Nearcritical Percolation and Crystallisation

Simon Aumann

Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
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Zusammenfassung

Diese Dissertation enthält Ergebnisse über Singularität von Skalen-Limiten nahkritischer Perkolation, eine Steifheitsabschätzung sowie ein Resultat zu spontaner Rotationssymmetriebrechung.

Zunächst wird gezeigt, dass auf dem Dreiecksgitter die Verteilungen der Skalen-Limiten nahkritischer Explorationspfade mit verschiedenen Parametern zueinander singulär sind. Dies verallgemeinert mit einer ähnlichen Technik ein Resultat von Nolin und Werner. Man kann folgern, dass die Singularität sogar schon an einem infesitimal kleinen Anfangsstück erkannt werden kann, und dass nahkritische Skalen-Limiten von Explorationspfaden unter Streckungen wechselseitig singulär werden.

Als zweites werden Skalen-Limiten der gesamten Konfiguration zweidimensionaler Perkolation in der sogenannten Quad-Crossing-Topologie untersucht, die von Schramm und Smirnov eingeführt wurde. Es wird gezeigt, dass zwei solche Limiten mit unterschiedlichen Parametern singulär zueinander sind. Dieses Resultat gilt für Perkolationsmodelle auf ziemlich allgemeinen Gittern, beispielsweise für Kanten-Perkolation auf dem Quadrad-Gitter und für Ecken-Perkolation auf dem Dreiecksgitter.

Drittens wird eine Steifheitsabschätzung für 1-Formen mit nichtverschwindender äußerer Ableitung gezeigt, die ein Theorem von Friesecke, James and Müller verallgemeinert.

Schließlich wird diese Abschätzung verwendet, um eine Variante spontaner Brechung der Rotationssymmetrie für Kristall-Modelle zu zeigen, die nahezu beliebige Defekte erlauben; dazu zählen unbeschränkte Defekte sowie Stufen-, Schraub- und gemischte Versetzungen, also Defekte mit Burgers Vektoren.

AMS Mathematics Subject Classification 2010:
60K35, 82B43, 60G30, 82B27, 53C24, 82D25, 82B21

Schlüsselwörter:
nahkritisch, Perkolation, Explorationspfad, voller Skalen-Limes, singulär, Steifheitsabschätzung, Kristall, spontane Symmetriebrechung, Burgers Vektor, beliebige Defekte

Abstract

This thesis contains results on singularity of nearcritical percolation scaling limits, on a rigidity estimate and on spontaneous rotational symmetry breaking.

First it is shown that – on the triangular lattice – the laws of scaling limits of nearcritical percolation exploration paths with different parameters are singular with respect to each other. This generalises a result of Nolin and Werner, using a similar technique. As a corollary, the singularity can even be detected from an infinitesimal initial segment. Moreover, nearcritical scaling limits of exploration paths are mutually singular under scaling maps.

Second full scaling limits of planar nearcritical percolation are investigated in the Quad-Crossing-Topology introduced by Schramm and Smirnov. It is shown that two nearcritical scaling limits with different parameters are singular with respect to each other. This result holds for percolation models on rather general lattices, including bond percolation on the square lattice and site percolation on the triangular lattice.

Third a rigidity estimate for 1-forms with non-vanishing exterior derivative is proven. It generalises a theorem on geometric rigidity of Friesecke, James and Müller.

Finally this estimate is used to prove a kind of spontaneous breaking of rotational symmetry for some models of crystals, which allow almost all kinds of defects, including unbounded defects as well as edge, screw and mixed dislocations, i.e. defects with Burgers vectors.

AMS Mathematics Subject Classification 2010:
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Keywords:
nearcritical, percolation, exploration path, full scaling limit, singular, rigidity estimate, crystal, spontaneous symmetry breaking, Burgers vector, arbitrary defects

Table of Contents

Chapter 1

Introduction

Statistical mechanics is an important branch of physics. Its goal is to describe the behaviour of large systems consisting of many microscopic objects. Since it is almost impossible to trace back the exact actions of all microscopic objects, one takes a statistical approach. The actions of a microscopic object is not described exactly, but probabilistic using some distribution. Then one wants to extract the behaviour of the whole system using some appropriate random variables and statistical tools like the mean or the variance, to give the simplest examples.

Besides giving physical explanations for various phenomena, statistical mechanics also raised models having mathematically rich and interesting structure. Physicists have many very good predictions based on heuristics and physical explanations, but the mathematical rigorous understanding of that models is well behind. This thesis adds some little pieces to the rigorous understanding.

One phenomenon of interest is the structure of a stone; let it be a porous stone or a crystal. Some models are described in this thesis, which can be used to describe such stones. Stones fit perfectly into the framework of statistical mechanics since it is very hard to determine the exact positions of single particles, but they can be described with some probability measure.

If someone has a porous stone in his hand, he may want to know whether it is permeable to water or not. Instead of looking at the exact structure of the holes in the stone, he only estimates which portion of the porous stone is rock and which portion is just air. The porous stone is partitioned into cells according to some fine grid. He assumes that each cell is empty with some probability which is given by the air portion of the porous stone. Otherwise it is filled up with rock. Moreover, he assumes that this happens for each cell independently. It may be surprising that it is possible to decide almost certain, whether the porous stone is permeable to water or not, i.e. whether water percolates through the stone or not. This yields to the mathematical model of percolation, which is the topic of the first part of the thesis.

Another interesting example for the structure of a stone is the crystallisation phenomenon. Crystals are built by molecules arranged in a regular pattern; but they also have defects. Though the microscopic forces on the molecules are rotational symmetric, whole crystals are not symmetric with respect to all rotations, but only with respect to some particular ones. In contrast, liquids are completely symmetric under rotations. So what is the difference between liquids and crystals? In order to describe crystals, one has to consider models which break rotational symmetry. The second part of the thesis describes such a model. Thereto we will need an estimate of geometric rigidity.

As already indicated, this thesis consists of two part, each of them consisting of two chapters. In the following, we introduce the topics of the chapters more precisely. We also state the main results of the thesis, but in simplified versions, since it is not reasonable to state the whole notation rigorously in the introduction.

All chapters can be understood independently of each other. The only interference between them is that the main result of Chapter 4 is needed in Chapter 5. Finally we state a word on the notation. Each chapter introduces its notation separately and independently. If the same symbol is used in different chapters, it usually denotes (almost) the same object, but the formal definition may be different. Constants probably have different values in different chapters, even if their labels are equal.

1.1 Nearcritical Percolation

The first part of the thesis is on percolation, more precisely on nearcritical percolation. Percolation is a widely spread model in statistical mechanics and is maybe the simplest one to state. It can be used, for instance, to model a porous stone as indicated above. In this thesis we always stick to two-dimensional percolation.

In the following, we explain one specific percolation model, namely face percolation on the (infinite) honeycomb lattice, which is equivalent to site percolation on the triangular lattice. The other archetypical example is bond percolation on the square lattice.

The model is simply the following: A coin, which shows heads with some fixed probability $p \in [0, 1]$, is thrown independently for each honeycomb. If the coin shows heads, the corresponding honeycomb is coloured blue; and if it shows tails, it is coloured yellow. Equivalently one can declare it either open or closed. That is all. Some percolation instances with different values of p are shown in Figures 1.1, 1.2 and 1.3.

Then one looks at the random blue and yellow clusters, i.e. connected hexagons with the same colour. We may consider the yellow clusters as land and the blue clusters as water. How does the scenario look like? Does it look like a big continent with lakes, or rather like a great ocean with islands, or like some mixture? This depends on p, of course, and probably also on the randomness. It is astonishing that there is a critical probability $p_c \in [0, 1]$ such that the qualitative picture only depends on the numeric order of p and p_c. If $p < p_c$, then there is a unique infinite yellow cluster and no blue infinite cluster, cf. Figure 1.1. Thus the image of a big continent with some lakes is correct. Of course, the lakes also have islands, which

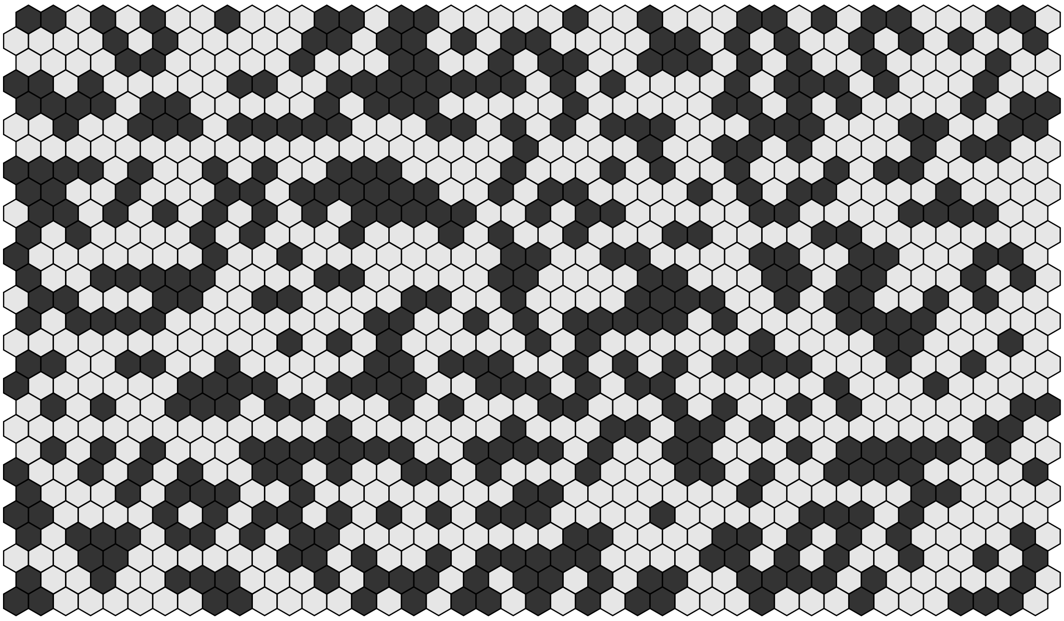


Figure 1.1: An instance of subcritical percolation, with $p = 0.4 < p_c$

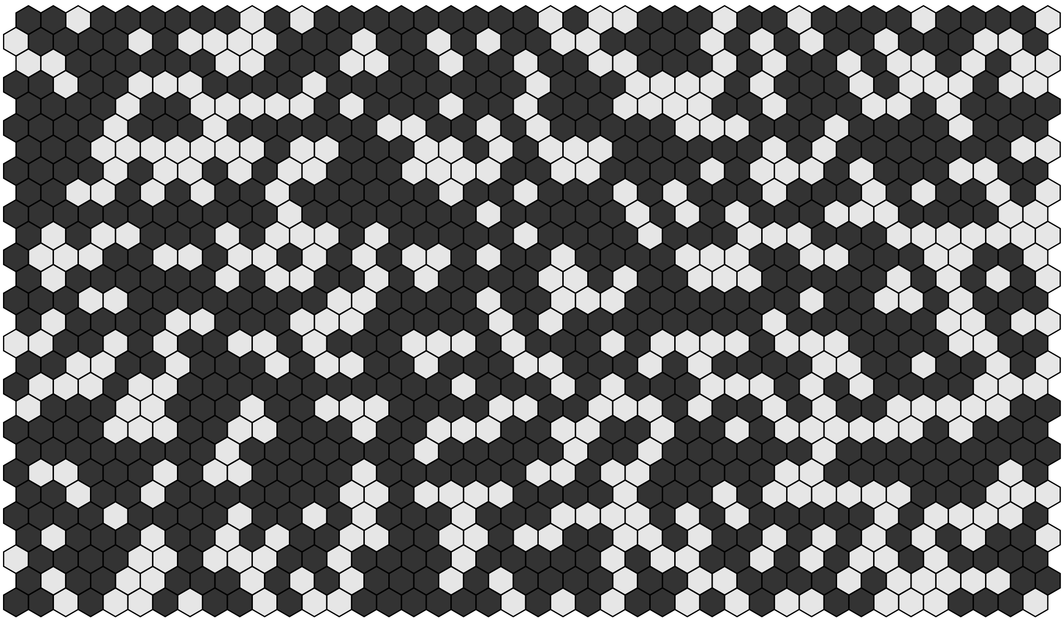


Figure 1.2: An instance of supercritical percolation, with $p = 0.6 > p_c$

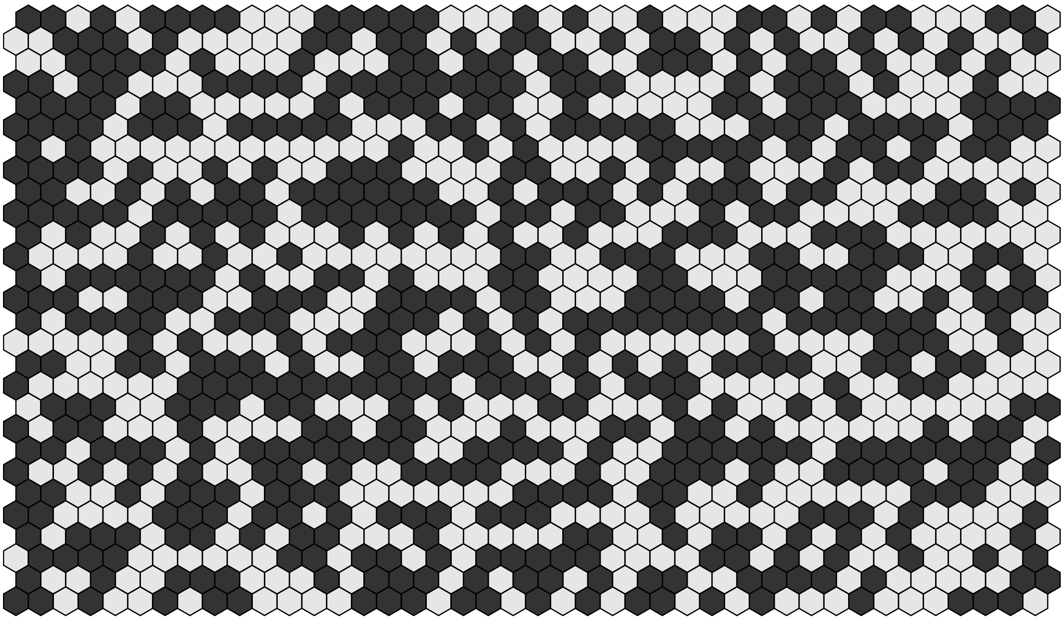


Figure 1.3: An instance of critical percolation, with $p = 0.5 = p_c$

have small lakes, and so on. And if $p > p_c$, it is the other way round, cf. Figure 1.2. There exists a great blue ocean with islands. The most interesting case is $p = p_c$, cf. Figure 1.3. Then there is no infinite cluster, such that there are arbitrarily large blue and yellow clusters surrounding each other. In that case, the "coasts", i.e. the lines separating the blue and yellow clusters, are especially interesting.

Instead of looking at percolation on the infinite lattice, we could also consider percolation in a fixed finite domain, but with variable mesh size $\eta > 0$. Then an interesting scenario is the limit $\eta \to 0$. If $p < p_c$, then, in the limit, the whole domain is basically yellow: the probability that any open set is coloured blue tends to zero. Conversely, if $p > p_c$, everything is basically blue in the limit. Thus the most interesting case is the critical case $p = p_c$.

In order to determine what happens in the limit if $p = p_c$, we trace back the coasts, i.e. the interfaces between the blue and the yellow clusters. For fixed mesh size, they will be some random polygonal lines. But in the limit, an interface will be a fractal curve, which touches itself many times. It was a major problem to determine this limit. In 2001, Smirnov proved the conformal invariance of the scaling limit of critical percolation interfaces on the triangular lattice. This paved the way for describing this limit by a Schramm-Loewner-Evolution and for determining various crossing probabilities. Thus nowadays the scaling limit of critical percolation is quite well understood.

Therefore, we know the limit in the subcritical case $p < p_c$, in the supercritical case $p > p_c$ as well as in the critical case $p = p_c$. Thus, one might think that everything is known. But this is not the case, since there are also nearcritical limits. These are obtained by choosing p depending on the mesh size and slightly different from p_c, but converging to p_c in a well-chosen speed. In fact, one has one free real parameter in the speed factor. These nearcritical limits are by far not as well understood as the others. In the first part of this thesis, we will show some facts on these nearcritical limits. We will explain them in the next two subsections.

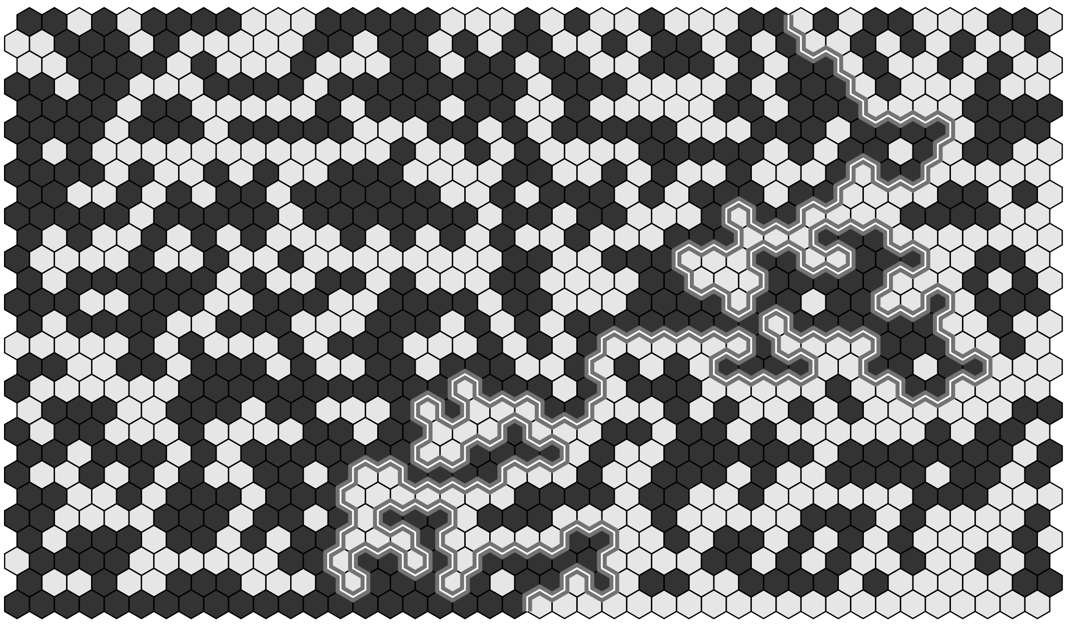
1.1.1 Singularity of Nearcritical Exploration Paths

In Chapter 2, we consider the scaling limit of nearcritical exploration paths. Garban, Pete and Schramm showed in [GPS-13b] that, in the quad-crossing space, there indeed exist nearcritical limits, not only limit points. But we do not use this fact, since we are interested in the exploration paths. For that, only the existence of limit points, not of a limit, is yet established.

Any percolation configuration induces its *exploration path* as follows. We deterministically colour the hexagons on the negative real axis blue and the hexagons on the positive axis yellow. Then there is a unique path starting at the origin, having blue hexagons to the left and having yellow hexagons to the right. This path is the exploration path. Figure 1.4 shows an example.

Nolin and Werner showed in [NW-09] that every nearcritical scaling limit point of exploration paths is singular with respect to an SLE_6 curve, i.e. to the critical limit. In the present thesis, we enhance this result by showing that two different

Figure 1.4: A critical exploration path



nearcritical scaling limits are singular with respect to each other (Theorem 2.1). It is even possible to detect the singularity by looking at an infinitesimal initial segment of the exploration path (Corollary 2.2). Applying the main result to conformal maps, we obtain that nearcritical scaling limits are in general not conformally invariant or absolutely continuous. In fact, under scaling maps, they are mutually singular (Corollary 2.12).

Let us now state these theorems a bit more precisely. We consider nearcritical face percolation on the honeycomb lattice with mesh size $\eta > 0$ in the upper half plain. It is nearcritical percolation, if each hexagon is blue with probability

$$p_\eta^\iota = \frac{1}{2} + \iota \cdot \frac{\eta^2}{\alpha_4^\eta},$$

where $\iota \in \mathbb{R}$ is a free real parameter and α_4^η is the probability that there exist four arms of alternating colours up to (Euclidean) distance 1 in critical site percolation on the triangular lattice with mesh size η.

Let Γ_η^ι denote the law of the exploration path. A simplified version of Theorem 2.1 reads as follows:

Theorem. *Let $\mu < \lambda$ be real numbers. Let further $(\eta_k)_{k\in\mathbb{N}}$ be a sequence converging to zero such that $\Gamma_{\eta_k}^\mu \to \Gamma^\mu$ and $\Gamma_{\eta_k}^\lambda \to \Gamma^\lambda$ weakly for some measures Γ^μ and Γ^λ, and such that the probability that a quad is crossed converges. Then the probability measures Γ^μ and Γ^λ are singular with respect to each other.*

Let us remark that there exist sequences $(\eta_k)_k$ fulfilling the hypothesis. The result of Nolin and Werner is the special case $\mu = 0$. If $\mathcal{A}$ denotes the σ-algebra of infinitesimal initial segments of paths starting at the origin, Corollary 2.2 is

Corollary. *Under the conditions of Theorem 2.1, the laws Γ^μ and Γ^λ restricted to $\mathcal{A}$ are singular with respect to each other.*

Interestingly, the proof of Nolin and Werner can be extended to our result. But one has to be careful. In fact, we also give a more detailed and self-contained version of their proof. Nevertheless, some modifications and slightly different approaches are needed. In particular, the non-existence of an analogue to Cardy's formula requires some work. Namely, we need the fact that the probability of crossing a quad with fractal boundary can be well approximated using rather weak approximations to the quad (Lemma 2.3).

These results are properly stated and proven in Chapter 2. This chapter, and also parts of this introduction, are almost literally already published in [A-14a].

1.1.2 Singularity of Full Scaling Limits

After determining the scaling limit of critical exploration paths, limits of full percolation configurations have been explored, too. In Chapter 3 we consider such full configuration limits. We already explained that, in order to obtain a scaling limit, one considers percolation on a lattice with mesh size $\eta > 0$ and lets η tend to 0. In the case of the full configuration limit, it is a-priori not clear, in what sense, or in what topology, the limit $\eta \to 0$ shall be taken. There are several possibilities, nine of them are explained in [SS-11, p. 1770ff]. It is highly non-trivial that these different approaches yield equivalent results. Camia and Newman established the full scaling limit of critical percolation on the triangular lattice as an ensemble of oriented loops, see [CN-06]. Schramm and Smirnov suggested to look at the set of quads which are crossed by the percolation configuration and constructed a nice topology for that purpose, the so-called Quad-Crossing-Topology, see [SS-11]. Since it is closely related to the original physical motivation of percolation and it yields the existence of limit points for free (by compactness), we choose to work with Schramm and Smirnov's set-up.

They considered percolation models on tilings of the plane, rather than on lattices. Each tile is either coloured blue or yellow, independently of each other. All site or bond percolation models can be handled in this way using appropriate tilings. The results of [SS-11] hold on a wide range of percolation models. In fact, two basic assumptions on the one-arm event and on the four-arm event are sufficient. The results of Chapter 3 also hold on rather general tilings, but a bit stronger assumptions are needed. Basically, we require the assumption of [SS-11] on the four-arm event and the Russo-Seymour-Welsh Theory (RSW). The exact conditions are presented in Chapter 3. In particular, we need the arm separation lemmas of [K-87] and [N-08]. They should hold on any graph which is invariant under reflection in one of the coordinate axes and under rotation around the origin by an angle $\phi \in (0, \pi)$, as stated in [K-87, p. 112]. But the proofs are written up only for bond or site percolation on the square lattice in [K-87] and for site percolation on the triangular lattice in [N-08]. Hence we choose to formulate the exact properties we need as conditions. We will first prove our results under that conditions and we will verify them for bond percolation on the square lattice and site percolation on the triangular lattice afterwards.

Again we want to consider nearcritical scaling limits. Nearcritical percolation is obtained by colouring a tile blue with a probability slightly different from the critical one. The difference depends on the mesh size, but converges to zero in a well-chosen speed. It includes one free real parameter. We may choose this parameter different for different tiles. The main result of Chapter 3 is the following: We consider two inhomogeneous nearcritical percolations such that the differences of their parameters are uniformly bounded away from zero in a macroscopic region. Then we show that any corresponding sub-sequential scaling limits are singular with respect to each other.

More precisely: Let $\eta > 0$ and let H_η be a locally finite tiling whose tiles have diameter at most η. For each tile t we choose a number $\iota_\eta(t)$ inside a fixed compact interval and colour the tile blue with probability

$$p_\eta^\iota(t) \;=\; p_c + \iota_\eta(t) \cdot \frac{\eta^2}{\alpha_4^\eta}\,,$$

where α_4^η is the probability of a four arm event again. This induces a probability measure P_η^ι on the Quad-Crossing space. Then a simplified version of Theorem 3.1 reads as follows.

Theorem. *Let H_η, $\eta > 0$, be locally finite tilings such that each tile has diameter at most η. We assume that the tilings satisfy some rather general conditions (RSW and the four-arm exponent being greater than 1 are enough). Let $\mu_\eta(t), \lambda_\eta(t)$ be real numbers in the fixed compact interval. They induce measures P_η^μ and P_η^λ. Considering weak limits with respect to the Quad-Crossing-Topology, let P^μ be any weak limit point of $\{P_\eta^\mu : \eta > 0\}$, let P^λ be any weak limit point of $\{P_\eta^\lambda : \eta > 0\}$ and let η_n, $n \in \mathbb{N}$, be a sequence converging to zero such that $P_{\eta_n}^\mu \to P^\mu$ and $P_{\eta_n}^\lambda \to P^\lambda$ weakly as $n \to \infty$.*

Assume that there exist $\sigma > 0$ and an open, non-empty set D such that

$$\lambda_\eta(t) - \mu_\eta(t) \geq \sigma$$

uniformly in $\eta \in \{\eta_n : n \in \mathbb{N}\}$ and all tiles which are contained in D.

Then the laws P^μ and P^λ are singular with respect to each other.

Lemma 3.5 shows

Lemma. *The conditions needed for Theorem 3.1 are fulfilled by tilings representing site percolation on the triangular lattice or bond percolation on the square lattice.*

Similarly to the results on the exploration paths, we can even detect the singularity if we restrict the probability measures to the σ-algebra of the infinitesimal neighbourhood of any point inside D. This is the content of Corollary 3.2.

In the preceding section we already explained the following. Nolin and Werner showed in [NW-09, Proposition 6] that – on the triangular lattice – any (subsequential) scaling limit of nearcritical exploration paths is singular with respect to an SLE_6 curve, i.e. to the limit of critical exploration paths. This is extended in Chapter 2, where it is shown that the limits of two nearcritical exploration paths with different parameters are singular with respect to each other.

The result of Chapter 3 is somewhat different to those results, as we will now explain. First, we consider different objects. While in [NW-09] and Chapter 2 the singularity of exploration paths is detected, here it is the singularity of the full configurations in the Quad-Crossing-Topology. As long as the equivalence of different descriptions of the limit object is not proven, these are independent results. In particular, it is – even on the triangular lattice – an open question, whether the exploration path as a curve is a random variable of the set of all crossed quads (cf. [GPS-13a, Question 2.14]). Though the trace of the exploration path can be recovered from the set of all crossed quads, it is not clear how to detect its behaviour at double points. Thus the present result is not an easy corollary to the singularity of the exploration paths. Second, the results of [NW-09] and Chapter 2 hold only for site percolation on the triangular lattice, whereas the results of Chapter 3 hold under rather general assumptions on the lattice, which are, for instance, also fulfilled by bond percolation on the square lattice. Last, and indeed least, the percolation may also be inhomogeneous here. Since the restriction to homogeneous percolation in [NW-09] and Chapter 2 has only technical, but not conceptual reasons, this is only a minor difference.

The proofs in Chapter 3 use ideas from [NW-09] and Chapter 2. In fact, they are technically simpler since there is no need to consider domains with fractal boundary. First we prove an estimate of a difference of the probability that a square is crossed. Then we show an abstract result, which is used to detect the singularity.

These results are properly stated and proven in Chapter 3. This chapter, and also parts of the current section, are almost literally already published in [A-14b].

1.2 Crystallisation

Condensed matters in solid state usually have the structure of a crystal: The molecules are arranged in some regular pattern. Real crystals are in fact not perfectly regular, but form a perturbation of the pattern. They also have defects. One can describe a crystal using the fundamental approach of statistical mechanics. Some probability distributions determine the location of the molecules. Their local interaction should specify the distribution. One wants to extract the global behaviour of the crystal from these local interactions. This is not well understood in a mathematically rigorous sense yet.

One question to tackle is whether the crystal globally preserves or breaks symmetries of the local interactions. Richthammer showed that the translational symmetry is preserved in a quite general two-dimensional setting, see [R-07]. But in the case of rotational symmetry one expects a different outcome: rotational symmetry should be broken. Merkl and Rolles showed this for a toy model of a crystal without defects in [MR-09]. This was extended by Heydenreich, Merkl and Rolles in [HMR-13] to a model which allows simple defects.

In the second part of this thesis, it is shown that the rotational symmetry is broken (in a weaker form) for a class of crystal models where almost all kinds of defects are allowed. This is the content of Chapter 5. Thereto we need a rigidity estimate. This analytic tool is derived in Chapter 4. In the following two subsections we introduce this topics more precisely. Most of this introduction and both chapters of the second part almost literally compose the article [A-14c], which is submitted for publication and also availably at arXiv.

1.2.1 A Rigidity Estimate

Results on geometric rigidity go back to a theorem of Liouville. It states that if the derivative of a smooth function $v : \mathbb{R}^d \supseteq M \to \mathbb{R}^d$ is point-wise a rotation, then the function is globally a rigid motion, i.e. its derivative is everywhere the same rotation. A major step further was the now classical rigidity estimate of Friesecke, James and Müller [FJM-02, Theorem 3.1]. They bounded the L^2-distance of the derivative from a constant rotation by a constant times the L^2-distance from the whole rotation group $\mathrm{SO}(d)$. This was further generalised by Müller, Scardia and Zeppieri to fields with non-zero curl, at least in dimension $d = 2$, see [MSZ-13, Theorem 3.3].

Here we consider matrix-valued functions $V : M \to \mathbb{R}^{d\times d}$ on an open, connected and bounded set $M \subset \mathbb{R}^d$ with smooth boundary in dimension $d \geq 2$. We also identify such a function line by line with a vector of 1-forms. We show that the L^2-distance of V from a single constant rotation $R \in \mathrm{SO}(d)$ is bounded by a constant times the sum of the L^2-distance of V from the rotation group $\mathrm{SO}(d)$ and the L^p-norm of the (component-wise) exterior derivative dV of V. More precisely, Theorem 4.1 reads as follows:

Theorem. *Let $d \geq 2$ and $M \subset \mathbb{R}^d$ be open, connected and bounded with smooth boundary. Let further $p \geq 2d/(2 + d)$. Then there exist constants $C_1 = C_1(M)$ and $C_2 = C_2(M, p)$ such that for all $V \in L^2(M, \mathbb{R}^{d\times d})$ with $dV \in L^p(M)$ there exists a rotation $R \in \mathrm{SO}(d)$ with*

$$\|V - R\|_{L^2(M)} \leq C_1\| \operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2(M)} + C_2\|dV\|_{L^p(M)}\,.$$

We also determine the scaling of the constants in Lemma 4.4. Note that one of them is not scale-invariant. If $V = dv$ for some function $v : M \to \mathbb{R}^d$ (which implies $dV = 0$), this estimate reduces to [FJM-02, Theorem 3.1]. It is also an extension of [MSZ-13, Theorem 3.3], which handles the case $d = 2$ and $p = 1$.

1.2.2 Spontaneous Symmetry Breaking

We will show a kind of rotational symmetry breaking for a class of crystals model where almost arbitrary defects are allowed. Let us describe this class informally. A model consists of a tessellation, some local Hamiltonians, a measure for the surface of the defects and some parameters. The crystal shall have a favourite structure, which depends on the considered matter and is described by the tessellation. Thus the molecules form locally a perturbation of the tessellation. A local perturbation costs some energy, which is described by the local Hamiltonians. As already mentioned, the crystal may have various defects. In particular, there may be edge, screw and mixed dislocations, i.e. defects with Burgers vectors, as well as large unbounded defects. We only require that the size of a defect is larger than an arbitrary small, but fixed number. A defect is punished proportional to the size of its surface. This can be interpreted as a surface tension. Moreover, there is a chemical potential which favours a large number of molecules.

Let us be a bit more precise. The crystal lives in a d-dimensional box ($d \geq 2$) of size N (with periodic boundary), and the centre of the molecules are given by a random set $\mathcal{P}$ of points in the box. A point configuration $\mathcal{P}$ determines a set $\mathcal{T}$ of tiles, which are locally a perturbation of the tessellation. Furthermore, it determines the quantity S measuring the surface of the defects. The local Hamiltonian $H_{\mathrm{loc}}(\square)$

gives the energy costs of the perturbed tile $\square$ in any way which fulfil a reasonable inequality. Then the global Hamiltonian is defined by

$$H_{\sigma,m,N}(\mathcal{P}) := \sum_{\square\in\mathcal{T}} H_{\text{loc}}(\square) + \sigma S - m|\mathcal{P}|$$

for $\sigma > 0$ and $m \in \mathbb{R}$. The three addends describe the local perturbation, the surface energy and a chemical potential. Using a Possion Point Process μ in the box as reference measure, the probability measure $P_{\beta,\sigma,m,N}$ is given by

$$dP_{\beta,\sigma,m,N} := \frac{1}{Z_{\beta,\sigma,m,N}} e^{-\beta H_{\beta,\sigma,m,N}}\, d\mu$$

with inverse temperature $\beta > 0$ and partition sum $Z_{\beta,\sigma,m,N}$. Now we can state a simplified version of Theorem 5.1, which is the main result of Chapter 5.

Theorem. *There exist m_0 and $\sigma_0(N,m) \asymp N^2 + m$ such that for all $m \geq m_0$*

$$\lim_{\beta\to\infty} \limsup_{N\to\infty} \sup_{\sigma\geq\sigma_0(N,m)} E_{\beta,\sigma,m,N}\left[\inf_{R\in\mathrm{SO}(d)} \frac{1}{|\mathcal{T}|} \sum_{\square\in\mathcal{T}} \|V - R\|^2_{L^2(\square)}\right] = 0\,,$$

where $V : \cup\mathcal{T} \to \mathbb{R}^{d\times d}$ measures point-wise the deformation (rotation and scaling) of the crystal.

Therefore the crystal is globally close to a constant rotation $R \in \mathrm{SO}(d)$, i.e. there is a long-range order in the crystal. But if the local Hamiltonians and the surface measure are chosen rotational invariant, which is possible and reasonable, the global Hamiltonian is rotational symmetric. Therefore the rotational symmetry is broken.

In order to prove this result, we follow the approach of Heydenreich, Merkl and Rolles. Their main ingredient is the theorem on geometric rigidity of Friesecke, James and Müller [FJM-02, Theorem 3.1]. We first prove a more general rigidity estimate described below and apply it to prove the result stated above, using a more or less similar technique as Heydenreich, Merkl and Rolles.

The main constraint of our theorem is that the limit is not uniform in the box size: σ_0 depends on N. But with the chosen method this is the best possible result, since one constant in the rigidity estimate is not scale-invariant. In order to get results uniform in the size of the box, one might have to use much more involved approaches like renormalisation.

Finally we give two examples of concrete models. First we consider the two-dimensional triangular lattice. This yields a model analogous to the model considered in [HMR-13]. Then we draw our attention to a crystal whose favourite structure is the d-dimensional cubic lattice.

Part I

Percolation

Chapter 2

Singularity of Nearcritical Percolation Exploration Paths

In this chapter we show that the scaling limits of nearcritical percolation exploration paths with different parameters are mutually singular on the triangular lattice. Apart from some linguistic changes, it is already published at ALEA and also available at arXiv.[1]

In Section 2.1 we introduce precisely the model and state the main theorem of this chapter. Before the proof, there is an expository section, namely Section 2.2. We first review some aspects of [NW-09]. Thereafter we give some heuristics why the result of Nolin and Werner as well as our theorem should be true. Of course, this does formally not prove anything, but it hopefully makes the proof more accessible. It could also be seen as an introduction in and an outline of the proof. Section 2.3 is devoted to the formal proof of the main theorem. Finally, in Section 2.4, we discuss a consequence for conformal maps.

2.1 Notation and Statement of the Main Theorem

Let us start with the basic definitions and notations. Let $H_r := \{z \in \mathbb{C} : |z| < r, \mathrm{Im}(z) > 0\}$ be the upper half circle around 0 with radius $r > 0$. We work on the hexagonal lattice with mesh size $\eta > 0$. Let H_r^η be all hexagons of size η which are entirely contained in $\overline{H_r}$.

We consider face percolation in H_r^η with different parameters p^μ and p^λ. Thereto let $\mu, \lambda \in \mathbb{R}$ and $\mu_\eta, \lambda_\eta \in \mathbb{R}$, $\eta > 0$, such that $\mu_\eta \to \mu$ and $\lambda_\eta \to \lambda$ as $\eta \to 0$. Each hexagon is independently of the others blue (open) with probability

$$p^\iota = p^\iota_\eta = \frac{1}{2} + \iota_\eta \cdot \frac{\eta^2}{\alpha_4^\eta}$$

[1] [A-14a] Simon Aumann: Singularity of Nearcritical Percolation Exploration Paths, *ALEA Lat. Am. J. Probab. Math. Stat.* **11** No. 1, 229-252, 2014 or arXiv:1110.4203

and otherwise yellow (closed), where we choose $\iota \in \{\mu, \lambda\}$ depending on the desired parameter. Here and in the following, α_4^η is the probability that there exists four arms of alternating colours up to (Euclidean) distance 1 in critical site percolation on the triangular lattice with mesh size η. Smirnov and Werner showed in [SW-01, Theorem 4] that $\alpha_4^\eta = \eta^{\frac{5}{4}+o(1)}$ as $\eta \to 0$. Therefore (or by using the five arm exponent) it follows that $p^\iota \to \frac{1}{2}$ as $\eta \to 0$. As we are interested in that limit, we may hence choose η small enough such that $p^\iota \in (0,1)$. Thus we work on the families of probability spaces

$$\Big(\Omega_\eta := \{\text{blue,yellow}\}^{H_r^\eta}, \quad \mathfrak{P}(\Omega_\eta), \quad P_\eta^\iota := \bigotimes_{H_r^\eta} \big(p^\iota \delta_{\text{blue}} + (1-p^\iota)\delta_{\text{yellow}}\big) \Big)_{\eta>0}$$

with $\iota \in \{\mu, \lambda\}$ and product-σ-algebra $\mathfrak{P}(\Omega_\eta)$. The choice of p^ι ensures that we are still in the critical window, but obtain scaling limits different from the critical one (if $\iota \neq 0$, of course). This follows from Kesten's scaling relations and can explicitly be deduced from [NW-09, Proposition 4] together with [N-08, Proposition 32], for example.

If we colour the negative real axis blue and the positive axis yellow, then there is a unique path, called *exploration path*, on the hexagonal lattice starting at the origin and stopping η-close to the upper boundary of H_r, which has blue hexagons to the left and yellow hexagons to the right. Let us denote this path by the random variable

$$\gamma_\eta : \ (\Omega_\eta, \mathfrak{P}(\Omega_\eta)) \to (\mathcal{S}_r, \mathcal{B}(\mathcal{S}_r))\,,$$

where $\mathcal{S}_r$ (with Borel-σ-algebra $\mathcal{B}(\mathcal{S}_r)$ induced by the metric below) is the space of curves in $\overline{H_r}$, i.e. equivalence classes of continuous functions $[0,1] \to \overline{H_r}$. Two such functions f, g represent the same curve if and only if $f = g \circ \phi$ for some increasing bijection $\phi : [0,1] \to [0,1]$. We introduce a topology on $\mathcal{S}_r$ via the metric

$$\text{dist}(f,g) := \inf_{\phi} \max_{t\in[0,1]} |f(t) - g\circ\phi(t)|$$

where the infimum is taken over all increasing bijections $\phi : [0,1] \to [0,1]$. Then $\mathcal{S}_r$ is a complete separable space. Let

$$\Gamma_\eta^\iota := \gamma_\eta(P_\eta^\iota)$$

denote the law of γ_η under P_η^ι, for $\eta > 0$ and $\iota \in \{\mu, \lambda\}$. Using a technique developed by Aizenman and Burchard in [AB-99], Nolin and Werner showed in [NW-09, Proposition 1] that the family $(\Gamma_\eta^\iota)_{\eta>0}$ is tight, i.e. for each sequence η_k there is a subsequence η_{k_l} such that $\Gamma_{\eta_{k_l}}^\iota$ converges weakly.

For the statement of the main theorem we need, in contrast to Nolin and Werner, a result using the Quad-Crossing Topology introduced by Schramm and Smirnov in [SS-11]. Therefore we review that concept very briefly. For a much more detailed account one should consult [SS-11, p. 1778f]. Let D be a domain. A *quad* q in D is a topological quadrilateral, i.e. a homeomorphism $q : [0,1]^2 \to q([0,1]^2) \subset D$. Let

$\mathcal{Q}_D$ be the set of all quads in D. A quad q is crossed by a percolation configuration, if the union of all blue (topologically closed) hexagons contains a connected closed subset of $\bar{q} := q([0,1]^2)$ which intersects both opposite sides $\partial_0 q := q(\{0\} \times [0,1])$ and $\partial_2 q := q(\{1\} \times [0,1])$. This event is denoted by $\boxminus q \subset \Omega_\eta$. We will further need the notations $\partial_1 q := q([0,1] \times \{0\})$ and $\partial_3 q := q([0,1] \times \{1\})$ for the other two sides of the quad. Moreover, let $q^\circ := q((0,1)^2)$ be the interior and ∂q be the whole boundary of q.

Using a partial order on $\mathcal{Q}_D$ induced by crossings, one can define the set $\mathcal{H}_D$ of all closed lower sets $S \subset \mathcal{Q}_D$. Schramm and Smirnov constructed a topology on $\mathcal{H}_D$, namely the Quad-Crossing-Topology. For our purposes the following facts are enough. There is a random variable $cr : \Omega_\eta \to \mathcal{H}_D$ which assigns each percolation configuration the set of all crossed quads. Thus each probability measure on Ω_η induces a probability measure on $\mathcal{H}_D$. Moreover, the space of all probability measures on $\mathcal{H}_D$ is tight ([SS-11, Corollary 1.15]). Finally, if $\mathbb{P}$ is any limit point of the measures $cr(P^\mu_\eta)$, $\eta > 0$, then $\mathbb{P}[\partial cr(\boxminus q)] = 0$ for every quad $q \in \mathcal{Q}_D$ ([SS-11, Lemma 5.1]). Therefore there exists a sequence $(\eta_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \eta_k = 0$ such that $P^\mu_{\eta_k}[\boxminus q]$ converges as $k \to \infty$ for all quads $q \in \mathcal{Q}_D$.

Now we are ready to state the main theorem of the present chapter.

Theorem 2.1. *Let $\mu < \lambda$ be real numbers, $\mu_\eta \to \mu$, $\lambda_\eta \to \lambda$ and $r > 0$. Let further $(\eta_k)_{k\in\mathbb{N}}$ be a sequence converging to zero such that $P^\mu_{\eta_k}[\boxminus q]$ converges for all quads $q \in \mathcal{Q}_{H_r}$ and such that $\Gamma^\mu_{\eta_k} \to \Gamma^\mu$ and $\Gamma^\lambda_{\eta_k} \to \Gamma^\lambda$ weakly for some measures Γ^μ and Γ^λ on $(\mathcal{S}_r, \mathcal{B}(\mathcal{S}_r))$ as $k \to \infty$.*

Then the probability measures Γ^μ and Γ^λ are singular with respect to each other.

Γ^μ and Γ^λ are distributions of the scaling limits of the discrete exploration paths (in the limit point sense). Let us remark that [NW-09, Proposition 6] is included in this theorem as the special case $\mu = \mu_\eta = 0$. In that case the hypothesis on the quad crossing probabilities is always fulfilled since it follows from Cardy's formula. But in our case, we unfortunately do not have any analogue; that is the reason for the additional condition.

The theorem also holds if $\mu > \lambda$, i.e. if the condition on the quad crossing probabilities holds for the larger value. In that case quite a few inequality signs have to be switched. Thus for better readability, we restrict ourselves to the case $\mu < \lambda$.

Actually we do not need to look at the whole exploration path to detect the singularity. In fact, it is enough to look at an infinitesimal initial segment as the following corollary shows. We consider the space $(\mathcal{S}_1, \mathcal{B}(\mathcal{S}_1))$ of curves in H_1. Let

$$\tau_n(\gamma) := \inf\{t \geq 0 : |\gamma(t)| = \tfrac{1}{n}\}$$

be the first exit time of $H_{\frac{1}{n}}$ and

$$\mathcal{A}_n := \sigma(\mathrm{id}[0, \tau_n], \mathrm{id}(0) = 0)$$

be the σ-algebra generated by curves starting at the origin until exiting $H_{\frac{1}{n}}$, $n \in \mathbb{N}$. Then $\mathcal{A}_n$, $n \in \mathbb{N}$, is decreasing. Let

$$\mathcal{A} := \bigcap_{n \in \mathbb{N}} \mathcal{A}_n$$

be their tail-σ-algebra, the σ-algebra of infinitesimal initial segments of paths starting at the origin. With that notation, Theorem 2.1 implies

Corollary 2.2. *Under the conditions of Theorem 2.1, the laws Γ^μ and Γ^λ restricted to $\mathcal{A}$ are singular with respect to each other.*

Proof. By Theorem 2.1 applied to $r = \frac{1}{n}$, there are sets $A_n \in \mathcal{A}_n$ with $\Gamma^\mu[A_n] = 0$ and $\Gamma^\lambda[A_n] = 1$. We set

$$A_* := \bigcup_{m \geq 1} \bigcap_{n \geq m} A_n \, .$$

Then $A_* \in \mathcal{A}$. Since countable unions or intersection of sets of probability zero respectively one have probability zero respectively one, it follows that $\Gamma^\mu[A_*] = 0$ and $\Gamma^\lambda[A_*] = 1$, which proves the corollary. □

We conjecture that Theorem 2.1 and its corollary also hold on other lattices. In fact, if we can apply RSW techniques, most elements of the proof work. We need the separation lemmas and other results of [N-08], which are delicate consequences of RSW ([N-08, Theorem 2]). Thus they remain true on other lattices, cf. [N-08, Section 8.1]. We further need the following bounds on arm events. Let $\alpha_2^\eta(\rho, R)$ and $\alpha_4^\eta(\rho, R)$ be the probabilities of the events that at critical percolation with mesh size η there exist two respectively four arms of alternating colours inside an annulus with radii ρ and R (i.e., in particular, $\alpha_4^\eta(\eta, 1) = \alpha_4^\eta$). We need that there are "exponents" $\hat{\alpha}_4, \check{\alpha}_2 > 0$ and constants $c, c' > 0$ such that

$$\alpha_2^\eta(\rho, R) \geq c(\rho/R)^{\check{\alpha}_2} \qquad \text{and} \qquad \alpha_4^\eta(\rho, R) \leq c'(\rho/R)^{\hat{\alpha}_4}$$

for all $0 < \eta \leq \rho \leq R$ and such that

$$2\hat{\alpha}_4 - \check{\alpha}_2 > 2 \, . \tag{2.1}$$

Since the two arm exponent in the half plane exists and is 1 as a consequence of RSW (see [N-08, Theorem 23], for instance), it follows that we can choose $\check{\alpha}_2 \leq 1$, which we also need. While the analogues to [N-08, Proposition 13] and [N-08, Theorem 10] yield the existence of such exponents also for other lattices, inequality (2.1) is yet proven only for site percolation on the triangular lattice (or equivalently, face percolation on the hexagonal lattice). Indeed, we can choose $\check{\alpha}_2 = \frac{1}{4} + \beta$ and $\hat{\alpha}_4 = \frac{5}{4} - \beta$ for any $\beta > 0$ there. Since the former inequality is the only needed special property of the triangular lattice, we choose to write up the proof with the exponents $\check{\alpha}_2$ and $\hat{\alpha}_4$ and not with the explicit values. Hence the results can immediately be enhanced to other lattices as soon as inequality (2.1) is established.

2.2 Heuristics

This section is of expository nature and therefore not rigorous. First we review some aspects of [NW-09]. Then we give a heuristic explanation why a nearcritical scaling limit should be singular with respect to the critical or to another nearcritical scaling limit. These heuristics could in fact also be seen as an outline of the proof. Formally, this section is not needed for the remainder of this chapter.

Let us recall some of our notation: P_η^ι denotes the probability measure of nearcritical percolation with parameter $\iota \in \mathbb{R}$, i.e. a site is open with probability

$$p^\iota = \frac{1}{2} + \iota \cdot \frac{\eta^2}{\alpha_4^\eta(\eta, 1)} .$$

Moreover, the random variable γ_η denotes the exploration path and Γ_η^ι its law under P_η^ι.

A basic concept of nearcritical percolation is the introduction of a characteristic length. Below that length, the Russo-Seymour-Welsh Theory (RSW) is still valid. This means that the probability that a set is crossed by the percolation configuration (in some specific way) does only depend on the shape of the set, but not on its size – as long as this size is below the characteristic length. In the set-up considered in this chapter, the mesh size of the lattice and the nearcritical probabilities are chosen such that the characteristic length is of order one. Thus RSW techniques are applicable.

The first result of Nolin and Werner [NW-09, Proposition 1] shows tightness of the laws of the exploration paths. We shortly outline their proof. It is an application of [AB-99, Theorem 1.2]. Let us denote the annulus around x with radii $\rho < R$ by $A(x, \rho, R)$. RSW considerations imply that there exist some constants $c, \alpha > 0$ such that

$$P_\eta^\iota\big[\gamma_\eta \text{ crosses } A(x, \rho, R)\big] \leq c(\rho/R)^\alpha$$

uniformly for all $\eta \leq \rho \leq R$. Using the BK Inequality, it follows that, for all $k \in \mathbb{N}$,

$$P_\eta^\iota\big[\gamma_\eta \text{ crosses } A(x, \rho, R) \ k \text{ times}\big] \leq c_k(\rho/R)^{\alpha k} .$$

Therefore the hypothesis of [AB-99, Theorem 1.2] is fulfilled and tightness follows. This means that for each sequence η_k there is a subsequence η_{k_l} such that $\Gamma_{\eta_{k_l}}^\iota$ converges weakly.

Nolin and Werner also determined the Hausdorff dimension of any sub-sequential scaling limit of the critical and nearcritical exploration paths. It is $7/4$ in both cases, see [NW-09, Proposition 3]. The proof is based on RSW techniques and the knowledge of the two-arm exponent of critical percolation.

The perhaps most important result of [NW-09] is Proposition 6. It states that the law of any nearcritical sub-sequential limit is singular with respect to the law of an SLE_6 curve, which is the critical limit. As already mentioned, we enhance this result and show that $\Gamma^\mu \perp \Gamma^\lambda$, where Γ^ι is a limit point of Γ_η^ι, $\iota \in \{\mu, \lambda\}$. In the following, we heuristically argue why these theorems hold.

Let us consider an equilateral triangle Δ of size δ. The scale δ should be an intermediate one, i.e $\eta \ll \delta \ll 1$. We assume that the exploration path γ_η entered the triangle somewhere in the middle of the triangle's bottom line and is at time σ somewhere in the middle of the triangle. If that is the case, we say that the triangle is *good* for γ_η. We even look at the following stronger event: Conditionally

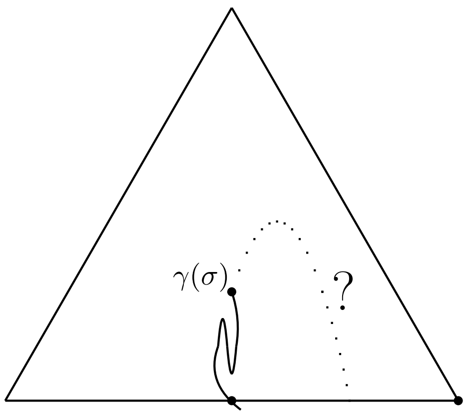


Figure 2.1: A (maybe very) good triangle

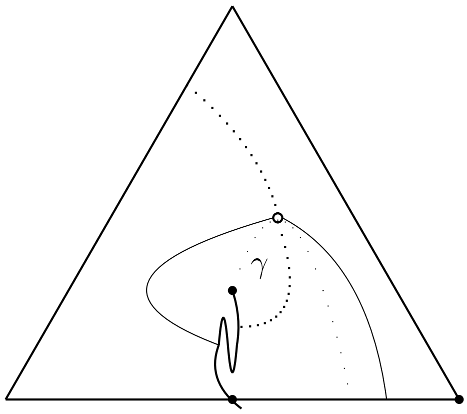


Figure 2.2: A pivotal site with four arms

on $\gamma_\eta[0,\sigma]$, we ask whether γ_η exists the triangle on the right part of the bottom line. In that case we call the triangle even *very good* for γ_η. This events are schematically drawn in Figure 2.1.

We estimate the difference of the probability of being very good, conditionally on $\gamma_\eta[0,\sigma]$, under P_η^λ and under P_η^μ. Thereto we use the standard monotone coupling of percolation with different parameters $p \in [0,1]$ (for all hexagons not discovered by $\gamma_\eta[0,\sigma]$). Thus the set $\omega(p)$ of blue hexagons at level p increases. If a good triangle Δ is very good for $\gamma_\eta(\omega(p^\lambda))$, but not for $\gamma_\eta(\omega(p^\mu))$, then there exists a site x in the triangle which is pivotal for some crossing event and switched from yellow to blue, cf. Figure 2.2. It is pivotal, iff there are four arms of alternating colours from x to some described parts of the boundary. Therefore we conclude

$$\begin{aligned} & P_\eta^\lambda\big[\Delta \text{ is very good for } \gamma_\eta \mid \gamma_\eta[0,\sigma]\big] - P_\eta^\mu\big[\Delta \text{ is very good for } \gamma_\eta \mid \gamma_\eta[0,\sigma]\big] \\ = \;& P\big[\Delta \text{ is very good for } \gamma_\eta(\omega(p^\lambda)) \text{ but not for } \gamma_\eta(\omega(p^\mu)) \mid \gamma_\eta[0,\sigma]\big] \\ \approx \;& P\big[\exists\, x \in \Delta \setminus \gamma_\eta[0,\sigma] : \text{ four arms from } x \text{ to } \partial\Delta, x \text{ switched between } p^\mu \text{ and } p^\lambda\big] \end{aligned}$$

Since the crossing event is increasing, the latter event can happen only for one x inside the triangle. Since there are around $(\delta/\eta)^2$ sites inside the triangle, we conclude

$$\begin{aligned} & P\big[\exists\, x \in \Delta \setminus \gamma_\eta[0,\sigma] : \text{ four arms from } x \text{ to } \partial\Delta,\, x \text{ switched between } p^\mu \text{ and } p^\lambda\big] \\ \approx \;& (\delta/\eta)^2 \alpha_4^\eta(\eta,\delta)(p^\lambda - p^\mu) \\ = \;& (\delta/\eta)^2 \alpha_4^\eta(\eta,\delta)\,(\lambda-\mu)\,\eta^2/\alpha_4^\eta(\eta,1)\,, \end{aligned}$$

where we used

$$p^\lambda - p^\mu = \tfrac{1}{2} + \lambda\eta^2/\alpha_4^\eta(\eta,1) - \tfrac{1}{2} - \mu\eta^2/\alpha_4^\eta(\eta,1) = (\lambda-\mu)\,\eta^2/\alpha_4^\eta(\eta,1)$$

in the last step. Now $\lambda - \mu \asymp 1$ and quasi-multiplicativity, that is $\alpha_4^\eta(\eta,1) \asymp \alpha_4^\eta(\eta,\delta)\alpha_4^\eta(\delta,1)$, and finally $\alpha_4^\eta(\delta,1) \to \delta^{5/4}$ yield

$$(\delta/\eta)^2\alpha_4^\eta(\eta,\delta)\,(\lambda-\mu)\,\eta^2/\alpha_4^\eta(\eta,1) \;\approx\; \delta^2/\alpha_4^\eta(\delta,1) \;\approx\; \delta^{3/4}\,.$$

Thus we established the estimate

$$P_\eta^\lambda\big[\Delta \text{ very good for } \gamma \mid \Delta \text{ good for } \gamma\big] - P_\eta^\mu\big[\Delta \text{ very good for } \gamma \mid \Delta \text{ good for } \gamma\big] \\ \approx \delta^{3/4}$$

for every triangle Δ of scale δ.

We will use this estimate to evaluate the expectation of the random variable

$$\begin{aligned} Z^\delta(\gamma) \;\; := \;\; & \#\{\text{very good triangles of scale } \delta \text{ for } \gamma\} \\ & - E^\mu[\#\{\text{very good triangles of scale } \delta \text{ for } \gamma\}]\,. \end{aligned}$$

Since the Hausdorff dimension of the exploration path is 7/4, it touches approximately $\delta^{-7/4}$ triangles. By RSW, the number of good triangles is of the same order of magnitude. Therefore we conclude

$$E^\mu[Z^\delta] = \delta^{-7/4}\cdot 0 = 0 \qquad \text{and} \qquad E^\lambda[Z^\delta] \approx \delta^{-7/4}\cdot\delta^{3/4} = \delta^{-1}\,.$$

Though the events being good or very good of different triangles are not independent, we can conclude using a martingale approach that

$$\mathrm{Var}^\mu[Z^\delta] \le \delta^{-7/4} \qquad \text{and} \qquad \mathrm{Var}^\lambda[Z^\delta] \le \delta^{-7/4}\,.$$

Now by Chebyshev's inequality, it follows that

$$P^\mu[Z^\delta > \delta^{-15/16}] \;\le\; \delta^{15/8}\,\mathrm{Var}^\mu[Z^\delta] \;\le\; \delta^{15/8}\delta^{-7/4} = \delta^{1/8}$$

and

$$\begin{aligned} P^\lambda[Z^\delta < \delta^{-15/16}] \;\; & \approx \;\; P^\lambda\big[Z^\delta - E^\lambda[Z^\delta] < \delta^{-15/16} - \delta^{-1}\big] \\ & \le \;\; \big(\delta^{-15/16}(1-\delta^{-1/16})\big)^{-2}\,\mathrm{Var}^\lambda[Z^\delta] \;\le\; \delta^{1/8}\,. \end{aligned}$$

Now we choose a sequence of scales $(\delta_n)_n$ such that $\delta_n^{1/8}$ is summable. Then the Borel-Cantelli Lemma implies

$$P^\mu[Z^{\delta_n}(\gamma) > \delta_n^{-15/16} \text{ for infinitely many } n] = 0$$

and

$$P^\lambda[Z^{\delta_n}(\gamma) < \delta_n^{-15/16} \text{ for infinitely many } n] = 0\,.$$

As the complements of these events are disjoint, the mutual singularity of $\Gamma^\mu = \gamma(P^\mu)$ and $\Gamma^\lambda = \gamma(P^\lambda)$ follows.

2.3 Proof of the Main Theorem

We partition the rigorous proof of Theorem 2.1 in four subsections. In Section 2.3.1 we prove a lemma which is also of independent interest. It states that we can approximate the probability of crossing a quad even if it has fractal boundary and if we use quite weak approximations to it. In Section 2.3.2 we look at one mesoscopic triangle, whereas in Section 2.3.3 we give estimates for many mesoscopic triangles. Finally, in Section 2.3.4, we consider the continuum limit to conclude the proof of Theorem 2.1.

2.3.1 A Quad Crossing Lemma

We say that a sequence $(q_n)_{n\in\mathbb{N}}$ of quads converges in the kernel (or Caratheodory) sense to a quad q with respect to some $z_0 \in \mathbb{C}$, if

- $z_0 \in q_n^\circ$ for all $n \in \mathbb{N}$ and $z_0 \in q^\circ$,
- for every $z \in q^\circ$ there exists a neighbourhood of z which is contained in all but finitely many q_n° (and in q°),
- for each $z \in \partial q$ there exist $z_n \in \partial q_n$ with $z_n \to z$ and
- $q_n(i,j) \to q(i,j)$ for $(i,j) \in \{0,1\}^2$.

This is the usual kernel convergence for domains with the additional requirement that the corners of the quads converge. We further need the following condition, which is illustrated in Figure 2.3:

$$\begin{aligned}&\forall\, \varepsilon > 0\; \exists\, n_0 \in \mathbb{N}\; \forall\, n \geq n_0, i \in \{0,1,2,3\} :\\ &\quad U_\varepsilon(\partial_i q) \cap \big(\overline{q} \cup U_\varepsilon(\partial_{i-1} q) \cup U_\varepsilon(\partial_{i+1} q)\big) \text{ contains a path connecting}\\ &\quad \partial_{i-1} q_n \text{ and } \partial_{i+1} q_n \text{ not intersecting } \partial_i q_n\end{aligned} \tag{2.2}$$

Here and in the following, $U_\varepsilon(\cdot)$ denotes the ε-neighbourhood. We use cyclic indexes, i.e. $3+1 \equiv 0$. The condition demands that $\partial_i q_n$ is not close to any other side of q_n or q inside the quad for a long time. Thus inside $\overline{q}$, $\partial_i q_n$ is close to $\partial_i q$. But note that there may be parts of $\partial_i q_n$ far away from $\partial_i q$ and even ∂q outside $\overline{q}$.

Lemma 2.3. *Let some quads $q_n, n \in \mathbb{N}$, converge in the kernel sense to a quad q as $n \to \infty$ (with respect to some z_0). Assume further that condition (2.2) is fulfilled. Let P_η, $\eta > 0$, be any (near-)critical probability measures, i.e. $P_\eta = P_\eta^\iota$ for any bounded sequence $(\iota_\eta)_\eta \subset \mathbb{R}$.*

Then for all $\rho > 0$ there exist $n_0 \in \mathbb{N}$ and $\eta_0 > 0$ such that for all $n \geq n_0$ and $\eta \leq \eta_0$

$$P_\eta\big[\boxminus q_n \triangle \boxminus q\big] \leq \rho\,,$$

where $\triangle$ denotes the symmetric difference.

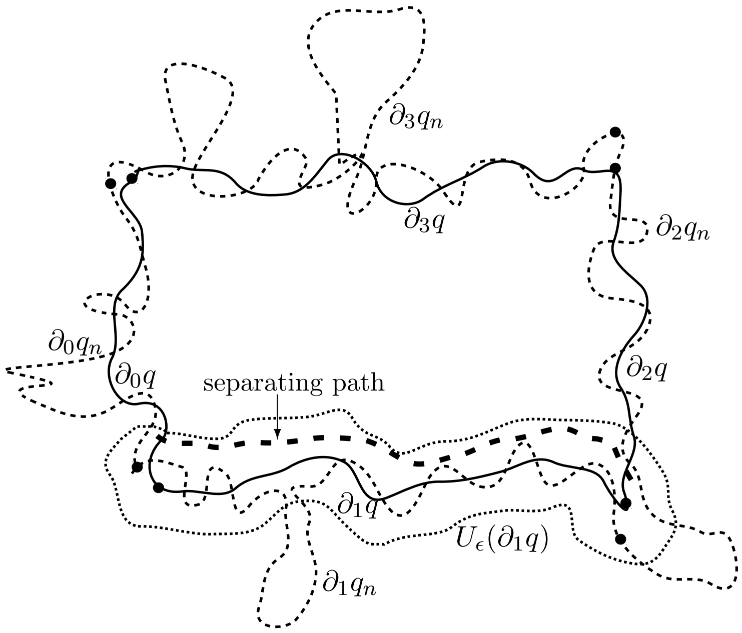


Figure 2.3: Quads q (solid) and q_n (dashed) satisfying condition (2.2) with the neighbourhood of $\partial_1 q$ (fine dotted) and a separating path (strong dashed)

Let us remark that we do not impose any smoothness conditions on the boundary of the quad. Otherwise, we could just use the 3-arm-exponent in the half plane. We further remark that the proof relies only on RSW techniques. Thus the lemma is valid on any lattice where RSW works.

In order to prove Lemma 2.3, we want to apply Lemma A.1 of [SS-11]. It states that if two quads differ only at one side by some ζ, then the probability of the symmetric difference of the corresponding crossing events is small. More precisely, a slightly simplified version reads as follows in our notation.

Let $d > 0$. There exists a positive function $\Delta(\zeta)$ such that $\Delta(\zeta) \to 0$ as $\zeta \to 0$ and the following estimates hold. If two quads q, q' of diameter at least d satisfy for some $\zeta < d/2$

(i) $[\dots]^2$ *or*

(ii) $\overline{q'} \subset \overline{q}$, $\partial_0 q' = \partial_0 q$, $\partial_1 q' \subset \partial_1 q$, $\partial_3 q' \subset \partial_3 q$ *and each point on* $\partial_2 q'$ *can be connected to* $\partial_2 q$ *by a path in* $\overline{q}$ *of diameter at most* ζ, *or*

(iii) $\overline{q'} \subset \overline{q}$, $\partial_0 q' \subset \partial_0 q$, $\partial_1 q' = \partial_1 q$, $\partial_2 q' \subset \partial_2 q$ *and and each point on* $\partial_3 q'$ *can be connected to* $\partial_3 q$ *by a path in* $\overline{q}$ *of diameter at most* ζ,

[2] We omit this item since we do not need it

then for all $\eta < \zeta$
$$P_\eta\big[\boxminus q \triangle \boxminus q'\big] \;\leq\; \Delta(\zeta)\,.$$

For the sake of completeness, we shortly outline how one can prove that. Let two quads q, q' satisfy condition (iii). If $\boxminus q \triangle \boxminus q'$ happens, there exists a yellow vertical crossing of q and two blue arms from a disk of radius ζ to $\partial_0 q'$ and $\partial_2 q'$. If we condition on the left-most yellow vertical crossing, percolation on the right of it is still unbiased. Therefore we can apply RSW, yielding that the probability of an arm from a disk of radius ζ to $\partial_2 q'$ tends to 0 as $\zeta \to 0$, as desired. The details are properly written up in [SS-11].

Proof of Lemma 2.3. First we claim that for each $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds:

- $|q_n(i,j) - q(i,j)| < \varepsilon$ for each $(i,j) \in \{0,1\}^2$
- for any $z \in \partial_i q$ there exist $z_n \in \partial_i q_n$ with $|z - z_n| < \varepsilon$, $i \in \{0,1,2,3\}$ and
- $\overline{q} \setminus U_\varepsilon(\partial q) \subset \overline{q_n}$

Note the uniformity and that z and z_n belong to the same side. Indeed, the first item is obvious from the kernel convergence. The second item can be fulfilled by covering $\partial_i q$ with finitely many balls of radius $\varepsilon/2$ (Condition (2.2) with $\varepsilon/2$ ensures that the z_n's belong to the correct side). Finally, using compactness, a finite sub-cover of the covering of $\overline{q} \setminus U_\varepsilon(\partial q)$ by the neighbourhoods used in the definition of the kernel convergence yields the third item.

Let $\varepsilon > 0$. We will specify ε depending on ρ later on. Let $n \geq n_0$, where n_0 is associated to ε such that the claim and condition (2.2) hold with this n_0. We need a further scale $\tilde{\varepsilon} = \varepsilon^\alpha \gg \varepsilon$ for some $\alpha > 0$ specified below. For $i \in \{0,1,2,3\}$, let $u_i^{\tilde{\varepsilon}}$ be a closed curve, homeomorphic to a circle, around $\partial_i q$, which stays between the $\tilde{\varepsilon}$- and the $2\tilde{\varepsilon}$-neighbourhood of $\partial_i q$. We try to avoid that some of the $u_i^{\tilde{\varepsilon}}$ intersect each other outside the $2\tilde{\varepsilon}$-neighbourhoods of the quad-corners. If this is not possible (for example, when q contains a slit), we treat the affected regions as different.

We label the corners of the quad q with $a = q(0,1)$, $b = q(0,0)$, $c = q(1,0)$ and $d = q(1,1)$. Now we define some points on the curves $u_i^{\tilde{\varepsilon}}$ near the corners. Starting at some point of $u_0^{\tilde{\varepsilon}}$ near b and moving along $u_0^{\tilde{\varepsilon}}$ outside q (i.e. in counter-clockwise direction), let a_b the first hit point of $u_0^{\tilde{\varepsilon}} \cap u_3^{\tilde{\varepsilon}}$. Similarly, let a_d the first hit point of $u_0^{\tilde{\varepsilon}} \cap u_3^{\tilde{\varepsilon}}$ starting near d and moving along $u_3^{\tilde{\varepsilon}}$ outside q (i.e. now in clockwise direction). Analogously we define the points b_a, b_c, c_b, c_d, d_c and d_a. The notation should be interpreted as follows: a point e_f (with $e, f \in \{a,b,c,d\}$) is near to the corner e, but on the way to f on the curve $u^{\tilde{\varepsilon}}$ outside q. These definitions are illustrated in Figure 2.4.

We use these points and curves to define the following quads. They are schematically drawn in Figure 2.5 below. We define the quads by giving the corners and the sides. We do not specify the parametrisations, since they are irrelevant. Let q^0 be

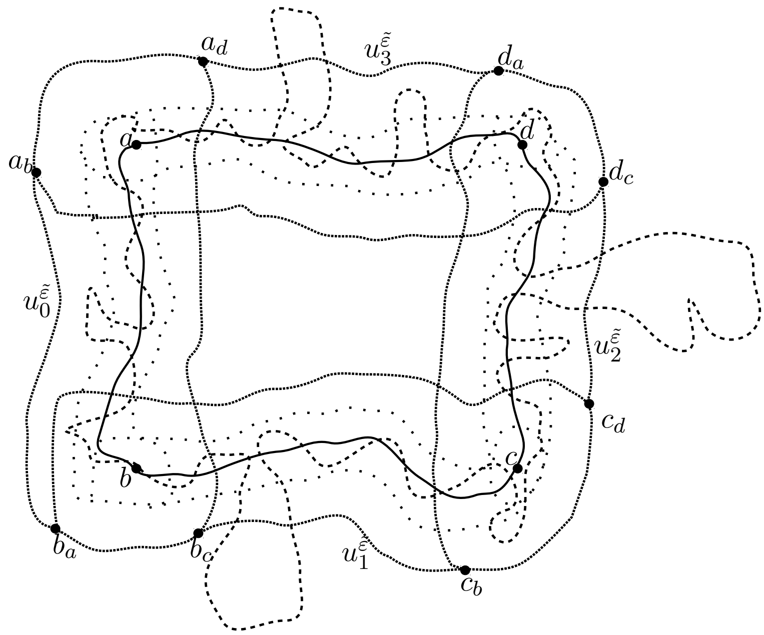


Figure 2.4: Quads q (solid) and q_n (dashed) with the ε-neighbourhood of q (wide dotted), the curves $u_i^{\tilde{\varepsilon}}$ (fine dotted) of q and the marked points

defined by the corners a_d, b_c, c_b and d_a with the following sides: Let $\partial_0 q^0$ consist of the part of $u_0^{\tilde{\varepsilon}}$ between a_d and b_c which intersects $\bar{q}$. The side $\partial_1 q^0$ consists of the part of $u_1^{\tilde{\varepsilon}}$ between b_c and c_b which stays outside $\bar{q}$. The side $\partial_2 q^0$ shall consist of the part of $u_2^{\tilde{\varepsilon}}$ between c_b and d_a which intersects $\bar{q}$. And finally, let $\partial_3 q^0$ consist of the part of $u_3^{\tilde{\varepsilon}}$ between d_a and a_d which stays outside $\bar{q}$. We abbreviate this definition by

$$q^0 = \big[a_d \text{ –i– } b_c \text{ –o– } c_b \text{ –i– } d_a \text{ –o–}\big]$$

Here we give the corners and the sides between them. An "–o–" indicates that the corresponding side consists of the part of $u_i^{\tilde{\varepsilon}}$ between the given corners which stays **o**utside $\bar{q}$, whereas an "–i–" denotes that the part of $u_i^{\tilde{\varepsilon}}$ which **i**ntersects $\bar{q}$ is used. With this notation we further define the quads

$$\begin{aligned} q^1 &= \big[a_d \text{ –i– } b_c \text{ –o– } c_d \text{ –o– } d_c \text{ –o–}\big] \\ q^2 &= \big[a_b \text{ –o– } b_a \text{ –o– } c_d \text{ –o– } d_c \text{ –o–}\big] \\ q^3 &= \big[a_b \text{ –o– } b_a \text{ –o– } c_d \text{ –o– } d_c \text{ –i–}\big] \\ q^4 &= \big[a_b \text{ –o– } b_a \text{ –i– } c_d \text{ –o– } d_c \text{ –i–}\big] \end{aligned}$$

which are schematically drawn in Figure 2.5.

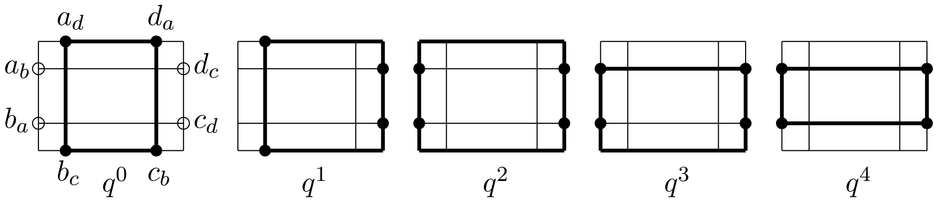


Figure 2.5: Schematic drawing of the quads q^0, q^1, q^2, q^3 and q^4

Then

$$\boxminus q^0 \triangle \boxminus q^4 \;\subseteq\; \bigcup_{i=0}^{3} \boxminus q^i \triangle \boxminus q^{i+1}$$

and each pair (q^i, q^{i+1}), $i = 0, 1, 2, 3$, satisfies condition (ii) or (iii) of Lemma A.1 in [SS-11] as cited above with $\zeta = 4\tilde{\varepsilon}$ (for $i = 1, 3$, the sides ∂_0 and ∂_2 as well as the sides ∂_1 and ∂_3 have to be interchanged). We conclude for $\eta < \tilde{\varepsilon}$

$$P_\eta\big[\boxminus q^0 \triangle \boxminus q^4\big] \;\leq\; f(\tilde{\varepsilon})$$

for some function f with $f(\tilde{\varepsilon}) \to 0$ as $\tilde{\varepsilon} \to 0$.

Now we want to link the previous observation to the event of interest. By the construction of the quads q^0 and q^4, every crossing of q^4 contains a crossing of q and every crossing of q contains one of q^0, i.e. $\boxminus q^4 \subseteq \boxminus q \subseteq \boxminus q^0$. This statement is only almost true, if we consider q_n instead of q, since q_n may have excursions outside $U_{\tilde{\varepsilon}}(\bar{q})$, i.e. in general $\boxminus q^4 \not\subseteq \boxminus q_n \not\subseteq \boxminus q^0$. But as ∂q_n will come 4ε-close to itself after leaving $U_\varepsilon(\partial q)$, we can control the events $\boxminus q_n \setminus \boxminus q^0$ and $\boxminus q^4 \setminus \boxminus q_n$, as follows. Mind that we now use the ε-neighbourhoods. We will need the distance between ε and $\tilde{\varepsilon}$ to control some arm events below.

Let us cover $U_\varepsilon(\partial q)$ with finitely many balls of radius ε centred at points z_j, $j \in J$. We need at most $c\varepsilon^{-2}$ many balls, with some numerical constant $c > 0$. Assume that there exists $x \in \partial_i q_n \setminus U_\varepsilon(\overline{q})$, i.e. some part of $\partial_i q_n$ is far away from $\overline{q}$. Then we claim that there exist $j \in J$ and $x_1, x_2 \in U_{2\varepsilon}(z_j) \cap \partial_i q_n$ such that x lies in between x_1 and x_2 on $\partial_i q_n$. Indeed, let $\partial_i q_n|^1$ respectively $\partial_i q_n|^2$ be the part of $\partial_i q_n \cap U_\varepsilon(\partial_i q)$ before respectively after x, and let $U^k := U_\varepsilon(\partial_i q_n|^k)$, $k \in \{1, 2\}$. Then $\partial_i q \subseteq U^1 \cup U^2$, since for all $z \in \partial_i q$ there exists $z_n \in \partial_i q_n$ with $|z_n - z| < \varepsilon$ (second item above), i.e. $z_n \in \partial_i q_n|^1 \cup \partial_i q_n|^2$. Thus $U^1 \cap U^2 \neq \emptyset$. Therefore there exists $j \in J$ with $U_\varepsilon(z_j) \cap U^1 \cap U^2 \neq \emptyset$. We conclude that there are $y_k \in U_\varepsilon(z_j) \cap U^k$ and $x_k \in \partial_i q_n|^k$ with $|y_k - x_k| < \varepsilon$, which implies $|x_k - z_j| < 2\varepsilon$, $k \in \{1, 2\}$, as claimed.

Now if $\boxminus q_n \setminus \boxminus q^0$ happens, each crossing of q_n must leave q^0 between b_c and c_b or between d_a and a_d. By the geometry of q_n, explained in the claim above, the crossing is forced to re-enter some ball $B_{2\varepsilon}(z_j)$ with $z_j \in \overline{q^0}$ after leaving q_0 (at least $\tilde{\varepsilon}$ away from ∂q). Furthermore, it must reach the paths whose existence is postulated in condition (2.2) for $i = 0, 2$. Thus it reaches the ε-neighbourhoods of $\partial_0 q$ and $\partial_2 q$, which are of distance at least $\tilde{\varepsilon} - 2\varepsilon$ of the ball. Thus the crossing induces four blue arms inside the annulus centred at z_j with radii 2ε and $\tilde{\varepsilon}-2\varepsilon$. Moreover, there must exist two yellow arms inside this annulus preventing q^0 being crossed. The event $\boxminus q^4 \setminus \boxminus q_n$ is treated similarly, or by duality, considering a yellow vertical crossing of q_n which does not induce a vertical crossing of q^4. Therefore, we conclude

$$\big(\boxminus q_n \setminus \boxminus q^0\big) \cup \big(\boxminus q^4 \setminus \boxminus q_n\big) \subseteq \bigcup_{j\in J} A_6(z_j, 2\varepsilon, \tilde{\varepsilon} - 2\varepsilon)\,,$$

where $A_6(z,\varrho,R)$ denotes the event that there exist six arms, not all of them of the same colour, inside the annulus centred at z of radii ϱ and R. By standard RSW techniques, we have for $\eta < \varrho$

$$P_\eta\big[A_6(z,\varrho,R)\big] \le (\varrho/R)^{2+\nu}$$

for some $\nu > 0$ (i.e. the polychromatic 6-arm-exponent is larger than 2). Recall that $\tilde{\varepsilon} = \varepsilon^\alpha$ for some $\alpha > 0$. Therefore $\tilde{\varepsilon} - 2\varepsilon \ge \frac{1}{2}\varepsilon^\alpha$ for small α. It follows that

$$P_\eta\Big[\bigcup_{j\in J} A_6(z_j, 2\varepsilon, \tilde{\varepsilon}-2\varepsilon)\Big] \;\le\; c\varepsilon^{-2}\cdot\big(\tfrac{2\varepsilon}{\tilde{\varepsilon}-2\varepsilon}\big)^{2+\nu} \;\le\; c\varepsilon^{\nu-2\alpha-\nu\alpha}\,,$$

which tends to zero as $\varepsilon \to 0$ for sufficiently small $\alpha > 0$.

Summing up, we have

$$\boxminus q_n \triangle \boxminus q \;\subseteq\; \big(\boxminus q^0 \triangle \boxminus q^4\big) \,\cup\, \bigcup_{j\in J} A_6(z_j, 2\varepsilon, \tilde{\varepsilon}-2\varepsilon)$$

and therefore for $\eta < \varepsilon$

$$P_\eta\big[\boxminus q_n \triangle \boxminus q\big] \;\le\; \tilde{f}(\varepsilon)$$

for some function $\tilde{f}$ with $\tilde{f}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

To conclude the proof, given $\rho > 0$, we choose $\varepsilon > 0$ such that $\tilde{f}(\varepsilon) \le \rho$, $\eta_0 = \frac{1}{2}\varepsilon$ and $n_0 \in \mathbb{N}$ associated to ε as above. □

Remark 2.4. Just convergence in the kernel sense is not enough, as the following counterexample shows. Let q be the quad $q : [0,1]^2 \to [0,1]^2$, $q(z) = z$, and let quads q_n be given by

$$\overline{q_n} := [0,1]^2 \setminus (\tfrac{1}{n},1] \times (\tfrac{1}{n},\tfrac{2}{n}), \qquad \partial_i q_n = \partial_i q, \quad i \in \{0,1,3\},$$

and $\partial_2 q_n$ consisting of the boundary part between $(1,0)$ and $(1,1)$.

Then q_n converge in the kernel sense to q. But if $P_\eta^{0.5}$ denotes the critical percolation measure, then

$$P_\eta^{0.5}(\boxminus q) = \tfrac{1}{2},$$

whereas

$$P_\eta^{0.5}(\boxminus q_n) \to 1$$

as $\eta \to 0$ with $\eta \asymp 1/n$. RSW yields the last assertion considering concentric (quarter-)annuli around $(0,0)$ with radii $2/n \cdot 2^k$ and $2/n \cdot 2^{k+1}$, $0 \le k \le c\log n$. Thus a condition like (2.2) is necessary for Lemma 2.3.

2.3.2 One Mesoscopic Triangle

Now we begin with the proof of Theorem 2.1. Using the same basic ideas, we more or less follow the set-up of the proof of [NW-09, Proposition 6]. But now and then we take slightly different approaches for various reasons. In particular, we work longer with the discrete exploration paths.

Let us fix a sequence $(\eta_k)_{k\in\mathbb{N}}$ fulfilling the hypothesis of the theorem. As explained before stating the theorem, such a sequence does exist. In the following, we omit the subscript k of η_k and simply write η for an element of the chosen sequence. The limit $\eta \to 0$ is always to be understood along the sequence $(\eta_k)_k$.

First we need some definitions. Consider a small equilateral triangle t of size δ which is contained in H_r. The size δ shall be some mesoscopic size, intermediate between the mesh size η and the size r of the domain.

According to Figure 2.6, we define the open rectangle $r = r(t)$ to be the whole dotted area, the closed segments $l = l(t)$, $m = m(t)$ and $b = b(t)$ to be the lower, the middle respectively the upper "line" of r as well as the smaller triangle t' just like in [NW-09, p. 814], to which we also refer for exact definitions. But note that the exact definitions are not that important for the proof.

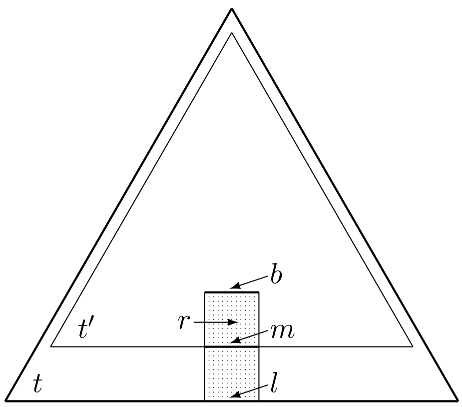


Figure 2.6: Definition of r, m, b, t'

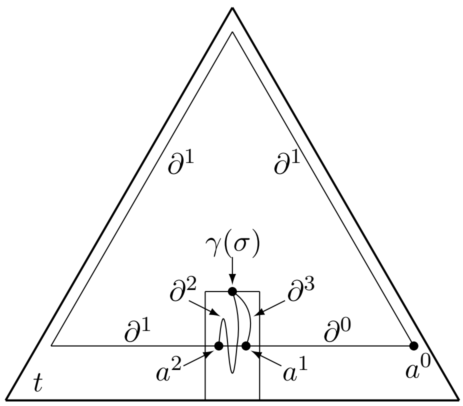


Figure 2.7: Definition of $\gamma(\sigma), a^i, \partial^i$

Given a curve $\gamma \in \mathcal{S}_r$, let $\sigma = \sigma(t, \gamma)$ be its first hitting time of $t \setminus r$ or the first hitting time of l after hitting m, whatever happens first. If $\gamma(\sigma) \in b$ we say that the triangle t is *good* for the curve γ. Let us denote this event by $G(t, \gamma)$.

If a triangle t is good for a curve γ, we define the following. Let $a^0 = a^0(t)$ be the right corner of t', $a^1 = a^1(t, \gamma)$ be the right-most point and $a^2 = a^2(t, \gamma)$ the left-most point on $m \cap \gamma[0, \sigma]$. We further define the set $d = d(t, \gamma)$ as the union of the connected component of $t' \setminus \gamma[0, \sigma]$ which has the top boundaries of t' on its boundary, and the components of $r \setminus (t' \cup \gamma[0, \sigma])$ which touch the former component between a^2 and a^1. Then $d(t, \gamma)$ is a simply connected set whose boundary consists of $\partial t' \setminus (a^2, a^1)$ and some points of $\gamma[0, \sigma]$. We partition its boundary as follows. Let $\partial^0(t, \gamma)$ be the part of the boundary between a^1 and a^0, $\partial^1(t, \gamma)$ the part between a^0

and a^2, $\partial^2(t,\gamma)$ the prime ends between a^2 and $\gamma(\sigma)$ and finally $\partial^3(t,\gamma)$ the prime ends between $\gamma(\sigma)$ and a^1 (all in counter-clockwise direction). With these boundary parts, one can consider $d(t,\gamma)$ as some quad. These definitions are illustrated in Figure 2.7. Note that they depend on the curve γ only up to time σ.

Now we define the event $VG(\gamma, t)$ that the triangle t is *very good* for γ: it holds if t is good for γ and if, after time σ, γ hits $\partial^0(t,\gamma)$ before $\partial^1(t,\gamma)$. Note that all these definitions are analogous to [NW-09, p. 814]. We only decreased the indices of ∂^i to be consistent with the quad notation introduced above. We further enlarged the set d a little bit to ensure the observation in the next paragraph.

When we apply these definitions to the discrete exploration paths γ_η, we adjust them to the discrete setting: All sets shall be unions of hexagons, a point is considered as a hexagon and $\gamma_\eta[0,\sigma]$ shall be the exploration path up to time σ together with the touching blue and yellow hexagons. If t is good for γ_η, the event $VG(t,\gamma_\eta)$ is equivalent to the existence of a blue crossing from ∂^0 to ∂^2 inside $d(t,\gamma_\eta)$, i.e. to $\boxminus d(t,\gamma_\eta)$. This observation is ensured by the slight enlargement of d. Without it, the exploration path could bypass some blue crossings using hexagons below m.

In the following lemma we estimate the difference between the P_η^λ- and the P_η^μ-probability of the event that the exploration path is very good for some triangle conditioned on the path up to time σ. We state (and use) this lemma only in the discrete setting. By this means, we avoid having to consider a limit simultaneously in the event and in the conditioning – which is tricky. Let us recall that δ is the mesoscopic size of the triangle t, that $\gamma_\eta : \Omega_\eta \to \mathcal{S}_r$ is the exploration path and that $\hat{\alpha}_4$ is the exponent bounding the probability of a four arm event from above.

Lemma 2.5. *The following estimate holds for all very small $\beta > 0$ and for all small enough δ and $\eta \ll \delta$ on the event $G(t,\gamma_\eta)$:*

$$P_\eta^\lambda\big[VG(t,\gamma_\eta) \mid \gamma_\eta[0,\sigma]\big] \;-\; P_\eta^\mu\big[VG(t,\gamma_\eta) \mid \gamma_\eta[0,\sigma]\big] \;\geq\; \delta^{2-\hat{\alpha}_4+\beta}\,.$$

Here and in the following, $\eta \ll \delta$ means for all $\eta < \eta_0$ where η_0 depends on δ. In fact, $\eta_0 = c\delta$ for some universal constant $c > 0$ will be enough.

Proof. We follow the corresponding part of the proof of Nolin and Werner, see [NW-09, p. 816]. Let $\eta > 0$ be small. We couple the percolation configurations in a monotone manner such that the set of blue hexagons increases. More precisely, let $\hat{P}$ be the uniform measure on $\hat{\Omega}_\eta := [0,1]^{H_r^\eta}$, and for $p \in [0,1]$ let the random variable $\omega(p) : \hat{\Omega}_\eta \to \Omega_\eta$ be defined by $(\omega(p)(\hat{\omega}))_x =$ blue iff $\hat{\omega}_x \leq p$ for $x \in H_r^\eta$ and $\hat{\omega} = (\hat{\omega}_x)_x \in \hat{\Omega}_\eta$.

Given $\gamma_\eta[0,\sigma]$ and $G(t,\gamma_\eta)$, the event $VG(t,\gamma_\eta)$ only depends on the hexagons inside $d(t,\gamma_\eta)$ since it is equivalent to $\boxminus d(t,\gamma_\eta)$. Moreover, given $\gamma_\eta[0,\sigma]$, percolation inside $d(t,\gamma_\eta)$ is still unbiased, i.e. we may use all percolation techniques there, for instance RSW and the separation lemmas.

Suppose now that t is good for γ_η. We conclude

$$P_\eta^\lambda\big[VG(t,\gamma_\eta) \mid \gamma_\eta[0,\sigma]\big] - P_\eta^\mu\big[VG(t,\gamma_\eta) \mid \gamma_\eta[0,\sigma]\big] = \hat{P}[E_\eta]\,,$$

where E_η is the event that there exists a blue crossing from $\partial^0(t,\gamma_\eta)$ to $\partial^2(t,\gamma_\eta)$ in $d(t,\gamma_\eta)$ for $\omega(p^\lambda)$, but not for $\omega(p^\mu)$.

In order to prove the proposed estimate, we can restrict ourselves to the following sub-event of E_η. For a hexagon x inside a deterministic rhombus of size 0.1δ inside $d(t,\gamma_\eta)$ (away from the boundary) and for $p \in [p^\mu, p^\lambda]$, let us consider the event that x is pivotal for the existence of the desired crossing. In that case there are four arms of alternating colours from x to the boundary of $d(t,\gamma_\eta)$. Its probability is bounded from below by $C\,\alpha_4^\eta(\delta)$ for some constant $C>0$, uniformly in x, p and $\eta \ll \delta$. This is a consequence of the separation lemmas, RSW and the uniform estimates for arm events, which are still valid in the nearcritical regime (cf. e.g. [N-08]). As the crossing event is increasing, the event that x is pivotal and switched from yellow to blue at p (i.e. $\hat\omega_x = p$), can happen only for one hexagon x and for one p. Therefore, the $\hat P$-probability that this occurs for some x in the rhombus and for some $p \in [p^\mu, p^\lambda]$, which is clearly a sub-event of E_η, is larger than

$$C\alpha_4^\eta(\delta)\,(\tfrac{0.1\delta}{\eta})^2\,(p^\lambda - p^\mu)\,.$$

Using

$$p^\lambda - p^\mu = \tfrac12 + \lambda_\eta \eta^2/\alpha_4^\eta - \tfrac12 - \mu_\eta\eta^2/\alpha_4^\eta = (\lambda_\eta - \mu_\eta)\,\eta^2/\alpha_4^\eta(\eta,1) \tag{2.3}$$

we estimate

$$\begin{aligned}
\hat P[E_\eta] &\geq C\,\alpha_4^\eta(\eta,\delta)(\tfrac{0.1\delta}{\eta})^2\,(p^\lambda - p^\mu)\\
&\overset{(2.3)}{=} C'\,\delta^2\eta^{-2}\,\alpha_4^\eta(\eta,\delta)\,[\alpha_4^\eta(\eta,1)]^{-1}\eta^2(\lambda_\eta-\mu_\eta)\\
&\geq C''\,\delta^2[\alpha_4^\eta(\delta,1)]^{-1}(\lambda-\mu+o(1))\\
&\geq \delta^{2-\hat\alpha_4+\beta}\,,
\end{aligned}$$

the latter if δ is small enough, depending on β, C'' and the $o(1)$-term. Quasi-multiplicativity yields the last but one line. The lemma follows. □

Remark 2.6. Using the ratio limit theorem [GPS-13a, Proposition 4.9.] (stating $\alpha_4^\eta(\eta,\delta)/\alpha_4^\eta(\eta,1) \to \delta^{-5/4}$) instead of quasi-multiplicativity, we could have concluded on the hexagonal lattice that

$$P_\eta^\lambda\big[VG(t,\gamma_\eta)\mid\gamma_\eta[0,\sigma]\big] \;-\; P_\eta^\mu\big[VG(t,\gamma_\eta)\mid\gamma_\eta[0,\sigma]\big] \;\geq\; C\delta^{\frac34}$$

for small enough δ and $\eta \ll \delta$ on $G(t,\gamma_\eta)$, for some constant $C>0$ independent of η and δ.

Remark 2.7. Though the proof of Lemma 2.5 is almost the same as the corresponding part of [NW-09], it contains the main reason, why [NW-09, Proposition 6] expands to Theorem 2.1: it is the quite trivial equation (2.3). This equation shows that the distance between two different nearcritical probabilities is – up to constants

– the same as the distance between a nearcritical and the critical probability. In fact,

$$p^{\lambda} - p^{\mu} \asymp \frac{\eta^2}{\alpha_4^{\eta}(\eta,1)} \asymp p_{\text{nearcritical}} - p_{\text{critical}}$$

as $\eta \to 0$.

2.3.3 Many Mesoscopic Triangles

We continue the proof of Theorem 2.1 similar to [NW-09, p. 816] by looking at a whole bunch of small triangles. Thereto let $\delta \gg \eta > 0$. Later on we will send η – and finally even δ – to zero, but in this subsection δ and η are fixed. Using a triangular grid of mesh size 4δ, we place a circle of radius δ at each site and put an equilateral triangle of size δ in its centre. This defines $N = N(\delta) \asymp \delta^{-2}$ deterministic triangles on the whole domain. We fix some very small $\beta > 0$ and set $M = M(\delta) := \lfloor \delta^{-2+\check{\alpha}_2+\beta} \rfloor$, where $\check{\alpha}_2$ is the exponent bounding the two arm probability from below.

Given the discrete exploration path γ_η, we assign each triangle t its hitting time $\sigma(t, \gamma_\eta)$ as defined at the beginning of the proof. If a triangle is not hit at all, we set $\sigma(t, \gamma_\eta) = 1$. We arrange the N triangles in the order $t_1, \ldots, t_N$ such that $\sigma_1 \le \sigma_2 \le \ldots \le \sigma_N$ where $\sigma_k = \sigma(t_k, \gamma_\eta)$. Note that these inequalities are strict unless $\sigma_k = \sigma_{k+1} = 1$. We further introduce the σ-Algebras on Ω_η

$$\mathcal{F}_k := \sigma(\gamma_\eta[0, \sigma_{k+1}])\,, \quad k \in \{0, \ldots, N-1\}$$

and $\mathcal{F}_N = \mathcal{F}_{N+1} := \sigma(\gamma_\eta[0,1])$. Note the shift in the index and the very different meaning of the two letters σ in that formula. Let us remark that we can already decide at time σ_k whether the triangle t_k is good or not, i.e. $G(t_k, \gamma_\eta) \in \mathcal{F}_{k-1}$. Moreover, $VG(t_k, \gamma_\eta) \in \mathcal{F}_k$ since if t_k is good, the status *very good* is decided at the next hitting of the triangle's boundary and thus before hitting the next triangle at time σ_{k+1}.

Instead of defining a random variable which resembles the quantity Z of [NW-09, p. 817] right now, we develop a discrete analogue. With that approach we can explicitly estimate some variances. To this end, we define the bounded random variables $\Omega_\eta \to \mathbb{R}$

$$X^{\delta,\iota}_{\eta,n} := \sum_{k=1}^{n} \mathbb{1}_{G(t_k,\gamma_\eta)} \big(\mathbb{1}_{VG(t_k,\gamma_\eta)} - P^{\iota}_{\eta}[VG(t_k,\gamma_\eta) \mid \mathcal{F}_{k-1}] \big)$$

for $n \in \{0, \ldots, N\}$ and $\iota \in \{\mu, \lambda\}$. Moreover, $X^{\delta,\iota}_{\eta,N+1} := X^{\delta,\iota}_{\eta,N}$. By the remark in the previous paragraph, $X^{\delta,\iota}_{\eta,n}$ is $\mathcal{F}_n$-measurable. In fact, it is a martingale with respect to P^{ι}_{η} since

$$\begin{aligned} E_{P^{\iota}_{\eta}}\big[\mathbb{1}_{G(t_n,\gamma_\eta)}\big(\mathbb{1}_{VG(t_n,\gamma_\eta)} - P^{\iota}_{\eta}[VG(t_n,\gamma_\eta) \mid \mathcal{F}_{n-1}]\big) \mid \mathcal{F}_{n-1}\big] = \\ = \mathbb{1}_{G(t_n,\gamma_\eta)}\big(E_{P^{\iota}_{\eta}}[\mathbb{1}_{VG(t_n,\gamma_\eta)} \mid \mathcal{F}_{n-1}] - P^{\iota}_{\eta}[VG(t_n,\gamma_\eta) \mid \mathcal{F}_{n-1}]\big) = 0\,. \end{aligned}$$

But we will need a slightly different martingale. To this end, we define for $a \in \mathbb{N}_0$

$$T_a := \inf\left\{n \in \mathbb{N}_0 : \sum_{k=1}^{n} \mathbb{1}_{G(t_k,\gamma_\eta)} \geq a\right\} \wedge (N+1)\,.$$

Then $\{T_a = n\} \in \mathcal{F}_{n-1}$ for all $n \in \{0, 1, \ldots, N+1\}$ and $a \in \mathbb{N}_0$ (with $\mathcal{F}_{-1} := \{\emptyset, \Omega_\eta\}$). Thus T_a is a "pre-visible stopping time", i.e. T_a is $\mathcal{F}_{T_a - 1}$-measurable. As $(T_a)_{a\in\mathbb{N}_0}$ is a non-decreasing sequence of bounded stopping times, the Optional Sampling Theorem implies that

$$\left(X^{\delta,\iota}_{\eta,T_a}\right)_{a\in\mathbb{N}_0} \text{ is an } (\mathcal{F}_{T_a})_{a\in\mathbb{N}_0}\text{-martingale with respect to } P^\iota_\eta\,.$$

It follows that

$$E_{P^\iota_\eta}\left[X^{\delta,\iota}_{\eta,T_a}\right] \;=\; 0$$

and

$$\mathrm{Var}_{P^\iota_\eta}\left[X^{\delta,\iota}_{\eta,T_a}\right] \;=\; \sum_{\tilde{a}=0}^{a-1} \mathrm{Var}_{P^\iota_\eta}\left[X^{\delta,\iota}_{\eta,T_{\tilde{a}+1}} - X^{\delta,\iota}_{\eta,T_{\tilde{a}}}\right] \;\leq\; \sum_{\tilde{a}=0}^{a-1} 1 \;=\; a$$

since the absolute value of the increments is at most one. Indeed, as T_a counts the number of good triangles, all addends between $T_{\tilde{a}}$ and $T_{\tilde{a}+1}$ are zero.

Now we look at the processes stopped at time T_M. By Chebyshev's inequality it follows that

$$P^\mu_\eta\left[X^{\delta,\mu}_{\eta,T_M} \geq \delta^{-1+\frac{1}{2}\check{\alpha}_2}\right] \leq \delta^{2-\check{\alpha}_2}\,\mathrm{Var}_{P^\mu_\eta}\left[X^{\delta,\mu}_{\eta,T_M}\right] \leq \delta^{2-\check{\alpha}_2}\cdot M \leq \delta^{2-\check{\alpha}_2}\cdot\delta^{-2+\check{\alpha}_2+\beta} = \delta^\beta.$$

Moreover, we have by Lemma 2.5 on the event that there are at least M good triangles, i.e. on $\{T_M \leq N\}$

$$\begin{aligned}
X^{\delta,\mu}_{\eta,T_M} &= X^{\delta,\lambda}_{\eta,T_M} + \sum_{k=1}^{T_M} \mathbb{1}_{G(t_k,\gamma_\eta)}\big(P^\lambda_\eta[VG(t_k,\gamma_\eta)\mid\mathcal{F}_{k-1}] - P^\mu_\eta[VG(t_k,\gamma_\eta)\mid\mathcal{F}_{k-1}]\big)\\
&\geq X^{\delta,\lambda}_{\eta,T_M} + \sum_{k=1}^{T_M} \mathbb{1}_{G(t_k,\gamma_\eta)}\cdot\delta^{2-\hat{\alpha}_4+\frac{\beta}{2}}\\
&= X^{\delta,\lambda}_{\eta,T_M} + M\cdot\delta^{2-\hat{\alpha}_4+\frac{\beta}{2}} \geq X^{\delta,\lambda}_{\eta,T_M} + \delta^{\check{\alpha}_2-\hat{\alpha}_4+2\beta}
\end{aligned}$$

for small enough δ and $\eta \ll \delta$. Therefore

$$\begin{aligned}
P^\lambda_\eta\left[X^{\delta,\mu}_{\eta,T_M} \leq \tfrac{1}{2}\delta^{\check{\alpha}_2-\hat{\alpha}_4+2\beta}, T_M \leq N\right] &\leq P^\lambda_\eta\left[X^{\delta,\lambda}_{\eta,T_M} + \delta^{\check{\alpha}_2-\hat{\alpha}_4+2\beta} \leq \tfrac{1}{2}\delta^{\check{\alpha}_2-\hat{\alpha}_4+2\beta}\right]\\
&= P^\lambda_\eta\left[X^{\delta,\lambda}_{\eta,T_M} \leq -\tfrac{1}{2}\delta^{\check{\alpha}_2-\hat{\alpha}_4+2\beta}\right]\\
&\leq 4\,\delta^{2\hat{\alpha}_4-2\check{\alpha}_2-4\beta}\;\mathrm{Var}_{P^\lambda_\eta}\left[X^{\delta,\lambda}_{\eta,T_M}\right]\\
&\leq 4\,\delta^{2\hat{\alpha}_4-2\check{\alpha}_2-4\beta}\cdot M \;\leq\; 4\,\delta^{2\hat{\alpha}_4-\check{\alpha}_2-2-3\beta}\,.
\end{aligned}$$

Thus we arrived at

Lemma 2.8. *The following estimates hold:*

$$P^{\mu}_{\eta}\big[X^{\delta,\mu}_{\eta,T_M} \ge \delta^{-1+\frac{1}{2}\check{\alpha}_2}\big] \;\le\; \delta^{\beta}$$

whereas

$$P^{\lambda}_{\eta}\big[X^{\delta,\mu}_{\eta,T_M} \le \tfrac{1}{2}\delta^{\check{\alpha}_2-\hat{\alpha}_4+2\beta}\big] \;\le\; 4\,\delta^{2\hat{\alpha}_4-\check{\alpha}_2-2-3\beta} + P^{\lambda}_{\eta}\big[T_M = N+1\big]$$

for all small enough δ and $\eta \ll \delta$. □

Let us remark that $2\hat{\alpha}_4 - \check{\alpha}_2 - 2 - 3\beta > 0$ for small β by inequality (2.1). The main ingredient to Lemma 2.8 was the estimate of the variance. For $\iota \in \{\mu, \lambda\}$, we estimated the variance of $X^{\delta,\iota}_{\eta,T_M}$ with respect to P^{ι}_{η} using a martingale structure. But this approach did not yield an estimate of the variance of $X^{\delta,\mu}_{\eta,T_M}$ with respect to P^{λ}_{η} (mind the λ and the μ), which, together with the corresponding expectation, would have been nice for the second statement of Lemma 2.8. Instead we used a point-wise estimate of $X^{\delta,\mu}_{\eta,T_M} - X^{\delta,\lambda}_{\eta,T_M}$.

One could be tempted to simply estimate the variance by independence since the considered triangles are disjoint. But this account is tricky since the exploration path obviously depends on its past, and therefore the events $G(t_k, \gamma_\eta)$ respectively $VG(t_k, \gamma_\eta)$, $k \in \{1, \dots, N\}$, are not independent. Moreover, the exploration path could enter, leave and re-enter the bottom half of the rectangle r of some triangle while making a different triangle good and possibly very good in the meantime. Hence we chose the martingale approach described above which does not use any geometric information. Alternatively, it may be possible to estimate the variance with some ideas used in the proof of Lemma 2.9 below.

We still have to look at the event $\{T_M = N+1\}$, i.e. at the event that there are less than $M = \lfloor \delta^{-2+\check{\alpha}_2+\beta} \rfloor$ good triangles to benefit from the second estimate of Lemma 2.8.

Lemma 2.9. *There are a function J with $J(\delta) \to \infty$ as $\delta \to 0$ and a numerical constant $C_0 \in (0,1)$ such that*

$$P^{\lambda}_{\eta}\big[T_M = N+1\big] \;=\; P^{\lambda}_{\eta}\Big[\sum_{k=1}^{N} \mathbb{1}_{G(t,\gamma_\eta)} < M\Big] \;\le\; (1-C_0)^{J(\delta)}$$

for small enough δ and $\eta \ll \delta$.

Proof. We follow the rough outline in [NW-09, p. 816f] using ideas of the proof of [NW-09, Proposition 2]. We choose $J = J(\delta)$ such that $\delta^{\beta} = c_5(r2^{-J})^{2-\check{\alpha}_2}$ with some constant $c_5 > 0$ specified below. The reason for that choice will become clear later on. Since $\check{\alpha}_2 < 2$, $J(\delta)$ tends to infinity as $\delta \to 0$.

We use the notation $f(\eta,\delta,j) \asymp g(\eta,\delta,j)$ to indicate that there are numerical constants $c, c' > 0$ such that $\exists\,\delta_0 > 0\;\forall\,\delta < \delta_0\;\exists\,\eta_0 > 0\;\forall\,\eta < \eta_0\;\forall\,j \in \{0,\dots,\lfloor J \rfloor\}$:

$$c\,f(\eta,\delta,j) \le g(\eta,\delta,j) \le c'\,f(\eta,\delta,j)\,.$$

Below we will need a statement similar to item (3) in [NW-09, p. 803], namely

$$\sum_{k=1}^{n} k\alpha_2^\eta(\delta, 4\delta k) \asymp n^2\alpha_2^\eta(\delta, 4\delta n)\,. \tag{2.4}$$

Since it is harder to cross a larger annulus, we have in one direction

$$\sum_{k=1}^{n} k\alpha_2^\eta(\delta, 4\delta k) \geq \sum_{k=1}^{n} k\alpha_2^\eta(\delta, 4\delta n) \geq \tfrac{1}{2}n^2\alpha_2^\eta(\delta, 4\delta n)\,.$$

The other direction follows using quasi-multiplicativity:

$$\sum_{k=1}^{n} k\frac{\alpha_2^\eta(\delta, 4\delta k)}{\alpha_2^\eta(\delta, 4\delta n)} \leq C\sum_{k=1}^{n} k\frac{1}{\alpha_2^\eta(4\delta k, 4\delta n)} \leq C\sum_{k=1}^{n} k\Big(\frac{4\delta n}{4\delta k}\Big)^{\breve{\alpha}_2} \leq C\sum_{k=1}^{n} k\frac{n}{k} = Cn^2$$

since $\breve{\alpha}_2 \leq 1$.

Now we begin with the actual proof. We choose $0 < \delta < \delta_0$ and $0 < \eta < \eta_0$ for some appropriate $\delta_0 > \eta_0 > 0$. Let us recall that our domain is the half-circle H_r with radius $r > 0$. We consider the following half-annuli:

$$B_j \;:=\; H_{r2^{-j}} \setminus H_{r2^{-j-1}} \qquad j \in \{0, \ldots, \lfloor J \rfloor\}\,,$$

If δ and β are small enough, then $\delta < 2^{-10}c_5^{-1}\delta^\beta = 2^{-10}(r2^{-J})^{2-\breve{\alpha}_2} \leq r2^{-J-10}$ by the choice of J and $\breve{\alpha}_2 \leq 1$. Thus there are some triangles in the half-annuli. Let $\mathcal{T}_j$ be the set of all triangles which are contained in B_j and whose distance from the boundary of B_j is at least $r2^{-j-3}$. $\mathcal{T}_j$ consists of $\asymp r^2 2^{-2j}\delta^{-2}$ triangles. For a triangle $t \in \mathcal{T}_j$, let $G'_j(t)$ be the event that there are a blue and a yellow arm originating at $b(t)$, crossing $r(t)$, staying inside B_j and finally ending at the negative respectively positive real axis. If $G'_j(t)$ is fulfilled, then t is good for γ_η, i.e. $G'_j(t) \subset G(t, \gamma_\eta)$.

Now we want to estimate the probability of $G'_j(t)$. Note that $G'_j(t)$ implies $A_2(t, \delta, r2^{-j-3})$, the event that there exist two arms of different colour inside the annulus with radii δ and $r2^{-j-3}$ centred at the centre of t. Conversely if $A_2(t, \delta, r2^{-j-3})$ with some specified separated landing sequences is fulfilled and if some deterministic rectangles of fixed aspect ratio are crossed, then $G'_j(t)$ occurs. By the arm separation lemmas and RSW it follows that

$$P_\eta^\lambda\big[G'_j(t)\big] \;\asymp\; P_\eta^\lambda\big[A_2(t, \delta, r2^{-j-3})\big] \;\asymp\; \alpha_2^\eta(\delta, r2^{-j})\,.$$

Let the random variable G_j be the number of triangles $t \in \mathcal{T}_j$ that fulfil $G'_j(t)$. We want to estimate the probability that G_j is quite small. Thereto we apply the second moment method. While the first moment is immediately estimated:

$$E_{P_\eta^\lambda}[G_j] = \sum_{t\in\mathcal{T}_j} P_\eta^\lambda[G'_j(t)] \asymp \sum_{t\in\mathcal{T}_j} \alpha_2^\eta(\delta, r2^{-j}) \asymp r^2 2^{-2j}\delta^{-2}\alpha_2^\eta(\delta, r2^{-j})\,,$$

the second moment is more involved. Let $t, \tilde{t} \in \mathcal{T}_j$ be two different triangles. Let $\|t;\tilde{t}\|$ denote the distance of their centres. If both events $G'_j(t)$ and $G'_j(\tilde{t})$ occur, then there are two crossings of different colour in each of the the following three annuli: the annulus around t with radii δ and $\frac{1}{2}\|t;\tilde{t}\|$, the annulus around $\tilde{t}$ with radii δ and $\frac{1}{2}\|t;\tilde{t}\|$, and finally the annulus around the centre between the two triangles with radii $2\|t;\tilde{t}\|$ and $r2^{-j-3}$. Since these annuli are disjoint, it follows that

$$P_\eta^\lambda[G'_j(t) \cap G'_j(\tilde{t})] \le c_1 \cdot \alpha_2^\eta(\delta, \tfrac{1}{2}\|t;\tilde{t}\|) \cdot \alpha_2^\eta(\delta, \tfrac{1}{2}\|t;\tilde{t}\|) \cdot \alpha_2^\eta(2\|t;\tilde{t}\|, r2^{-j-3}) \, .$$

Here and in the following, $c_1, c_2, \ldots, c_7 > 0$ are numerical constants. Using quasi-multiplicativity, we conclude

$$\begin{aligned} E_{P_\eta^\lambda}\big[G_j^2\big] &= \sum_{t \in \mathcal{T}_j} P_\eta^\lambda[G'_j(t)] + \sum_{t \neq \tilde{t} \in \mathcal{T}_j} P_\eta^\lambda[G'_j(t) \cap G'_j(\tilde{t})] \\ &\le E_{P_\eta^\lambda}\big[G_j\big] + c_2 \sum_{t \neq \tilde{t} \in \mathcal{T}_j} \alpha_2^\eta(\delta, 4\delta \big\lfloor \tfrac{1}{8\delta}\|t;\tilde{t}\| \big\rfloor) \cdot \alpha_2^\eta(\delta, r2^{-j}) \, . \end{aligned}$$

Since the triangles were placed using a triangular grid of mesh size 4δ, there are at most $c_3 \cdot k$ triangles in T_j at distance $4\delta k$ from some fixed triangle for $k \in \{1, \ldots, \lfloor r2^{-j}/\delta \rfloor\}$ and no triangles further away. This, equation (2.4) and the estimate of the first moment imply

$$\begin{aligned} E_{P_\eta^\lambda}\big[G_j^2\big] - E_{P_\eta^\lambda}\big[G_j\big] &\le c_2 \sum_{t \in \mathcal{T}_j} \sum_{k=1}^{\lfloor r2^{-j}/\delta \rfloor} c_3 \, k \, \alpha_2^\eta(\delta, 4\delta\, k) \cdot \alpha_2^\eta(\delta, r2^{-j}) \\ &\asymp E_{P_\eta^\lambda}[G_j] \cdot \lfloor r2^{-j}/\delta \rfloor^2 \alpha_2^\eta(\delta, 4\delta \lfloor r2^{-j}/\delta \rfloor) \asymp E_{P_\eta^\lambda}[G_j]^2 \, . \end{aligned}$$

As (note that $\frac{1}{2}$ and M will become relevant later on)

$$\begin{aligned} \tfrac{1}{2} E_{P_\eta^\lambda}[G_j] &\ge c_4 r^2 2^{-2j} \delta^{-2} \alpha_2^\eta(\delta, r2^{-j}) \ge c_5 r^2 2^{-2j} \delta^{-2} \big(\delta/(r2^{-j})\big)^{\check{\alpha}_2} \\ &= c_5 \big(r2^{-j}\delta^{-1}\big)^{2-\check{\alpha}_2} \ge c_5 \big(r2^{-J}\delta^{-1}\big)^{2-\check{\alpha}_2} = \delta^\beta \delta^{-2+\check{\alpha}_2} \ge M \ge 1 \end{aligned}$$

by our choice of J and $M = \lfloor \delta^{-2+\check{\alpha}_2+\beta} \rfloor$, we conclude

$$E_{P_\eta^\lambda}\big[G_j^2\big] \le c_6 \, E_{P_\eta^\lambda}\big[G_j\big]^2 + E_{P_\eta^\lambda}\big[G_j\big] \le c_7 \, E_{P_\eta^\lambda}\big[G_j\big]^2 \, .$$

Since

$$E[X] = E\big[X \mathbb{1}_{X < \frac{1}{2}E[X]} + X \mathbb{1}_{X \ge \frac{1}{2}E[X]}\big] \le \tfrac{1}{2}E[X] + E\big[X \mathbb{1}_{X \ge \frac{1}{2}E[X]}\big]$$

and therefore

$$\big(\tfrac{1}{2}E[X]\big)^2 \le E\big[X \mathbb{1}_{X \ge \frac{1}{2}E[X]}\big]^2 \le E\big[X^2\big] \cdot P\big[X \ge \tfrac{1}{2}E[X]\big]$$

holds for any non-negative random variable X, we conclude

$$P_\eta^\lambda\big[G_j \geq \tfrac{1}{2} E_{P_\eta^\lambda}[G_j]\big] \geq \frac{E_{P_\eta^\lambda}[G_j]^2}{4\, E_{P_\eta^\lambda}[G_j^2]} \geq C_0$$

for the numerical constant $C_0 := (4c_7)^{-1} \in (0,1)$.

As G_j depends only on the hexagons inside B_j and as these sets are pairwise disjoint, it follows that

$$P_\eta^\lambda\big[G_j < \tfrac{1}{2} E_{P_\eta^\lambda}[G_j] \text{ for all } j \in \{0, \dots, \lfloor J \rfloor\}\big] \leq (1 - C_0)^{J+1}\,.$$

Now we link the former event to the event of interest to conclude the proof. On the one hand, we have

$$G_j \;\leq\; \sum_{k=1}^{N} \mathbb{1}_{G(t_k, \gamma_\eta)}$$

for all $j \leq J$ since every triangle t with $G'_j(t)$ is good for γ_η. On the other hand, we already estimated for all $j \leq J$:

$$\tfrac{1}{2} E_{P_\eta^\lambda}[G_j] \;\geq\; M\,.$$

Therefore we conclude

$$P_\eta^\lambda\big[\sum_{k=1}^{N} \mathbb{1}_{G(t,\gamma_\eta)} < M\big] \;\leq\; P_\eta^\lambda\big[G_j < \tfrac{1}{2} E_{P_\eta^\lambda}[G_j] \text{ for all } j \leq J\big] \;\leq\; (1 - C_0)^{J+1}\,,$$

which completes the proof. □

In fact, this lemma is the only place where we used the fact that we have a straight boundary near the starting point of the exploration path. Therefore it was possible to define the sets B_j such that the estimates above hold uniformly for all j. A smooth boundary would also have been sufficient, but for a fractal boundary additional ideas are necessary.

2.3.4 Continuum Limit

Now we want to pass to the limit. Thereto we will need the following convergence lemma. Let us remark, that Nolin and Werner could just rely on Cardy's formula for their convergence results whereas we will have to use Lemma 2.3.

Lemma 2.10. *Let $\mathcal{T}$ be a finite set of triangles in H_r. Then there exists a set $\mathcal{N} \subset \mathcal{S}_r$ with*

$$\Gamma^\iota[\mathcal{N}] = 0\,, \quad \iota \in \{\mu, \lambda\}\,,$$

and such that for all $\gamma \in \mathcal{N}^c$ the following holds: If γ^n, $n \in \mathbb{N}$, is a sequence in $\mathcal{S}_r$ with $\operatorname{dist}(\gamma^n, \gamma) \to 0$ *as $n \to \infty$, then for all triangles $t \in \mathcal{T}$*

$$\mathbb{1}_{G(t,\gamma^n)} \;\to\; \mathbb{1}_{G(t,\gamma)}\,, \qquad \mathbb{1}_{VG(t,\gamma^n)} \;\to\; \mathbb{1}_{VG(t,\gamma)}$$

as $n \to \infty$ and for all $\rho > 0$ there exist $n_0 \in \mathbb{N}$ and $\eta_0 > 0$ such that for all $n \geq n_0$ and $\eta \leq \eta_0$

$$\left|\mathbb{1}_{G(t,\gamma^n)} P^\iota_\eta\big[\boxminus d(t,\gamma^n)\big] - \mathbb{1}_{G(t,\gamma)} P^\iota_\eta\big[\boxminus d(t,\gamma)\big]\right| \leq \rho$$

for $\iota \in \{\mu, \lambda\}$.

Proof. Let $\mathcal{N}$ be the set of all curves γ which – for some triangle $t \in \mathcal{T}$ – hit an end point of $b(t)$ or $a^0(t)$ or only touch $b(t)$ or the boundary of t' without crossing them. Then we claim that RSW implies that $\Gamma^\iota[\mathcal{N}] = 0$, $\iota \in \{\mu, \lambda\}$. Indeed, considering concentric annuli around any deterministic point yields that γ hits that point with Γ^ι-probability zero. And if γ touches any deterministic straight line (without crossing it), then there are three macroscopic (i.e. of size r) arms of alternating colours originating at some point on the line going to one of its sides. Since the 3-arm half-plane exponent is larger than 1 (in fact, it is 2 by RSW considerations, see [N-08, Theorem 23], for instance), this event has Γ^ι-probability zero. As $\mathcal{N}$ consists of finitely many such events, the claim follows.

For the remainder of the proof let $\gamma \in \mathcal{N}^c$, let γ^n converge to γ in the dist-metric and let $t \in \mathcal{T}$.

Suppose that t is good for γ. Since $\mathrm{dist}(\gamma^n, \gamma) \to 0$, i.e. $\gamma^n[0,1] \to \gamma[0,1]$ in the Hausdorff sense, and since γ crosses b at σ and does not hit an end point of b (because of $\gamma \in \mathcal{N}^c$), t is also good for γ^n for all large enough n. Conversely, if t is good for γ^n for all large n, it is also good for γ. Now let t be good for γ and for γ^n for all large n. Since γ crosses $\partial^0 \cup \partial^1$ at the first hitting and since γ does not hit a^0, the status of being very good is identical for γ and for γ^n for all large enough n. Thus we have shown that $\mathbb{1}_{G(t,\gamma^n)} \to \mathbb{1}_{G(t,\gamma)}$ and $\mathbb{1}_{VG(t,\gamma^n)} \to \mathbb{1}_{VG(t,\gamma)}$ as $n \to \infty$.

For the last assertion let $\rho > 0$. We can assume that t is good for γ and for γ_n for all large n. Since $d(t,\gamma)$ is defined as the connected component of $t' \setminus \gamma[0,\sigma]$ which contains a point near the tip of t together with some components also defined by $\gamma[0,\sigma]$ and as $\mathrm{dist}(\gamma^n,\gamma) \to 0$, we conclude that $d(t,\gamma^n)$ converge in the kernel sense to $d(t,\gamma)$. Furthermore, $\mathrm{dist}(\gamma^n,\gamma) \to 0$ implies condition (2.2). Thus Lemma 2.3 yields that there are $n_0 \in \mathbb{N}$ and $\eta_0 > 0$ such that for all $n \geq n_0$ and $\eta \leq \eta_0$

$$\Big|P^\iota_\eta\big[\boxminus d(t,\gamma^n)\big] - P^\iota_\eta\big[\boxminus d(t,\gamma)\big]\Big| \;\leq\; P^\iota_\eta\big[\boxminus d(t,\gamma^n) \bigtriangleup \boxminus d(t,\gamma)\big] \;\leq\; \rho$$

which implies the last assertion since $G(t,\gamma^n)$ and $G(t,\gamma^n)$ for all large n simultaneously hold. □

Inspired by the random variables T_a and X defined on Ω_η, we define the following random variables, but on $\mathcal{S}_r$ this time. We still have $\eta \ll \delta$ fixed and we use the triangles defined above. Given a curve $\gamma \in \mathcal{S}_r$ we arrange them in the order $t_1, \ldots, t_N$ according to their hitting time as above. Recall that $M = \lfloor \delta^{-2+\check{\alpha}_2+\beta} \rfloor$. We define

$$T \;:=\; \inf\{n \in \mathbb{N} : \sum_{k=1}^{n} \mathbb{1}_{G(t_k,\cdot)} \geq M\} \wedge (N+1)$$

and

$$Z_\eta^{\delta,\mu} := \sum_{k=1}^{T} \mathbb{1}_{G(t_k,\cdot)}\big(\mathbb{1}_{VG(t_k,\cdot)} - P_\eta^\mu\big[\boxminus\, d(t_k,\cdot)\big]\big)$$

on $\mathcal{S}_r$. Finally we define, letting $\eta \to 0$ now,

$$Z^{\delta,\mu} := \lim_{\eta\to 0} Z_\eta^{\delta,\mu} = \sum_{k=1}^{T} \mathbb{1}_{G(t_k,\cdot)}\big(\mathbb{1}_{VG(t_k,\cdot)} - \lim_{\eta\to 0} P_\eta^\mu\big[\boxminus\, d(t_k,\cdot)\big]\big)\,,$$

which resembles the quantity Z in [NW-09, p. 817]. The limit exists for all curves $\gamma \in \mathcal{S}_r$ since we have chosen the subsequence $(\eta_k)_{k\in\mathbb{N}}$ with the property that the limit of the crossing probabilities of any quad exists. Note that we defined these random variables only for the parameter μ and not for λ, since we will only need the versions with μ.

Lemma 2.11. *The laws $Z_\eta^{\delta,\mu}(\Gamma_\eta^\iota)$ converge weakly to $Z^{\delta,\mu}(\Gamma^\iota)$ as $\eta \to 0$, for $\iota \in \{\mu,\lambda\}$.*

Proof. Let $\iota \in \{\mu,\lambda\}$. We use Skorokhod's representation theorem to construct the following coupling. Let $(\bar\Omega, \bar{\mathcal{A}}, \bar P)$ be a suitable probability space and let $\bar\gamma, \bar\gamma_\eta : \bar\Omega \to \mathcal{S}_r$, $\eta > 0$, random variables such that $\bar\gamma_\eta \to \bar\gamma$ $\bar P$-a.s. (in the dist-metric) as $\eta \to 0$ and such that $\Gamma^\iota = \bar\gamma(\bar P)$ and $\Gamma_\eta^\iota = \bar\gamma_\eta(\bar P)$, $\eta > 0$. From Lemma 2.10 it follows that

$$Z_\eta^{\delta,\mu} \circ \bar\gamma_\eta \;\to\; Z^{\delta,\mu} \circ \bar\gamma \quad \bar P\text{-a.s.}$$

since $\bar P\big[\bar\gamma^{-1}[\mathcal{N}^c]\big] = \Gamma^\iota[\mathcal{N}^c] = 1$ and since every ingredient converges on $\bar\gamma^{-1}[\mathcal{N}^c]$. In particular, Lemma 2.10 implies that if we choose any sequence $n(\eta)$ such that $n(\eta) \to \infty$ as $\eta \to 0$, then

$$\lim_{\eta\to 0} \mathbb{1}_{G(t,\gamma^{n(\eta)})} P_\eta^\mu\big[\boxminus\, d(t,\gamma^{n(\eta)})\big] = \lim_{n\to\infty}\lim_{\eta\to 0} \mathbb{1}_{G(t,\gamma^n)} P_\eta^\mu\big[\boxminus\, d(t,\gamma^n)\big]\,,$$

since the double limit is uniform in n and η. Therefore $\mathbb{1}_{G(t,\bar\gamma_\eta)} P_\eta^\iota[\boxminus d(t,\bar\gamma_\eta)]$ converges on $\bar\gamma^{-1}[\mathcal{N}^c]$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. By the Dominated Convergence Theorem, we conclude

$$\int f\, d\big(Z_\eta^{\delta,\mu}(\Gamma_\eta^\iota)\big) = \int f(Z_\eta^{\delta,\mu} \circ \bar\gamma_\eta)\, d\bar P \to \int f(Z^{\delta,\mu} \circ \bar\gamma)\, d\bar P = \int f\, d\big(Z^{\delta,\mu}(\Gamma^\iota)\big)$$

as $\eta \to 0$. Thus the Portmanteau Theorem yields the desired weak convergence. □

For that Lemma it is crucial that the limit of $Z_\eta^{\delta,\mu}$ does exist, which is ensured by the choice of the sequence η_k in the very beginning. For the definition of $Z^{\delta,\mu}$, in principle, it is possible to use the limes superior. But then there are problems showing the weak convergence since the sequence used to determine the limes superior may depend on γ. The results in [SS-11] allowed us to choose the same sequence for all curves.

Now we give the link between the results on the discrete paths and the convergence lemmas to conclude the proof of Theorem 2.1. The key is the following connection between the random variables $X^{\delta,\mu}_{\eta,T_M}$, $Z^{\delta,\mu}_\eta$ and γ_η. On the event that a triangle t is good for the discrete exploration path γ_η, it is very good if and only if the quad $d(t,\gamma_\eta)$ is crossed. Therefore

$$X^{\delta,\mu}_{\eta,T_M} \;=\; Z^{\delta,\mu}_\eta \circ \gamma_\eta$$

by their definitions. We conclude $Z^{\delta,\mu}_\eta(\Gamma^\iota_\eta) = Z^{\delta,\mu}_\eta(\gamma_\eta(P^\iota_\eta)) = (Z^{\delta,\mu}_\eta \circ \gamma_\eta)(P^\iota_\eta) = X^{\delta,\mu}_{\eta,T_M}(P^\iota_\eta)$. Now Lemma 2.8 yields

$$\Gamma^\mu_\eta\big[Z^{\delta,\mu}_\eta \geq \delta^{-1+\frac12\check\alpha_2}\big] \;=\; P^\mu_\eta\big[X^{\delta,\mu}_{\eta,T_M} \geq \delta^{-1+\frac12\check\alpha_2}\big] \;\leq\; \delta^\beta$$

and

$$\Gamma^\lambda_\eta\big[Z^{\delta,\mu}_\eta \leq \tfrac12\delta^{\check\alpha_2-\hat\alpha_4+2\beta}\big] \;\leq\; 4\delta^{2\hat\alpha_4-\check\alpha_2-2-3\beta} + P^\lambda_\eta\big[T_M = N+1\big]\,.$$

With Lemma 2.11 and the Portmanteau Theorem we conclude

$$\Gamma^\mu\big[Z^{\delta,\mu} > \delta^{-1+\frac12\check\alpha_2}\big] \;\leq\; \liminf_{\eta\to0}\Gamma^\mu_\eta\big[Z^{\delta,\mu}_\eta > \delta^{-1+\frac12\check\alpha_2}\big] \;\leq\; \delta^\beta$$

and

$$\begin{aligned}\Gamma^\lambda\big[Z^{\delta,\mu} < \tfrac12\delta^{\check\alpha_2-\hat\alpha_4+2\beta}\big] \;&\leq\; \liminf_{\eta\to0}\Gamma^\lambda_\eta\big[Z^{\delta,\mu}_\eta < \tfrac12\delta^{\check\alpha_2-\hat\alpha_4+2\beta}\big]\\ &\leq\; 4\delta^{2\hat\alpha_4-\check\alpha_2-2-3\beta} + (1-C_0)^{J(\delta)}\,,\end{aligned}$$

where $J(\delta)$ and C_0 are chosen according to Lemma 2.9.

Because of inequality (2.1), namely $2\hat\alpha_4 - \check\alpha_2 > 2$, we can now choose a sequence δ_n, $n\in\mathbb{N}$, converging fast enough to zero such that the bounds on the right hand sides are summable. Then the Borel-Cantelli Lemma yields

$$\Gamma^\mu[Z^{\delta_n,\mu} > \delta_n^{-1+\frac12\check\alpha_2} \text{ for infinitely many } n] \;=\; 0$$

and

$$\Gamma^\lambda[Z^{\delta_n,\mu} < \tfrac12\delta_n^{\check\alpha_2-\hat\alpha_4+2\beta} \text{ for infinitely many } n] \;=\; 0\,.$$

Because of inequality (2.1) again, we have $1-\frac12\check\alpha_2 < \hat\alpha_4 - \check\alpha_2 - 2\beta$, which implies

$$\delta^{-1+\frac12\check\alpha_2} \;<\; \tfrac12\delta^{\check\alpha_2-\hat\alpha_4+2\beta}$$

for $\delta < 1$ small enough. Thus we conclude

$$\Gamma^\lambda[Z^{\delta_n,\mu} > \delta_n^{-1+\frac12\check\alpha_2} \text{ for infinitely many } n] \;=\; 1\,.$$

Therefore we detected an event which has probability zero under Γ^μ, but probability one under Γ^λ. This concludes the proof of Theorem 2.1.

Let us remark that we used inequality (2.1) only in the very last paragraph. In fact, this is the only place where we need a property proven only for site percolation on the triangular lattice, namely the values of two critical exponents.

2.4 Consequences for Conformal Maps

The critical scaling limit is conformally invariant. Does a similar statement hold for nearcritical limits? We can use the result above to give a negative answer to that question.

Let D be a domain and $f : D \to \tilde{D}$ be a conformal map. We consider percolation with $p_\eta^\mu = \frac{1}{2} + \mu \cdot \eta^2 / \alpha_4^\eta$ in both domains. Let $a \in \partial D$ and $\tilde{a} := f(a)$. We impose some corresponding boundary colours near a and $\tilde{a}$. Let γ_η respectively $\tilde{\gamma}_\eta$ be the discrete exploration paths starting at a respectively at $\tilde{a}$. If $\gamma_\eta(P_\eta^\mu) \to \Gamma^\mu$ and $\tilde{\gamma}_\eta(\tilde{P}_\eta^\mu) \to \tilde{\Gamma}^\mu$ weakly, we consider the following question: How are the laws $f(\Gamma^\mu)$ and $\tilde{\Gamma}^\mu$ related? We give an answer in the special case considering a scaling map on H_r for some $r > 0$.

Corollary 2.12. *Let $D = H_r$ for some $r > 0$ and let $f : D \to \tilde{D}$ be the scaling map with factor $\sigma \in \mathbb{R}^+$, i.e. $f(z) = \sigma z$. Assume $\gamma_\eta(P_\eta^\mu) \to \Gamma^\mu$, $\tilde{\gamma}_\eta(\tilde{P}_\eta^\mu) \to \tilde{\Gamma}^\mu$ weakly and that $\tilde{P}_\eta^\mu(\boxminus\tilde{q})$ converge as $\eta \to 0$ for every quad $\tilde{q}$ in $\tilde{D}$.*

If $\sigma = 1$ or $\mu = 0$, $f(\Gamma^\mu)$ and $\tilde{\Gamma}^\mu$ are identically distributed. But if $\sigma \neq 1$ and $\mu \neq 0$, the laws $f(\Gamma^\mu)$ and $\tilde{\Gamma}^\mu$ are singular with respect to each other.

Proof. The statement is clear if $\sigma = 1$ since then f is the identity map. If $\mu = 0$ we are in the well-known critical case. Thus we may assume $\sigma \neq 1$ and $\mu \neq 0$. Let ω_η be a realization of percolation in D with mesh size η and $p_\eta^\mu = \frac{1}{2} + \mu\eta^2/\alpha_4^\eta$. Then $f(\omega_\eta)$ is a realization of percolation in $\tilde{D}$ with mesh size $\sigma\eta =: \zeta$. Each hexagon of $f(\omega_\eta)$ is blue with probability

$$p'_\zeta = \frac{1}{2} + \mu\frac{\eta^2}{\alpha_4^\eta} = \frac{1}{2} + \mu\frac{\alpha_4^{\sigma\eta}(\sigma\eta, 1)}{\sigma^2\alpha_4^\eta(\eta, 1)} \cdot \frac{(\sigma\eta)^2}{\alpha_4^{\sigma\eta}(\sigma\eta, 1)} = \frac{1}{2} + \mu\sigma^{\frac{5}{4}}(1 + o(1))\sigma^{-2} \cdot \frac{\zeta^2}{\alpha_4^\zeta(\zeta, 1)},$$

where we used $\alpha_4^{\sigma\eta}(\sigma\eta, 1) = \alpha_4^\eta(\eta, \sigma^{-1})$ and the ratio limit theorem [GPS-13a, Proposition 4.9.] stating $\lim_{\eta\to 0} \alpha_4^\eta(\eta, \delta)/\alpha_4^\eta(\eta, 1) = \delta^{-5/4}$. Therefore $f(\omega_\eta)$ is a realization of percolation in $\tilde{D}$ with mesh size ζ and $p_\zeta^\lambda = \frac{1}{2} + \lambda_\zeta \cdot \zeta^2/\alpha_4^\zeta$ where $\lambda_\zeta \to \mu\sigma^{-3/4} =: \lambda \neq \mu$. Therefore $f(P_{\zeta/\sigma}^\mu) = \tilde{P}_\zeta^\lambda$. Note that $\tilde{\gamma}_\zeta \circ f = f \circ \gamma_{\zeta/\sigma}$ by the definition of the exploration paths. Thus $\tilde{\gamma}_\zeta(\tilde{P}_\zeta^\lambda) = \tilde{\gamma}_\zeta(f(P_{\zeta/\sigma}^\mu)) = f(\gamma_{\zeta/\sigma}(P_{\zeta/\sigma}^\mu))$. As we assumed that $\gamma_\eta(P_\eta^\mu)$ converge weakly to Γ^μ, it follows that $\tilde{\gamma}_\zeta(\tilde{P}_\zeta^\lambda)$ converge weakly to $f(\Gamma^\mu) =: \tilde{\Gamma}^\lambda$ since f is continuous.

On the other hand, $\tilde{\gamma}_\eta$ is the discrete exploration path of percolation in $\tilde{D}$ with $p_\eta^\mu = \frac{1}{2} + \mu \cdot \eta^2/\alpha_4^\eta$, whose law converges weakly to $\tilde{\Gamma}^\mu$. By Theorem 2.1, $\tilde{\Gamma}^\mu$ and $f(\Gamma^\mu)$ are singular with respect to each other (even if $\mu\sigma^{-\frac{3}{4}} < \mu$ by the remark in the second paragraph after stating the theorem). $\square$

Chapter 3

Singularity of Full Scaling Limits of Planar Nearcritical Percolation

In this chapter we show that the scaling limits of full nearcritical percolation configurations with different parameters are mutually singular. We prove this result for a class of percolation models satisfying some reasonable conditions. Finally we show that site percolation on the triangular lattice as well as bond percolation on the square lattice lie in this class. With some linguistic changes and an introduction, it is already published at SPA and also available at arXiv.[1] First in Section 3.1 we formally introduce the model. Thereafter we state our results including all needed lemmas, which will be proved in Section 3.2.

3.1 Model and Results

As already mentioned in the introduction, we use the set-up of [SS-11]. Therefore we consider percolation on tilings of the plane rather than on lattices. A tiling is a collection of polygonal, topologically closed tiles such that the tiles may intersect each other only at their boundary and such that their union is the whole plane. We further require that the tilings are locally finite, i.e. any bounded set contains only finitely many tiles, and trivalent, i.e. any point belongs to at most three tiles.

For $\eta > 0$, let H_η be a locally finite trivalent tiling such that the diameter of each tile is at most η. A percolation model is obtained by colouring every tile either blue or yellow. Some tiles may have a deterministic colour, while each tile $t \in H'_\eta \subseteq H_\eta$ is coloured randomly blue with some probability $\mathbf{p}(t) \in [0, 1]$ and otherwise yellow, independently of each other. Any site or bond percolation model can be realized using such a tiling, cf. [SS-11, p. 1774f]. Colouring some tiles deterministically

[1] [A-14b] Simon Aumann: Singularity of Full Scaling Limits of Planar Nearcritical Percolation, *Stoch. Proc. Appl.* **124** No. 11, 3807-3818, 2014 or arXiv:1301.5175

ensures that the tiling is trivalent. For each $\eta > 0$, we therefore obtain the probability space

$$\Big(\bar{\Omega}_\eta := \{\text{blue,yellow}\}^{H'_\eta}, \quad \bar{\mathcal{A}}_\eta\,, \quad \bar{P}^{\mathbf{p}}_\eta := \bigotimes_{t\in H'_\eta} \big(\mathbf{p}(t)\delta_{\text{blue}} + (1-\mathbf{p}(t))\delta_{\text{yellow}}\big)\Big)$$

with product-σ-algebra $\bar{\mathcal{A}}_\eta$ and $\mathbf{p} : H'_\eta \to [0,1]$.

But we want to describe all discrete processes as well as the scaling limit by different probability measures on the same space. Thereto we use the space $\mathcal{H}$ of all closed lower sets of quads introduced by Schramm and Smirnov in [SS-11, Section 1.3]. As the exact construction is not important for understanding the present chapter (but it is important for the properties derived in [SS-11] we need), we explain it only very briefly. A quad Q is a homeomorphism $Q : [0,1]^2 \to Q([0,1]^2) \subset \mathbb{C}$. A crossing of Q is a connected closed subset of $Q([0,1]^2)$ which intersects the images of the left and the right side of $[0,1]^2$. The question, whether every crossing of a quad contains a crossing of a second quad, provides a partial order on the quads. If a set of quads also contains all smaller quads (in the sense of the partial order), it is called a lower set of quads. Then $\mathcal{H}$ is the space of all closed lower sets of quads. For a quad Q, we define the event $\boxminus Q \subset \mathcal{H}$ that the quad Q is crossed: it is the set of all lower sets which contain Q. The space $\mathcal{H}$ is equipped with the so-called Quad-Crossing-Topology, which is the minimal topology containing all $(\boxminus Q)^c$ and other certain lower sets of quads. The induced Borel-σ-algebra $\mathcal{B}(\mathcal{H})$ is generated by the events $\boxminus Q$. For $D \subset \mathbb{C}$, let $\mathcal{B}_D$ be the restriction of $\mathcal{B}(\mathcal{H})$ to lower sets of quads inside D.

Any configuration $\bar{\omega}_\eta \in \bar{\Omega}_\eta$ induces an element of $\mathcal{H}$, namely the set ω_η of all quads, which contain a blue crossing, i.e. a crossing which is a subset of the union of all blue tiles. Note that this is a closed lower set. Thus, for all $\eta > 0$ and $\mathbf{p} : H'_\eta \to [0,1]$, the measure $\bar{P}^{\mathbf{p}}_\eta$ induces a probability measure $P^{\mathbf{p}}_\eta$ on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. We will mainly work with these probability measures.

Now we define a special measure on $\mathcal{H}$, namely the critical measure P^0_η. It is induced by $\bar{P}^{\mathbf{p}}_\eta$ with $\mathbf{p}(t) = p^{\text{crit}}_\eta$ for all tiles $t \in H'_\eta$. There p^{crit}_η is the critical probability of the tiling H_η, i.e.

$$p^{\text{crit}}_\eta := \sup\{p \in [0,1] \mid P^{\mathbf{p}}_\eta[\text{There is an infinite blue cluster}] = 0, \mathbf{p}(t) = p\, \forall t \in H'_\eta\}\,.$$

In fact, we do not use criticality. Thus p^{crit}_η could be any number in $(0,1)$ such that the conditions below are satisfied. But they usually hold only if p^{crit}_η is indeed the critical probability.

For $z \in \mathbb{C}$ and $0 < \eta \le r < R$, let $A_4(z, r, R)$ be the event that there are four crossings of alternating colour inside the annulus centred at z with radii r and R.

We fix some $R_0, N_0 > 0$ and $z_0 \in \mathbb{C}$ for the remainder of this chapter. We want to define the nearcritical models. We abbreviate

$$\alpha^\eta_4 := P^0_\eta[A_4(z_0, \eta, R_0)]$$

and define the set

$$\Pi_\eta := \left\{ P_\eta^{\mathbf{p}} \mid \mathbf{p}(t) = (p_\eta^{\text{crit}} + \iota_\eta(t) \cdot \frac{\eta^2}{\alpha_4^\eta}) \vee 0 \wedge 1,\ \iota_\eta(t) \in [-N_0, N_0],\ t \in H'_\eta \right\},$$

the set of all probability measures on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ which are in the critical window. If we want to specify the chosen parameter $\iota = (\iota_\eta(t))_{t \in H'_\eta}$, we write P_η^ι for the corresponding measure. We therefore use the speed factor η^2/α_4^η for the convergence of the nearcritical probabilities to the critical one. This rate is inspired by [K-87, Theorem 4], [N-08, Proposition 32] and the results of [GPS-13a]. From Lemma 3.6 below and [NW-09, Proposition 4], it follows that η^2/α_4^η is indeed the correct rate.

Conditions. We impose the following basic conditions on the tilings H_η, $\eta > 0$. The constants $\eta_0, c_1, c_2, c_3 > 0$ as well as the functions Δ_4 and Δ_1 may depend on R_0 and N_0. The words in *italic* are only headings without any formal meaning.

1. *The following multi-scale bound on the four arm event holds:*

There exists a positive function $\Delta_4(r, R)$ such that for all fixed $R \leq R_0$

$$\lim_{r \to 0} \Delta_4(r, R) = 0$$

and such that for all $\eta \leq r < R \leq R_0$

$$P_\eta^0[A_4(z_0, r, R)] \leq \frac{r}{R} \Delta_4(r, R)\,.$$

2. *The probabilities in the critical window are eventually strictly between 0 and 1:*

There exists $\eta_0 > 0$ such that for all $\eta \in (0, \eta_0)$:

$$0 < p_\eta^{\text{crit}} - N_0 \frac{\eta^2}{\alpha_4^\eta} < p_\eta^{\text{crit}} + N_0 \frac{\eta^2}{\alpha_4^\eta} < 1\,.$$

3. *The probabilities of the four-arm events are comparable on the whole plane over all (near)critical measures:*

There are constants $c_1, c_2 > 0$ such that for all $\eta \leq r < R \leq R_0$, $z \in \mathbb{C}$ and $P_\eta \in \Pi_\eta$ the following inequalities hold:

$$c_1\, P_\eta^0[A_4(z_0, \eta, R)] \leq P_\eta[A_4(z, \eta, R)] \quad \text{and}$$
$$P_\eta[A_4(z, r, R)] \leq c_2\, P_\eta^0[A_4(z_0, r, R)]\,.$$

(Note that we need the first inequality for $r = \eta$ only.)

4. *The probability of the four arm event is uniformly comparable to the probability of the following modified four arm event:*

For $R > 0$ and $z \in \mathbb{C}$, let $Q(z, R)$ be the square with side length R centred at z. For a tile t in $Q(z, R)$ whose distance from z is at most $R/4$, let $A'_4(t, \partial Q(z, R))$

be the event that there are four arms of alternating colour from t to the left, lower, right and upper boundary of $Q(z,R)$, respectively.

There exists a constant $c_3 > 0$ such that for all $4\eta \leq R \leq R_0$, $z \in \mathbb{C}$, $P_\eta \in \Pi_\eta$ and all tiles t in $Q(z,R)$ whose distance from z is at most $R/4$:

$$P_\eta[A_4'(t, \partial Q(z,R))] \geq c_3\, P_\eta[A_4(z,\eta,R)]\,.$$

5. *There is the following bound on the one arm event:*

There exists a positive function $\Delta_1(r,R)$ such that for all fixed $R \leq R_0$

$$\lim_{r\to 0} \Delta_1(r,R) = 0$$

and such that for all $\eta \leq r < R \leq R_0$, $z \in \mathbb{C}$, $P_\eta \in \Pi_\eta$ and $col \in \{\text{blue}, \text{yellow}\}$

$$P_\eta[A_1^{col}(z,r,R)] \leq \Delta_1(r,R)\,,$$

where $A_1^{col}(z,r,R)$ is the event that there exists a crossing of colour *col* inside the annulus centred at z with radii r and R.

Conditions 3 and 1 imply

$$P_\eta[A_4(z,r,R)] \leq \frac{r}{R} c_2 \Delta_4(r,R)$$

for all $z \in \mathbb{C}$ and $\eta \leq r < R \leq R_0$ and $P_\eta \in \Pi_\eta$. This and condition 5 are Assumptions 1.1. of [SS-11]. Therefore we can apply most results of that article, including [SS-11, Corollary 1.16], yielding that any family $P_\eta \in \Pi_\eta$, $\eta > 0$, is tight. Thus there exist nearcritical scaling limits, at least along subsequences.

Now we are ready to state the main theorem of the present chapter.

Theorem 3.1. *Let H_η, $\eta > 0$, be locally finite trivalent tilings such that each tile of H_η has diameter at most η, and such that conditions 1-5 are fulfilled. For $\eta > 0$, let measures $P_\eta^\mu, P_\eta^\lambda \in \Pi_\eta$ be given by $\mu_\eta(t), \lambda_\eta(t) \in [-N_0, N_0]$, $t \in H_\eta'$. Considering weak limits with respect to the Quad-Crossing-Topology, let P^μ be any weak limit point of $\{P_\eta^\mu : \eta > 0\}$, let P^λ be any weak limit point of $\{P_\eta^\lambda : \eta > 0\}$ and let η_n, $n \in \mathbb{N}$, be a sequence converging to zero such that $P_{\eta_n}^\mu \to P^\mu$ and $P_{\eta_n}^\lambda \to P^\lambda$ weakly as $n \to \infty$.*

Assume that there exist $\sigma > 0$ and an open, non-empty set $D \subset \mathbb{C}$ such that

$$\lambda_\eta(t) - \mu_\eta(t) \;\geq\; \sigma$$

uniformly in $\eta \in \{\eta_n : n \in \mathbb{N}\}$ and all tiles $t \in H_\eta'$ which are contained in D.

Then the laws P^μ and P^λ – even restricted to $\mathcal{B}_D$ – are singular with respect to each other.

Similarly to Corollary 2.2 for the exploration paths, we can even detect the asymmetry by only looking at an infinitesimal neighbourhood of a point inside D, more precisely:

Corollary 3.2. *Let the conditions of Theorem 3.1 be fulfilled. Let* $z \in D$. *Let* $\mathcal{B}_z := \bigcap_{n\in\mathbb{N}} \mathcal{B}_{B_{1/n}(z)}$ *be the tail-*σ*-algebra of the restrictions of* $\mathcal{B}(\mathcal{H})$ *to lower sets of quads in the ball* $B_{1/n}(z)$.

Then the laws P^μ *and* P^λ *restricted to* $\mathcal{B}_z$ *are singular with respect to each other.*

We base the proof of Theorem 3.1 on the following two lemmas. The first one is specific for the model. The second one is rather abstract to detect the singularity.

Lemma 3.3. *Under the conditions of Theorem 3.1, there exists a function* $\Delta_\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\Delta_\sigma(\delta) \to 0$ *as* $\delta \to 0$ *such that for any square* Q *of side length* $\delta \leq R_0$ *inside* D*:*

$$P^\lambda[\boxminus Q] - P^\mu[\boxminus Q] \geq \frac{\delta}{\Delta_\sigma(\delta)},$$

where $\boxminus Q$ *denotes the event that there exists a horizontal blue crossing of the square* Q.

Lemma 3.4. *Let* P *and* P' *be two probability measures on a space* $(\Omega, \mathcal{A})$. *Let* $a, b > 0$ *and let* $(\Delta_n)_{n\in\mathbb{N}}$ *be a positive sequence converging to infinity. Set* $K_n := \lceil an^2 \rceil$, $n \in \mathbb{N}$. *For large enough* $n \in \mathbb{N}$, *let* X_k^n, $k \in \{1, \ldots, K_n\}$, *be random variables which are uncorrelated in* k *with respect to* P *and* P', *absolutely bounded by* b, *and satisfy*

$$E_{P'}[X_k^n] - E_P[X_k^n] \geq \tfrac{1}{n}\Delta_n, \qquad k \in \{1, \ldots, K_n\}.$$

Then P *and* P' *are singular with respect to each other.*

Using results of [SW-01], [SS-11, Appendix B], [N-08] and [K-87] as well as standard techniques, we can easily verify conditions 1-5 in the two most important cases:

Lemma 3.5. *Conditions 1 to 5 are fulfilled by tilings representing site percolation on the triangular lattice or bond percolation on the square lattice.*

Thereto we will need the following converse of [N-08, Proposition 32], which estimates the characteristic length. For the remainder of this section, we consider site percolation on the triangular lattice or bond percolation on the square lattice, each with mesh size 1. Let $p_c = \frac{1}{2}$ be the critical probability. For $\varepsilon \in (0, \frac{1}{2})$ and $p \in (0, 1)$, let $L_\varepsilon(p)$ be the corresponding characteristic length as defined in [N-08, Section 3.1] or [K-87, Equation (1.21)], respectively, i.e.

$$L_\varepsilon(p) := \begin{cases} \inf\{n \in \mathbb{N} : P_p[\boxminus(n \times n)] \leq \varepsilon\} & \text{if } p < p_c \\ \inf\{n \in \mathbb{N} : P_p[\boxminus(n \times n)] \geq 1 - \varepsilon\} & \text{if } p > p_c \end{cases}$$

and $L_\varepsilon(p_c) = \infty$, where P_p denotes the product measure with probability p for blue, and $\boxminus(m \times n)$ denotes the event that there is a horizontal blue crossing of a an $m \times n$ rectangle.

Moreover, for $m < n$, let $\alpha_4(m, n)$ be the probability that at critical percolation there exist four arms of alternating colour inside the annulus centred at the origin with radii m and n. We abbreviate $\alpha_4(n) := \alpha_4(1, n)$.

Lemma 3.6. *For all $\varepsilon \in (0, \frac{1}{2})$ and $C_1, C_2 > 0$, there exist $C_3, C_4 > 0$ such that for all $p \in (0,1)$ and $n \geq 1$ the following implication holds:*

$$C_1 \leq |p - p_c| n^2 \alpha_4(n) \leq C_2 \quad \Longrightarrow \quad C_3 \leq \frac{n}{L_\varepsilon(p)} \leq C_4 .$$

Finally, we need the following lemma, which restates Remark 36 of [N-08]. Since the author is not aware of a formal statement in the literature, it is included here for the sake of completeness.

Lemma 3.7. *For all $\varepsilon_0 \in (0, \frac{1}{2})$ and all $K \geq 1$, there exists an $\varepsilon \in (0, \varepsilon_0)$ such that for all $0 < p < p_c$:*

$$L_\varepsilon(p) \geq K \cdot L_{\varepsilon_0}(p) .$$

3.2 Proofs

In this section, we give the proofs of all stated assertions.

Proof of Lemma 3.3. Let Q be a square of side length $\delta \leq R_0$ inside D. Let $\eta \in \{\eta_n : n \in \mathbb{N}\}$ be small enough such that $4\eta < \delta$ and $\eta < \eta_0$, where η_0 is chosen according to condition 2.

We construct a coupling $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P})$ as follows. Let

$$\hat{\Omega} := \big(\{\text{blue,yellow}\} \times \{\text{blue,yellow}\}\big)^{H'_\eta}$$

with product-σ-algebra $\hat{\mathcal{A}}$. Informally, let $\hat{P}$ be the probability measure which has marginal distributions $\bar{P}^\mu_\eta$ and $\bar{P}^\lambda_\eta$ such that the set of blue tiles in Q increases. More precisely, we define the random variables

$$f_I : \hat{\Omega} \to \mathcal{H}, \quad I \in \{1,2\}^{H'_\eta} .$$

For $\hat{\omega} = (\hat{\omega}_1(t), \hat{\omega}_2(t))_{t \in H'_\eta} \in \hat{\Omega}$, let $f_I(\hat{\omega})$ be the set of all quads which contain a blue crossing if tile $t \in H'_\eta$ is coloured with colour $\hat{\omega}_{I(t)}(t)$. We abbreviate $\langle 1 \rangle := (1, \dots, 1)$ and $\langle 2 \rangle := (2, \dots, 2)$. Then let $\hat{P}$ be a probability measure on $\hat{\Omega}$ such that

$$f_{\langle 1 \rangle}\big(\hat{P}\big) = P^\mu_\eta , \qquad f_{\langle 2 \rangle}\big(\hat{P}\big) = P^\lambda_\eta \qquad \text{and}$$
$$\hat{P}\big[\hat{\omega} : \hat{\omega}_1(t) = \text{blue},\, \hat{\omega}_2(t) = \text{yellow for some tile } t \text{ in } Q\big] = 0 .$$

Such a coupling can be obtained, for example, from the standard monotone coupling using independent, uniformly on $[0,1]$ distributed random variables as $\lambda_\eta(t) > \mu_\eta(t)$ in Q.

It follows that

$$\begin{aligned} P^\lambda_\eta[\boxminus Q] - P^\mu_\eta[\boxminus Q] &= \hat{P}\big[f^{-1}_{\langle 2 \rangle}[\boxminus Q] \setminus f^{-1}_{\langle 1 \rangle}[\boxminus Q]\big] - \hat{P}\big[f^{-1}_{\langle 1 \rangle}[\boxminus Q] \setminus f^{-1}_{\langle 2 \rangle}[\boxminus Q]\big] \\ &= \hat{P}\big[f^{-1}_{\langle 1 \rangle}[\boxminus Q]^c \cap f^{-1}_{\langle 2 \rangle}[\boxminus Q]\big] - 0 , \end{aligned}$$

since $\hat{\omega} \in f_{\langle 1\rangle}^{-1}[\boxminus Q] \setminus f_{\langle 2\rangle}^{-1}[\boxminus Q]$ implies that there is a tile t in Q with $\hat{\omega}_1(t) =$ blue and $\hat{\omega}_2(t) =$ yellow. Thus we have to estimate the probability of the event of all $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2)$ such that $\hat{\omega}_2$ induces a blue crossing of Q, but $\hat{\omega}_1$ does not.

Let $\mathcal{T} = \{t_1, \ldots, t_K\}$ be the set of all tiles in Q whose distance from the centre z_Q of Q is at most $\delta/4$ – arranged in any (but fixed) order. In order to prove the proposed estimate, we restrict ourselves to the event that the crossing arises out of switches from yellow to blue of some tiles in $\mathcal{T}$. Thereto we change the coordinates of $\hat{\omega}$ we use for the tiles in $\mathcal{T}$ one by one. Formally, for $k = 0, \ldots, K$, let $I_k \in \{1,2\}^{H'_\eta}$ be defined by $I_k(t) = 1$ if $t \in H'_\eta \setminus \{t_1, \ldots, t_k\}$, and $I_k(t) = 2$ if $t \in \{t_1, \ldots, t_k\}$. Then

$$\hat{P}\big[f_{\langle 1\rangle}^{-1}[\boxminus Q]^c \cap f_{\langle 2\rangle}^{-1}[\boxminus Q]\big] \geq \hat{P}\big[\bigcup_{k=1}^{K} f_{I_{k-1}}^{-1}[\boxminus Q]^c \cap f_{I_k}^{-1}[\boxminus Q]\big] .$$

As the crossing event is increasing, the event $f_{I_{k-1}}^{-1}[\boxminus Q]^c \cap f_{I_k}^{-1}[\boxminus Q]$ can happen only for one $k \in \{1, \ldots, K\}$. This is the case if and only if the following two events occur: first, the event $f_{I_k}^{-1}[A'_4(t_k, \partial Q)]$ that there are four arms of alternating colour from t_k to the left, lower, right and upper boundary of Q, respectively, which means that t_k is pivotal for the crossing event; second, the event that the colour of t_k switches from $\hat{\omega}_1(t_k) =$ yellow to $\hat{\omega}_2(t_k) =$ blue, which we denote by $Sw(t_k)$. Note that they are independent events. Using the described disjointness and independence, we get

$$\hat{P}\big[\bigcup_{k=1}^{K} f_{I_{k-1}}^{-1}[\boxminus Q]^c \cap f_{I_k}^{-1}[\boxminus Q]\big] = \sum_{k=1}^{K} \hat{P}\big[f_{I_k}^{-1}[A'_4(t_k, \partial Q)]\big] \cdot \hat{P}[Sw(t_k)] .$$

Now we estimate these probabilities. Elementary probability calculus and the construction of the coupling yield

$$\begin{aligned}
\hat{P}[Sw(t_k)] &= \hat{P}\big[\{\hat{\omega} : \hat{\omega}_2(t_k) = \text{blue}\} \setminus \{\hat{\omega} : \hat{\omega}_1(t_k) = \text{blue}\}\big] \\
&= \hat{P}\big[\{\hat{\omega} : \hat{\omega}_2(t_k) = \text{blue}\}\big] - \hat{P}\big[\{\hat{\omega} : \hat{\omega}_1(t_k) = \text{blue}\}\big] + \\
&\quad + \hat{P}\big[\{\hat{\omega} : \hat{\omega}_1(t_k) = \text{blue}\} \setminus \{\hat{\omega} : \hat{\omega}_2(t_k) = \text{blue}\}\big] \\
&= P_\eta^\lambda[t_k \text{ blue}] - P_\eta^\mu[t_k \text{ blue}] + \hat{P}[\hat{\omega} : \hat{\omega}_1(t_k) = \text{blue}, \, \hat{\omega}_2(t_k) = \text{yellow}] \\
&= \big(p_\eta^{\text{crit}} + \lambda_\eta(t_k) \cdot \frac{\eta^2}{\alpha_4^\eta}\big) - \big(p_\eta^{\text{crit}} + \mu_\eta(t_k) \cdot \frac{\eta^2}{\alpha_4^\eta}\big) + 0 \\
&= \big(\lambda_\eta(t_k) - \mu_\eta(t_k)\big) \frac{\eta^2}{\alpha_4^\eta} \; \geq \; \sigma \cdot \frac{\eta^2}{\alpha_4^\eta} ,
\end{aligned}$$

because of $\eta < \eta_0$ (such that, by condition 2, the probabilities are given by the used formulas) and because of the assumption in Theorem 3.1.

Let $P_\eta^{I_k}$ denote the image law of $\hat{P}$ under f_{I_k}. Then $P_\eta^{I_k} \in \Pi_\eta$. Using conditions 4 and 3, we conclude

$$\hat{P}\big[f_{I_k}^{-1}[A'_4(t_k, \partial Q)]\big] \; \geq \; c_3 P_\eta^{I_k}[A_4(z_Q, \eta, \delta)] \; \geq \; c_3 c_1 P_\eta^0[A_4(z_0, \eta, \delta)] .$$

As there are $K \geq c_4(\delta/\eta)^2$ tiles in $\mathcal{T}$ (for some numerical constant $c_4 > 0$), the equations above imply

$$P_\eta^\lambda[\boxminus Q] - P_\eta^\mu[\boxminus Q] \geq c_4 \frac{\delta^2}{\eta^2} \cdot c_3 c_1 P_\eta^0[A_4(z_0,\eta,\delta)] \cdot \sigma \frac{\eta^2}{\alpha_4^\eta} = \sigma c_1 c_3 c_4 \cdot \frac{\delta^2 P_\eta^0[A_4(z_0,\eta,\delta)]}{P_\eta^0[A_4(z_0,\eta,R_0)]} .$$

Using first $A_4(z_0,\eta,R_0) \subseteq A_4(z_0,\eta,\delta) \cap A_4(z_0,\delta,R_0)$ and independence of the latter two events and then condition 1, we conclude

$$\begin{aligned} P_\eta^\lambda[\boxminus Q] - P_\eta^\mu[\boxminus Q] &\geq \sigma c_1 c_3 c_4 \cdot \frac{\delta^2}{P_\eta^0[A_4(z_0,\delta,R_0)]} \\ &\geq \sigma c_1 c_3 c_4 \cdot \frac{\delta^2 R_0}{\delta \Delta_4(\delta,R_0)} = \frac{\delta}{\Delta_\sigma(\delta)} \end{aligned}$$

with $\Delta_\sigma(\delta) := (\sigma c_1 c_3 c_4 R_0)^{-1} \Delta_4(\delta,R_0)$. Condition 1 implies $\Delta_\sigma(\delta) \to 0$ as $\delta \to 0$.

For $\iota \in \{\mu,\lambda\}$, Lemma 5.1 of [SS-11] (implying $P^\iota[\partial \boxminus Q] = 0$) and the weak convergence of $P_{\eta_n}^\iota$ yield $P_{\eta_n}^\iota[\boxminus Q] \to P^\iota[\boxminus Q]$ as $n \to \infty$, which concludes the proof. □

Proof of Lemma 3.4. We define for large enough $n \in \mathbb{N}$

$$Z_n := \sum_{k=1}^{K_n} \left(X_k^n - E_P[X_k^n] \right) .$$

It follows that $E_P[Z_n] = 0$ and that

$$E_{P'}[Z_n] = \sum_{k=1}^{K_n} \left(E_{P'}[X_k^n] - E_P[X_k^n] \right) \geq K_n \cdot \tfrac{1}{n}\Delta_n \geq an\Delta_n ,$$

because of the assumption and $K_n = \lceil an^2 \rceil$. Since the random variables are uncorrelated and bounded, we can estimate the variance of Z_n under P or under P' as follows:

$$\mathrm{Var}[Z_n] = \sum_{k=1}^{K_n} \mathrm{Var}\left[X_k^n - E_P[X_k^n] \right] = \sum_{k=1}^{K_n} \mathrm{Var}[X_k^n] \leq K_n b^2 \leq (a+1) b^2 n^2 .$$

Using Chebyshev's inequality, we estimate

$$P[Z_n \geq \tfrac{a}{2} n \Delta_n] \leq \frac{4}{a^2 n^2 \Delta_n^2} \mathrm{Var}_P[Z_n] \leq \frac{4(a+1)b^2 n^2}{a^2 n^2 \Delta_n^2} = \frac{4(a+1)b^2}{a^2} \cdot \Delta_n^{-2}$$

and

$$\begin{aligned} P'[Z_n < \tfrac{a}{2} n\Delta_n] &= P'\left[(E_{P'}[Z_n] - Z_n) > (E_{P'}[Z_n] - \tfrac{a}{2} n\Delta_n) \right] \\ &\leq P'\left[|E_{P'}[Z_n] - Z_n| > (an\Delta_n - \tfrac{a}{2} n\Delta_n) \right] \\ &\leq \frac{4}{a^2 n^2 \Delta_n^2} \mathrm{Var}_{P'}[Z_n] \leq \frac{4(a+1)b^2}{a^2} \cdot \Delta_n^{-2} . \end{aligned}$$

If we now choose a sparse enough sub-sequence n_l, $l \in \mathbb{N}$, i.e. such that $\sum_l \Delta_{n_l}^{-2} < \infty$, the Borel-Cantelli Lemma yields

$$\begin{array}{rrcl} & P'\big[Z_{n_l} < \frac{a}{2} n_l \Delta_{n_l} \text{ for infinitely many } l\big] & = & 0 \\ \text{implying} & P'\big[Z_{n_l} \geq \frac{a}{2} n_l \Delta_{n_l} \text{ for infinitely many } l\big] & = & 1\,, \\ \text{while} & P\big[Z_{n_l} \geq \frac{a}{2} n_l \Delta_{n_l} \text{ for infinitely many } l\big] & = & 0\,. \end{array}$$

Therefore we detected an event which has P-probability zero, but P'-probability one. □

Proof of Theorem 3.1. We want to apply Lemma 3.4. Let $P' = P^\lambda$ and $P = P^\mu$. We set $\delta_n = 1/n$, $n \in \mathbb{N}$, and choose an appropriate $a > 0$ (depending on the size of D) such that, for sufficiently large n, we can place $K_n = \lceil an^2 \rceil$ disjoint squares $Q_1^n, \ldots, Q_{K_n}^n$ of size δ_n in D. We define the random variables $X_k^n : \mathcal{H} \to \mathbb{R}$ by

$$X_k^n = \mathbb{1}_{\boxminus Q_k^n}\,, \qquad k \in \{1, \ldots, K_n\}\,.$$

Since the disjointness of the squares yields independence of the crossing events for all $P_{\eta_m}^\iota$, since $P_{\eta_m}^\iota \to P^\iota$ weakly and since $P^\iota[\partial \boxminus Q_k^n] = 0$ by [SS-11, Lemma 5.1], the random variables X_k^n, $k \in \{1, \ldots, K_n\}$, are independent for P^ι, $\iota \in \{\mu, \lambda\}$. Moreover, $|X_k^n| \leq 1$, and Lemma 3.3 yields

$$E_{P^\lambda}[X_k^n] - E_{P^\mu}[X_k^n] = P^\lambda[\boxminus Q_k^n] - P^\mu[\boxminus Q_k^n] \geq \frac{\delta_n}{\Delta_\sigma(\delta_n)} = \tfrac{1}{n}\Delta_n$$

with $\Delta_n := \Delta_\sigma(\delta_n)^{-1} \to \infty$ as $n \to \infty$. Thus Lemma 3.4 yields that P^μ and P^λ are singular with respect to each other. Since all random variables X_k^n are $\mathcal{B}_D$-measurable, we can also apply Lemma 3.4 when P^μ and P^λ are restricted to $\mathcal{B}_D$. □

Proof of Corollary 3.2. The proof is analogous to the proof of Corollary 2.2. Let $m_0 \in \mathbb{N}$ such that $B_{\frac{1}{m_0}}(z) \subseteq D$. Let $n \geq m_0$. By Theorem 3.1 – applied inside $B_{\frac{1}{n}}(z)$ – there are sets $B_n \in \mathcal{B}_{B_{\frac{1}{n}}(z)}$ with $P^\mu[B_n] = 0$ and $P^\lambda[B_n] = 1$. We set

$$B_* := \bigcup_{m \geq m_0} \bigcap_{n \geq m} B_n\,.$$

Then $B_* \in \mathcal{B}_z$. Since countable unions or intersection of sets of probability zero respectively one have probability zero respectively one, it follows that $P^\mu[B_*] = 0$ and $P^\lambda[B_*] = 1$, which proves the corollary. □

Proof of Lemma 3.5. As it is proven on the triangular lattice that the 4-arm-exponent is 5/4, see [SW-01, Theorem 4], condition 1 holds. For bond percolation on the square lattice, this condition is proven by Christophe Garban in [SS-11, Lemma B.1].

Now we claim that R_0 is below a characteristic length of $p_\eta^{-N_0} = p_\eta^{\mathrm{crit}} - N_0\eta^2/\alpha_\eta^4$, i.e. there is some $\varepsilon \in (0, \frac{1}{2})$ such that $R_0/\eta \leq L_\varepsilon(p_\eta^{-N_0})$ for all $\eta > 0$. Thereto we provisionally fix some $\varepsilon_0 \in (0, \frac{1}{2})$. Since

$$\left|p_\eta^{-N_0} - p_\eta^{\mathrm{crit}}\right| (R_0/\eta)^2 P_\eta^0[A_4(z_0, \eta, R_0)] \;=\; R_0^2 N_0\,,$$

Lemma 3.6 (for $n = (R_0/\eta)$ and $p = p_\eta^{-N_0}$) yields that $R_0/\eta \leq C_4 L_{\varepsilon_0}(p_\eta^{-N_0})$ for some $C_4 = C_4(R_0, N_0, \varepsilon_0) > 0$. By Lemma 3.7, we find an $\varepsilon \in (0, \varepsilon_0)$ such that the claim holds.

Now we fix this $\varepsilon > 0$. Since every $P_\eta \in \Pi_\eta$ is between $P_\eta^{-N_0}$ and $P_\eta^{+N_0}$, the claim above allows us to use arguments of RSW style and to apply most of the results of [N-08] and [K-87] as long as we use radii $R \leq R_0$. In fact, all of the remaining conditions easily follow from the results of these papers.

The following reasoning is a standard technique. By RSW, there is a constant $c > 0$ such that for all $col \in \{\text{blue}, \text{yellow}\}$, $z \in \mathbb{C}$, $\eta \leq r \leq R_0/2$ and $P_\eta \in \Pi_\eta$

$$c \leq P_\eta[A_1^{col}(z, r, 2r)] \leq 1 - c\,.$$

Let $R \leq R_0$ be fixed. For $r \in (\eta, R/2)$, let $K_r \in \mathbb{N}$ be the largest number such that $2^{K_r} \leq R/r$. Then $K_r \to \infty$ as $r \to 0$. It follows that

$$\begin{aligned} P_\eta[A_1^{col}(z, r, R)] &\leq P_\eta[\forall k = 1, \ldots, K_r : A_1^{col}(z, r2^{k-1}, r2^k)] \\ &\leq \prod_{k=1}^{K_r} P_\eta[A_1^{col}(z, r2^{k-1}, r2^k)] \leq (1-c)^{K_r} \to 0 \end{aligned}$$

as $r \to 0$, which shows condition 5 (on both lattices).

By [SSt-10, Corollary A.8] (stating that the 5-arm-exponent is 2) and Reimer's Inequality, it follows that (for some $\tilde{c} > 0$)

$$\tilde{c} R_0^{-2} \eta^2 \leq P_\eta^0[A_5(z_0, \eta, R_0)] \leq P_\eta^0[A_4(z_0, \eta, R_0)] \cdot P_\eta^0[A_1(z_0, \eta, R_0)]\,. \tag{3.1}$$

Thus condition 5 yields $\eta^2/\alpha_4^\eta \to 0$ as $\eta \to 0$, which, together with $p_\eta^{\mathrm{crit}} = \frac{1}{2}$, implies condition 2 (on both lattices).

Since the considered lattices are transitive, the estimates in conditions 3 and 4 hold uniformly in $z \in \mathbb{C}$, if they hold for $z = 0$. Thus we consider only this case. Condition 3 on the triangular lattice is included in Theorem 26 of [N-08]. On the square lattice, condition 3 is a consequence of [K-87, Lemma 8] (with $v = 0$) and [K-87, Lemma 4]. These two lemmas (with $\kappa = 0.5$) also imply condition 4 on the square lattice. On the triangular lattice, it is a special case of equation (4.20) in [N-08]. □

Note that we are considering site percolation on the triangular lattice or bond percolation on the square lattice with mesh size 1 in the remaining two lemmas.

Proof of Lemma 3.6. We fix some $\varepsilon \in (0, \frac{1}{2})$ and abbreviate $L(p) := L_\varepsilon(p)$. We will use the following facts. First,

$$\exists\, \overline{C_1}(\varepsilon), \overline{C_2}(\varepsilon) > 0 \,\forall\, p \in (0,1) :\ \overline{C_1} \ \le\ |p - p_c| L(p)^2\, \alpha_4\big(L(p)\big) \ \le\ \overline{C_2}\,, \tag{3.2}$$

which is [N-08, Proposition 32] for the triangular lattice and [K-87, Theorem 4] for the square lattice. Second, we need quasi-multiplicativity [SSt-10, Proposition 4]:

$$\exists\, C_5 > 0 \,\forall\, m < \tilde{n} :\ \alpha_4(m) \cdot \alpha_4(m, \tilde{n}) \ \le\ C_5\, \alpha_4(\tilde{n})\,. \tag{3.3}$$

Finally, we need an estimate of the four arm event, namely

$$\exists\, \beta, C_6 > 0 \,\forall\, m < \tilde{n} :\ \alpha_4(m, \tilde{n}) \ \ge\ C_6 \Big(\frac{m}{\tilde{n}}\Big)^{2-\beta}\,. \tag{3.4}$$

Its proof is analogous to the proof of equation (3.1) above. Note that we can a-priori apply the RSW theory for (3.4), since there we consider only critical percolation.

Let $C_1, C_2 > 0$. We define $C_3, C_4 > 0$ by

$$C_4 \ :=\ \max\Big\{ \big(\tfrac{C_2 C_5}{\overline{C_1} C_6}\big)^{\frac{1}{\beta}}, 1 \Big\} \quad \text{and} \quad \frac{1}{C_3} \ :=\ \max\Big\{ \big(\tfrac{\overline{C_2} C_5}{C_1 C_6}\big)^{\frac{1}{\beta}}, 1 \Big\}\,.$$

Let $p \in (0,1)$ and $n \ge 1$ with

$$C_1 \ \le\ |p - p_c|\, n^2 \alpha_4(n) \ \le\ C_2\,. \tag{3.5}$$

First, we show that $n/L(p) \le C_4$. We can assume $n > L(p)$, since otherwise $n/L(p) \le 1 \le C_4$. Facts (3.3) and (3.4) with $m = L(p)$ and $\tilde{n} = n$ imply

$$\frac{\alpha_4(n)}{\alpha_4(L(p))} \ge \tfrac{1}{C_5}\, \alpha_4\big(L(p), n\big) \ge \tfrac{C_6}{C_5} \Big(\frac{L(p)}{n}\Big)^{2-\beta}\,.$$

Combined with the left inequality of (3.2) and the right inequality of (3.5), we conclude

$$\tfrac{C_2}{\overline{C_1}} \ \ge\ \frac{|p - p_c| n^2 \alpha_4(n)}{|p - p_c|\, L(p)^2 \alpha_4(L(p))} \ \ge\ \Big(\frac{n}{L(p)}\Big)^2 \tfrac{C_6}{C_5} \Big(\frac{L(p)}{n}\Big)^{2-\beta} \ =\ \tfrac{C_6}{C_5} \Big(\frac{n}{L(p)}\Big)^{\beta}$$

and therefore $n/L(p) \le C_4$.

An analogous reasoning with interchanged roles of $L(p)$ and n yields the other estimate, i.e. $L(p)/n \le 1/C_3$. Thereto we may assume $L(p) > n$, since otherwise $L(p)/n \le 1 \le 1/C_3$. Using facts (ii) and (iii) with $m = n$ and $\tilde{n} = L(p)$, we get

$$\frac{\alpha_4(L(p))}{\alpha_4(n)} \ge \tfrac{1}{C_5}\, \alpha_4\big(n, L(p)\big) \ge \tfrac{C_6}{C_5} \Big(\frac{n}{L(p)}\Big)^{2-\beta}\,.$$

Now we apply the right inequality of (3.2) and the left inequality of (3.5) to conclude

$$\frac{\overline{C_2}}{C_1} \ \ge\ \frac{|p - p_c|\, L(p)^2 \alpha_4(L(p))}{|p - p_c|\, n^2 \alpha_4(n)} \ \ge\ \Big(\frac{L(p)}{n}\Big)^2 \tfrac{C_6}{C_5} \Big(\frac{n}{L(p)}\Big)^{2-\beta} \ =\ \tfrac{C_6}{C_5} \Big(\frac{L(p)}{n}\Big)^{\beta}$$

and therefore $L(p)/n \le 1/C_3$. □

Proof of Lemma 3.7. Let $\varepsilon_0 \in (0, \frac{1}{2})$ and $K \geq 2$. The RSW Theorem (see [N-08, Theorem 2], for instance) states that there is a universal positive function $f_K(\cdot)$, such that, for all $m \in \mathbb{N}$, if the probability of crossing an $m \times m$ rectangle is at least δ, then the probability of crossing an $Km \times m$ rectangle is at least $f_K(\delta)$. We set $\varepsilon := f_K(\varepsilon_0)/2$. Then $\varepsilon \in (0, \varepsilon_0)$ as $f_k(\delta) \leq \delta$. Let $p \in (0, p_c)$. We abbreviate $L := L_{\varepsilon_0}(p)$. We have to show that $L_\varepsilon(p) \geq KL$. By the definition of $L_\varepsilon(p)$, it suffices to show that $P_p[\boxminus(n \times n)] > \varepsilon$ for all $n < KL$. If $n \leq L$, then $P_p[\boxminus(n \times n)] \geq \varepsilon_0 > \varepsilon$ by the definition of $L = L_{\varepsilon_0}(p)$. Now let $n \in (L, KL)$. Since every crossing of a $KL \times L$ rectangle induces a crossing of an $n \times n$ rectangle (if the rectangles are matched on the upper left corner), it follows that $P_p[\boxminus(n \times n)] \geq P_p[\boxminus(KL \times L)] \geq f_K(\varepsilon_0) > \varepsilon$, which completes the proof. $\square$

Part II

Crystallisation

Chapter 4

A Rigidity Estimate

In this relatively short chapter we show a rigidity estimate for 1-forms with non-vanishing exterior derivative. It is part of a paper submitted for publication. A preprint is available on arXiv.[1] The rigidity estimate is stated in Section 4.1 and proven in Section 4.2.

4.1 Statement of the Rigidity Estimate

Let $d \geq 2$. We work with functions mapping to $\mathbb{R}^{d\times d}$ defined on an open, connected and bounded set $M \subset \mathbb{R}^d$ with smooth boundary. We identify such a matrix-valued function $V = (V_{ij})_{1\leq i,j\leq d}$ line by line with a vector $V = (V_i)_{1\leq i\leq d}$ of 1-forms $V_i = \sum_{j=1}^d V_{ij} dx_j$. Then the exterior derivative $dV = (dV_i)_{1\leq i\leq d}$ is a vector of 2-forms with components $dV_i = \sum_{k<l}(\partial_k V_{il} - \partial_l V_{ik}) dx_k \wedge dx_l$ if the derivatives exist. For $p \geq 1$, its p-norm is defined by

$$\|dV_i\|_{L^p(M)}^p := \sum_{k<l} \left\|\partial_k V_{il} - \partial_l V_{ik}\right\|_{L^p(M)}^p \qquad \text{and} \qquad \|dV\|_{L^p(M)}^p := \sum_{i=1}^d \|dV_i\|_{L^p(M)}^p .$$

We say that $V \in L^2(M, \mathbb{R}^{d\times d})$ satisfies $dV \in L^p(M)$ for some $p \geq 1$ if there exist smooth functions $V^n \in L^2(M, \mathbb{R}^{d\times d})$, $n \in \mathbb{N}$, such that $V^n \to V$ in L^2 as $n \to \infty$ and such that $(dV^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in L^p. In that case we define $dV := L^p\text{-}\lim_{n\to\infty} dV^n$. This limit is well-defined by the following remark.

Remark. For $k \in \mathbb{N}_0$, let $L^2\Omega^k(M)$ denote the space of k-forms on M whose coefficients are in $L^2(M)$. Other spaces of k-forms are defined analogously. Let $\nu \in L^2\Omega^1(M)$ be a 1-form. Then a 2-form ω is the exterior derivative of ν in the weak sense, i.e. $d\nu = \omega$, if $\langle \nu, \delta\chi\rangle = \langle \omega, \chi\rangle$ holds for all 2-forms $\chi \in C_c^\infty\Omega^2(M)$, where the codifferential δ is the adjoint operator to d. Therefore the weak exterior derivative is unique. In particular, if there are smooth 1-forms $\nu_n \in C^\infty\Omega^1(M)$ such

[1] [A-14c] Simon Aumann: Spontaneous Breaking of Rotational Symmetry with Arbitrary Defects and a Rigidity Estimate, *submitted*, arXiv:1408.5375, 2014

that $\nu_n \to \nu$ in L^2 and $d\nu_n \to \psi$ in L^p for a 2-form $\psi \in L^p\Omega^2(M)$, then $\psi = \omega = d\nu$. Thus the limit is well-defined.

Note that we did not require that the weak exterior derivative ω of ν is in L^p, but we imposed the possibly stronger condition that we can approximate ν with smooth 1-forms whose exterior derivatives converge in L^p. It is not relevant for our purposes whether these two conditions are equivalent.

Now we can state the rigidity estimate of this chapter.

Theorem 4.1. *Let $d \geq 2$ and $M \subset \mathbb{R}^d$ be open, connected and bounded with smooth boundary. Let further $p \geq 2d/(2+d)$. Then there exist constants $C_1 = C_1(M)$ and $C_2 = C_2(M,p)$ such that for all $V \in L^2(M, \mathbb{R}^{d\times d})$ with $dV \in L^p(M)$ there exists a rotation $R \in \mathrm{SO}(d)$ with*

$$\|V - R\|_{L^2(M)} \leq C_1 \| \operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2(M)} + C_2 \|dV\|_{L^p(M)} .$$

If V is the Jacobi-matrix of some function $v : M \to \mathbb{R}^d$ and is point-wise in $\mathrm{SO}(d)$, i.e. if $V = dv \in \mathrm{SO}(d)$, then the right hand side is zero and Liouville's Theorem is recovered. And if only $V = dv$ for some function $v : M \to \mathbb{R}^d$, then $dV = 0$ and the theorem above reduces to the rigidity estimate [FJM-02, Theorem 3.1]. Moreover, the special case $d = 2$ and $p = 1$ is already covered by [MSZ-13, Theorem 3.3].

Theorem 4.1 also holds if M is a finite box with periodic boundary conditions:

Corollary 4.2. *Let $[M]$ be a d-dimensional torus with $d \geq 2$. Let further $p \geq 2d/(2+d)$. Then there exist constants $C_1 = C_1([M])$ and $C_2 = C_2([M],p)$ such that for all $V \in L^2([M], \mathbb{R}^{d\times d})$ with $dV \in L^p([M])$ there exists a rotation $R \in \mathrm{SO}(d)$ with*

$$\|V - R\|_{L^2([M])} \leq C_1 \| \operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2([M])} + C_2 \|dV\|_{L^p([M])} .$$

Remark 4.3. The formulation of Theorem 4.1 is not the most general one. It should also hold if M is an open, connected and bounded set with a more general boundary. In the proof, we will apply Lemma 3.2.1 in the book [S-95] of Schwarz. He considers manifolds with smooth boundary. Though not formally stated, his results also hold if the boundary is only piecewise smooth. In [MMM-08] Mitrea, Mitrea and Monniaux considered similar problems as in [S-95], but for domains with Lipschitz boundary. Unfortunately they do not state the exact lemma we need. Since a smooth boundary is sufficient for our purposes, we stick to that case, where the needed lemma is explicitly stated in the literature.

It is also possible to generalise Theorem 4.1 in another direction. If M is a flat manifold, which means that all transition maps are just translations, then it makes sense to speak about global rotations. Theorem 4.1 immediately generalises to compact connected flat manifolds using a straightforward generalisation of Lemma 4.7 to such manifolds.

We also determine the scaling of the constants in the theorem and in the corollary above.

Lemma 4.4. *Assume that Theorem 4.1 holds on $M \subseteq \mathbb{R}^d$, $d \geq 2$, for some $p \geq 1$ with constants $C_1(M)$ and $C_2(M,p)$. Let $\eta > 0$. Then Theorem 4.1 holds on ηM for p with constants*

$$C_1(\eta M) = C_1(M) \qquad \text{and} \qquad C_2(\eta M, p) = \eta^{\frac{d}{2}-\frac{d}{p}+1}\, C_2(M,p)\,.$$

The same statement is true if $M \equiv [M]$ is a torus as in Corollary 4.2.

Therefore C_1 is scale invariant, but C_2 is not (except if $p = 2d/(2+d)$).

Remark 4.5. The assumption $p \geq 2d/(2+d)$ is best possible. Indeed, if we had $p < 2d/(2+d)$, then $dp - 2d + 2p < 0$, which is equivalent to $\frac{d}{2} - \frac{d}{p} + 1 < 0$. Thus, by Lemma 4.4, the constant $C_2(\eta M, p)$ would tend to zero as $\eta \to \infty$. But the latter is impossible.

Indeed, consider some smooth $V : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ such that first $V(x) \in \mathrm{SO}(d)$ for all $x \in \mathbb{R}^d$, second $V(x) = R_0$ for all $x \in \mathbb{R}^d$ with $|x| \geq 1$ (for some fixed $R_0 \in \mathrm{SO}(d)$) and third V being not constant on $B_1(0)$. Then $\|dV\|_{L^p(B_1(0))} > 0$ by Liouville's Theorem. Let $M = B_1(0)$ and let η be large. Then $\inf_{R\in \mathrm{SO}(d)} \|V - R\|_{L^2(\eta M)} \geq c$ for some constant $c > 0$ since its argmin converges to R_0. Moreover, $dV(x) = 0$ for $|x| > 1$, which implies that $\|dV\|_{L^p(\eta M)} = \|dV\|_{L^p(B_1(0))} \in (0,\infty)$ is constant (for $\eta > 1$). Theorem 4.1 states that $0 < c/\|dV\|_{L^p(B_1(0))} \leq C_2(\eta M, p)$. Therefore $C_2(\eta M, p) \to 0$ as $\eta \to \infty$ is indeed impossible.

4.2 Proof of the Rigidity Estimate

Let $A \subseteq \mathbb{R}^d$ such that $B \subseteq A \subseteq \overline{B}$ for an open set $B \subseteq \mathbb{R}^d$ (where $\overline{B}$ denotes the closure of B). Let further $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $p \geq 1$. Then $W^{k,p}(A,\mathbb{R}^n)$ denotes the Sobolev space of functions $f : A \to \mathbb{R}^n$ such that all partial derivatives up to order k exist in the weak sense and have finite p-norm. In particular, $W^{0,2}(A,\mathbb{R}^n) = L^2(A,\mathbb{R}^n)$.

For the proof of the rigidity estimate, we use a covering argument. Therefore we need

Lemma 4.6. *Let $A_1, A_2 \subseteq \mathbb{R}^d$ such that $B_j \subseteq A_j \subseteq \overline{B_j}$ for an open set $B_j \subseteq \mathbb{R}^d$, $j \in \{1,2\}$, and $\lambda(A_1 \cap A_2) > 0$ and $\lambda(A_2) < \infty$, where λ denotes the Lebesgue-measure. Assume that, for $j \in \{1,2\}$, there exists a constant $c_j > 0$ such that for all $V \in W^{1,2}(A_j, \mathbb{R}^{d\times d})$ with $dV = 0$ there exists a rotation $R_j \in \mathrm{SO}(d)$ with*

$$\|V - R_j\|_{L^2(A_j)} \leq c_j \|\operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2(A_j)}\,.$$

Then there exists a constant $C > 0$ such that for all $V \in W^{1,2}(A_1 \cup A_2, \mathbb{R}^{d\times d})$ with $dV = 0$ there exists a rotation $R \in \mathrm{SO}(d)$ with

$$\|V - R\|_{L^2(A_1\cup A_2)} \leq C \|\operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2(A_1\cup A_2)}\,.$$

Proof. We set

$$C = \sqrt{\Big(\frac{4\lambda(A_2)}{\lambda(A_1\cap A_2)}+2\Big)\big(c_1^2+c_2^2\big)} \;<\; \infty\,.$$

Let $V \in W^{1,2}(A_1\cup A_2,\mathbb{R}^{d\times d})$ with $dV=0$ and let R_1 and R_2 be rotations associated to the restriction of V to A_1 and A_2, respectively. In the following calculation, we first use that R_1-R_2 is constant. Then we apply the inequality $(a+b)^2\le 2(a^2+b^2)$ and the fact that the L^2-norm on increasing sets increases. Finally we plug in the assumptions. This yields

$$\begin{aligned}
\|R_2-R_1\|^2_{L^2(A_2)} &= \lambda(A_2)|R_2-R_1|^2 = \frac{\lambda(A_2)}{\lambda(A_1\cap A_2)}\|R_2-R_1\|^2_{L^2(A_1\cap A_2)}\\
&\le \frac{\lambda(A_2)}{\lambda(A_1\cap A_2)}\cdot 2\Big(\|R_2-V\|^2_{L^2(A_1\cap A_2)}+\|V-R_1\|^2_{L^2(A_1\cap A_2)}\Big)\\
&\le \frac{2\lambda(A_2)}{\lambda(A_1\cap A_2)}\Big(\|R_2-V\|^2_{L^2(A_2)}+\|V-R_1\|^2_{L^2(A_1)}\Big)\\
&\le \frac{2\lambda(A_2)}{\lambda(A_1\cap A_2)}\Big(c_2^2\|\operatorname{dist}(V,\mathrm{SO}(d))\|^2_{L^2(A_2)}+\\
&\qquad\qquad + c_1^2\|\operatorname{dist}(V,\mathrm{SO}(d))\|^2_{L^2(A_1)}\Big)\\
&\le \frac{2\lambda(A_2)}{\lambda(A_1\cap A_2)}\big(c_1^2+c_2^2\big)\|\operatorname{dist}(V,\mathrm{SO}(d))\|^2_{L^2(A_1\cup A_2)}\,.
\end{aligned}$$

We set $R=R_1$ and estimate using again elementary inequalities, the assumptions and finally the just obtained estimate of $\|R_2-R_1\|_{L^2(A_2)}$

$$\begin{aligned}
\|V-R_1\|^2_{L^2(A_1\cup A_2)} &\le \|V-R_1\|^2_{L^2(A_1)}+\|V-R_1\|^2_{L^2(A_2)}\\
&\le \|V-R_1\|^2_{L^2(A_1)}+2\big(\|V-R_2\|^2_{L^2(A_2)}+\|R_2-R_1\|^2_{L^2(A_2)}\big)\\
&\le c_1^2\|\operatorname{dist}(V,\mathrm{SO}(d))\|^2_{L^2(A_1)}+2c_2^2\|\operatorname{dist}(V,\mathrm{SO}(d))\|^2_{L^2(A_2)}\\
&\quad +2\|R_2-R_1\|^2_{L^2(A_2)}\\
&\le 2\big(c_1^2+c_2^2\big)\|\operatorname{dist}(V,\mathrm{SO}(d))\|^2_{L^2(A_1\cup A_2)}+2\|R_2-R_1\|^2_{L^2(A_2)}\\
&\le \Big(\frac{4\lambda(A_2)}{\lambda(A_1\cap A_2)}+2\Big)\big(c_1^2+c_2^2\big)\|\operatorname{dist}(V,\mathrm{SO}(d))\|^2_{L^2(A_1\cup A_2)}\,,
\end{aligned}$$

which proves the lemma. □

The case $dV=0$ of Theorem 4.1 is preponed into the following lemma. It looks almost like the rigidity estimate of Friesecke et al., but it handles closed 1-forms. In contrast, [FJM-02, Theorem 3.1] considers only exact 1-forms.

Lemma 4.7. *Let $d\ge 2$ and $M\subset\mathbb{R}^d$ be open, connected and bounded with Lipschitz boundary. Then there exists a constant $C(M)$ such that for all $V\in W^{1,2}(M,\mathbb{R}^{d\times d})$ with $dV=0$ there exists a rotation $R\in\mathrm{SO}(d)$ with*

$$\|V-R\|_{L^2(M)} \;\le\; C(M)\|\operatorname{dist}(V,\mathrm{SO}(d))\|_{L^2(M)}\,.$$

Proof. We show this lemma by a covering argument. For $x \in \overline{M}$, let $A_x \subseteq \overline{M}$ be a contractible open neighbourhood of x in $\overline{M}$. Since $\overline{M}$ is compact, there exists a finite subcover of $(A_x)_{x \in \overline{M}}$ of $\overline{M}$. Since M is connected, we can arrange the subcover $A_1, \ldots, A_K$ such that $A_k \cap \bigcup_{l=1}^{k-1} A_l \neq \emptyset$ for all $k \in \{2, \ldots, K\}$. These sets have positive Lebesgue measure. Moreover, for each $k \in \{1, \ldots, K\}$, there is an open set B_k such that $B_k \subseteq A_k \subseteq \overline{B_k}$. Let $C_k = C(B_k)$ be the constant in the rigidity estimate [FJM-02, Theorem 3.1] of Friesecke, James and Müller associated to B_k. Note that it does not matter whether we use A_k or B_k in their rigidity estimate.

Let $V \in W^{1,2}(M, \mathbb{R}^{d \times d})$ with $dV = 0$. Let $k \in \{1, \ldots, K\}$. Since A_k is contractible, there exist $v_k \in W^{2,2}(A_k, \mathbb{R}^d)$ with $V = Dv_k$ on A_k. Of course, the functions v_k need not fit together to a global function v. Nevertheless, for each $k \in \{1, \ldots, K\}$, there exists a rotation $R_k \in \mathrm{SO}(d)$ such that

$$\|Dv_k - R_k\|_{L^2(A_k)} \leq C_k \|\operatorname{dist}(Dv_k, \mathrm{SO}(d))\|_{L^2(A_k)} .$$

Using Lemma 4.6, we show by induction on k, that there exist constants $\tilde{C}_k$ (independent of V) and rotations $\tilde{R}_k$ such that

$$\|V - \tilde{R}_k\|_{L^2(\bigcup_{l=1}^{k} A_l)} \leq \tilde{C}_k \|\operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2(\bigcup_{l=1}^{k} A_l)}$$

for all $k \in \{1, \ldots, K\}$, which implies the theorem since $\bigcup_{l=1}^{K} A_l = \overline{M}$. □

Now we are ready to prove the main rigidity estimate.

Proof of Theorem 4.1. The case $d = 2$ and $p = 1$ is already covered by Müller et. al. in [MSZ-13, Theorem 3.3]. Therefore we may assume $p > 1$.

First we prove the Theorem for $V \in W^{1,p}(M, \mathbb{R}^{d \times d})$. We claim that this implies $V \in L^2(M, \mathbb{R}^{d \times d})$. Indeed, M is bounded, and if $2d/(2 + d) \leq p \leq 2$ then $1 \leq p \leq 2 \leq dp/(d - 1p)$. Therefore Sobolev's Lemma (see [S-95, Theorem 1.3.3(b)], for instance) states that

$$\|V\|_{L^2(M)} \; = \; \|V\|_{W^{0,2}(M)} \; \leq \; C_3 \|V\|_{W^{1,p}(M)} \tag{4.1}$$

for some constant $C_3 = C_3(M, p) > 0$.

Let $i \in \{1, \ldots, d\}$. Considering the ith line V_i as a 1-form, we look for 1-forms W_i which solve of the equation

$$dW_i = dV_i$$

Obviously, $W_i = V_i$ is a solution. Moreover, $dV_i \in W^{0,p}\Omega^2(\overline{M})$, which is the space of 2-forms with coefficients in $W^{0,p}(\overline{M})$. According to Lemma 3.2.1 of [S-95] we choose a solution $W_i \in W^{1,p}\Omega^1(\overline{M})$ such that

$$\|W_i\|_{W^{1,p}\Omega^1(\overline{M})} \leq C_4 \|dV_i\|_{W^{0,p}\Omega^2(\overline{M})} \tag{4.2}$$

for some constant $C_4 = C_4(M, p) > 0$. Note that [S-95, Lemma 3.2.1] requires $p > 1$. Therefore this was assumed in the beginning of the proof. Since this lemma is stated for compact ∂-manifolds[2], we worked on $\overline{M}$. Note that $\overline{M}$ is a compact ∂-manifold since M is open and bounded with smooth boundary.

Now we define $U_i := V_i - W_i$. Then $dU_i = dV_i - dW_i = 0$. We set $W = (W_i)_{1\le i\le d}$, $U = (U_i)_{1\le i\le d}$. By Lemma 4.7, there exist a constant C_1, only depending on M, and a rotation $R \in \mathrm{SO}(d)$ such that

$$\|U - R\|_{L^2(M)} \le C_1 \|\operatorname{dist}(U, \mathrm{SO}(d))\|_{L^2(M)} .$$

Using the triangle inequality twice and in between the assertion just above, we estimate

$$\begin{aligned}
\|V - R\|_{L^2(M)} &= \|W + U - R\|_{L^2(M)} \\
&\le \|U - R\|_{L^2(M)} + \|W\|_{L^2(M)} \\
&\le C_1 \|\operatorname{dist}(U, \mathrm{SO}(d))\|_{L^2(M)} + \|W\|_{L^2(M)} \\
&= C_1 \|\operatorname{dist}(V - W, \mathrm{SO}(d))\|_{L^2(M)} + \|W\|_{L^2(M)} \\
&\le C_1 \|\operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2(M)} + (C_1 + 1)\|W\|_{L^2(M)}
\end{aligned}$$

Combining estimate (4.1) for W, i.e. Sobolev's Lemma, and estimate (4.2) yields

$$\|W\|_{L^2(M)} \le C_3 \|W\|_{W^{1,p}(M)} \le C_3 C_4 \|dV\|_{W^{0,p}(\overline{M})} = C_3 C_4 \|dV\|_{L^p(M)} .$$

By setting $C_2 = (C_1 + 1) C_3 C_4$, we arrive at

$$\|V - R\|_{L^2(M)} \le C_1 \|\operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2(M)} + C_2 \|dV\|_{L^p(M)} .$$

which proves the theorem in the case $V \in W^{1,p}(M, \mathbb{R}^{d\times d})$.

For general $V \in L^2(M, \mathbb{R}^{d\times d})$ with $dV \in L^p(M)$, we use a sequence $V^m \in C^\infty(M, \mathbb{R}^{d\times d})$, $m \in \mathbb{N}$, which converges point-wise almost everywhere and with $\|V - V^m\|_{L^2(M)} \to 0$ and $\|dV - dV^m\|_{L^p(M)} \to 0$ as $m \to \infty$. Then also

$$\|\operatorname{dist}(V, \mathrm{SO}(d)) - \operatorname{dist}(V^m, \mathrm{SO}(d))\|_{L^2(M)} \to 0$$

and the theorem follows. □

Proof of Corollary 4.2. Let $v_1, \dots, v_d \in \mathbb{R}^d$ be vectors such that $[M] = \mathbb{R}^d / \{z_1 v_1 + \dots + z_d v_d \mid z_1, \dots, z_d \in \mathbb{Z}\}$ and define $M := \{\lambda_1 v_1 + \dots + \lambda_d v_d \mid \lambda_1, \dots, \lambda_d \in [0, 1)\}$. We choose a ball $B \subseteq \mathbb{R}^d$ such that $B \supseteq M$. Moreover, let $\widetilde{M}$ be the union of n translated copies of M such that $\widetilde{M} \supseteq B$ (with some suitable $n \in \mathbb{N}$). We identify any function on $[M]$ with the function on M evaluated at the corresponding representatives and extend it periodically to $\widetilde{M}$. Applying Theorem 4.1 to the ball B yields

$$\begin{aligned}
\|V - R\|_{L^2([M])} &\le \|V - R\|_{L^2(B)} \\
&\le C_1(B) \|\operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2(B)} + C_2(B, p) \|dV\|_{L^p(B)} \\
&\le C_1(B) \|\operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2(\widetilde{M})} + C_2(B, p) \|dV\|_{L^p(\widetilde{M})} \\
&= \sqrt{n} C_1(B) \|\operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2([M])} + \sqrt[p]{n} C_2(B, p) \|dV\|_{L^p([M])} ,
\end{aligned}$$

[2] A ∂-manifold is a complete manifold with boundary equipped with an oriented smooth atlas, see [S-95, Definition 1.1.2]

where we used $M \subseteq B \subseteq \widetilde{M}$ and the facts that all functions are periodically extended to $\widetilde{M}$ and that $\widetilde{M}$ consists of n copies of M. Therefore the corollary follows with $C_1([M]) = \sqrt{n} C_1(B)$ and $C_2([M], p) = \sqrt[p]{n} C_2(B, p)$. □

Finally we proof the behaviour of the constants under scaling.

Proof of Lemma 4.4. Let $\widetilde{M} := \eta M$ be the scaled domain. Let $\widetilde{V} \in L^2(\widetilde{M}, \mathbb{R}^{d\times d})$ with $d\widetilde{V} \in L^p(\widetilde{M})$. We define $V \in L^2(M, \mathbb{R}^{d\times d})$ by $V(x) := \widetilde{V}(\eta x)$, $x \in M$.

A change of variables yields

$$\int_M |V(x) - R|^2 \, dx = \int_M |\widetilde{V}(\eta x) - R|^2 \, dx = \eta^{-d} \int_{\widetilde{M}} |\widetilde{V}(y) - R|^2 \, dy$$

and therefore

$$\|V - R\|_{L^2(M)} = \eta^{-\frac{d}{2}} \|\widetilde{V} - R\|_{L^2(\widetilde{M})} .$$

Analogously,

$$\| \operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2(M)} = \eta^{-\frac{d}{2}} \| \operatorname{dist}(\widetilde{V}, \mathrm{SO}(d))\|_{L^2(\widetilde{M})} .$$

Moreover, $dV(x) = \eta \, d\widetilde{V}(\eta x)$ and thus

$$\int_M |dV(x)|^p \, dx = \int_M \eta^p \, |d\widetilde{V}(\eta x)|^p \, dx = \eta^{p-d} \int_{\widetilde{M}} |d\widetilde{V}(y)|^p \, dy ,$$

which implies $dV \in L^p(M)$ and

$$\|dV\|_{L^p(M)} = \eta^{1-\frac{d}{p}} \|d\widetilde{V}\|_{L^p(\widetilde{M})} .$$

Using Theorem 4.1 on M, we conclude

$$\begin{aligned} \|\widetilde{V} - R\|_{L^2(\widetilde{M})} &= \eta^{\frac{d}{2}} \|V - R\|_{L^2(M)} \\ &\leq \eta^{\frac{d}{2}} C_1(M) \| \operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2(M)} + \eta^{\frac{d}{2}} C_2(M,p) \|dV\|_{L^p(M)} \\ &= C_1(M) \| \operatorname{dist}(\widetilde{V}, \mathrm{SO}(d))\|_{L^2(\widetilde{M})} + C_2(M,p) \eta^{\frac{d}{2}+1-\frac{d}{p}} \|d\widetilde{V}\|_{L^p(\widetilde{M})} . \end{aligned}$$

Since $\widetilde{V}$ was arbitrary, we can choose $C_1(\eta M) = C_1(M)$ as well as $C_2(\eta M, p) = \eta^{\frac{d}{2}-\frac{d}{p}+1} C_2(M,p)$, as desired. The proof for the torus is analogous. □

Chapter 5

Spontaneous Rotational Symmetry Breaking

In this chapter we show a kind of spontaneous breaking of rotational symmetry for some models of crystals, which allow almost all kind of defects, including unbounded defects as well as edge, screw and mixed dislocation defects, i.e. defects with Burgers vectors. It is part of a paper submitted for publication. A preprint is available on arXiv.[1]

Let us start with an informal description of the crystal. The crystal is given by random points in a box Λ_N, which are the centres of the molecules. Thus there is no reference lattice. We assume that the crystal has a favourite structure which should be interpreted as a property of the considered material. This structure is given by a fixed tessellation of $\mathbb{R}^d$. The random points $\mathcal{P}$ determine a set $\mathcal{T}$ of tiles such that each tile in $\mathcal{T}$ is an enlarged ε-perturbation of a standard tile and such that $\mathcal{T}$ locally looks like the given tessellation. The perturbed tiles need not cover the whole box Λ_N. The remaining "holes" are the defects. Almost all defects are feasible. We only require that each defect has a minimum size, i.e. the boundary of a defect does not come closer than 3ρ to itself (for some fixed $\rho \in (0, 1)$). But the defects may be arbitrarily large and may also have Burgers vectors. Thus there may exist edge, screw and also mixed dislocations. We assume that the crystal is connected and sufficiently large, i.e. its size is comparable to the size of the box.

The distribution of the points is given in the Gibbsian setting using a Poisson Point Process as reference measure. The Hamiltonian consists of three parts. The first part is given by some local Hamiltonians which measures the energy costs due to local deformations of the crystal. These local Hamiltonians are part of the model and shall fulfil a reasonable inequality. They can be given by a pair-potential using adjacent points, for instance (cf. Section 5.4). The second part can be interpreted as a surface energy. It punishes defects proportional to their surface. The last part of the Hamiltonian can be thought as a chemical potential; increasing it favours more

[1] [A-14c] Simon Aumann: Spontaneous Breaking of Rotational Symmetry with Arbitrary Defects and a Rigidity Estimate, *submitted*, arXiv:1408.5375, 2014

points. Then we show that, in an appropriate limit, the local deformation of the crystal is close to a constant rotation.

The organisation of this chapter is as follows. In Section 5.1 we define the model in detail. After an overview we describe first the tessellation and then the crystal. Thereafter, we define the local deformation of the crystal as well as the Hamiltonian and the corresponding probability measure. Then we state the main theorem in Section 5.2, which will be proved in Section 5.3. The structure of the proof is explained in the beginning of that section. Finally, we give two examples of concrete models in Section 5.4.

5.1 Definition of the Model

First we outline the components of our model.

1. A periodic locally finite tessellation of $\mathbb{R}^d$, whose tiles are closed polytopes (maybe of different types).
2. A parameter $\varepsilon > 0$, which measures the size of the allowed deformation of the crystal.
3. A parameter $\rho \in (0, \rho_{\max})$, which is a lower bound of the size of a defect.
4. A constant $c_0 > 0$, which is a relative lower bound on the number of the tiles of the crystal.
5. Some local Hamiltonians, which measure the local deformation of a tile, and constants $c_1 > 0$, $c_2^{\mathbb{R}} \in \mathbb{R}$ satisfying a certain inequality (cf. (5.3) below).
6. A function S, which measures the surface of the defects, and a constant $c_3 > 0$ satisfying a certain condition (cf. (5.4) below).

In the following subsections, we describe the model accurately.

5.1.1 The Underlying Tessellation

We choose a tessellation $\mathcal{M}$ of the space $\mathbb{R}^d$, $d \geq 2$, with the following properties. Each tile $\boxtimes \in \mathcal{M}$ is a closed polytope. There are finitely many different types $i \in I$ of tiles. If two tiles have the same type, then their geometric shape and size as well as the types and the placement of their neighbouring tiles are identical. We allow different tile types since they naturally arise if one considers a densest sphere packing in dimension $d \geq 3$, for instance. The tessellation shall be locally finite and B_0-periodic for a finite box B_0 which is the image of the cube $[0,1]^d$ under some linear map L. Thus the vectors Le_j, $j = 1, \ldots, d$, span the box B_0 (where e_j denotes the jth unit vector).

Throughout we fix some $\varepsilon > 0$, $\rho \in (0, \rho_{\max})$ and $c_0 > 0$, where $\rho_{\max} := 1 \wedge \min\{\operatorname{dist}(\boxtimes, \tilde{\boxtimes}) \mid \boxtimes, \tilde{\boxtimes} \in \mathcal{M}, \boxtimes \cap \tilde{\boxtimes} = \emptyset\}/3$.

For each $i \in I$, we choose a fixed tile of type i in B_0, which we denote by $\boxtimes^i$. Denoting its corners by $s_1, \ldots, s_{n_i}$, we define the set

$$\begin{aligned}\mathcal{N}_\varepsilon(\boxtimes^i) \quad := \quad & \big\{\square = \operatorname{hull}\{x_1, \ldots, x_{n_i}\} \mid x_1, \ldots, x_{n_i} \in \mathbb{R}^d \\ & \text{such that } \forall\, 1 \leq l \leq n_i : |x_l - s_l| \leq \varepsilon \,\wedge\, \lambda(\square) \geq \lambda(\boxtimes^i)\big\}\end{aligned}$$

of all enlarged perturbed tiles. In the following, a “standard” tile (as in $\mathcal{M}$) is denoted by $\boxtimes$, while a perturbed tile is denoted by $\square$.

5.1.2 The Crystal

Let $N \in \mathbb{N}$. Let the torus

$$\Lambda_N := \mathbb{R}^d \Big/ \big\{ N(z_1 L e_1 + \ldots + z_d L e_d) \mid z_1, \ldots, z_d \in \mathbb{Z} \big\}$$

be the “universe” of the crystal, with periodic boundary conditions. Moreover, let $\tilde{\Omega}, \mathcal{F}, \mu$ be a suitable probability space and for $\omega \in \Omega$ let

$$\mathcal{P} = \mathcal{P}(\omega) = \{X_1, \ldots, X_{|\mathcal{P}|}\} \subset \Lambda_N$$

be Poisson points, which shall model the centres of the molecules of the crystal; this means that $X_1, X_2, \ldots$ is a sequence of iid random variables which are uniformly distributed on Λ_N and independent of $|\mathcal{P}|$, and $\mu(|\mathcal{P}| = k) = e^{-\lambda(\Lambda_N)} \lambda(\Lambda_N)^k / k!$, $k \in \mathbb{N}_0$. Note that we suppress the N-dependency of $\tilde{\Omega}$ and $\mathcal{P}$ (and of Ω and $\mathcal{T}$ defined later) to simplify the notation as N is clear from the context.

The molecules of the crystal shall compose a perturbation of the tessellation which may have all kinds of defects. We will define the set $\mathcal{T} = \mathcal{T}(\omega)$ of perturbed tiles. The following construction is a bit complicated, but has the advantage that an upcoming condition is quite simple; the condition ensures that a point configuration is admitted. First we define a set $\hat{\mathcal{T}}$ which contains all possibly perturbed tiles whose corners are taken from the point configuration. Here we do not impose any condition on the relative locations of the perturbed tiles to each other. But we do impose such conditions in the next step, in which we define when a subset $\tilde{\mathcal{T}} \subseteq \hat{\mathcal{T}}$ is called a perturbation of $\mathcal{M}$: locally, the relative locations of the tiles must be such as in $\mathcal{M}$. Finally we define a particular perturbation $\mathcal{T}$, which is the set containing all perturbed tiles of the crystal. It is a maximal perturbation of $\mathcal{M}$ under the conditions that it is connected and that the tiles are not too close to each other (at the boundary of the defects).

Before stating the precise definitions, we give an example. The underlying tessellation is just the two-dimensional triangular lattice. We start with a random point configuration (with periodic boundary condition) which is illustrated in Figure 5.1. Then the set $\hat{\mathcal{T}}$ of all possibly perturbed tiles contains all triangles in Figure 5.2, regardless whether they are white or grey shaded. In the grey shaded regions, $\hat{\mathcal{T}}$ does not look like the triangular lattice since the triangles do overlap or there is an interior vertex with five or seven adjacent triangles. Thus a perturbation of $\mathcal{M}$ is any set of the triangles in Figure 5.2 which contains not all grey shaded triangle such that it looks everywhere like $\mathcal{M}$. The crystal $\mathcal{T}$ is drawn in Figure 5.3. It contains all white triangles. The grey shaded regions (including the grey triangle) are the defects of the crystal. Note that the triangle formed by the three points inside the huge defect is not included since the crystal must be connected. Furthermore, the

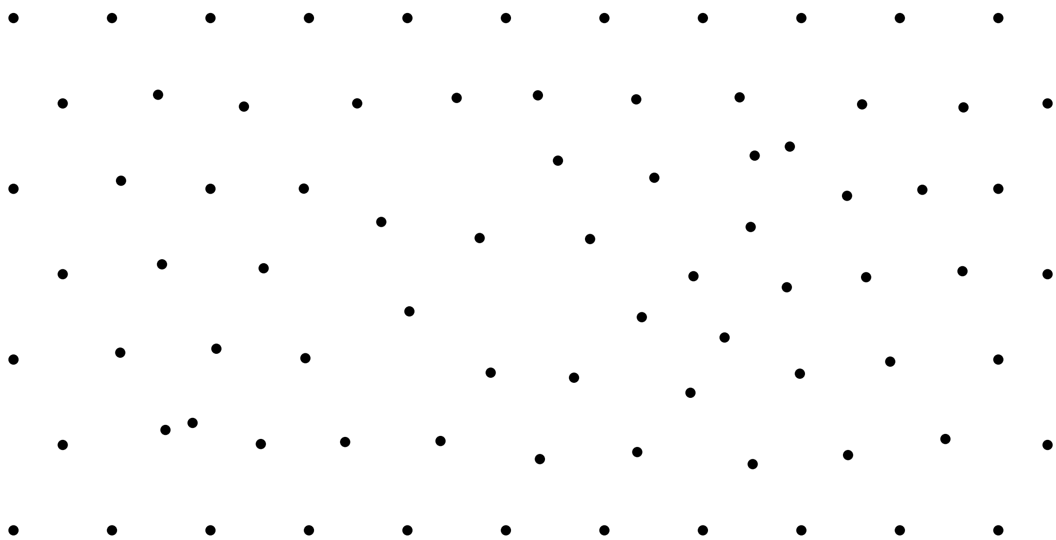


Figure 5.1: A random point configuration (with periodic boundary condition)

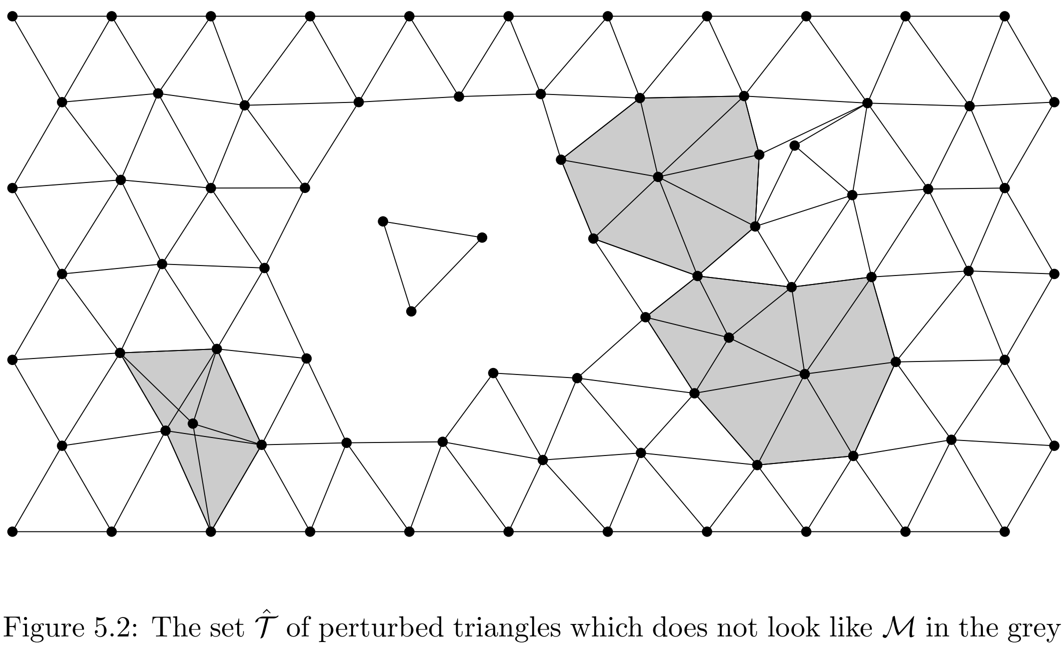


Figure 5.2: The set $\hat{\mathcal{T}}$ of perturbed triangles which does not look like $\mathcal{M}$ in the grey shaded regions

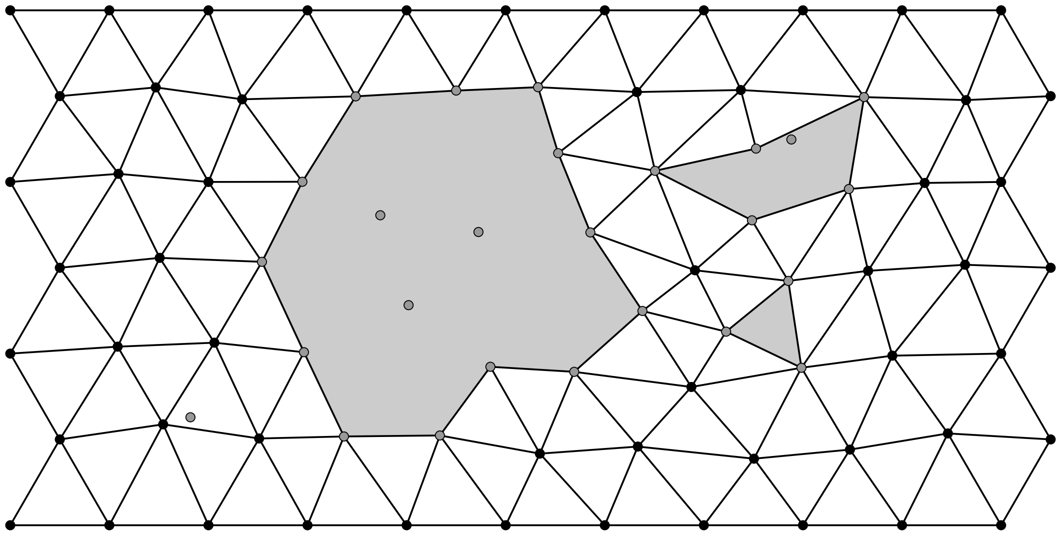


Figure 5.3: The crystal $\mathcal{T}$ with defects grey shaded and the surface points $\partial\mathcal{P}$ in dark grey

triangles which contain the point inside the upper right defect are not included since otherwise some triangles would be too close to each other.

Now we state the precise definitions. Let

$$\begin{aligned}\hat{\mathcal{T}} \;:=\; & \big\{\square = \operatorname{hull}\{X_{j_1},\dots,X_{j_k}\} \mid \{j_1,\dots,j_k\} \subset \{1,\dots,|\mathcal{P}|\}, \\ & \exists\, R \in \mathrm{SO}(d)\, \exists\, a \in \mathbb{R}^d\, \exists\, i \in I : \square \in a + R \cdot \mathcal{N}_\varepsilon(\boxtimes^i)\big\}\end{aligned}$$

be the set of all possibly perturbed tiles. Any subset $\tilde{\mathcal{T}} \subset \hat{\mathcal{T}}$ is called a perturbation of $\mathcal{M}$, if for $j = 1,\dots,j_{\tilde{\mathcal{T}}}$ (with some $j_{\tilde{\mathcal{T}}} \in \mathbb{N}$), there are are sets $\tilde{\mathcal{T}}_j \subset \hat{\mathcal{T}}$, $\mathcal{M}_j \subset \mathcal{M}$ and continuous bijective maps

$$v_j : \cup\tilde{\mathcal{T}}_j \to \cup\mathcal{M}_j,$$

mapping each tile $\square \in a + R \cdot \mathcal{N}_\varepsilon(\boxtimes^i)$ to a tile $\tilde{a} + \boxtimes^i \in \mathcal{M}_j$ (with some $a, \tilde{a} \in \mathbb{R}^d$ and $R \in \mathrm{SO}(d)$) such that

$$\tilde{\mathcal{T}} = \bigcup_{j=1}^{j_{\tilde{\mathcal{T}}}} \tilde{\mathcal{T}}_j$$

and the sets $\tilde{\mathcal{T}}_j$ do overlap, i.e. all intersections $\cup\tilde{\mathcal{T}}_j \cap \cup\tilde{\mathcal{T}}_{j'}$ consist of whole tiles if they are not empty. Thus $\tilde{\mathcal{T}}$ is a perturbation of $\mathcal{M}$ if it looks locally like $\mathcal{M}$. Now we define $\mathcal{T} = \mathcal{T}(\omega)$ to be a largest subset of $\hat{\mathcal{T}}$ such that

(i) $\mathcal{T}$ is a perturbation of $\mathcal{M}$,

(ii) $\cup\mathcal{T}$ is connected,

(iii) if $\square \cap \tilde{\square} = \emptyset$ then even $\operatorname{dist}(\square, \tilde{\square}) > 3\rho$ holds for all $\square, \tilde{\square} \in \mathcal{T}$ and

(iv) for all $\Box \in \mathcal{T}$, all faces F of $\Box$ and for all $\tilde{\Box} \in \mathcal{T}$ with $F \nsubseteq \tilde{\Box}$ there exists a point $x \in F$ such that $\mathrm{dist}(x, \tilde{\Box}) > 3\rho$.

Here "a largest subset" is understood as a subset whose cardinality (number of tiles) is maximal under all subsets with these properties. In fact, there need not exist a unique largest subset. In that case, we choose one of them according to some fixed rule.

A tile $\Box \in \mathcal{T}$ inherits its type from the corresponding tile in $\mathcal{M}$ using the bijections introduced above. We denote it by $\imath(\Box)$.

Furthermore, we define the set of surface points of $\mathcal{P}$ as follows:

$$\partial\mathcal{P} := \left\{x \in \mathcal{P} \mid x \in \partial\cup\mathcal{T} \text{ or } x \notin \mathcal{V}(\mathcal{T})\right\}, \tag{5.1}$$

where $\partial\cup\mathcal{T}$ denotes the topological boundary of the set $\cup\mathcal{T} = \{x \in \Lambda_N \mid \exists \Box \in \mathcal{T} : x \in \Box\}$ and $\mathcal{V}(\mathcal{T})$ is the set of points of $\mathcal{P}$, which are vertices of any tile $\Box \in \mathcal{T}$. In the example above, the surface points are drawn in grey in Figure 5.3. Note that there are surface points which are not vertices of any tile. We will call such surface points also exterior points (though they can also lie inside the crystal, as one of them does in the example). Such points are possible, but will be unlikely.

We need only one condition on the set $\mathcal{P}$. We namely require that the crystal has a minimum size. Thereto we define the space of admitted configuration to be

$$\Omega := \{\omega \in \tilde{\Omega} \mid |\mathcal{T}| \geq c_0 N^d\}.$$

Then $\Omega \neq \emptyset$ for large enough N (even for all $N \in \mathbb{N}$ if $c_0 \leq 1$) as restricting $\mathcal{M}$ to Λ_N yields an allowed point configuration. Thereto we had to choose $\rho < \rho_{\max} \leq \min\{\mathrm{dist}(\boxtimes, \tilde{\boxtimes}) \mid \boxtimes, \tilde{\boxtimes} \in \mathcal{M}, \boxtimes \cap \tilde{\boxtimes} = \emptyset\}/3$. Otherwise even the points of $\mathcal{M}$ would not compose a huge crystal.

Note that we do not require a minimal distance between two points and that there may exist points inside a tile which do not belong to the tile. But all such points are included in the surface points $\partial\mathcal{P}$, which consists not only of the surface vertices of $\mathcal{T}$, but also of the points not belonging to any tile.

5.1.3 The Local Deformation of the Crystal

Now we define a random function $V = V(\omega) \in L^2(\cup\mathcal{T}, \mathbb{R}^{d\times d})$ which measures the local deformation (rotation and scaling) of the crystal. Thereto, for $i \in I$, we partition the tile $\boxtimes^i$ into simplices $\boxtimes^{i,1}, \ldots, \boxtimes^{i,J_i}$. For any $\Box \in a + R \cdot \mathcal{N}_\varepsilon(\boxtimes^i)$ (with some $a \in \mathbb{R}^d$ and $R \in \mathrm{SO}(d)$) we define the bijective map

$$v_\Box : \Box \to \boxtimes^i \tag{5.2}$$

such that its restriction to $v_\Box^{-1}[\boxtimes^{i,j}]$ is affine linear for each $j \in \{1, \ldots, J_i\}$. Using these maps, we define

$$V : \cup\mathcal{T} \to \mathbb{R}^{d\times d}, \quad x \mapsto \nabla v_\Box(x) \quad \text{if } x \in \Box.$$

Note that the Jacobi matrix $\nabla v_\square$ is not well-defined on the boundary of the preimage of a simplex; but since these boundaries have zero Lebesgue measure, this is irrelevant. Then V is a piecewise constant function on $\cup\mathcal{T}$. Though it is locally defined as a derivative, it is, in general, globally not a derivative, since there may be defects with Burgers vectors.

5.1.4 The Hamiltonian

We assume that some local Hamiltonians

$$H^i_{\text{loc}} : \mathcal{N}_\varepsilon(\boxtimes^i) \to \mathbb{R}\,, \quad i \in I\,,$$

are given which are continuous and fulfil

$$\begin{aligned}&\exists\, c_1 > 0\, \exists\, c_2^{\mathbb{R}} \in \mathbb{R}\; \forall\, i \in I\, \forall\, \square \in \mathcal{N}_\varepsilon(\boxtimes^i) :\\ &\quad H^i_{\text{loc}}(\square) - H^i_{\text{loc}}(\boxtimes^i) \;\geq\; c_1 \|\operatorname{dist}(\nabla v_\square, \mathrm{SO}(d))\|^2_{L^2(\square)} + c_2^{\mathbb{R}}\big(\lambda(\square) - \lambda(\boxtimes^i)\big)\,. \end{aligned} \tag{5.3}$$

A tile $\square \in \mathcal{T}$ satisfies $\square = a + R \cdot \tilde{\square}$ for some $a \in \mathbb{R}^d$, $R \in \mathrm{SO}(d)$ and $\tilde{\square} \in \mathcal{N}_\varepsilon(\boxtimes^{\imath(\square)})$ by definition. If there are several choices of a, R and $\tilde{\square}$ possible, we choose one of them according to some fixed rule. We extend H^i_{loc}, $i \in I$, to $\mathcal{T}$ by setting $H^{\imath(\square)}_{\text{loc}}(\square) := H^{\imath(\square)}_{\text{loc}}(\tilde{\square})$.

Let further a quantity $S : \Omega \to \mathbb{R}$ be given which measures the number of surface points of the crystal in the following sense:

$$\exists\, c_3 > 0\, \forall\, N \in \mathbb{N}\, \forall\, \omega \in \Omega : \quad c_3|\partial\mathcal{P}| \;\leq\; S \quad \text{and} \quad \partial\mathcal{P} = \emptyset \;\Rightarrow\; S = 0\,. \tag{5.4}$$

Now we define the Hamiltonian

$$H_{\sigma,m,N}(\omega) \;:=\; \sum_{\square \in \mathcal{T}} H^{\imath(\square)}_{\text{loc}}(\square) + \sigma S - m|\mathcal{P}| \tag{5.5}$$

for $\sigma > 0$, $m \in \mathbb{R}$ and $N \in \mathbb{N}$. The first addend measures the local energy of the crystal caused by the perturbation of $\mathcal{M}$. The term σS represents the surface energy. Finally, m can be interpreted as a chemical potential. Using this Hamiltonian we define for $\beta > 0$, $\sigma > 0$, $m \in \mathbb{R}$ and $N \in \mathbb{N}$ the partition sum

$$Z_{\beta,\sigma,m,N} \;:=\; \int_\Omega e^{-\beta H_{\sigma,m,N}}\, d\mu \tag{5.6}$$

and the probability measure $P_{\beta,\sigma,m,N}$ via

$$\frac{dP_{\beta,\sigma,m,N}}{d\mu} \;:=\; \frac{1}{Z_{\beta,\sigma,m,N}} e^{-\beta H_{\sigma,m,N}}\,. \tag{5.7}$$

Let $E_{\beta,\sigma,m,N}$ denote the expectation with respect to $P_{\beta,\sigma,m,N}$.

Note that $P_{\beta,\sigma,m,N}$ is well-defined as $Z_{\beta,\sigma,m,N} \in (0,\infty)$, at least for large enough N. Indeed, the lower bound on $H_{\sigma,m,N}$ provided by Lemma 5.8 below implies $Z_{\beta,\sigma,m,N} < \infty$ (cf. the remark after that lemma). Furthermore, Lemma 5.11 below implies $Z_{\beta,\sigma,m,N} > 0$ for large enough N.

5.2 The Main Result

Now we are ready to state the main result.

Theorem 5.1. *There exist $m_0 \in \mathbb{R}$ and constants $c_4, c_5 > 0$ and $c_6^{\mathbb{R}} \in \mathbb{R}$ depending only on the model, but not on m, σ, β or N, such that the rotational symmetry of the crystal is broken in the following sense:*

$$\forall\, m \geq m_0 : \quad \lim_{\beta\to\infty} \limsup_{N\to\infty} \sup_{\sigma \geq \sigma_0(N,m)} E_{\beta,\sigma,m,N}\left[\inf_{R\in \mathrm{SO}(d)} \frac{1}{|\mathcal{T}|} \sum_{\square\in\mathcal{T}} \|V - R\|^2_{L^2(\square)} \right] = 0$$

where $\sigma_0(N,m) := c_4 N^2 + c_5 m + c_6^{\mathbb{R}}$.

The main constraint of this theorem is that the estimate is not uniform in the size of the box since σ_0 depends on N. Thus it does not carry over to infinite-volume limits. The reason for that N-dependency lies in the scaling behaviour of the constants in Theorem 4.1 as stated in Lemma 4.4. It is not possible to get better results using the chosen method.

Another constraint is that we assumed or rather conditioned on the event that the size of the crystal is comparable to the box size, i.e. $|\mathcal{T}| \geq c_0 N^d$. Whether this event has large probability is a different topic and not discussed in this thesis. But one might expect that its probability is large if the chemical potential m is large enough. Then more points are more likely and they should form more tiles, since otherwise they are surface points which are punished with $\sigma \geq m$.

Let us further remark, that the crystal consists only of enlarged perturbed tiles, i.e. the Lebesgue measure of any perturbed tile must not be smaller than the Lebesgue measure of the corresponding standard tile. Therefore, it is not possible to cover the whole box with more tiles than the standard tessellation would need. This may be considered as a hard-core condition. Furthermore, the whole perturbed tile must be ε-close to a standard tile. For instance, the postulate that only the edge lengths are close to the corresponding standard edge lengths might not be enough.

Moreover, we assume in the definition of $\mathcal{T}$ that each defect has a minimum size: non-adjacent tiles must have distance larger than 3ρ. This condition is crucial to extend V into the defects.

We also assume by definition that the crystal is connected. This assumption is necessary. If the crystal consists of two components, for example, there is no reason why one could use the same rotation R for both components. Indeed, the second component could be a rotated copy of the first one.

Finally, we equipped the box Λ_N with periodic boundary conditions. This has in particular the advantage that configurations without defects have no boundary, which is a technical relaxation, especially in Lemma 5.11. Otherwise, the periodic boundary is not essentially used.

Despite these constraints, especially the non-uniformity in N, Theorem 5.1 has the feature that it handles almost all kinds of defects, including unbounded and dislocation defects. Up to the author's knowledge, it is the first result on spontaneous symmetry breaking allowing such general defects.

5.3 Proof of the Main Result

Before we start the proof, we give an overview. Generally, we prove Theorem 5.1 using more or less the same approach as Heydenreich, Merkl and Rolles used in [HMR-13]. But the implementation of that approach is different.

One main difference is that we work directly on the level of the derivatives: Indeed V is matrix-valued and locally the derivative of a function $v_\square$. But globally, V need not be any derivative. Moreover, $v_\square$ is the inverse of the corresponding function in [HMR-13]. This is due to the fact that there is no reference lattice.

First we extend the function V into the defects in Subsection 5.3.1. Thereto we use a tube-neighbourhood of $\cup\mathcal{T}$. This extension is different to the extension in [HMR-13] since we consider different kinds of defects. In Subsection 5.3.2 we define the standard configuration and estimate the cardinality of some subsets of $\mathcal{P}$ and $\mathcal{T}$; this section has no counterpart in [HMR-13]. Afterwards, in Subsection 5.3.3, we prove an estimate for the Hamiltonian, which is an analogue to [HMR-13, Lemma 3.2]. Though its proof is different, it uses the same general idea, namely to apply a rigidity estimate. In Subsection 5.3.4 a lower bound for the partition sum is proven, which is used in Subsection 5.3.5 to receive an upper bound for the internal energy. The proofs of these results, which are analogues to [HMR-13, Lemma 3.1] and [HMR-13, Lemma 3.2], respectively, use ideas from their proofs. Finally, in Subsection 5.3.6, we prove a corollary which states the main result in different forms and also implies Theorem 5.1.

In the following we need quite a lot different constants. Unless explicitly stated, they are all uniform constants. Almost all of them depend on the model, i.e. on the tessellation, the local Hamiltonians, the surface measure S or on the constants $\varepsilon, \rho, c_0, c_1, c_2^{\mathbb{R}}, c_3$. But they are independent of m, σ, β, N and ω.

The constants in the lemmas and in the proofs are numbered separately. The constants in the lemmas are needed globally. Though we need the constants in the proofs only locally, they are numbered in ascending order to avoid confusion. Most of the constants are positive, but some can be any real number. In that case the constant has a little $^{\mathbb{R}}$ as superscript.

5.3.1 Extension into the Defects

First we want to extend the random function $V = V(\omega) \in L^2(\cup\mathcal{T}, \mathbb{R}^{d\times d})$, which measures the local deformation of the crystal, into the defects. We receive a random function also denoted by $V = V(\omega)$ with $V \in L^2(\Lambda_N, \mathbb{R}^{d\times d})$ and $dV \in L^p(\Lambda_N)$, $p \geq 1$.

We define a ρ-tube-neighbourhood $\partial^{\overline{0\rho}}\cup\mathcal{T}$ of $\cup\mathcal{T}$ using a homeomorphism

$$g = (g_\partial, g_t) : \partial^{\overline{0\rho}}\cup\mathcal{T} \to \partial\cup\mathcal{T} \times [0,1]$$

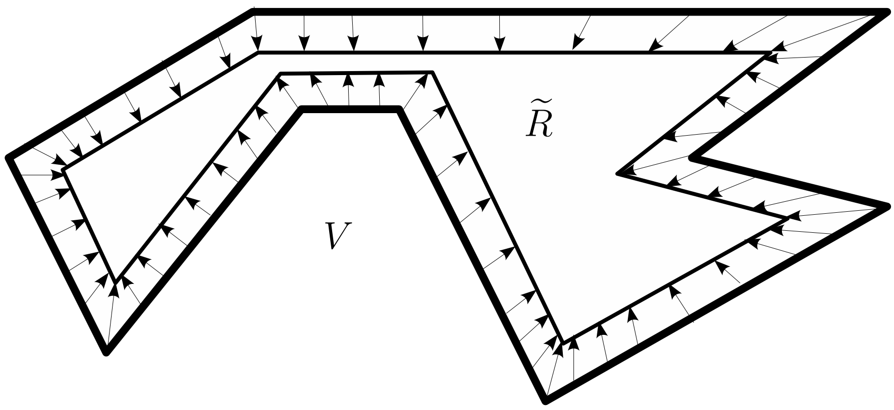


Figure 5.4: A defect (with arrows or hatched) of the crystal (white, outside), the ρ-tube-neighbourhood (with arrows) and the vector field w (the arrows)

such that $\partial^{\overline{0\rho}} \cup \mathcal{T} \subseteq \Lambda_N \setminus \mathrm{int}(\cup \mathcal{T})$, $g(x) = (x, 0)$ for all $x \in \partial \cup \mathcal{T}$ and such that dg_t exists and is uniformly bounded. Though not formally required, one can imagine $\partial^{\overline{0\rho}} \cup \mathcal{T}$ as the set of points whose distance from $\cup \mathcal{T}$ is at most ρ. Then g is some parametrisation of this set. This is also the reason for the notation. The proof of the existence of such a homeomorphism is given in Lemma 5.2 below. The main ingredient is a vector field w defined on $\partial \cup \mathcal{T}$, which exists since the distance of two disjoint tiles is greater than 3ρ by the definition of $\mathcal{T}$.

This construction is schematically drawn in Figure 5.4. The crystal is the white area outside and the defect consists of the hatched area and of the area with arrows. The latter one is the ρ-tube-neighbourhood $\partial^{\overline{0\rho}} \cup \mathcal{T}$. We will extend the function V, which is already defined in the white area, into the defects by setting it constant inside the hatched area and by interpolating inside the area with arrows.

In order to extend V, we choose a rotation $\widetilde{R} = \widetilde{R}(\omega) \in \mathrm{SO}(d)$ uniformly at random, independently of $\mathcal{P}$. We could also use a fixed rotation; but if it is chosen uniformly at random, the random variable V is rotational invariant. Moreover, let $\tilde{V}^n : \cup \mathcal{T} \to \mathbb{R}^{d \times d}$, $n \in \mathbb{N}$, be smooth functions which converge to V on $\cup \mathcal{T}$. First we extend $\tilde{V}^n$ to V^n as follows:

$$V^n(x) := \begin{cases} \tilde{V}^n(x) & \text{if } x \in \cup \mathcal{T} \\ (1 - g_t(x)) \tilde{V}^n(g_\partial(x)) + g_t(x) \widetilde{R} & \text{if } x \in \partial^{\overline{0\rho}} \cup \mathcal{T} \\ \widetilde{R} & \text{if } x \in \Lambda_N \setminus \left(\cup \mathcal{T} \cup \partial^{\overline{0\rho}} \cup \mathcal{T} \right). \end{cases}$$

Finally, we define V as the L^2-limit of V^n. This limit exists and is independent of the choice of the sequence $\tilde{V}^n$. Moreover, Lemma 5.3 below implies that $dV \in L^p(\Lambda_N)$, $p \geq 1$.

Now we prove the existence of the homeomorphism g.

Lemma 5.2. *There exists a constant $c_7 > 0$ such that for all $N \in \mathbb{N}$ and $\omega \in \Omega$, there exists a Lipschitz-continuous homeomorphism*

$$g = (g_\partial, g_t) : \partial^{\overline{0\rho}} \cup \mathcal{T} \to \partial \cup \mathcal{T} \times [0,1]$$

with Lipschitz-continuous inverse such that first $\partial^{\overline{0\rho}} \cup \mathcal{T} \subseteq \Lambda_N \setminus \mathrm{int}(\cup \mathcal{T})$, second $g(x) = (x, 0)$ for all $x \in \partial \cup \mathcal{T}$ and finally dg_t exists with $|dg_t| \le c_7$.

Proof. For any $z \in \mathbb{R}^d \setminus 0$, we can decompose a vector $w \in \mathbb{R}^d$ into

$$w = w_{\perp z} + w_{\| z}$$

where $w_{\| z}$ is the orthogonal projection of w onto $z\mathbb{R}$ and $w_{\perp z} := w - w_{\| z}$. This decomposition is linear in w.

In order to construct the homeomorphism, we will define a vector field $w : \partial \cup \mathcal{T} \to \mathbb{R}^d$. The boundary of $\cup \mathcal{T}$ is Lipschitz as it consists of $(d-1)$-dimensional polytopes. Thus there exist open sets $W_j \subset \mathbb{R}^d$ covering $\partial \cup \mathcal{T}$, open sets $\tilde{U}_j \subset \mathbb{R}^{d-1}$ and compatible Lipschitz continuous bijective maps $h_j : (-2,2) \times \tilde{U}_j \to W_j$ mapping $\{0\} \times \tilde{U}_j$ to $\partial \cup \mathcal{T}$, $(-2,0) \times \tilde{U}_j$ to $\mathrm{int}(\cup \mathcal{T})$ and $(0,2) \times \tilde{U}_j$ to $\Lambda_N \setminus \cup \mathcal{T}$, $j \in J$. We can further assume that for all $x, y \in \partial \cup \mathcal{T}$ with $|x - y| \le 2\rho$, there exists $j \in J$ with $x, y \in W_j$, because $|x - y| \le 2\rho$ implies that x and y belong to the same tile or to adjacent tiles (the distance of non-adjacent tiles is greater than 3ρ by the definition of $\mathcal{T}$). Note that the angles between two adjacent polytopes are uniformly bounded away from zero. Indeed, if the defect is locally due to a missing tile, this follows from the fact that all tiles are ε-perturbations of the given tessellation; and if the defect is locally an inserted wedge, i.e. it comes locally from a slit, then the angle of that wedge is bounded away from zero by condition (iv) in the definition of $\mathcal{T}$. Therefore the Lipschitz constants of $(h_j)_{j \in J}$ can be uniformly bounded for all $N \in \mathbb{N}$ and $\omega \in \Omega$.

We define the vector field $\tilde{w} : \partial \cup \mathcal{T} \to \mathbb{R}^d$ by pushing the field $u(z) = (1, 0, \ldots, 0)$, $z \in \{0\} \times \tilde{U}_j$ forward with h_j, i.e. $\tilde{w}(x) := h_j[(1, 0, \ldots, 0) + h_j^{-1}(x)] - x$ for suitable j, $x \in \partial \cup \mathcal{T}$. Then $\tilde{w}$ is uniformly Lipschitz and $|\tilde{w}|$ is uniformly bounded away from zero and infinity (in ω and x). Now we scale $\tilde{w}$ to lower its Lipschitz constant and size. This yields a vector field $w : \partial \cup \mathcal{T} \to \mathbb{R}^d$ such that for all $x, y \in \partial \cup \mathcal{T}$:

(i) $x + tw(x) \notin \cup \mathcal{T}$ for all $t \in (0, 1]$

(ii) $|w(x) - w(y)| \le c_{27}|x - y|$

(iii) $c_{28} \le |w(x)| \le \rho$

(iv) $|w(y)_{\perp (y-x)}| \ge |w(y)|/c_{26}$ if $0 < |x - y| \le 2\rho$.

for some universal constants $c_{26}, c_{27}, c_{28} > 0$ satisfying

$$(1 + c_{26})c_{27} < 1 \qquad \text{and} \qquad \rho + \sqrt{2}c_{27} < 1\,. \tag{5.8}$$

Condition (iv), which is scale-invariant, already holds for $\tilde{w}$: since $|x - y| \le 2\rho$ implies $x, y \in U_j$ for some j, we can use $u(h_j^{-1}(y)) = (1, 0, \ldots, 0) \perp h_j^{-1}(x) - h_j^{-1}(y)$ and the Lipschitz property of h_j to derive (iv). Conditions (iii) and (ii) and Equation

(5.8) are be fulfilled by scaling (c_{26} and $\rho < \rho_{\max} \le 1$ are already fixed). Condition (i) follows from (iii) since the distance between two disjoint tiles is greater than 3ρ by the definition of $\mathcal{T}$.

Using the vector field w, we define the function

$$\begin{aligned} f : \partial\cup\mathcal{T} \times [0,1] &\to \Lambda_N \\ (x,t) &\mapsto x + tw(x)\,, \end{aligned}$$

which will be the inverse of the homeomorphism g. It is Lipschitz-continuous since

$$\big|f(x,t)-f(y,s)\big| \;=\; \big|(x-y)+t(w(x)-w(y))+(t-s)w(y)\big| \;\le\; (1+c_{27})|x-y|+\rho|t-s|$$

by properties (ii) and (iii) of w.

We will also derive a reverse Lipschitz condition to conclude that f is injective and its inverse is also Lipschitz-continuous. Thereto let $x, y \in \partial\cup\mathcal{T}$ and $t, s \in [0,1]$. First we assume $x \neq y$. We estimate using the triangle inequality and the Lipschitz continuity of w

$$\begin{aligned} \big|x-y+(t-s)w(y)\big| &\le \big|x+tw(x)-y-sw(y)\big| + t\big|w(y)-w(x)\big| \\ &\le \big|f(x,t)-f(y,s)\big| + c_{27}|x-y|\,. \end{aligned} \tag{5.9}$$

Pythagoras' Theorem yields that

$$\begin{aligned} \big|x-y+(t-s)w(y)\big|^2 &= \big|x-y+(t-s)w(y)_{\|(x-y)}\big|^2 + \big|(t-s)w(y)_{\perp(x-y)}\big|^2 \\ &\ge \big((1-\rho)|x-y|\big)^2 + \big|(t-s)w(y)_{\perp(x-y)}\big|^2 \end{aligned} \tag{5.10}$$

since $|(t-s)w(y)_{\|(x-y)}| \le |w(y)| \le \rho$.

The inequality $\sqrt{2}\sqrt{a^2+b^2} \ge (a+b)$ yields (5.10) without the squares, but with an additional $\sqrt{2}$ on the left hand side. Combing this with (5.9) yields

$$\begin{aligned} \sqrt{2}\big|f(x,t)-f(y,s)\big| &\ge (1-\rho)|x-y| + \big|(t-s)w(y)_{\perp(x-y)}\big| - \sqrt{2}c_{27}|x-y| \\ &= \big(1-(\rho+\sqrt{2}c_{27})\big)\big|x-y\big| + \big|w(y)_{\perp(x-y)}\big|\big|t-s\big|\,. \end{aligned} \tag{5.11}$$

Note that $\rho + \sqrt{2}c_{27} < 1$ by (5.8).

Now if $|x-y| \le 2\rho$, then $|w(y)_{\perp(x-y)}| \ge |w(y)|/c_{26} \ge c_{28}/c_{26}$. Otherwise $|w(y)_{\perp(x-y)}| \ge 0$ and $\frac{1}{2}|x-y| \ge \rho \ge \rho|t-s|$. Therefore, in both cases (5.11) implies

$$\big|f(x,t)-f(y,s)\big| \;\ge\; c_{29}\big(|x-y|+|t-s|\big) \tag{5.12}$$

for some constant $c_{29} > 0$. Now we consider the case $x = y$. Then

$$\big|f(x,t)-f(y,s)\big| \;=\; \big|x+tw(x)-y-sw(y)\big| \;=\; |t-s||w(x)| \ge c_{28}|t-s|$$

by property (iii). Thus (5.12) also holds in that case.

Inequality (5.12) implies that f is indeed injective. Moreover, property (i) implies $\partial^{\overline{0\rho}}\cup\mathcal{T} := \operatorname{im} f \subseteq \Lambda_N \setminus \operatorname{int}(\cup\mathcal{T})$. We define

$$g : \partial^{\overline{0\rho}}\cup\mathcal{T} \to \partial\cup\mathcal{T} \times [0,1], \quad z \mapsto f^{-1}(z)$$

as the inverse of f. Then $g(x) = (x, 0)$ for all $x \in \cup\mathcal{T}$ holds by definition. Furthermore, (5.12) implies that g is Lipschitz continuous. Thus the existence of dg_t as well as the bound $|dg_t| \leq c_7$ for some $c_7 > 0$ follow. □

Finally in this section, we prove a bound of $\operatorname{dist}(V, \mathrm{SO}(d))$ and dV.

Lemma 5.3. *There exists a constant $c_8 > 0$ such that for all $N \in \mathbb{N}$ and $\omega \in \Omega$ and λ-almost all $x \in \Lambda_N$*

$$\operatorname{dist}(V(x), \mathrm{SO}(d))^2 \leq c_8 \quad \textit{and} \quad |dV(x)| \leq c_8 .$$

Proof. First we note that V and V^n are uniformly bounded on $\cup\mathcal{T}$ and therefore also on $\partial\cup\mathcal{T}$ since any tile $\square \in \mathcal{T}$ is, up to translation and rotation, ε-close to $\boxtimes^{\imath(\square)}$. Moreover, $\widetilde{R}$ is uniformly bounded since $\mathrm{SO}(d)$ is compact. Thus V^n and therefore V is uniformly bounded on the whole Λ_N, which implies the first inequality.

For the second inequality, we first note that since $V\restriction_{\cup\mathcal{T}}$ is locally the derivative of a continuous piecewise affine linear function, we could also choose $\tilde{V}^n$ locally as a derivative. Therefore $dV = 0$ on $\cup\mathcal{T}$. Moreover, $dV = 0$ on $\Lambda_N \setminus \left(\cup\mathcal{T} \cup \partial^{\overline{0\rho}}\cup\mathcal{T}\right)$ since $\widetilde{R}$ is constant. Finally, we calculate for $x \in \partial^{\overline{0\rho}}\cup\mathcal{T}$

$$\begin{aligned} dV^n(x) &= (1 - g_t(x))d\tilde{V}^n(g_\partial(x)) - dg_t(x) \wedge \tilde{V}^n(g_\partial(x)) + g_t(x)d\widetilde{R} + dg_t(x) \wedge \widetilde{R} \\ &= -dg_t(x) \wedge \left(V^n(g_\partial(x)) - \widetilde{R}\right) \end{aligned}$$

since $d\tilde{V}^n = 0$ on $\cup\mathcal{T}$ since $\tilde{V}^n$ is locally a derivative. Since $|dg_t| \leq c_7$ by Lemma 5.2, since V^n and $\widetilde{R}$ are uniformly bounded and since $V^n \to V$, the second inequality follows. □

5.3.2 Cardinality of Subsets of $\mathcal{P}$ and $\mathcal{T}$

In this section, we give some definitions and some lemmas, which estimate the cardinality of several subsets of $\mathcal{P}$ and $\mathcal{T}$.

First we define the *standard configuration* $\varphi \in \Omega$ with points $\mathcal{Q}$ and tiles $\mathcal{U}$ as a fixed element of Ω such that the crystal is exactly the tessellation $\mathcal{M}$. More precisely, using the notation $\mathcal{V}(\mathcal{M}\restriction_{\Lambda_N})$ for the vertices of $\mathcal{M}$ inside Λ_N, we require

$$\mathcal{Q} := \mathcal{P}(\varphi) = \mathcal{V}(\mathcal{M}\restriction_{\Lambda_N}) \quad \text{and thus} \quad \mathcal{U} := \mathcal{T}(\varphi) = \mathcal{M}\restriction_{\Lambda_N} .$$

The choice of $\rho < \rho_{\max}$ ensures the last equation and $\varphi \in \Omega$ (if N is large enough, depending on c_0).

We will need some subsets of $\mathcal{T}$ and $\mathcal{P}$. We define the set of boundary tiles by

$$\partial\mathcal{T} := \left\{\square \in \mathcal{T} : \square \cap \partial\cup\mathcal{T} \neq \emptyset\right\}$$

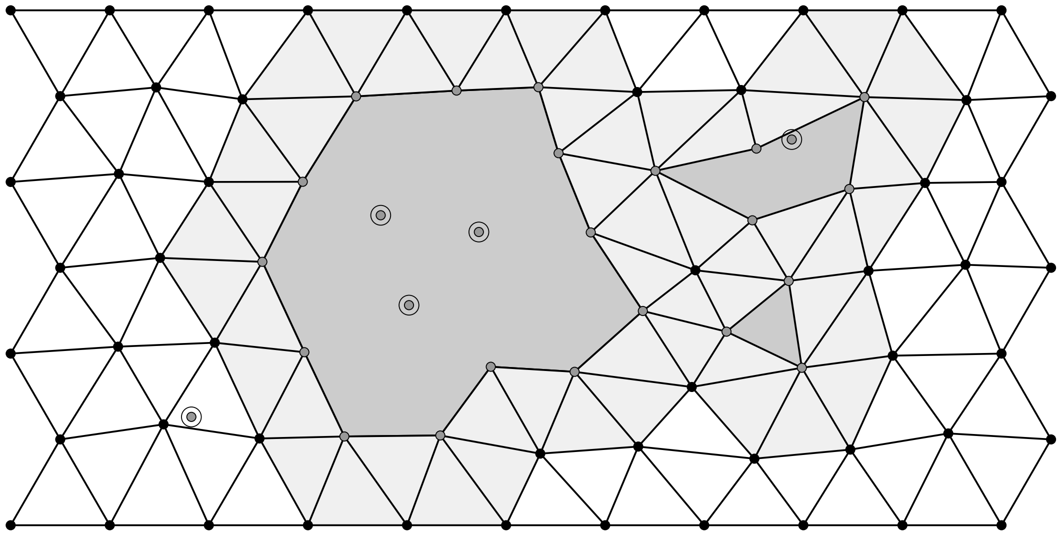


Figure 5.5: The boundary tiles (light grey), the surface points (grey) and the exterior points (with circle) of a crystal with defects (dark grey area)

and for $i \in I$ the set

$$\mathcal{T}^i := \{\Box \in \mathcal{T} : \imath(\Box) = i\},$$

which consists of all tiles of type i (recall that $\imath(\Box)$ denotes the type of $\Box$). Obviously, $\mathcal{U}^i$ denotes the set of all tiles of type i in $\mathcal{U}$, $i \in I$. Let us further recall that we already defined the surface points $\partial\mathcal{P}$ in (5.1) as follows:

$$\partial\mathcal{P} := \{x \in \mathcal{P} \mid x \in \partial\cup\mathcal{T} \text{ or } x \notin \mathcal{V}(\mathcal{T})\},$$

where $\partial\cup\mathcal{T}$ denotes the topological boundary and $\mathcal{V}(\mathcal{T})$ is the set of points of $\mathcal{P}$, which are vertices of any tile $\Box \in \mathcal{T}$. Furthermore, we need the notation

$$\mathcal{P}^{\text{ext}} := \{x \in \mathcal{P} \mid x \notin \mathcal{V}(\mathcal{T})\}$$

for the exterior points. Note that the exterior points, which are not contained in any perturbed tile of $\mathcal{T}$, are contained in the set of surface points. Note further that the standard configuration has empty boundary, i.e. $\partial\mathcal{Q} = \emptyset$ and $\partial\mathcal{U} = \emptyset$.

These sets are illustrated in Figure 5.5. It shows the example of a crystal used in Section 5.1.2. The defects are shaded in dark grey. The boundary tiles are light grey shaded. All surface points are drawn in grey. The five surface points which also are exterior points are marked with a circle. Note that one of the exterior points is inside the crystal but is not a vertex of any tile.

Similarly to the tile types, we may also partition the vertices $\mathcal{V}(\mathcal{M})$ of $\mathcal{M}$ into types $j \in J$, depending on their adjacent tiles. The assignment of the types to the tiles and vertices shall in particular imply that, for all $i, l \in I, j \in J$ the quantities

$$b_{i,l} := \sum_{\substack{\tilde{\boxtimes} \in \mathcal{M} \\ \imath(\tilde{\boxtimes}) = l}} \mathbb{1}_{\boxtimes^i \cap \tilde{\boxtimes} \neq \emptyset}\,, \qquad e_{ij} := \sum_{\substack{x \in \mathcal{V}(\mathcal{M}) \\ \jmath(x) = j}} \mathbb{1}_{x \in \boxtimes^i}\,, \qquad f_{ij} := \sum_{\substack{\boxtimes \in \mathcal{M} \\ \imath(\boxtimes) = i}} \mathbb{1}_{x^j \in \boxtimes} \tag{5.13}$$

are well-defined, finite and independent of the choice of $\boxtimes^i$ of type i and $x^j \in \mathcal{V}(\mathcal{M})$ of type j, respectively. These quantities are interpreted as follows: $b_{i,l}$ denotes the number of neighbouring tiles of type l to a tile of type i, and e_{ij} denotes the number of adjacent vertices of type j to a tile of type i, and finally f_{ij} denotes the number of adjacent tiles of type i to a vertex of type j.

In fact, we need the different vertex types only in this section; therefore the letter j may denote various index variables later. But the letter i will only be used for a tile type.

The following lemma shows that the number of tiles of type i is bounded by the number of such tiles in the standard configuration, up to an error in terms of the number of boundary tiles.

Lemma 5.4. *There exists a constant $c_9 > 0$ such that for all $N \in \mathbb{N}$, $\omega \in \Omega$ and $i \in I$ the following inequality holds:*

$$|\mathcal{T}^i| \leq |\mathcal{U}^i| + c_9\,|\partial\mathcal{T}|\,.$$

Proof. First we show that there exist constants $c_{i,l} > 0$, $i, l \in I$, and $c_9 > 0$ such that for all $N \in \mathbb{N}$ and $\omega \in \Omega$

$$\big|c_{i,l}|\mathcal{T}^l| - |\mathcal{T}^i|\big| \leq c_9\,|\partial\mathcal{T}|\,. \tag{5.14}$$

Let $i, l \in I$. We define the quantity

$$A := \sum_{\square \in \mathcal{T}^i} \sum_{\tilde{\square} \in \mathcal{T}^l} \mathbb{1}_{\square \cap \tilde{\square} \neq \emptyset}\,.$$

By the definition of $b_{i,l}$ in equation (5.13), it follows that, for all $\square \in \mathcal{T}^i$,

$$0 \leq \sum_{\tilde{\square} \in \mathcal{T}^l} \mathbb{1}_{\square \cap \tilde{\square} \neq \emptyset} \leq b_{i,l} \quad \text{and even} \quad \sum_{\tilde{\square} \in \mathcal{T}^l} \mathbb{1}_{\square \cap \tilde{\square} \neq \emptyset} = b_{i,l} \quad \text{if } \square \in \mathcal{T}^i \setminus \partial\mathcal{T}\,.$$

Summing over all $\square \in \mathcal{T}^i$ yields

$$0 \leq b_{i,l}|\mathcal{T}^i| - A = \sum_{\square \in \mathcal{T}^i} \Big(b_{i,l} - \sum_{\tilde{\square} \in \mathcal{T}^l} \mathbb{1}_{\square \cap \tilde{\square} \neq \emptyset}\Big) \leq b_{i,l}|\partial\mathcal{T} \cap \mathcal{T}^i| \leq b_{i,l}|\partial\mathcal{T}|\,.$$

Analogously, it follows that

$$-b_{l,i}|\partial\mathcal{T}| \leq A - b_{l,i}|\mathcal{T}^l| \leq 0\,.$$

Adding these two inequalities, we get

$$-b_{l,i}|\partial\mathcal{T}| \le b_{i,l}|\mathcal{T}^i| - b_{l,i}|\mathcal{T}^l| \le b_{i,l}|\partial\mathcal{T}| \tag{5.15}$$

Now we observe that either $b_{i,l} = 0 = b_{l,i}$ or $b_{i,l} \neq 0 \wedge b_{l,i} \neq 0$ since $b_{i,l}$ counts the tiles of type l adjacent to a tile of type i. In the latter case, we can define $c_{i,l} := b_{l,i}/b_{i,l} \in (0,\infty)$ and receive

$$\big||\mathcal{T}^i| - c_{i,l}|\mathcal{T}^l|\big| \le \max\{1, c_{i,l}\}\,|\partial\mathcal{T}| \tag{5.16}$$

by equation (5.15).

In the general case, there is a sequence $i = i_0, i_1, \ldots, i_n = l$ with some $n \le |I|$ such that $b_{i_{k-1},i_k} \neq 0$ for all $k \in \{1,\ldots,n\}$ since the tessellation $\mathcal{M}$ is connected. Therefore we can define $c_{i,l} := \prod_{k=1}^n c_{i_{k-1},i_k} \in (0,\infty)$. Using a telescope sum it follows that

$$\begin{aligned}
\big||\mathcal{T}^i| - c_{i,l}|\mathcal{T}^l|\big| &= \Big||\mathcal{T}^i| - \prod_{k=1}^n c_{i_{k-1},i_k}\,|\mathcal{T}^l|\Big| \\
&= \Big|\sum_{j=1}^n \Big(\prod_{k=1}^{j-1} c_{i_{k-1},i_k}\,|\mathcal{T}^{i_{j-1}}| - \prod_{k=1}^{j} c_{i_{k-1},i_k}\,|\mathcal{T}^{i_j}|\Big)\Big| \\
&\le \sum_{j=1}^n \prod_{k=1}^{j-1} c_{i_{k-1},i_k} \cdot \Big||\mathcal{T}^{i_{j-1}}| - c_{i_{j-1},i_j}|\mathcal{T}^{i_j}|\Big| \\
&\overset{(5.16)}{\le} \sum_{j=1}^n \prod_{k=1}^{j-1} c_{i_{k-1},i_k} \max\{1, c_{i_{j-1},i_j}\}\,|\partial\mathcal{T}| \le c_9\,|\partial\mathcal{T}|\,,
\end{aligned}$$

where c_9 is the supremum of the last sum over all possible choices of the sequence $i_0, i_1, \ldots, i_n$. For the last line, we used the already covered case $b_{i_{j-1},i_j} \neq 0$. Thus claim (5.14) follows.

Using $\lambda(\square) \ge \lambda(\boxtimes^i)$ for all $i \in I$ and $\square \in \mathcal{N}_\varepsilon(\boxtimes^i)$, we estimate

$$\sum_{i\in I} |\mathcal{T}^i|\lambda(\boxtimes^i) = \sum_{\square\in\mathcal{T}} \lambda(\boxtimes^{\imath(\square)}) \le \sum_{\square\in\mathcal{T}} \lambda(\square) \le \lambda(\Lambda_N) = \lambda(\cup\mathcal{U}) = \sum_{i\in I} |\mathcal{U}^i|\lambda(\boxtimes^i)$$

since the standard configuration covers the whole box with standard tiles. Therefore there exists $i_0 = i_0(\omega) \in I$ with $|\mathcal{T}^{i_0}| \le |\mathcal{U}^{i_0}|$.

Let $i \in I$. Using claim (5.14) it follows that

$$|\mathcal{T}^i| \le c_{i,i_0}|\mathcal{T}^{i_0}| + c_9|\partial\mathcal{T}| \le c_{i,i_0}|\mathcal{U}^{i_0}| + c_9|\partial\mathcal{T}| = |\mathcal{U}^i| + c_9|\partial\mathcal{T}|\,.$$

The last equality follows again from claim (5.14), applied to $\mathcal{U}$, since $\partial\mathcal{U} = \emptyset$. $\square$

In the next two lemmas, we use the relation $\asymp$ to indicate that the quotient of the left and of the right is uniformly in $N \in \mathbb{N}$ and $\omega \in \Omega$ bounded away from zero and infinity. But we also state the inequalities we need in the sequel explicitly. First we show that different measurements of the boundary have approximately equal size.

Lemma 5.5. *There are constants $\gamma_i > 0$, $i \in I$, such that*

$$|\partial\mathcal{P} \setminus \mathcal{P}^{ext}| \asymp |\mathcal{P} \setminus \mathcal{P}^{ext}| - \sum_{i\in I} \gamma_i |\mathcal{T}^i| \asymp |\partial\mathcal{T}| \asymp \lambda(\overline{\partial^{0\rho}} \cup \mathcal{T}) .$$

In particular it is shown that there are constants $c_{10} > 0$, $c_{11} > 0$ and $\gamma_i > 0$, $i \in I$, such that for all $N \in \mathbb{N}$ and $\omega \in \Omega$

(a) $|\partial\mathcal{P}| \geq |\mathcal{P}| - \sum_{i\in I} \gamma_i |\mathcal{T}^i| \geq 0$,

(b) $|\partial\mathcal{T}| \leq c_{10}\big(|\mathcal{P}| - \sum_{i\in I} \gamma_i |\mathcal{T}^i|\big)$ *and*

(c) $\lambda(\overline{\partial^{0\rho}} \cup \mathcal{T}) \leq c_{11} |\partial\mathcal{T}|$.

Proof. First we show $|\partial\mathcal{P} \setminus \mathcal{P}^{\text{ext}}| \asymp |\mathcal{P} \setminus \mathcal{P}^{\text{ext}}| - \sum_{i\in I} \gamma_i |\mathcal{T}^i|$. Thereto, we partition all points $\mathcal{P}$ into the points $\mathcal{P}^j$ of type $j \in J$ and into the exterior points $\mathcal{P}^{\text{ext}}$. Of course, a point in $\mathcal{P} \setminus \mathcal{P}^{\text{ext}}$ inherits its type from the corresponding point in $\mathcal{M}$. Since $e_{ij} = \sum_{x\in\mathcal{P}^j} \mathbb{1}_{x\in\square}$ is the number of vertices of type j adjacent to any tile $\square \in \mathcal{T}$ of type $\imath(\square) = i$ (including the boundary tiles), see equation (5.13), it follows that

$$e_{ij}\, |\mathcal{T}^i| = \sum_{x\in\mathcal{P}^j} \sum_{\square\in\mathcal{T}^i} \mathbb{1}_{x\in\square} . \tag{5.17}$$

We observe that $e_{ij} = 0$ iff $f_{ij} = 0$ and define

$$\gamma_i := \sum_{j\in J} \frac{1}{|I_j|} \frac{e_{ij}}{f_{ij}} \mathbb{1}_{f_{ij}\neq 0} \tag{5.18}$$

with $I_j := \{i \in I \mid f_{ij} \neq 0\}$. Note that $I_j \neq \emptyset$. Therefore

$$\begin{aligned} |\mathcal{P}| - |\mathcal{P}^{\text{ext}}| - \sum_{i\in I} \gamma_i |\mathcal{T}^i| &= \sum_{j\in J} |\mathcal{P}^j| - \sum_{i\in I} \sum_{j\in J} \frac{1}{|I_j|} \frac{e_{ij}}{f_{ij}} \mathbb{1}_{f_{ij}\neq 0} |\mathcal{T}^i| \\ &= \sum_{j\in J} \Big(|\mathcal{P}^j| - \sum_{i\in I_j} \frac{1}{|I_j|} \frac{e_{ij}}{f_{ij}} |\mathcal{T}^i| \Big) = \sum_{j\in J} \sum_{i\in I_j} \frac{1}{|I_j|} \Big(|\mathcal{P}^j| - \frac{e_{ij}}{f_{ij}} |\mathcal{T}^i| \Big) \\ &\overset{(5.17)}{=} \sum_{j\in J} \sum_{i\in I_j} \frac{1}{|I_j|} \sum_{x\in\mathcal{P}^j} \Big(1 - \frac{1}{f_{ij}} \sum_{\square\in\mathcal{T}^i} \mathbb{1}_{x\in\square} \Big) . \end{aligned} \tag{5.19}$$

Now we examine the expression

$$A_i(x) := 1 - \frac{1}{f_{ij}} \sum_{\square\in\mathcal{T}^i} \mathbb{1}_{x\in\square}$$

for $x \in \mathcal{P}^j$. Since f_{ij} counts number of tiles of type i adjacent to a vertex of type j, it follows that $A_i(x) = 0$ if $x \notin \partial\mathcal{P}$ and $A_i(x) \leq 1$ in general. Therefore we can continue (5.19) as follows:

$$|\mathcal{P} \setminus \mathcal{P}^{\text{ext}}| - \sum_{i\in I} \gamma_i |\mathcal{T}^i| \overset{(5.19)}{=} \sum_{j\in J} \sum_{i\in I_j} \frac{1}{|I_j|} \sum_{x\in\mathcal{P}^j} A_i(x) \leq \sum_{j\in J} \sum_{i\in I_j} \frac{1}{|I_j|} |\mathcal{P}^j \cap \partial\mathcal{P}| = |\partial\mathcal{P} \setminus \mathcal{P}^{\text{ext}}| ,$$

which is one of the two desired inequalities. By adding $|\mathcal{P}^{\text{ext}}|$, this also shows the main inequality of Assertion (a) since $\mathcal{P}^{\text{ext}} \subset \partial\mathcal{P}$; "$\geq 0$" follows from (b).

For the other inequality, we define $\mathcal{P}_*^j := \{x \in \mathcal{P}^j \mid \exists\, i \in I_j : \sum_{\Box \in \mathcal{T}^i} \mathbb{1}_{x\in\Box} < f_{ij}\}$. We observe that $x \in \mathcal{P}_*^j$ for some j if a tile is missing which should be adjacent to x. Thus $\mathcal{P}_*^j \subseteq \partial\mathcal{P}$. If $x \in \partial\mathcal{P} \setminus \bigcup_j \mathcal{P}_*^j$, then the defect at x is induced by a slit. In that case there is a vertex $y \in \partial\mathcal{P} \setminus \mathcal{P}^{\text{ext}}$ adjacent to x such that a tile is missing at y, i.e. $y \in \mathcal{P}_*^j$ for some j. Since the vertex degree is uniformly bounded, we conclude

$$\sum_{j\in J} |\mathcal{P}_*^j| \;\geq\; c_{30} |\partial\mathcal{P} \setminus \mathcal{P}^{\text{ext}}| \tag{5.20}$$

for some $c_{30} > 0$. For $x \in \mathcal{P}_*^j$, let $i_0(x)$ be the smallest $i \in I_j$ with $\sum_{\Box\in\mathcal{T}^i} \mathbb{1}_{x\in\Box} < f_{ij}$. It follows that

$$A_i(x) \geq |I| c_{31} \mathbb{1}_{i=i_0(x)}$$

for $x \in \mathcal{P}^j \cap \partial\mathcal{P}$ with $c_{31} := \frac{1}{|I|} \min\left\{\frac{1}{f_{ij}} \mid i \in I_j, j \in J\right\}$. Plugging this into (5.19) yields

$$\begin{aligned}
|\mathcal{P} \setminus \mathcal{P}^{\text{ext}}| - \sum_{i\in I} \gamma_i |\mathcal{T}^i| &= \sum_{j\in J}\sum_{i\in I_j} \frac{1}{|I_j|} \sum_{x\in\mathcal{P}^j} A_i(x) \;\geq\; \sum_{j\in J}\sum_{i\in I_j} \frac{1}{|I_j|} \sum_{x\in\mathcal{P}_*^j} |I| c_{31} \mathbb{1}_{i=i_0(x)} \\
&= c_{31} \sum_{j\in J} \frac{|I|}{|I_j|} \sum_{x\in\mathcal{P}_*^j} \sum_{i\in I_j} \mathbb{1}_{i=i_0(x)} = c_{31} \sum_{j\in J} \frac{|I|}{|I_j|} \sum_{x\in\mathcal{P}_*^j} 1 \\
&\geq c_{31} \sum_{j\in J} |\mathcal{P}_*^j| \;\geq\; c_{30} c_{31}\, |\partial\mathcal{P} \setminus \mathcal{P}^{\text{ext}}|\,,
\end{aligned} \tag{5.21}$$

as desired. We used (5.20) in the last step.

Second we show $|\partial\mathcal{P} \setminus \mathcal{P}^{\text{ext}}| \asymp |\partial\mathcal{T}|$. For all $x \in \partial\mathcal{P} \setminus \mathcal{P}^{\text{ext}}$ there exists at least one $\Box \in \partial\mathcal{T}$ with $x \in \Box$. Therefore

$$\begin{aligned}
|\partial\mathcal{P} \setminus \mathcal{P}^{\text{ext}}| \quad &\leq \quad \sum_{x\in\partial\mathcal{P}\setminus\mathcal{P}^{\text{ext}}} \sum_{\Box\in\partial\mathcal{T}} \mathbb{1}_{x\in\Box} \;\leq\; \sum_{i\in I} \sum_{\Box\in(\mathcal{T}^i\cap\partial\mathcal{T})} \sum_{j\in J} \sum_{x\in\mathcal{P}^j} \mathbb{1}_{x\in\Box} \\
&\overset{(5.13)}{=} \quad \sum_{i\in I} \sum_{\Box\in(\mathcal{T}^i\cap\partial\mathcal{T})} \sum_{j\in J} e_{ij} \;\leq\; |\partial\mathcal{T}| \max_{i\in I} \left\{ \textstyle\sum_{j\in J} e_{ij} \right\}.
\end{aligned}$$

Conversely, for each $\Box \in \partial\mathcal{T}$, there exists at least one $x \in \partial\mathcal{P} \setminus \mathcal{P}^{\text{ext}}$ with $x \in \Box$. Therefore

$$\begin{aligned}
|\partial\mathcal{T}| \quad &\leq \quad \sum_{\Box\in\partial\mathcal{T}} \sum_{x\in\partial\mathcal{P}\setminus\mathcal{P}^{\text{ext}}} \mathbb{1}_{x\in\Box} \;\leq\; \sum_{j\in J} \sum_{x\in(\mathcal{P}^j\cap\partial\mathcal{P})} \sum_{i\in I} \sum_{\Box\in\mathcal{T}^i} \mathbb{1}_{x\in\Box} \\
&\overset{(5.13)}{\leq} \quad \sum_{j\in J} \sum_{x\in(\mathcal{P}^j\cap\partial\mathcal{P})} \sum_{i\in I} f_{ij} \;\leq\; |\partial\mathcal{P} \setminus \mathcal{P}^{\text{ext}}| \max_{j\in J} \left\{ \textstyle\sum_{i\in I} f_{ij} \right\}.
\end{aligned}$$

This and (5.21) imply Assertion (b) with $c_{10} := \max_{j\in J} \sum_{i\in I} f_{ij}/(c_{30}c_{31})$.

Finally, we show $|\partial\mathcal{T}| \asymp \lambda(\partial^{\overline{0\rho}}\cup\mathcal{T})$. For a set $A \subset \mathbb{R}^d$, let $\mathcal{O}(A) = \lambda_{d-1}(\partial A)$ denote the surface area of A. Using the Lipschitz continuous homeomorphism g, we conclude that

$$\rho\,\mathcal{O}(\cup\mathcal{T}) \asymp \lambda(\partial^{\overline{0\rho}}\cup\mathcal{T}).$$

Moreover,

$$\mathcal{O}(\cup\mathcal{T}) \le \sum_{\square\in\partial\mathcal{T}} \mathcal{O}(\square) \le c_{32}\,|\partial\mathcal{T}|$$

for some constant $c_{32} > 0$ since the surface area of a tile is uniformly bounded. But for the other direction one has to be careful, since there may exists boundary tiles which do not have a face which is part of $\partial\cup\mathcal{T}$. But let $\partial^*\mathcal{T}$ denote the set of boundary tiles having a face which is contained in $\partial\cup\mathcal{T}$. Since for each tile $\square \in \partial\mathcal{T} \setminus \partial^*\mathcal{T}$ there exists a tile $\tilde{\square} \in \partial^*\mathcal{T}$ with $\square \cap \tilde{\square} \neq \emptyset$ and since each tile $\tilde{\square} \in \partial^*\mathcal{T}$ intersects at most $\max_{i,l\in I} b_{i,l}$ other tiles, there is a constant $c_{33} > 0$ such that $|\partial^*\mathcal{T}| \ge c_{33}|\partial\mathcal{T}|$. Since the area of a face of a tile is at least $c_{34} > 0$ (say),

$$\mathcal{O}(\cup\mathcal{T}) \ge c_{34}\,|\partial^*\mathcal{T}| \ge c_{34}c_{33}\,|\partial\mathcal{T}|$$

follows. Combining all three displayed formulas in this paragraph yields the claim, which in particular implies Assertion (c). □

Now we observe that the size of the crystal is comparable to the size of the box, where we can understand each size in two different senses.

Lemma 5.6. *It is true that*

$$|\mathcal{T}| \asymp |\mathcal{P} \setminus \mathcal{P}^{ext}| \asymp N^d \asymp \lambda(\Lambda_N).$$

In particular, there are constants c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, $c_{17} > 0$ such that for all $N \in \mathbb{N}$ and $\omega \in \Omega$

(d) $c_{12}\,N^d \le |\mathcal{P} \setminus \mathcal{P}^{ext}| \le c_{13}\,N^d$,

(e) $|\mathcal{T}| \le c_{14}\,N^d$,

(f) $|\mathcal{T}| \ge c_{15}\,\lambda(\Lambda_N)$ *and* $\lambda(\Lambda_N) \ge c_{16}\,|\mathcal{T}|$,

(g) $\lambda(\Lambda_N) = c_{17}N^d$.

Proof. Since Λ_N consists of N^d copies of the box B_0, Assertion (g) follows with $c_{17} = \lambda(B_0) > 0$.

Now note that $c_0N^d \le |\mathcal{T}|$ holds by the definition of Ω. Moreover, $\lambda(\square) \ge \lambda(\boxtimes^{\imath(\square)})$ for all $\square \in \mathcal{T}$ implies

$$\min_{i\in I} \lambda(\boxtimes^i)\,|\mathcal{T}| \le \sum_{\square\in\mathcal{T}} \lambda(\square) \le \lambda(\Lambda_N).$$

Thus we have shown that $|\mathcal{T}| \asymp N^d \asymp \lambda(\Lambda_N)$ as well as Assertions (e) and (f).

Finally, Lemma 5.5 implies $|\mathcal{P} \setminus \mathcal{P}^{\text{ext}}| \geq \sum_{i\in I} \gamma_i |\mathcal{T}^i| \geq \min_{i\in I} \gamma_i \,|\mathcal{T}|$. The other direction follows from the two facts that each point in $\mathcal{P} \setminus \mathcal{P}^{\text{ext}}$ is a corner of a tile of $\mathcal{T}$ and that the number of vertices per tile is bounded. This also yields Assertion (d). □

5.3.3 Estimates for the Hamiltonian

The goal of this subsection is to prove the following estimate for the Hamiltonian, which is an analogue to [HMR-13, Lemma 3.2]. Thereto we define

$$m_0 := \max_{i\in I} \left\{ \left(H^i_{\text{loc}}(\boxtimes^i) - (c_2^{\mathbb{R}} - |c_2^{\mathbb{R}}|)\lambda(\boxtimes^i) \right) / \gamma^i \right\}, \tag{5.22}$$

where the constants $\gamma_i > 0$ depend only on the tessellation and are specified in (5.18) above.

Lemma 5.7. *There exist $c_4, c_5 > 0$, $c_6^{\mathbb{R}} \in \mathbb{R}$ and $c_{20} > 0$ such that for all $m \geq m_0$, $N \in \mathbb{N}$ and $\sigma \geq \sigma_0(N,m) = c_4 N^2 + c_5 m + c_6^{\mathbb{R}}$ and for all $\omega \in \Omega$ there exists a random rotation $R = R(\omega) \in \mathrm{SO}(d)$ with*

$$H_{\sigma,m,N}(\omega) - H_{\sigma,m,N}(\varphi) \;\geq\; c_{20} \, \|V - R\|^2_{L^2(\Lambda_N)} \,.$$

We partition the proof of Lemma 5.7 into several lemmas. For better readability and shorter formulas, we omit the indexes σ, m, N of $H_{\sigma,m,N}$ sometimes in the proofs, but not in the statements of the lemmas.

Lemma 5.8. *There exist constants $c_{18} > 0$ and $c_{19}^{\mathbb{R}} \in \mathbb{R}$ such that for all $m \geq m_0, \sigma > 0, N \in \mathbb{N}$ and $\omega \in \Omega$ it is true that*

$$H_{\sigma,m,N}(\omega) - H_{\sigma,m,N}(\varphi) \geq c_1 \|\mathrm{dist}(V, \mathrm{SO}(d))\|^2_{L^2(\cup\mathcal{T})} + \left(\sigma c_3 - c_{18} m - c_{19}^{\mathbb{R}}\right)\left(|\mathcal{P}| - \sum_{i\in I} \gamma_i |\mathcal{T}^i|\right).$$

Remark. Lemma 5.8 and Lemma 5.5(a) imply $H_{\sigma,m,N} \geq H_{\sigma,m,N}(\varphi) + \alpha|\mathcal{P}|$ with $\alpha = \min\{\sigma c_3 - c_{18} m - c_{19}^{\mathbb{R}}, 0\} \leq 0$. Therefore

$$Z_{\beta,\sigma,m,N} \;\leq\; e^{-\beta H_{\sigma,m,N}(\varphi)} \int_\Omega e^{-\beta\alpha|\mathcal{P}|} d\mu \;<\; \infty$$

since the exponential moment of the Poisson distributed random variable $|\mathcal{P}|$ exists. The conclusion also holds for $m < m_0$ as $H_{\sigma,m,N} = H_{\sigma,m_0,N} - (m - m_0)|\mathcal{P}|$.

Proof. Using first the definition (5.5) of $H_{\sigma,m,N}$, second the assumption (5.3) on the local Hamiltonians H^i_{loc} and assumption (5.4) on the quantity S (note $\partial\mathcal{Q} = \emptyset$) and finally Lemma 5.5(a), we estimate

$$\begin{aligned} H(\omega) - H(\varphi) \;&=\; \sum_{\square\in\mathcal{T}} \left(H^{\imath(\square)}_{\text{loc}}(\square) - H^{\imath(\square)}_{\text{loc}}(\boxtimes^{\imath(\square)}) \right) + \sum_{i\in I} |\mathcal{T}^i| H^i_{\text{loc}}(\boxtimes^i) \\ &\quad - \sum_{i\in I} |\mathcal{U}^i| H^i_{\text{loc}}(\boxtimes^i) + \sigma S(\omega) - \sigma S(\varphi) - m|\mathcal{P}| + m|\mathcal{Q}| \end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{\square\in\mathcal{T}} \Big(c_1 \| \operatorname{dist}(\nabla v_\square, \mathrm{SO}(d))\|^2_{L^2(\square)} + c_2^{\mathbb{R}} \big(\lambda(\square) - \lambda(\boxtimes^{\imath(\square)})\big)\Big) \\
&\quad + \sum_{i\in I} \big(|\mathcal{T}^i| - |\mathcal{U}^i|\big) H^i_{\mathrm{loc}}(\boxtimes^i) + \sigma c_3 |\partial\mathcal{P}| - m\big(|\mathcal{P}| - |\mathcal{Q}|\big) \\
&\geq c_1 \| \operatorname{dist}(V, \mathrm{SO}(d))\|^2_{L^2(\cup\mathcal{T})} + c_2^{\mathbb{R}} \sum_{\square\in\mathcal{T}} \big(\lambda(\square) - \lambda(\boxtimes^{\imath(\square)})\big) \\
&\quad + \sum_{i\in I} \big(|\mathcal{T}^i| - |\mathcal{U}^i|\big) H^i_{\mathrm{loc}}(\boxtimes^i) + \sigma c_3 \big(|\mathcal{P}| - \sum_{i\in I}\gamma_i |\mathcal{T}^i|\big) \\
&\quad - m\big(|\mathcal{P}| - \sum_{i\in I}\gamma_i|\mathcal{T}^i|\big) + m\big(|\mathcal{Q}| - \sum_{i\in I}\gamma_i |\mathcal{T}^i|\big) .
\end{aligned} \tag{5.23}
$$

Now we bound the term $c_2^{\mathbb{R}} \sum_{\square\in\mathcal{T}} \big(\lambda(\square) - \lambda(\boxtimes^{\imath(\square)})\big)$ from below using the fact $\lambda(\square) \geq \lambda(\boxtimes^{\imath(\square)})$ for all $\square \in \mathcal{T}$. If $c_2^{\mathbb{R}} \geq 0$, we are done with bounding by 0, but $c_2^{\mathbb{R}} < 0$ is also possible. The just mentioned fact implies

$$
\lambda(\cup\mathcal{T}) + \sum_{i\in I} |\mathcal{T}^i| \lambda(\boxtimes^i) \;\leq\; 2\lambda(\cup\mathcal{T}) \;\leq\; 2\lambda(\Lambda_N) \;=\; 2\sum_{i\in I} |\mathcal{U}^i| \lambda(\boxtimes^i) .
$$

Subtracting $2\sum_{i\in I} |\mathcal{T}^i|\lambda(\boxtimes^i)$ from this inequality yields

$$
\sum_{\square\in\mathcal{T}} \big(\lambda(\square) - \lambda(\boxtimes^{\imath(\square)})\big) \;\leq\; 2\sum_{i\in I} \big(|\mathcal{U}^i| - |\mathcal{T}^i|\big)\lambda(\boxtimes^i) .
$$

Altogether, it follows that

$$
c_2^{\mathbb{R}} \sum_{\square\in\mathcal{T}} \big(\lambda(\square) - \lambda(\boxtimes^{\imath(\square)})\big) \;\geq\; (c_2^{\mathbb{R}} - |c_2^{\mathbb{R}}|) \sum_{i\in I} \big(|\mathcal{U}^i| - |\mathcal{T}^i|\big)\lambda(\boxtimes^i) \tag{5.24}
$$

since $c_2^{\mathbb{R}} - |c_2^{\mathbb{R}}| = -2|c_2^{\mathbb{R}}|$ if $c_2^{\mathbb{R}} < 0$ and $c_2^{\mathbb{R}} - |c_2^{\mathbb{R}}| = 0$ if $c_2^{\mathbb{R}} \geq 0$.

Moreover, Lemma 5.5(a) for φ yields $|\mathcal{Q}| = \sum_{i\in I}\gamma_i |\mathcal{U}^i|$ since $\partial\mathcal{Q} = \emptyset$. Therefore

$$
m\big(|\mathcal{Q}| - \sum_{i\in I}\gamma_i|\mathcal{T}^i|\big) \;=\; m\sum_{i\in I}\gamma_i\big(|\mathcal{U}^i| - |\mathcal{T}^i|\big) . \tag{5.25}
$$

Plugging (5.24) and (5.25) into (5.23) yields

$$
\begin{aligned}
H(\omega) - H(\varphi) \;&\geq\; c_1 \| \operatorname{dist}(V, \mathrm{SO}(d))\|^2_{L^2(\cup\mathcal{T})} + (\sigma c_3 - m)\big(|\mathcal{P}| - \sum_{i\in I}\gamma_i|\mathcal{T}^i|\big) \\
&\quad + \sum_{i\in I} \big(|\mathcal{U}^i| - |\mathcal{T}^i|\big)\big(m\gamma_i + (c_2^{\mathbb{R}} - |c_2^{\mathbb{R}}|)\lambda(\boxtimes^i) - H^i_{\mathrm{loc}}(\boxtimes^i)\big)
\end{aligned} \tag{5.26}
$$

Since $m\gamma_i + (c_2^{\mathbb{R}} - |c_2^{\mathbb{R}}|)\lambda(\boxtimes^i) - H^i_{\mathrm{loc}}(\boxtimes^i) \geq 0$ for $m \geq m_0$ by the choice of m_0 in (5.22), we can first use Lemma 5.4 and then Lemma 5.5(b) and receive

$$
\begin{aligned}
&\sum_{i\in I} \big(|\mathcal{U}^i| - |\mathcal{T}^i|\big)\big(m\gamma_i + (c_2^{\mathbb{R}} - |c_2^{\mathbb{R}}|)\lambda(\boxtimes^i) - H^i_{\mathrm{loc}}(\boxtimes^i)\big) \;\geq \\
&\quad\geq\; -c_9|\partial\mathcal{T}| \sum_{i\in I} \big(m\gamma_i + (c_2^{\mathbb{R}} - |c_2^{\mathbb{R}}|)\lambda(\boxtimes^i) - H^i_{\mathrm{loc}}(\boxtimes^i)\big) \;=\; -c_9|\partial\mathcal{T}|(mc_{35} + c_{36}^{\mathbb{R}}) \\
&\quad\geq\; -c_9 c_{10}\big(|\mathcal{P}| - \sum_{i\in I}\gamma_i|\mathcal{T}^i|\big)(mc_{35} + c_{36}^{\mathbb{R}})
\end{aligned}
$$

for constants $c_{35} > 0$ and $c_{36}^{\mathbb{R}} \in \mathbb{R}$ with $mc_{35} + c_{36}^{\mathbb{R}} > 0$ for $m \geq m_0$. Inserting this into (5.26) yields the claim, namely

$$H(\omega) - H(\varphi) \geq c_1 \| \operatorname{dist}(V, \mathrm{SO}(d))\|^2_{L^2(\cup\mathcal{T})} + \big(\sigma c_3 - c_{18} m - c_{19}^{\mathbb{R}}\big)\big(|\mathcal{P}| - \sum_{i\in I} \gamma_i |\mathcal{T}^i|\big)$$

with constants $c_{18} := 1 + c_9 c_{10} c_{35} > 0$ and $c_{19}^{\mathbb{R}} := c_9 c_{10} c_{36}^{\mathbb{R}} \in \mathbb{R}$. □

Lemma 5.9. *For all $p \in [2d/(2+d), 2]$, there exist constants $c_{20} > 0$ and $c_{21}(p) > 0$ such that for all $N \in \mathbb{N}$ and $\omega \in \Omega$ there exists a random rotation $R = R(\omega) \in \mathrm{SO}(d)$ with*

$$c_1 \| \operatorname{dist}(V, \mathrm{SO}(d))\|^2_{L^2(\cup\mathcal{T})} \;\geq\; c_{20} \|V - R\|^2_{L^2(\Lambda_N)} \;-\; c_{21}(p)\, N^{2+d-\frac{2d}{p}} \lambda\big(\partial^{\overline{0\rho}} \cup \mathcal{T}\big)^{\frac{2}{p}} .$$

Proof. By Corollary 4.2 and Lemma 4.4, there exists a random rotation $R = R(\omega) \in \mathrm{SO}(d)$ such that

$$\|V - R\|_{L^2(\Lambda_N)} \;\leq\; C_1 \| \operatorname{dist}(V, \mathrm{SO}(d))\|_{L^2(\Lambda_N)} + N^{\frac{d}{2} - \frac{d}{p} + 1} C_2(p) \|dV\|_{L^p(\Lambda_N)} \tag{5.27}$$

with scale-invariant constants $C_1 = C_1(\Lambda_1)$ and $C_2(p) = C_2(\Lambda_1, p)$.

Since $V = \widetilde{R} \in \mathrm{SO}(d)$ on $\Lambda_N \setminus (\cup\mathcal{T} \cup \partial^{\overline{0\rho}} \cup \mathcal{T})$ and Lemma 5.3, it follows that

$$\begin{aligned} \| \operatorname{dist}(V, \mathrm{SO}(d))\|^2_{L^2(\Lambda_N)} &= \| \operatorname{dist}(V, \mathrm{SO}(d))\|^2_{L^2(\cup\mathcal{T})} + \| \operatorname{dist}(V, \mathrm{SO}(d))\|^2_{L^2(\partial^{\overline{0\rho}} \cup \mathcal{T})} \\ &\leq \| \operatorname{dist}(V, \mathrm{SO}(d))\|^2_{L^2(\cup\mathcal{T})} + c_8\, \lambda(\partial^{\overline{0\rho}} \cup \mathcal{T}) \end{aligned} \tag{5.28}$$

and, also using $dV = 0$ on $\cup\mathcal{T}$,

$$\|dV\|^2_{L^p(\Lambda_N)} = \|dV\|^2_{L^p(\partial^{\overline{0\rho}} \cup \mathcal{T})} = \Big(\int_{\partial^{\overline{0\rho}} \cup \mathcal{T}} |dV|^p \, d\lambda \Big)^{\frac{2}{p}} \;\leq\; c_8^2 \lambda\big(\partial^{\overline{0\rho}} \cup \mathcal{T}\big)^{\frac{2}{p}} . \tag{5.29}$$

Using $\frac{2}{p} - 1 \geq 0$ (since $p \leq 2$) at $*$ yields for all $y \geq 0$:

$$y = 0 \vee y > \rho^d \;\Leftrightarrow\; y = 0 \vee \rho^{-d} y > 1 \;\overset{*}{\Leftrightarrow}\; y = 0 \vee \rho^{d - \frac{2d}{p}} y^{\frac{2}{p} - 1} > 1 \;\Leftrightarrow\; \rho^{d - \frac{2d}{p}} y^{\frac{2}{p}} \geq y .$$

With $y = \lambda(\partial^{\overline{0\rho}} \cup \mathcal{T})$ (note $y \geq \rho^d$ if $y \neq 0$) it follows that

$$\lambda(\partial^{\overline{0\rho}} \cup \mathcal{T}) \;\leq\; \rho^{d - \frac{2d}{p}} \lambda\big(\partial^{\overline{0\rho}} \cup \mathcal{T}\big)^{\frac{2}{p}} . \tag{5.30}$$

Inserting the combination of (5.28) and (5.30) as well as (5.29) into the squared version of (5.27) yields

$$\|V - R\|^2_{L^2(\Lambda_N)} \;\leq\; 2C_1^2 \| \operatorname{dist}(V, \mathrm{SO}(d))\|^2_{L^2(\cup\mathcal{T})} + 2N^{d - \frac{2d}{p} + 2} c_{37}(p) \lambda\big(\partial^{\overline{0\rho}} \cup \mathcal{T}\big)^{\frac{2}{p}}$$

for some constant $c_{37}(p) > 0$. Thus the lemma follows by a little rearrangement and renaming of constants. □

Lemma 5.10. *For all $m \geq m_0$, $\sigma > 0$, $N \in \mathbb{N}$ and $\omega \in \Omega$ there exists a random rotation $R = R(\omega) \in \mathrm{SO}(d)$ such that*

$$H_{\sigma,m,N}(\omega) - H_{\sigma,m,N}(\varphi) \geq c_{20} \|V - R\|^2_{L^2(\Lambda_N)} + c_3\big(\sigma - \sigma_0(N,m)\big)\big(|\mathcal{P}| - \sum_{i\in I}\gamma_i|\mathcal{T}^i|\big)$$

with $\sigma_0(N,m) = c_4 N^2 + c_5 m + c_6^{\mathbb{R}}$ for some constants $c_4, c_5 > 0$ and $c_6^{\mathbb{R}} \in \mathbb{R}$.

Proof. Lemma 5.8 and Lemma 5.9 together state that

$$\begin{aligned} H(\omega) - H(\varphi) \quad \geq \quad & c_{20}\|V - R\|^2_{L^2(\Lambda_N)} + \big(\sigma c_3 - c_{18} m - c_{19}^{\mathbb{R}}\big)\big(|\mathcal{P}| - \sum_{i\in I}\gamma_i|\mathcal{T}^i|\big) \\ & - c_{21}(p) N^{2+d-\frac{2d}{p}} \lambda\big(\partial^{\overline{0\rho}} \cup \mathcal{T}\big)^{\frac{2}{p}} \end{aligned}$$

for all $p \in [2d/(2+d), 2]$. Therefore we have to estimate $\lambda(\partial^{\overline{0\rho}} \cup \mathcal{T})^{\frac{2}{p}}$ from above. We start the estimate with Lemma 5.5(c) to get a bound in terms of $|\partial\mathcal{T}|$. Then we use two different bounds: On the one hand we use Lemma 5.5(b), i.e. $|\partial\mathcal{T}| \leq c_{10}(|\mathcal{P}| - \sum_{i\in I}\gamma_i|\mathcal{T}^i|)$, and on the other hand we use the bound $|\partial\mathcal{T}| \leq |\mathcal{T}| \leq c_{14} N^d$, provided by Lemma 5.6(e). This yields

$$\lambda\big(\partial^{\overline{0\rho}} \cup \mathcal{T}\big)^{\frac{2}{p}} \leq \big(c_{11}|\partial\mathcal{T}|\big)^{1+(\frac{2}{p}-1)} \leq c_{11}{}^{\frac{2}{p}}\, c_{10}\big(|\mathcal{P}| - \sum_{i\in I}\gamma_i|\mathcal{T}^i|\big)\big(c_{14}N^d\big)^{\frac{2}{p}-1}$$

Since $N^{2+d-\frac{2d}{p}} N^{\frac{2d}{p}-d} = N^2$ for all p, the choice of p does not matter. We choose $p = 2$ (i.e. $\frac{2}{p} = 1$). Setting $c_4 := c_{21}(2) c_{11} c_{10}/c_3$, we conclude

$$H(\omega) - H(\varphi) \geq c_{20}\|V - R\|^2_{L^2(\Lambda_N)} + \big(\sigma c_3 - c_{18} m - c_{19}^{\mathbb{R}} - c_4 c_3 N^2\big)\big(|\mathcal{P}| - \sum_{i\in I}\gamma_i|\mathcal{T}^i|\big)\,,$$

which implies the lemma with $c_5 := c_{18}/c_3 > 0$ and $c_6^{\mathbb{R}} := c_{19}^{\mathbb{R}}/c_3 \in \mathbb{R}$. □

Let us remark that a choice $p > 2$ would give a worse result. Though Lemma 5.9 would also work with an additional $\lambda(\partial^{\overline{0\rho}} \cup \mathcal{T})^1$-term, the factor $N^{2+d-\frac{2d}{p}}$ would be worse than N^2 and could not be compensated by $|\partial\mathcal{T}|^{\frac{2}{p}-1}$ since $|\partial\mathcal{T}|$ may be small.

Proof of Lemma 5.7. This lemma is an immediate corollary to Lemma 5.10 since $\sigma - \sigma_0(N,m) \geq 0$ and $|\mathcal{P}| - \sum_{i\in I}\gamma_i|\mathcal{T}^i| \geq 0$ by Lemma 5.5(a). □

5.3.4 A Lower Bound for the Partition Sum

In this section we prove the following lower bound of the partition sum, which is an analogue to [HMR-13, Lemma 3.1].

Lemma 5.11. *For all $\gamma > 0$ and $m \in \mathbb{R}$ there exist a constant $c_{22}(\gamma,m) > 0$ and an $N_0(\gamma,m) \in \mathbb{N}$ such that for all $N \geq N_0(\gamma,m)$, $\beta > 0$ and $\sigma > 0$ one has*

$$Z_{\beta,\sigma,m,N} \geq e^{-N^d[\beta\gamma + c_{22}(\gamma,m)]} e^{-\beta H_{\sigma,m,N}(\varphi)}\,.$$

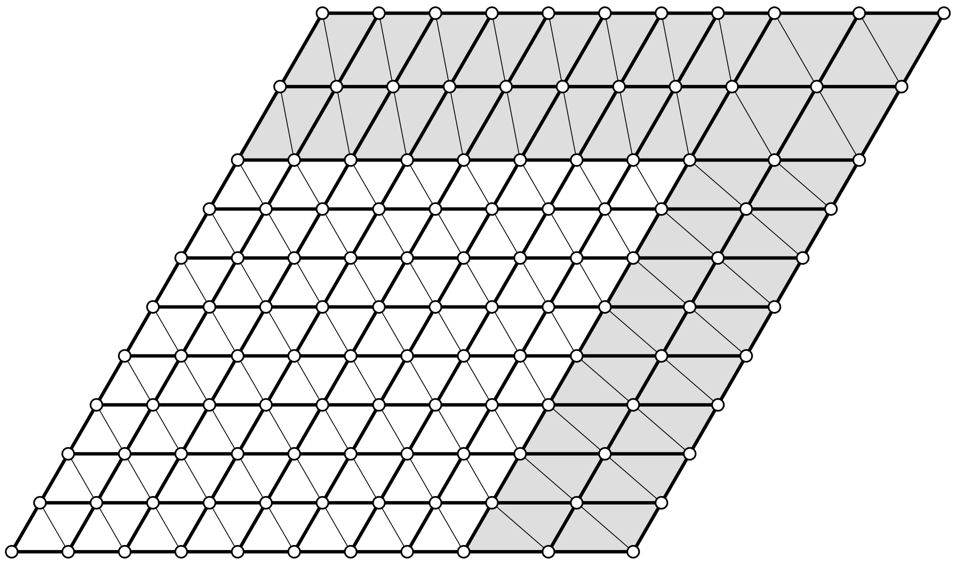


Figure 5.6: The configuration φ_r for the triangular lattice

Proof. The proof uses the idea of the proof of [HMR-13, Lemma 3.1], namely to restrict the integral to a set of blurred configurations. But we have to blur a configuration slightly differently to the standard configuration since we have to ensure that the Lebesgue measure of the blurred tiles is not smaller than the Lebesgue measure of the corresponding standard tile.

We start the proof with some preliminaries. Let us recall that $\mathcal{M}$ is B_0-periodic for some box B_0, which is the image of the cube $[0,1]^d$ under some linear map L. For $r \in (0, \frac{\varepsilon}{4})$ and $N \in \mathbb{N}$ such that $\lfloor \frac{N}{1+r} \rfloor \geq \lceil \frac{4}{\varepsilon} \rceil =: n_0$ we define a configuration φ_r with vertices $\mathcal{Q}_r$ and tiles $\mathcal{U}_r$ as follows (we suppress the N-dependency in the notation). It looks almost like the given tessellation $\mathcal{M}$, but is a bit enlarged. The domain Λ_N is partitioned into boxes $B_{\mathbf{k}}$, $\mathbf{k} \in \{1, \ldots, \lfloor \frac{N}{1+r} \rfloor\}^d$ which are slight enlargements of $B_0 = L\big[[0,1]^d\big]$. In each box-direction Le_j (with unit vector e_j), there are $\lfloor \frac{N}{1+r} \rfloor - n_0$ boxes scaled by the factor $(1+r)$, followed by n_0 boxes scaled by O/n_0, with "off-cut" $O := N-(1+r)(\lfloor \frac{N}{1+r} \rfloor - n_0)$. Thus box $B_{\mathbf{k}}$, with $\mathbf{k} \in \{1, \ldots, \lfloor \frac{N}{1+r} \rfloor\}^d$, has length $\mathbf{l}^{\mathbf{k}}_j$ in box direction Le_j, $j \in \{1, \ldots, d\}$, where $\mathbf{l}^{\mathbf{k}}_j = |Le_j|(1+r)$ if $1 \leq \mathbf{k}_j \leq \lfloor \frac{N}{1+r} \rfloor - n_0$ and $\mathbf{l}^{\mathbf{k}}_j = |Le_j|\, O/n_0$ else. Now φ_r is defined such that $\mathcal{Q}_r\lceil_{B_{\mathbf{k}}} = \mathcal{V}(\mathbf{l}^{\mathbf{k}} \cdot \mathcal{M})$ and similarly for $\mathcal{U}_r$. At the separation hyperplanes between the scales, the points are moved a little bit, such that all tiles, which intersects such a separation hyperplane are scaled like the box which is to the "left" in the corresponding coordinate direction.

Figure 5.6 illustrates the configuration φ_r in the case where $\mathcal{M}$ is the triangular lattice. In that case, B_0 is a rhombus consisting of two triangles. The white boxes are the boxes scaled by $1+r$ and build the "bulk". In contrast, the grey shaded boxes are in the "off-cut" and scaled by larger factors which may also differ in different directions. We have to use this "off-cut-boxes" to ensure that $\mathcal{T}(\varphi_r)$ completely fills the domain Λ_N, whose size is a natural number times the size of B_0 (in each direction).

Moreover, we blur the configuration φ_r a little bit and define the set

$$A_r := \left\{\omega \in \tilde{\Omega} \mid \exists \text{bijective } f : \mathcal{P}(\omega) \to \mathcal{Q}_r : \forall x \in \mathcal{P}(\omega) : |x - f(x)| < \tfrac{r}{2}\right\}$$

of all configurations whose points are $r/2$-close to $\mathcal{Q}_r$. Then we claim that all configurations in A_r are admitted configurations without any defect, i.e. $A_r \subset \Omega$ and $\partial\mathcal{P} = \emptyset$ on A_r. Since

$$O = N - (1+r)(\lfloor \tfrac{N}{1+r} \rfloor - n_0) \begin{cases} \le N - (1+r)(\frac{N}{1+r} - 1 - n_0) = (1+r)(1+n_0) \\ \ge N - (1+r)(\frac{N}{1+r} - n_0) = (1+r)n_0 \end{cases}$$

we conclude $1 + r \le \frac{O}{n_0}$ and

$$1 \le 1 + r - 2 \cdot \tfrac{r}{2} \le \tfrac{O}{n_0} + 2 \cdot \tfrac{r}{2} \le (1+r)(1 + \tfrac{1}{n_0}) + r = 1 + 2r + \tfrac{1}{n_0} + \tfrac{r}{n_0} \le 1 + \varepsilon$$

as $n_0 \ge \frac{4}{\varepsilon}$ and $r \le \frac{\varepsilon}{4}$. By the definition of the set A_r, the distance between two points in $\mathcal{P}(\omega)$ for any $\omega \in A_r$ is in $[1 + r - 2 \cdot \frac{r}{2}, \frac{O}{n_0} + 2 \cdot \frac{r}{2}]$ times the distance of the corresponding points in $\mathcal{Q}$. Thus the estimate above shows that all tiles are, up to translation, in $\mathcal{N}_\varepsilon(\boxtimes^i)$ for some $i \in I$ and the claim follows.

Furthermore, there exists a constant $c_{38} > 0$ such that

$$\left|H^i_{\text{loc}}(\square) - H^i_{\text{loc}}(\boxtimes^i)\right| \le c_{38} \tag{5.31}$$

for all $i \in I$ and $\square \in \mathcal{N}_\varepsilon(\boxtimes^i)$ since the image of the compact set $\mathcal{N}_\varepsilon(\boxtimes^i)$ is compact as H^i_{loc} is continuous.

Now we begin with the actual proof. Let $\gamma > 0$ and $m \in \mathbb{R}$. We choose $r \in (0, \frac{\varepsilon}{4}) \cap (0, \frac{1}{2})$ so small that

$$r(mdc_{39} + c_{40}) \le \tfrac{\gamma}{3} \tag{5.32}$$

for some constants $c_{39}, c_{40} > 0$ defined below and that for all $i \in I$ and $\square \in \mathcal{N}_{2r}(\boxtimes^i)$

$$\left|H^i_{\text{loc}}(\square) - H^i_{\text{loc}}(\boxtimes^i)\right| \le \tfrac{\gamma}{3c_{14}}, \tag{5.33}$$

which is possible since H^i_{loc}, $i \in I$, are continuous. Furthermore, we choose $N_0 \in \mathbb{N} \setminus \{1\}$ large enough such that $\lfloor \frac{N_0}{1+r} \rfloor \ge \lceil \frac{4}{\varepsilon} \rceil = n_0$ and such that

$$\tfrac{1}{N_0}\left(c_{38}c_{41} + mdc_{39} + c_{40}\right) \le \tfrac{\gamma}{3} \tag{5.34}$$

for some constant $c_{41} > 0$ defined below. Let $N \ge N_0$. Now we estimate $\mu(A_r)$, where A_r is the set of blurred configurations defined above. Since the number of points is Poisson distributed and independent from the location of the points, which are iid and uniformly distributed, it follows that

$$\begin{aligned}\mu(A_r) &= \mu(|\mathcal{P}| = |\mathcal{Q}_r|) \cdot \frac{\lambda(U_{\frac{r}{2}}(0))}{\lambda(\Lambda_N)} |\mathcal{Q}_r| \cdot \frac{\lambda(U_{\frac{r}{2}}(0))}{\lambda(\Lambda_N)} (|\mathcal{Q}_r| - 1) \cdot \ldots \cdot \frac{\lambda(U_{\frac{r}{2}}(0))}{\lambda(\Lambda_N)} 1 \\ &= e^{-\lambda(\Lambda_N)} \frac{\lambda(\Lambda_N)^{|\mathcal{Q}_r|}}{|\mathcal{Q}_r|!} \cdot \Big(\frac{\lambda(U_{\frac{r}{2}}(0))}{\lambda(\Lambda_N)}\Big)^{|\mathcal{Q}_r|} \cdot |\mathcal{Q}_r|! \;=\; e^{-c_{17}N^d + |\mathcal{Q}_r| \log \lambda(U_{\frac{r}{2}}(0))} \\ &\geq e^{-N^d \cdot c_{42}(r)} \end{aligned} \tag{5.35}$$

for some constant $c_{42}(r) > 0$ only depending on r since $\lambda(\Lambda_N) = c_{17}N^d$ and $|\mathcal{Q}_r| \leq c_{13}N^d$ by Lemma 5.6, Assertions (g) and (d). Note that $c_{42}(r) \to \infty$ since $\lambda(U_{\frac{r}{2}}(0)) \to 0$ as $r \to 0$.

In the following, we estimate the difference of the Hamiltonians of any configuration in A_r and the standard configuration. Thereto we call $\mathcal{T}_{\text{bulk}}$ the set of tiles which are in a box which is scaled by $(1+r)$ in all directions. The set of all other tiles is called $\mathcal{T}_{\text{off}}$. Since n_0 is fixed, there is a uniform constant $c_{41} > 0$ such that $|\mathcal{T}_{\text{off}}| \leq c_{41}N^{d-1}$. It follows that, for all $\omega \in \mathcal{A}_r$,

$$\begin{aligned} &H_{\sigma,m,N}(\omega) - H_{\sigma,m,N}(\varphi) = \\ &= \sum_{\square \in \mathcal{T}_{\text{bulk}}} \big(H_{\text{loc}}^{\imath(\square)}(\square) - H_{\text{loc}}^{\imath(\square)}(\boxtimes^{\imath(\square)})\big) + \sum_{\square \in \mathcal{T}_{\text{off}}} \big(H_{\text{loc}}^{\imath(\square)}(\square) - H_{\text{loc}}^{\imath(\square)}(\boxtimes^{\imath(\square)})\big) \\ &\quad + \sum_{i \in I} |\mathcal{U}_r^i| H_{\text{loc}}^i(\boxtimes^i) - \sum_{i \in I} |\mathcal{U}^i| H_{\text{loc}}^i(\boxtimes^i) + \sigma 0 - \sigma 0 - m|\mathcal{Q}_r| + m|\mathcal{Q}| \\ &\leq \tfrac{\gamma}{3c_{14}} |\mathcal{U}_r| + c_{38}c_{41}N^{d-1} - \sum_{i \in I} \big(|\mathcal{U}^i| - |\mathcal{U}_r^i|\big) H_{\text{loc}}^i(\boxtimes^i) + m(|\mathcal{Q}| - |\mathcal{Q}_r|) \end{aligned} \tag{5.36}$$

using also the estimates (5.33), $|\mathcal{T}_{\text{bulk}}| \leq |\mathcal{U}_r|$ and (5.31). Let $c_{39} := |\mathcal{V}(\mathcal{M}\restriction_{B_0})|$ and $c_{43}^i := |\mathcal{U}^i \cap \mathcal{M}\restriction_{B_0}|$ be the number of vertices and tiles of type i, respectively, in B_0 (of the standard configuration φ). Then:

$$|\mathcal{Q}| = c_{39}N^d, \quad |\mathcal{Q}_r| = c_{39}\big\lfloor \tfrac{N}{1+r} \big\rfloor^d, \quad |\mathcal{U}^i| = c_{43}^i N^d \quad \text{and} \quad |\mathcal{Q}_r^i| = c_{43}^i \big\lfloor \tfrac{N}{1+r} \big\rfloor^d.$$

Therefore we estimate using $\frac{1}{1+r} \geq (1-r)$ and $(1-x)^d \geq 1 - dx$ for $x = r + \frac{1}{N} \in (0,1)$, which can be derived with Taylor expansions,

$$\begin{aligned} |\mathcal{Q}| - |\mathcal{Q}_r| &= c_{39}\big(N^d - \lfloor \tfrac{N}{1+r} \rfloor^d\big) \;\leq\; c_{39}N^d\big(1 - \big(\tfrac{1}{1+r} - \tfrac{1}{N}\big)^d\big) \\ &\leq c_{39}N^d\big(1 - \big(1 - r - \tfrac{1}{N}\big)^d\big) \;\leq\; c_{39}N^d\big(1 - \big(1 - d\big(r + \tfrac{1}{N}\big)\big)\big) \\ &= dc_{39}\big(r + \tfrac{1}{N}\big)N^d. \end{aligned} \tag{5.37}$$

Analogously, we receive

$$|\mathcal{U}^i| - |\mathcal{U}_r^i| \;\leq\; dc_{43}^i\big(r + \tfrac{1}{N}\big)N^d,$$

which implies

$$-\sum_{i\in I}\big(|\mathcal{U}^i|-|\mathcal{U}^i_r|\big)H^i_{\mathrm{loc}}(\boxtimes^i)\;\le\;\sum_{i\in I}\big(|\mathcal{U}^i|-|\mathcal{U}^i_r|\big)\big|H^i_{\mathrm{loc}}(\boxtimes^i)\big|$$
$$\le\sum_{i\in I}dc^i_{43}\big(r+\tfrac{1}{N}\big)N^d\big|H^i_{\mathrm{loc}}(\boxtimes^i)\big|\;=\;c_{40}\big(r+\tfrac{1}{N}\big)N^d \tag{5.38}$$

with $c_{40}:=\sum_{i\in I}dc^i_{43}|H^i_{\mathrm{loc}}(\boxtimes^i)|>0$. Using $|\mathcal{U}_r|\le c_{14}N^d$, (5.38) and (5.37) for the first inequality and (5.34) and (5.32) for the second inequality, we continue the estimate (5.36) as follows:

$$\begin{aligned}H(\omega)-H(\varphi)&\le\tfrac{\gamma}{3c_{14}}c_{14}N^d+c_{38}c_{41}N^{d-1}+c_{40}\big(r+\tfrac{1}{N}\big)N^d+mdc_{39}\big(r+\tfrac{1}{N}\big)N^d\\&=N^d\big(\tfrac{\gamma}{3}+\tfrac{1}{N}\big(c_{38}c_{41}+mdc_{39}+c_{40}\big)+r(mdc_{39}+c_{40})\big)\\&\le N^d\big(\tfrac{\gamma}{3}+\tfrac{\gamma}{3}+\tfrac{\gamma}{3}\big)\;=\;\gamma N^d\,.\end{aligned}\tag{5.39}$$

Finally, we estimate the partition sum using first (5.39) and then (5.35) to conclude the proof:

$$\begin{aligned}Z_{\beta,\sigma,m,N}&=\int_\Omega e^{-\beta H(\omega)}\,\mu(d\omega)\;\ge\;e^{-\beta H(\varphi)}\int_{A_r}e^{-\beta(H(\omega)-H(\varphi))}\,\mu(d\omega)\\&\ge e^{-\beta H(\varphi)}e^{-\beta\gamma N^d}\mu(A_r)\\&\ge e^{-\beta H(\varphi)}e^{-\beta\gamma N^d}e^{-N^d\cdot c_{42}(r)}\;=\;e^{-N^d[\beta\gamma+c_{22}(\gamma,m)]}e^{-\beta H(\varphi)}\end{aligned}$$

with $c_{22}(\gamma,m):=c_{42}(r(\gamma,m))>0$. Note that $c_{22}(\gamma,m)\to\infty$ as $\gamma\to 0$ or $m\to\infty$. □

5.3.5 An Upper Bound for the Internal Energy

In this section, we obtain an estimate of $E_{\beta,\sigma,m,N}[\frac{1}{|\mathcal{T}|}(H_{\sigma,m,N}(\cdot)-H_{\sigma,m,N}(\varphi)]$.

Thereto we will need some labelled spanning trees of $\mathcal{T}$. We define an index set Σ as the union of all edges of $\boxtimes^i$, $i\in I$, regarded as vectors in $\mathbb{R}^d$ (each edge induces two vectors with opposite orientation). Let $\mathbf{T}^\Sigma_n$ be the set of trees with n vertices labelled by elements of Σ. We denote the label of a vertex k by ξ_k. We can consider a tree $T\in\mathbf{T}^\Sigma_n$ as a rooted tree with root 1. Then, for each $l\in\{2,\dots,n\}$, there exists a unique $k_T(l)\in\{1,\dots,l-1\}$ such that $k_T(l)\sim l$ in T. For $\omega\in\Omega$, we define the function $\eta:\{1,\dots,|\mathcal{P}\setminus\mathcal{P}^{\mathrm{ext}}|\}\to\{k\in\mathbb{N}\mid X_k\in\mathcal{P}\setminus\mathcal{P}^{\mathrm{ext}}\}$ as the unique increasing bijection between these sets.

For a labelled tree $T\in\mathbf{T}^\Sigma_n$ and $\omega\in\Omega$ with $n=|\mathcal{P}\setminus\mathcal{P}^{\mathrm{ext}}|$, we define the graph $G(T,\mathcal{T})$ as follows: The vertex set is $\{X_{\eta(k)},k=1,\dots,n\}$; two such vertices $X_{\eta(k)}$ and $X_{\eta(l)}$, $1\le k<l\le n$, form an edge, if $k=k_T(l)$ (i.e. $k\sim l$ in T) and if there is a tile $\square\in\mathcal{T}$ such that $X_{\eta(k)},X_{\eta(l)}\in\square$ and $\xi_l=v_\square(X_{\eta(l)})-v_\square(X_{\eta(k)})$, where $v_\square:\square\to\boxtimes^{\imath(\square)}$ is the affine linear map defined in (5.2). Thus $G(T,\mathcal{T})$ can be viewed as a graph isomorphic to a sub-graph of T using vertices of $\mathcal{P}\setminus\mathcal{P}^{\mathrm{ext}}$ such that the label of a vertex coincide with the role of an adjacent edge in $\mathcal{T}$.

A labelled tree $T \in \mathbf{T}^{\Sigma}_{|\mathcal{P}\setminus\mathcal{P}^{\text{ext}}|}$ is called a *labelled spanning tree* of $\mathcal{T}$ if $G(T,\mathcal{T})$ is a spanning tree of $\cup\mathcal{T}$, viewed as a graph with vertices $\mathcal{P}\setminus\mathcal{P}^{\text{ext}}$ and edges formed by the edges of the tiles. In that case we write $T \bowtie \mathcal{T}$. Since $\cup\mathcal{T}$ is connected, there exists a labelled spanning tree $T \in \mathbf{T}^{\Sigma}_{\mathcal{P}\setminus\mathcal{P}^{\text{ext}}}$: just take any spanning tree and label the vertices accordingly level by level, beginning with the vertices adjacent to the root (whose label is irrelevant).

Lemma 5.12. *There is a constant $c_{23} > 0$ such that for all $N \in \mathbb{N}$, $R \in \mathrm{SO}(d)$, $\omega \in \Omega$ and $T \in \mathbf{T}^{\Sigma}_{|\mathcal{P}\setminus\mathcal{P}^{ext}|}$ with $T \bowtie \mathcal{T}$ the following estimate holds:*

$$\|V - R\|^2_{L^2(\Lambda_N)} \geq c_{23} \sum_{l=2}^{|\mathcal{P}\setminus\mathcal{P}^{ext}|} \left|(X_{\eta(l)} - X_{\eta(k_T(l))}) - R^t\xi_l\right|^2$$

Proof. Let $\square \in \mathcal{T}$. Let $\mathrm{Sim}(\square) := \{v_\square^{-1}[\boxtimes^{\imath(\square),j}] \mid j = 1,\dots,J_{\imath(\square)}\}$ be the set of simplices on which $v_\square$ is affine linear. For a simplex $\triangle \in \mathrm{Sim}(\square)$, let $\overset{\triangle}{\eta} : \{0,\dots,d\} \to \{k \in \mathbb{N} \mid X_k \text{ is a vertex of } \triangle\}$ be the unique increasing bijection between these sets.

In the following estimate for a single simplex we simple write x_k for $X_{\overset{\triangle}{\eta}(k)}$, $k = 0,\dots,d$. We have

$$v_\square(x) = V_\triangle x + z_\triangle\,, \qquad x \in \triangle\,,$$

for some $V_\triangle \in \mathbb{R}^{d\times d}$ and $z_\triangle \in \mathbb{R}^d$ since $v_\square$ is affine linear on $\triangle$. Using this and $|Ry| = |y|$ because of $R \in \mathrm{SO}(d)$, it follows that

$$\begin{aligned}
\sum_{0\le k<l\le d} \left|(x_l - x_k) - R^t(v_\square(x_l) - v_\square(x_k))\right|^2 &= \sum_{0\le k<l\le d} \left|R(x_l - x_k) - (v_\square(x_l) - v_\square(x_k))\right|^2 \\
&= \sum_{0\le k<l\le d} \left|R(x_l - x_k) - ((V_\triangle x_l + z_\triangle) - (V_\triangle x_k + z_\triangle))\right|^2 \\
&\le \sum_{0\le k<l\le d} |R - V_\triangle|^2 |x_l - x_k|^2 \le c_{44}|R - V_\triangle|^2 \qquad (5.40)
\end{aligned}$$

for some uniform constant $c_{44} > 0$ since the size of a tile is uniformly bounded.

Therefore we can estimate using the fact that the size of a simplex is uniformly bounded

$$\begin{aligned}
\|V - R\|^2_{L^2(\Lambda_N)} &\geq \sum_{\square\in\mathcal{T}} \|V - R\|^2_{L^2(\square)} = \sum_{\square\in\mathcal{T}} \sum_{\triangle\in\mathrm{Sim}(\square)} \lambda(\triangle)|V_\triangle - R|^2 \\
&\overset{(5.40)}{\geq} \sum_{\square\in\mathcal{T}} \sum_{\triangle\in\mathrm{Sim}(\square)} \frac{\lambda(\triangle)}{c_{44}} \sum_{0\le k<l\le d} \left|(X_{\overset{\triangle}{\eta}(l)} - X_{\overset{\triangle}{\eta}(k)}) - R^t(v_\square(X_{\overset{\triangle}{\eta}(l)}) - v_\square(X_{\overset{\triangle}{\eta}(k)}))\right|^2 \\
&\geq c_{23} \sum_{l=2}^{n} \left|(X_{\eta(l)} - X_{\eta(k_T(l))}) - R^t\xi_l\right|^2
\end{aligned}$$

for some $c_{23} > 0$. We obtained the last inequality by restricting the sum, which is taken over all edges of all simplices, to edges in $G(T,\mathcal{T})$; note that $\xi_{\eta^{-1}(\overset{\triangle}{\eta}(l))} = v_\square(X_{\overset{\triangle}{\eta}(l)}) - v_\square(X_{\overset{\triangle}{\eta}(k)})$ if $\{X_{\overset{\triangle}{\eta}(k)}, X_{\overset{\triangle}{\eta}(l)}\}$ is an edge of $G(T,\mathcal{T})$ since $T \bowtie \mathcal{T}$. $\square$

The following lemma is an analogue to [HMR-13, Lemma 3.3].

Lemma 5.13. *There exist constants $c_{24} > 0$ and $\beta_0 > 0$ such that for all $m \geq m_0$ and all $\delta > 0$ there exist $N_0(\delta, m) \in \mathbb{N}$ and $c_{25}^{\mathbb{R}}(\delta, m) \in \mathbb{R}$ such that for all $N \geq N_0$, $\sigma \geq \sigma_0(N, m)$ and $\beta \geq \beta_0$ the following estimate holds:*

$$E_{\beta,\sigma,m,N}\left[\tfrac{1}{|\mathcal{T}|}\big(H_{\sigma,m,N}(\cdot) - H_{\sigma,m,N}(\varphi)\big)\right] \;\leq\; \delta + \tfrac{1}{\beta}\exp\!\big(-N^d\big[\beta\tfrac{c_0}{8}\delta + c_{24}\log\beta - c_{25}^{\mathbb{R}}(\delta,m)\big]\big)$$

Proof. We use some ideas of the proof of [HMR-13, Lemma 3.3]. Let $\delta > 0$ and $m \geq m_0$. We set $N_0(\delta, m) = N_0(\gamma, m)$ as in Lemma 5.11 with $\gamma = \frac{c_0}{8}\delta$. Let $N \geq N_0(\delta, m)$ and $\sigma \geq \sigma_0(N, m)$. We set

$$\begin{aligned} \Omega^{>\delta} &:= \big\{\omega \in \Omega : H_{\sigma,m,N}(\omega) - H_{\sigma,m,N}(\varphi) > \delta|\mathcal{T}|\big\} \qquad \text{and} \\ \Omega^{\leq\delta} &:= \big\{\omega \in \Omega : H_{\sigma,m,N}(\omega) - H_{\sigma,m,N}(\varphi) \leq \delta|\mathcal{T}|\big\}. \end{aligned}$$

First we estimate

$$E_{\beta,\sigma,m,N}\left[\tfrac{1}{|\mathcal{T}|}\big(H(\cdot) - H(\varphi)\big)\mathbb{1}_{\Omega^{\leq\delta}}\right] \;\leq\; E_{\beta,\sigma,m,N}\left[\tfrac{1}{|\mathcal{T}|}\delta|\mathcal{T}|\mathbb{1}_{\Omega^{\leq\delta}}\right] \;\leq\; \delta\,. \tag{5.41}$$

The estimate on $\Omega^{>\delta}$ is much more involved. Using the inequality $xe^{-x} \leq e^{-x/2}$ for $x = \beta(H(\omega) - H(\varphi))$ and $|\mathcal{T}| \geq 1$, we estimate similarly as in the proof of Markov's Inequality :

$$\begin{aligned} &E_{\beta,\sigma,m,N}\left[\tfrac{1}{|\mathcal{T}|}\big(H(\cdot) - H(\varphi)\big)\mathbb{1}_{\Omega^{>\delta}}\right] = \\ &= \frac{e^{-\beta H(\varphi)}}{Z_{\beta,\sigma,m,N}} \int_{\Omega^{>\delta}} \tfrac{1}{|\mathcal{T}|}\big(H(\omega) - H(\varphi)\big)e^{-\beta(H(\omega)-H(\varphi))}\,\mu(d\omega) \\ &\leq \frac{e^{-\beta H(\varphi)}}{\beta Z_{\beta,\sigma,m,N}} \int_{\Omega^{>\delta}} e^{-\frac{\beta}{2}(H(\omega)-H(\varphi))} e^{\frac{\beta}{4}(H(\omega)-H(\varphi)-\delta|\mathcal{T}|)}\,\mu(d\omega) \\ &\leq \frac{e^{-\beta H(\varphi)}}{\beta Z_{\beta,\sigma,m,N}} \int_{\Omega} e^{-\frac{\beta}{4}(H(\omega)-H(\varphi)+\delta|\mathcal{T}|)}\,\mu(d\omega) \\ &\leq \frac{e^{-\beta H(\varphi)}}{\beta Z_{\beta,\sigma,m,N}} e^{-\frac{\beta}{4}\delta c_0 N^d} \int_{\Omega} e^{-\frac{\beta}{4}c_{20}\|V-R\|^2_{L^2(\Lambda_N)}}\,d\mu\,, \end{aligned} \tag{5.42}$$

where we used Lemma 5.7 and $|\mathcal{T}| \geq c_0 N^d$ in the last step. Now we partition Ω into $\Omega_n := \{\omega \in \Omega : |\mathcal{P} \setminus \mathcal{P}^{\mathrm{ext}}| = n\}$, $n \in \mathbb{N}$. Using Lemma 5.12, we estimate the integral in the last line restricted to Ω_n

$$\begin{aligned} &\int_{\Omega_n} e^{-\frac{\beta}{4}c_{20}\|V-R\|^2_{L^2(\Lambda_N)}}\,d\mu \leq \\ &\leq \sum_{T\in\mathbf{T}_n^{\Sigma}} \int_{\Omega_n} \mathbb{1}_{T\bowtie\mathcal{T}}\, e^{-\frac{\beta}{4}c_{20}c_{23}\sum_{l=2}^{n}\left|(X_{\eta(l)}-X_{\eta(k_T(l))})-R^t\xi_l\right|^2}\,d\mu \\ &\leq \sum_{T\in\mathbf{T}_n^{\Sigma}} \int_{\Lambda_N^n} e^{-\frac{\beta}{4}c_{20}c_{23}\sum_{l=2}^{n}\left|(x_l-x_{k_T(l)})-R^t\xi_l\right|^2}\,\frac{dx_1}{\lambda(\Lambda_N)}\cdots\frac{dx_n}{\lambda(\Lambda_N)} \end{aligned} \tag{5.43}$$

where we used $\mathbb{1}_{T \bowtie \mathcal{T}} \leq 1$ and the fact that $X_{\eta(k)}$, $1 \leq k \leq n$, are independent and uniformly distributed on Λ_N. For each tree T, we define the matrix $M_T = (M_{kl})_{kl} \in \mathbb{R}^{n \times n}$ as follows: $M_{kk} = 1$, $M_{kl} = -1$ if $k = k_T(l)$ and $M_{kl} = 0$ else. Then $\det M_T = 1$ since all diagonal entries are 1 and M_T is a lower triangular matrix as $k_T(l) < l$. Using the transformation

$$y = M_T x - R^t \xi$$

with $x = (x_1, \ldots, x_n)^t$, $y = (y_1, \ldots, y_n)^t$ and $\xi = (0, \xi_2, \ldots, \xi_n)^t$, we continue

$$\begin{aligned}(5.43) &= \sum_{T \in \mathbf{T}_n^\Sigma} \frac{1}{\lambda(\Lambda_N)^n} \int_{y[\Lambda_N^n]} \exp\Big[-\tfrac{\beta}{4} c_{20} c_{23} \sum_{l=2}^n |y_l|^2\Big]\, dy_1 \ldots dy_n \\ &\leq \sum_{T \in \mathbf{T}_n^\Sigma} \frac{\lambda(y_1[\Lambda_N])}{\lambda(\Lambda_N)^n} \Big(\int_{\mathbb{R}^d} e^{-\frac{\beta}{4} c_{20} c_{23} |y_2|^2}\, dy_2 \Big)^{n-1} \\ &= \sum_{T \in \mathbf{T}_n^\Sigma} \frac{1}{\lambda(\Lambda_N)^{n-1}} \big(\beta c_{45}\big)^{-\frac{d}{2}(n-1)} = n^{n-2} |\Sigma|^n \big(\lambda(\Lambda_N)(\beta c_{45})^{\frac{d}{2}}\big)^{-(n-1)} \end{aligned} \tag{5.44}$$

with $c_{45} := c_{20} c_{23}/(8\pi)$. In the last line we used first $y_1[\Lambda_N] = \Lambda_N$ and second $|\mathbf{T}_n^\Sigma| = n^{n-2} |\Sigma|^n$ by Cayley's formula.

Lemma 5.6, Assertions (e) and (d), state that $\lambda(\Lambda_N) = c_{17} N^d$ and $\Omega_n = \{|\mathcal{P} \setminus \mathcal{P}^{\text{ext}}| = n\} = \emptyset$ if $n \notin A := [c_{12} N^d, c_{13} N^d] \cap \mathbb{N}$, respectively. Therefore (5.44) implies, with $c_{46} = c_{45}^{\frac{d}{2}} c_{17}/(c_{13} \Sigma)$,

$$\begin{aligned}\int_\Omega e^{-\frac{\beta}{4} c_{20} \|V - R\|^2_{L^2(\Lambda_N)}}\, d\mu &\leq \sum_{n \in A} n^{n-2} |\Sigma|^n \big(\lambda(\Lambda_N)(\beta c_{45})^{\frac{d}{2}}\big)^{-(n-1)} \\ &\leq \sum_{n \in A} |\Sigma|^{-1} \Big(\frac{c_{13} N^d |\Sigma|}{c_{17} N^d (\beta c_{45})^{\frac{d}{2}}} \Big)^{n-1} = \sum_{n \in A} |\Sigma|^{-1} \big(c_{46} \beta^{\frac{d}{2}}\big)^{-(n-1)} \\ &\leq (c_{13} - c_{12}) N^d |\Sigma|^{-1} \big(c_{46} \beta^{\frac{d}{2}}\big)^{-(c_{12} N^d - 1)} \\ &\leq e^{-N^d [c_{24} \log \beta - c_{47}^{\mathbb{R}}]} \end{aligned} \tag{5.45}$$

for $\beta \geq \beta_0 := c_{46}^{-\frac{2}{d}} > 0$ and some constants $c_{24} > 0$ and $c_{47}^{\mathbb{R}} \in \mathbb{R}$.

Using Lemma 5.11 and (5.45), we estimate (5.42) further:

$$\begin{aligned} & E_{\beta,\sigma,m,N}\Big[\tfrac{1}{|\mathcal{T}|}\big(H_{\sigma,m,N}(\cdot) - H_{\sigma,m,N}(\varphi)\big) \mathbb{1}_{\Omega^{>\delta}}\Big] \leq \\ &\leq \tfrac{1}{\beta} e^{+N^d[\beta\gamma + c_{22}(\gamma,m)]} e^{-\frac{\beta}{4}\delta c_0 N^d} e^{-N^d[c_{24} \log \beta - c_{47}^{\mathbb{R}}]} \\ &= \tfrac{1}{\beta} \exp\big(- N^d \big[\beta \tfrac{c_0}{4} \delta - \beta\gamma + c_{24} \log \beta - c_{47}^{\mathbb{R}} - c_{22}(\gamma, m)\big]\big) \\ &= \tfrac{1}{\beta} \exp\big(- N^d \big[\beta \tfrac{c_0}{8} \delta + c_{24} \log \beta - c_{25}^{\mathbb{R}}(\delta, m)\big]\big) \end{aligned} \tag{5.46}$$

with $\gamma = \frac{c_0}{8}\delta$ and $c_{25}^{\mathbb{R}}(\delta, m) = c_{47}^{\mathbb{R}} + c_{22}(\frac{c_0}{8}\delta, m) \in \mathbb{R}$. The combination of (5.41) and (5.46) yields the conclusion of the lemma. □

5.3.6 Results

Corollary 5.14. *The following statements hold for all $m \geq m_0$:*

$$\lim_{\beta\to\infty} \limsup_{N\to\infty} \sup_{\sigma\geq\sigma_0(N,m)} E_{\beta,\sigma,m,N}\big[\tfrac{1}{|\mathcal{T}|}\big(H_{\sigma,m,N}(\cdot) - H_{\sigma,m,N}(\varphi)\big)\big] \;=\; 0 \tag{5.47}$$

$$\lim_{\beta\to\infty} \limsup_{N\to\infty} \sup_{\sigma\geq\sigma_0(N,m)} E_{\beta,\sigma,m,N}\big[\tfrac{1}{|\mathcal{T}|} \inf_{R\in\mathrm{SO}(d)} \sum_{\square\in\mathcal{T}} \|V - R\|^2_{L^2(\square)}\big] \;=\; 0 \tag{5.48}$$

$$\lim_{\beta\to\infty} \limsup_{N\to\infty} \sup_{\sigma\geq\sigma_0(N,m)} E_{\beta,\sigma,m,N}\big[\tfrac{1}{\lambda(\Lambda_N)} \inf_{R\in\mathrm{SO}(d)} \|V - R\|^2_{L^2(\Lambda_N)}\big] \;=\; 0 \tag{5.49}$$

Proof. Let $\delta > 0$. We define

$$f(\beta,\delta,m) := \beta\tfrac{c_0}{8}\delta + c_{24}\log\beta - c^{\mathbb{R}}_{25}(\delta,m)\,.$$

Then $\lim_{\beta\to\infty} f(\beta,\delta,m) = \infty$ for fixed δ and m. Lemma 5.13 states that for all $\beta \geq \beta_0$ and $N \geq N_0(\delta,m)$

$$\sup_{\sigma\geq\sigma_0(N,m)} E_{\beta,\sigma,m,N}\big[\tfrac{1}{|\mathcal{T}|}\big(H_{\sigma,m,N}(\cdot)-H_{\sigma,m,N}(\varphi)\big)\big] \;\leq\; \delta+\tfrac{1}{\beta}e^{-N^d f(\beta,\delta,m)} \;\leq\; \delta+\tfrac{1}{\beta}e^{-f(\beta,\delta,m)}$$

if $f(\beta,\delta,m) > 0$ (which is fulfilled for large enough β). Therefore

$$\sup_{N\geq N_0(\delta,m)} \sup_{\sigma\geq\sigma_0(N,m)} E_{\beta,\sigma,m,N}\big[\tfrac{1}{|\mathcal{T}|}\big(H_{\sigma,m,N}(\cdot) - H_{\sigma,m,N}(\varphi)\big)\big] \;\leq\; \delta + \tfrac{1}{\beta}e^{-f(\beta,\delta,m)}\,,$$

which implies

$$\begin{aligned}
\limsup_{\beta\to\infty} \limsup_{N\to\infty} \sup_{\sigma\geq\sigma_0(N,m)} & E_{\beta,\sigma,m,N}\big[\tfrac{1}{|\mathcal{T}|}\big(H_{\sigma,m,N}(\cdot) - H_{\sigma,m,N}(\varphi)\big)\big] \\
& \leq\; \lim_{\beta\to\infty} \big(\delta + \tfrac{1}{\beta}e^{-f(\beta,\delta,m)}\big) \;=\; \delta
\end{aligned}$$

and therefore claim (5.47) with "≤ 0" instead of "$= 0$" and with "lim sup" instead of "lim" since $\delta > 0$ was arbitrary.

Lemma 5.7 states that there exists $R(\omega)$ such that

$$\inf_{R\in\mathrm{SO}(d)} \|V(\omega) - R\|^2_{L^2(\Lambda_N)} \;\leq\; \|V(\omega) - R(\omega)\|^2_{L^2(\Lambda_N)} \;\leq\; \tfrac{1}{c_{20}}\big(H_{\sigma,m,N}(\omega) - H_{\sigma,m,N}(\varphi)\big)\,.$$

Thus we can estimate using also Lemma 5.6(f)

$$\begin{aligned}
0 \;&\leq\; E_{\beta,\sigma,m,N}\big[\tfrac{1}{|\mathcal{T}|} \inf_{R\in\mathrm{SO}(d)} \sum_{\square\in\mathcal{T}} \|V - R\|^2_{L^2(\square)}\big] \\
&\leq\; E_{\beta,\sigma,m,N}\big[\tfrac{1}{c_{15}\lambda(\Lambda_N)} \inf_{R\in\mathrm{SO}(d)} \|V - R\|^2_{L^2(\Lambda_N)}\big] \\
&\leq\; E_{\beta,\sigma,m,N}\big[\tfrac{1}{c_{15}c_{16}|\mathcal{T}|} \inf_{R\in\mathrm{SO}(d)} \|V - R\|^2_{L^2(\Lambda_N)}\big] \\
&\leq\; \tfrac{1}{c_{15}c_{16}c_{20}} E_{\beta,\sigma,m,N}\big[\tfrac{1}{|\mathcal{T}|}\big(H_{\sigma,m,N}(\cdot) - H_{\sigma,m,N}(\varphi)\big)\big]\,.
\end{aligned}$$

Therefore, the already proven version of claim (5.47), namely the one with "≤ 0" and "lim sup", implies the real claim (5.47) as well as claims (5.48) and (5.49). □

Proof of Theorem 5.1. It is exactly statement (5.48) of Corollary 5.14 above. □

5.4 Some Concrete Models

In this section, we want to give two concrete models to which we can apply the results of the previous sections. Thereto we have to choose all components stated in the beginning of Section 5.1. First we consider a model on the triangular lattice which is an analogue to the model considered in [HMR-13]. Then we work with the d-dimensional cubic lattice. Other models can be constructed similarly.

5.4.1 Two-dimensional Triangular Lattice

As already stated, the following model is an analogue to [HMR-13]. Thus we work with their set-up and fix

(a) a real-valued potential function ϕ defined in an open interval containing 1 such that ϕ is twice continuously differentiable with $\phi'' > 0$ and $\phi'(1) = 0$,

(b) an $\alpha > 0$ so small that ϕ is defined on $[1-\alpha, 1+\alpha]$ and that [HMR-13, Corollary 2.4] holds and

(c) an $\ell \in (1-\alpha/2, 1+\alpha/2)$.

This are almost literally the same assumptions as in [HMR-13, page 2]. We only use the letter ϕ for the potential since V has a different meaning here.

We identify $\mathbb{C}$ and $\mathbb{R}^2$ and work on the triangular lattice $A_2 = \mathbb{Z} + \tau\mathbb{Z}$ with $\tau = e^{i\pi/3}$ and edges formed by nearest neighbours. In the following, we choose the components of our model.

1. Let us define the tessellation $\mathcal{M}$ of $\mathbb{R}^2$ first. All tiles will have the same type, i.e. $I = \{1\}$. Therefore we omit the superscript $i = 1$ in the following. Let the standard tile $\boxtimes$ be the triangle with vertices $s_1 := 0$, $s_2 := \ell 1$ and $s_3 := \ell\tau$, i.e.
$$\boxtimes := \left\{ \lambda_1 \ell + \lambda_2 \ell\tau \mid \lambda_1 . \lambda_2 \geq 0,\ \lambda_1 + \lambda_2 \leq 1 \right\}.$$
Then the tessellation $\mathcal{M}$ is given by
$$\mathcal{M} := \left\{ \square = z + \xi\boxtimes \mid z \in \ell A_2, \xi \in \{1, \tau\} \right\}.$$

2. We choose the parameter $\varepsilon \in (0, \frac{\alpha}{4})$ arbitrary.

3. We choose the parameter $\rho \in (0, \frac{\ell}{3})$ arbitrary.

4. We choose the parameter $c_0 > 0$ arbitrary.

5. The local Hamiltonian is induced by the potential ϕ and defined by
$$\begin{aligned} H_{\text{loc}}: \quad & \mathcal{N}_\varepsilon(\boxtimes) \to \mathbb{R} \\ \square = \text{hull}\{x_1, x_2, x_3\} & \mapsto \tfrac{1}{2}\big(\phi(|x_1 - x_2|) + \phi(|x_2 - x_3|) + \phi(|x_3 - x_1|)\big) \end{aligned}$$

where x_1, x_2, x_3 are the corners of $\square$. Since $|x_1 - x_2| \le |x_1 - s_1| + |s_1 - s_2| + |s_2 - x_2| \le \ell + 2\varepsilon < 1 + \alpha$ and similarly $|x_1 - x_2| \ge 1 - \alpha$ and for the other vertex-pairs we conclude that H_{loc} is well-defined. Moreover, it inherits continuity from ϕ. Lemma 5.15 below shows that inequality (5.3) is fulfilled.

6. Finally we define the quantity S measuring the surface of the crystal by

$$S := |\partial \mathcal{P}|$$

such that condition (5.4) is obviously fulfilled (with $c_3 = 1$).

The upcoming lemma shows that the local Hamiltonian indeed fulfils inequality (5.3).

Lemma 5.15. *There are constants $c_1 > 0$ and $c_2^{\mathbb{R}} \in \mathbb{R}$ (depending on ϕ) such that inequality* (5.3) *holds for all $\square \in \mathcal{N}_\varepsilon(\boxtimes)$, i.e.*

$$H_{\text{loc}}(\square) - H_{\text{loc}}(\boxtimes) \;\ge\; c_1 \| \operatorname{dist}(\nabla v_\square, \mathrm{SO}(2))\|^2_{L^2(\square)} + c_2^{\mathbb{R}}\big(\lambda(\square) - \lambda(\boxtimes)\big)\,,$$

where $v_\square$ is the affine linear map mapping $\square$ to $\boxtimes$.

Proof. This is a more or less direct consequence of Corollary 2.4 in [HMR-13]. Let x_1, x_2 and x_3 be the corners of $\square \in \mathcal{N}_\varepsilon(\boxtimes^i)$. By the definition of H_{loc}, we have

$$H_{\text{loc}}(\square) - H_{\text{loc}}(\boxtimes) = \tfrac{1}{2}\big(\phi(|x_1 - x_2|) + \phi(|x_2 - x_3|) + \phi(|x_3 - x_1|) - 3\phi(\ell)\big)\,. \quad (5.50)$$

Moreover, [HMR-13, Corollary 2.4] states (in our notation)

$$\begin{aligned} &\phi(|x_1 - x_2|) + \phi(|x_2 - x_3|) + \phi(|x_3 - x_1|) - 3\phi(\ell) - p(\ell)\big(\lambda(\square) - \lambda(\boxtimes)\big) \\ &\quad \asymp_\phi \operatorname{dist}(\ell^{-1}\nabla\omega, \mathrm{SO}(2))^2 \end{aligned} \quad (5.51)$$

where $p(\ell) = 2\sqrt{3}\phi'(\ell)/\ell$ and ω is the affine linear map mapping $0 \mapsto x_1$, $1 \mapsto x_2$ and $\tau \mapsto x_3$. Since $v_\square$ is the affine linear map mapping $x_1 \mapsto 0$, $x_2 \mapsto \ell$ and $x_3 \mapsto \ell\tau$, we conclude

$$v_\square \circ \omega = \ell \,\mathrm{Id} \qquad \text{and therefore} \qquad \ell^{-1}\nabla\omega = (\nabla v_\square)^{-1}\,. \quad (5.52)$$

Now we use the following fact: For all $A \in \mathbb{R}^{2\times 2}$ which are close to SO(2) one has

$$\operatorname{dist}(A^{-1}, \mathrm{SO}(2))^2 \asymp \operatorname{dist}(A, \mathrm{SO}(2))^2\,.$$

Applying this fact to $A = \nabla v_\square$, which is close to SO(2) as $|x_j - s_j| \le \varepsilon$ $(j = 1, 2, 3)$, yields

$$\operatorname{dist}(\ell^{-1}\nabla\omega, \mathrm{SO}(2))^2 \asymp \operatorname{dist}(\nabla v_\square, \mathrm{SO}(2))^2\,. \quad (5.53)$$

since $(\nabla v_\square)^{-1} = \ell^{-1}\nabla\omega$ by (5.52). Combining equations (5.50), (5.51), (5.53) and $\lambda(\square) \asymp 1$ yields the lemma. $\square$

We recall the definition of the Hamiltonian and the probability measure in the end of Section 5.1. Thereto let $\beta > 0$, $\sigma > 0$, $m \geq m_0$ and $N \in \mathbb{N}$. We define the Hamiltonian

$$H_{\sigma,m,N} := \sum_{\square\in\mathcal{T}} H_{\mathrm{loc}}(\square) + \sigma S - m|\mathcal{P}|$$

and the probability measure $P_{\beta,\sigma,m,N}$ via

$$\frac{dP_{\beta,\sigma,m,N}}{d\mu} := \frac{1}{Z_{\beta,\sigma,m,N}} e^{-\beta H_{\sigma,m,N}} \qquad \text{with} \quad Z_{\beta,\sigma,m,N} := \int_\Omega e^{-\beta H_{\sigma,m,N}}\, d\mu\,.$$

One may be bothered by the fact that edges inside the crystal $\mathcal{T}$ appear twice in the Hamiltonian whereas boundary edges appear only once. But this disturbance can be fixed using the following alternative tilde-versions. Let us define the Hamiltonian

$$\tilde{H}_{\sigma,m,N} := \sum_{\substack{x,y\in\mathcal{P} \\ x\sim y \text{ in } \mathcal{T}}} \phi\big(|x-y|\big) + \sigma|\partial\mathcal{P}| - m|\mathcal{P}|\,.$$

where $x \sim y$ in $\mathcal{T}$ iff there exists $\square \in \mathcal{T}$ with $x, y \in \square$ and $x \neq y$. Then the probability measure $\tilde{P}_{\beta,\sigma,m,N}$ is defined via

$$\frac{d\tilde{P}_{\beta,\sigma,m,N}}{d\mu} := \frac{1}{\tilde{Z}_{\beta,\sigma,m,N}} e^{-\beta \tilde{H}_{\sigma,m,N}} \qquad \text{with} \quad \tilde{Z}_{\beta,\sigma,m,N} := \int_\Omega e^{-\beta \tilde{H}_{\sigma,m,N}}\, d\mu\,.$$

We denote the expectation with respect to $P_{\beta,\sigma,m,N}$ with $E_{\beta,\sigma,m,N}$ and the expectation with respect to $\tilde{P}_{\beta,\sigma,m,N}$ with $\tilde{E}_{\beta,\sigma,m,N}$.

Then we have the following corollary to Theorem 5.1.

Corollary 5.16. *There exist $m_0 \in \mathbb{R}$ and $\sigma_0(N,m) \asymp N^2 + m$ such that the rotational symmetry of the crystal is broken in the following sense:*

$$\forall\, m \geq m_0 : \quad \lim_{\beta\to\infty} \limsup_{N\to\infty} \sup_{\sigma\geq\sigma_0(N,m)} E_{\beta,\sigma,m,N}\left[\inf_{R\in\mathrm{SO}(2)} \frac{1}{|\mathcal{T}|} \sum_{\square\in\mathcal{T}} \|V-R\|^2_{L^2(\square)}\right] = 0$$

as well as

$$\forall\, m \geq m_0 : \quad \lim_{\beta\to\infty} \limsup_{N\to\infty} \sup_{\sigma\geq\sigma_0(N,m)} \tilde{E}_{\beta,\sigma,m,N}\left[\inf_{R\in\mathrm{SO}(2)} \frac{1}{|\mathcal{T}|} \sum_{\square\in\mathcal{T}} \|V-R\|^2_{L^2(\square)}\right] = 0$$

holds.

Proof. For $E_{\beta,\sigma,m,N}$, this is exactly the statement of Theorem 5.1. For $\tilde{E}_{\beta,\sigma,m,N}$, we observe that

$$\big|H_{\sigma,m,N} - \tilde{H}_{\sigma,m,N}\big| \leq \Bigg|\sum_{\substack{x,y\in\partial\mathcal{P} \\ x\sim y \text{ in } \mathcal{T}}} \tfrac{1}{2}\phi\big(|x-y|\big)\Bigg| \leq c_{48}|\partial\mathcal{P}| = c_{48} S$$

with $c_{48} := 6 \sup_{t\in[1-\alpha,1+\alpha]} |\phi(t)|$. Therefore

$$\tilde{H}_{\sigma,m,N} \;\geq\; H_{\sigma,m,N} - c_{48} S \;=\; \sum_{\square\in\mathcal{T}} H_{\text{loc}}(\square) + (\sigma - c_{48}) S - m|\mathcal{P}| \;=\; H_{\sigma-c_{48},m,N}$$

and analogously

$$\tilde{H}_{\sigma,m,N} \;\leq\; H_{\sigma+c_{48},m,N}\,.$$

Thus $\tilde{Z}_{\beta,\sigma,m,N} \geq Z_{\beta,\sigma+c_{48},m,N}$ and

$$\begin{aligned} &\tilde{E}_{\beta,\sigma,m,N}\big[\inf_{R\in\text{SO}(2)} \tfrac{1}{|\mathcal{T}|} \sum_{\square\in\mathcal{T}} \|V-R\|^2_{L^2(\square)}\big] \\ &\leq \int_\Omega \frac{e^{-\beta H_{\sigma-c_{48},m,N}}}{Z_{\beta,\sigma+c_{48},m,N}} \inf_{R\in\text{SO}(2)} \tfrac{1}{|\mathcal{T}|} \sum_{\square\in\mathcal{T}} \|V-R\|^2_{L^2(\square)}\, d\mu \end{aligned}$$

Moreover, we observe that the lower bound of the partition sum in Lemma 5.11 does not depend on σ since $H_{\sigma,m,N}(\varphi)$ is independent of σ. Thus we can apply the proof of Theorem 5.1 for $P_{\beta,\sigma-c_{48},m,N}$ if $\sigma - c_{48} \geq \sigma_0(N,m)$ to conclude that the appropriate limit of the right hand side is 0. Therefore the corollary for $\tilde{E}_{\sigma,m,N}$ follows if we enlarge $\sigma_0(N,m)$ by c_{48}. $\square$

5.4.2 Cubic Lattice in d Dimensions

Finally we give an example on the cubic lattice in dimension $d \geq 2$. First we note that a cube is not stabilized by fixing all its edge lengths: it can be arbitrarily flat. Thus there is no chance to be close to SO(d) if only the edge lengths are specified. Therefore we specify the lengths of the diagonals, too. Though not required, we use all diagonals in order to simplify the presentation. The following model is quite similar to the model on the triangular lattice; thus we do not present all technical details.

We define $D := \{A \subset \{1,\ldots,2^d\} : |A| = 2\}$. This set is used to index a pair or "double" of vertices of a cube, or the corresponding edge or diagonal. We shortly write $kj \in D$ for $\{k,j\} \in D$. Similarly as for the model on the triangular lattice, we fix

(a) a tuple of real-valued potential functions ϕ_{kj}, $kj \in D$, defined in an open interval containing 1 such that each ϕ_{kj} is twice continuously differentiable with $\phi''_{kj} > 0$ and $\phi'_{kj}(1) = 0$,

(b) an $\alpha > 0$ so small that each ϕ_{kj} is defined on $[1-\alpha, 1+\alpha]$ and that Lemma 5.17 below holds and

(c) an $\ell \in (1-\alpha/2, 1+\alpha/2)$.

Using this input, we define the model according to the set-up in Section 5.1. First we choose the parameters $\varepsilon \in (0, \frac{\alpha}{4})$, $\rho \in (0, \frac{\ell}{3})$ and $c_0 > 0$ arbitrary. The tessellation

$\mathcal{M}$ will be induced by the lattice $\ell\mathbb{Z}^d$. Again there is only one tile type such we can omit the superscript i. The standard tile $\boxtimes$ is the cube $\boxtimes = \{(z_1, \ldots, z_d) \in \mathbb{R}^d \mid 0 \leq z_1, \ldots, z_d \leq \ell\}$; its corners are denoted by $s_1, \ldots, s_{2^d}$. Then $\mathcal{M} := \{z + \boxtimes \mid z \in \ell\mathbb{Z}^d\}$. If a perturbed cube $\square \in \mathcal{N}_\varepsilon(\boxtimes)$ has corners $x_1, \ldots, x_{2^d}$, we define its local Hamiltonian using the given potential functions as follows:

$$H_{\mathrm{loc}}(\square) := \sum_{kj\in D} \phi_{kj}\left(\frac{|x_k - x_j|}{\ell^{-1}|s_k - s_j|}\right).$$

Thus we allow different potentials for different edges or diagonals. Similarly to the example on the triangular lattice we conclude that H_{loc} is well-defined and continuous; Lemma 5.17 below shows that inequality (5.3) is fulfilled. Again we define the quantity S measuring the surface of the crystal by $S := |\partial\mathcal{P}|$ such that condition (5.4) is obviously fulfilled. We still need

Lemma 5.17. *For sufficiently small $\alpha > 0$, there are constants $c_1 > 0$ and $c_2^{\mathbb{R}} \in \mathbb{R}$ such that inequality* (5.3) *holds for all $\square \in \mathcal{N}_\varepsilon(\boxtimes)$, i.e.*

$$H_{\mathrm{loc}}(\square) - H_{\mathrm{loc}}(\boxtimes) \geq c_1 \| \operatorname{dist}(\nabla v_\square, \mathrm{SO}(d))\|^2_{L^2(\square)} + c_2^{\mathbb{R}}\big(\lambda(\square) - \lambda(\boxtimes)\big),$$

where $v_\square$ is the affine linear map mapping $\square$ to $\boxtimes$.

Proof. The proof is quite similar to the proofs of Lemma 2.2, Lemma 2.3 and Corollary 2.4 in [HMR-13]. In fact, it generalises their arguments to higher dimensions. Therefore, we present not all technical details.

Let a tile $\square \in \mathcal{N}_\varepsilon(\boxtimes)$ with corners $x_1, \ldots, x_{2^d}$ be given. We abbreviate

$$\xi_{kj} := \frac{|x_k - x_j|}{\ell^{-1}|s_k - s_j|}$$

for $kj \in D$. There exists a twice continuously differentiable function

$$f : \mathbb{R}_+^{|D|} \to \mathbb{R} \qquad \text{with} \qquad \lambda(\square) = f\big(\xi_{kj} : kj \in D\big)$$

for $\square \in \mathcal{N}_\varepsilon(\boxtimes)$. Using a Taylor expansion around $(\ell, \ldots, \ell)$, we conclude

$$\lambda(\square) - \lambda(\boxtimes) = \sum_{kj\in D} \partial_{kj} f(\ell, \ldots, \ell)\,(\xi_{kj} - \ell) + O\big(\textstyle\sum(\xi_{kl} - \ell)^2\big).$$

Note that $b := \inf_{kl} \partial_{kj} f(\ell, \ldots, \ell) > 0$ since increasing an edge length increases the volume. It follows that

$$b \sum_{kj\in D} |\xi_{kj} - \ell| \leq \sum_{kj\in D} \partial_{kj} f(\ell, \ldots, \ell)\,|\xi_{kj} - \ell| \leq |\lambda(\square) - \lambda(\boxtimes)| + O\big(\textstyle\sum(\xi_{kl} - \ell)^2\big).$$

Now use $\sup|\phi'_{kj}(l)| \leq \alpha \sup|\phi''_{kj}(l)|$, where the suprema are taken over all $kj \in D$ and $l \in [1 - \alpha/2, 1 + \alpha/2]$, to conclude

$$\begin{aligned}\sum_{kj\in D} \phi'(\ell)(\xi_{kj} - \ell) &\geq -\sup|\phi'_{kj}(l)| \sum_{kj\in D} |\xi_{kj} - \ell| \\ &\geq -\tfrac{\alpha}{b} \sup|\phi''_{kj}(l)| \Big(\lambda(\square) - \lambda(\boxtimes) + O\big(\textstyle\sum(\xi_{kl} - \ell)^2\big)\Big).\end{aligned}$$

Note that we can ignore the absolute value of $|\lambda(\square)-\lambda(\boxtimes)|$ since $\lambda(\square) \geq \lambda(\boxtimes)$ holds for all $\square \in \mathcal{N}_\varepsilon(\boxtimes)$. Applying Taylor's Theorem to ϕ_{kj}, $kj \in D$, yields with the just obtained estimate

$$\begin{aligned} H_{\text{loc}}(\square) - H_{\text{loc}}(\boxtimes) &= \sum_{kj\in D} \Big(\phi_{kj}(\xi_{kj}) - \phi_{kj}(\ell)\Big) \\ &= \sum_{kj\in D} \Big(\phi'_{kj}(\ell)\,(\xi_{kj}-\ell) + \tfrac{1}{2}\phi''_{kj}(\ell)\,(\xi_{kj}-\ell)^2 + o\big((\xi_{kj}-\ell)^2\big)\Big) \\ &\geq -\tfrac{\alpha}{b}\sup|\phi''_{kj}(l)|\big(\lambda(\square)-\lambda(\boxtimes)\big) + \inf\big[\tfrac{1}{2}\phi''_{kj}(l)\big]\sum_{kj\in D}(\xi_{kj}-\ell)^2 \\ &\quad + o\big(\textstyle\sum(\xi_{kj}-\ell)^2\big) - \tfrac{\alpha}{b}\sup|\phi''_{kj}(l)|\,O\big(\textstyle\sum(\xi_{kl}-\ell)^2\big) \\ &\geq c_2^{\mathbb{R}}\big(\lambda(\square)-\lambda(\boxtimes)\big) + c_{49}\sum_{kj\in D}(\xi_{kj}-\ell)^2 \end{aligned} \tag{5.54}$$

with $c_2^{\mathbb{R}} = -\alpha\sup|\phi''_{kj}(l)|/b \in \mathbb{R}$ and some constant $c_{49} > 0$ for small enough $\alpha > 0$ since $\inf \frac{1}{2}\phi''_{kj}(l) > 0$, where the infimum is taken over all $kj \in D$ and $l \in [1-\alpha/2, 1+\alpha/2]$.

It remains to bound $\sum_{kj}(\xi_{kj}-\ell)^2$ in terms of $\|\operatorname{dist}(\nabla v_\square, \mathrm{SO}(2))\|^2_{L^2(\square)}$. Thereto we consider any simplex $\triangle \subset \square$ such that $v_\square$ is affine linear on $\triangle$. Let $\tilde{D} = \tilde{D}_\triangle \subset D$ denote the corresponding set of vertex pairs of the simplex. Let $kj \in \tilde{D}$. Setting $M := (\nabla v_\square)^{-1}$, which is constant on $\triangle$, yields $x_k - x_j = M(s_k - s_j)$ as $v_\square$ maps x_k to s_k and x_j to s_j. Using also $\ell^{-2}|s_k - s_j|^2 \asymp 1$ we conclude

$$\begin{aligned} \xi_{kj} - \ell &\asymp \ell^{-2}|s_k-s_j|^2(\xi_{kj}^2-\ell^2) \;=\; |x_k-x_j|^2 - |s_k-s_j|^2 \\ &= |M(s_k-s_j)|^2 - |s_k-s_j|^2 \;=\; \langle (s_k-s_j), (M^*M - \mathrm{Id})(s_k-s_j)\rangle \end{aligned}$$

We define a norm $\|Q\|_s$ of a symmetric $d\times d$-matrix Q by

$$\|Q\|_s := \sqrt{\sum_{kj\in\tilde{D}} \langle (s_k-s_j), Q(s_k-s_j)\rangle^2}\,.$$

This is obviously a semi-norm; since $(s_k - s_j)$, $kj \in \tilde{D}$, are the edges of a simplex, it even is a norm. As in the proof of [HMR-13, Lemma 2.3] we conclude $\|M^*M - \mathrm{Id}\,\|_s \asymp \operatorname{dist}(M, \mathrm{SO}(d))$. Thus we have shown that

$$\sum_{kj\in\tilde{D}} (\xi_{kj}-\ell)^2 \asymp \|M^*M - \mathrm{Id}\,\|_s^2 \asymp \operatorname{dist}(M,\mathrm{SO}(d))^2 \asymp \operatorname{dist}(\nabla v_\square, \mathrm{SO}(d))^2$$

since $\nabla v_\square = M^{-1}$ is close to $\mathrm{SO}(d)$ (for small α) because $\square$ is an ε-perturbation of $\boxtimes$. Using the facts that the Lebesgue measure of any simplex of $\square$ is of order 1 and that each diagonal belongs only to a finite number of simplexes, we conclude

$$\begin{aligned} \sum_{kj\in D} (\xi_{kj}-\ell)^2 &\gtrsim \sum_{\triangle}\sum_{kj\in\tilde{D}_\triangle} (\xi_{kj}-\ell)^2 \asymp \sum_{\triangle} \|\operatorname{dist}(\nabla v_\square, \mathrm{SO}(d))\|^2_{L^2(\triangle)} \\ &= \|\operatorname{dist}(\nabla v_\square, \mathrm{SO}(d))\|^2_{L^2(\square)}\,. \end{aligned}$$

Inserting this inequality into (5.54) completes the proof. □

It follows that all assumptions in Section 5.1 are fulfilled. Therefore the very last corollary needs no further proof.

Corollary 5.18. *The rotational symmetry of the crystal model on the cubic lattice introduced above is broken in the sense of Theorem 5.1.* □

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, §8, Abs. 2 Pkt. 5)

Hiermit erkläre ich an Eides statt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

München, den 1. September 2014

Simon Aumann