O(d,d) Target-Space Duality in String Theory

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit Aspekten von Zielraumdualitäten innerhalb der Theorie geschlossener bosonischer Strings. Zu Beginn wird eine Einführung in generalisierte Geometrie gegeben, die das primäre mathematische Gerüst dieser Arbeit bildet. Analog zur allgemeinen Relativitätstheorie als Theorie von Riemannschen Metriken wird eine vereinheitlichte Theorie zu formulieren versucht, die Hintergründe der String-Theorie – Riemannsche Metriken sowie Kalb-Ramond Zweiformen – mithilfe von Courant-Algebroiden auf dem generalisierten Tangentialbündel beschreibt. Die duale Konfiguration von Hintergründen, gegeben durch eine Metrik und ein Bivektorfeld, wird durch das generalisierte Kotangentialbündel beschrieben. Das Fehlen eines konventionellen Krümmungstensors und die damit ausbleibende Möglichkeit, generalisierte Gravitationstheorien auf Courant-Algebroiden zu formulieren, wird im Detail studiert. Dies führt zum Begriff der Lie-Algebroide, deren Differentialgeometrie sich als adäquat zur Formulierung generalisierter Gravitationstheorien herausstellt. Verschiedene solcher Theorien stehen durch geeignete Homomorphismen in direkter Beziehung zueinander. Dies erweist sich als hilfreich für die Beschreibung nicht-geometrischer Hintergründe.

Zielraumdualität wird durch die sogenannte $O(d,d)$-Dualität beschrieben, welche zweidimensionale nicht-lineare Sigmamodelle für verschiedene Stringhintergründe als identisch enthüllt, deren Hintergründe und Koordinaten durch $O(d,d)$-Transformationen miteinander in Beziehung stehen. Dabei werden insbesondere die Integrabilitätsbedingungen an die dualen Koordinaten mithilfe von Courant-Algebroiden studiert. Neben (nicht-abelscher) T-Dualität beinhaltet $O(d,d)$-Dualität die neuartige Poisson-Dualität, welche von einer Poissonstruktur induziert wird. T- und Poisson-Dualität werden auf den Drei-Torus mit konstantem H-Fluss angewandt, was die Existenz von nicht-geometrischen Hintergründen offenbart. Diese übersteigen konventionelle geometrische Konzepte aufgrund des Fehlens einer globalen Beschreibung.

Das Problem der Beschreibung nicht-geometrische Hintergründe wird mithilfe von generalisierter Geometrie behandelt. Eine vereinheitlichte Erfassung T-dualer Hintergründe basierend auf proto-Lie Bialgebroiden für geometrische und nicht-geometrische Hintergründe wird vorgestellt. Zusammen bilden sie einen Courant-Algebroiden, dessen anomale Jacobi-Identität Bedingungen für das gleichzeitige Auftreten dualer Flüsse liefert. Das Fehlen genereller Gravitationstheorien führt zur Beschränkung auf Lie-Algebroide. Deren Gravitationstheorien ermöglichen eine globale Beschreibung nicht-geometrischer Hintergründe durch eine genaue Vorschrift der Kartenwechsel auf diesen Räumen. Diese Beschreibung lässt sich auf alle möglichen Supergravitationstheorien übertragen.

Die Frage nach einer vereinheitlichten Beschreibung dualer Hintergründe wird durch

einen Zugang mittels einer konformen Feldtheorie, die invariant ist unter T-Dualitäten, wiederaufgenommen, indem duale Koordinaten als gleichwertig betrachtet werden. Die modulare Invarianz der Zustandsumme auf dem Torus sowie die Prämisse der physikalischen intermediären Zustände in der Streuung vierer Tachyonen führt zwangsläufig zur Auftreten der starken Zwangsbedingung der Doppelfeldtheorie für nicht-kompakte Räume. Dies steht im Gegensatz zu torisch kompaktifizierten Räumen, die diese Zwangsbedingung nicht erfordern. Damit werden das Auftreten der starken Zwangsbedingung aufgeklärt und mögliche Abschwächungen plausibel gemacht.

Abstract

In this thesis various aspects of target-space duality in closed bosonic string theory are studied. It begins by introducing generalized geometry as the main mathematical framework. In analogy to general relativity with the Riemannian metric as dynamical quantity, a unified description for string backgrounds – Riemannian metrics together with Kalb-Ramond two-form fields – is approached via Courant algebroids on the generalized tangent bundle equipped with a generalized metric. The dual background configuration, i.e. a metric and a bivector field, is described by the generalized cotangent bundle. The absence of a conventional curvature tensor and consequently the problem of defining generalized gravity theories on Courant algebroids is investigated in detail. This leads to the introduction of Lie algebroids whose differential geometry is suitable for the formulation of gravity theories. Different such theories are shown to be interrelated by appropriate homomorphisms. This proves to be useful for describing non-geometric backgrounds.

Target-space duality is introduced in terms of $O(d,d)$-duality which identifies two-dimensional non-linear sigma models for different string backgrounds as physically equivalent under certain conditions: The backgrounds and coordinates of the dual theories have to be related by certain $O(d,d)$ transformations. In particular, integrability conditions of the dual coordinates are formulated in terms of Courant algebroids. Apart from (non-abelian) T-duality, $O(d,d)$-duality contains the novel Poisson-duality induced by Poisson structures. T- and Poisson-duality are applied to the three-torus with constant H-flux which shows the existence of non-geometric backgrounds. The latter exceed conventional conceptions of geometry as they cannot be described globally.

The problem of describing non-geometric backgrounds is approached with generalizes geometry. A unified description of T-dual backgrounds is given in terms of proto-Lie bialgebroids – one for the geometric sector and another for the non-geometric one. They combine into a Courant algebroid whose anomalous Jacobi identity provides conditions for the concurrent appearance of dual fluxes. The absence of a gravity theory leads to the restriction to Lie algebroids. Their gravity theories allow for a global description of non-geometric backgrounds by an exact prescription for the patching of these backgrounds. The description extends to all possible supergravity theories.

The question whether a unified description of dual backgrounds is possible is reconsidered in a manifestly T-duality invariant conformal field theory approach. Dual coordinates are treated on equal footing. Modular invariance of the one-loop partition function together with the premise of physical intermediate states in four-tachyon scattering inevitably leads to the appearance of the strong constraint of double field theory on non-compact spaces. Toroidally compactified directions do not require a constraint. This explains the appearance of the strong constraint and justifies possible attenuations.

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Contents

Chapter 1

Introduction

Gravity in conjunction with the weak, the electromagnetic, and the strong force constitute the fundamental interactions in nature. Gravity is considerably weaker than the other three and therefore irrelevant in sub-atomic processes described by the standard model of particle physics – a particular quantum field theory. Yet, opposed to the weak and the strong force it is believed to have an infinite range. Thus, along with the electromagnetic force, gravity is predominant in the large scale universe.

At the Planck scale at approximately $10^{19} GeV$ however, general relativity and quantum field theory become both, equally important and individually invalid: A photon used to measure objects smaller then the Planck length $l_{\text{Pl}} \approx 10^{-35} m$ would collapse into a black hole.[1] Then quantum theory – responsible for the particle-like nature of a photon – is inappropriate as no meaningful measurement can be made and general relativity – responsible for the imprint of matter on the shape of the space – is not able to describe the emerging singularity. Consequently, the understanding of the Planck regime and in particular the description of the very early universe (up to 10^{-43} seconds after the big bang singularity) as well as the physics close to a black hole singularity requires the formulation of a quantum theory of gravity.

Unfortunately experimental indications for pursuing certain directions towards quantum gravity are hardly amenable owing to the magnitude of the Planck scale. As a comparison, the Large Hadron Collider works only at the electroweak scale of about $10^3 GeV$. Measurements related to inflation may allow access to energies around $10^{16} GeV$ with first possible imprints of quantum gravity.[2] So far, however, only theoretical principles are to be followed.

The quest for finding a quantum gravity theory is closely connected with the conception of space. Spacetime is dynamical in the general theory of relativity. Its shape is

[1] The reduced Compton wavelength is $\lambda = \hbar (Mc)^{-1}$ with c the speed of light and the Planck length is $l_{\text{Pl}} = \sqrt{\hbar G c^{-3}}$ with G the gravitational constant. Resolving Planck length requires a wavelength $\lambda \leq l_{\text{Pl}}$, which corresponds to a particle of mass $M \geq M_{\text{Pl}} = \sqrt{\hbar c G^{-1}}$. The Schwarzschild radius for a particle of Planck mass is $r_S = 2c^{-2} G M_{\text{Pl}} = 2 l_{\text{Pl}}$. Hence $\lambda \leq r_S$, i.e. the corresponding particle would be smaller than its own Schwarzschild radius. This results in a black hole.

[2] $10^{16} GeV$ is the expected energy scale at which inflation occurs. Recently, hints for primordial B-modes have been found [1]. A well-established theoretical explanation are primordial gravitational waves arising from quantum fluctuations of the gravitational field amplified during inflation (see e.g. [2]). However, also cosmic dust might be responsible for these findings [3].

determined by the energy and matter content through the energy-momentum tensor with gravity manifesting itself as geodesic motion of a free-falling object on this potentially curved spacetime. The standard model of particle physics, on the other hand, is formulated on a fixed background – the four-dimensional flat Minkowski space – and describes the fundamental particles as well as the mediation of forces by point-like objects. Yet, incorporating a dynamical spacetime into quantum field theory complicates the causal description of physical processes: Near the Planck scale, quantum fluctuations cause a permanent change of the matter content and thereby an incessantly altering spacetime which prevents a well-defined notion of causality. Along with the loss of the predictive power of a perturbative quantization of gravity due to an infinite number of undetermined parameters (non-renormalizability), this calls for approaches beyond general relativity and quantum field theory to describe quantum gravity.

1.1 String theory

The failure of a conventional perturbative approach via point particles to quantize gravity might indicate a mathematical inconsistency of the concept of space as a continuum of points at very high energies or equivalently at very small length scales. So far, two strategies have been adopted: Either studying mathematical structures such as non-commutative geometries[3] as well as certain "discretizations" of space[4] which dismiss the notion of points, or changing the probe from a point particle to higher-dimensional objects.

In this thesis the latter approach will be followed by using string theory[5], where the probes are given by strings, i.e. one-dimensional objects. A string probes spacetime very differently compared to point particles as it can wind around compact directions. Thereby it is able to resolve global properties of a space invisible to a point particle.

String theory can be formulated as a two-dimensional non-linear sigma model which describes the embedding of the two-dimensional surface stretched by a moving string – a membrane – into a target-space, spacetime itself. This membrane is called worldsheet, following the notion of a worldline for particles: The action describes the minimal surface between two strings on a given spacetime analogous to the worldline describing the shortest path between two points. This inconspicuous description has far reaching consequences when it is quantized. The string admits infinitely many vibrational modes with every single one constituting a state in the theory. Thus particles are described by the different excitations of a single string, which generically include a graviton and gauge bosons. This is remarkable since it constitutes a quantum theory including quantum gauge theory and quantum gravity. So far, string theory is the only framework not only providing a consistent quantum gravity but also unifying the two distinct fundamental theories.

The combination of gauge and gravity theory is a generic feature of string theory. However, realistic physical models have to be encountered within an enormous set of solutions. Prior to quantization, string theory is unique, but there are four broad classes of consistent quantum theories:

[3]Conne's non-commutative geometry is introduced in [4] and several others are reviewed in [5].

[4]Loop quantum gravity and causal dynamical triangulation are reviewed for example in [6] and [7], respectively.

[5]See for instance the introductory textbooks [8, 9, 10, 11, 12, 13].

- the *bosonic string.* Consistency requires a critical spacetime dimension of 26. Containing no fermions and showing a tachyonic state it is unphysical. Nonetheless, it will be of main interest here as the major geometrical features are included.
- the *type II superstring.* The critical dimension is 10 and it includes fermions in a supersymmetric manner without having a tachyon. The appearance of non-chiral and chiral massless fermions further distinguishes type IIA and type IIB respectively.
- the *heterotic superstring.* It constitutes another supersymmetric theory in ten spacetime dimensions. Two types can be distinguished: one containing a $SO(32)$ gauge field and another containing an $E_8 \times E_8$ gauge theory.
- the *type I superstring.* The only ten dimensional supersymmetric theory containing open strings.

The classical theory admits two-dimensional conformal invariance which has to be preserved during quantization. This restricts the allowed spacetimes. Particularly the high dimensionality of spacetime predicted by string theory has to be reduced in order to meet the observation of a four-dimensional spacetime at the accessible energy scales; the remaining dimensions are compactified. As a result the superstring is considered on a space $M = M_4 \times X_6$ with M_4, a four-dimensional spacetime, and X_6, a compact six-dimensional space sufficiently small to remain hidden. For example, to recover the standard model, M_4 must be the Minkowski space, yet for describing cosmology, M_4 has to be the de Sitter space. But still the possibilities for viable compact six-dimensional spaces – also known as vacua – are vast. This is known as the *string landscape.*

Seen from the four-dimensional perspective, compactification introduces lots of scalar fields describing for instance the size and shape of the internal space. However, there are only two (or a few more) virtually confirmed scalar particles: the Higgs boson and the inflaton(s). The remaining scalars have to be moved to currently undetectable regimes. This procedure is called *moduli stabilization* and can be partially achieved by compactifying on spaces equipped with additional fields and associated *fluxes.*[6]

Thus, despite the predictive power of the initial framework, consistent solutions are far from unique and due to their abundance, finding realistic models proves difficult. In principle, all known vacua can be scanned for realistic models, but the suitable solution might not even be known. Nevertheless, string theory is not understood completely and conceptual progress could achieve both, extension of the possibilities for finding realistic models and narrowing down the possibilities by undiscovered consistency requirements.

1.2 Dualities

String theory is abounding in symmetries, including the remarkable equivalence of different string theories. For example, type IIA theory compactified on a circle of radius R describes the same physics as type IIB theory compactified on a circle of Radius $1/R$. This *T-duality* is just one instance in a web of dualities identifying all five superstring theories with each

[6] See [14] for a review.

other (including M-theory). The present work investigates generalizations of T-duality due to its geometric nature of relating string theories on very different spacetimes.

The massless sector common to all string theories is the bosonic NS-NS sector containing a symmetric tensor field G, a two form field B called *Kalb-Ramond field* and a scalar field, the *dilaton* ϕ. The Kalb-Ramond field can be considered the higher-dimensional analogue of a gauge field and the dilaton is in some sense a quantum correction. In the worldsheet description, these fields determine the shape of the spacetime on which the string resides. To be more precise, the two-dimensional non-linear sigma model describing the worldsheet Σ of a string embedded into the spacetime M is given by

$$S(G,B) = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{|\det h|}\left(h^{\alpha\beta}\,G_{ab} + \varepsilon^{\alpha\beta}\,B_{ab}\right)\partial_\alpha X^a\,\partial_\beta X^b\,.$$

From the two-dimensional perspective[7] the fields X^a are bosons while G and B are non-constant couplings. From the spacetime point of view, however, X^a are the coordinates of spacetime pulled-back to Σ, the symmetric tensor field G is a metric on M, and B induces a torsion[8] through the *H-flux* $H = dB$. Because the metric and the Kalb-Ramond field constitute the data describing the spacetime on which the string evolves, the pair (G,B) is called the *background*.

Now, T-duality identifies different backgrounds (G,B) and (g,b) related by the *Buscher rules* [17, 18] as describing the same quantum physics, i.e. $S(G,B)$ and $S(g,b)$ provide the same quantum theory [19]. For this reason it is referred to as *target-space duality*. This is a genuine feature of string theory caused by the extended nature of the probe in use and has far reaching implications. It has revealed the existence of higher dimensional dynamical objects called *D-branes* [20], the mathematical connection between certain Calabi-Yau manifolds called *mirror symmetry* [21], as well as exotic geometries including non-commutative spaces[9] called *non-geometries* [26, 27].

The unification of quantum theory with gravity is among the most intriguing problems in fundamental physics for which string theory provides a promising framework. In order to be able to utilize its rich structure for constructing realistic theories, further conceptual progress is necessary.

Accordingly, this thesis focuses on the implications of target-space symmetries as they serve the twofold desire phrased at the end of section 1.1: They extend the possibilities of finding realistic models by revealing non-geometric backgrounds as possible vacua while narrowing them down by identification. Moreover, they provide new insights into the geometry of spacetime near the Planck scale. A complete description of quantum gravity requires a background-independent theory accounting for the permanent change of spacetime due to quantum fluctuations which is beyond the scope of the thesis.[10]

[7] Σ carries local coordinates with indices α,β and is equipped with a metric h. ε denotes the Levi-Civita tensor with $\sqrt{|\det h|}\varepsilon \in \{0,\pm 1\}$.

[8] This is only true in the classical case. The failure of H contributing as torsion to quantum corrections is discussed in [15, 16].

[9] In the case of open string theory, non-commutative geometry is well established [22, 23]. It can be indirectly attributed to T-duality as being confined to D-branes. Non-commutative or even non-associative geometry in closed string theory is still speculative [24, 25].

[10] A *String field theory* might achieve this goal [28, 29, 30].

1.3 Thematic scope

In the following the main problems treated in the present work will be explained in more detail.

1.3.1 T-duality

The section is devoted to a more extensive discussion of T-duality [31]. To begin with, its appearance in the mass spectrum of the closed bosonic string compactified on a circle of radius R is reviewed. The circle compactifications amounts to the choice of one direction, say $X^{25}(\tau, \sigma)$ with τ being the time direction on the worldsheet and σ its circular direction which is not periodic. Instead, the string can wind around the circular direction as $X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi R W$ with $W \in \mathbb{Z}$ being the number of windings. In effect this means that the X^{25} direction has to be circled W times in order to return to the initial point. Starting with the sigma model $S(G = \delta, B = 0)$, the dynamics governed by the wave equation is solved by splitting the coordinates into left- and right-movers $X^a(\tau, \sigma) = X^a_L(\tau + \sigma) + X^a_R(\tau - \sigma)$ which makes quantization simple. The mass M of the N^{th} excited state is then given by

$$\alpha' M^2 = \frac{\alpha'}{R^2} P^2 + \frac{R^2}{\alpha'} W^2 + 2(N - 2) \,,$$

with $P \in \mathbb{Z}$ denoting the quantized momentum in the circular direction. Hence, the spectrum is invariant under inversion of the radius of the circle $R \to \frac{\alpha'}{R}$, accompanied with a simultaneous interchange of momentum and winding modes $P \leftrightarrow W$. This symmetry of the spectrum extends to a symmetry of the whole theory by reflecting the right-moving coordinate, i.e. $X^{25}_R \to -X^{25}_R$, while leaving the left-moving part untouched.[11] This procedure also applies to toroidal compactifications – in particular Narain compactifications [32] – and unveils the T-duality group $O(d, d; \mathbb{Z})$. The flaw of this method is its confinement to particular backgrounds and the requirement of the explicit solution of the theory.

In order to investigate the scope of T-duality, a sigma-model approach is taken [17, 19]. T-duality identifies the actions $S(G, B)$ and $S(g, b)$ for two different backgrounds as equivalent. This can be shown by constructing an intermediate action from which both possibilities of describing the same physics can be derived. In principle, the intermediate model is obtained by gauging isometries: The initial background (G, B) admits isometries when the action $S(G, B)$ is invariant under spacetime diffeomorphisms in certain directions which is a global symmetry from the two-dimensional worldsheet perspective. The global symmetry is promoted to a local gauge symmetry. This requires the addition of an auxiliary gauge field A with adjusted transformation properties.[12] In order to be able to return to the initial model, a Lagrange multiplier λ is introduced. The result is a gauged non-linear sigma model $S(G, B; A, \lambda)$. This model can be solved for λ, giving back the initial theory $S(G, B)$ by returning the solution back to the action. This procedure is termed *integrating-out* λ. However, integrating-out A gives back a different theory which turns out to be the

[11] This reflection also applies to the fermionic coordinates for the superstring. In particular, reflecting the right-moving fermion in the circular direction changes the chirality, giving rise to the interchange of IIA and IIB theories.

[12] A is a one-form with values in the Lie algebra associated to the gauged (subgroup of the) isometry group.

sigma model $S(g,b)$ in terms of a different background. This can also be realized in an Hamiltonian approach via canonical transformations [33].

This method works well in the case of abelian isometries.[13] However, gauging isometries seems inappropriate for non-abelian isometries. Even if a background admitting non-abelian isometries could be gauged consistently [36], the two emerging theories will not be dual. This can be traced back to non-trivial holonomies of the gauge field, i.e. parallel transport of a field around a loop in spacetime does not give back the initial field. The possible holonomies must vanish in order to obtain an equivalent dual theory. In the abelian case, the Lagrange multipliers do not transform under gauge transformations – they form a gauge singlet – and can therefore be chosen periodically without difficulties. Choosing the right periodicities then constrains the holonomies to vanish [19]. In the non-abelian case however, the Lagrange multipliers transform under gauge transformations which makes it impossible to introduce such periodicities constraining the holonomies to vanish [37].[14] Even if the procedure is followed in spite of these subtleties, there is neither the possibility to recover the initial model from integrating-out the Lagrange multipliers, nor is the dual model really equivalent. Examples for this observation are studied in [38, 34, 37] and a canonical approach can be found in [39]. Nevertheless, non-abelian T-duality was used recently as solution generating technique in supergravity.[15]

Circumventing the approach of gauging isometries and thereby avoiding the introduction of a problematic auxiliary gauge field might lead to a more thorough understanding of target space dualities. This will be addressed following [49], where a new method for approaching target-space duality is developed. In particular, it contains non-abelian T-dualities and a novel duality termed *Poisson duality*.

1.3.2 Non-geometric backgrounds

T-duality leads to backgrounds which extend the conventional notion of geometry. In the following, this will be explained by means of simple examples whilst touching upon its potential utility for constructing physical theories.

As mentioned before, abelian T-duality in a single direction, say the k$^{\text{th}}$, interchanges the initial background (G,B) with (g,b) via the Buscher rules

$$\begin{aligned} g_{kk} = \frac{1}{G_{kk}}\,, \qquad g_{ka} &= -\frac{B_{ka}}{G_{kk}}\,, \qquad g_{ab} = G_{ab} - \frac{G_{ak}G_{kb} + B_{ak}B_{kb}}{G_{kk}}\,, \\ b_{ka} &= -\frac{G_{ka}}{G_{kk}}\,, \qquad b_{ab} = B_{ab} - \frac{G_{ak}B_{kb} + B_{ak}G_{kb}}{G_{kk}} \end{aligned}$$

with $a, b \neq k$. To get an idea of topology changes due to T-duality, the spacetime $M = \mathbb{R}^2 \times \mathbb{R}^{1,d-3}$ is considered. It is assumed to be equipped with the metric $ds^2 = dr^2 + r^2 d\phi^2 + d\mathbf{x}^2$, where (r,ϕ) denote spherical coordinates on the plane $\mathbb{R}^2$ and $d\mathbf{x}^2$ is the flat Minkowski metric on $\mathbb{R}^{1,d-3}$. In particular, this space is flat and has a vanishing Ricci scalar $R = 0$. In a next step T-duality is performed along the isometric angular ϕ-direction. The Buscher

[13] Global issues are discussed in [34] and are revisited in [35].

[14] Even if consistent periodicities for the Langrange multipliers are introduced, they can not contribute to a local action [37].

[15] See [40, 41, 42, 43, 44, 45, 46, 47, 48].

rules give rise to the new metric $d\tilde{s}^2 = dr^2 + r^{-2}d\tilde{\phi}^2 + d\mathbf{x}^2$. Thereby T-duality has inverted the radial dependence of the angular direction. Although the initial metric was well-defined even at $r = 0$, the dual metric is singular at that point. Furthermore the new Ricci scalar $\widetilde{R} = -2r^{-2}$ is singular at the origin as well. Hence, a singular space was obtained from a simple flat one. This shows the peculiarity of T-duality: Although the dual space admits singularities, it has to be considered a proper string background as it provides the same quantum theory as the initial background.

This argument also applies to the aforementioned non-geometric backgrounds, but their idiosyncrasy is of global nature. As the prototypical example, three spacetime directions are assumed to be compactified on a flat, rectangular torus with metric $ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. Since $\{x^1, x^2, x^3\}$ parametrize directions on the torus, they are periodic and well-defined geometric entities have to be invariant under $x^i \to x^i + 2\pi n$ for some integer n. Moreover, the torus is equipped with a linear Kalb-Ramond field $B = hx^3dx^1 \wedge dx^2$ with $h \in \mathbb{Z}$.[16] Going around the x^3-direction n times is non-trivial; it changes the Kalb-Ramond field by $2\pi hndx^1 \wedge dx^2 = d(2\pi hnx^1dx^2)$. Thus, the change is an exact one-form, i.e. the field changes by a gauge transformation $B \to B + d\xi$ constituting a symmetry of the theory. As a result the background just described is well-defined upon invoking the target-space symmetries of string theory. The background admits isometries along x^1 and x^2 which will be T-dualized together. The resulting background is described by the metric $d\tilde{s}^2 = f(x^3)[(dx^1)^2 + (dx^2)^2] + (dx^3)^2$ and the Kalb-Ramond field $b = -hf(x^3)x^3dx^1 \wedge dx^2$. The function f is given by $f(x) = (1 - hx^2)^{-1}$. As $f(x)$ is not 2π-periodic, neither are g nor b. Also, the change of both g and b can not be compensated by a coordinate transformation or a gauge transformation like in the previous case. Since these two transformations build the target-space symmetry group of the bosonic string – sometimes referred to as the *geometric group* – the latter background is called non-geometric [27]. It is not well-defined globally but nevertheless provides a viable string background. However, so-called *β-transformations* are the proper transition functions for this background. They are not contained in the geometric group and can be considered as a mixture of T-duality and gauge transformations. Nonetheless, it is important to notice that there actually is a transformation serving as proper transition function[17] even for non-geometric backgrounds, which is not arbitrary: coordinate changes, gauge transformations, β-transformations and T-duality in d-dimensions generate the indefinite orthogonal group $O(d, d)$.

The non-geometric background encountered above is known as the *Q-flux* background or *T-fold* [50, 51]. The Q-flux is the analogue of the H-flux on the initial background. The latter is given by $H_{123} = (dB)_{123} = h$ while the former is given in terms of the derivative of a bivector field β as $Q^{12}{}_3 = \partial_3\beta^{12} = h$ [27, 52]. The significance of the fluxes is their appearance in the four-dimensional low energy effective theory associated with the string: gauged supergravity.[18] From the point of view of these theories, the existence of non-geometric fluxes is inevitable in a complete description consistent with T-duality. Especially in the example given above, there must exist yet another flux. The Q-flux

[16] This is only an approximate string vacuum as it is only valid up to linear order in H.

[17] Here the term transition function is understood in the precise sense of changing the local patch of the vector bundle involved. In the case at hand, circling the x^3-direction includes two local patches.

[18] Reviews are found in [53, 54].

background is not isometric in the x^3-direction and therefore T-duality is not applicable. However, type IIA and type IIB supergravity only match completely under T-duality if a flux associated to the forbidden third T-duality is introduced as well. Then, starting from the flat torus with constant H-flux, the following chain of fluxes emerges:

$$H_{abc} \xleftrightarrow{\mathcal{T}_{T(1)}} f^{a}{}_{bc} \xleftrightarrow{\mathcal{T}_{T(2)}} Q^{ab}{}_{c} \xleftrightarrow{\mathcal{T}_{T(3)}} R^{abc} .$$

The f-flux, also known as *geometric flux*, describes the *twisted torus* arising from a single T-duality. The second T-duality gives the Q-flux introduced above and the *R-flux* is conjectured from matching the supergravity theories [27, 55].

The R-flux was argued to elude a geometric description by conventional methods even locally [27]. This seems to be connected to speculations about non-associative structures on the corresponding spaces [56, 24, 25, 57]. However, string theory vacua, whose target-spaces are inaccessible geometrically, are well-known in terms of asymmetric orbifold constructions in conformal field theory [58]: The target space coordinates are described by the pulled-back coordinates X^a, which split into a left and a right moving part (cf. the beginning of section 1.3.1). The left- and right-movers can also be treated asymmetrically which obstructs their forming of a sensible coordinate. The relation between the R-flux and asymmetric strings was first noticed in [57] and further elaborated on in [59, 60].[19]

The immediate question concerning the benefit of non-geometric fluxes arises. One could argue that although they are associated with formerly unknown geometries, they have a well-understood dual counterpart providing the same physical theory. However, the fluxes do appear in the superpotential of the low energy supergravity theories compactified to four dimensions, providing additional parameters [27]. Nevertheless they do not comprise additional degrees of freedom; the restrictions among the simultaneous appearance of the geometric and non-geometric are studied in [27, 62, 63, 64].[20] Two prominent applications of non-geometric configurations are the following:

- Prior to the usage of non-geometric fluxes, stable four-dimensional de Sitter vacua could not be found in string theory[21] because there have always been tachyonic directions. Yet the cosmological constant is measured to be positive [68] and our universe is spatially flat, so the large-scale spacetime geometry used in the cosmological standard model – the Λ-CDM model – is de Sitter. Consequently the existence of such vacua is of utmost importance to build cosmological models from string theory. Only recently a scalar potential induced from geometric as well as non-geometric fluxes have been used to stabilize the moduli in a meta-stable de Sitter minimum [69, 70, 71, 72, 73].

- Many gauged supergravity theories cannot stem from geometric string compactifications. This discrepancy between the landscape of string theory vacua and the possible consistent effective field theories is known as the *swampland* [74, 75], which raises the question whether some potentially realistic theories might be inaccessible via string theory. However, at least part of the swampland can be explained by non-geometric string theory compactifications [76, 77, 60].

[19] See also [61, 26] for general considerations.
[20] In the context of double field theory, the restrictions are studied in [65, 66].
[21] See e.g. [67]

Apart from being an important step for understanding string theory more completely, non-geometric fluxes introduce new possibilities for constructing realistic models.

A simultaneous mathematical description of (non-constant) geometric and non-geometric fluxes [64] as well as the local geometric structure of the associated low energy effective theories in ten dimensions will be presented in this thesis [78, 79, 80]. In particular, the limitations upon a global description of non-geometric backgrounds will be addressed.

1.3.3 Double field theory

The geometric group of string theory consists of coordinate transformations as well as gauge-transformations of the Kalb-Ramond fields. As mentioned in the previous section, non-geometric backgrounds require transition functions beyond the geometric group. For example, the Q-flux background is patched-up by β-transformations. In general, all possible transition functions emerging from target space duality are elements of $O(d,d)$. On that account the conventional gravity theories arising as low-energy effective theories from string theory are not suitable for a global description of such backgrounds.

Along with the T-duality group $O(d,d;\mathbb{Z})$ for string theory compactified on a d-dimensional torus [31], this motivated various attempts to construct duality or $O(d,d)$ invariant theories. In [81, 82], T-duality was realized as a world-sheet symmetry by treating left- and right-moving degrees of freedom on equal footing and by considering objects invariant under reflection of the right-moving coordinates. A geometric target space approach to the problem was pursued in [50, 83, 51, 84]: The usual and the winding coordinates were considered as coordinates of a doubled manifold, termed *doubled geometry*. Many quantum aspects of this theory were studied further in [85, 86, 87], where in particular an $O(d,d)$ invariant target space effective action was presented. Arguably the most prevalent theory now is *double field theory* (DFT).[22] It was developed in [91, 92, 93, 94] as a covariant doubled target space approach to duality symmetries. Whereas in the doubled geometry approach the compact part of space is doubled, in double field theory the whole space-time manifold is doubled.

All this approaches have in common a doubling of degrees of freedom, which have to be reduced to the physical ones. Thus the imposing of constraints is necessary. Since in DFT one treats the massless modes of the closed string, the level matching condition $L_0 - \overline{L}_0 = 0$ must be satisfied. This leads to the so-called weak constraint

$$\partial_a \tilde{\partial}^a f = 0\,, \tag{1.1}$$

where ∂_a and $\tilde{\partial}^a$ denote derivatives with respect to the standard coordinates x^a and the winding coordinates $\tilde{x}_a$, respectively. For consistency of DFT, i.e. in particular for the closure of the symmetry algebra of generalized diffeomorphisms, a stronger version of this constraint has been imposed [92, 94], namely

$$\partial_a f\, \tilde{\partial}^a g + \tilde{\partial}^a f\, \partial_a g = 0 \tag{1.2}$$

for f, g physical fields depending on the doubled coordinates. But it turned out that this *ad hoc* introduced *strong constraint* (1.2) is merely a sufficient condition for consistency. In

[22] Recent reviews can be found in [88, 89, 90]

the so-called flux formulation of the DFT [95, 96, 97, 98, 99, 65], motivated by the vielbein formalism of general relativity and the early work [91, 92], it was shown that a weaker constraint, namely the so-called *closure constraint*, is also sufficient for consistency of DFT. This is supported by the observation that Scherk-Schwarz reductions [100] of DFT lead to consistent gauged supergravity theories in lower dimensions [99] without implementing the strong constraint along the compact directions.[23]

Taking this into consideration, two fundamental aspects will be addressed in this thesis.

- DFT is not only invariant under the duality group $O(d, d; \mathbb{Z})$ but includes general, even non-constant $O(d, d)$ transformations. However, evidence for duality beyond abelian duality in toroidal backgrounds is scarce. Following [49], an extended symmetry structure encompassing in particular non-constant $O(d, d)$ transformations will be presented.

- The origin of the ad-hoc constraints is not clear: they are always introduced a posteriori for consistency. It would be interesting to deduce the precise form of the constraints and to systematically distinguish the non-compact from the compact case. This will be achieved in a duality-invariant worldsheet approach following [102].

1.3.4 Generalized geometry

General relativity is a theory of gravity in conformity with the fundamental principle of relativity. The laws of nature are the same for any observer, i.e. in any coordinate system. In mathematical terms, this is accounted for by the theory being formulated as differential geometry on the tangent bundle[24] equipped with a metric tensor, which is the dynamical object in the theory. The tangent bundle is special in the following sense. For any choice of local coordinates $\{x^a\}$ of the d-dimensional spacetime manifold, there is a local frame $\{\partial/\partial x^a\}$, the coordinate frame, on the tangent bundle. This frame changes by a $GL(d)$-rotation if the coordinates are changed by a diffeomorphism. Hence, although every vector bundle has this $GL(d)$-*structure*, the one of the tangent bundle is intimately connected to changes of coordinates on the underlying manifold. Tensors such as the metric or the curvature on the (co-)tangent bundle are invariant under coordinate changes. This is reflected locally by a $GL(d)$-rotation associated to the diffeomorphism[25] of the components of the tensor, called the *transition function*.

As mentioned above, the manifest symmetry group of string theory is the geometric group consisting of coordinate changes and gauge transformations of the Kalb-Ramond field. Its effective theory is supergravity combining general relativity and Yang-Mills theory. But string theory also allows for non-geometric backgrounds whose transition functions are general elements of $O(d, d)$. An associated geometrical theory analogous to general relativity would therefore require an underlying bundle with the structure group $O(d, d)$. This is the *generalized tangent bundle*[26] defined in *generalized geometry* [103, 104, 105].

[23] See [101] for a recent discussion about compactification of DFT on non-geometric backgrounds.

[24] Using the vielbein formalism provides an even more instructive picture: General relativity is formulated on the frame bundle of the spacetime. The tangent bundle is the associated vector bundle to the frame bundle, which is a $GL(d)$-principle bundle.

[25] The $GL(d)$-matrix associated to a diffeomorphism is the pullback of the latter.

[26] Analogous to the tangent bundle, it is the associated vector bundle to an $O(d, d)$-principle bundle.

Therefore generalized geometry is a candidate for a unified description of dualities by mimicking the constructions in general relativity.

The dynamical object in general relativity is the Riemannian metric which provides a measure of distance on the tangent space. By its appearance in the Levi-Civita connection and its ensuing appearance in the curvature tensor, it dynamically determines the shape of spacetime. In string theory, the spacetime background is determined by the metric and the Kalb-Ramond field and so the analogue of the Riemannian metric in generalized geometry is expected to combine these two fields. This *generalized metric*

$$\mathcal{H}(G,B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

appears in various contexts in string theory. It characterizes for example the Hamiltonian density of the bosonic string sigma model and more specifically, it describes the contribution of the zero modes to the mass spectrum of the string. It is also the most efficient description of the action of duality on the background by conjugation with an $O(d,d)$ transformation: The dual background (g,b) is determined by $\mathcal{H}(g,b) = \mathcal{T}^t\mathcal{H}(G,B)\mathcal{T}$ with $\mathcal{T} \in O(d,d)$. From a more mathematical point of view, the Riemannian metric corresponds to a reduction of the structure group of the tangent bundle $GL(d)$ to the orthogonal group $O(d)$. Similarly, the generalized metric reduces the $O(d,d)$-structure of the generalized cotangent bundle to $O(d) \times O(d)$.

Thus, the generalized tangent bundle equipped with a generalized metric is the bosonic string analogue of the tangent bundle with a Riemannian metric. The next step is to set up a gravity theory on the generalized tangent bundle with $\mathcal{H}$ the dynamical field in order to obtain a unified description of the bosonic string. This amounts to the definition of a covariant derivative compatible with the changes of frame and a curvature tensor. More specifically, Riemannian geometry is build on the tangent bundle equipped with a metric, the Lie bracket and the partial derivative. By demanding torsion-freeness and compatibility with the metric, the Levi-Civita connection is completely determined by the partial derivative, the metric and the Lie bracket. Since the Riemann curvature tensor is defined in terms of the Lie bracket and the Levi-Civita connection, its construction does not require extra data. The Lie bracket is distinguished by its conservation under diffeomorphisms – they form the unique automorphisms of the bracket. Accordingly the ingredients for a gravity theory are

- a vector bundle with reduced structure group by a metric,
- a bracket conserved by the desired symmetries,
- a partial derivative mapping any smooth function to a section in the vector bundle.

The first prerequisite has already been discussed. As to the second and third point, the suitable structure on the generalized tangent bundle is a *Courant algebroid* [106]. Apart from the bundle, it contains an *anchor map* which relates the generalized tangent bundle with the tangent bundle. This allows for defining a partial derivative related to the conventional one. The *Courant bracket* is uniquely determined by conservation under diffeomorphism and so-called exact *B-transformations*.[27] The latter comply with gauge transformations of the Kalb-Ramond field. With these structures at hand, a gravity theory governing the dynamics of the generalized metric can be approached. However, it is neither possible to find the analogue of the uniquely determined Levi-Civita connection nor to define a proper curvature tensor.[28] Both of these idiosyncrasies derive from to the anomalous structure of the unique[29] Courant bracket, i.e. the failure of the Jacobi identity and of the Leibniz rule.

A notion capable of carrying a consistent theory of gravity are *Lie algebroids* [108] which can be obtained from Courant algebroids by restriction to *Dirac structures*. A Lie algebroid is defined on a conventional vector bundle equipped with a non-anomalous bracket and an anchor map relating it to the tangent bundle. Unfortunately, Lie algebroids suffice neither for a unified description of the bosonic string, nor for incorporating the whole duality group. Nevertheless they are well suited for describing the geometry and therefore the gravity theory of non-geometric backgrounds by characterizing them patch-wise [80]. Moreover, they provide interesting connections between bosonic string theory and Poisson geometry [79].

The present work provides an introduction to generalized geometry with the focus on its inability to underlie generalized theories of gravity. This naturally leads to the introduction of Lie algebroids, the geometry of which will be discussed in detail. These two theories constitute the main mathematical framework of this thesis.

1.4 Structure of the thesis

The thesis is based on the papers [57, 16, 64, 78, 79, 80, 102, 49] with strong emphasis on the last three.[30]

The thesis is organized as follows. In chapter 2 generalized geometry is introduced. First, the generalized tangent bundle as well as the generalized cotangent bundle are defined and equipped with a generalized metric. Then Courant algebroids are introduced and explicit brackets for the generalized (co-)tangent bundle are constructed by implementing their structure groups as automorphisms. The covariant derivative and the torsion tensor are defined. By elaborating on the malfunction of the exterior covariant derivative, Dirac structures and with them Lie algebroids appear naturally. The latter are discussed in detail by describing their cohomology theory, their connection to Courant algebroids via (proto-)Lie bialgebroids and their differential geometry. The main result of this section is theorem 2.17 which describes the relation between gravity theories on different Lie algebroids.

Chapter 3 focusses on the description of the target-space duality structure of closed bosonic string theory. The background dependent constrained non-linear sigma model description is introduced and reformulated in a Hamiltonian description. This reveals the classical $O(d,d)$-duality structure to whose exploration the rest of the chapter is devoted.

[27] Although desired in the context of duality, a bracket with $O(d,d)$ as automorphism group is not known. It most likely requires a structure beyond the generalized cotangent bundle. Partially, this is achieved in DFT by the C-bracket upon imposing constraints (the strong constraint or possible attenuations) .

[28] The Ricci tensor was defined indirectly in [107].

[29] Unique with respect to its automorphism group and the bundle.

[30] The first five papers are covered in [109]. In particular, [78] and [79] are special cases of [80].

Its consistency requirements are described in terms of a Courant algebroid bridging this novel approach to the common method of performing dualities. Examples of $O(d,d)$-duality are given which contain T-duality and the new Poisson duality. Quantum corrections are accounted for by shifting the dilaton in order to preserve conformal invariance. Finally, T- and Poisson duality are applied to the flat rectangular three-torus with constant H-flux. This gives rise to non-geometric backgrounds including the Q-flux T-fold.

The structure of non-geometric backgrounds is described in chapter 4. First, a mathematical formalism in terms of proto-Lie bialgebroids and Courant algebroids is developed in order to simultaneously describe all T-dual fluxes – including the non-geometric ones. In particular, conditions for their concurrent appearance are given. The lack of general gravity theories on Courant algebroids enforces the restriction to Lie algebroids. It is shown that transition functions of backgrounds related by duality, including the non-geometric backgrounds, are elements of $O(d,d)$. Although the transitions can be complicated, the geometric structure of the backgrounds is described efficiently in terms of Lie algebroids. This allows to construct (super-)gravity theories on every patch. The (super-)gravity theories on the different patches are related by theorem 2.17 which thereby provides a global description of non-geometric backgrounds.

Chapter 5 aims at finding a duality-invariant theory using different methods. A simple T-duality invariant conformal field theory arising from the free boson is constructed. Its one-loop partition function as well as four-point scattering of T-duality invariant operators is studied in order to check consistency of the theory. In non-compact directions the strong constraint of DFT (1.2) is derived. It is a consequence of modular invariance of the partition function and the premise of having physical intermediate states in the scattering of four tachyons. Finally, the scattering of three gravitons is considered in order to determine the effective theory associated to the T-duality invariant CFT. This is shown match with the action of DFT.

The thesis closes with concluding remarks and an outlook in chapter 6.

Chapter 2

Generalized geometry

A string is considered moving on a background determined by a (pseudo-)Riemannian metric G, the Kalb-Ramond two-form B and the dilaton ϕ. The pair (G, B) is called a *background* with the dilaton being a quantum correction. As will be shown in chapter 3, T-duality or more generally $O(d,d)$-duality mixes the metric and the Kalb-Ramond field. Hence a unified description of duality requires both fields to be treated on equal footing. This also applies to the associated symmetries: The metric and its dynamical theory – the general theory of relativity – are tightly connected to coordinate transformations. The Kalb-Ramond field is a higher order gauge connection associated to a gerbe [110] with gauge transformations its associated symmetry.

The combination of both concepts within one framework is the subject of generalized geometry [103, 104, 111, 105], which is introduced in this chapter. A particular focus lies on the formulation of a geometrical theory combining diffeomorphisms and gauge transformations.

2.1 The generalized tangent bundle

The infinitesimal generators of diffeomorphisms are vector fields, i.e. sections of the tangent bundle TM. On the other hand, gauge transformations of the Kalb-Ramond field are generated by one-forms, i.e. sections of the cotangent bundle T^*M. The combination of both can be achieved by defining the *generalized tangent bundle* as an extension

$$0 \longrightarrow T^*M \longrightarrow E \longrightarrow TM \longrightarrow 0 \tag{2.1}$$

in the following way. Locally on a patch $U_i \subset M$, $E|_{U_i} = TU_i \oplus T^*U_i$. Therefore the sections $A \in \Gamma(E)$ – called *generalized vectors* – can locally be written as $A_{(i)} = X_{(i)} + \xi_{(i)}$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(T^*M)$. A change of frame[1] within TM or T^*M is achieved by $\mathrm{GL}(d)$-transformations. Additionally, $TM \oplus T^*M$ on U_i and $TM \oplus T^*M$ on U_j are related on the overlap $U_i \cap U_j$ by a *B-transformation*

$$e^{-\mathsf{B}}(X+\xi) = X + \xi - \iota_X \mathsf{B} \equiv \begin{pmatrix} \mathbb{1} & 0 \\ -\mathsf{B} & \mathbb{1} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} \tag{2.2}$$

[1]Diffeomorphism are special changes of frame as they affect the coordinate frame.

for $\mathsf{B} = d\omega_{(ij)}$ a two-form with $\omega_{(ij)} \in \Gamma(T^*M)$ a connective structure on a gerbe [111, 112]. The group of B-transformations (2.2) will be denoted G_B. The convenient representation of generalized vectors as $2d$-vectors has been introduced. Moreover, the entries of a $d \times d$-matrix $\mathcal{T}$ acting on a generalized vector $(X, \xi)^t$ have to be interpreted as linear maps:

$$\mathcal{T} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \qquad \text{with} \qquad \begin{array}{l} t_{11} : TM \to TM\,, \\ t_{12} : T^*M \to TM\,, \\ t_{21} : TM \to T^*M\,, \\ t_{22} : T^*M \to T^*M\,. \end{array} \tag{2.3}$$

In total, the general transition function for the vector bundle E is given by

$$g_{(ij)} = \begin{pmatrix} \mathbb{1} & 0 \\ -d\omega_{(ij)} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathsf{A}_{(ij)} & 0 \\ 0 & \mathsf{A}_{(ij)}^{-t} \end{pmatrix}\,, \tag{2.4}$$

which is an element of the semi-direct product $G_{d\omega} \rtimes \mathrm{GL}(d)$ of exact B-transformations (2.2) and changes of frame. Therefore the generalized tangent bundle can equivalently be defined by giving the cocycles $g_{(ij)}$ for the patches $\{U_i\}$. In addition, E is equipped with the natural inner product

$$\langle X + \xi, Y + \eta \rangle \equiv \begin{pmatrix} X \\ \xi \end{pmatrix}^t \underbrace{\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}}_{\equiv \eta} \begin{pmatrix} Y \\ \eta \end{pmatrix} = \iota_X \eta + \iota_Y \xi\,. \tag{2.5}$$

This inner product defines the indefinite orthogonal group $O(d,d)$: $\mathcal{T} \in O(d,d)$ if and only if $\langle \mathcal{T}A, \mathcal{T}B \rangle = \langle A, B \rangle$. Just taking the bundle E defined by (2.1) together with (2.5) actually defines an $O(d,d)$-structure bundle. However, the generalized tangent bundle is defined by reducing $O(d,d)$ to $G_{d\omega} \rtimes \mathrm{GL}(d)$. The subgroup $SO(d,d;C^\infty(M))$ is generated by the following three transformations

$$\begin{array}{llll} \text{change of frames} & \mathcal{T}_A = \begin{pmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{A}^{-t} \end{pmatrix} & \text{with} & \mathsf{A} \in \mathrm{GL}(d) \otimes C^\infty(M)\,, \\ \text{B-transformations} & \mathcal{T}_B = \begin{pmatrix} \mathbb{1} & 0 \\ -\mathsf{B} & \mathbb{1} \end{pmatrix} & \text{with} & \mathsf{B} \in G_B \equiv \Gamma(\Lambda^2 T^*M)\,, \\ \beta\text{-transformations} & \mathcal{T}_\beta = \begin{pmatrix} \mathbb{1} & -\beta \\ 0 & \mathbb{1} \end{pmatrix} & \text{with} & \beta \in G_\beta \equiv \Gamma(\Lambda^2 TM)\,. \end{array} \tag{2.6}$$

The first two have been encountered in the structure group of the generalized tangent bundle. As they are not contained in the latter, β-transformations play a special role. The consequence of replacing B-transformations by β-transformations in the definition of the generalized tangent bundle will be discussed in section 2.1.2. For generating the entire group $O(d,d)$ a further element of negative determinant is needed in addition to (2.6) which changes the connected components [80]; being of major importance for the discussion of duality it is taken to be

$$\text{T-duality in } k^{\text{th}} \text{ direction} \qquad \mathcal{T}_{\mathrm{T}(k)} = \begin{pmatrix} \mathbb{1} - 1_k & 1_k \\ 1_k & \mathbb{1} - 1_k \end{pmatrix} \tag{2.7}$$

with 1_k the $d \times d$-matrix with 1 as k^{th} diagonal entry. In section 3.2.3, (2.7) is shown to provide the description of the Buscher rules within $O(d,d)$.

To summarize, the generalized tangent bundle E is defined by having structure group $G_{d\omega} \rtimes \mathrm{GL}(d)$ and is equipped with the $O(d,d)$-invariant inner product $\langle \cdot, \cdot \rangle$. In the next subsection a reduction of the structure group of E is introduced.

2.1.1 Generalized metrics

In Riemannian geometry, which can be considered as differential geometry on the tangent bundle, the structure group associated to changes of frames is $\mathrm{GL}(d)$. The implementation of additional structure on the tangent bundle gives rise to a reduction of the structure group. For example, the introduction of a Riemannian metric can equivalently be considered a reduction of the structure group to the orthogonal group $O(d)$. As being the dynamical object in a geometrical theory of gravity, the introduction of a metric is inevitable.

Following this logic, the reduction of the structure group of the generalized tangent bundle analogously to the reduction $\mathrm{GL}(d) \to O(d)$ gives rise to the dynamical object desired in the geometrical theory for the massless degrees of freedom of string theory. This is achieved by reducing $O(d,d) \to O(d) \times O(d)$ via the following object.

Definition 2.1. A *generalized metric* is a splitting of the generalized tangent bundle E into two rank-d subbundles C_+ and C_-, which are orthogonal as well as positive and negative definite with respect to the inner product $\langle \cdot, \cdot \rangle$ (2.5), respectively. Then E is the direct sum $E = C_+ \oplus C_-$.

The $O(d) \times O(d)$-structure is defined by the conservation of $\langle \cdot, \cdot \rangle|_{C_\pm}$. The subbundles can explicitly be constructed from the background (G, B) by taking its graph:

$$C_\pm = \mathrm{graph}_{TM}(B \pm G) = \{X + (B \pm G)(X)\} \subset E \,. \tag{2.8}$$

Here the metric and the Kalb-Ramond field are considered as maps $TM \to T^*M$. This connects to the conventional idea of a metric as a ruler on the space by defining the positive definite symmetric bilinear form

$$\mathcal{G}(A, B) = A^t \mathcal{H} B = \langle A, \eta \mathcal{H} B \rangle = \langle A, B \rangle|_{C_+} - \langle A, B \rangle|_{C_-} \tag{2.9}$$

with $\mathcal{H} : E \to E$ a symmetric automorphism. Thus $\mathcal{H}$ encodes the subbundles as $C_\pm = \frac{1}{2}(\mathrm{id}_E \pm \eta\mathcal{H})(E)$, i.e. as ± 1-eigenspaces of $\eta\mathcal{H}$, which requires $(\eta\mathcal{H})^2 = \mathbb{1}$. Using (2.8), this allows to determine the automorphism:

$$\mathcal{H}(G, B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} . \tag{2.10}$$

Its specification being equivalent to the definition given above, $\mathcal{H}$ is referred to as generalized metric as well. In particular, $\mathcal{G} = \langle \cdot, \cdot \rangle|_{C_+} - \langle \cdot, \cdot \rangle|_{C_-}$. The group preserving $\langle \cdot, \cdot \rangle|_{C_\pm}$ is $O(d)$. Hence, this establishes the reduction of the structure group to $O(d) \times O(d)$ by the introduction of the generalized metric (2.10), which plays an important role for the discussion of dualities in string theory and provides a unified description of the metric and the Kalb-Ramond field.

2.1.2 The generalized cotangent bundle

Since duality puts the metric and the Kalb-Ramond field on similar footing, it seems equally reasonable to consider the extension

$$0 \longrightarrow TM \longrightarrow E^* \longrightarrow T^*M \longrightarrow 0 \tag{2.11}$$

of the cotangent bundle instead of the extension of the tangent bundle (2.1). The *generalized cotangent bundle* E^* is also equipped with the inner product (2.5) but its transition functions are comprised of changes of frames and β-transformations (2.6)

$$e^{-\beta}(X+\xi) = X - \iota_\xi \beta + \xi \equiv \begin{pmatrix} \mathbb{1} & -\beta \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} . \tag{2.12}$$

The difficulty is to find the appropriate notion of *exact* β-transformations suitable for string theory. The gauge transformations of the Kalb-Ramond field determine the de Rham cohomology to be the right notion. This is only a particular example in the class of Lie-algebroid cohomologies discussed in section 2.3.1. The appropriate cohomology for β-transformations will be discovered in chapter 3 – for now, the associated nilpotent derivative acting on multivector fields will be denoted $d_{\mathcal{A}}$. Then the transition functions are given by

$$g^*_{(ij)} = \begin{pmatrix} \mathbb{1} & -d_{\mathcal{A}} X_{(ij)} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathsf{A}_{(ij)} & 0 \\ 0 & \mathsf{A}^{-t}_{(ij)} \end{pmatrix} \tag{2.13}$$

with $X_{(ij)}$ a vector field and $d_{\mathcal{A}} X_{(ij)}$ a bi-vector field. Hence the structure group of E^* is $G_{d_{\mathcal{A}}X} \rtimes \mathrm{GL}(d)$. It can be reduced to $O(d) \times O(d)$ analogous to the reduction presented above. A splitting into the subbundles

$$C^*_\pm = \mathrm{graph}_{T^*M}(\beta \pm g) = \{(\beta \pm g)(\xi) + \xi\} \subset E^* \tag{2.14}$$

is introduced with g a metric on the cotangent bundle and β a bi-vector. Then the generalized metric on E^* can equivalently be determined by

$$\mathcal{H}^*(g,\beta) = \begin{pmatrix} g^{-1} & -g^{-1}\beta \\ \beta g^{-1} & g - \beta g^{-1} \beta \end{pmatrix} . \tag{2.15}$$

The role of this metric can be appreciated in the context of non-geometric frames, which are studied in chapter 4.

2.2 Courant algebroids

Lie algebras with the Lie bracket are the algebraic structure of vector fields, the infinitesimal generators of diffeomorphisms $\mathrm{diff}(M)$. Now the algebraic structure of the infinitesimal generators of $G_{d\omega} \rtimes \mathrm{diff}(M)$ is discussed in a similar fashion. The restriction of the general linear group to the group of diffeomorphism is justified by the goal of describing the symmetries of string theory (cf. section 3.1). A Lie algebra is a vector space equipped with the Lie bracket $[\cdot,\cdot]$, which satisfies the Jacobi identity $\mathrm{Jac}_{\mathrm{Lie}}(X,Y,Z)=0$ for the Jacobiator

$$\mathrm{Jac}_{\mathrm{Lie}}(X,Y,Z) = [[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] . \tag{2.16}$$

This notion can be generalized as follows.

Definition 2.2 ([106])**.** A *Courant algebroid* is a quadruple $\mathcal{C} = (C, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], \alpha)$ consisting of a vector bundle $C \to M$ equipped with an inner product $\langle\cdot,\cdot\rangle$, an antisymmetric bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(C)$ and a smooth bundle map $\alpha : C \to TM$ called the *anchor*. Introducing the *Nijenhuis tensor*

$$T = \tfrac{1}{6}\left(\langle [\![c_1, c_2]\!], c_3\rangle + \langle [\![c_3, c_1]\!], c_2\rangle + \langle [\![c_2, c_3]\!], c_1\rangle\right) \tag{2.17}$$

for $c_1, c_2, c_3 \in \Gamma(C)$ and the differential $\mathcal{D} : C^\infty(M) \to \Gamma(C)$ via $\langle \mathcal{D}f, c_1\rangle = \alpha(c_1)f$, the Courant algebroid has to satisfy the following properties:

- anchor property: $\alpha([\![c_1, c_2]\!]) = [\alpha(c_1), \alpha(c_2)]$
- Jacobi identity: $\mathrm{Jac}_{\mathcal{C}}(c_1, c_2, c_3) = \mathcal{D}T(c_1, c_2, c_3)$
- Leibniz rule: $[\![c_1, fc_2]\!] = f[\![c_1, c_2]\!] + [\alpha(c_1)f]c_2 - \frac{1}{2}\langle c_1, c_2\rangle \mathcal{D}f$
- $\alpha \circ \mathcal{D} = 0$
- $\alpha(c_1)\langle c_2, c_3\rangle = \left\langle [\![c_1, c_2]\!] + \frac{1}{2}\mathcal{D}\langle c_1, c_2\rangle, c_3\right\rangle + \left\langle c_2, [\![c_1, c_3]\!] + \frac{1}{2}\mathcal{D}\langle c_1, c_3\rangle\right\rangle$

The Jacobi identity and the Leibniz rule have unusual defects whose consequences will be encountered in the next section. However, the definition is tailor-made for serving as the algebraic structure for the generators of the structure group of the generalized tangent bundle much like the Lie bracket. This will be discussed in the following.

2.2.1 The Courant bracket and its symmetries

As the major example the natural Courant algebroid structure on the generalized tangent bundle is presented. The action of infinitesimal diffeomorphisms is given by the Lie derivative of vector fields; $X.T = L_X T$ for a tensor field T. In particular, on vector fields and one-forms it acts as $L_X Y = [X, Y]$ and $L_X \xi = \iota_X d\xi + d\iota_X \xi$, respectively. For the generalized tangent bundle the infinitesimal action of B-transformations (2.2) given by $\mathsf{B}.(X + \xi) = \iota_X \mathsf{B}$ has to be considered as well. The Lie algebra of the group $G_{d\xi} \rtimes \mathrm{diff}(M)$ consists of sections $X - d\xi \in \Gamma(TM \oplus \Lambda^2 T^* M)$ whose action on a generalized vector $Y + \eta$ is therefore given by

$$(X - d\xi).(Y + \eta) = L_X(Y + \eta) - \iota_Y d\xi = [X, Y] + L_X\eta - \iota_Y d\xi\,. \tag{2.18}$$

This is sometimes referred to as *generalized Lie derivative* or *Dorfman bracket* $(X + \xi) \bullet (Y + \eta)$. The Dorfman bracket satisfies similar properties as a Courant algebroid; the major difference being its lack of antisymmetry. Given the Dorfman bracket (2.18) a Courant bracket can be defined by [113]

$$[\![A, B]\!] = \tfrac{1}{2}(A \bullet B - B \bullet A) = [X, Y] + L_X\eta - L_Y\xi - \tfrac{1}{2}d(\iota_X\eta - \iota_Y\xi)\,. \tag{2.19}$$

This is known as the *Courant bracket*. The associated Courant algebroid is $\mathcal{E} = (TM \oplus T^*M, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], \mathrm{pr}_{TM})$ with anchor pr_{TM} the projection on the tangent bundle. In particular, $\mathcal{D}f = df$.

The Lie bracket is preserved by diffeomorphisms, i.e. $f^*[X,Y] = [f^*X, f^*Y]$ for f a diffeomorphism. In fact, this is the only bundle automorphism of TM with this property (cf. [104], proposition 3.22). As to the Courant bracket, the only two automorphisms of E preserving it as well as the inner product are diffeomorphisms and B-transformations (2.2) (cf. [104], proposition 3.23 & 3.24):

Proposition 2.3 ([104]). *The group of bundle automorphisms $E \to E$ orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$ and preserving the bracket $[\![\cdot,\cdot]\!]$ is $G_{d\omega} \rtimes \mathrm{diff}(M)$.*

In this sense, the Courant bracket relates to the generalized tangent bundle as the Lie bracket relates to the tangent bundle, justifying the notion of Courant algebroids.

The Courant bracket for the generalized cotangent bundle

As will be explained in detail in section 2.3.1, a nilpotent derivative is associated to a Lie-algebroid bracket. A particular example is the exterior derivative d associated to the Lie bracket. In this way, a proper notion of exactness for β-transformations is related to a Lie-algebroid bracket $[\cdot,\cdot]_{\mathcal{A}}$ on the cotangent bundle. The brackets are also preserved by diffeomorphisms. Thus the construction above can be repeated for the generalized cotangent bundle, which yields the bracket

$$[\![A,B]\!]^* = [\xi,\eta]_{\mathcal{A}} + \mathcal{L}_\xi Y - \mathcal{L}_\eta X - \tfrac{1}{2} d_{\mathcal{A}}(\iota_\xi Y - \iota_\eta X) \tag{2.20}$$

with $\mathcal{L}_\xi X = \iota_\xi d_{\mathcal{A}} X + d_{\mathcal{A}} \iota_\xi X$. This is a Courant algebroid $\mathcal{E}^*$ on $TM \oplus T^*M$ with anchor $\alpha = \rho \circ \mathrm{pr}_{T^*M}$ and $\mathcal{D}f = d_{\mathcal{A}} f$. The injective homomorphism $\rho : T^*M \to TM$ satisfies $\rho([\xi,\eta]_{\mathcal{A}}) = [\rho(\xi), \rho(\eta)]$ and is the anchor for the Lie algebroid $\mathcal{A}$ (cf. section 2.3). As above, the bracket (2.20) is preserved under the structure group of E^*.

Proposition 2.4. *The group of bundle automorphisms $E^* \to E^*$ orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$ and preserving the bracket $[\![\cdot,\cdot]\!]^*$ is $G_{d_{\mathcal{A}} X} \rtimes \mathrm{diff}(M)$.*

Proof. The coordinate-free notation makes invariance under changes of frames of (2.20) manifest. Preservation under exact β-transformations (2.12) follows from

$$\begin{aligned}
[\![e^\beta(X+\xi), e^\beta(Y+\eta)]\!]^* &= [\![X+\xi, Y+\eta]\!]^* + \mathcal{L}_\xi \iota_\eta \beta - \mathcal{L}_\eta \iota_\xi \beta + d_{\mathcal{A}} \iota_\eta \iota_\xi \beta \\
&= [\![X+\xi, Y+\eta]\!]^* + \mathcal{L}_\xi \iota_\eta \beta - \iota_\eta d_{\mathcal{A}} \iota_\xi \beta \\
&= [\![X+\xi, Y+\eta]\!]^* + [\mathcal{L}_\xi, \iota_\eta]\beta + \iota_\eta \iota_\xi d_{\mathcal{A}} \beta \\
&= e^\beta([\![X+\xi, Y+\eta]\!]^*) + \iota_\eta \iota_\xi d_{\mathcal{A}} \beta \,.
\end{aligned}$$

In the first step antisymmetry of β was used and in the last step the identity $[\mathcal{L}_\xi, \iota_\eta] = \iota_{[\xi,\eta]_{\mathcal{A}}}$ (2.40). Therefore e^β preserves the bracket if β is $\mathcal{A}$-closed, i.e. in particular for $\beta = d_{\mathcal{A}} X$. For proving that these two groups are the only automorphisms, the existence of an orthogonal bundle automorphism (f, F) preserving the bracket is assumed. f is an automorphism of M and F an automorphism of E^*. Considering the change of frame given by $f_c = \mathrm{diag}(f^*, (f^*)^{-t})$ the pair $(\mathrm{id}_M, G = f_c^{-1} \circ F)$ is orthogonal and bracket-preserving as well. Then for $A, B \in \Gamma(E^*)$ and $h \in \mathcal{C}^\infty(M)$ this yields on the one hand

$$\begin{aligned}
G([\![A, hB]\!]^*) &= G\left(h[\![A,B]\!]^* + (\alpha(A)h)B - \tfrac{1}{2}\langle A,B\rangle d_{\mathcal{A}} h\right) \\
&= hG([\![A,B]\!]^*) + (\alpha(A)h)G(B) - \tfrac{1}{2}\langle A,B\rangle G(d_{\mathcal{A}} h)
\end{aligned}$$

and on the other hand

$$\begin{aligned}[\![G(A), G(hB)]\!]^* &= h[\![G(A), G(B)]\!]^* + (\alpha\,(G(A))\,h)\,G(B) - \tfrac{1}{2}\langle G(A), G(B)\rangle d_{\mathcal{A}}h \\ &= hG([\![A, B]\!]^*) + (\alpha\,(G(A))\,h)\,G(B) - \tfrac{1}{2}\langle A, B\rangle d_{\mathcal{A}}h\end{aligned}$$

by orthogonality. Comparing these evaluations gives

$$(\alpha(A)h)G(B) - \tfrac{1}{2}\langle A, B\rangle G(d_{\mathcal{A}}h) = (\alpha\,(G(A))\,h)\,G(B) - \tfrac{1}{2}\langle A, B\rangle d_{\mathcal{A}}h\,.$$

Suppose that $A = \xi$, $B = \eta$ are one-forms such that $\langle A, B\rangle = 0$. This gives $\alpha(G(\xi) - \xi) = 0$. This implies $G(\xi) = \xi$ for all one-forms and therefore sets $G = \left(\begin{smallmatrix} * & * \\ * & 1\end{smallmatrix}\right)$. Taking only vector fields restricts the automorphism to $G = \left(\begin{smallmatrix} * & * \\ 0 & 1\end{smallmatrix}\right)$. Then for general sections A and B the last equation reduces to $\langle A, B\rangle G(d_{\mathcal{A}}h) = \langle A, B\rangle d_{\mathcal{A}}h$. This further reduces the matrix to $G = \left(\begin{smallmatrix} 1 & * \\ 0 & 1\end{smallmatrix}\right)$. Orthogonality with respect to the inner product forces the remaining unknown entry to be an antisymmetric bivector field and preservation of the bracket demands $\mathcal{A}$-exactness. Hence $G = f_c^{-1} \circ F = e^{d_{\mathcal{A}}X}$, or equivalently $F = f_c \circ e^{d_{\mathcal{A}}X}$: Every orthogonal, bracket-preserving automorphism is a composition of diffeomorphisms and $\mathcal{A}$-exact β-transformations. □

Having established the natural algebraic structures on the generalized (co-)tangent bundle, these can be used for constructing a dynamical theory for the generalized metric $\mathcal{H}$ ($\mathcal{H}^*$).

2.2.2 Towards a differential geometry

General relativity is a dynamical theory for a Riemannian metric with coordinate changes a manifest symmetry. The main ingredient is the tangent bundle itself with its Lie algebra structure for vector fields and the metric. The bracket and partial derivative are used to explicitly construct the Levi-Civita connection via the Koszul formula, from which curvature and torsion can be defined – they determine the shape of the space. The analogous procedure in generalized geometry will be followed as far as possible in the present section. In particular, a consistent definition of torsion and curvature within generalized geometry would provide a unified description of string backgrounds.

The generalized covariant derivative

The main object in differential geometry for describing dynamics is the covariant derivative or connection as it connects the different patches of a vector bundle. The aim of the following is to find the right notions for derivatives in order to define torsion and curvature.

Definition 2.5. Let $\mathcal{C} = (C, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], \alpha)$ be a Courant algebroid and V a vector bundle. A *generalized covariant derivative* ,or *$\mathcal{C}$-connection* for short, on V is a bilinear map $\nabla : \Gamma(V) \times \Gamma(C) \to \Gamma(V)$ satisfying

$$\begin{aligned}\nabla_{fc}s &= f\nabla_c s\,, \\ \nabla_c(fs) &= [\alpha(c)f]\,s + f\nabla_c s\end{aligned} \tag{2.21}$$

for all $c \in \Gamma(C)$, $s \in \Gamma(V)$ and $f \in C^\infty(M)$.

The generalized covariant derivative can also be interpreted as a connection $\nabla : \Gamma(V) \to \Gamma(C^* \otimes V)$ where C^* is the dual bundle. A simple way of formulating the curvature in differential geometry is using the *exterior covariant derivative.* To define it, the derivative $\mathcal{D}$ has to be extended to sections $\Gamma(\Lambda^\bullet C^*)$ analogous to the exterior derivative d. This is achieved by defining

$$\begin{aligned} d_{\mathcal{C}}\gamma(c_0, \ldots, c_k) = & \sum_{i=0}^{k} (-1)^i\, \alpha(c_i)\gamma\big(c_0, \ldots, \widehat{c_i}, \ldots, c_k\big) \\ & + \sum_{i<j} (-1)^{i+j}\, \gamma\big([\![c_i, c_j]\!], c_0, \ldots, \widehat{c_i}, \ldots, \widehat{c_j}, \ldots, c_k\big) \end{aligned} \tag{2.22}$$

with $\gamma \in \Gamma(\Lambda^k C^*)$ and the hat indicating omission of the entry. This is the usual way of constructing the exterior derivative given a bracket and a partial derivative. However, already nilpotency is a problem: Whereas for functions f it still satisfies

$$d_{\mathcal{C}}^2 f(c_0, c_1) = \big([\alpha(c_0), \alpha(c_1)] - \alpha\big([\![c_0, c_1]\!]\big)\big)\, f = 0$$

by the anchor property, for a $\mu \in \Gamma(C^*)$ one obtains

$$d_{\mathcal{C}}^2 \mu(c_0, c_1, c_2) = \mu\big(\mathrm{Jac}_{\mathcal{C}}(c_0, c_1, c_2)\big) = \mu\big(\mathcal{D}T(c_0, c_1, c_2)\big)\,.$$

So the naive construction for an exterior derivative fails to be nilpotent. Even worse, the exterior derivative of a section is not a tensor any more as it lacks function-linearity:

$$\begin{aligned} d_{\mathcal{C}}\gamma(f c_0, \ldots, c_k) = & f d_{\mathcal{C}}\gamma(c_0, \ldots, c_k) \\ & + \frac{1}{2}\sum_{i=1}^{k} (-1)^i \langle c_0, c_i\rangle\, \gamma(\mathcal{D}f, c_1, \ldots, \widehat{c_i}, \ldots, c_k)\,. \end{aligned}$$

The defect is due to the anomalous Leibniz rule for the bracket $[\![\cdot,\cdot]\!]$. However, inspired by [114] the bracket can be modified.

Proposition 2.6. *The* modified Courant bracket *for a Courant algebroid $\mathcal{C} = (C, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], \alpha)$ with generalized connection ∇ on C is given by*

$$(\!(c_1, c_2)\!) = [\![c_1, c_2]\!] - \tfrac{1}{2}\big(\langle\nabla c_1, c_2\rangle - \langle c_1, \nabla c_2\rangle\big) \tag{2.23}$$

with $\langle\langle\nabla c_1, c_2\rangle, c_3\rangle = \langle\nabla_{c_3} c_1, c_2\rangle$ for $c_1, c_2, c_3 \in \Gamma(C)$. It satisfies the Leibniz rule $(\!(c_1, f c_2)\!) = f(\!(c_1, c_2)\!) + [\alpha(c_1)f]c_2$.

The claim can readily be checked from the definition and the Leibniz rule for Courant algebroids. The disadvantage of this bracket is a complicated Jacobiator as well as the lack of the anchor property. For the latter to hold, the anchor applied to the new term has to vanish. However, for an arbitrary function f it satisfies

$$\alpha(\langle\nabla c_1, c_2\rangle)f = \langle\mathcal{D}f, \langle\nabla c_1, c_2\rangle\rangle = \langle\nabla_{\mathcal{D}f} c_1, c_2\rangle\,, \tag{2.24}$$

which is neither vanishing in general nor symmetric.

Using the formula (2.22) allows for defining a derivative $d_{(\!(\cdot,\cdot)\!)} : \Gamma(\Lambda^k C^*) \to \Gamma(\Lambda^{k+1}C^*)$ as it is antisymmetric and satisfies the Leibniz rule. However, it is also not nilpotent due to the failure of the Jacobi identity. Since it maps tensors to tensors this derivative is suitable for extending the generalized covariant derivative[2] to a map $\nabla : \Gamma(\Lambda^k C^* \otimes V) \to \Gamma(\Lambda^{k+1}C^* \otimes V)$ uniquely via[3]

$$\nabla(\gamma \otimes s) = d_{(\!(\cdot,\cdot)\!)}\gamma \otimes s + (-1)^k \gamma \wedge \nabla(s) \tag{2.25}$$

for $\gamma \in \Gamma(\Lambda^k C^*)$ and $s \in \Gamma(V)$. The extension can explicitly be constructed using a formula analogous to (2.22):

Proposition 2.7. *For $T \in \Gamma(\Lambda^k C^* \otimes V)$ the* generalized exterior covariant derivative *(2.25) is given by*

$$\begin{aligned}\nabla T(c_0, \dots, c_k) = \; & \sum_{i=0}^{k} (-1)^i \, \nabla_{c_i} T\big(c_0, \dots, \widehat{c_i}, \dots, c_k\big) \\ & + \sum_{i<j} (-1)^{i+j} \, T\big((\!(c_i, c_j)\!), c_0, \dots, \widehat{c_i}, \dots, \widehat{c_j}, \dots, c_k\big)\end{aligned} \tag{2.26}$$

with $c_i \in \Gamma(C)$. ∇T is $C^\infty(M)$-linear and satisfies the Leibniz rule $\nabla(fT) = d_{(\!(\cdot,\cdot)\!)} f \wedge T + f \nabla T$.

Proof. Consider $T = \gamma \otimes s$. By the Leibniz rule for the generalized covariant derivative the first line in (2.26) gives

$$\begin{aligned}\nabla_{c_i} T\big(c_0, \dots, \widehat{c_i}, \dots, c_k\big) = & \big[\alpha(c_i)\gamma\big(c_0, \dots, \widehat{c_i}, \dots, c_k\big)\big] \, s \\ & + \gamma\big(c_0, \dots, \widehat{c_i}, \dots, c_k\big) \nabla_{c_i} s \,,\end{aligned}$$

whose first term combines with the second line in (2.26) to $d_{(\!(\cdot,\cdot)\!)}$ via (2.22). The sign in (2.25) arises from the possibilities of inserting $k+1$ sections into $\gamma \wedge \nabla s$ in comparison to the last term in the last equation. The linearity follows from the Leibniz rule of $(\!(\cdot,\cdot)\!)$. The Leibniz rule is a direct consequence of (2.25) and the Leibniz rule for the generalized covariant derivative. □

Dirac structures

From now on ∇ denotes an $\mathcal{E}$-connection on the generalized tangent bundle E with its natural Courant algebroid structure $\mathcal{E}$ and is assumed to coincide with the connection in the definition of the modified Courant bracket (2.23). In particular, generalized connections on a Courant algebroid are assumed to be compatible with the inner product, i.e.

$$\alpha(A)\langle B, C\rangle = \langle \nabla_A B, C\rangle + \langle B, \nabla_A C\rangle \tag{2.27}$$

for $A, B, C \in \Gamma(E)$. In this case it is important to notice that the inner product (2.5) identifies E with its dual.

[2]Note that the generalized covariant derivative in the definition of $(\!(\cdot,\cdot)\!)$ may differ from this one.

[3]To be precise, the $\wedge$ in (2.25) refers to an exterior product between γ and the $\Gamma(C)$-component of ∇s which is tensored with the $\Gamma(V)$-component of ∇s: $\gamma \wedge \nabla s = (\gamma \wedge \mu) \otimes t$ for $\nabla s = \mu \otimes t$.

In Riemannian geometry the curvature can now be defined as composition of the exterior covariant derivative, i.e. via $\nabla \circ \nabla$ (cf. def. 2.15). A crucial property of the curvature is its function linearity in order to be tensorial in every entry. However the generalized exterior covariant derivative satisfies

$$\nabla \circ \nabla(fT) = d^2_{\langle\!\langle \cdot,\cdot \rangle\!\rangle} f \wedge T + f \nabla \circ \nabla T \tag{2.28}$$

for $T \in \Gamma(\Lambda^k E \otimes E)$ by using the Leibniz rule and (2.25). Thus function-linearity fails by the absence of nilpotency. Using the $\langle \cdot, \cdot \rangle$-compatibility of the generalized covariant derivative and the axiom $\alpha \circ \mathcal{D} = 0$, the nilpotency of $d_{\langle\!\langle \cdot,\cdot \rangle\!\rangle}$ is spoiled as

$$d^2_{\langle\!\langle \cdot,\cdot \rangle\!\rangle} f(A, B) = \alpha\big(\langle \nabla A, B \rangle\big) f = \langle \nabla_{\mathcal{D}f} A, B \rangle \, . \tag{2.29}$$

In order for this to vanish and being able to define a proper curvature, one can proceed in two ways:

- Imposing $\nabla_{\mathcal{D}f} A = 0$. To this end, let $\{e_M\}$ be a frame for E. Then the $\mathcal{E}$-connection can locally be written as $\nabla e_M = \omega_M^N \otimes e_N$ with $\omega_M^N \in \Gamma(E)$. By the Leibniz rule, the covariant derivative of an arbitrary section $A = A^M e_M$ is $\nabla A = (\mathcal{D} A^M + A^N \omega_N^M) \otimes e_M$. In particular, using $\alpha \circ \mathcal{D} = 0$, the case of interest becomes $\nabla_{\mathcal{D}f} A = A^N [\alpha(\omega_N^M) f] e_M$. Since A and f are arbitrary, $\alpha(\omega_N^M) = 0$ has to be demanded. However, for $\mathcal{E}$ with $\alpha = \mathrm{pr}_{TM}$, this restricts the connection coefficient to be a one-form. This implies $\nabla_\xi A = 0$ for all one-forms ξ, which in turn restricts the generalized covariant derivative to be a conventional TM-covariant derivative. Thus this option is not interesting.

- For $\nabla_{\mathcal{D}f} A \neq 0$ the remaining possibility is $\langle A, B \rangle = 0$ with A, B arbitrary. This case is of particular interest as it renders the Courant bracket a bracket satisfying the Jacobi identity and a regular Leibniz rule, cf. definition 2.2.

The second case is associated to the following structure for Courant algebroids.

Definition 2.8. Let $\mathcal{C} = (C, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!], \alpha)$ be a Courant algebroid. A maximal-rank subbundle $D \subset C$ with $[\![d_1, d_2]\!] \in \Gamma(D)$ for $d_i \in D$ is called a *Dirac structure* if it is isotropic, i.e. if $\langle A, B \rangle = 0$ for all $A, B \in D$.

Any defect to common construction from Courant algebroids encountered so far have been associated to the inner product. Hence isotropic subbundles allow to mimic the usual constructions of differential geometry and in particular support the definition of a tensorial curvature. Dirac structures are special case of Lie algebroids, which are discussed in the next section.

2.3 Lie algebroids

Riemannian geometry and accordingly general relativity are concepts based on the mathematical structures of the tangent bundle with its Lie algebra of vector fields. In principle, differential geometry is determined by a vector bundle, a bracket and a derivative. This can be considered the leitmotif for introducing Lie algebroids [108]:

Definition 2.9. A *Lie algebroid* is a triple $\mathcal{A} = (A, [\cdot,\cdot]_A, \rho)$ consisting of a vector bundle $A \to M$, an antisymmetric, bilinear bracket $[\cdot,\cdot]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ satisfying the Jacobi identity and a homomorphism $\rho : A \to TM$ called the *anchor*. Moreover, it satisfies

- the anchor property $\rho([s_1, s_2]_A) = [\rho(s_1), \rho(s_2)]$,
- the Leibniz rule $[s_1, fs_2]_A = f[s_1, s_2]_A + [\rho(s_1)f]s_2$

for all $s_1, s_2 \in \Gamma(A)$ and $f \in C^\infty(M)$.

The anchor property is redundant as it is a consequence of the Leibniz rule and the Jacobi identity. However, this property is crucial as it connects a Lie algebroid to the tangent bundle – the trivial Lie algebroid $(TM, [\cdot,\cdot], \mathrm{id}_{TM})$ (see figure 2.1).

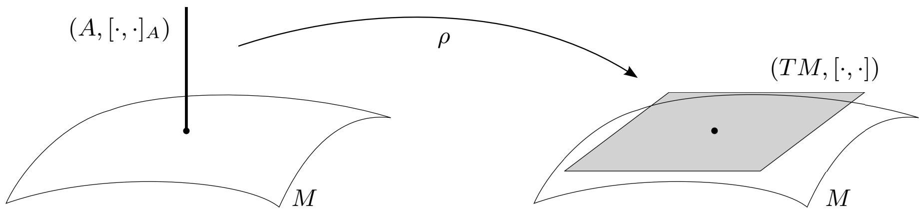


Figure 2.1: The anchor map $\rho : A \to TM$ connects the Lie algebroid to the tangent bundle with Lie bracket.

The bracket can be uniquely extended to $\Gamma(\Lambda^\bullet A)$ by the Gerstenhaber properties

$$\begin{aligned} [s_1, s_2 \wedge s_3]_A &= [s_1, s_2]_A \wedge s_3 + (-1)^{(k-1)l}\, s_2 \wedge [s_1, s_3]_A \\ [s_1, s_2]_A &= -(-1)^{(k-1)(l-1)}[s_2, s_1]_A \end{aligned} \tag{2.30}$$

for $s_1 \in \Gamma(\Lambda^k A)$ and $s_2 \in \Gamma(\Lambda^l A)$. In particular, on functions it is defined by $[f, g]_A = 0$ and $[s_1, f]_A = \rho(s_1)f$.

The concept is illustrated by the following three examples:

- $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \mathrm{id}_\mathfrak{g})$ with $\mathfrak{g}$ the Lie algebra to the Lie group G and $[g_i, g_j]_\mathfrak{g} = f^k{}_{ij} g_k$ its Lie bracket with structure constants $f^k{}_{ij}$. Since $\mathfrak{g} = T_0G$, the anchor is the identity map. In this sense, the Lie algebra is considered a vector bundle over a single point $\mathfrak{g} \to \{0\}$. This allows to interpret a Lie algebroid as a "bundle of Lie algebras" which in particular allows for non-constant structure coefficients. The associated structures analogous to Lie groups are Lie groupoids.
- $(D, [\![\cdot,\cdot]\!], \alpha)$ with D a Dirac structure for a Courant algebroid $(C, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!], \alpha)$. By definition 2.8 and 2.2, a Dirac structure is a Lie algebroid. For example, TM is a Dirac structure for the generalized tangent bundle $\mathcal{E}$: it gives the trivial Lie algebroid $(TM, [\cdot,\cdot], \mathrm{id}_{TM})$. The second instance is T^*M as Dirac structure for the generalized cotangent bundle $\mathcal{E}^*$ and gives the Lie algebroid $(T^*M, [\cdot,\cdot]_\mathcal{A}, \rho)$. An example for the latter structure is given next.
- $(T^*M, [\cdot,\cdot]_\mathrm{K}, \beta^\sharp)$ with $\beta \in \Gamma(\Lambda^2 TM)$ inducing the homomorphism $\beta^\sharp : T^*M \to TM; \xi \mapsto \xi_m \beta^{ma} e_a$ and the *Koszul bracket*

$$[\xi, \eta]_\mathrm{K} = L_{\beta^\sharp \xi}\eta - \iota_{\beta^\sharp \eta} d\xi\,. \tag{2.31}$$

For this bracket to satisfy the Jacobi identity, β has to be a Poisson structure, i.e. $\{f,g\} = \beta(df, dg)$ has to be a Poisson bracket for all $f, g \in C^\infty(M)$. This in turn is the case if and only if $\Theta \in \Gamma(\Lambda^3 TM)$ given by

$$\Theta = \tfrac{1}{2}[\beta, \beta]_{\mathrm{SN}} \tag{2.32}$$

vanishes. Above the Lie bracket was extended by the Gerstenhaber properties (2.30) to the Schouten-Nijenhuis bracket. The Koszul bracket is the natural geometric structure in Poisson geometry.

The anchor of a Lie algebroid is a particular example of bundle homomorphisms preserving the Lie algebroid structure. In general, they are defined as follows.

Definition 2.10. Let $\mathcal{A}_1 = (A_1, [\cdot,\cdot]_{A_1}, \rho_1)$ and $\mathcal{A}_2 = (A_2, [\cdot,\cdot]_{A_2}, \rho_2)$ be Lie algebroids with $A_{1/2}$ vector bundles over M. A bundle homomorphism $\Phi : A_1 \to A_2$ is called a *Lie algebroid homomorphism* if it satisfies

$$\rho_2 \circ \Phi = \rho_1 \qquad \text{and} \qquad \Phi([s_1, s_2]_{A_1}) = [\Phi(s_1), \Phi(s_2)]_{A_2} \tag{2.33}$$

for $s_1, s_2 \in \Gamma(A_1)$. The Lie algebroids are *isomorphic* – $\mathcal{A}_1 \cong \mathcal{A}_2$ – if the Lie algebroid homomorphism is invertible.

Having introduced the basic notions for Lie algebroids, more advanced structures can be discussed.

2.3.1 Lie-algebroid cohomology

As mentioned above, gauge transformations of the Kalb-Ramond field are described in terms of the de Rham exterior derivative. However, also an alternating bivector field appears in generalized geometry by considering β-transformations (2.12). The right notion of exactness for alternating multivector fields can be described in terms of the cohomology of a Lie algebroid on the cotangent bundle. Introducing the appropriate concepts is the purpose of this section.

Definition 2.11. For a Lie algebroid $\mathcal{A} = (A, [\cdot,\cdot]_A, \rho)$ the *exterior derivative* $d_{\mathcal{A}} : \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^{k+1} A^*)$ is given by

$$\begin{aligned} d_{\mathcal{A}}\sigma(s_0, \dots, s_k) = & \sum_{i=0}^{k} (-1)^i \, \rho(s_i)\sigma\big(s_0, \dots, \widehat{s_i}, \dots, s_k\big) \\ & + \sum_{i<j} (-1)^{i+j} \, \sigma\big([s_i, s_j]_A, s_0, \dots, \widehat{s_i}, \dots, \widehat{s_j}, \dots, s_k\big) \end{aligned} \tag{2.34}$$

with $s_i \in \Gamma(A)$ and $\sigma \in \Gamma(\Lambda^k A^*)$.

This definition in analogous to (2.22). However, due to the Jacobi identity this derivative is nilpotent. Moreover, the triple $(\Gamma(\Lambda^\bullet A^*), d_{\mathcal{A}}, \wedge)$ is a differential graded algebra – this is equivalent to the definition of a Lie algebroid [115]. The k^{th} *Lie algebroid cohomology* on $\mathcal{A}$ is defined as

$$H^k_{\mathcal{A}} = \frac{\ker\left\{d_{\mathcal{A}} : \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^{k+1} A^*)\right\}}{\operatorname{im}\left\{d_{\mathcal{A}} : \Gamma(\Lambda^{k-1} A^*) \to \Gamma(\Lambda^k A^*)\right\}} . \tag{2.35}$$

Sections in the class of zero in $H^\bullet_{\mathcal{A}}$, i.e. ones of the form $d_{\mathcal{A}}\sigma$ will be called $\mathcal{A}$-*exact*. Lie algebroid homomorphisms allow for a switching between the two associated cohomologies. To this end the transpose of a linear map $\Phi : V_1 \to V_2$ is introduced as

$$\Phi^t : V_2^* \to V_1^*;\, \omega \mapsto \omega \circ \Phi\,, \tag{2.36}$$

which naturally extends to linear maps between multisections of the same order. In particular, $(\otimes^k \Phi^t)(\sigma)(s_1, \ldots, s_k) = \sigma(\Phi(s_1), \ldots, \Phi(s_k))$ for $\sigma \in \Gamma(\Lambda^k V_2^*)$ and $s_i \in \Gamma(V_1)$.

Proposition 2.12. *Let $\Phi : A_1 \to A_2$ be a Lie algebroid homomorphism between $\mathcal{A}_1 = (A_1, [\cdot,\cdot]_{A_1}, \rho_1)$ and $\mathcal{A}_2 = (A_2, [\cdot,\cdot]_{A_2}, \rho_2)$. Then the corresponding exterior derivatives are associated via*

$$(\otimes^{\bullet+1}\Phi^t) \circ d_{\mathcal{A}_2} = d_{\mathcal{A}_1} \circ (\otimes^\bullet \Phi^t)\,. \tag{2.37}$$

Applying the properties of a Lie algebroid homomorphism within the defining formula (2.34) proves the claim immediately. As particular example, the anchor of any Lie algebroid connects its cohomology to the de Rham cohomology.

2.3.2 Lie bialgebroids and Courant algebroids

Lie algebroids can arise as Dirac structures of Courant algebroids. However, the converse is possible as well [106, 113].

Definition 2.13. Let $\mathcal{A} = (A, [\cdot,\cdot]_A, \rho)$ and $\mathcal{A}^* = (A^*, [\cdot,\cdot]_{A^*}, \rho_*)$ be two Lie algebroids with A^* the dual vector bundle to A. Then the pair $(\mathcal{A}, \mathcal{A}^*)$ is called a *Lie bialgebroid* if

$$d_{\mathcal{A}}([\sigma_1, \sigma_2]_{A^*}) = [d_{\mathcal{A}}\sigma_1, \sigma_2]_{A^*} + (-1)^k\, [\sigma_1, d_{\mathcal{A}}\sigma_2]_{A^*} \tag{2.38}$$

for $\sigma_1 \in \Gamma(\Lambda^k A^*)$ and $\sigma_2 \in \Gamma(\Lambda^\bullet A^*)$,i.e. if $d_{\mathcal{A}}$ is a graded derivation for $[\cdot,\cdot]_{A^*}$.

In particular, if $(\mathcal{A}, \mathcal{A}^*)$ is a Lie bialgebroid, so is $(\mathcal{A}^*, \mathcal{A})$. In the definition, the Lie algebroid bracket $[\cdot,\cdot]_{A^*}$ has been uniquely extended to a bracket on arbitrary alternating multisections by the Gerstenhaber properties (2.30).

Another useful notion is the Lie derivative on a Lie algebroid $\mathcal{A}$; it can be defined by setting

$$\begin{aligned} L^{\mathcal{A}}_s f &= \rho(s) f \\ L^{\mathcal{A}}_{s_1} s_2 &= [s_1, s_2]_A \\ L^{\mathcal{A}}_s \sigma &= \iota_s \circ d_{\mathcal{A}}\sigma + d_{\mathcal{A}} \circ \iota_s \sigma \end{aligned} \tag{2.39}$$

with $f \in C^\infty(M)$, $s, s_1, s_2 \in \Gamma(A)$ and $\sigma \in \Gamma(A^*)$. It extends to multisections by demanding the product rule with respect to tensor products. The Lie derivative satisfies

$$\begin{aligned} \left[L^{\mathcal{A}}_{s_1}, L^{\mathcal{A}}_{s_2}\right] &= L^{\mathcal{A}}_{[s_1,s_2]_A} \\ \left[L^{\mathcal{A}}_{s_1}, \iota_{s_2}\right] &= \iota_{[s_1,s_2]_A} \\ L^{\mathcal{A}}_s\big|_{\Gamma(\Lambda^\bullet A^*)} \circ d_{\mathcal{A}} &= d_{\mathcal{A}} \circ L^{\mathcal{A}}_s\big|_{\Gamma(\Lambda^\bullet A^*)} \\ L^{\mathcal{A}}_{fs}\sigma &= d_{\mathcal{A}} f \wedge \iota_s\sigma + f\, L^{\mathcal{A}}_s\sigma \,. \end{aligned} \tag{2.40}$$

The proofs are completely analogous to the proofs for the standard Lie derivative. Then a Courant algebroid can be found as follows.

Proposition 2.14 ([106])**.** *Let $(\mathcal{A}, \mathcal{A}^*)$ be a Lie bialgebroid with anchor ρ and ρ_*, respectively. Then the quadruple $(A \oplus A^*, \langle\cdot,\cdot\rangle_+, [\![\cdot,\cdot]\!], \alpha)$ with*

$$\begin{aligned} \alpha(s+\sigma) &= \rho(s) + \rho_*(\sigma) \\ \langle s+\sigma, t+\tau\rangle_\pm &= \iota_s\tau \pm \iota_t\sigma \\ [\![s+\sigma, t+\tau]\!] &= [s,t]_A + L^{\mathcal{A}}_s\tau - L^{\mathcal{A}}_t\sigma - \tfrac{1}{2}d_{\mathcal{A}}\langle s+\sigma, t+\tau\rangle_- \\ &\quad [\sigma,\tau]_{A^*} + L^{\mathcal{A}^*}_\sigma t - L^{\mathcal{A}^*}_\tau s + \tfrac{1}{2}d_{\mathcal{A}^*}\langle s+\sigma, t+\tau\rangle_- \end{aligned} \tag{2.41}$$

for $s, t \in \Gamma(A)$ and $\sigma, \tau \in \Gamma(A^*)$ is a Courant algebroid.

The compatibility condition (2.38) between the two Lie algebroids is important for proving the Jacobi identity for the Courant bracket. Moreover, the converse is also true: If $\mathcal{C}$ is a Courant algebroid with A_1 and A_2 transversal Dirac structures, i.e. $C = A_1 \oplus A_2$, then $(\mathcal{A}_1, \mathcal{A}_2)$ with the associated Lie algebroid structures is a Lie bialgebroid. A_1 and A_2 are dual under the inner product on $\mathcal{C}$.

The most immediate example is (TM, T^*M) with $(TM, [\cdot,\cdot], \mathrm{id}_{TM})$ and T^*M equipped with the trivial structure (bracket and anchor being zero-maps); it gives the Courant bracket (2.19). Similarly, taking a Lie algebroid structure on T^*M and TM with trivial structure, the bracket (2.20) is obtained.

Proposition 2.14 was generalized to the case of *proto-Lie bialgebroids* in [113]. A proto-Lie bialgebroid consists of two quasi-Lie algebroids with complicated compatibility conditions. A *quasi-Lie algebroid* $\mathcal{A}$ is a Lie algebroid whose anchor property admits a defect

$$\Delta_{\mathcal{A}}(s_1, s_2) = \rho\left([s_1, s_2]_A\right) - [\rho(s_1), \rho(s_2)] \,. \tag{2.42}$$

The compatibility conditions are most efficiently described in terms of supermanifolds and A_∞-structures, which are not covered in this thesis. A particular example of proto-Lie bialgebroids is introduced in section 4.1 for studying non-geometric fluxes.

2.3.3 Differential geometry

As opposed to the Courant algebroid, curvature and torsion can be introduced as usual. Let $\mathcal{A} = (A, [\cdot,\cdot]_A, \rho)$ be a Lie algebroid and let $\nabla : \Gamma(A) \to \Gamma(A^* \otimes A)$ be an $\mathcal{A}$-connection – the connection is defined as in 2.5. As above, the connection extends to the exterior connection $\nabla : \Gamma(\Lambda^k A^* \otimes A) \to \Gamma(\Lambda^{k+1} A^* \otimes A)$ via (2.26) with respect to $\mathcal{A}$. First, curvature and torsion are defined.

Definition 2.15. With the identity $\mathrm{id}_A : A \to A$ considered as a section $\mathbb{1}_A \in \Gamma(A \otimes A^*)$, *curvature* and *torsion* are defined as

$$\begin{aligned} R &= \nabla \circ \nabla \qquad &&\in \Gamma\big(\Lambda^2 A^* \otimes \mathrm{End}(A)\big)\,, \\ T &= \nabla \mathbb{1}_A &&\in \Gamma\big(\Lambda^2 A^* \otimes A\big)\,, \end{aligned} \tag{2.43}$$

respectively. $\mathrm{End}(A) = A^* \otimes A$ denotes the endomorphism bundle. Moreover, the *Ricci tensor* $\mathrm{Ric} \in \Gamma(A^* \otimes A^*)$ is defined as $\mathrm{Ric}(s_1, s_2) = \mathrm{tr}(t \mapsto R(t, s_2)s_1)$. If A is additionally equipped with a metric g, the *Ricci scalar* is given by $S = \mathrm{tr}_g \mathrm{Ric} \in C^\infty(M)$.

The endomorphism-valuedness of the curvature is a consequence of (2.26) and the properties of the Lie algebroid bracket (cf. section 2.2.2). Using (2.26) the definition of the curvature can be cast into the more familiar form

$$R(s_1, s_2)(s_3) = [\nabla_{s_1}, \nabla_{s_2}]s_3 - \nabla_{[s_1,s_2]_{\mathcal{A}}} s_3\,. \tag{2.44}$$

Similarly, the torsion can be written as

$$T(s_1, s_2) = \nabla_{s_1} s_2 - \nabla_{s_2} s_1 - [s_1, s_2]_{\mathcal{A}}\,. \tag{2.45}$$

In a local frame $\{e_\alpha\}$ for A with $\{e^\alpha\}$ its dual, curvature and torsion are written as

$$R^\alpha{}_{\beta\gamma\delta} = \langle e^\alpha, R(e_\gamma, e_\delta)e_\beta\rangle \qquad \text{and} \qquad T^\alpha{}_{\beta\gamma} = \langle e^\alpha, T(e_\beta, e_\gamma)\rangle\,, \tag{2.46}$$

respectively. Returning to the leitmotif of formulating geometry in particular in terms of a bracket, the connection will be specified. A connection is said to be *compatible with* g for $g \in \Gamma(\bigodot^2 A^*)$ a metric on A if

$$\rho(s_1)g(s_2, s_3) = g(\nabla_{s_1} s_2, s_3) + g(s_2, \nabla_{s_1} s_3) \tag{2.47}$$

for all $s_1, s_2, s_3 \in \Gamma(A)$. This allows to determine the connection in terms of the bracket, the metric and the torsion as follows.

Proposition 2.16. *Let g be a metric on A and ∇ an $\mathcal{A}$-connection compatible with g. For a given torsion T the connection is uniquely determined by the* Koszul formula

$$\begin{aligned} g(\nabla_{s_1} s_2, s_3) = \tfrac{1}{2}\big[&\rho(s_1)g(s_2, s_3) + \rho(s_2)g(s_3, s_1) - \rho(s_3)g(s_1, s_2) \\ &+ g([s_1, s_2]_{\mathcal{A}}, s_3) + g([s_2, s_3]_{\mathcal{A}}, s_1) - g([s_3, s_1]_{\mathcal{A}}, s_2) \\ &+ g(T(s_1, s_2), s_3) + g(T(s_2, s_3), s_1) - g(T(s_3, s_1), s_2)\big] \end{aligned} \tag{2.48}$$

for all $s_1, s_2, s_3 \in \Gamma(A)$. This connection is called Bismut *if $g(T(s_1, s_2), s_3)$ is totally antisymmetric and* Levi-Civita *in the case of vanishing torsion.*

The Koszul formula follows from successive application of (2.45) and (2.47). Uniqueness is a consequence of the Koszul formula and non-degeneracy of the metric.

Having introduced the basic ingredients for formulating a theory of gravity on arbitrary Lie algebroids, the relation between different such gravity theories can be studied. The detailed analysis of these relations is done in chapter 4. The main mathematical input is the following.

Theorem 2.17. *Let $\mathcal{A}_1 = (A_1, [\cdot,\cdot]_{A_1}, \rho_1)$ and $\mathcal{A}_2 = (A_2, [\cdot,\cdot]_{A_2}, \rho_2)$ be two Lie algebroids and $\Phi : A_1 \to A_2$ an injective Lie algebroid homomorphism. Moreover, let $\nabla_{1/2}$ be the Levi-Civita $\mathcal{A}_{1/2}$-connection compatible with the metric $g_{1/2}$ and let the metrics be related via $\otimes^2\Phi^t(g_2) = g_1$. Then the connections are related as*

$$\Phi\left[(\nabla_1)_{s_1}s_2\right] = (\nabla_2)_{\Phi(s_1)}\Phi(s_2) \tag{2.49}$$

for all $s_i \in \Gamma(A_1)$. For the curvature $R_{1/2}$ and torsion $T_{1/2}$ on $\mathcal{A}_{1/2}$ this implies

$$\begin{aligned} \Phi\left[R_1(s_1,s_2)s_3\right] &= R_2(\Phi(s_2),\Phi(s_2))\Phi(s_3)\,,\\ \Phi\left[T_1(s_1,s_2)\right] &= T_2(\Phi(s_2),\Phi(s_2))\,. \end{aligned} \tag{2.50}$$

Proof. By using the relation between the metrics and the Koszul formula one finds

$$\begin{aligned} g_2\big(\Phi[(\nabla_1)_{s_1}s_2],\Phi(s_3)\big) &= g_1\big((\nabla_1)_{s_1}s_2,s_3\big)\\ &= \tfrac{1}{2}\left[\rho_2\big(\Phi(s_1)\big)g_2\big(\Phi(s_2),\Phi(s_3)\big) + g_2\big(\Phi([s_1,s_2]_{\mathcal{A}_1}),\Phi(s_3)\big) + \ldots\right]\\ &= g_2\big((\nabla_2)_{\Phi(s_1)}\Phi(s_2),\Phi(s_3)\big) \end{aligned}$$

with the dots indicating the remaining permuted terms in the Koszul formula. In the second step the Koszul formula for ∇_1 was used together with the relations between the metrics and $\rho_2 \circ \Phi = \rho_1$. In the last step $\Phi([s_1,s_2]_{\mathcal{A}_1}) = [\Phi(s_1),\Phi(s_2)]_{\mathcal{A}_2}$ was used to finally obtain the Koszul formula for ∇_1. The first claim follows from non-degeneracy of the metric. Using (2.44) and (2.45), the last claim readily follows from the first one and the properties of Φ. □

The relation to Riemannian geometry

The case of Φ being an invertible anchor is of particular interest in chapter 4. Then theorem 2.17 relates the geometry on Lie algebroids to Riemannian geometry:

$$\begin{aligned} \widehat{\nabla}_{s_1}s_2 &= \rho^{-1}\left[\nabla_{\rho(s_1)}\rho(s_2)\right]\,,\\ \widehat{R}(s_1,s_2)s_3 &= \rho^{-1}\left[R(\rho(s_2),\rho(s_2))\rho(s_3)\right]\,,\\ \widehat{T}(s_1,s_2) &= \rho^{-1}\left[T(\rho(s_2),\rho(s_2))\right]\,. \end{aligned} \tag{2.51}$$

The hat indicates the objects on the Lie algebroid $\mathcal{A}$ whereas the un-hatted objects refer to the tangent bundle. Moreover, if the $\mathcal{A}$-connection is compatible with the insertion $\langle\cdot,\cdot\rangle : A \times A^* \to \mathbb{R}; (s,\sigma) \mapsto \iota_s\sigma$, i.e.

$$\rho(t)\langle s,\sigma\rangle = \langle\widehat{\nabla}_t s,\sigma\rangle + \langle s,\widehat{\nabla}_t\sigma\rangle \tag{2.52}$$

for $s,t \in \Gamma(A)$ and $\sigma \in \Gamma(A^*)$, the connection can be extended to dual sections; also the relation between the two connections extends to

$$\widehat{\nabla}_s\sigma = \rho^t\left[\nabla_{\rho(s)}\rho^{-t}(\sigma)\right]\,. \tag{2.53}$$

This can be seen as follows: Since $\langle \rho(s), \rho^{-t}(\sigma)\rangle = \sigma(\rho^{-1}\circ\rho(s)) = \langle s,\sigma\rangle$ by definition of the transposition and the insertion, compatibility of ∇ and $\widehat{\nabla}$ with the latter gives

$$\begin{aligned}\langle \widehat{\nabla}_t s,\sigma\rangle + \langle s, \widehat{\nabla}_t\sigma\rangle &= \big\langle \nabla_{\rho(t)}\rho(s), \rho^{-t}(\sigma)\big\rangle + \big\langle \rho(s), \nabla_{\rho(t)}\rho^{-t}(\sigma)\big\rangle \\ &= \big\langle \rho(\widehat{\nabla}_t s), \rho^{-t}(\sigma)\big\rangle + \big\langle \rho(s), \nabla_{\rho(t)}\rho^{-t}(\sigma)\big\rangle \\ &= \big\langle \widehat{\nabla}_t s,\sigma\big\rangle + \big\langle s, \rho^t\big[\nabla_{\rho(t)}\rho^{-t}(\sigma)\big]\big\rangle\,. \end{aligned} \tag{2.54}$$

Comparing the second term on the left-hand side with the last term gives the desired relation.

It is useful to give the local formulas. To this end, let $\{e_\alpha\}$ and $\{e_a\}$ be a frames for A and TM respectively. The anchor is written locally as

$$\begin{aligned} \langle \rho(e_\alpha), e^a\rangle &\equiv \rho^a{}_\alpha\ , \qquad & \langle \rho^{-1}(e_a), e^\alpha\rangle &\equiv \rho^\alpha{}_a\,, \\ \langle e_\alpha, \rho^t(e^a)\rangle &\equiv \rho_\alpha{}^a\ , \qquad & \langle e_a, \rho^{-t}(e^\alpha)\rangle &\equiv \rho_a{}^\alpha\,. \end{aligned} \tag{2.55}$$

The local expressions for curvature and torsion are

$$\langle e^\alpha, \widehat{R}(e_\gamma,e_\delta)e_\beta\rangle = \widehat{R}^\alpha{}_{\beta\gamma\delta} \qquad \text{and} \qquad \langle e^\alpha, \widehat{T}(e_\beta,e_\gamma)\rangle = \widehat{T}^\alpha{}_{\beta\gamma} \tag{2.56}$$

and accordingly for the tangent bundle. With these assignments the relations (2.51) for the curvatures and torsions locally read

$$\begin{aligned} \widehat{R}^\alpha{}_{\beta\gamma\delta} &= \rho^\alpha{}_a\,\rho^b{}_\beta\,\rho^c{}_\gamma\,\rho^d{}_\delta\,R^a{}_{bcd} \\ \widehat{R}_{\alpha\beta} &= \rho^a{}_\alpha\,\rho^b{}_\beta\,R_{ab} \\ \widehat{R} &= R \\ \widehat{T}^\alpha{}_{\beta\gamma} &= \rho^\alpha{}_a\,\rho^b{}_\beta\,\rho^c{}_\gamma\,T^a{}_{bc}\,. \end{aligned} \tag{2.57}$$

Here $\widehat{R}_{\alpha\beta} = \widehat{R}^\gamma{}_{\alpha\gamma\beta} = \mathrm{Ric}(e_\alpha,e_\beta)$ denotes the Ricci tensor and $\widehat{R} = g^{\alpha\beta}\widehat{R}_{\alpha\beta} = S$ the Ricci scalar. By (2.57), all the well-known properties of the Riemannian objects – (anti-)symmetries and Bianchi identities – carry over to the objects on the Lie algebroid.

2.4 Summary

The generalized tangent bundle has been described as a vector bundle with $G_{d\omega} \rtimes \mathrm{GL}(d)$-structure and an $O(d,d)$-invariant inner product. In analogy to Riemannian geometry, the structure group was reduced to $O(d)\times O(d)$ by the introduction of a generalized metric $\mathcal{H}$. In order to approach a dynamical theory for the generalized metric – a generalized gravity theory analogous to general relativity as dynamical theory of a Riemannian metric – the algebraic structure of the infinitesimal structure group has been formulated in terms of Courant algebroids. As a novel result, the same has been done for the generalized cotangent bundle – a vector bundle with $G_{d_A X} \rtimes \mathrm{GL}(d)$-structure. A connection on a Courant algebroid has been introduced in order to define a curvature tensor. The obstructions for constructing it have been systematically traced back to the unusual properties of the Courant bracket. Circumventing these obstructions non-trivially has lead to the restriction to Dirac structures.

Dirac structures are particular examples of Lie algebroids. The latter have been introduced and their cohomology as well as a consistent differential geometry on them has been discussed briefly. In particular, the precise relation between the geometries on different Lie algebroids connected by a Lie algebroid homomorphism has been formulated in the novel theorem 2.17. Finally, the relation between Riemannian geometry and Lie algebroid geometry has been discussed in some detail.

In the next two chapters, the formalism presented above is used to reveal the geometrical structures of dualities in string theory. Courant algebroids turn out to be the suitable structure for combining the geometry of T-dual quantities, while Lie algebroids are used to govern the dynamical theory for a string backgrounds locally. But first, a geometric approach to target-space dualities is presented to substantiate the necessity for a generalized geometric approach to string theory.

Chapter 3

$O(d, d)$-duality

The main subject of this thesis is a geometric description of target-space dualities in string theory and the treatment of unconventional implications thereof. To set the stage, this chapter is devoted to the target-space duality structure of string theory and aims to provide the necessary structures and most instructive examples. The approach followed here was developed recently in [49]. Apart from the well-known T-duality, a new duality called Poisson duality is discovered within this novel approach.

The chapter is organized as follows. In section 3.1 classical features of the string sigma model are recapitulated. In particular, the appearance of $O(d, d)$ is extracted from the constraints in a Hamiltonian formulation followed by a brief review of the conventinal approach to T-duality. Section 3.2 is devoted to the detailed discussion of $O(d, d)$-duality. It includes the study of the integrability conditions for the mapping of coordinate one-forms manifest in the isometry algebra, the main elements of $O(d, d)$ and the special role of the dilaton for duality on the quantum level. The section closes with an example providing a new approximate non-geometric background.

3.1 The bosonic string sigma model

String theory is described in a background dependent fashion by a two-dimensional non-linear sigma model. For discussing closed bosonic strings, Σ is a two-dimensional manifold with metric[1] $h = \mathrm{diag}(-1, 1)$ and $\partial\Sigma = \emptyset$. The worldsheet Σ is embedded into a d-dimensional Riemannian manifold M via $X : \Sigma \hookrightarrow M$. Having local coordinates $\{x^a\}_{a=1}^d$ for M, their pull-back to Σ is denoted $X^a = X^* x^a$. With $\star$ the Hodge operator with respect to h, the action can be written as[2]

$$S(X; G, B) = \frac{1}{4\pi\alpha'} \int_\Sigma \Big[G(X)_{ab}\, dX^a \wedge \star dX^b + B(X)_{ab}\, dX^a \wedge dX^b \Big] \,. \tag{3.1}$$

[1]The sigma model (3.1) is invariant under two-dimensional Weyl rescalings and two-dimensional diffeomorphisms. Hence conformal gauge can be chosen.

[2]The conventions are as follows: The coordinates on Σ are $\{\tau, \sigma\}$ and the orientation is given by the volume element $d\tau \wedge d\sigma$. Then the Hodge operator is given by $\alpha \wedge \star\beta = h(\alpha, \beta) d\tau \wedge d\sigma$ for arbitrary $\alpha, \beta \in \Gamma(\Lambda^n T^*\Sigma)$.

G is a Riemannian metric[3] on the target-space M and B a two-form; the pair (G, B) will be called the *background*. The dilaton will be discussed separately in section 3.2.4 as it contributes as quantum correction and breaks Weyl invariance already classically. The immediate classical features of (3.1) are the following.

- Varying the action with respect to X^a yields the equation of motion

$$d\star dX^a + \Gamma^a{}_{bc}\, dX^b \wedge \star dX^c = \tfrac{1}{2}\, G^{am}\, H_{mbc}\, dX^b \wedge dX^c \tag{3.2}$$

  with $H = dB$ and $\Gamma^a{}_{bc} = \frac{1}{2}G^{am}(\partial_b G_{mc} + \partial_c G_{mb} - \partial_m G_{bc})$ the coefficients of the Levi-Civita connection on TM. Possible boundary terms are neglected. For $H = 0$, (3.2) is the generalization of the geodesic equation for a worldsheet. In the presence of the H-term, (3.2) can be interpreted as geodesic motion of a membrane in Einstein-Cartan theory with Bismut connection $\Gamma^a{}_{bc} - \frac{1}{2}G^{am}H_{mbc}$.

- The equation of motion for a general worldsheet metric h is vanishing of the energy-momentum tensor, $T_{\alpha\beta} = 0$. In the conformal gauge chosen here, this has to be considered as constraints which read

$$\begin{aligned} G_{ab}(\partial_\tau X^a \partial_\tau X^b + \partial_\sigma X^a \partial_\sigma X^b) = 0\,, \\ G_{ab}\, \partial_\tau X^a \partial_\sigma X^b = 0\,. \end{aligned} \tag{3.3}$$

  Hence the dynamics of the theory is determined by the equation of motion (3.2) accompanied by the constraints (3.3).

From this – and especially from the interpretation of the equations of motion (3.2) – it can be seen that the sigma model (3.1) describes the motion of a one-dimensional string by describing the membrane it draws on the d-dimensional background.

Hamiltonian description

The Hamiltonian density can be determined from the Lagrangian density in (3.1) by performing a Legendre transformation with respect to the canonical momentum and τ-derivative of the coordinate fields X^a. In principle there are two possibilities for canonically conjugate variables to the coordinate field X^a, which will become important for the discussion of duality:

- the *canonical momentum* $P_a = \frac{\partial L}{\partial \partial_\tau X^a} = \frac{1}{2\pi\alpha'}(-G_{ab}\partial_\tau X^b + B_{ab}\partial_\sigma X^b)$,
- the *canonical winding* $W_a = \frac{\partial L}{\partial \partial_\sigma X^a} = \frac{1}{2\pi\alpha'}(G_{ab}\partial_\sigma X^b - B_{ab}\partial_\tau X^b)$.

However, by virtue of the first constraint in (3.3), the Hamiltonian density arising from a Legendre transformation with respect to P and $\partial_\tau X$ coincides with the one resulting from a transformation with respect to W and $\partial_\sigma X$ since

$$\partial_\tau X^a\, P_a = \partial_\sigma X^a\, W_a. \tag{3.4}$$

[3] Positive definiteness of the metric is a crucial assumption for the following discussion.

Performing the transformation $L \to \partial_\tau X^a P_a - L = \text{Ham}$, the Hamiltonian density can be written as

$$\begin{aligned}\text{Ham}(X;G,B) &= -\frac{1}{4\pi\alpha'}\begin{pmatrix}\partial_\sigma X\\ 2\pi\alpha' P\end{pmatrix}^t \mathcal{H}(G,B)\begin{pmatrix}\partial_\sigma X\\ 2\pi\alpha' P\end{pmatrix}\\ &= \frac{1}{4\pi\alpha'}\begin{pmatrix}\partial_\tau X\\ -2\pi\alpha' W\end{pmatrix}^t \mathcal{H}(G,B)\begin{pmatrix}\partial_\tau X\\ -2\pi\alpha' W\end{pmatrix},\end{aligned} \tag{3.5}$$

where $\mathcal{H}$ denotes the generalized metric (2.10). Defining the generalized vectors

$$\begin{aligned}A_P(X) &= \partial_\sigma X^a \frac{\partial}{\partial x^a} + 2\pi\alpha'\, P_a\, dx^a\,,\\ A_W(X) &= \partial_\tau X^a \frac{\partial}{\partial x^a} - 2\pi\alpha'\, W_a\, dx^a\end{aligned} \tag{3.6}$$

in $TM \oplus T^*M$, the Hamiltonian density (3.5) is proportional to the squared length of A_P and A_W as measured by the generalized metric (2.10):

$$\text{Ham}(X;G,B) = -\tfrac{1}{4\pi\alpha'}||A_P||^2_{\mathcal{H}} = \tfrac{1}{4\pi\alpha'}||A_W||^2_{\mathcal{H}}\,. \tag{3.7}$$

Hence the Hamiltonian density of the sigma model for closed string theory can be interpreted as "kinetic energy" in generalized geometry with respect to the generalized "velocities" $A_{P/W}$. In this sense, the sigma model (3.1) is simple: It describes the geodesic motion of a membrane on a background whose shape is determined by the minimizing kinetic energy in generalized geometry.

3.1.1 Review of T-duality

The conventional procedure for obtaining T-dual sigma models outlined in section 1.3.1 by gauging isometries will be reviewed briefly [19]. For simplicity, a single isometry generated by a vector field k is considered. In the case of multiple non-abelian isometries the gauging procedure can be found in [36]. With respect to the infinitesimal coordinate transformation

$$X^a \to X^a + \epsilon\, k^a \tag{3.8}$$

the sigma model (3.1) transforms as $S \to S + \delta S$ with

$$\delta S(X;G,B) = \frac{\epsilon}{4\pi\alpha'}\int_\Sigma \Big[(L_k G)_{ab}\, dX^a \wedge \star dX^b + (L_k B)_{ab}\, dX^a \wedge dX^b\Big]\,. \tag{3.9}$$

Thus k generates an isometry if it satisfies

$$L_k G = 0 \qquad \& \qquad L_k B = d\nu \quad \text{for}\, \nu \in \Gamma(T^*M)\,. \tag{3.10}$$

By using that a gauge transformation $B \to B + d\omega$ induces the transformation $\nu \to \nu + L_k\omega$, a gauge in which $\nu = 0$ can be found. Assuming this gauge to be chosen in adapted coordinates $k = \frac{\partial}{\partial X^1}$ allows to gauge the isometry generated by k via minimal coupling: Introducing the gauge field $A \in \Gamma(T^*\Sigma)$ which transforms under the local version of (3.8) as $\delta A = -d\epsilon$, minimal coupling amounts to the substitution $dX^1 \to DX^1 = dX^1 + A$. Choosing the gauge $A \to A - dX^1$, the gauged sigma model takes the form $S_{\text{gauged}} = S(X^m; G, B) + S_g$ with

$$S_g = \frac{1}{4\pi\alpha'} \int_\Sigma \big(G_{11}\, A \wedge \star A + 2\, G_{1m}\, A \wedge \star dX^m + 2\, B_{1m}\, A \wedge dX^m - 2\, A \wedge d\lambda\big) \tag{3.11}$$

for $m \neq 1$. Integrating out the Lagrange multiplier λ yields $A = dX^1$ locally and gives back the initial sigma model (3.1). Integrating out the gauge field yields

$$\star A = -\frac{1}{G_{11}}\big(G_{1m} \star dX^m + B_{1m}\, dX^m - d\lambda\big)\,. \tag{3.12}$$

Plugging this back into the gauged action and considering $d\lambda = d\widetilde{X}^1$ as a new coordinate, the resulting action can be written as (3.1) with the new background (g, b) given by the *Buscher rules* [17]

$$\begin{aligned} g_{11} &= \frac{1}{G_{11}}\,, & g_{1m} &= -\frac{B_{1m}}{G_{11}}\,, & g_{mn} &= G_{mn} - \frac{G_{m1}G_{1n} + B_{m1}B_{1n}}{G_{11}}\,, \\ & & b_{1m} &= -\frac{G_{1m}}{G_{11}}\,, & b_{mn} &= B_{mn} - \frac{G_{m1}B_{1n} + B_{m1}G_{1n}}{G_{11}}\,. \end{aligned} \tag{3.13}$$

Hence, T-duality can be performed along the direction of an isometry and the dual backgrounds are related by (3.13). It also introduces a new coordinate one-form $d\widetilde{X}$ which can be related to dX^1 on-shell by (3.12): Identifying $A = dX^1$ and $d\lambda = d\widetilde{X}^1$, (3.12) can be written as

$$d\widetilde{X}^1 = G_{1a} \star dX^a + B_{1a} dX^a\,. \tag{3.14}$$

This is the conserved current associated to the isometry (3.8) generated by $k = \frac{\partial}{\partial X^1}$. For gauging multiple isometries $\{k^i\}$, further conditions apart from (3.10) arise [36]: With $\kappa_i = \nu_i - \iota_{k_i} B$ such that $\iota_{k_i} H = d\kappa_i$ and $[k_i, k_j] = F^m{}_{ij} k_m$, also

$$L_{k_i}\kappa_j = F^m{}_{ij}\,\kappa_m \qquad \text{and} \qquad \langle k_i + \kappa_i, k_j + \kappa_j\rangle = 0 \tag{3.15}$$

have to be satisfied. The second condition ensures the gauged sigma model to be free of anomalies. These anomalies arise from the introduction of the auxiliary gauge field A. It also causes problems for obtaining genuinely dual theories due to possible holonomies [19, 34]. In the abelian case they can be compensated by assigning appropriate periodicities to the Langrange multipliers λ_i. This is possible since λ_i does not transform under gauge transformations if (3.15) is satisfied which allows to choose any periodicity in a consistent manner. However, in the non-abelian case the Lagrange multipliers transform as $\delta\lambda_i = -F^m{}_{ni}\lambda_m\epsilon^n$ and periodicities can not be assigned consistently anymore.

In the next chapter a different approach to duality is developed and the Buscher rules (3.52) with (5.13) as well as the conditions (3.10), (3.15) are encountered as special cases.

3.2 $O(d,d)$-duality

In this section a new way of performing duality is proposed by redefining the background and identifying dual coordinates directly. This avoids the procedure of gauging and accordingly circumvents the problem of anomalies and holonomies caused by the introduction of an auxiliary gauge field.

Already on the classical level the indefinite orthogonal group $O(d,d)$ appears naturally. In terms of the generalized vector A_P (3.6), the constraints (3.3), i.e. the components of the energy momentum tensor can be rewritten as

$$A_P^t\,\mathcal{H}(G,B)\,A_P = 0 \qquad \text{and} \qquad A_P^t\,\eta\,A_P = \langle A_P, A_P\rangle = 0 \tag{3.16}$$

with $\langle\cdot,\cdot\rangle$ the inner product (2.5). As the first constraint sets the Hamiltonian density to zero, the constrained dynamics is completely governed by (3.4). The second constraint is preserved by $O(d,d)$-transformations. Therefore all admissible generalized vectors solving the second constraint in (3.16) are related by an $O(d,d)$-transformation via $A'_P = \mathcal{T}A_P$. For A'_P to solve the first constraint as well, a compensating $O(d,d)$-conjugation with $\mathcal{T}^{-1}$ has to be applied to the generalized metric (2.10). This, in turn, leaves the Hamiltonian density (3.5) and the energy momentum tensor (3.16) invariant. This classical duality will be described in detail in the following.

3.2.1 Field redefinitions and duality

The admissible generalized vector A_P will be transformed by[4]

$$\mathcal{T} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22}\end{pmatrix} \in O(d,d) \qquad \text{with} \qquad \begin{array}{l} t_{11} \equiv (t_{11})^a{}_{\bar{a}} \,:\, TM \;\to\; TM \\ t_{12} \equiv (t_{12})^{a\bar{a}} \,:\, T^*M \to TM \\ t_{21} \equiv (t_{21})_{a\bar{a}} \,:\, TM \;\to T^*M \\ t_{22} \equiv (t_{22})_a{}^{\bar{a}} \,:\, T^*M \to T^*M \end{array} \tag{3.17}$$

as $A_P \to \mathcal{T}^{-1}A_P$ (cf. (2.3)). In order for the first constraint in (3.16) to remain satisfied the generalized metric has to be conjugated with $\mathcal{T}$ simultaneously:

$$\mathcal{H}(G,B) \to \mathcal{T}^t\,\mathcal{H}(G,B)\,\mathcal{T} \equiv \mathcal{H}(g,b)\,. \tag{3.18}$$

By (3.5), this simultaneous transformation leaves the Hamiltonian density invariant, giving an equivalent theory. This specific equivalence will be called *$\mathcal{T}$-duality*. First, the conjugation of the generalized metric is discussed in order to extract the dual background. Then the rotation of the generalized vector is used to determine the dual phase space.

In (3.18), $\mathcal{H}(g,b)$ refers to a redefinition of the background in order for the generalized metric to have the standard form (2.10) as follows.

Field redefinition ([80])**.** *An $O(d,d)$-rotated generalized metric $\mathcal{T}^t\mathcal{H}(G,B)\mathcal{T}$ takes the form $\mathcal{H}(g,b)$ (2.10) with respect to the new background (g,b). In terms of the automorphism*

$$\gamma = t_{22} + (G-B)t_{12} : T^*M \to T^*M \tag{3.19}$$

[4]The bar over the index indicates the one associated to the domain. As to operations with linear maps, inversion swaps indices (e.g. $t_{11}^{-1} \equiv (t_{11})^{\bar{a}}{}_a : TM \to TM$) and transposition commutes them (e.g. $t_{11}^t \equiv (t_{11})_{\bar{a}}{}^a : T^*M \to T^*M$). The combination $f^{-t} = (f^{-1})^t$ is used as well.

and $\delta = t_{21} + (G-B)t_{11}$, the new background (g,b) is given by

$$g = \gamma^{-1}\, G\, \gamma^{-t} \qquad \textit{and} \qquad b = \gamma^{-1}(\gamma\, \delta^t - G)\gamma^{-t}\,. \tag{3.20}$$

Proof. First, the invertibility of γ will be shown. Since $O(d,d)$ is generated by the matrices (2.6) and (2.7), one can proceed case-by-case. Since $\gamma_A = \mathsf{A}^{-t}$ and $\gamma_B = \mathbb{1}$ for changes of frame and B-transformations, respectively, invertibility is obvious. For β-transformations the map becomes $\gamma_\beta = \mathbb{1} - (G-B)\beta$. Since G is positive definite by assumption, $(G-B)$ is positive definite by antisymmetry of B as well. As such, $(G-B)$ has a positive definite inverse which allows to write $\gamma_\beta = (G-B)[(G-B)^{-1} - \beta]$. Since β is also antisymmetric, the matrix $[(G-B)^{-1} - \beta]$ is also positive definite and therefore invertible. Hence γ_β is invertible as a product of invertible matrices. The remaining generator is T-duality in the k^{th} direction (2.7) with $\gamma_{T(k)} = \mathbb{1} - 1_k + (G-B)1_k = \mathbb{1} - 1_k + G_{kk}1_k$ by antisymmetry of B. Since $\det(\gamma_{T(k)}) = G_{kk} = G(e_k, e_k) \neq 0$ by positive definiteness, it is invertible as well. The relations (3.20) directly follow from the evaluation of (3.18). □

The simultaneous rotation of the generalized vectors (3.6) gives rise to redefined phase space coordinates. They can be read-off from the transformation

$$A_P \to \mathcal{T}^{-1}\, A_P(X) \equiv A_{\widetilde{P}}(\widetilde{X}) = \begin{pmatrix} \partial_\sigma \widetilde{X} \\ 2\pi\alpha'\widetilde{P} \end{pmatrix} \tag{3.21}$$

and analogously for the winding vector. Using $\mathcal{T}^{-1} = \eta\mathcal{T}^t\eta$ for all $\mathcal{T} \in O(d,d)$, the dual pair becomes

$$\begin{aligned} \partial_\sigma \widetilde{X}^{\bar a} &= -(t_{12})^{\bar a m} G_{ma}\, \partial_\tau X^a + \big[(t_{22})^{\bar a}{}_a + (t_{12})^{\bar a m} B_{ma}\big]\partial_\sigma X^a\,, \\ \widetilde{P}_{\bar a} &= \tfrac{1}{2\pi\alpha'}\left\{-(t_{11})_{\bar a}{}^m G_{ma}\, \partial_\tau X^a + \big[(t_{21})_{\bar a a} + (t_{11})_{\bar a}{}^m B_{ma}\big]\partial_\sigma X^a\right\}\,. \end{aligned} \tag{3.22}$$

For determining the dual coordinates the τ-derivative of X^a is required as well. In principle, $\partial_\tau \widetilde{X}^{\bar a}$ can be computed from the first Hamilton equation. Since the Hamiltonian densities with respect to momentum and winding coincide by (3.4), it is easier to deduce it directly from the winding vector $A_{\widetilde{W}}(\widetilde{X})$ as above:

$$\begin{aligned} \partial_\tau \widetilde{X}^{\bar a} &= -(t_{12})^{\bar a m} G_{ma}\, \partial_\sigma X^a + \big[(t_{22})^{\bar a}{}_a + (t_{12})^{\bar a m} B_{ma}\big]\partial_\tau X^a\,, \\ \widetilde{W}_{\bar a} &= \tfrac{-1}{2\pi\alpha'}\left\{-(t_{11})_{\bar a}{}^m G_{ma}\, \partial_\sigma X^a + \big[(t_{21})_{\bar a a} + (t_{11})_{\bar a}{}^m B_{ma}\big]\partial_\tau X^a\right\}\,. \end{aligned} \tag{3.23}$$

Having determined both worldsheet derivatives of the dual coordinates[5] $\widetilde{X}^{\bar a}$, the main result of this chapter can be formulated.

$O(d,d)$**-duality.** *Let $\{e_a\}_{a=1}^d$ be a frame for TM. For $\mathcal{T} \in O(d,d;\mathcal{C}^\infty(M))$, the sigma model $S(X;G,B)$ (3.1) is $\mathcal{T}$-dual to $S(\widetilde{X};g,b)$ with the coordinates related via*

$$d\widetilde{X}^{\bar a} = \big(\iota_{t_{12}^\sharp e^{\bar a}} G\big)_a \star dX^a + \big(t_{22}^\sharp e^{\bar a} + \iota_{t_{12}^\sharp e^{\bar a}} B\big)_a\, dX^a \tag{3.24}$$

[5] Using the relations between the elements of the $O(d,d)$-matrix $\mathcal{T}^{-1}$, (3.22) and (3.23) satisfy the constraint $\partial_\tau \widetilde{X}^{\bar a}\widetilde{P}_{\bar a} = \partial_\sigma \widetilde{X}^{\bar a}\widetilde{W}_{\bar a}$ (3.4) as well.

and the backgrounds related by the field redefinition (3.20)*, provided*

$$L_{t_{12}^\sharp e^{\bar a}} G = 0 \qquad \text{and} \qquad L_{t_{12}^\sharp e^{\bar a}} B = -d\big(t_{22}^\sharp e^{\bar a}\big)\,. \tag{3.25}$$

Here $t_{12}^\sharp e^{\bar a} = (t_{12})^{\bar a m} e_m$ *and* $t_{22}^\sharp e^{\bar a} = (t_{22})^{\bar a}{}_m e^m$*. The requirement* (3.25) *is the integrability condition for* (3.24)*.*

Proof. Equation (3.24) is the combination of (3.22) and (3.23). The integrability condition (3.25) can be deduced by differentiating (3.24) and using the equations of motion (3.2) as well as $\iota_v H = L_v B - d\iota_v B$ for any vector field v: (3.2) gives

$$\begin{aligned}(t_{12})^{\bar a m} G_{ma}\, d\star dX^a = X^*\big(L_{t_{12}^\sharp e^{\bar a}} B - d\iota_{t_{12}^\sharp e^{\bar a}} B\big) + \tfrac{1}{2}(t_{12})^{\bar a m}\partial_m G_{ab} dX^a \wedge \star dX^b \\ - (t_{12})^{\bar a m} dG_{ma} \wedge \star dX^a\,.\end{aligned}$$

Plugging this into $d(d\widetilde{X}^{\bar a})$ gives

$$d\big(d\widetilde{X}^{\bar a}\big) = X^*\big[L_{t_{12}^\sharp e^{\bar a}} B + d\big(t_{22}^\sharp e^{\bar a}\big)\big] + \tfrac{1}{2}\big(L_{t_{12}^\sharp e^{\bar a}} G\big)_{ab} dX^a \wedge \star dX^b\,.$$

As the symmetric and antisymmetric parts have to vanish separately, (3.25) follows. □

Further restrictions arise from the algebra spanned by the vectors $t_{12}^\sharp e^{\bar a}$, which will be discussed in section 3.2.2. $O(d,d)$-duality can be described in terms of the *duality map* as follows. By defining $dX = dX^a e_a \in \Gamma(TM \otimes T^*\Sigma)$, the duality automorphism

$$\mathfrak{D} : \Gamma(TM \otimes T^*\Sigma) \to \Gamma(TM \otimes T^*\Sigma)\,;\; dX \mapsto d\widetilde{X} = \mathfrak{D}(dX) \tag{3.26}$$

follows from (3.24). In matrix notation it can be written globally as

$$\mathfrak{D} = t_{12}^t\, G \otimes \star + \big(t_{22}^t + t_{12}^t\, B\big) \otimes \mathrm{id}_{T^*\Sigma}\,. \tag{3.27}$$

Indeed, the inverse of the duality map (3.27) can be easily determined by the inverse procedure and reads

$$\mathfrak{D}^{-1} = t_{12}\, g \otimes \star + \big(t_{11} + t_{12}\, b\big) \otimes \mathrm{id}_{T^*\Sigma} \tag{3.28}$$

in terms of the dual background (3.20). Hence $O(d,d)$-duality is invertible.

Duality and isometries

The dual coordinates (3.24) and the integrability conditions (3.25) can be interpreted as follows. As shown in section 3.1.1, the sigma model (3.1) transforms under the infinitesimal diffeomorphism $X^a \to X^a + \epsilon k^a$, generated by the vector field k as $S \to S + \delta S$ with δS given by (3.9). Thus, k generates an isometry if it satisfies (3.10), i.e.

$$L_k G = 0 \qquad \text{and} \qquad L_k B = d\nu \quad \text{for } \nu \in \Gamma(T^*M)\,.$$

Comparing this with the integrability condition (3.25), $t_{12}^\sharp e^{\bar a}$ is seen to generate an isometry for (3.1). In particular, the one-form ν is explicitly determined as $-t_{22}^\sharp e^{\bar a}$. These special isometries will be called *duality isometries*. Assuming ϵ to be non-constant, the conserved current associated to the isometry generated by k can be computed; it reads

$$J_k = (\iota_k G)_a \star dX^a + (\iota_k B - \nu)_a dX^a.$$

Therefore the dual coordinates $d\widetilde{X}^{\bar{a}}$ (3.24) coincide with the conserved currents $J_{t^\sharp_{12} e^{\bar{a}}}$. Hence the integrability condition (3.25) for the dual coordinates (3.24) ensures the vector $t^\sharp_{12} e^{\bar{a}}$ to generate an isometry and the duality map (3.27) interchanges the TM-valued coordinate one-form dX with the TM-valued conserved current $J = d\widetilde{X}$.

Is (3.24) a coordinate transformation?

By using the Poincaré lemma and the integrability conditions (3.25), (3.24) is locally exact. Then the local primitive for $d\widetilde{X}^{\bar{a}}$ might be interpreted as dual pulled-back coordinate $\widetilde{X}^{\bar{a}}$. First, this raises the question whether the coordinates on the target-space are changed, i.e. $\widetilde{X}^{\bar{a}} = X^*(\tilde{x}^{\bar{a}})$, or the embedding is changed, i.e. $\widetilde{X}^{\bar{a}} = \widetilde{X}^*(x^{\bar{a}})$. Second, it is not clear if the resulting relation $\widetilde{X}^{\bar{a}}(X)$ is invertible, i.e. if $X^a(\widetilde{X})$ can be found. In particular, both questions are important for the interpretation of the field redefinition (3.20) due to (3.18), since the new background still depends on the initial coordinates.[6] This also effects the interpretation of (3.28).

In the case of constant $O(d,d)$-transformations and constant backgrounds, (3.24) can be integrated directly and the relation between the dual coordinates is invertible: The equations of motion (3.2) reduces to the wave equation and is solved by $X^a(\tau,\sigma) = X^a_+(\sigma^+) + X^a_-(\sigma^-)$ with the light-cone coordinates $\sigma^\pm = \tau \pm \sigma$. Using that $O(d,d)$-duality with respect to the unit matrix leaves everything invariant, (3.24) can be integrated to give

$$\begin{aligned} \widetilde{X}^{\bar{a}}_+ &= \left[(t_{22})^{\bar{a}}{}_a + (t_{12})^{\bar{a}m}(B_{ma} - G_{ma})\right] X^a_+ \\ \widetilde{X}^{\bar{a}}_- &= \left[(t_{22})^{\bar{a}}{}_a + (t_{12})^{\bar{a}m}(B_{ma} + G_{ma})\right] X^a_- \,. \end{aligned} \tag{3.29}$$

Invertibility of $t_{22} + t_{12}(B \pm G)$ is equivalent to the invertibility of (3.19). Thus in this case (3.24) gives rise to a proper change of coordinates. Keeping the necessity of a positive definite metric for invertibility of (3.19) in mind, this shows that $O(d,d)$-duality includes the well-known case of the T-duality group $O(d,d;\mathbb{Z})$ for toroidal target-spaces; the transformations have to be integer in order for the periodicities to be preserved (see e.g. [31]).[7] The novelty is that $O(d,d)$-duality gives the precise relation between the dual coordinates.

Comment on global issues

In string theory the metric G on the whole target space has Lorentzian signature $(d-1,1)$. However, for the map (3.19) responsible for the field redefinition to be invertible, the signature was assumed to be Euclidean since positive definiteness was crucial in the proof of invertibility. Thus $O(d,d)$-duality is specifically applicable to the compact space C of

[6] I thank the referee for pointing out this problem of interpretation.

[7] See sections 3.2.2 and 3.2.3 for further relations to the known cases.

a string compactification $M_d = M_{d-\dim C} \times C$. The integrability conditions (3.25) ensure $d\widetilde{X}^{\bar{a}}$ to be closed; exactness might be spoiled by the winding

$$\tilde{c}_{\text{wind}}\big(\widetilde{X}^{\bar{a}}\big) = \oint_\gamma d\widetilde{X}^{\bar{a}} \tag{3.30}$$

of $\widetilde{X}^{\bar{a}}$ around a compact direction in C; γ is a closed curve in Σ. This is related to the winding number of the initial coordinate one-forms by (3.24).

3.2.2 The algebra of isometries and consistency

In this section the consistency of the integrability condition (3.25) is studied in terms the associated isometry algebra. The aim is to formulate a Courant algebroid 2.2 taking into account both integrability conditions at once.

$O(d,d)$-duality is feasible if $t^\sharp_{12}e^{\bar{a}}$ generates isometries for (3.1). Moreover, the isometry algebra has to close and has to satisfy the Jacobi identity. For their part, Killing vector fields are closed: $[t^\sharp_{12}e^{\bar{a}}, t^\sharp_{12}e^{\bar{b}}]$ is a Killing vector field as well. However, consistent duality isometries require $[t^\sharp_{12}e^{\bar{a}}, t^\sharp_{12}e^{\bar{b}}]$ to be a linear combination of the generators $t^\sharp_{12}e^{\bar{a}}$. This is described in terms of Lie algebroids 2.9 in the following.

The vector fields $t^\sharp_{12}e^{\bar{a}}$ generate non-abelian isometries with algebra

$$\begin{aligned}\big[t^\sharp_{12}e^{\bar{a}}, t^\sharp_{12}e^{\bar{b}}\big] &= \big[D^{\bar{a}}(t_{12})^{\bar{b}p} - D^{\bar{b}}(t_{12})^{\bar{a}p} + (t_{12})^{\bar{a}m}\,(t_{12})^{\bar{b}n}\,f^p{}_{mn}\big]e_p \\ &= F_{\bar{m}}{}^{\bar{a}\bar{b}}\,t^\sharp_{12}e^{\bar{m}} + \Theta^{m\bar{a}\bar{b}}\,e_m\end{aligned} \tag{3.31}$$

with the differential $D^{\bar{a}} = (t_{12})^{\bar{a}m}e_m$. Hence the duality isometries do not span a closed algebra in general. The defect is given by $\Theta \in \Gamma(\bigotimes^3 TM)$, which can locally be written as

$$\begin{aligned}\Theta^{abc} = {} & (t_{12})^{bm}\partial_m(t_{12})^{ca} - (t_{12})^{cm}\partial_m(t_{12})^{ba} \\ & - \tfrac{1}{2}\big[(t_{12})^{ma}\partial_m(t_{12})^{bc} - (t_{12})^{ma}\partial_m(t_{12})^{cb}\big] \\ & + (t_{12})^{bm}\,(t_{12})^{cn}\,f^a{}_{mn} - (t_{12})^{ma}\,(t_{12})^{bn}\,f^c{}_{mn} - (t_{12})^{cm}\,(t_{12})^{na}\,f^b{}_{mn}\,.\end{aligned} \tag{3.32}$$

The structure constants $F_{\bar{a}}{}^{\bar{b}\bar{c}}$ can be determined in terms of the structure constants $[e_a, e_b] = f^m{}_{ab}e_m$:

$$F_a{}^{bc} = \tfrac{1}{2}\big[\partial_a(t_{12})^{bc} - \partial_a(t_{12})^{cb}\big] + (t_{12})^{bm}\,f^c{}_{am} - (t_{12})^{cm}\,f^b{}_{am}\,. \tag{3.33}$$

Thus the isometry algebra (3.31) closes if the defect (3.32) vanishes[8], which is assumed in the following. This condition can conveniently be studied in terms of Lie algebroids. $t^\sharp_{12}$ maps T^*M to TM and can therefore be applied to general one-forms ξ, η: $t^\sharp_{12}\xi = \xi_{\bar{a}}(t_{12})^{\bar{a}m}e_m$. Then the Lie bracket gives

$$\big[t^\sharp_{12}\xi, t^\sharp_{12}\eta\big] = \big[\xi_{\bar{m}}\,D^{\bar{m}}\eta_{\bar{a}} - \eta_{\bar{m}}\,D^{\bar{m}}\xi_{\bar{a}} + \xi_{\bar{m}}\,\eta_{\bar{n}}\,F_{\bar{a}}{}^{\bar{m}\bar{n}}\big]t^\sharp_{12}e^{\bar{a}}\,. \tag{3.34}$$

[8]This condition is only sufficient. However, for an antisymmetric t_{12} this is a natural construction.

From this a Lie algebroid $(T^*M, [\cdot,\cdot]_{\text{iso}}, t_{12}^\sharp)$ can be deduced. As can readily be seen from (3.34) and the properties of the Lie bracket, the bracket $[\cdot,\cdot]_{\text{iso}}$ is given by

$$[\xi,\eta]_{\text{iso}} = \left[\xi_{\bar m}\, D^{\bar m}\eta_{\bar a} - \eta_{\bar m}\, D^{\bar m}\xi_{\bar a} + \xi_{\bar m}\,\eta_{\bar n}\, F_{\bar a}{}^{\bar m\bar n}\right] e^{\bar a} \qquad \forall \xi,\eta \in \Gamma(T^*M)\,, \tag{3.35}$$

which fulfills the anchor property by construction. This construction of a Lie algebroid is analogous to the one introduced in [80]. From the anchor property it follows that if the Lie algebroid bracket satisfies the Jacobi identity, the isometry algebra (3.31) satisfies it as well. It is more instructive to study the Jacobi identity for $[\cdot,\cdot]_{\text{iso}}$. To this end, two cases are distinguished.

- t_{12} **antisymmetric:** The bracket (3.35) can be written as

  $$[\xi,\eta]_{\text{iso}} = L_{t_{12}^\sharp\xi}\eta - \iota_{t_{12}^\sharp\eta} d\xi = [\xi,\eta]_K\,, \tag{3.36}$$

  i.e. it coincides with the Koszul bracket (2.31). It satisfies the Jacobi identity and anchor property (the anchor being $t_{12}^\sharp$) if t_{12} is a Poisson bi-vector; this is equivalent to the vanishing of Θ (3.32), which now agrees with (2.32). Hence for t_{12} antisymmetric, the isometry algebra (3.31) is a Lie algebra if and only if t_{12} is a Poisson bi-vector.

- t_{12} **symmetric:** The structure constant becomes very simple such that the Lie algebroid bracket (3.35) reduces to

  $$[\xi,\eta]_{\text{iso}} = \iota_{t_{12}^\sharp\xi} d\eta - \iota_{t_{12}^\sharp\eta} d\xi\,. \tag{3.37}$$

  The Jacobi identity can be checked by using vanishing of (3.32) and the Jacobi identity for the Lie bracket.

The case of an antisymmetric t_{12} is of particular importance as it covers the case of β-transformations (2.12) discussed in section 3.2.3.

Now the second condition in (3.25) will be discussed. Assuming $\Theta = 0$, consistency of the integrability conditions (3.25) with the algebra (3.31) requires the two ways of evaluating $L_{[t_{12}^\sharp e^{\bar a}, t_{12}^\sharp e^{\bar b}]}B$ to coincide:

$$dF_{\bar m}{}^{\bar a\bar b} \wedge \iota_{t_{12}^\sharp e^{\bar m}} B - F_{\bar m}{}^{\bar a\bar b}\, d\big(t_{22}^\sharp e^{\bar m}\big) = -d\left[L_{t_{12}^\sharp e^{\bar a}}\big(t_{22}^\sharp e^{\bar b}\big) - L_{t_{12}^\sharp e^{\bar b}}\big(t_{22}^\sharp e^{\bar a}\big)\right]. \tag{3.38}$$

This in turn is only consistent if the left-hand-side is closed, which is equivalent to

$$dF_{\bar m}{}^{\bar a\bar b} \wedge \iota_{t_{12}^\sharp e^{\bar m}} H = 0\,. \tag{3.39}$$

The two immediate solutions are as follows:

- $F_{\bar m}{}^{\bar a\bar b}$ constant. This depends on the choice of frame $\{e_a\}_{a=1}^d$ for TM. Choosing a holonomic frame such as the coordinate frame, $F = 0$ for t_{12} symmetric. For t_{12} antisymmetric, $\partial_d\partial_a(t_{12})^{bc}$ has to vanish. Thus the components are restricted to be at most linear in the coordinates.

- $\iota_{t^\sharp_{12}e^{\bar m}}H = 0$. This is equivalent to $\iota_{t^\sharp_{12}e^{\bar m}}B + t^\sharp_{22}e^{\bar m}$ being closed. Since this requirement is not met in the simplest examples of duality (see [27] or section 3.2.5), this option will be discarded.

For a constant F, the consistency condition (3.38) reduces up to exact terms to

$$L_{t^\sharp_{12}e^{\bar a}}\big(t^\sharp_{22}e^{\bar b}\big) - L_{t^\sharp_{12}e^{\bar b}}\big(t^\sharp_{22}e^{\bar a}\big) = F_{\bar m}{}^{\bar a\bar b}\, t^\sharp_{22}e^{\bar m}\,. \tag{3.40}$$

To summarize, the algebra of the generators of isometries $t^\sharp_{12}e^{\bar a}$ closes and satisfies the Jacobi identity if the defect Θ (3.32) vanishes. This is required by consistency of the integrability condition $L_{t^\sharp_{12}e^{\bar a}}G = 0$. Consistency of the condition $L_{t^\sharp_{12}e^{\bar a}}B = -d(t^\sharp_{22}e^{\bar a})$ is ensured by having constant structure coefficients $F_{\bar a}{}^{\bar b\bar c}$ for the isometry algebra with the condition (3.40). These two consistency conditions can be combined coherently into a Courant algebroid.

The Courant algebroid of isometries

Above the consistency conditions on $t^\sharp_{12}e^{\bar a}$ and $t^\sharp_{22}e^{\bar a}$ have been formulated. For the former this was accomplished by the introduction of the Lie algebroid $(T^*M, [\cdot,\cdot]_{\text{iso}}, t^\sharp_{12})$. For the latter the condition (3.40) has to be satisfied. Now both conditions are combined into a Courant algebroid 2.2. The purpose for this is to bridge to the well-known approaches to T-duality via gauging of (multiple) dualities [36].

It is convenient to introduce $\kappa \in \Gamma(T^*M \otimes TM)$ given by $\kappa^{\bar a} = t^\sharp_{22}e^{\bar a} + \iota_{t^\sharp_{12}e^{\bar a}}B \in \Gamma(T^*M)$. Then the dual coordinates (3.24) read

$$d\widetilde{X}^{\bar a} = \big(\iota_{t^\sharp_{12}e^{\bar a}}G\big)_a \star dX^a + \kappa^{\bar a}_a\, dX^a \tag{3.41}$$

and with $H = dB$ the integrability condition (3.25) becomes

$$L_{t^\sharp_{12}e^{\bar a}}G = 0 \qquad \text{and} \qquad \iota_{t^\sharp_{12}e^{\bar a}}H = -d\kappa^{\bar a}\,. \tag{3.42}$$

Evaluating the second condition for the commutator gives

$$\iota_{[t^\sharp_{12}e^{\bar a}, t^\sharp_{12}e^{\bar b}]}H = -d\big(L_{t^\sharp_{12}e^{\bar a}}\kappa^{\bar b}\big) \tag{3.43}$$

As one can see, the one-form $L_{t^\sharp_{12}e^{\bar a}}\kappa^{\bar b}$ corresponds to the vector $[t^\sharp_{12}e^{\bar a}, t^\sharp_{12}e^{\bar b}]$. This suggests to combine $t^\sharp_{12}e^{\bar a}$ and $\kappa^{\bar a}$ to a generalized vector with Dorfman bracket (2.18)

$$[\![t^\sharp_{12}e^{\bar a} + \kappa^{\bar a}, t^\sharp_{12}e^{\bar b} + \kappa^{\bar b}]\!]_D = [t^\sharp_{12}e^{\bar a}, t^\sharp_{12}e^{\bar b}] + L_{t^\sharp_{12}e^{\bar a}}\kappa^{\bar b} - \iota_{t^\sharp_{12}e^{\bar b}}d\kappa^{\bar a} + \iota_{t^\sharp_{12}e^{\bar a}}\iota_{t^\sharp_{12}e^{\bar b}}H\,, \tag{3.44}$$

where the last two terms add-up to zero by the integrability conditions. The bracket (3.44) is the H-twisted Dorfman bracket introduced in [116]. In [117] and more recently in [35], this bracket was studied in the context of isometries. Assuming $\Theta = 0$ and using that the last two terms of (3.44) vanish by integrability, closure of the bracket requires

$$L_{t^\sharp_{12}e^{\bar a}}\kappa^{\bar b} = F_{\bar m}{}^{\bar a\bar b}\,\kappa^{\bar m}\,. \tag{3.45}$$

Using the definition of $\kappa^{\bar{a}}$, this can be seen to agree with the consistency condition (3.40) up to exact terms. Hence the closure of the bracket (3.44) is equivalent to closure of the isometry algebra (3.31) and the consistency condition (3.40). Therefore its consistency summarizes the consistency of the isometry algebra with the integrability conditions by a Courant algebroid $(TM \oplus T^*M, [\![\cdot,\cdot]\!]_D, \mathrm{pr}_{TM})$.

The Courant-algebroid perspective provides the connection to the conventional approach to target space dualities presented in section 3.1.1. Gauging multiple (non-abelian) isometries requires the additional conditions (3.15). Here the first condition arises as the requirement (3.40) of closure of the Dorfman bracket (3.44). The second condition in (3.15) ensures the absence of anomalies caused by the auxiliary gauge field. In the present case, using the generalized vectors $t_{12}^\sharp e^{\bar{a}} + \kappa^{\bar{a}}$, it reads

$$\langle t_{12}^\sharp e^{\bar{a}} + \kappa^{\bar{a}}, t_{12}^\sharp e^{\bar{b}} + \kappa^{\bar{b}} \rangle = 0\,. \tag{3.46}$$

This condition forces the subbundle spanned by $t_{12}^\sharp e^{\bar{a}} + \kappa^{\bar{a}}$ to be a Dirac structure 2.8 for the Courant algebroid defined by (3.44). The Dirac condition (3.46) is non-physical since the problem of anomalies is absent in the present approach, but it provides a mathematical interpretation of the second condition in (3.15). By the duality map (3.27), (3.44) can be considered the algebra of the conserved currents (3.24). Then (3.46) and (3.40) ensure anomaly freedom and closure of the current algebra respectively [117].

As the present approach avoids gauging the isometries, anomaly free currents and thereby the Dirac structure are not needed. In this sense, $O(d,d)$-duality requires less conditions than the conventional procedure. However, the need for isometries and the first condition in (3.15) in the conventional approach is recovered in terms of the integrability conditions (3.25) as well as the closure conditions $\Theta = 0$ and (3.40).

3.2.3 Examples of $O(d,d)$-duality: the prototypes

This section is devoted to the explicit construction of duality for the generators of $O(d,d)$-transformations (2.6) and (2.7). The coordinate frame $\{\frac{\partial}{\partial x^a}\}_{a=1}^d$ is considered for simplicity.

Changes of frame

Given an invertible $d \times d$-matrix A, the $O(d,d)$-matrix

$$\mathcal{T}_A = \begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix} \tag{3.47}$$

is considered. Applied to the generalized metric it gives

$$\mathcal{T}_A^t\,\mathcal{H}(G,B)\,\mathcal{T}_A = \mathcal{H}(\mathsf{A}^t G \mathsf{A}, \mathsf{A}^t B \mathsf{A})\,. \tag{3.48}$$

Therefore $\mathcal{T}_A$ gives rise to a change of frame of the tangent bundle. In respect of $O(d,d)$-duality, the integrability conditions (3.25) are satisfied trivially and the dual coordinates are given by the change of frame $d\widetilde{X}^{\bar{a}} = \mathsf{A}^{\bar{a}}{}_a\, dX^a$. Since the background transforms with the inverse, the dual action coincides with the initial one; $S(\widetilde{X}; g, b) = S(X; G, B)$.

B-transformations

Given an antisymmetric $d \times d$-matrix B corresponding to a two-form, a B-transformation (2.2) is given by the matrix

$$\mathcal{T}_{\mathsf{B}} = \begin{pmatrix} \mathbb{1} & 0 \\ -\mathsf{B} & \mathbb{1} \end{pmatrix} . \tag{3.49}$$

Conjugating the generalized metric with it results in

$$\mathcal{T}_{\mathsf{B}}^t \, \mathcal{H}(G,B) \, \mathcal{T}_{\mathsf{B}} = \mathcal{H}(G, B + \mathsf{B}) . \tag{3.50}$$

It corresponds to a gauge transformation for an exact B, i.e. a symmetry of (3.1). The $O(d,d)$-duality is again trivial with dual coordinate one-form $d\widetilde{X}^a = dX^a$. Therefore the dual action becomes $S(\widetilde{X}; g, b) = S(X; G, B + \mathsf{B})$.

T-duality

The T-duality matrix (2.7) in the k$^{\text{th}}$ direction

$$\mathcal{T}_{T(k)} = \begin{pmatrix} \mathbb{1} - 1_k & 1_k \\ 1_k & \mathbb{1} - 1_k \end{pmatrix} \tag{3.51}$$

is considered [112]. From the field redefinition (3.20) the components of the new metric and two-form can be determined. A tedious calculation leads to

$$\begin{aligned} g_{kk} &= \frac{1}{G_{kk}} , & g_{ka} &= -\frac{B_{ka}}{G_{kk}} , & g_{ab} &= G_{ab} - \frac{G_{ak}G_{kb} + B_{ak}B_{kb}}{G_{kk}} , \\ & & b_{ka} &= -\frac{G_{ka}}{G_{kk}} , & b_{ab} &= B_{ab} - \frac{G_{ak}B_{kb} + B_{ak}G_{kb}}{G_{kk}} \end{aligned} \tag{3.52}$$

for $a, b \neq k$. These are the *Buscher rules* in the k^{th} direction [17] (cf. (3.13)). For the integrability condition (3.25) to be satisfied, the vector field e_k has to be Killing with $L_{e_k} B = 0$. Moreover, vanishing of (3.32) and the Jacobi identity for the Killing algebra (3.31) are trivial for a single T-duality. Then the dual coordinate one-forms are

$$d\widetilde{X}^k = G_{ka} \star dX^a + B_{ka} dX^a \qquad \& \qquad d\widetilde{X}^a = dX^a \quad \text{for } a \neq k . \tag{3.53}$$

Indeed, (3.14) is recovered. Hence $O(d,d)$-duality yields T-duality presented in section 3.1.1 as a special case. In section 3.2.5 it is discussed in more detail.

β-transformations

For an antisymmetric bivector field $\beta \in \Gamma(\Lambda^2 TM)$ corresponding to an antisymmetric $d \times d$-matrix,

$$\mathcal{T}_\beta = \begin{pmatrix} \mathbb{1} & -\beta \\ 0 & \mathbb{1} \end{pmatrix} \tag{3.54}$$

is defined. The transformed background (3.20) induced by this β-transformation (2.12) is given in terms of $\gamma_\beta = \mathbb{1} - (G-B)\beta$ (3.19) as

$$\begin{aligned} g &= \gamma_\beta^{-1}\, G\, \gamma_\beta^{-t} \\ b &= \gamma_\beta^{-1}\left[B - (G-B)\beta(G-B)^t\right]\gamma_\beta^{-t} \,. \end{aligned} \tag{3.55}$$

Moreover, $O(d,d)$-duality is non-trivial: (3.25) requires $\beta^\sharp e^a$ to be a Killing vector with $L_{\beta^\sharp e^a} B = 0$ and consistency of the Killing algebra demands β to be a Poisson bi-vector at most linear in the coordinates. The dual coordinate one-forms (3.24) are

$$d\widetilde{X}^{\bar a} = \beta^{\bar a m} G_{ma} \star dX^a + (\beta^{\bar a m} B_{ma} + \delta^{\bar a}_a) dX^a \,. \tag{3.56}$$

Hence, $O(d,d)$-duality establishes the classical equivalence between the sigma models $S(X; G,B)$ (3.1) and $S(\widetilde{X}; g, b)$ with the coordinates and the backgrounds related by a β-transformation, provided β is a Poisson structure. This duality will be called *Poisson duality* and is applied in section 3.2.5.

The four particular $O(d,d)$-transformations considered here span $O(d,d;C^\infty(M))$ [80]. Thus, by composition non-abelian dualities are covered as well. The question of conformality of $O(d,d)$-dual backgrounds will be addressed in the next section.

3.2.4 Quantum aspects of $O(d,d)$-dual backgrounds

As discussed so far, $O(d,d)$-duality is a classical equivalence of constrained sigma models. For being a duality of string theory, it has to preserve conformality of the backgrounds. Of course, dual sigma models (3.1) arising from $O(d,d)$-duality of a given background admit two-dimensional diffeomorphism and Weyl invariance as well. However, Weyl invariance gets lost in the process of quantization. In order to allow for a larger class of backgrounds admitting conformal invariance at the quantum level, the action for the dilaton ϕ given by

$$S_{\text{dil}}(X;\phi) = \frac{1}{4\pi}\int_\Sigma \phi(X)\, R^{(2)} \star 1 \tag{3.57}$$

is added to the sigma model action (3.1). $R^{(2)}$ denotes the Ricci scalar on $T\Sigma$. This term does not admit Weyl invariance classically, but is considered a quantum correction as it is of higher order in the string length α' as compared to (3.1). Then the classical lack of Weyl invariance can be compensated by a one-loop contribution from $S(X;G,B)$. This gives rise to the lowest order of the equations

$$\begin{aligned} 0 &= R_{ab} + 2\,\nabla_a\nabla_b\phi - \tfrac{1}{4}\, H_{amn}\, H_b{}^{mn} + \mathcal{O}(\alpha')\,, \\ 0 &= G^{ab}\,\nabla_a\phi\,\nabla_b\phi - \tfrac{1}{2} G^{ab}\,\nabla_a\nabla_b\phi - \tfrac{1}{24}\, H_{abc}\, H^{abc} + \mathcal{O}(\alpha')\,, \\ 0 &= \tfrac{1}{2}\,\nabla^m H_{mab} - \nabla^m\phi\, H_{mab} + \mathcal{O}(\alpha')\,, \end{aligned} \tag{3.58}$$

with R_{ab} the Ricci tensor on TM with respect to the Levi-Civita connection $\nabla_a \equiv \nabla_{e_a}$ and $H = dB$. Hence a background (G,B) with dilaton ϕ provides a conformal quantum theory if it satisfies the string equations of motion (3.58) (see [118]).

In order for $O(d,d)$-duality to be a duality of the full quantum theory, the dual background (g,b) has to satisfy the string equations of motion as well. Therefore dualization of a background without dilaton might require its introduction on the dual space. This is discussed for the prototypes of duality.

- The string equations of motion are generally shown to consist of the Riemann tensor, the three-form flux H, the dilaton and covariant derivatives as well as contractions thereof. Thus the equations are invariant under coordinate transformations (3.47) and exact B-transformations (3.49). Therefore they retain conformality trivially.

- For T-duality (2.7), the Buscher rules (3.52) have to be supplemented with a shift of the dilaton by $-\ln G^2_{kk}$ [17, 18]. This is shown by gauging the isometry associated to T-duality in the k^{th} direction and carefully integrating-out the thereby introduced gauge fields. More generally, if T-duality is applied in multiple directions $\{e_i\}_{i=1}^k$ for $k \leq d$, the shift of the dilaton is given by

$$\phi \to \phi - \tfrac{1}{2}\ln\det(G+B)_{ij}\,; \tag{3.59}$$

  see for instance [119]. The subscript indicates the dualized directions.

- β-transformations destroy conformality of the initial background. This can be seen most easily by considering a background with G Ricci-flat and vanishing dilaton and Kalb-Ramond field. This certainly satisfies (3.58). As will be detailed in chapter 4, the map $\gamma_\beta^{-t} : TM \to TM$ (3.19) is a Lie algebroid automorphism. Then, according to theorem 2.17 and (2.57), the Ricci tensor associated to the redefined metric (3.20) vanishes as well. However, (3.20) introduces a non-vanishing Kalb-Ramond field whose presence in (3.58) can only be accounted for by introducing a compensating dilaton. To determine the necessary shift of the dilaton, the following relation between β-transformations and B-transformations via T-duality is utilized:

$$\begin{pmatrix} 0 & \mathbb{1}^* \\ \mathbb{1}_* & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ -\mathsf{B} & \mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1}^* \\ \mathbb{1}_* & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & -\mathbb{1}^*\mathsf{B}\mathbb{1}^* \\ 0 & \mathbb{1} \end{pmatrix}. \tag{3.60}$$

  For simplicity, T-duality in every direction is considered. The unit matrices $\mathbb{1}^*$ and $\mathbb{1}_*$ have to be understood in a formal manner; they act as unit on the component matrices but interchange TM and T^*M (cf. (3.17)). In particular, $\mathbb{1}^*\mathsf{B}\mathbb{1}^*$ is a bivector field $\sum_{a,b} \mathsf{B}_{ab} e_a \wedge e_b$. With the Killing vectors being $\{e_a\}_{a=1}^d$, every direction has to be isometric. This is only necessary if β has full rank. For a β of lower rank, T-duality in the linearly independent directions is sufficient and accordingly fewer isometries are required. The chain (3.60) of $O(d,d)$-transformations will be performed successively.

  1. The first complete T-duality gives the background $\mathbb{1}_*(G+B)^{-1}\mathbb{1}_*$ (cf. (3.20)) and the dilaton has to be shifted by $-\frac{1}{2}\ln\det(G+B)$ (3.59).
  2. The next step in the chain (3.60) is the B-transformation. For this to be a duality, B has to be exact – $\mathsf{B} = d\omega$ with ω a one-form. This gives the background $\mathbb{1}_*(G+B)^{-1}\mathbb{1}_* + d\omega$.

3. The final background arising from the last T-duality can be written as

$$g + b = (G+B)\big[\mathbb{1} - (G-B)\mathbb{1}^* d\omega \mathbb{1}^*\big]^{-t} = \delta_\beta^t\, \gamma_\beta^{-t}\,.$$

By comparison with the fied redefinition (3.20) and (3.19), this reproduces the correct background arising from a β-transformation (2.12) with $\beta = \mathbb{1}^* d\omega \mathbb{1}^*$. Moreover, it induces an additional dilaton shift by $-\frac{1}{2}\ln\det[\mathbb{1}_*(G+B)^{-1}\mathbb{1}_* + d\omega]$.

Hence, the procedure shows that this particular β-transformation gives dual quantum theories if the dilaton

$$\phi = -\tfrac{1}{2}\ln\det(G+B) - \tfrac{1}{2}\ln\det[\mathbb{1}_*(G+B)^{-1}\mathbb{1}_* + d\omega] = -\tfrac{1}{2}\ln\det(\gamma_{\mathbb{1}^* d\omega \mathbb{1}^*})^t$$

is introduced. Since γ_β is positive definite as G is assumed to be Riemannian, the logarithm is well-defined.

It is no coincidence that the shift of the dilaton for β-transformations is given by the transpose of the associated automorphism γ (3.19). Indeed, one can show that

$$\det(G+B)_{ij} = \det\left[\mathbb{1} - \textstyle\sum_{i=1}^k 1_i + (G-B)\sum_{i=1}^k 1_i\right] = \det\gamma_T^t\,;$$

thus the shift of the dilaton for T-duality (3.59) is also given by the logarithm of $\det\gamma_T^t > 0$. For B-transformations (2.2) the automorphism is $\gamma_B = \mathbb{1}$; hence $\ln\det\gamma_B = 0$. This leads to conjecture that for $O(d,d)$-duality to be a duality on the quantum level, the dilaton has to be shifted as

$$\phi \to \phi - \tfrac{1}{2}\ln\det\gamma^t\,. \tag{3.61}$$

The redefinition (3.61) leaves the measure factor $\sqrt{|\det G|}e^{-2\phi}$, which is related to the string coupling constant, invariant; this follows from

$$\sqrt{|\det G|}e^{-2\phi} \to \sqrt{|\det g|}e^{-2\phi+\ln\det\gamma} = \sqrt{|\det G||\det\gamma^{-1}|}(\det\gamma)e^{-2\phi} = \sqrt{|\det G|}e^{-2\phi}$$

by (3.20). This also validates a perturbative approach to the dual theories.

Remark 1. Coordinate changes on the target space have to be considered an exception to (3.61). They imply a change of frame (3.47) with A the pullback of a diffeomorphism. The dilaton is a scalar and therefore does not change under diffeomorphisms. Moreover, there is no need to compensate the determinant $|\det\gamma_A|$ arising in the measure factor $\sqrt{|\det g|}$ as it will be compensated by the Jacobian determinant from the change of coordinates.

A more rigorous way to derive the dilaton shift is to study the change of the path integral measure $[\mathcal{D}X] \to [\mathcal{D}\widetilde{X}]$ by (3.24). In particular, up to the worldsheet operations the duality map (3.27) comprises γ^t, which enters the Jacobian determinant. A more detailed study is beyond the scope of this work.

Exact β-transformations

B-transformations only give rise to equivalent theories if they are exact. An analogous notion for β-transformations based on the analysis above is presented now. The bivectors found above can be considered exact in the Lie algebroid $\mathcal{A} = (T^*M, [\cdot,\cdot]_{\mathcal{A}}, \mathbb{1}^*)$ with bracket

$$[\xi,\eta]_{\mathcal{A}} = \big(\xi_m\, \delta^{mn}\, \partial_n\eta_a - \eta_m\, \delta^{mn}\, \partial_n\xi_a + \xi_m\, \eta_n\, \delta^{mp}\, \delta^{nq}\, f^k{}_{pq}\, \delta_{ka}\big)e^a\,. \tag{3.62}$$

The components of $\mathbb{1}^*$ and its inverse $\mathbb{1}_*$ are written as δ^{ab} and δ_{ab} respectively. Using (2.34), the Lie algebroid induces a nilpotent exterior derivative $d_{\mathcal{A}}$ on $\Gamma(\Lambda^\bullet TM)$. Then it follows from proposition 2.12 that $\mathbb{1}^* d\omega \mathbb{1}^* = d_{\mathcal{A}}(\mathbb{1}^*\omega)$. Therefore, an admissible bivector β is Poisson and of the form

$$\beta = d_{\mathcal{A}}\alpha \quad \text{with} \quad \alpha = \mathbb{1}^*\omega\,,\ \omega \in \Gamma(T^*M)\,, \tag{3.63}$$

and consequently exact with respect to $\mathcal{A}$.

3.2.5 T- and Poisson duality for $\mathbb{T}^3$ with H-flux

Duality is illustrated by applying it to the flat euclidean three-torus $\mathbb{T}^3$ with

$$G = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \qquad \& \qquad B = h\, x^3\, dx^1 \wedge dx^2\,. \tag{3.64}$$

Since the metric is flat, $H = h dx^1 \wedge dx^2 \wedge dx^3$ and the dilaton is zero, this background satisfies the string equations of motion (3.58) only up to linear order in H. However, in particular because of its simplicity and non-trivial global structure it serves as a good instance for discussing the peculiarities of duality. The global structure is given by demanding the background to be periodic in every direction. This corresponds to the periodicities of the three cycles spanning the torus, which are assumed to have circumference 1. To be more precise, (3.64) describes the space in one particular patch. The only non-trivial direction is x^3; encircling this direction via $x^3 \to x^3 + n$ for $n \in \mathbb{N}$ leaves the metric invariant but shifts the Kalb-Ramond field as $B \to B + nh dx^1 \wedge dx^2$. The latter shift can be compensated by the gauge transformation $B \to B - d(nhx^1 dx^2)$. Thus the symmetries of the sigma model (3.1) for the background (3.64) suffice to describe it in every patch.

T-duality

The background admits two duality isometries associated to $\mathcal{T}_{T(1)}$ and $\mathcal{T}_{T(2)}$, namely $\partial/\partial x^1$ and $\partial/\partial x^2$ respectively. Hence T-duality will be performed along this two directions:

- Performing the duality via $\mathcal{T}_{T(1)}$ amounts to apply the Buscher rules (3.52) along the first direction of (3.64). This yields the new background

$$G = \big(dx^1 - h\, x^3 dx^2\big)^2 + (dx^2)^2 + (dx^3)^2 \qquad \& \qquad B = 0\,, \tag{3.65}$$

  which is known as *twisted torus*. The change of the metric from going around x^3 n times can be compensated by a diffeomorphism (3.47) given by

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & 0 \\ nh & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,.$$

Hence the symmetries suffice to describe this background globally as well. The metric can be diagonalized by introducing the new frame

$$e^1 = dx^1 - hx^3 dx^2\,, \quad e^2 = dx^2\,, \quad e^3 = dx^3\,. \tag{3.66}$$

This is a non-holonomic frame whose structure constant $f^a{}_{bc}$ can be determined using (2.34):

$$de^a = -\tfrac{1}{2}\, f^a{}_{bc}\, e^b \wedge e^c \quad \Longrightarrow \quad f^1{}_{23} = -h\,. \tag{3.67}$$

Thus the first T-duality has changed the background with non-vanishing H-flux $H_{123} = h$ to the twisted torus with vanishing H-flux but non-vanishing *geometric flux* $f^1{}_{23} = -h$.

- Now $\mathcal{T}_{T(2)}$ is applied to (3.65). The resulting background reads

$$\begin{aligned} G &= \frac{1}{1+(hx^3)^2}\left[(dx^1)^2 + (dx^2)^2\right] + (dx^3)^2\,, \\ B &= \frac{-hx^3}{1+(hx^3)^2}\, dx^1 \wedge dx^2\,. \end{aligned} \tag{3.68}$$

The change of the background by going n times around x^3 cannot be compensated by a symmetry transformation anymore. It requires a β-transformation (2.12) defined by

$$\beta_t = -nh\, \tfrac{\partial}{\partial x^1} \wedge \tfrac{\partial}{\partial x^2}\,. \tag{3.69}$$

For this reason the background is called *non-geometric*: it is patched-up by transformations beyond the symmetries of the sigma model (3.1). This background is an example of a *T-fold* [50, 26, 84]. The metric is diagonalized by the dreibein

$$e^1 = \frac{1}{\sqrt{1+(hx^3)^2}}\, dx^1\,, \quad e^2 = \frac{1}{\sqrt{1+(hx^3)^2}}\, dx^2\,, \quad e^3 = dx^3\,. \tag{3.70}$$

In this frame the H-flux becomes $H_{123} = -h$ and the non-vanishing structure constants of the frame are $f^1{}_{13} = f^2{}_{23} = \frac{-hx^3}{1+(hx^3)^2}$. In particular the latter are ill-defined quantities. To obtain a well-defined flux, a field redefinition $(g+\beta) = (G+B)^{-1}$ is performed. Then the background is described by a metric on T^*M and a bivector given by (cf. [52])

$$g = \left(\tfrac{\partial}{\partial x^1}\right)^2 + \left(\tfrac{\partial}{\partial x^2}\right)^2 + \left(\tfrac{\partial}{\partial x^3}\right)^2 \qquad \& \qquad \beta = -hx^3\, \tfrac{\partial}{\partial x^1} \wedge \tfrac{\partial}{\partial x^2}\,. \tag{3.71}$$

In this background[9], encircling x^3 is just a constant shift in β. The well-defined flux associated to β has the non-vanishing component $Q^{12}{}_3 = \partial_3 \beta^{12} = -h$. This is called the non-geometric *Q-flux*. This background can be described by the generalized cotangent bundle.

[9] The simplicity of this background is no surprise: the redefinition induced by $(g+\beta) = (G+B)^{-1}$ is a complete T-duality with $t_{12} = \mathbb{1}^*$ and $t_{21} = \mathbb{1}_*$ as introduced in section 3.2.4, which gives the background $(\widetilde{g}+b) = \mathbb{1}_*(G+B)^{-1}\mathbb{1}_*$. Then the field redefinition above is given by multiplying both sides with $\mathbb{1}^*$ and defining $g = \mathbb{1}^*\widetilde{g}\mathbb{1}^*$ and $\beta = \mathbb{1}^* b \mathbb{1}^*$. Moreover, formally applying the (forbidden) T-duality in the third direction does not change (3.64). Thus the redefinition just revokes the two T-dualities and interprets the resulting components on the cotangent bundle.

The background (3.68) has no isometries left to perform a last T-duality along x^3. However, in [27] it was argued that the third T-dual has to exist. They compared flux compactifications of type IIA and type IIB theory and discovered a mismatch between certain coefficients in the superpotentials. This was cured by the formal introduction of the *R-flux* R. This flux was argued to be associated to a non-associative spacetime structure.[10] In particular, in [57] a CFT for the approximate background (3.64) was developed in order to study the R-flux.

Finally, the argument given above completes the chain of T-dualities existent for the background (3.64); it reads

$$H_{abc} \xleftrightarrow{\mathcal{T}_{T(1)}} f^a{}_{bc} \xleftrightarrow{\mathcal{T}_{T(2)}} Q^{ab}{}_c \xleftrightarrow{\mathcal{T}_{T(3)}} R^{abc}\,.$$

Poisson duality

Now the Poisson dual background to (3.64) will be determined. Using $L_{fv}\xi = fL_v\xi + df \wedge \iota_v\xi$ for any vector field v and due to (3.25), it turns out that the only admissible β-transformations (2.12) for duality are given by constant Poisson structures with all components but β^{12} vanishing. Thus the only possibility is

$$\beta = -c\,\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \tag{3.72}$$

with $c \in \mathbb{R}$. This is a trivial Poisson structure with the only non-trivial Poisson bracket reading $\{x^1, x^2\} = -c$. Being constant, it is $\mathcal{A}$-exact as well. The duality isometry is generated by the vectors $\beta^\sharp dx^1 = -c\frac{\partial}{\partial x^2}$ and $\beta^\sharp dx^2 = c\frac{\partial}{\partial x^1}$. These are Killing vectors for the metric G and satisfy $L_{\beta^\sharp e^a} B = 0$. Performing the duality, the dual coordinate one-forms (3.24) are

$$\begin{aligned} d\widetilde{X}^1 &= (1 + chX^3)dX^1 - c\star dX^2\,, \\ d\widetilde{X}^2 &= (1 + chX^3)dX^2 + c\star dX^1\,, \\ d\widetilde{X}^3 &= dX^3\,. \end{aligned} \tag{3.73}$$

The new background is determined by the field redefinition (3.20) and reads

$$\begin{aligned} g &= \frac{1}{c^2 + (1 + c\,h\,x^3)^2}\left[(dx^1)^2 + (dx^2)^2\right] + (dx^3)^2\,, \\ b &= \frac{1}{2}\,\frac{c + 2\,h\,x^3 + c(h\,x^3)^2}{c^2 + (1 + c\,h\,x^3)^2}\,dx^1 \wedge dx^2\,. \end{aligned} \tag{3.74}$$

The procedure of section 3.2.4 can be applied to this case by using a T-duality along x^1 and x^2. Hence for preserving (approximate) conformality the dilaton (3.61)

$$\phi = -\tfrac{1}{2}\ln\left[c^2 + (1 + c\,h\,x^3)^2\right] \tag{3.75}$$

has to be introduced. The following observations are made.

[10] See [56, 120, 121] for target-space approaches and [24, 25, 57, 59] for CFT approaches

- For $c = 1$, this is equivalent to the Q-flux background (3.68) with a translation $x^3 \to x^3 - \frac{1}{h}$, which is not a symmetry.
- In general, the monodromy upon $x^3 \to x^3+1$ for (3.74) is given by the $O(3,3)$-matrix

$$\mathcal{T}_{\text{mono}} = \left(\begin{array}{ccc|ccc} 1-ch & 0 & 0 & 0 & -c^2h & 0 \\ 0 & 1-ch & 0 & c^2h & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -h & 0 & 1+ch & 0 & 0 \\ h & 0 & 0 & 0 & 1+ch & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right), \tag{3.76}$$

  which is a combination of β- and B-transformations and therefore not a symmetry of the theory. This means that $x^3 \to x^3 + 1$ gives the same background as $\mathcal{T}^t_{\text{mono}}\mathcal{H}(g,b)\mathcal{T}_{\text{mono}}$; thus $\mathcal{T}_{\text{mono}}$ is the transition function for (3.74).

As being inequivalent to the Q-flux background (3.68), (3.74) with (3.75) is an example of a new approximate non-geometric background.

Actually, Poisson-duality induced by (3.72) is also admissible for the backgrounds T-dual to (3.64). For the twisted torus (3.65) Poisson duality yields

$$\begin{aligned} g_{tt} &= \frac{1}{1+c^2}\left(dx^1 - h\,x^3dx^2\right)^2 + (dx^2)^2 + (dx^3)^2 \\ b_{tt} &= \frac{c}{1+c^2}\,dx^1 \wedge dx^2\,, \end{aligned} \tag{3.77}$$

which is a twisted torus again with a well-defined Kalb-Ramond field. Here duality acts as gauge transformation by $\xi = \frac{x^1}{1+c^2}dx^2$ accompanied by a rescaling. Starting from the T-fold (3.68) is similar; one obtains

$$\begin{aligned} g_{Tf} &= \frac{1}{1+(c-hx^3)^2}\left[(dx^1)^2 + (dx^2)^2\right] + (dx^3)^2\,, \\ b_{Tf} &= \frac{c-hx^3}{1+(c-hx^3)^2}\,dx^1 \wedge dx^2\,. \end{aligned} \tag{3.78}$$

Again, this is the Q-flux background up to a translation $x^3 \to x^3 - \frac{c}{h}$ – Poisson duality merely acts as translations of x^3. Hence Poisson duality with (3.72) preserves the global structure of the backgrounds T-dual to (3.64).

3.3 Summary and discussion

In this chapter the target space symmetry structure of string theory has been explored. A field redefinition (3.20) of the background accompanied with a transformation of the coordinates (3.24) – both induced by a given $O(d,d)$-transformation – provides dual descriptions of the classical system. For the new coordinates to be consistent, the integrability conditions (3.25) have to be satisfied which guarantee the existence of certain isometries of the background. The analysis of the isometry algebra revealed a Courant algebroid with bracket (3.44) unifying the integrability conditions. The generators of $O(d,d)$-transformations have been analyzed in detail. In particular, well-known T-duality [31] is contained in $O(d,d)$-duality as well as Poisson duality induced by β-transformations.

Since the method is not restricted to constant $O(d,d)$-transformations, non-abelian dualities can be treated as well. The present findings allow for decomposing them into the four generating classes – diffeomorphisms, B-transformations, T-dualities and β-transformations. It would be interesting to study non-abelian duality more detailed in this context. Related to this, the connection to Poisson-Lie T-duality [122, 123] deserves further attention. There, the condition for the existence of isometries present here is relaxed by having currents which are not conserved but obey an extremal surface condition.

Although being an evident classical duality, the quantum aspects of $O(d,d)$-duality are barely studied. In particular, the conjecture for the general shift of the dilaton needs to be verified more thoroughly. Moreover, the discussion lacks a clear criterion for conformality of a dual background. The arguments presented here rely on the symmetries and T-duality. A discussion of global aspects of the procedure from the quantum field theory point of view might be helpful.

A related global question concerns the values of the entries of the $O(d,d)$ transformations; so far they can be non-constant, i.e. function-valued. For example, if periodicities of certain compact directions are required, the entries might be restricted to integers or integer-valued functions. However, up to now it is not clear if $O(d,d)$-duality allows for "decompactifications" of dualized directions in a consistent way. In the conventional approach to dualities, the preservation of compactness and periodic directions during dualization seems to stem from the constraint of holonomies of the auxiliary gauge field by the Lagrange multiplier, which serves as the dual coordinate [119]. Since this issue is absent in the method presented above, the question has to be approached differently.

Due to the problem of invertibility of the primitive of (3.24) discussed in section 3.2.1, it is not clear yet whether $O(d,d)$-duality goes beyond the well-known $O(d,d;\mathbb{Z})$-duality for toroidal backgrounds. However, it avoids the procedure of gauging isometries with the associated problem of possible non-trivial holonomies and provides a direct relation between the dual coordinates via (3.24) (cf. (3.29)). Moreover, all the conditions known from the conventional approach of gauging isometries are recovered and interpreted in a geometric fashion in terms of Lie and Courant algebroids. Furthermore, the approach of $O(d,d)$-duality has lead to the construction of a new (approximate) non-geometric background. Thus it seems to provide a fertile (at least) alternative approach to target-space dualities.

As an application, T- and Poisson duality for the flat rectangular three-torus with constant H-flux was presented. This revealed the existence of exotic string backgrounds eluding a conventional geometric description. These non-geometric backgrounds are characterized by transition functions beyond the symmetries of the theory. They will be studied within the low energy effective description of (3.1) in the next chapter. However, the proper description of non-geometric backgrounds requires an extension of string theory which includes dualities as manifest symmetries. A modest step in this direction is taken in chapter 5.

Chapter 4

Geometric aspects of non-geometric backgrounds

In the previous chapter the possibility for non-geometric string backgrounds has been shown by the application of duality. As was described in the introduction, the T-dual fluxes discovered in the chain (3.2.5) on the one hand allow for new string theory backgrounds as they introduce additional possibilities for stabilizing moduli. On the other hand, they describe the "swampland" of gauged supergravity to some extend. In any case they cannot be neglected in a holistic understanding of string theory.

In this section some properties of these fluxes and the associated backgrounds are studied. First, the algebraic structure of the fluxes is described within a Courant algebroid in a unified manner. The resulting Bianchi identities constitute important constraints among the simultaneous presence of different fluxes, which is important for non-geometric phenomenology. Since this structure describes all the fluxes at once it is a natural candidate for formulating a gravity theory for all fluxes simultaneously. However, as has been shown in section 2.2.2, this is not possible in general and requires the restriction to Lie algebroids.

Second, the low energy effective theory for the string sigma model (3.1) is formulated in terms of Lie algebroids. This circumvents the technical problems of an approach by means of Courant algebroids. Beside its autonomous interestingness, the connection between gravity theories on different Lie algebroids is shown, which enables a patch-wise description of non-geometric string backgrounds. The approach straight-forwardly generalizes to supergravities as well as to quantum corrections.

4.1 The Courant algebroid for dual fluxes

The H-flux emerges as higher abelian field strength to the Kalb-Ramond field B – $H = dB$ – and the geometric flux f is the structure constant of a non-holonomic frame on the tangent bundle. Thus the geometric part of the duality chain (3.2.5) is well-understood. However, the nature of the non-geometric Q- and R-flux is not clear;

$$\underbrace{H_{abc} \xleftrightarrow{\mathcal{T}_{T(1)}} f^a{}_{bc}}_{\text{geometric}} \xleftrightarrow{\mathcal{T}_{T(2)}} \underbrace{Q^{ab}{}_c \xleftrightarrow{\mathcal{T}_{T(3)}} R^{abc}}_{\text{non-geometric}} .$$

In [27] an ad-hoc gauge algebra is given which reproduces the Bianchi identities for constant fluxes H, f, Q and R previously found by applying T-duality to the immediate identity $dH = 0$ in a non-holonomic frame. In [52, 124] the gauge algebra is related to the Courant bracket by computing the algebra of conserved charges and explicit expressions for Q and R are provided.

To generalize these results, in [64] the non-geometric fluxes are described by a quasi-Lie algebroid on the cotangent bundle. This can be seen as the complement to the quasi-Lie algebroid on the tangent bundle describing the geometric fluxes. Analogous to f describing a non-holonomic frame on TM with respect to the Lie bracket, Q describes a non-holonomic frame on T^*M with respect to the Koszul bracket (2.31). The Lie bracket is twisted such that the H-flux is incorporated as the defect of the anchor property. Similarly, the R is incorporated as defect from twisting the Koszul bracket. A unified mathematical description of all the fluxes is found by combining the twisted brackets to a Courant algebroid analogous to proposition 2.14. This Courant bracket is a global version of the one found in [52] and is used to derive the Bianchi identities for general non-constant fluxes, generalizing the work [27].

4.1.1 The proto-Lie bialgebroid for dual fluxes

The utilization of a twisted Koszul bracket for describing Q and R necessitates the introduction of an alternating bivector β in addition to the background data (G, B). Does β has to be considered an additional datum to describe space-time? For introducing the Q-flux in section 3.2.5, a T-duality has been employed to convert the background to a background on the cotangent bundle via $(g + \beta) = (G + B)^{-1}$ (cf. footnote 9 and [125] for an approach via quasi-Poisson sigma models). Hence β for the Q-flux is related to the background (G, B) via

$$\beta = -(G + B)^{-1}\, B\, (G + B)^{-t} \tag{4.1}$$

and does not comprise additional information. In the following, two quasi-Lie algebroids for describing the geometric and non-geometric part of the chain (3.2.5) are formulated. To this end, the background is assumed to be (G, B) with the bivector β given by (4.1). The appropriate brackets are determined as follows.

- The geometric flux arises as structure coefficient of a non-holonomic frame $\{e_a\}$ for TM, i.e

  $$[e_a, e_b] = f^m{}_{ab}\, e_m \qquad \Longleftrightarrow \qquad de^a = -\tfrac{1}{2}\, f^a{}_{bc}\, e^b \wedge e^c\,,$$

  where the equivalence follows from (2.34). For this to be a Lie bracket, the Jacobi identity has to be satisfied; this gives

  $$\partial_{[c|} f^p{}_{|ab]} = f^m{}_{[ab|} f^p{}_{m|c]} \tag{4.2}$$

  which is always assumed in the following. Therefore the Lie bracket is appropriate for describing the f-flux as structure coefficient.

- In section 3.2.5 the Q-flux associated to two T-dualities of (3.64) was introduced by a partial derivative of (4.1). This can be reproduced by the structure coefficient of the Koszul bracket (2.31) with respect to (4.1). In a holonomic frame the structure coefficients read

$$[e^a, e^b]_K = L_{\beta^{am} e_m} e^b = d\beta^{ab} = \partial_c \beta^{ab}\, e^c$$

since $de^a = 0$. This legitimizes the Koszul bracket for describing the Q-flux. However, β is not assumed to be a Poisson structure, i.e. $(T^*M, [\cdot,\cdot]_K, \beta^\sharp; \Theta)$ is a quasi-Lie algebroid with the defect given by Θ (2.32).

Thus the Lie and Koszul bracket describe the f- and Q-flux as structure coefficients of non-holonomic bases respectively. As invoked above, the H- and R-flux will be introduced as a defect to the Lie algebroid properties by twisting:

- The most natural twist of the Lie bracket by $H \in \Gamma(\Lambda^3 T^*M)$ gives rise to the H-*twisted Lie bracket*

$$[X, Y]_H = [X, Y] - \beta^\sharp \left(\iota_Y \iota_X H\right) . \tag{4.3}$$

This is the quasi-Lie algebroid $\mathcal{L}_H = (TM, [\cdot,\cdot]_L, \mathrm{id}_{TM}; H)$ with the defect to the anchor property (2.42) given by

$$\Delta_{\mathcal{L}_H}(X, Y) = [X, Y]_H - [X, Y] = \beta^\sharp \left(\iota_X \iota_Y H\right) \tag{4.4}$$

since the anchor is the identity. In the non-holonomic frame introduced above the bracket evaluates to

$$[e_a, e_b]_H = \left(f^c{}_{ab} - H_{abm}\, \beta^{mc}\right) e_c \equiv \mathcal{F}^c{}_{ab}\, e_c\,. \tag{4.5}$$

- Similarly the Koszul bracket can be twisted by H, giving the H*-twisted Koszul bracket*

$$[\xi, \eta]_K^H = [\xi, \eta]_K + \iota_{\beta^\sharp \eta} \iota_{\beta^\sharp \xi} H\,. \tag{4.6}$$

This is the quasi-Lie algebroid $\mathcal{K}_H = (T^*M, [\cdot,\cdot]_K^H, \beta^\sharp; \mathcal{R})$. Since β is not assumed to be a Poisson structure, the defect to the anchor property of the twisted bracket adds to the defect (2.42) of the Koszul bracket:

$$\Delta_{\mathcal{K}_H}(\xi, \eta) = \beta^\sharp [\xi, \eta]_K^H - [\beta^\sharp \xi, \beta^\sharp \eta] = \iota_\xi \iota_\eta \left[\Theta + \otimes^3 \beta^\sharp (H)\right] \equiv \iota_\xi \iota_\eta \mathcal{R}\,. \tag{4.7}$$

Θ is given in (2.32) and locally $[\otimes^3 \beta^\sharp(H)]^{abc} = \beta^{am} \beta^{bn} \beta^{ck} H_{mnk}$. In the non-holonomic frame the structure coefficients become

$$[e^a, e^b]_K^H = \left(\partial_c \beta^{ab} + 2 f^{[a|}{}_{cm} \beta^{m|b]} + \beta^{am} \beta^{bn} H_{mnc}\right) e^c \equiv \mathcal{Q}^{ab}{}_c\, e^c\,. \tag{4.8}$$

Therefore the algebra for the geometric sector is determined by the structure coefficients $\mathcal{F}$ with a defect H and the algebra for the non-geometric sector is determined by the structure coefficients $\mathcal{Q}$ with a defect $\mathcal{R}$. In particular, the precise form of the R-flux is a consequence of the mathematical structures and analogy to the geometric sector. Indeed, both structures combine to a proto-Lie bialgebroid [113], which is a generalization of a Lie bialgebroid 2.13. Apart from diverse compatibility conditions and the duality of the underlying vector bundles, a defining feature is the reciprocal relation between the defects and the anchors: $\Delta_{\mathcal{L}_H}(X, Y) = \beta^\sharp(\iota_X \iota_Y H)$ and $\Delta_{\mathcal{K}_H}(\xi, \eta) = \mathrm{id}_{TM}(\iota_\xi \iota_\eta \mathcal{R})$.

4.1.2 The Courant algebroid and Bianchi identities

Analogous to proposition 2.14, the proto-Lie bialgebroid $(\mathcal{L}_H, \mathcal{K}_H)$ can be combined to a Courant algebroid.

Proposition 4.1 ([64]). *The proto-Lie bialgebroid $(\mathcal{L}_H, \mathcal{K}_H)$ gives rise to the Courant algebroid $\mathcal{C}_{\text{dual}} = (TM \oplus T^*M, \langle\cdot,\cdot\rangle_+, [\![\cdot,\cdot]\!]_{\text{dual}}, \alpha)$ with*

$$\begin{aligned}\alpha(X+\xi) &= X + \beta^\sharp\xi \\ \langle X+\xi, Y+\eta\rangle_\pm &= \iota_X\eta \pm \iota_Y\xi\end{aligned} \tag{4.9}$$

and with the bracket given in terms of the associated Lie derivatives (2.39) as

$$\begin{aligned}[\![X+\xi, Y+\eta]\!]_{\text{dual}} = [X,Y]_H &+ L_X^{\mathcal{L}_H}\eta - L_Y^{\mathcal{L}_H}\xi - \tfrac{1}{2}\, d_{\mathcal{L}_H}\langle X+\xi, Y+\eta\rangle_- + \iota_Y\iota_X H \\ [\xi,\eta]_K^H &+ L_\xi^{\mathcal{K}_H} Y - L_\eta^{\mathcal{K}_H} X + \tfrac{1}{2}\, d_{\mathcal{K}_H}\langle X+\xi, Y+\eta\rangle_- + \iota_\eta\iota_\xi\mathcal{R}\,.\end{aligned} \tag{4.10}$$

Although not nilpotent, the derivatives $d_{\mathcal{L}_H}$ and $d_{\mathcal{K}_H}$ are defined by (2.34) with respect to the associated brackets (4.3) and (4.6) respectively.

Before proving the proposition, the algebra defined by the bracket (4.10) is considered. For a non-holonomic frame $\{e_a\}$ for TM and its dual $\{e^a\}$, the bracket becomes

$$\begin{aligned}[\![e_a, e_b]\!]_{\text{dual}} &= \mathcal{F}^c{}_{ab}\, e_c + H_{abc}\, e^c \\ [\![e_a, e^b]\!]_{\text{dual}} &= \mathcal{Q}^{bc}{}_a\, e_c - \mathcal{F}^b{}_{ac}\, e^c \\ [\![e^a, e^b]\!]_{\text{dual}} &= \mathcal{R}^{abc}\, e_c + \mathcal{Q}^{ab}{}_c\, e^c\,.\end{aligned} \tag{4.11}$$

In particular, the second bracket is determined by the definition of the Lie derivative (2.39) and the relation between the exterior derivative and the Lie algebroid bracket (2.34). The algebra defined by (4.11) is a generalization of the ad-hoc gauge algebra given in [27] to non-constant fluxes H, $\mathcal{F}$, $\mathcal{Q}$ and $\mathcal{R}$.

Proof of proposition 4.1. The anchor property is a lengthy but straight-forward evaluation. The only difference to the case of a Lie bialgebroid 2.14 is the defect Δ for the quasi-Lie algebroids which is taken into account by the explicit appearance of H and $\mathcal{R}$ in the definition of the bracket (4.10). The Leibniz rule can be evaluated directly: the last property of (2.40) and the Leibniz rules for $\mathcal{L}_H$ and $\mathcal{K}_H$ give rise to

$$\begin{aligned}[\![X+\xi, f(Y+\eta)]\!]_{\text{dual}} &= f[\![X+\xi, Y+\eta]\!]_{\text{dual}} + \big[(X+\beta^\sharp\xi)f\big](Y+\eta) \\ &\quad - \tfrac{1}{2}\langle X+\xi, Y+\eta\rangle_-(d_{\mathcal{L}_H} - d_{\mathcal{K}_H})f - \iota_Y\xi\, d_{\mathcal{L}_H}f - \iota_X\eta\, d_{\mathcal{K}_H}f \\ &= f[\![X+\xi, Y+\eta]\!]_{\text{dual}} + \big[(X+\beta^\sharp\xi)f\big](Y+\eta) \\ &\quad - \tfrac{1}{2}\langle X+\xi, Y+\eta\rangle_+(d_{\mathcal{L}_H} + d_{\mathcal{K}_H})f\,.\end{aligned}$$

Since $(d_{\mathcal{L}_H} + d_{\mathcal{K}_H})f(X+\xi) = (X+\beta^\sharp\xi)f = \alpha(X+\xi)f$, $\mathcal{D} = d_{\mathcal{L}_H} + d_{\mathcal{K}_H}$ on functions and therefore the above calculation reproduces the correct Leibniz rule. Since the derivative $\mathcal{D}$ is identified now, $\alpha\circ\mathcal{D} = 0$ follows from

$$\alpha(\mathcal{D}f)(\xi) = \big(d_{\mathcal{K}_H}f + \beta^\sharp d_{\mathcal{L}_H}f\big)(\xi) = (\beta^\sharp\xi)f + \beta^{ab}\,\partial_a f\,\xi_b = (\beta^\sharp\xi)f - (\beta^\sharp\xi)f = 0\,.$$

Proving the fifth property in 2.2 is again long and straight-forward. It remains to show the Jacobi identity. For a particular frame the Jacobiator becomes

$$\begin{aligned}
\mathrm{Jac}_{\mathcal{C}_{\mathrm{dual}}}(e_a, e_b, e_c) =& -3\Big(\partial_{[c|}\,\mathcal{F}^d{}_{|ab]} + \mathcal{F}^m{}_{[ab|}\,\mathcal{F}^d{}_{|c]m} + H_{[ab|m}\,\mathcal{Q}^{md}{}_{|c]}\Big)e_d \\
& -3\Big(\partial_{[c|}\,H_{|ab]d} - 2\mathcal{F}^m{}_{[ab|}\,H_{|cd]m}\Big)e^d + \tfrac{3}{2}\,\mathcal{D}H_{abc}\,, \\
\mathrm{Jac}_{\mathcal{C}_{\mathrm{dual}}}(e_a, e_b, e^c) =& -\Big(\beta^{cm}\partial_m\mathcal{F}^d{}_{ab} + 2\partial_{[a|}\,\mathcal{Q}^{cd}{}_{|b]} - H_{mab}\mathcal{R}^{mcd} - \mathcal{F}^m{}_{ab}\mathcal{Q}^{cd}{}_m \\
& + 4\mathcal{Q}^{[c|m}{}_{[a|}\mathcal{F}^{|d]}{}_{m|b]}\Big)e_d - \Big(\beta^{cm}\partial_m H_{abd} - 2\partial_{[a|}\mathcal{F}^c{}_{|b]d} \\
& - 3H_{m[ab|}\mathcal{Q}^{mc}{}_{|d]} + 3\mathcal{F}^m{}_{[ab|}\mathcal{F}^c{}_{m|d]}\Big)e^d + \tfrac{3}{2}\,\mathcal{D}\mathcal{F}^c{}_{ab}\,, \\
\mathrm{Jac}_{\mathcal{C}_{\mathrm{dual}}}(e_a, e^b, e^c) =& +\Big(-\partial_a\mathcal{R}^{bcd} - 2\beta^{[c|m}\partial_m\mathcal{Q}^{|b]d}_a + 3\mathcal{Q}^{[b|m}{}_a\mathcal{Q}^{|cd]}{}_m \\
& - 3\mathcal{F}^{[b|}{}_{am}\mathcal{R}^{|cd]m}\Big)e_d + \Big(2\beta^{[c|m}\partial_m\mathcal{F}^{|b]}{}_{ad} - \partial_a\mathcal{Q}^{bc}{}_d + \mathcal{Q}^{bc}{}_m\mathcal{F}^m{}_{ad} \\
& + \mathcal{R}^{bcm}H_{mad} - 4\mathcal{Q}^{[b|m}{}_{[a|}\mathcal{F}^{|c]}_{m|d]}\Big)e^d + \tfrac{3}{2}\,\mathcal{D}\mathcal{Q}^{bc}{}_a\,, \\
\mathrm{Jac}_{\mathcal{C}_{\mathrm{dual}}}(e^a, e^b, e^c) =& -3\Big(\beta^{[c|m}\,\partial_m\mathcal{R}^{|ab]d} - 2\mathcal{R}^{[ab|m}\,\mathcal{Q}^{|cd]}{}_m\Big)e_d - 3\Big(\beta^{[c|m}\,\partial_m\mathcal{Q}^{|ab]}{}_d \\
& + \mathcal{R}^{[ab|m}\,\mathcal{F}^{|c]}{}_{md} + \mathcal{Q}^{[ab|}{}_m\,\mathcal{Q}^{|c]m}{}_d\Big)e^d + \tfrac{3}{2}\,\mathcal{D}\mathcal{R}^{abc}\,.
\end{aligned}$$

To simplify the Jacobiators further, the anchor can be applied. Since

$$\alpha\left([\![[\![A,B]\!]_{\mathrm{dual}},c]\!]_{\mathrm{dual}}\right) = \left[\alpha\left([\![A,B]\!]_{\mathrm{dual}}\right),\alpha(C)\right] = \left[[\alpha(A),\alpha(B)],\alpha(C)\right]\,,$$

the Jacobiators above become Jacobiators of the Lie bracket, i.e. trivial identites. Using these identities together with

$$dH = 0 \qquad \Longleftrightarrow \qquad \partial_{[a|}H_{|bcd]} - \tfrac{3}{2}\,\mathcal{F}^m{}_{[ab|}H_{m|cd]} = 0\,,$$

the Jacobiators can be written in terms of the Nijenhuis tensor (2.17) as

$$\begin{aligned}
\mathrm{Jac}_{\mathcal{C}_{\mathrm{dual}}}(e_a, e_b, e_c) &= \tfrac{1}{2}\,\mathcal{D}H_{abc} = \mathcal{D}T(e_a, e_b, e_c)\,, \\
\mathrm{Jac}_{\mathcal{C}_{\mathrm{dual}}}(e_a, e_b, e^c) &= \tfrac{1}{2}\,\mathcal{D}\mathcal{F}^c{}_{ab} = \mathcal{D}T(e_a, e_b, e^c)\,, \\
\mathrm{Jac}_{\mathcal{C}_{\mathrm{dual}}}(e^a, e^b, e^c) &= \tfrac{1}{2}\,\mathcal{D}\mathcal{Q}^{bc}{}_a = \mathcal{D}T(e_a, e^b, e^c)\,, \\
\mathrm{Jac}_{\mathcal{C}_{\mathrm{dual}}}(e^a, e^b, e^c) &= \tfrac{3}{2}\,\mathcal{D}\mathcal{R}^{abc} = \mathcal{D}T(e^a, e^b, e^c)\,.
\end{aligned}$$

Hence the Jacobi identity of a Courant algebroid is satisfied. This completes the proof as all properties in definition 2.2 are verified. □

Bianchi identities

Bianchi identities are trivial identities which encode restrictions among the involved quantities. The trivial identities for the four fluxes can be obtained from the Jacobi identity of the Courant bracket above by anchoring, as was done in the proof. Since $\alpha \circ \mathcal{D} = 0$ they read

$$\begin{aligned} \alpha\left(\mathrm{Jac}_{\mathcal{C}_{\mathrm{dual}}}(e_a,e_b,e_c)\right) &= \mathrm{Jac}_{\mathrm{Lie}}(e_a,e_b,e_c) &&= 0\,,\\ \alpha\left(\mathrm{Jac}_{\mathcal{C}_{\mathrm{dual}}}(e_a,e_b,e^c)\right) &= \mathrm{Jac}_{\mathrm{Lie}}(e_a,e_b,\beta^\sharp e^c) &&= 0\,,\\ \alpha\left(\mathrm{Jac}_{\mathcal{C}_{\mathrm{dual}}}(e_a,e^b,e^c)\right) &= \mathrm{Jac}_{\mathrm{Lie}}(e_a,\beta^\sharp e^b,\beta^\sharp e^c) &&= 0\,,\\ \alpha\left(\mathrm{Jac}_{\mathcal{C}_{\mathrm{dual}}}(e^a,e^b,e^c)\right) &= \mathrm{Jac}_{\mathrm{Lie}}(\beta^\sharp e^a,\beta^\sharp e^b,\beta^\sharp e^c) &&= 0\,, \end{aligned} \tag{4.12}$$

together with

$$dH = 0 \qquad \Longleftrightarrow \qquad \partial_{[a|}H_{|bcd]} - \tfrac{3}{2}\,\mathcal{F}^m{}_{[ab|}H_{m|cd]} = 0\,. \tag{4.13}$$

The precise form of the identities (4.12) can be read off from the Jacobiators given in the proof of proposition 4.1. In particular, for constant fluxes the identities (4.12) coincide with those previously derived in [27, 62, 63]. The identities (4.12) are generalized to double field theory in [126, 97, 98, 66], where in particular the R-flux is accessible through conventional dualities.

The Bianchi identities restrict the possibilities for the concurrent appearance of the dual fluxes.

4.2 The structure of non-geometric patches

The dynamics of a string background is described by the equations of motion (3.58), which retain conformal invariance of the quantized theory. They can also be formulated by varying the action

$$\mathcal{S} = -\frac{1}{2\kappa^2}\int_M d^dx\,\sqrt{\det G}\,e^{-2\phi}\left(R - \tfrac{1}{12}\,H_{abc}\,H^{abc} + 4\,\partial_a\phi\,\partial^a\phi\right) + \mathcal{O}(\alpha')\,. \tag{4.14}$$

with respect to G, B and ϕ. κ denotes a normalization constant and the metric is assumed to be positive definite. This action is the low energy effective action for the massless fields G, B and ϕ in the spectrum of string theory and can therefore also be derived from scattering of these states. Moreover, the effective action for any $O(d,d)$-dual background (g,b) (3.20) with appropriate dilaton (3.61) is of the form (4.14) because duality preserves the form of the sigma model (3.1). The action is invariant under spacetime diffeomorphisms and Kalb-Ramond field gauge transformations

$$B \to B + d\xi \qquad \text{for} \quad \xi \in \Gamma(T^*M)\,, \tag{4.15}$$

i.e. under the geometric group $G_{d\xi} \rtimes \mathrm{diff}(M)$. In section 3.2.5 string backgrounds have been discovered whose global description requires patching with transformations beyond the symmetries of the underlying theory. These changes of the transition functions are a general feature of $O(d,d)$-duality: Let $\mathcal{M}_{(ij)} \in O(d,d)$ be the transition function for a background (G,B) from a patch $U_i \subset M$ to the patch $U_j \subset M$; it is an element of $O(d,d)$ since it arises from the structure group of the generalized (co)tangent bundle. Denoting the generalized metric (2.10) in a local patch by $\mathcal{H}_{(i)}(G,B)$, the transition function acts as

$$\mathcal{H}_{(j)}(G,B) = \mathcal{M}^t_{(ij)}\,\mathcal{H}_{(i)}(G,B)\,\mathcal{M}_{(ij)}\,. \tag{4.16}$$

Performing the $O(d,d)$-duality induced by $\mathcal{T} \in O(d,d)$ amounts to a conjugation of the generalized metric (cf. (3.18)), giving the field redefinition (3.20). Hence, if the new background in the patch U_i is encoded in $\mathcal{T}^t \mathcal{H}_{(i)} \mathcal{T}$, the new background in the patch U_j becomes

$$\begin{aligned} \mathcal{T}^t\, \mathcal{H}_{(j)}(G,B)\, \mathcal{T} &= \mathcal{T}^t\, \mathcal{M}^t_{(ij)}\, \mathcal{H}_{(i)}(G,B)\, \mathcal{M}_{(ij)}\, \mathcal{T} \\ &= \left(\mathcal{T}^{-1}\mathcal{M}_{ij}\mathcal{T}\right)^t \left(\mathcal{T}^t\, \mathcal{H}_{(i)}(G,B)\, \mathcal{T}\right) \left(\mathcal{T}^{-1}\mathcal{M}_{ij}\mathcal{T}\right) . \end{aligned} \tag{4.17}$$

Therefore the new transition function for $\mathcal{T}$-dual backgrounds reads

$$\mathcal{M}'_{(ij)} = \mathcal{T}^{-1}\mathcal{M}_{ij}\mathcal{T} \in O(d,d)\,. \tag{4.18}$$

In particular, it is possible that $\mathcal{M}'_{(ij)} \notin G_{d\xi} \rtimes \mathrm{GL}(d)$.[1] This happened for example for the Q-flux background (3.68), where two T-dualities have changed a B-transformation to a β-transformation. In summary, (4.18) attest $O(d,d)$-duality the general possibility for producing non-geometric backgrounds.

If the transition function is not an element of $G_{d\xi} \rtimes \mathrm{GL}(d)$, the action (4.14) can only be understood locally. The purpose of this section is the description of the theory in every patch by providing the patch-wise interpretation discovered in [80]: On every patch the theory is described by an action of the form (4.14). The actions on different patches differ if the transition functions are non-geometric, i.e. if they are T-dualities, β-transformations or more general transformations outside $G_{d\xi} \rtimes \mathrm{GL}(d)$. To achieve this, Lie algebroids are constructed whose differential geometries (cf. section 2.3.3) are appropriate for formulating these actions. Then they are constructed explicitly with a particular emphasis on the gauge transformations. Two examples of the recent literature are presented as well as the extension of the procedure to all higher-order corrections and supergravities.

4.2.1 The Lie algebroid for $O(d,d)$ transitions

The field redefinition (3.20) describing the change of the metric and the Kalb Ramond field under duality is given by conjugation with the automorphism (3.19)

$$\gamma = t_{22} + (G-B)t_{12} : T^*M \to T^*M$$

for an $O(d,d)$-matrix $\mathcal{T}$ with the four $d \times d$-submatrices t_{ij}; $i,j \in \{1,2\}$. More precisely, defining[2] $\mathfrak{b} = \gamma^{-1}\delta^{-t} - g$, the redefined background (g,b) is related to the old background (G,B) via

$$g = (\otimes^2\gamma^{-1})(G) \qquad \text{and} \qquad \mathfrak{b} = (\otimes^2\gamma^{-1})(B)\,. \tag{4.19}$$

As compared to (3.20), $\gamma^{-1} : T^*M \to T^*M$ is considered a Lie algebroid homomorphism and acts as described in section 2.3.1:

$$(\otimes^2\gamma^{-1})(G)(X,Y) = G\big(\gamma^{-t}(X), \gamma^{-t}(Y)\big)$$

[1] From the perspective of the structure group of the tangent bundle, $\mathrm{diff}(M) \subset \mathrm{GL}(d)$. Of course, as opposed to mere changes of frame for TM, diffeomorphisms also change the coordinates of M.

[2] The inverse of (3.19) is $\gamma^{-1} = t^t_{11} + (g-b)t^t_{12}$ and $\delta^{-1} = t^t_{21} + (g-b)t^t_{22}$, which follows from (3.20).

and analogous for $\mathfrak{b}$. Hence (4.19) is equivalent to (3.20), but more convenient for the following purposes. In particular, g is the metric on a Lie algebroid, which is related to G as in theorem 2.17 and $\mathfrak{b}$ turns out to be the right object for formulating the flux in the different patches. To employ the full strength of this theorem, this Lie algebroid has to be specified. There are two possibilities.

- $\mathfrak{t} = (TM, [\cdot,\cdot]_{\mathfrak{t}}, \rho = \gamma^{-t})$: (4.19) suggests to interpret $\gamma^{-t} : TM \to TM$ as anchor of a Lie algebroid on the tangent bundle. As was done in section 3.2.2 the bracket can be determined by computing the Lie bracket of anchored vector fields. With the abbreviation[3] $D_{\bar{a}} = \gamma^{-t}(e_{\bar{a}}) = \gamma^m{}_{\bar{a}}\partial_m$ one obtains

$$[\gamma^{-t}(X), \gamma^{-t}(Y)] = \big(X^{\bar{m}}\, D_{\bar{m}} Y^{\bar{a}} - Y^{\bar{m}}\, D_{\bar{m}} X^{\bar{a}} + X^{\bar{m}}\, Y^{\bar{n}}\, F^{\bar{a}}{}_{\bar{m}\bar{n}}\big)\gamma^{-t}(e_{\bar{a}})\,.$$

With the structure coefficient for the frame given by $[e_a, e_b] = f^c{}_{ab}e_c$, F reads

$$F^{\bar{a}}{}_{\bar{b}\bar{c}} = \gamma^{\bar{a}}{}_m\big(D_{\bar{b}}\gamma^m{}_{\bar{c}} - D_{\bar{c}}\gamma^m{}_{\bar{b}} + \gamma^p{}_{\bar{b}}\,\gamma^q{}_{\bar{c}}\, f^m{}_{pq}\big)\,. \tag{4.20}$$

As the anchor property equates this with $\gamma^{-t}([X,Y]_{\mathfrak{t}})$, the Lie algebroid bracket can be read-off:

$$[X, Y]_{\mathfrak{t}} = \big(X^{\bar{m}} D_{\bar{m}} Y^{\bar{a}} - Y^{\bar{m}} D_{\bar{m}} X^{\bar{a}} + X^{\bar{m}}\, Y^{\bar{n}}\, F^{\bar{a}}{}_{\bar{m}\bar{n}}\big)e_{\bar{a}}\,. \tag{4.21}$$

The requirements of definition 2.9 are satisfied by construction.

- $\mathfrak{t}^* = (T^*M, [\cdot,\cdot]_{\mathfrak{t}^*}, \widetilde{\rho} = G^{-1}\circ\gamma)$: The field redefinition (3.20) for the metric can be rewritten as

$$g = \gamma^{-1}\, G\, \gamma^{-t} = \big(G^{-1}\gamma\big)^{-1}\, G^{-1}\, \big(G^{-1}\gamma\big)^{-t}\,. \tag{4.22}$$

Inverting this relation is interpreted as giving a metric $\widetilde{g}$ on the cotangent bundle via

$$\widetilde{g} = \big(G^{-1}\gamma\big)^t\, G\, \big(G^{-1}\gamma\big) \equiv \Big[\otimes^2 \big(G^{-1}\circ\gamma\big)^t\Big]\,(G)\,. \tag{4.23}$$

Using the redefinition (3.20) for the Kalb-Ramond field and $\widetilde{g}$, the equivalent to the B-field on the cotangent bundle reads

$$\widetilde{\beta} = \Big[\otimes^2 \big(G^{-1}\circ\gamma\big)^t\Big]\,(B) \qquad\Longleftrightarrow\qquad \widetilde{\beta} = \widetilde{g}\,\mathfrak{b}\,\widetilde{g}\,. \tag{4.24}$$

Thus the field redefinition (3.20) implies an equivalent field redefinition to a background $(\widetilde{g}, \widetilde{\beta})$ on the cotangent bundle. Hence, mimicking the construction of $\mathfrak{t}$, the anchor is indeed given by $\widetilde{\rho} = G^{-1}\circ\gamma : T^*M \to TM$ and the bracket reads

$$[\xi, \eta]_{\mathfrak{t}^*} = \big(\xi_{\bar{m}} D^{\bar{m}}\eta_{\bar{a}} - \eta_{\bar{m}} D^{\bar{m}}\xi_{\bar{a}} + \xi_{\bar{m}}\,\eta_{\bar{n}}\, Q^{\bar{m}\bar{n}}{}_{\bar{a}}\big)e^{\bar{a}}\,, \tag{4.25}$$

with $D^{\bar{a}} = \widetilde{\rho}(e^{\bar{a}})$ and the structure coefficient

$$Q^{\bar{a}\bar{b}}{}_{\bar{c}} = \widetilde{\rho}_{\bar{c}m}\big(D^{\bar{a}}\widetilde{\rho}^{m\bar{b}} - D^{\bar{b}}\widetilde{\rho}^{m\bar{a}} + \widetilde{\rho}^{p\bar{a}}\,\widetilde{\rho}^{q\bar{b}}\, f^m{}_{pq}\big)\,. \tag{4.26}$$

[3] The notations of section 3.2.1 are employed. In particular, $\gamma \equiv (\gamma_a{}^{\bar{a}})$, $\gamma^{-1} \equiv (\gamma_{\bar{a}}{}^a)$, $\gamma^t \equiv (\gamma^{\bar{a}}{}_a)$ and $\gamma^{-t} \equiv (\gamma^a{}_{\bar{a}})$.

Therefore the field redefinition (3.20) together with the Riemannian structure of the manifold M gives rise to two Lie algebroids $\mathfrak{t}$ and $\mathfrak{t}^*$ associated to the redefined background (g, b). Indeed, they are equivalent as Lie algebroids.

Proposition 4.2. *There exists a Lie algebroid isomorphism such that* $\mathfrak{t} \cong \mathfrak{t}^*$. *More precisely, the Lie algebroid isomorphism is given by* g *(4.19).*

Proof. The conditions in definition 2.10 have to be checked. The redefined metric is a bundle isomorphism $g : TM \to T^*M$. Using the definition of $\widetilde{\rho}$ and the field redefinition (3.20) or equivalently (4.19) gives

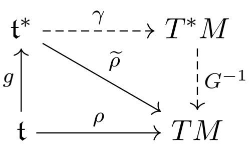
$$\widetilde{\rho} \circ g = \left(G^{-1} \circ \gamma\right) \circ \left(\gamma^{-1} \circ G \circ \gamma^{-t}\right) = \gamma^{-t} = \rho\,.$$

To prove $g([X,Y]_{\mathfrak{t}}) = [g(X), g(Y)]_{\mathfrak{t}^*}$, the relation $\rho = \widetilde{\rho} \circ g$ is employed. The anchor properties can be evaluated as follows:

$$\widetilde{\rho} \circ g\big([X,Y]_{\mathfrak{t}}\big) = \rho\big([X,Y]_{\mathfrak{t}}\big) = [\rho(X), \rho(Y)] = [\widetilde{\rho} \circ g(X), \widetilde{\rho} \circ g(Y)] = \widetilde{\rho}\big([g(X), g(Y)]_{\mathfrak{t}^*}\big)\,.$$

Since $\widetilde{\rho}$ is invertible as composition of the invertible maps γ and G^{-1}, this implies the desired relation between the brackets. Hence g is a Lie algebroid isomorphism. □

The relation between the two Lie algebroids induced by the field redefinition can be summarized in the following commuting diagram:



The Lie algebroid $\mathfrak{t}$ is associated to the background (g, b) in (3.20) via $\mathfrak{b}$. For the interpretation of $\mathfrak{t}^*$ the Lie algebroid isomorphism g is used instead of decomposing $\widetilde{\beta}$ analogous to $\mathfrak{b}$. Noting that $\widetilde{g} = (\otimes^2 g^{-1})g$ since $\widetilde{g} = g^{-1}$, the new Kalb-Ramond field similarly converts to $\widetilde{b} = (\otimes^2 g^{-1})b$. Hence $\mathfrak{t}^*$ describes the background $(\widetilde{g}, \widetilde{b})$ on T^*M, which is related to the new background (g, b) by the isomorphism g.

4.2.2 The patch-wise effective theory: Lie-algebroid gravity

In this section the proper analogue of the effective action (4.14) in different $O(d,d)$-patches is formulated. Since $O(d,d)$-transformations act on the background by the field redefinition (3.20), the transitioned action has to be formulated in terms of the resulting background.[4] In particular, all the symmetries have to be retained in the transition and the actions must coincide in order to describe the same theory. The difficulty in achieving this lies in the mixing of metric and Kalb-Ramond field induced by (3.20), which implies a mixing of the initial gauge transformations and diffeomorphisms. Thus the aim is to identify the correct symmetries in the new patch.

[4] The new action is not the low energy effective action associated to the $O(d,d)$-dual sigma model as the transition is described by a mere change of the background, which is not a duality.

In the previous section, two Lie algebroids have been identified whose anchors relate the initial background and the redefined one. Hence the general structure of the redefinition is the existence of a Lie algebroid isomorphism $\rho : A \to TM$ between the two Lie algebroids $\mathcal{A} = (A, [\cdot,\cdot]_{\mathcal{A}}, \rho)$ and $(TM, [\cdot,\cdot], \mathrm{id}_{TM})$ whose respective metrics g and G are related via

$$g = \otimes^2 \rho^t(G) \quad \in \Gamma(\bigodot^2 A^*)\,; \tag{4.27}$$

these are the assumptions for theorem 2.17. In particular, the relations (2.51) and (2.57) are used to translate the connection, the curvature and the torsion: Indicating the geometric objects on $\mathcal{A}$ with a hat, they relate to those on TM by

$$\begin{aligned} \widehat{R}^{\alpha}{}_{\beta\gamma\delta} &= \rho^{\alpha}{}_{a}\, \rho^{b}{}_{\beta}\, \rho^{c}{}_{\gamma}\, \rho^{d}{}_{\delta}\, R^{a}{}_{bcd} \\ \widehat{R}_{\alpha\beta} &= \rho^{a}{}_{\alpha}\, \rho^{b}{}_{\beta}\, R_{ab} \\ \widehat{R} &= R \\ \widehat{T}^{\alpha}{}_{\beta\gamma} &= \rho^{\alpha}{}_{a}\, \rho^{b}{}_{\beta}\, \rho^{c}{}_{\gamma}\, T^{a}{}_{bc} \end{aligned}$$

with $\{e_\alpha\}$ a frame for A and $\{e_a\}$ a frame for TM.

Symmetries

By construction, also the geometric objects on the Lie algebroids $\mathcal{A}$ are proper tensors, i.e. behave in the ordinary way under diffeomorphisms: On the tangent bundle, diffeomorphisms induce special changes of frame related to the coordinate frame and changes of coordinates on the underlying manifold. For Lie algebroids whose underlying bundle has the same rank as the dimension of the manifold and an invertible anchor – which is in particular the case for $\mathfrak{t}$ and $\mathfrak{t}^*$ – there is a proper notion of coordinate frame as well. Let $\{x^a\}$ be a basis for M and $\{\frac{\partial}{\partial x^a}\}$ the coordinate frame for TM with its dual $\{dx^a\}$. Then $\{\rho^{-1}(\frac{\partial}{\partial x^a}) \equiv \partial^{\mathcal{A}}_a\}$ is a frame for the vector bundle A with dual $\{d_{\mathcal{A}} x^a\}$ since ρ is invertible. This follows from

$$d_{\mathcal{A}} x^a\left(\partial^{\mathcal{A}}_b\right) = \rho\left(\rho^{-1}\left(\tfrac{\partial}{\partial x^b}\right)\right) x^a = \frac{\partial x^a}{\partial x^b} = \delta^a_b \tag{4.28}$$

by (2.34). Hence $\{d_{\mathcal{A}} x^a\}$ is indeed the dual frame. Since $d_{\mathcal{A}} x^a = \rho^{-t}(dx^a)$ by proposition 2.12 and since the anchor is linear, the frame behaves in the ordinary way under a change of basis $x^a \to y^{a'}(x)$:

$$\partial^{\mathcal{A}}_a = \frac{\partial y^{a'}}{\partial x^a}\, \partial^{\mathcal{A}}_{a'} \qquad \text{and} \qquad d_{\mathcal{A}} x^a = \frac{\partial x^a}{\partial y^{a'}}\, d_{\mathcal{A}} y^{a'}\,. \tag{4.29}$$

The transformation behavior for arbitrary tensors then follows from their multilinearity.

This shows that any tensor on A transforms as usual under diffeomorphisms. It remains to discuss gauge transformations. The Lie algebroids $\mathfrak{t}$ and $\mathfrak{t}^*$ constructed from the field redefinition also satisfy

$$\mathfrak{b} = \otimes^2 \rho^t(B) \quad \in \Gamma(\Lambda^2 A^*)\,, \tag{4.30}$$

which is the analogue of the Kalb-Ramond field from the perspective of the action. Indeed, by using proposition 2.12, $\mathfrak{b}$ inherits the gauge transformations from B:

$$\otimes^2\rho^t(B+d\xi) = \mathfrak{b} + \otimes^2\rho^t(d\xi) = \mathfrak{b} + d_{\mathcal{A}}\left(\rho^t\xi\right) . \tag{4.31}$$

Thus the $\mathfrak{b}$-field gauge transformations are $\mathfrak{b} \to \mathfrak{b} + d_{\mathcal{A}}\sigma$ for $\sigma \in \Gamma(A^*)$. The associated gauge invariant object is

$$\Theta = d_{\mathcal{A}}\mathfrak{b} = \otimes^3\rho^t(H) ; \tag{4.32}$$

the relation to the H-flux $H = dB$ follows from proposition 2.12 and (4.30). In particular, similar to the connections, the curvatures and the torsions according to theorem 2.17, also the gauge invariant fluxes are related to each other by applying the anchor; B and $\mathfrak{b}$ are related by anchoring as well, but are gauge dependent quantities. These observations lead to the following notion for tensors on $\mathcal{A}$, which distinguishes objects with a gauge dependence stemming from the B-dependence of the anchor from those with an inherent gauge dependence.

Definition 4.3. A section $\tau \in \Gamma(\bigotimes^r A \otimes \bigotimes^s A^*)$ of the Lie algebroid $\mathcal{A} = (A, [\cdot,\cdot]_{\mathcal{A}}, \rho)$ is called a *ρ-tensor of type* (r,s) if

$$\left[(\otimes^r\rho)\otimes(\otimes^s\rho^{-t})\right](\tau) \quad \in \Gamma\left(\bigotimes^r TM \otimes \bigotimes^s T^*M\right)$$

is gauge invariant. A *ρ-gauge transformation* of a k-form $\tau \in \Gamma(\Lambda^k A^*)$ is given by

$$\tau \to \tau + d_{\mathcal{A}}\sigma \tag{4.33}$$

for a $(k-1)$-form $\sigma \in \Gamma(\Lambda^{k-1}A^*)$.

In other words, a ρ-tensor is characterized as a section whose image under the anchor is a conventional, gauge invariant tensor. Written in components with $\rho \equiv (\rho^a{}_\alpha)$, the section $\tau^{\alpha_1\dots\alpha_r}{}_{\beta_1\dots\beta_s}$ is a ρ-tensor if there exists a gauge invariant (r,s)-tensor T with

$$T^{a_1\dots a_r}{}_{b_1\dots b_s} = \rho_{\alpha_1}{}^{a_1}\dots\rho_{\alpha_r}{}^{a_r}\,\rho^{\beta_1}{}_{b_1}\dots\rho^{\beta_s}{}_{b_s}\,\tau^{\alpha_1\dots\alpha_r}{}_{\beta_1\dots\beta_s} .$$

Moreover, any contractions or traces of ρ-tensors are again ρ-tensors as an anchor always contracts with its inverse. In particular, (2.51) shows that the Levi-Civita connection $\widehat{\nabla}$ and the associated curvature $\widehat{R}$ and torsion $\widehat{T}$ on $\mathcal{A}$ are ρ-tensors as well as Θ due to (4.32). On the other hand, $\mathfrak{b}$ is not a ρ-tensor as (4.30) relates it to B, which is gauge dependent. This gauge dependence amounts to a ρ-gauge dependence of $\mathfrak{b}$ in terms of $\mathcal{A}$-exact two-forms.

The action

As has been shown above, $\widehat{R}$ and Θ are the analogues of R and H on the Lie algebroid and related to the latter by applying the anchor. To formulate the Lagrangian analogous to the one appearing in (4.14), the dilaton ϕ is assumed to be unchanged. This can be understood as extension of the principle observed for the metric, the Kalb-Ramond field, the connection and the curvature: The geometric objects on $\mathcal{A}$ are related to those on TM by applying the anchor. Since ϕ is a scalar, the action of the anchor is trivial. This completes the list of ingredients for a ρ-scalar Lagrangian analogous to (4.14).

For the integration, the duality invariant measure $\sqrt{\det G}e^{-2\phi}$ is taken as in (4.14). However, using (4.27) the first factor can be written in terms of the new metric as

$$\sqrt{\det G} = \sqrt{\det\left(\otimes^2\rho^{-t}g\right)} = \sqrt{\det g}\,|\det\rho^{-1}|\,. \tag{4.34}$$

This allows to formulate the diffeomorphism and $\mathcal{A}$-gauge invariant action

$$\widehat{\mathcal{S}}_{\mathcal{A}} = -\frac{1}{2\kappa^2}\int_M d^dx\,\sqrt{\det g}\,|\det\rho^{-1}|\,e^{-2\phi}\left(\widehat{R} - \tfrac{1}{12}\,\Theta_{\alpha\beta\gamma}\,\Theta^{\alpha\beta\gamma} + 4\,D_\alpha\phi\,D^\alpha\phi\right)\,. \tag{4.35}$$

In particular, ρ-scalars and conventional scalars coincide. For the Ricci scalar this was already observed in (2.57), and for contracted terms it is a consequence of the contraction of anchors. For example

$$\Theta_{\alpha\beta\gamma}\,g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}\,\Theta_{\mu\nu\rho} = H_{abc}\,G^{am}G^{bn}G^{ck}\,H_{mnk} \tag{4.36}$$

by (4.32) and the inverse of (4.27). Together with (4.34) this implies the equivalence of $\widehat{\mathcal{S}}_{\mathcal{A}}$ (4.35) and $\mathcal{S}$ (4.14):

Proposition 4.4 ([80])**.** *Let $\mathcal{A} = (A, [\cdot,\cdot]_{\mathcal{A}}, \rho)$ with* $\operatorname{rank}(A) = \dim(M)$ *and an invertible anchor. Let $\mathcal{A}$ and TM be equipped with a metric and a two-form $(g, \mathfrak{b})$ and (G, B) respectively, which are related by $g = \otimes^2\rho^t(G)$ and $\mathfrak{b} = \otimes^2\rho^t(B)$. Then the theories $\widehat{\mathcal{S}}_{\mathcal{A}}$ (4.35) and $\mathcal{S}$ (4.14) coincide to all orders in α':*

$$\widehat{\mathcal{S}}_{\mathcal{A}}(g,\mathfrak{b}) \xleftrightarrow[\mathfrak{b}=\otimes^2\rho^t(B)]{g=\otimes^2\rho^t(G)} \mathcal{S}(G,B)\,.$$

The result applies to all α'-corrections since the effective action for the bosonic string consists of contractions and covariant derivatives of the curvature tensor, H and the dilaton. Since they are related to the quantities on $\mathcal{A}$ by anchoring, the above procedure extends to any action comprising these fields.

Interpretation

The transition functions, whose impact on the background is given by the field redefinitions (4.27) and (4.30), are elements of $O(d,d)$. In section 4.2.1 the isomorphic Lie algebroids $\mathfrak{t} = (TM, [\cdot,\cdot]_{\mathfrak{t}}, \rho = \gamma^{-t})$ and $\mathfrak{t}^* = (T^*M, [\cdot,\cdot]_{\mathfrak{t}^*}, \widetilde{\rho} = G^{-1}\circ\gamma)$ have been constructed from the redefinition (3.20) which provide the geometry for the redefined theory. Now the change of the background under the transition between patches is considered by taking a closer look to the generators of $O(d,d)$ as discussed in section 3.2.3. Let (G, B) be a background requiring the following transition functions:

- Change of frame (3.47): The map (3.19) is given by $\gamma_A = \mathsf{A}^{-t}$. Then $\rho = \mathsf{A}$ and $(g,\mathfrak{b}) = (\mathsf{A}^tG\mathsf{A}, \mathsf{A}^tB\mathsf{A})$. This conserves the Lagrangian, but the measure $\sqrt{\det G}$ receives an additional factor $|\det\mathsf{A}^{-1}|$. If A stems from a diffeomorphism, this compensates the Jacobian determinant of the change of coordinates. Hence $\widehat{\mathcal{S}}_{\mathfrak{t}} = \mathcal{S}$ for diffeomorphisms.

- B-transformations (2.2): Since γ is just the identity, also $\rho = \mathbb{1}$ and the background in the new patch coincides with the background in the initial patch[5]; $(g, \mathfrak{b}) = (G, B)$. Again $\widehat{\mathcal{S}}_{\mathfrak{t}} = \mathcal{S}$

- β-transformations (2.12): Here the anchor reads $\rho = [\mathbb{1} - (G-B)\beta]^{-t}$. Thus the transition is very complicated as it in particular involves the background itself, which causes the mixing of coordinate and gauge transformations.

- T-duality (2.7): Similar to β-transformations, the anchor $\rho = \mathbb{1} - 1_k + (G-B)1_k$ leads to a complicated transition with a B-gauge dependence in every tensor.

The complicated forms of the anchor for the non-geometric transition functions related to β-transformations and T-duality illustrate the difficulty of finding the action (4.35) by direct computations as opposed to the approach using Lie algebroids followed here.

It is also always possible to change from the tangent bundle picture $\mathfrak{t}$ to the cotangent bundle picture $\mathfrak{t}^*$ by successively applying proposition 4.2, theorem 2.17 and proposition 4.4.

In total, the theory on different patches related by $O(d,d)$-valued transition functions can be summarized by figure 4.1.

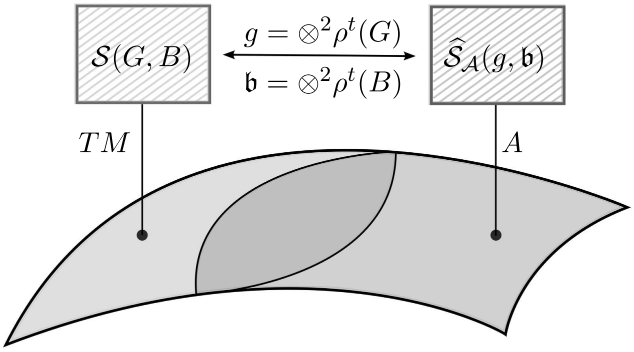


Figure 4.1: Every patch is described by a different action

Hence the suggested prescription provides a global understanding in particular of the theory on non-geometric backgrounds as being governed by the different actions (4.35) on different patches.

Moreover, the equations of motion for the redefined action $\widehat{\mathcal{S}}_{\mathcal{A}}$ (4.35) read

$$\begin{aligned}
0 &= \widehat{R}_{\alpha\beta} + 2\,\widehat{\nabla}_\alpha \widehat{\nabla}_\beta \phi - \tfrac{1}{4}\,\Theta_{\alpha\mu\nu}\,\Theta_\beta{}^{\mu\nu} + \mathcal{O}(\alpha')\,, \\
0 &= g^{\alpha\beta}\,\widehat{\nabla}_\alpha\phi\,\widehat{\nabla}_\beta\phi - \tfrac{1}{2} g^{ab}\,\widehat{\nabla}_\alpha \widehat{\nabla}_\beta \phi - \tfrac{1}{24}\,\Theta_{\alpha\beta\gamma}\,\Theta^{\alpha\beta\gamma} + \mathcal{O}(\alpha')\,, \\
0 &= \tfrac{1}{2}\,\widehat{\nabla}^\mu \Theta_{\mu\alpha\beta} - \widehat{\nabla}^\mu\phi\,\Theta_{\mu\alpha\beta} + \mathcal{O}(\alpha')\,,
\end{aligned} \tag{4.37}$$

which follows directly from applying the anchor to the ordinary equations of motion (3.58) and proposition 4.4 or by direct variation with respect to g, $\mathfrak{b}$ and ϕ. Hence $\widehat{\mathcal{S}}_{\mathcal{A}}$ can in particular be considered as extended gravitational theory on a Lie algebroid, which can be studied on its own right, i.e. without a relation to the theories on TM.

[5] In this case the difference to actual duality is most apparent: Whereas the Kalb-Ramond field of the new background for duality is shifted, the Kalb-Ramond field in a different patch remains the same.

Remark 2. The initial goal for formulating the theory for redefined backgrounds was a global description of non-geometric backgrounds by a single action. This was motivated by the field redefinition $(G+B)^{-1} = g^{-1} + \beta$ for the Q-flux presented in section 3.2.5 and introduced in [52]. The new background (3.71) appears to be globally well-defined as the metric is well-defined and β changes under the transition by a gauge transformation. However, the redefinition of the action from (4.14) to (4.35) also changes the symmetry group according to (4.18). In particular, what appears to be a gauge transformation for the background (3.71) is not a symmetry of the redefined action (4.35) as it stems from β-transformations for the initial background, which are not symmetries for (4.14) either. As advocated in [127, 128, 129, 77], the redefined action (4.35) related to the new background might be globally well-defined if certain terms are neglected and if one restricts to very special backgrounds. This approach is pursued further in [130, 131].

4.2.3 Examples

The construction developed in the previous sections generalizes two particular constructions appearing in the recent literature, which are briefly presented for the sake of completeness. The first example provides a Lie algebroid structure $\mathfrak{t}$ on TM and the second instance provides a Lie algebroid $\mathfrak{t}^*$ on T^*M (see section 4.2.1).

An example for $\mathfrak{t}$: The Q-flux redefinition

The first example follows the setting employed in [127, 128, 77]. The initial background (G, B) is related to the new background (g, β) via

$$(G+B)^{-1} = g^{-1} + \beta\,. \tag{4.38}$$

Hence, g is a metric on TM and β an alternating bivector. This redefinition is related to the $O(d,d)$-transformation

$$\mathcal{T}_I = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}\,. \tag{4.39}$$

Using (3.20) with $\mathcal{T}_I$ gives the background (g, b) with b related to β defined above by $\beta = g^{-1} b g^{-1}$. This gives rise to a Lie algebroid $\mathfrak{t}$ on TM with anchor $\rho_I = \mathbb{1} - G^{-1}B$ and bracket given in terms of the anchor by (4.20) and (4.21). Hence the procedure for determining the new action developed above is applicable; although the redefined quantities are cumbersome if written down explicitly (cf. [80]), (4.35) retains all the desired symmetries.

This construction is particularly useful for the Q-flux background (3.68) as it becomes very simple: In this case $g = \mathbb{1}_*$ and (4.39) describes a complete T-duality; see (3.71), footnote 9 in chapter 3 and remark 2 for details.

An example for $\mathfrak{t}^*$: Quasi-Poisson geometry

The second example was constructed in [78, 79]. Starting from the background (G, B), the underlying $O(d,d)$-transformation for the redefinition (4.27) and (4.30) is

$$\mathcal{T}_{II} = \mathcal{T}_{-2B}\,\mathcal{T}_\beta = \begin{pmatrix} \mathbb{1} & -\beta \\ 2B & -\mathbb{1} \end{pmatrix}\,, \tag{4.40}$$

i.e. a composition of B- and β-transformations, provided that the Kalb-Ramond field is invertible with $B = \beta^{-1}$. The anchor is given by $\widetilde{\rho} = -\beta \equiv \beta^\sharp$.[6] The fields (4.27) and (4.30) on the cotangent bundle are given by

$$\widetilde{g} = -\beta\, G\, \beta \qquad \text{and} \qquad \widetilde{\beta} = -\beta \tag{4.41}$$

respectively. The bracket on T^*M is determined by the structure coefficient (4.26), which on a holonomic frame reads

$$\begin{aligned} Q^{\bar{a}\bar{b}}{}_{\bar{c}} &= -B_{\bar{c}m}\left(\beta^{n\bar{a}}\partial_n\beta^{m\bar{b}} - \beta^{n\bar{b}}\partial_n\beta^{m\bar{a}}\right) \\ &= -\beta^{\bar{a}m}\beta^{\bar{b}n}\left(\partial_n B_{\bar{c}m} - \partial_m B_{\bar{c}n}\right) \\ &= \partial_{\bar{c}}\beta^{\bar{a}\bar{b}} - \beta^{\bar{a}m}\beta^{\bar{b}n}H_{mn\bar{c}}\,, \end{aligned} \tag{4.42}$$

where $\partial M^{-1} = -M^{-1}(\partial M)M^{-1}$ was used multiple times. This is the structure constant of the $(-H)$-twisted Koszul bracket (4.6) given in (4.8). The bracket does not satisfy the anchor property a priori; the defect is given by (4.7) as

$$\Delta_{\mathcal{K}_{-H}} = \iota_\eta\iota_\xi\left[\otimes^3\beta^\sharp(H) - \Theta\right]\,, \tag{4.43}$$

with Θ introduced in (2.32). However, in a holonomic frame and using $\beta = B^{-1}$, Θ can be evaluated to

$$\Theta^{abc} = \beta^{am}\partial_m\beta^{bc} + \beta^{cm}\partial_m\beta^{ab} + \beta^{bm}\partial_m\beta^{ca} = \beta^{am}\beta^{bn}\beta^{ck}H_{mnk}\,; \tag{4.44}$$

this implies $\Delta_{\mathcal{K}_{-H}} = 0$. Thus the Lie algebroid $\mathfrak{t}^*$ coincides with the Lie algebroid $\mathcal{K}_{-H} = (T^*M, [\cdot,\cdot]_K^{-H}, \beta^\sharp)$. The vanishing of the defect, i.e.

$$\Theta = \otimes^3\beta^\sharp(H) \tag{4.45}$$

means that β is a *twisted* or *quasi-Poisson structure* [116, 132]: The Poisson bracket is given by $\{f,g\} = \beta(df,dg)$ and its Jacobiator reads

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = \iota_{dh}\iota_{dg}\iota_{df}\Theta = \iota_{dh}\iota_{dg}\iota_{df}\left[\otimes^3\beta^\sharp(H)\right]$$

by using (4.45). Thus the Poisson bracket defined by β does not satisfy the Jacobi identity; the defect is given in terms of the twist of the Koszul bracket.

Since this redefinition gives a gravity theory (4.35) whose underlying structure is the $(-H)$-twisted Koszul bracket, it is reasonable to expect a relation to the non-geometric sector of the duality chain (3.2.5) – In section 4.1.1 this structure was identified as being suitable for describing the Q- and R-flux.

Even more interesting, it naturally trades the Kalb-Ramond field for a quasi-Poisson structure. Since the Jacobi identity is not satisfied for $H \neq 0$, this might hint towards non-associative structures in string theory.[7] Moreover, since the work of Kontsevich on the quantization of Poisson manifolds [135] there has been a lot of progress in quantizing quasi-Poisson structures as well [136, 137, 138, 139]. Thus the action (4.35) in terms of the quasi-Poisson structure β might enable a direct quantization of this class of gravitational theories.

[6] The difference in the convention between matrix multiplication and the $\sharp$-prescription is a transposition.
[7] See [56, 120, 121, 24, 25, 57, 59, 133, 134].

4.2.4 Lie-algebroid supergravity

Proposition 4.4 describes the redefinition for the massless Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector of closed bosonic string theory to all orders in α'. However, the superstring spectrum, which can be obtained by adding fermions to (3.1), contains additional massless bosonic states – the Ramond-Ramond (R-R) sector – and massless fermionic states in the R-NS and NS-R sectors (see e.g. [13]). The general rule for translating geometric objects to the Lie algebroid $\mathcal{A}$ is by applying the anchor: For T an (r, s)-tensor on TM,

$$\tau = \left[\left(\otimes^r \rho^{-1}\right) \otimes \left(\otimes^s \rho^t\right)\right] T \tag{4.46}$$

is an (r, s) ρ-tensor according to definition 4.3. This has been shown explicitly for the objects related to the metric and the Kalb-Ramond field and will assumed to be a general rule for the remaining fields.

In the following, the bosonic ten-dimensional supergravities arising from closed string theory are considered. The translation of fermionic terms reduces to the problem of redefining the spin connection done in (4.61); details can be found in [79, 80].

Type II theories

Focusing on type II supergravities, the massless field content given in table 4.1 has to be considered. The difference between IIA and IIB is that the chiralities of the left- and right-moving Ramond ground states differ for the former but coincides for the latter. Here

type	bosonic		fermionic
	NS-NS	R-R	NS-R/R-NS
IIA	G, B, ϕ	C_1, C_3	$2\times\ \psi, \lambda$
IIB	G, B, ϕ	C_0, C_2, C_4	$2\times\ \psi, \lambda$

Table 4.1: The massless spectrum of IIA and IIB theories.

C_n is an n-form, ψ is a spin 3/2 fermion – the gravitino – and λ is a spin 1/2 fermion – the dilatino. While the transition for the metric, the Kalb-Ramond field and the dilaton has been established above, a rule must be found for treating the R-R fields.

The forms C_n appear in the effective action for the type II string via the field strengths

$$\begin{array}{ll} \underline{\text{IIA:}} & \underline{\text{IIB:}} \\ F_2 = dC_1\,, & F_1 = dC_0\,, \\ F_4 = dC_3 - dB\wedge C_1\,, & F_3 = dC_2 - C_0\, dB\,, \\ & F_5 = dC_4 - \frac{1}{2}C_2\wedge dB + \frac{1}{2}B\wedge dC_2\,. \end{array} \tag{4.47}$$

These field strengths are proper tensors as they are invariant under B-field gauge transformations $B \to B + d\xi$ and the C-field gauge transformations

$$\begin{array}{ll} \underline{\text{IIA:}} & \underline{\text{IIB:}} \\ C_1 \to C_1 + d\Lambda_0\,, & C_0 \to C_0\,, \\ C_3 \to C_3 + \Lambda_0\, d\Lambda_2 - dB\,, & C_2 \to C_2 + d\Lambda_1\,, \\ & C_4 \to C_4 + d\Lambda_3 - \frac{1}{2}dB\wedge\Lambda_1 + \frac{1}{2}dC_2\wedge\xi \end{array} \tag{4.48}$$

for Λ_n arbitrary n-forms. In the previous sections the anchor translated between the geometric quantities on the tangent bundle and the Lie algebroid. Since the field strengths defined above are tensors, they can be translated to the Lie algebroid via (4.46), giving ρ-tensors. The same can be done for the associated gauge fields: Defining

$$\widehat{C}_n = \otimes^n \rho^t(C_n) \qquad \in \Gamma(\Lambda^n A^*)\,,$$

the translated field strengths retain their form. For F_5 for example, this means

$$\begin{aligned}\widehat{F}_5 &= \otimes^5 \rho^t(F_5) = \otimes^5 \rho^t \left(dC_4 - \tfrac{1}{2} C_2 \wedge dB + \tfrac{1}{2} B \wedge dC_2\right)\\ &= d_{\mathcal{A}} \widehat{C}_4 - \tfrac{1}{2}\widehat{C}_2 \wedge d_{\mathcal{A}} \mathfrak{b} + \tfrac{1}{2}\mathfrak{b} \wedge d_{\mathcal{A}}\widehat{C}_2\,,\end{aligned}$$

where proposition 2.12 and (4.30) was used as well as the compatibility of Lie algebroid homomorphisms with tensor products – in this case $\rho^t(\xi \wedge \eta) = \rho^t(\xi) \wedge \rho^t(\eta)$ for ξ and η one-forms. Thus, apart from $\widehat{R}$, Θ and ϕ, the following quantities appear on the Lie algebroid side:

$$\begin{array}{ll} \underline{\text{IIA on } \mathcal{A}:} & \underline{\text{IIB on } \mathcal{A}:} \\ \widehat{F}_2 = d_{\mathcal{A}}\widehat{C}_1\,, & \widehat{F}_1 = d_{\mathcal{A}}\widehat{C}_0\,, \\ \widehat{F}_4 = d_{\mathcal{A}}\widehat{C}_3 - d_{\mathcal{A}}\mathfrak{b} \wedge \widehat{C}_1\,, & \widehat{F}_3 = d_{\mathcal{A}}\widehat{C}_2 - \widehat{C}_0 d_{\mathcal{A}}\mathfrak{b}\,, \\ & \widehat{F}_5 = d_{\mathcal{A}}\widehat{C}_4 - \tfrac{1}{2}\widehat{C}_2 \wedge d_{\mathcal{A}}\mathfrak{b} + \tfrac{1}{2}\mathfrak{b} \wedge d_{\mathcal{A}}\widehat{C}_2\,. \end{array} \tag{4.49}$$

Also the gauge transformations (4.48) translate to the Lie algebroid consistently. Denoting $\widehat{\Lambda}_n = \otimes^n \rho^{-1}(\Lambda_n)$, the gauge transformation for C_4 becomes

$$\begin{aligned}\widehat{C}_n &\to \widehat{C}_n + \otimes^4 \rho^t \left(d\Lambda_3 - \tfrac{1}{2} dB \wedge \Lambda_1 + \tfrac{1}{2} dC_2 \wedge \xi\right)\\ &= \widehat{C}_n + d_{\mathcal{A}}\widehat{\Lambda}_3 - \tfrac{1}{2} d_{\mathcal{A}}\mathfrak{b} \wedge \widehat{\Lambda}_1 + \tfrac{1}{2} d_{\mathcal{A}}\widehat{C}_2 \wedge \rho^t(\xi)\,.\end{aligned}$$

In particular, it involves the $\mathfrak{b}$-field gauge transformation $\mathfrak{b} \to \mathfrak{b} + d_{\mathcal{A}}\rho^t(\xi)$. In total, the new gauge transformations read

$$\begin{array}{ll} \underline{\text{IIA on } \mathcal{A}:} & \underline{\text{IIB on } \mathcal{A}:} \\ \widehat{C}_1 \to \widehat{C}_1 + d_{\mathcal{A}}\widehat{\Lambda}_0\,, & \widehat{C}_0 \to \widehat{C}_0\,, \\ \widehat{C}_3 \to \widehat{C}_3 + \widehat{\Lambda}_0\, d_{\mathcal{A}}\widehat{\Lambda}_2 - d_{\mathcal{A}}\mathfrak{b}\,, & \widehat{C}_2 \to \widehat{C}_2 + d_{\mathcal{A}}\widehat{\Lambda}_1\,, \\ & \widehat{C}_4 \to \widehat{C}_4 + d_{\mathcal{A}}\widehat{\Lambda}_3 - \tfrac{1}{2} d_{\mathcal{A}}\mathfrak{b} \wedge \widehat{\Lambda}_1 + \tfrac{1}{2} d_{\mathcal{A}}\widehat{C}_2 \wedge \rho^t(\xi)\,. \end{array} \tag{4.50}$$

Then gauge invariance of the translated field strengths (4.49) under the translated gauge transformations (4.50) follows in particular from nilpotency of $d_{\mathcal{A}}$.

The bosonic part of the ten-dimensional type II actions includes the following actions apart from the universal NS-NS actions (4.14) (see e.g. [13]):

$$\begin{array}{ll} \mathcal{S}^{\text{A}}_{\text{RR}} \sim \int d^{10}x \sqrt{\det G}\left(|F_2|^2 + |F_4|^2\right) & ,\quad \mathcal{S}^{\text{A}}_{\text{CS}} \sim \int B \wedge dC_3 \wedge dC_3\,, \\ \mathcal{S}^{\text{B}}_{\text{RR}} \sim \int d^{10}x \sqrt{\det G}\left(|F_1|^2 + |F_3|^2 + \tfrac{1}{2}|F_5|^2\right) & ,\quad \mathcal{S}^{\text{B}}_{\text{CS}} \sim \int C_4 \wedge H \wedge F_3 \end{array} \tag{4.51}$$

with $|F_n|^2 = \frac{1}{n!}(F_n)_{a_1 \ldots a_n}(F_n)^{a_1 \cdots a_n}$. In addition, in IIB theory the five-form F_5 is self dual, i.e. $\star F_5 = F_5$; this has to be considered an additional constraint. The R-R Lagrangian densities $L_{\text{RR}}^{\text{A/B}}$ are invariant under the gauge transformations (4.48) by definition. Their translation is therefore analogous to the translation performed in section 4.2.2, i.e. F_n has to be exchanged with $\widehat{F}_n$ and the measure factor $\sqrt{\det G}$ with $\sqrt{\det g}|\det \rho^{-1}|$ as in (4.34). The Chern-Simons terms are more subtle. Under gauge transformations they transform as

$$\begin{aligned} \mathcal{S}_{\text{CS}}^{\text{A}} &\to \mathcal{S}_{\text{CS}}^{\text{A}} + \textstyle\int d\left(\xi \wedge dC_3 \wedge dC_3\right) , \\ \mathcal{S}_{\text{CS}}^{\text{B}} &\to \mathcal{S}_{\text{CS}}^{\text{B}} + \textstyle\int d\left(\Lambda_3 \wedge H \wedge F_3\right) , \end{aligned} \tag{4.52}$$

where evaluation of the second line requires the anomalous Bianchi identity

$$dF_3 = H \wedge F_1 . \tag{4.53}$$

Thus the Chern-Simons terms are invariant up to total derivatives. For translating this terms, it is useful to give them locally. In the coordinate frame $\{dx^a\}$ they read

$$\begin{aligned} B \wedge dC_3 \wedge dC_3 &= \tfrac{1}{2!4!4!}\, \varepsilon^{a_1 \ldots a_{10}} \left(B \wedge dC_3 \wedge dC_3\right)_{a_1 \ldots a_{10}} \text{vol} \equiv \mathcal{L}_{\text{CS}}^{\text{A}}\, \text{vol} , \\ C_4 \wedge H \wedge F_3 &= \tfrac{1}{4!3!3!}\, \varepsilon^{a_1 \ldots a_{10}} \left(C_4 \wedge H \wedge F_3\right)_{a_1 \ldots a_{10}} \text{vol} \equiv \mathcal{L}_{\text{CS}}^{\text{B}}\, \text{vol} , \end{aligned} \tag{4.54}$$

with the volume form $\text{vol} = \sqrt{\det G} dx^1 \wedge \cdots \wedge dx^{10}$ and ε the Levi-Civita tensor, which is related to the Levi-Civita symbol $\epsilon \in \{0, \pm 1\}$ by $\sqrt{\det G}\varepsilon = \epsilon$. The Levi-Civita tensor can also be translated using (4.46); thus the Chern-Simons Lagrangian densities $\mathcal{L}_{\text{CS}}^{\text{A/B}}$ are ordinary scalars and translate directly to ρ-scalars. In particular, the gauge transformations of the redefined Chern-Simons terms are top degree exact form on $\mathcal{A}$, i.e. contribute as $\int d_{\mathcal{A}}\sigma$ for $\sigma \in \Gamma(\Lambda^9 A^*)$. Due to the redefined Levi-Civita tensor this, however, agrees with an exact ten-form: $\int d_{\mathcal{A}}\sigma = \int d(\otimes^9 \rho^{-t} \sigma)$. Moreover, according to (4.34) the redefined volume form is

$$\widehat{\text{vol}} = \sqrt{\det g}\, |\det \rho^{-1}|\, dx^1 \wedge \cdots \wedge dx^{10} . \tag{4.55}$$

Thus, in total the translated bosonic sector of type II supergravities is governed by the universal action $\widehat{\mathcal{S}}_{\mathcal{A}}$ (4.35) together with

$$\begin{aligned} \widehat{\mathcal{S}}_{\text{RR}}^{\text{A}} &\sim \textstyle\int \widehat{\text{vol}} \left(|\widehat{F}_2|^2 + |\widehat{F}_4|^2\right) &,\quad \widehat{\mathcal{S}}_{\text{CS}}^{\text{A}} &\sim \textstyle\int \mathfrak{b} \wedge d_{\mathcal{A}}\widehat{C}_3 \wedge d_{\mathcal{A}}\widehat{C}_3 , \\ \widehat{\mathcal{S}}_{\text{RR}}^{\text{B}} &\sim \textstyle\int \widehat{\text{vol}} \left(|\widehat{F}_1|^2 + |\widehat{F}_3|^2 + \tfrac{1}{2}|\widehat{F}_5|^2\right) &,\quad \widehat{\mathcal{S}}_{\text{CS}}^{\text{B}} &\sim \textstyle\int \widehat{C}_4 \wedge \Theta \wedge \widehat{F}_3 \end{aligned} \tag{4.56}$$

with Θ given in (4.32) and the Chern-Simons terms have to be understood as in (4.54) with respect to the volume form (4.55) and the translated Levi-Civita tensor $\widehat{\varepsilon}$ satisfying $\sqrt{\det g}|\det \rho^{-1}|\widehat{\varepsilon} = \epsilon$. This concludes the translation of the bosonic sector.

Heterotic theories

From the string theory perspective the heterotic string arises from considering a 26-dimensional bosonic left-moving sector and a 10-dimensional fermionic right-moving sector. The overlapping 16 dimensions have to be compactified on a 16-dimensional even self-dual Euclidean lattice, which only leaves two options: either the root lattice of $E_8 \times E_8$ or the root lattice of $SO(32)$. Then, apart from the background (G, B, ϕ), the massless bosonic fields of heterotic supergravity include a gauge connection $A = A^{\mathsf{a}} T^{\mathsf{a}} \in \Gamma(T^*M \otimes \mathfrak{g})$ with $\mathfrak{g}$ the Lie algebra for either $E_8 \times E_8$ or $SO(32)$ with generators T^{a}. Due to gauge anomaly cancellation the H-term in (4.14) receives corrections; it reads

$$H = dB - \tfrac{\alpha'}{4}(\Omega_{\rm YM} - \Omega_{\rm L})\,, \tag{4.57}$$

where $\Omega_{\rm YM}$ and $\Omega_{\rm L}$ denote the Yang-Mills and Lorentz Chern-Simons three-forms

$$\begin{aligned} \Omega_{\rm YM} &= {\rm tr}\left(A \wedge dA - \tfrac{2i}{3}\, A \wedge A \wedge A\right) \quad \in \Gamma\left(\Lambda^3 T^*M \otimes \mathfrak{g}\right)\,, \\ \Omega_{\rm L} &= {\rm tr}\left(\omega \wedge d\omega - \tfrac{2}{3}\, \omega \wedge \omega \wedge \omega\right) \quad \in \Gamma\left(\Lambda^3 T^*M \otimes so(10)\right)\,. \end{aligned} \tag{4.58}$$

The wedge product of gauge connections includes the commutator of the Lie-algebra part – $A \wedge A = A^{\mathsf{a}} \wedge A^{\mathsf{b}} \otimes [T^{\mathsf{a}}, T^{\mathsf{b}}]$ – and the trace is normalized such that ${\rm tr}(T^{\mathsf{a}} T^{\mathsf{b}}) = \delta^{\mathsf{ab}}$. Moreover, $\omega \in \Gamma(T^*M \otimes so(10))$ denotes the spin connection.[8] It is defined by being given with respect to an orthonormal frame in which $G = \delta_{\mu\nu} e^\mu \otimes e^\nu$. For the bases of TM being related via $e_\mu = e_\mu^a e_a$ and denoting the vector-valued connection one form by $\omega_a = \omega^b{}_a \otimes e_b = \nabla e_a$, the spin connection $\omega^\mu{}_\nu$ is given in terms of an arbitrary connection $\omega^a{}_b$ by

$$\omega^\mu{}_\nu = e^\mu_a\, de^a_\nu + e^\mu_a e^b_\nu\, \omega^a{}_b\,. \tag{4.59}$$

Then $\omega = \omega^\mu{}_\nu e_\mu e^\nu \in \Gamma(T^*M \otimes {\rm End}(TM))$. The respective trace is normalized as above with respect to the generators of $so(10)$. The field strength for the Yang-Mills gauge connection is given by $F = dA - iA \wedge A$ and the action is

$$\mathcal{S}_{\rm het} = -\frac{1}{2\kappa^2} \int d^{10}x \sqrt{\det G} e^{-2\phi} \left(R + 4(\partial\phi)^2 - \tfrac{1}{2}\,|H|^2 - \tfrac{\alpha'}{4}\,{\rm tr}|F|^2\right)\,. \tag{4.60}$$

The translation to the Lie algebroid is again straight-forward using (4.46). In particular, the gauge connection $\widehat{A} \in \Gamma(A^* \otimes \mathfrak{g})$ on the Lie algebroid is related to the one on the tangent bundle by applying the anchor to the form-part:

$$\widehat{A} = \widehat{A}^{\mathsf{a}} T^{\mathsf{a}} = \rho^t(A^{\mathsf{a}}) T^{\mathsf{a}}\,.$$

Then $\widehat{F} = d_{\mathcal{A}}\widehat{A} - i\widehat{A} \wedge \widehat{A}$ which implies $|\widehat{F}|^2 = |F|^2$. This procedure also leaves the trace unaltered. Moreover, the spin connection on the Lie algebroid with $g = \delta_{\mu\nu}\widehat{e}^\mu \otimes \widehat{e}^\nu$, which implies $\widehat{e}^\mu = \rho^t e^\mu$, is given by

$$\widehat{\omega}^\mu{}_\nu = \widehat{e}^\mu_\alpha\, d_{\mathcal{A}}\widehat{e}^\alpha_\nu + \widehat{e}^\mu_\alpha \widehat{e}^\beta_\nu\, \widehat{\omega}^\alpha{}_\beta = \rho^t(\omega^\mu{}_\nu) \tag{4.61}$$

with $\widehat{\omega}_\alpha = \widehat{\nabla} e_\alpha$. Then, the redefined action is given by

$$\widehat{\mathcal{S}}_{\rm het} = -\frac{1}{2\kappa^2} \int \widehat{\rm vol}\, e^{-2\phi} \left(\widehat{R} + 4(D\phi)^2 - \tfrac{1}{2}\,|\widehat{H}|^2 - \tfrac{\alpha'}{4}\,{\rm tr}|\widehat{F}|^2\right) \tag{4.62}$$

with the hatted quantities defined in the straight-forward manner.

This completes the translation of supergravity theories to Lie algebroids. They exemplify the general procedure of formulating geometric theories on Lie algebroids.

[8] A connection one-form on an n-dimensional vector bundle V is a ${\rm End}(V)$-valued one form. In the language of principle bundles ${\rm End}(V)$ can be considered the Lie algebra associated to the general structure group $GL(n)$. Since the manifold is assumed to be oriented and equipped with a metric the structure group reduces to $SO(n)$. Hence the spin connection takes values in its Lie algebra $so(n)$.

4.3 Summary and discussion

This chapter was devoted to the geometrical description of non-geometric backgrounds. In the first part the algebraic structure for describing the T-dual fluxes appearing in the duality chain (3.2.5) was developed. It is composed of two quasi-Lie algebroids – one, the H-twisted Lie bracket, for the geometric sector of duality and another, the H-twisted Koszul bracket, for the non-geometric sector. They combine into a proto-Lie bialgebroid which allows to construct a Courant algebroid including both sectors at once. In particular, the Bianchi identities for the fluxes follow from the Jacobi identity of the Courant algebroid. However, as has been shown is section 2.2.2, it is not possible to derive a unified gravity theory for all these fluxes based on the Courant algebroid structure.

Consequently, the second part approached the question of describing non-geometric backgrounds from the Lie algebroid perspective, which – as opposed to a description in terms of Courant algebroids – allows for formulating consistent gravity theories. Since transition functions for a background represented by the generalized metric are elements of $O(d,d)$, the field redefinition (3.20) can be used to describe the background in different patches. It induces Lie algebroids which are suitable for describing the geometry in the different patches. Most importantly, they govern a thorough transition of all the symmetries of the model and render the immediate identification of the theories on different patches on account of theorem 2.17. Against this background the description of transitions in terms of Lie algebroids appears to be the most natural and efficient organizing principle. In particular, the formalism extends to supergravities and higher-order corrections.

The connection between the two parts of this chapter is the appearance of a geometric and a non-geometric sector. Whereas the approach via Courant algebroids unifies both sectors, they can be formulated separately in the approach by means of Lie algebroids. The geometric sector is given by the H-twisted Lie-bracket (4.3). It describes the Lie algebroid $(TM, [\cdot,\cdot], \mathrm{id}_{TM})$ if $H = 0$. Is is important to note that the three-form responsible for the twist does not have to coincide with the field strengths of the Kalb-Ramond field; the naming is conventional. Thus the geometric sector is described by the standard action (4.14), which is the Lie-algebroid gravity (4.35) on the latter Lie algebroid. The non-geometric sector is described by the H-twisted Koszul bracket (4.6), which describes the Lie algebroid $(T^*M, [\cdot,\cdot]_K^H, \beta^\sharp)$ if the defining bivector field is a quasi-Poisson structure. The geometry is described in the second example of section 4.2.3 and its Lie-algebroid gravity is again described by (4.35). Hence in the Lie algebroid approach either the geometric or the non-geometric sector is described.

In general, the lack of a global description of non-geometric backgrounds is closely connected with the absence of a proper differential geometry for Courant algebroids. In the next chapter the problem is embarked on from a field theoretic perspective by imposing duality invariance from the very beginning.

Chapter 5

T-duality invariant CFT

A background preserves the classical world-sheet symmetries – in particular the conformal symmetry of the sigma model (3.1) in conformal gauge – during quantization if it satisfies the equations (3.58). Hence a proper string background gives rise to a two-dimensional conformal quantum field theory (CFT) (see [140, 141] for introductions). In particular, two-dimensional CFT's admit an infinite-dimensional symmetry algebra imposing strong restrictions which leave these theories rather simple.

In this chapter the powerful language of two-dimensional CFTs is employed to study first simple features of manifest duality invariant theories. This bottom-up approach is complementary to the geometric approach taken in the previous chapters and follows [102]. In particular, a T-duality invariant theory will be given in terms of the conserved currents, the propagators and duality invariant operators, which is inspired by [81, 82].

First the CFT description of the sigma model (3.1) on a spherical world-sheet Σ and a flat target-space M is given and its behavior under $O(d,d)$-duality is explored. Then, a proposal for the duality invariant theory including the study of its simplest states is presented. From the spacetime perspective, manifest duality is implemented by the introduction of a second, dual set of coordinates. This can be interpreted as "doubling" of the spacetime, which engenders additional degrees of freedom. Hence a consistent analysis is expected to reveal constraints to treat these unphysical redundancies. For finding restrictions to the space of states, the partition function of the theory on a toroidal world-sheet is studied which corresponds to a string-theoretic one-loop analysis. Then the lightest states of the theory are scattered. The premise of having physical intermediate states consistent with the one-loop analysis then leads to the strong constraint (1.2) of double field theory for an uncompactified theory. This provides a derivation of this ad-hoc restriction. The procedure is repeated assuming certain directions of the target space being compactified to a torus. Finally the associated effective theory is related to double field theory [91, 142, 143].

5.1 The free bosonic CFT and duality

The CFT to be studied arises from the sigma model (3.1) by the following restrictions. The world-sheet is assumed to be the two-sphere $\Sigma = S^2$ and the background is taken to be flat without a Kalb-Ramond field, i.e. G constant and $B = 0$; by (3.58) this is a conformal background. Moreover, the world-sheet is Euclideanized by Wick-rotating the τ

coordinate to $t = i\tau$. Then another change of coordinates is imposed in order to map the sphere to the completed complex plane: $z = \exp(t - i\sigma)$. This allows to write the sigma model (3.1) as

$$S(X;G) = \frac{1}{2\pi\alpha'} \int_\Sigma dz d\bar{z}\, G_{ab}\, \partial X^a\, \bar{\partial} X^b \tag{5.1}$$

with the abbreviations $\partial \equiv \frac{\partial}{\partial z}$ and $\bar{\partial} \equiv \frac{\partial}{\partial \bar{z}}$. The classical features of the action simplify significantly under the current assumptions.

- The equations of motion (3.2) become the wave equation

$$\partial\bar{\partial} X^a(z,\bar{z}) = 0 \tag{5.2}$$

  with the general solution being the splitting into a holomorphic (left-moving) and an anti-holomorphic (right-moving) part: $X^a(z,\bar{z}) = X_L^a(z) + X_R^a(\bar{z})$; it can be expanded in modes as

$$\begin{aligned} X_L^a(z) &= q_L^a - \tfrac{i\alpha'}{2}\, k_L^a\, \ln z + i\sqrt{\tfrac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^a}{n z^n}\,, \\ X_R^a(\bar{z}) &= q_R^a - \tfrac{i\alpha'}{2}\, k_R^a\, \ln \bar{z} + i\sqrt{\tfrac{\alpha'}{2}} \sum_{n\neq 0} \frac{\bar{\alpha}_n^a}{n \bar{z}^n}\,. \end{aligned} \tag{5.3}$$

  $k_{L/R} \equiv (p \pm w)$ denotes the left/right-moving momentum with $p^a \sim \int d\sigma P^a$ and $w^a \sim \int d\sigma W^a$ the canonical momentum and winding introduced in section 3.1.[1] The fundamental solution $\mathcal{G}^{ab}(z_1, z_2)$ to (5.2) has to satisfy $\partial_{z_1}\bar{\partial}_{z_1}\mathcal{G}^{ab}(z_1,z_2) = -\pi\alpha' G^{ab}\delta(z_1 - z_2)$. Using $\partial(1/\bar{z}) = 2\pi\delta(z)$ in the sense of distributions the fundamental solution becomes

$$\mathcal{G}^{ab}(z_1,z_2) = \langle X^a(z_1,\bar{z}_1)\, X^b(z_2,\bar{z}_2)\rangle = -\tfrac{\alpha'}{2}\, G^{ab}\, \ln|z_{12}|^2 \tag{5.4}$$

  with $z_{ij} = z_i - z_j$. The bracket $\langle \dots \rangle$ abbreviates the path integral with respect to (5.1).

- The non-vanishing components of the energy-momentum tensor imposing the constraints (3.3) are

$$T(z) = -\tfrac{1}{\alpha'}\, G_{ab}\, \partial X^a(z)\, \partial X^b(z) \qquad \text{and} \qquad \bar{T}(\bar{z}) = -\tfrac{1}{\alpha'}\, G_{ab}\, \bar{\partial} X^a(\bar{z})\, \bar{\partial} X^b(\bar{z}) \tag{5.5}$$

  with the prefactors set for convenience. It gives rise to an infinite set of conserved currents $f(z)T(z)$ for any holomorphic function f and similarly for the anti-holomorphic part which are associated to infinitesimal conformal transformations $z \to z + f(z)$. Since the world-sheet is assumed to be a sphere, the holomorphic functions are not arbitrary but have to be invariant under $\sigma \to \sigma + 2\pi n$; for the complex coordinates

[1] This can be seen by inserting (5.3) into P_a and W_a and an integration over σ. In complex coordinates, this translates to two integrations according to $d\sigma = \frac{i}{2}(dz/z - d\bar{z}/\bar{z})$ over a closed contour around the origin and requires careful treatment of the orientation and Cauchy's theorem.

this amounts to $z \to e^{-2\pi n i} z = z$. Thus the functions restrict to polynomials in z which are spanned by monomials $\{z^n | n \in \mathbb{Z}\}$. The associated conserved charges are given by the *Virasoro generators*

$$L_n = \oint_{C_0} \frac{dz}{2\pi i} z^{n+1} T(z) \qquad \Longleftrightarrow \qquad T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}} \,. \tag{5.6}$$

C_0 denotes a closed curve around the origin and the equivalence follows from Cauchy's theorem. The modes associated to $\bar{T}(\bar{z})$ are denoted $\bar{L}_n$.

Classically the energy momentum tensor (5.5) has to vanish; this can not be imposed for the associated operator. Instead, a primary field $\phi(z, \bar{z})$ of conformal weight $(h, \bar{h})$[2] is *physical* if $h = \bar{h} = 1$; in particular, $h = \bar{h}$ which is called *level matching*. These physical states satisfy the quantum analogue of the constraint (3.3).[3]

Duality

Now the effect of $O(d,d)$-duality on the free bosonic CFT is studied. In particular, the dual propagator (5.4) and the dual energy momentum tensor (5.5) are highlighted as they determine the physical field content and correlation functions of the theory. Since G is constant and $B = 0$, duality is simple: The integrability conditions (3.25) only allow for $O(d,d)$-transformations $\mathcal{T}$ with the components t_{12} and t_{22} constant. Therefore the dual coordinates (3.24) can be integrated to give

$$\begin{aligned} \widetilde{X}^{\bar{a}}_L(z) &= \left[(t_{22})^{\bar{a}}{}_a + (t_{12})^{\bar{a}m} G_{ma}\right] X^a_L(z) \,, \\ \widetilde{X}^{\bar{a}}_R(\bar{z}) &= \left[(t_{22})^{\bar{a}}{}_a - (t_{12})^{\bar{a}m} G_{ma}\right] X^a_R(\bar{z}) \,; \end{aligned} \tag{5.8}$$

cf. (3.29).This can be seen by noting that $\star dz = dz$ and $\star d\bar{z} = -d\bar{z}$ for the complex coordinates. The new background (3.20) is comprised of a metric g and a constant Kalb-Ramond field. The latter can be neglected as it contributes a total derivative to the action and the former is related to the initial metric by

$$\begin{aligned} g^{\bar{a}\bar{b}} &= \left[(t_{22})^{\bar{a}}{}_a + (t_{12})^{\bar{a}m} G_{ma}\right]\left[(t_{22})^{\bar{b}}{}_b + (t_{12})^{\bar{b}n} G_{nb}\right] G^{ab} \\ &= \left[(t_{22})^{\bar{a}}{}_a - (t_{12})^{\bar{a}m} G_{ma}\right]\left[(t_{22})^{\bar{b}}{}_b - (t_{12})^{\bar{b}n} G_{nb}\right] G^{ab} \,. \end{aligned} \tag{5.9}$$

[2] An operator $\phi(z, \bar{z})$ is primary if its operator product expansion (OPE) with the energy momentum tensor reads

$$T(z_1)\phi(z_2, \bar{z}_2) = \frac{h\,\phi(z_2, \bar{z}_2)}{z_{12}^2} + \frac{\partial\phi(z_2, \bar{z}_2)}{z_{12}} + \text{reg.} \tag{5.7}$$

and similar for the anti-holomorphic part. The equation has to be understood as being inserted radially ordered into the path integral with further arbitrary operators. h denotes the conformal weight. Primary fields transform correctly under conformal transformations; thus they are studied in CFT.

[3] Inserting the mode expansion (5.6) into (5.7) gives $[L_n, \phi(z, \bar{z})] = z^{n+1}\partial\phi + (n+1)hz^n\phi$ upon using Cauchy's theorem for ϕ primary. The state $|\phi\rangle$ associated to a primary operator ϕ is given by $\lim_{z,\bar{z}\to 0} \phi(z, \bar{z})|0\rangle$ with $|0\rangle$ the ground state. Regularity of $T(z)$ at $z = 0$ requires $L_n|0\rangle = 0 \;\forall n > -2$. Using the commutator above then implies $L_0|\phi\rangle = h|\phi\rangle$ and $L_n|\phi\rangle = 0 \;\forall n > 0$. Thus demanding in particular $L_0|\phi\rangle = 0$ in order to satisfy the constraint (3.3) is not consistent. Instead, level matching $(L_0 - \bar{L}_0)|\phi\rangle = 0$ as well as $(L_0 - 1)|\phi\rangle = 0$ is required.

The inverse is given for convenience. The equality of the upper and the lower line can be seen from their difference

$$2\big[(t_{22})^{\bar{a}}{}_m (t_{12})^{\bar{b}m} + (t_{12})^{\bar{a}m}(t_{22})^{\bar{b}}{}_m\big] = 2\big[t^t_{22}t_{12} + t^t_{12}t_{22}\big] = 2\big[(\mathcal{T}^t\eta\mathcal{T})_{\mathrm{lr}}\big] = 2\,\eta_{\mathrm{lr}} = 0$$

with the index lr denoting the lower right $d\times d$-block of the matrix which vanishes because $\mathcal{T}\in O(d,d)$. Then on the dual background the propagator (5.4) reads

$$\big\langle \widetilde{X}^{\bar{a}}_L\,\widetilde{X}^{\bar{b}}_L\big\rangle = -\tfrac{\alpha'}{2}\,g^{\bar{a}\bar{b}}\ln(z-w) \qquad \Longleftrightarrow \qquad \big\langle X^a_L\,X^b_L\big\rangle = -\tfrac{\alpha'}{2}\,G^{ab}\ln(z-w)\,. \tag{5.10}$$

The propagator (5.4) has been split into a holomorphic and an anti-holomorphic part. The equivalence follows from combining the first lines of (5.8) and (5.9). The dual energy-momentum tensor reads

$$\widetilde{T}(z) = -\tfrac{1}{\alpha'}\,g_{\bar{a}\bar{b}}\,\partial\widetilde{X}^{\bar{a}}(z)\,\partial\widetilde{X}^{\bar{b}}(z) = -\tfrac{1}{\alpha'}\,G_{ab}\,\partial X^a(z)\,\partial X^b(z) = T(z)\,, \tag{5.11}$$

i.e. it remains unchanged. The same holds for the anti-holomorphic parts. Hence the CFT on the initial background is equivalent to its $O(d,d)$-dual.

From now on $G=\delta=\mathrm{diag}(1,\dots,1)$ is assumed.[4] In total, duality interchanges the coordinates X^a with the dual coordinates $\widetilde{X}^a = \widetilde{X}^a_L + \widetilde{X}^a_R$ (5.8) while interchanging the canonical momentum $p^a = \frac{1}{2}(k_L + k_R)$ with its dual

$$\widetilde{p}^{\bar{a}} = (t_{22})^{\bar{a}}{}_a p^a + (t_{12})^{\bar{a}m}\delta_{ma}w^a\,. \tag{5.12}$$

For the generating classes of $O(d,d)$ (2.6) and (2.7) the present observations read as follows.

- For changes of frame (3.47) and B-transformations (2.2) nothing changes.
- For T-duality in the k$^{\text{th}}$ direction (2.7) the coordinate changes as

  $$\widetilde{X}^k(z,\bar{z}) = \widetilde{X}^k_L(z) + \widetilde{X}^k_R(\bar{z}) = X^k_L(z) - X^k_R(\bar{z}) \tag{5.13}$$

  by (5.8). The other directions remain unchanged. Using (5.9) also the metric $g=\delta$ is unaltered. Thus T-duality acts by reflecting the right-moving coordinate in the dualized direction while interchanging p and w due to (5.12): $\widetilde{p}^k = w^k$.
- For a β-transformation (2.12) the dual coordinates (5.8) become

  $$\widetilde{X}^{\bar{a}}(z,\bar{z}) = X^{\bar{a}}(z,\bar{z}) + \beta^{\bar{a}m}\delta_{ma}\left[X^a_L(z) - X^a_R(\bar{z})\right]\,. \tag{5.14}$$

  The new inverse metric can be determined by (5.9); it reads

  $$g^{\bar{a}\bar{b}} = \delta^{\bar{a}\bar{b}} - \beta^{\bar{a}m}\delta_{mn}\beta^{n\bar{b}}\,. \tag{5.15}$$

  The canonical momentum (5.12) becomes $\widetilde{p}^{\bar{a}} = p^{\bar{a}} + \beta^{\bar{a}m}\delta_{mn}w^n$.

Although $O(d,d)$-duality leaves the CFT invariant in general, T-duality is particularly simple. Thus in the following T-duality will be considered for simplicity. Remarks about the generalization can be found in the conclusion 6.

[4] Fluctuations around this trivial metric are included in the CFT by certain vertex operators and their scattering. Hence this is not a restriction in the end.

5.2 T-duality invariant CFT

T-duality acts by reflecting the right-moving coordinate, interchanges momentum and winding and leaves the metric on the present simple background invariant. In particular, it does not alter the propagators for the left- and right-moving coordinates as well as the energy-momentum tensor. Therefore, treating X and $\widetilde{X}$ or equivalently left- and right-movers on equal footing makes T-duality manifest. The resulting theory is not governed by the sigma model (5.1) anymore. The propagators for the standard and dual coordinates are

$$\begin{aligned} \big\langle X^a(z_1,\bar{z}_1)X^b(z_2,\bar{z}_2)\big\rangle &= -\frac{\alpha'}{2}\,\delta^{ab}\ln|z_{12}|^2\,, \\ \big\langle \widetilde{X}^a(z_1,\bar{z}_1)\widetilde{X}^b(z_2,\bar{z}_2)\big\rangle &= -\frac{\alpha'}{2}\,\delta^{ab}\ln|z_{12}|^2\,, \\ \big\langle X^a(z_1,\bar{z}_1)\widetilde{X}^b(z_2,\bar{z}_2)\big\rangle &= -\frac{\alpha'}{2}\,\delta^{ab}\ln\frac{z_{12}}{\bar{z}_{12}}\,; \end{aligned} \tag{5.16}$$

this follows from (5.4) and (5.13). In the following, the most elementary physical states of this theory are determined without referring to any compactified directions. The presence of compact directions will be studied in section 5.5. The difference is that in contrast to the former case, the latter requires momentum and winding to be quantized.

Vertex operators and descendants

The manifest duality-invariant primary field solely containing the coordinate fields is

$$V_{p,w}(z,\bar{z}) =: e^{ip_aX^a(z,\bar{z})}\,e^{iw_a\widetilde{X}^a(z,\bar{z})}:\,, \tag{5.17}$$

which will be called tachyon in the following. It is a primary field of weight

$$(h,\bar{h}) = \left(\frac{\alpha'}{4}(p+w)^2, \frac{\alpha'}{4}(p-w)^2\right) \tag{5.18}$$

which results in the mass

$$M^2 = -\frac{2}{\alpha'}(h+\bar{h}) = -(p^2+w^2)\,. \tag{5.19}$$

The OPE of two such fields is

$$\begin{aligned} V_{p_1,w_1}(z_1,\bar{z}_1)\,V_{p_2,w_2}(z_2,\bar{z}_2) =& |z_{12}|^{\alpha'(p_1\cdot p_2+w_1\cdot w_2)}\left(\frac{z_{12}}{\bar{z}_{12}}\right)^{\frac{\alpha'}{2}(p_1\cdot w_2+w_1\cdot p_2)} \\ &\times V_{p_1+p_2,w_1+w_2}(z_2,\bar{z}_2)+\dots\,, \end{aligned} \tag{5.20}$$

and admits a logarithmic branch point whose absence (locality) requires the quantization condition

$$\alpha'(p_1\cdot w_2+w_1\cdot p_2)\in\mathbb{Z}\,. \tag{5.21}$$

The first descendant states of (5.17) are as follows:

- At the first excited level one has a *form field* $\mathcal{A}_{p,w}$ and its complex conjugate $\bar{\mathcal{A}}_{p,w}$

$$\begin{aligned}\mathcal{A}_{p,w}(z,\bar{z}) &= A_a :\partial X^a(z)\, V_{p,w}(z,\bar{z}): \,, \\ \bar{\mathcal{A}}_{p,w}(z,\bar{z}) &= \bar{A}_a :\bar{\partial} X^a(\bar{z})\, V_{p,w}(z,\bar{z}):\end{aligned} \tag{5.22}$$

with A and $\bar{A}$ one-forms. For heterotic torus compactifications these states give rise to the well-known enhancement of the gauge group. $\mathcal{A}$ is primary with conformal weight $(h,\bar{h}) = (1+\frac{\alpha'}{4}(p+w)^2, \frac{\alpha'}{4}(p-w)^2)$ if it is transversely polarized in the sense $A_a(p^a+w^a)=0$. Similarly, $\bar{\mathcal{A}}$ is primary with $(h,\bar{h}) = (\frac{\alpha'}{4}(p+w)^2, 1+\frac{\alpha'}{4}(p-w)^2)$ for $\bar{A}_a(p^a-w^a)=0$.

- At the next level one finds a $(0,2)$-*tensor field* $\mathcal{E}_{p,w}$

$$\mathcal{E}_{p,w}(z,\bar{z}) = E_{ab} :\partial X^a(z)\, \bar{\partial} X^b(\bar{z})\, V_{p,w}(z,\bar{z}): \tag{5.23}$$

with the polarization E_{ab}. It is a primary field with $(h,\bar{h}) = (1+\frac{\alpha'}{4}(p+w)^2, 1+\frac{\alpha'}{4}(p-w)^2)$ for transverse polarization in the sense $E_{ab}(p^a+w^a)=0=E_{ab}(p^b-w^b)$.

In section 5.6 it is shown that string scattering amplitudes of three such states (5.23) can be matched precisely with interactions in DFT. This gives credence to the usage of this duality invariant CFT as a two-dimensional world-sheet model of DFT.

In order to be consistent with the constraints, the states considered above have to be physical. The resulting restrictions are shown in table 5.1.

state	level-matching	primary	mass
$V_{p,w}$	$p\cdot w=0$	—	$M^2=-\frac{4}{\alpha'}$
$\mathcal{A}_{p,w}$	$p\cdot w=-\frac{1}{\alpha'}$	$A_m(p^m+w^m)=0$	$M^2=-\frac{2}{\alpha'}$
$\bar{\mathcal{A}}_{p,w}$	$p\cdot w=\frac{1}{\alpha'}$	$\bar{A}_m(p^m-w^m)=0$	$M^2=-\frac{2}{\alpha'}$
$\mathcal{E}_{p,w}$	$p\cdot w=0$	$E_{mn}(p^m+w^m)=0=E_{mn}(p^n-w^n)$	$M^2=0$

Table 5.1: The physical state condition requires the operators to be level-matched primaries of conformal weight $(1,1)$. This sets the mass of the states.

Clearly $V_{p,w}$ corresponds to a negative mass^2 state, i.e., as expected, it is a tachyon. Moreover, the two states $\mathcal{A}_{p,w}$ and $\bar{\mathcal{A}}_{p,w}$ are tachyonic as well. Finally, $\mathcal{E}_{p,w}$ is massless and therefore, depending on the polarization, gives the graviton, the B-field and the dilaton. Next the one-loop partition function is considered whose modular invariance imposes additional constraints by relating the holomorphic with the anti-holomorphic sector.

5.3 The one-loop partition function

In this section the torus partition function for the CFT introduced above is computed with a particular emphasis on the modular properties. For a CFT defined on the world sheet torus with modular parameter τ and Hilbert space $\mathfrak{H}$, the partition function is given by

$$\begin{aligned}Z(\tau,\bar{\tau}) &= \mathrm{tr}_{\mathfrak{H}}\big(q^{L_0-\frac{c}{24}}\,\bar{q}^{\bar{L}_0-\frac{c}{24}}\big) \\ &= e^{\frac{\pi c}{6}\mathrm{Im}(\tau)}\,\mathrm{tr}_{\mathfrak{H}}\big(e^{-2\pi(L_0+\bar{L}_0)\mathrm{Im}(\tau)}e^{2\pi i(L_0-\bar{L}_0)\mathrm{Re}(\tau)}\big)\end{aligned} \tag{5.24}$$

with $q = e^{2\pi i\tau}$. The trace is taken over the whole Hilbert space $\mathfrak{H}$ which beyond the oscillator modes also includes the continuous trace over momenta and windings. $\mathfrak{H}$ itself is built upon the highest weight state $|p,w\rangle = \lim_{z,\bar{z}\to 0} V_{p,w}(z,\bar{z})|0\rangle$ by acting with the Virasoro generators L_n (5.6). Using $L_0|p,w\rangle = \frac{\alpha'}{4}(p+w)^2|p,w\rangle$ and $\bar{L}_0|p,w\rangle = \frac{\alpha'}{4}(p-w)^2|p,w\rangle$, the continuous trace can be evaluated as

$$\begin{aligned} f(\tau,\bar{\tau}) &= \int \frac{d^dp}{(2\pi)^d} \int \frac{d^dw}{(2\pi)^d} \langle p,w| e^{-\pi\alpha'(p^2+w^2)\mathrm{Im}(\tau)} e^{2\pi i\alpha' p\cdot w \mathrm{Re}(\tau)} |p,w\rangle \\ &= \langle p,w|p,w\rangle \, \frac{1}{2} \left(\int \frac{d^dk_L}{(2\pi)^d} \, e^{i\frac{\pi}{2}\alpha'\, k_L^2\, \tau} \right) \left(\int \frac{d^dk_R}{(2\pi)^d} \, e^{-i\frac{\pi}{2}\alpha'\, k_R^2\, \bar{\tau}} \right) . \end{aligned} \tag{5.25}$$

The evaluation of the trace over the oscillator part is as usual so that altogether one obtains

$$Z(\tau,\bar{\tau}) = \frac{f(\tau,\bar{\tau})}{|\eta(\tau)|^{2d}} \, . \tag{5.26}$$

As a Riemannian surface the two-torus is invariant under the modular group $\mathrm{PSL}(2,\mathbb{Z})$ of integer-valued two-dimensional projective special linear transformations. They act on the modular parameter τ as

$$\tau \to \frac{a+b\tau}{c+d\tau} \qquad \text{with } a,b,c,d \in \mathbb{Z}\,; \; ac-bd=1\,. \tag{5.27}$$

It suffices to check modular invariance for its generators, the T-transformation $\tau \to \tau+1$ and the S-transformation $\tau \to -\frac{1}{\tau}$. For $\mathrm{Im}(\tau) > 0$ the integral (5.25) can be evaluated to be proportional to $|\tau|^{-d}$; this is not invariant under a modular T-transformation while $|\eta(\tau)|$ is invariant itself. T-invariance yields the level matching condition

$$\alpha'\, p\cdot w \in \mathbb{Z} \qquad \Longleftrightarrow \qquad \frac{\alpha'}{4}(k_L^2 - k_R^2) \in \mathbb{Z}\,, \tag{5.28}$$

i.e. the two integrals are not independent. Writing $k_R^2 = k_L^2 - \frac{4}{\alpha'}m$ for an integer m, level matching can be imposed by including a factor $\delta(k_L^2 - k_R^2 - \frac{4}{\alpha'}m)$ in (5.25). Then, to evaluate the remaining integral in (5.25), d-dimensional spherical coordinates with radius $|k_R|$ are introduced; up to constant factors one is left with

$$f(\tau,\bar{\tau}) \sim e^{2\pi i\, m\,\tau} \int \frac{d^dk_L}{(2\pi)^d} \, |k_L|^{d-1} \, e^{-\pi\alpha'\, k_L^2\, \mathrm{Im}(\tau)} \sim \frac{\Gamma\left(d-\frac{1}{2}\right)}{\mathrm{Im}(\tau)^{\frac{d}{2}}} \, \frac{e^{2\pi i\, m\,\tau}}{\mathrm{Im}(\tau)^{\frac{d-1}{2}}} \tag{5.29}$$

for $\mathrm{Im}(\tau) > 0$, which is T-invariant. However, realizing that $\mathrm{Im}(\tau)^{\frac{d}{2}}|\eta(\tau)|^{2d}$ is already S-invariant, invariance under a modular S-transformation is spoiled by the second factor $e^{2\pi i\, m\,\tau}\, \mathrm{Im}(\tau)^{\frac{1-d}{2}}$ in (5.29).

For the unwanted factor in (5.29) to be absent and for obtaining a modular invariant result, the second integral in (5.25) has to evaluate to

$$\int \frac{d^dk_R}{(2\pi)^d} \, e^{-i\frac{\pi}{2}\alpha'\, k_R^2\, \bar{\tau}} \, \delta(k_L, k_R) = g(\bar{\tau}) \, e^{-i\frac{\pi}{2}\alpha'\, k_L^2\, \bar{\tau}} \, . \tag{5.30}$$

$\delta(k_L, k_R)$ implements relations between the momenta to be determined and $g(\bar\tau)$ is a modular function independent of the momenta. Thus the modular function is given by

$$g(\bar\tau) = \int \frac{d^d k_R}{(2\pi)^d}\, e^{i\frac{\pi}{2}\alpha'(k_L^2 - k_R^2)\bar\tau}\, \delta(k_L, k_R) = e^{2\pi i m\,\bar\tau} \int \frac{d^d k_R}{(2\pi)^d}\, \delta(k_L, k_R)\,, \tag{5.31}$$

where level matching was used. In (5.31), $g(\bar\tau)$ factorizes into a $\bar\tau$-dependent factor and a momentum dependent one. The former is not modular invariant unless $m = 0$. The remaining integral over the momentum must be constant, i.e. $\delta(k_L, k_R)$ has to be of the form $\delta^d(k_R - F(k_L))$, with F a vector-valued function.

F can be determined as follows. Since $m = 0$, level-matching (5.28) can be written as

$$\begin{pmatrix} k_L \\ k_R \end{pmatrix}^t \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} k_L \\ k_R \end{pmatrix} \equiv \langle K, K\rangle_d = 0 \tag{5.32}$$

and is invariant under $O(d,d)$-transformations[5] of the vector $K = (k_L, k_R)^t$. Hence, to maintain level-matching while having a relation between the left- and right-moving momentum requires them to be related by an $O(d,d)$-transformation. Thus the general form of F can be constructed by rotating the most simple solution $k_R = k_L$. An $O(d,d)$ transformation $\mathcal{T} \in O(d,d)$ satisfies $\mathcal{T}^t \mathrm{diag}(\mathbb{1}, -\mathbb{1})\mathcal{T} = \mathrm{diag}(\mathbb{1}, -\mathbb{1})$ so that in particular the transpose satisfies

$$\mathcal{T}^t = \begin{pmatrix} t_{11}^t & t_{21}^t \\ t_{12}^t & t_{22}^t \end{pmatrix} \in O(d,d) \quad \Longleftrightarrow \quad \begin{cases} t_{11}t_{11}^t - t_{12}t_{12}^t = \mathbb{1} \\ t_{21}t_{21}^t - t_{22}t_{22}^t = -\mathbb{1} \\ t_{11}t_{21}^t - t_{12}t_{22}^t = 0 \end{cases} . \tag{5.33}$$

Acting with $\mathcal{T}$ on $(k_L, k_R)^t$ modifies the simple solution according to

$$\{k_R = k_L\} \mapsto \{t_{21}k_L + t_{22}k_R = t_{11}k_L + t_{12}k_R\} \Longleftrightarrow \{k_R = (t_{22} - t_{12})^{-1}(t_{11} - t_{21})k_L\}\,.$$

Using the conditions (5.33) for the matrix elements of $\mathcal{T}^t$ gives $(t_{22}-t_{12})^{-1}(t_{11}-t_{21}) \in O(d)$. Therefore, the conditions for modular T- and S-invariance imply that the right and left momenta are related by an $O(d)$ transformation as

$$k_R = \mathcal{M}\, k_L \quad \text{with} \quad \mathcal{M} \in O(d)\,. \tag{5.34}$$

Having shown that modular invariance requires the insertion of $(2\pi)^d\delta^d(k_R - \mathcal{M}k_L)$ and denoting $\langle p, w|p, w\rangle = V_d$, the final torus partition function reads

$$Z(\tau, \bar\tau) = \frac{V_d/2}{(2\pi\sqrt{\alpha'})^d\, \mathrm{Im}(\tau)^{\frac{d}{2}}\, |\eta(\tau)|^{2d}}\,. \tag{5.35}$$

This section closes with the following three remarks:

- In terms of momentum and winding, (5.34) enforces $w = 0$ for $\mathcal{M} = \mathbb{1}$ and $p = 0$ for $\mathcal{M} = -\mathbb{1}$.

[5] In the variables $k_{L/R}$ the bilinear form η defining $O(d,d)$ is diagonalized. In terms of p and w the group is defined as in the previous chapters.

- Invariance under modular T-transformation implied $\alpha' p \cdot w \in \mathbb{Z}$, while only the additional invariance under the modular S-transformation really led to the weak constraint $p \cdot w = 0$.
- The latter truncates the spectrum as only those states are allowed whose number of left- and right- oscillator excitations match. Comparison with table 5.1 therefore shows that in particular $\mathcal{A}_{p,w}$ and its complex conjugate are forbidden.

In string theory, the partition function is related to the one-loop vacuum polarization diagram with all string excitations running in the loop. In order to detect further constraints, one also needs to consider string diagrams containing momenta and winding of many states. For this reason, in the next section the string scattering amplitude of four tachyons is considered.

5.4 Tachyon scattering and the strong constraint

In the T-duality invariant CFT the correlation function of N tachyon vertex operators $V_{p_i,w_i}(z_i, \bar{z}_i) \equiv V_i$ can be straightforwardly computed as

$$\big\langle V_1 \dots V_N \big\rangle = \prod_{1 \le i < j \le N} |z_{ij}|^{\alpha'(p_i \cdot p_j + w_i \cdot w_j)} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\frac{\alpha'}{2}(p_i \cdot w_j + w_i \cdot p_j)} \delta\Big(\sum p_i\Big)\delta\Big(\sum w_i\Big)\,. \tag{5.36}$$

The difference to the standard tachyon correlator is the $\frac{z_{ij}}{\bar{z}_{ij}}$-factor. As being an amplitude on the two-sphere, it has to be invariant under the conformal group $SL(2,\mathbb{C})$. This means that (5.36) has to be independent of the order of the inserted operators. Although obvious from an abstract point of view, $SL(2,\mathbb{C})$-invariance will be checked explicitly for the orders of interest in the following. After having confirmed the consistency, the duality invariant Virasoro-Shapiro amplitude is computed and its pole structure studied.

$SL(2,\mathbb{C})$-invariance

The correlator (5.36) was computed using primary fields; therefore it must be invariant under the conformal group. Due to the $\frac{z_{ij}}{\bar{z}_{ij}}$-factors this is not obvious anymore and will be illustrated by discussing the first three orders. Non-vanishing of (5.36) implies momentum as well as winding conservation

$$\sum_{i=1}^{N} p_i = 0 = \sum_{i=1}^{N} w_i \tag{5.37}$$

as one-point functions vanish generally. Moreover, level matching $p_i \cdot w_i = 0$ as well as the mass-shell condition $(p_i + w_i)^2 = \frac{4}{\alpha'}$ are invoked. The orders of interest are as follows:

- The two-point function of two tachyons reads upon using (5.37) and the mass-shell condition:

$$\begin{aligned} \big\langle V_1\, V_2 \big\rangle &= |z_{12}|^{\alpha'(p_1 \cdot p_2 + w_1 \cdot w_2)} \left(\frac{z_{12}}{\bar{z}_{12}}\right)^{\frac{\alpha'}{2}(p_1 \cdot w_2 + w_1 \cdot p_2)} \big\langle V_{p_1+p_2, w_1+w_2} \big\rangle \\ &= |z_{12}|^{-4} \left(\frac{z_{12}}{\bar{z}_{12}}\right)^{\frac{\alpha'}{2}(-p_1 \cdot w_1 - w_1 \cdot p_1)} \big\langle \mathbb{1} \big\rangle\,. \end{aligned} \tag{5.38}$$

Employing level matching as well, the correlator is proportional to $|z_{12}|^{-4}$ which is expected from conformal invariance.

- The three-point function becomes

$$\left\langle V_1\, V_2\, V_3 \right\rangle = |z_{12}|^{-2}|z_{13}|^{-2}|z_{23}|^{-2}\Big(\frac{z_{12}}{z_{13}\, z_{23}}\,\frac{\bar{z}_{13}\,\bar{z}_{23}}{\bar{z}_{12}}\Big)^{\frac{\alpha'}{2}(p_1\cdot w_2+w_1\cdot p_2)}\left\langle \mathbb{1}\right\rangle . \tag{5.39}$$

Again, the first factor is expected from conformal invariance. The exponent of the inconsistent term is $p_1\cdot w_2 + w_1\cdot p_2 = (p_1+p_2)\cdot(w_1+w_2) = p_3\cdot w_3 = 0$, i.e. it vanishes.

- The 4-point function

$$\left\langle V_1\, V_2\, V_3\, V_4 \right\rangle = \prod_{1\le i,j\le 4} |z_{ij}|^{\alpha'(p_i\cdot p_j + w_i\cdot w_j)}\Big(\frac{z_{ij}}{\bar{z}_{ij}}\Big)^{\frac{\alpha'}{2}(p_i\cdot w_j+w_i\cdot p_j)}\left\langle V_{1+2+3+4}\right\rangle . \tag{5.40}$$

can be simplified using $p_1\cdot w_2 + w_1\cdot p_2 = p_3\cdot w_4 + w_3\cdot p_4$ as well as $p_1\cdot p_2 + w_1\cdot w_2 = p_3\cdot p_4 + w_3\cdot w_4$ and similarly for all the other combinations. The correlator becomes

$$\begin{aligned}\left\langle V_1\, V_2\, V_3\, V_4 \right\rangle = \Big(\frac{\mathsf{X}^{12}_{43}}{\bar{\mathsf{X}}^{12}_{43}}\Big)^{\frac{\alpha'}{2}(p_1\cdot w_2+w_1\cdot p_2)}&\Big(\frac{\mathsf{X}^{14}_{23}}{\bar{\mathsf{X}}^{14}_{23}}\Big)^{\frac{\alpha'}{2}(p_1\cdot w_4+w_1\cdot p_4)}\\ &\times\big|\mathsf{X}^{12}_{43}\big|^{\alpha'(p_1\cdot p_2+w_1\cdot w_2)}\big|\mathsf{X}^{14}_{23}\big|^{\alpha'(p_1\cdot p_4+w_1\cdot w_4)}\left\langle \mathbb{1}\right\rangle\end{aligned} \tag{5.41}$$

with $\mathsf{X}^{ij}_{kl} = \frac{z_{ij}z_{kl}}{z_{il}z_{kj}}$ the $SL(2,\mathbb{C})$-invariant cross ratio, which implies the $SL(2,\mathbb{C})$-invariance of the 4-point function.

This verifies the conformal invariance of the relevant orders.

The duality invariant Virasoro-Shapiro amplitude

Now the full string-theoretic amplitude for tachyons is considered. For N tachyons it is given by

$$\begin{aligned}A_N(p_i,w_i) = g_s^N\, C_{S^2}\int \prod_{i=1}^{N} d^2 z_i\, \textstyle\prod_{j=1}^3 \delta(z_j - z_j^0)\, |z_{12}z_{13}z_{23}|^2 \\ \times \left\langle V_1 \dots V_N\right\rangle(z_1,\dots z_N)\,.\end{aligned} \tag{5.42}$$

Here the conformal group $SL(2,\mathbb{C})$ has been used to fix three of the N insertion points on the sphere. The standard choice is $z_1 = 0$, $z_2 = 1$ and $z_3 \to \infty$. Moreover, (5.42) includes the three c-ghost correlator $\big|\left\langle c(z_1)\, c(z_2)\, c(z_3)\right\rangle\big|^2 = |z_{12}\, z_{23}\, z_{13}|^2$. The latter are included to avoid over-counting of gauge orbits. The prefactors are a factor of the closed string coupling constant g_c for every closed string vertex operator and C_{S^2} accounting for various normalizations (see e.g. [13]).

Three-point amplitude

The three-tachyon amplitude is given by

$$A_3(p_i, w_i) = g_c^3\, C_{S^2} \left\langle (c\,\bar{c}\,V_1)(c\,\bar{c}\,V_2)(c\,\bar{c}\,V_3)\right\rangle = g_c^3\, C_{S^2}\,, \tag{5.43}$$

where the δ-distributions implementing momentum and winding conservation have to be understood as implicit. The three-point amplitude is therefore identical to the standard one for three tachyons without a winding dependence.

Four-point amplitude

Using (5.36) and reordering the monomials, the four-point amplitude reads

$$\begin{aligned} A_4(p_i, w_j) &= g_c^4\, C_{S^2} \int d^2z \left\langle (c\,\bar{c}\,V_1)\,(c\,\bar{c}\,V_2)\,(c\,\bar{c}\,V_3)\,V_4\right\rangle \\ &= g_c^4\, C_{S^2} \int d^2z \Big\{ z^{\alpha'(p_1\cdot w_4 + w_1\cdot p_4)} (1-z)^{\alpha'(p_2\cdot w_4 + w_2\cdot p_4)} \\ &\qquad\qquad \times |z|^{\alpha'(p_1-w_1)\cdot(p_4-w_4)}\, |1-z|^{\alpha'(p_2-w_2)\cdot(p_4-w_4)} \Big\}\,. \end{aligned} \tag{5.44}$$

It is convenient to introduce two sets of Mandelstam variables

$$\begin{aligned} s &= -(k_{L3}+k_{L4})^2 \quad , \qquad \mathfrak{s} = -(k_{R3}+k_{R4})^2 \\ t &= -(k_{L2}+k_{L4})^2 \quad , \qquad \mathfrak{t} = -(k_{R2}+k_{R4})^2 \\ u &= -(k_{L1}+k_{L4})^2 \quad , \qquad \mathfrak{u} = -(k_{R2}+k_{R4})^2 \end{aligned} \tag{5.45}$$

with $s+t+u = \mathfrak{s}+\mathfrak{t}+\mathfrak{u} = -\frac{16}{\alpha'}$ by level matching and the mass-shell condition. The relation between the two sets is given by

$$(k_{Li}+k_{Lj})^2 - (k_{Ri}+k_{Rj})^2 = 4(p_i\cdot w_j + w_i\cdot p_j) \in \frac{4}{\alpha'}\,\mathbb{Z}\,. \tag{5.46}$$

Defining the function $\alpha(s) = -1 - \frac{\alpha'}{4}s$, the amplitude is integrated to

$$A_4(p_i, w_j) = 2\pi\, g_c^4\, C_{S^2} \frac{\Gamma\big(\alpha(s)\big)\,\Gamma\big(\alpha(t)\big)\,\Gamma\big(\alpha(u)\big)}{\Gamma\big(\alpha(\mathfrak{t})+\alpha(\mathfrak{u})\big)\,\Gamma\big(\alpha(\mathfrak{s})+\alpha(\mathfrak{u})\big)\,\Gamma\big(\alpha(\mathfrak{s})+\alpha(\mathfrak{t})\big)}\,. \tag{5.47}$$

Using (5.46), the α's can be related as $\alpha(\mathfrak{s}) = \alpha(s) - n_{34}$, where

$$n_{ij} = \alpha'(p_i\cdot w_j + w_i\cdot p_j) \quad \text{with} \quad n_{14}+n_{24}+n_{34} = 0\,. \tag{5.48}$$

Then, in terms of the left-moving variables the amplitude becomes

$$A_4(p_i, w_j) = \frac{2\pi\, g_c^4\, C_{S^2}\; \Gamma\big(\alpha(s)\big)\,\Gamma\big(\alpha(t)\big)\,\Gamma\big(\alpha(u)\big)}{\Gamma\big(\alpha(t)+\alpha(u)+n_{34}\big)\,\Gamma\big(\alpha(s)+\alpha(u)+n_{24}\big)\,\Gamma\big(\alpha(s)+\alpha(t)+n_{14}\big)}\,. \tag{5.49}$$

A similar expression can be found in terms of right-moving variables.

In contrast to the standard form of the Virasoro-Shapiro amplitude, (5.49) is not symmetric in the s-, t- and u-channel. Channel duality can be retained by requiring $n_{14} = n_{24} = n_{34}$, which due to (5.48) implies $n_{ij} = 0$. In the following, this constraint is argued for in a more rigorous fashion.

Pole structure and the strong constraint

In string theory the poles of the four-tachyon amplitude appear where physical states become on-shell. Thus, they encode the mass spectrum of the theory. Now, $\Gamma(x)$ has no zeros but single poles at $x = -n$ for $n \in \mathbb{N}$ with residue $\frac{(-1)^n}{n!}$. Therefore the n^{th} pole in the s-channel of (5.49) is located at

$$s = \frac{4}{\alpha'}(n-1) \quad \Longleftrightarrow \quad \mathfrak{s} = \frac{4}{\alpha'}(n + n_{34} - 1)\,. \tag{5.50}$$

Hence, $s = -(k_{L3} + k_{L4})^2 \equiv -(k_L^{\text{int}})^2$ and $\mathfrak{s} \equiv -(k_R^{\text{int}})^2$ with $k_{L/R}^{\text{int}} = p^{\text{int}} \pm w^{\text{int}}$ can be considered as describing a physical intermediate state with mass and level-matching condition given by

$$(M^{int})^2 = -\big((p^{\text{int}})^2 + (w^{\text{int}})^2\big) = \frac{4}{\alpha'}\big(n + \frac{n_{34}}{2} - 1\big) \quad \text{and} \quad p^{\text{int}} \cdot w^{\text{int}} = \frac{n_{34}}{\alpha'}\,, \tag{5.51}$$

respectively. This corresponds to an asymmetrically excited state with the difference between the number of right- and left-excitations being n_{34}. However, the condition (5.34) for modular invariance forbids asymmetrically excited states. Since the same argument holds for the t- and u-channel, consistency of the poles with the physical spectrum requires $n_{ij} = 0$. This is nothing else than the *strong constraint* (in momentum space)

$$p_i \cdot w_j + p_j \cdot w_i = 0 \quad \forall i, j\,. \tag{5.52}$$

Indeed, defining the functions as $f_i(x, \tilde{x}) = \exp(ip_i \cdot x + iw_i \cdot \tilde{x})$, the relation (5.52) translates into

$$\partial_a f_i\, \tilde{\partial}^a f_j + \tilde{\partial}^a f_i\, \partial_a f_j = 0 \tag{5.53}$$

which is the strong constraint (1.2) of DFT [92, 142, 94].

To summarize, while modular invariance of the partition function determined the physical spectrum, consistency with the pole structure of the Virasoro-Shapiro amplitude allowed to derive the strong constraint. In terms of left- and right-moving momenta $K_i = (k_{Li}, k_{Ri})^t$, the strong constraint (5.52) reads $\langle K_i, K_j \rangle_d = 0\ \forall i, j$. Combining it with $k_{R_i} = \mathcal{M}_i k_{Li}$ (5.34), one obtains the joint condition

$$k_{Li}{}^t \big(\mathbb{1} - \mathcal{M}_i^t \mathcal{M}_j\big) k_{Lj} = 0 \tag{5.54}$$

which for fixed i, j must hold for all left-moving momenta. This implies $\mathcal{M}_i = \mathcal{M}_j$ for all i, j so that both constraints can be summarized by the consistency condition

$$k_{Ri} = \mathcal{M}\, k_{Li} \quad \text{with } \mathcal{M} \in O(d)\ \forall i\,. \tag{5.55}$$

This means that the solution to the strong constraint is chosen independently of the concrete functions f, g in (1.2).

5.5 Constraints from torus compactifications

In the previous discussion momentum and winding were continuous. This is different in the presence of compact directions. The purpose of this section is to show that the strong constraint (5.52) relaxes for toroidal compactifications. Hints for this expectation come from Scherk-Schwarz reductions of DFT [99]. Hence in the following the analysis from the previous two sections is repeated for the case of $k < d$ compact directions.

Torus compactification

A general compactifications on a k-dimensional torus $T^k = \mathbb{R}^k/2\pi\Lambda_k$ with Λ_k a k-dimensional lattice is considered. Since the coordinates X^a and $\widetilde{X}^a$ are independent, they can be compactified on different tori T^k and $\widetilde{T}^k$. With indices $I, J, \dots$ indicating the internal directions, the coordinates X^I and $\widetilde{X}^I$ acquire new boundary conditions

$$\begin{aligned} X^I(e^{-2\pi i}z, e^{2\pi i}\bar{z}) &= X^I(z,\bar{z}) + 2\pi\sqrt{\alpha'}\, t^I\,, \\ \widetilde{X}^I(e^{-2\pi i}z, e^{2\pi i}\bar{z}) &= \widetilde{X}^I(z,\bar{z}) + 2\pi\sqrt{\alpha'}\, \tilde{t}^I\,, \end{aligned} \tag{5.56}$$

with t^I and $\tilde{t}^I$ vector fields on the internal tori, i.e. $t \in \Lambda_k$ and $\tilde{t} \in \widetilde{\Lambda}_k$ lattice vectors. The factors $\sqrt{\alpha'}$ are introduced for convenience.[6] Using the mode expansion (5.3), in order to satisfy the boundary conditions, the internal winding and momentum are $w^I = \frac{1}{\sqrt{\alpha'}}\, t^I$ and $p^I = \frac{1}{\sqrt{\alpha'}}\, \tilde{t}^I$. Then the basic vertex operator (5.17) is of the form

$$V^{\mathrm{c}}_{p,w}(z,\bar{z}) =: e^{ip_\mu X^\mu}\, e^{\frac{i}{\sqrt{\alpha'}}\tilde{t}_I X^I}\, e^{iw_\mu \widetilde{X}^\mu}\, e^{\frac{i}{\sqrt{\alpha'}} t_I \widetilde{X}^I} : \,. \tag{5.57}$$

Small Greek indices $\mu, \nu, \dots$ now denote the external coordinates. The physical state condition for (5.57) can be deduced from the conformal weight (5.18) as before; it reads $p^\mu w_\mu = -\frac{1}{\alpha'}\, t_I \tilde{t}^I$. For the $V^{\mathrm{c}}_{p,w} V^{\mathrm{c}}_{p',w'}$-OPE to be single-valued $t^I \tilde{t}'_I + \tilde{t}^I t'_I \in \mathbb{Z}$ is needed. Hence the tori are not independent but their lattices are contained in each others dual lattices.

It is convenient to introduce the lattice vectors $t^I_L = \frac{1}{\sqrt{2}}(\tilde{t}^I + t^I)$ and $t^I_R = \frac{1}{\sqrt{2}}(\tilde{t}^I - t^I)$ as well as the bilinear form $\langle\cdot,\cdot\rangle_k$ defined by $\mathrm{diag}(\mathbb{1}_k, -\mathbb{1}_k)$. With the $2k$-dimensional vector $L = (t_L, t_R)^t$ the above condition for single-valuedness becomes $\langle L, L'\rangle_k \in \mathbb{Z}$. Denoting the lattice spanned by the L's as Γ_{2k}, this means $\Gamma_{2k} \subset \Gamma^*_{2k}$, i.e. the lattice is integral. Further restrictions on the lattice Γ_{2k} will arise from the partition function.

The one-loop partition function

The partition function can be evaluated as before. The only difference is the zero-mode contribution from the internal momenta and windings. Using (5.35) and (5.25) for the internal part one obtains

$$Z_{\mathrm{c}}(\tau,\bar{\tau}) = \frac{V_{d-k}/2}{\left(2\pi\sqrt{\alpha'}\right)^{d-k}} \frac{1}{\mathrm{Im}(\tau)^{\frac{d-k}{2}}\, |\eta(\tau)|^{2d}} \sum_{(t_L,t_R)\in\Gamma_{2k}} e^{i\pi\, t_L^2\, \tau}\, e^{-i\pi\, t_R^2\, \bar{\tau}}\,. \tag{5.58}$$

[6] To make the conventions clear, note that for a circle they are such that the radius comes with a factor $\sqrt{\alpha'}$. Then the internal momentum comes with $\sqrt{\alpha'}$ and internal winding with the inverse.

Under a modular T-transformation, all but the last term is invariant. Thus the lattice vectors have to satisfy $\langle L, L\rangle_k \in 2\mathbb{Z}$. This means that T-invariance implies that Γ_{2k} has to be an even lattice. Moreover, using Poisson resummation twice, the partition function is shown to be invariant under a modular S-transformations if $\Gamma_{2k} = \Gamma^*_{2k}$. This reproduces the well known result that modular invariance requires the lattice Γ_{2k} to be even and self-dual [32]. No additional constraints have to be put on the internal sector.

Moreover, the external momenta still have to satisfy the condition (5.34), i.e. $k_R^\mu = \mathcal{M}^\mu{}_\nu\, k_L^\nu$ for $\mathcal{M} \in O(d-k)$. Thus, the physical spectrum in the internal sector is less constrained compared to the non-compact case. In particular, asymmetrically excited states are allowed.

Pole structure

Again the scattering of four vertex operators (5.57) is examined. The only difference to the analysis in section 5.4 is that the contractions of momenta and windings split into separate contractions of external and internal momenta and windings. The n^{th} pole in the s-channel seen from the external point of view is

$$s^e = \frac{4}{\alpha'}\left[n + \frac{1}{2}(t_{L3} + t_{L4})^2 - 1\right] \tag{5.59}$$

and the difference between the external left- and right-movers is $\mathfrak{s}^e - s^e = \frac{4}{\alpha'}(n_{34} - \frac{1}{2}\langle L_3 + L_4, L_3 + L_4\rangle_k)$. Splitting n_{34} and using level matching allows to write this difference as

$$\mathfrak{s}^e - s^e = \langle K_3^e + K_4^e, K_3^e + K_4^e\rangle_{d-k} \tag{5.60}$$

with $K_i^e = ((k_L^\mu), (k_R^\mu))^t$ collecting the external momenta. As before, the pole corresponds to an asymmetrically excited state. However, the external part still has to satisfy the condition (5.34) for modular invariance, i.e. (5.60) has to vanish. This implies $\langle K_i^e, K_j^e\rangle_{d-k} = 0$, which is equivalent to (5.52). Then the difference between left- and right-excitations of the intermediate states is $\langle L_3, L_4\rangle_k$. As asymmetric excitations are valid, this describes a physical state. Therefore, the strong constraint still applies to the external directions whereas no further constraint arises for the internal momenta and windings.

5.6 The low energy effective theory and DFT

The dual coordinates $\widetilde{X}$ (5.8) can be interpreted as the pullback of spacetime coordinates $\tilde{x}$ analogous to $X^a = X^* x^a$. The underlying spacetime is therefore doubled, i.e. has dimension $2d$. This engenders unphysical degrees of freedom which are reduced by the constraint (5.55) or the strong constraint (5.52). A proposal for a manifestly $O(d,d)$-invariant spacetime is provided by DFT, which will be connected to the approach pursued above in the following.

In this section the scattering amplitude of three massless states represented by the vertex operators (5.23) is rederived [82]. The result is compared to the action of double field theory [94] by expanding the latter into third order in fluctuations – they match. This computation is meant to provide evidence for the relevance of this T-duality invariant CFT for DFT.

5.6.1 3-Graviton scattering from CFT

Calculating an N-point function of insertions of graviton vertex operators $\mathcal{E}_{p,w}(z,\bar{z})$ (5.23) is combinatorially more involved than a tachyon amplitude. For taking care of that one conveniently defines

$$\mathcal{V}_i(z_i,\bar{z}_i) = :e^{\kappa_i\cdot\partial X(z_i)-\lambda_i\cdot\bar{\partial}\widetilde{X}(\bar{z}_i)}e^{ip_i\cdot X(z_i,\bar{z}_i)}\,e^{iw_i\cdot\widetilde{X}(z_i,\bar{z}_i)}: \tag{5.61}$$

with κ_i and λ_i auxiliary parameters. One can derive the vertex operators corresponding to the first excited states simply by acting on (5.61) with derivatives with respect to both κ_i and λ_i. This operator is related to a massless graviton vertex operator $\mathcal{E}_{p_i,w_i}$ by

$$\mathcal{E}_{p_i,w_i}(z_i,\bar{z}_i) = E_{iab}\frac{\partial}{\partial\kappa_{ia}}\frac{\partial}{\partial\lambda_{ib}}\mathcal{V}_i\Big|_{\kappa_i=\lambda_i=0}. \tag{5.62}$$

The N point correlation function can be written as

$$\begin{aligned}\langle\prod_{i=1}^{N}\mathcal{V}_i(z_i,\bar{z}_i)\rangle = &\prod_{1\le i<j\le N}|z_i-z_j|^{\alpha'(p_i\cdot p_j+w_i\cdot w_j)}\left(\frac{z_i-z_j}{\bar{z}_i-\bar{z}_j}\right)^{\frac{\alpha'}{2}(p_i\cdot w_j+w_i\cdot p_j)}\\ &\times F_{ij}(z_{ij},\bar{z}_{ij})\,\delta\Big(\sum p_i\Big)\,\delta\Big(\sum w_i\Big)\end{aligned} \tag{5.63}$$

with

$$\begin{aligned}F_{ij}(z_{ij},\bar{z}_{ij}) = \exp\bigg(-\frac{\alpha'}{2}\Big[&\frac{\kappa_i\cdot\kappa_j}{(z_i-z_j)^2}+2i\frac{(p_{[i|}+w_{[i|})\cdot\kappa_{|j]}}{z_i-z_j}\\ &+\frac{\lambda_i\cdot\lambda_j}{(\bar{z}_i-\bar{z}_j)^2}+2i\frac{(p_{[i|}-w_{[i|})\cdot\lambda_{|j]}}{\bar{z}_i-\bar{z}_j}\Big]\bigg).\end{aligned} \tag{5.64}$$

The full 3-graviton amplitude is then given by

$$\begin{aligned}\mathcal{A}_3(p_i,w_i,E_i) &= g_c^3\,C_{S^2}\,\langle\prod_{i=1}^{3}(c\,\bar{c}\,\mathcal{E}_{p_i,w_i})\rangle\\ &=g_c^3\,C_{S^2}\,A(\vec{z},\vec{\bar{z}})\prod_{k=1}^{3}E_{kab}\frac{\partial}{\partial\kappa_{ka}}\frac{\partial}{\partial\lambda_{kb}}\prod_{1\le i<j\le 3}F_{ij}(z_{ij},\bar{z}_{ij})|_{\kappa_i=\lambda_i=0}\,,\end{aligned} \tag{5.65}$$

where $A(\vec{z},\vec{\bar{z}})$ collects the contractions of the remaining exponentials (5.36). Notice that the derivatives with respect to κ and the ones with respect to λ can be treated separately. Denoting $F(\vec{z},\vec{\bar{z}}) := \prod_{1\le i<j\le 3}F_{ij}(z_{ij},\bar{z}_{ij})$ and taking three derivatives with respect to κ, one finds

$$\begin{aligned}\prod_{k=1}^{3}\frac{\partial}{\partial\kappa_{ka}}\,F|_{\kappa_i=\lambda_i=0} = &\frac{\alpha'^2}{4}\frac{\eta^{ac}k_{1L}^b+\eta^{bc}k_{3L}^a+\eta^{ab}k_{2L}^c}{z_{12}z_{13}z_{23}}\\ &+\frac{\alpha'}{2}\left(\frac{k_{1L}^a}{z_{12}}-\frac{k_{3L}^a}{z_{23}}\right)\left(\frac{k_{2L}^b}{z_{12}}+\frac{k_{3L}^b}{z_{13}}\right)\left(\frac{k_{1L}^c}{z_{13}}+\frac{k_{2L}^c}{z_{23}}\right)\,,\end{aligned} \tag{5.66}$$

where momentum and winding conservation was used as well as the transverse polarization of E_{mn}. The λ-derivatives can be worked out analogously. The two parts can now be contracted with the corresponding polarization tensors of the massless vertex operators to get the full three-point amplitude. Restricting to second order in momentum and winding and imposing the correct normalization of the graviton vertex operator by including a factor of $\frac{2}{\alpha'}$ in each $\mathcal{E}$, the three-graviton scattering amplitude reads

$$\mathcal{A}_3(p_i, w_i, E_i) = 4\pi g_c\, E_{1ad}E_{2be}E_{3cf}t^{abc}\tilde{t}^{def} + \mathcal{O}(p^4, p^3w, \ldots, w^4)\;, \tag{5.67}$$

with

$$\begin{aligned} t^{abc} &= \eta^{ca}k^b_{1L} + \eta^{ba}k^c_{2L} + \eta^{cb}k^a_{3L}\,, \\ \tilde{t}^{abc} &= \eta^{ca}k^b_{1R} + \eta^{ba}k^c_{2R} + \eta^{cb}k^a_{3R}\,. \end{aligned} \tag{5.68}$$

Here $C_{S^2} = \frac{8\pi}{\alpha' g_c^2}$ was used which can be determined from unitarity by factorizing the four-point amplitude (5.49) over the tachyonic pole. This result was first presented in [82] and consistently reduces to the well-known three-graviton scattering amplitude [100] for vanishing B-field and zero winding.

5.6.2 3-point interaction from DFT

Double field theory formulated in terms of the background field $\mathcal{E}_{ab} = G_{ab} + B_{ab}$ and the dilaton field d is considered [94]:

$$\begin{aligned} S = \int d^dx\, d^d\tilde{x}\, e^{-2d}\Big[&-\tfrac{1}{4}G^{am}G^{bn}G^{pq}\mathcal{D}_p\mathcal{E}_{mn}\mathcal{D}_q\mathcal{E}_{ab} + \tfrac{1}{4}G^{mn}(\mathcal{D}^b\mathcal{E}_{am}\mathcal{D}^a\mathcal{E}_{bn} \\ &+\overline{\mathcal{D}}^b\mathcal{E}_{ma}\overline{\mathcal{D}}^a\mathcal{E}_{nb}) + (\mathcal{D}^a d\overline{\mathcal{D}}^b\mathcal{E}_{ab} + \overline{\mathcal{D}}^a d\mathcal{D}^b\mathcal{E}_{ab}) + 4\mathcal{D}^a d\mathcal{D}_a d\Big] \end{aligned} \tag{5.69}$$

with $\mathcal{D}_a = \partial_a - \mathcal{E}_{am}\tilde{\partial}^m$ and $\overline{\mathcal{D}}_a = \partial_a + \mathcal{E}_{ma}\tilde{\partial}^m$ as well as $\tilde{\partial}$ the derivative with respect to $\tilde{x}$. The inverse metric G^{ab} is used to raise indices and $2\kappa_d^2 = 1$ is set. To compare (5.69) with the CFT result (5.67), the background $\mathcal{E}$ is expanded around the flat Minkowski background as follows (see [144, 142]):

$$\mathcal{E}_{ab} = E_{ab} + f_{ab}(e,d)\;, \quad f_{ab}(e,d) = e_{ab} + \tfrac{1}{2}e_a{}^m e_{mb} + \mathcal{O}(e^3)\,. \tag{5.70}$$

Here E_{ab} denotes the constant background, which for vanishing B-field reduces to the Minkowski metric η_{ab} and e_{ab} denote the fluctuations around this background. It is important to take the higher-order fluctuation into account in the expansion of the different objects. Thus (5.69) is expanded up to cubic order in the fluctuation e_{ab} (see [94]). The metric G_{ab} is simply given by $G_{ab} = \frac{1}{2}(\mathcal{E}_{ab} + \mathcal{E}_{ba})$ and hence, for example, the expansion of the inverse metric takes the form

$$G^{ab} = \eta^{ab} - e^{(ab)} + \tfrac{1}{4}e^{am}e^b{}_m + \tfrac{1}{4}e^{ma}e_m{}^b + \mathcal{O}(e^3)\,. \tag{5.71}$$

Then, up to a total derivative, the action to cubic order in the fluctuation reads

$$\begin{aligned} S = \int dx\, d\tilde{x}\, \Big[&\tfrac{1}{4}e_{ab}\Box e^{ab} + \tfrac{1}{4}(D^a e_{ab})^2 + \tfrac{1}{4}(\overline{D}^b e_{ab})^2 - 2dD^a\overline{D}^b e_{ab} - 4d\Box d \\ &+\tfrac{1}{4}e_{ab}\Big((D^a e_{mn}(\overline{D}^b e^{mn}) - (D^a e_{mn})(\overline{D}^n e^{mb}) - (D^m e^{an})(\overline{D}^b e_{mn})\Big) \\ &+\tfrac{1}{2}d\big((D^a e_{ab})^2 + (\overline{D}^b e_{ab})^2 + \frac{1}{2}(D^m e_{ab})^2 + \tfrac{1}{2}(\overline{D}^m e_{ab})^2 \\ &+2e^{ab}(D_a D^m e_{mb} + \overline{D}_b\overline{D}^m e_{am})\big) + 4e_{ab}dD^a\overline{D}^b d + 4d^2\Box d\Big]\,, \end{aligned} \tag{5.72}$$

which was first derived in [142]. The derivatives are given by

$$\begin{aligned} D_a &= \partial_a - E_{am}\tilde{\partial}^m \,, \\ \overline{D}_a &= \partial_a + E_{ma}\tilde{\partial}^m \,, \end{aligned} \qquad \square = \tfrac{1}{2}(D^a D_a + \overline{D}^a \overline{D}_a) \,. \tag{5.73}$$

In order to compare with the three-point amplitude from the CFT side, the constant κ_d is introduced by modifying the fluctuation to $2\kappa_d e_{ab}$. In this way the match with the expansion of the standard Einstein-Hilbert action to third order in the metric fluctuation h_{ab} is obtained. Then, from the second line in (5.72) and after a partial integration, the interaction term for three e_{ab}'s is identified to be

$$\begin{aligned} &\kappa_d e_{ab}\Big((D^a e_{mn}(\overline{D}^b e^{mn}) - (D^a e_{mn})(\overline{D}^n e^{mb}) - (D^m e^{an})(\overline{D}^b e_{mn})\Big) \\ =& -\kappa_d e_{ab}\Big(e^{mn} D^a \overline{D}^b e_{mn} + (D^a e_{mn})(\overline{D}^n e^{mb}) + (D^m e^{an})(\overline{D}^b e_{mn})\Big) + (\text{tot. der.})\,. \end{aligned} \tag{5.74}$$

The missing term from the partial integration vanishes because of $D^m e_{ma} = 0$, following from the polarization constraint as listed in table 5.1. Next the value of the three-graviton vertex in momentum space is read off by using $\partial_a \to ip_a$ and $\tilde{\partial}^a \to iw^a$, which translates derivatives to momenta and winding modes. Keeping track of possible permutations results in

$$\begin{aligned} A^{eee} = 4\pi g_c\Big(& k^a_{3R}\, e_{1ab}\, k^b_{3L}\, e^{mn}_2\, e_{3mn} + k^a_{3R}\, e_{1ab}\, e^{mb}_2\, e_{3mn}\, k^n_{2L} + k^m_{3R}\, e_{1mn}\, e^{an}_3\, e_{2ab}\, k^b_{1L} \\ & + (\text{cyclic permutations})\Big)\,, \end{aligned} \tag{5.75}$$

where $g_c = \frac{\kappa_d}{2\pi}$. This result matches with the string scattering amplitude (5.67). The difference in the left- and right-moving momenta can be cured by switching the sign of the B-field. Therefore, at least up to second order in derivatives the action (5.69) serves as effective theory for the massless sector of the T-duality invariant CFT.

5.7 Summary and discussion

In this chapter a T-duality symmetric CFT was analyzed whose tree-level string scattering amplitudes at the two-derivative level are described by DFT. From studying one-loop modular invariance and the pole structure of the four tachyon amplitude it was deduced that the strong constraint (5.52) must be imposed in all the *non-compact* directions, whereas *compact* toroidal directions are not subject to any further constraint beyond those following from modular invariance.

These observations are in agreement with the possibility of relaxing the strong constraint on the internal space in Scherk-Schwarz compactifications [99], in light of which the torus is a special case. The additional constraints found there apply to the possible fluxes, saying that they are constant and subject to quadratic constraints. Since fluxes are absent in the torus compactifications studied here, no constraints are expected.

It is important to classify the findings of this section. First of all, the T-duality invariant CFT can still be considered as a string theory. Although it goes beyond the closed bosonic string and does not have an obvious sigma-model description, it is still a two-dimensional conformal field theory. This might be compared to (asymmetric) orbifold conformal field theories, which are legitimate string theory constructions. They also lack a clear spacetime interpretation by their asymmetric treatment of left- and right-movers similar to the T-duality invariant CFT. Thus, as a string theory it makes sense to consider the T-duality invariant CFT on the two-torus as one-loop contribution to the perturbative expansion of the theory. Therefore modular invariance of the torus partition function is necessary. This also justifies the construction of full-fledged string theory scattering amplitudes from the CFT correlation functions performed in section 5.4. In particular, the premise of having physical intermediate states in the scattering of four tachyon states is valid. Thus the final constraint (5.55) summarizing the relation between left- and right-movers (5.34) and the strong constraint (5.52) is a genuine prediction in string theory.

The perspective of T-duality invariant CFT being a valid string theory might also shed some light on the interpretation of double field theory. As being coincident at least to lowest orders with the effective theory for massless states in the T-duality invariant CFT, it is indeed a string theory. This is supported by the importance of the constraints in DFT, which lead to the reduction to string theory effective actions when applied directly. The origin of DFT also has to be kept in mind: it arose in particular from string field theory [142]. Thus DFT might not go beyond string theory but serves as an efficient tool to describe target-space dualities. In particular, the T-duality invariant CFT can be used to study higher order correction to the DFT action.

Chapter 6

Conclusion and Outlook

The thesis closes with two final thoughts.

Geometry of duality. The aspects of target-space dualities explored in this thesis illustrate their rich geometrical structure. Yet, the appropriate language for a unified description of dualities is still missing. In principle, generalized geometry provides a versatile approach for incorporating symmetries. For example, *exceptional generalized geometry* [145, 146] is based on a bundle whose structure group comprises the exceptional groups of the E-series and is in particular applied to M-theory. Nevertheless, T-duality, or the more general $O(d,d)$-duality, requires a Courant bracket with automorphism group $O(d,d)$. The problem is the simultaneous incorporation of exact B- and β-transformations. In section 2.1 and section 2.1.2 the generalized tangent and cotangent bundle with their associated Courant brackets (see section 2.2) have been introduced. The former describes diffeomorphisms together with B-transformations while the latter describes diffeomorphisms together with β-transformations. Moreover, the automorphisms of the Courant bracket introduced in proposition 4.1 are diffeomorphisms and a mixture of B- and β-transformations. The three examples hint towards the capability of only describing "half" of B- plus β-transformations. This is supported by a naive consideration of the infinitesimal generators: diffeomorphisms are generated by vector fields, exact B-transformations by one-forms and exact β-transformations again by vector fields. Hence, the local form of the bundle is expected to be $TM \oplus T^*M \oplus TM$ which goes beyond the generalized tangent bundle. In contrast, double field theory assumes the generalized tangent bundle of an extended spacetime manifold, yet consistency always requires the implementation of constraints. As was shown in detail in section 2.2.2, even if such a generalized structure is found, gravity theories in the conventional sense are inaccessible due to the seeming absence of an endomorphism-valued curvature tensor. This stems from the anomalous properties of the Courant bracket: the Jacobi identity and the Leibniz rule admit defects. A framework for treating such defects in a systematic manner is given by *strongly-homotopy algebras* or *A_∞-structures* [147]. As they also appear in the formulation of (closed) string field theory [30], a pursuit in this direction might be helpful. In total, finding a geometry appropriate for describing duality remains an open problem.

Duality and non-commutative geometry. The scope of $O(d,d)$-duality requires further investigation. Apart from open questions concerning aspects of this duality in the quantum theory, it might give new insights into non-commutative geometry in closed string theory. It has been argued to be related to non-geometric backgrounds. However, since T-duality is a canonical transformation which does not change the classical Poisson structure, it is impossible to obtain non-canonical Poisson structures from canonical ones by its application. In $O(d,d)$-duality this is even more apparent as duality leaves the classical Hamiltonian density invariant which especially preserves the phase space. Nevertheless, global effects through winding might be responsible for the occurrence of non-commutativity.

For instance, the derivation of the equations of motion (3.2) was performed under the assumption of the periodicity of the closed string in order to remove total derivative terms. But if the variation δX^a is only constraint to vanish at infinite time the term

$$\int_{-\infty}^{\infty} d\tau \big[W_a\, \delta X^a\big]_{\sigma=0}^{\sigma=2\pi}\,, \tag{6.1}$$

with $W_a = \frac{1}{2\pi\alpha'}(G_{ab}\partial_\sigma X^b - B_{ab}\partial_\tau X^b)$ the canonical winding of the string (3.1), remains. Its vanishing might result in boundary conditions reminiscent of Dirichlet boundary conditions in open string theory. For example, constraining the canonical winding to vanish at 0 and 2π gives rise to the boundary condition

$$\big(\partial_\sigma X^a - G^{am} B_{mn} \partial_\tau X^n\big)\big|_{\sigma\in\{0,2\pi\}} = 0\,. \tag{6.2}$$

This is analogous to the open string case [22] and would cause a non-vanishing equal time commutator $[X^a, X^b]\big|_{\sigma=0,2\pi}$ as long as $B \neq 0$. Here $O(d,d)$-duality only comes into play due to its ability of generating a B-field from backgrounds lacking it. But the implementation of such a boundary condition is unreasonable for constant winding and if $\delta X^a(\sigma=0) = \delta X^a(\sigma=2\pi)$ is assumed.

$O(d,d)$-duality or more specifically Poisson duality potentially gives rise to non-vanishing commutators between the ordinary and the dual coordinate in a doubled approach. Using the dual coordinates (5.8) arising from the Poisson duality induced by the Poisson vector β and splitting the propagator (5.4) into its holomorphic and anti-holomorphic part yields

$$\big\langle \widetilde{X}^a(z_1,\bar z_1), X^b(z_2,\bar z_2)\big\rangle = -\frac{\alpha'}{2}\left(G^{ab}\,\ln|z_{12}|^2 - \beta^{ab}\,\ln\frac{z_{12}}{\bar z_{12}}\right)\,. \tag{6.3}$$

The equal time commutator can be obtained in a barely rigorous manner from this propagator as follows [23]: Radial ordering – which corresponds to time ordering in the coordinates $z = \exp(t - i\sigma)$ – is implicit in the expression above. Keeping track of it and writing $z_i = r_i e^{-i\sigma_i}$, the equal time commutator with $r_1 = r_2$ becomes

$$\begin{aligned}\big\langle \big[\widetilde{X}^a(r,\sigma_1), X^b(r,\sigma_2)\big]\ldots\big\rangle &= \lim_{\delta\to 0}\big\langle \widetilde{X}^a(r+\delta,\sigma_1)X^b(r,\sigma_2) - X^b(r+\delta,\sigma_2)\widetilde{X}^a(r,\sigma_1)\big\rangle \\ &= \alpha'\,\beta^{ab}\,\ln\frac{e^{-i\sigma_1}-e^{-i\sigma_2}}{e^{i\sigma_1}-e^{i\sigma_2}}\,.\end{aligned} \tag{6.4}$$

For $r_1 = r_2 = r$ the difference between the worldsheet points is $z_{12} = 2r\sin[(\sigma_2 - \sigma_1)/2]\exp[i(\pi - \sigma_1 - \sigma_2)/2]$. Then, omitting the path integral, the commutator becomes

$$\big[\widetilde{X}^a(t,\sigma_1), X^b(t,\sigma_2)\big] = i\,\beta^{ab}(\pi - \sigma_1 - \sigma_2)\,. \tag{6.5}$$

Thus the equal time commutator of a Poisson-dual coordinate with an ordinary one is proportional to the Poisson structure. Unfortunately, a worldsheet dependence remains which makes the commutator ill-defined on the spacetime. A potential remaining worldsheet dependence of equal time commutators of closed string coordinates was also discussed in [24]. Comparison with the open string case in [22] shows that the same world-sheet dependence appears, but it is canceled by boundary terms leaving a proper target-space quantity. This hints at the inclusion of global boundary condition – possibly as discussed above – in order for this to be consistent.

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