B.30}$$

and, in particular,

$$S^{\pm} : s^{\pm} = 0 \,, \tag{B.31}$$

where the divisors S^+ and S^- do not necessarily need to have the same homology class. This is expected to hold generally for smooth, $SU(3)$ holonomy Calabi-Yau threefolds, since the group of their 4-cycles is completely specified topologically to be $H^{1,1}(Z)$, and thus all 4-cycles are algebraic anywhere in the complex structure moduli space. We can therefore write

$$\begin{aligned} \Delta'_{SU(2N+1)} &\sim s\left(a_{4,N+1} + \frac{\xi^+}{2s} a_{3,N}\right)\left(a_{4,N+1} + \frac{\xi^-}{2s} a_{3,N}\right) \\ &\sim \left(a_{4,N+1}s^- + \frac{r^-}{2} a_{3,N}\right)\left(a_{4,N+1}s^+ + \frac{r^+}{2} a_{3,N}\right) \,. \end{aligned} \tag{B.32}$$

Having brought $\Delta'_{SU(2N+1)}$ in this form, one can easily read off the homology classes of $W^{\pm}$:

$$\begin{aligned} W^{\pm}_{SU(2N+1)} &= 4c_1 - (N+1)\pi^* S + S^{\mp} \\ &= 4c_1 - (N+1)S^{\pm} - NS^{\mp} \end{aligned} \tag{B.33}$$

Note that the two irreducible components of the Whitney umbrella have different homology classes if and only if the classes of the $SU(2N+1)$ brane stack and image brane stack are different as well.

[12] In [159] X was written as a complete intersection of two equations in an ambient fivefold.

B.3 Computational strategies and survey

After introducing the geometric objects relevant in the weak-coupling picture of our F-theory set-ups, we turn to the actual derivation of our main result, equations (5.41) and (5.42). In principle, it is possible to derive these two formulas analytically. To do so, one can write down a general Tate model, engineer singularities by restricting coefficients accordingly, resolve them and use known intersection relations to reduce $c_3(Y_4) \wedge J$ to an expression in terms of quantities on the base manifold B_3. Once one has an expression for $c_3(Y_4) \wedge J$ in terms of quantities on B_3, this can then be lifted to X.

In practice, this approach quickly becomes very cumbersome. As a way out, we automated the calculation and used an algorithm to calculate $\mathcal{C}$ for a range of examples. Let us go into a bit more detail and outline the algorithm that we applied. The basic idea is as follows: For D7-branes located along a certain set of divisors $\{S_1, \dots, T_1, \dots\}$, one expects the curve $\mathcal{C}$ to be given by a linear combination of all the curves one can obtain from taking intersections between the D7-brane divisors and the divisor Poincaré-dual to $c_1(B_3)$. One can thus write down the most general ansatz, consisting of said $\binom{n_{SU}+n_{USp}+2}{2}$ terms. Next, one chooses a base manifold B_3 and selects the gauge groups hosted on the D7-brane divisors. In toric language, choosing a gauge group corresponds to determining a set of tops [160, 115] sharing the same generic fiber space. After requiring flatness in codimension 2 on B_3 [161, 88], see also [162, 92], one makes an explicit choice for the D7-brane divisors S_i and T_i. This choice fixes the location of the tops over the base manifold. Using the methods developed in [88][13], one can then construct all Calabi-Yau fourfolds containing the given base and tops. After choosing one of these fourfolds, it is straightforward to compute its third Chern class and to calculate intersection numbers with a base of divisors. By demanding

$$\int_{Y_4} c_3 \wedge \omega^{(0)}{}_\alpha \stackrel{!}{=} \mathcal{C} \cdot D^b_\alpha \qquad \text{(B.34)}$$

one thus obtains a set of linear constraints that the expansion coefficients for $\mathcal{C}$ have to satisfy.

Instead of using a single basis, one can use a set of base manifolds, find all homologically inequivalent tuples of base divisors and then enforce (B.34) for all such manifolds Y_4. In creating such a large number of manifolds, we heavily relied on the methods and code developed in [161, 116, 88, 163]. [14] Let us emphasize here that while the algorithm described here deals with computing the image of c_3 under the F-theory limit, it is straightforward to generalize this set-up to compute other quantities that might be challenging to obtain analytically.

Unlike the analytic computation, this is of course by no means a rigorous proof. Nevertheless, the above procedure quickly produces highly overconstrained systems of linear equations for a variety of

[13] In [90, 91] an equivalent method for determining all fibrations of a top over a base was presented. While in [88] one computes the set of fourfold completions by using convexity arguments for the fiber polygon, the authors of [90, 91] demand that the fiber coordinates must be sections of certain line bundles, thereby enforcing restrictions on the line bundle classes.

[14] Note that similar methods for constructing global F-theory compactifications have recently been under intensive investigation in [89, 87, 162, 90, 92, 164, 91].

bases. In all of these cases, we verified that there exist unique solutions fitting furthermore into the logic of equations (5.41) and (5.42). We therefore believe that our findings are relatively robust.

Last, but not least, let us close this section with a concrete survey of the gauge groups that we studied in order to verify (5.41) and (5.42). For models with purely non-Abelian gauge groups we studied simple gauge groups with rank ≤ 10 and gauge groups with two or three simple factors and rank ≤ 7. Furthermore, we examined $U(1)$-restricted models with simple non-Abelian gauge groups of rank ≤ 10. For those cases, we found the following expressions to hold:

$$\begin{aligned} \mathcal{C}_{\mathbb{P}_{231}} &= -60 c_1 (B_3)^2 \\ &\quad + 16 \sum_{i=1}^{n_{SU}} N_i c_1 \cdot S_i - \sum_{i=1}^{n_{SU}} N_i (N_i + 1) S_i^2 - \sum_{i \neq j} N_i N_j S_i \cdot S_j \\ &\quad + 15 \sum_{i=1}^{n_{USp}} 2 M_i c_1 \cdot T_i - \sum_{i=1}^{n_{USp}} 2 M_i (2 M_i + 1) T_i^2 - \sum_{i \neq j} 4 M_i M_j T_i \cdot T_j \qquad \text{(B.35)} \\ \mathcal{C}_{F_{11}, SU(2N)} &= -36 c_1 + 18 N S \cdot c_1 - 2N(N+1) S^2 \qquad \text{(B.36)} \\ \mathcal{C}_{F_{11}, SU(2N+1)} &= -36 c_1 + (18N + 10) S \cdot c_1 - 2(N^2 + 2N + 1) S^2 \qquad \text{(B.37)} \end{aligned}$$

Using the expressions for the Whitney umbrella and the pullbacks of the gauge group divisors to the double cover X given in subsections B.1 and B.2, one can confirm that the formulas are equivalent to equations (5.41) and (5.42).

C Results of the dimensional reduction

C.1 Results of chapter II

In the following we give the results of the dimensional reduction of the higher-derivative corrections in (3.3). We consider only terms which have two external derivatives and hence the various index summations reduce to those ones where two indices of each $\hat{G}$ are external and the remaining summed indices are purely internal. In this spirit, the reduction of $t_8 t_8 \hat{G}^2 \hat{R}^3$ yields

$$t_8 t_8 \hat{G}^2 \hat{R}^3 \hat{*} 1 \supset \text{sgn}(\circ \cdots \circ)\, G^{\circ\circ}{}_{\mu_1\mu_2} G^{\mu_1\mu_2}{}_{\circ\circ} \hat{R}^{\circ\circ}{}_{\circ\circ} \hat{R}^{\circ\circ}{}_{\circ\circ} \hat{R}^{\circ\circ}{}_{\circ\circ} *_{11} 1 = 14 \text{ terms} := X_{t_8 t_8}\,. \qquad \text{(C.1)}$$

The symbols $\circ$ schematically represent all appearing permutations of internal indices due to the index structure of the t_8 tensor. One then reduces $\epsilon_{11}\epsilon_{11}\hat{G}^2\hat{R}^3$ and finds

$$\frac{1}{96}\epsilon_{11}\epsilon_{11}\hat{G}^2\hat{R}^3 *_{11} 1 \supset \text{sgn}(\circ \cdots \circ) G^{\circ\circ}{}_{\mu_1\mu_2} G^{\mu_1\mu_2}{}_{\circ\circ} R^{\circ\circ}{}_{\circ\circ} R^{\circ\circ}{}_{\circ\circ} R^{\circ\circ}{}_{\circ\circ} *_{11} 1 = 8 \text{ terms } - X_{t_8 t_8}\,. \qquad \text{(C.2)}$$

Thus one has

$$-\left(t_8t_8\hat{G}^2R^3+\frac{1}{96}\epsilon_{11}\epsilon_{11}\hat{G}^2R^3\right)*_{11}1=2^7\left[F^i\wedge *_3F^j\right] \tag{C.3}$$
$$\times\Big[R^{(0)m_2}{}_{m_1m_3}{}^{m_4}R^{(0)m_1}{}_{m_2m_5}{}^{m_6}R^{(0)m_3}{}_{m_4m_6}{}^{m_5}(\omega_i^{(0)}{}_m{}^{m_0}(\omega_j^{(0)})_{m_0}{}^m$$
$$+R^{(0)m_2}{}_{m_1m_3}{}^{m_4}R^{(0)m_5}{}_{m_2m_4}{}^{m_6}R^{(0)m_1}{}_{m_5m_6}{}^{m_3}(\omega_i^{(0)})_m{}^{m_0}(\omega_j^{(0)})_{m_0}{}^m$$
$$+6R^{(0)m}{}_{m_1m_2}{}^{m_3}R^{(0)m_4}{}_{m_3m_5}{}^{m_6}R^{(0)m_2}{}_{m_4m_6}{}^{m_5}(\omega_i^{(0)})_m{}^{m_0}(\omega_j^{(0)})_{m_0}{}^{m_1}$$
$$-6R^{(0)m_2}{}_{m_1m_3}{}^{m_4}R^{(0)m}{}_{m_2m_5}{}^{m_6}R^{(0)m_3}{}_{m_4m_6}{}^{m_5}(\omega_i^{(0)})_m{}^{m_0}(\omega_j^{(0)})_{m_0}{}^{m_1}$$
$$-6R^{(0)m_2}{}_{m_1m_3}{}^{m_4}R^{(0)m_5}{}_{m_2m_4}{}^{m_6}R^{(0)m}{}_{m_5m_6}{}^{m_3}(\omega_i^{(0)})_m{}^{m_0}(\omega_j^{(0)})_{m_0}{}^{m_1}\Big]*^{(0)}1.$$

These eight terms, each containing different index summations between three Riemann tensors and the components of two $(1,1)$-forms, can be rewritten using three curvature two-forms and two $(1,1)$-forms as in (4.23).

Let us shortly command on a relation between the Euler density and the higher-derivative correction $\epsilon_{11}\epsilon_{11}\hat{R}^4$ which can be written as

$$\epsilon_{11}\epsilon_{11}R^4=3!E_8(M_{11})\,, \tag{C.4}$$

where one uses the Euler density

$$E_n(M_d)=\frac{1}{(d-n)!}\epsilon_{N_1\cdots N_n}\epsilon^{N_1\cdots N_{d-n}M_{d-n+1}\cdots M_n}R^{N_{d-n+1}N_{d-n+2}}{}_{M_{d-n+1}M_{d-n+2}}\cdots R^{N_{d-1}N_d}{}_{M_{d-1}M_d}\,, \tag{C.5}$$

where $n>0$ and D being the real dimension of the manifold M_D. Let us first compute $E_8(M_3\times M_8)$ for a generic product space. By using the definition (C.5), splitting indices and applying Schouten identities it is straightforward to show that

$$E_8(M_3\times M_8)=-E_8(M_8)+4\,E_2(M_3)E_6(M_8)\,, \tag{C.6}$$

where $E_2(M_3)=-2R^{(3)}_{\text{sc}}$ and

$$E_6(M_8)=6!R^{(0)m_1m_2}{}_{m_1m_2}\cdots R^{(0)m_5m_6]}{}_{m_5m_6}. \tag{C.7}$$

Evaluating this in the case of $M_8=Y_4$, one finds

$$E_6(Y_4)*_8\mathbf{1}=3\cdot 2^7\;\;c_3^{(0)}\wedge J^{(0)}\,. \tag{C.8}$$

which results in the volume correction in (4.13). While

$$E_8(Y_4)=-2^8c_4^{(0)}\;\;. \tag{C.9}$$

C.2 Results of chapter III - $11d$ two derivative terms

The reduction of the lowest order part of the action (2.61) gives the following contribution to the Kinetic terms of the $3d$ theory

$$
\begin{aligned}
S^{(0)}|_{\text{kin}} = \frac{1}{2\kappa_{11}} \int_{\mathcal{M}_3} R * 1 \int_{Y_4} \Big[& e^{\alpha^2(3W^{(2)}-768Z)} \Big(1 + i\delta v^i \omega^{(0)\,m}_{im} + \frac{1}{2}\delta v^i \delta v^j \left(\omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j - \omega^{(0)\,m}_{im} \omega_{jn}{}^n \right) \Big) \\
& + 3\alpha^2 \delta v^i \partial_i W^{(2)}| + 3i\alpha^2 \delta v^i \delta v^j \, \partial_{(i} W^{(2)}| \, \omega_{j)m}{}^m + \frac{3}{2}\alpha^2 \delta v^i \delta v^j \partial_i \partial_j W^{(2)}| + 1536\alpha^2 \delta v^i Z_{m\bar{n}} \omega^{(0)\bar{n}m}_i \\
& + i768\alpha^2 Z \delta v^i \omega^{(0)\,m}_{im} + 384\alpha^2 Z \delta v^i \delta v^j \omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j - 384\alpha^2 \delta v^i \delta v^j Z \omega^{(0)\,m}_{im} \omega^{(0)\,n}_{jn} \Big] *^{(0)} 1 \\
+ \frac{1}{2\kappa_{11}} \int_{\mathcal{M}_3} d\delta v^i \wedge * d\delta v^j \int_{Y_4} \Big[& e^{\alpha^2(3W^{(2)}-768Z)} \left(\frac{1}{2} \omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j - \omega^{(0)\,m}_{im} \omega_{jn}{}^n \right) \\
& + 3i\alpha^2 \partial_{(i} W^{(2)}| \, \omega^{(0)\ \ m}_{j)m} + 3072\alpha^2 i Z_{m\bar{n}} \omega^{(0)\bar{n}m}_i \omega^{(0)\,s}_{js} - 1536\alpha^2 Z \omega^{(0)\,m}_{im} \omega^{(0)\,n}_{jn} \Big] *^{(0)} 1 \\
+ \frac{1}{2\kappa_{11}} \frac{1}{2} \int_{\mathcal{M}_3} F^i \wedge * F^j \int_{Y_4} & e^{\alpha^2(3W^{(2)}-256Z)} \omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j *^{(0)} 1 + \alpha \frac{1}{2\kappa_{11}} \int_{\mathcal{M}_3} F^i \wedge A^j \int_{Y_4} \frac{1}{2} G^{(1)} \wedge \omega^{(0)}_i \wedge \omega^{(0)}_j \, .
\end{aligned}
\tag{C.10}
$$

It is interesting to note that in these terms the value of $\tilde{F}$, $\rho^{(\mathrm{s})}_i$ and $\rho^{(\mathrm{v})}_i$ drop out of these expressions as they contribute only internal space total derivatives to the $3d$ effective theory.

C.3 Results of chapter III - $11d$ eight-derivative terms

Let us record the reduction of certain higher-derivative terms which are used as intermediate results in deriving the effective action (7.27). These results were computed using the mathematica package xAct and required the use of several internal space total derivative and schouten identities.

$$
\begin{aligned}
\int \hat{t}_8 \hat{t}_8 \hat{R}^4 \hat{*} 1|_{\text{kin}} &= \frac{1}{2\kappa_{11}} \int_{\mathcal{M}_3} d\delta v^i \wedge * d\delta v^j \int_{Y_4} 384 \left(Z \omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j + 4 Z_{m\bar{n}r\bar{s}} \omega^{(0)\bar{n}m}_i \omega^{(0)\bar{s}r}_j \right) *^{(0)} 1 \, , \\
-\frac{1}{24} \int \hat{\epsilon}_{11} \hat{\epsilon}_{11} \hat{R}^4 \hat{*} 1|_{\text{kin}} &= \frac{1}{2\kappa_{11}} \int_{\mathcal{M}_3} R * 1 \int_{Y_4} \left(768 Z - 1536 \delta v^i Z_{m\bar{n}} \omega^{(0)\bar{n}m}_i \right) *^{(0)} 1 \\
&\quad + \frac{1}{2\kappa_{11}} \int_{\mathcal{M}_3} d\delta v^i \wedge * d\delta v^j \int_{Y_4} 1536 Z_{m\bar{n}r\bar{s}} \omega^{(0)\bar{n}m}_i \omega^{(0)\bar{s}r}_j *^{(0)} 1 \, , \\
\int_{Y_4} 3^2 2^{13} \hat{C} \wedge \hat{X}_8|_{\text{kin}} &= 0 \, .
\end{aligned}
\tag{C.11}
$$

Similarly we note that the reduction of the $\hat{G}^2 \hat{R}^3$ terms uses the identities

$$
\begin{aligned}
-\int \hat{t}_8 \hat{t}_8 \hat{G}^2 \hat{R}^3 \hat{*} 1|_{\text{kin}} = \frac{1}{2\kappa_{11}^2} 384 \int_{\mathcal{M}_3} F^i \wedge * F^j \int_{Y_4} \Big[& Z \omega^{(0)}_{im\bar{n}} \omega^{(0)\bar{n}m}_j \\
& - 4i Z_{m\bar{n}} \omega^{(0)\bar{r}m}_i \omega^{(0)\bar{n}}_{j\ \ \bar{r}} - 4 Z_{m\bar{n}r\bar{s}} \omega^{(0)\bar{n}m}_{\ \ i} \omega^{(0)\bar{s}r}_{\ \ j} \Big] *^{(0)} 1,
\end{aligned}
$$

$$-\frac{1}{96}\int \hat{\epsilon}_{11}\hat{\epsilon}_{11}\hat{G}^2\hat{R}^3\hat{*}1|_{\rm kin} = \frac{1}{2\kappa_{11}^2}1536\int_{\mathcal{M}_3} F^i\wedge *F^j\int_{Y_4} Z_{m\bar{n}r\bar{s}}\,\omega_i^{(0)\bar{n}m}\omega_j^{(0)\bar{s}r} *^{(0)} 1. \tag{C.12}$$

Finally reducing the $(\hat{\nabla}\hat{G})^2\hat{R}^2$ terms in (2.61) gives

$$\begin{aligned}
\int \hat{s}_{18}(\hat{\nabla}\hat{G})^2\hat{R}^2\hat{*}1|_{\rm kin} = \frac{1}{2\kappa_{11}^2}\int_{\mathcal{M}_3}\frac{1}{2}F^i\wedge *F^j\int_{Y_4}\Big[&-96(1+a_2)\omega_i^{(0)\bar{m}n}\omega_j^{(0)\bar{r}r}R^{(0)\bar{s}}{}_{\bar{r}n}{}^{s}R^{(0)\bar{t}}{}_{\bar{s}r}{}^{t}R^{(0)}{}_{s\bar{m}t\bar{t}}\\
&-48(2+a_1+a_2)\omega_i^{(0)\bar{m}n}\omega_j^{(0)\bar{r}r}R^{(0)\bar{s}s}{}_{t}{}^{u}R^{(0)}{}_{n\bar{m}r}{}^{t}R^{(0)}{}_{s\bar{r}u\bar{s}}+48(1+a_1)\omega_i^{(0)\bar{m}n}\omega_j^{(0)\bar{r}r}R^{(0)\bar{s}}{}_{\bar{m}r}{}^{s}R^{(0)\bar{t}}{}_{\bar{r}n}{}^{t}R^{(0)}{}_{s\bar{s}t\bar{t}}\\
&+48(1+a_2)\omega_i^{(0)\bar{m}n}\omega_j^{(0)\bar{r}r}R^{(0)\bar{s}}{}_{\bar{m}n}{}^{s}R^{(0)\bar{t}}{}_{\bar{r}r}{}^{t}R^{(0)}{}_{s\bar{s}t\bar{t}}-48(2+a_1+a_2)\omega_i^{(0)\bar{m}n}\omega_j^{(0)\bar{r}r}R^{(0)\bar{s}}{}_{\bar{m}n\bar{r}}R^{(0)\bar{t}s}{}_{r}{}^{t}R^{(0)}{}_{s\bar{s}t\bar{t}}\\
&+24(1+a_1)\omega_i^{(0)\bar{m}n}\omega_j^{(0)\bar{r}r}R^{(0)\bar{s}s\bar{t}t}R^{(0)}{}_{n\bar{m}r\bar{r}}R^{(0)}{}_{s\bar{s}t\bar{t}}+48(1+a_2)\omega_i^{(0)\bar{m}n}\omega_j^{(0)\bar{r}r}R^{(0)\bar{s}s}{}_{n}{}^{t}R^{(0)}{}_{r\bar{s}s}{}^{u}R^{(0)}{}_{t\bar{m}u\bar{r}}\\
&+48(a_1-a_2)\omega_i^{(0)\bar{m}n}\omega_j^{(0)\bar{r}r}R^{(0)\bar{s}}{}_{\bar{r}n}{}^{s}R^{(0)}{}_{r}{}^{t}{}_{s}{}^{u}R^{(0)}{}_{t\bar{m}u\bar{s}}-48(1+a_1)\omega_i^{(0)\bar{m}n}\omega_j^{(0)}{}_{n}{}^{r}R^{(0)\bar{r}}{}_{\bar{m}s}{}^{t}R^{(0)\bar{s}s}{}_{r}{}^{u}R^{(0)}{}_{t\bar{r}u\bar{s}}\\
&+48(1+a_1)\omega_i^{(0)\bar{m}n}\omega_j^{(0)}{}_{n}{}^{r}R^{(0)\bar{r}}{}_{\bar{m}r}{}^{s}R^{(0)\bar{s}t}{}_{s}{}^{u}R^{(0)}{}_{t\bar{r}u\bar{s}}+48(1+a_2)\omega_i^{(0)\bar{m}n}\omega_j^{(0)\bar{r}r}R^{(0)\bar{s}}{}_{\bar{m}s}{}^{t}R^{(0)}{}_{n}{}^{s}{}_{r}{}^{u}R^{(0)}{}_{t\bar{r}u\bar{s}}\\
&+96(1+a_2)\omega_i^{(0)\bar{m}n}\omega_j^{(0)\bar{r}r}R^{(0)\bar{s}}{}_{\bar{m}n}{}^{s}R^{(0)}{}_{r}{}^{t}{}_{s}{}^{u}R^{(0)}{}_{t\bar{r}u\bar{s}}-48(1+a_1)\omega_i^{(0)\bar{m}n}\omega_j^{(0)}{}_{n}{}^{r}R^{(0)\bar{r}s}{}_{r}{}^{t}R^{(0)}{}_{s}{}^{u}{}_{t}{}^{v}R^{(0)}{}_{u\bar{m}v\bar{r}}\\
&+48(1+a_2)\omega_i^{(0)\bar{m}n}\omega_j^{(0)\bar{r}r}R^{(0)}{}_{n}{}^{s}{}_{r}{}^{t}R^{(0)}{}_{s}{}^{u}{}_{t}{}^{v}R^{(0)}{}_{u\bar{m}v\bar{r}}\Big] *^{(0)} 1
\end{aligned} \tag{C.13}$$

Where we see directly that in the reduction $\mathcal{Z}_3 = \mathcal{Z}_4 = \mathcal{Z}_5 = \mathcal{Z}_6 = 0$. The result above represents the only terms in the reduction result that can not be expressed in terms of $Z_{m\bar{m}n\bar{n}}$ for arbitrary choice of the parameters a_1 and a_2. For this reason we now make the choice $a_1 = a_2$ which then allows the result to be rewritten as

$$\int \hat{s}_{18}(\hat{\nabla}\hat{G})^2\hat{R}^2\hat{*}1|_{\rm kin} = \frac{1}{2\kappa_{11}^2}192(1+a_1)\int_{\mathcal{M}_3} F^i\wedge *F^j\int_{Y_4}(iZ_{m\bar{n}}\omega_i{}^{(0)\bar{r}m}\omega_j{}^{(0)\bar{n}}{}_{\bar{r}} + 2Z_{m\bar{n}r\bar{s}}\omega_i{}^{(0)\bar{n}m}\omega_j^{(0)\bar{s}r}) *^{(0)} 1\,. \tag{C.14}$$

Furthermore we note that if the basis (A.36) is reduced with and arbitrary set of coefficients and then we demand that the result can be expressed in terms of $Z_{m\bar{m}n\bar{n}}$, then only a multiple of the linear combination

$$\int_{\mathcal{M}_3} F^i\wedge *F^j\int_{Y_4}(iZ_{m\bar{n}}\omega_i^{(0)\bar{r}m}\omega_j^{(0)\bar{n}}{}_{\bar{r}} + 2Z_{m\bar{n}r\bar{s}}\omega_i^{(0)\bar{n}m}\omega_j^{(0)\bar{s}r}) *^{(0)} 1\,, \tag{C.15}$$

is produced.

D Outlook details

As we suggest in section 9.2, there are various directions in which one can naturally extend the study of higher-derivative corrections to $3d$ effective physics. Firstly, we introduce the discussion of complex structure deformations in section D.0.1. Secondly, we allow for $h^{2,1}\neq 0$ and give a preliminary results in section D.0.2.

D.0.1 Complex structure deformations at α^2

Another very interesting direction is to generalize the study of chapter III by allowing for complex structure deformations of the metric, thus at the same time one deforms the background geometry w.r.t. the Kähler deformations given by $h^{1,1}$ harmonic $(1,1)$ -forms $\omega_i^{(0)}$

$$g_{m\bar{n}} = g^{(0)}_{m\bar{n}} + i\delta v^i \omega^{(0)}_{im\bar{n}} \quad , \tag{D.1}$$

and

$$g_{mn} = \delta\bar{z}^I \Xi_{Imn} \quad , \qquad\qquad g_{\bar{m}\bar{n}} = \delta z^I \bar{\Xi}_{I\bar{m}\bar{n}} \quad , \tag{D.2}$$

the $h^{3,1}$ complex structure deformations with $\Xi_I, \bar{\Xi}_I$ given by

$$\Xi_{Imn} = \frac{1}{3|\Omega^{(0)}|^2} \xi^{(0)}_{Im\bar{r}\bar{s}\bar{t}} \Omega^{(0)\bar{r}\bar{s}\bar{t}}_{n} \quad , \qquad\qquad \bar{\Xi}_{I\bar{m}\bar{n}} = -\frac{1}{3|\Omega^{(0)}|^2} \bar{\xi}^{(0)}_{Irst\bar{m}} \bar{\Omega}^{(0)rst}_{\bar{n}} \quad , \tag{D.3}$$

with $|\Omega^{(0)}|^2 = \frac{1}{4!}\Omega^{(0)}_{mnrs}\bar{\Omega}^{(0)mnrs}$ and where $\xi^{(0)}_I, \bar{\xi}^{(0)}_J$ $I, J = 1, \ldots, h^{3,1}$ are harmonic $(1,3), (3,1)$ forms of the zeroth order Calabi-Yau metric, respectively, and $\Omega^{(0)}$ is the holomorphic four-form. One expects α^2-corrections to the kinetic terms for the complex structure moduli arising form the $(\hat{t}_8\hat{t}_8 - \frac{1}{24}\hat{\epsilon}_{11}\hat{\epsilon}_{11})\hat{R}^4$ terms in (2.63). However, this shall be studied in future work. Here we want to state a more peculiar result. At the level of the α^2-corrections one experiences a mixing between the Kähler and complex structure deformations as

$$\begin{aligned}(\hat{t}_8\hat{t}_8 - \frac{1}{24}\hat{\epsilon}_{11}\hat{\epsilon}_{11})\hat{R}^4 \supset \frac{1}{2\kappa^2_{11}} 3^2 2^{10} \int_{\mathcal{M}_3} \Big[& d\delta v^i \wedge * d\delta z^I \int_{Y_4} Y_{Iim\bar{m}} g^{\bar{m}m} *^{(0)} 1 \\ & + d\delta v^i \wedge * d\delta \bar{z}^I \int_{Y_4} \overline{Y_{Iim\bar{m}}} g^{\bar{m}m} *^{(0)} 1 \Big] \quad ,\end{aligned} \tag{D.4}$$

where we find a similar structure as in chapter III given by

$$\begin{aligned} Y_{Iim\bar{m}} &= \frac{i}{4!} \epsilon^{(0)}_{m\bar{m}m_1\bar{m}_1m_2\bar{m}_2m_3\bar{m}_3} \epsilon^{(0)}_{n\bar{n}n_1\bar{n}_1n_2\bar{n}_2n_3\bar{n}_3} \nabla^{(0)m_1} \Xi^{(0)\bar{m}_1\bar{n}}_I \nabla^{(0)n} \omega^{(0)\bar{n}_1n_1}_i R^{(0)\bar{m}_2m_2\bar{n}_2n_2} R^{(0)\bar{m}_3m_3\bar{n}_3n_3} \quad , \\ \overline{Y_{Iim\bar{m}}} &= \frac{i}{4!} \epsilon^{(0)}_{m\bar{m}m_1\bar{m}_1m_2\bar{m}_2m_3\bar{m}_3} \epsilon^{(0)}_{n\bar{n}n_1\bar{n}_1n_2\bar{n}_2n_3\bar{n}_3} \nabla^{(0)\bar{m}_1} \bar{\Xi}^{(0)m_1n}_I \nabla^{(0)\bar{n}} \omega^{(0)\bar{n}_1n_1}_i R^{(0)\bar{m}_2m_2\bar{n}_2n_2} R^{(0)\bar{m}_3m_3\bar{n}_3n_3} \quad , \end{aligned} \tag{D.5}$$

where $Y_{Iim\bar{m}}$ and its complex conjugate is obviously antisymmetric in its indices. Note that (D.4) is real since the RHS is a complex expression plus its complex conjugate. This mixing of the Kähler and complex structure sector is intriguing and it will be very interesting to study its implications on the variables of the effective action in particular the Kähler potential.

D.0.2 $h^{2,1}$ chiral multplets at α^2

In the following we perform the reduction of the $\hat{G}^2\hat{R}^3$ structure in (2.68) allowing only for $h^{2,1}$ fluctuations of the M-theory three-form $\hat{C}$ and considering only terms in $3d$ with two external derivatives, which are of maximal order α^2. The four-form field strength $\hat{G}$ gives rise to complex scalars $N^K, \bar{N}^L$ as

$$\hat{G}_{\mu m n \bar{r}} = \partial_\mu N^K \alpha^{(0)}_{K m n \bar{r}} \quad \text{and} \quad \hat{G}_{\mu m \bar{n} \bar{r}} = \partial_\mu \bar{N}^K \bar{\alpha}^{(0)}_{K m \bar{n} \bar{r}} \ , \tag{D.6}$$

with $\alpha^{(0)}_k, \bar{\alpha}^{(0)}_L$ $K, L = 1, \ldots, h^{2,1}$ are harmonic $(2,1)$ and $(1,2)$-forms of the zeroth order Calabi-Yau metric, respectively. Using the closure and co-closure of $\alpha^{(0)}_k, \bar{\alpha}^{(0)}_L$ one finds that

$$\begin{aligned} \nabla_m \alpha^{(0)\ \ \ r}_{Knr} &= \nabla_n \alpha^{(0)\ \ \ r}_{Kmr} \ , \qquad \nabla_m \alpha^{(0)\ \ \ m}_{Knr} = 0 \ , \\ \nabla_{\bar{m}} \alpha^{(0)}_{Knr\bar{s}} &= \nabla_{\bar{s}} \alpha^{(0)}_{Knr\bar{m}} \ , \qquad \nabla_{\bar{m}} \alpha^{(0)\ \ \bar{m}n}_{Kn} = 0 \ , \end{aligned} \tag{D.7}$$

and that

$$\begin{aligned} \nabla_m \bar{\alpha}^{(0)}_{Kn\bar{r}\bar{s}} &= \nabla_n \bar{\alpha}^{(0)}_{Km\bar{r}\bar{s}} \ , \qquad \nabla_m \bar{\alpha}^{(0)\ \ mn}_{Kn} = 0 \ , \\ \nabla_{\bar{m}} \bar{\alpha}^{(0)\bar{n}}_{K\ \ \ \bar{n}\bar{r}} &= \nabla_{\bar{r}} \bar{\alpha}^{(0)\bar{n}}_{K\ \ \ \bar{n}\bar{m}} \ , \text{ and} \qquad \nabla_{\bar{m}} \bar{\alpha}^{(0)\bar{m}}_{K\ \ \ \bar{n}\bar{r}} = 0 \ . \end{aligned} \tag{D.8}$$

Upon using (D.6) one finds the higher-curvature contributions arising from $\hat{G}^2\hat{R}^3$ at order α^2 to the effective action to be

$$\begin{aligned} S^{(2)}_{\hat{G}^2\hat{R}^3}\Big|_{h^{2,1}} = \frac{1}{2\kappa_{11}^2} 384 \int_{\mathcal{M}_3} dN^K \wedge *d\bar{N}^L \int_{Y_4} \big(&- Z \alpha^{(0)}_{Kmn\bar{r}} \bar{\alpha}^{(0)\bar{r}mn}_L + 4i Z_{m\bar{n}} \alpha^{(0)\ \ \ m}_{Krs} \bar{\alpha}^{(0)\bar{n}rs}_L \\ &- 8 Z_{m\bar{n}r\bar{s}} \alpha^{(0)\bar{n}m\bar{o}}_K \bar{\alpha}^{(0)\bar{s}r}_{L\ \ \ \ \bar{o}} \big) *^{(0)} 1 \end{aligned} \ ,$$

which remarkably involves the same $Z_{m\bar{m}n\bar{n}}$ like structures.[15] Note that the $\hat{G}^2\hat{R}^3$ terms in the case of the $(1,1)$ expansion in chapter III did not give rise to the intrinsically non-topological term $Z_{m\bar{m}n\bar{n}}$ with all indices not contracted on the metric, but it originated only from the more involved structure $(\hat{\nabla}\hat{G})^2\hat{R}^2$. One may speculate that fluctuating the $(\hat{\nabla}\hat{G})^2\hat{R}^2$ terms w.r.t. to the $h^{2,1}$-forms requires new structures with six free indices $Z^6_{m\bar{m}n\bar{n}r\bar{r}}$, such that $\alpha^{(0)}_K, \bar{\alpha}^{(0)}_L$ can be entirely hooked upon it. However, we conclude that this sector involves the same Z-like structures we have encountered before but also has the potential to be the playground for interesting studies on its own right, as of among other things to identify new fundamental building blocks.

[15] Note that we restrict ourselves to the sector where $\hat{C}$ only is expanded in harmonic $(2,1)$ and $(1,2)$-forms.

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