# Higher Brauer Groups

**Thomas Sebastian Alexander Benedikt Jahn**

Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
der Ludwig–Maximilians–Universität München

vorgelegt von
Thomas Sebastian Alexander Benedikt Jahn
am 2. Juni 2015

# Higher Brauer Groups

**Thomas Sebastian Alexander Benedikt Jahn**

Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
der Ludwig–Maximilians–Universität München

vorgelegt von
Thomas Sebastian Alexander Benedikt Jahn
am 2. Juni 2015

Erstgutachter: Prof. Dr. Andreas Rosenschon, München

Zweitgutachter: Prof. Dr. Alexander Schmidt, Heidelberg

Tag der mündlichen Prüfung: 21. Juli 2015

# Contents

# Zusammenfassung

Die klassische kohomologische Brauergruppe $\mathrm{Br}(X) = \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m)$ einer glatten algebraischen Varietät $X$ ist in verschiedenen Kontexten der algebraischen Geometrie von Interesse. Beispielsweise steht die kohomologische Brauergruppe in Zusammenhang mit Werten von Zeta-Funktionen und einer Vermutung von Tate über die Surjektivität gewisser Zyklenabbildungen. Für eine glatte projektive Fläche $X$ über einem endlichen Körper gilt die Tate Vermutung für Divisoren an der Primzahl $\ell$ ungleich der Charakteristik genau dann, wenn die $\ell$-primär Gruppe $\mathrm{Br}(X)(\ell)$ endlich ist. Weiter hat die Zeta-Funktion von $X$ eine Darstellung als eine rationale Funktion, wobei nach einer Vermutung von Tate und Artin sich bestimmte Werte einer der auftretenden Faktoren durch eine Formel approximieren lassen, die die Ordnung $|\mathrm{Br}(X)|$ der Brauergruppe involviert. Urabe hat gezeigt, dass für eine solche Fläche $X$ und eine Primzahl $\ell$ ungleich der Charakteristik, die Gruppenordnung $|\mathrm{Br}(X)(\ell)_{\mathrm{nd}}|$ der $\ell$-primär Gruppe $\mathrm{Br}(X)(\ell)$ modulo ihrer maximalen divisiblen Untergruppe eine Quadratzahl ist.

Blochs Zyklenkomplex definiert für jedes $n \in \mathbb{N}$ einen Komplex $\mathbb{Z}(n)_{\mathrm{et}}$ von étale Garben, wobei $\mathbb{Z}(1)_{\mathrm{et}} \sim \mathbb{G}_m[-1]$. Die Gruppen $\mathrm{Br}^r(X) = \mathbb{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}(r)_{\mathrm{et}})$ definieren 'höhere' Brauergruppen, deren Eigenschaften diejenigen der klassischen Brauergruppe verallgemeinern. Zum Beispiel gilt die Tate Vermutung für eine glatte projektive Varietät $X$ über einem endlichen Körper in Kodimension $r$ an der Primzahl $\ell$ genau dann, wenn $\mathrm{Br}^r(X)(\ell)$ endlich ist.

In diesem Dissertationsprojekt verallgemeinern wir Urabes Resultat über die Ordnung von Brauergruppen von Flächen auf 'höhere' Brauergruppen von Varietäten der Dimension $2r$, d.h. wir zeigen: Ist $X$ eine glatte projektive Varietät über einem endlichen Körper der Dimension $2r$, so ist $|\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}}|$ eine Quadratzahl.

# Abstract

The classical cohomlogical Brauer group $\mathrm{Br}(X) = \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m)$ of a smooth algebraic variety $X$ is of interest in various aspects of algebraic geometry. For example, the cohomological Brauer group is related to values of zeta functions and a conjecture by Tate about the surjectivity of certain cycle maps. For a smooth projective surface $X$ over a finite field the Tate conjecture for divisors at a prime $\ell$ not equal to the characteristic holds if and only if the $\ell$-primary subgroup $\mathrm{Br}(X)(\ell)$ is finite. Moreover, the zeta function of $X$ can be written as a rational function, where according to a conjecture by Artin and Tate one of the appearing factors admits an approximation by a formula involving the order $|\mathrm{Br}(X)|$ of the Brauer group. Urabe has shown that for such a surface $X$ and a prime $\ell$ not equal to the characteristic, the order $|\mathrm{Br}(X)(\ell)_{\mathrm{nd}}|$ of the $\ell$-primary subgroup $\mathrm{Br}(X)(\ell)$ modulo its maximal divisible subgroup is a square number.

Bloch's cycle complex defines for each $n \in \mathbb{N}$ a complex $\mathbb{Z}(n)_{\mathrm{et}}$ of étale sheaves, where $\mathbb{Z}(1)_{\mathrm{et}} \sim \mathbb{G}_m[-1]$. The groups $\mathrm{Br}^r(X) = \mathbb{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}(r)_{\mathrm{et}})$ define 'higher' Brauer groups, whose properties generalise those of the classical Brauer group. For example, the Tate conjecture for a smooth projective variety $X$ over a finite field in codimension $r$ at a prime $\ell$ holds if and only if $\mathrm{Br}^r(X)(\ell)$ is finite.

In this dissertation we generalise Urabe's result about the oder of Brauer groups of surfaces to 'higher' Brauer groups of varieties of dimension $2r$, i.e. we show: If $X$ is a smooth projective variety over a finite field of dimension $2r$, then $|\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}}|$ is a square number.

# Chapter 1.

# Introduction

In the late Fifties Birch and Swinnerton-Dyer related the rank of the group of rational points of an elliptic curve over a number field with the order of the poles of its $L$-series. Based on a their significant set of empirical data, they conjectured a relation between the order of the poles of L-series and the number of non-torsion generators of the group of rational points. Later they refined their methods which allowed them to deduce from their data a more detailed description of the asymptotic behaviour of the L-series. These conjectures relating geometric and arithmetic objects gave rise to much research in algebraic geometry and algebraic number theory, and led to a number of other conjectures.

For example, Artin and Tate stated a variant of this conjecture for smooth projective surfaces over finite fields. More precisely, they gave a conjectural formula for the asymptotical behaviour of the zeta function of the surface $X$ at 1, where it has a pole. This formula involves the order of the cohomological Brauer group $\mathrm{Br}(X) = \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m)$ (the finiteness of this group is part of the conjecture). The conjectures of Birch and Swinnerton-Dyer also motivated the Tate conjecture about the image of certain cycle maps and its relation to the order of certain poles of zeta functions. In case of a surface, Tate and Artin have shown that the Brauer group provides the obstruction for the Tate conjecture for divisors to hold.

In an attempt to generalise this type of relation between the description of the zeta function at 1 and the Brauer group $\mathrm{Br}(X)$ to describe the zeta function at other values, Lichtenbaum conjectured that one should replace the single étale sheaf $\mathbb{G}_m$ with a suitable (finite) complex of étale sheaves satisfying certain axioms. In weight 2, corresponding to the zeta function at 2, he constructed such a complex and showed that it satisfies many of the expected properties.

Bloch's cycle complex gives rise to (unbounded) complexes of étale sheaves $\mathbb{Z}(n)_{\mathrm{et}}$, which satisfies many of the axioms stated by Lichtenbaum; in fact, one expects that $\mathbb{Z}(n)_{\text{et}}$ should satisfy all of these axioms. We refer to the hypercohomology groups of $\mathbb{Z}(n)_{\text{et}}$ as the Lichtenbaum cohomology groups. As a special case, we set $\mathrm{Br}^r(X) = \mathbb{H}^{2r+1}_{\text{et}}(X, \mathbb{Z}(r))$; since $\mathbb{Z}(1)_{\text{et}} \sim \mathbb{G}_m[-1]$, we have $\mathrm{Br}(X) \cong \mathrm{Br}^1(X)$, and we may view the $\mathrm{Br}^r(X)$ as higher Brauer groups. These higher Brauer groups have properties analogous to the ones of the classical Brauer group. For example, if $X$ is a smooth projective variety over a finite field, the Tate conjecture holds for $X$ in codimension $r$ at the prime $\ell$ if and only if $\mathrm{Br}^r(X)(\ell)$ is finite.

In this dissertation we consider the question whether one can generalise Urabe's theorem to these higher Brauer groups. In our main result we show that for a smooth projective variety $X$ over a finite field of dimension $2r$ and a prime $\ell$ not equal to the characteristic, the group $\mathrm{Br}^r(X)(\ell)_{\text{nd}}$ is a square number; this generalises Urabe's result for $r = 1$ to all $r \geq 1$.

In the following chapter we will explain in detail how the classical Brauer group is related to various conjectures, and how the higher Brauer groups provide a natural generalisation of this setting. In Chapter 3 we will give the proof of our main result:

**Theorem 1.0.1.** *Let $X$ be a smooth projective variety of dimension $2r$ over a finite field $k$. Then $|\mathrm{Br}^r(X)(\ell)_{\text{nd}}|$ is a square number for every $\ell \neq \mathrm{char}(k)$.*

Urabe's proof for $r = 1$ mainly uses cohomological methods such as pairings induced by the cup product, the Hochschild-Serre spectral sequence and Steenrod operations. Moreover, an important role in Urabe's proof is played by the Wu formula for étale cohomology (also proven in [Ura96]). We basically use the analogous methods in our proof. However, the cohomology class of the canonical divisor, which plays a crucial role in Urabe's proof, had to be replaced by suitable cohomology classes for $r > 1$. This classes are made up of Chern classes and their construction is the crux of our generalisation.

Theorem 1.0.1 has been published in Mathematische Annalen 362 No. 1 (2015), 43–54. The content of Chapter 3 is an expanded and more detailed version of the proof of this result.

## Notations and Conventions

Let $G$ be a group. The order of $G$ is denoted by the symbol $|G|$. For each integer $n$ we have a morphism $m_n : G \to G$ that is multiplication by $n$. The kernel of this map is denoted by $G_n = \ker(m_n)$.

The torsion subgroup $G_{\text{tor}} \subseteq G$ is $G_{\text{tor}} = \bigcup_n G_n$. We denote by $G_{\text{free}}$ the quotient $G/G_{\text{tor}}$. Given a prime $\ell$ we let $G(\ell) = \bigcup_{n\in\mathbb{N}} G_{\ell^n}$ to which we refer as *$\ell$-primary torsion subgroup*.

A subgroup $H$ of $G$ is called divisible, if for each positive integer $n$ we have $nH = H$. The maximal divisible subgroup of a group $G$ is denoted by $G_{\text{div}}$; the quotient $G/G_{\text{div}}$ is referred to as non-divisible subgroup $G_{\text{nd}}$.

Let $k$ be a field. Its algebraic closure is denoted by $\overline{k}$. The absolute Galois group of $k$, i.e. the Galois group $\text{Gal}(\overline{k}/k)$ of the algebraic closure $\overline{k}$ over $k$ is denoted by $G_k$ (in Chapter 3 we simply put $G$). For a scheme $X$ over $k$ we put $\overline{X} = X \times_k \overline{k}$.

## Acknowledgement

There are many people who in one way or another deserve many thanks. Due to space limitation I am only able to mention a few of them.

First of all I am grateful to my advisor Professor Andreas Rosenschon for suggesting the problem, his helpful advice and many fruitful conversations.

Then I would like to express my gratitude to my colleagues and friends at Munich University, in particular Thomas Götzer and Martin Hofer, for our useful discussions and their support – both mathematical and non-mathematical.

I also want to thank my family for their encouragement and unconditional support throughout my life.

Finally, and above all I thank my affectionate fiancée Sabrina for everything.

# Chapter 2.

# Higher Brauer Groups

After having produced a set of empirical data relating the rank of the group of rational points of an elliptic curve over a number field with the order of the poles of its $L$-series, Birch and Swinnerton-Dyer conjectured that the L-series $L(s)$ of an elliptic curve behaves asymptotically as $c(1-s)^r$ at $s = 1$ (Conjecture 2.2.1) for some constant $c$. Using a refinement of their methods they also were able to correctly predict the value of that constant in a number of examples. Therefore, they conjectured a precise formula for the constant $c$ involving the order of the Tate-Shafarevich group, and the group of rational points of the curve (Conjecture 2.2.2). Of course, using the order of the Tate-Shafarevich group only make sense if it is finite; this is conjectured to be true but is still an open problem.

These conjectural formulas can be reformulated as formulas that describe the asymptotic behaviour of a factor of the zeta function $\zeta(X,s)$ of a smooth projective surface $X$ over a finite field (Conjecture 2.2.4). Here the role of the Tate-Shafarevich group is played by the cohomological Brauer group $\mathrm{Br}(X) = \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m)$ and the group of rational points is replaced by the group of cycles of codimension 1 modulo homological equivalence. Again the formula only makes sense if the cohomological Brauer group $\mathrm{Br}(X)$ is finite; this is also conjectured but not known.

Thinking about the Birch and Swinnerton-Dyer conjectures, Tate was led to conjecture that for every smooth projective $k$-variety $X$ with $k$ finitely generated over its prime field, the cycle class map $\mathrm{CH}^1(X) \otimes \mathbb{Q}_\ell \to \mathrm{H}^2_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell(1))^{G_k}$ is surjective. In fact, if $X$ is a smooth projective surface over a finite field, this conjecture holds if and only if $\mathrm{Br}(X)(\ell)_{\mathrm{nd}}$ is finite (Theorem 2.2.18).

The (cohomological) Brauer group of $X$ is defined using the étale sheaf $\mathbb{G}_m$ which is known to be related to the behaviour of $\zeta(X,s)$ at $s = 1$ (and the constant sheaf $\mathbb{Z}$ describes the behaviour at $s = 0$). One therefore looked for sheaves describing the poles at $s = 2, 3, \ldots$ Lichtenbaum suggested that such sheaves might not exist and one should rather look for complexes of étale sheaves $\Gamma(2), \Gamma(3), \ldots$ for that purpose. Moreover, he predicted a number of properties which complexes that admit a generalisation of the known formulas for the poles at $s = 0, 1$ should satisfy. Under some conditions Milne proved that, if there exist complexes of étale sheaves satisfying Lichtenbaum's axioms, the expected formulas would indeed hold (p. 24).

Bloch's cycles complexes are complexes of étale sheaves $\mathbb{Z}(2)_{\text{et}}, \mathbb{Z}(3)_{\text{et}}, \ldots$ which seem to satisfy the properties predicted by Lichtenbaum. This complexes are used to define Higher Brauer groups (Definition 2.4.8). These higher Brauer group seem to be a good generalisation of Brauer groups as $\text{Br}(X) \cong \text{Br}^1(X)$ and for example (Theorem 2.4.12), if $X$ is a smooth projective variety over a finite field $k$, finiteness of $\text{Br}^r(X)(\ell)_{\text{nd}}$ is equivalent to the Tate conjecture in codimension $r$, i.e. the assertion that canonical cycle map $\text{CH}^r(X) \otimes \mathbb{Q}_\ell \to \text{H}^{2r}_{\text{et}}(\overline{X}, \mathbb{Q}_\ell(r))^{G_k}$ is surjective.

## 2.1. Brauer groups

Before motivating and introducing higher Brauer groups we recall the definition and basic properties of cohomological Brauer groups of a scheme $X$.

The presheaf $\mathbb{G}_m$ given by $U \mapsto \text{Hom}_X(U, X \times \text{Spec } \mathbb{Z}[T, T^{-1}])$ is in fact a sheaf for the étale site on $X$. Etale cohomology in degree 1 with coefficients in $\mathbb{G}_m$ is isomorphic to the Chow group in codimension 1. In degree 2 we get the Brauer group of $X$:

**Definition 2.1.1.** *The* (cohomological) Brauer group *of a scheme $X$ is the étale cohomology group* $\text{Br}(X) := \text{H}^2_{\text{et}}(X, \mathbb{G}_m)$.

In the literature one also finds Brauer groups defined using similarity classes of Azumaya algebras, see for example [Gro68a]. This definition is related, but in general not equivalent, to our definition. In order to make this more precise we give the Azumaya algebra definition of Brauer groups following [Mil80, IV].

Let $R$ be a commutative local ring and $A$ an $R$-algebra containing an identity element. We assume that $R$ is identified via $R \to A$, $r \mapsto r1$ with a subring of the center of $A$. Denote by $A^\circ$ the opposite algebra of $A$, i.e. if $A$ is an algebra with addition $+$ and multiplication $\bullet$, $A^\circ$ is the algebra with addition $+$ and multiplication $*$ given by $a * b = b \bullet a$. The $R$-algebra $A$ is called an *Azumaya algebra* if the following conditions are satisfied.

(i) $A$ is free of finite rank as $R$-module.

(ii) The map $f : A \otimes_R A^\circ \to \mathrm{End}(A)$ which is given by $f(a \otimes a')(x) = axa'$ is an isomorphism.

Two such Azumaya algebras $A, A'$ are called *similar* if there exist integers $n, n'$ such that $A \otimes_R M_n(R) \cong A' \otimes_R M_{n'}(R)$, where $M_n(R)$ denotes the $R$-algebra of $n \times n$ matrices with coefficients in $R$. Similarity is in fact an equivalence relation and we denote by $[A]$ the similarity class of $A$. Moreover, the set of similarity classes of Azumaya algebras equipped with the group law given by $[A][A'] = [A \otimes_R A']$ is a group with identity $[R]$ and inverse $[A^\circ]$. This group is called the *Brauer group* $\mathrm{Br}(R)$ of the ring $R$.

The Azumaya Brauer group of a scheme $X$ is defined as follows. A coherent $\mathcal{O}_X$-module $\mathcal{A}$ is an *Azumaya algebra* over $X$ if for any closed point $x \in X$ the algebra $\mathcal{A}_x$ is an Azumaya algebra over $\mathcal{O}_{X,x}$. It immediately follows that $\mathcal{A}$ is locally free of finite rank and that $\mathcal{A}_x$ is an Azumaya algebra over $\mathcal{O}_{X,x}$ at any point $x \in X$ (not necessarily closed). If there exist locally free $\mathcal{O}_X$-modules $\mathcal{E}, \mathcal{E}'$ together with an isomorphism $\mathcal{A} \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}')$, two Azumaya algebras $\mathcal{A}$, $\mathcal{A}'$ over $\mathcal{O}_X$ are said to be *similar*. Again, similarity is an equivalence relation and the set of similarity classes equipped with the product $[\mathcal{A}][\mathcal{A}'] = [\mathcal{A} \otimes \mathcal{A}']$ forms a group – the *Azumaya Brauer group* $\mathrm{Br}_{\mathrm{Az}}(X)$ of $X$.

We will see shortly that for any scheme $X$ the Azumaya Brauer group can be regarded as a subgroup of the cohomological Brauer group. For this result we need the definition of the sheaves $\mathcal{GL}_n$ and $\mathcal{PGL}_n$. For any scheme $U$ we set $\mathrm{GL}_n(U) = \mathrm{GL}_n(\Gamma(U, \mathcal{O}_U))$ and $\mathrm{PGL}_n(U) = \mathrm{Aut}(M_n(\mathcal{O}_U))$. These functors are representable and therefore $U \mapsto \mathrm{GL}_n(U)$ and $U \mapsto \mathrm{PGL}_n(U)$ define sheaves $\mathcal{GL}_n$ and $\mathcal{PGL}_n$ for the flat, and therefore for the étale topology.

Let $\mathcal{A}$ be an Azumaya algebra over $\mathcal{O}_X$. The Skolem-Noether theorem (the theorem is stated and proven e.g. in [Mil80, IV. Proposition 2.3]) yields for each automorphism $\varphi$ the existence of an étale open covering $U_i$ together with elements $u_i \in \Gamma(U_i, \mathcal{A})^*$ such that the restriction $\varphi|_{U_i}$ is given by $a \mapsto u_i a {u_i}^{-1}$. It follows immediately, that the sequence of étale sheaves $1 \to \mathbb{G}_m \to \mathcal{GL}_n \to \mathcal{PGL}_n \to 1$ is a short exact sequence.

Let us assume that for each étale sheaf $\mathcal{F}$ the étale cohomology groups $\mathrm{H}^i_{\mathrm{et}}(X, \mathcal{F})$ are isomorphic to the étale Čech cohomology groups $\check{\mathrm{H}}^i_{\mathrm{et}}(X, \mathcal{F})$ in each degree $i$. For example, this assumption is satisfied for any quasi-projective scheme $X$ over an affine scheme.

One can identify the set of isomorphism classes of Azumaya algebras of rank $n^2$ with $\check{\mathrm{H}}^i_{\mathrm{et}}(X, \mathcal{PGL}_n)$. Then one verifies that the connecting homomorphisms $d_n$ in

$$\check{\mathrm{H}}^i_{\mathrm{et}}(X, \mathcal{GL}_n) \to \check{\mathrm{H}}^i_{\mathrm{et}}(X, \mathcal{PGL}_n) \xrightarrow{d_n} \check{\mathrm{H}}^2_{\mathrm{et}}(X, \mathbb{G}_m)$$

are compatible (for different $n$) with the group law of the Azumayah Brauer group $\mathrm{Br}_{\mathrm{Az}}(X)$ and induce a monomorphism $\mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}(X)$.

Although we have only sketched a proof that works under certain assumptions regarding Čech cohomology, the result holds for any scheme [Gro68a, no. 2], see also [Mil80, IV Theorem 2.5].

**Theorem 2.1.2.** *For a scheme $X$ there exists a canonical map* $\mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}(X)$ *that is a monomorphism.*

There exist examples where this monomorphism is not surjective, see [Gro68b, no. 2]. However, in some cases this monomorphism is in fact an isomorphism. Most notably is the following statement [Gro68b, Corollaire 2.2].

**Theorem 2.1.3.** *Let $X$ be a noetherian scheme of dimension* $\dim(X) \leq 1$ *or a noetherian smooth scheme of dimension* $\dim(X) \leq 2$*. Then* $\mathrm{Br}_{\mathrm{Az}}(X)$ *and* $\mathrm{Br}(X)$ *are isomorphic.*

*Remark* 2.1.4. In the literature the term 'Brauer group' mostly refers to the group defined using Azumaya algebras. However, as we do not use this definition here, we refer to the cohomological Brauer groups simply as Brauer groups.

An important tool for the discussion of Brauer groups is Kummer theory which we will discuss next. For this denote by $\mu_a$ the subsheaf of $\mathbb{G}_m$ such that $\mu_a(U)$ is the group of $a$-th roots of 1 in the ring $\Gamma(U, \mathcal{O}_X)$. The *Kummer sequence*

$$0 \to \mu_a \to \mathbb{G}_m \xrightarrow{e_a} \mathbb{G}_m \to 0\,,$$

where we denote by $e_a : \mathbb{G}_m \to \mathbb{G}_m$ the map given by $e_a(U) : u \mapsto u^a$ in $\mathrm{End}(\mathbb{G}_m(U))$, is an exact sequence of étale sheaves. We consider the associated long exact sequence

$$\ldots \to \mathrm{H}^1_{\mathrm{et}}(X, \mathbb{G}_m) \to \mathrm{H}^1_{\mathrm{et}}(X, \mathbb{G}_m) \to \mathrm{H}^2_{\mathrm{et}}(X, \mu_a) \to \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m) \to \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m) \to \ldots$$

and note that the maps induced by $e_a$ are multiplication by $a$. Using the isomorphism $\mathrm{H}^1_{\mathrm{et}}(X, \mathbb{G}_m) \cong \mathrm{Pic}(X)$ we get the short exact sequence

$$0 \to \mathrm{Pic}(X) \otimes \mathbb{Z}/a\mathbb{Z} \to \mathrm{H}^2_{\mathrm{et}}(X, \mu_a) \to \mathrm{Br}(X)_a \to 0\,. \tag{2.1}$$

*Example* 2.1.5. Computing Brauer groups is generally hard and only few computations are known. Here we will compute the Brauer group of a smooth projective variety $X$ over an algebraically closed field $k$ of characteristic zero.

By $\rho(X)$ we denote the rank of the Neron-Severi group $\mathrm{NS}^1(X)$ and by $b_2$ the second Betti number. We fix a prime $\ell$ and consider the exact sequence (2.1) for $a = \ell^n$. Using the isomorphism $\mathrm{NS}^1(X) \otimes \mathbb{Z}/\ell^n\mathbb{Z} \cong \mathrm{Pic}(X) \otimes \mathbb{Z}/\ell^n\mathbb{Z}$ and going to the projective limit over all $n$ we get the exact sequence

$$0 \to \mathrm{NS}^1(X) \otimes \mathbb{Z}_\ell \to \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{Z}_\ell(1)) \to \mathrm{T}_\ell\mathrm{Br}(X) \to 0$$

where $\mathrm{T}_\ell M = \varprojlim_n M_{\ell^n}$ denotes the Tate module of the module $M$. As the Tate module $\mathrm{T}_\ell\mathrm{Br}(X)$ is torsion-free the torsion subgroups of the two first groups are isomorphic and hence we get an exact sequence of free $\mathbb{Z}_\ell$-modules

$$0 \to (\mathrm{NS}^1(X) \otimes \mathbb{Z}_\ell)_{\mathrm{free}} \to \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{Z}_\ell(1))_{\mathrm{free}} \to \mathrm{T}_\ell\mathrm{Br}(X) \to 0\,.$$

Using this exact sequence tensored with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ and the fact that $\mathrm{Br}(X)(\ell)_{\mathrm{div}}$ is isomorphic to $\mathrm{T}_\ell(\mathrm{Br}(X)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ [Gro68c, Section 8.1] we get an isomorphism $\mathrm{Br}(X)_{\mathrm{div}} \cong (\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$. On the other hand, considering again (2.1) for $a = \ell^n$ we apply the direct limit over all $n$ and get – using that $\mathrm{Pic}(X) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is a divisible group – that $\mathrm{Br}(X)_{\mathrm{nd}} \cong \oplus_\ell \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))_{\mathrm{nd}} \cong \oplus_\ell \mathrm{H}^3_{\mathrm{et}}(X, \mathbb{Z}_\ell(1))_{\mathrm{nd}}$.

A more concrete computation of the Brauer group of a surface over a finite field will be given in Example 2.2.7. There we will use the link between the Brauer group and the zeta function of a variety, which is discussed in Section 2.2.

—

It is conjectured that Brauer groups of varieties over finite fields are finite or that at least the $\ell$-primary part ($\ell$ different from the characteristic of the field) is finite. The first assertion is part of Conjecture 2.2.4; the second statement is equivalent to the Tate conjecture for divisors, which is discussed below (see Theorem 2.2.18). Assuming that the Brauer group is finite, it is of interest to exhibit its order; in particular, as there exist conjectural statements involving the order of Brauer groups such as Conjecture 2.2.4 below. A first result in this direction is the following theorem. Note that $\mathrm{Br}(X)(\ell)_{\mathrm{nd}}$ is always finite and that $\mathrm{Br}(X)(\ell) = \mathrm{Br}(X)(\ell)_{\mathrm{nd}}$ if $\mathrm{Br}(X)(\ell)$ is finite.

**Theorem 2.1.6** (Tate [Tat66b, Theorem 5.1]). *Let $X$ be a smooth projective geometrically connected surface over a finite field $k$. For any prime $\ell \neq \mathrm{char}(k)$ there exists a skew-symmetric bilinear form*

$$\mathrm{Br}(X)(\ell)_{\mathrm{nd}} \times \mathrm{Br}(X)(\ell)_{\mathrm{nd}} \to \mathbb{Q}/\mathbb{Z}\,.$$

*Therefore, the order of* $\mathrm{Br}(X)(\ell)_{\mathrm{nd}}$ *is a square or twice a square.*

*Sketch of Tate's proof.* By Poincaré duality the cup-product pairing is a perfect pairing of Galois modules $\mathrm{H}^2_{\mathrm{et}}(\overline{X}, \mu_m) \times \mathrm{H}^2_{\mathrm{et}}(\overline{X}, \mu_m) \to \mathrm{H}^4_{\mathrm{et}}(\overline{X}, \mu_m^{\otimes 2}) \cong \mathbb{Z}/m\mathbb{Z}$. From the Hochschild-Serre spectral sequence $E_2^{i,j} = \mathrm{H}^i(k, \mathrm{H}^j_{\mathrm{et}}(\overline{X}, \mu_m)) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{et}}(X, \mu_m)$ we get the short exact sequence $0 \to \mathrm{H}^i_{\mathrm{et}}(\overline{X}, \mu_m)_{G_k} \to \mathrm{H}^{i+1}_{\mathrm{et}}(X, \mu_m) \to \mathrm{H}^{i+1}_{\mathrm{et}}(\overline{X}, \mu_m)^{G_k} \to 0$ which implies that the cup-product pairings

$$\mathrm{H}^i_{\mathrm{et}}(X, \mu_m) \times \mathrm{H}^{5-i}_{\mathrm{et}}(X, \mu_m) \to \mathrm{H}^5_{\mathrm{et}}(X, \mu_m^{\otimes 2}) \cong \mathbb{Z}/m\mathbb{Z} \tag{2.2}$$

are again perfect pairings.

We consider the long exact cohomology sequence associated with the canonical short exact sequence $0 \to \mu_m \to \mu_{m^2} \to \mu_m \to 0$ and define the two groups $C := \mathrm{coker}(\mathrm{H}^2_{\mathrm{et}}(X, \mu_{m^2}) \to \mathrm{H}^2_{\mathrm{et}}(X, \mu_m))$ and $K := \ker(\mathrm{H}^3_{\mathrm{et}}(X, \mu_m) \to \mathrm{H}^3_{\mathrm{et}}(X, \mu_{m^2}))$. These groups are isomorphic, where the isomorphism is induced by the connecting homomorphism $\delta : \mathrm{H}^2_{\mathrm{et}}(X, \mu_m) \to \mathrm{H}^3_{\mathrm{et}}(X, \mu_m)$. Moreover, $C$ is isomorphic to the group $B = \mathrm{Br}(X)_m / m\mathrm{Br}(X)_{m^2}$.

Using these isomorphisms the pairing (2.2) induces a pairing $B \times B \to \mathbb{Z}/m\mathbb{Z}$ that is skew-symmetric. This can be seen by a short computation. Applying the limit yields a form $\mathrm{Br}(X)(\ell) \times \mathrm{Br}(X)(\ell) \to \mathbb{Q}/\mathbb{Z}$ with kernels $\mathrm{Br}(X)(\ell)_{\mathrm{div}}$. □

Tate moreover conjectured this bilinear form to be alternating, which would imply that $\mathrm{Br}(X)(\ell)_{\mathrm{nd}}$ is always a square number. That the order of $\mathrm{Br}(X)(\ell)_{\mathrm{nd}}$ is indeed a square was proven by Urabe [Ura96] using a different bilinear form and a different method for $\ell = 2$.

We remark that it has been shown by Liu, Lorenzini and Raynaud that if for some prime $\ell$ the $\ell$-primary part $\mathrm{Br}(X)(\ell)$ is finite, the oder of the whole Brauer group $|\mathrm{Br}(X)|$ is a square [LLR05, Theorem 1].

## 2.2. Zeta functions and Tate's conjecture

Let $A$ be an abelian variety of dimension $d$ over a number field $k$. Let $S$ be a finite set of primes of $k$ containing the archimedean ones and those primes at which $A$ does not have good reduction. For each prime $\nu \notin S$, the reduction of $A$ at $\nu$ is an abelian variety $A_\nu$ over the finite field $k(\nu)$. We denote by $N_\nu$ the cardinality of $k(\nu)$. There exists a polynomial with integral coefficients $P_\nu(A,T) = \prod_{i=1}^{2d}(1-\alpha_{i,\nu}T)$ where the complex numbers $\alpha_{i,\nu}$ have absolute value $N_\nu^{1/2}$.

The Euler product $L_S(A,s) = \prod_{\nu\notin S} P_\nu(A, N_\nu^{-s})^{-1}$ converges in the complex half plane of numbers with real part greater than $\frac{3}{2}$. Conjecturally, it admits an analytic continuation to the whole complex plane.

After having produced a lot of numerical evidence, Birch and Swinnerton-Dyer conjectured a relation between the multiplicity of the zero of $L_S(A,s)$ at $s=1$ and the finite rank of the group $A(k)$ of $k$-rational points for elliptic curves [BSD63], [BSD65]. Extended to abelian varieties over number fields their conjecture could be stated as follows [Tat66b, Conjecture A].

**Conjecture 2.2.1.** *The L-function $L_S(A,s)$ has a zero of order equal to the rank of $A(k)$.*

Following the exposition of [Tat66b, § 1], see also [Gor79], we explain how this conjecture is refined. Conjecture 2.2.1 says that $L_S(A,s)$ behaves asymptotically like $c(1-s)^r$ at $s=1$ where $r = \mathrm{rk}A(k)$ and $c$ is some constant. The next aim was to (conjecturally) describe this constant $c$. One of the difficulties in that task was that the constant is not independent from the chosen set $S$ (whereas the above conjecture is). Incorporating work from Tamagawa, Birch and Swinnerton-Dyer were able to resolve these difficulties and finally stated a refined conjecture again supported by numerical evidence.

For each prime $\nu$ of $k$ we denote by $k_\nu$ the completion of $k$ at $\nu$ with $\nu$-adic valuation $|\cdot|_\nu$. We choose for each $\nu$ a Haar measure $\mu_\nu$ on $k_\nu$ such that the ring of $\nu$-integers $\mathcal{O}_\nu$ has measure 1. Let $\omega$ be a non-vanishing holomorphic differential form of degree $d$ on $A$ over $k$. A prime $\nu$ on $k$ is called *bad* if $A$ has bad reduction or the reduction of $\omega$ is not nonzero or not regular. Finally, we denote by the symbol $|\mu|$ the $(\prod_\nu \mu_\nu)$-measure of the quotient of the adèle ring of $A_k$ by the discrete subring $k$. For a finite set $S$ of primes of $k$ that contains the bad ones Birch and Swinnerton-Dyer have defined the L-series

$$L_S^*(A,s) = |\mu|^d \left( \prod_{\nu\notin S} P_\nu(A, N_\nu^{-s}) \prod_{\nu\in S} \int_{A(k_\nu)} |\omega|_\nu \mu_\nu^d \right)^{-1}.$$

For any prime $\nu$ which is not bad the equalities $P_\nu(A, N_\nu^{-1}) = |A_\nu(k(\nu))| \cdot N_\nu^{-d} = \int_{A(k_\nu)} |\omega|_\nu \mu_\nu^d$ hold and therefore, the asymptotic behaviour at $s=1$ is independent of the choice of the set of primes $S$.

The conjectural description of the asymptotic behaviour of this function also involves the Tate-Shafarevich group $\text{Ш}(A,k) = \bigcap_v \ker(\mathrm{H}^1(k,A) \to \mathrm{H}^1(k_v, A_v))$. Finally, note that the groups of rational points $A(k)$ and $A^\vee(k)$ (here $A^\vee$ denotes the dual abelian variety) are both of the same finite rank $r$. We denote by $\{a_1, \dots, a_r\}$ and $\{a'_1, \dots, a'_r\}$ bases of the torsion free quotients of $A(k)$ and $A^\vee(k)$ and by $\langle\, , \rangle$ the canonical height pairing.

Birch and Swinnerton-Dyer conjectured the following refinement of the first conjecture (again extended to abelian varieties) [Tat66b, Conjecture B].

**Conjecture 2.2.2.** *Let $A$ and $S$ be as above.*

1. *The Tate-Shafarevich group $\text{Ш}(A,k)$ is finite.*

2. *The series $L^*_S(A,s)$ asymptotically behaves like*

$$(s-1)^r \frac{|\text{Ш}(A,k)| \cdot |\det\langle a_i, a'_j\rangle|}{|A(k)_{\mathrm{tor}}| \cdot |A^\vee(k)_{\mathrm{tor}}|} \quad \text{as } s \to 1\,.$$

Artin and Tate reformulated this conjecture as a conjecture for smooth projective surfaces over finite fields. We will introduce their conjecture and sketch this reformulation shortly. As their conjecture involves zeta functions we first shall recall their definition and fundamental properties. See e.g. [Ser65] for a more detailed and exhaustive introduction of zeta functions.

For a scheme $X$ of finite type over $\mathbb{Z}$ its zeta function is defined by

$$\zeta(X,s) = \prod_{x \in X^\circ} \frac{1}{1 - N_x^{-s}}\,,$$

where $X^\circ$ denotes the set of closed points $x$ in $X$ and $N_x$ is the number of elements of the (finite) residue field $k(x)$. This product converges absolutely if the real part of $s$ is greater than the dimension $\dim X$ and it can be continued analytically in the complex half-plane where the real part of $s$ is greater than $\dim X - \frac{1}{2}$.

If in particular $X$ is a smooth projective variety of dimension $d$ over the finite field $\mathbb{F}_q$ with $q$ elements, one considers the following function defined by Weil

$$Z(X,T) := \exp\left(\sum_{m=1}^{\infty} |X(\mathbb{F}_{q^m})| \frac{T^m}{m}\right)\,,$$

which satisfies $Z(X, q^{-s}) = \zeta(X,s)$. Regarding this function Weil has stated the following conjectures [Wei49] which have later been proven by Deligne [Del74].

**Theorem 2.2.3** (Weil, Deligne)**.** *Let $X$ be a smooth projective variety of dimension $d$ over a finite field $\mathbb{F}_q$. Then the following assertions hold.*

*(i)* *$Z(X,T)$ is a rational function.*

*(ii)* *$Z(X,T)$ satisfies the functional equation $Z(X,q^{-dT}) = \pm q^{\frac{dE}{2}} T^E Z(X,T)$ where $E$ is the Euler characteristic of $X$.*

*(iii)* *$Z(X,T)$ may be written as the rational function*

$$\frac{P_1(X,T)P_3(X,T)\dots P_{2d-1}(X,T)}{P_0(X,T)P_2(X,T)\dots P_{2d}(X,T)}$$

*with $P_0(X,T) = 1 - T$, $P_{2d}(X,T) = 1 - q^d T$, $P_i(X,T) \in \mathbb{Z}[T]$ and $P_i(X,T) = \prod_{j=1}^{b_i}\left(1-\alpha_j^{(i)}T\right)$ in $\mathbb{C}[T]$ such that $|\alpha_j^{(i)}| = q^{i/2}$.*

*(iv)* *If $X$ is the reduction modulo $p$ of a smooth projective variety $Y$ defined over a number field then $\deg(P_i(X,T))$ is the $i$-th Betti number of $Y(\mathbb{C})$.*

Before Deligne proved the Weil conjectures the rationality of $Z(X,T)$ was proven by Dwork [Dwo60]. It follows from the rationality of $Z(X,T)$ that the zeta function is a meromorphic function in the whole complex plane. Also preceding the proof of Deligne, Grothendieck [Gro64] has shown that the functions $P_i(X,T)$ are the characteristic polynomials of the action of the Frobenius on the cohomology groups $\mathrm{H}^i_{\mathrm{et}}(\overline{X},\mathbb{Q}_\ell)$.

For a surface $X$ over a finite field $\mathbb{F}_q$ Artin and Tate conjectured the following formula for $P_2(X,T)$ involving the Brauer group of $X$.

**Conjecture 2.2.4** (Artin and Tate [Tat66b, Conjecture C])**.** *Let $X$ be a smooth projective surface over a finite field $\mathbb{F}_q$. The Brauer group $\mathrm{Br}(X)$ is finite and*

$$P_2(X,q^{-s}) \sim \left(1-q^{1-s}\right)^{\rho(X)} \frac{|\mathrm{Br}(X)|\cdot|\det(D_i\cdot D_j)|}{q^{\alpha(X)}\cdot|\mathrm{NS}^1(X)_{\mathrm{tor}}|^2}, \quad \textit{as } s\to 1\,. \tag{2.3}$$

The definitions of the quantities occurring in (2.3) are as follows: Recall that we denote the rank of $\mathrm{NS}^1(X)$ by $\rho(X)$. We choose a base $\{D_1,\dots,D_{\rho(X)}\}$ of the torsion-free quotient $\mathrm{NS}^1(X)_{\mathrm{free}}$; by $D\cdot D'$ we denote the intersection product on $\mathrm{NS}^1(X)$. The number $\alpha(X)$ is defined by $\alpha(X) = \chi(X,\mathcal{O}_X) - 1 + \dim_{\mathbb{Q}_\ell}\mathrm{H}^1_{\mathrm{et}}(X,\mathbb{Q}_\ell)$ (this definition agrees with the one given in [Tat66b], cp. [Mil88, Remark 6.5]).

We next sketch how this conjecture is related to Conjecture 2.2.2. Let $C$ be an irreducible smooth curve over a perfect field with function field $k(C)$. Associated to each closed point $v \in C^\circ$ is a valuation and the completion of $k(C)$ with respect to this valuation is denoted by $k(C)_v$. We define the Tate-Shafarevich group of an abelian variety $A$ over $k(C)$ by

$$Ш(C, A) = \bigcap_{v \in C^\circ} \ker(\mathrm{H}^1(k(C), A) \to \mathrm{H}^1(k(C)_v, A_v)) .$$

Using this definition and the usual analogy between number fields and function fields of curves one can state a conjecture analogous to Conjecture 2.2.2 for abelian varieties over function fields.

Artin and Tate started 'deriving' their conjecture from (the function field analogon of) the conjecture of Birch and Swinnerton-Dyer after Artin realised the following connection between Brauer groups and Tate-Shafarevich groups.

**Theorem 2.2.5** (Artin [Tat66b, Theorem 3.1], [Gro68c, No. 4])**.** *Let $X$ be a regular surface and let $f : X \to C$ be a proper morphism with fibres of dimension one. Assume that the geometric fibres are connected and that the generic fibre $X_\eta$ is smooth. If $f$ admits a section, then there exists a short exact sequence*

$$0 \to \mathrm{Br}(C) \to \mathrm{Br}(X) \to Ш(C, \mathrm{Jac}(X_\eta)) \to 0 .$$

The Brauer group of a complete curve is trivial and in that case we have an isomorphism $\mathrm{Br}(X) \cong Ш(C, \mathrm{Jac}(X_\eta))$.

Artin and Tate considered the setting where $f : X \to C$ is a smooth proper morphism from a surface to a curve over a finite field $k = \mathbb{F}_q$ such that the geometric fibres are connected and the generic fibre $X_\eta$ is smooth. Assume moreover that $f$ has a section. As mentioned above we have in that case $\mathrm{Br}(X) \cong Ш(C, A)$ where $A = \mathrm{Jac}(X_\eta)$ is the Jacobian of the generic fibre. For each closed point $v \in C^\circ$ we denote by $X_v$ the fibre $f^{-1}(v)$ and denote by $m_v$ the number of its irreducible components. Moreover, we define a polynomial $P_v$ such that $P_v(N_v^{-s}) = \zeta(X_v, s)(1 - N_v^{-s})(1 - N_v^{1-s})^{m_v}$. We call a closed point $v$ *good* if $f$ is smooth at $v$ and $A$ has good reduction at $v$. For such good $v$ this definition of the polynomial $P_v$ is analogous, in the sense of the usual dictionary between number fields and function fields of a curve, to the definition given for an abelian variety over a number field at the beginning of this section. Define $L(s) := \prod_{v \in C^\circ} P_v(N_v^{-s})^{-1}$.

One easily checks the equation

$$\zeta(X, s) = \prod_{v \in C^\circ} \zeta(X_v, s) = \zeta(C, s)\zeta(C, s-1)L(s)^{-1} \prod_{v \in C^\circ} (1 - N_v^{1-s})^{1-m_v} .$$

Writing the zeta functions in the form given by Theorem 2.2.3(iii) one gets

$$\frac{P_1(X, q^{-s})P_3(X, q^{-s})}{(1-q^{-s})P_2(X, q^{-s})(1-q^{2-s})}$$

on the left hand side and on the right

$$\frac{P_1(C, q^{-s})}{(1-q^{-s})(1-q^{1-s})}\frac{P_1(C, q^{1-s})}{(1-q^{1-s})(1-q^{2-s})}L(s)^{-1}\prod_{v\in C^\circ}(1-N_v^{1-s})^{1-m_v}.$$

Solving this equation gives us that $P_2(X, q^{-s})$ equals

$$\frac{P_1(X, q^{-s})}{P_1(C, q^{-s})}\cdot\frac{P_3(X, q^{-s})}{P_1(C, q^{1-s})}(1-q^{1-s})^2L(s)\prod_{v\in C^\circ}(1-N_v^{1-s})^{m_v-1}.$$

Assuming the asymptotical behaviour of $L(s)$ at $s = 1$ predicted by the function field analogon of Conjecture 2.2.2 we get that $P_2(X, q^{-s})$ asymptotically behaves like

$$\frac{P_1(X, q^{-s})}{P_1(C, q^{-s})}\cdot\frac{P_3(X, q^{-s})}{P_1(C, q^{1-s})}(1-q^{1-s})^{(2+r-\sum_v(m_v-1))}\cdot\frac{|Ш(C, A)|\cdot|\det(D_i\cdot D_j)|}{|A(k(C))_{\mathrm{tor}}|^2}$$

as $s \to 1$. Let $B$ be the cokernel of the embedding of the Picard variety of $C$ into the Picard variety of $X$. One has that $P_1(X, T)/P_1(C, T) = P_1(B, T)$ which, together with $P_1(B, q^{-1})P_1(B, q) = |B(k(C))|^2 \cdot q^{-\dim B}$ we use to again rewrite the formula:

$$(1-q^{1-s})^{(2+r-\sum_v(m_v-1))}\cdot|Ш(C, A)|\cdot\frac{|B(k(C))|^2}{|A(k(C))_{\mathrm{tor}}|^2}\cdot\frac{|\det(D_i\cdot D_j)|}{q^{\dim B}}.$$

We already know that $|Ш(C, A)| = |\mathrm{Br}(X)|$; deeper investigation of $\mathrm{NS}^1(X)$ in this setting (cp. [Tat66b, p. 428f.]) gives us the equations $\rho(X) = 2 + r - \sum_v(m_v - 1)$, $|\mathrm{NS}^1(X)_{\mathrm{tor}}|^2 = (|B(k(C))|/|A(k(C))_{\mathrm{tor}}|)^2$ and $|\det(D_i \cdot D_j)| \cdot q^{\dim B} = |\det\langle a_i, a_j\rangle|$. We therefore end up with the asymptotic behaviour of $P_2(X, q^{-s})$ as predicted by Conjecture 2.2.4. In fact it was this translation which led Artin and Tate state their conjecture [Tat66b, pp. 427–430].

Artin and Tate also conjectured that this connection should hold in greater generality, i.e., they conjectured [Tat66b, Conjecture d] that for more general fibrations $f : X \to C$ ($X$ a surface and $C$ a curve over a finite field and $f$ with connected geometric fibres and smooth generic fibre) the function field analogon of the conjecture of Birch and Swinnerton-Dyer holds for the Jacobian of the generic fibre if and only if Conjecture 2.2.4 holds for $X$. This *Conjecture d* by Artin and Tate has been proven by Liu, Lorenzini and Raynaud [LLR05, Theorem 2].

*Example* 2.2.6. For example, let $X = C_1 \times C_2$ be the product of two smooth projective connected curves and let $f : X \to C_1$ be the canonical projection and denote by $A$ the Jacobian of the generic fibre of $f$. In this case the conjecture of Birch and Swinnerton-Dyer and hence the conjecture of Artin and Tate are known to hold, see [Mil68, Corollary of Theorem 3].

We remark that for rational surfaces Conjecture 2.2.4 is known to be true. This is used in the following concrete example.

*Example* 2.2.7 (Milne [Mil70a, p. 306–307]). Let $k$ be finite field containing the cube roots of one and having characteristic $\mathrm{char}(k) \neq 3$. We fix an element $a \in k$ that is not a cube in $k$ and consider the rational surface $X$ that is given by the zeros of $X_0^3 + X_1^3 + X_2^3 - aX_3^3$.

Consider the field extension $k' = k(\sqrt[3]{a})$ of $k$ and the base change $X' = X \times_k k'$ (which is isomorphic to $\mathbb{P}^2_k$ with six points blown up. We have that $\mathrm{NS}^1(X)$ has rank 1 and therefore the rank of $\mathrm{NS}^1(\overline{X})$ (which is equal to the rank of $\mathrm{NS}'(X')$) equals 7. Thus, the characteristic polynomial $P_2(X, T)$ is of degree 7 and is known to have a zero of multiplicity 1. Moreover, the Frobenius action is already given by the action of $\mathrm{Gal}(k'/k) \cong \mathbb{Z}/3\mathbb{Z}$. From these considerations (and that $P_2(X, T)$ has integral coefficients) it follows that

$$P_2(X, T) = (1 - qT)(1 - zqT)^3(1 - z^2qT)^3$$

where $z$ is a primitive third root of unity. As $(1 - z)^3(1 - z^2)^3 = 3^3 = 27$ the asymptotic behaviour of $P_2(X, q^{-s})$ at $s = 1$ is like $27(1 - q^{1-s})$.

Since we know that the formula of Conjecture 2.2.4 holds in this case and moreover $\alpha(X) = 0$ and $\mathrm{NS}^1(X)_{\mathrm{tor}} = 0$ for rational surfaces, we have the equation $27 = |\mathrm{Br}(X)| \cdot |\det(D_i \cdot D_j)|$. From Noether's formula we get $|\det(D_i \cdot D_j)| = 3$ and therefore $|\mathrm{Br}(X)| = 9$. In Theorem 2.1.6 we have constructed a skew-symmetric pairing $\mathrm{Br}(X)(3)_{\mathrm{nd}} \times \mathrm{Br}(X)(3)_{\mathrm{nd}} \to \mathbb{Q}/\mathbb{Z}$ and we therefore have that $\mathrm{Br}(X)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ rather than $\mathbb{Z}/9\mathbb{Z}$ (here we use that in our case $\mathrm{Br}(X)$ is isomorphic to $\mathrm{Br}(X)(3)_{\mathrm{nd}}$).

—

We turn towards Tate's conjecture. Our approach is to state Tate's conjecture (and related conjectures) only after having it 'derived' heuristically from a certain variant of the conjecture of Birch and Swinnerton-Dyer. Afterwards we give another evidence for Tate's conjecture and finally show how Brauer groups provide obstructions to Tate's conjecture.

We first extend our consideration of zeta functions to varieties over fields that are finitely generated over their prime field (but not necessarily finite). Let $k$ be such a field and let $V$ be an irreducible smooth projective $k$-scheme. We can construct a morphism $f : X \to Y$ of schemes over $\mathbb{Z}$ with $X$ irreducible and $Y$ regular such that the regular fibre of $f$ is $V$. For each closed point $y \in Y^\circ$ the fibre $X_y := f^{-1}(y)$ is a scheme over the function field $k(y)$. This function field is finite and we define the integer $N_y = |k(y)|$. We have that $\zeta(X,s) = \prod_{y \in Y^\circ} \zeta(X_y, s)$ and using Theorem 2.2.3 we can write the zeta function as

$$\zeta(X,s) = \frac{\Phi_0(s)\Phi_2(s)\cdots\Phi_{2d}(s)}{\Phi_1(s)\Phi_3(s)\cdots\Phi_{2d-1}(s)}$$

where $\Phi_i(s) = \prod_{y \in Y^\circ} P_i(X_y, N_y^{-s})^{-1}$ and $d = \dim(V)$.

The order of the zeros of $\Phi_1(s)$ at $s = 1$ is subject to the following conjecture [Tat65, p. 104], which is an extension of Conjecture 2.2.1 of Birch and Swinnerton-Dyer.

**Conjecture 2.2.8.** *Let $V$ and $f : X \to Y$ be as above. The order of the zero of $\Phi_1(s)$ at $s = 1$ (and by duality the order of the zero of $\Phi_{2d-1}(s)$ at $s = \dim X - 1$) and the rank of the group of $k$-rational points of the Picard variety of $Y$ are equal.*

This conjecture led Tate to state the following conjectural relation between the rank of $\mathrm{NS}^i(V)_{\mathrm{free}}$ and the pole order of $\Phi_{2i}(s)$ at $s = \dim Y + 1$ [Tat65, Conjecture 2].

**Conjecture 2.2.9.** *Let $V$ and $f : X \to Y$ be as above. The rank of $\mathrm{NS}^i(V)$ and the order of the pole of $\Phi_{2i}(s)$ at $s = \dim Y + i$ (and hence by duality the order of the pole of $\Phi_{2d-2i}(s)$ at $s = \dim X - i$) are equal.*

Let in particular $i = 1$ and $k$ be finite. If $V \to C$ is a morphism with general fibre $V_c$ over $k(c)$ such that the conjectures make sense and if $C$ and $V_c$ are curves, Conjecture 2.2.8 for $V_c/k(c)$ and Conjecture 2.2.9 for $V/k$ are equivalent [Tat63, Section 4].

From this last conjecture the famous Tate conjecture (and other conjectures due to Tate) can be 'derived' heuristically (at least for finite fields) as we will sketch shortly. Before that recall that there exist cycle maps $\mathrm{Z}^i(V) \otimes \mathbb{Q}_\ell \to \mathrm{H}^{2i}_{\mathrm{et}}(\overline{V}, \mathbb{Q}_\ell(i))$ where $k$ is any field, $\ell \neq \mathrm{char}(k)$ a prime and $V$ a $k$-variety; $\mathrm{Z}^i(V)$ denote the free abelian group generated by the irreducible subschemes of codimension $i$ of $V$. Each class in $\mathrm{H}^{2i}_{\mathrm{et}}(\overline{V}, \mathbb{Q}_\ell(i))$ that is the image of an element of $\mathrm{Z}^i(V)$ is invariant under the action of the Galois group $G_k$ on $\mathrm{H}^{2i}_{\mathrm{et}}(\overline{V}, \mathbb{Q}_\ell(i))$. Moreover, the cycle map passes to rational equivalence. We therefore have cycle maps

$$c^i_{\mathbb{Q}_\ell} : \mathrm{CH}^i(V) \otimes \mathbb{Q}_\ell \to \mathrm{H}^{2i}_{\mathrm{et}}(\overline{V}, \mathbb{Q}_\ell(i))^{G_k}.$$

Let $V$ be a variety over a finite field $k$ such that Conjecture 2.2.9 holds in codimension $i = 1$. We give a heuristic deduction of Tate's conjectures. From Theorem 2.2.3(iii) it follows that $\zeta(V, s)$ has a pole at $s = i$ if and only if $P_{2i}(V, q^{-i}) = 0$. Moreover, the order of the pole at $s = i$ is equal to the multiplicity of the zero of $P_{2i}(V, q^{-s})$ at $s = i$. This amounts to saying that the pole order at $s = i$ is equal to the multiplicity of the factor $1 - q^i T$ in $P_{2i}(V, T)$. Recall that the polynomial $P_{2i}(V, T)$ is the characteristic polynomial of the Frobenius acting on $\mathrm{H}^{2i}_{\mathrm{et}}(\overline{V}, \mathbb{Q}_\ell)$. If we assume that it acts semisimply the multiplicity of $1 - q^i T$ is equal to the dimension of the eigenspace corresponding to $q^i$ of the Frobenius acting on $\mathrm{H}^{2i}_{\mathrm{et}}(\overline{V}, \mathbb{Q}_\ell)$ for some $\ell \neq \mathrm{char}(k)$. After twisting the vector space we finally get that the pole order is equal to the dimension of $\mathrm{H}^{2i}_{\mathrm{et}}(\overline{V}, \mathbb{Q}_\ell(i))^{G_k}$. Since we assume that Conjecture 2.2.9 holds in this setting, we have that the rank of $\mathrm{NS}^1(V)$ and the dimension of $\mathrm{H}^{2}_{\mathrm{et}}(\overline{V}, \mathbb{Q}_\ell(1))^{G_k}$ agree. This means that the cycle map $c^1_{\mathbb{Q}_\ell} : \mathrm{CH}^1(V) \otimes \mathbb{Q}_\ell \to \mathrm{H}^{2}_{\mathrm{et}}(\overline{V}, \mathbb{Q}_\ell(1))^{G_k}$ is surjective.

Because of such reasonings Tate formulated the following “optimistic conjectural statements” [Tat65], [Tat94].

**Conjecture 2.2.10** (Tate's conjecture)**.** *Let $k$ be a field finitely generated over its prime field and let $X$ be a smooth projective $k$-variety. The Tate conjecture $\mathrm{TC}^r(X)_{\mathbb{Q}_\ell}$ in codimension $r$ at the prime $\ell \neq \mathrm{char}(k)$ is the statement that the cycle map*

$$c^r_{\mathbb{Q}_\ell} : \mathrm{CH}^r(X) \otimes \mathbb{Q}_\ell \to \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell(r))^{G_k}$$

*is surjective.*

**Conjecture 2.2.11.** *Let $k$ and $X$ be as in Conjecture 2.2.10. For each $r$ the Galois group $G_k$ acts semisimply on $\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell(r))$.*

**Conjecture 2.2.12** (Strong Tate conjecture)**.** *Let $k$ and $X$ be as in Conjecture 2.2.10. The pole order of $\zeta(X, s)$ at $s = r$ equals the dimension of the subspace of $\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell(r))$ spanned by the image of $c^r_{\mathbb{Q}_\ell}$.*

We remark that in codimension 1 Tate's conjecture implies the Strong Tate conjecture (e.g. [Mil07, Theorem 1.4]). In general, we have the following equivalence.

**Theorem 2.2.13.** *Let $k$ be a finite field and fix an integer $r$.*

*(i) If Tate's conjecture holds in codimension $r$ for some prime $\ell \neq \mathrm{char}(k)$ it holds for every prime different from the characteristic. Similarly, if the assertion of Conjecture 2.2.11 holds for $r$ and some $\ell \neq \mathrm{char}(k)$ it holds for $r$ and any $\ell \neq \mathrm{char}(k)$.*

*(ii) The Strong Tate conjecture for $r$ holds if and only if Tate's conjecture holds in codimensions $r$ and $\dim X - r$ and the assertion of Conjecture 2.2.11 holds for $r$.*

*Proof.* This is part of [Tat94, Theorem 2.9]. □

*Example* 2.2.14. In some cases the Tate conjecture in codimension one is known. These cases include the following.

(a) Any abelian variety $A$ (see Corollary 2.2.17 below).

(b) All K3 surfaces in zero characteristic [Tat94, Theorem 5.6].

(c) Any K3 surface over a finite field $k$ of characteristic $\mathrm{char}(k) \geq 5$ [Cha13].

Moreover, for a variety $X$ in a certain class of varieties over a finite field $k$ the Tate conjecture was established by Soulé [Sou84] in codimensions 0, 1, $\dim X - 1$ and $\dim X$; this class contains products of geometrically irreducible curves, and abelian varieties over $k$.

Another reason for believing in Tate's conjecture (for $r = 1$) is that $\mathrm{TC}^1(A)_{\mathbb{Q}_\ell}$ for an abelian variety $A$ is implied by the following theorem (see Theorem 2.2.16 below). However, note that the following theorem was itself only a conjecture when Tate stated his conjectures.

**Theorem 2.2.15** (Tate, Zarhin and Faltings)**.** *Let $k$ a field finitely generated over its prime field. For abelian varieties $A$ and $B$ over $k$ the canonical map*

$$\mathrm{Hom}_k(A, B) \otimes \mathbb{Q}_\ell \to \mathrm{Hom}_{G_k}(\mathrm{H}^1_{\mathrm{et}}(\overline{A}, \mathbb{Q}_\ell), \mathrm{H}^1_{\mathrm{et}}(\overline{B}, \mathbb{Q}_\ell)) \tag{2.4}$$

*is an isomorphism.*

*Proof.* This theorem was proven by Tate for $k$ being a finite field [Tat66a]. Later Zarhin gave a proof for function fields over finite fields [Zar74a], [Zar74b]. And finally the theorem was proven for number fields by Faltings [Fal83]. The methods used in the proofs can be extended to prove the result for arbitrary fields finitely generated over its prime field. □

**Theorem 2.2.16** (Tate [Tat66a, Theorem 4]). *Let $A$ be an abelian variety over a field finitely generated over its field. The bijectivity of (2.4) for $B = A^\vee$ the dual abelian variety implies* $\mathrm{TC}^1(A)_{\mathbb{Q}_\ell}$.

Before proving the theorem we need to recall some facts about abelian varieties and introduce some notation. Let $\overline{A}$ be an abelian variety over an algebraically closed field. The variety has group law morphism $\mu : \overline{A} \times \overline{A} \to \overline{A}$ and it therefore makes sense to speak about multiplication $n : \overline{A} \to \overline{A}$, $a \mapsto na$ for each $n \in \mathbb{Z}$. We denote the kernel $\ker(n)$ of this multiplication by $\overline{A}_n$. For a given prime $\ell$ the Tate module $\mathrm{T}_\ell \overline{A} = \varprojlim_n \overline{A}_{\ell^n}$ is defined. Recall also, that we have a canonical isomorphism $\mathrm{H}^r_{\mathrm{et}}(\overline{A}, \mathbb{Q}_\ell(s)) \cong \bigwedge^r \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell \overline{A}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell(s)$.

Let $X$ and $Y$ be varieties over a field $k$. The divisors on $X \times Y$ which are of the type $D \times Y + X \times E + (\varphi)$ where $D$ is a divisor on $X$, $E$ is a divisor on $Y$ and $\varphi$ is a function on $X \times Y$ span the subgroup of *trivial correspondences*. The group of *divisorial correspondences* $\mathrm{DC}_k(X, Y)$ is defined to be the quotient of the group of divisors on $X \times Y$ modulo trivial correspondences. We have a canonical isomorphism $\mathrm{Hom}(X, Y) \cong \mathrm{DC}_k(\mathrm{Alb}(X), \mathrm{Pic}(Y))$ [Lan59, Ch. VI, Theorem 2] where $\mathrm{Alb}(X)$ denotes the Albanese variety of $X$ and $\mathrm{Pic}(Y)$ denotes the Picard variety of $Y$. In particular, $\mathrm{Hom}(A, A^\vee) \cong \mathrm{DC}_k(A, A)$.

*Proof (following [Tat66a]).* Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{NS}^1(A) \otimes \mathbb{Q}_\ell & \xrightarrow{\mu^* - p_1^* - p_2^*} & \mathrm{DC}_k(A, A) \otimes \mathbb{Q}_\ell \\ \downarrow c^1_\ell & & \downarrow c^1_\ell \\ \mathrm{H}^2_{\mathrm{et}}(\overline{A}, \mathbb{Q}_\ell(1)) & \xrightarrow{\mu^* - p_1^* - p_2^*} & \mathrm{H}^1_{\mathrm{et}}(\overline{A}, \mathbb{Q}_\ell) \otimes \mathrm{H}^1_{\mathrm{et}}(\overline{A}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(1) \end{array}$$

where $p_1, p_2 : A \times A \to A$ are the projections. Denote by $\Delta : A \to A \times A$ is the diagonal map; the diagram stays commutative when we replace the horizontal maps by $\Delta^*$ in the opposite direction. The pullback $(\mu\Delta)^* \in \mathrm{End}(\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell \overline{A}, \mathbb{Z}_\ell))$ is multiplication by 2 and thus, using the canonical isomorphism $\mathrm{H}^2_{\mathrm{et}}(\overline{A}, \mathbb{Q}_\ell(1)) \cong \bigwedge^2 \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell \overline{A}, \mathbb{Z}_\ell)$, the pullback $(\mu\Delta)^* \in \mathrm{End}(\mathrm{H}^2_{\mathrm{et}}(\overline{A}, \mathbb{Q}_\ell(1)))$ is multiplication by 4. It follows that $(\mu^* - p_1^* - p_2^*)\Delta^* = 4 - 1 - 1 = 2$ (in both rows), i.e. the objects on left are direct summands of the corresponding objects on the right.

We also have a commutative diagram with the horizontal arrows being the canonical $G_k$-equivariant isomorphisms and the right vertical map being the evident morphism coming from (2.4).

$$\begin{array}{ccc} \mathrm{DC}_k(A,A)\otimes\mathbb{Q}_\ell & \xrightarrow{\cong} & \mathrm{Hom}(A,A^\vee)\otimes\mathbb{Q}_\ell \\ \downarrow c_\ell^1 & & \downarrow \\ \mathrm{H}^1_{\mathrm{et}}(\overline{A},\mathbb{Q}_\ell)\otimes\mathrm{H}^1_{\mathrm{et}}(\overline{A},\mathbb{Q}_\ell)\otimes\mathbb{Q}_\ell(1) & \xrightarrow{\cong} & \mathrm{Hom}(\mathrm{H}^1_{\mathrm{et}}(\overline{A},\mathbb{Q}_\ell),\mathrm{H}^1_{\mathrm{et}}(\overline{A},\mathbb{Q}_\ell))\otimes\mathbb{Q}_\ell(1) \end{array}$$

Since the image of (2.4) is $\mathrm{Hom}_G(\mathrm{H}^1_{\mathrm{et}}(\overline{A},\mathbb{Q}_\ell),\mathrm{H}^1_{\mathrm{et}}(\overline{A},\mathbb{Q}_\ell))$ it follows that $c_\ell^1$ yields an isomorphism $\mathrm{NS}^1(A)\otimes\mathbb{Q}_\ell \xrightarrow{\cong} \mathrm{H}^2_{\mathrm{et}}(\overline{A},\mathbb{Q}_\ell(1))^G$, i.e. $\mathrm{TC}^1(A)_{\mathbb{Q}_\ell}$ holds. □

**Corollary 2.2.17.** *It follows from the last two theorems that* $\mathrm{TC}^1(A)_{\mathbb{Q}_\ell}$ *holds for any abelian variety A over a field k finitely generated over its prime field.*

For surfaces over finite fields the Brauer group provides obstructions to the Tate conjecture in codimension 1 to hold:

**Theorem 2.2.18** (Tate [Tat66b, Theorem 5.2])**.** *Let X be a smooth projective surface over the finite field* $\mathbb{F}$*. For any prime* $\ell \neq \mathrm{char}\,\mathbb{F}$ *the Tate conjecture* $\mathrm{TC}^1(X)_{\mathbb{Q}_\ell}$ *holds if and only if the* $\ell$*-primary part of the Brauer group* $\mathrm{Br}(X)(\ell)$ *is finite.*

As we shall see later, higher Brauer groups provide similar obstructions to the Tate conjecture in any codimension for varieties over finite fields of any dimension.

—

There are also 'integral' cycle maps $\mathrm{CH}^r(X)\otimes\mathbb{Z}_\ell \to \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X},\mathbb{Z}_\ell(r))^{G_k}$ and one might ask whether these are surjective. However, they are not as there are examples where there exist torsion elements in $\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X},\mathbb{Z}_\ell(r))^{G_k}$ that do not come from any cycle. In the following we sketch how such counterexamples are being constructed.

For this recall, that the Hodge conjecture for a smooth projective variety $X$ over the field of complex numbers $\mathbb{C}$, is the statement that the image of the cycle map

$$c^r_{\mathbb{Q}} : \mathrm{CH}^r(X)\otimes\mathbb{Q} \to \mathrm{H}^r_{\mathrm{B}}(X,\mathbb{Q}(r))$$

equals a certain group $\mathrm{Hdg}^{2r}(X,\mathbb{Q})$; here $\mathrm{H}^r_{\mathrm{B}}(X,\mathbb{Q}(r))$ denotes singular cohomology. The corresponding integral statement is known to be false as Atiyah and Hirzebruch have provided counterexamples [AH62].

These counterexamples are constructed by observing that each $x \in \mathrm{im}(c^r_{\mathbb{Q}})$ annihilates the Steenrod operations defined by Steenrod for simplicial complexes [Ste47] (see also [Ste62]). On the other hand it is possible to construct classes which by the integral Hodge conjecture should be in the image of the integral cycle map but which do not annihilate the Steenrod operations.

Steenrod operations have been defined in the language of derived functors and hence are available for étale cohomology (see [Eps66]). In particular, for a smooth projective variety $X$ there exist for each pair $(i,j)$ of integers and any prime $\ell$ homomorphisms

$$\mathrm{P}^i : \mathrm{H}^j_{\mathrm{et}}(X,\mathbb{Z}/\ell\mathbb{Z}) \to \mathrm{H}^{i+j}_{\mathrm{et}}(X,\mathbb{Z}/\ell\mathbb{Z}) \quad \text{for } \ell \neq 2$$
$$\mathrm{Sq}^i : \mathrm{H}^j_{\mathrm{et}}(X,\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^{i+j}_{\mathrm{et}}(X,\mathbb{Z}/2\mathbb{Z}) \quad \text{for } \ell = 2\,.$$

The Steenrod operations for $\ell = 2$ are referred to as *Steenrod squares*; we will discuss them in more detail in Section 3.3 as they will be used in the proof of our main result.

That Steenrod operations are available in étale cohomology allows one to construct counterexamples of Atiyah Hirzebruch type for the surjectivity of the integral cycle maps as we will see next. In the subsequent we put $\mathrm{P}^i = \mathrm{Sq}^i$ if $\ell = 2$ in order to achieve a better readability.

**Theorem 2.2.19** (Colliot-Thélène, Szamuely [CTS10, Théorème 2.1])**.**

a) *Let $X$ be a smooth projective variety over an algebraically closed field $k$. For each integer $i$ and each prime $\ell \neq \mathrm{char}(k)$ such that $\ell \geq i$ the restrictions of the odd Steenrod operations $\mathrm{P}^{2i+1} : \mathrm{H}^{2r}_{\mathrm{et}}(X,\mathbb{Z}/\ell\mathbb{Z}) \to \mathrm{H}^{2r+2i+1}_{\mathrm{et}}(X,\mathbb{Z}/\mathbb{Z}\ell)$ to the 'algebraic classes', i.e. classes that come from cycles, are trivial.*

b) *For each algebraically closed field $k$ and each prime $\ell \neq \mathrm{char}(k)$ there exists a smooth projective $k$-variety $X$ and a $\ell$-torsion class $c \in \mathrm{H}^4_{\mathrm{et}}(X,\mathbb{Z}_\ell(2))$ whose image in the cohomology group $\mathrm{H}^4_{\mathrm{et}}(X,\mathbb{Z}/\ell\mathbb{Z})$ is not mapped to zero by at least one operation $\mathrm{P}^{2i+1}$.*

**Corollary 2.2.20.** *Let $k$ be a finite field and let $\ell \neq \mathrm{char}(k)$ be a prime. There exists a smooth projective variety $X$ over $k$ such that $CH^2(X) \otimes \mathbb{Z}_\ell \to \mathrm{H}^4_{\mathrm{et}}(\overline{X},\mathbb{Z}_\ell(2))^{G_k}$ is not surjective.*

*Proof.* By Theorem 2.2.19 b) there exists a smooth projective $\overline{k}$-variety $Y$, a $\ell$-torsion class $c$ in $\mathrm{H}^4_{\mathrm{et}}(Y,\mathbb{Z}_\ell(2))$ and a Steenrod operation $\mathrm{P}^{2i+1}$ such that $\mathrm{P}^{2i+1}(\widehat{c}) \neq 0$, where $\widehat{c}$ is the image of $c$ in $\mathrm{H}^4_{\mathrm{et}}(Y,\mu_\ell^{\otimes 2})$.

Since the class $\widehat{c}$ is torsion, there exists an open subgroup $H \subseteq G_k$ such that $\widehat{c}$ is an element of $\mathrm{H}^4_{\mathrm{et}}(Y,\mu_\ell^{\otimes 2})^H$ and thus, $\widehat{c} \in \mathrm{H}^4_{\mathrm{et}}(Y,\mu_\ell^{\otimes 2})^{G_k}$. If the integral cycle map was surjective, $\widehat{c}$ would be algebraic and thus $\mathrm{P}^{2i+1}(\widehat{c}) = 0$ by Theorem 2.2.19 b). □

In view of this counterexample one could still ask whether the image of the integral cycle map is $\mathrm{H}^{2i}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(i))^{G_k}_{\mathrm{free}}$. Counterexamples provided by Kollár [BCC92, p. 134] show that the corresponding claim for the integral form of the Hodge conjecture is false [Voi07, Section 2.2]. This might suggest that there exist varieties $X$ such that the image of the integral cycle map is a strict subset of $\mathrm{H}^{2i}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(i))^{G_k}_{\mathrm{free}}$. However, Kollár's methods can not be used in an arithmetic setting.

## 2.3. Lichtenbaum's complex

Let $X$ be a smooth projective geometrically connected scheme of dimension $d$ over a finite field with $q = p^n$ elements. We want to understand the behaviour of the zeta function $\zeta(X,s) = Z(X, q^{-s})$ at $s = 0, 1, 2, \ldots$

We first consider $s = 0$, i.e. $t = 1$. It is known [Mil86, Theorem 0.4] that the function $(1-t)Z(X,t)$ converges to

$$\frac{|\mathrm{H}^2_{\mathrm{et}}(X,\mathbb{Z})_{\mathrm{cotor}}| \cdot |\mathrm{H}^4_{\mathrm{et}}(X,\mathbb{Z})| \cdots}{|\mathrm{H}^3_{\mathrm{et}}(X,\mathbb{Z})| \cdot |\mathrm{H}^5_{\mathrm{et}}(X,\mathbb{Z})| \cdots} \quad \text{as } t \to 1\,. \tag{2.5}$$

Note here that the cohomology groups $\mathrm{H}^i_{\mathrm{et}}(X,\mathbb{Z})$ vanish for large $i$ and are finite except for $i = 1, 2$. Recall that an abelian group $G$ is cotorsion if $\mathrm{Ext}(F, G)$ is trivial for each free abelian group $F$. For any abelian group we denote by $G_{\mathrm{cotor}}$ the cotorsion subgroup of $G$.

For the pole at $s = 1$ we have a similar result assuming the Tate conjecture $\mathrm{TC}^1(X)_{\mathbb{Q}_\ell}$ for one (and hence for all, see [Mil86, Proposition 0.3]) prime $\ell \neq p$ [Mil86, Theorem 0.4]. The cohomology groups $\mathrm{H}^i_{\mathrm{et}}(X, \mathbb{G}_m)$ vanish for large $i$ and are finite for $i \neq 1, 3$; in addition $\mathrm{H}^1_{\mathrm{et}}(X,\mathbb{G}_m)_{\mathrm{tor}} \cong \mathrm{CH}^1(X)_{\mathrm{tor}}$ and $\mathrm{H}^3_{\mathrm{et}}(X,\mathbb{G}_m)_{\mathrm{cotor}}$ are finite. We also assume that $\mathrm{Br}(X)$ is finite.

The function $(1-qt)^{a_1(X)} Z(X,t)$ converges to

$$q^{\chi(X,\mathcal{O}_X)} \cdot \frac{|\mathrm{H}^1_{\mathrm{et}}(X,\mathbb{G}_m)_{\mathrm{tor}}| \cdot |\mathrm{H}^3_{\mathrm{et}}(X,\mathbb{G}_m)_{\mathrm{cotor}}| \cdot |\mathrm{H}^5_{\mathrm{et}}(X,\mathbb{G}_m)| \cdots}{|\mathrm{H}^0_{\mathrm{et}}(X,\mathbb{G}_m)| \cdot |\mathrm{H}^2_{\mathrm{et}}(X,\mathbb{G}_m)| \cdots R_1(X)} \tag{2.6}$$

as $t \to q^{-1}$. We do not define the regulator term $R_1(X)$ here (see e.g. [Lic84] for a definition). However, we remark that it agrees – at least under the assumptions of Theorem 2.3.2 below – with the regulator $\det(\delta^1)$ whose definition is sketched below. The number $a_1(X)$ is the order of the pole.

In summary, the constant sheaf $\mathbb{Z}$ is related to the behaviour of the zeta function at $s = 0$ and the sheaf $\mathbb{G}_m$ is related to the behaviour at $s = 1$. This motivates the question which sheaves are related to the behaviour at integers $s \geq 2$.

Instead of a single sheaf for each $s$ (which probably do not exist) Lichtenbaum [Lic84] conjectured the existence of complexes of étale sheaves $\Gamma(r)$ satisfying the following list of axioms. We state the axioms in the derived category of the category of étale sheaves on a scheme $X$.

**(L0)** $\Gamma(0) = \mathbb{Z}$ and $\Gamma(1) = \mathbb{G}_m[-1]$. (Here we regard a sheaf $\mathcal{F}$ as the complex of sheaves that has zero at each degree but $\mathcal{F}$ in degree zero; if $\mathcal{C}$ is a complex of sheaves we denote by $\mathcal{C}[n]$ the same complex with shifted degrees.)

**(L1)** For all $r \geq 1$ the complex $\Gamma(r)$ is acyclic outside of $[1, r]$, i.e. $\mathrm{H}^i(\Gamma(r)) = 0$ for $i < 1$ and $i > r$.

**(L2)** If $\alpha : X_{\mathrm{et}} \to X_{\mathrm{Zar}}$ is the morphism of sites given by the identity map then $\mathrm{R}^{r+1}\alpha_*\Gamma(r) = 0$ ('Hilbert's Theorem 90').

**(L3)** There exists an exact triangle $\Gamma(r) \xrightarrow{n} \Gamma(r) \to \mu_n^{\otimes r} \to \Gamma(r)[1]$.

**(L4)** There exist products $\Gamma(r) \otimes^{\mathrm{L}} \Gamma(s) \to \Gamma(r+s)$.

**(L5)** The sheaf $\mathcal{H}^i(X, \Gamma(r))$ is isomorphic to the sheaf $\mathrm{gr}^r_\Gamma \mathcal{K}_{2r-i}$ (which is the graded quotient with respect to the filtration on Quillen's K-groups [Sou85], see [Mil88, p. 63]).

**(L6)** For a field $F$, the cohomology group $\mathbb{H}^r_{\mathrm{et}}(F, \Gamma(r))$ is canonically isomorphic to $\mathrm{K}^M_r(F)$ (where $\mathrm{K}^M_r(F)$ denotes the $r$th Milnor K-group of the field $F$, see [Mil70b]).

*Remark* 2.3.1. We mention that Beilinson has conjectured the existence of complexes of Zariski sheaves satisfying similar axioms [Bei82].

Assume for the remaining part of this section that $X$ is a smooth projective variety of dimension $d$ over a finite field. Milne proved under some further assumptions that if such complexes $\Gamma(n)$ exist, they admit the expected descriptions of the behaviour of the zeta function at $s = r$, see Theorems 2.3.2 and 2.3.3 below.

Before stating the theorems recall some definitions. The groups $A^r(X)$ are the images of $\mathrm{CH}^r(X)$ in $\mathbb{H}^{2r}_{\mathrm{et}}(\overline{X}, \widehat{\Gamma}(r))$ (see [Mil88, p. 69] for a precise definition of $\mathbb{H}^{2r}_{\mathrm{et}}(\overline{X}, \widehat{\Gamma}(r))$).[1] Denote by $\rho_r(X)$ the rank of $A^r(X)$. There is one homomorphism $G_k \to \widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ that sends the Frobenius to 1. This homomorphism defines

[1] One might think of $\mathbb{H}^{2r}_{\mathrm{et}}(X, \widehat{\Gamma}(r))$ as being defined as the product over all primes $\prod_\ell \mathbb{H}^{2r}_{\mathrm{et}}(X, \Gamma_\ell(r))$ where $\mathbb{H}^{2r}_{\mathrm{et}}(X, \Gamma_\ell(r)) = \varprojlim_n \mathbb{H}^{2r}_{\mathrm{et}}(X, (\Gamma/\ell^n\Gamma)(r))$ for $\ell \neq \operatorname{char} k$. For $\ell = \operatorname{char} k$ one has to change definitions.

a canonical element of $\mathrm{H}^1(k,\widehat{\mathbb{Z}}) \subseteq \mathrm{H}^1_{\mathrm{et}}(X,\widehat{\mathbb{Z}})$ and cupping with this element is a homomorphism $\varepsilon^{2r} : \mathbb{H}^{2r}_{\mathrm{et}}(X,\widehat{\Gamma}(r)) \to \mathbb{H}^{2r+1}_{\mathrm{et}}(X,\widehat{\Gamma}(r))$ for each integer $r$. The homomorphism $\delta^r$ is defined by the commutativity of the following diagram with exact rows (see [Mil88, Equation 3.4.3]).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{H}^{2r}_{\mathrm{et}}(X,\Gamma(r))^{\wedge} & \longrightarrow & \mathbb{H}^{2r}_{\mathrm{et}}(X,\widehat{\Gamma}(r)) & \longrightarrow & \varprojlim_n \mathbb{H}^{2r}_{\mathrm{et}}(X,\Gamma/n\Gamma(r)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \varepsilon^{2r}} & & \downarrow{\scriptstyle \delta^{r}} & & \\ 0 & \longrightarrow & \mathbb{H}^{2r+1}_{\mathrm{et}}(X,\Gamma(r))^{\wedge} & \longrightarrow & \mathbb{H}^{2r+1}_{\mathrm{et}}(X,\widehat{\Gamma}(r)) & \longrightarrow & \varprojlim_n \mathbb{H}^{2r+1}_{\mathrm{et}}(X,\Gamma/n\Gamma(r)) & \longrightarrow & 0 \end{array}$$

Next, we set $\chi(X,\mathcal{O}_X,r) = \sum_{i=0}^{r}(r-i)\chi(X,\Omega^i_X)$. Finally, let $s^i(r)$ be the dimension of the perfect group scheme $\mathcal{H}^i(X,\mathbb{Z}_p(r))$ (see [Mil86]), $a_{i,1},\ldots,a_{i,n}$ the inverse roots of $P_i(X,t)$ and set $\alpha_r(X) = s^{2r+1}(r) - 2s^{2r}(r) + \sum_{\mathrm{ord}_q(a_{2r,j})<r}(r - \mathrm{ord}_q(a_{2r,j}))$.

**Theorem 2.3.2** (Milne [Mil88, Theorem 4.3])**.** *Let $X$ be a smooth projective variety over the finite field $\mathbb{F}_q$. Let $\Gamma(r)$ be a complex on $X$ satisfying Lichtenbaum's axiom (L3) and let $\mathrm{CH}^r(X) \to \mathrm{H}^{2r}_{\mathrm{et}}(X,\Gamma(r))$ be a cycle map that is compatible with the étale cycle maps through the cohomology sequence arising from the exact triangle of axiom (L3). Assume that $\mathrm{H}^{2r+1}_{\mathrm{et}}(X,\Gamma(r))_{\mathrm{nd}}$ is torsion. If the strong Tate conjecture holds for $r$ and all primes $\ell$, then*

$$\chi'(X,\Gamma(r)) := \prod_{i\neq 2r} |\mathbb{H}^i_{\mathrm{et}}(X,\Gamma(r))_{\mathrm{nd}}|^{(-1)^i} \cdot \frac{|\mathbb{H}^{2r}_{\mathrm{et}}(X,\Gamma(r))_{\mathrm{tor}}|}{\det(\delta^r)} \tag{2.7}$$

*is defined, and as $s \to r$*

$$\zeta(X,s) \sim \pm\chi'(X,\Gamma(r)) \cdot q^{\chi(X,\mathcal{O},r)}(1-q^{r-s})^{-\rho_r}.$$

For $s = 0$ (in which case all the assumptions of the theorem are satisfied) we recover (2.5), since $\rho_0 = 0$ and $\chi(X,\mathcal{O}_X,0) = 0$. Similarly, for $s = 1$ we get (2.6), since $\chi(X,\mathcal{O}_X,1) = \chi(X,\mathcal{O}_X)$ and $R_1(X) = \det(\delta^1)$ as noted above. Moreover, $a_1(X) = \rho_1$ by the strong Tate conjecture.

The existence of $\Gamma(r)$ would also give the following results which includes Conjecture 2.2.4 above. More precisely, for $r = 1$ equation (2.8) below is the same as equation (2.3) of the conjecture mentioned.

**Theorem 2.3.3** (Milne [Mil88, Theorem 6.6])**.** *Let $X$ be a smooth projective variety of dimension $d = 2r$ over the finite field $\mathbb{F}_q$ and assume that there exist complexes $\Gamma(n)$ satisfying the following:*

1. *The Lichtenbaum axioms (L3) and (L4).*

2. *There exist natural cycle maps* $\mathrm{CH}^r(X) \to \mathbb{H}^{2r}_{\mathrm{et}}(X, \Gamma(r))$ *which are compatible with the étale cycle maps through the cohomology sequence arising from the exact triangle of axiom (L3) and which are also compatible with the product structure of axiom (L4).*

3. *There exists a degree map* $\mathbb{H}^{2d}_{\mathrm{et}}(\overline{X}, \Gamma(d)) \to \mathbb{Z}$ *compatible with the degree isomorphism* $\mathrm{H}^{2d}_{\mathrm{et}}(\overline{X}, \mu_m^{\otimes d}) \xrightarrow{\cong} \mathbb{Z}/m\mathbb{Z}$ *through the cohomology sequence from the (L3) triangle.*

4. *The groups* $\mathbb{H}^{2r}_{\mathrm{et}}(X, \Gamma(r))$ *and* $\mathbb{H}^{2d-2r}_{\mathrm{et}}(X, \Gamma(d-r))$ *are finitely generated.*

5. *The group* $\mathbb{H}^{2r+1}_{\mathrm{et}}(X, \Gamma(r))$ *is torsion.*

*If moreover the strong Tate conjecture holds in codimension r for all primes* $\ell \neq \mathrm{char}(k)$ *and the cycle map* $\mathrm{CH}^r(X) \to \mathbb{H}^{2r}_{\mathrm{et}}(X, \Gamma(r))$ *is surjective, then*

$$P_{2r}(X, q^{-s}) \sim \pm \frac{|\mathbb{H}^{2r+1}_{\mathrm{et}}(X, \Gamma(r))| \cdot |\det(D_i \cdot D_j)|}{q^{\alpha_r(X)} |A^r(X)_{\mathrm{tor}}|^2} (1 - q^{r-s})^{\rho_r} \quad \text{as } s \to r\,, \tag{2.8}$$

*where the* $D_i$ *form a basis for* $A^r(X)_{\mathrm{free}}$.

Another consequence of the existence of such a complex would be the following obstruction to the Tate conjecture which generalises Theorem 2.2.18.

**Theorem 2.3.4** (Milne [Mil88, Remark 4.5(g)])**.** *Let X be a smooth projective variety over a finite field k. The Tate conjecture* $\mathrm{TC}^r(X)_{\mathbb{Q}_\ell}$ *for a prime* $\ell \neq \mathrm{char}(k)$ *and an integer r is equivalent to the nullity of the divisible subgroup of* $\mathbb{H}^{2r+1}_{\mathrm{et}}(X, \Gamma(r))$.

In view of the last three theorems and the fact that $\mathbb{H}^3_{\mathrm{et}}(X, \Gamma(1)) \cong \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m) = \mathrm{Br}(X)$ one could be tempted to call the groups $\mathbb{H}^{2r+1}_{\mathrm{et}}(X, \Gamma(r))$ 'higher Brauer groups' but although Lichtenbaum has proposed a candidate for the complex $\Gamma(2)$ [Lic87], [Lic90] the existence of such complexes for each $r$ is yet not known and hence these 'higher Brauer groups' would depend on the construction of the complexes $\Gamma(r)$.

However, in the following section we will introduce complexes of sheaves for the étale topology which are conjectured to satisfy the axioms stated by Lichtenbaum. We will later use these complexes to define the higher Brauer groups considered in this dissertation.

## 2.4. Lichtenbaum and motivic cohomology

In this dissertation we use Lichtenbaum cohomology (also referred to as étale motivic cohomology) and motivic cohomology. In particular, the higher Brauer groups discussed in this dissertation are certain Lichtenbaum cohomology groups. We will define Lichtenbaum and motivic cohomology groups using Bloch's cycle complex [Blo86] (see also [Blo94], [Lev94]) which is to be introduced next.

Roughly speaking, the definition of Bloch's complex mimics the construction of simplicial cohomology in algebraic topology.

Let $k$ be a field and $X$ an equi-dimensional $k$-scheme. First, we consider the analogon to $n$-simplices and their faces in topology. For each integer $n \geq 0$ the $n$-simplex is the affine $k$-scheme

$$\Delta^n = \operatorname{Spec}\left(k[t_0,\ldots,t_n]/\left(\sum_{i=0}^{n} t_i - 1\right)\right) \cong \mathbb{A}^n_k\,.$$

Given a map $\rho : \{1,\ldots,m\} \to \{1,\ldots,n\}$ that is non-decreasing, i.e. a map such that $\rho(i) \leq \rho(j)$ for $i < j$, we get an induced map

$$\widetilde{\rho} : \Delta^m \to \Delta^n, \widetilde{\rho}^*(t_i) = \sum_{\rho(j)=i} t_j\,.$$

If $\rho$ is injective, we call $\widetilde{\rho}$ a *face map* and the image $\widetilde{\rho}(\Delta^m) \subseteq \Delta^n$ a *face*. If $\rho$ is surjective, $\widetilde{\rho}$ is called a *degeneracy map*.

The second step in algebraic topology would be to define boundary maps on the free abelian groups generated by $n$-simplices. Bloch's analogon is as follows.

For each index $n \geq 0$ let $z^n(X \times_k \Delta^i)$ be the free abelian group generated by the irreducible closed subvarieties of $X \times_k \Delta^i$ of codimension $n$.

**Definition 2.4.1.** *The group of* codimension $n$ Bloch cycles $z^n(X,i)$ *is the subgroup of* $z^n(X \times_k \Delta^i)$ *generated by the irreducible subvarieties which intersect all faces of* $X \times_k \Delta^i$ *properly.*

One checks that for each face map $\partial_j : \Delta^{i-1} \to \Delta^i$ we get an induced homomorphism $\partial_j^* : z^n(X,i) \to z^n(X,i-1)$ by mapping an irreducible subscheme $T \in z^n(X,i)$ to the intersection (with multiplicities) $T \cap (X \times \Delta^{i-1})$. We consider the alternating sums $d^i = \sum_j (-1)^j \partial_j^* : z^n(X,i) \to z^n(X,i-1)$; as the compositions $d^{i-1}d^i$ are trivial we get a complex:

**Definition 2.4.2.** *We denote by* $z^n(X,\bullet)$ *the complex of abelian groups*

$$\cdots \to z^n(X,i) \xrightarrow{d^i} z^n(X,i-1) \to \cdots \to z^n(X,0) \to 0\,.$$

*By $z^*(X,\bullet)$ we denote the complex of graded abelian groups $\sum_n z^n(X,\bullet)$.*

*Remarks* 2.4.3. (a) There exists an isomorphism $\mathrm{H}_0(z^n(X,\bullet)) \cong \mathrm{CH}^n(X)$. More generally one defines the *m-th higher Chow group of codimension n* to be homology group $\mathrm{H}_m(z^n(X,\star))$.

(b) The complex is covariant functorial for proper maps and contravariant functorial for flat maps [Blo86, Proposition 1.3].

**Lemma 2.4.4.** *The presheaves $z^n(-,i) : U \mapsto z^n(U,i)$ are sheaves for the flat and hence for the Zariski and the étale site on X. Therefore, $z^n(-,\bullet)$ is a complex of sheaves on the small étale site and the small Zariski site of X.*

*Proof.* See e.g. [Gei04, Lemma 3.1]. □

**Definition 2.4.5.** *For an abelian group A and $\tau$ either the Zariski or étale topology we set*

$$A_X(n)_\tau := (z^n(-,\bullet)_\tau \otimes A)[2n-\bullet] .$$

Note that these complexes are unbounded on the left. Lichtenbaum cohomology is defined by taking hypercohomology of the complex $A_X(n)_{\mathrm{et}}$. Note also that we have indexed the complex such that it is a complex of degree $+1$ and Definition A.0.7 is applicable.

**Definition 2.4.6.** *The* Lichtenbaum cohomology group $\mathrm{H}^m_{\mathrm{L}}(X, A(n))$ *with coefficients in A in degree m and weight n (resp.* motivic cohomology groups $\mathrm{H}^m_{\mathrm{M}}(X, A(n))$ *with coefficients in A in degree m and weight n) is the hypercohomology group $\mathbb{H}^m_{\mathrm{et}}(X, A(n)_{\mathrm{et}})$ (resp. $\mathbb{H}^m_{\mathrm{Zar}}(X, A(n)_{\mathrm{Zar}})$).*

Bloch conjectures [Blo86, Section 11] that the complexes $\mathbb{Z}_X(r)_{\mathrm{et}}$ are the complexes $\Gamma(r)$ whose existence was predicted by Lichtenbaum (see Section 2.3 above), i.e. they satisfy the axioms (L0) to (L6). That the complexes $\mathbb{Z}_X(r)_{\mathrm{et}}$ satisfy the axioms (L0) and (L4) has already been proven by Bloch [Blo86]. We note that here the axiom (L1) might be the most difficult to verify.

The following theorem by Geisser and Levine link Lichtenbaum cohomology and étale cohomology and thereby establishes axiom (L3).

**Theorem 2.4.7** (Geisser and Levine [GL01, Theorem 1.5])**.** *Let X be a smooth variety over a field k. For each integer m prime to* char *k and each weight n there exists a quasi-isomorphism $(\mathbb{Z}/m\mathbb{Z})_X(n)_{\mathrm{et}} \xrightarrow{\sim} \mu_m^{\otimes n}$, i.e. there exists for each degree i an isomorphism $\mathrm{H}^i_{\mathrm{L}}(X, \mathbb{Z}/m\mathbb{Z}(n)) \xrightarrow{\cong} \mathrm{H}^i_{\mathrm{et}}(X, \mu_m^{\otimes n})$.*

From this we get the following generalisation of the exact sequence (2.1) coming from Kummer theory. For each prime $\ell$ we have exact sequences

$$0 \to \mathbb{Z}_X(n)_{\mathrm{et}} \xrightarrow{\ell^m} \mathbb{Z}_X(n)_{\mathrm{et}} \to (\mathbb{Z}/\ell^m\mathbb{Z})_X(n)_{\mathrm{et}} \to 0\,.$$

From these sequences we get for each $\ell \neq \mathrm{char}(k)$ with help of Theorem 2.4.7 exact universal coefficient sequences

$$0 \to \mathrm{H}^i_{\mathrm{L}}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}/\ell^m\mathbb{Z} \to \mathrm{H}^i_{\mathrm{et}}(X, \mu_{\ell^m}^{\otimes n}) \to \mathrm{H}^{i+1}_{\mathrm{L}}(X, \mathbb{Z}(n))_{\ell^m} \to 0\,. \qquad (2.9)$$

Finally, we give the definition of higher Brauer groups. Although we motivated the definition for varieties over finite fields only, the definition is given for varieties over arbitrary fields.

**Definition 2.4.8.** *Let $X$ be a smooth quasi-projective variety over a field $k$.*

*(i) For each $r$ the $r$-th higher Brauer group is* $\mathrm{Br}^r(X) := \mathrm{H}^{2r+1}_{\mathrm{L}}(X, \mathbb{Z}(r))$.

*(ii) For each $r$ the $r$-th Chow-L group is* $\mathrm{CH}^r_{\mathrm{L}}(X) := \mathrm{H}^{2r}_{\mathrm{L}}(X, \mathbb{Z}(r))$.

Our terminology is justified by the following observation: The complex $\mathbb{Z}_X(1)_{\mathrm{et}}$ is quasi-isomorphic to the complex $\mathbb{G}_m[-1]$ and therefore we have isomorphisms

$$\mathrm{Br}^1(X) = \mathrm{H}^3_{\mathrm{L}}(X, \mathbb{Z}(1)) \cong \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m) = \mathrm{Br}(X) \quad \text{and}$$
$$\mathrm{CH}^1_{\mathrm{L}}(X) = \mathrm{H}^2_{\mathrm{L}}(X, \mathbb{Z}(1)) \cong \mathrm{H}^1_{\mathrm{et}}(X, \mathbb{G}_m) = \mathrm{CH}^1(X)\,.$$

Moreover, $\mathrm{H}^{2r}_{\mathrm{M}}(X, \mathbb{Z}(r)) \cong \mathrm{CH}^r(X)$.

We shall also remark that the universal coefficient sequence (2.9) for $n = 2$ is the Kummer sequence (2.1). Using this sequence (2.9) the non-divisible quotient of the $\ell$-primary part of the Brauer group can be expressed in terms of étale cohomology:

**Lemma 2.4.9.** *Let $k$ be a field and let $X$ be a smooth quasi-projective variety over $k$. For primes $\ell \neq \mathrm{char}(k)$ there are isomorphisms*

$$\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}} \xleftarrow{\cong} \mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))_{\mathrm{nd}} \xrightarrow{\cong} \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r))_{\mathrm{tor}}\,.$$

*Proof.* From (2.9) we obtain by taking the evident direct limit the exact sequence

$$0 \to \mathrm{H}^{2r}_{\mathrm{L}}(X, \mathbb{Z}(r)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \to \mathrm{Br}^r(X)(\ell) \to 0\,.$$

The restriction of the surjection to the direct summand $\mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{\mathrm{nd}}$ induces a surjection $\mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{\mathrm{nd}} \to \mathrm{Br}^r(X)(\ell)_{\mathrm{nd}}$ (having an epimorphism

$f : A \to B$ an element $a \in A$ is divisible if and only if $f(a) \in B$ is divisible). The kernel is a subgroup of $\mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_{\mathrm{nd}}$, hence non-divisible. But each element in $\mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is divisible. This implies that the restriction of the surjection already is first of the two claimed isomorphism.

For the second isomorphism, consider the long exact cohomology sequence associated with $0 \to \mathbb{Z}_\ell(r) \to \mathbb{Q}_\ell(r) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell(r) \to 0$

$$\mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell(r)) \xrightarrow{\varphi} \mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) \xrightarrow{\delta} \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r)) \xrightarrow{\psi} \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Q}_\ell(r))\,.$$

The image $\operatorname{im}\varphi$ is the maximal divisible subgroup of $\mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))$: Certainly each element in $\operatorname{im}\varphi$ is divisible, since étale cohomology groups with $\mathbb{Q}_\ell$ coefficients are divisible. In addition, if $x \in \mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ be divisible then $\delta(x)$ is divisible in $\mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r))$ which is non-divisible; this implies $\delta(x) = 0$, i.e. $x \in \operatorname{im}\varphi$.

The kernel $\ker\psi$ is the torsion subgroup of $\mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r))$: First, the group $\mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Q}_\ell(r))$ is a $\mathbb{Q}_\ell$ vectorspace and thus torsion free; i.e. each torsion element in mapped under $\psi$ to 0. Second, let $x$ be some element such that $\psi(x) = 0$, i.e. it comes from an element of $\mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ which is a torsion group.

It follow from these observations that the boundary map $\delta$ induces an isomorphism $\mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))_{\mathrm{nd}} \to \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_\ell)_{\mathrm{tor}}$ as claimed. □

*Example* 2.4.10. Let $k$ be a field of characteristic zero that is algebraically closed and let $X$ be a smooth projective variety over $k$. We adapt the computation of the Brauer group of $X$ given in Example 2.1.5 to its higher Brauer groups.

Let $\rho_r$ be the rank of $\mathrm{CH}^r_{\mathrm{L}}(X)$ and $b_{2r}$ the $2r$-th Betti number of $X$. For a prime $\ell$ we consider the exact sequence (2.9) for $n = 2r$ and apply the projective limit over all $m$. Again we eventually end up with the exact sequence of free $\mathbb{Z}_\ell$-modules

$$0 \to (\mathrm{CH}^r_{\mathrm{L}}(X) \otimes \mathbb{Z}_\ell)_{\mathrm{free}} \to \mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r))_{\mathrm{free}} \to \mathrm{T}_\ell\mathrm{Br}^r(X) \to 0\,.$$

This leads to $\mathrm{Br}(X)_{\mathrm{div}} \cong (\mathbb{Q}/\mathbb{Z})^{b_{2r}-\rho_r}$ and we get $\mathrm{Br}^r(X)_{\mathrm{nd}} \cong \oplus_\ell \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r))_{\mathrm{nd}}$ by Lemma 2.4.9. Note that this coincides with our result in Example 2.1.5 for $r = 1$.

—

Let $k$ be an arbitrary fields and $X$ a smooth quasi-projective $k$-scheme. We have already seen that $\mathrm{CH}^1(X) \cong \mathrm{CH}^1_{\mathrm{L}}(X)$ and are now interested in the relation between $\mathrm{CH}^2(X)$ and $\mathrm{CH}^2_{\mathrm{L}}(X)$. Kahn has shown [Kah12, Proposition 2.9] that there exists a short exact sequence

$$0 \to \mathrm{CH}^2(X) \to \mathrm{CH}^2_{\mathrm{L}}(X) \to \mathrm{H}^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \to 0$$

where $\mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))$ is the Zariski sheaf associated with $U \mapsto \mathrm{H}^3_{\mathrm{et}}(U, \mathbb{Q}/\mathbb{Z}(2))$. Here the coefficient group $\mathbb{Q}/\mathbb{Z}(n) = \bigoplus_\ell \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$ where $\mathbb{Q}_\ell/\mathbb{Z}_\ell(n) = \varinjlim_i \mu_{\ell^i}^{\otimes n}$ for primes $\ell \neq \mathrm{char}(k)$ and for $\ell = \mathrm{char}(k)$ one has to change the definition and use the Hodge-Witt logarithmic sheaf $\nu_i(n)[-n]$ instead of $\mu_{\ell^i}^{\otimes n}$, see [Kah12, Définition 2.7]. The last group $\mathrm{H}^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$ is the unramified cohomology group (in degree 3) studied for example in [CTK13].

In general, $\mathrm{H}^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))$ can be non-trivial, and even infinite. For example, let $k$ be a algebraically closed field of characteristic zero, $\ell$ a prime such that $\ell = 1 \mod 3$ and $E \subseteq \mathbb{P}^2_k$ the Fermat curve given by $X^3 + Y^3 + Z^3 = 0$. It follows from work by Schoen that $\mathrm{H}^0_{\mathrm{Zar}}(E^3, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$ is in fact infinite [Sch02, Theorem 0.2 and Remark 14.2].

Over finite fields the situation seems to be somewhat better. For any smooth projective variety over a finite field it is conjectured that $\mathrm{H}^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))$ is finite [CTK13, conjecture 5.1]. For a certain class $B_{\mathrm{Tate}}(k)$ of smooth projective varieties over $k$ this has been established in [Kah03], see also [CTK13, Théorème 3.18]. We do not give a definition of the class $B_{\mathrm{Tate}}(k)$ but remark that by definition for each variety $X \in B_{\mathrm{Tate}}(k)$ the strong form of the Tate conjecture is assumed to hold in each codimension.

*Remark* 2.4.11. Although it is true in codimensions $i = 1, 2$ that the canonical map $\mathrm{CH}^i(X) \to \mathrm{CH}^i_{\mathrm{L}}(X)$ is injective this is not true for any codimension, see [RS15].

—

Next, we describe how Theorem 2.2.18 on the relation between Brauer groups and Tate conjecture in codimension 1 is being generalised. What the reader should expect to hold is a result similar to Milne's Theorem 2.3.4 with the conjectural group $\mathbb{H}^{2r+1}(X, \Gamma(r))$ replaced by $\mathrm{Br}^r(X)$.

But the result we are going to state contains more. Recall Corollary 2.2.20 which provides us with counterexamples to the 'naive integral Tate conjecture', i.e. the statement that the maps $\mathrm{CH}^r(X) \otimes \mathbb{Z}_\ell \to \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(r))^{G_k}$ are surjective. By basically replacing $\mathrm{CH}^r(X)$ by $\mathrm{CH}^r_{\mathrm{L}}(X)$ (and defining the corresponding cycle maps) Rosenschon and Srinivas formulated an integral L-Tate conjecture and showed that it is indeed equivalent to the usual Tate conjecture.

Their construction is as follows: Let $k$ be a field finitely generated over its prime field. Then for each prime $\ell \neq \mathrm{char}\, k$ and each codimension $r$ there exists an integral cycle map

$$c^r_{\mathbb{Z}_\ell} : \mathrm{CH}^r_{\mathrm{L}}(X) \otimes \mathbb{Z}_\ell \to \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(r))^{G_k}.$$

which is defined (see [RS15, Section 6]) as the composition of the cycle map to continuous étale cohomology

$$\mathrm{H}^m_{\mathrm{L}}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_\ell \to \mathrm{H}^m_{\mathrm{cont}}(X, \mathbb{Z}_\ell(n))$$

(see [Kah02, Section 1.4] and [Kah12, Section 3.1]) with the map

$$\mathrm{H}^m_{\mathrm{cont}}(X, \mathbb{Z}_\ell(n)) \to \mathrm{H}^m_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(n))^{G_k}$$

from the Hochschild-Serre spectral sequence [Jan88, (0.3)]. The L-Tate conjecture $\mathrm{TC}^r(X)_{\mathbb{Z}_\ell}$ in codimension $r$ at prime $\ell$ is the statement that $c^r_{\mathbb{Z}_\ell}$ is surjective.

We are finally in position to state the theorem that links the Tate conjecture, L-Tate conjecture and higher Brauer groups.

**Theorem 2.4.12** (Rosenschon and Srinivas [RS15, Theorem 1.4])**.** *Let $X$ be a smooth projective geometrically integral variety over a finite field $k$. For each prime $\ell \neq \mathrm{char}\, k$ and integer $r \geq 0$ we have the equivalences*

$$\mathrm{TC}^r(X)_{\mathbb{Q}_\ell} \Leftrightarrow \mathrm{TC}^r(X)_{\mathbb{Z}_\ell} \Leftrightarrow \mathrm{Br}^r(X)(\ell) < \infty\,.$$

—

Let $X$ be a smooth projective variety of even dimension $d = 2r$ over a finite field $k$. Recall that Theorem 2.3.3 gives a formula for $P_{2r}(X, q^{-s})$ involving the order of $\mathbb{H}^{2r+1}(X, \Gamma(r))$ (if such a complex $\Gamma(r)$ exists and under other conditions). As the complex $\mathbb{Z}(r)_{\mathrm{et}}$ is conjectured to satisfy Lichtenbaum's axiom, it seems plausible that such a formula can also be given involving the order of $\mathrm{Br}^r(X)$. This motivates to exhibit the order of $\mathrm{Br}^r(X)$. We at least have results on the order of $\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}}$ for any prime $\ell \neq \mathrm{char}(k)$. Note that if $\mathrm{Br}^r(X)(\ell)$ is finite (which happens precisely if the Tate conjecture holds in codimension $r$) it is equal to $\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}}$. For surfaces Urabe [Ura96] has proven that the order of $\mathrm{Br}(X)(\ell)_{\mathrm{nd}}$ is a square; our Theorem 1.0.1 extends his result to varieties of higher dimensions.

# Chapter 3.

# The Order of Higher Brauer Groups

In this chapter we proof Theorem 1.0.1.

Our setting is as follows. Let $X$ be a smooth projective variety of dimension $d = 2r$ over a finite field of characteristic $p$. Denote the algebraic closure of $k$ by $\overline{k}$ and put $\overline{X} = X \times_k \overline{k}$. By $G$ we denote the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$.

In a nutshell our proof goes as follows: For $\ell \neq 2$ we first construct a non-degenerate skew-symmetric bilinear form

$$\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}} \times \mathrm{Br}^r(X)(\ell)_{\mathrm{nd}} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

which essentially comes from the cup product

$$\mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) \times \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r)) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

in étale cohomology (here we use the isomorphisms from Lemma 2.4.9). Next, we prove that the order of $\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}}$ is odd. Given this, we use the general facts that each skew-symmetric bilinear form $A \times A \to \mathbb{Q}/\mathbb{Z}$ on an abelian group $A$ of odd order with values in $\mathbb{Q}/\mathbb{Z}$ is alternating, and that the existence of a non-degenerate alternating bilinear form $A \times A \to \mathbb{Q}/\mathbb{Z}$ on a finite group $A$ implies that the order of $A$ is a square (cf. [Ura96, Introduction]), which implies our claim.

Unfortunately, this method fails for $\ell = 2$. However, for $H := \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(r))_{\mathrm{free}}$ we will prove that there exists a group $D$ such that $|\mathrm{Br}^r(X)(2)_{\mathrm{nd}}| = |(H_G)_{\mathrm{tor}}| \cdot |D|^2$; it therefore suffices to prove that $(H_G)_{\mathrm{tor}}$ has square order. This is proven by constructing a non-degenerate alternating bilinear form

$$(H_G)_{\mathrm{tor}} \times (H_G)_{\mathrm{tor}} \to \mathbb{Q}_2/\mathbb{Z}_2\,.$$

In order to establish that this form is alternating we show the existence of a cohomology class $\omega_r$ in $\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(r))^G$ with certain properties using Steenrod operations and the Wu formula. In Urabe's proof for surfaces (i.e. $r = 1$) $\omega_1$ was chosen to be the cohomology class of the canonical divisor. The classes $\omega_r$ (for arbitrary $r$) constructed in the course of our proof are build up of Chern classes of the normal bundle.

## 3.1. Bilinear form on $\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}}$

In this section we prove Theorem 1.0.1 for primes $\ell \neq 2$. As sketched before we will construct a bilinear form $\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}} \times \mathrm{Br}^r(X)(\ell)_{\mathrm{nd}} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$.

For this we start with the usual cup product pairings in étale cohomology

$$\mathrm{H}^i_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r}) \times \mathrm{H}^{4r-i}_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r}) \to \mathrm{H}^{4r}_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes 2r}) \cong \mathbb{Z}/\ell^m\mathbb{Z}$$

which by Poincaré duality are non-degenerate pairings of finite $G$-modules. Denote by $\varphi_i : \mathrm{H}^i_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r}) \to \mathrm{Hom}(\mathrm{H}^{4r-i}_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r}), \mathbb{Z}/\ell^m\mathbb{Z})$ the isomorphism associated with the pairing. Since $G$ operates trivially on $\mathbb{Z}/\ell^m\mathbb{Z}$ we get the well-defined homomorphism $\psi_i : \mathrm{H}^i_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r})_G \to \mathrm{Hom}(\mathrm{H}^{4r-i}_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r})^G, \mathbb{Z}/\ell^m\mathbb{Z})$ given by $\psi_i(x)(\overline{y}) = \varphi_i(x)(y)$ for all $x \in \mathrm{H}^i_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r})_G$ and $y \in \mathrm{H}^{4r-i}_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r})^G$. These maps fit into the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^i_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r})_G & \xrightarrow{\psi_i} & \mathrm{Hom}(\mathrm{H}^{4r-i}_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r})^G, \mathbb{Z}/\ell^m\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{H}^i_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r}) & \xrightarrow{\varphi_i} & \mathrm{Hom}(\mathrm{H}^{4r-i}_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r}), \mathbb{Z}/\ell^m\mathbb{Z}) \end{array}$$

with the vertical maps being the inclusion on the left side and the injection coming from the canonical surjection $\mathrm{H}^{4r-i}_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r}) \to \mathrm{H}^{4r-i}_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r})^G$ on the right side. It follows from this diagram that the bilinear form

$$\mathrm{H}^i_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r})_G \times \mathrm{H}^{4r-i}_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes r})^G \to \mathbb{Z}/\ell^m\mathbb{Z}$$

associated with $\psi_i$ is non-degenerate.

From the Hochschild-Serre spectral sequence we get the short exact sequences

$$0 \to \mathrm{H}^i_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes n})_G \to \mathrm{H}^{i+1}_{\mathrm{et}}(X, \mu_{\ell^m}^{\otimes n}) \to \mathrm{H}^{i+1}_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes n})^G \to 0\,.$$

In particular, since $\dim X = 2r$, we get an isomorphism $\mathbb{Z}/\ell^m\mathbb{Z} \cong \mathrm{H}^{4r+1}_{\mathrm{et}}(X, \mu_{\ell^m}^{\otimes 2r})$; this follows from $\mathrm{H}^{4r}_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes 2r}) \cong \mathbb{Z}/\ell^m\mathbb{Z}$ and $\mathrm{H}^{4r+1}_{\mathrm{et}}(\overline{X}, \mu_{\ell^m}^{\otimes 2r}) = 0$ (cf. [Mil80, Ch. VI, Lemma 11.3 and Theorem 1.1]).

**Lemma 3.1.1.** *The cup-product pairing*

$$\mathrm{H}^{2r}_{\mathrm{et}}(X, \mu_{\ell^m}^{\otimes r}) \times \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mu_{\ell^m}^{\otimes r}) \to \mathrm{H}^{4r+1}_{\mathrm{et}}(X, \mu_{\ell^m}^{\otimes 2r})\,. \tag{3.1}$$

*is a non-degenerate bilinear form.*

*Proof.* From the short exact sequences given by the Hochschild-Serre spectral sequence (in bidegrees $(i,r) = (2r-1,r)$ and $(i,r) = (2r,r)$) we obtain the following commutative diagram with exact columns where the horizontal maps come from the cup-products.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{H}^{2r-1}_{\mathrm{et}}(\overline{X},\mu_{\ell^m}^{\otimes r})_G & \hookrightarrow & \mathrm{Hom}(\mathrm{H}^{2r+1}_{\mathrm{et}}(\overline{X},\mu_{\ell^m}^{\otimes r})^G,\mathbb{Z}/\ell^m\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathrm{H}^{2r}_{\mathrm{et}}(X,\mu_{\ell^m}^{\otimes r}) & \longrightarrow & \mathrm{Hom}(\mathrm{H}^{2r+1}_{\mathrm{et}}(X,\mu_{\ell^m}^{\otimes r}),\mathbb{Z}/\ell^m\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X},\mu_{\ell^m}^{\otimes r})^G & \hookrightarrow & \mathrm{Hom}(\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X},\mu_{\ell^m}^{\otimes r})_G,\mathbb{Z}/\ell^m\mathbb{Z}) \\
\downarrow & & \\
0 & &
\end{array}$$

Since the first and the third horizontal arrow are injective as the corresponding forms are non-degenerate, the middle arrow is injective. □

We continue the construction of a non-degenerate skew-symmetric bilinear form $\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}} \times \mathrm{Br}^r(X)(\ell)_{\mathrm{nd}} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$. For $m \geq n > 0$ the canonical diagram

$$\begin{array}{ccccc}
\mathrm{H}^{2r}_{\mathrm{et}}(X,\mu_{\ell^n}^{\otimes r}) & \times & \mathrm{H}^{2r+1}_{\mathrm{et}}(X,\mu_{\ell^n}^{\otimes r}) & \longrightarrow & \mathrm{H}^{4r+1}_{\mathrm{et}}(X,\mu_{\ell^n}^{\otimes 2r}) \\
\downarrow & & \uparrow & & \downarrow \\
\mathrm{H}^{2r}_{\mathrm{et}}(X,\mu_{\ell^m}^{\otimes r}) & \times & \mathrm{H}^{2r+1}_{\mathrm{et}}(X,\mu_{\ell^m}^{\otimes r}) & \longrightarrow & \mathrm{H}^{4r+1}_{\mathrm{et}}(X,\mu_{\ell^m}^{\otimes 2r})
\end{array}$$

commutes. After passing to the limit we obtain a bilinear form

$$\mathrm{H}^{2r}_{\mathrm{et}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) \times \mathrm{H}^{2r+1}_{\mathrm{et}}(X,\mathbb{Z}_\ell(r)) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \tag{3.2}$$

which fits into the following commutative diagram for all integers $m$.

$$\begin{array}{ccccc}
\mathrm{H}^{2r}_{\mathrm{et}}(X,\mu_{\ell^m}^{\otimes r}) & \times & \mathrm{H}^{2r+1}_{\mathrm{et}}(X,\mu_{\ell^m}^{\otimes r}) & \longrightarrow & \mathbb{Z}/\ell^m\mathbb{Z} \\
\downarrow & & \uparrow & & \downarrow \\
\mathrm{H}^{2r}_{\mathrm{et}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) & \times & \mathrm{H}^{2r+1}_{\mathrm{et}}(X,\mathbb{Z}_\ell(r)) & \longrightarrow & \mathbb{Q}_\ell/\mathbb{Z}_\ell
\end{array} \tag{3.3}$$

It follows from Lemma 3.1.1 that the bilinear form (3.2) is non-degenerate. Denote by $\varphi : \mathrm{H}^{2r+1}_{\mathrm{et}}(X,\mathbb{Z}_\ell(r)) \to \mathrm{Hom}(\mathrm{H}^{2r}_{\mathrm{et}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(r)),\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ the monomorphism associated with (3.2). By setting $\psi(x)([y]) := \varphi(x)(y)$ for all $x \in \mathrm{H}^{2r+1}_{\mathrm{et}}(X,\mathbb{Z}_\ell(r))_{\mathrm{tor}}$ and all classes $[y] \in \mathrm{H}^{2r}_{\mathrm{et}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(r))_{\mathrm{nd}}$ we get a well-defined homomorphism

$$\psi : \mathrm{H}^{2r+1}_{\mathrm{et}}(X,\mathbb{Z}_\ell(r))_{\mathrm{tor}} \to \mathrm{Hom}(\mathrm{H}^{2r}_{\mathrm{et}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(r))_{\mathrm{nd}},\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

which provides us with a bilinear form

$$\mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))_{\mathrm{nd}} \times \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r))_{\mathrm{tor}} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell\,. \tag{3.4}$$

Furthermore, the homomorphisms $\varphi$ and $\psi$ fit into the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r))_{\mathrm{tor}} & \xrightarrow{\ \psi\ } & \mathrm{Hom}(\mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))_{\mathrm{nd}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ \downarrow & & \downarrow \\ \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r)) & \underset{\varphi}{\hookrightarrow} & \mathrm{Hom}(\mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r)), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \end{array}$$

where the horizontal maps are the canonical injections. It follows that the form (3.4) is non-degenerate.

We we use the isomorphisms from Lemma 2.4.9 to rewrite this bilinear form as

$$\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}} \times \mathrm{Br}^r(X)(\ell)_{\mathrm{nd}} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell\,. \tag{3.5}$$

**Proposition 3.1.2.** *The form (3.5) is non-degenerate and skew-symmetric.*

*Proof.* Non-degeneracy follows immediately from the non-degeneracy of (3.4).

For skew-symmetry we use that the boundary maps $\delta'$ and $\delta$ of the long exact sequences associated with the short exact sequences $0 \to \mu_{\ell^m}^{\otimes r} \to \mu_{\ell^{2m}}^{\otimes r} \to \mu_{\ell^m}^{\otimes r} \to 0$ and $0 \to \mathbb{Z}_\ell(r) \to \mathbb{Q}_\ell(r) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell(r) \to 0$ fit into the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^{2r}_{\mathrm{et}}(X, \mu_{\ell^m}^{\otimes r}) & \xrightarrow{\ \delta'\ } & \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mu_{\ell^m}^{\otimes r}) \\ \downarrow & & \uparrow \\ \mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) & \xrightarrow{\ \delta\ } & \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r))\,. \end{array}$$

Since the diagram (3.3) is commutative, this also holds for the diagram

$$\begin{array}{ccc} \mathrm{H}^{2r}_{\mathrm{et}}(X, \mu_{\ell^m}^{\otimes r}) \times \mathrm{H}^{2r}_{\mathrm{et}}(X, \mu_{\ell^m}^{\otimes r}) & \xrightarrow{(x,y) \mapsto x \cup \delta'(y)} & \mathbb{Z}/\ell^m\mathbb{Z} \\ \downarrow \qquad\qquad \downarrow & & \downarrow \\ \mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) \times \mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) & \xrightarrow{(x,y) \mapsto x \cup \delta(y)} & \mathbb{Q}_\ell/\mathbb{Z}_\ell\,. \end{array}$$

If $x, y \in \mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))$, we need to verify that $x \cup \delta(y) + y \cup \delta(x) = 0$. Choose $m$ large enough so that there exist preimages $x', y' \in \mathrm{H}^{2r}_{\mathrm{et}}(X, \mu_{\ell^m}^{\otimes r})$; then it suffices to show that $x' \cup \delta'(y') + y' \cup \delta'(x') = \delta'(x' \cup y') = 0$. But $\delta'$ in degree $4r$ is trivial since $\mathrm{H}^{4r+1}_{\mathrm{et}}(X, \mu_{\ell^m}^{\otimes 2r}) \cong \mathbb{Z}/\ell^m\mathbb{Z} \to \mathbb{Z}/\ell^{2m}\mathbb{Z} \cong \mathrm{H}^{4r+1}_{\mathrm{et}}(X, \mu_{\ell^{2m}}^{\otimes 2r})$ is injective, which proves our claim. □

From the bilinear form (3.5) we obtain Theorem 1.0.1 in case $\ell \neq 2$.

**Proposition 3.1.3.** *For $\ell \neq 2$ the order of* $\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}}$ *is a square.*

*Proof.* The group $\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}}$ is isomorphic to $\mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r))_{\mathrm{tor}}$ which is finite as a quotient of a finite group (cf. [Mil80, Ch. VI, Corollary 2.8]). Since the order of every element $x \in \mathrm{Br}^r(X)(\ell)_{\mathrm{nd}}$ is a power of $\ell$, $\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}}$ cannot contain a subgroup of even order and the order of $\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}}$ must be odd. Hence (3.5) is a skew-symmetric non-degenerate bilinear form on a finite group of odd order. This implies that the order of $\mathrm{Br}^r(X)(\ell)_{\mathrm{nd}}$ is a square number, as already mentioned at the beginning of this section. □

## 3.2. Alternating form

In the remaining sections we consider the case $\ell = 2$. For simplicity, we write $H$ for the group $\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(r))_{\mathrm{free}}$. We will show in the final section of this chapter that the order of $\mathrm{Br}^r(X)(2)_{\mathrm{nd}}$ can be written as a product $|(H_G)_{\mathrm{tor}}| \cdot |D|^2$. To prove that $|(H_G)_{\mathrm{tor}}|$ is a square, we construct a non-degenerate alternating bilinear form $\langle\, , \rangle_4 : (H_G)_{\mathrm{tor}} \times (H_G)_{\mathrm{tor}} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$ which again is induced by the cup product using auxiliary bilinear forms $\langle\, , \rangle_1$, $\langle\, , \rangle_2$ and $\langle\, , \rangle_3$. We construct these bilinear forms in the first part of this section. For convenience of the reader, the bilinear forms are summarised in Table 3.1. To show that the bilinear form $\langle\, , \rangle_4$ is alternating, we need to exhibit a cohomology class with certain properties, which is done in Section 3.3.

We consider the cup product pairing $\cup : \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(r)) \times \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(r)) \to \mathbb{Z}_\ell$, i.e. the usual cup product in étale cohomology composed with the canonical isomorphism $\mathrm{H}^{4r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(2r)) \xrightarrow{\cong} \mathbb{Z}_\ell$. By restriction to the free subgroup $H$, we obtain the bilinear form

$$\langle\, , \rangle_1 : H \times H \to \mathbb{Z}_\ell, \quad \langle x, y \rangle_1 = x \cup y$$

which is unimodular, see for example [Zar12, Corollary 1.3].

Using this form we define

$$\langle\, , \rangle_2 : (H \otimes (\mathbb{Q}_\ell/\mathbb{Z}_\ell)) \times H \to \mathbb{Q}_\ell/\mathbb{Z}_\ell, \quad (x \otimes q, y) \mapsto \langle x, y \rangle_1 \otimes q\,.$$

Denote by $\alpha : H \to \mathrm{Hom}((H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ the linear map associated with the form $\langle , \rangle_2$ and denote by $\iota : H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to (H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G$ the inclusion map. If $h \in H$, then $\iota^*\alpha((1-\sigma)h)$ is the map $x \otimes q \mapsto \langle x, (1-\sigma)h \rangle_1 \otimes q$ and for each

$x \in H$ and $q \in \mathbb{Q}_\ell/\mathbb{Z}_\ell$ we have $\langle x, (1-\sigma)h\rangle_1 \otimes q = \langle (1-\sigma)x, h\rangle_1 \otimes q = 0$, i.e. the map $\iota^* \circ \alpha$ factors through $H_G$ and thus yields the bilinear form

$$(\,,\,) : (H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G \times H_G \to \mathbb{Q}_\ell/\mathbb{Z}_\ell, \quad (x, [y]) = \langle x, y\rangle_2 .$$

Next, consider the linear map $\beta : (H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G \to \mathrm{Hom}(H_G, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ associated with $(\,,\,)$ and denote by $\kappa : (H_G)_{\mathrm{tor}} \to H_G$ the canonical injection. Obviously, if $h \in (H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G$ is divisible, then $\beta(h)(\kappa(x)) = (h, x)$ vanishes for each torsion element $x \in (H_G)_{\mathrm{tor}}$. We therefore have our next induced bilinear form

$$\langle\,,\,\rangle_3 : ((H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G)_{\mathrm{nd}} \times (H_G)_{\mathrm{tor}} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell, \quad \langle [x], y\rangle_3 = \langle x, y\rangle_2$$

which we rewrite as

$$\langle\,,\,\rangle_4 : (H_G)_{\mathrm{tor}} \times (H_G)_{\mathrm{tor}} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell, \quad (x, y) \mapsto \langle \overline{\delta}^{-1}(x), y\rangle_3$$

using the isomorphism $\overline{\delta} : ((H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G)_{\mathrm{nd}} \to (H_G)_{\mathrm{tor}}$ of the following lemma.

**Lemma 3.2.1.** *There is an isomorphism* $\overline{\delta} : ((H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G)_{\mathrm{nd}} \to (H_G)_{\mathrm{tor}}$.

*Proof.* Since $H$ is torsion free, tensoring $0 \to \mathbb{Z}_\ell \to \mathbb{Q}_\ell \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0$ with $H$ yields the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{\varphi} & H \otimes \mathbb{Q}_\ell & \xrightarrow{\psi} & H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma-1} & & \downarrow{\scriptstyle \sigma-1} & & \downarrow{\scriptstyle \sigma-1} & & \\ 0 & \longrightarrow & H & \xrightarrow{\varphi} & H \otimes \mathbb{Q}_\ell & \xrightarrow{\psi} & H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & 0 \end{array}$$

and the exact sequence $(H \otimes \mathbb{Q}_\ell)^G \to (H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G \xrightarrow{\delta} H_G \to (H \otimes \mathbb{Q}_\ell)_G$. Since $H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is torsion, the image of $\delta$ equals $(H_G)_{\mathrm{tor}}$, and since $(H \otimes \mathbb{Q}_\ell)^G$ is divisible, the image of $\psi : (H \otimes \mathbb{Q}_\ell)^G \to (H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G$ equals $(H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G_{\mathrm{div}}$. Thus, the boundary map $\delta$ induces an isomorphism $\overline{\delta}$. □

We still have to prove that $\langle\,,\,\rangle_4$ is non-degenerate and alternating.

**Lemma 3.2.2.** *The bilinear forms* $\langle\,,\,\rangle_2$, $(\,,\,)$, $\langle\,,\,\rangle_3$ *and* $\langle\,,\,\rangle_4$ *are non-degenerate.*

*Proof.* Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & H \otimes \mathbb{Q}_\ell & \longrightarrow & H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(H, \mathbb{Z}_\ell) & \longrightarrow & \mathrm{Hom}(H, \mathbb{Q}_\ell) & \longrightarrow & \mathrm{Hom}(H, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & & \end{array}$$

| Form | | Definition |
|---|---|---|
| $\langle\,,\rangle_1:$ | $H \times H \to \mathbb{Z}_\ell$ | $\langle x,y\rangle_1 = x \cup y$ |
| $\langle\,,\rangle_2:$ | $(H \otimes (\mathbb{Q}_\ell/\mathbb{Z}_\ell)) \times H \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$ | $\langle x \otimes q, y\rangle_2 = \langle x,y\rangle_1 \otimes q$ |
| $(\,,):$ | $(H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G \times H_G \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$ | $(x,[y]) = \langle x,y\rangle_2$ |
| $\langle\,,\rangle_3:$ | $((H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G)_{\mathrm{nd}} \times (H_G)_{\mathrm{tor}} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$ | $\langle [x],y\rangle_3 = \langle x,y\rangle_2$ |
| $\langle\,,\rangle_4:$ | $(H_G)_{\mathrm{tor}} \times (H_G)_{\mathrm{tor}} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$ | $\langle x,y\rangle_4 = \langle \overline{\delta}^{-1}(x), y\rangle_3$ |
| | $(H \otimes \mathbb{Q}_\ell) \times H \to \mathrm{H}^{4r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(2r)) \otimes \mathbb{Q}_\ell$ | $(x \otimes q, y) \mapsto \langle x,y\rangle_1 \otimes q$ |
| $\langle\,,\rangle_5:$ | $(H \otimes \mathbb{Q}_\ell) \times (H \otimes \mathbb{Q}_\ell) \to \mathbb{Q}_\ell$ | $\langle u \otimes p, v \otimes q\rangle_5 = \langle u,v\rangle_1 \otimes pq$ |

Table 3.1.: The bilinear forms

where the outer vertical homomorphisms are induced by the bilinear forms $\langle\,,\rangle_1$ and $\langle\,,\rangle_2$ respectively, and the middle vertical homomorphism is induced by the bilinear form $(H \otimes \mathbb{Q}_\ell) \times H \to \mathrm{H}^{4r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(2r)) \otimes \mathbb{Q}_\ell$, $(x \otimes q, y) \mapsto \langle x,y\rangle_1 \otimes q$ that is unimodular. Since $\langle\,,\rangle_1$ is unimodular, the first vertical map is an isomorphism. Thus it follows from the snake lemma that the last vertical homomorphism is injective, i.e. $\langle\,,\rangle_2$ is non-degenerate.

For the proof of non-degeneracy of $(\,,)$ let $x \in (H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G$ such that $(x,[y])$ is trivial for all $[y] \in H_G$. Then $\langle x,y\rangle_2 = 0$ for all $y \in H$, and since $\langle\,,\rangle_2$ is non-degenerate, we have $x = 0$. Thus $(\,,)$ is non-degenerate as well.

Finally, let $y \in (H_G)_{\mathrm{tor}}$ with $\langle [x],y\rangle_3 = 0$ for every $[x] \in ((H \otimes \mathbb{Q}_\ell/Z_\ell)^G)_{\mathrm{nd}}$. This means $(x,y) = 0$ for all $x \in H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$. Hence, $y = 0$ and thus $\langle\,,\rangle_3$ is non-degenerate. The non-degeneracy of $\langle\,,\rangle_4$ follows immediately from this. □

**Proposition 3.2.3.** *The form $\langle\,,\rangle_4$ is alternating.*

*Proof.* We have to verify that $\langle z,z\rangle_4 = 0$ for each $z \in (H_G)_{\mathrm{tor}}$. We show first that to prove $\langle z,z\rangle_4 = 0$ it suffices to show that the cup product of two particular classes lies in $2\mathbb{Z}_\ell$; for this we reverse the construction of $\langle\,,\rangle_4$.

First, since $\operatorname{im}\delta = (H_G)_{\mathrm{tor}}$, we find a $y \in (H \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G$ such that $\delta(y) = z$ and therefore $\langle z,z\rangle_4 = \langle y,z\rangle_3$. Second, let $w \in H$ be a representative of the class $z$; we obtain $\langle y,z\rangle_3 = \langle y,w\rangle_2$. Third, we use the form $\langle\,,\rangle_1$ to define our last form

$$\langle\,,\rangle_5 : (H \otimes \mathbb{Q}_\ell) \times (H \otimes \mathbb{Q}_\ell) \to \mathbb{Q}_\ell, \quad \langle u \otimes p, v \otimes q\rangle_5 = \langle u,v\rangle_1 \otimes pq$$

and show by a short computation that $\langle y,w\rangle_2 = \langle x,(\sigma-1)(x)\rangle_5 + \mathbb{Z}_\ell$ in $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ for a $x \in H \otimes \mathbb{Q}_\ell$ such that $\psi(x) = y$. Hence in order to prove $\langle z,z\rangle_4 = 0$ it is sufficient to show that $\langle x,(\sigma-1)(x)\rangle_5 \in \mathbb{Z}_\ell$.

Because of $\langle w, w\rangle_1 = \langle(\sigma-1)x, (\sigma-1)x\rangle_5 = -2\cdot\langle x, (\sigma-1)x\rangle_5$ it is even enough to prove $\langle w, w\rangle_1 \in 2\mathbb{Z}_\ell$. For $\ell \neq 2$ this follows from $\mathbb{Z}_\ell = 2\mathbb{Z}_\ell$ and we are left with the case $\ell = 2$.

For that case we need the following lemma which is proved in the next section; a similar result has been stated and used in a different context in [EJ12].

**Lemma 3.2.4.** *For every integer $r \geq 0$ there exists a $\omega_r \in \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(r))^G$ such that $\omega_r \cup x + x \cup x \in 2\mathbb{Z}_2$ for each $x \in \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(r))$.*

We continue with the proof of Proposition 3.2.3 in the case $\ell = 2$. As before it suffices to prove $\langle w, w\rangle_1 \in 2\mathbb{Z}_2$ and since $\langle w, w\rangle_1 = w \cup w$, it even suffices to prove that $\omega_r \cup w = 0$.

Let $\pi : H \to H_G = H/(\sigma-1)H$ denote the canonical projection and consider the preimage $\pi^{-1}((H_G)_{\mathrm{tor}}) = ((\sigma-1)H \otimes \mathbb{Q}_\ell) \cap H$. We have $w \in \pi^{-1}((H_G)_{\mathrm{tor}})$ since $\pi(w) = z \in (H_G)_{\mathrm{tor}}$. Because of the orthogonality of $(\sigma-1)H$ and $\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(r))^G$, considered as subgroups of $\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(r))$, we also have the orthogonality of the preimage $\pi^{-1}((H_G)_{\mathrm{tor}}) = ((\sigma-1)H \otimes \mathbb{Q}_\ell) \cap H$ and $\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(r))^G$. In particular $\omega_r \cup w = 0$ holds. □

## 3.3. Steenrod squares and $\omega_r$

We still have to prove Lemma 3.2.4. The proof of the lemma uses Steenrod squares which we will discuss first. Steenrod squares where first defined for simplicial complexes by Steenrod [Ste47] (see also [Ste62]). Epstein then introduced Steenrod squares into the world of derived functors [Eps66]. Here we will use them in étale cohomology (with supports).

Let $Z$ be a scheme and a $Y$ a closed subscheme. The Steenrod squares are homomorphisms between étale cohomology groups with support

$$\mathrm{Sq}^m : \mathrm{H}^n_{\mathrm{et},Y}(Z, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^{m+n}_{\mathrm{et},Y}(Z, \mathbb{Z}/2\mathbb{Z})$$

constructed in [Eps66]. Among the basic properties established in [Eps66] are

**(S)** For $y \in \mathrm{H}^n_{\mathrm{et}}(\overline{X}, \mathbb{Z}/2\mathbb{Z})$ we have the formula $\mathrm{Sq}^n(y) = y \cup y$.

**(I)** The operation $\mathrm{Sq}^0(-)$ is the identity.

**(C)** The Cartan formula

$$\mathrm{Sq}^j(x \cup y) = \sum_{i=0}^{j} \mathrm{Sq}^i(x) \cup \mathrm{Sq}^{j-i}(y)$$

for each $x, y \in \mathrm{H}^*_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z})$.

**(B)** The homomorphism $\mathrm{Sq}^1 : \mathrm{H}^n_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^{n+1}_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z})$ agrees with the Bockstein homomorphism, i.e. the boundary map of the long exact cohomology sequence coming from $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$.

**(V)** The homomorphism $\mathrm{Sq}^m : \mathrm{H}^n_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^{m+n}_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z})$ vanish for integers $m > n$.

Using the Poincaré duality we moreover get:

**(P)** There exists a class $v_{2r} \in \mathrm{H}^{2r}_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z})$ such that the maps $\mathrm{Sq}^{2r}(-)$ and $v_{2r} \cup -$ coincide in $\mathrm{Hom}(\mathrm{H}^{2r}_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z}), \mathrm{H}^{4r}_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z}))$.

Lemma 3.2.4 is a statement about elements of étale cohomology groups with coefficients in the sheaf $\mathbb{Z}_2(r)$ whereas the Steenrod squares require coefficients in the sheaf $\mathbb{Z}/2\mathbb{Z}$. We therefore will use the canonical map

$$\varepsilon : \sum_i \mathrm{H}^{2i}_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}_2(i)) \to \sum_i \mathrm{H}^{2i}_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z}),$$

find some $v \in \mathrm{H}^{2r}_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z})$ such that $v \cup \varepsilon(x) = \varepsilon(x) \cup \varepsilon(x)$ and finally find some $\varepsilon$-preimage $\omega_r$ of $v$. For this last step will use the following notation: Let $\mathcal{E}$ be a vector bundle on $\overline{\mathrm{X}}$ with total space $T$. In each degree $i$ we have a canonical isomorphism $\varphi : \mathrm{H}^i_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^{i+2r}_{\mathrm{et},\overline{\mathrm{X}}}(T, \mathbb{Z}/2\mathbb{Z})$. We write 1 for the generator of $\mathrm{H}^0_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and set $w_i(\mathcal{E}) = \varphi^{-1}(\mathrm{Sq}^i(\varphi(1))) \in \mathrm{H}^i_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z})$. Let $c_k(\mathcal{E})$ be the $k$-th Chern class of $\mathcal{E}$ and let $\varepsilon(c_k(\mathcal{E}))$ be its class in $\mathrm{H}^{2k}_{\mathrm{et}}(\overline{\mathrm{X}}, \mathbb{Z}/2\mathbb{Z})$.

These classes $w_i(\mathcal{E})$ have some important properties:

**Lemma 3.3.1** ([Ura96, Proposition 2.8]). *We have that*

$$w_i(\mathcal{E}) = \begin{cases} 0, & i \text{ is odd} \\ \varepsilon(c_{i/2}(\mathcal{E})), & i \text{ is even}. \end{cases}$$

**Lemma 3.3.2** (Wu formula, [Ura96, Theorem 0.5]). *Denote by $\mathcal{N} = \mathcal{N}_{\overline{\mathrm{X}}/\overline{\mathrm{X}} \times_{\mathbb{F}} \overline{\mathrm{X}}}$ the normal bundle. The class $w_{2r} = w_{2r}(\mathcal{N})$ is related to the classes $v_i$ by the Wu formula*

$$w_{2r}(\mathcal{N}) = \sum_{s=0}^{r} \mathrm{Sq}^{2s}(v_{2r-2s}).$$

For the following lemma cf. [MS74, Problem 8-A].

**Lemma 3.3.3.** *For each vector bundle $\mathcal{E}$ and every pair of natural numbers $i, j$ we have*

$$\mathrm{Sq}^j w_i(\mathcal{E}) = \sum_{k=0}^{j} \binom{i-j}{k} w_{j-k}(\mathcal{E}) \cup w_{i+k}(\mathcal{E})\,, \tag{3.6}$$

*where $\binom{n}{k}$ denotes the class of $\frac{n(n-1)\cdots(n-k+1)}{k!}$ in $\mathbb{Z}/2\mathbb{Z}$.*

*Proof.* This is shown by induction on the rank of the vector bundle.

If $\mathcal{E}$ is a line bundle, then we can prove the lemma by a number of computations using the elementary properties of Steenrod squares stated above and the classes $w_i(\mathcal{E})$. We omit $\mathcal{E}$ in the notation for a better readability. Note first that $w_0 = \varepsilon(c_0)$, $w_2 = \varepsilon(c_1)$ and $w_i = 0$ for $i \neq 0, 2$. We have $\mathrm{Sq}^0(\varepsilon(c_0)) = \varepsilon(c_0)$ by property (I) and $\binom{0}{0}\varepsilon(c_0) \cup \varepsilon(c_0) = \varepsilon(c_0)$. For $j > 0$ the binomial coefficients on the right hand side are trivial which agrees with property (V).

Next, we have to compute Steenrod squares of $w_2 = \varepsilon(c_1)$. First, $\mathrm{Sq}^0 w_2 = w_2$ by property (I) and $\binom{2}{0} w_0 \cup w_2 = w_2$. Second, $\mathrm{Sq}^1 w_2 = \beta(w_2) = \beta(\varepsilon(c_1)) = 0$ by (B) an the fact that the image of $c_1$ in $\mathrm{H}^2_{\mathrm{et}}(\overline{X}, \mathbb{Z}/4\mathbb{Z})$ is mapped to $\varepsilon(c_1)$; on the right hand side we have $\binom{1}{0} w_1 \cup w_2 + \binom{1}{1} w_0 \cup w_3 = 0$. Third, $\mathrm{Sq}^2 w_2 = w_2^2$ by (S) and $\binom{0}{0} w_2 \cup w_2 + \binom{0}{1} w_1 \cup w_3 + \binom{0}{2} w_0 \cup w_4 = w_2^2$. Finally, for $j > 2$ both sides are trivial by (V) and the vanishing of the binomial coefficients.

Having proved the lemma for line bundles we now consider an vector bundle $\mathcal{E}$ of arbitrary rank. For an exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of vector bundles it is known that $w_i(\mathcal{E}) = \sum_{k=0}^{i} w_k(\mathcal{E}') \cup w_{i-k}(\mathcal{E}'')$ [Ura96, Lemma 2.6]. A straightforward computation involving this shows that if the equation (3.6) is satisfied for $\mathcal{E}'$ and $\mathcal{E}''$, it also is satisfied for $\mathcal{E}$. Now the assertion follows from the splitting principle (cf. [Gro58], [Ful98, p. 51]). □

We finally prove the lemma.

*Proof of Lemma 3.2.4.* The kernel of the map $\varepsilon$ intersected with $\mathrm{H}^{4r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(2r))$ is $2\mathbb{Z}_2$. If the required $G$-invariant element$\omega_r \in \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(r))$ exists, it satisfies the equation $\varepsilon(\omega_r) \cup \varepsilon(x) + \varepsilon(x) \cup \varepsilon(x) = 0$ for every $x \in \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(r))$. Since this is an equation in $\mathbb{Z}/2\mathbb{Z}$, it even satisfies $\varepsilon(\omega_r) \cup \varepsilon(x) = \varepsilon(x) \cup \varepsilon(x)$.

From the above properties of Steenrod squares we have for each element $x$ in $\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(r))$ the equation $v_{2r} \cup \varepsilon(x) = \mathrm{Sq}^{2r}(\varepsilon(x)) = \varepsilon(x) \cup \varepsilon(x)$.

Denote by $\mathcal{N} = \mathcal{N}_{\overline{X}/\overline{X}\times_{\overline{\mathbb{F}}}\overline{X}}$ the normal bundle. Using that $\mathrm{Sq}^0$ is the identity and the Wu formula, we get that $v_{2r}$ equals $w_{2r}(\mathcal{N}) + \sum_{s=1}^{r} \mathrm{Sq}^{2s}(v_{2r-2s})$. It is therefore possible to compute the $v_{2r}$ recursively, using the Lemmas 3.3.1 and 3.3.3, in terms of polynomials in the $\varepsilon(c_k(\mathcal{E}))$'s. We therefore find the desired element $\omega_r$. □

*Example* 3.3.4. We compute $\omega_1$: $v_2 = w_2(\mathcal{N}) + \mathrm{Sq}^2(v_0) = \varepsilon(c_1(\mathcal{N}))$. Therefore, $\omega_1 = c_1(\mathcal{N})$ is a canonical choice. Urabe proved in [Ura96, Proposition 2.1] that the class $\omega_1$ of the canonical line bundle $K_{\overline{X}} \in \mathrm{Pic}(\overline{X})$ in $\mathrm{H}^2_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(1))$ also has the desired property. In particular, the element $\omega_1$ considered by him and our $\omega_1$ constructed here coincide up to a sign.

For $\omega_2$ we compute $v_4 = w_4(\mathcal{N}) + \mathrm{Sq}^2(v_2) = w_4(\mathcal{N}) + v_2 \cup v_2$ and we therefore take $\omega_2 = c_2(\mathcal{N}) + c_1(\mathcal{N})^2$. Finally, $v_6 = w_6(\mathcal{N}) + \mathrm{Sq}^2(v_4)$ and we compute $\mathrm{Sq}^2(v_4) = \mathrm{Sq}^2(w_4(\mathcal{N})) + \mathrm{Sq}^2(w_2(\mathcal{N}) \cup w_2(\mathcal{N})) = w_2(\mathcal{N}) \cup w_4(\mathcal{N}) + w_6(\mathcal{N})$. It follows that $v_6 = w_2(\mathcal{N}) \cup w_4(\mathcal{N})$ and thus, $\omega_3 = c_1(\mathcal{N})c_2(\mathcal{N})$ is a canonical choice.

## 3.4. Proof of the theorem

We have shown that the order $|(H_G)_{\mathrm{tor}}|$ is a square, hence our final goal is to determine the group $D$ with the property that $|\mathrm{Br}^r(X)(2)_{\mathrm{nd}}| = |(H_G)_{\mathrm{tor}}| \cdot |D|^2$.

The following diagram will turn out to be helpful.

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & (\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))_{\mathrm{div}})^G & & \\
 & & & & & & \downarrow{\scriptstyle\alpha} & & \\
0 & \to & \mathrm{H}^{2r-1}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))_G & \xrightarrow{\tilde{\rho}} & \mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) & \xrightarrow{\tilde{\pi}} & \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))^G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\delta'} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta''} & & \\
0 & \longrightarrow & \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(r))_G & \xrightarrow{\rho} & \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_\ell(r)) & \xrightarrow{\pi} & \mathrm{H}^{2r+1}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(r))^G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\beta} & & & & & & \\
 & & (\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_\ell(r))_{\mathrm{free}})_G & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

**Lemma 3.4.1.** *The above diagram is commutative with exact rows and columns.*

*Proof.* Both rows are induced from the Hochschild-Serre spectral sequence (with divisible and $\ell$-adic coefficients) and are exact. The outer columns are exact without applying the functors $\bullet_G$ and $\bullet^G$ and these functors are right- and left-exact, respectively. The right square is commutative since the homomorphism $\pi$ and $\tilde{\pi}$ are induced by the covering $\overline{X} \to X$.

For the commutativity of the left square, we consider the Hochschild-Serre spectral sequence $E_2^{p,q}(\star) = \mathrm{H}^p_{\mathrm{et}}(G, \mathrm{H}^q_{\mathrm{et}}(\overline{X}, \mu_{\ell^\star}^{\otimes r})) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{et}}(X, \mu_{\ell^\star}^{\otimes r})$. The connecting homomorphism associated with $0 \to \mu_{\ell^m}^{\otimes r} \to \mu_{\ell^{m+n}}^{\otimes r} \to \mu_{\ell^n}^{\otimes r} \to 0$ induces a homomorphism $\varinjlim_n E_2^{1,2r-1}(n) \to \varprojlim_m E_2^{1,2r}(m)$ which coincidences with $\delta'$. Similarly, there exists a homomorphism $\varinjlim_n E_0^{2r}(n) \to \varprojlim_m E_0^{2r+1}(m)$ that agrees with $\delta$. Since the cohomological dimension being at most one and thus $E_2^{p,q}(\star) = 0$ for $p \neq 0, 1$ we gain the compositions (we omit $(\star)$ for a moment)

$$\mathrm{H}^{q-1}_{\mathrm{et}}(\overline{X}, \mu_{\ell^\star}^{\otimes r})_G = E_2^{1,q-1} = E_\infty^{1,q-1} \cong E_1^q / E_2^q \xleftarrow{\cong} E_1^q \subseteq E_0^q = \mathrm{H}^q_{\mathrm{et}}(X, \mu_{\ell^\star}^{\otimes r}) .$$

For $q = 2r$ and $q = 2r+1$ these compositions are the horizontal maps of the commutative diagram

$$\begin{array}{ccc} E_2^{1,2r-1}(n) & \longrightarrow & E_0^{2r}(n) \\ \downarrow & & \downarrow \\ E_2^{1,2r}(m) & \longrightarrow & E_0^{2r+1}(m) . \end{array} \tag{3.7}$$

Application of the direct limit over all $n$ to this composition with $q = 2r$ induces a homomorphism $\mathrm{H}^{2r-1}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))_G \to \mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))$ that coincidences with $\tilde{\rho}$. Similarly, we get a homomorphism that equals $\rho$ when applying an inverse limit over all $m$ with $q = 2r+1$. We can therefore deduce the commutativity of the left square from the diagram (3.7) above. □

We define the two groups

$$C := \mathrm{im}\left(\mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(r))_G \xrightarrow{\rho} \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_2(r))\right) \cap \mathrm{H}^{2r+1}_{\mathrm{et}}(X, \mathbb{Z}_2(r))_{\mathrm{tor}} \quad \text{and}$$
$$D := \mathrm{im}\left(\mathrm{H}^{2r-1}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_2/\mathbb{Z}_2(r))_G \xrightarrow{\tilde{\rho}} \mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_2/\mathbb{Z}_2(r)) \to \mathrm{H}^{2r}_{\mathrm{et}}(X, \mathbb{Q}_2/\mathbb{Z}_2(r))_{\mathrm{nd}}\right)$$

which we consider as subgroups of $B = \mathrm{Br}^r(X)(2)_{\mathrm{nd}}$ in view of Lemma 2.4.9. It follows directly from the above diagram that $D \subseteq C \subseteq B$.

**Lemma 3.4.2.** *There is an isomorphism $C/D \cong (H_G)_{\mathrm{tor}}$.*

*Proof.* The composition $\varphi : C \to \mathrm{im}(\rho) \xrightarrow{\rho^{-1}} \mathrm{H}^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Z}_2(r))_G \to H_G$ has kernel $D$ and image $(H_G)_{\mathrm{tor}}$ and therefore induces the desired isomorphism. The assertions about kernel and cokernel follow from the diagram at the beginning of this section. □

We are now ready to prove the following theorem which in addition with Proposition 3.1.3 implies Theorem 1.0.1.

**Theorem 3.4.3.** *If* char $\mathbb{F} \neq 2$*, the order of* $\mathrm{Br}^r(X)(2)_{\mathrm{nd}}$ *is a square.*

*Proof.* The group $B$ is finite as we have seen in the proof of Proposition 3.1.3. By Proposition 3.1.2 there exists a non-degenerate bilinear form on $B$; this form induces a non-degenerate bilinear form $C \times (B/D) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$ (recall the diagram before Lemma 3.4.1) which implies $|C| = |B/D| = |B|/|D|$. Therefore, the order of $B$ equals the product $|C/D| \cdot |D|^2$. By Lemma 3.4.2 the first factor is the order of $(H_G)_{\mathrm{tor}}$ and thus a square by Proposition 3.2.3. □

# Appendix A.

# Hypercohomology

In this thesis we use hypercohomology of unbounded complexes. In particular, higher Brauer groups are defined as certain hypercohomology groups with coefficients in Bloch's cycle complexes; these complexes are not bounded below. Here we provide a definition of hypercohomology of unbounded complexes; see also [SV00, p. 121] and [GL01].

Let $\mathfrak{A}$ be any abelian category with enough injectives. The two categories we have in mind are the categories of sheaves on the small Zariski and on the small étale site of a variety. By $\mathfrak{Ab}$ we denote the category of abelian groups. We also fix a left exact additive functor $F : \mathfrak{A} \to \mathfrak{Ab}$; in our applications we will take the global section functors.

Let $C^\bullet$ be a complex of degree $+1$ (i.e. the differentials of $C^\bullet$ are of the type $\delta : C^i \to C^{i+1}$) in the category $\mathfrak{A}$. We emphasise that we do not assume that $C^\bullet$ is bounded below. As we use injective resolutions to define cohomology we need some appropriate notion of a resolution of complexes in order to define hypercohomology.

Before defining those resolutions we fix some notations. A bicomplex $I^{\bullet,\bullet}$ is a grid of objects $I^{i,j}$ and morphisms $I^{i,j} \to I^{i+1,j}$ and $I^{i,j} \to I^{i,j+1}$ in $\mathfrak{A}$ such that each column and each row forms a complex. By the symbol $I^{i,\bullet}$ we denote the complex

$$\cdots \to I^{i,j-1} \to I^{i,j} \to I^{i,j+1} \to \dots ;$$

similarly for $I^{\bullet,j}$. Denote by $B^\bullet(C^\bullet)$ the complex with $B^i(C^\bullet) = \ker(C^i \to C^{i+1})$. For a bicomplex we denote by $B^{\bullet,\bullet}(I^{\bullet,\bullet})$ the bicomplex with $B^{\bullet,j}(I^{\bullet,\bullet}) = B^\bullet(I^{\bullet,j})$. Similarly, we denote by $Z^\bullet(C^\bullet)$ and $H^\bullet(C^\bullet)$ the complexes defined by $Z^i(C^\bullet) = \mathrm{im}(C^{i-1} \to C^i)$ and $H^i(C^\bullet) = C^i/Z^i(C^\bullet)$. The definitions of the bicomplexes $Z^{\bullet,\bullet}$ and $H^{\bullet,\bullet}$ are analogous to the definition of $B^{\bullet,\bullet}$.

**Definition A.0.4.** *Let $C^\bullet$ be a complex in $\mathfrak{A}$. A* Cartan-Eilenberg resolution *$I^{\bullet,\bullet}$ of $C^\bullet$ is a bicomplex together with a map $\varepsilon : C^\bullet \to I^{\bullet,0}$ of complexes such that*

1. *If* $C^p = 0$ *the column* $I^{p,\bullet}$ *is trivial.*

2. *For each* $i$ *the complexes* $B^{i,\bullet}(I^{\bullet,\bullet})$ *resp.* $H^{i,\bullet}(I^{\bullet,\bullet})$ *are injective resolutions of* $B^i(C^\bullet)$*, resp.* $H^i(C^\bullet)$.

For the proof that each complex in $\mathfrak{A}$ has a Cartan-Eilenberg resolution we need the following classical result from homological algebra.

**Lemma A.0.5** (Horseshoe Lemma)**.** *Let* $0 \to A' \to A \to A'' \to 0$ *be an exact sequence in* $\mathfrak{A}$ *and let* $I'^\bullet$ *and* $I''^\bullet$ *be injective resolutions of* $A'$ *and* $A''$*. Then there exists an injective resolution* $I^\bullet$ *of* $A$ *and maps of complexes such that* $0 \to I'^\bullet \to I^\bullet \to I''^\bullet \to 0$ *is an exact sequence of complexes.*

*Proof.* The proof is dual to [CE56, Proposition V.2.2]. □

The following lemma is dual to [CE56, Proposition XVII.1.2].

**Lemma A.0.6.** *Each complex* $C^\bullet$ *has a Cartan-Eilenberg resolution* $I^{\bullet,\bullet}$.

*Proof.* For each $i$ we fix injective resolutions $B^i(C^\bullet) \to I_B^{i,\bullet}$ and $H^i(C^\bullet) \to I_H^{i,\bullet}$. The sequence $0 \to B^i(C^\bullet) \to Z^i(C^\bullet) \to H^i(C^\bullet) \to 0$ is exact and the Horseshoe Lemma applied to it yields an injective resolution $Z^i(C^\bullet) \to I_Z^{i,\bullet}$. Applying again the Horseshoe Lemma to the exact sequence $0 \to Z^i(C) \to C^i \to B^{i+1}(C) \to 0$ we get an injective resolution $C^i \to I_C^{i,\bullet}$.

Now define $I^{\bullet,\bullet}$ to be the complex whose $i$-th column is the complex $I_C^{i,\bullet}$ but with the differentials replaced by their $(-1)^i$ multiple. The vertical differentials are given by the composites $I_C^{i,\bullet} \to I_B^{i+1,\bullet} \to I_Z^{i+1,\bullet} \to I_C^{i+1,\bullet}$. □

Having chosen a Cartan-Eilenberg resolution $A^\bullet \to I^{\bullet,\bullet}$ of the given complex $A^\bullet$ we apply the functor $F$ at each component. This leads to a bicomplex which we denote simply by $F(I^{\bullet,\bullet})$. Next, we compute the total product complex $\mathrm{Tot}(F(I^{\bullet,\bullet}))$, i.e. the complex in $\mathfrak{Ab}$ with components $\mathrm{Tot}(F(I^{\bullet,\bullet}))^r = \prod_{m+n=r} F(I^{m,n})$. Finally, the cohomology groups of this complex give the $i$-th hyper right derived functor of $F$: $\mathbb{R}^i F(A^\bullet) := \mathrm{H}^i\mathrm{Tot}(F(I^{\bullet,\bullet}))$.

**Definition A.0.7.** *Let* $X$ *be a smooth projective variety and let* $\mathcal{F}^\bullet$ *be a complex of sheaves on* $X$ *for any Grothendieck topology* $\tau$*. We define* $\tau$*-hypercohomology with coefficients in the complex* $\mathcal{F}^\bullet$ *by*

$$\mathbb{H}^i_\tau(X, \mathcal{F}^\bullet) = \mathrm{H}^i\mathrm{Tot}(\Gamma(X, I^{\bullet,\bullet}))\,.$$

# Bibliography


[AH62] M. F. Atiyah and F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology **1** (1962), 25–45.

[BCC92] E. Ballico, F. Catanese, and C. Ciliberto (eds.), *Classification of irregular varieties*, Lecture Notes in Mathematics, vol. 1515, Springer-Verlag, Berlin, 1992, Minimal models and abelian varieties.

[Bei82] A. Beilinson, *Letter to Soule*, available at K-theory Preprint Archives, http://www.math.uiuc.edu/K-theory/0694, 1982.

[Blo86] S. Bloch, *Algebraic Cycles and Higher K-Theory*, Adv. in Math **61** (1986), no. 3, 267–304.

[Blo94] ———, *The moving lemma for higher Chow groups*, J. Algebraic Geom. **3** (1994), no. 3, 537–568.

[BSD63] B. J. Birch and H. P. F. Swinnerton-Dyer, *Notes on elliptic curves. I*, J. Reine Angew. Math. **212** (1963), 7–25.

[BSD65] ———, *Notes on elliptic curves. II*, J. Reine Angew. Math. **218** (1965), 79–108.

[CE56] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956.

[Cha13] F. Charles, *The Tate conjecture for K3 surfaces over finite fields*, Invent. Math. **194** (2013), no. 1, 119–145.

[CTK13] J.-L. Colliot-Thélène and B. Kahn, *Cycles de codimension 2 et $H^3$ non ramifié pour les variétés sur les corps finis*, J. K-Theory **11** (2013), no. 1, 1–53.

[CTS10] J.-L. Colliot-Thélène and T. Szamuely, *Autour de la conjecture de Tate à coefficients $\mathbf{Z}_\ell$ pour les variétés sur les corps finis*, The geometry of algebraic cycles, Clay Math. Proc., vol. 9, Amer. Math. Soc., Providence, RI, 2010, pp. 83–98.

[Del74] P. Deligne, *La conjecture de Weil. I*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 43, 273–307.

[Dwo60] B. Dwork, *On the rationality of the zeta function of an algebraic variety*, Amer. J. Math. **82** (1960), 631–648.

[EJ12] A.-S. Elsenhans and J. Jahnel, *On the characteristic polynomial of the Frobenius on étale cohomology*, arXiv:1106.3953v2, April 2012.

[Eps66] D. B. A. Epstein, *Steenrod operations in homological algebra*, Invent. Math. **1** (1966), 152–208. MR 0199240 (33 #7389)

[Fal83] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), no. 3, 349–366.

[Ful98] W. Fulton, *Intersection Theory*, 2 ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 2, Springer, 1998.

[Gei04] T. Geisser, *Motivic cohomology over Dedekind rings*, Math. Z. **248** (2004), no. 4, 773–794.

[GL01] T. Geisser and M. Levine, *The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky*, J. Reine Angew. Math. **530** (2001), 55–103.

[Gor79] W. J. Gordon, *Linking the conjectures of Artin-Tate and Birch-Swinnerton-Dyer*, Compositio Math. **38** (1979), no. 2, 163–199.

[Gro58] A. Grothendieck, *La théorie des classes de Chern*, Bull. Soc. Math. France **86** (1958), 137–154.

[Gro64] ———, *Formule de Lefschetz et rationalité des fonctions L*, Séminaire Bourbaki, Vol. 9, Soc. Math. France, Paris, 1964, pp. 41–55.

[Gro68a] ———, *Le groupe de Brauer. I. Algèbres d'Azumaya et interprétations diverses*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam; Masson, Paris, 1968, pp. 46–66.

[Gro68b] ———, *Le groupe de Brauer. II. Théorie cohomologique*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam; Masson, Paris, 1968, pp. 67–87.

[Gro68c] ______, *Le groupe de Brauer. III. Exemples et compléments*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam; Masson, Paris, 1968, pp. 88–188.

[Jah15] T. Jahn, *The order of higher Brauer groups*, Math. Ann. **362** (2015), no. 1-2, 43–54.

[Jan88] U. Jannsen, *Continuous étale cohomology*, Math. Ann. **280** (1988), no. 2, 207–245.

[Kah02] B. Kahn, *The Geisser-Levine method revisited and algebraic cycles over a finite field*, Math. Ann. **324** (2002), no. 3, 581–617.

[Kah03] ______, *Équivalences rationnelle et numérique sur certaines variétés de type abélien sur un corps fini*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 6, 977–1002.

[Kah12] ______, *Classes de cycles motiviques étales*, Algebra Number Theory **6** (2012), no. 7, 1369–1407.

[Lan59] S. Lang, *Abelian varieties*, Interscience Tracts in Pure and Applied Mathematics. No. 7, Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London, 1959.

[Lev94] M. Levine, *Bloch's higher Chow groups revisited*, Astérisque (1994), no. 226, 10, 235–320, *K*-theory (Strasbourg, 1992).

[Lic84] S. Lichtenbaum, *Values of zeta-functions at nonnegative integers*, Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983), Lecture Notes in Math., vol. 1068, Springer, Berlin, 1984, pp. 127–138.

[Lic87] ______, *The construction of weight-two arithmetic cohomology*, Invent. Math. **88** (1987), no. 1, 183–215.

[Lic90] ______, *New results on weight-two motivic cohomology*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 35–55.

[LLR05] Q. Liu, D. Lorenzini, and M. Raynaud, *On the Brauer group of a surface*, Invent. Math. **159** (2005), no. 3, 673–676.

[Mil68] J. S. Milne, *The Tate-Šafarevič group of a constant abelian variety*, Invent. Math. **6** (1968), 91–105.

[Mil70a] ———, *The Brauer group of a rational surface*, Invent. Math. **11** (1970), 304–307.

[Mil70b] J. W. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1970), 318–344.

[Mil80] J. S. Milne, *Etale Cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, 1980.

[Mil86] ———, *Values of zeta functions of varieties over finite fields*, Amer. J. Math. **108** (1986), no. 2, 297–360.

[Mil88] ———, *Motivic cohomology and values of zeta functions*, Compositio Math **68** (1988), 59–102.

[Mil07] ———, *The Tate conjecture over finite fields (AIM talk)*, arXiv:0709.3040, September 2007.

[MS74] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, no. 76, Princeton University Press, 1974.

[RS15] A. Rosenschon and V. Srinivas, *Étale motivic cohomology and algebraic cycles*, to appear in Journal of the Inst. of Math. Jussieu, 2015.

[Sch02] C. Schoen, *Complex varieties for which the Chow group mod n is not finite*, J. Algebraic Geom. **11** (2002), no. 1, 41–100.

[Ser65] Jean-Pierre Serre, *Zeta and L Functions*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963) (O. F. G. Schilling, ed.), Harper & Row, New York, 1965, pp. 82–92.

[Sou84] C. Soulé, *Groupes de Chow et K-théorie de variétés sur un corps fini*, Math. Ann. **268** (1984), no. 3, 317–345.

[Sou85] ———, *Opérations en K-théorie algébrique*, Canad. J. Math. **37** (1985), no. 3, 488–550.

[Ste47] N. E. Steenrod, *Products of cocycles and extensions of mappings*, Ann. of Math. (2) **48** (1947), 290–320.

[Ste62] ______, *Cohomology operations*, Lectures by N. E. Steenrod written and revised by D. B. A. Epstein. Annals of Mathematics Studies, No. 50, Princeton University Press, Princeton, N.J., 1962.

[SV00] A. Suslin and V. Voevodsky, *Bloch-Kato Conjecture and Motivic Cohomology with Finite Coefficients*, The Arithmetic and Geometry of Algebraic Cycles, NATO Science Series, vol. 548, Springer Netherlands, 2000, pp. 117–189.

[Tat63] J. Tate, *Duality theorems in Galois cohomology over number fields*, Proc. Internat. Congr. Mathematicians (Stockholm, 1962), Inst. Mittag-Leffler, Djursholm, 1963, pp. 288–295.

[Tat65] ______, *Algebraic cycles and poles of zeta functions*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), Harper & Row, New York, 1965, pp. 93–110.

[Tat66a] ______, *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144.

[Tat66b] ______, *On the conjectures of Birch and Swinnerton-Dyer and a geometric analogon*, Séminaire Bourbaki **18** (1966), no. 306, 415–440.

[Tat94] ______, *Conjectures on algebraic cycles in l-adic cohomology*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 71–83.

[Ura96] T. Urabe, *The bilinear form of the Brauer group of a surface*, Invent. Math. **125** (1996), no. 3, 557–585.

[Voi07] C. Voisin, *Some aspects of the Hodge conjecture*, Jpn. J. Math. **2** (2007), no. 2, 261–296.

[Wei49] A. Weil, *Numbers of solutions of equations in finite fields*, Bull. Amer. Math. Soc **55** (1949), no. 5, 497–508.

[Zar74a] Ju. G. Zarhin, *Isogenies of abelian varieties over fields of finite characteristic*, Math. USSR Sbornik **24** (1974), no. 3, 451–461.

[Zar74b] ______, *A remark on endomorphisms of abelian varieties over function fields of finite characteristic*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 471–474.

[Zar12] ______, *Poincaré duality and unimodularity*, arXiv:1112.1429v3, May 2012.

# Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, § 8, Abs. 2 Pkt. 5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

Jahn, Thomas
München

............................