Mathematical Models for Financial Bubbles

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Dissertation an der Fakultät für Mathematik,
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Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständing, ohne unerlaubte Beihilfe angeferteigt ist.

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Abstract

Financial bubbles have been present in the history of financial markets from the early days up to the modern age. An asset is said to exhibit a bubble when its market value exceeds its fundamental valuation. Although this phenomenon has been thoroughly studied in the economic literature, a mathematical martingale theory of bubbles, based on an absence of arbitrage has only recently been developed. In this dissertation, we aim to further contribute to the developement of this theory.
In the first part we construct a model that allows us to capture the birth of a financial bubble and to describe its behavior as an initial submartingale in the *build-up phase*, which then turns into a supermartingale in the *collapse phase*. To this purpose we construct a flow in the space of equivalent martingale measures and we study the shifting perception of the fundamental value of a given asset.
In the second part of the dissertation, we study the formation of financial bubbles in the valuation of defaultable claims in a reduced-form setting. In our model a bubble is born due to investor heterogeneity. Furthermore, our study shows how changes in the dynamics of the defaultable claim's market price may lead to a different selection of the martingale measure used for pricing. In this way we are able to unify the classical martingale theory of bubbles with a constructive approach to the study of bubbles, based on the interactions between investors.

Zusammenfassung

Finanz-Blasen sind seit der Entstehung der Finanzmärkte bis zur heutigen Zeit gegenwärtig. Es gilt, dass ein Vermögenswert eine Finanzblase aufweist, sobald dessen Marktwert die fundamentale Bewertung übersteigt. Obwohl dieses Phänomen in der Wirtschaftsliteratur ausgiebig behandelt wurde, ist eine mathematische Martingaltheorie von Blasen, die auf der Abwesenheit von Arbitragemöglichkeiten beruht, erst in letzter Zeit entwickelt worden. Das Ziel dieser Dissertation ist es einen Beitrag zur Weiterentwicklung dieser Theorie zu leisten.

Im ersten Abschnitt konstruieren wir ein Model mit Hilfe dessen man die Entstehung einer Finanz-Blase erfassen und deren Verhalten anfänglich als Submartingal in der *build-up phase* beschrieben werden kann, welches dann in der *collapse phase* zu einem Supermartingal wird. Zu diesem Zweck entwickeln wir einen Zahlungsstrom im Raum der äquivalenten MartingalmaÃe und wir untersuchen die zu dem Vermögenswert passende Verschiebung des fundamentalen Werts.

Der zweite Teil der Dissertation beschäftigt sich mit der Bildung von Finanz-Blasen bei der Bewertung von Forderungen, die mit Ausfallrisiken behaftet sind, in einer reduzierten Marktumgebung. In unserem Model ist die Entstehung einer Blase die Folge der Heterogenität der Investoren. Des Weiteren zeigen unsere Untersuchungen, inwieweit Veränderungen der Dynamik des Marktpreises einer risikobehafteten Forderung zu einer Veränderung des zur Bewertung verwendeten Martingalmaßes führen kann. Dadurch sind wir in der Lage die klassische Martingaltheorie von Finanz-Blasen mit einem konstruktivem Ansatz zur Untersuchung von Finanz-Blasen zu vereinigen, der auf den Interaktionen zwischen Marktteilnehmern basiert.

Contents

Chapter 1

Introduction

1.1 Motivation

Bubbles and financial crises have been recorded throughout modern history, with probably the earliest documented events going back to the 17th and 18th century: the Dutch tulipmania of 1634-1637, the Mississippi bubble of 1719-1720 and the South Sea bubble of 1720, see Garber [28], [29]. All of these historical events exhibit two phases: a run-up phase characterized by a sharp increase in the price of an asset(price of tulips, shares of the Compagnie des Indes, shares of the South Sea company respectively), followed by a decline phase -the collapse of the price. Let us illustrate this fact with a short expose of the last two historical events mentioned above.

The Mississippi bubble was created by the rise and fall of the Compagnie des Indes, vehicle through which John Law, France's Controller General of Finance tried to refinance France's debt. The share prices of Compagnie d'Indes rose from 1800 livres in July 1719, to 3000 livres in October 1719 and reached a peak value of 10000 livres at the end of the year. The rise in the price was partly motivated by large-scale takeovers of other commercial companies (acquisition of the East India Company and China Company in May 1719, acquisition of the Senegalese Company in September 1719) which lead to a monolopy over all French trade outside Europe, together with a takeover of government functions (such as the mint and the collection of taxes). Law's idea was to establish a "fund of credit" which, when leveraged, could finance and help expand the commercial enterprise: "finance of the operation came first; expanded commercial activity would result naturally once the financial structure was in place"(Garber [28]). In January 1720, due to increasing attempts of shareholders to convert capital gains into gold form, the share prices began to fall below 10000 livres. Despite Law's attempts to prevent the stock price's decline, the price fell to 2000 livres in September 1720 and to 500 livres in September 1721, the same value the stock had in May 1719.

The South Sea bubble was caused by a similar plan, though much simpler, of the South Sea Company to acquire British government debt. The Company's share price experienced a sharp increase from about 120 pounds in January 1720 to 775 pounds in August 1720. The speculation around the Company's shares lead to an increase in the prices of other Companies and to the creation of numerous, mostly fraudulent, "bubble companies". The South Sea Company's share prices fell from its peak value of 775 pounds to 290 pounds in October 1720. The collapse is generally attributed to a liquidity crisis caused by the "Bubble Act" passed by the Parliament in June 1720, and by the collapse of Law's Compagnie d'Indes in September 1720, see Garber [28], Neal [47], Neal and Schubert [48].

More recent examples of bubbles include the ones that occured around the US banking crises of 1837, 1873 and 1907, The Great Crash of 1929, the Japanese housing bubble of 1970-1989, the Dot-Com bubble of 1997-2002 and the recent US housing bubble. For an overwiew of the causes that triggered these events, see Brunnermeier and Oehmke [11] and Protter [50], [51].

Several characteristics of financial bubbles have been pointed out as reccurent in the economical literature. Asset price bubbles usually occur in periods of technological and financial innovation, which leads to increasing expectations of economic growth and profit among investors, see for example Brunnermeier and Oehmke [11], Protter [51], Scheinkman [54]. The developement of trade between Europe and its colonies can be seen as a possible trigger for the Mississippi and South Sea Bubbles, the advent of railroads, the internet, and of sophisticated financial instruments and hedging techniques can be seen as triggers for the Panic of 1873, the Dot-Com Bubble and the recent credit bubble, respectively. Bubbles arise simultaneously with an increasing trading volume of the asset, and bubbles burst once the asset's supply is increased or a liquidity crisis appears, see for example Brunnermeier [10]. Furthermore, in an environment with several risky assets, asset price bubbles can implode due to contagion, if the assets are held by the same investors and are exposed to the same funding liquidity constraint.

An asset price bubble is defined to be the difference between the asset's market price and the lowest superreplicating price of its future dividents. From the economic perspective, one has to connect the appearance and disappearance of an asset price bubble at the microeconomic level with the interactions between the market participants. In the economic literature, investor heterogeneity and limits to arbitrage are often indicated as possible factors behing the creation of asset price bubbles. Limits to arbitrage can result from short-selling constraints, see Miller [46], or shocks to funding liquidity, see e.g. Schleifer and Vishny [56]).

Since most financial investments are exposed to risk and knightian uncertainty, this can originate divergence of opinions among investors, or investor heterogeneity (investors *agree to disagree*). Harrison and Kreps [30] point out that agents may disagree on the value of future dividends. In Scheinkman and Xiong [55], each investor has a set of signals and observations on which he bases his estimation of the asset's fundamental value. Overconfidence, or the agents' tendency to exaggerate the importance of certain signals, can also lead to investor heterogeneity. In a similar way, in the model proposed in Föllmer et. al.[25], investors may use different predictors when forecasting future prices and this may create heterogeneous beliefs in the market. For other interesting references on the topic of bubble formation, see for instance Tirole [60], DeLong, Shleifer, Summers and Waldmann [21], Abreu and Brunnermeier [1], and the references therein.

In mathematical finance, the characterization of price bubbles in terms martingales/strict local martingales, in the setting of a market satisfying the "no free lunch with vanishing risk hypothesis ($NFLVR$)" was first introduced by Loewenstein and Willard [44]. In Cox and Hobson [18], an asset contains a bubble within its price process, if the discounted price process follows under the risk neutral measure used for pricing a strict local martingale, that is, a local martingale that is not a martingale. For a set of examples of strict local martingales and references to additional literature on the topic, see Appendix A.

This approach to the study of financial bubbles, referred to as *the martingale theory of bubbles*, was fundamented in the complete and incomplete market settings in the seminal works of Jarrow, Protter and Shimbo [39],[40]. In Jarrow et .al [40], [36], the authors also provide a study of derivatives written on assets exhibiting price bubbles, such as European call and put options, American options, forward and future contracts. An interesting property that has been highlighted in the above works is that the put-call parity usually does not hold when the asset's price process is driven by a strict local martingale. The presence of bubbles in foreign exchange rates has been studied in Jarrow and Protter [37] and Carr, Fisher and Ruf [13]. In the model constructed in [37], exchange rate bubbles are caused by the existence of a bubble in the price of either or both the domestic and foreign market currencies. Furthermore, in contrast to asset price bubbles that are always positive, exchange rate bubbles can be negative. More precisely, if a foreign currency exhange rate is positive, then its inverse exchange rate is negative. Carr, Fisher and Ruf [13] use the Föllmer measure [24] to construct a pricing operator for complete models where the exhange rate is driven by a strict local martingale. This construction allows to preserve the put-call parity and also provides the minimal joint replication price for a contingent claim.

Further connections between bubbles and the prices of derivatives written on assets whose price process is driven by a strict local martingale have been studied in Pal and Protter [49]. Karatzas, Kreher and Nikeghbali [42] extend the results of [49], by providing the decomposition of the price of certain clases of path-dependent options (modified call-options and chooser options) and barrier options into a "non-bubble" term and a default term. Since the martingale theory of financial bubbles does not allow for bubbles in the price of bounded asset prices, in Bilina-Falafala, Jarrow and Protter [8] the novel concept of a relative asset bubble is introduced, which allows the study of risky assets with bounded payoffs within the current theory. With the help of the volatility-based criteria developed by Carr et.al [12] for identifying strict local martingales, in Jarrow, Kchia and Protter[34] a methodology for identifying bubbles (in real time) has been developed. In a recent article, Herdegen and Herrmann [31] discuss the stochastic investment opportunities in market model with bubbles. For a comprehensive survey of the recent mathematical literature on financial bubbles we refer to Protter [51].

The present thesis can be divided into two parts. In the first part, which covers Chapter 2 and Chapter 3, we construct a model that allows for the slow birth of an asset price bubble starting at zero, in the sense that its initial behavior is described by a submartingale. To this purpose we fix two local martingale measures Q and R, under which the asset's wealth process follows a uniformly integrable martingale and a strict local martingale, respectively. In Section 2.3 of Chapter 2, we construct a flow $\mathcal{R} = (\mathcal{R}_t)_{t\geq 0}$ in the space of martingale measures which moves from the initial measure Q to the measure R, via convex combinations of the two. We provide sufficient conditions under which the resulting $\mathcal{R}$-bubble perceived under the flow $\mathcal{R}$ is a local R-submartingale. Thus, we are able to capture in a realistic way the birth and subsequent behavior of the $\mathcal{R}$-bubble under the reference measure R as follows: the $\mathcal{R}$- bubble starts from its initial value as a submartingale and then turns into a supermartingale before it finally falls back to zero. In Section 2.4, we examine in the setting of a slight extension of the Delbaen-Schachermayer example, the behavior of the bubble process constructed above. We show that the sufficient conditions under which the bubble process is a submartingale are satisfied. In the final Section 2.5 of Chapter 2, we change our point of view and instead of using R as reference measure, we take Q as reference. Here again, the birth of the bubble can be described as a initial submartingale. However, its behavior is more delicate to examine, as illustrated in the context of the Delbaen-Schachermayer example.

Chapter 3 is dedicated to the study of the $\mathcal{R}$-bubble in stochastic volatility models. We verify that the sufficient conditions for our $\mathcal{R}$-bubble process to be a local R-submartingale hold for a variant of the stochastic volatility model discussed by Sin [57]. Moreover, our model can be modified in such a way that the condition no longer holds. A similar analysis of the bubble process is done in Section 3.2 in a modified variant of the Andersen-Piterbarg volatility model [3].

In Chapter 4 we examine the formation of financial bubbles in the valuation of defaultable claims in a reduced form credit risk model. The birth of a bubble is generated by the impact of the heterogeneous beliefs of investors on the defaultable claim's market wealth. A large group of investors whose trades can influence the asset price may consider the claim to be a safe investment under certain circumstances. Their trading actions create changes in the dynamics of the asset's market wealth process, and these lead to subsequent shifts in the selection of the martingale measures used for pricing. Therefore, our model also provides an explanation how microeconomic interactions between agents may at an aggregate level determine a shift in the martingale measure, via a change in the dynamics of the market wealth process. In this way we establish a connection between the impact of different views within the groups of investors on asset prices and the classical results of the martingale theory of bubbles, see Biagini, Föllmer and Nedelcu [4] and Jarrow and Protter [40]. The Chapter concludes with a characterization of the space of equivalent local martingale measures with the help of measure pasting, characterization which allows to rigurously capture how changes in the dynamics of the asset price process lead to different selections of local martingale measures used for pricing.

1.2 Contributing Manuscripts

This thesis is based on the following manuscripts, which were developed by the thesis' author S. Nedelcu, in cooperation with coauthors:

1. F. Biagini, H. Föllmer, and S. Nedelcu [4]: *Shifting martingale measures and the birth of a bubble as a submartingale. Finance and Stochastics, 18(2): 297-326, 2014.*
   The results of this paper are the product of a joint work of S. Nedelcu with two coauthors, Prof. F. Biagini and Prof. H. Föllmer. Most of the work was developed at the LMU Munich. Certain parts where finalized during S. Nedelcu's visits to Humboldt University and University

of Luxembourg (at the invitation of Prof. H. Föllmer). After a suggestion of H. Föllmer, S. Nedelcu and F. Biagini started to study a convex flow on the space of equivalent martingale measures and proved most of the results in Section 3, in particular Theorem 2.3.9 and Corollary 2.3.10. The argumentation concerning the economical significance of the model which is contained in Section 2, and Section 4 which contains the Delbaen-Schachermayer example, was developed by S. Nedelcu together with his 2 coauthors. Section 5 was developed independently by S. Nedelcu. Section 6 was developed by S. Nedelcu together with Prof. H. Föllmer. However, the final version of this part is mainly due to S. Nedelcu.

2. F. Biagini and S. Nedelcu [5]: *The formation of financial bubbles in defaultable markets. Preprint, Available at http://www.fm.mathematik.uni-muenchen.de/download/publications, 2014.*
   The construction of the reduced-form credit risk framework in which the formation of bubbles is studied, was developed by S. Nedelcu together with Prof. F. Biagini. Thus, Section 2, in which the model's framework is constructed, Section 3, which contains the connection with the classical martingale theory of bubbles, and Section 5, which contains a case study, are the result of this joint work. A significant part of the computations contained in the proofs of these Chapters was developed by S. Nedelcu. Furthermore S. Nedelcu suggested the use of the predictable version of Girsanov's theorem in the proofs concerning measure changes and the existence of equivalent local martingale measures. The main idea of Section 4, the use of the concept of *measure pasting* to the study of asset price bubbles, was developed by S. Nedelcu. This plays a fundamental role in unifying a constructive approach to the study of asset price bubbles with the martingale theory of bubbles.

The following list indicates in which way the two publications contribute to each Chapter of the Thesis and which sections represent unpublished manuscripts. The formulation of the statements of the Propositions, Lemmas, Theorems, etc. is the same as in the two manuscripts. However, the author, who has been involved in the developement of all the results contained in the two publications, provides in the present thesis a more detailed version for most of the proofs.

1. Chapter 1 provides a presentation of the existing mathematical literature on financial bubbles. The summary of each article was written

independently by S. Nedelcu. The Chapter concludes with a summary of the Thesis.

2. Chapter 2 is based the Sections $2, 3, 4$ and 6 of Biagini, Föllmer and Nedelcu [4].

3. Section 3.1 of Chapter 3 is based on Section 5 of Biagini, Föllmer and Nedelcu [4]. Section 3.2 of Chapter 3 was developed independently by S. Nedelcu, and is based on a manuscript which is not yet published.

4. Chapter 4 is based on Biagini and Nedelcu [5]. Section 4.4.1, developed independently by S. Nedelcu as part of an earlier draft version of Biagini and Nedelcu [4], and is based on a manuscript which is not yet published.

Chapter 2

Shifting martingale measures

The contents of this Chapter are based on the author's joint work with F. Biagini and H. Föllmer, which is contained in the article F. Biagini, H. Föllmer and **S. Nedelcu** [4]. More precisely, the present Chapter is based on Sections $2, 3, 4$ and 6 of [4]. The detailed description of the author's personal contribution is presented in Section 1.2.

2.1 Motivation

An asset price bubble is defined as the difference between two components: the observed *market price* of a given financial asset, which represents the amount that the marginal buyer is willing to pay, and the asset's intrinsic or *fundamental value*, which is defined as the expected sum of future discounted dividends. As a result, *Knightian/model uncertainty* may arise due to the fact that the asset's fundamental value definition is made in terms of a conditional expectation. The fundamental component may be differently perceived by investors according to their choice of the pricing measure. Thus, some agents may consider that a bubble exists in the asset price if their choice of the pricing measure does not lead to an equality between the corresponding fundamental value and the asset's market value. However, as shown in experimental economics, see Smith, Suchanek and Williams [58], bubbles may arise even if the probabilistic structure is perfectly known and the economic agents are kept informed across all times about the fundamental values of the assets.

In the present chapter we examine the perception of the fundamental value. A first study on how this perception is connected to bubbles is done in a complete market setting in [39], by Jarrow, Protter, and Shimbo, who however point out the following inconvenient of the model: due to the uniqueness of the martingale measure, bubbles cannot be born in this setting. They either exist from the start of the model (and may disappear in time), or not. In order to overcome this difficulty, Jarrow, Protter, and Shimbo consider in their paper on *Asset price bubbles in incomplete financial markets* [40] an incomplete market model, i.e. a setting where there exist an infinite number of local martingale measures. Hence the possible martingale measure that can be used for pricing is not unique anymore and each measure provides a market consistent view of the future.

In our setting the discounted price process of a liquid financial asset follows a semimartingale S under the real-world measure and D denotes the associated cumulative discounted dividend process. We assume the existence of an equivalent local martingale measure which turns the wealth process $W = S + D$ into a local martingale. By following an argument of Harrison and Kreps [30] we prove that any martingale measure can be seen as a prediction scheme that is consistent with the observed price process S from a speculative point of view which takes into account future dividends and the possibility of selling the asset at some future time.

However, different choices of the martingale measures may give different assessments of the fundamental value, which is defined as the conditional expectation of future discounted dividends under the chosen equivalent local martingale measure. Hence, if we denote S^R to be the instrinsic value process under a martingale measure R, the bubble is defined as the difference $\beta^R = S - S^R$, and this will be a non-negative local martingale under R. Nevertheless, in this incomplete model, if the bubble is defined in terms of one fixed martingale measure R, then it either exists from the start of the model (if R allows the existence of bubble) or it is zero all the time. In order to allow the birth of a bubble in this framework after the model starts, we eliminate the time-consistency assumption.

We consider the definition of time consistency as provided in Föllmer and Schied [26]. *Time consistency* amounts to the requirement that all conditional probability distributions $R_t(\cdot|\mathcal{F}_t)$, where R_t is an equivalent martingale measure for all $t > 0$ belong to the same martingale measure R. Here $R_t(\cdot|\mathcal{F}_t)$ represents the market's view at time t. In particular, a complete market model is automaticaly time-consistent, due to the uniqueness of the martingale measure. While time consistency is taked for granted in the mathematical literature, in the real financial world various interactions between investors at the microeconomic level, like herding behavior of heterogeneous agents with interacting preferences and expectations, actions of large traders (see Chapter 4), or regime changes in the economy, may cause a shift of the martingale measure. This in turn leads to a dynamics in the space of equivalent local martingale measures and corresponding shifting perceptions of the fundamental value.

In [40], Jarrow, Protter and Shimbo consider a market model where regime shifts occur in the underlying economy (due to risk aversion, institutional structures, technological innovations etc.) at different random times. This allows for the construction of a *dynamic market model* where a bubble suddenly appears in the price of an asset at a stopping time and disappears again at a later stopping time.

The main objective of this chapter is to provide a realistic mathematical model which captures the two stages of a financial bubble: *the build-up stage*, when the market price process diverges from the fundamental value and *the collapse phase* when the market price drops and returns to the fundamental value. To this purpose we wish to capture in the proposed setting the slow birth of a perceived bubble starting at zero and to describe it as an initial submartingale, which then turns into a supermartingale before it falls back to the initial value zero. We consider two martingale measures Q and R representing the views of two (groups of) agents. A martingale measure is often interpreted as a price equilibrium corresponding to the subjective preferences and expectations of some representative agent; see for example Föllmer and Schied [27], Section 3.1. In the case of the martingale measure Q, the wealth process W is a uniformly integrable martingale, we have $S = S^Q$. Hence, this subjective view can be interpreted as "optimistic" or "exuberant", since the market price is seen to coincide with the asset's perceived intrinsic value computed under the pricing measure Q. In particular, under the measure Q there is no perception of a bubble contained in the asset's price. Under the measure R, the market wealth process W is no longer uniformly integrable and we have $S > S^R$. Hence, under this view that can be characterized as "pessimistic" or "sober", the market price is not justified from a fundamental point of view and is affected by a bubble.

We provide examples of incomplete financial market models where such martingale measures Q and R coexist, see Section 2.4 of the present Chapter and Chapter 3 for examples which concern stochastic volatility models. Furthermore, we prove in the Delbaen-Schachermayer example, see Section 2.4 as well as in the stochastic volatility model of Section 3.1, that the following condition is satisfied: The fundamental wealth $W^R = S^R + D$ perceived under the "sober" measure R behaves as a submartingale under the "optimistic" measure Q. In terms of the agents' perspectives, this behavior of W^R under the optimistic view represented by Q suggests that the pessimistic assessment W^R is expected to be corrected through an upward trend.

In Section 2.3, we study a flow $\mathcal{R} = (R_t)_{t\geq 0}$ in the space of martingale measures that moves from the initial uniformly integrable martingale measure Q to the non-uniformly integrable martingale measure R via convex combinations of Q and R by putting an increasing weight on R. At each time t, the fundamental value process is computed with respect to the martingale measure R_t. Consequently, the asset's fundamental value is described by the process $S^{\mathcal{R}}$. A further consequence of this construction is a shifting perception of the asset's fundamental value. We denote by $\beta^{\mathcal{R}} = S - S^{\mathcal{R}}$ the resulting $\mathcal{R}$-bubble perceived under the flow $\mathcal{R}$. In Theorem 2.3.9, we provide a crucial condition under which the birth and the subsequent behavior of the $\mathcal{R}$-bubble under the reference measure R can be described as follows: The $\mathcal{R}$-bubble starts from its initial value as a submartingale (*the build-up stage*) and then turns into a supermartingale before it finally falls back to zero (*the collapse phase*). In Remark 2.3.3, we provide a possible economic intepretation for the shifting perception of the asset's fundamental value by refering to a microeconomic model of interacting agents described in Föllmer et al. [25].

In Section 2.4, we consider a slight extension of the classical Delbaen-Schachermayer setting. Instead of defining the price process along with the measures Q and R in terms of two independent geometric Brownian motions, we consider a more general case where the price process S and the Radon-Nikodym density process of Q with respect to R are defined in terms of two independent continuous martingales. We are able to explicitly compute the processes W^R and $\beta^{\mathcal{R}}$ and show that the necessary condition for W^R to be a local submartingale under Q is satisfied.

Section 2.5 provides a study of our model from the Q-perspective. To this purpose we compute the canonical decomposition of the $\mathcal{R}$-bubble under the measure Q and provide conditions when the the birth of the bubble can be described as an initial submartingale with respect to the measure Q. However, as we show in the context of our extension of the Delbaen-Schachermayer example, the modelling of the $\mathcal{R}$-bubble's behavior is not straighforward and requires more technical effort.

The present chapter complements the study of successive regime switching of Jarrow et al.[40] by allowing the capture of a submartingale behaviour in the birth of a perceived bubble and also provides a framework that allows for shifting martingale measures. This setting can be used as basis for a study of the space of equivalent local martingale measures where the topological characterization of this space can be related to the size of a bubble. Furthermore, these contributions allow us to provide an interesting model also from a practitioner point of view, since a dynamic in the space of martingale measures can be connected with the interactions of the market participants at the microeconomic level. We illustrate this in Chapter 4, by showing how possible changes in the dynamics of the asset price process can lead to different selections of the martingale measures used for pricing. Moreover, Chapter 4 provides an example of an incomplete market setting where time-consistency fails and an asset price bubble can be born after the start of the model.

2.2 The Setting

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions: $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous i.e. $\mathcal{F}_t = \cap_{u>t}\mathcal{F}_u$ for all $t \geq 0$. We consider a market model that contains a risky asset and a money market account. We take the money market accout as numéraire and consider directly a discounted setting, i.e. it is supposed to be constantly equal to 1. Let $D = (D_t)_{t\geq 0}$ be a non-negative increasing and adapted right-continuous process representing the uncertain cumulative cash flow generated by the risky asset. We assume that the filtration is such that all martingales have continuous paths.

If ζ is a stopping time representing the maturity date or default time of the risky asset and the process D is seen as a cumulative dividend process, one can recover the setting of Jarrow et al. [40] by setting $D_t := D_\zeta$ on $\{\zeta \leq t\}$ and defining the terminal payoff or liquidation value of the asset as $X_\zeta := (D_\zeta - D_{\zeta-})1_{\{\zeta<\infty\}}$.

Let $S = (S_t)_{t\geq 0}$ be a non-negative, adapted càdlàg process representing the market price of the risky asset and denote by $W = (W_t)_{t\geq 0}$ the corresponding *wealth process* defined by

$$W_t = S_t + D_t,\ t \geq 0.$$

Definition 2.2.1. *A probability measure Q equivalent to P under which the wealth process W is a Q-local martingale is called an* equivalent local martingale measure*.*

We denote by $\mathcal{M}_{loc}(W)$ the class of all probability measures $Q \approx P$ such that W is a local martingale under Q, and we asssume that

$$\mathcal{M}_{loc}(W) \neq \emptyset. \tag{2.2.1}$$

The existence of an equivalent local martingale measure implies, via the First Fundamental Theorem, that S satisfies the *No Free Lunch with Vanishing Risk* ($NFLVR$) condition. The converse implication is also true, see Delbaen and Schachermayer [19]. In economic terms, $NFLVR$ amounts to the exclusion of all self-financing trading strategies that start with zero initial investment and generate a non-negative cash flow for sure and a strictly positive cash flow with positive probability (arbitrage opportunities). We provide examples of markets satisfying the $NFLVR$ condition in Section 2.4, Chapter 3 and Section 4.4 in Chapter 4.

Furthermore, the existence of an equivalent local martingale measure implies that the process W follows a semimartingale under the real world measure P, i.e. W can be written as the sum between a càdlàg local martingale and a càdlàg finite variation process.

Although the cumulative cash flow D associated to the risky asset is exogenously given, the price S_t can be justified at any time t from the perspective of any probability measure $Q \in \mathcal{M}_{loc}(W)$ in the following way: the investors determine the value S_t at time t by taking into account the expectation of the future cumulative cash-flows together with the option to sell the asset at some future time τ. As explained in Harrison and Kreps [30], this quantity represents the maximum amount that the risky asset is worth for any investor pricing under the measure Q at time t. This reasoning is expressed in rigorous mathematical way in the following Lemma, more precisely in equation (2.2.2) below.

Lemma 2.2.2. *For any $Q \in \mathcal{M}_{loc}(W)$, the limits $S_\infty := \lim_{t\to\infty} S_t$, $W_\infty := \lim_{t\to\infty} W_t$ and $D_\infty := \lim_{t\to\infty} D_t$ exist a.s. and in $L^1(Q)$, and*

$$\begin{aligned} S_t &= \text{ess sup}_{\tau\geq t}\mathbb{E}_Q[D_\tau - D_t + S_\tau|\mathcal{F}_t] \\ &= \text{ess sup}_{\tau\geq t}\mathbb{E}_Q[D_\tau - D_t + S_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t], \end{aligned} \tag{2.2.2}$$

where the essential supremum is taken over all stopping times $\tau \geq t$.

Proof. Since W is a non-negative local martingale, it follows from Fatou's lemma that W is a supermartingale under Q since

$$\begin{aligned} \mathbb{E}_Q[W_t|\mathcal{F}_s] &= \mathbb{E}_Q[\lim_{n\to\infty} W_{t\wedge\sigma_n}|\mathcal{F}_s] \leq \liminf_{n\to\infty} \mathbb{E}_Q[W_{t\wedge\sigma_n}|\mathcal{F}_s] \\ &= \liminf_{n\to\infty} W_{s\wedge\sigma_n} = W_s, \end{aligned}$$

for any $s \leq t$, where $(\sigma_n)_{n\geq 0}$ represents a localizing sequence for the local martingale W. By following the same reasoning we have

$$\sup_{t\geq 0} \mathbb{E}_Q[|W_t|] \leq \mathbb{E}_Q[W_0] < \infty.$$

Therefore an application of the Martingale Convergence Theorem yields the existence of the limit $W_\infty := \lim_{t\to\infty} W_t$ Q-a.s. and in $L^1(Q)$. So does $S_\infty := \lim_{t\to\infty} S_t$, since the limit $D_\infty := \lim_{t\to\infty} D_t$ exists by monotonicity. Thus the right side of equation (2.2.2) is well defined. Moreover, it follows from the optional sampling theorem that

$$W_t \geq \mathbb{E}_Q[W_\tau|\mathcal{F}_t] \tag{2.2.3}$$

for any stopping time $\tau \geq t$, due to the fact that W is a right-continuous closed supermartingale. Since $W_t = S_t + D_t$ for all $t \geq 0$, (2.2.3) translates into

$$S_t \geq \mathbb{E}_Q[D_\tau - D_t + S_\tau|\mathcal{F}_t] \geq \mathbb{E}_Q[D_\tau - D_t + S_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t], \tag{2.2.4}$$

for any stopping time $\tau \geq t$. Let now $\zeta = (\zeta_n)_{n\geq 0}$ be a localizing sequence for the Q-local martingale W. Then we get equality in (2.2.3), and hence in (2.2.4), for $n > t$ and $\tau = \zeta \wedge n$.So we have proved (2.2.2). □

Let

$$S_t^Q := \mathbb{E}_Q[D_\infty - D_t|\mathcal{F}_t], \quad t \geq 0, \tag{2.2.5}$$

be the potential generated by the increasing process D under the measure Q. For the sake of convenience we remind the reader the definition of a potential.

Definition 2.2.3. *An adapted càdlàg process $X = (X_t)_{t\geq 0}$ is called a potential if it is a non-negative supermartingale satisfying $\lim_{t\to\infty} \mathbb{E}[X_t] = 0$.*

An important consequence of Lemma 2.2.2 is that

$$S_t \geq S_t^Q = \mathbb{E}_Q[D_\infty - D_t|\mathcal{F}_t], \tag{2.2.6}$$

Now we can define precisely the fundamental price of an asset as the expected future cumulative cash-flow under a given equivalent local martingale measure.

Definition 2.2.4. *For $Q \in \mathcal{M}_{loc}(W)$ the potential S^Q defined in* (2.2.5) *will be called the* fundamental price *of the asset* perceived under the measure Q.

For any martingale measure $Q \in \mathcal{M}_{loc}(W)$ and given the possibility of selling the asset at some future time, Lemma 2.2.2 shows that the given price of an asset can be justified from a speculative point of view. In this sense different martingale measures provide the same assessment of the price S. However, if one considers each martingale measure Q to represent the views of a certain class of investors, then members of different classes may disagree on the asset's fundamental value S^Q. In order to illustrate this point we write the set $\mathcal{M}_{loc}(W)$ as the following reunion:

$$\mathcal{M}_{loc}(W) = \mathcal{M}_{UI}(W) \cup \mathcal{M}_{NUI}(W),$$

where $\mathcal{M}_{UI}(W)$ is the space of measures $Q \approx P$ under which W a uniformly integrable martingale and $\mathcal{M}_{NUI}(W) = \mathcal{M}_{loc}(W) \setminus \mathcal{M}_{UI}(W)$. There exist frameworks where the classes $\mathcal{M}_{UI}(W)$ and $\mathcal{M}_{NUI}(W)$ can be simultaneously non-empty, see the examples of Section 2.4 and Chapter 3. From now on we assume that this is the case:

Assumption 2.2.5. *$\mathcal{M}_{UI}(W) \neq \emptyset$ and $\mathcal{M}_{NUI}(W) \neq \emptyset$.*

Lemma 2.2.6. *A measure $Q \in \mathcal{M}_{loc}(W)$ belongs to $\mathcal{M}_{UI}(W)$ if and only if*

$$S_t = \mathbb{E}_Q[D_\infty - D_t + S_\infty|\mathcal{F}_t], \quad t \geq 0. \tag{2.2.7}$$

Proof. If $Q \in \mathcal{M}_{UI}(W)$ then W is a Q-uniformly integrable martingale and

$$W_t = \mathbb{E}_Q[W_\infty|\mathcal{F}_t], \tag{2.2.8}$$

for all $t \geq 0$. and this gives

$$S_t + D_t = \mathbb{E}_Q[S_\infty + D_\infty|\mathcal{F}_t], \quad t \geq 0.$$

which is equivalent to (2.2.7). Conversely, condition (2.2.7) implies (2.2.8), and so W is a uniformly integrable martingale under Q. □

The following Assumption guarantees that the given market price S is justified also from a fundamental point of view i.e. there exists an equivalent local martingale measure under which S is perceived as a fundamental price.

Assumption 2.2.7. *There exists $Q \in \mathcal{M}_{loc}(W)$ such that*

$$S = S^Q, \tag{2.2.9}$$

where S^Q is the fundamental price perceived under Q as defined in (2.2.6)*.*

Lemma 2.2.8. *Assumption 2.2.7 holds if and only if $S_\infty = 0$ a.s., and in this case equation* (2.2.9) *is satisfied if and only if $Q \in \mathcal{M}_{UI}(W)$.*

Proof. By (2.2.2) the equality $S = S^Q$ implies $S_\infty = 0$ a.s. Conversely, if $S_\infty = 0$ a.s. then (2.2.7) shows that $S = S^Q$ holds if and only if $Q \in \mathcal{M}_{UI}(W)$, which by Assumption 2.2.5 contains at least one element. □

From now on we assume that Assumption 2.2.7 is satisfied, and so we have $W_\infty = D_\infty$ a.s.

Definition 2.2.9. *Let $Q \in \mathcal{M}_{UI}(W)$. The process $W^Q = S^Q + D$, defined by*

$$W_t^Q := \mathbb{E}_Q[D_\infty|\mathcal{F}_t], \quad t \geq 0, \tag{2.2.10}$$

will be called the fundamental wealth *of the asset* perceived under *Q.*

The above results lead to the following definition of a bubble.

Definition 2.2.10. *For any $Q \in \mathcal{M}_{loc}(W)$ the non-negative adapted process β^Q defined by*

$$\beta^Q = S - S^Q = W - W^Q \geq 0 \tag{2.2.11}$$

will be called the bubble perceived under *Q or the Q-*bubble*.*

The following result summarizes our findings and provides a characterization of the Q-bubble.

Corollary 2.2.11. *A measure $Q \in \mathcal{M}_{loc}(W)$ belongs to $\mathcal{M}_{UI}(W)$ if and only if the Q-bubble reduces to the trivial case $\beta^Q = 0$. For $Q \in \mathcal{M}_{NUI}(W)$ the Q-bubble β^Q is a non-negative local martingale such that $\beta_0^Q > 0$ and*

$$\lim_{t \to \infty} \beta_t^Q = 0, \text{ a.s. and in } L^1(Q). \tag{2.2.12}$$

Proof. Since $\beta^Q = W - W^Q$, then it is easy to see that β^Q is a Q-local martingale as the difference between the Q-local martingale W and the Q-uniformly integrable martingale W^Q given by $W_t^Q = \mathbb{E}_Q[D_\infty | \mathcal{F}_t]$, $t \geq 0$. By Lemma 2.2.2 and Lemma 2.2.8 we have that S and S^Q converge to 0 almost surely and in $L^1(Q)$, we obtain (2.2.12). □

For $Q \in \mathcal{M}_{NUI}(W)$ the Q-bubble β^Q appears immediately at time 0 and follows the dynamic of a non-negative local martingale. It follows from Fatou's lemma that β^Q is a Q-supermartingale and therefore

$$\mathbb{E}_Q[\beta_0^Q] \geq \mathbb{E}_Q[\beta_t^Q],$$

for all $t \geq 0$. In order to overcome this model drawback, in the following section we consider a flow in the space $\mathcal{M}_{loc}(W)$ that begins in $\mathcal{M}_{UI}(W)$ and then enters the class $\mathcal{M}_{NUI}(W)$. This allows us to describe the slow birth of a bubble starting from an initial value 0 and which follows the dynamics of a submartingale process in the first phase.

2.3 The birth of a bubble as a submartingale

In this section we consider a flow $\mathcal{R} = (R_t)_{t\geq 0}$ in the space of equivalent local martingale measures, i.e. $R_t \in \mathcal{M}_{loc}(W)$ for any $t \geq 0$. The market's view of the future at each time t will be expressed as the conditional expectation under the measure R_t of the future cumulative cash flows. We assume that $\mathcal{R}$ is càdlàg in the simple sense that the adapted process $W^{\mathcal{R}}$ defined by

$$W_t^{\mathcal{R}} := \mathbb{E}_{R_t}[D_\infty|\mathcal{F}_t], \quad t \geq 0, \tag{2.3.1}$$

admits a càdlàg version. This implies that the adapted process S^R defined by

$$S_t^{\mathcal{R}} = W_t^{\mathcal{R}} - D_t = \mathbb{E}_{R_t}[D_\infty|\mathcal{F}_t] = \mathbb{E}_{R_t}[D_\infty - D_t|\mathcal{F}_t], \ t \geq 0.$$

has also a càdlàg version. This property is for example satisfied in the dynamic market setting described in Jarrow et al.[40], and holds if the flow consists in switching from one martingale measure to another at certain stopping times.

Definition 2.3.1. *For a càdlàg flow $\mathcal{R} = (R_t)_{t\geq 0}$ we define the $\mathcal{R}$*-bubble *as the non-negative, adapted, càdlàg process*

$$\beta^{\mathcal{R}} := W - W^{\mathcal{R}} = S - S^{\mathcal{R}} \geq 0.$$

It is easy to see that the definitions of the processes $W^{\mathcal{R}}, S^{\mathcal{R}}$ and of the associated bubble process $\beta^{\mathcal{R}}$ depend on the conditional probability distributions

$$R_t[\cdot|\mathcal{F}_t], \quad t \geq 0. \tag{2.3.2}$$

In the classical mathematical literature, these conditional probability distributions which quantify the market's view of the future at each time t are assumed to be *time consistent*. This amounts to the requirement that the conditional probability distributions $\{R_t[\cdot|\mathcal{F}_t];\ t \geq 0\}$ belong to the same martingale measure $R_0 \in \mathcal{M}_{loc}(W)$. From an economic perspective, if

$$\pi_t(H) = \int H dR_t[\cdot|\mathcal{F}_t] = \mathbb{E}_{R_t}[H|\mathcal{F}_t], \quad t \geq 0$$

represents the prediction of the value of a bounded contingent claim H at time t, then time consistency requires

$$\pi_s(\pi_t(H)) = \pi_s(H) \tag{2.3.3}$$

or equivalently

$$\mathbb{E}_{R_s}[\mathbb{E}_{R_t}[H|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}_{R_s}[H|\mathcal{F}_s].$$

for any $s \leq t$. It is easy to see that (2.3.3) is satisfied if all the conditional distributions in (2.3.2) belong to the same martingale measure $R_0 \in \mathcal{M}_{loc}(W)$. The converse holds as well, as shown by the following proposition.

Proposition 2.3.2. *If $R_t[\cdot|\mathcal{F}_t] \neq R_0[\cdot|\mathcal{F}_t]$ for some $t > 0$ then time consistency fails.*

Proof. If $R_t[\cdot|\mathcal{F}_t] \neq R_0[\cdot|\mathcal{F}_t]$, then for some $A \in \mathcal{F}$ and some $t > 0$, the event

$$B_t = \{R_t[A|\mathcal{F}_t] > R_0[A|\mathcal{F}_t]\}$$

has positive probability $R_0[B_t] > 0$. In particular

$$1_{B_t}(R_t[A|\mathcal{F}_t] - R_0[A|\mathcal{F}_t]) \geq 0.$$

We consider the bounded contingent claim $H := I_{A\cap B_t}$. Then H satisfies

$$\pi_t(H) = \mathbb{E}_{R_t}[H|\mathcal{F}_t] \geq \mathbb{E}_{R_0}[H|\mathcal{F}_t],$$

and the inequality is strict on B_t. Thus we get

$$\begin{aligned}\pi_0(H) = \mathbb{E}_{R_0}[H|\mathcal{F}_0] = \mathbb{E}_{R_0}[H] &= \mathbb{E}_{R_0}[\mathbb{E}_{R_0}[H|\mathcal{F}_t]] \\ &< \mathbb{E}_{R_0}[\mathbb{E}_{R_t}[H|\mathcal{F}_t]] = \mathbb{E}_{R_0}[\pi_t(H)] = \pi_0(\pi_t(H)),\end{aligned}$$

which contradicts the time consistency condition (2.3.3). □

In the time consistent case when the conditional probability distributions $R_t[\cdot|\mathcal{F}_t]$ belong to the same local martingale measure $R_0 \in \mathcal{M}_{loc}(W)$, we are in the setting described by Corollary 2.2.11. More precisely, if the pricing measure R_0 belongs to the set $\mathcal{M}_{UI}(W)$, then the asset price does not contain a bubble, or a bubble already exists at time $t = 0$ if $R_0 \in \mathcal{M}_{NUI}(W)$.

As soon as the flow $\mathcal{R}$ is not constant, it describes a shifting system of predictions $(R_t[\cdot|\mathcal{F}_t])_{t\geq 0}$ that is not time consistent. Let us focus now on the time inconsistent case. As pointed in Lemma 2.2.8, the $\mathcal{R}$-bubble reduces to the trivial case at time t if the market's forward looking view given by a measure $\mathcal{R}_t \in \mathcal{M}_{UI}(W)$ and it will be positive when the flow passes through $\mathcal{M}_{NUI}(W)$. Our objective is to allow the flow to move from some initial measure Q in $\mathcal{M}_{UI}(W)$ to some measure R in $\mathcal{M}_{NUI}(W)$ via adapted convex combinations. We consider

$$Q \in \mathcal{M}_{UI}(W) \text{ and } R \in \mathcal{M}_{NUI}(W) \tag{2.3.4}$$

and let $\xi = (\xi_t)_{t\geq 0}$ be an adapted càdlàg process with values in $[0, 1]$ starting in $\xi_0 = 0$. On the space of equivalent local martingale measures $\mathcal{M}_{loc}(W)$ we define the flow $\mathcal{R} = (R_t)_{t\geq 0}$ such that at each time $t \geq 0$, we have

$$R_t[\cdot|\mathcal{F}_t] := \xi_t R[\cdot|\mathcal{F}_t] + (1 - \xi_t)Q[\cdot|\mathcal{F}_t]. \tag{2.3.5}$$

Remark 2.3.3. *A possible economic interpretation of our model is provided in Föllmer et al. [25]: there are two financial "gurus", one optimistic whose views are captured by the measure Q, and one pessimistic whose view is captured by the measure R. The agents are divided into two groups, each group following the predictions indicated by one of the gurus. Based on these predictions, they forecast the future prices of the asset. However, the agents may change their affiliation from one group to another, according to the accuracy of the predictions indicated by each guru. In consequence, the size of the two groups will shift in time, as agents become "chartists" or "trend-chasers". Therefore, at any time t, the temporary price equilibrium is given by some martingale measure R_t, which can be regarded as weighted average of Q and R, with the size of each weight depending on the size of the corresponding group of agents.*

Lemma 2.3.4. *For the flow $\mathcal{R} = (R_t)_{t\geq 0}$ defined by (2.3.5), the $\mathcal{R}$-bubble $\beta^{\mathcal{R}} = S - S^{\mathcal{R}}$ is given by*

$$\beta_t^{\mathcal{R}} = \xi_t(S_t - S_t^R) = \xi_t \beta_t^R, \quad t \geq 0. \tag{2.3.6}$$

The $\mathcal{R}$-bubble starts at $\beta_0^{\mathcal{R}} = 0$, and it dies out in the long run:

$$\lim_{t\to\infty} \beta_t^{\mathcal{R}} = 0 \text{ a.s. and in } L^1(R).$$

Proof. At time $t = 0$, since $\xi_0 = 0$ we have $R_0 = Q \in \mathcal{M}_{UI}(W)$. Therefore the $\mathcal{R}$-bubble starts at the initial value 0, since

$$\beta_0^{\mathcal{R}} = W_0 - W_0^Q = W_0 - \mathbb{E}_Q[D_\infty] = W_0 - \mathbb{E}_Q[W_\infty] = 0.$$

Since

$$\begin{aligned} W_t^{\mathcal{R}} &= \xi_t \mathbb{E}_R[W_\infty|\mathcal{F}_t] + (1-\xi_t)\mathbb{E}_Q[W_\infty|\mathcal{F}_t] \\ &= \xi_t W_t^R + (1-\xi_t)W_t, \end{aligned} \tag{2.3.7}$$

we obtain

$$\begin{aligned} \beta_t^{\mathcal{R}} &= W_t - W_t^{\mathcal{R}} = W_t - \xi_t W_t^R - (1-\xi_t)W_t \\ &= \xi_t(W_t - W_t^R) = \xi_t(S_t - S_t^R) = \xi_t \beta_t^R. \end{aligned} \tag{2.3.8}$$

Hence $\lim_{t\to\infty} \beta_t^{\mathcal{R}} = 0$ *a.s.* and in $L^1(R)$, since β^R converges to 0 by Corollary 2.2.11 and ξ is bounded. □

By allowing the placement of an increasing weight ξ_t on the prediction provided by R in (2.3.5), we are able to describe the initial behavior of the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ as an local R-submartingale starting from the initial value 0.

Proposition 2.3.5. *If the process ξ is increasing then the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ is a local submartingale under R. If ξ remains constant after some stopping time τ_1, then $\beta^{\mathcal{R}}$ is a local martingale under R, and hence an R-supermartingale, after time τ_1.*

Proof. By Corollary 2.2.11, the R-bubble $\beta^R = W - W^R$ is a local martingale under R. Let σ be a localizing stopping time for β^R under R, i.e. the stopped process $(\beta^R)^\sigma_t := \beta^R_{t\wedge\sigma}$ is an R-martingale. Then the stopped process $(\beta^{\mathcal{R}})^\sigma = (\xi\beta^R)^\sigma$ is an R-submartingale since

$$\begin{aligned}(\xi\beta^R)^\sigma_s &= \xi_{s\wedge\sigma}\beta^R_{s\wedge\sigma} = \xi_{s\wedge\sigma}\mathbb{E}_R[\beta^R_{t\wedge\sigma}|\mathcal{F}_s] = \mathbb{E}_R[\xi_{s\wedge\sigma}\beta^R_{t\wedge\sigma}|\mathcal{F}_s]\\ &\le \mathbb{E}_R[\xi_{t\wedge\sigma}\beta^R_{t\wedge\sigma}|\mathcal{F}_s] = \mathbb{E}_R[(\xi\beta^R)^\sigma_t|\mathcal{F}_s]\end{aligned}$$

for $s \le t$. To show that $\beta^{\mathcal{R}}$ is a local R-martingale after time τ_1, we prove that the stopped process $(\beta^{\mathcal{R}})^\sigma$ satisfies

$$\mathbb{E}_R[(\beta^{\mathcal{R}})^\sigma_\tau] = \mathbb{E}_R[(\beta^{\mathcal{R}})^\sigma_{\tau_1}]$$

for any stopping time $\tau \ge \tau_1$. By (2.3.6) we obtain

$$\begin{aligned}\mathbb{E}_R[\beta^{\mathcal{R}}_{\tau\wedge\sigma}] &= \mathbb{E}_R[\xi_{\tau\wedge\sigma}\beta^R_{\tau\wedge\sigma}] = \mathbb{E}_R[\xi_{\tau_1\wedge\sigma}\mathbb{E}_R[\beta^R_{\tau\wedge\sigma}|\mathcal{F}_{\tau_1\wedge\sigma}]]\\ &= \mathbb{E}_R[\xi_{\tau_1\wedge\sigma}\beta^R_{\tau_1\wedge\sigma}] = \mathbb{E}_R[\beta^{\mathcal{R}}_{\tau_1\wedge\sigma}],\end{aligned}$$

since $\xi_{\tau\wedge\sigma} = \xi_{\tau_1\wedge\sigma}$. □

Let us consider now the general case when the process ξ is a special semimartingale under R taking values in $[0, 1]$. Therefore ξ admits the unique decomposition

$$\xi = M^\xi + A^\xi, \tag{2.3.9}$$

where M^ξ represents the local R-martingale part and A^ξ is a predictable process with paths of bounded variation. We want to find conditions under which initial behavior of the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ is described by an R-submartingale starting from 0. As in (2.3.8), the bubble $\beta^{\mathcal{R}}$ is given by

$$\beta^{\mathcal{R}}_t = \xi_t(S_t - S^R_t) = \xi_t\beta^R_t.$$

Remember that β^R is an local R-martingale. By applying the integration by parts formula, we obtain the canonical decomposition of $\beta^{\mathcal{R}}$

$$\begin{aligned}d\beta^{\mathcal{R}}_t &= d(\xi_t\beta^R_t) = \xi_t d\beta^R_t + \beta^R_t d\xi_t + d[\xi, \beta^R]_t\\ &= (\xi_t d\beta^R_t + \beta^R_t dM^\xi_t) + \beta^R_t dA^\xi_t + d[\xi, \beta^R]_t\\ &= (\xi_t d\beta^R_t + \beta^R_t dM^\xi_t) + dA^{\mathcal{R}}_t,\end{aligned} \tag{2.3.10}$$

where we have denoted by $A^{\mathcal{R}}$ the predictable process with paths of bounded variation

$$A_t^{\mathcal{R}} = \int_0^t \beta_s^R dA_s^{\xi} + [\xi, \beta^R]_t, \quad t \geq 0. \tag{2.3.11}$$

The following proposition provides necessary and sufficient conditions under which $\beta^{\mathcal{R}}$ is a local R-submartingale.

Proposition 2.3.6. *The $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ is a local R-submartingale if and only if $A^{\mathcal{R}}$ is an increasing process. If ξ is a submartingale, then the local R-submartingale property for $\beta^{\mathcal{R}}$ holds whenever the process $[\xi, \beta^R]$ is increasing.*

Proof. The first part is a direct consequence of (2.3.10). If ξ is a submartingale then A^{ξ} is an increasing process. Hence also

$$\int_0^t \beta_s^R dA_s^{\xi}, \quad t \geq 0,$$

is increasing, because $\beta^R \geq 0$. Thus $A^{\mathcal{R}}$ increases whenever $[\xi, \beta^R]$ is increasing. □

From now on we specify the form of the flow $\mathcal{R} = (R_t)_{t \geq 0}$ and assume

$$R_t = (1 - \lambda_t)Q + \lambda_t R, \tag{2.3.12}$$

where $(\lambda_t)_{t \geq 0}$ is a deterministic càdlàg process of bounded variation that takes values in $[0, 1]$ with $\lambda_0 = 0$. Let us denote by $M = (M_t)_{t \geq 0}$ the Radon-Nikodym density process of Q with respect to R

$$M_t = \mathbb{E}_R[\frac{dQ}{dR}|\mathcal{F}_t], \quad t \geq 0. \tag{2.3.13}$$

Lemma 2.3.7. *The conditional distributions $R_t[\cdot|\mathcal{F}_t]$ are of the form* (2.3.5) *where the adapted process ξ is given by*

$$\xi_t = \frac{\lambda_t}{\lambda_t + (1 - \lambda_t)M_t}, \quad t \geq 0. \tag{2.3.14}$$

Proof. For any $\mathcal{F}$-measurable $Z \geq 0$ and any $A_t \in \mathcal{F}_t$ we have

$$\begin{aligned}
\mathbb{E}_{R_t}[Z; A_t] &= \mathbb{E}_R\Big[\frac{dR_t}{dR} Z; A_t\Big] = \mathbb{E}_R[(\lambda_t + (1 - \lambda_t)M_\infty)Z; A_t] \\
&= \mathbb{E}_R[\lambda_t \mathbb{E}_R[Z|\mathcal{F}_t]; A_t] + \mathbb{E}_R[(1 - \lambda_t)\mathbb{E}_R[M_\infty Z|\mathcal{F}_t]; A_t] \\
&= \mathbb{E}_R[\lambda_t \mathbb{E}_R[Z|\mathcal{F}_t]; A_t] + \mathbb{E}_R\Big[(1 - \lambda_t)M_t \frac{1}{\mathbb{E}_R[M_\infty|\mathcal{F}_t]} \mathbb{E}_R[M_\infty Z|\mathcal{F}_t]; A_t\Big] \\
&= \mathbb{E}_R[\lambda_t \mathbb{E}_R[Z|\mathcal{F}_t] + (1 - \lambda_t)M_t \mathbb{E}_Q[Z|\mathcal{F}_t]; A_t].
\end{aligned}$$

Since

$$\frac{dR_t}{dR}|_{\mathcal{F}_t} = \lambda_t + (1-\lambda_t)M_t,$$

we have

$$\lambda_t \frac{dR}{dR_t}|_{\mathcal{F}_t} = \frac{\lambda_t}{\lambda_t + (1-\lambda_t)M_t} = \xi_t$$

and

$$\begin{aligned}(1-\lambda_t)M_t \frac{dR}{dR_t}|_{\mathcal{F}_t} &= \frac{(1-\lambda_t)M_t}{\lambda_t + (1-\lambda_t)M_t} \\ &= 1 - \frac{\lambda_t}{\lambda_t + (1-\lambda_t)M_t} = 1 - \xi_t.\end{aligned}$$

Thus we can write

$$\begin{aligned}\mathbb{E}_{R_t}[Z; A_t] &= \mathbb{E}_R[\lambda_t \mathbb{E}_R[Z|\mathcal{F}_t] + (1-\lambda_t)M_t \mathbb{E}_Q[Z|\mathcal{F}_t]; A_t] \\ &= \mathbb{E}_R\Big[\frac{dR_t}{dR}(\xi_t \mathbb{E}_R[Z|\mathcal{F}_t] + (1-\xi_t)\mathbb{E}_Q[Z|\mathcal{F}_t]); A_t\Big] \\ &= \mathbb{E}_{R_t}[\xi_t \mathbb{E}_R[Z|\mathcal{F}_t] + (1-\xi_t)\mathbb{E}_Q[Z|\mathcal{F}_t]; A_t],\end{aligned}$$

and this amounts to the representation (2.3.5) of the conditional distribution $R_t[\cdot|\mathcal{F}_t]$. □

Lemma 2.3.8. *If λ is increasing, then the process $(\xi_t)_{t\geq 0}$ defined in* (2.3.14) *is an R-submartingale with values in* $[0,1]$*, and its Doob-Meyer decomposition* (2.3.9) *is given by*

$$M_t^\xi = -\int_0^t \frac{\lambda_s(1-\lambda_s)}{(\lambda_s + (1-\lambda_s)M_s)^2} dM_s \tag{2.3.15}$$

and

$$A_t^\xi = \int_0^t \frac{M_s}{(\lambda_s + (1-\lambda_s)M_s)^2} d\lambda_s + \int_0^t \frac{\lambda_s(1-\lambda_s)^2}{(\lambda_s + (1-\lambda_s)M_s)^3} d[M,M]_s \tag{2.3.16}$$

Proof. We have that $\xi_t = g(M_t, \lambda_t)$, where the function g on $(0,\infty)\times[0,1]$ defined by

$$g(x,y) = \frac{y}{y + (1-y)x} \tag{2.3.17}$$

is convex in x and increasing in y since

$$g_{xx}(x,y) = \frac{2y(1-y)^2}{(y+(1-y)x)^3} \geq 0,$$

and

$$g_y(x,y) = \frac{x}{(y+(1-y)x)^2} \geq 0.$$

By applying Jensen's inequality, we obtain

$$\begin{aligned}\xi_s &= g(M_s,\lambda_s) = g(\mathbb{E}_R[M_t|\mathcal{F}_s],\lambda_s) \leq \mathbb{E}_R[g(M_t,\lambda_s)|\mathcal{F}_s] \\ &\leq \mathbb{E}_R[g(M_t,\lambda_t)|\mathcal{F}_s] = \mathbb{E}_R[\xi_t|\mathcal{F}_t],\end{aligned}$$

for any $s \leq t$. Hence ξ is an R-submartingale. We apply the Itô's formula to $\xi_t = g(M_t,\lambda_t)$ in order to obtain the Doob-Meyer decomposition (2.3.9) of ξ. We have

$$\begin{aligned}\xi_t &= \xi_0 + \int_0^t g_x(M_s,\lambda_s)dM_s + \int_0^t g_y(M_s,\lambda_s)d\lambda_s + \frac{1}{2}\int_0^t g_{xx}(M_s,\lambda_s)d[M,M]_s \\ &= -\int_0^t \frac{\lambda_s(1-\lambda_s)}{(\lambda_s+(1-\lambda_s)M_s)^2}dM_s + \int_0^t \frac{M_s}{(\lambda_s+(1-\lambda_s)M_s)^2}d\lambda_s \\ &\quad + \int_0^t \frac{\lambda_s(1-\lambda_s)^2}{(\lambda_s+(1-\lambda_s)M_s)^3}d[M,M]_s\end{aligned}$$

Therefore the local martingale part M^ξ is given by

$$M_t^\xi = \int_0^t g_x(M_s,\lambda_s)dM_s, \quad t \geq 0,$$

and the finite variation part A^ξ is given by

$$A_t^\xi = \int_0^t \frac{1}{2}g_{xx}(M_s,\lambda_s)d[M,M]_s + \int_0^t g_y(M_s,\lambda_s)d\lambda_s, \quad t \geq 0$$

and this proves (2.3.15) and (2.3.16). □

Theorem 2.3.9. *Consider a flow $\mathcal{R} = (R_t)_{t\geq 0}$ of the form* (2.3.12)*, where λ is an increasing, right-continuous function on $[0,\infty)$ with values in $[0,1]$ and initial value $\lambda_0 = 0$. Assume that*

$$W^R \text{ is a local submartingale under } Q \tag{2.3.18}$$

or, equivalently, that

$$[W^R, M] \text{ is an increasing process.} \tag{2.3.19}$$

Then the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ is a local submartingale under R with initial value $\beta_0^{\mathcal{R}} = 0$. After time $t_1 = \inf\{t; \lambda_t = 1\}$, $\beta^{\mathcal{R}}$ is a local martingale under R, and hence an R-supermartingale.

Proof. Remember that W^R and M are both martingales under R. An application of Itô's product formula provides us with the following canonical decomposition of the semimartingale $W^R M$ under R

$$d(W^R M) = W^R dM + M dW^R + d[W^R, M]$$

Therefore the quadratic covariation $[W^R, M]$, which represents the predictable process of bounded variation in the canonical decomposition of $W^R M$, is an increasing process if and only if $W^R M$ is a local submartingale under R. It follows from Girsanov's theorem that this is equivalent to W^R being a local submartingale under Q.
Since W is a martingale under Q, this implies that WM is an R-local martingale. Due to the fact that W and M are both continuous R-local martingales, and in particular locally square integrable local martingales, Corollary II.2 in [52] implies that the unique process $[W, M]$ that compensates WM must be equal to zero. However, note that this result holds also is we assume continuity for just one of the processes W and M. Since W is an R-local martingale, the semimartingale decomposition of W under Q is given by

$$W = \Big(W - \frac{1}{M} d[M, Z]\Big) + \frac{1}{M} d[M, Z]. \tag{2.3.20}$$

By (2.3.20) and the fact that W is a Q-martingale, we obtain $[W, M] \equiv 0$. Therefore

$$[\beta^R, M] = [W - W^R, M] = -[W^R, M] \tag{2.3.21}$$

is a decreasing process. Let us compute the quadratic covariation between ξ and β^R. It follows from Lemma 2.3.7 that

$$d[\xi, \beta^R] = d[M^\xi, \beta^R] + d[A^\xi, \beta^R] = g_x(M, \lambda) d[M, \beta^R].$$

Since $g(x, y)$ is decreasing in x, we have $g_x(M, \lambda) \leq 0$. Therefore $[\xi, \beta^R]$ is an increasing process. The local submartingale property of $\beta^{\mathcal{R}}$ under R follows from Proposition 2.3.6. The rest follows as in Proposition 2.3.5 since $\xi_t = 1$ for $t \geq t_1$. □

Suppose that the wealth process W is strictly positive. Then there exists a unique semimartingale L such that W is the solution of the equation

$$W_t = W_0 + \int_0^t W_s dL_s, \quad t \geq 0,$$

or equivalently, W can be written as the Doléans exponential

$$W = \mathcal{E}(L) = \exp(L - \frac{1}{2}[L, L]). \tag{2.3.22}$$

The process L is called the stochastic logarithm and is given by

$$L_t = \int_0^t \frac{1}{W_s} dW_s, \quad t \geq 0.$$

It is easy to see that L is a local martingale under R. Using the representation (2.3.22) of W, we factorize the fundamental wealth process W^R perceived under R as follows

$$\begin{aligned} W_t^R &= \mathbb{E}_R[W_\infty^R | \mathcal{F}_t] = \mathbb{E}_R[W_\infty | \mathcal{F}_t] = W_t \mathbb{E}_R\Big[\frac{W_\infty}{W_t} | \mathcal{F}_t\Big] \\ &= W_t \mathbb{E}_R[\exp(L_\infty - \frac{1}{2}[L, L]_\infty) \exp(-L_t + \frac{1}{2}[L, L]_t) | \mathcal{F}_t]. \end{aligned}$$

Therefore

$$W_t^R = W_t C_t, \quad t \geq 0, \tag{2.3.23}$$

where $C = (C_t)_{t \geq 0}$ is a semimartingale given by

$$C_t := \mathbb{E}_R[\exp\{L_\infty - L_t - \frac{1}{2}([L, L]_\infty - [L, L]_t)\} | \mathcal{F}_t], \quad t \geq 0. \tag{2.3.24}$$

The martingale property of W under Q implies $[W, M] \equiv 0$, and so the factorization 2.3.23 yields:

$$d[W^R, M] = W d[C, M] + C d[W, M] = W d[C, M]. \tag{2.3.25}$$

Since W is strictly positive, the criterion in Theorem 2.3.9 now takes the following form:

Corollary 2.3.10. *The $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ is a local R-submartingale if $[C, M]$ is an increasing process, where C is defined by the factorization $W^R = WC$ in* (2.3.23) *and* (2.3.24)*.*

2.4 The Delbaen-Schachermayer example

In the present section we provide an example of an incomplete financial market model where Assumption (2.2.5) is satisfied and where the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ exhibits a local martingale behavior under R. More precisely, we show that Condition (2.3.19) of Theorem 2.3.9 is satisfied.

Our model is a slight extension of the classical Delbaen-Schachermayer setting, see [20]. Instead of defining the price process along with the measures Q and R in terms of two independent geometric Brownian motions, we consider a more general case where the price process S and the Radon-Nikodym density process of Q with respect to R are defined in terms of two independent continuous martingales.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions and let $X^{(1)}$ and let $X^{(2)}$ be two independent and strictly positive continuous martingales such that $X_0^{(1)} = X_0^{(2)} = 1$ and

$$\lim_{t\uparrow\infty} X_t^{(1)} = \lim_{t\uparrow\infty} X_t^{(2)} = 0, \quad P - a.s.$$

We fix constants $a \in (0,1)$ and $b \in (1,\infty)$ and define the stopping times

$$\tau_1 := \inf\{t > 0; X_t^{(1)} = a\}, \quad \tau_2 := \inf\{t > 0; X_t^{(2)} = b\} \tag{2.4.1}$$

and $\tau := \tau_1 \wedge \tau_2$. Note that $\tau_1 < \infty$ P-a.s. We fix $N \in \mathbb{N}$. An application of Doob's stopping time theorem to the martingale $X^{(2)}$ yields

$$\mathbb{E}_P[X^{(2)}_{\tau_2\wedge N}|\mathcal{F}_t] = X^{(2)}_{t\wedge\tau_2\wedge N}.$$

By passing to the limit with the help of the Lebesgue's dominated convergence theorem we obtain

$$\begin{aligned}\mathbb{E}_P[X^{(2)}_{\tau_2}|\mathcal{F}_t] = \mathbb{E}_P[\lim_{n\to\infty} X^{(2)}_{\tau_2\wedge N}|\mathcal{F}_t] &= \lim_{n\to\infty} \mathbb{E}_P[X^{(2)}_{\tau_2\wedge N}|\mathcal{F}_t]\\ &= \lim_{n\to\infty} X^{(2)}_{t\wedge\tau_2\wedge N} = X^{(2)}_{t\wedge\tau_2}.\end{aligned}$$

Therefore

$$P[\tau_2 < \infty|\mathcal{F}_t] = \frac{1}{b} X^{(2)}_{t\wedge\tau_2}. \tag{2.4.2}$$

Now consider an asset that generates a single payment $X_\tau^{(1)}$ at time τ, and whose price process S is given by $S_t = X_t^1 1_{\{\tau>t\}}$, $t \geq 0$. Therefore the cumulative dividend process D is given by

$$D_t = X_\tau^{(1)} 1_{\{\tau\leq t\}}, \quad t \geq 0,$$

and the corresponding wealth process W is given by the process $X^{(1)}$ stopped at τ:

$$W_t = S_t + D_t = X^{(1)}_{\tau\wedge t}, \quad t \geq 0.$$

Hence W is a P-martingale lower bounded by a since

$$\begin{aligned} W_t = X^{(1)}_{t\wedge\tau} &= X^{(1)}_{\tau_1} 1_{\{\tau_1 \leq \tau_2\wedge t\}} + X^{(1)}_{\{\tau_2\wedge t\}} 1_{\{\tau_2\wedge t < \tau_1\}} \\ &> a 1_{\{\tau_1\leq\tau_2\wedge t\}} + a 1_{\{\tau_2\wedge t<\tau_1\}} = a. \end{aligned}$$

However, W it is not uniformly integrable, as shown in [20]. More precisely:

Lemma 2.4.1. *We have*

$$\mathbb{E}_P[W_\infty|\mathcal{F}_t] = a(1 - \frac{1}{b}X^{(2)}_{t\wedge\tau}) + \frac{1}{b}X^{(1)}_{t\wedge\tau}X^{(2)}_{t\wedge\tau}, \tag{2.4.3}$$

and this is strictly smaller than $W_t = X^{(1)}_t$ on the set $\{\tau > t\}$.

Proof. Equation (2.4.3) is satisfied on the set $\{\tau \leq t\}$, since

$$\begin{aligned} a(1 - \frac{1}{b}X^{(2)}_{t\wedge\tau}) + \frac{1}{b}X^{(1)}_{t\wedge\tau}X^{(2)}_{t\wedge\tau} &= a(1 - \frac{1}{b}X^{(2)}_{\tau}) + \frac{1}{b}X^{(1)}_{\tau}X^{(2)}_{\tau} \\ &= a(1 - \frac{1}{b}X^{(2)}_{\tau_1}1_{\{\tau_1<\tau_2\}} - \frac{1}{b}X^{(2)}_{\tau_2}1_{\{\tau_2\leq\tau_1\}}) \\ &\quad + \frac{1}{b}X^{(1)}_{\tau_1}X^{(2)}_{\tau_1}1_{\{\tau_1<\tau_2\}} + \frac{1}{b}X^{(1)}_{\tau_2}X^{(2)}_{\tau_2}1_{\{\tau_2\leq\tau_1\}} \\ &= a - \frac{a}{b}X^{(2)}_{\tau_1}1_{\{\tau_1<\tau_2\}} - \frac{a}{b}b1_{\{\tau_2\leq\tau_1\}} \\ &\quad + \frac{a}{b}X^{(2)}_{\tau_1}1_{\{\tau_1<\tau_2\}} + X^{(1)}_{\tau_2}1_{\{\tau_2\leq\tau_1\}} \\ &= a - a1_{\{\tau_2\leq\tau_1\}} + X^{(1)}_{\tau_2}1_{\{\tau_2\leq\tau_1\}} \\ &= X^{(1)}_{\tau_1}1_{\{\tau_2>\tau_1\}} + X^{(1)}_{\tau_2}1_{\{\tau_2\leq\tau_1\}} \\ &= X^{(1)}_{\tau} \end{aligned}$$

and W_∞ coincide with $X^{(1)}_\tau$. On the set $\{\tau > t\}$ we have

$$\begin{aligned} \mathbb{E}_P[W_\infty|\mathcal{F}_t] &= \mathbb{E}_P[X^{(1)}_\tau|\mathcal{F}_t] \\ &= \mathbb{E}_P[X^{(1)}_{\tau_1}1_{\{\tau_2=\infty\}}|\mathcal{F}_t] + \mathbb{E}_P[X^{(1)}_{\tau}1_{\{\tau_2<\infty\}}|\mathcal{F}_t] \\ &= aP[\tau_2 = \infty|\mathcal{F}_t] + \mathbb{E}_P[\mathbb{E}_P[X^{(1)}_{\tau_1\wedge\tau_2}|\mathcal{F}_t \vee \sigma(\tau_2)]1_{\{\tau_2<\infty\}}|\mathcal{F}_t]. \end{aligned} \tag{2.4.4}$$

Since τ_2 is independent of $X^{(1)}$, the last term reduces to

$$X^{(1)}_t P[\tau_2 < \infty|\mathcal{F}_t],$$

and by (2.4.2) we obtain (2.4.3). It follows from definition (2.4.1) of τ_1 and τ_2 that $X_t^{(1)} > a$ and $X_t^{(2)} < b$ on on $\{\tau > t\}$. Therefore $\mathbb{E}_R[W_\infty|\mathcal{F}_t] < W_t = X_t^{(1)}$ on $\{\tau > t\}$. □

Consider the bounded martingale M defined by

$$M_t := X^{(2)}_{t\wedge\tau}, \quad t \geq 0,$$

and denote by $Q \approx P$ the probability measure with the Radon-Nikodym density process

$$\frac{dQ}{dP} = M_\infty = X_\tau^{(2)} > 0.$$

We now show that W is a uniformly integrable martingale under Q. It follows from Corollary II.2 and Exercise III.21 in Protter [52] that W is a Q-local martingale since $[W, M] \equiv 0$. Furthermore $\mathbb{E}_P[X_\tau^{(1)}|\tau_2] = 1$ on $\{\tau_2 < \infty\}$ and $X_\tau^{(2)} = \mathbb{E}_P[X_{\tau_2}^{(2)} 1_{\{\tau_2<\infty\}}|\mathcal{F}_\tau]$, hence

$$\begin{aligned}
\mathbb{E}_Q[W_\infty] &= \mathbb{E}_P[X_\tau^{(1)} X_\tau^{(2)}] = \mathbb{E}_P[X_\tau^{(1)} X_{\tau_2}^{(2)} 1_{\{\tau_2<\infty\}}] \\
&= b\mathbb{E}_P[\mathbb{E}_P[X_\tau^{(1)}|\tau_2] 1_{\{\tau_2<\infty\}}] = bP(\tau_2 < \infty) \\
&= b\frac{1}{b}\mathbb{E}_P[X^{(2)}_{t\wedge\tau_2}] = 1 = W_0,
\end{aligned} \tag{2.4.5}$$

and this implies uniform integrability of W under Q.

Set $R := P$. Then

$$R \in \mathcal{M}_{NUI}(W) \text{ and } Q \in \mathcal{M}_{UI}(W).$$

As in Section 3 we now consider a flow $\mathcal{R} = (R_t)_{t\geq 0}$ of the form (2.3.12). By (2.4.3), the fundamental wealth process W^R perceived under R is given by

$$W_t^R = \mathbb{E}_R[W_\infty|\mathcal{F}_t] = a(1 - \frac{1}{b}M_t) + \frac{1}{b}W_t M_t, \quad t \geq 0. \tag{2.4.6}$$

Condition (2.3.19) of Theorem 2.3.9 is satisfied in our present case as shown below.

Proposition 2.4.2. *W^R is a local submartingale under Q.*

Proof. Since $[W, M] = 0$, we obtain

$$d[W^R, M] = \frac{1}{b} d[(W - a)M, M] = \frac{1}{b}(W - a)d[M, M].$$

This implies that $[W^R, M]$ is an increasing process, i.e. W^R is a local submartingale under Q. □

We now consider the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$. By (2.4.6), the R-bubble takes the form

$$\beta^R = W - W^R = (W - a)(1 - \frac{1}{b}M), \tag{2.4.7}$$

and so the $\mathcal{R}$-bubble is given by

$$\beta^{\mathcal{R}} = \xi\beta^R = \xi(W - W^R) = \xi(W - a)(1 - \frac{1}{b}M).$$

In particular the $\mathcal{R}$-bubble vanishes at time τ, that is, $\beta_t^{\mathcal{R}} = 0$ for $t \geq \tau$. Since condition (2.3.18) is satisfied, the $\mathcal{R}$-bubble starts from its initial value 0 as a R-submartingale, which then turns into a supermartingale before it finally returns to 0. More precisely:

Corollary 2.4.3. *The behavior of the $\mathcal{R}$-bubble under the measure R is described by Theorem 2.3.9.*

2.5 The behavior of the $\mathcal{R}$-bubble under Q

We consider the setting of Section 2.3, where the flow $\mathcal{R}$ consists in moving from a measure $Q \in \mathcal{M}_{UI}(W)$ to a measure $R \in \mathcal{M}_{NUI}(W)$ such that, at any time $t > 0$, the market's forward-looking view is given by the conditional distribution

$$R_t[\cdot|\mathcal{F}_t] = \xi_t R[\cdot|\mathcal{F}_t] + (1 - \xi_t)Q[\cdot|\mathcal{F}_t],$$

where $\xi = (\xi_t)_{t\geq 0}$ is an adapted, càdlàg process with values in $[0, 1]$ starting from $\xi_0 = 0$. It follows from Lemma 2.3.4 that the $\mathcal{R}$-bubble is of the form

$$\beta^{\mathcal{R}} = W - W^{\mathcal{R}} = \xi\beta^R.$$

The aim of this section is to examine the $\mathcal{R}$-bubble under the measure Q. We first examine the R-bubble $\beta^R = W - W^R = S - S^R$. We consider that Condition (2.3.18) is satisfied i.e. W^R is a local submartingale under Q. Therefore W^R admits the following Doob-Meyer decomposition

$$W^R = M^Q + A^Q, \tag{2.5.1}$$

where M^Q is a Q-local martingale and A^Q is an increasing continuous process of bounded variation.

Proposition 2.5.1. *Under condition* (2.3.18) *the R-bubble β^R is a uniformly integrable supermartingale under Q. More precisely, β^R is the Q-potential generated by the increasing process A^Q, that is,*

$$\beta_t^R = \mathbb{E}_R[A_\infty^Q - A_t^Q|\mathcal{F}_t], \quad t \geq 0. \tag{2.5.2}$$

Proof. Since W is uniformly integrable under Q such that $W > M^Q$ and $W > \beta^R$, the R-bubble

$$\beta^R = W - W^R = (W - M^Q) - A^Q$$

is a uniformly integrable Q-supermartingale. Since

$$\mathbb{E}_Q[M^Q_\infty|\mathcal{F}_t] = \mathbb{E}_Q[W_\infty - A_\infty|\mathcal{F}_t] = W_t - \mathbb{E}_Q[A^R_\infty|\mathcal{F}_t], \tag{2.5.3}$$

we obtain (2.5.2). $\square$

We denote by $\tilde{M} = (\tilde{M}_t)_{t\geq 0}$ the Radon-Nikodym density process of R with respect to Q i.e.

$$\tilde{M}_t := \frac{dR}{dQ}|_{\mathcal{F}_t} = \frac{1}{M_t}, \quad t \geq 0,$$

where M is defined in (2.3.13). We know that $\tilde{M}$ is a Q-martingale. Moreover, the $\mathcal{R}$-bubble can be written under the form

$$\beta^{\mathcal{R}} = \tilde{\xi}\tilde{\beta}^R,$$

where $\tilde{\xi} := \xi M$ and $\tilde{\beta}^R := \beta^R \tilde{M}$.

Lemma 2.5.2. *The process $\tilde{\beta}^R = \beta^R\tilde{M}$ is a local martingale under Q. Under condition* (2.3.18)*, the processes $[\tilde{\beta}^R, \tilde{M}]$ and $[\beta^R, \tilde{M}]$ are both increasing.*

Proof. Since β^R is a local R-martingale, it follows from Exercise III.21 in Protter [52] that $\tilde{\beta}^R$ is a local martingale under Q. Under condition (2.3.18) the process $[\beta^R, M]$ is decreasing, see (2.3.21). Applying Itô's formula together with the integration by parts formula to $\tilde{\beta}^R = \beta^R\tilde{M}$ and $\tilde{M} = M^{-1}$ we have

$$\begin{aligned} d[\tilde{\beta}^R, \tilde{M}] &= d[\beta^R\tilde{M}, \tilde{M}] = \beta^R d[\tilde{M}, \tilde{M}] + \tilde{M} d[\beta^R, \tilde{M}] \\ &= -\frac{1}{M^3} d[\beta^R, M] + \frac{1}{M^4}\beta^R d[M, M] \end{aligned}$$

and so $[\tilde{\beta}^R, \tilde{M}]$ is increasing. Moreover an application of Itô's formula yields

$$d[\beta^R, \tilde{M}] = -\frac{1}{M^2} d[\beta^R, M].$$

Hence also $[\beta^R, \tilde{M}]$ is increasing. $\square$

Let $(\lambda_t)_{t\geq 0}$ be an increasing càdlàg function that takes values in $[0,1]$ and starts in $\lambda_0 = 0$. In the following, we focus on the special case where the flow $\mathcal{R} = (R_t)_{t\geq 0}$ is of the form (2.3.12), i.e.

$$R_t = (1-\lambda_t)Q + \lambda_t R,$$

In particular, the process ξ is now given by

$$\xi_t = \frac{\lambda_t}{\lambda_t + (1-\lambda_t)M_t}, \quad t \geq 0, \tag{2.5.4}$$

as shown in Lemma 2.3.7.

Proposition 2.5.3. *The process $\tilde{\xi} = \xi M$ is a submartingale under Q. More precisely, the Doob-Meyer decomposition of $\tilde{\xi}$ under Q is given by*

$$\tilde{\xi}_t = \tilde{M}^\xi + \tilde{A}^\xi \tag{2.5.5}$$

with

$$d\tilde{M}^\xi = -\frac{\lambda^2}{(\lambda\tilde{M} + (1-\lambda))^2} d\tilde{M}$$

and

$$d\tilde{A}^\xi = \frac{1}{(\lambda\tilde{M} + (1-\lambda))^2} d\lambda + \frac{\lambda^3}{(\lambda\tilde{M} + (1-\lambda))^3} d[\tilde{M},\tilde{M}]. \tag{2.5.6}$$

Proof. Note that

$$\tilde{\xi}_t = \tilde{g}(\tilde{M}_t, \lambda_t),$$

where the function $\tilde{g}(x,y)$ is defined by

$$\tilde{g}(x,y) := \frac{y}{xy + (1-y)}$$

and has the following partial derivatives

$$\tilde{g}_x(x,y) = -\frac{y^2}{(xy+(1-y))^2}, \quad \tilde{g}_y(x,y) = \frac{1}{(xy+(1-y))^2} \tag{2.5.7}$$

and

$$\tilde{g}_{xx}(x,y) = \frac{2y^3}{(xy+(1-y))^3}. \tag{2.5.8}$$

Therefore $\tilde{g}(x,y)$ is convex in $x \in (0,\infty)$ and increasing in $y \in [0,1]$.

As in the proof of Lemma 2.3.8, it follows that $\tilde{\xi}$ is a Q-submartingale. An application of Itô's formula yields the Doob-Meyer decomposition of $\tilde{\xi}$

$$\begin{aligned}\tilde{\xi}_t = \tilde{g}(\tilde{M}_t, \lambda_t) &= \int_0^t \tilde{g}_y(\tilde{M}_s, \lambda_s) d\lambda_s + \int_0^t g_x(\tilde{M}_s, \lambda_s) d\tilde{M}_s \\ &+ \frac{1}{2} \int_0^t g_{xx}(\tilde{M}_s, \lambda_s) d[\tilde{M}, \tilde{M}]_s \\ &= \int_0^t \frac{1}{(\lambda_s \tilde{M}_s + (1-\lambda_s))^2} d\lambda_s - \int_0^t \frac{\lambda_s^2}{(\lambda_s \tilde{M}_s + (1-\lambda_s))^2} d\tilde{M}_s \\ &+ \int_0^t \frac{\lambda_s^3}{(\lambda_s \tilde{M}_s + (1-\lambda_s))^3} d[\tilde{M}, \tilde{M}]_s.\end{aligned}$$

□

We investigate the behavior of the $\mathcal{R}$-bubble $\beta^{\mathcal{R}} = \xi \beta^R = \tilde{\xi} \tilde{\beta}^R$ under the measure Q.

Proposition 2.5.4. *Under Q the $\mathcal{R}$-bubble has the canonical decomposition*

$$\beta^{\mathcal{R}} = \tilde{M}^{\mathcal{R}} + \tilde{A}^{\mathcal{R}},$$

where the local martingale $\tilde{M}^{\mathcal{R}}$ is given by

$$d\tilde{M}^{\mathcal{R}} = \tilde{\xi} d\tilde{\beta}^R + \tilde{\beta}^R dM^{\tilde{\xi}}.$$

The process $\tilde{A}^{\mathcal{R}}$ takes the form

$$d\tilde{A}^{\mathcal{R}} = \frac{\tilde{M}}{\lambda \tilde{M} + (1-\lambda)} (\beta^R d\lambda - dD), \tag{2.5.9}$$

where D denotes the increasing process given by

$$dD = \frac{\lambda^2 (1-\lambda) \beta^R}{\tilde{M}(\lambda \tilde{M} + (1-\lambda))} d[\tilde{M}, \tilde{M}] + \lambda^2 d[\beta^R, \tilde{M}].$$

Proof. Applying integration by parts to $\beta^{\mathcal{R}} = \tilde{\xi} \tilde{\beta}^R$ and using the Doob-Meyer decomposition (2.5.5) of $\tilde{\xi}$, we obtain

$$\begin{aligned}d\beta^{\mathcal{R}} &= \tilde{\xi} d\tilde{\beta}^R + \tilde{\beta}^R d\tilde{\xi} + d[\tilde{\beta}^R, \tilde{\xi}] \\ &= (\tilde{\xi} d\tilde{\beta}^R + \tilde{\beta}^R d\tilde{M}^{\xi}) + (\tilde{\beta}^R d\tilde{A}^{\xi} + d[\tilde{\beta}^R, \tilde{\xi}]) \\ &=: d\tilde{M}^{\mathcal{R}} + d\tilde{A}^{\mathcal{R}},\end{aligned}$$

where we have denoted

$$d\tilde{M}^{\mathcal{R}} = \tilde{\xi} d\tilde{\beta}^R + \tilde{\beta}^R d\tilde{M}^{\xi},$$

and

$$d\tilde{A}^{\mathcal{R}} = \tilde{\beta}^R d\tilde{A}^{\xi} + d[\tilde{\beta}^R, \tilde{\xi}].$$

Since $\tilde{\beta}^R$ is a Q-local martingale by Lemma 2.5.2 and $\tilde{M}^{\xi}$ is the Q-local martingale part of $\tilde{\xi}$, it follows that $\tilde{M}^{\mathcal{R}}$ is a local martingale under Q. The finite-variation part is given by $\tilde{A}^{\mathcal{R}}$ and the decomposition is unique since $\beta^{\mathcal{R}}$ is a special semimartingale due its continuity. Since $\tilde{\xi} = \tilde{g}(\tilde{M}, \lambda)$ and $\tilde{\beta}^{\mathcal{R}} = \beta^R \tilde{M}$, we obtain

$$\begin{aligned} d[\tilde{\beta}^R, \tilde{\xi}] &= d[\tilde{g}(\tilde{M}, \lambda), \tilde{\beta}^R] = \tilde{g}_x(\tilde{M}, \lambda) d[\tilde{\beta}^R, \tilde{M}] \\ &= \tilde{g}_x(\tilde{M}, \lambda) d[\beta^R \tilde{M}, \tilde{M}] \\ &= \tilde{g}_x(\tilde{M}, \lambda)(\beta^R d[\tilde{M}, \tilde{M}] + \tilde{M}[\beta^R, \tilde{M}]). \end{aligned}$$

By (2.5.7) and (2.5.6), we obtain

$$\begin{aligned} d\tilde{A}^{\mathcal{R}} &= \frac{\beta^R \tilde{M}}{(\lambda \tilde{M} + (1-\lambda))^2} d\lambda + \frac{\beta^R \tilde{M} \lambda^3}{(\lambda \tilde{M} + (1-\lambda))^3} d[\tilde{M}, \tilde{M}] \\ &\quad - \frac{\beta^R \lambda^2}{(\lambda \tilde{M} + (1-\lambda))^2} d[\tilde{M}, \tilde{M}] - \frac{\lambda^2 \tilde{M}}{(\lambda \tilde{M} + (1-\lambda))^2} d[\beta^R, \tilde{M}] \\ &= \frac{\beta^R \tilde{M}}{(\lambda \tilde{M} + (1-\lambda))^2} d\lambda - \frac{\beta^R \lambda^2 (1-\lambda)}{(\lambda \tilde{M} + (1-\lambda))^3} d[\tilde{M}, \tilde{M}] \\ &\quad - \frac{\lambda^2 \tilde{M}}{(\lambda \tilde{M} + (1-\lambda))^2} d[\beta^R, \tilde{M}] \\ &= \frac{\tilde{M}}{(\lambda \tilde{M} + (1-\lambda))^2} (\beta^R d\lambda - dD). \end{aligned}$$

It follows from Lemma 2.5.2 that the process D is increasing. □

Therefore the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ can exhibit a supermartingale behavior under Q in periods when the process λ stays constant and $\beta^{\mathcal{R}}$ is a strict Q-submartingale, if the increase in λ is strong enough to compensate for the increase in D, as it may happen in the build-up phase of the bubble.

Definition 2.5.5. *We say that the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ behaves locally as a strict Q-submartingale in a given random period if $\tilde{A}^{\mathcal{R}}$ is strictly increasing in that period.*

We conclude this section by examining in the setting of Section 2.4 the qualitative behavior of the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ from the measure Q-perspective. According to (2.4.7), the R-bubble now takes the form

$$\beta^R = (W - a)(1 - \frac{1}{b}M). \tag{2.5.10}$$

We have $d[W, \tilde{M}] = -M^{-2}d[W, M] = 0$ and $d[M, \tilde{M}] = -\tilde{M}^{-2}d[\tilde{M}, \tilde{M}]$. Therefore the increasing process $[\beta^R, \tilde{M}]$ is given by

$$\begin{aligned} d[\beta^R, \tilde{M}] &= d[(W - a)(1 - \frac{1}{b}M), \tilde{M}] \\ &= -\frac{1}{b}(W - a)d[M, \tilde{M}] + (1 - \frac{1}{b}M)d[W, \tilde{M}] \\ &= \frac{1}{b}(W - a)\tilde{M}^{-2}d[\tilde{M}, \tilde{M}]. \end{aligned} \tag{2.5.11}$$

Let

$$\phi := \frac{d\lambda}{d[\tilde{M}, \tilde{M}]}$$

be the density of the absolute continuous part of λ with respect to $[\tilde{M}, \tilde{M}]$.

Corollary 2.5.6. *The $\mathcal{R}$-bubble behaves locally as a strict Q-submartingale in periods where*

$$\phi_t > \lambda_t^2(1 - \lambda_t(1 - \frac{1}{b}))(\tilde{M}_t - \frac{1}{b})^{-1}(\lambda_t\tilde{M}_t + (1 - \lambda_t))^{-1}. \tag{2.5.12}$$

Proof. In this setting, by using (2.5.9), (2.5.10) and (2.5.11), the finite variation part $\tilde{A}^{\mathcal{R}}$ of $\beta^{\mathcal{R}}$ is equal to

$$\begin{aligned}
d\tilde{A}^{\mathcal{R}}_t &= \frac{\tilde{M}_t}{\lambda_t\tilde{M}_t + (1-\lambda_t)}(\beta^{\mathcal{R}}_t d\lambda_t - dD_t) \\
&= \frac{\tilde{M}_t}{\lambda_t\tilde{M}_t + (1-\lambda_t)}\beta^R_t d\lambda_t - \frac{(1-\lambda_t)\lambda_t^2}{(\lambda_t\tilde{M}_t + (1-\lambda_t))^2}\beta^R_t d[\tilde{M},\tilde{M}]_t \\
&\quad - \frac{\lambda^2}{\lambda\tilde{M} + (1-\lambda)} d[\beta^R,\tilde{M}]_t \\
&= \frac{\tilde{M}_t\beta^R_t}{\lambda_t\tilde{M}_t + (1-\lambda_t)}\phi_t d[\tilde{M},\tilde{M}]_t - \frac{\lambda_t^2(1-\lambda_t)}{(\lambda_t\tilde{M}_t + (1-\lambda_t))^2}\beta^R_t d[\tilde{M},\tilde{M}]_t \\
&\quad - \frac{\lambda_t^2}{b(\lambda_t\tilde{M}_t + (1-\lambda_t))}(W_t - a)\tilde{M}_t^{-2} d[\tilde{M},\tilde{M}]_t \\
&= \frac{\tilde{M}_t}{\lambda_t\tilde{M}_t + (1-\lambda_t)}(W_t - a)(1 - \frac{1}{b}M_t)\phi_t d[\tilde{M},\tilde{M}] \\
&\quad - \frac{\lambda_t^2(1-\lambda_t)}{(\lambda_t\tilde{M}_t + (1-\lambda_t))^2}(W_t - a)(1 - \frac{1}{b}M_t) d[\tilde{M},\tilde{M}]_t \\
&\quad - \frac{\lambda_t^2}{b(\lambda_t\tilde{M}_t + (1-\lambda_t))}(W_t - a)\tilde{M}_t^{-2} d[\tilde{M},\tilde{M}]_t.
\end{aligned}$$

Hence, the condition $d\tilde{A}^{\mathcal{R}} > 0$ is equivalent to

$$\frac{\tilde{M}_t}{\lambda_t\tilde{M}_t + (1-\lambda_t)}(1-\frac{1}{b}M_t)\phi_t \geq \frac{\lambda_t^2(1-\lambda_t)}{(\lambda_t\tilde{M}_t + (1-\lambda_t))^2}(1-\frac{1}{b}M_t) + \frac{\lambda_t^2}{b(\lambda_t\tilde{M}_t + (1-\lambda_t))}\tilde{M}_t^{-2}$$

Multiplying by $(\lambda_t\tilde{M}_t + (1-\lambda_t))^2$ we obtain

$$(\tilde{M}_t - \frac{1}{b})(\lambda_t\tilde{M}_t + (1-\lambda_t))\phi_t \geq \lambda_t^2(1 - \lambda_t(1 - \frac{1}{b})).$$

□

We now focus on the special case where the martingale $X^{(2)}$ in Section 2.4 is of the form $dX^{(2)} = X^{(2)}dB$ for some Brownian motion B. Then the quadratic covariation of $\tilde{M}$ is equal to $d[\tilde{M},\tilde{M}] = \tilde{M}^2 dt$ up to the stopping time τ introduced in Section 2.4.

Let λ be continuous and piecewise differentiable with right-continuous derivative λ'. Then the density ϕ is given by $\phi = \tilde{M}^{-2}\lambda'$. We define the functions

$$f(x,t) := (1 - \frac{1}{b}x)(\lambda(t) + (1-\lambda(t))x)\lambda'(t)$$

and

$$h(t) := \lambda^2(t)(1 - \lambda(t))(1 - \frac{1}{b}).$$

The following Corollary provides a characterization of the behavior of the $\mathcal{R}$-bubble under Q.

Corollary 2.5.7. *Up to time τ, the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ behaves locally as a strict Q-submartingale as long as the process (M_t, t) stays in the domain*

$$D_+ := \{(x,t); f(x,t) > h(t)\},$$

and as a strict supermartingale under Q as long as it stays in

$$D_- := \{(x,t); f(x,t) < h(t)\}.$$

In particular, if $\lambda'(0) > 0$ then $\beta^{\mathcal{R}}$ behaves as a strict Q-submartingale up to the exit time

$$\sigma := \inf\{t > 0; (M_t, t) \notin D_+\} > 0$$

from D_+.

Proof. In this setting, (2.5.12) is equivalent to the condition $f(M_t, t) > h(t)$, and the condition $f(M_t, t) < h(t)$ is equivalent to $d\tilde{A}^{\mathcal{R}} < 0$. Note that $\lambda'(0) > 0$ implies $(1, 0) \in D_+$, hence $(M_t, t) \in D_+$ for small enough t, and this implies that the exit time from D_+ is strictly positive. □

Chapter 3

Stochastic volatility models

The contents of Section 3.1 of this Chapter are based on Section 5 of Biagini, Föllmer and **S. Nedelcu** [4], which was developed independently by the author. Also Section 3.2 of the present Chapter was developed independently by the author and is based on a manuscript which is not yet published.

In the present chapter, we provide two examples within the framework of stochastic volatility models, where we can compute explicitly the processes W^R and $\beta^{\mathcal{R}}$ and verify our condition on the submartingale behavior of W^R under Q. In Section 3.1 we present a version of the stochastic volatility model introduced by Sin [57]. We also show that the model can be modified in such a way the condition no longer holds. In Section 3.2 we show that $\beta^{\mathcal{R}}$ follows a local R-submartingale in a modification of the Andersen-Piterbarg stochastic volatility model. The Andersen-Piterbarg model represents a generalisation of the model of Sin [57] by allowing correlation between the Brownian motions driving the asset price process and the volatility, respectively.

3.1 Stochastic model with independent Brownian motions

Let $B = (B^1, B^2, B^3)$ be a 3-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We consider the stochastic volatility model

$$\begin{aligned} dX_t &= \sigma_1 v_t X_t dB_t^1 + \sigma_2 v_t X_t dB_t^2, \;\; X_0 = x, \\ dv_t &= a_1 v_t dB_t^1 + a_2 v_t dB_t^2 + a_3 v_t dB_t^3, \;\; v_0 = 1, \end{aligned} \tag{3.1.1}$$

where the vectors $a = (a_1, a_2)$ and $\sigma = (\sigma_1, \sigma_2)$ are not parallel and satisfy $(a \cdot \sigma) > 0$, and that $a_3 \in \{0, 1\}$.

The model (3.1.1) is obtained by doing two important modifications to the model studied by C.A.Sin [57]. Firstly, we drop the drift term that existed in the equation of v under P. This modification is made in order to compute the fundamental value W^R in Proposition 3.1.2. Secondly, we extend the model, by allowing it to be driven by a 3-dimensional Brownian motion instead of a 2-dimensional Brownian motion. This extension allows us to construct a counterexample to Condition (2.3.18).

The following theorem provides the corresponding variant of Theorem 3.9 in [57].

Theorem 3.1.1. *There exists a unique solution* (X, v) *of* (3.1.1).

For any $T > 0$*, the process* $(X_t)_{t\in[0,T]}$ *is a strict local martingale under* P*. Moreover, there exists an equivalent martingale measure* Q *for* X *such that the densities*

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = M_t, \quad 0 \leq t \leq T,$$

are given by

$$M_t = \mathcal{E}\left(-\int_0^\cdot \frac{v_s(a\cdot\sigma)}{a\cdot\sigma^\perp}\sigma_1^\perp dB_s^1 - \int_0^\cdot \frac{v_s(a\cdot\sigma)}{a\cdot\sigma^\perp}\sigma_2^\perp dB_s^2 + |\alpha|^2 B_t^3\right)_t, \quad (3.1.2)$$

where $\mathcal{E}(Z) = \exp(Z - \frac{1}{2}[Z,Z])$ *denotes the stochastic exponential of a continuous semimartingale* Z*, the vector* $\sigma^\perp = (\sigma_1^\perp, \sigma_2^\perp) \neq 0$ *satisfies*

$$\sigma\cdot\sigma^\perp = \sigma_1\sigma_1^\perp + \sigma_2\sigma_2^\perp = 0,$$

and where we put $|\alpha| = \sqrt{a_1^2 + a_2^2 + a_3^2}$*. More precisely, the process* $(X_t)_{t\in[0,T]}$ *is a martingale under* Q *satisfying*

$$dX_t = \sigma_1 v_t X_t dB_t^{Q,1} + \sigma_2 v_t X_t dB_t^{Q,2}, \; X_0 = x,$$
$$dv_t = a_1 v_t dB_t^{Q,1} + a_2 v_t dB_t^{Q,2} + a_3 v_t dB_t^{Q,3} - (a\cdot\sigma)v_t^2 dt + a_3|\alpha|^2 v_t dt, \; v_0 = 1,$$

where $B^Q = (B^{Q,1}, B^{Q,2}, B^{Q,3})$ *is a 3-dimensional Brownian motion under* Q*.*

Proof. We proceed as in the proof of Theorem 3.3 in [57]. We start by showing that there exists a unique solution (X, v) of equation (3.1.1). We define the process $W = (W_t)_{t\geq 0}$ by

$$W_t = |\alpha|^{-1}(\alpha_1 B_t^1 + \alpha_2 B_t^2 + \alpha_3 B_t^3), \quad t \geq 0. \quad (3.1.3)$$

Then W is continuous local martingale. Moreover

$$\begin{aligned} d[W,W]_t &= |\alpha|^{-2}(\alpha_1^2 d[B^1,B^1]_t + \alpha_2^2 d[B^2,B^2]_t + \alpha_t^3 d[B^3,B^3]_t) \\ &= |\alpha|^{-2}(\alpha_1^2+\alpha_2^2+\alpha_3^2)dt = dt. \end{aligned}$$

Therefore it follows from Levy's Theorem, see Theorem II.39 in [52], that W is a Brownian motion under P. The process v satisfies the 1-dimensional stochastic differential equation

$$dv_t = |\alpha| v_t dW_t, \quad 0 \le t \le T. \tag{3.1.4}$$

Due to the linearity of the coefficients, equation (3.1.4) admits a unique solution $v = \mathcal{E}(|\alpha| W)$. Therefore X is uniquely determined as the Doléans exponential of the square integrable process

$$\int_0^t \sigma_1 v_s dB_s^1 + \int_0^t \sigma_2 v_s dB_s^2.$$

We prove that $(X_t)_{t\in[0,T]}$ is a strict local martingale under P. To this purpose, it is sufficient to prove that $\mathbb{E}_P[X_T] < X_0$. It follows from Lemma 4.2 of [57] that the expectation of the local martingale X under P is equal to

$$\mathbb{E}_P[X_T] = X_0 P(\{w_t \text{ does not explode on } [0,T]\}),$$

where the auxiliary process $(w_t)_{t\in[0,T]}$ is given by

$$dw_t = a_1 w_t dB_t^1 + a_2 w_t dB_t^2 + a_3 w_t dB_t^3 + (a \cdot \sigma) w_t^2 dt, \quad w_0 = 1.$$

Moreover, w is the solution of the 1-dimensional stochastic differential equation

$$dw_t = |\alpha| w_t dW_t + (a \cdot \sigma) w_t^2 dt, \quad t \in [0,T], \tag{3.1.5}$$

where W is a P-Brownian motion defined in (3.1.3). It follows from Lemma 4.3 of [57] that the unique solution of equation (3.1.5) explodes to $+\infty$ in finite time with positive probability. This implies that $\mathbb{E}_P[X_T] < X_0$.
Let us show that the process $M = (M_t)_{t\in[0,T]}$ is a well defined Radon-Nikodym density process, i.e. is a true martingale under the measure P. It follows from Lemma 4.2. of [57] that the expectation under P of M_T can be written as

$$\mathbb{E}_P[M_T] = M_0 P(\{\hat{v}_t \text{ does not explode on } [0,T]\}) \tag{3.1.6}$$

where $\hat{v} = (\hat{v}_t)_{t\in[0,T]}$ satisfies

$$d\hat{v}_t = a_1 \hat{v}_t dB_t^1 + a_2 \hat{v}_t dB_t^2 + a_3 \hat{v}_t dB_t^3 - (a \cdot \sigma)(\hat{v}_t)^2 dt + a_3 |\alpha|^2 \hat{v}_t dt,$$

for all $t \in [0,T]$. The process $\hat{v}$ solves the 1-dimensional stochastic differential equation

$$d\hat{v}_t = |\alpha| \hat{v}_t dW_t - (a \cdot \sigma)(\hat{v}_t)^2 dt + a_3 |\alpha|^2 \hat{v}_t dt \quad t \in [0,T].$$

The explosion time of the process $\hat{v}$, where by explosion we mean that the process $\hat{v}$ escapes the interval $(0, \infty)$ on which is defined, is given by

$$\tau_\infty = \inf\{t \geq 0; \hat{v}_t \notin (0, \infty)\}.$$

We apply Feller's test to $\hat{v}$ (see Chapter 5, section 5.5 of Karatzas and Shreve [41]) in order to prove that

$$P(\{\tau_\infty = +\infty\}) = P(\{\hat{v}_t \text{ does not explode on } [0, T]\}) = 1.$$

As explained in Chapter 5, section 5.5 of Karatzas and Shreve [41], the Feller's test goes as follows: in order to determine whether a process $\hat{v}$ explodes or not in finite time, one has first to examine the behavior of the scale function $p(x)$ by computing the limits $\lim_{x \to 0} p(x)$ and $\lim_{x \to \infty} p(x)$. If

$$\lim_{x \to 0} p(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} p(x) = +\infty, \tag{3.1.7}$$

Then Proposition 5.22 in [41] implies that $P(\tau_\infty = +\infty) = 1$. If at least one of the equalities in is not satisfied, then we proceed to compute the function $u(x)$ defined in Problem 5.28 of [41] ad to examine the limits

$$\lim_{x \to \infty} u(x) \quad \text{and} \quad \lim_{x \to 0} u(x).$$

Theorem 5.29 in [41] states that $P(\tau_\infty = +\infty) = 1$ or $P(\tau_\infty = +\infty) < 1$ according to whether the equality

$$\lim_{x \to 0} u(x) = \lim_{x \to \infty} u(x) = +\infty.$$

is satisfied or not.

The scale function is equal in our case to

$$p(x) = \int_1^x \exp(-2 \int_1^y \frac{-(a \cdot \sigma)z^2 + a_3|\alpha|^2 z}{|\alpha|^2 z^2} dz) dy,$$

and examine the limits $\lim_{x \downarrow 0} p(x)$ and $\lim_{x \uparrow \infty} p(x)$. Here we distinguish between two cases:

Case 1: $a_3 = 0$. We have

$$\begin{aligned} p(x) &= \int_1^x \exp(\frac{2(a \cdot \sigma)}{|\alpha|^2} \int_1^y dz) dy \\ &= k \int_1^x \exp(\frac{2(a \cdot \sigma)y}{|\alpha|^2}) dy \\ &= k_1 \frac{|\alpha|^2}{2(a \cdot \sigma)} \exp(\frac{2(a \cdot \sigma)x}{|\alpha|^2}) - k_2 \end{aligned}$$

with $k, k_1, k_2 \in \mathbb{R}_+$. Since $a \cdot \sigma > 0$, we have

$$\lim_{x \uparrow \infty} p(x) = +\infty,$$

By Problem 5.27 of [41] we obtain that

$$u(\infty) = +\infty,$$

where

$$u(x) = \int_1^x p'(y) \int_1^y \frac{2}{p'(z)|\alpha|^2 z^2} dz dy.$$

Moreover

$$\lim_{x \to 0+} p(x) = k_1 \frac{|\alpha|^2}{2(a \cdot \sigma)} - k_2 > -\infty$$

As required by Feller's test, we now compute

$$\begin{aligned}
\lim_{x \to 0+} u(x) &= \lim_{x \to 0+} \int_1^x p'(y) \int_1^y \frac{2}{|\alpha|^2 z^2 p'(z)} dz dy \\
&= \lim_{x \to 0+} \int_1^x \frac{2}{|\alpha|^2 z^2 p'(z)} \int_z^x p'(y) dy dz \\
&= \lim_{x \to 0+} \int_1^x \frac{2}{|\alpha|^2 z^2} \exp(-\frac{2(a \cdot \sigma) z}{|\alpha|^2}) \int_z^x \exp(\frac{2(a \cdot \sigma) y}{|\alpha|^2}) dy dz \\
&\geq \lim_{x \to 0+} e^{-\frac{2(a \cdot \sigma)}{|\alpha|^2}} \int_1^x \frac{2}{|\alpha|^2 z^2} \int_z^x dy dz \\
&= \lim_{x \to 0+} \left(e^{-\frac{2(a \cdot \sigma)}{|\alpha|^2}} \frac{2}{|\alpha|^2} \int_1^x \frac{1}{z^2} (x - z) dz \right) \\
&= e^{-\frac{2(a \cdot \sigma)}{|\alpha|^2}} \frac{2}{|\alpha|^2} \lim_{x \to 0+} (-\log x - x + 1) = +\infty
\end{aligned}$$

It follows from Feller's Test for Explosions, see Theorem 5.29 of [41], that

$$P(\tau_\infty = +\infty) = 1.$$

Hence $\hat{v}$ does not explode on $[0, T]$ and

$$\mathbb{E}_P[M_T] = M_0 P(\{\hat{v}_t \text{ does not explode on } [0, T]\}) = M_0.$$

This implies that the positive local martingale M has constant expectation and therefore is a true P-martingale. We denote by Q the probability measure having the Radon-Nikodym density process with respect to P given by M.

It follows from Girsanov's theorem that the process $(B_t^{Q,i})_{t\in[0,T]}$, $i=1,2$ given by

$$B_t^{Q,i} = B_t^i - \int_0^t \frac{1}{M_s} d[M,B^i]_s = B_t^i + \int_0^t \frac{v_s(a\cdot\sigma)}{(a\cdot\sigma^\perp)}\sigma_i^\perp ds,$$

for all $t\in[0,T]$ is a Brownian motion under Q Hence the canonical semimartingale decomposition of v under Q can be written in the following way

$$\begin{aligned} dv_t &= \big(a_1 v_t dB_t^1 + a_2 v_t dB_t^2 - \frac{1}{M_t} d[M,v]_t\big) + \frac{1}{M_t} d[M,v]_t \\ &= \big(a_1 v_t dB_t^1 + a_2 v_t dB_t^2 + a_1 \frac{v_t^2(a\cdot\sigma)}{a\cdot\sigma^\perp}\sigma_1^\perp dt + a_2 \frac{v_t^2(a\cdot\sigma)}{a\cdot\sigma^\perp}\sigma_2^\perp dt\big) - v_t^2(a\cdot\sigma)dt \\ &= a_1 v_t dB_t^{Q,1} + a_2 v_t dB_t^{Q,2} - v_t^2(a\cdot\sigma)dt. \end{aligned}$$

Analogously, one can obtain the canonical decomposition of X with respect to Q

$$\begin{aligned} dX_t &= \big(\sigma_1 v_t X_t dB_t^1 + \sigma_2 v_t X_t dB_t^2 - \frac{1}{M_t} d[M,X]_t\big) + \frac{1}{M_t} d[M,X]_t \\ &= \big(\sigma_1 v_t X_t dB_t^1 + \sigma_2 v_t X_t dB_t^2 + \sigma_1 \frac{v_t^2(a\cdot t)}{(a\cdot\sigma^\perp)}\sigma_1^\perp X_t dt \\ &\quad + \sigma_2 \frac{v_t^2(a\cdot t)}{(a\cdot\sigma^\perp)}\sigma_2^\perp X_t dt\big) + \frac{v_t^2(a\cdot\sigma)}{(a\cdot\sigma^\perp)}(\sigma\sigma^\perp)X_t dt \\ &= \sigma_1 v_t X_t dB_t^{Q,1} + \sigma_2 v_t X_t dB_t^{Q,2}. \end{aligned}$$

Hence, under the measure Q the bivariate process (X,v) satisfies

$$\begin{aligned} dX_t &= \sigma_1 v_t X_t dB_t^{Q,1} + \sigma_2 v_t X_t dB_t^{Q,2},\ X_0 = x, \\ dv_t &= a_1 v_t dB_t^{Q,1} + a_2 v_t dB_t^{Q,2} - (a\cdot\sigma)v_t^2 dt,\ v_0 = 1. \end{aligned}$$

It is easy to see that X is a positive local Q-martingale, hence a Q-supermartingale. To show that is a true Q-martingale it is enough to show that it has constant expectation, i.e. $\mathbb{E}_Q[X_T] = X_0$. It follows from Lemma 4.2 from [57] that

$$\mathbb{E}_Q[X_T] = X_0 Q(\{\bar{v}_t \text{ does not explode on } [0,T]\}),$$

where the auxiliary process $(\bar{v}_t)_{t\in[0,T]}$ satisfies the stochastic differential equation

$$d\bar{v}_t = a_1\bar{v}_t dB_t^1 + a_2\bar{v}_t dB_t^2. \tag{3.1.8}$$

Since the equation (3.1.8) has linear coefficients, it follows from Remark 5.19 [41] that it has a non-exploding solution. Therefore $(X_t)_{t\in[0,T]}$ is a Q-martingale.

Case 2: $a_3 = 1$. We follow the same steps as in Case 1. The scale function is in this case equal to:

$$\begin{aligned} p(x) &= \int_0^x \exp(-2\int_1^y \frac{-(a\cdot\sigma)z^2 + |\alpha|_3^2 z}{|\alpha|^2 z^2} dz) dy \\ &= k\int_1^x \exp(2\frac{(a\cdot\sigma)y}{|\alpha|^2}) y^{-2} dy, \end{aligned}$$

where $k \in \mathbb{R}_+$. We examine the limits $\lim_{x\downarrow 0} p(x)$ and $\lim_{x\uparrow\infty} p(x)$. We have that

$$\lim_{x\downarrow 0} p(x) = \lim_{x\downarrow 0} k\int_1^x \exp(2\frac{(a\cdot\sigma)y}{|\alpha|^2}) y^{-2} dy = -\infty$$

By Problem 5.27 of [41] we have that

$$u(0+) = +\infty,$$

where

$$u(x) = \int_1^x p'(y)\int_1^y \frac{2}{p'(z)|\alpha|^2 z^2} dz dy.$$

Moreover, we have that

$$\begin{aligned} \lim_{x\uparrow\infty} p(x) &= \lim_{x\uparrow\infty} k\int_1^x \exp(2\frac{(a\cdot\sigma)y}{|\alpha|^2}) y^{-2} dy \\ &= +\infty. \end{aligned}$$

Then it follows from Problem 5.27 of [41] that

$$u(\infty) = +\infty.$$

It follows from Feller's Test for Explosions, see Theorem 5.29 of [41], that

$$P(\tau_\infty = +\infty) = 1.$$

Therefore $\hat{v}$ does not explode on $[0,T]$ and

$$\mathbb{E}_P[M_T] = M_0 P(\{\hat{v}_t \text{ does not explode on } [0,T]\}) = M_0.$$

Thus the positive local martingale M has constant expectation and therefore is a true P-martingale.

It follows from Girsanov's Theorem that the Brownian motion $(B_t^i)_{t\in[0,T]}$, $i = 1,2$ admits the following canonical decomposition under Q

$$\begin{aligned} dB_t^i &= \Big(dB_t^i - \frac{1}{M_t} d[M, B^i]_t\Big) + \frac{1}{M_t} d[M, B^i]_t \\ &= \Big(B_t^i + \frac{v_t(a\cdot\sigma)}{a\cdot\sigma^\perp}\sigma_1^\perp dt\Big) - \frac{v_t(a\cdot\sigma)}{a\cdot\sigma^\perp}\sigma_1^\perp dt \\ &= B_t^{Q,i} - \frac{v_t(a\cdot\sigma)}{a\cdot\sigma^\perp}\sigma_1^\perp dt, \end{aligned}$$

where $(B_t^{Q,i})_{t\in[0,T]}$ is a Q-Brownian motion,for $i = 1, 2$. Analogously, one obtains that the canonical semimartingale decomposition of B^3 under Q is given by

$$\begin{aligned} dB_t^3 &= \Big(dB_t^3 - \frac{1}{M_t} d[M, B^3]_t \Big) + \frac{1}{M_t} d[M, B^3]_t \\ &= (dB_t^3 - |\alpha|^2 dt) + |\alpha|^2 dt = dB_t^{Q,3} + |\alpha|^2 dt, \end{aligned}$$

where $(B_t^{Q,3})_{t\in[0,T]}$ is a Q-Brownian motion. We see that X follows under Q the dynamics

$$\begin{aligned} dX_t &= \sigma_1 v_t X_t dB_t^1 + \sigma_2 v_t X_t dB_t^2 + \sigma_1 \frac{(a\cdot\sigma)v_t^2}{a\cdot\sigma^\perp} \sigma_1^\perp X_t dt \\ &+ \sigma_2 \frac{(a\cdot\sigma)v_t^2}{a\cdot\sigma^\perp} \sigma_2^\perp X_t dt - \frac{v_t^2(a\cdot\sigma)}{a\cdot\sigma^\perp}(\sigma\cdot\sigma^\perp) X_t dt \\ &= \sigma_1 v_t X_t dB_t^{1,Q} + \sigma_2 v_t X_t dB_t^{2,Q}, \end{aligned}$$

and v has the following dynamics under Q

$$\begin{aligned} dv_t &= \big(a_1 v_t dB_t^1 + a_2 v_t dB_t^2 + v_t dB_t^3 - \frac{1}{M_t} d[M, v]_t \big) + \frac{1}{M_t} d[M, v]_t \\ &= a_1 v_t dB_t^1 + a_2 v_t dB_t^2 + v_t dB_t^3 + \frac{1}{M_t} \frac{v_t(a\cdot\sigma)}{a\cdot\sigma^\perp} \sigma_1^\perp a_1 v_t M_t dt \\ &+ \frac{1}{M_t} \frac{v_t(a\cdot\sigma)}{a\cdot\sigma^\perp} \sigma_2^\perp a_2 v_t M_t dt - \frac{1}{M_t} |\alpha|^2 v_t M_t dt - (a\cdot\sigma) v_t^2 dt + |\alpha|^2 v_t dt \\ &= \big(a_1 v_t dB_t^{Q,1} + a_2 v_t dB_t^{Q,2} + v_t dB_t^{Q,3} \big) - (a\cdot\sigma) v_t^2 dt + |\alpha|^2 v_t dt. \end{aligned}$$

Under the measure Q, the bivariate process (X, v) satisfies

$$\begin{aligned} dX_t &= \sigma_1 v_t X_t dB_t^{1,Q} + \sigma_2 v_t X_t dB_t^{2,Q}, \ t \in [0, T], \\ dv_t &= a_1 v_t dB_t^{1,Q} + a_2 v_t dB_t^{2,Q} + v_t dB_t^{3,Q} - (a\cdot\sigma) v_t^2 dt + |\alpha|^2 v_t dt. \end{aligned}$$

Thus X is a positive local Q-martingale. As in the previous case, in oder to show that is a true martingale it is enough to show that it has constant expectation. By applying Lemma 4.2 from [57] we obtain

$$\mathbb{E}_Q[X_T] = X_0 Q(\{\hat{w}_t \text{ does not explode on } [0, T]\}),$$

where

$$d\hat{w}_t = a_1 \hat{w}_t dB_t^{1,Q} + a_2 \hat{w}_t dB_t^{2,Q} + \hat{w}_t dB_t^{3,Q} + |\alpha|^2 \hat{w}_t dt. \tag{3.1.9}$$

Due to the linearity of the coefficients, it follows from Remark 5.19 in [41] that equation (3.1.9) has a non-exploding solution. Therefore $(X_t)_{t\in[0,T]}$ is a Q-martingale. □

In the setting of Chapter 2, we now consider a financial market model with a financial asset that generates a single payment X_T at time T and whose price process S is given by $S_t := X_t$ for $t < T$ and $S_T = 0$. Then the wealth process is given by $W = X$. Theorem 3.1.1 shows that W is a uniformly integrable martingale under Q, and so we have

$$Q \in \mathcal{M}_{UI}(W).$$

Moreover, it follows from Theorem 3.1.1 that the wealth process W is a strict local martingale under P. We put

$$R := P \in \mathcal{M}_{NUI}(W).$$

The fundamental value W^R perceived under R is given by

$$W_t^R = \mathbb{E}_R[W_T|\mathcal{F}_t] = \mathbb{E}_R[X_T|\mathcal{F}_t], \quad t \in [0,T].$$

Proposition 3.1.2. *The process W^R admits the factorization $W^R = W \cdot C$, where the semimartingale C is of the form*

$$C_t = 1 + (\sigma_1 c_1(t) + \sigma_2 c_2(t))v_t, \quad t \in [0,T].$$

The time-dependent coefficients are given by

$$\begin{aligned} c_1(t) &= \mathbb{E}_R\left[\frac{1}{v_t}\int_0^{T-t} e^{X_u} v_{u+t} d\tilde{B}_u^1\right], \\ c_2(t) &= \mathbb{E}_R\left[\frac{1}{v_t}\int_0^{T-t} e^{X_u} v_{u+t} d\tilde{B}_u^2\right], \end{aligned} \tag{3.1.10}$$

and satisfy

$$\sigma_1 c_1(t) + \sigma_2 c_2(t) < 0 \tag{3.1.11}$$

for any $t \in [0,T)$.

Proof. Since process X is given by the Doléans exponential

$$X_t = \mathcal{E}\left(\int_0^\cdot \sigma_1 v_s dB_s^1 + \int_0^\cdot \sigma_2 v_s dB_s^2\right)_t, \ t \in [0,T].$$

we have

$$\begin{aligned} \frac{X_T}{X_t} &= \exp\left(\int_t^T \sigma_1 v_s dB_s^1 + \int_t^T \sigma_2 v_s dB_s^2 - \frac{1}{2}\int_t^T (\sigma_1^2+\sigma_2^2)v_s^2 ds\right) \\ &= \exp\left(v_t\int_t^T \sigma_1 \frac{v_s}{v_t} dB_s^1 + v_t\int_t^T \sigma_2 \frac{v_s}{v_t} dB_s^2 - \frac{1}{2}v_t^2\int_t^T (\sigma_1^2+\sigma_2^2)(\frac{v_s}{v_t})^2 ds\right). \end{aligned}$$

We write W under the following factorial form

$$W_t^R = \mathbb{E}_R[X_T|\mathcal{F}_t] = X_t \mathbb{E}_R[\frac{X_T}{X_t}|\mathcal{F}_t] = W_t C_t,$$

where the semimartingale C is given by

$$C_t := \mathbb{E}_R[\frac{X_T}{X_t}|\mathcal{F}_t] \tag{3.1.12}$$

for $t \in [0,T]$. The fact that the Brownian motion has increments independent of the past implies that

$$\frac{v_u}{v_t} = \exp(a_1(B_u^1 - B_t^1) + a_2(B_u^2 - B_t^2) + a_3(B_u^3 - B_t^3) - \frac{1}{2}|\alpha|^2(t-u))$$

is independent of $\mathcal{F}_t$ for $T \geq u \geq t$. We fix $y := v_t$ and define

$$Y_u = \sigma_1 y \int_t^{t+u} \frac{v_s}{v_t} dB_s^1 + \sigma_2 y \int_t^{t+u} \frac{v_s}{v_t} dB_s^2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) y^2 \int_t^{t+u} \left(\frac{v_s}{v_t}\right)^2 ds,$$

for $u \geq 0$. In particular, we have $Y_0 = 0$ and

$$Y_{T-t} = \sigma_1 y \int_t^T \frac{v_s}{v_t} dB_s^1 + \sigma_2 y \int_t^T \frac{v_s}{v_t} dB_s^2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) y^2 \int_t^T \left(\frac{v_s}{v_t}\right)^2 ds.$$

An application of Itô's formula for the function $f(x) = e^x$ yields

$$\begin{aligned} e^{Y_{T-t}} &= e^{Y_0} + \int_0^{T-t} e^{Y_u} dY_u + \frac{1}{2} \int_0^{T-t} e^{Y_u} d[Y,Y]_u \\ &= e^{Y_0} + \sigma_1 y \int_0^{T-t} \frac{v_{u+t}}{v_t} e^{Y_u} d\tilde{B}_u^1 + \sigma_2 y \int_0^{T-t} \frac{v_{u+t}}{v_t} e^{Y_u} d\tilde{B}_u^2, \end{aligned} \tag{3.1.13}$$

where the Brownian motion $\tilde{B} = (\tilde{B}^1, \tilde{B}^2)$ defined by $\tilde{B}_u^i := B_{t+u}^i - B_t^i$, $i = 1,2$, is independent of $\mathcal{F}_t$. For fixed $v_t = y$, we have

$$C_t = \mathbb{E}_R[\frac{X_T}{X_t}|\mathcal{F}_t] = \mathbb{E}_R[e^{Y_{T-t}}] = 1 + (\sigma_1 c_1(t) + \sigma_2 c_2(t))y, \tag{3.1.14}$$

where $c_1(t)$ and $c_2(t)$ are given by (3.1.10). It follows from an application of Feller's Test for Explosions, see [57], that $W_t^R < W_t$ for any $t \in [0,T)$. Therefore

$$W_t(1 + (\sigma_1 c_1(t) + \sigma_2 c_2(t))v_t) < W_t, \quad t \in [0,T),$$

and this implies (3.1.11). □

We consider the flow of equivalent local martingale measures $\mathcal{R} = (R_t)_{t\geq 0}$ defined by (2.3.12). Lemma 2.3.4 implies that the resulting bubble $\beta^{\mathcal{R}}$ is of the form

$$\beta^{\mathcal{R}} = W - W^{\mathcal{R}} = \xi(W - W^R).$$

Corollary 3.1.3. *If $a_3 = 0$, the process W^R is a submartingale under the measure Q, and so the behavior of the bubble $\beta^{\mathcal{R}}$ is again described by Theorem 2.3.9.*

Proof. We verify the sufficient condition in Corollary 2.3.10. Since

$$\begin{aligned} dC_t &= (\sigma_1 c_1(t) + \sigma_2 c_2(t))dv_t + \sigma_1 v_t dc_1(t) + \sigma_2 v_t dc_2(t) \\ &= (\sigma_1 c_1(t) + \sigma_2 c_2(t))a_1 v_t dB_t^1 + (\sigma_1 c_1(t) + \sigma_2 c_2(t))a_2 v_t dB_t^2 \\ &\quad + \sigma_1 v_t dc_1(t) + \sigma_2 v_t dc_2(t). \end{aligned} \tag{3.1.15}$$

the local martingale part of the semimartingale C is given by

$$M_t^C = \int_0^t a_1(\sigma_1 c_1(s) + \sigma_2 c_2(s))v_s dB_s^1 + \int_0^t a_2(\sigma_1 c_1(s) + \sigma_2 c_2(s))v_s dB_s^2.$$

Since M is the stochastic exponential defined in (3.1.2), M satisfies the stochastic differential equation

$$M_t = -\int_0^t \frac{v_s(a.\sigma)}{(a.\sigma^\perp)}\sigma_1^\perp M_s dB_s^1 - \int_0^t \frac{v_s(a.\sigma)}{(a.\sigma^\perp)}\sigma_2^\perp M_s dB_s^2.$$

Therefore the quadratic covariation $[M, C]$ is equal to

$$[M, C]_t = [M, M^c]_t = \int_0^t -(\sigma_1 c_1(s) + \sigma_2 c_2(s))(a.\sigma)v_s^2 M_s ds.$$

It follows from (3.1.11) that the integrand is strictly positive, and therefore $[M, C]$ is an increasing process. Thus $\beta^{\mathcal{R}}$ is a local submartingale under R, as a consequence of Corollary 2.3.10. □

We now provide a modification of the model in such a way that Condition (2.3.19) is no longer satisfied. To this aim we choose the parameters such that

$$\frac{|\alpha|^2}{(a\cdot\sigma)} > 1,$$

and introduce the stopping time

$$\tau := \inf\{t > 0; v_t = \frac{|\alpha|^2}{(a\cdot\sigma)}\}. \tag{3.1.16}$$

Consider a financial asset that generates a single payment X_{τ_0} at time $\tau_0 := T \wedge \tau$ and whose price process S is given again by $S_t := X_t$ for $t < \tau_0$ and $S_t := 0$ for $t \geq \tau_0$. The wealth process is then given again by $W = X$.

Proposition 3.1.4. *If $a_3 = 1$, the quadratic covariation $[M, C]$ is a decreasing process, and so condition* (2.3.19) *is no longer satisfied.*

Proof. By using the same computations as in the proof of Proposition 3.1.2 we obtain

$$\begin{aligned} dC_t &= (\sigma_1 c_1(t) + \sigma_2 c_2(t))dv_t + \sigma_1 v_t dc_1(t) + \sigma_2 v_t dc_2(t) \\ &= (\sigma_1 c_1(t) + \sigma_2 c_2(t))(a_1 v_t dB_t^1 + a_2 v_t dB_t^2 + v_t dB_t^3) \\ &\quad + \sigma_1 v_t dc_1(t) + \sigma_2 v_t dc_2(t), \end{aligned}$$

where $c_1(t)$ and $c_2(t)$ are given by (3.1.10). It follows that the local martingale part of C is given by

$$dM_t^C = (\sigma_1 c_1(t) + \sigma_2 c_2(t))(a_1 v_t dB_t^1 + a_2 v_t dB_t^2 + v_t dB_t^3).$$

and

$$\begin{aligned} d[M^C, M]_t &= -(\sigma_1 c_1(t) + \sigma_2 c_2(t))(a \cdot \sigma)v_t^2 M_t dt \\ &\quad + (\sigma_1 c_1(t) + \sigma_2 c_2(t))|a|^2 v_t M_t dt \\ &= -(\sigma_1 c_1(t) + \sigma_2 c_2(t))(-|\alpha|^2 + (a \cdot \sigma)v_t)v_t M_t dt. \end{aligned}$$

In view of (3.1.11) the process is decreasing on $[0, \tau_0]$, since $(a \cdot \sigma)v_t - |\alpha|^2 \leq 0$ on $[0, \tau_0]$ by the definition of τ in (3.1.16) and since $v_0 = 1$. □

3.2 Stochastic model with correlated Brownian motions

The following example is derived from the Anderson-Piterbarg model [3], which is generalisation of the previous case, by allowing correlation between the Brownian motions driving the processes. Our example consists of a process driven by a 3-dimensional Brownian motion, instead of a 2-dimensional Brownian motion, as is the case in [3].
Let $B = (B^1, B^2, B^3)$ be a 3-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and let $\rho \in (-1, 1)$. We assume that $(\mathcal{F}_t)_{t\geq 0}$ is the canonical filtration generated by $B = (B^1, B^2, B^3)$. We define the following Brownian motions:

$$\begin{aligned} dW_t^1 &= \rho dB_t^3 + \sqrt{1-\rho^2} dB_t^1 = \rho dB_t^3 + \bar{\rho} dB_t^1, \\ dW_t^2 &= \rho dB_t^3 + \sqrt{1-\rho^2} dB_t^2 = \rho dB_t^3 + \bar{\rho} dB_t^2 \end{aligned}$$

where we have denoted $\bar{\rho} = \sqrt{1-\rho^2}$.
We consider the following stochastic volatility model

$$\begin{aligned} dX_t &= \lambda_1 X_t \sqrt{v_t} dW_t^1 + \lambda_2 X_t \sqrt{v_t} dW_t^2, \quad X_0 = 1 \\ dv_t &= -k v_t dt + \epsilon v_t dB_t^3, \end{aligned} \tag{3.2.1}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and k, ϵ are positive constants.

Theorem 3.2.1. *If $\rho > 0$ and $\lambda_2 < 0 < \lambda_1 \leq -\frac{2-\rho^2}{\rho^2}\lambda_2$, satisfying $\lambda_1+\lambda_2 > 0$, there exists a unique solution (X, v) of (3.2.1) and for any $T > 0$, the process $X = (X_t)_{t\in[0,T]}$ is a strict local martingale under P. Moreover there exists an equivalent martingale measure Q for X with the Radon-Nikodym density process*

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = M_t, \quad 0 \leq t \leq T,$$

given by

$$M_t = \mathcal{E}\left(\int_0^{\cdot} \alpha_1 \sqrt{v_s} dW_s^1 + \int_0^{\cdot} \alpha_2 \sqrt{v_s} dW_s^2\right)_t,$$

where the coefficieints α_1 and α_2 are equal to

$$\begin{aligned} \alpha_1 &= \lambda_2 + \rho^2 \lambda_1 \\ \alpha_2 &= -\lambda_1 - \rho^2 \lambda_2 \end{aligned} \tag{3.2.2}$$

Then the process X is a martingale under Q satisfying

$$\begin{aligned} dX_t &= \lambda_1 \sqrt{v_t} X_t dW_t^{1,Q} + \lambda_1 \sqrt{v_t} X_t dW_t^{2,Q} \\ dv_t &= (-k v_t + \epsilon\rho(\alpha_1 + \alpha_2) v_t^{\frac{3}{2}}) dt + \epsilon v_t dB_t^{3,Q}. \end{aligned}$$

where the $W^{1,Q}$, $W^{2,Q}$ and $B^{3,Q}$ are Brownian motions under Q given by

$$\begin{aligned} W_t^{1,Q} &= W_t^1 - \int_0^t (\alpha_1 \sqrt{v_s} + \alpha_2 \sqrt{v_s} \rho^2) ds, \\ W_t^{2,Q} &= W_t^2 - \int_0^t (\alpha_1 \sqrt{v_s} \rho^2 + \alpha_2 \sqrt{v_s}) ds, \\ B_t^{3,Q} &= B_t^3 - \int_0^t (\alpha_1 + \alpha_2) \rho \sqrt{v_s} ds, \end{aligned}$$

for all $t \in [0, T]$.

Theorem 3.2.1 is a consequence of the following Lemma. The techniques used in Lemma 3.2.2 are derived from Piterbarg and Anderson [2] and adapted to our case.

Lemma 3.2.2. *If $\lambda_1, \lambda_2 \in \mathbb{R}$ fullfil the assumptions of Theorem 3.2.1, then* (3.2.1) *admits a unique solution and*

$$\mathbb{E}_P[X_T] = \tilde{P}(\tau_\infty^v > T), \tag{3.2.3}$$

where $\tilde{P}$ is a probability measure such that the process v satisfies under $\tilde{P}$ the equation

$$dv_t = \epsilon v_t d\tilde{B}_t^3 - kv_t dt + (\lambda_1 + \lambda_2)\epsilon\rho v_t^{\frac{3}{2}} dt$$

where $(\tilde{B}_t^3)_{t\in[0,T]}$ is a $\tilde{P}$-Brownian motion and τ_∞^v represents the explosion time of the process v, i.e. the first time v exits the interval $(0,\infty)$ at $+\infty$. More precisely $\tau_\infty^v = \lim_{n\to\infty} \tau_n$, where

$$\tau_n = \inf\{t \geq 0; |v_t| \geq n\}, \quad n \in \mathbb{N}. \tag{3.2.4}$$

Proof. First we show the uniqueness of the solutions. The process v is a geometric Brownian motion having the form

$$v_t = \exp(\epsilon B_t^3 - (k + \frac{1}{2}\epsilon^2)t), \quad 0 \leq t \leq T.$$

Therefore X is uniquely determined as the stochastic exponential of the process

$$\int_0^t \lambda_1 \sqrt{v_s} dW_s^1 + \int_0^t \lambda_2 \sqrt{V_s} dW_s^2.$$

Then the solution of (3.2.1) is given by

$$\begin{aligned} X_t &= \exp(\int_0^t \lambda_1 \sqrt{v_s} dW_s^1 + \int_0^t \lambda_2 \sqrt{v_s} dW_s^2 - \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2\rho^2) v_s ds) \\ &= \exp\Big(\int_0^t (\lambda_1 + \lambda_2)\rho\sqrt{v_s} dB_s^3 + \int_0^t \lambda_1\bar{\rho}\sqrt{v_s} dB_s^1 + \int_0^t \lambda_2\bar{\rho}\sqrt{v_s} dB_s^2 \\ &\quad - \frac{1}{2}\int_0^t (\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2\rho^2) v_s ds \Big). \end{aligned}$$

Let $\mathcal{F}_T^3 = \sigma(B_s^3; s \leq T)$. We compute $\mathbb{E}_P[X_T 1_{\{\tau_n > T\}}]$, where τ_n is defined in

(3.2.4):

$$
\begin{aligned}
\mathbb{E}_P[X_T 1_{\{\tau_n>T\}}] &= \mathbb{E}_P[\mathbb{E}_P[X_T 1_{\{\tau_n>T\}}|\mathcal{F}_T^3]] = \mathbb{E}_P[1_{\{\tau_n>T\}}\mathbb{E}_P[X_T|\mathcal{F}_T^3]] \\
&= \mathbb{E}_P\Big[1_{\{\tau_n>T\}} \exp\Big(\int_0^T (\lambda_1+\lambda_2)\rho\sqrt{v_s}dB_s^3 - \frac{1}{2}\int_0^T (\lambda_1^2+\lambda_2^2 \\
&\quad + 2\lambda_1\lambda_2\rho^2)v_s ds\Big)\mathbb{E}_P\Big[\exp\Big(\int_0^T \lambda_1\bar{\rho}\sqrt{v_s}dB_s^1 + \int_0^T \lambda_2\bar{\rho}\sqrt{v_s}dB_s^2\Big)|\mathcal{F}_T^3\Big]\Big] \\
&= \mathbb{E}_P\Big[1_{\{\tau_n>T\}} \exp\Big(\int_0^T (\lambda_1+\lambda_2)\rho\sqrt{v_s}dB_s^3 - \frac{1}{2}\int_0^T (\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2\rho^2)ds\Big) \\
&\quad \mathbb{E}_P\Big[\exp\Big(\int_0^T \lambda_1\bar{\rho}\sqrt{v_s}dB_s^1\Big)\exp\Big(\int_0^T \lambda_2\bar{\rho}\sqrt{v_s}dB_s^2\Big)|\mathcal{F}_T^3\Big]\Big] \\
&= \mathbb{E}_P\Big[1_{\{\tau_n>T\}} \exp\Big(\int_0^T (\lambda_1+\lambda_2)\rho\sqrt{v_s}dB_s^3 - \frac{1}{2}\int_0^T (\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2\rho^2)v_s ds\Big) \\
&\quad \exp\Big(\frac{1}{2}\int_0^T (\lambda_1^2+\lambda_2^2)\bar{\rho}^2 v_s ds\Big)\Big] \\
&= \mathbb{E}_P\Big[1_{\{\tau_n>T\}} \exp\Big(\int_0^T (\lambda_1+\lambda_2)\rho\sqrt{v_s}dB_s^3 - \frac{1}{2}\int_0^T (\lambda_1+\lambda_2)^2\rho^2 v_s ds\Big)\Big].
\end{aligned}
$$

We consider the local martingale

$$
\xi_t := \exp\Big(\int_0^t (\lambda_1+\lambda_2)\rho\sqrt{v_s}dB_s^3 - \frac{1}{2}\int_0^T (\lambda_1+\lambda_2)^2\rho^2 v_s ds\Big), \quad t \in [0,T].
$$

By replacing ξ_T in the above computations, we obtain

$$
\mathbb{E}_P[X_T 1_{\{\tau_n>T\}}] = \mathbb{E}_P\Big[1_{\{\tau_n>T\}}\xi_T\Big] = \mathbb{E}_P\Big[1_{\{\tau_n>T\}}\xi_{T\wedge\tau_n}\Big]. \tag{3.2.5}
$$

Let $\xi^n := (\xi_t^{\tau_n})_{t\in[0,T]}$. Then ξ^n is a true P-martingale and we can define a measure $\tilde{P}^n$ with Radon-Nikodym density with respect to P given by

$$
\frac{d\tilde{P}^n}{dP}|_{\mathcal{F}_t} = \xi_t^n, \quad t \in [0,T].
$$

Therefore (3.2.5) becomes

$$
\mathbb{E}_P[X_T 1_{\{\tau_n>T\}}] = \mathbb{E}_{\tilde{P}^n}[1_{\{\tau_n>T\}}].
$$

By applying the monotone convergence theorem and using the fact $P(\tau_\infty^v = +\infty) = 1$ we have

$$
\begin{aligned}
\mathbb{E}_P[X_T] = \mathbb{E}_P[X_T 1_{\{\tau_\infty^v>T\}}] &= \lim_{n\to\infty}\mathbb{E}_P[X_T 1_{\{\tau_n>T\}}] \\
&= \lim_{n\to\infty}\mathbb{E}_{\tilde{P}_n}[1_{\{\tau_n>T\}}] = \lim_{n\to\infty}\mathbb{E}_{\tilde{P}}[1_{\{\tau_n>T\}}] \\
&= \mathbb{E}_{\tilde{P}}[1_{\{\tau_\infty^v>T\}}].
\end{aligned}
$$

The third equality follows from the fact that v follows the same law under $\tilde{P}_n$ as under $\tilde{P}$ up to the explosion time τ_∞^v, see also Lemma 4.2 in Sin [57]. We now examine the behavior of the process v under $\tilde{P}$. We apply Feller's test to v (see Ch.5 in Karatzas and Shreve [41]) in order to show

$$\tilde{P}(\tau_\infty^v > T) < 1,$$

which implies $\mathbb{E}_P[X_T] < 1$, or equivalently, X is a strict P-local martingale. We compute the scale function

$$\begin{aligned} p_1(x) &= \int_1^x \exp\Big(-2\int_1^y \frac{-kz + \rho\epsilon(\lambda_1+\lambda_2)z^{\frac{3}{2}}}{\epsilon^2 z^2}dz\Big)dy \\ &= \int_1^x \exp\Big(\frac{2k}{\epsilon^2}\int_1^y \frac{1}{z}dz - \frac{2\rho(\lambda_1+\lambda_2)}{\epsilon}\int_1^y z^{-\frac{1}{2}}dz\Big)dy \\ &= K_1\int_1^x y^{\frac{2k}{\epsilon^2}} e^{-\frac{4\rho(\lambda_1+\lambda_2)}{\epsilon}y^{\frac{1}{2}}}dy, \end{aligned}$$

where $K_1 > 0$ is some positive constant. For simplicity we denote

$$\alpha := \frac{2k}{\epsilon^2} > 0 \text{ and } \beta := \frac{4\rho(\lambda_1+\lambda_2)}{\epsilon} > 0$$

Let n be the smallest positive integer such that $n \geq \alpha$.
Then

$$\int_1^\infty y^\alpha e^{-\beta y^{\frac{1}{2}}}dy \leq \int_1^\infty y^n e^{-\beta y^{\frac{1}{2}}}dy.$$

Since $\int_1^\infty y^n e^{-\beta y^{\frac{1}{2}}}dy$ is convergent this implies

$$p_1(+\infty) = \lim_{x\to\infty} K_1\int_1^x y^\alpha e^{-\beta y^{\frac{1}{2}}}dy < +\infty.$$

We consider the function

$$u_1(x) = \int_1^x p_1'(y)\int_1^y \frac{2}{p_1'(z)\epsilon^2 z^2}dzdy.$$

and we examine the limit $\lim_{x\to\infty} u_1(x)$. By applying Fubini's theorem we obtain

$$u_1(x) = \int_1^x \frac{2}{p_1'(z)\epsilon^2 z^2}\int_z^x p_1'(y)dydz = \frac{2K_2}{\epsilon^2}\int_1^x z^{-2-\alpha}e^{\beta z^{\frac{1}{2}}}\int_z^x y^\alpha e^{-\beta y^{\frac{1}{2}}}dydz,$$

where $K_2 > 0$ is a positive constant. We examine first the inner integral. By doing the substitution $w = y^{\frac{1}{2}}$ we get

$$\int_z^x y^\alpha e^{-\beta y^{\frac{1}{2}}} dy = \int_{z^{\frac{1}{2}}}^{x^{\frac{1}{2}}} w^{2\alpha} e^{-\beta w} 2w dw = 2 \int_{z^{\frac{1}{2}}}^{x^{\frac{1}{2}}} w^{2\alpha+1} e^{-\beta w} dw$$
$$\leq 2 \int_{z^{\frac{1}{2}}}^{\infty} w^{2\alpha+1} e^{-\beta w} dw.$$

Let $n_0 \in \mathbb{N}$ be the smallest positive integer satisfying $n_0 \geq 2\alpha + 1$. In particular $2\alpha + 2 > n_0 \geq 2\alpha + 1$. Then for $z > 1$

$$\begin{aligned}
2 \int_{z^{\frac{1}{2}}}^{\infty} w^{2\alpha+1} e^{-\beta w} dw &\leq 2 \int_{z^{\frac{1}{2}}}^{\infty} w^{n_0} e^{-\beta w} dw = \frac{2}{\beta} z^{\frac{n_0}{2}} e^{-\beta z^{\frac{1}{2}}} + \frac{2n_0}{\beta} \int_{z^{\frac{1}{2}}}^{\infty} w^{n_0-1} e^{-\beta w} dw \\
&= \frac{2}{\beta} z^{\frac{n_0}{2}} e^{-\beta z^{\frac{1}{2}}} + \frac{2n_0}{\beta^2} e^{-\beta z^{\frac{1}{2}}} z^{\frac{n_0-1}{2}} \\
&\quad + \frac{2n_0(n_0-1)}{\beta^2} \int_{z^{\frac{1}{2}}}^{\infty} w^{n_0-2} e^{-\beta w} dw \\
&= \frac{2}{\beta} z^{\frac{n_0}{2}} e^{-\beta z^{\frac{1}{2}}} + \frac{2n_0}{\beta^2} z^{\frac{n_0-1}{2}} e^{-\beta z^{\frac{1}{2}}} \\
&\quad + ... + \frac{2n_0(n_0-1)...(n_0-(n_0-2))}{\beta^{(n_0-2)+1}} \int_{z^{\frac{1}{2}}}^{\infty} w^{n_0-(n_0-1)} e^{-\beta w} dw \\
&= \frac{2}{\beta} z^{\frac{n_0}{2}} e^{-\beta z^{\frac{1}{2}}} + \frac{2n_0}{\beta^2} z^{\frac{n_0-1}{2}} e^{-\beta z^{\frac{1}{2}}} \\
&\quad + ... + \frac{2n_0(n_0-1)...(n_0-(n_0-2))}{\beta^{(n_0-2)+1}} \frac{1}{\beta} z^{\frac{n_0-(n_0-1)}{2}} e^{-\beta z^{\frac{1}{2}}} \\
&\quad + \frac{2n_0(n_0-1)...(n_0-(n_0-1))}{\beta^{(n_0-2)+2}} \int_{z^{\frac{1}{2}}}^{\infty} e^{-\beta w} dw \\
&= \frac{2}{\beta} e^{-\beta z^{\frac{1}{2}}} z^{\frac{n_0}{2}} + \sum_{k=0}^{n_0-1} \frac{2n_0(n_0-1)...(n_0-k)}{\beta^{k+1}} z^{n_0-k-1} e^{-\beta z^{\frac{1}{2}}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
u_1(x) &= \frac{2K_2}{\epsilon^2} \int_1^x z^{-2-\alpha} e^{\beta z^{\frac{1}{2}}} \Big(\frac{2}{\beta} z^{\frac{n_0}{2}} e^{-\beta z^{\frac{1}{2}}} + \sum_{k=0}^{n_0-1} \frac{2n_0(n_0-1)...(n_0-k)}{\beta^{k+1}} z^{\frac{n_0-k-1}{2}} e^{-\beta z^{\frac{1}{2}}} \Big) dz \\
&= \frac{4K_2}{\epsilon^2 \beta} \int_1^x z^{\frac{n_0}{2}-2-\alpha} dz + \frac{4K_2}{\epsilon^2} \sum_{k=0}^{n_0-1} \frac{n_0(n_0-1)...(n_0-k)}{\beta^{k+1}} \int_1^x z^{\frac{n_0-k-1}{2}-\alpha-2} dz \\
&= \frac{4K_2}{\epsilon^2 \beta} \frac{1}{\frac{n_0}{2} - \alpha - 1} x^{\frac{n_0}{2}-\alpha-1} \\
&\quad + \frac{4K_2}{\epsilon^2} \sum_{k=0}^{n_0-1} \frac{n_0(n_0-1)...(n_0-k)}{\beta^{k+1}} \frac{1}{\frac{n_0}{2} - \alpha - 1 - \frac{k+1}{2}} x^{\frac{n_0}{2}-\alpha-1-\frac{k+1}{2}} - K_3
\end{aligned}$$

Therefore,$u_1(+\infty) = \lim_{x\to\infty} u_1(x) < \infty$, since $\frac{n_0}{2} - \alpha - 1 < 0$. Hence, via Theorem 5.29 in Karatzas and Shreve [41] we have that $\tilde{P}(\tau = +\infty) < 1$, where $\tau = \inf\{t \geq 0; v_t \notin (0,\infty)\}$. In order to conclude that $\tilde{P}(\tau^v_\infty = \infty) < 1$, we show that the process v doesn't reach 0 in finite time. To this purpose we employ a comparison result, namely Proposition 2.18 of Karatzas and Shreve [41]. We consider the process $\bar{v} = (\bar{v})_{t\in[0,T]}$ satisfying under $\tilde{P}$ the equation

$$d\bar{v}_t = -k\bar{v}_t dt + \epsilon\bar{v}_t d\tilde{B}^3_t.$$

We know that $\tilde{P}(\tau_0 = \infty) = 1$, where τ_0 is the first time $\bar{v}$ exits the interval $(0,\infty)$. It follows from Proposition 2.18 of Karatzas and Shreve [41] that

$$\bar{v}_t \leq v_t \quad \tilde{P} - a.s.$$

for all $t \in [0,T]$. Hence v doesn't reach 0 in finite time.
Therefore X is a strict P-local martingale and this concludes our proof. □

We now proceed with the proof of Theorem 3.2.1

Proof. We start by showing that the process M is a well defined Radon-Nikodym density process i.e. a true martingale under the measure P. By using the same reasoning as in Lemma 3.2.2 we obtain

$$\mathbb{E}_P[M_T 1_{\{\tau_n > T\}}] = \mathbb{E}_P[1_{\{\tau_n > T\}}\mathbb{E}_P[M_T|\mathcal{F}^3_T]] = \mathbb{E}_P\Big[1_{\{\tau_n > T\}} \exp\Big(\int_0^T (\alpha_1+\alpha_2)\rho\sqrt{v_s}dB^3_s - \frac{1}{2}\int_0^T (\alpha_1+\alpha_2)^2\rho^2 v_s ds\Big)\Big].$$

We define the local martingale $(\bar{\xi})_{t\in[0,T]}$ by

$$\bar{\xi}_t := \exp\Big(\int_0^t (\alpha_1+\alpha_2)\rho\sqrt{v_s}dB^3_s - \frac{1}{2}\int_0^t (\alpha_1+\alpha_2)^2\rho^2 v_s ds\Big),$$

for all $t \in [0,T]$. As in Lemma 3.2.2, we can define the measures $\bar{P}^n$ with the Radon-Nikodym density processes with respect to P given by

$$\frac{d\bar{P}^n}{dP}|_{\mathcal{F}_t} = \bar{\xi}^n_t, \quad t \in [0,T].$$

where $\bar{\xi}^n_t := \bar{\xi}_{t\wedge\tau_n}$ for all $t \in [0,T]$. By passing to the limit and using the arguments Lemma 4.2 in [57], we obtain

$$\mathbb{E}_P[M_T] = \bar{P}(\tau^v_\infty > T). \tag{3.2.6}$$

The process v follows under the measure $\bar{P}$ the following equation:

$$dv_t = (-kv_t + \epsilon\rho(\alpha_1 + \alpha_2)v_t^{\frac{3}{2}})dt + \epsilon v_t d\bar{B}_t^3,$$

where $(\bar{B}_t^3)_{t\in[0,T]}$ is a $\bar{P}$-Brownian motion.
We apply Feller's test to v to in order to show that v does not explode at $+\infty$ in finite time. We denote for simplicity

$$\gamma := -\frac{4\rho(\alpha_1 + \alpha_2)}{\epsilon} \geq 0.$$

The scale function is equal to

$$p_2(x) = K_4 \int_1^x y^{\frac{2k}{\epsilon^2}} e^{-\frac{4\rho(\alpha_1+\alpha_2)}{\epsilon} y^{\frac{1}{2}}} dy = K_4 \int_1^x y^{\alpha} e^{\gamma y^{\frac{1}{2}}} dy.$$

where $K_4 > 0$ is some positive constant. Since

$$\lim_{x\to\infty} \int_1^x y^{\alpha} dy = +\infty,$$

and

$$0 \leq \int_1^x y^{\alpha} dy \leq \int_1^x y^{\alpha} e^{\gamma y^{\frac{1}{2}}} dy,$$

it follows that $\lim_{x\to\infty} p_2(x) = +\infty$. Hence, via Problem 5.27 in [41] we also have that

$$\lim_{x\to\infty} u_2(x) = +\infty,$$

where $u_2(x)$ is given by

$$u_2(x) = \int_1^x p_2'(y) \int_1^y \frac{2}{p_2'(z)\epsilon^2 z^2} dz dy.$$

Furthermore $\lim_{x\to 0+} p_2(x) > -\infty$ since

$$-\infty < -\frac{e^{\gamma}}{\alpha + 1} = -e^{\gamma} \lim_{x\to 0+} \int_x^1 y^{\alpha} dy \leq -\lim_{x\to 0+} \int_x^1 y^{\alpha} e^{\gamma y^{\frac{1}{2}}} dy.$$

As requested by Feller's test we compute

$$\begin{aligned}
\lim_{x\to 0+} u_2(x) &= \frac{2}{\epsilon^2} \lim_{x\to 0+} \int_1^x z^{-2-\alpha} e^{-\gamma y^{\frac{1}{2}}} \int_z^x y^{\alpha} e^{\gamma y^{\frac{1}{2}}} dy dz \\
&\geq \frac{2e^{-\gamma}}{\epsilon^2} \lim_{x\to 0+} \int_1^x z^{-2-\alpha} \int_z^x y^{\alpha} dy dz \\
&= \frac{2e^{-\gamma}}{\epsilon^2(\alpha+1)} \lim_{x\to 0+} \int_1^x z^{-2-\alpha}(x^{\alpha+1} - z^{\alpha+1}) dz \\
&= \frac{2e^{-\gamma}}{\epsilon^2(\alpha+1)} \lim_{x\to 0+} x^{\alpha+1} \int_1^x z^{-2-\alpha} dz - \frac{2e^{-\gamma}}{\epsilon^2(\alpha+1)} \lim_{x\to 0+} \int_1^x \frac{1}{z} dz \\
&= +\infty.
\end{aligned}$$

Therefore it follows from Theorem 5.29 in [41] that v does not exit the interval $(0,\infty)$ in finite time, hence $\bar{P}(\tau_\infty^v = +\infty) = 1$. It follows from (3.2.6) that $(M_t)_{t\in[0,T]}$ is a P-martingale. This allows us to define a measure Q equivalent to P having M as density process

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = M_t, \quad t \in [0,T].$$

Applying Girsanov's theorem, we see that under the measure Q the bivariate process (X, v) satisfies

$$\begin{aligned} dX_t &= \lambda_1\sqrt{v_t}X_t dW_t^{1,Q} + \lambda_2\sqrt{v_t}X_t dW_t^{2,Q} \quad t \in [0,T], \\ dv_t &= (-kv_t + \epsilon\rho(\alpha_1+\alpha_2)v_t^{\frac{3}{2}})dt + \epsilon v_t dB_t^{3,Q}, \end{aligned}$$

where $W^{1,Q}$, $W^{2,Q}$ and $\tilde{B}^{3,Q}$ are Q-Brownian motions satisfying

$$d[W^{1,Q}, W^{2,Q}]_t = \rho^2 dt \quad \text{and} \quad d[W^{1,Q}, B^3]_t = d[W^{2,Q}, B^3]_t = \rho dt,$$

for all $t \in [0,T]$. Thus X remains a positive local martingale under the measure Q. We show that X is a true Q-martingale by proving that X has constant expectation under Q. By following the same technique as before we obtain

$$\mathbb{E}_Q[X_T] = \tilde{Q}(\tau_\infty^v > T),$$

where $\tilde{Q}$ is an equivalent measure to Q under which v satisfies the equation

$$dv_t = \epsilon v_t d\tilde{B}_t^{3,\tilde{Q}} + (-kv_t + \epsilon\rho(\alpha_1+\alpha_2+\lambda_1+\lambda_2)v_t^{\frac{3}{2}})dt,$$

and $(\tilde{B}_t^{3,\tilde{Q}})_{t\in[0,T]}$ is a $\tilde{Q}$-Brownian motion. By replacing α_1 and α_2 from (3.2.2) into the equation of v we obtain

$$dv_t = \epsilon v_t d\tilde{B}_t^{3,\tilde{Q}} + [-kv_t + \epsilon\rho((2-\rho^2)\lambda_2 + \rho^2\lambda_1)v_t^{\frac{3}{2}}]dt,$$

and $(2-\rho^2)\lambda_2 + \rho^2\lambda_1 \leq 0$ since $\lambda_1 \leq -\frac{2-\rho^2}{\rho^2}\lambda_2$. For the sake of brevity we denote $\eta := (2-\rho^2)\lambda_2 + \rho^2\lambda_1$. The scale function of v is equal to

$$p_3(x) := \int_1^x \exp\Big(-2\int_1^y \frac{-kz + \eta\rho z^{\frac{3}{2}}}{\epsilon^2 z^2}dz\Big)dy = \tilde{K}\int_1^x y^{\frac{2k}{\epsilon^2}} e^{-\frac{\rho\eta}{\epsilon}y^{\frac{1}{2}}}dy.$$

Since

$$0 \leq \int_1^x y^{\frac{2k}{\epsilon^2}}dy \leq \int_1^x y^{\frac{2k}{\epsilon^2}} e^{-\frac{\rho\eta}{\epsilon}y^{\frac{1}{2}}}dy,$$

for all $x \geq 1$, it follows

$$p_3(+\infty) = \lim_{x\to\infty} \tilde{K} \int_1^x y^{\frac{2k}{\epsilon^2}} e^{-\frac{\rho n}{\epsilon} y^{\frac{1}{2}}} dy = +\infty,$$

which implies via Problem 5.27 in [41] that $\lim_{x\to\infty} u_3(x) = +\infty$. As before, one can show that $\lim_{x\to 0+} u_3(x) = +\infty$. Hence the boundary $+\infty$ is unattainable in finite time, therefore $\tilde{Q}(\tau^v_\infty = +\infty) = 1$, which implies $\mathbb{E}_Q[X_T] = 1$, and this concludes our proof. □

As in the previous setting, we consider a financial asset that pays no dividends and gives a final payoff X_T at time T and whose price process S is given by $S_t = X_t$ for all $t < T$ and $S_T = 0$.
The wealth process is equal to $W_t = X_t$ for all $t \in [0, T]$. Theorem 3.2.1 provides us with the two measures $Q \in \mathcal{M}_{UI}(W)$ and $R \in \mathcal{M}_{NUI}(W)$ and we check if the bubble process $\beta^{\mathcal{R}}$ exhibits a submartingale behavior under R. We start by computing the fundamental value W^R under R, given by

$$W^R_t = \mathbb{E}_R[W_T|\mathcal{F}_t] = \mathbb{E}_R[X_T|\mathcal{F}_t].$$

Proposition 3.2.3. *The process W^R admits the multiplicative decomposition $W^R = W \cdot C$, where the semimartingale C defined as $C_t = \mathbb{E}_P[\frac{X_T}{X_t}|\mathcal{F}_t]$ is equal to*

$$C_t = 1 + (\lambda_1 g_1(t) + \lambda_2 g_2(t))v_t, \quad t \in [0, T],$$

with time dependendent coefficients $g_i(t)$ which satisfy

$$\lambda_1 g_1(t) + \lambda_2 g_2(t) < 0, \quad t \in [0, T).$$

Proof. The desired result is obtained by following the same steps in the proof of Proposition 3.1.2. □

Let us now find the canonical semimartingale decomposition of C. We know that

$$C_t = 1 + (\lambda_1 g_1(t) + \lambda_2 g_2(t))v_t, \quad t \in [0, T].$$

Therefore

$$\begin{aligned} dC_t &= (\lambda_1 g_1(t) + \lambda_2 g_2(t))dv_t + \lambda_1 v_t dg_1(t) + \lambda_2 v_t dg_2(t) \\ &= \epsilon(\lambda_1 g_1(t) + \lambda_2 g_2(t))v_t dB^3_t + (\lambda_1 v_t dg_1(t) + \lambda_2 v_t dg_2(t) \\ &\quad - k(\lambda_1 g_1(t) + \lambda_2 g_2(t))v_t)dt \end{aligned}$$

Hence, the local martingale part of C is being given by

$$M^C_t = \int_0^t \epsilon(\lambda_1 g_1(s) + \lambda_2 g_2(s))v_s dB^3_s, \quad t \in [0, T].$$

Proposition 3.2.4. *The bubbles process $\beta^{\mathcal{R}}$ is a local R-submartingale.*

Proof. According to Corollary 2.3.10, in order for the bubble to exhibit a local R-submartingale behavior it is sufficient that $[C, M]$ is an increasing process. We remind that in this example the density process M is given by

$$M_t = 1 + \int_0^t \alpha_1 \sqrt{v_s} M_s dW_s^1 + \int_0^t \alpha_2 \sqrt{v_s} M_s dW_s^2, \quad t \in [0, T].$$

Hence

$$[M, C]_t = [M, M^C]_t = \int_0^t (\lambda_1 g_1(s) + \lambda_2 g_2(s)) \epsilon \rho (\alpha_1 + \alpha_2) v_s^{\frac{3}{2}} M_s ds$$

and since $(\lambda_1 g_1(s) + \lambda_2 g_2(s))(\alpha_1 + \alpha_2) > 0$, this concludes our proof. $\square$

Chapter 4

The formation of financial bubbles in defaultable markets

The contents of Sections 4.2, 4.3 and 4.4 of this Chapter are based on the author's joint work with F. Biagini, and are contained in the manuscript F. Biagini and **S. Nedelcu** [5]. The detailed description of the author's personal contribution is presented in Section 1.2. Section 4.4.1 of the present Chapter was developed independently by the author and is based on a manuscript which is not yet published.

4.1 Introduction

Aim of this Chaper is to construct a mathematical model that allows for the formation of bubbles in the valuation of defaultable contingent claims in a reduced form credit risk model. Furthermore, we establish a connection between the classical martingale theory of bubbles and the constructive approach to the modelling of asset price bubbles which we propose in this Chapter.

Credit risk models have been introduced to describe financial markets affected by default risk, see Bielecki and Rutkowski [7] for an overview of all the main approaches. A default risk represents the possibility that a counterparty is not able to fullfill its obligations that are stated in a financial contract. An example of financial intruments that are affected by default risk are corporate bonds. These are financial products issued by a firm,that, in exchange of a fee for this commitment, take the obligation to make specific payments at future dates to the buyers of the corporate bonds. The firm may be forced to default on its commitment due to different circumstances (for example bankruptcy). This event causes losses for the bondholders if it occurs during the lifetime of the bond i.e. the period between the moment when it is issued and its maturity. A corporate bond represents an example of *defaultable claim.*

In a reduced form setting, see for example Bielecki and Rutkowski [7], the firm default time is represented by a totally inaccessible stopping time, which implies that the market cannot predict the time of the default. However, the default time may not be a stopping time with respect to the restricted filtration. The distinction between the information that is available to the market and information that is known within the firm (where the moment of default is an accessible stopping time) has been studied and pointed out in the literature. Duffie and Lando [22] consider that the bond investors do not observe the bond issuer's assets directly, but receive instead only periodic accounting reports. Thus, investors draw their information from delayed financial reports and from other publicly available data. In this way the market observes the values of the firm's assets plus additional noise, due to the lack of perfect information.

Cetin et al. [14] provide a different approach, where they construct a reduced form model starting from a structural model. They consider two information sets. In their approach the firm's management can anticipate the default model by examining the firm's cash flows. In constrast, the market observes only a partition of the manager's information set. This structure has the default time as being an accessible stopping time with respect to the filtration representing the information available for the firm's manager and a totally inaccesible stopping time with respect to the filtration representing the information available to the market.

We consider here a classical reduced form setting, as discussed in Chapter 8 of Bielecki and Rutkowski [7], and consider a market model with constant money market accout and the possibility of investing in defaultable claims, i.e. contracts traded over the counter between default-prone parties.

Our objective is to provide an explanation for the mechanism that leads to a change of the martingale measure used for pricing. We do this by modelling the market price, which is influenced by the asset's probability of default as perceived by the investor. As we point out in Chapter 2, the study of asset price bubbles in the mathematical literature has been done in the framework of the martingale approach, see Jarrow, Protter et.al [39], [40], [36], [34], [37] and Biagini, Föllmer and Nedelcu [4]. In an incomplete financial market model a bubble is generated after the start of the model if a switch from a uniformly integrable martingale measure Q to a non-uniformly integrable martingale measure R occurs. Thus the dynamics of the discounted wealth process is described by a uniformly integrable martingale under Q and afterwards by a non-uniformly integrable martingale/strict local martingale under

R.

As pointed out in Chapter 1, in the microeconomic theory of bubble formation, factors such as limits of arbitrage and investor heterogeneity are often seen as triggering the formation of asset price bubbles. Here we focus on the investor heterogeneity concept as possible explanation for the creation of bubbles in our setting. Divergence of opinions may arise due to the fact that investors have different estimations of the value of future dividends (see e.g.Harrison and Kreps [30]), overconfidence (see Scheinkman and Xiong [55]), or due to their use of different economical indicators to forecast the future price of the asset (see Föllmer et al.[25]).

A first attempt to explain the formation of asset price bubbles with a constructive approach in the arbitrage-free pricing methodology is presented in Jarrow, Protter and Roch [38]. Here a bubble is generated in the price of a liquid financial asset traded through a limit order book via market trading activity (volume of market orders, liquidity, etc). However, unlike in the classical martingale theory of bubbles, in [38] the asset's intrinsic value is exogenously determined while the asset's price bubble (and thus also the asset's market price) is endogenously determined by the impact of liquidity risk. The aim of the present Chapter is to costruct a mathematical model that can identify possible triggers generating a bubble in the price of defaultable claims. Therefore, our setting and methods are completely different from the ones in [38]. For the first time bubble generation is examined in the context of defaultable markets. Furthermore, we are also able to show how microeconomic factors may at an aggregate level determine a shift of the martingale measure by characterizing the set of equivalent martingale measures with the help of measure pasting, see Theorem 4.5.6. Moreover, our model allows for the succesive creation and dissappearance of bubbles within the price of the asset until the time of maturity.

The present Chapter has the following outline. In Section 4.2 we describe the setting. We place ourselves in a reduced form setting credit risk model, see Bielecki and Rutkowski [7]. Our model includes the possibility of investing in defaultable claims, i.e. contingent agreements traded over-the-counter between default-prone parties. A defaultable contingent claim is a triplet $H = (X, R, \tau)$, where X is the promised claim, R is the recovery process and τ is the default time. For the sake of simplicity, the money market account is supposed to be constantly equal to one.

We model the impact of investors' heterogeneous beliefs on the market wealth of a defaultable claim in the following way: at the starting moment of our model, a given defaultable contingent claim $H = (X, R, \tau)$ is evaluated by using the underlying pricing measure, as it is usual in the reduced form setting, see Definition 8.1.2 in [7]. After a certain time, the claim will be to be considered a safe investment, if the conditional probability of default in the remaining time interval goes below a certain threshold $p \in (0, 1)$, with $p < P(0 < \tau \le T)$. The trading activity of the investors willing to buy the claim determines a deviation from the initially estimated wealth via a factor f, which is a function of time and of the credibility process introduced in Definition 4.2.4. This construction also allows us to define the novel concept of *observed asset price bubble*, which represents the difference between the modified wealth process (called market wealth process) and the initially estimated wealth process.

Section 4.3 integrates our model in the classical martingale theory of bubbles. We provide conditions when an increase in the market wealth, due to investors' trading activity can lead to an increase in the asset's fundamental value, computed with respect to a corresponding martingale measure.

In Section 4.4, we construct a reduced -form model where the stochastic intensity associated to the default time τ is driven by a Cox-Ingersoll-Ross model. Within this model we illustrate the results of the previous sections.

Section 4.5 establishes a connection between our approach and the martingale theory of bubbles, as described by Cox and Hobson [18], Jarrow, Protter and Shimbo [40] and Biagini, Föllmer and Nedelcu [4]. To this purpose we characterize the set of equivalent martingale measures for the market wealth process W of a defaultable claim via measure pasting, see Definition 4.5.2. This concept, as introduced in Section 6.4 of Föllmer and Schied [27], allows us to provide a characterization of the equivalent martingale measures for W in the following way: if σ_1 denotes the starting moment of the influence of the credibility process on the contract value, Theorem 4.5.6 shows that all equivalent martingale measures for W are given by the pasting in σ_1 of an equivalent martingale measure for the initially estimated wealth up to σ_1 with an equivalent martingale measure for $(W - W_{\sigma_1})1_{\{\cdot \ge \sigma_1\}}$, if $0 < \sigma_1 < T$. This result justifies in a rigorous mathematical way how a shift in the underlying equivalent martingale measure is determined by a change in the wealth process due to the impact of microeconomic factors.

4.2 The Setting

Let $(\Omega, \mathcal{G}, \mathbb{F}, P)$ be a filtered probability space where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions of right-continuity and completeness. Let $T > 0$ be a fixed time horizon. We consider a market model that contains a defaultable asset with maturity date T and a money market account. We will use the money market account as numéraire, and so we may assume that it is constantly equal to 1. The default time of the asset is represented by a non-negative $\mathcal{G}$-measurable random variable $\tau : \Omega \to [0, +\infty]$, with $P(\tau = 0) = 0$ and $P(\tau > t) > 0$, for each $t \in [0, T]$. Note that the random time τ may not be an $\mathbb{F}$-stopping time.

For the default time τ, we introduce the associated jump process $H = (H_t)_{t\in[0,T]}$ given by $H_t = 1_{\{\tau\le t\}}$, $t \in [0, T]$. We refer to H as the *default process* associated to the default time τ. It is obvious that H is right-continuous. We denote by $\mathbb{H} = (\mathcal{H}_t)_{0\le t\le T}$ the filtration generated by the process H, i.e. $\mathcal{H}_t = \sigma(H_u; u \le t)$ for any $t \in [0, T]$.

Let $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ be the filtration obtained by progressively enlarging the filtration $\mathbb{F}$ with the random time τ, i.e. $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. For the sake of simplicity we assume $\mathcal{G}_0 = \mathcal{F}_0 = \mathcal{H}_0 = \{\emptyset, \Omega\}$ and $\mathcal{G} = \mathcal{G}_T = \mathcal{F}_T \vee \mathcal{H}_T$. The filtration $\mathbb{G}$ is the smallest filtration satisfying the usual conditions containing the original filtration $\mathbb{F}$ and for which τ is a stopping time. In the credit risk literature $\mathbb{G}$ is the filtration containing the information relevant to the market and is used for the pricing and hedging of defaultable claims.
We denote by $\mathcal{Z} = (\mathcal{Z}_t)_{t\in[0,T]}$ the $\mathbb{F}$-*survival process* of τ with respect to the filtration $\mathbb{F}$ defined by

$$\mathcal{Z}_t = P(\tau > t|\mathcal{F}_t), \quad t \in [0, T],$$

and chosen to be càdlàg. The process $\mathcal{Z}$ is a bounded, non-negative $\mathbb{F}$-supermartingale under P. In the credit risk literature the process Z is also referred to as the Azéma $\mathbb{F}$-supermartingale, see Coculescu et al.[16]. We assume that $\mathcal{Z}_t > 0$ for all $t \in [0, T]$.

Definition 4.2.1. *The process* $\Gamma = (\Gamma_t)_{t\in[0,T]}$ *defined by*

$$\Gamma_t = -\ln \mathcal{Z}_t = -\ln P(\tau > t|\mathcal{F}_t), \quad t \in [0, T]$$

is called the $\mathbb{F}$-hazard process *of* τ *under* P.

Since $\mathcal{Z}_t > 0$ this implies that Γ is well defined for every $t \in [0, T]$. We make the following Assumptions for the rest of the Chapter:

Assumption 4.2.2. *We consider that:*

i) The immersion property holds under the measure P, i.e. all $(\mathbb{F}, P)$-martingales are also $(\mathbb{G}, P)$-martingales.

ii) The hazard process Γ admits the representation

$$\Gamma_t = \int_0^t \mu_s ds, \quad t \in [0,T],$$

where $\mu = (\mu_t)_{t\in[0,T]}$ is an $\mathbb{F}$-adapted process such that $\int_0^t \mu_s ds < \infty$ a.s. for all $t \in [0,T]$. The process μ is called the stochastic intensity *or* hazard rate *of τ*

Note that the existence of the intensity implies that τ is a totally inaccessible $\mathbb{G}$-stopping time. If the immersion property holds, by Corollary 3.9 of Coculescu et al.[16] we obtain that the Azéma supermartingale $\mathcal{Z}$ is a decreasing process. Hence Γ is increasing, which implies that $(\mu_t)_{t\in[0,T]}$ is a non-negative process. Furthermore, since the Azéma supermartingale $\mathcal{Z}$ is continuous and decreasing, by Corollary 3.4 of Coculescu and Nikeghbali [17] we have that τ avoids all $\mathbb{F}$-stopping times.
The fact that $\mathcal{Z}$ is continuous and decreasing also implies that the process $(\Gamma_{t\wedge\tau})_{t\geq 0}$ represents the P-compensator for the default process H, see Proposition 5.1.3 of Bielecki and Rutkowski [7] and Section 1.3 of Coculescu and Nikeghbali [17]. Therefore the compensated process $\hat{M} = (\hat{M}_t)_{t\in[0,T]}$ defined by

$$\hat{M}_t := H_t - \int_0^{t\wedge\tau} \mu_s ds = H_t - \int_0^t \hat{\mu}_s ds, \quad t \in [0,T].$$

is a $\mathbb{G}$-martingale, see Proposition 5.1.3 of Bielecki and Rutkowski [7]. Notice that for the sake of brevity we put $\hat{\mu}_t := \mu_t 1_{\{\tau\geq t\}}$.

Definition 4.2.3. *A defaultable claim is given by a triplet $H = (X, R, \tau)$, where:*

1. *the* promised contingent claim *$X \in L^1(\mathcal{F}_T)$ represents the non-negative payoff received by the owner of the claim at time T, if there was no default prior to or at time T.*

2. *the* recovery process *R represents the recovery payoff at time τ of default if default occurs prior to or at the maturity date T, and it is assumed to be a strictly positive, continuous, $\mathbb{F}$-adapted process that satisfies*

$$\mathbb{E}_P\Big[\sup_{t\in[0,T]} R_t\Big] < \infty. \tag{4.2.1}$$

3. the $\mathbb{G}$-stopping time τ represents the default time.

We assume that the underlying probability measure P is a martingale measure. The existence of P implies that there is no free-lunch with vanishing risk, see [19]. We consider an intensity-based model for the valuation of defaultable contingent claims, as described in Chapter 8 of Bielecki and Rutkowski [7]. By using Definition 8.1.2 of [7] we set

$$W_t^e = \mathbb{E}_P[X1_{\{\tau>T\}} + R_\tau 1_{\{\tau\leq T\}}|\mathcal{G}_t], \quad t \in [0,T],$$

to be the *estimated wealth process* associated to a defaultable claim $H = (X, R, \tau)$ under P. Given

$$\Lambda_t := \mathbb{E}_P[X1_{\{\tau>T\}} + R_\tau 1_{\{\tau\leq T\}}|\mathcal{G}_t]1_{\{\tau>t\}}, \quad t \in [0,T]. \tag{4.2.2}$$

then we can rewrite W^e as

$$W_t^e = \Lambda_t + R_\tau 1_{\{\tau\leq t\}}, \quad t \in [0,T].$$

In the sequel, we model the impact of the trading activity of heterogeneous investors on the initial estimated wealth process of the asset. The investors may consider the defaultable claim to be a safe investment if some circumstances are verified, as we explain more in detail below. To this aim we first introduce the following notion of *credibility process* which will play a crucial role in our discussion.

Definition 4.2.4. *For any $t \leq T$ the credibility process $F = (F_t)_{t\in[0,T]}$ is defined as*

$$F_t = P(t < \tau \leq T|\mathcal{G}_t),$$

for all $t \in [0,T]$.

It is easy to see that credibility process has the following dynamics:

Lemma 4.2.5. *The process $F = (F_t)_{t\in[0,T]}$ is a $(\mathbb{G}, P)$-supermartingale.*

Proof. This immediately follows since F can be written in the form

$$F_t = \mathbb{E}_P[1_{\{\tau\leq T\}}|\mathcal{G}_t] - 1_{\{\tau\leq t\}}.$$

□

Let

$$f : [0,T] \times (0,1] \to [1,\infty) \tag{4.2.3}$$

be a deterministic function in $C^{1,2}([0,T] \times (0,1])$. Fix $p \in (0,1)$ with $p < P(0 < \tau \leq T)$. We assume that

i) For all $t \in [0,T]$, $f(t,x) = 1$ for all $x \geq p$ and $f(t,x) > 1$ for $x < p$,

ii) f is strictly decreasing in both arguments for $x < p$.

Definition 4.2.6. *The* market wealth process $W = (W_t)_{t\in[0,T]}$ *of the defaultable asset is defined as*

$$W_t = f(t, F_t)\Lambda_t + R_\tau 1_{\{\tau \leq t\}}, \tag{4.2.4}$$

for all $t \in [0,T]$.

The process Λ represents the defaultable claims' initial price estimation made by the investors. However, if at time t, the claim's conditional probability of default F_t goes below the threshold $p \in (0,1)$, then the asset is perceived as a safe investment (i.e. the asset becomes "credible" enough). The credibility process F can be seen as an indicator capturing the views of a large investor who purchases the claim when F goes below the threshold p and whose trades can affect the stock price. Everyone in the market will follow the large investor, thus generating a possible bubble.

The impact of the fluctuations of the credibility process F on the initial value estimation Λ is quantified by the function $f(t, F_t)$. Note that in our model, Λ is also influenced at a time t when $F_t < p$, by the length of the remaining time interval $[t,T]$ to maturity. Since the investors are aware of the supermartingale property of F, see Proposition 4.2.5, the perception of the asset as being safe at an earlier date impacts the price in a more significant way than at a later date, i.e. if $F_{t_1} = F_{t_2}$ for $t_1 < t_2$, then $f(t_1, F_{t_1}) > f(t_2, F_{t_2})$.

Hence, our model is able to capture the fact that the views of optimistic investors are expressed more fully than the ones of pessimists and that *prices are biased upwards*, see Scheinkman [54]. Of course, other explanations for this model are of course possible, see for example Brunnermeyer and Oehmke[11], Hugonnier [32], Scheinkman [54].

An important consequence of this construction is the fact that the changes in the dynamics of the underlying price process may lead to a different selection of the martingale measure used for pricing, as we point out in Section 4.3.

Remark 4.2.7. *The impact of the credibility affects the value of the defaultable asset only strictly prior to the default time τ. If τ occurs before or at T, the recovery payment R_τ will be paid, because this is settled a priori in the contractual agreement underlying the claim. Hence R is not influenced by the credibility process. Furthermore, at the time of maturity T, when the* *promised claim X must be delivered according to previously agreed contractual obligations if no default occurs, we have*

$$W_T = X1_{\{\tau>T\}} + R_\tau 1_{\{\tau\le T\}} = W_T^e,$$

i.e . at time $t = T$ there will be no difference between the asset's market wealth and its initial estimated wealth.

Remark 4.2.8. *The martingale theory of bubbles does not allow for the existence of bubbles in the price of assets with bounded payoffs. Therefore, in our setting, the payoff of the defaultable claim (and of the corresponding wealth process associated to the claim) must not be upper bounded. A possible way of avoiding this model limitation in order to include the treament of defaultable bonds is by introducing the concept of* relative asset price bubble, *see Bilina-Falafala, Jarrow and Protter [8].*

In the sequel we denote by σ_1 the starting moment of the influence of F on Λ, i.e.

$$\sigma_1 := \inf\{t \in [0,T]; F_t < p\}. \qquad (4.2.5)$$

Note that $\sigma_1 \le \tau$, see also Proposition 4.3.4.

4.3 Bubbles in defaultable claim valuation

As in Chapter 2, we denote by $\mathcal{M}_{loc}(W)$ the space of probability measures $Q \approx P$ defined on $(\Omega, \mathbb{G})$, under which the market wealth process W is a $(\mathbb{G}, Q)$-local martingale and consider

$$\mathcal{M}_{loc}(W) = \mathcal{M}_{UI}(W) \cup \mathcal{M}_{NUI}(W),$$

where $\mathcal{M}_{UI}(W)$ denotes the collection of measures $Q \approx P$ that render W a uniformly integrable martingale and $\mathcal{M}_{NUI}(W) = \mathcal{M}_{loc}(W) \setminus \mathcal{M}_{UI}(W)$. Typically, there exist frameworks where the classes $\mathcal{M}_{UI}(W)$ and $\mathcal{M}_{NUI}(W)$ can be simultaneously non-empty, see Section 2.4 and the examples in Chapter 3.

Remark 4.3.1. *Here $\mathcal{M}_{loc}(W) \cap \mathcal{M}_{loc}(W^e) = \emptyset$ if $0 < \sigma_1 < T$. By contradiction we assume that there exists $Q \in \mathcal{M}_{loc}(W) \cap \mathcal{M}_{loc}(W^e)$. Then, since $W_T = W_T^e$ and $W_0^e = W_0$, this would imply*

$$0 = \mathbb{E}_Q[W_T - W_T^e] < \mathbb{E}_Q[W_{\sigma_1+\epsilon} - W_{\sigma_1+\epsilon}^e] \leq \mathbb{E}_Q[W_0 - W_0^e] = 0,$$

for any $\epsilon > 0$ such that $\sigma_1 + \epsilon < T$, which is of course not possible.

In particular the measure P is an equivalent local martingale measure only for the process W^e and not for the process W.

Definition 4.3.2. *Let $Q \in \mathcal{M}_{loc}(W)$. The process W^Q defined by*

$$W_t^Q = \mathbb{E}_Q[X 1_{\{\tau > T\}} + R_\tau 1_{\{\tau \leq T\}} | \mathcal{G}_t], \quad t \in [0, T],$$

is called the fundamental wealth process *of the defaultable claim perceived under the measure Q.*

Note that

$$W_t \geq \mathbb{E}_Q[W_T | \mathcal{G}_t] = \mathbb{E}_Q[W_T^Q | \mathcal{G}_T] = W_t^Q$$

for any $Q \in \mathcal{M}_{loc}(W)$, with strict inequality if W is a strict local martingale under Q. We introduce the definition of a Q-bubble, which coincides with Definition 2.2.11, see also Jarrow et al.[40].

Definition 4.3.3. *For any $Q \in \mathcal{M}_{loc}(W)$, the non-negative adapted process $\beta^Q = (\beta_t^Q)_{t \in [0,T]}$ defined by*

$$\beta_t^Q = W_t - W_t^Q \geq 0,$$

is called the bubble *perceived under the measure Q or Q-bubble.*

As in Chapter 2, the existence and the size of the Q-bubble β^Q depends on the choice of the martingale measure. If $Q \in \mathcal{M}_{UI}(W)$, then $\beta^Q = 0$. For $Q \in \mathcal{M}_{NUI}(W)$, the Q-bubble is a non-negative local martingale with $\beta^Q_T = 0$. Furthermore it is also clear that there is no bubble at time T, since at time of maturity the asset X must be delivered according to contractual obligations, see also Remark 4.2.7. The following Proposition shows that the market wealth process exhibits no bubbles after the time of defaults default.

Proposition 4.3.4. *Let $\tau < T$ a.s. Then for any $Q \in \mathcal{M}_{loc}(W)$*

$$\beta^Q_t = W_t - W^Q_t = 0, \quad t \in [\tau, T].$$

Proof. Let $Q \in \mathcal{M}_{loc}(W)$. Then

$$\begin{aligned} W^Q_t 1_{\{\tau \le t\}} &= \mathbb{E}_Q[X 1_{\{\tau > T\}} + R_\tau 1_{\{\tau \le T\}} | \mathcal{G}_t] 1_{\{\tau \le t\}} \\ &= \mathbb{E}_Q[X 1_{\{\tau > T\}} | \mathcal{G}_t] 1_{\{\tau \le t\}} + \mathbb{E}_Q[R_\tau 1_{\{\tau \le T\}} | \mathcal{G}_t] 1_{\{\tau \le t\}} \\ &= \mathbb{E}_Q[R_\tau 1_{\{\tau \le t\}} | \mathcal{G}_t] = R_\tau 1_{\{\tau \le t\}} = W_t 1_{\{\tau \le t\}}. \end{aligned} \tag{4.3.1}$$

□

Definition 4.3.3 introduces the concept of bubble as in the approach of Jarrow et al.[40]. We now propose an alternative way of defining a bubble for defaultable claims, which captures the market wealth's divergence from the initial estimation W^e.

Definition 4.3.5. *For any $t \in [0, T]$, we define the* observed bubble $\beta^o = (\beta^o_t)_{t \in [0,T]}$ *by*

$$\beta^o_t = W_t - W^e_t = (f(t, F_t) - 1)\Lambda_t 1_{\{\tau > t\}}, \quad t \in [0, T]. \tag{4.3.2}$$

The *observed bubble* represents the difference between the market wealth W and the estimated wealth W^e, which is induced by the trading behavior of a large and influential group of investors. This behavior is triggered by the sentiment of safe investment due to the perception of a low default risk associated to the claim.

Remark 4.3.6. *The choice of the function f is arbitrary. It could be also chosen to include the appearance of negative observed bubbles. It may happen that for some reason a particular asset is not seen as trustworthy by a consistent number of investors. In this case there could be a decrease in the asset's market value that may not be motivated by the underlying economic and financial conditions.*

In the following, we examine the connection between the observed bubble β^o and a Q-bubble for a given $Q \in \mathcal{M}_{loc}(W)$. By using Definition 4.3.3, we rewrite (4.3.2) as the following sum

$$\begin{aligned}\beta_t^o &= W_t - W_t^e = (W_t - W_t^Q) + (W_t^Q - W_t^e) \\ &= \beta_t^Q + (W_t^Q - W_t^e) \geq 0.\end{aligned}$$

In particular if $Q \in \mathcal{M}_{UI}(W)$, then $\beta_t^Q = 0$ and

$$\beta_t^o = W_t^Q - W_t^e = W_t - W_t^e \geq 0, \quad t \in [0, T].$$

A change in the dynamics of the market wealth from its initial dynamics represented by W^e leads to the creation of an observed bubble. However, this may not create a bubble in the martingale sense if the market selects as pricing measure corresponding to the wealth process W, a measure from the set $\mathcal{M}_{UI}(W)$.

The observed bubble can be regarded as the sum of two components: the possible bubble generated by a choice of a martingale measure belonging to the set $\mathcal{M}_{NUI}(W)$ and the difference between the intrinsic value of the asset and its initial estimated value. Under certain conditions, an increase in the asset's market wealth can lead to an increase of its' fundamental value making the second component of the bubble non-negative. The following proposition provides a sufficient condition under which the estimated value of the asset will be surpassed by its fundamental value.

Proposition 4.3.7. *Let $Q \in \mathcal{M}_{loc}(W)$ with Radon-Nikodym density process $Z = (Z_t)_{t\in[0,T]}$ i.e $Z_t = \frac{dQ}{dP}|_{\mathcal{G}_t}$, $t \in [0, T]$. If the process $W^e Z$ is a P-submartingale, then*

$$W_t^Q \geq W_t^e, \tag{4.3.3}$$

for all $t \in [0, T]$.

Proof. By applying Bayes' theorem we obtain

$$\begin{aligned}W_t^Q - W_t^e &= \mathbb{E}_Q[X 1_{\{\tau>T\}} + R_\tau 1_{\{\tau\leq T\}}|\mathcal{G}_t] - \mathbb{E}_P[X 1_{\{\tau>T\}} + R_\tau 1_{\{\tau\leq T\}}|\mathcal{G}_t] \\ &= \frac{1}{Z_t}\mathbb{E}_P[(X 1_{\{\tau>T\}} + R_\tau 1_{\{\tau\leq T\}}) Z_T|\mathcal{G}_t] \\ &- \mathbb{E}_P[X 1_{\{\tau>T\}} + R_\tau 1_{\{\tau\leq T\}}|\mathcal{G}_t] \\ &= \frac{1}{Z_t}(\mathbb{E}_P[W_T^e Z_T|\mathcal{G}_t] - W_t^e Z_t).\end{aligned}$$

Therefore it is sufficient that $W^e Z$ is a $(\mathbb{G}, P)$-submartingale for (4.3.3) to hold. □

4.4 The case of CIR default intensity

We now illustrate the concepts presented in Sections 4.2 and Section 4.3 by considering a specific setting.

Let $(\Omega, \mathcal{G}, \mathbb{F}, P)$ be a filtered probability space endowed with a 2-dimensional $\mathbb{F}$-Brownian motion $B = (B^1, B^2)$ and we assume that $\mathbb{F} = \mathbb{F}^1 \vee \mathbb{F}^2$, where $\mathbb{F}^1$ and $\mathbb{F}^2$ are the natural filtrations associated to B^1 and B^2 respectively. As in the previous section, let the random time of default be represented by a non-negative $\mathcal{G}$-measurable random variable $\tau : \Omega \to [0, +\infty]$, with $P(\tau = 0) = 0$ and $P(\tau > t) > 0$, for each $t \in [0, T]$. We introduce the right-continuous process $(H_t)_{t\in[0,T]}$ by setting $H_t = 1_{\{\tau \le t\}}$ and we denote by $\mathbb{H} = (\mathcal{H}_t)_{0\le t\le T}$ the associated filtration. Let $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ be the filtration obtained by progressively enlarging the filtration $\mathbb{F}$ with the random time τ, i.e. $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. For the sake of simplicity we assume $\mathcal{G}_0 = \mathcal{F}_0 = \mathcal{H}_0 = \{\emptyset, \Omega\}$ and $\mathcal{G} = \mathcal{G}_T = \mathcal{F}_T \vee \mathcal{H}_T$.

Since the immersion property holds, the process $(B^i_t)_{t\ge 0}$ is a continuous $\mathbb{G}$-martingale, for $i = 1, 2$. Furthermore, an application of Levy's characterization theorem (see Theorem II.39 in Protter [52]) yields that $(B^i_t)_{t\ge0}$ is a $\mathbb{G}$-Brownian motion, for $i = 1, 2$. Therefore $B = (B^1, B^2)$ remains a 2-dimensional Brownian motion in the enlarged filtration $\mathbb{G}$.

Let $X = (X_t)_{t\in[0,T]}$ be a process satisfying the following dynamics

$$dX_t = \sigma X_t dB^2_t, \quad X_0 = x_0, \tag{4.4.1}$$

with $x_0 \in \mathbb{R}_+$. It is easy to obtain the closed-form expression of X. The (unique) process X satisfying (4.4.1) is the Doléans-Dade exponential

$$X_t = x_0 \exp(\sigma B^2_t - \frac{1}{2}\sigma^2 t), \quad t \in [0, T].$$

We consider the defaultable contingent claim given by the triplet (X, R, τ), where the promised contingent claim is given by $X = X_T$ and the recovery process $R = (R_t)_{t\in[0,T]}$ is defined by setting $R_t = cX_t$ for all $t \in [0, T]$ and some $c \in (0, 1)$. By applying Doob's maximal inequality to the martingale X we obtain

$$\begin{aligned}
\mathbb{E}_P[\sup_{t\in[0,T]} R_t] &\le c\gamma P(\sup_{t\in[0,T]} X_t \le \gamma) + c\mathbb{E}_P[\sup_{t\in[0,T]} X_t 1_{\{\sup_{t\in[0,T]} X_t > \gamma\}}] \\
&\le c\gamma(1 - P(\sup_{t\in[0,T]} X_t > \gamma)) + c\mathbb{E}_P[\sup_{t\in[0,T]} X_t^2]^{\frac{1}{2}} P(\sup_{t\in[0,T]} X_t > \gamma)^{\frac{1}{2}} \\
&\le c\gamma\big(1 - \frac{1}{\gamma^2} x_0 e^{\sigma^2 T}\big) + \frac{4c}{\gamma^2} x_0 e^{\sigma^2 T} < \infty.
\end{aligned}$$

for some $\gamma > 0$. Hence the recovery process R satisfies (4.2.1). All through this section we assume that $0 < \sigma_1 < T$, P-a.s.

We assume that the stochastic intensity μ associated to the default time τ is given by a Cox-Ingersoll-Ross model

$$\begin{aligned} d\mu_t &= (a + b\mu_t)dt + \theta\sqrt{\mu_t}dB_t^1, \\ \mu_0 &= \tilde{\mu}, \end{aligned} \tag{4.4.2}$$

where $a, \theta \in \mathbb{R}_+$, $\tilde{\mu} > 0$ and $b \in \mathbb{R}$. The process μ is an affine process.

Let us compute the dynamics of the credibility process F under P within this framework. To this purpose, we start with the following auxiliary results.

Lemma 4.4.1. *The process $(\tilde{F}_t)_{t\geq 0}$ given by*

$$\tilde{F}_t = P(t < \tau \leq T|\mathcal{F}_t), \quad t \in [0, T],$$

satisfies under P the following equation

$$d\tilde{F}_t = -\psi_t\sqrt{\mu_t}dB_t^1 - e^{-\Gamma_t}\mu_t dt, \quad t \in [0, T], \tag{4.4.3}$$

where $(\psi_t)_{t\in[0,T]}$ is given by the formula

$$\psi_t = \theta\beta(t)e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t}, \tag{4.4.4}$$

with

$$\alpha(t) = \frac{2a}{c^2}\ln\Big(\frac{2\lambda e^{\frac{(\lambda-b)(T-t)}{2}}}{(\lambda - b)(e^{\lambda(T-t)} - 1) + 2\lambda}\Big), \tag{4.4.5}$$

and

$$\beta(t) = -\frac{2(e^{\lambda(T-t)} - 1)}{(\lambda - b)(e^{\lambda(T-t)} - 1) + 2\lambda}, \tag{4.4.6}$$

for all $t \in [0, T]$ with $\lambda := \sqrt{b^2 + 2\theta^2}$.

Proof. By using the definition of the $\mathbb{F}$-hazard process, F can be be written under the form

$$\begin{aligned} \tilde{F}_t &= P(t < \tau \leq T|\mathcal{F}_t) = P(\tau \leq T|\mathcal{F}_t) - P(\tau \leq t|\mathcal{F}_t) \\ &= 1 - P(\tau > T|\mathcal{F}_t) - 1 + P(\tau > t|\mathcal{F}_t) = e^{-\Gamma_t} - \mathbb{E}_P[e^{-\Gamma_T}|\mathcal{F}_t] \\ &= e^{-\Gamma_t}(1 - \mathbb{E}_P[e^{-(\Gamma_T-\Gamma_t)}|\mathcal{F}_t]). \end{aligned}$$

Since μ is an affine process, e.g. by Filipovic [23], we have that

$$\tilde{F}_t = e^{-\Gamma_t}(1 - \mathbb{E}_P[e^{-(\Gamma_T-\Gamma_t)}|\mathcal{F}_t^1]) = -e^{-\Gamma_t}e^{\alpha(t)+\beta(t)\mu_t} + e^{-\Gamma_t}, \tag{4.4.7}$$

where $\alpha(t)$ and $\beta(t)$ are given by (4.4.5) and (4.4.6) respectively. By applying Itô's formula and using (4.4.2) we obtain that $\tilde{F}$ satisfies the equation

$$\begin{aligned} d\tilde{F}_t &= -e^{-\Gamma_t}e^{\alpha(t)+\beta(t)\mu_t}\theta\sqrt{\mu_t}\beta(t)dB_t^1 - e^{-\Gamma_t}\mu_t dt \\ &= -\psi_t\sqrt{\mu_t}dB_t^1 - e^{-\Gamma_t}\mu_t dt \end{aligned} \tag{4.4.8}$$

for all $t \in [0, T]$, where ψ is given by (4.4.4). □

Lemma 4.4.2. *The process $L = (L_t)_{t\in[0,T]}$ given by*

$$L_t = (1 - H_t)e^{\Gamma_t}, \quad t \in [0, T],$$

follows a $\mathbb{G}$-martingale and solves under P the following stochastic differential equation

$$dL_t = -L_t d\hat{M}_t \quad t \in [0, T]. \tag{4.4.9}$$

Proof. See Proposition 5.1.3 in Bielecki and Rutkowski [7]. □

Proposition 4.4.3. *The credibility process F satisfies under P the following equation*

$$dF_t = -\psi_t L_t \sqrt{\mu_t} dB^1_t - \tilde{F}_t L_{t-} d\hat{M}_t - e^{-\Gamma_t} L_t \mu_t dt, \tag{4.4.10}$$

for all $t \in [0, T]$.

Proof. It follows from Corollary 5.1.1 of Bielecki and Rutkowski [7] that

$$\begin{aligned} F_t &= P(t < \tau \le T|\mathcal{G}_t) = 1_{\{\tau > t\}} \mathbb{E}_P[1_{\{t<\tau\le T\}} e^{\Gamma_t}|\mathcal{F}_t] \\ &= L_t P(t < \tau \le T|\mathcal{F}_t) = L_t \tilde{F}_t. \end{aligned}$$

By applying the integration by parts formula and using (4.4.3) and (4.4.9), we obtain

$$\begin{aligned} dF_t &= L_t d\tilde{F}_t + \tilde{F}_t dL_t + d[F, L]_t \\ &= -\psi_t L_t \sqrt{\mu_t} dB^1_t - \tilde{F}_t L_{t-} d\hat{M}_t - e^{-\Gamma_t} L_t \mu_t dt. \end{aligned}$$

□

We examine the dynamics of the process Λ in this setting.

Theorem 4.4.4. *The process Λ satisfies the following equation*

$$d\Lambda_t = \Lambda_{t-}(\xi^1_t dB^1_t + \sigma dB^2_t - d\hat{M}_t + \xi^2_t dt), \tag{4.4.11}$$

for all $t \in [0, T]$, where ξ^1 and ξ^2 are given by

$$\xi^1_t = \frac{(1-c)\psi_t \sqrt{\mu_t} e^{\Gamma_t}}{(1-c)e^{\alpha(t)+\beta(t)\mu_t} + c} \quad \text{and} \quad \xi^2_t = -\frac{c\mu_t}{(1-c)e^{\alpha(t)+\beta(t)\mu_t} + c} \tag{4.4.12}$$

for all $t \in [0, T]$.

Proof. By (4.2.2) we have

$$\begin{aligned}\Lambda_t &= \mathbb{E}_P[X1_{\{\tau>T\}} + R_\tau 1_{\{\tau\le T\}}|\mathcal{G}_t]1_{\{\tau>t\}} \\ &= \mathbb{E}_P[X_T 1_{\{\tau>T\}}|\mathcal{G}_t] + \mathbb{E}_P[cX_\tau 1_{\{\tau\le T\}}|\mathcal{G}_t]1_{\{\tau>t\}}.\end{aligned} \tag{4.4.13}$$

Let us compute the first component of the above sum. It follows from Corollary 5.1.1. of Bielecki and Rutkowski [7] that

$$\mathbb{E}_P[X_T 1_{\{\tau>T\}}|\mathcal{G}_t] = 1_{\{\tau>t\}}\mathbb{E}_P[1_{\{\tau>T\}}e^{\Gamma_t}X_T|\mathcal{F}_t] = L_t\mathbb{E}_P[X_T 1_{\{\tau>T\}}|\mathcal{F}_t]. \tag{4.4.14}$$

where $L_t = (1-H_t)e^{\Gamma_t}$, $t \in [0,T]$. By using the definition of the $\mathbb{F}$-hazard process together with the fact that X_T is $\mathcal{F}_T$-measurable, we obtain

$$\begin{aligned}\mathbb{E}_P[X_T 1_{\{\tau>T\}}|\mathcal{G}_t] &= L_t\mathbb{E}_P[1_{\{\tau>T\}}X_T|\mathcal{F}_t] = L_t\mathbb{E}_P[\mathbb{E}_P[X_T 1_{\{\tau>T\}}|\mathcal{F}_T]|\mathcal{F}_t] \\ &= L_t\mathbb{E}_P[X_T P(\tau>T|\mathcal{F}_T)|\mathcal{F}_t] = L_t\mathbb{E}_P[X_T e^{-\Gamma_T}|\mathcal{F}_t] \\ &= L_t\mathbb{E}_P[X_T|\mathcal{F}_t^2]\mathbb{E}_P[e^{-\Gamma_T}|\mathcal{F}_t^1] \\ &= L_t X_t e^{-\Gamma_t}\mathbb{E}_P[e^{-(\Gamma_T-\Gamma_t)}|\mathcal{F}_t^1] \\ &= L_t X_t e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t},\end{aligned}$$

where the last equality follows from the properties of the affine process μ. Let us focus now on the second term of (4.4.13). By following the arguments of Lemma 4.1. of Biagini and Schreiber [6] we obtain

$$\begin{aligned}\mathbb{E}_P[cX_\tau 1_{\{\tau\le T\}}|\mathcal{G}_t]1_{\{\tau>t\}} &= 1_{\{\tau>t\}}\Big(c1_{\{\tau\le t\}}X_\tau + cL_t\mathbb{E}_P[\int_0^T X_s e^{-\Gamma_s}d\Gamma_s|\mathcal{F}_t] \\ &\quad - cL_t\int_0^t X_s e^{-\Gamma_s}d\Gamma_s\Big) \\ &= cL_t\Big(\mathbb{E}_P[\int_0^T X_s e^{-\Gamma_s}\mu_s ds|\mathcal{F}_t] - \int_0^t X_s e^{-\Gamma_s}\mu_s ds\Big) \\ &= cL_t\mathbb{E}_P[\int_t^T X_s e^{-\Gamma_s}\mu_s ds|\mathcal{F}_t] \\ &= cL_t\int_t^T \mathbb{E}_P[X_s e^{-\Gamma_s}\mu_s|\mathcal{F}_t]ds \\ &= cL_t\int_t^T \mathbb{E}_P[X_s|\mathcal{F}_t^2]\mathbb{E}_P[e^{-\Gamma_s}\mu_s|\mathcal{F}_t^1]ds \\ &= cL_t X_t\int_t^T \mathbb{E}_P[e^{-\Gamma_s}\mu_s|\mathcal{F}_t^1]ds \\ &= cL_t X_t(\mathbb{E}_P[\int_0^T e^{-\Gamma_s}\mu_s ds|\mathcal{F}_t^1] - \int_0^t e^{-\Gamma_s}\mu_s ds).\end{aligned}$$

By using the affine structure of the stochastic intensity μ we obtain

$$\begin{aligned}\mathbb{E}_P[\int_0^T e^{-\Gamma_s}\mu_s ds|\mathcal{F}_t] &= \mathbb{E}_P[1-e^{-\Gamma_T}|\mathcal{F}_t] = 1-\mathbb{E}_P[e^{-\Gamma_T}|\mathcal{F}_t] \\ &= 1-e^{-\Gamma_t}\mathbb{E}_P[e^{-(\Gamma_T-\Gamma_t)}|\mathcal{F}_t] = 1-e^{-\Gamma_t}e^{\alpha(t)+\beta(t)\mu_t}.\end{aligned}$$

Therefore

$$\begin{aligned}\Lambda_t &= L_tX_te^{\alpha(t)+\beta(t)\mu_t-\Gamma_t} + cL_tX_t\Big(1-e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t}-\int_0^t e^{-\Gamma_s}\mu_s ds\Big) \\ &= L_tX_t\Big(e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t}+c-ce^{\alpha(t)+\beta(t)\mu_t-\Gamma_t}-c\int_0^t e^{-\Gamma_s}\mu_s ds\Big) \\ &= L_tX_t\Big(c+(1-c)e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t}-c\int_0^t e^{-\Gamma_s}\mu_s ds\Big) \\ &= L_tX_tD_t.\end{aligned} \tag{4.4.15}$$

Let us denote by $D=(D_t)_{t\geq 0}$ the process

$$\begin{aligned}D_t &:= c+(1-c)e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t}-c\int_0^t e^{-\Gamma_s}\mu_s ds \\ &= (1-c)e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t}+ce^{-\Gamma_t}>0,\end{aligned} \tag{4.4.16}$$

a.s. for all $t\in[0,T]$. Since D is strictly positive, we rewrite D under the form of a stochastic exponential. By Itô's formula we obtain

$$d\Big(e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t}\Big) = \theta\beta(t)\sqrt{\mu_t}e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t}dB_t^1 = \psi_t\sqrt{\mu_t}dB_t^1,$$

where ψ is defined in (4.4.4). Therefore we have

$$\begin{aligned}dD_t &= (1-c)\psi_t\sqrt{\mu_t}dB_t^1 - ce^{-\Gamma_t}\mu_t dt \\ &= D_t\Big(\frac{(1-c)\psi_t\sqrt{\mu_t}}{D_t}dB_t^1-\frac{ce^{-\Gamma_t}\mu_t}{D_t}dt\Big) = D_t(\xi_t^1 dB_t^1+\xi_t^2 dt),\end{aligned}$$

where

$$\xi_t^1 = \frac{(1-c)\psi_t\sqrt{\mu_t}}{D_t} = \frac{(1-c)\psi_t\sqrt{\mu_t}e^{\Gamma_t}}{(1-c)e^{\alpha(t)+\beta(t)\mu_t}+c}\leq 0.$$

and

$$\xi_t^2 = -\frac{ce^{-\Gamma_t}\mu_t}{D_t} = -\frac{c\mu_t}{(1-c)e^{\alpha(t)+\beta(t)\mu_t}+c}\leq 0.$$

An application of Itô's product formula yields

$$
\begin{aligned}
d\Lambda_t &= L_t d(XD)_t + X_t D_t dL_t + d[XD, L]_t \\
&= L_t X_t dD_t + L_t D_t dX_t + L_t d[D, X]_t + X_t D_t dL_t + X_t d[D, L]_t + D_t d[X, L]_t \\
&= L_t X_t dD_t + L_t D_t dX_t + X_t D_t dL_t \\
&= L_t X_t D_t(\xi_t^1 dB_t^1 + \xi_t^2 dt) + L_t D_t \sigma X_t dB_t^2 - X_t D_t L_t d\hat{M}_t \\
&= \Lambda_{t-}(\xi_t^1 dB_t^1 + \sigma dB_t^2 - d\hat{M}_t + \xi_t^2 dt),
\end{aligned}
$$

and this concludes the proof. □

The following Theorem provides us with the dynamics of the market wealth process W. We remind that in our setting the immersion property holds under the measure P, i.e. all $\mathbb{F}$-martingales are also $\mathbb{G}$-martingales.

Theorem 4.4.5. *The market wealth process W is a $(\mathbb{G}, P)$-semimartingale that admits the canonical decomposition*

$$W_t = M_t + A_t,$$

for all $t \in [0, T]$, where the local martingale part M is given by

$$
\begin{aligned}
dM_t &= (f(t, F_t)\xi_t^1 - f_x(t, F_t)L_t\psi_t\sqrt{\mu_t})\Lambda_t dB_t^1 + \sigma f(t, F_t)\Lambda_t dB_t^2 \\
&\quad + (cX_t - f(t, F_t)\Lambda_{t-})d\hat{M}_t,
\end{aligned}
\tag{4.4.17}
$$

and the finite variation part A is given by

$$
\begin{aligned}
dA_t = \Big\{&[f(t, F_t)\xi_t^2 + f_t(t, F_t) - f_x(t, F_t)L_t e^{-\Gamma_t}\mu_t + \frac{1}{2}f_{xx}(t, F_t)L_t^2\psi_t^2\mu_t \\
&+ f_x(t, F_t)L_t\tilde{F}_t\hat{\mu}_t - f_x(t, F_t)L_t\psi_t\sqrt{\mu}_t\xi_t^1]\Lambda_t + cX_t\hat{\mu}_t\Big\}dt.
\end{aligned}
\tag{4.4.18}
$$

Proof. By applying the integration by parts formula we obtain

$$dW_t = f(t, F_t)d\Lambda_t + \Lambda_t df(t, F_t) + d[\Lambda, f(\cdot, F)]_t + cX_t dH_t. \tag{4.4.19}$$

First we compute the dynamics of $f(t, F_t)$. Using Itô's formula (see Theorem

II.32 in Protter [52]) and (4.4.10) we have

$$
\begin{aligned}
f(t,F_t) &= f(0,F_0) + \int_0^t f_s(s,F_s)ds + \int_0^t f_x(s,F_{s-})dF_s + \frac{1}{2}\int_0^t f_{xx}(s,F_{s-})d[F,F]^c_s \\
&\quad + \sum_{0<s\le t}\{f(s,F_s) - f(s,F_{s-}) - f_x(s,F_{s-})\Delta F_s)\} \\
&= f(0,F_0) + \int_0^t f_s(s,F_s)ds - \int_0^t f_x(s,F_s)L_s\psi_s\sqrt{\mu_s}dB^1_s \\
&\quad - \int_0^t f_x(s,F_{s-})\tilde{F}_{s-}L_{s-}d\hat{M}_s - \int_0^t f_x(s,F_s)L_s e^{-\Gamma_s}\mu_s ds \\
&\quad + \frac{1}{2}\int_0^t f_{xx}(s,F_s)L_s^2\psi_s^2\mu_s d[B^1,B^1]_s \\
&\quad + \sum_{0<s\le t}\{f(s,F_s) - f(s,F_{s-}) - f_x(s,F_{s-})\Delta F_s)\} \\
&= f(0,F_0) + \int_0^t f_s(s,F_s)ds - \int_0^t f_x(s,F_s)L_s\psi_s\sqrt{\mu_s}dB^1_s \\
&\quad - \int_0^t f_x(s,F_{s-})\tilde{F}_{s-}L_{s-}d\hat{M}_s - \int_0^t f_x(s,F_s)L_s e^{-\Gamma_s}\mu_s ds \\
&\quad + \frac{1}{2}\int_0^t f_{xx}(s,F_s)L_s^2\psi_s^2\mu_s ds \\
&\quad + \int_0^t \{f(s,F_s) - f(s,F_{s-}) + f_x(s,F_{s-})\tilde{F}_{s-}L_{s-}\}d\hat{M}_s \\
&\quad + \int_0^t f_x(s,F_s)\tilde{F}_s L_s\hat{\mu}_s ds,
\end{aligned}
$$

where we wrote the sum of jumps as a stochastic integral as shown below

$$
\begin{aligned}
&\sum_{0<s\le t}\{f(s,F_s) - f(s,F_{s-}) - f_x(s,F_{s-})\Delta F_s\} \\
&\quad = \sum_{0<s\le t}\{f(s,F_s) - f(s,F_{s-}) + f_x(s,F_{s-})\tilde{F}_s L_{s-}\Delta H_s\} \\
&\quad = \sum_{0<s\le t}\{f(s,F_s) - f(s,F_{s-})\}\Delta H_s + \sum_{0<s\le t} f_x(s,F_{s-})\tilde{F}_s L_{s-}\Delta H_s \\
&\quad = \int_0^t (f(s,F_s) - f(s,F_{s-}))dH_s + \int_0^t f_x(s,F_{s-})\tilde{F}_s L_{s-}dH_s \\
&\quad = \int_0^t \{f(s,F_s) - f(s,F_{s-}) + f_x(s,F_{s-})\tilde{F}_s L_{s-}\}d\hat{M}_s + \int_0^t f_x(s,F_{s-})\tilde{F}_s L_{s-}\hat{\mu}_s ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f(t,F_t) = f(0,F_0) &- \int_0^t f_x(s,F_s)L_s\psi_s\sqrt{\mu_s}dB_s^1 + \int_0^t \Big(f(s,F_s) - f(s,F_{s-})\Big)d\hat{M}_s \\
&+ \int_0^t \Big(f_s(s,F_s) - f_x(s,F_s)L_s e^{-\Gamma_s}\mu_s + \frac{1}{2}f_{xx}(s,F_s)L_s^2\psi_s^2\mu_s \\
&+ f_x(s,F_s)\tilde{F}_s L_s\hat{\mu}_s\Big)ds.
\end{aligned} \tag{4.4.20}
$$

Hence by Theorem 4.4.4 and (4.4.20) we have that the quadratic covariation $[\Lambda, f(\cdot,F)]$ is given by

$$
\begin{aligned}
d[\Lambda, f(\cdot,F)]_t &= -f_x(t,F_t)L_t\psi_t\xi_t^1\sqrt{\mu_t}\Lambda_t d[B^1,B^1]_t \\
&\quad - (f(t,F_t) - f(t,F_{t-}))\Lambda_{t-}d[\hat{M},\hat{M}]_t \\
&= -(f(t,F_t) - f(t,F_{t-}))\Lambda_{t-}dH_t - f_x(t,F_t)L_t\psi_t\xi_t^1\sqrt{\mu_t}\Lambda_t dt \\
&= -(f(t,F_t) - f(t,F_{t-}))\Lambda_{t-}d\hat{M}_t - f_x(t,F_t)L_t\psi_t\xi_t^1\sqrt{\mu_t}\Lambda_t dt.
\end{aligned}
$$

By replacing the expressions of $[\Lambda, f(\cdot,F)]$ and $f(t,F_t)$ in (4.4.19) and by using (4.4.11) we obtain

$$
\begin{aligned}
dW_t &= f(t,F_t)\xi_t^1\Lambda_t dB_t^1 + \sigma f(t,F_t)\Lambda_t dB_t^2 - f(t,F_{t-})\Lambda_{t-}d\hat{M}_t \\
&\quad + f(t,F_t)\xi_t^2\Lambda_t dt - f_x(t,F_t)L_t\psi_t\sqrt{\mu_t}\Lambda_t dB_t^1 \\
&\quad + (f(t,F_t) - f(t,F_{t-}))\Lambda_{t-}d\hat{M}_t + \Big(f_t(t,F_t) - f_x(t,F_t)L_t e^{-\Gamma_t}\mu_t \\
&\quad + \frac{1}{2}f_{xx}(t,F_t)L_t^2\psi_t^2\mu_t + f_x(t,F_t)\tilde{F}_t L_t\hat{\mu}_t\Big)\Lambda_t dt \\
&\quad - f_x(t,F_t)L_t\psi_t\sqrt{\mu_t}\xi_t^1\Lambda_t dt - (f(t,F_t) - f(t,F_{t-}))\Lambda_{t-}d\hat{M}_t \\
&\quad + cX_t d\hat{M}_t + cX_t\hat{\mu}_t dt \\
&= \Big(f(t,F_t)\xi_t^1 - f_x(t,F_t)L_t\psi_t\sqrt{\mu_t}\Big)\Lambda_t dB_t^1 + \sigma f(t,F_t)\Lambda_t dB_t^2 \\
&\quad + (cX_t - f(t,F_{t-})\Lambda_{t-})d\hat{M}_t \\
&\quad + \Big\{\Big(f(t,F_t)\xi_t^2 + f_t(t,F_t) - f_x(t,F_t)L_t e^{-\Gamma_t}\mu_t \\
&\quad + \frac{1}{2}f_{xx}(t,F_t)L_t^2\psi_t^2\mu_t + f_x(t,F_t)L_t\tilde{F}_t\hat{\mu}_t \\
&\quad - f_x(t,F_t)L_t\psi_t\xi_t^1\sqrt{\mu_t}\Big)\Lambda_t + cX_t\hat{\mu}_t\Big\}dt.
\end{aligned}
$$

This concludes the proof. □

Let us assume $\mathcal{M}_{loc}(W) \neq \emptyset$. We are interested in deriving a general form for the Radon-Nikodym density process of a measure $Q \in \mathcal{M}_{loc}(W)$ with respect to the measure P. To this purpose, we consider a setting where the predictable version of Girsanov's theorem can be applied. This allows us to use the conditional quadratic covariation in the computation of the canonical semimartingale decomposition of W with respect to Q, see Theorem III.40 in Protter [52].

Proposition 4.4.6. *Let $Q \in \mathcal{M}_{loc}(W)$ with Radon-Nikodym density process $Z = (Z_t)_{t\in[0,T]}$ i.e. $Z_t = \frac{dQ}{dP}|_{\mathcal{G}_t}$, $t \in [0,T]$. We assume that the quadratic covariation $[Z, M]$ is locally integrable. Then Z admits the representation*

$$dZ_t = Z_{t-}(b_t^{(1)} dB_t^1 + b_t^{(2)} dB_t^2 + b_{t-}^{(3)} d\hat{M}_t), \quad t \in [0,T], \tag{4.4.21}$$

where $(b_t^{(i)})_{t\in[0,T]}$ are $\mathbb{G}$-predictable processes for all $i = 1, 2, 3$, satisfying

$$\begin{aligned} 0 = dA_t + \Big[\Big(b_t^{(1)}(\xi_t^1 f(t,F_t) - f_x(t,F_t) L_t \psi_t \sqrt{\mu_t}) \\ + \sigma b_t^{(2)} f(t,F_t)\Big)\Lambda_t + b_t^{(3)}(cX_t - f(t,F_t)\Lambda_{t-})\hat{\mu}_t\Big] dt \end{aligned} \tag{4.4.22}$$

$dt \otimes dP$-almost surely on $[0,T] \times \Omega$.

Proof. Since the process $Z = (Z_t)_{t\in[0,T]}$ is a $\mathbb{G}$-martingale and the hazard process Γ admits the representation

$$\Gamma_t = \int_0^t \mu_s ds, \quad t \in [0,T],$$

it follows from Corollary 5.2.4. of Bielecki and Rutkowski [7] that Z can be written under the form

$$Z_t = Z_0 + \int_0^t \zeta_s^{(1)} dB_s^1 + \int_0^t \zeta_s^{(2)} dB_s^2 + \int_0^t \zeta_s^{(3)} d\hat{M}_s,$$

for all $t \in [0,T]$, where the processes $(\zeta_t^{(i)})_{t\in[0,T]}$ are $\mathbb{G}$-predictable. Furthermore, since Z is a stricly positive martingale, Z can be written as the stochastic exponential

$$Z_t = \mathcal{E}(K)_t, \quad t \in [0,T],$$

where $K = (K_t)_{t\in[0,T]}$ is a local martingale given by

$$K_t = K_0 + \int_0^t b_s^{(1)} dB_s^1 + \int_0^t b_s^{(2)} dB_s^2 + \int_0^t b_s^{(3)} d\hat{M}_s, \quad t \in [0,T],$$

with

$$b_t^{(i)} = \frac{\zeta_t^{(i)}}{Z_t}, \quad t \in [0,T].$$

Hence representation (4.4.21) follows.

Since the quadratic covariation $[Z,M]$ is locally integrable, it follows from Chapter III in Protter [52], that the conditional quadratic covariation process $\langle Z,M\rangle$ exists and is defined as the compensator of $[Z,M]$. Therefore, we can apply the predictable version of Girsanov's Theorem. The local P-martingale process $(M_t)_{t\in[0,T]}$ given by (4.4.17) admits under Q the following canonical semimartingale decomposition

$$M_t = \Big(M_t - \int_0^t \frac{1}{Z_{s-}} d\langle Z,M\rangle_s\Big) + \int_0^t \frac{1}{Z_{s-}} d\langle Z,M\rangle_s, \quad t \in [0,T].$$

Since the process $(M_t^Q)_{t\in[0,T]}$ given by

$$M_t^Q = M_t - \int_0^t \frac{1}{Z_{s-}} d\langle Z,M\rangle_s, \quad t \in [0,T].$$

is a local Q-martingale, it follows that W admits under Q the canonical semimartingale decomposition

$$W_t = M_t^Q + \int_0^t \frac{1}{Z_{s-}} d\langle Z,M\rangle_s + A_t, \quad t \in [0,T].$$

Since $Q \in \mathcal{M}_{loc}(W)$, this implies that

$$A_t + \int_0^t \frac{1}{Z_{s-}} d\langle Z,M\rangle_s = 0$$

$dt \otimes dP$-a.s. Therefore (4.4.22) follows from the fact that

$$\begin{aligned}
A_t + \int_0^t \frac{1}{Z_{s-}} d\langle Z,M\rangle_s &= A_t + \int_0^t \frac{1}{Z_s} Z_s b_s^{(1)} (\xi_t^1 f(t,F_t) - f_x(t,F_t) L_t \psi_t \sqrt{\mu_t}) \Lambda_t d[B^1,B^1]_s \\
&\quad + \int_0^t \frac{1}{Z_s} Z_s b_s^{(2)} \sigma f(s,F_s) \Lambda_s d[B^2,B^2]_s \\
&\quad + \int_0^t \frac{1}{Z_{s-}} Z_{s-} (cX_s - f(s,F_s)\Lambda_{s-}) d\langle \hat{M},\hat{M}\rangle_s \\
&= A_t + \int_0^t \Big[\Big(b_s^{(1)} (\xi_s^1 f(s,F_s) - f_x(s,F_s) L_s \psi_s \sqrt{\mu_s}) \\
&\quad + \sigma b_s^{(2)} f(s,F_s)\Big)\Lambda_s + b_s^{(3)} (cX_s - f(s,F_s)\Lambda_{s-})\hat{\mu}_s\Big] ds.
\end{aligned}$$

□

We now state the following auxiliary result, that will be used later in the proofs.

Lemma 4.4.7. *On the set* $\{\tau > t\}$ *we have*

$$R_t < \Lambda_t. \tag{4.4.23}$$

Proof. Since the recovery process $(R_t)_{t\geq 0}$ is given by $R_t = cX_t$ for all $t \geq 0$, with $c \in (0, 1)$, we obtain

$$\begin{aligned} R_t 1_{\{\tau>t\}} &= cX_t 1_{\{\tau>t\}} = c\mathbb{E}_P[X_T|\mathcal{G}_t]1_{\{\tau>t\}} \\ &= c1_{\{\tau>t\}}\mathbb{E}_P[X_T 1_{\{\tau>T\}} + X_T 1_{\{\tau\leq T\}}|\mathcal{G}_t] \\ &= c(1_{\{\tau>t\}}\mathbb{E}_P[X_T 1_{\{\tau>T\}}|\mathcal{G}_t] + 1_{\{\tau>t\}}\mathbb{E}_P[\mathbb{E}_P[X_T 1_{\{t<\tau\leq T\}}|\mathcal{G}_\tau]|\mathcal{G}_t]) \\ &= c1_{\{\tau>t\}}\mathbb{E}_P[X_T 1_{\{\tau>T\}} + X_\tau 1_{\{t<\tau\leq T\}}|\mathcal{G}_t] < \Lambda_t 1_{\{\tau>t\}}. \end{aligned}$$

□

We consider the following form for the function $f(t, x)$. Let

$$f(t,x) := \begin{cases} 1 + k(T-t)(1-\frac{x}{p}) & \text{if } x \leq p,\ t \in [0,T], \\ 1 & \text{if } x > p,\ t \in [0,T], \end{cases}$$

where $k > 0$ is a positive constant. The partial derivatives of $f(t, x)$ will be equal to

$$f_t(t,x) := \begin{cases} -k(1-\frac{x}{p}) & \text{if } x \leq p,\ t \in [0,T], \\ 0 & \text{if } x > p,\ t \in [0,T]. \end{cases}$$

and

$$f_x(t,x) := \begin{cases} -\frac{k}{p}(T-t) & \text{if } x \leq p,\ t \in [0,T], \\ 0 & \text{if } x > p,\ t \in [0,T]. \end{cases}$$

Note that $f(t, x)$ has bounded first and second order partial derivatives. Furthermore the impact of the credibility process F on the wealth process W is bounded. However the wealth process W is not bounded. For a comment on price bubbles in the case of assets with a bounded wealth process, we refer to Remark 4.2.8.

For this specific example of $f(t, x)$ we show that there exist a measure $Q \in \mathcal{M}_{loc}(W)$ and compute its associated Radon-Nikodym density process with respect to P.

Theorem 4.4.8. *Suppose that the intensity process* $(\mu_t)_{t\in[0,T]}$ *satisfies the condition*

$$\mathbb{E}_P\Big[\exp(\frac{1}{\sigma^2}\int_0^T \mu_s^2 ds)\Big] < \infty. \tag{4.4.24}$$

Then $\mathcal{M}_{loc}(W) \neq \emptyset$.

Proof. We start by bringing the expression of the finite variation part A of W, as given in (4.4.18), to a simpler form. Note that $f_{xx}(t,x) = 0$. From (4.4.4), (4.4.7) and (4.4.12) we obtain

$$\begin{aligned}
dA_t &= \Big\{[f(t,F_t)\xi_t^2 + f_t(t,F_t) - f_x(t,F_t)L_t e^{-\Gamma_t}\mu_t + \frac{1}{2}f_{xx}(t,F_t)L_t^2\psi_t^2\mu_t \\
&\quad + f_x(t,F_t)L_t\tilde{F}_t\hat{\mu}_t - f_x(t,F_t)L_t\psi_t\sqrt{\mu}_t\xi_t^1]\Lambda_t + cX_t\hat{\mu}_t\Big\}dt \\
&= \Big\{[f(t,F_t)\xi_t^2 + f_t(t,F_t) - f_x(t,F_t)\hat{\mu}_t + f_x(t,F_t)(1 - e^{\alpha(t)+\beta(t)\mu_t})\hat{\mu}_t \\
&\quad - f_x(t,F_t)\frac{(1-c)\psi_t\sqrt{\mu_t}e^{\Gamma_t}}{(1-c)e^{\alpha(t)+\beta(t)\mu_t} + c}L_t\psi_t\sqrt{\mu_t}]\Lambda_t \\
&\quad + cX_t\hat{\mu}_t\Big\}dt \\
&= \Big\{f(t,F_t)\xi_t^2 + f_t(t,F_t)\Big\}\Lambda_t dt + \Big\{[-f_x(t,F_t)\hat{\mu}_t - f_x(t,F_t)e^{\alpha(t)+\beta(t)\mu_t}\hat{\mu}_t \\
&\quad + f_x(t,F_t)\hat{\mu}_t - f_x(t,F_t)\frac{1-c}{(1-c)e^{\alpha(t)+\beta(t)\mu_t} + c}\theta^2\beta^2(t)e^{2\alpha(t)+2\beta(t)\mu_t}\hat{\mu}_t]\Lambda_t \\
&\quad + cX_t\Big\}\hat{\mu}_t dt \\
&= \Big\{f(t,F_t)\xi_t^2 + f_t(t,F_t)\Big\}\Lambda_t dt + \Big\{ - f_x(t,F_t)e^{\alpha(t)+\beta(t)\mu_t} \\
&\quad \cdot \Big(1 + \frac{1-c}{(1-c)e^{\alpha(t)+\beta(t)\mu_t} + c}\theta^2\beta^2(t)e^{\alpha(t)+\beta(t)\mu_t}\Big)\Lambda_t + cX_t\Big\}\hat{\mu}_t dt.
\end{aligned}$$

We denote

$$\delta_t := -f_x(t,F_t)e^{\alpha(t)+\beta(t)\mu_t}\Big(1 + \frac{(1-c)e^{\alpha(t)+\beta(t)\mu_t}}{(1-c)e^{\alpha(t)+\beta(t)\mu_t} + c}\theta^2\beta^2(t)\Big) \qquad (4.4.25)$$

for all $t \in [0,T]$. It is easy to see that $0 \le \delta_t < C_\delta$ for some constant $C_\delta > 0$ a.s. for all $t \in [0,T]$. Therefore

$$dA_t = \Big\{f(t,F_t)\xi_t^2 + f_t(t,F_t)\Big\}\Lambda_t dt + \Big(\delta_t\Lambda_t + cX_t\Big)\hat{\mu}_t dt.$$

We define the measure Q^1 by

$$\frac{dQ^1}{dP}|_{\mathcal{G}_T} = Z_T^{(1)}, \qquad (4.4.26)$$

with the Radon-Nikodym density process $Z^{(1)} = (Z_t^{(1)})_{t\in[0,T]}$ given by

$$dZ_t^{(1)} = b_t^{(1)}Z_t^{(1)}dB_t^2,$$

and

$$
\begin{aligned}
b_t^{(1)} &= -\frac{f(t,F_t)\xi_t^2 + f_t(t,F_t) + \frac{cX_t}{2\Lambda_t}\mu_t 1_{\{\tau>t\}}}{\sigma f(t,F_t)} \\
&= -\frac{1}{\sigma}\xi_t^2 - \frac{f_t(t,F_t)}{\sigma f(t,F_t)} - \frac{cX_t}{2\sigma f(t,F_t)\Lambda_t}\mu_t 1_{\{\tau>t\}} \\
&= \frac{1}{\sigma}\frac{c}{(1-c)e^{\alpha(t)+\beta(t)\mu_t}+c}\mu_t + k(1-\frac{F_t}{p})\frac{1}{\sigma f(t,F_t)}1_{\{\sigma_1\le t\}} \\
&\quad - \frac{1}{2\sigma f(t,F_t)}\frac{c\mu_t}{(1-c)e^{\alpha(t)+\beta(t)\mu_t}+c}1_{\{\tau>t\}} \\
&= \frac{1}{\sigma}\frac{c\mu_t}{(1-c)e^{\alpha(t)+\beta(t)\mu_t}+c}\Big(1-\frac{1}{2f(t,F_t)}1_{\{\tau>t\}}\Big) + k(1-\frac{F_t}{p})\frac{1}{\sigma f(t,F_t)}1_{\{\sigma_1\le t\}},
\end{aligned}
$$

where we have used the fact that $\Lambda_t = X_t L_t D_t$ with D_t defined in (4.4.16). We prove that $Z^{(1)}$ is a true P-martingale. To this purpose we show that Novikov's condition is satisfied i.e.

$$
\mathbb{E}_P\Big[\exp\Big(\frac{1}{2}\int_0^T (b_t^{(1)})^2 d[B^2,B^2]_t\Big)\Big] < \infty, \quad P-\text{a.s.}.
$$

We have

$$
\begin{aligned}
\mathbb{E}_P\Big[\exp(\frac{1}{2}\int_0^T (b_t^{(1)})^2 dt)\Big] &\le \mathbb{E}_P\Big[\exp\Big(\int_0^T \frac{k^2}{\sigma^2 f(t,F_t)^2}\Big(1-\frac{F_t}{p}\Big)^2 1_{\{\sigma_1\le t\}}dt \\
&\quad + \int_0^T \frac{c^2\mu_t^2}{\sigma^2(c+(1-c)e^{\alpha(t)+\beta(t)\mu_t})^2}\Big(1-\frac{1}{2f(t,F_t)}1_{\{\tau>t\}}\Big)^2 dt\Big)\Big] \\
&\le C\mathbb{E}_P\Big[\exp\Big(\int_0^T \frac{c^2\mu_t^2}{\sigma^2(c+(1-c)e^{\alpha(t)+\beta(t)\mu_t})^2}dt\Big)\Big] \\
&\le C\mathbb{E}_P\Big[\exp\Big(\frac{1}{\sigma^2}\int_0^T \mu_t^2 dt\Big)\Big] < \infty,
\end{aligned}
$$

where $C = C(T) > 0$ is some positive constant depending on T and the last inequality follows from (4.4.24). Therefore $Z^{(1)}$ is indeed a Radon-Nikodym density process. It follows from Girsanov's theorem that the process

$$
\tilde{B}_t^2 = B_t^2 - \int_0^t \frac{1}{Z_s^{(1)}} d[Z^{(1)}, B^2] = B_t^2 - \int_0^t b_s^{(1)} ds, \quad t \in [0,T],
$$

is a Q^1-local martingale for all $t \in [0,T]$. It follows from Levy's characterization theorem that $(\tilde{B}_t^2)_{t\in[0,T]}$ is a Q^1-Brownian motion with respect to the filtration $\mathbb{G}$. By applying Girsanov's theorem we obtain the canonical semimartingale decomposition of W under Q^1:

$$
W_t = M_t^1 + A_t^1, \quad t \in [0,T],
$$

where the local martingale part $(M^1_t)_{t\in[0,T]}$ is equal to

$$
\begin{aligned}
M^1_t &= M_t - \int_0^t \frac{1}{Z^{(1)}_s} d[Z^{(1)}, M]_s \\
&= M_0 + \int_0^t (f(s,F_s)\xi^1_s - f_x(s,F_s)L_s\psi_s\sqrt{\mu_s})\Lambda_s dB^1_s + \int_0^t \sigma f(s,F_s)\Lambda_s dB^2_s \\
&\quad + \int_0^t (cX_s - f(s,F_s)\Lambda_{s-})d\hat{M}_s - \int_0^t \sigma f(s,F_s)\Lambda_s b^{(2)}_s ds \\
&= M_0 + \int_0^t \Big(f(s,F_s)\xi^1_s - f_x(s,F_s)L_s\psi_s\sqrt{\mu_s}\Big)\Lambda_s dB^1_s + \int_0^t \sigma f(s,F_s)\Lambda_s d\tilde{B}^2_s \\
&\quad + \int_0^t (cX_s - f(s,F_s)\Lambda_s)d\hat{M}_s,
\end{aligned}
$$

and the finite variation part $(A^1_t)_{t\in[0,T]}$ is equal by

$$
\begin{aligned}
A^1_t &= A_t + \int_0^t \frac{1}{Z^{(1)}_s} d[Z^{(1)}, B^2]_s \\
&= \int_0^t (f(s,F_s)\xi^2_s + f_s(s,F_s))\Lambda_s ds + \int_0^t (\delta_s\Lambda_s + cX_s)\hat{\mu}_s ds \\
&\quad + \int_0^t \sigma f(s,F_s) b^{(1)}_s \Lambda_s ds \\
&= \int_0^t (\frac{c}{2}X_s + \delta_s\Lambda_s)\hat{\mu}_s ds.
\end{aligned}
$$

We define the process $Z^{(2)} = (Z^{(2)})_{t\in[0,T]}$ as the stochastic exponential satisfying the equation

$$dZ^{(2)}_t = b_t Z^{(2)}_t d\hat{M}_t, \quad t \in [0,T], \tag{4.4.27}$$

with

$$b_t = \frac{\frac{c}{2}X_t + \delta_t\Lambda_t}{f(t,F_t)\Lambda_t - cX_t}, \quad t \in [0,T]. \tag{4.4.28}$$

We prove that $Z^{(2)}$ is a Q^1-martingale. First we show that the Q^1-local martingale $Z^{(2)}$ is positive. For this it is sufficient to show that $\Delta(b.\hat{M})_t > -1$ for all $t \in [0,T]$, where $b.\hat{M}$ denotes the stochastic integral of b with respect to $\hat{M}$. We start by showing that $(b_t)_{t\in[0,T]}$ defined in (4.4.28) satisfies $b_t > -1$ for all $t \in [0,T]$. We have

$$
\begin{aligned}
b_t &= \frac{\frac{c}{2}X_t + \delta_t\Lambda_t}{f(t,F_t)\Lambda_t - cX_t} 1_{\{\tau\le t\}} + \frac{\frac{c}{2}X_t + \delta_t\Lambda_t}{f(t,F_t)\Lambda_t - cX_t} 1_{\{\tau>t\}} \\
&= -\frac{1}{2} 1_{\{\tau\le t\}} + \frac{\frac{c}{2}X_t + \delta_t\Lambda_t}{f(t,F_t)\Lambda_t - cX_t} 1_{\{\tau>t\}}.
\end{aligned}
\tag{4.4.29}
$$

On the set $\{\tau > t\}$ we have by Lemma 4.4.7, (4.2.3) and (4.4.25) that

$$cX_t < \Lambda_t \leq f(t, F_t)\Lambda_t, \tag{4.4.30}$$

so

$$b_t \geq 0 \quad \text{on} \quad \{\tau > t\}. \tag{4.4.31}$$

Moreover $b_t > -1$ for all $t \in [0, T]$. Therefore the Q^1-local martingale $Z^{(2)}$ is positive since

$$\Delta(b.\hat{M})_t = b_t \Delta \hat{M}_t = b_t \Delta H_t > -1,$$

for all $t \in [0, T]$. Since $Z^{(2)}$ is defined as the unique solution of (4.4.27), it follows from Theorem II.37 in Protter [52] that $Z^{(2)}$ is given by

$$Z_t^{(2)} = \exp\Big(\int_0^t b_s d\hat{M}_s\Big)\Pi_{s\leq t}(1 + b_s \Delta H_s)\exp(-b_s \Delta H_s),$$

for all $t \in [0, T]$. We have by (4.4.29) and (4.4.31) that

$$\begin{aligned}
Z_t^{(2)} &= \exp\Big(\int_0^t b_s dH_s - \int_0^{t\wedge\tau} b_s \mu_s ds\Big)(1 + b_\tau 1_{\{\tau\leq t\}})\exp(-b_\tau 1_{\{\tau\leq t\}}) \\
&= 1_{\{\tau>t\}}\exp\Big(-\int_0^{t\wedge\tau} b_s \mu_s ds\Big) + \frac{1}{2}1_{\{\tau\leq t\}}\exp\Big(-\int_0^{t\wedge\tau} b_s \mu_s ds\Big) \\
&\leq 1_{\{\tau>t\}} + \frac{1}{2}1_{\{\tau\leq t\}} < \frac{3}{2},
\end{aligned}$$

for all $t \in [0, T]$. Therefore the positive Q^1-local martingale $Z^{(2)}$ is also upper bounded, hence $Z^{(2)}$ is a Q^1-uniformly integrable martingale, see Theorem I.51 in Protter [52]. Analogously, we obtain that

$$|\Delta Z_t^{(2)}| = |Z_t^{(2)} - Z_{t-}^{(2)}| \leq \mathcal{K}_\Delta,$$

for some $\mathcal{K}_\Delta > 0$. Hence $Z^{(2)}$ has bounded jumps. It follows from Lemma 3.14 in [33] that $[Z^{(2)}, M^1]$ has locally integrable variation. Therefore its Q^1-compensator $\langle Z^{(2)}, M^1\rangle$ exists and is well defined.

Since $Z^{(2)}$ is a Q^1-martingale, we can define the measure Q^2 with the Radon-Nikodym density process w.r.t. Q^1 given by

$$\frac{dQ^2}{dQ^1}|_{\mathcal{G}_T} = Z_T^{(2)}. \tag{4.4.32}$$

It follows from Proposition 5.3.1 in Bielecki and Rutkowski [7] that the process

$$\tilde{M}_t = \hat{M}_t - \int_0^t b_s \hat{\mu}_s ds, \quad t \in [0, T]$$

is a $\mathbb{G}$-martingale under Q^2. Since the conditional quadratic covariation $\langle Z^{(2)}, M^1\rangle$ exists under Q^1, we can apply the predictable version of Girsanov's theorem (see Theorem III.40 in [52]) in order to compute the canonical semimartingale decomposition of W under Q^2. We have

$$W_t = M_t^2 + A_t^2, \quad t \in [0,T],$$

Q^2-a.s., where the local martingale part $(M_t^2)_{t\in[0,T]}$ is given by

$$\begin{aligned}
M_t^2 &= M_t^1 - \int_0^t \frac{1}{Z_{s-}^{(2)}} d\langle Z^{(2)}, M^1\rangle_s \\
&= M_0 + \int_0^t \Big(f(s,F_s)\xi_s^1 - f_x(s,F_s)L_s\psi_s\sqrt{\mu_s}\Big)\Lambda_s dB_s^1 + \int_0^t \sigma f(s,F_s)\Lambda_s d\tilde{B}_s^2 \\
&\quad + \int_0^t (cX_s - f(s,F_s)\Lambda_s) d\hat{M}_s - \int_0^t (cX_s - f(s,F_s)\Lambda_s) b_s \hat{\mu}_s ds \\
&= M_0 + \int_0^t \Big(f(s,F_s)\xi_s^1 - f_x(s,F_s)L_s\psi_s\sqrt{\mu_s}\Big)\Lambda_s dB_s^1 + \int_0^t \sigma f(s,F_s)\Lambda_s d\tilde{B}_s^2 \\
&\quad + \int_0^t (cX_s - f(s,F_s)\Lambda_s) d\tilde{M}_s
\end{aligned}$$

and the finite variation part $(A_t^2)_{t\in[0,T]}$ is given by by

$$\begin{aligned}
A_t^2 &= A_t^1 + \int_0^t \frac{1}{Z_{s-}^{(2)}} d\langle Z^{(2)}, M^1\rangle_s \\
&= \int_0^t (\frac{c}{2}X_s + \delta_s\Lambda_s)\hat{\mu}_s ds + \int_0^t b_s d\langle \hat{M}, M^1\rangle_s \\
&= \int_0^t (\frac{c}{2}X_s + \delta_s\Lambda_s)\hat{\mu}_s ds + \int_0^t b_s (cX_s - f(s,F_s)\Lambda_s)\hat{\mu}_s ds \\
&= 0,
\end{aligned}$$

where we have used (4.4.28). Hence W is a local martingale under Q^2. Then the equivalent probability measure $Q \approx P$ defined by

$$\frac{dQ}{dP} = \frac{dQ^1}{dP}\frac{dQ^2}{dQ^1} \tag{4.4.33}$$

where Q^1 and Q^2 are defined in (4.4.26) and (4.4.32) respectively, belongs to $\mathcal{M}_{loc}(W)$, i.e $\mathcal{M}_{loc}(W) \neq \emptyset$. □

The following proposition provides us with a sufficient condition under which an equivalent local martingale measure $Q \in \mathcal{M}_{loc}(W)$ belongs to the subset $\mathcal{M}_{NUI}(W)$.

Proposition 4.4.9. *Let $Q \in \mathcal{M}_{loc}(W)$ and we assume that $0 < \sigma_1 < T$. If the estimated wealth process $(W_t^e)_{t\in[0,T]}$ is a Q-supermartingale with respect to the filtration $\mathbb{G}$, then $Q \in \mathcal{M}_{NUI}(W)$.*

Proof. Let $Q \in \mathcal{M}_{loc}(W)$ such that $(W_t^e)_{t\in[0,T]}$ is a $(\mathbb{G}, Q)$-supermartingale. Let $\epsilon > 0$ such that $\sigma_1 + \epsilon \le T$. We have

$$\begin{aligned}\mathbb{E}_Q[W_T] = \mathbb{E}_Q[W_T^e] = \mathbb{E}_Q[\mathbb{E}_Q[W_T^e|\mathcal{G}_{\sigma_1+\epsilon}]] \le \mathbb{E}_Q[W_{\sigma_1+\epsilon}^e] \\ < \mathbb{E}_Q[W_{\sigma_1+\epsilon}] \le \mathbb{E}_Q[W_0].\end{aligned}$$

Therefore W is a Q-strict local martingale on $[0, T]$. □

We now provide sufficient conditions under which Proposition 4.3.7 holds in this framework.

Proposition 4.4.10. *Let $Q \in \mathcal{M}_{loc}(W)$ whose Radon-Nikodym density process with respect to P, $Z = (Z_t)_{t\in[0,T]}$ i.e. $Z_t = \frac{dQ}{dP}|_{\mathcal{G}_t}$ for all $t \in [0, T]$, satisfies*

i) $E_P[\sup_{t\in[0,T]} Z_t^2] < +\infty$.

ii) The quadratic covariation process $[W^e, Z]$ is increasing.

Then W^eZ is a P-submartingale.

Proof. By applying the integration by parts formula we obtain the semimartingale decomposition of W^eZ

$$d(W^eZ)_t = W^e dZ_t + Z_t dW_t^e + d[W^e, Z] = dm_t + da_t, \quad t \in [0, T],$$

where the local martingale part $(m_t)_{t\in[0,T]}$ is given by

$$dm_t = Z_{t-}dW_t^e + W_{t-}^e dZ_t,$$

and the finite variation part $(a_t)_{t\in[0,T]}$ is equal to

$$da_t = d[W^e, Z]_t.$$

Hence the process W^eZ is a local P-submartingale. In order to show that W^eZ is a P-submartingale, we prove

$$\mathbb{E}_P[\sup_{s\in[0,T]} |m_s|] < \infty.$$

Then it follows from Theorem I.51 in Protter [52] that $(m_t)_{t\in[0,T]}$ is a P-martingale. We have

$$0 < W_t^e = \mathbb{E}_P[X_T 1_{\{\tau>T\}} + cX_\tau 1_{\{\tau\le T\}}|\mathcal{G}_t] < X_{t\wedge\tau},$$

for all $t \in [0,T]$, since $c < 1$. Hence

$$\mathbb{E}_P[(W_t^e)^2] \leq \mathbb{E}_P[X_{t\wedge\tau}^2] < \infty. \tag{4.4.34}$$

for all $t \in [0,T]$. Therefore W^e is a square integrable P-martingale and satisfies $\mathbb{E}_P[(W_T^e)^2] = \mathbb{E}_P[[W^e, W^e]_T]$. Let

$$m_t = m_t^+ - m_t^-, \quad t \in [0,T],$$

be the decomposition of the local martingale m into its positive part m^+ and its negative part m^-. Since

$$(W^e Z)_t = m_t + a_t = m_t^+ - m_t^- + a_t \geq 0,$$

for all $t \in [0,T]$, we have $m_t^- \leq a_t$, for all $t \in [0,T]$. Moreover

$$m_t^+ - m_t^- + a_t \leq X_{t\wedge\tau} Z_t.$$

Therefore $m_t^+ \leq X_{t\wedge\tau} Z_t$ for all $t \in [0,T]$ and

$$|m_t| = m_t^+ + m_t^- \leq X_{t\wedge\tau} Z_t + a_t.$$

Hence

$$\begin{aligned}
\mathbb{E}_P[\sup_{t\in[0,T]} |m_t|] &= \mathbb{E}_P[\sup_{t\in[0,T]} |X_{t\wedge\tau} Z_t + a_t|] \\
&\leq \mathbb{E}_P[\sup_{t\in[0,T]} |X_{t\wedge\tau} Z_t|] + \mathbb{E}_P[\sup_{t\in[0,T]} [W^e, Z]_t] \\
&= 2\mathbb{E}_P[\sup_{t\in[0,T]} (X_{t\wedge\tau}^2 + Z_t^2)] + \mathbb{E}_P[[W^e, Z]_T] \\
&\leq 2\mathbb{E}_P[\sup_{t\in[0,T]} X_t^2] + 2\mathbb{E}_P[\sup_{t\in[0,T]} Z_t^2] + \mathbb{E}_P[[W^e, Z]_T].
\end{aligned}$$

By the Kunita-Watanabe inequality, see Theorem II.25 in [52], we have

$$\mathbb{E}_P[[W^e, Z]_T] \leq \mathbb{E}_P[[W^e, W^e]_T]^{\frac{1}{2}} \mathbb{E}_P[[Z, Z]_T]^{\frac{1}{2}} = \mathbb{E}_P[(W_T^e)^2]^{\frac{1}{2}} \mathbb{E}_P[Z_T^2]^{\frac{1}{2}}.$$

Therefore by (4.4.34) and by Doob's maximal inequality it follows

$$\mathbb{E}_P[\sup_{t\in[0,T]} |m_t|] \leq 8\mathbb{E}_P[X_T^2] + 2\mathbb{E}_P[\sup_{t\in[0,T]} Z_t^2] + \mathbb{E}_P[X_T^2]^{\frac{1}{2}} \mathbb{E}_P[Z_T^2]^{\frac{1}{2}} < \infty.$$

We can conclude that $(m_t)_{t\in[0,T]}$ is a P-martingale. □

Corollary 4.4.11. *If $Q \in \mathcal{M}_{loc}(W)$ satisfies the assumptions of Proposition 4.4.10, then*

$$W_t^Q - W_t^e \geq 0,$$

for all $t \in [0,T]$

Proof. It is a consequence of Propositions 4.3.7 and 4.4.10. □

We now obtain sufficient conditions under which the process W^e is a R-supermartingale on the whole interval $[0,T]$ for a measure $R \in \mathcal{M}_{loc}(W)$.

Proposition 4.4.12. *Let $R \in \mathcal{M}_{loc}(W)$ with Radon-Nikodym density process $Z = (Z_t)_{t\in[0,T]}$, where $Z_t = \frac{dR}{dP}|_{\mathcal{G}_t}$, for $t \in [0,T]$. We assume that the quadratic covariation process $[Z,M]$ is locally integrable.*
If the processes $b^{(i)}$, $i = 1,2,3$, in the representation (4.4.21) of Z satisfy the inequality

$$(\xi_t^1 b_t^{(1)} + \sigma b_t^{(2)})\Big(1 + \frac{c}{1-c} e^{-\alpha(t)-\beta(t)\mu_t}\Big) \le b_t^{(3)}\mu_t, \tag{4.4.35}$$

on the set $\{\tau > t\}$, then W^e is a R-supermartingale on $[0,T]$. In particular $R \in \mathcal{M}_{NUI}(W)$.

Proof. By Theorem 4.4.4 it follows that W^e satisfies under P the equation

$$\begin{aligned}
dW_t^e &= d\Lambda_t + R_t dH_t \\
&= \Lambda_{t-}(\xi_t^1 dB_t^1 + \sigma dB_t^2 - d\hat{M}_t + \xi_t^2 dt) + R_{t-} d\hat{M}_t + R_t \hat{\mu}_t dt \\
&= \Lambda_t(\xi_t^1 dB_t^1 + \sigma dB_t^2) + (cX_t - \Lambda_{t-}) d\hat{M}_t + (\xi_t^2 \Lambda_t + cX_t\hat{\mu}_t)dt \\
&= \Lambda_t(\xi_t^1 dB_t^1 + \sigma dB_t^2) + (cX_t - \Lambda_{t-}) d\hat{M}_t + (cX_t\hat{\mu}_t - \frac{ce^{-\Gamma_t}\mu_t}{D_t} L_t X_t D_t)dt \\
&= \Lambda_t(\xi_t^1 dB_t^1 + \sigma dB_t^2) + (cX_t - \Lambda_{t-}) d\hat{M}_t + (cX_t\hat{\mu}_t - ce^{-\Gamma_t}\mu_t(1-H_t)e^{\Gamma_t}X_t)dt \\
&= \Lambda_t(\xi_t^1 dB_t^1 + \sigma dB_t^2) + (cX_t - \Lambda_{t-}) d\hat{M}_t + (cX_t\hat{\mu}_t - c\hat{\mu}_t X_t)dt \\
&= \Lambda_t(\xi_t^1 dB_t^1 + \sigma dB_t^2) + (cX_t - \Lambda_{t-}) d\hat{M}_t,
\end{aligned}$$

where we have used (4.4.12) and (4.4.15). Since $[Z,M]$ is locally integrable, its P-compensator process $\langle Z, M\rangle$ exists and it is well defined. Therefore one can apply the predictable version of Girsanov's theorem in order to determine the canonical semimartingale decomposition of W^e under R. We have

$$W_t^e = \mathcal{N}_t + \mathcal{A}_t, \quad t \in [0,T],$$

where the local martingale part $\mathcal{N} = (\mathcal{N}_t)_{t\in[0,T]}$ is given by

$$\mathcal{N}_t = W_t^e - \int_0^t \frac{1}{Z_s} d\langle Z, W^e\rangle, \quad t \in [0,T],$$

and the finite variation part $\mathcal{A} = (\mathcal{A}_t)_{t\in[0,T]}$ is equal to

$$\begin{aligned}
\mathcal{A}_t &= \int_0^t \frac{1}{Z_s} d\langle Z, W^e\rangle \\
&= \int_0^t \frac{1}{Z_s}\Big((\xi_s^1 b_s^{(1)} + \sigma b_s^{(2)})\Lambda_s + (cX_s - \Lambda_s) b_s^{(3)}\hat{\mu}_s\Big) ds.
\end{aligned}$$

Therefore $\mathcal{A}$ is decreasing if

$$(\xi_t^1 b_t^{(1)} + \sigma b_t^{(2)})\Lambda_t + (cX_t - \Lambda_t) b_t^{(3)} \hat{\mu}_t \leq 0, \quad t \in [0,T].$$

This can be reformulated as follows on $\{\tau > t\}$

$$(\xi_t^1 b_t^{(1)} + \sigma b_t^{(2)})\Lambda_t \leq (\Lambda_t - cX_t) b_t^{(3)} \mu_t \tag{4.4.36}$$

By replacing $\Lambda_t = L_t D_t X_t$, where $(D_t)_{t\in[0,T]}$ is defined in (4.4.16), (4.4.36) becomes

$$(\xi_t^1 b_t^{(1)} + \sigma b_t^{(2)}) L_t D_t \leq (L_t D_t - c) b_t^{(3)} \mu_t$$

on $\{\tau > t\}$, or

$$(\xi_t^1 b_t^{(1)} + \sigma b_t^{(2)}) \Big(1 + \frac{c}{1-c} e^{-\alpha(t) - \beta(t)\mu_t}\Big) \leq b_t^{(3)} \mu_t,$$

on the set $\{\tau > t\}$. Hence W^e is a local R-supermartingale. Since $W_t^e = \mathcal{N}_t + \mathcal{A}_t \geq 0$, this implies $\mathcal{N}_t \geq -\mathcal{A}_t$, for all $t \in [0,T]$. Therefore $\mathcal{N}$ is a positive local R-martingale. The fact that $\mathcal{A}$ is a decreasing process together with the supermartingale property of $\mathcal{N}$ imply

$$\mathbb{E}_R[W_t^e|\mathcal{G}_s] = \mathbb{E}_R[\mathcal{N}_t|\mathcal{G}_s] + \mathbb{E}_R[\mathcal{A}_t|\mathcal{G}_s] \leq \mathcal{N}_s + \mathcal{A}_s,$$

for any $s,t \in [0,T]$, with $s \leq t$, i.e. W^e is an R-supermartingale. By Proposition 4.4.9 we also obtain that $R \in \mathcal{M}_{NUI}(W)$. □

4.4.1 Reduced Information Setting

The credibility process F can be alternatively defined in a reduced information setting, by conditioning the probability of default on the remaining time interval $(t,T]$ with respect to the smaller filtration $\mathbb{F}$. Thus F is defined under the form

$$F_t = \mathbb{P}(t < \tau \leq T|\mathcal{F}_t), \tag{4.4.37}$$

for all $t \in [0,T]$. This would correspond to a model where the investors cannot estimate the time of default using the larger information set $\mathcal{G}_t$. The dynamics of F are given by

Proposition 4.4.13. *The credibility process F satisfies under P the following equation*

$$dF_t = -\psi_t \sqrt{\mu_t} dB_t^1 - e^{-\Gamma_t} \mu_t dt,$$

where

$$\psi_t = \theta\beta(t) e^{\alpha(t) + \beta(t)\mu_t - \Gamma_t}, \tag{4.4.38}$$

with

$$\alpha(t) = \frac{2a}{c^2} \ln\Big(\frac{2\lambda e^{\frac{(\lambda-b)(T-t)}{2}}}{(\lambda-b)(e^{\lambda(T-t)}-1)+2\lambda}\Big), \tag{4.4.39}$$

and

$$\beta(t) = -\frac{2(e^{\lambda(T-t)}-1)}{(\lambda-b)(e^{\lambda(T-t)}-1)+2\lambda}, \tag{4.4.40}$$

for all $t \in [0,T]$ *with* $\lambda := \sqrt{b^2+2\theta^2}$.

Proof. See the proof of Proposition 4.4.3. □

We can compute the canonical decomposition of the market wealth process W in the new setting. Note that the structure of the process Λ, as computed in Theorem 4.4.4, remains unchanged.

Theorem 4.4.14. *The market wealth process* W *is a* $(\mathbb{G}, P)$*-semimartingale that admits the canonical decomposition*

$$W_t = M_t + A_t,$$

for all $t \in [0,T]$*, where the local martingale part* M *is given by*

$$\begin{aligned} dM_t &= (f(t,F_t)\xi_t^1 - f_x(t,F_t)\psi_t\sqrt{\mu_t})\Lambda_t dB_t^1 + \sigma f(t,F_t)\Lambda_t dB_t^2 \\ &\quad + (cX_t - f(t,F_t)\Lambda_{t-})d\hat{M}_t, \end{aligned} \tag{4.4.41}$$

and the finite variation part A *is given by*

$$\begin{aligned} dA_t = \Big\{ &[f_t(t,F_t) + f(t,F_t)\xi_t^2 - f_x(t,F_t)\psi_t\xi_t^1\sqrt{\mu_t}]\Lambda_t \\ &+ [(\frac{1}{2}f_{xx}(t,F_t)\psi_t^2 - f_x(t,F_t)e^{-\Gamma_t})\Lambda_t + cX_t]\hat{\mu}_t \Big\} dt. \end{aligned} \tag{4.4.42}$$

Proof. By applying the integration by parts formula and using Theorem 4.4.4 we obtain

$$\begin{aligned} dW_t &= f(t,F_t)d\Lambda_t + \Lambda_t df(t,F_t) + d[\Lambda, f(\cdot,F)]_t + R_t dH_t \\ &= f(t,F_t)d\Lambda_t + \Lambda_t df(t,F_t) + d[\Lambda, f(\cdot,F)]_t + cX_t d\hat{M}_t + cX_t\hat{\mu}_t dt \\ &= f(t,F_t)\Lambda_{t-}(\xi_t^1 dB_t^1 + \sigma dB_t^2 - d\hat{M}_t + \xi_t^2 dt) \\ &+ \Lambda_t(f_t(t,F_t)dt - f_x(t,F_t)\psi_t\sqrt{\mu_t}dB_t^1 - f_x(t,F_t)e^{-\Gamma_t}\mu_t dt + \frac{1}{2}f_{xx}(t,F_t)\psi_t^2\mu_t dt) \\ &- f_x(t,F_t)\Lambda_t\psi_t\xi_t^1\sqrt{\mu_t}dt + cX_t d\hat{M}_t + cX_t\hat{\mu}_t dt \end{aligned}$$

By rearranging the terms, equations (4.4.41) and (4.4.42) follow. □

We now assume that $\mathcal{M}_{loc}(W) \neq \emptyset$ and derive first a general form for the Radon-Nikodym density process associated to a measure $Q \in \mathcal{M}_{loc}(W)$.

Proposition 4.4.15. *Let $Q \in \mathcal{M}_{loc}(W)$ with Radon-Nikodym density process $Z = (Z_t)_{t\in[0,T]}$ i.e. $Z_t = \frac{dQ}{dP}|_{\mathcal{G}_t}$, $t \in [0,T]$. Furthermore, we assume that the quadratic covariation $[Z, M]$ is locally integrable. Then Z admits the representation*

$$dZ_t = Z_t(b_t^{(1)} dB_t^1 + b_t^{(2)} dB_t^2 + b_t^{(3)} d\hat{M}_t), \quad t \in [0,T], \tag{4.4.43}$$

where $(b_t^{(i)})_{t\in[0,T]}$ are $\mathbb{G}$-predictable processes for all $i = 1, 2, 3$, satisfying

$$\begin{aligned} 0 = dA_t + \Big[\Big(b_t^{(1)}(\xi_t^1 f(t,F_t) - f_x(t,F_t)\psi_t\sqrt{\mu_t}) \\ + \sigma b_t^{(2)} f(t,F_t)\Big)\Lambda_t + b_t^{(3)}(cX_t - f(t,F_t)\Lambda_{t-})\hat{\mu}_t\Big] dt \end{aligned} \tag{4.4.44}$$

$dt \otimes dP$-almost surely on $[0,T] \times \Omega$.

Proof. Since the process Z is strictly positive, representation (4.4.43) follows by the martingale representation theorem with respect to $(\mathbb{G}, P)$, see [7]. From the predictable version of Girsanov's Theorem we obtain that W admits under Q the following decomposition

$$W_t = M_t - \int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s + A_t + \int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s, \quad t \in [0,T].$$

Since $Q \in \mathcal{M}_{loc}(W)$, this implies that

$$A_t + \int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s = 0$$

$dt \otimes dP$-a.s. which is equivalent to (4.4.44). □

One can obtain within this setting the analogous results of Theorem 4.4.8 and Propositions 4.4.10 and 4.4.12 by following similar steps to the ones in their proofs, Section 4.4.

4.5 Characterization of $\mathcal{M}_{loc}(W)$ by measure pasting

In this section we characterise $\mathcal{M}_{loc}(W)$, by using the *pasting of measures*, a concept which plays an important role in mathematical finance in the study of American options, in particular in the construction of the Upper and Lower Snell envelopes, as presented in Section 6.4 of Föllmer and Schied [27].

This technique allows us to connect the classical martingale theory of bubbles with a constructive approach to the study of bubbles. The changes that occur in the dynamics of the asset's wealth process are captured through pasting and lead to changes of dynamics in the space of equivalent local martingale measures. Furthermore, pasting allows for a construction of an equivalent local martingale measure for W on the whole interval $[0,T]$, see Theorem 4.5.6. To this purpose we decompose the market wealth process W as the following sum:

$$\begin{aligned} W_t &= W_t^{\sigma_1} + (W_t - W_{\sigma_1})1_{\{t\geq\sigma_1\}} \\ &= W_t^{(1)} + W_t^{(2)}, \end{aligned} \tag{4.5.1}$$

where we denote $W_t^{(1)} := W_t^{\sigma_1}$ and $W_t^{(2)} := (W_t - W_{\sigma_1})1_{\{t\geq\sigma_1\}}$ for $t \in [0,T]$. Note that before the optimistic investors start to influence the price, i.e. before the credibility process goes below the threshold p, the asset's market wealth coincides with the estimated wealth, and in particular, $W_t^{(1)} = W_t^{\sigma_1} = W_t^e$ on $\{\sigma_1 > t\}$. We start our study with a model where there will be only one shift in the dynamics of the asset's market price at time $\sigma_1 < T$.

Assumption 4.5.1. *The following assumptions hold:*

i) We have $0 < \sigma_1 < T$.

ii) For $i = 1,2$, $\mathcal{M}_{loc}(W^{(i)}) \neq \emptyset$.

We wish to describe the structure of the equivalent local martingale measures $Q \in \mathcal{M}_{loc}(W)$. The market wealth process W coincides with the estimated wealth W^e until the starting time σ_1 of the bubble. After σ_1 the underlying price process is affected by the impact of the credibility and starts to change, producing also an alteration of the total wealth process W that deviates from W^e. Hence an equivalent measure $Q \in \mathcal{M}_{loc}(W)$ must take into account the change in the market wealth's dynamics that occurs after σ_1. We can interpret this as a switch of measures created by the investors' heterogeneity or overconfidence. The endogenous construction of the asset's wealth process W allows us to provide a possible explanation to the dynamic in the space of equivalent martingale measures as in the approach of Biagini et al. [4] and Jarrow et al. [40]. Moreover, we show that an equivalent local martingale measure Q for W on the whole interval $[0,T]$ can be obtained through the pasting in σ_1 of two equivalent local martingale measures Q_1 and Q_2 corresponding to the processes W^{σ_1} and $(W - W_{\sigma_1})1_{\{\cdot \geq \sigma_1\}}$ respectively.

We start by recalling some results concerning measure pasting. In the sequel let Q_1 and Q_2 be two equivalent measures on $(\Omega, \mathcal{G}_T)$ and η be a $\mathcal{G}$-stopping time with $0 \leq \eta \leq T$.

Definition 4.5.2. *The probability measure Q*

$$Q(A) := \mathbb{E}_{Q_1}[Q_2(A|\mathcal{G}_\eta)], \quad A \in \mathcal{G}_T,$$

is called the pasting of Q_1 and Q_2 in η.

We remind the reader of the following result which relates the conditional expectation with respect to the measure Q with the conditional expectations with respect to the initial measures Q_1 and Q_2.

Lemma 4.5.3. *If Q is the pasting of Q_1 and Q_2 in η, then for all stopping times ξ and all $\mathcal{G}_T$-measurable random variables $Y \geq 0$ it holds that*

$$\mathbb{E}_Q[Y|\mathcal{G}_\xi] = \mathbb{E}_{Q_1}[\mathbb{E}_{Q_2}[Y|\mathcal{G}_{\eta\vee\xi}]|\mathcal{G}_\xi].$$

Proof. See Lemma 6.40 in [27]. □

The following series of technical results will be used to obtain the desired representation of the elements in the set $\mathcal{M}_{loc}(W)$. For $i = 1, 2$ let $Z^{(i)} := (Z_t^{(i)})_{t\in[0,T]}$ be the corresponding Radon-Nikodym density process

$$Z_t^{(i)} = \frac{dQ_i}{dP}|_{\mathcal{G}_t}, \quad t \in [0,T]. \tag{4.5.2}$$

We put

$$U_t = \frac{dQ_2}{dQ_1}|_{\mathcal{G}_t}, \ t \in [0,T]. \tag{4.5.3}$$

Lemma 4.5.4. *The pasting Q of Q_1 and Q_2 in η is equivalent to Q_1 and satisfies*

$$\frac{dQ}{dQ_1}|_{\mathcal{G}_t} = \frac{U_t}{U_{\eta\wedge t}}, \quad t \in [0,T], \tag{4.5.4}$$

where U is introduced in (4.5.3).

Proof. By Lemma 6.39 in [27] we have

$$\frac{dQ}{dQ_1} = \frac{U_T}{U_\eta}.$$

Furthermore

$$\begin{aligned}
\mathbb{E}_{Q_1}[\frac{dQ}{dQ_1}|\mathcal{G}_t] &= \mathbb{E}_{Q_1}[\frac{U_T}{U_\eta}1_{\{\eta\le t\}}|\mathcal{G}_t] + \mathbb{E}_{Q_1}[\frac{U_T}{U_\eta}1_{\{t<\eta\}}|\mathcal{G}_t] \\
&= \frac{1}{U_\eta}U_t 1_{\{\eta\le t\}} + \mathbb{E}_{Q_1}[\mathbb{E}_{Q_1}[\frac{U_T}{U_\eta}1_{\{t<\eta\}}|\mathcal{G}_\eta]|\mathcal{G}_t] \\
&= \frac{1}{U_\eta}U_t 1_{\{\eta\le t\}} + \mathbb{E}_{Q_1}[1_{\{t<\eta\}}\frac{1}{U_\eta}\mathbb{E}_{Q_1}[U_T|\mathcal{G}_\eta]|\mathcal{G}_t] \\
&= \frac{1}{U_\eta}U_t 1_{\{\eta\le t\}} + \mathbb{E}_{Q_1}[1_{\{t<\eta\}}|\mathcal{G}_t] \\
&= \frac{U_t}{U_{t\wedge\eta}}1_{\{\eta\le t\}} + \frac{U_t}{U_{t\wedge\eta}}1_{\{t<\eta\}} \\
&= \frac{U_t}{U_{t\wedge\eta}},
\end{aligned} \tag{4.5.5}$$

□

where the third equality follows from an application of Doob's optional stopping time theorem and the fourth equality follows from fact that $\{\eta > t\} \in \mathcal{G}_t$, since η is a $\mathbb{G}$-stopping time.

This allows us to obtain the Radon-Nykodim density process $Z = (Z_t)_{t\in[0,T]}$ of the measure Q obtained through pasting, i.e. $Z_t = \frac{dQ}{dP}|_{\mathcal{G}_t}$, $t \in [0,T]$.

Corollary 4.5.5. *The process* $Z = (Z_t)_{t\in[0,T]}$ *given by*

$$Z_t = Z_t^{(1)}\frac{U_t}{U_{t\wedge\eta}}, \quad t \in [0,T], \tag{4.5.6}$$

is a P-martingale with respect to the filtration $\mathbb{G}$ and $\frac{dQ}{dP}|_{\mathcal{G}_t} = Z_t$, $t \in [0,T]$. *Furthermore*

$$Z_t = Z_{t\wedge\eta}^{(1)}\frac{Z_t^{(2)}}{Z_{t\wedge\eta}^{(2)}}, \quad t \in [0,T],$$

where $Z^{(1)}$, $Z^{(2)}$ *are given in* (4.5.2).

Proof. By Lemma 4.5.4 and by applying Bayes formula we obtain

$$\begin{aligned}
Z_t = Z_t^{(1)}\frac{U_t}{U_{t\wedge\eta}} &= Z_t^{(1)}\frac{\mathbb{E}_P[\frac{dQ_2}{dP}|\mathcal{G}_t]}{\mathbb{E}_P[\frac{dQ_1}{dP}|\mathcal{G}_t]}\frac{\mathbb{E}_P[\frac{dQ_1}{dP}|\mathcal{G}_{t\wedge\eta}]}{\mathbb{E}_P[\frac{dQ_2}{dP}|\mathcal{G}_{t\wedge\eta}]} \\
&= Z_t^{(1)}\frac{Z_t^{(2)}}{Z_t^{(1)}}\frac{Z_{t\wedge\eta}^{(1)}}{Z_{t\wedge\eta}^{(2)}} = Z_{t\wedge\eta}^{(1)}\frac{Z_t^{(2)}}{Z_{t\wedge\eta}^{(2)}}, \quad t \in [0,T].
\end{aligned}$$

□

The following result represents the central theorem of this section and provides a way of constructing a local martingale measure Q for the process W on the whole interval $[0,T]$, under Assumption 4.5.1. This new measure takes into account the possible changes that occur in the dynamics of W. Furthermore we can determine whether the resulting measure Q belongs to one of the sets $\mathcal{M}_{UI}(W)$ or $\mathcal{M}_{NUI}(W)$.

Theorem 4.5.6. *We assume that Assumption 4.5.1 holds.*

i) Let $Q_i \in \mathcal{M}_{loc}(W^i)$, $i = 1, 2$, and let $Q \approx P$ be the measure obtained by the pasting of Q_1 and Q_2 in σ_1, with Radon-Nikodym density process $Z_t = \frac{dQ}{dP}|_{\mathcal{G}_t}$, $t \in [0,T]$. Then $Q \in \mathcal{M}_{loc}(W)$.
In addition, if $Q_1 \in \mathcal{M}_{NUI}(W^{(1)})$ or $Q_2 \in \mathcal{M}_{NUI}(W^{(2)})$, then $Q \in \mathcal{M}_{NUI}(W)$.

ii) On the other hand, let $Q \in \mathcal{M}_{loc}(W)$ with Radon-Nikodym density $Z = (Z_t)_{t\in[0,T]}$, i.e. $Z_t = \frac{dQ}{dP}|_{\mathcal{G}_t}$, $t \in [0,T]$. There exist $Q_i \in \mathcal{M}_{loc}(W^{(i)})$, $i = 1, 2$, with corresponding Radon-Nikodym density processes $Z_t^{(i)} = \frac{dQ_i}{dP}|_{\mathcal{G}_t}$ given by $Z_t^{(1)} = Z_{t\wedge\sigma_1}$ and $Z_t^{(2)} = \frac{Z_t}{Z_{t\wedge\sigma_1}}$ for all $t \in [0,T]$, such that Z can be written in the form

$$Z_t = Z_t^{(1)} Z_t^{(2)}, \quad t \in [0,T],$$

and Q is the pasting of Q_1 and Q_2 in σ_1.

Proof. *i*) Let $(\tau_n^i)_{n\geq 0}$ be a localizing sequence such that $W^{(i),\tau_n^i}$ is a Q_i-martingale on $[0,T]$ for $i = 1, 2$. We now construct a sequence of $\mathbb{G}$-stopping times $(\tau_n)_{n\geq 0}$ such that W^{τ_n} is a Q-martingale on $[0,T]$, where Q is the pasting of Q_1 and Q_2 in σ_1. We define the sequence of stopping times $(\tau_n)_{n\geq 0}$ by

$\tau_n := \tau_n^1 \wedge \tau_n^2$, $n \geq 0$. For any $s \leq t$, it follows from Lemma 4.5.3 that

$$\begin{aligned}
\mathbb{E}_Q[W_{t\wedge\tau_n}|\mathcal{G}_s] &= \mathbb{E}_{Q_1}[\mathbb{E}_{Q_2}[W_{t\wedge\tau_n}|\mathcal{G}_{s\vee\sigma_1}]|\mathcal{G}_s] \\
&= \mathbb{E}_{Q_1}[\mathbb{E}_{Q_2}[W_{t\wedge\tau_n\wedge\sigma_1} + W^{(2)}_{t\wedge\tau_n}|\mathcal{G}_{s\vee\sigma_1}]|\mathcal{G}_s] \\
&= \mathbb{E}_{Q_1}[W_{t\wedge\tau_n\wedge\sigma_1} + W^{(2)}_{(t\wedge\tau_n)\wedge(s\vee\sigma_1)}|\mathcal{G}_s] \\
&= W_{s\wedge\tau_n\wedge\sigma_1} + \mathbb{E}_{Q_1}[W^{(2)}_{(t\wedge\tau_n)\wedge(s\vee\sigma_1)}|\mathcal{G}_s] \\
&= W_{s\wedge\tau_n\wedge\sigma_1} + \mathbb{E}_{Q_1}[1_{\{s\wedge\tau_n<\sigma_1\}}W^{(2)}_{(t\wedge\tau_n\wedge\sigma_1)\vee(s\wedge\tau_n)}|\mathcal{G}_s] \\
&\quad + \mathbb{E}_{Q_1}[1_{\{s\wedge\tau_n\geq\sigma_1\}}W^{(2)}_{(t\wedge\tau_n\wedge\sigma_1)\vee(s\wedge\tau_n)}|\mathcal{G}_s] \\
&= W_{s\wedge\tau_n\wedge\sigma_1} + \mathbb{E}_{Q_1}[1_{\{s\wedge\tau_n<\sigma_1\}}W^{(2)}_{t\wedge\tau_n\wedge\sigma_1}|\mathcal{G}_s] \\
&\quad + \mathbb{E}_{Q_1}[1_{\{s\wedge\tau_n\geq\sigma_1\}}W^{(2)}_{s\wedge\tau_n}|\mathcal{G}_s] \\
&= W_{s\wedge\tau_n\wedge\sigma_1} + W^{(2)}_{s\wedge\tau_n}1_{\{s\wedge\tau_n\geq\sigma_1\}} \\
&= W^{(1)}_{s\wedge\tau_n} + W^{(2)}_{s\wedge\tau_n} = W_{s\wedge\tau_n},
\end{aligned} \tag{4.5.7}$$

since $W^{(2)}_{t\wedge\tau_n\wedge\sigma_1} = (W_{t\wedge\tau_n\wedge\sigma_1} - W_{\sigma_1})1_{\{t\wedge\tau_n\wedge\sigma_1\geq\sigma_1\}} = 0$. The second and last equality in (4.5.7) follow from (4.5.1). In these computations, we have used the measurability properties of the stopping times and Doob's optional stopping time theorem.

Hence $Q \in \mathcal{M}_{loc}(W)$. For the second part of i) we use the fact that if $Q \in \mathcal{M}_{NUI}(W)$, then there exists $t \in [0,T]$ such that

$$W_t > \mathbb{E}_Q[W_T|\mathcal{G}_t].$$

This is a consequence of the fact that W is a positive local martingale and hence a supermartingale. Therefore, the supermartingale inequality holds

$$W_t \geq \mathbb{E}_Q[W_T|\mathcal{G}_t], \quad t \in [0,T].$$

for all $Q \in \mathcal{M}_{loc}(W)$, with equality for all $Q \in \mathcal{M}_{loc}(W)$. If W is not a uniformly integrable martingale, then the inequality must be strict at least for some $t \in [0,T]$.

By applying Lemma 4.5.3 we obtain

$$\begin{aligned}
\mathbb{E}_Q[W_T|\mathcal{G}_t] &= \mathbb{E}_{Q_1}[\mathbb{E}_{Q_2}[W_T|\mathcal{G}_{\sigma_1\vee t}]|\mathcal{G}_t] \\
&\leq \mathbb{E}_{Q_1}[W_{\sigma_1\vee t}|\mathcal{G}_t] \leq W_t,
\end{aligned} \tag{4.5.8}$$

where one of the inequalities is strict for some t if Q_1 or Q_2 belong to the set $\mathcal{M}_{NUI}(W)$.

ii) Consider the set

$$\mathcal{Z}(W) := \{Z; ZW \text{ is a } P-\text{local martingale}\}.$$

Then the Radon-Nikodym density process Z belongs to $\mathcal{Z}(W)$. By Lemma 2.3. of Stricker and Yan [59], since $\sigma_1 < T$, we have that $Z \in \mathcal{Z}(W^{\sigma_1})$ i.e. ZW^{σ_1} is a P-local martingale on $[0,T]$. We define the measure $Q_1 \approx P$ by

$$\frac{dQ_1}{dP} = Z_{\sigma_1}.$$

and prove that $\frac{Z_t}{Z_{t\wedge\sigma_1}}$ is a Radon-Nikodym density process for a measure $Q_2 \approx P$ such that $W^{(2)}$ is a Q_2-local martingale on $[0,T]$. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for the Q-local martingale W. Then

$$\begin{aligned}
\mathbb{E}_P\Big[\frac{Z_t}{Z_{t\wedge\sigma_1}} W^{(2)}_{t\wedge\tau_n}|\mathcal{G}_s\Big] &= \mathbb{E}_P\Big[\frac{Z_t}{Z_{t\wedge\sigma_1}}(W_{t\wedge\tau_n} - W_{\sigma_1})1_{\{t\wedge\tau_n\geq\sigma_1\}}|\mathcal{G}_s\Big] \\
&= \mathbb{E}_P\Big[\frac{Z_t}{Z_{t\wedge\sigma_1}}(W_{t\wedge\tau_n} - W_{\sigma_1})1_{\{t\wedge\tau_n\geq\sigma_1\}}(1_{\{s\wedge\tau_n\geq\sigma_1\}} \\
&\quad + 1_{\{s\wedge\tau_n<\sigma_1\leq t\wedge\tau_n\}} + 1_{\{t\wedge\tau_n<\sigma_1\}})|\mathcal{G}_s\Big] \\
&= \mathbb{E}_P\Big[\frac{Z_t}{Z_{t\wedge\sigma_1}}(W_{t\wedge\tau_n} - W_{\sigma_1})1_{\{s\wedge\tau_n\geq\sigma_1\}}|\mathcal{G}_s\Big] \\
&\quad + \mathbb{E}_P\Big[\frac{Z_t}{Z_{t\wedge\sigma_1}}(W_{t\wedge\tau_n} - W_{\sigma_1})1_{\{s\wedge\tau_n<\sigma_1\leq t\wedge\tau_n\}}|\mathcal{G}_s\Big] \\
&= 1_{\{s\wedge\tau_n\geq\sigma_1\}}\frac{1}{Z_{\sigma_1}}(\mathbb{E}_P[Z_t W_{t\wedge\tau_n}|\mathcal{G}_s] - \mathbb{E}_P[Z_t W_{\sigma_1}|\mathcal{G}_s]) \\
&\quad + \mathbb{E}_P\Big[\mathbb{E}_P\Big[\frac{Z_t}{Z_{t\wedge\sigma_1}}(W_{t\wedge\tau_n} - W_{\sigma_1})1_{\{s\wedge\tau_n<\sigma_1\leq t\wedge\tau_n\}}|\mathcal{G}_{s\vee\sigma_1}\Big]|\mathcal{G}_s\Big] \\
&= \frac{Z_s}{Z_{\sigma_1\wedge s}}(W_{s\wedge\tau_n} - W_{\sigma_1})1_{\{s\wedge\tau_n\geq\sigma_1\}} \\
&\quad + \mathbb{E}_P\Big[1_{\{s\wedge\tau_n<\sigma_1\leq t\wedge\tau_n\}}\frac{1}{Z_{\sigma_1}}(\mathbb{E}_P[Z_t W_{t\wedge\tau_n}|\mathcal{G}_{s\vee\sigma_1}] \\
&\quad - \mathbb{E}_P[W_{\sigma_1} Z_t|\mathcal{G}_{s\vee\sigma_1}])|\mathcal{G}_s\Big] \\
&= \frac{Z_s}{Z_{\sigma_1\wedge s}} W^{(2)}_{s\wedge\tau_n} + \mathbb{E}_P\Big[1_{\{s\wedge\tau_n<\sigma_1\leq t\wedge\tau_n\}}\frac{1}{Z_{\sigma_1}}(Z_{t\wedge(\sigma_1\vee s)} W_{(t\wedge\tau_n)\wedge(\sigma_1\vee s)} \\
&\quad - W_{\sigma_1} Z_{t\wedge(\sigma_1\vee s)})|\mathcal{G}_s\Big] \\
&= \frac{Z_s}{Z_{\sigma_1\wedge s}} W^{(2)}_{s\wedge\tau_n} + \mathbb{E}_P\Big[1_{\{s\wedge\tau_n<\sigma_1\leq t\wedge\tau_n\}}\frac{Z_{t\wedge(\sigma_1\vee s)}}{Z_{\sigma_1}}(W_{t\wedge\tau_n\wedge\sigma_1} - W_{\sigma_1})|\mathcal{G}_s\Big] \\
&= \frac{Z_s}{Z_{\sigma_1\wedge s}} W^{(2)}_{s\wedge\tau_n}.
\end{aligned}$$

These computations are a consequence of the definition of $W^{(2)}$, see 4.5.1, of the stopping times' measurability properties, as well as of Doob's optional stopping time theorem. Hence $(\frac{1}{Z_{\sigma_1 \wedge t}} Z_t W_t^{(2)})_{t\in[0,T]}$ is a P-local martingale. Thus we can define

$$\frac{dQ_2}{dP}|_{\mathcal{G}_t} := \frac{Z_t}{Z_{\sigma_1 \wedge t}}, \quad t \in [0,T],$$

and by Corollary 4.5.5 it follows that Q is the pasting of Q_1 and Q_2 in σ_1. □

Theorem 4.5.6 shows how the selection as pricing measure of a martingale measure belonging to $\mathcal{M}_{NUI}(W)$ on a stochastic time interval leads to the formation of a bubble in the sense of Definition 4.3.3. This is made clear by the fact that the pricing measure corresponding to the time interval $[0,T]$, and which is obtained through pasting, preserves the strict local martingale property of the wealth process W. Reciprocally, one can decompose such a measure into two pricing measures corresponding to smaller time intervals.

Remark 4.5.7. *Note however that this model can be extended to include succesive changes in the dynamics of the market wealth process* W*. Let* $(\sigma_n)_{n\in\mathbb{N}}$ *be a sequence of stopping times marking the moment when the process* W *is influenced by diverse microeconomic factors (risk-aversion, liquidity etc.). Then one can consider successive measure pastings in order to construct a measure* Q *belonging to* $\mathcal{M}_{loc}(W)$ *on the whole interval* $[0,T]$*.*

Appendix A

Strict local martingales

This section is intended to provide the reader an introduction to strict local martingales and a (very) short list of references to the mathematical literature on this topic. The question whether a local martingale process is a true martingale or a strict local martingale has generated growing interest for stochastic analysis and also mathematical finance, especially due to their importance in the modelling of financial bubbles.

A strict local martingale is a local martingale which is not a martingale. One of the most known examples of strict local martingales is the inverse 3-dimensional Bessel process.

Example A.0.8. *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *be a filtered probability space and* $(W_t)_{t\geq 0}$ *is a* 3*-dimensional Brownian motion starting in* $x_0 \neq 0$*, we define the process*

$$X_t = \|W_t\|^{-1}, \quad t \geq 0.$$

One can show that $(X_t)_{t\geq 0}$ *a local martingale with the corresponding localizing sequence* $\tau_n = \inf\{t \geq 0; \|W_t\| \leq \frac{1}{n}\}$*. Moreover, we have that*

$$\mathbb{E}_P[X_0] = \frac{1}{\|x_0\|}$$

while $\lim_{t\to\infty} \mathbb{E}_P[X_t] = 0$*. Hence* X *cannot be a martingale, since it does not have constant expectation. For a detailed proof we refer to Chung and Williams [15].*

We now provide an easy intuitive example of strict local martingale, taken from Kazamaki [43]. For the sake of completeness, we also present the corresponding proof.

Lemma A.0.9. *[43] Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *be a filtered probability space and* $(B_t)_{t\geq 0}$ *be a* 1*-dimensional Brownian motion with* $B_0 = 0$*. We define the* *stopping time* $\tau = \inf\{t \geq 0; B_t \leq t-1\}$. *The process* $(L_t)_{t\geq 0}$ *given by*

$$L_t = \mathcal{E}\left(\alpha B^{\tau}\right)_t, \quad t \geq 0$$

is a strict P*-local martingale, for any* $\alpha > 1$.

Proof. It follows from the continuity of B that

$$\mathcal{E}\left(B^{\tau}\right)_{\infty} = \exp(\frac{\tau}{2} - 1),$$

since $\tau < \infty$. Therefore $\mathbb{E}_P[\exp(\frac{\tau}{2})] \leq e$, which implies by Novikov's criterion that $\mathbb{E}_P[\mathcal{E}\left(B^{\tau}\right)] = 1$ and in particular $\mathbb{E}_P[\exp(\frac{\tau}{2})] = e$. We have that

$$\begin{aligned}\mathbb{E}_P[L_{\infty}] &= \mathbb{E}_P[\mathcal{E}\left(\alpha B^{\tau}\right)_{\infty}] = \mathbb{E}_P[\exp(\alpha\tau - \alpha - \frac{1}{2}\alpha^2\tau)] \\ &\leq e^{-\alpha}\mathbb{E}_P[\exp(\frac{\tau}{2})] = e^{-\alpha+1} < 1.\end{aligned}$$

Therefore L is a strict local martingale. □

We conclude this section by presenting the reader a list of recent results concerning strict local martingales.

Stochastic exponentials of continuous local martingales, often used as density processes for absolutely continuous measure changes, have been extensively studied and several criterias for determining whether a process is a strict local martingale or a true martingale have been developed, see Carr et al.[12], Mijatovic and Urusov [45].

A more general approach (which doesn't require the continuity property of the process) for the study of right-continuous positive local martingales involves the use of the Föllmer measure [24]. For each positive right-continuous local martingale X one can associate a measure P^X such that

$$\mathbb{E}_P[X_t] = P^X(\tau^X > t),$$

where $\tau^X = \lim_{n\to\infty}(\inf\{t \geq 0; X_t > n\} \wedge n)$. The interested reader may consult Jarrow and Larsson [35], Carr et al.[13], Kardaras et al.[42] for a series of interesting results concerning this method. For an alternative way of constructing strict local martingales with jumps and other examples, see Protter [53].

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